A Survey of Knot Theory

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Preface

This book is an expanded English version of "Knot Theory" which I edited and which was published in the original Japanese by Springer-Verlag Tokyo in 1990. This version covers many research methods and results of knot theory in classical dimensions which were developed before 1995. The purpose is to inform advanced undergraduates and graduate students in mathematics as well as researchers in other disciplines about what knot theory is and how to study it. In addition, I hope that some parts of this book can be read by less advanced undergraduates, and that other parts will be useful to knot theorists as a reference. Since the study of knot theory is now undergoing rapid progress and uses many areas of modern mathematics (as seen in this book), I thought that it was a good idea to have several knot theorists quickly develop a book which surveys the entire scope of knot theory. Thus, in preparing the Japanese version, I asked my colleagues in the KOOK Seminar to write basic materials on the subject that I had selected (the KOOK seminar is a seminar on geometric topology organized by members of Kobe University, Osaka University, Osaka City University, and Kwansei Gakuin University that has been held monthly for the last ten years). In making this expanded English version, my colleagues in the KOOK seminar also helped me in translating some parts of the book into English. Here is a list of the contributors and their roles, where * denotes a contribution to the English translation:

Dr. Hiroshi Goda: Chapter 4*
Dr. Toshio Harikae: Chapter 15*
Dr. Daniel J. Heath: English advisor*
Dr. Fujitsugu Hosokawa: Prelude
Dr. Seiichi Kamada: Chapter 14*, References, References*
Dr. Taizo Kanenobu: Chapter 2, Chapter 2*
Dr. Shin'ichi Kinoshita: Chapter 15
Dr. Masako Kobayashi: Appendix C*
Dr. Tsuyoshi Kobayashi: Chapter 4, Chapter 9, Chapter 9*
Dr. Toru Maeda: Chapter 6
Dr. Yoshihiko Marumoto: Chapter 13, Chapter 13*
Dr. Yasuyuki Miyazawa: Chapter 11*
Dr. Kanji Morimoto: Appendix C
Dr. Hitoshi Murakami: Chapter 8, Chapter 11
Dr. Jun Murakami: Chapter 9
Dr. Yasutaka Nakanishi: Chapter 3, Chapter 3*, Appendix F
Dr. Makoto Sakuma: Chapter 7, Chapter 7*, Chapter 10, Chapter 10*, Appendix F, Appendix F*
Dr. Tetsuo Shibuya: Chapter 13
Dr. Junzo Tao: Appendix B
Dr. Masakazu Teragaito: Chapter 8*

Dr. Yoshiaki Uchida: Chapter B*
Dr. Shuji Yamada: Chapter 1, Chapter 1*
Dr. Katsuyuki Yoshikawa: Chapter 14

All of the figures in the book were illustrated by Dr. Yasutaka Nakanishi. Since I revised the contents of most of the chapters extensively, I bear sole responsibility for the accuracy of the contents.

In the final stages of development, Dr. Taizo Kanenobu, Dr. Yasutaka Nakanishi, and Dr. Makoto Sakuma kindly checked the contents of the book.

Hirozumi Fujii, Shin'ichi Sugihara, and Makoto Tamura, who are doctoral students at Osaka City University and Osaka University, helped me greatly in preparing the references. The graduate students Teruhisa Kadokami, Yoshihiko Tsujii, Shigeru Nagamatsu, and Makoto Soma, and a research associate, Naoko Kamada, helped me in editing the book. Also, Etsuko Miyahara, Miho Sakuma, Masae Shiomi, Tatsuyuki Shiomi, and Hiroshi Yokota helped me in various ways in making this book.

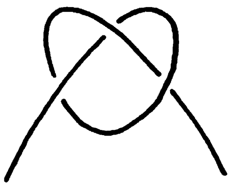
Dr. John Dean of the University of Texas at Austin made many valuable linguistic improvements.

I would like to thank all of them for their invaluable contributions.

May 8, 1996
Akio Kawauchi

A prelude to the study of knot theory

When we think of a knot, we imagine a string as shown in figure 1; we do not imagine an untied string as in figure 2. From the topological viewpoint, however, these strings are the same. This is because it depends on the mathematical viewpoint we adopt whether or not the pictures are the same one.



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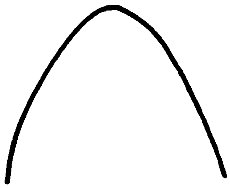

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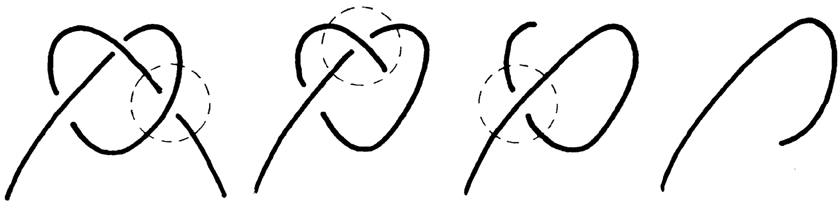
In planar geometry, we consider the pictures of figures 3 and 4 to be distinct. In this case, the mathematical viewpoint we have adopted is that two pictures in the plane are regarded as the same when they are congruent, i.e., when one can be transformed into the other by a congruence transformation of the plane. To see that the pictures of figures 3 and 4 are distinct, we use the property that length and angle are invariant under any congruence transformation. In figure 3 the angle at any point except the end points is 180°. In figure 4 there is a point whose angle is not 180°, so we can conclude that they are distinct.

3

4

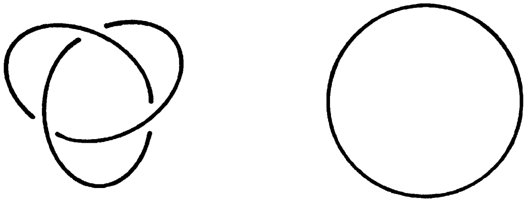
The topological viewpoint is one in which two objects in Euclidean 3-space $\mathbf{R}^3$, such as the strings in figures 1 and 2, are regarded as the same when one can be sent to the other by an auto-homeomorphism h of $\mathbf{R}^3$. There is a restricted version of this viewpoint which imposes on h the condition that h should preserve an orientation of $\mathbf{R}^3$ or that it should preserve the orientations of both $\mathbf{R}^3$ and the objects (when the objects are oriented). Adopting this restricted viewpoint, we can develop a different mathematical theory. For example, in planar geometry, a reflection (in a line) reverses the orientation of the plane and hence is different from a parallel translation or a rotation, which are orientation preserving. In planar geometry, the requirement that congruences preserve orientation corresponds to whether or not reflections are included in our congruence transformations. That the strings of figures 1 and 2 are the same from the topological viewpoint is shown by the illustrations in figures 5–8. Figure 5 is deformed into figure 6 by an auto-homeomorphism of $\mathbf{R}^3$ which contracts an arc neighborhood of the right endpoint of the string. The deformations are similar in figures 6, 7 and 8.

Now let us consider the embedded circles in figures 9 and 10 obtained from figures 1 and 2 respectively by joining their endpoints together. The two embedded circles do not appear to be the same from the topological viewpoint. As a matter of fact, it can be shown that they are distinct.



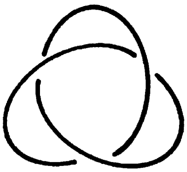
5 6 7 8

By a *knot*, we will mean a circle embedded in $\mathbf{R}^3$ (or in S^3) such as in figures 9 and 10. Knot theory is, in a sense, the study of how to determine whether or not two given knots are the same. In Chapter 0, the precise definition of a knot and related basic concepts are stated. We used the notion of angle in order to show that the pictures in figures 3 and 4 are distinct in planar geometry. Angle and length are numbers which are invariant under congruence transformations. Similarly, in knot theory, in order to distinguish two knots, we find and compare a number (or more generally an algebraic system) which is invariant under auto-homeomorphisms of $\mathbf{R}^3$ (or S^3). Such a number or algebraic system is called a *knot invariant.* Knot invariants play an important role in knot theory. In planar geometry, we know three necessary and sufficient conditions for two triangles to be congruent, which are stated in terms of the invariants of congruence transformations, namely, the angle and the length. The problem of finding a necessary and sufficient condition for two knots to be the same in terms of computable knot invariants is not yet solved, however. This is one reason why many researchers pay attention to knot theory. In addition, there is the fact that knot theory has recently come to have applications to other areas of science as well as to mathematics.



9 10

When we consider a knot in $\mathbf{R}^3$, the first problem is how to describe the knot on paper. When we see an object in $\mathbf{R}^3$ with our eyes, we project $\mathbf{R}^3$ radially from a point into the sight plane, and use the difference in view between our two eyes to judge far and near. In drawing, the perspective method of describing an object in $\mathbf{R}^3$, the perspective representation is well-known. The method of describing a knot in the plane which we will use is called a *regular presentation.* It is similar to the perspective representation except that our eye is placed at the point at infinity in $\mathbf{R}^3$ and we use as the projection an orthogonal projection such that no two segments overlap and three or more segments do not meet at one point. Further, in the regular presentation, the upper-lower relation at every crossing point of two segments is marked, as can be seen in figure 9. However, if we change the direction of orthogonal projection or deform the knot itself by an auto-homeomorphism of $\mathbf{R}^3$, then the regular presentation may change so as to appear to be a regular presentation of a distinct knot.



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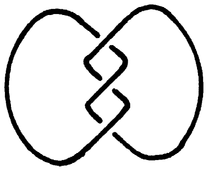

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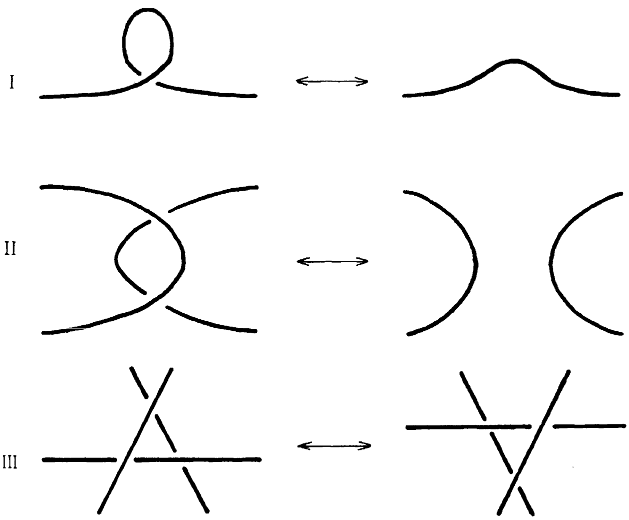
For example, the knots of figures 11 and 12 appear to represent distinct knots though they are the same. For any two regularly presented knots to be the same, it is known as a necessary and sufficient condition that one can be deformed into the other by a finite number of three kinds of moves, called the *Reidemeister moves.* The notion of knot presentations as well as this argument are discussed in Chapter 1. In Appendix A, we show that several notions of knot equivalence are the same.

Once we know about knot presentations, several knot invariants come to mind. For example, the minimal crossing number of all regular presentations of a knot, the *crossing number* of the knot, is the easiest understandable invariant, though the determination of the crossing number of a given knot is not easy. A list of knots with up to 10 crossings was known by R. H. Fox and the knots with up to 9 crossings were known earlier. Nowadays, a nearly complete list of the knots with up to 13 crossings is known. In Appendix F, we list the knots with up to 10 crossings together with the data which are now known. The knot in figure 11 is a knot with 3 crossings. There are just two knots with crossing number 3, the other knot of which is its mirror image (the image by a reflection of $\mathbf{R}^3$ in a plane). Since reflections are orientation-reversing auto-homeomorphisms of $\mathbf{R}^3$, the mathematical viewpoint we have adopted determines whether or not we regard two knots which are related by a mirror reflection as the same knot. The question of whether or not the knot in figure 11 and its mirror image can be transformed into

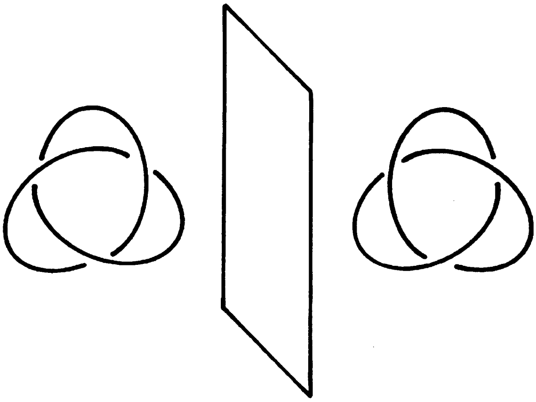


Reidemeister moves

one another by an orientation-preserving auto-homeomorphism of $\mathbf{R}^3$ had been a very difficult question until it was answered negatively by M. Dehn around 1930. The knot tables in the appendix consider two knots related by a mirror reflection to be the same. Thus, we list only one knot with crossing number 3.

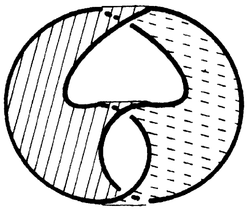
We discuss in Chapter 2 standard examples of knots appearing very often in knot theory, and in Chapter 3 basic methods of construction and decomposition. One of the classically known knot invariants is the *Alexander polynomial.* This is a polynomial derived from the fundamental group of the complement $\mathbf{R}^3 - K$ of a knot K in $\mathbf{R}^3$. For example, the Alexander polynomial of the knot of figure 9 is $t^2 - t + 1$ and the Alexander polynomial of the knot in figure 10 is 1, so that we can conclude that these knots are distinct. The fundamental group of the complement of a knot in $\mathbf{R}^3$ has group-theoretically interesting structures. This is discussed in Chapter 6. The book by Crowell and Fox [1963] contains an excellent account of a method of how to compute the Alexander polynomial from the fundamental group of the knot complement. In Chapters 5 and 7 the calculation of the Alexander polynomial using covering space theory is discussed.

In 1984, a new polynomial invariant, called the *Jones polynomial* was discovered by V. F. R. Jones. It is defined by using a braid presentation of a knot, discussed in Chapter 1, and then by analyzing when two braids represent the same knot in the Hecke algebra for the braid group. Prior to the appearance of Jones polynomial, J. H. Conway modified the Alexander polynomial into a new polynomial, called the *Conway polynomial* with a geometric identity formula, called a *skein relation.* Using the fact that the Jones polynomial has a similar skein relation, several related Laurent polynomial invariants have since been discovered.



The mirror image

These are discussed in Chapters 8 and 9. We also discuss in Chapter 16 an algebra of certain knot invariants, which were invented by V. A. Vassiliev and include in a sense all of these polynomial invariants.



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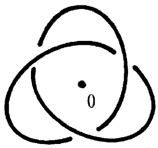
It is obvious that the knot of figure 10 is untied. Mathematically, we can say that it is the boundary of a disk. Such a knot is called a *trivial knot* or an *unknot.* In general, it is known that any knot is the boundary of a surface, so that a non-trivial knot is the boundary of a surface which is not a disk. There are many interesting questions about what properties these surfaces have, how we can determine them, what knot invariants we can derive from them, etc. For example, the surface of figure 13 is such a surface for the knot of figure 12 and is seen to be different from a disk. In fact, this surface is a compact surface of genus 1 with connected boundary. Such a compact orientable surface bounded by a knot is called a *Seifert surface* for the knot. Seifert surfaces for a given knot are not uniquely determined, but we can consider the minimal genus of such Seifert surfaces, which is a knot invariant, called the *genus* of the knot. The genus of a trivial knot is 0, but we know that the genera of the knots in figures 11 and 12 (which are the same) is 1. In this way, we can find several properties of knots by



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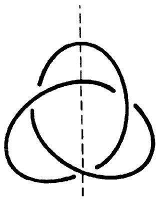

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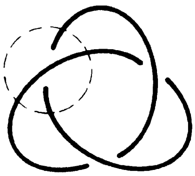
investigating the topological and algebraic aspects of Seifert surfaces. These ideas are discussed in Chapters 4 and 5.

A knot can be represented in various forms. Although it depends on our feelings whether or not such a form is beautiful, a sense of balance or symmetry is one reason why we may feel it to be beautiful. From such a sense of symmetry of a knot, we can derive a feature of the knot. For example, if we move the knot of figure 11 by the 120° rotation around the point 0 shown in figure 14, then it is unchanged. Similarly, if we move it by the 180° rotation around the dotted line as shown in figure 15, then it again remains unchanged. The mathematical theory that uses this idea is developed in Chapter 10.



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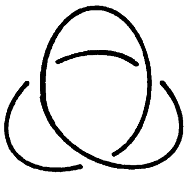

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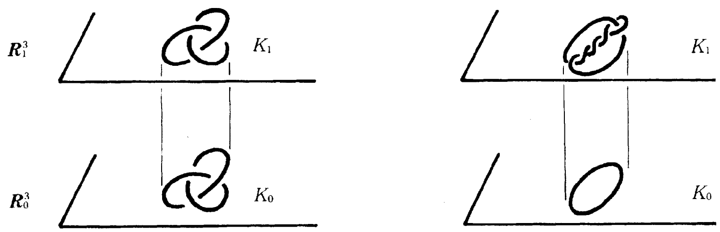
There is an old problem asking the difference in complexity between a given knot and the trivial knot. For example, when we change the upper-lower relation at the crossing point of the knot of figure 11 encircled by the dotted circle in figure 16, we obtain a trivial knot, shown in figure 17. In other words, the knot of figure 11 is not trivial, as mentioned before, but we can obtain a trivial knot by changing the upper-lower relation at one crossing point of the knot. Then we may have the question of how many such crossing changes are needed to obtain a trivial knot from a given knot. Not only are there many (clever or unclever) methods of finding the places to make crossing changes, but the places we can make crossing changes depend on the regular presentation of the knot we are working with. Thus, we see that this question is not simple. However, we can consider, as a knot invariant, the minimum of the numbers of such places for all possible regular presentations of the knot. This number is called the *unknotting number* of the knot. The unknotting number of the knot of figure 11 is not 0 since it isn't trivial, so it must be 1 since, as mentioned before, a crossing change at one place makes the knot trivial. The unknotting number may seem to be computable for any knot, but the computation is actually very difficult. This problem is discussed in Chapter 11.

A knot itself is a circle, which is a 1-dimensional closed manifold. The only 1-dimensional compact connected manifolds are arcs and circles. Considering the possible ways to embed a 1-dimensional manifold into $\mathbf{R}^3$, we see that the embedding is unique for an arc, but not unique for a circle; from this fact knot theory emerges as a mathematical problem. One generalization of knot theory is the study of embeddings of an m-dimensional manifold into an n-dimensional manifold, namely, how an m-dimensional manifold can be situated in an n-dimensional manifold, for positive integers m, n with $m < n$. In this sense, knot theory might also be called *situation analysis*. For example, we consider two parallel Euclidean 3-spaces $\mathbf{R}_0^3$ and $\mathbf{R}_1^3$ in $\mathbf{R}^4$ and a knot K_0 in $\mathbf{R}_0^3$ and a knot K_1 in $\mathbf{R}_1^3$. If K_0 and K_1 are the same knot, then we have a cylinder bounded by K_0 and K_1 and embedded in the region $\mathbf{R}^3 \times [0,1]$ between $\mathbf{R}_0^3$ and $\mathbf{R}_1^3$. Even if K_0 and K_1 are distinct knots, it is possible that we have a cylinder bounded by K_1 and K_0 and embedded in $\mathbf{R}^3 \times [0,1]$. In this case, we say that the knots K_0 and K_1 are *cobordant* and we can construct a mathematical theory that considers when two knots are cobordant. This theory is discussed in Chapter 12.

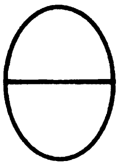
As a high-dimensional generalization, the embedding problem of the m-dimensional sphere S^m into the n-dimensional sphere S^n has been studied considerably. In the cases where $n - m = 1$ and $n - m \geq 3$, it is shown in [Brown 1960] and [Zeeman 1960] respectively that the embeddings are unique from the topological viewpoint. In the case of $n - m = 2$, many research results are known and the field is referred to as high-dimensional knot theory. In particular, 2-dimensional knot theory, which deals with embeddings of S^2 in Euclidean 4-space $\mathbf{R}^4$ or S^4, is still being studied by many researchers. This is discussed in Chapters 13 and 14. Further, the study of embeddings of a 2-dimensional closed manifold in $\mathbf{R}^4$ has also progressed during the last 20 years; some of these results are included in Chapters 13 and 14.



Cobordism

In knot theory, we have restricted ourselves to the case embedded circles, but we can develop a similar theory for other objects embedded in $\mathbf{R}^3$. For example, we may consider a circle with a diameter in figure 18, called a *θ-curve*. The object in figure 19 is also a θ-curve (called *Kinoshita's θ-curve*), but the embedding into $\mathbf{R}^3$

is different from that of figure 18. The study of how to distinguish this difference from the topological viewpoint is an interesting mathematical theory which is a generalization of knot theory. In this theory, we consider a more complicated object than a θ-curve, called a *graph* in general. This theory is now developing, but some of its contents and results are reported in Chapter 15.



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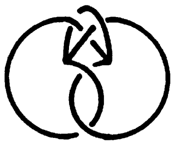

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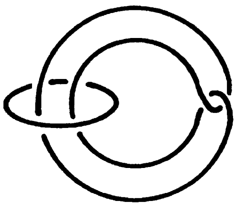
As another generalization of knot theory different from the generalization to graphs, we can also consider simultaneous embeddings of several circles into $\mathbf{R}^3$ or S^3. For example, figure 20 illustrates two entangled circles in $\mathbf{R}^3$, called the *Whitehead link* and figure 21 illustrates three entangled circles in $\mathbf{R}^3$, called the *Borromean rings.* When we embed several circles into $\mathbf{R}^3$ in this way, the embedded image is called a *link.* The theory of links is included in knot theory. We often consider a link exterior, the Dehn surgery manifold of a link (i.e., a closed manifold obtained from a link exterior by attaching solid tori) and a branched covering manifold with branch set a link, rather than the same concepts for a knot. This is because the study of 3-dimensional manifolds happens to be easier in this context. Much research is done from this viewpoint. In this book, we allow the term links to include knots. We make a survey of covering space in Appendix B and two surveys topics in 3-dimensional manifold theory, including the theory of canonical decompositions, Dehn surgery and Heegaard splittings in Appendices C and D.



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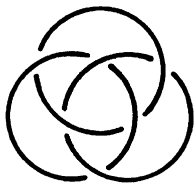

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Links exist not only in the mathematical world but also in the natural world. Recently, molecular biology is rapidly developing owing to the recent development of technology which allows us to see and photograph the structure of DNA and cells by a highly efficient electron microscope. Though the structure of DNA is a double helix when we observe a small piece of it, the whole of this DNA structure may form a circle which we call a *DNA knot.* The existence of not only a trivial DNA knot but also several DNA knots such as the *trefoil knot* of figure 9 and the *figure eight knot*, listed as 4_1 in the knot table of the appendix were confirmed and they could in turn be photographed. Also, in the field of chemistry, we usually distinguish compounds by a molecular structure expressing a covalent bond of atoms. However, in the case of high molecular weight compounds, long twisted chains of molecules form a link or a graph to which knot theory can also be applied. The topological viewpoint may be insufficient to distinguish compounds, but a wonderful feature of mathematics is that we can change our mathematical viewpoint, or consider a suitable device fitting to our needs. Knot theory is also becoming useful in elementary particle theory, an area of theoretical physics, through braid representations, etc. and there is much research collaboration between knot theorists and physicists.

We can trace the history of knot theory back to the 19th century. In Japan, it was introduced about 40 years ago by Hidetaka Terasaka, who was a professor of Osaka University.

Recently, the number of researchers in knot theory is increasing together with those in 3-dimensional manifold theory (in which the Poincaré conjecture is the most famous unsolved problem). This is because knot theory is necessary for the study of 3-dimensional manifolds, for example, because of the following facts: link exteriors give interesting, concrete examples of 3-dimensional manifolds, and every closed connected orientable 3-dimensional manifold is obtained as a Dehn surgery manifold along a link and as a branched covering manifold over S^3 with branch set a knot.

Mathematics appears to be loosely related to the other natural sciences, but there are many demands on mathematics from not only mathematics itself but also the other fields of natural science. These demends influence the mathematical viewpoint we adopt and the direction of our research. From such demands, new mathematical theories are created. Knot theory is an area of mathematics which is expected to develop much more in the future and we would be happy if this book is the origin of your study.

Notes on research conventions and notations

(1) In this book, unless otherwise specified, spaces and maps are considered to be in the PL category, which we discuss in Chapter 0. We omit "PL" after Chapter 0. Thus, PL spaces, PL manifolds, PL maps, PL homeomorphisms, PL links (knots), etc. are simply written as spaces, manifolds, maps, homeomorphisms, links (knots), etc., respectively.

(2) Unless otherwise specified, both $\mathbf{R}^3$ and S^3 are considered as the ambient spaces of knots and links.

(3) The notation $\cong$ is used for PL homeomorphisms, to indicate the same link (knot) types, and for group (module) isomorphisms when the meanings are obvious.

(4) Homology groups are with integral coefficients, unless otherwise stated. By A^n, we mean the n-fold direct sum $A \oplus A \oplus \cdots \oplus A$ of an abelian group (or a module) A.

(5) The smallest normal subgroup containing elements $x_1, x_2, \ldots, x_m$ in a group G is denoted by $\langle\langle x_1, x_2, \ldots, x_m \rangle\rangle^G$.

(6) A free group F with a basis $x_1, x_2, \ldots, x_r$ is denoted by $\langle x_1, x_2, \ldots, x_r \rangle$. For words $R_1, R_2, \ldots, R_s$ in $x_1, x_2, \ldots, x_r$, the quotient group G of F by the normal subgroup $\langle\langle R_1, R_2, \ldots, R_s \rangle\rangle^G$ is denoted by $\langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_s \rangle$. We call it a *presentation* (or a *finite presentation* when r and s are finite) of the group G. R_i is called a *relation*, and instead of R_i we also write $R_i = 1$ or $U_i = V_i$ when $R_i = U_i V_i^{-1}$ (or $U_i^{-1} V_i$).

(7) In a finite presentation $\langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_s \rangle$ of a group G, we call $r - s$ the *deficiency* of the presentation and the maximum of the deficiencies of all finite presentations of G is called the *deficiency* of the group G and denoted by def$G$.

(8) A finite presentation $\langle x_1, x_2, \ldots, x_r \mid R_1, R_2, \ldots, R_s \rangle$ is called a *Wirtinger presentation* if each relation R_i is in the form $x_h^{-1} w x_k w^{-1}$ for some letters x_h, x_k and a word w in $x_1, x_2, \ldots, x_r$.

(9) The transpose of a matrix A is denoted by A'. The determinant and the trace (i.e., the sum of diagonal entries) of a square matrix A are denoted by det$A$ and tr$A$, respectively. E^n denotes the identity matrix of size n.

(10) The identity map and the empty set are respectively denoted by id and $\emptyset$.

(11) For a topological space X and a subspace Y, we denote the closure of $X - Y$ in X by $\mathrm{cl}(X - Y)$.

(12) $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}, \mathbf{S}_k$ denote respectively the set of natural numbers, the ring of integers, the rational number field, the real number field, the complex number field and the symmetric group on k letters.

(13) Base points of fundamental groups are omitted unless confusion might occur.

(14) $\mathbf{R}^n$ denotes Euclidean space of dimension n. S^{n-1} and D^n denote the $(n-1)$-dimensional sphere $\{x \in \mathbf{R}^n \mid \|x\| = 1\}$ and the n-dimensional ball $\{x \in \mathbf{R}^n \mid \|x\| \leq 1\}$, respectively.

(15) By a *surface*, we mean a connected 2-manifold except for a "Seifert surface" for a link, which we allow to be disconnected.

(16) In the references, the symbols *, **, ***, etc., are attached to papers whose publishing data are insufficient.

Chapter 0
Fundamentals of knot theory

In this chapter, we first explain the PL category in which we consider spaces. Next, PL manifolds and related matters are defined. Finally, PL knots and PL links are defined together with related basic concepts.

0.1 Spaces

A *simplicial complex* is a (finite or infinite) set K of simplices in Euclidean space $\mathbf{R}^N$ of a large dimension N which satisfies the following conditions (1),(2) and (3):

(1) For each pair $A_1, A_2 \in K$, the intersection $A_1 \cap A_2$ is a face of A_1 and of A_2 unless it is $\emptyset$.
(2) All faces of each $A \in K$ are contained in K.
(3) For each $A \in K$, there are only finitely many elements of K meeting A.

The union of all simplices in K is called the *polyhedron* of K and denoted by $|K|$. For two simplicial complexes K_1, K_2, we say that a map $f : |K_1| \to |K_2|$ is *PL(=piecewise linear)* if f defines a simplicial map $K_1' \to K_2'$ under suitable simplicial subdivisions K_1' and K_2' of K_1 and K_2. For any two simplicial complexes K_1 and K_2 with $|K_1| = |K_2|$, the identity map $\mathrm{id} : |K_1| \to |K_2|$ is PL. The composite map of any two PL maps is also a PL map. By a *triangulation* of a topological space X, we mean a pair (K, t) of a simplicial complex K and a homeomorphism $t : |K| \cong X$.

Definition 0.1.1 A non-empty collection $\mathcal{T}$ of triangulations of a topological space X is called a *PL structure* on X if we have the following:

(1) For any $(K_i, t_i) \in \mathcal{T}, i = 1, 2$, the homeomorphism $t_2^{-1} t_1 : |K_1| \to |K_2|$ is PL.
(2) A triangulation (K, t) of X belongs to $\mathcal{T}$ if $t_0^{-1} t : |K| \to |K_0|$ is PL for some $(K_0, t_0) \in \mathcal{T}$.

We call a topological space X together with a PL structure $\mathcal{T}$ a *PL space* and each $(K, t) \in \mathcal{T}$ a *triangulation* of the PL space X. The *dimension* of X is defined to be the dimension of K. A one-dimensional PL space is called a *graph*. Given a triangulation (K, t) of X, there is a unique PL structure $\mathcal{T}$ on X containing (K, t). Unless otherwise stated, the polyhedron $|K|$ of a simplicial complex K is considered to be a PL space with PL structure containing (K, id). In particular, a simplex A and its boundary ∂A are PL spaces since $A = |K(A)|$ for the simplicial complex $K(A)$ consisting of all faces of A and $\partial A = |K(\partial A)|$ for the simplicial complex $K(\partial A) = K(A) - \{A\}$. For PL spaces $X_i, i = 1, 2$, we say that a map $f : X_1 \to X_2$ is *PL* if there is a triangulation (K_i, t_i) of X_i such that $t_2^{-1} f t_1 : |K_1| \to |K_2|$ is PL. Further, when f is a homeomorphism, the map f is called a *PL homeomorphism*. In this case, f^{-1} is also a PL homeomorphism. For example, if there is a triangulation

(K_i, t_i) of X_i for $i = 1, 2$, with $|K_1| = |K_2|$, then the map $t_2 t_1^{-1} : X_1 \to X_2$ is a PL homeomorphism.

As the category of topological spaces and (continuous) maps is called TOP category, the category of PL spaces and PL maps is called *PL category.* See [Hudson 1969], [Rourke-Sanderson 1972], [Homma 1980] etc. for fundamental techniques in the PL category. A topological subspace Y of a PL space X is called a *PL subspace of* X (or a *PL space in* X) if Y is a PL space and the inclusion $i : Y \subset X$ is PL. In this case, the pair (X, Y) is called a *PL space pair.* Any open set O of a PL space X is not a PL subspace in general, but O has a unique PL structure so that every PL subspace Y of X with $Y \subset O$ is a PL subspace of O. This open set O together with this PL structure is called a *PL open subspace* of X.

For an n-dimensional simplex (or simply, n-simplex) A and a PL space pair $(X, \dot{X})$, we assume that there is a surjective PL map $f : (A, \partial A) \to (X, \dot{X})$ inducing a PL homeomorphism $A - \partial A \cong X - \dot{X}$. Then we call X an *n-dimensional cell* (or simply an *n-cell*) and $X - \dot{X}$ the *interior*, denoted by $\mathrm{int}X$ and $\dot{X}$ the *boundary.* In this book, a set K of PL cells in a PL space, PL homeomorphic to $\mathbf{R}^N$ for some N is called a *cell complex* if K has the following (1), (2) and (3):

(1) If $X_1 \neq X_2$ for any $X_1, X_2 \in K$, then $\mathrm{int}X_1 \cap \mathrm{int}X_2 = \emptyset$.
(2) If the dimension of $X \in K$ is n, then $\dot{X}$ is contained in the union of PL cells of dimensions$\leq n - 1$ in K.
(3) For each $X \in K$, there are only finitely many PL cells in K meeting X.

The union of all PL cells in a cell complex K is also called the *polyhedron* of K and denoted by $|K|$.

0.2 Manifolds and submanifolds

An n-dimensional *PL ball* (or simply, a PL *n-ball* or a *PL arc* for $n = 1$ or a *PL disk* for $n = 2$) is a PL space which is PL homeomorphic to an n-dimensional simplex A. The n-dimensional ball D^n is regarded as a PL n-ball by a PL structure that includes a triangulation $(K(A), t)$ of D^n. An n-dimensional *PL sphere* (or simply, a PL *n-sphere*) is a PL space which is PL homeomorphic to the boundary ∂A of an $(n+1)$-dimensional simplex A. It is also called a *PL circle* when $n = 1$ and a *PL sphere* when $n = 2$. The n-dimensional sphere S^n is regarded as a PL n-sphere by a PL structure that includes a triangulation $(K(\partial A), t)$ of S^n. Up to PL homeomorphism, such PL structures on D^n and S^n are unique. For each simplex A in a simplicial complex K, the *link of* A *in* K, denoted by $\mathrm{Link}(A, K)$ is the subcomplex of K consisting of all simplices not meeting A which are faces of simplices containing A in K. A PL space M is called an n-dimensional *PL manifold* (or *PL n-manifold*) if M has a triangulation (K, t) with the following property: For each vertex v of K, $|\mathrm{Link}(v, K)|$ is a PL $(n-1)$-ball or a PL $(n-1)$-sphere. Such a K is also called an n-dimensional *combinatorial manifold* (or *combinatorial n-manifold*) and (K, t), a *combinatorial manifold triangulation* of M. For example, differentiable manifolds are PL manifolds (cf. [Munkres 1961]). For a combinatorial n-manifold K and a simplex A with $\mathrm{dim}A < n$, $|\mathrm{Link}(A, K)|$

is a PL $(n - \dim A - 1)$-ball or PL $(n - \dim A - 1)$-sphere. We denote by ∂K the subcomplex of K consisting of all simplices A such that $|\mathrm{Link}(A, K)|$ is a PL ball. Then ∂K is a combinatorial $(n-1)$-manifold unless it is $\emptyset$. $\partial M = t(\partial K)$ is an $(n-1)$-dimensional PL manifold, unless it is $\emptyset$, and is independent of the choice of combinatorial manifold triangulation (K, t) of M. It is called the *boundary* of M. A PL manifold M without boundary (i.e., such that $\partial M = \emptyset$) is said to be *closed* or *open* according to whether M is compact or non-compact. We set $\mathrm{int}M = M - \partial M$ and call it the *interior* of M. A PL subspace of a PL manifold M is called a *PL submanifold* (or a *PL manifold in M*) if its PL structure has a combinatorial manifold triangulation. For example, ∂M is a PL submanifold of M (if it is not $\emptyset$). Any PL open subspace of a PL manifold is a PL manifold, which we call a *PL open submanifold.* For example, $\mathrm{int}M$ is a PL open submanifold of M. In particular, the interior of a PL n-ball is called a *PL open n-ball.* It is PL homeomorphic to $\mathbf{R}^n$. A PL open submanifold of S^n obtained by removing one point from S^n is also PL homeomorphic to $\mathbf{R}^n$. A PL submanifold L of a PL manifold M is said to be *proper* if $\partial L = L \cap \partial M$. A *PL loop* in a PL manifold M is the image of a PL map from S^1 into M. In particular, it is called a *PL simple loop* if the map is one-to-one. A PL space pair is called a *trivial PL sphere pair* if it is PL homeomorphic to the boundary pair $(\partial A, \partial A')$ of a simplex A and a face A'. It is also called a *trivial PL ball pair* if it is PL homeomorphic to a cone over $(\partial A, \partial A')$. A proper PL submanifold L in a PL manifold M is said to be *locally flat* if there are combinatorial manifold triangulations (K_M, t_M) and (K_L, t_L) of M and L, respectively, such that K_L is a subcomplex of K_M and $t_L = t_M |_{|K_L|}$ and for each vertex v of K_L, $(|\mathrm{Link}(v, K_M)|, |\mathrm{Link}(v, K_L)|)$ is a trivial PL sphere or ball pair.

An *orientation* of an n-simplex is an equivalence class of orderings of the $n+1$ vertices modulo even permutations. By $[v_0, v_1, \ldots, v_n]$, we denote an oriented simplex with vertices ordered as $v_0, v_1, \ldots, v_n$ and by $-[v_0, v_1, \ldots, v_n]$, the simplex with opposite orientation. By the induced orientation of the face of $[v_0, v_1, \ldots, v_n]$ opposite to v_i, we mean the orientation given by $(-1)^i[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$. An *orientation* of a PL n-manifold M is an orientation of each n-simplex of K_M for a combinatorial manifold triangulation (K_M, t_M) of M such that the orientation of A_0 induced from the orientation of A_1 is opposite to that of A_0 induced from the orientation of A_2 for any n-simplices A_1, A_2, in K_M with $A_0 = A_1 \cap A_2$ an $(n-1)$-simplex. In terms of homology, we can describe it as follows: An orientation of M is a system $\{z_x \mid x \in \mathrm{int}M\}$ such that $z_x \in H_n(M, M-x) \cong \mathbf{Z}$ is a generator and for any points x, y connected with a PL arc α, z_x is sent to z_y under the natural composite isomorphism

$$H_n(M, M-x) \cong H_n(M, M-\alpha) \cong H_n(M, M-y).$$

According to whether or not such an orientation exists, we say that M is *orientable* or *non-orientable.* When M is orientable and an orientation is specified, M is said to be *oriented.* Any open PL submanifold, any PL n-submanifold, and the boundary ∂M (if it is not $\emptyset$) of an oriented n-manifold M are orientable with orientations induced from the orientation of M. (Unless otherwise mentioned, such PL manifolds are considered to be oriented by such orientations.) D^n, S^n and $\mathbf{R}^n$ are orientable. Let (M, L) be a pair such that M is a PL manifold M and L is a PL submanifold of M or $\emptyset$. Two PL auto-homeomorphisms h, h' of (M, L) are said to be *PL ambient isotopic* if there is a PL auto-homeomorphism family $\{h_t \mid 0 \leq t \leq 1\}$ of (M, L) such that $h_0 = h, h_1 = h'$ and the map $(M, L) \times [0,1] \to (M, L) \times [0,1]$ defined by this family is a PL map. This family $\{h_t \mid 0 \leq t \leq 1\}$ is called a *PL ambient isotopy* from h to h'. PL spaces N and N' in M are said to be *PL ambient isotopic* if there is a PL auto-homeomorphism h of M such that h and id are PL ambient isotopic and $h(N) = N'$. (When N and N' are oriented PL manifolds, the PL homeomorphism $h\,|_N \colon N \cong N'$ is understood to be orientation-preserving.) For example, any orientation-preserving PL auto-homeomorphisms of D^n, S^n and $\mathbf{R}^n$ are known to to be PL ambient isotopic to id.

0.3 Knots and links

Here, we denote S^{n+2} or $\mathbf{R}^{n+2}$ by M. An n-dimensional *PL link* (or simply a *PL n-link*) is a locally flat compact PL submanifold L of M each component of which is PL homeomorphic to S^n. In particular, it is called an n-dimensional *PL knot* (or *PL n-knot*) when L is connected. In this book, we discuss in detail the case when $n = 1$ or 2. A PL 1-link or 1-knot is usually called a PL link or knot, respectively. We assume that M and L are oriented unless otherwise stated.

Definition 0.3.1 Two PL n-links L and L' are *equivalent* if there is a PL auto-homeomorphism h of M with $h(L) = L'$. More strictly, if h is orientation-preserving or -reversing, then they are *positive-equivalent* or *negative-equivalent*, respectively. Further, L and L' *belong to the same type* and we denote it by $L \cong L'$ if h and $h\,|_L \colon L \to L'$ are orientation-preserving.

For a PL n-link L, the *type* of L is the collection of all PL n-links which contains L as a member and any two members of which belong to the same type. The same n-link as L but with the opposite orientations on all the components of L is denoted by $-L$. When $L \cong -L$, the link L is said to be *invertible*. We denote by L^* the image of L under an orientation-reversing PL auto-homeomorphism g of M, where we orient L^* with orientation induced from L by g. In this case, the type of L^* is independent of a choice of g and determined only by L. L^* is called the *mirror image* of L. $-(L^*) = (-L)^*$ is simply denoted by $-L^*$. We say that L is *amphicheiral* if $L \cong \rho L^*$ for a PL n-link ρL^* which is L^* or obtained from L^* by reversing the orientations of some components of L^*. In particular, L is $(+)$*amphicheiral* when $L \cong L^*$ and $(-)$*amphicheiral* when $L \cong -L^*$.

Let a PL n-link L have r components. L is *trivial* if L is the boundary of the union of r mutually disjoint PL $(n+1)$-balls in M. L is *split* if there are two disjoint PL $(n+2)$-balls $D_i^{n+2}, i = 1, 2$, in M with $L \cap \partial D_i^{n+2} = \emptyset$ and $L \cap D_i^{n+2} \neq \emptyset$ for $i = 1$ and 2. (Otherwise, L is *non-splittable*.) L is *completely splittable* if there are just r mutually disjoint PL $(n+2)$-balls $D_i^{n+2} (i = 1, 2, \ldots, r)$ in M such that $D_i^{n+2} \cap L$ is one component of L. (Otherwise, L is *not completely splittable.*) For any regular neighborhood $N(L)$ of L in M, there is a PL homeomorphism $f : (L \times D^2, L \times 0) \cong (N(L), L)$. We call $M - \mathrm{int}N(L)$ the *exterior* of L in M and denote it by $E(L, M)$ (or by $E(L)$ unless confusion will arise). For arbitrary regular neighborhoods $N(L)$ and $N(L')$ of PL n-links L and L' of the same type in M, there is a PL auto-homeomorphism h of M giving the same type of L and L' such that $h(N(L)) = N(L')$. This follows from the uniqueness of regular neighborhoods (a fact known to hold for a considerably general PL subspace in a PL space). In particular, $E(L, M)$ and $E(L', M)$ are orientation-preservingly PL homeomorphic. Let $L_i (i = 1, 2, \ldots, r)$ be the components of L. We say that $f(p_i \times D^2)$ and $f(p_i \times \partial D^2)$ for a point $p_i \in L_i$ are a *meridian disk* and a *meridian* of L_i, respectively, where the orientation of the meridian disk is chosen so that the intersection number of the meridian disk and L_i is $+1$. The set of these meridians for $i = 1, 2, \ldots, r$ is called a *meridian system* of L. This has the following special feature: For arbitrary meridian systems m and m' of PL n-links L and L' of the same type in M, there is a PL auto-homeomorphism h of M giving the same type of L and L' such that $h(m) = m'$.

We take $n = 1$. There is an oriented PL simple loop ℓ_i in $\partial N(L)$ such that ℓ_i is homologous to L_i in $N(L)$ and null-homologous in $M - L_i$. Such an ℓ_i is called a *longitude* of the component L_i. The set of these longitudes for $i = 1, 2, \ldots, r$ is called a *longitude system* of L. The pair (m, ℓ) of a meridian system m and a longitude system ℓ of L in $\partial N(L)$ such that each component of m meets ℓ in a single point is called a *meridian-longitude system pair* of L. This has the following special feature: For arbitrary meridian-longitude system pairs (m, ℓ) and (m', ℓ') of PL links L and L' of the same type in M, there is a PL auto-homeomorphism h of M giving the same type of L and L' such that $h(m) = m'$ and $h(\ell) = \ell'$.

A fundamental problem in n-dimensional knot theory is to determine when two PL n-links with the same number of components belong to the same type. For an r-component PL n-link L in $M = S^{n+2}$, the homology of the exterior $E(L)$ is determined only by n and r:

$$H_q(E(L)) \cong \begin{cases} \mathbf{Z}^{r-1} \, (q = n+1) \\ \mathbf{Z}^r \, (q = 1) \\ \mathbf{Z} \, (q = 0) \\ 0 \, (q \neq 0, 1, n+1). \end{cases}$$

Thus, it is an important problem in manifold theory to determine when two oriented PL manifolds with the same dimension, the same homology, and the same boundary are orientation-preservingly PL homeomorphic, since it is intimately related to the fundamental problem of n-dimensional knot theory.

Supplementary notes for Chapter 0

Let L be a compact topologically embedded submanifold of $M = \mathbf{R}^{n+2}$ (or S^{n+2}) each component of which is homeomorphic to S^n. This submanifold L is called an

n-dimensional *tame link* in M if there is a topological auto-homeomorphism h of M such that $h(L)$ is a PL subspace of M. Otherwise, L is called an n-dimensional *wild link* in M. If L has a topological neighborhood N in M such that (N, L) is homeomorphic to $L \times (D^2, 0)$, then L is called an n-dimensional *link in the TOP category.* In the case that $n = 1$, the concept of PL links coincides with that of tame links and with that of links in the TOP category. In the case that $n \geq 2$, there are non-locally flat compact PL submanifolds of M, each component of which is PL homeomorphic to S^n. We call them *non-locally flat PL links* in M. The notion of a slice knot which will be discussed in Chapter 12 is motivated by a 2-dimensional non-locally flat PL knot with just one non-locally flat point (cf. [Fox-Milnor 1966]).

Chapter 1
Presentations

In this chapter, we discuss regular presentations, braid presentations and bridge presentations for links.

1.1 Regular presentations

We call a line segment of a polygonal link L in $\mathbf{R}^3$ an *edge* of L and an end point of the line segment a *vertex* of L. Let $\mathbf{R}^2$ be a plane in $\mathbf{R}^3$ and $p : \mathbf{R}^3 \to \mathbf{R}^2$ be an orthogonal projection. Let L be a link in $\mathbf{R}^3$ and $p(L)$ the projection of L. We call a point c of the image $p(L)$ a *multiple point* if $p^{-1}(c) \cap L$ contains more than one point. The cardinality of $p^{-1}(c) \cap L$ is called the *order* of c and c is called an *n-multiple point* if the order of c is n. A two-multiple point is called a *double point.* We say that p is a *regular projection* for L if we have the following two conditions:

(1) The set of multiple points of the image $p(L)$ consists of finitely many double points.

(2) No point in the preimage $p^{-1}(c) \cap L$ of any double point $c \in p(L)$ is a vertex of L.

Any multiple point of any regular projection image $p(L)$ is like figure 1.1.1 a. Hence any multiple point like figure 1.1.1 b, c or d is not contained in any regular projection image $p(L)$. The following is a fundamental fact of combinatorial knot theory:

Proposition 1.1.1 *For any polygonal link L, there exists a regular projection for L.*

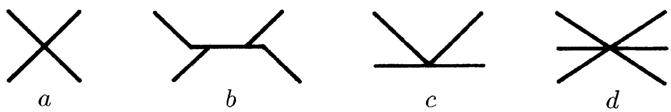


Fig. 1.1.1

Proof. Since the projections of a link L on two parallel planes coincide, any projection is determined by a straight line which goes through the origin and is orthogonal to the projecting plane. The space of straight lines going through the origin in $\mathbf{R}^3$ is the two-dimensional projective plane $\mathbf{R}P^2$. Define S to be the set of straight lines going through the origin which do not determine regular projections for L. It is enough to show that S is nowhere dense in $\mathbf{R}P^2$. We show that S is a one dimensional subset of $\mathbf{R}P^2$. Define S_1 to be the set of straight lines going through the origin which are parallel to a line going through a vertex of L and another point of L. Define S_2 to be the set of straight lines going through the origin which are parallel to a line going through more than two points of L. We can easily see that $S = S_1 \cup S_2$. In fact, a projection has multiple points of type b or c in figure 1.1.1 if and only if the projection is determined by a straight line in S_1. A projection has multiple points whose order is greater than 2 if and only if the projection is determined by a straight line in S_2. It can be easily checked that the set S_1 consists of finite line segments in $\mathbf{R}P^2$. An elementary calculation also enables us to show that S_2 consists of finitely many curve segments of second order. Therefore S is a one dimensional subset of $\mathbf{R}P^2$ (cf. [Crowell-Fox 1963]). □

For convenience, we assume that the projection determined by the z-axis is a regular projection for a link L. Each double point c of the regular projection image $p(L)$ is called a *crossing*. For the points c_+, c_- of $p^{-1}(c) \cap L$, we say that c_+ is an *overcrossing* and c_- is an *undercrossing* if the z-coordinate of c_+ is greater than that of c_-. The line segment of L that contains the overcrossing or undercrossing of c is called the *overpass* or the *underpass* of c, respectively. A *regular presentation* or simply *diagram* of a link L is a regular projection image $p(L)$ such that the overcrossing and the undercrossing at each crossing of $p(L)$ are distinguished. (Usually, we denote the diagram by erasing a small neighborhood of each undercrossing in order to distinguish between over and under.) If L has an orientation, then a regular projection $p(L)$ has the induced orientation. The crossing number of a regular projection $p(L)$ is the number of the crossings of $p(L)$. The *minimal crossing number* or simply *crossing number* of a link L is the smallest crossing number of all regular projections of all links with the same type as L. The minimal crossing number is one of the most general quantities that reflects the complexity of links. Let D_1 and D_2 be two diagrams. We say that D_1 is *identical* to D_2 if there is an orientation preserving auto-homeomorphism of the plane $\mathbf{R}^2$ that maps D_1 onto D_2 and makes the over–under relations coincide and makes the orientations coincide (if they have orientations). We do not distinguish these identical diagrams.

Exercise 1.1.2 Confirm that any two links with identical diagrams belong to the same type.

Exercise 1.1.3 For any positive integer n, show that there are only finitely many knot types whose minimal crossing numbers are less than n. [*Hint*: It is enough to show that there are only finitely many diagrams whose crossing numbers are less than n.]

The local moves of a diagram shown in figure 1.1.2 are called the *Reidemeister moves* of type I, II and III. We say that two link diagrams are *R-isotopic* if they can be transformed into each other by a finite sequence of the Reidemeister moves. The following theorem is a fundamental fact:

Theorem 1.1.4 *Let D_1 and D_2 be diagrams of L_1 and L_2, respectively. Then L_1 and L_2 belong to the same type if and only if D_1 and D_2 are R-isotopic.*

The proof of this theorem needs careful consideration. (See Appendix A for the details.) We add another type of move, the type IV, shown in figure 1.1.3, to the

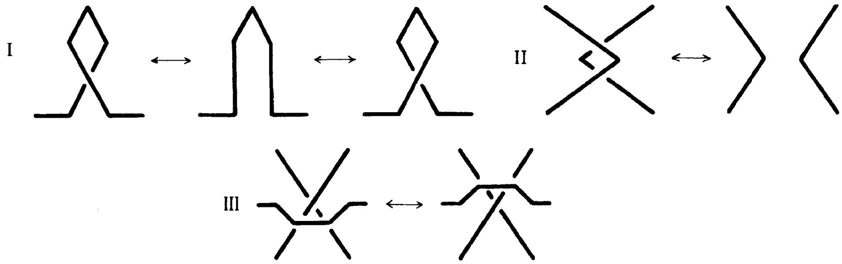


Fig. 1.1.2

Reidemeister moves. We say that any two diagrams are *regularly isotopic* if they can be transformed into each other by a finite sequence of the Reidemeister moves of type II, type III or type IV. We call such a deformation a *regular isotopy*. For a crossing c of an oriented link diagram D, we define the *sign* of c, $\operatorname{sign}(c)$ as follows: If the underpass of c goes through from the right side to the left side of the overpass of c, then $\operatorname{sign}(c) = +1$ (see figure 1.1.5); otherwise, $\operatorname{sign}(c) = -1$. We denote by $w(D)$ the sum of the signs of all crossings of the diagram D, namely,

$$w(D) = \sum_{c \in c(D)} \operatorname{sign}(c),$$

where $c(D)$ denotes the set of crossings of D. We call $w(D)$ the *writhe* of D. Let $D = D_1 \cup D_2 \cup \cdots \cup D_m$ be a diagram of m-components link with $D_i (i = 1, 2, \ldots, m)$ the diagrams of the knot components. We set $t(D) = \sum_{i=1}^{m} w(D_i)$ and we call it the *twisting number* of D.

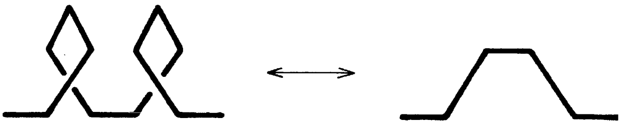


Fig. 1.1.3

Example 1.1.5. Let D be the diagram shown in figure 1.1.4. Then $w(D) = -4$ and $t(D) = 0$.

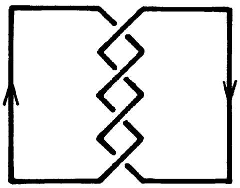


Fig. 1.1.4

Let the crossing in figure 1.1.5 be a crossing c of an oriented diagram D. Assume that the overcrossing and the undercrossing of c are on the same component D_i of D. Let D' be the same diagram as D but with the opposite orientation on D_i. Then the crossing c of D as in figure 1.1.5 changes a crossing of D' as in figure 1.1.6. Since the signs of the crossings of figures 1.1.5 and 1.1.6 are equal, we see that the twisting number of the diagram D is independent of the orientation of D. This quantity, the twisting number, is closely related to the notion of regular isotopy of diagrams.

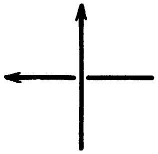


Fig. 1.1.5

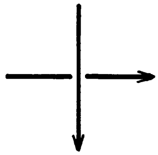


Fig. 1.1.6

Proposition 1.1.6 *Any two regularly isotopic link diagrams have the same twisting number.*

Exercise 1.1.7 Show Proposition 1.1.6. [*Hint*: Show that the Reidemeister moves of type II, III and IV do not change the twisting number.]

The following theorem means that the converse of this Proposition holds in part.

Theorem 1.1.8 *If two knot diagrams are R-isotopic and have the same twisting number, then they are regularly isotopic.*

Fig. 1.1.7

Proof. Classify the type I Reidemeister moves into the type I_+, I_-, I_+^* or I_-^* as they are shown in figure 1.1.7. Obviously, the moves of type I_+^* and type I_-^* are generated by the moves of type IV, type I_+ or type I_-. Therefore any R-isotopic diagrams can be transformed into each other by a finite sequence of the Reidemeister moves of type I_+, type I_-, type II, type III or type IV. Then we can postpone the Reidemeister moves of type I_+ and type I_- until the end of the sequence, although after this change, the length of the sequence may be longer than the original sequence. Let D and D' be two diagrams satisfying the assumption of the theorem. Consider a sequence of the Reidemeister moves stated above realizing the R-isotopy between D and D'. Postpone the moves of type I_+ and type I_- of the sequence until the end of the sequence. Since $t(D) = t(D')$, the number of moves of type

I_+ in the sequence is equal to that of I_-. Therefore we can use the Reidemeister move of type IV for a pair of the types I_+ and I_- in the sequence. So D and D' are regularly isotopic. $\square$

Exercise 1.1.9 Generalize Theorem 1.1.8 to links and prove it.

Let $L = K_1 \cup K_2$ be a 2-component link. Let $D = D_1 \cup D_2$ be a diagram of L with $p(K_i) = D_i$, $i = 1, 2$. We define the *linking number* of D_1 and D_2 to be

$$\frac{1}{2} \sum_{c \in D_1 \cap D_2} \mathrm{sign}(c)$$

and denote it by $\mathrm{Link}(D_1, D_2)$ or $\mathrm{Link}(D)$. For example, the linking number of the diagram shown in figure 1.1.4 is -2.

Exercise 1.1.10 Show that the linking number of any 2-component link diagram is an integer.

Exercise 1.1.11 Show that the linking number is invariant under the R-isotopy of diagrams.

According to this exercise and Theorem 1.1.4, the linking number is an invariant of 2-component links. So, we call $\mathrm{Link}(D)$ the *linking number* of the link L denote it by $\mathrm{Link}(K_1, K_2)$ or $\mathrm{Link}(L)$. (See Supplementary notes of this chapter.) For a diagram $D = p(L)$ of a link L with components K_i $(i = 1, 2, \ldots, m)$ and $D_i = p(K_i)$ $(i = 1, 2, \ldots, m)$, we define the *total linking number* to be

$$\sum_{i<j} \mathrm{Link}(D_i, D_j)$$

and denote it by $\mathrm{Link}(L)$ or $\mathrm{Link}(D)$. Obviously, we have

$$w(D) = t(D) + 2\mathrm{Link}(D).$$

For a link L in $S^3 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 1\}$, we consider the regular presentations, i.e., the diagrams as follows: Choose a pair of antipodal points $\{p_+, p_-\}$ of S^3 that do not intersect L. For simplification, we assume that $p_+ = (0, 0, 0, 1)$ and $p_- = (0, 0, 0, -1)$. Set $S^2 = \{(x, y, z, w) \in S^3 \mid w = 0\}$ and define the projection $p : S^3 - \{p_+, p_-\} \to S^2$ by

$$p(x, y, z, w) = (x, y, z, 0)/\|(x, y, z, 0)\|.$$

By using this projection, the regular projection, the regular presentation and the Reidemeister moves are defined on the sphere similarly to the planar case. The Reidemeister moves on the sphere are more natural than those on the plane, because the Reidemeister move of type IV is not needed for the definition of regular isotopy on the sphere.

1.2 Braid presentations

Let I^3 be the cube $\{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$, and let n be an integer. Take the points $P_i = (\frac{i}{n+1}, \frac{1}{2}, 1)$ and $Q_i = (\frac{i}{n+1}, \frac{1}{2}, 0)$, $i = 1, 2, \ldots n$, on the top and bottom of the cube I^3. Let $s_1, s_2, \ldots, s_n$ be n mutually disjoint polygonal arcs having the following properties:

(1) $\partial(s_1 \cup \cdots \cup s_n) = \{P_1, \ldots, P_n, Q_1, \ldots, Q_n\}$.

(2) Each arc s_i is monotone with respect to the z-coordinate.

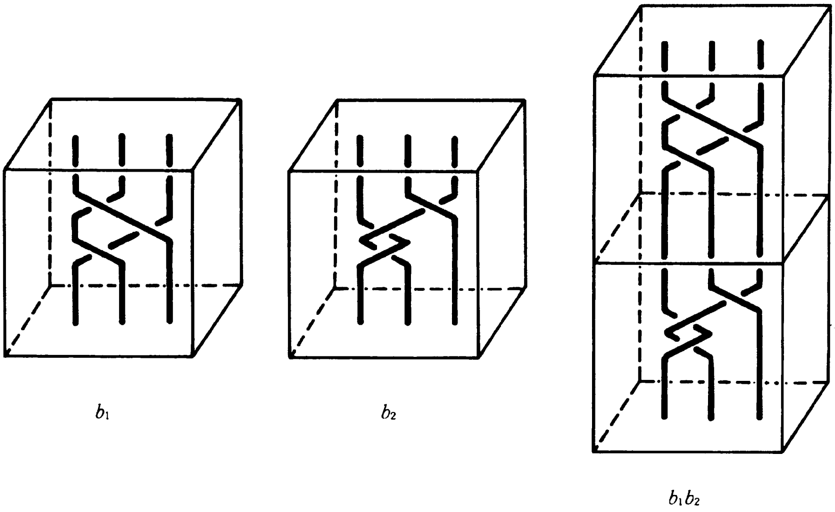


Fig. 1.2.1

We call $b = s_1 \cup s_2 \cup \cdots \cup s_n$ an *n-string braid* and each s_i a *string* of b. We say that two braids b_0 and b_1 are *equivalent* if there is an ambient isotopy $f_t : I^3 \to I^3 (0 \leq t \leq 1)$ such that $f_t \mid_{\partial I^3} = \text{id}$ $(0 \leq t \leq 1)$, $f_0 = \text{id}$ and $f_1(b_0) = b_1$. We say that two braids are *strongly equivalent* if there is an ambient isotopy as above with the extra condition that for each level t, $f_t(b_0)$ is a braid. These two equivalence relations on braids are actually the same equivalence relation ([Artin 1947]). Let $b_1 \subset I_1^3$ and $b_2 \subset I_2^3$ be two n-string braids. We construct a new braid $b_1 b_2$ in the cube $I_1^3 \cup I_2^3$ by attaching the bottom face of I_1^3 to the top face of I_2^3 naturally (see figure 1.2.1). (To make this more rigorous, we have to contract the height of $I_1^3 \cup I_2^3$ to 1/2.) This braid $b_1 b_2$ is called the *product* of b_1 and b_2. The quotient space of the set of n-string braids modulo the equivalence relation above becomes a group with this product operation. The identity element of this group is the braid which consists of n vertical straight line segments connecting the P_i's and the Q_i's. The inverse element of a braid b is the mirror image of b with respect to the plane $z = \frac{1}{2}$. This group is called the *n-string braid group*, and denoted by B_n. Let σ_i be the element of B_n shown in figure 1.2.2. Then the following theorem holds.

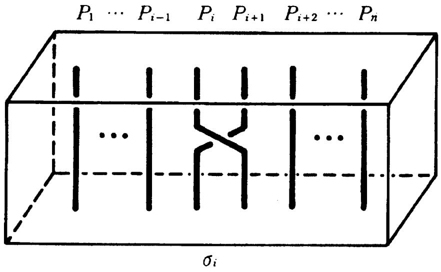


Fig. 1.2.2

Theorem 1.2.1 *The n-string braid group B_n has the following presentation:*

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_k = \sigma_k\sigma_i(|i-k| \geq 2), \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}(i \leq n-2)\rangle$$

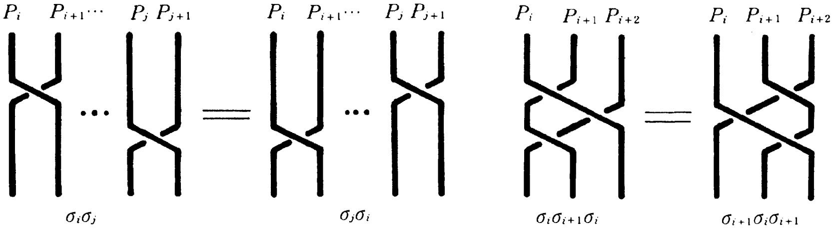


Fig. 1.2.3

See [Birman 1974] for the proof of this theorem. The geometric meaning of the relations in this group presentation is shown in figure 1.2.3.

Next, we discuss a relationship between braids and links. Let $b \subset I^3 \subset \mathbf{R}^3$ be an n-string braid in the cube located in $\mathbf{R}^3$. If we connect the end points of the braid b with mutually disjoint n polygonal arcs in the exterior of I^3, the braid b becomes a link in $\mathbf{R}^3$. We call this operation a *closing* of a braid. One of the most natural ways to close a braid is to connect P_i to Q_i $(1 \leq i \leq n)$ with trivial arcs as they are shown in figure 1.2.4. The link given by this closing is called a *vertically closed braid*, or simply a *closed braid* and denoted by $\hat{b}$. The orientation of $\hat{b}$ is given by the downward direction of the braid b. Another natural way to closing of a braid is to connect P_{2i-1} to P_{2i} and Q_{2i-1} to Q_{2i} as in figure 1.2.5. To do this, n must be an even integer. We call the link given by this closing a *horizontally closed braid.* In this case, the orientation of the resulting link is undefined. If $\hat{b}$ has the same link type as a link L, we call $\hat{b}$ a *braid presentation* of L. If a horizontally closed braid is positive-equivalent to a link K, then we call the closed braid a *plat presentation* of K. Do all links have braid presentations and plat presentations?

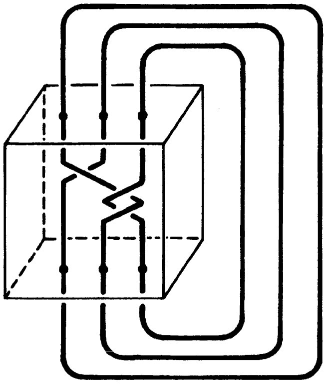


Fig. 1.2.4

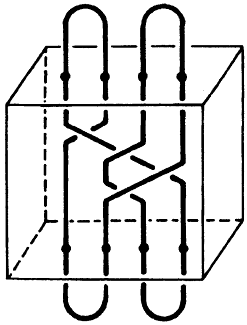


Fig. 1.2.5

The answer is 'yes' (cf. Supplementary notes of this chapter). To prove this, it is enough to show that any link has a braid presentation, because an n-string braid presentation can be regarded as a $2n$-string plat presentation.

Let D be a link diagram and c_i a crossing of D. A smoothing at a vertex c_i is a deformation of the diagram as in figure 1.2.6. Smoothing at all the vertices of D makes D a union of mutually disjoint simple loops $S_1 \cup \cdots \cup S_n$ in the plane. These simple loops are called the *Seifert circles* of D. We connect with an arc a_i two points in the Seifert circles obtained by the smoothing at c_i . Such an arc is called a *connecting arc* of the Seifert circles. The orientation of the Seifert circles is induced from that of the diagram. Then each connecting arc looks like figure 1.2.7 and there is no connecting arc like figure 1.2.8. The set of Seifert circles and connecting arcs $\{S_1, \ldots, S_n; a_1, \ldots, a_r\}$ is called the *system of Seifert circles* of the diagram D. The system of Seifert circles of a diagram of the figure eight knot is shown in figure 1.2.9.

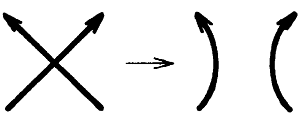


Fig. 1.2.6

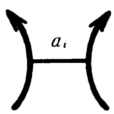


Fig. 1.2.7

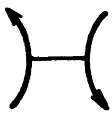


Fig.1.2.8

Now we introduce two types of deformations of a system of Seifert circles. We say that two oriented simple loops in the plane are *coherent* if they have the same rotation number. Let S_i and S_k be two Seifert circles in the system of Seifert circles of a link diagram D. Assume that S_k is inside S_i and their orientations are not coherent. Moreover, assume that there is a band b connecting S_i and S_k that does not intersect the other part of the system as in figure 1.2.10. Then we deform D by Reidemeister moves as follows: first, stretch out $S_k \cap b$ along b until it is near S_i and apply the Reidemeister move of type IV (see figure 1.2.11). Next,

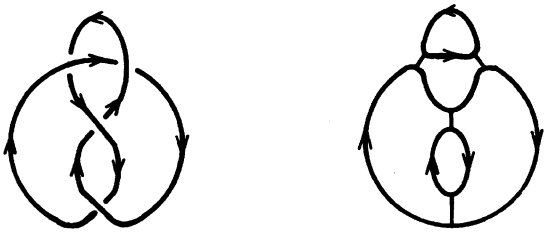


Fig. 1.2.9

expand this part into a big circle just inside S_i (see figure 1.2.12). This new circle may intersect some connecting arcs. The situation at these points is shown in figure 1.2.13. The resulting Seifert circles and connecting arcs are given in figure 1.2.14. So, the resulting system of Seifert circles becomes as in figure 1.2.15. This deformation of the system of Seifert circles is called a *concentric deformation of type I.*

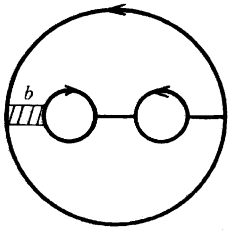


Fig. 1.2.10

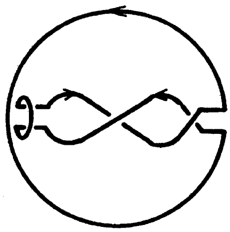


Fig. 1.2.11

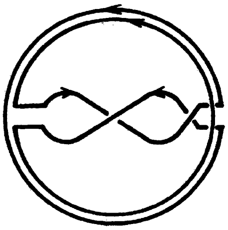


Fig. 1.2.12

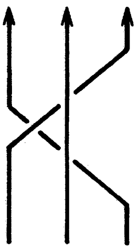


Fig. 1.2.13

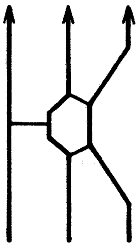


Fig. 1.2.14

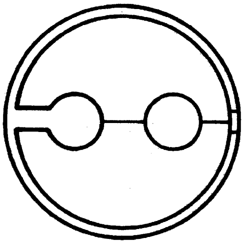


Fig. 1.2.15

Let S_i and S_k be two Seifert circles in the system of Seifert circles of a link diagram D. Assume that S_k is outside S_i and S_i is outside S_k and their orientations are coherent. Moreover, assume that there is a band b connecting S_i and S_k that does not intersect the other part of the system, as in figure 1.2.16. Then we deform D by Reidemeister moves as follows: At first, stretch out $S_k \cap b$ along b until it is near S_i. Next, expand this part into a big circle just outside S_i (see figure 1.2.17.) Then the new system of Seifert circles becomes as in figure 1.2.18. This deformation of the system of Seifert circles is called a *concentric deformation of type II.* If we consider the system of Seifert circles on the sphere, the concentric deformation of type II is nothing but the concentric deformation of type I. We note that a concentric deformation may increase the number of connecting arcs, but never changes the number of Seifert circles. Here we give an answer to the question mentioned before.

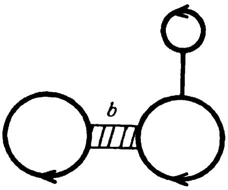


Fig. 1.2.16

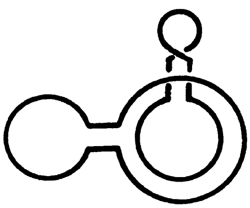


Fig. 1.2.17

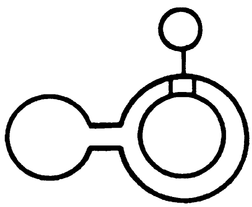


Fig. 1.2.18

Theorem 1.2.2 *Any link diagram can be deformed into a braid presentation by a finite sequence of concentric deformations of types I and II.*

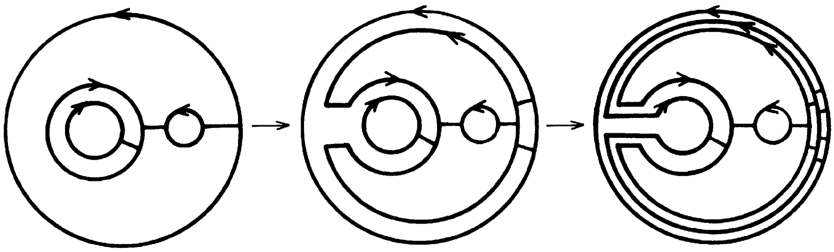


Fig. 1.2.19

Proof. A diagram whose Seifert circles are concentric is a braid presentation. So, we show that any link diagram can be deformed into such diagrams. Let D be a link diagram and $\mathcal{S}$ be the system of Seifert circles of D. If $\mathcal{S}$ has a Seifert circle that contains all other Seifert circles inside, then let S_0 denote that Seifert circle. Otherwise, we add a new trivial circle S_0 to $\mathcal{S}$ so that S_0 contains $\mathcal{S}$ inside. We shall deform all the Seifert circles into concentric circles parallel to S_0 by the following procedure: Firstly, we apply the concentric deformation of type I between S_0 and another Seifert circle until we cannot do it any more. (See figure 1.2.19.) After this deformation, if there is more than one outermost circle inside S_0, then we apply the concentric deformation of type II between the outermost circles inside S_0 as many times as possible. (See figure 1.2.20.) Then there is only one outermost circle, say S_1, inside S_0, whose orientation is coherent with S_0. Secondly, we do the same procedure for the circle S_1. Continuing this procedure inductively, we have concentric Seifert circles $S_0, \ldots, S_n$. If we added S_0 to the diagram D at the beginning, remove it. Thus, we have a braid presentation. □

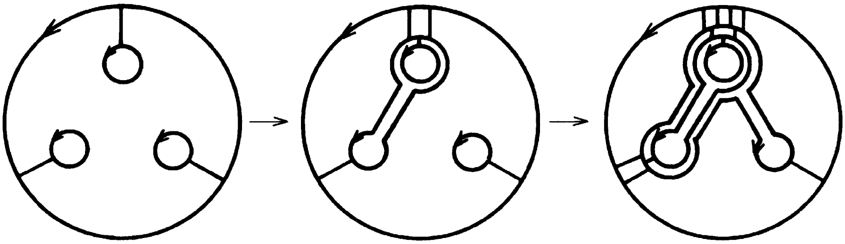


Fig. 1.2.20

The *braid index* of a link is the minimal number of braid strings among all braid presentations for the link. Theorem 1.2.2 implies the following corollary:

Corollary 1.2.3 *The minimal number of Seifert circles of all diagrams of a given link is equal to the braid index of the link.*

Exercise 1.2.4 Show that there are infinitely many link types of braid index 2.

Next, we discuss a necessary and sufficient condition for two closed braids to belong to the same link type. Let B_n be the n-string braid group. For any two integers m, n with $m < n$, we consider that $B_m \subset B_n$ by identifying each generator $\sigma_i \in B_m$ with $\sigma_i \in B_n$ $(i = 1, \ldots, m-1)$. Set

$$B = \{(b, n) \mid b \in B_n, n = 1, 2, 3, \ldots\}.$$

We define *Markov moves* of type I and II as follows:

$$\text{I} \quad (b_1 b_2, n) \leftrightarrow (b_2 b_1, n).$$
$$\text{II} \quad (b, n) \leftrightarrow (b\sigma_n^{\pm 1}, n+1).$$

We also call the move of type I a *conjugacy move* (see figure 1.2.21) and the move of type II a *stabilization* (see figure 1.2.22). We say that two elements of B are *Markov equivalent* if they can be deformed into each other by a finite sequence of Markov moves. Then we have the following theorem:

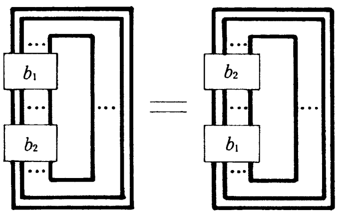


Fig. 1.2.21

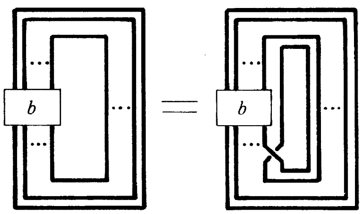


Fig. 1.2.22

Theorem 1.2.5 *For two braids (b, n) and (b', n'), the vertically closed braids $\hat{b}$ and $\hat{b'}$ belong to the same link type if and only if (b, n) and (b', n') are Markov equivalent.*

See [Birman 1974] for the proof of this theorem. According to this theorem and Theorem 1.2.2, it may be said that knot theory is the study of the Markov equivalence classes of the braid groups. The word problem in the braid group is solvable, i.e., there is an algorithm to determine whether or not two given words are the same element in the braid group. The conjugacy problem in the braid group is also solvable, i.e., there is an algorithm to determine whether or not two given words are conjugate in the braid group. However, the Markov equivalence problem has not yet been solved.

1.3 Bridge presentations

Let $D = p(L)$ be a link diagram of a link L in $\mathbf{R}^3$. Let $B_1 \cup \cdots \cup B_m$ be a union of mutually disjoint arcs in L that contains all overcrossings but not any undercrossings of D. We call $B_1, \ldots B_m$ *overbridges* of L and $p(B_1), \ldots p(B_m)$ *overbridges* of D. Then $\mathrm{cl}(L - (B_1 \cup \cdots \cup B_m))$ consists of m mutually disjoint arcs $C_1, \ldots, C_m$, which we call *underbridges* of L and whose projections we call *underbridges* of D. For a given diagram D, there are many choices of overbridges of D. If each overbridge contains at least one overcrossing and each underbridge contains at least one undercrossing then the number m of overbridges is minimal in the diagram D. We call such a number m the *bridge number* of D and we say that D is an *m-bridge diagram.* The *bridge number* $b(L)$ of a link L is the minimum of the bridge numbers of all diagrams of all links with the same link type as L. The diagram shown in figure 1.3.1 is a 2-bridge diagram for the figure-eight knot, whose bridge number is 2.

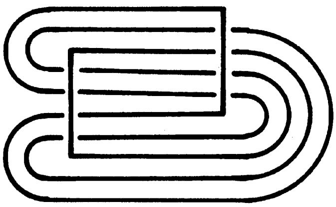


Fig. 1.3.1

Exercise 1.3.1 Show that any 1-bridge link is a trivial knot.

Exercise 1.3.2 Show that a link L is a b-bridge link if and only if L has a $2b$-plat presentation.

Supplementary notes for Chapter 1

Alexander proved that any link type can be presented by a closed braid in [Alexander 1923]. The proof of Theorem 1.2.2 is due to [Yamada 1987]. Usually, the linking number is defined for simplicial cycles by using the intersection numbers of simplicial chains. The intersection number of simplicial chains and the linking number for simplicial cycles are described in [Seifert-Threlfall 1980]. See [Kawauchi 1980] for the definitions of the intersection number of singular chains and the linking number of singular cycles. Some account of the word problem for the braid group is given in [Murasugi 1982]. Some account of the conjugacy problem for the braid group is given in [Birman 1974]. On the other hand, it is known that there is a finitely presented group with no algorithm to determine whether or not a word represents the unit element (cf. [Magnus-Karrass-Solitar 1966]).

Chapter 2
Standard examples

In this chapter, we discuss 2-bridge links, torus links and pretzel links. These links appear very often in studies on knot theory.

2.1 Two-bridge links

The 2-bridge links are first discussed using Schubert's normal form and then using Conway's normal form.

Schubert's Normal Form We consider the projection $p : S^3 - \{p_+, p_-\} \to S^2$ given in 1.1, where $p_+ = (0,0,0,1)$, $p_- = (0,0,0,-1)$, and $S^2 = \{(x,y,z,w) \in S^3 \mid w = 0\}$. By putting $B^3_+ = \{(x,y,z,w) \in S^3 \mid w \geq 0\}$ and $B^3_- = \{(x,y,z,w) \in S^3 \mid w \leq 0\}$, a 2-bridge knot or link K in S^3 (cf. 1.3) can be presented as follows: $K \cap \{p_+, p_-\} = \phi$ and each of $K \cap B^3_+$ and $K \cap B^3_-$ consists of two arcs which are mapped injectively into S^2 by p. The arc components $w_i (i = 1,2)$ of $K \cap B^3_+$ and $v_i (i = 1,2)$ of $K \cap B^3_-$ are the overbridges and the underbridges of K, respectively. We assume that K meets S^2 in four points A, B, C, and D, where the initial point and the terminal point of w_1 are A and B, respectively, the initial point and the terminal point of w_2 are C and D, respectively, the initial point of v_1 is B, and the initial point of v_2 is D. Further, we can deform K by an ambient isotopy of S^3 so that the overbridges $p(w_1)$ and $p(w_2)$ are straight (i.e. geodesic) lines in S^2, and each of the underbridges $p(v_1)$ and $p(v_2)$ intersects the overbridges transversally and alternately. More precisely, for a 2-bridge link K, there is a pair of coprime integers (α, β) satisfying

$$(2.1.1) \qquad \alpha > 0, \quad -\alpha < \beta < \alpha, \quad \beta \text{ is odd},$$

and K has the following regular projection: each bridge is divided into α segments and numbered from 0 to $2\alpha - 1$ modulo 2α as shown in figure 2.1.1. Thus B and D are numbered 0, and A and C are numbered α. Along the underbridge $p(v_1)$, one starts from 0 of the overbridge $p(w_1)$, which is B, and meets $p(w_2)$ at β, and next meets $p(w_1)$ at 2β, and then meets $p(w_2)$ at 3β. This is to be repeated until one reaches either $\alpha\beta$ of $p(w_2)$ $(= C)$ or α of $p(w_1)$ $(= A)$ according to whether α is odd or even. Similarly, along the underpass $p(v_2)$, one starts from 0 of $p(w_2)$, which is D, and meets $p(w_1)$ at β, and next meets $p(w_2)$ at 2β, and then meets $p(w_1)$ at 3β. This is to be repeated until one reaches either $\alpha\beta$ of $p(w_1)$ $(= A)$ or α of $p(w_2)$ $(= C)$ according to whether α is odd or even. We call this regular projection *Schubert's normal form* of a 2-bridge link and denote it by $S(\alpha, \beta)$, which is a knot or a 2-component link according to whether α is odd or even. For example, $S(5, -3)$ and $S(2, \pm 1)$ are shown in figures 2.1.2 and 2.1.3, respectively; the latter is called the *Hopf link*.

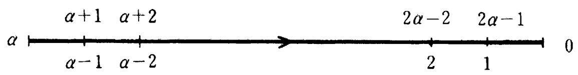


Fig. 2.1.1

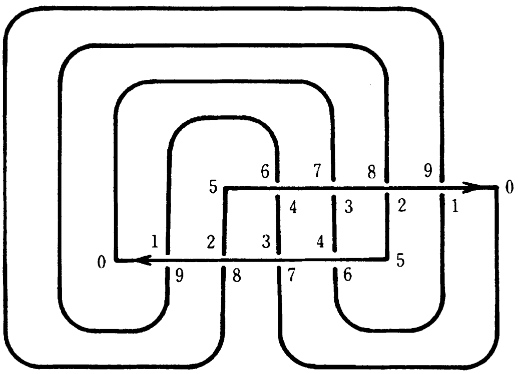


Fig. 2.1.2

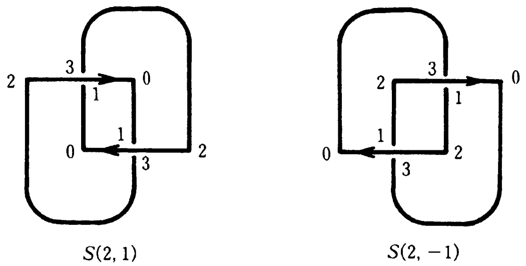


Fig. 2.1.3

Theorem 2.1.1 *The two-fold branched covering space over S^3 with branch set the 2-bridge link $S(\alpha, \beta)$ is the lens space $L(\alpha, \beta)$.*

Proof. Since both $(B_+^3, w_1 \cup w_2)$ and $(B_-^3, v_1 \cup v_2)$ are trivial tangles (cf. Chapter 3), the two-fold branched covering spaces over B_+^3 and B_-^3 with branch sets $w_1 \cup w_2$ and $v_1 \cup v_2$, respectively, are solid tori V_+ and V_-. Thus the two-fold branched covering space over S^3 with branch set a 2-bridge link is a lens space (cf. Appendix D). Each of the lifts of $p(w_1)$, $p(w_2)$ to V_+ is a meridian of V_+. Also, each of the lifts of $p(v_1)$ and $p(v_2)$ to V_- is a meridian of V_-. Thus, if the lift of $p(w_1)$ (or $p(w_2)$) is a standard meridian, then that of $p(v_1)$ (or $p(v_2)$) is a characteristic curve of the lens space. Let us consider $S(3, 1)$, which is shown in figure 2.1.4 a. Cut S^2 along $p(w_1) \cup p(w_2)$ to get the annulus as shown in figure 2.1.4 b. Then the two-fold branched covering space over S^2 with branch set $\{A, B, C, D\}$ is a torus which is obtained by gluing the two boundaries of the annulus given in figure 2.1.4

c. Since each of the lifts of $p(w_1)$ and $p(w_2)$) is a meridian of V_+, each of the lifts of $p(v_1)$ and $p(v_2)$ is homologous to 3(meridian)+1(longitude) in ∂V_+ with respect to a meridian-longitude system of V_+. Hence the two-fold branched covering space over S^3 with branch set $S(3,1)$ is the lens space $L(3,1)$. For other $S(\alpha,\beta)$, the assertion can be proved in the same way. □

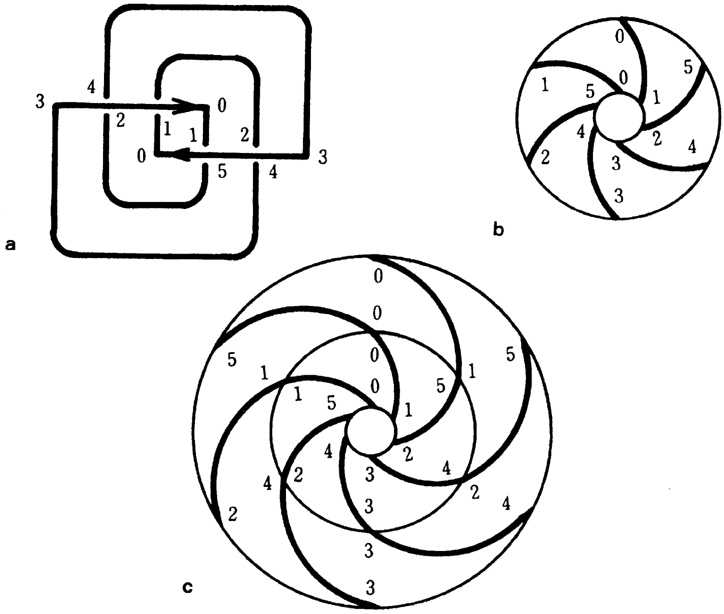


Fig. 2.1.4

Exercise 2.1.2 *Prove that $S(\alpha,\beta)$ is invertible.*

Theorem 2.1.3 (1) *The 2-bridge knots $S(\alpha,\beta)$ and $S(\alpha',\beta')$ belong to the same type if and only if*

$$\alpha = \alpha', \quad \beta^{\pm 1} \equiv \beta' \pmod{\alpha}.$$

(2) *The 2-component 2-bridge links $S(\alpha,\beta)$ and $S(\alpha',\beta')$ belong to the same type if and only if*

$$\alpha = \alpha', \quad \beta^{\pm 1} \equiv \beta' \pmod{2\alpha}.$$

If we consider positive-equivalence instead of the link type, then the condition of (2) *reduces to that of* (1).

Proof. When we ignore the orientation, the proof follows from the classification of the lens space (cf. Appendix D). For the oriented case, we refer to [Schubert 1956]. □

Exercise 2.1.4 Prove that the mirror image of $S(\alpha,\beta)$ is $S(\alpha,-\beta)$, so that the 2-bridge knot $S(\alpha,\beta)$ is amphicheiral if and only if $\beta^2 \equiv -1 \pmod{\alpha}$.

Exercise 2.1.5 Show that if we change the orientation of one of the components of the 2-component 2-bridge link $S(\alpha,\beta)$, then this becomes $S(\alpha,\beta')$, where $\beta' \equiv \alpha+\beta \pmod{2\alpha}$.

Example 2.1.6. $S(32,7)$ and $S(32,-25)$ are positive-equivalent as unoriented links, but do not belong to the same type as oriented links. In this example, we see that the linking numbers are both zero.

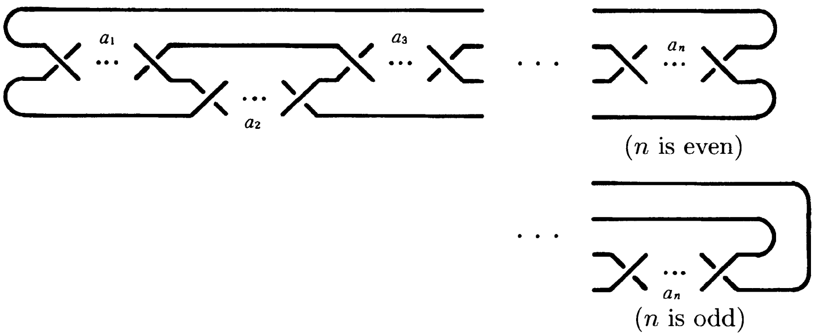


Fig. 2.1.5

Conway's Normal Form Any 2-bridge link has a 4-plat presentation, which can be further deformed as in figure 2.1.5, where a_i indicates $|a_i|(\neq 0)$ crossing points with sign $\varepsilon_i = a_i/|a_i| = \pm 1$. We denote the unoriented 2-bridge link with this regular projection by $C(a_1, a_2, \ldots, a_n)$, which is called *Conway's normal form.* For example, $C(3)$ and $C(3,-2,2,3)$ are shown in figures 2.1.6 a and b, respectively.

Exercise 2.1.7 Show that $C(a_1, a_2, \ldots, a_n)$ and $C(a_1, a_2, \ldots, a_n+\varepsilon, -\varepsilon)$, $\varepsilon = \pm 1$, are positive-equivalent.

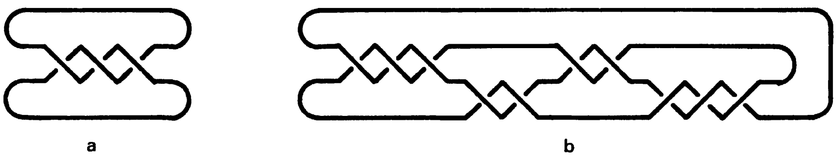


Fig. 2.1.6

Exercise 2.1.8 *Show that the mirror image of* $C(a_1,\ a_2,\ \ldots,\ a_n)$ *is* $C(-a_1,\ -a_2,\ \ldots,\ -a_n)$.

Exercise 2.1.9 Show that $C(a_n, \ldots, a_2, a_1)$ and $C(\varepsilon a_1, \varepsilon a_2, \ldots, \varepsilon a_n)$, where $\varepsilon = (-1)^{n-1}$, are positive-equivalent.

Theorem 2.1.10 *Let $\alpha(> 0)$ and β be coprime integers obtained by the continued fraction:*

(2.1.10.1)
$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}$$

Then the two-fold covering space over S^3 with branch set $C(a_1, a_2, \ldots, a_n)$ is the lens space $L(\alpha, \beta)$.

See [Burde-Zieschang 1985] for proof.

Let (α, β') be a pair of integers satisfying (2.1.1) and $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$. Then by Theorem 2.1.10, $C(a_1, a_2, \ldots, a_n)$ and $S(\alpha, \beta')$ are positive-equivalent (as unoriented links). Conversely, given a 2-bridge link $S(\alpha, \beta)$ in Schubert's normal form, using the continued fraction (2.1.10.1) we can deform it into Conway's normal form. Furthermore, we may suppose that all of the $a_1, a_2, \ldots, a_n$ are positive or negative according to whether β is positive or negative. Thus a 2-bridge link is alternating (cf. Definition 8.4.11). By Exercises 2.1.7 and 2.1.9, we can deform $S(\alpha, \beta)$ into $C(b_1, b_2, \ldots, b_m)$ so that all the b_i's are either positive or negative and neither $|b_1|$ nor $|b_m|$ is equal to one. We know that such a presentation is unique from the uniqueness of the continued fraction (Exercises 2.1.12 and 2.1.13):

Theorem 2.1.11 *Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$ be sets of integers such that all the a_i's or b_j's are positive or negative and none of $|a_1|$, $|a_n|$, $|b_1|$, $|b_m|$ is one. Then the 2-bridge links $C(a_1, a_2, \ldots, a_n)$ and $C(b_1, b_2, \ldots, b_m)$ are positive-equivalent if and only if $m = n$ and either $a_i = b_i$ or $a_i = \varepsilon b_{m-i}$ with $\varepsilon = (-1)^{m-1}$, for all i.*

Exercise 2.1.12 Let the a_i's be positive integers, and (α, β) and (α', β') the pairs of integers satisfying the condition (2.1.1) obtained from the continued fractions:

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}, \quad \frac{\alpha'}{\beta'} = a_n + \cfrac{1}{a_{n-1} + \cdots + \cfrac{1}{a_1}}.$$

Then prove that $\alpha = \alpha'$ and $\beta\beta' \equiv (-1)^{n-1} \pmod{\alpha}$.

Exercise 2.1.13 Let the a_i's and b_j's be positive integers. Suppose that

$$\frac{\alpha}{\beta} = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}} = b_1 + \cfrac{1}{b_2 + \cdots + \cfrac{1}{b_m}}$$

where $n \geq m$. Then prove that either

$$n = m \text{ and } (a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m), \text{ or}$$
$$n = m + 1 \text{ and } (a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m - 1, 1).$$

Exercise 2.1.14 (2.1.1) Let (α, β) be a pair of integers satisfying (1) and set $\beta' = \beta \pm \alpha$ ($|\beta'| < \alpha$) if α is odd, or $\beta' = \beta$ if α is even. Then prove that α/β' has a continued fraction:

$$\frac{\alpha}{\beta'} = 2b_1 + \cfrac{1}{2b_2 + \cdots + \cfrac{1}{2b_n}},$$

where each b_i is a non-zero integer and n is even or odd according to whether α is odd or even.

(2) Let $D(b_1, b_2, \ldots, b_n)$ be the oriented 2-bridge knot (if n is even) or 2-component 2-bridge link (if n is odd) with the corresponding diagram as shown in figure 2.1.7. Then prove that $S(\alpha, \beta)$ has the same type as $D(b_1, b_2, \ldots, b_n)$. Further, prove that this presentation is unique up to the relation $D(b_1, b_2, \ldots, b_n) = D(-b_n, \ldots, -b_2, -b_1)$.

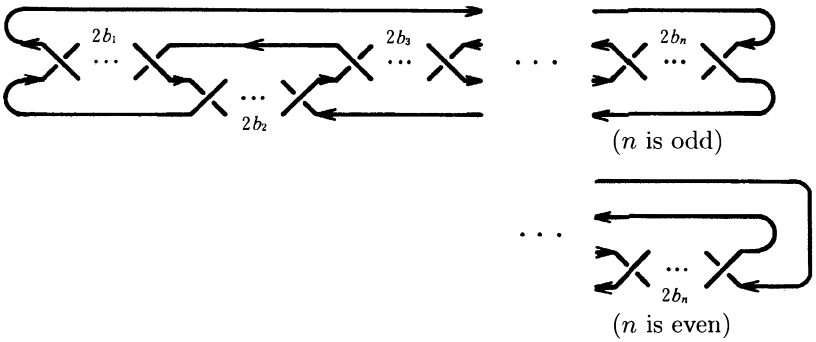


Fig. 2.1.7

Example 2.1.15. If we reverse the orientation of one of the two components of $S(4, 1) = D(-1, 1, -1)$, then we obtain $S(4, -3) = D(2)$.

Exercise 2.1.16 Prove that there exists an ambient isotopy of S^3 which interchanges the components of a 2-component 2-bridge link so that the link orientation remains as it was.

2.2 Torus links

A *torus link* is a link embedded in the standard torus T in S^3. Regarding T as the boundary of a tubular neighborhood of a trivial knot in S^3, we take a meridian-longitude system (m, ℓ) of the trivial knot on T. A torus knot on T is said to be *of type* (p, q) and denoted by $T(p, q)$ if it is homologous to $pm + q\ell$ in T for some coprime integers p and q. The torus link of type (np, nq), denoted by $T(np, nq)$, is the n-component parallel link of such loops which are oriented in the same direction. In other words, $T(np, nq)$ is the closed braid of the nq-braid $(\sigma_1\sigma_2 \ldots \sigma_{nq-1})^{np}$. The proof of the following proposition is not difficult:

$T(3,4)$

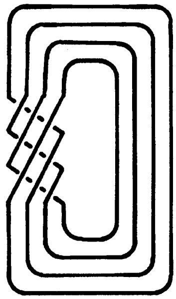


Fig. 2.2.1

Proposition 2.2.1

(1) *$T(\pm 1, q)$ and $T(p, \pm 1)$ are trivial knots.*
(2) *$T(np, nq)$, $T(-np, -nq)$, and $T(nq, np)$ belong to the same link type.*
(3) *The mirror image of $T(np, nq)$ is $T(np, -nq)$.*
(4) *$T(np, nq)$ is invertible.*

The torus knots are classified as follows:

Theorem 2.2.2

(1) *A torus knot $T(p, q)$ is trivial if and only if either $p = \pm 1$ or $q = \pm 1$.*
(2) *Two non-trivial torus knots $T(p, q)$ and $T(p', q')$ belong to the same type if and only if (p', q') is equal to one of (p, q), (q, p), $(-p, -q)$, and $(-q, -p)$.*
(3) *A non-trivial torus knot is not amphicheiral.*

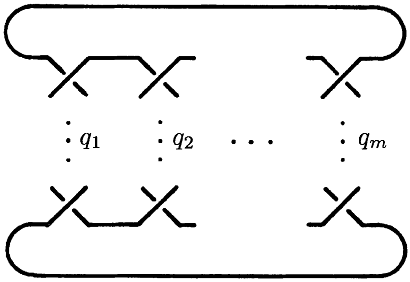
The proof of this theorem may be found in 6.1.17 and 12.2.15.

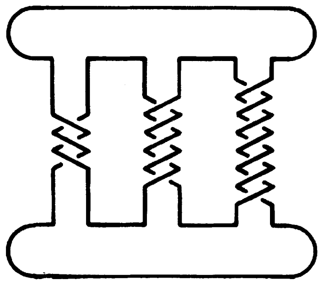
Exercise 2.2.3 Establish a similar classification for torus links.

Exercise 2.2.4 Find all of the torus links with crossing number ≤ 10.

2.3 Pretzel links

For non-zero integers $q_1, q_2, \ldots, q_m$, the link with the regular projection shown in figure 2.3.1 is called the *pretzel link* and denoted by $P(q_1, q_2, \ldots, q_m)$, where q_i indicates $|q_i|$ crossing points with sign $\varepsilon = q_i/|q_i| = \pm 1$. Suppose that $(q_1', q_2', \ldots, q_m')$ is a cyclic permutation of $(q_1, q_2, \ldots, q_m)$. Then $P(q_1', q_2', \ldots, q_m')$ and $P(q_1, q_2, \ldots, q_m)$ are positive-equivalent. If $q_i = \pm 1$, then $P(q_1, \ldots, q_i, \ldots, q_m)$ is positive-equivalent to $P(q_i, q_1, \ldots, \hat{q}_i, \ldots, q_m)$. So any pretzel link can be deformed into the form of $P(\varepsilon, \ldots, \varepsilon, p_1, p_2, \ldots, p_n)$, $\varepsilon = \pm 1$ and $|p_i| > 1$, which we denote by $P(-\varepsilon b;\ p_1, p_2, \ldots, p_n)$, with b the number of ε. If $b > 0$ and $p_i = -2\varepsilon$, then $P(-\varepsilon b; p_1, p_2, \ldots, p_n)$ has the same type as $P(-\varepsilon(b-1); p_1, p_2, \ldots, -p_i, \ldots, p_n)$. Thus we can assume that none of p_i is equal to -2ε when $b > 0$. Then the condition for the pretzel link $P(-\varepsilon b; p_1, p_2, \ldots, p_n)$ to be a knot is that either $n \geq 0$ and all of the p_i's and $n + b$ are odd or that $n \geq 1$ and just one of the p_i's is even. We call it a *pretzel knot of odd type* in the former case, and a *pretzel knot of even type* in the latter case. The pretzel knot is oriented by the orientation of the top arc running from right to left. The following classification theorem for pretzel knots is well-known:





$P(-3,5,7)$

Fig. 2.3.1

Theorem 2.3.1

(1) *A pretzel knot $P(-\varepsilon b; p_1, p_2, \ldots, p_n)$ is a 2-bridge knot (or possibly a trivial knot) if and only if $n \le 2$ and $P(-b; p_1, p_2)$ has the same type as $C(p_1, b, p_2)$.*

(2) *Two pretzel knots $P(-b; p_1, p_2, \ldots, p_n)$ and $P(-c; q_1, q_2, \ldots, q_m)$ which are neither 2-bridge nor trivial belong to the same type if and only if $m = n$, $b = c$ and one of the following conditions is satisfied:*

(a) *Both are of odd type and $(q_1, q_2, \ldots, q_n)$ is a cyclic permutation of $(p_1, p_2, \ldots, p_n)$.*

(b) *Both are of even type and $(q_1, q_2, \ldots, q_n)$ is a cyclic permutation of either $(p_1, p_2, \ldots, p_n)$ or $(p_n, \ldots, p_2, p_1)$.*

Further, the following is also known:

Theorem 2.3.2 *A pretzel knot $P(-\varepsilon b; p_1, p_2, \ldots, p_n)$ which is neither a 2-bridge knot nor a trivial knot is a torus knot if and only if $b = 0$, $n = 3$, and (p_1, p_2, p_3) is a cyclic permutation of either $(3\varepsilon, 3\varepsilon, -2\varepsilon)$ or $(3\varepsilon, 5\varepsilon, -2\varepsilon)$, where $\varepsilon = \pm 1$.*

Exercise 2.3.3 What are the types of the torus knots appearing in Theorem 2.3.2?

Exercise 2.3.4 Find pretzel links which are also torus links.

Supplementary notes for Chapter 2

The 2-bridge links were first studied in [Bankwitz-Schumann 1934] as 4-plat presentations, which is just Conway's normal form. They were classified by Schubert [Schubert 1956] via the classification of lens spaces. Another proof was given in [Burde 1975] by using the linking numbers of branched covering spaces. Conway's normal form was re-introduced in [Conway 1970] through the tangle theory (cf. Chapter 3). See also [Siebenmann *] on this matter. For Exercise 2.1.14, see [Kanenobu-Miyazawa 1992]. The classification of the torus knots followed from that of the free product $\mathbf{Z}_p * \mathbf{Z}_q$ which is the quotient group of the group of the torus knot (cf.[Schreier 1924]). Now the Alexander polynomial is an easy tool for solving this problem (cf. Exercise 7.4.4). The torus knots are characterized as the only knots whose groups have non-trivial centers (cf. Corollary 6.3.6). It is also known that the bridge index of a torus knot $T(p,q)$ with $|p| > |q|$ is $|q|$ (see [Schubert 1954]) and the minimal crossing number is $|p|(|q|-1)$ (cf. [Murasugi 1991] where the Jones polynomial discussed in Chapter 3 is used). The unknotting number of $T(p,q)$ is $(|p|-1)(|q|-1)/2$, which is the affirmative answer to Milnor's conjecture [Milnor 1968]. This last result was proved by F. B. Kronheimer and T. S. Mrowka in [Kronheimer-Mrowka 1993], who determined the 4-dimensional genus of $T(p,q)$ (defined in 12.3) by applying gauge theory to an embedded surface in a 4-manifold. The pretzel knot first appeared in the book of Reidemeister [Reidemeister 1932] as an example of a knot with trivial Alexander polynomial, and thereafter was often treated by many authors. A special feature of this knot is that the two-fold branched covering space over S^3 with this knot as the branch set is a Seifert manifold. From this point of view, J. M. Montesinos generalized the pretzel links to a class of links called the *Montesinos links* (cf. [Montesinos 1973', *]). The Montesinos link $M(-\varepsilon b;(p_1,q_1),(p_2,q_2),\dots,(p_n,q_n))$ is obtained from the pretzel link $P(-\varepsilon b;p_1,p_2,\dots,p_n)$ by replacing each 2-string braid of p_i-half twists with a rational tangle with slope q_i/p_i (see 3.3 for rational tangles). The classification of the Montesinos links is stated in [Burde-Zieschang 1985] (see also 10.7). The pretzel links are not in general simple links (see Chapter 3 for this definition). For example, $P(0;2,-2)$ is a 2-component trivial link. However, every pretzel knot is a simple knot. See, for example, [Kawauchi 1985'] for this proof and the proofs of Theorems 2.3.1 and 2.3.2. It should be noted that the π-orbifold group (defined in 10.6.7) of any pretzel link is a reflection group in 2-dimensional spherical, Euclidean or hyperbolic space, which has been known since the appearance of [Reidemeister 1932].

Chapter 3
Compositions and decompositions

In this chapter, we discuss how to construct a new link from given links by various compositions. Then we discuss decompositions, which are the inverse operations of compositions. After that, compositions of tangles are discussed. Throughout this chapter, links are understood to be links in S^3.

3.1 Compositions of links

We explain here the concepts of connected sum, band sum and companionship. For two oriented knots (S^3, K_1) and (S^3, K_2), let P_i be a point on K_i, and (B_i^3, B_i^1) a regular neighborhood of P_i in (S^3, K_i), which is a trivial ball pair $(i = 1, 2)$. Then we have the following definition:

Definition 3.1.1 The *connected sum* (or *composition*) of the knots K_1 and K_2, denoted by $K_1 \sharp K_2$, is an oriented knot obtained from the disjoint union of the manifold pairs $(S^3 - \mathrm{int}B_i^3, K_i - \mathrm{int}B_i^1)$ $(i = 1, 2)$ by pasting their boundaries along an orientation-reversing homeomorphism $\varphi : (\partial B_2^3, \partial B_2^1) \to (\partial B_1^3, \partial B_1^1)$.

The knots K_1 and K_2 are called the *factors* of the connected sum $K_1 \sharp K_2$. The construction can be simply described as follows: $K_1 \sharp K_2$ is a knot obtained by connecting any diagram of K_1 with that of K_2, as shown in figure 3.1.1.

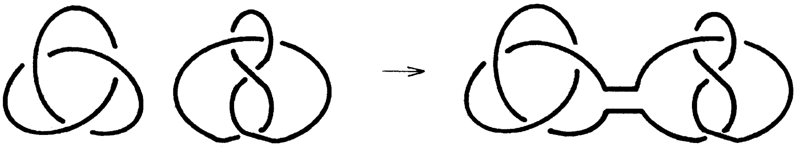


Fig. 3.1.1

Exercise 3.1.2 For two given knots K_1 and K_2, show that the knot type of the connected sum $K_1 \sharp K_2$ is uniquely determined.

Exercise 3.1.3 Show that the set of knot types forms an abelian semi-group with unit element under the connected sum operation.

For a 2-component split link $K_1 \cup K_2$, we take a disk B so that $b_1 = K_1 \cap B$ and $b_2 = K_2 \cap B$ are arcs in ∂B and some orientation of B is coherent with the orientations of K_1 and K_2.

Definition 3.1.4 A *band sum* of K_1 and K_2, denoted by $K_1 \sharp_b K_2$ is the knot $K_1 \cup K_2 \cup (\partial B - (\mathrm{int}b_1 \cup \mathrm{int}b_2))$ for such a disk B.

The orientation of $K_1 \sharp_b K_2$ is chosen to coincide with that of $K_1 - b_1$ (and also that of $K_2 - b_2$). The band sum operation is a special case of a *hyperbolic transformation* of a link (in 12.3) and also of a *fusion* of a link (in 13.1).

Exercise 3.1.5 Show that the knot type of a band sum $K_1 \sharp_b K_2$ is not uniquely determined by K_1 and K_2.

Exercise 3.1.6 Give the definitions of connected sum and band sum for links.

Let $V^* = N(K)$ be a regular neighborhood of a knot K in S^3, and let L be a link in a solid torus $V = D^2 \times S^1$ such that L is not contained in any 3-ball in V.

Definition 3.1.7 A *satellite* of the knot K is a link which is the image $L^* = \varphi(L) \subset V^* \subset S^3$ for a homeomorphism $\varphi : V \to V^*$.

In this definition, the knot K is also called a *companion* of the link L^*.

Exercise 3.1.8 Show that each link in a solid torus V as illustrated in figure 3.1.2 is not contained in any 3-ball in V.

a b c

Fig. 3.1.2

Definition 3.1.9 A *double* of a knot K is a satellite of K obtained by an image of the knot shown in figure 3.1.2a. In particular, a *twist knot* is a double of a trivial knot.

In the definition 3.1.7 above, L^* is not uniquely determined by K and L. Usually, we take the homeomorphism φ to send the the standard meridian-longitude system (ℓ_0, m_0) of V to a meridian-longitude system (ℓ, m) of K on V^*, which we call a *faithful* homeomorphism.

Exercise 3.1.10 When we use a faithful homeomorphism φ, show that L^* is uniquely determined by K and L.

Definition 3.1.11 A *cable knot* of a knot K is a satellite L^* of K in the definition 3.1.7 such that K is a non-trivial knot and L is a knot on ∂V.

More precisely, a cable knot L^* of a knot K is called the (p,q)-*cable knot* of K if φ is taken to be a faithful homeomorphism and L is homologous to $p\ell_0 + qm_0$ in ∂V.

Exercise 3.1.12 Show that both knots K_1 and K_2 are companions of the connected sum $K_1 \sharp K_2$.

Exercise 3.1.13 Show that, if both knots K_1 and K_2 are companions of each other, then $K_1 \cong \pm K_2$.

3.2 Decompositions of links

To decompose a link into links with respect to the connected sum or companionship, it is important to determine which links are not decomposable. In this section, such links are introduced. The following theorem is important for the connected sum decomposition:

Theorem 3.2.1 (Non-cancellation theorem) *A connected sum $L_1 \sharp L_2$ of any two links L_1 and L_2 is not a trivial link unless both links L_1 and L_2 are trivial links.*

The proof (whose details are left to the reader) is essentially obtained from the following two facts:

(1) If L_1 and L_2 are non-split links, then $L_1 \sharp L_2$ is a non-split link.
(2) If $L_1 \sharp L_2$ is a trivial knot, then L_1 and L_2 are trivial knots.

(1) is directly proved by a cut-and-paste argument of combinatorial topology. (2) is usually obtained from Schubert's result on the additivity of the knot genus (cf. 4.1.5) under the connected sum, i.e., $g(L_1 \sharp L_2) = g(L_1) + g(L_2)$ (which is also proved by a cut-and-paste argument).

Definition 3.2.2 A link L is *locally trivial* if any 2-sphere S in S^3 which intersects L transversally in two points bounds a 3-ball intersecting L in a trivial arc.

Exercise 3.2.3 Show the following statements:

(1) A trivial knot is locally trivial.
(2) A trivial 2-component link is locally trivial.

Definition 3.2.4 A link is *prime* if it is locally trivial, non-split and non-trivial.

Exercise 3.2.5 Show that each link shown in figure 3.2.1 is prime.

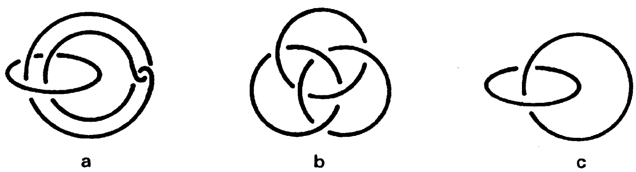


Fig. 3.2.1

Two-bridge knots and torus knots are well-known examples of prime knots.

Theorem 3.2.6 (Unique factorization theorem) *A non-split link can be decomposed into finitely many prime links with respect to the connected sum. Further, the decomposition is unique in the following sense: if*

$$L_1 \sharp L_2 \sharp \cdots \sharp L_m \cong L'_1 \sharp L'_2 \sharp \cdots \sharp L'_n$$

for prime links $L_i (i = 1, 2, ..., m)$ and $L'_j (j = 1, 2, ..., n)$, then we have $m = n$ and $L_i \cong L'_i$ $(i = 1, 2, \ldots, n)$ after permuting the indices suitably.

The proof is given by cut-and-paste argument in [Schubert 1949] and [Hashizume 1958].

Definition 3.2.7 A link L is *atoroidal* if any torus T in the interior of the link exterior $E(L) = \mathrm{cl}(S^3 - N(L))$ of L is compressible or ∂-parallel in $E(L)$ (see the terminology of Appendix C).

Definition 3.2.8 A link is *simple* if it is prime and atoroidal.

Exercise 3.2.9 Show the following statements:

(1) The link shown in figure 3.2.2, which is a connected sum of two Hopf links, is atoroidal.
(2) Non-trivial torus knots are simple.

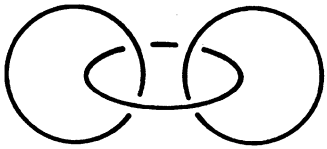


Fig. 3.2.2

Definition 3.2.10 A link L is *anannular* if any annulus A properly embedded in the exterior $E(L)$ of L is compressible or ∂-parallel (see the terminology in Appendix C).

Definition 3.2.11 A link L is *hyperbolic* if it is simple and anannular.

Remark. The word "hyperbolic"in Definition 3.2.11 originates from the fact that, due to Thurston's hyperbolization theorem, the exterior $E(L)$ is a hyperbolic manifold of finite volume in the sense of Appendix C (cf. Theorems C.7.2 and 6.1.13).

Exercise 3.2.12 Show that no torus link is hyperbolic.

Definition 3.2.13 A link L *has only trivial companions* if L has no companion except its components and a trivial knot.

Exercise 3.2.14 Check the differences among the following properties:

(1) A link is simple.
(2) A link is hyperbolic.
(3) A link has only trivial companions.

[*Hint*: In the case of knots, (1) and (3) are equivalent. If a knot is simple but not hyperbolic, then it is a torus knot.]

3.3 Definition of a tangle and examples

In this section, the concept of a tangle is introduced.

Definition 3.3.1 A *tangle* is the pair consisting of a 3-ball B^3 and a (possibly disconnected) proper 1-submanifold t with $\partial t \neq \emptyset$. In particular, it is an *n-string tangle* if t consists of n arcs.

Note that we do not consider the case $\partial t = \emptyset$ as a tangle (B, t).

Definition 3.3.2 A *trivial* (n-string) *tangle* is a tangle homeomorphic to the pair $(D^2, \{a_1, a_2, \ldots, a_n\}) \times [0,1]$ for some interior points $a_1, a_2, \ldots, a_n$ of D^2.

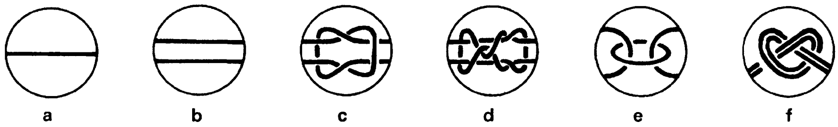
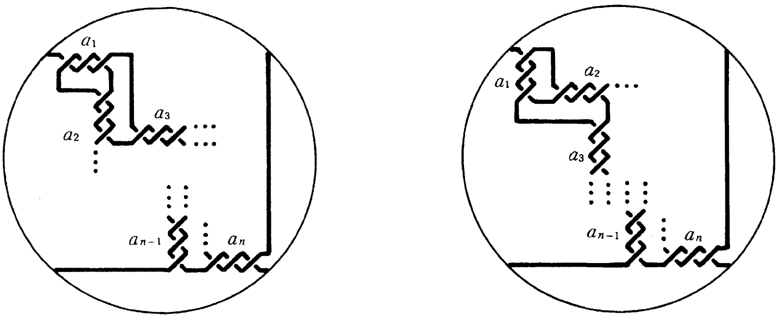


Fig. 3.3.1

In the tangles shown in figure 3.3.1, a and b are trivial tangles. We say that the tangles a, c, d and e are respectively a *trivial arc*, the *clasp*, the *K-T tangle* and the *chain tangle.* In the following definition, we give a notation for trivial 2-string tangles, which was introduced by [Conway 1970].

Definition 3.3.3 The *Conway notation* $a_1 a_2 \cdots a_n$ means the diagram of a trivial 2-string tangle obtained from a sequence of non-zero integers $a_1, a_2, \ldots, a_n$, as it is indicated in figure 3.3.2 where a_i denotes $|a_i|$ crossings with sign $\varepsilon_i = a_i/|a_i| = \pm 1$ $(i = 1, 2, \ldots, n)$. (We consider the signs of the crossings in figure 3.3.2 to be all positive.)



(n is odd) (n is even)

Fig. 3.3.2

Exercise 3.3.4 Show that every tangle shown in figure 3.3.2 is a trivial 2-string tangle.

We note that the link obtained from the tangle above by gluing the ears as shown in figure 3.3.3 is the 2-bridge knot $C(a_{n-1}, a_{n-2}, \ldots, a_2, a_1)$ (cf. 2.1). The trivial 2-string tangle with Conway notation $a_1 a_2 \cdots a_n$ is also called a *rational tangle* with *slope*

$$a_n + \cfrac{1}{a_{n-1} + \cdots + \cfrac{1}{a_1}},$$

which is a rational number or ∞. We see that the rational tangles with slopes 0 and ∞ are respectively deformed into the tangles of figures 3.3.1b and 3.4.1a by ambient isotopies keeping the boundary fixed.

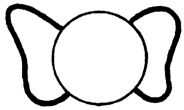


Fig. 3.3.3

For two tangles (A, s) and (B, t), suppose that the numbers of points in ∂s and in ∂t are equal. Then we have the following definition:

Definition 3.3.5 A *tangle sum* of (A, s) and (B, t) is the link $(A, s) \cup_\varphi (B, t)$ obtained by gluing them together via a homeomorphism $\varphi : \partial(B, t) \to \partial(A, s)$.

Example 3.3.6. As shown in figure 3.3.4, the link type of a tangle sum is not uniquely determined by the tangles.

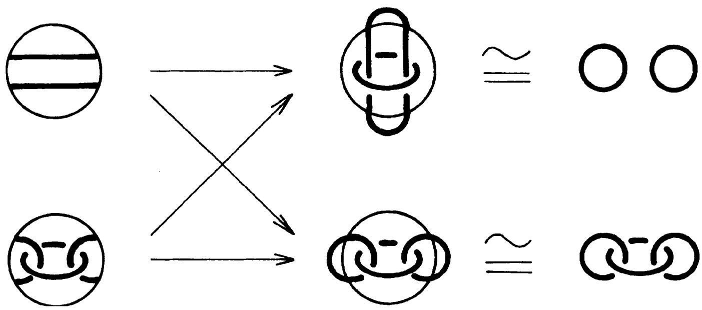


Fig. 3.3.4

Exercise 3.3.7 Construct a pair of distinct knots from the same pair of tangles by taking distinct tangle sums.

3.4 How to judge the non-splittability of a link

In this section, we give a condition for a link presented by a tangle sum to be a non-split link.

Definition 3.4.1 A tangle (B, t) is *non-split* if any proper disk D in B does not split t in B.

Example 3.4.2. The tangle shown in figure 3.4.1a is split, but those in figures 3.4.1b and 3.4.1c are non-split. In fact, there is a splitting disk shown in figure 3.4.1a. In figure 3.4.1b, both arc components link the loop component which represents a

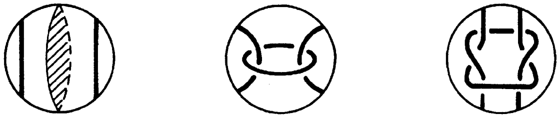


Fig. 3.4.1

generator of the fundamental group of the complement of each arc in B. In figure 3.4.1c, if the arc components are split, then the tangle must be a trivial 2-string tangle by the triviality of the arcs, and the knot obtained from this tangle by gluing the ears shown in figure 3.3.3 must be a 2-bridge knot, in particular, a prime knot. But the resulting knot is the connected sum of the trefoil knot and its mirror image (which is called the *square knot*), and hence is not prime, a contradiction.

The following theorem is useful for constructing a non-split link:

Theorem 3.4.3 *Any link obtained by any tangle sum of two non-split tangles is non-split.*

Exercise 3.4.4 Prove Theorem 3.4.3. [*Hint*: Suppose that the resulting link is split. Then consider the intersection of the splitting sphere and the bounding sphere of tangles. Cf.[Nakanishi 1981'].]

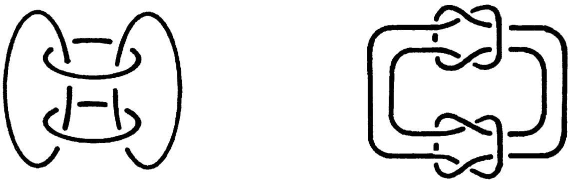


Fig. 3.4.2

Example 3.4.5. The links shown in figure 3.4.2 are non-split.

Here is a method for constructing a non-split tangle.

Theorem 3.4.6 *Let (C, v) be a tangle and D be a disk properly embedded in C such that D divides (C, v) into two tangles (A, s) and (B, t). We assume the following:*

(1) *The numbers of points in $(\partial A - D) \cap v$, $(\partial B - D) \cap v$ and $D \cap v$ are all greater than or equal to one.*
(2) *Any disk Δ properly embedded in A with $\Delta \cap \partial D = \emptyset$ and $\Delta \cap s = \emptyset$ does not split s in A.*
(3) *(B, t) is non-split.*

Then (C, v) is a non-split tangle.

Corollary 3.4.7 *In Theorem 3.4.6 above, we assume the following condition instead of condition (2):*

(2') The tangle (A, s) is non-split.

Then (C, v) is also non-split.

For the proof, see [Nakanishi 1981']. Here is one example.

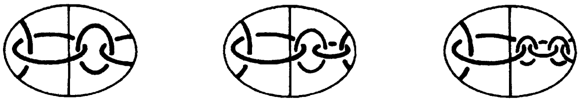


Fig. 3.4.3

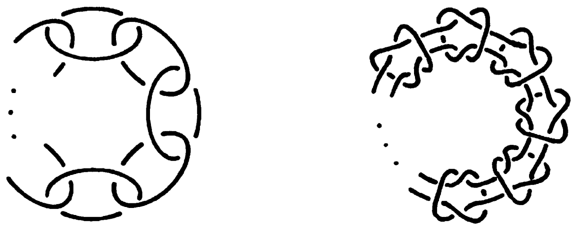


Fig. 3.4.4

Example 3.4.8. By Theorem 3.4.6, all the tangles in figure 3.4.3 can be shown to be non-split. Therefore, by Theorem 3.4.3, the chain links shown in figure 3.4.4 are non-split if their component numbers are greater than three in the case of 3.4.4a, and one in the case of 3.4.4b.

The link of figure 3.4.4b has the property that *every proper sublink is trivial.* A link with this property is called a *Brunnian* (or an *almost-trivial*) link.

Exercise 3.4.9 Prove the assertion in Example 3.4.8.

3.5 How to judge the primeness of a link

In this section, we give a condition for a link presented by a tangle sum to be a prime link.

Definition 3.5.1 A tangle (B, t) is *locally trivial* if any 2-sphere S in B which intersects t in two points transversely bounds a 3-ball in B which intersects t in a trivial arc.

Example 3.5.2. The tangles shown in figures 3.5.1a and 3.5.1b are not locally trivial, but those in figures 3.5.1c and d are locally trivial. In fact, some non-trivial 2-spheres are shown in figures 3.5.1a and 3.5.1b. In figure 3.5.1c, there is no

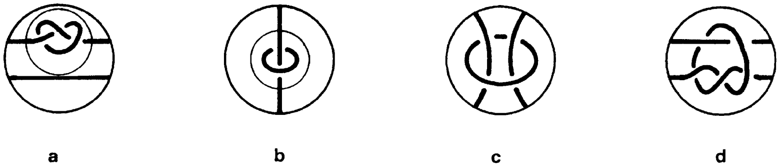


Fig. 3.5.1

2-sphere which bounds a 3-ball intersecting t in a knotted arc by the triviality of the arcs and the loop. If there is a 2-sphere which bounds a 3-ball containing the loop, then we must have an arc outside the ball, since the sphere intersects t in two points. This contradicts the fact that both of the arcs are linking the loop. In figure 3.5.1d, we remark that there is no loop. One arc is a trivial arc and the other is an arc of a trefoil knot. If there is a 2-sphere which bounds a 3-ball intersecting t in a knotted arc, then it must be an arc of the trefoil knot. By the non-cancellation theorem (Theorem 3.2.1), any link obtained from the tangle by gluing ears must have a trefoil knot as a factor. But the knot obtained from this tangle by gluing the ears as shown in figure 3.3.3 is a trivial knot, which is a contradiction.

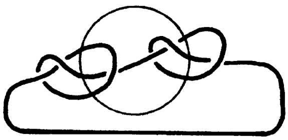


Fig. 3.5.2

Remark 3.5.3 As shown in figure 3.5.2, a link obtained from two locally trivial tangles by a tangle sum need not be locally trivial.

Definition 3.5.4 A tangle (B, t) is *indivisible* if any proper disk D in B which intersects t transversely in a single point divides (B, t) into two tangles, at least one of which is the trivial 1-string tangle shown in figure 3.3.1a.

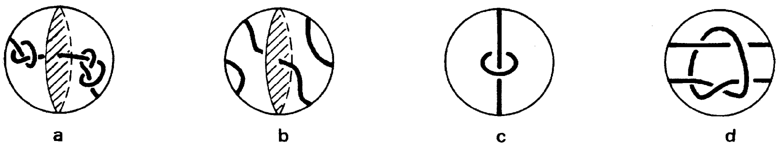


Fig. 3.5.3

Example 3.5.5. The tangles shown in figures 3.5.3a and 3.5.3b are divisible, but those in figures 3.5.3c and 3.5.3d are indivisible. In fact, some dividing disks are shown in figures 3.5.3a and 3.5.3b. In figure 3.5.3c or 3.5.3d, for any dividing disk

D, we consider two tangles obtained by dividing by D. In figure 3.5.3c, one of them is trivial, since the arc in the original 3-ball is trivial. In figure 3.5.3d, one of them has a single arc, since t consists of two arcs, and by the locally triviality, it is the trivial 1-string tangle shown in figure 3.3.1a.

Definition 3.5.6 A tangle is *prime* if it is non-split, locally trivial, and indivisible and if it is not a trivial 1-string tangle.

Exercise 3.5.7 Show that the tangles shown in figure 3.5.4 are not prime.

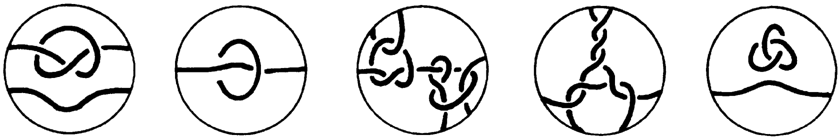


Fig. 3.5.4

Example 3.5.8. The tangles shown in figures 3.3.1c-f and 3.4.3 are prime tangles.

Exercise 3.5.9 Show the primeness of the tangles in Example 3.5.8.

Exercise 3.5.10 For any n-string tangle, show that indivisibility implies local triviality.

Exercise 3.5.11 For any n-string tangle with $n \geq 2$ except the trivial 2-string tangle, show that indivisibility implies primeness.

The following theorem is useful for constructing a prime link:

Theorem 3.5.12 *A link obtained from two prime tangles by any tangle sum is prime.*

The proof is in [Nakanishi 1981']. For example, the links shown in figure 3.4.4 are prime.

Exercise 3.5.13 Give a method of constructing a prime tangle by examining Theorem 3.4.6.

For example, as an answer to this exercise, we have the following theorem:

Theorem 3.5.14 *Let (C, v) be a tangle and D be a disk properly embedded in C such that D divides (C, v) into two tangles (A, s) and (B, t). We assume the following:*

(1) *The numbers of points in $(\partial A - D) \cap v$, $(\partial B - D) \cap v$ and $D \cap v$ are all greater than or equal to two.*
(2) *(A, S) is prime.*
(3) *(B, t) is prime.*

Then (C, v) is a prime tangle.

The proof is in [Nakanishi 1981']. There is also another method as follows:

Theorem 3.5.15 *Let (B, t) be a prime tangle. Let t^* be the union of t and an arc (or loop) parallel to a component of t in B. Then (B, t^*) is a prime tangle.*

Exercise 3.5.16 Prove Theorem 3.5.15 (cf. [Nakanishi 1981']).

The following gives a useful criterion for a tangle to be prime:

Theorem 3.5.17 *A tangle (B, t) is prime if and only if the double branched covering space over B with branch set t is irreducible and boundary-irreducible.*

The proof is an application of the equivariant sphere and loop theorems (cf. Appendix C) and may be found in [Nakanishi 1981']. The following theorem shows that Theorem 3.5.12 is not always almighty for showing primeness.

Theorem 3.5.18 *No 2-bridge knot or torus knot can be presented as a tangle sum of two prime tangles.*

The proof follows from Theorem 3.5.17 and [Magnus-Karras-Solitar 1966(p. 211)].

3.6 How to judge the hyperbolicity of a link

In this section, we give a condition for a link presented by a tangle sum to be a hyperbolic link.

Definition 3.6.1 A tangle (B, t) is *atoroidal* if there is no essential torus in it, namely, any torus T in the interior of the exterior $E(t; B) = \mathrm{cl}(B - N(t))$ is compressible or ∂-parallel to a component of $\mathrm{Fr}N(t) = \mathrm{cl}(\partial N(t) - (\partial N(t) \cap \partial B))$ in $E(t; B)$.

Definition 3.6.2 A tangle (B, t) is *anannular* if there is no essential annulus in it, namely, any proper annulus A in $E(t; B)$ with $\partial A \cap \mathrm{Fr}N(t) = \emptyset$ is compressible or ∂-parallel to a component of $\mathrm{Fr}N(t)$ or $\mathrm{cl}(\partial E(t; B) - \mathrm{Fr}N(t))$ in $E(t; B)$.

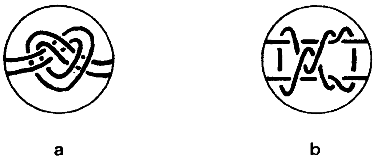


Fig. 3.6.1

Exercise 3.6.3 Show that the tangle shown in figure 3.6.1a is neither atoroidal nor anannular but the tangle shown in figure 3.6.1b is both atoroidal and anannular.

Exercise 3.6.4 Show that a 2-string tangle is anannular if it is locally trivial and atoroidal.

Definition 3.6.5 A tangle (B, t) is *hyperbolic* if it is prime, atoroidal and anannular.

The following theorem is useful for constructing hyperbolic links:

Theorem 3.6.6 *A link obtained from two hyperbolic tangles by any tangle sum is hyperbolic.*

Exercise 3.6.7 Prove Theorem 3.6.6.

3.7 Non-triviality of a link

In this section, we discuss some results due to Y. Nakanishi on the non-triviality of a link containing a given tangle.

Theorem 3.7.1 *For a non-trivial* 2*-string tangle* (B,t) *and two* 2*-string tangles* (A_1,s_1) *and* (A_2,s_2)*, we assume that the tangle sums* $(B,t)\cup_\varphi(A_1,s_1)$ *and* $(B,t)\cup_\psi(A_2,s_2)$ *are trivial knots for two homeomorphisms* $\varphi:\partial(A_1,s_1)\to\partial(B,t)$ *and* $\psi:\partial(A_2,s_2)\to\partial(B,t)$*. Then there exists a homeomorphism* $h:(A_1,s_1)\to(A_2,s_2)$ *such that* $\varphi=\psi h$*.*

Theorem 3.7.2 *For a non-trivial* 2*-string tangle* (B,t) *and two* 2*-string tangles* (A_1,s_1) *and* (A_2,s_2)*, we assume that the tangle sums* $(B,t)\cup_\varphi(A_1,s_1)$ *and* $(B,t)\cup_\psi(A_2,s_2)$ *are trivial* 2*-component links for two homeomorphisms* $\varphi:\partial(A_1,s_1)\to\partial(B,t)$ *and* $\psi:\partial(A_2,s_2)\to\partial(B,t)$*. Then there exists a homeomorphism* $h:(A_1,s_1)\to(A_2,s_2)$ *such that* $\varphi=\psi h$*.*

Theorem 3.7.3 *For a* 2*-string tangle* (B,t) *and two* 2*-string tangles* (A_1,s_1) *and* (A_2,s_2)*, we assume that some tangle sums* $(B,t)\cup_\varphi(A_1,s_1)$ *and* $(B,t)\cup_\psi(A_2,s_2)$ *are a trivial knot and a trivial* 2*-component link, respectively. Then the tangle* (B,t) *is a trivial* 2*-string tangle.*

Outlines of the proofs of these theorems are as follows: We consider the double covering spaces over S^3 with, as branch sets, the trivial knot and the 2-component trivial link given in these theorems. We denote the liftings of B and $A_i, i=1,2$, to these covering spaces by $\tilde{B}$ and $\tilde{A}_i, i=1,2$, respectively. If one of $(B,t),(A_1,s_1)$ and (A_2,s_2) is not locally trivial, i.e., has a local knot, then the resulting knots or links have a local knot as a factor, a contradiction. Hence $(B,t),(A_1,s_1)$ and (A_2,s_2) are all locally trivial, and (B,t) is a prime tangle. If (A_1,s_1) and (A_2,s_2) are non-split, then they are prime, and so the resulting knots or links are prime by Theorem 3.5.12. Therefore, (A_1,s_1) and (A_2,s_2) are all trivial 2-string tangles, and so $\tilde{A}_i(i=1,2)$ are solid tori. For Theorem 3.7.1, we apply the solution of the knot exterior conjecture by [Bleiler-Scharlemann 1988] (for a strongly invertible knot) or [Gordon-Luecke 1989] (cf. 6.1.12) to $\tilde{B}$ which is a non-trivial knot exterior. Then we can conclude that the boundaries of the splitting disks in the trivial tangles (A_1,s_1) and (A_2,s_2) are sent to isotopic curves in $\partial(B,t)$ by φ and ψ. For Theorem 3.7.2, the same conclusion can be obtained from a homological argument. This completes the outlined proofs of Theorems 3.7.1 and 3.7.2. For Theorem 3.7.3, suppose that (B,t) is non-trivial. By the arguments stated above, $\tilde{A}_i, i=1,2$, are solid tori and $\tilde{B}$ is a non-trivial knot exterior. Because $S^1\times S^2$ is a union of $\tilde{B}$ and $\tilde{A}_2$, $\tilde{B}$ must be a solid torus by the solution of the property R conjecture by [Gabai 1987'] (cf. Supplementary notes for Chapter 4), which is a contradiction.

There is another proof for Theorem 3.7.3 using the super-additivity under band sum (cf. [Gabai 1987"], [Scharlemann 1989]). This completes the outlined proof of Theorem 3.7.3.

Exercise 3.7.4 Complete the proofs of Theorems 3.7.1-3.

These theorems tell us that the non-triviality of a link, which is a global property, happens to be determined by a property of an embedded tangle, which is a local property. It would be interesting to study which global properties of links determined by properties of embedded tangles. At the end of this section, we raise a notion and an open problem due to Y. Nakanishi.

Definition 3.7.5 A tangle is a $*$-*tangle* if every link obtained from that tangle and any other tangle by any tangle sum is not trivial.

Open Problem 3.7.6 If there exist mutually disjoint diagrams of n $*$-tangles in a diagram of a knot K, then the unknotting number $u(K)$ (cf. Chapter 11) has $u(K) \geq n$. Is it true or not?

3.8 Conway mutation

In this last section, we explain Conway's mutation of a link, in [Conway 1970]. For a link L in S^3, we take a 2-sphere S in S^3 meeting L transversely in just 4 points. Such a sphere S is called a *Conway sphere* for L. Let ρ be a self-map of $(S, L \cap S)$ of period 2 such that the fixed point set of ρ on S is a two-point set disjoint from $L \cap S$, which we call a *symmetry* on $(S, L \cap S)$. There are three kinds of symmetries on $(S, L \cap S)$.

Definition 3.8.1 A link L' is an *elementary Conway mutant* of a link L if (S^3, L') is obtained from (S^3, L) by splitting along a Conway sphere S and re-gluing by using a symmetry ρ on $(S, L \cap S)$. A link L' is a *Conway mutant* of a link L if L' is obtained from L by a finite sequence of elementary Conway mutants.

We note that for an elementary Conway mutant L' of an oriented link L, there are two canonical ways to orient L'. The following observation gives evidence of difficulty in distinguishing between a link and its Conway mutant (cf. [Viro 1977]):

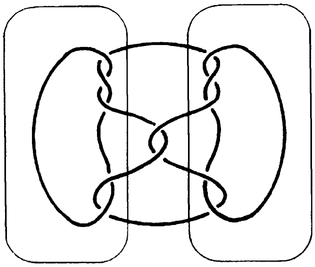
Proposition 3.8.2 *If L' is a Conway mutant of a link L, then the double covering spaces over S^3 with branch sets L and L' are orientation-preservingly homeomorphic.*

Proof. Let S be the Conway sphere S for L and ρ be the symmetry on $(S, S \cap L)$, used for constructing the Conway mutant L' of L. Let $(B_i, L_i)(i = 1, 2)$ with $L_i = B_i \cap L$ be the tangles obtained from (S^3, L) by splitting it along S. The double branched covering space M over S^3 with branch set L is obtained by gluing the double branched covering spaces $\tilde{B}_i(i = 1, 2)$ over $B_i(i = 1, 2)$ with branch sets $L_i(i = 1, 2)$. Let t be the non-trivial covering transformation of M. Note that the lift $\tilde{\rho}$ of ρ to the torus $\partial \tilde{B}_2$ is (non-equivariantly) ambient isotopic to $t\,|_{\partial \tilde{B}_2}$. Then we see that the double branched covering space M' over S^3 with branch set L' is orientation-preservingly (and non-equivariantly) homeomorphic to the manifold M^* obtained from $\tilde{B}_1$ and $\tilde{B}_2$ by re-gluing via $t\,|_{\partial\tilde{B}_2}$ instead of the identity on $\tilde{B}_2$. M^* is orientation-preservingly (and equivariantly) homeomorphic to M by a homeomorphism defined by id : $\tilde{B}_1 \to \tilde{B}_1$ and $t\,|_{\tilde{B}_2}: \tilde{B}_2 \to \tilde{B}_2$. □

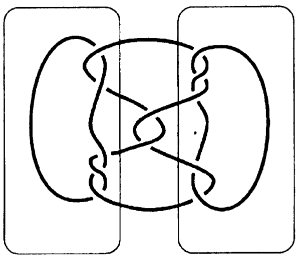
We also observe here another fact in [Cooper 1982] which you can understand after you know the facts in Chapter 5. Namely, the Seifert matrices of any knot and its Conway mutant are S-equivalent, so that their Alexander polynomials and their signatures are equal, respectively. Using Proposition 3.8.2 and this fact, we see that no two knots with up to 10 crossings in the knot table of this book are mutually Conway mutants. Here are some examples of inequivalent knots which are mutually Conway mutants.

Example 3.8.3. Let $P(q_1, q_2, \ldots, q_m)$ be a pretzel knot as described in 2.3. For any permutation $(q_1', q_2', \ldots, q_m')$ of $(q_1, q_2, \ldots, q_m)$, the pretzel knot $P(q_1', q_2', \ldots, q_m')$ is a Conway mutant of $P(q_1, q_2, \ldots, q_m)$, since any permutation is a composition of transpositions. By Theorem 2.3.1, we have finitely many inequivalent pretzel knots which are mutually Conway mutants.

Example 3.8.4. The Conway knot K_C shown in 3.8.1b is an elementary Conway mutant of the Kinoshita-Terasaka knot K_{KT} shown in figure 3.8.1a. These knots are known as non-trivial knots with 11 crossings and with trivial Alexander polynomials. The inequivalence of these knots was first observed by [Riley 1971]. This can be also shown by examining the torus decompositions of their double covering spaces (cf. 10.6) or by examining certain twisted Alexander polynomials of them (cf. [Wada 1994]). It is also observed in [Gabai 1984] that the genera (defined in 4.1.5) of K_{KT} and K_C are 2 and 3, respectively. Although K_{KT} is a ribbon knot defined in 13.1.9, it is unknown whether or not K_C is a ribbon knot or more generally a slice knot (defined in 12.1).



a



b

Fig. 3.8.1

Exercise 3.8.5 For any two links $L_i, i = 1, 2$, any two connected sums $L_1 \sharp L_2$ and $L_1 \sharp - L_2$ are mutually Conway mutants.

It is also easily checked that any Conway mutant of a trivial link, a torus link or a 2-bridge link is positive-equivalent to itself, because any Conway sphere for it necessarily bounds a split tangle.

Supplementary notes for Chapter 3

The notion of a connected sum or more generally that of a companion was introduced by [Schubert 1949], and the notion of primeness is naturally induced from it as a prime factor. For example, the unique factorization theorem for a link ([Schubert 1949], [Hashizume 1958]) makes a great step forward in the study of links. As for prime numbers in elementary number theory, it is a useful approach to consider a prime link as a prime factor. In the old days, only special types of links could be judged to be prime by a technical reason. Since [Kirby-Lickorish 1979], tangle theory has been developed by several knot theorists to be a great machine which enables us to judge primeness and hyperbolicity, even for complicated links (cf. [Myers 1983], [Nakanishi 1981', 1983], [Soma 1983]), though the tangle theory is not all powerful for such judgment. Conway mutation was generalized to the mutation of a 3-manifold in [Ruberman 1987]. Finally, we note that we can construct from any given link (S^3, L) a new link (S^3, L^*) with a map $q : (S^3, L^*) \to (S^3, L)$ such that q is close to a homeomorphism in several senses as an application of the tangle and mutation theories (cf. [Kawauchi 1993]).

Chapter 4
Seifert surfaces I: a topological approach

In this chapter, Seifert surfaces for links are introduced. The notions of an incompressible Seifert surface, a minimal genus Seifert surface, and a fiber Seifert surface are discussed with respect to Murasugi sums.

4.1 Definition and existence of Seifert surfaces

In this section, we define a Seifert surface for a link and discuss Seifert's algorithm for demonstrating the existence of a Seifert surface for any link and some properties of minimal genus Seifert surfaces.

Definition 4.1.1 A *Seifert surface* for a link L in $\mathbf{R}^3$ (or S^3) is a compact oriented 2-manifold S embedded in $\mathbf{R}^3$ (or S^3) such that $\partial S = L$ as an oriented link and S does not have any closed surface components.

Exercise 4.1.2 Let a, b, c be surfaces in $\mathbf{R}^3$ shown in figure 4.1.1 whose boundaries belong to the same trefoil knot type. Show that a is not orientable, thus it is not a Seifert surface, and that b is a Seifert surface which is ambient isotopic to c in $\mathbf{R}^3$.

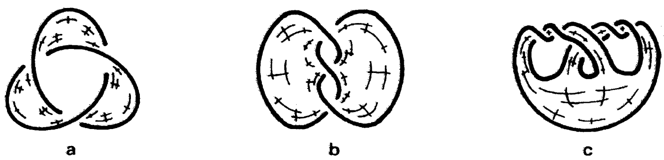


Fig. 4.1.1

Theorem 4.1.3 *For any oriented link L in $\mathbf{R}^3$, there exists a Seifert surface for L.*

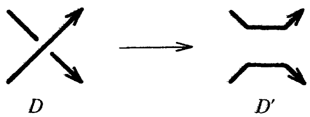


Fig. 4.1.2

Proof (Seifert's algorithm). Let D be a diagram of L in the $z = 0$ plane $\mathbf{R}^2$ in $\mathbf{R}^3$. Let D' be a diagram in $\mathbf{R}^2$, obtained by modifying a neighborhood of each crossing in D into two disjoint arcs, as shown in figure 4.1.2. Then D' has no crossings and hence is the boundary of a collection of oriented disks in $\mathbf{R}^2$. We deform these disks into mutually disjoint disks by slightly pushing their interiors into the upper half space. Then we paste half-twisted bands to the union S_0 of these disjoint disks, as shown in figure 4.1.3, to obtain a compact surface S whose boundary represents the same diagram as D in $\mathbf{R}^2$. Then we can orient S_0 by the orientation determined by the orientation of $D' \cap D$ (which comes from the orientation of L). This orientation of S_0 is naturally extended to the orientation of S as shown in figure 4.1.3. Hence, we have a Seifert surface S for L. □

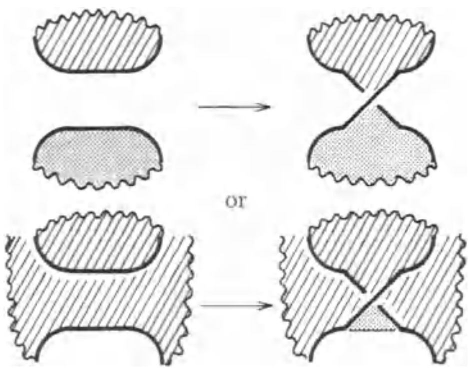


Fig. 4.1.3

Example 4.1.4. The torus link of type (2,4) has two components. We introduce two kinds of orientations on it, as shown in figure 4.1.4 and then apply Seifert's algorithm to the resulting oriented links. The genera of the resulting Seifert surfaces are 0 and 1 and we can see that 0 and 1 are the minimal genera of Seifert surfaces for these links, respectively (cf. 5.4.3). This means that the Seifert surfaces of a link are much affected by the link orientation.

Definition 4.1.5 A Seifert surface S for a link L is a *minimal genus Seifert surface* for L if

$$\chi(S) = \max\{\chi(F) \mid F \text{ is a Seifert surface for } L\},$$

i.e., S is the simplest 2-manifold in the sense of Euler characteristic. Further, the *genus* of a knot K is the genus of the minimal genus Seifert surface for K.

Exercise 4.1.6 The Seifert surface in figure 4.1.5 is obtained by applying Seifert's algorithm to the torus knot of type (3,4). Confirm that the genus of this surface is 3. In a similar way, show that the torus knot of type (p, q) has a Seifert surface with genus $(|p| - 1)(|q| - 1)/2$.

Let $E(= E(L))$ be the exterior of a link L in S^3. Let $S_E = S \cap E (\cong S)$ for a Seifert surface S of L.

Definition 4.1.7 A Seifert surface S for a link L in S^3 is *incompressible* if each component of S_E is incompressible in E (cf. Appendix C).

The following theorem is obtained from the loop theorem (cf. Appendix C):

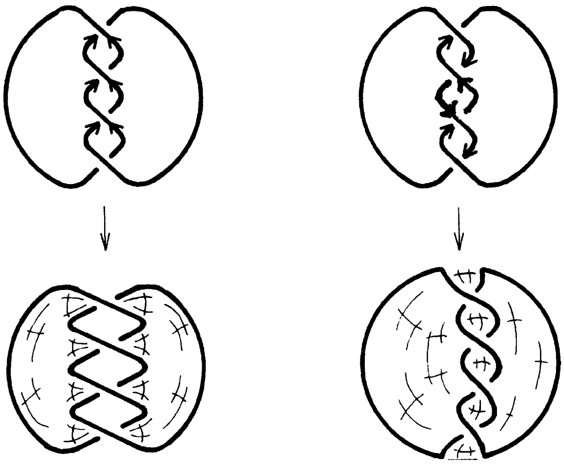


Fig. 4.1.4

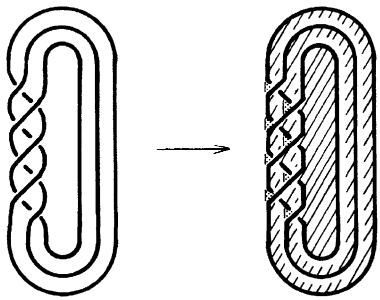


Fig. 4.1.5

Theorem 4.1.8 *A Seifert surface is incompressible if it is of minimal genus.*

In general, the converse of Theorem 4.1.8 is not true. For instance, Lyon [Lyon 1971] showed that there exist knots which have incompressible Seifert surfaces with arbitrarily large genera. Let $(E', (\partial E)')$ be the manifold pair obtained from $(E, \partial E)$ by cutting E along S_E.

Definition 4.1.9 A link L in S^3 is a *fibered link* if there is a Seifert surface S for L such that $(E', (\partial E)')$ is homeomorphic to $(S_E, \partial S_E) \times [0, 1]$.

In this definition, the Seifert surface S is called a *fiber surface.* Since E' is connected, a fiber surface S is connected. In general, we can show that every Seifert surface for a fibered link is connected (see 5.4.4).

Theorem 4.1.10 *The following three conditions on a Seifert surface S for a fibered link L are mutually equivalent:*

(1) *S is a minimal genus Seifert surface.*
(2) *S is an incompressible Seifert surface.*
(3) *S is a fiber surface.*

Exercise 4.1.11 Prove Theorem 4.1.10 [*Hint*: see Theorem 6.3.2].

Fig. 4.1.6

We say that an unknotted annulus ($\subset S^3$) with one full twist is a *Hopf band* (see figure 4.1.6).

Exercise 4.1.12 Show that a Hopf band is a fiber surface.

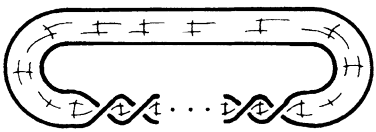


Fig. 4.1.7

Exercise 4.1.13 Let A be an unknotted annulus in S^3 with n full twists ($|n| \neq 0, 1$) as in figure 4.1.7. Show that A is a minimal genus Seifert surface. Show also that A is not a fiber surface (cf. 5.4.4).

4.2 The Murasugi sum

The Murasugi sum of surfaces in S^3 is a powerful machine for the study of various properties of Seifert surfaces. Here, we define the Murasugi sum and state some basic properties and applications.

Definition 4.2.1 A compact oriented surface R embedded in S^3 is a $2n$-*Murasugi sum* (or a $2n$-*generalized plumbing*) of two compact oriented surfaces R_1 and R_2 if we have the following:

(1) $R = R_1 \cup R_2$ with $R_1 \cap R_2$ a disk D such that

(1.1) ∂D is a $2n$-gon with edges $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ enumerated in this order (cf. figure 4.2.1),

(1.2) a_i is contained in ∂R_1 and is a proper arc in R_2 for all i,

(1.3) b_i is contained in ∂R_2 and is a proper arc in R_1 for all i.

(2) There exist 3-balls B_1, B_2 in S^3 (as shown in figure 4.2.2) such that
(2.1) $B_1 \cup B_2 = S^3$, $B_1 \cap B_2 = \partial B_1 = \partial B_2 = S^2$,
(2.2) $B_i \supset R_i \, (i = 1, 2)$,
(2.3) $\partial B_1 \cap R_1 = \partial B_2 \cap R_2 = D$.

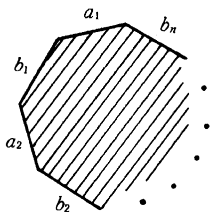


Fig. 4.2.1

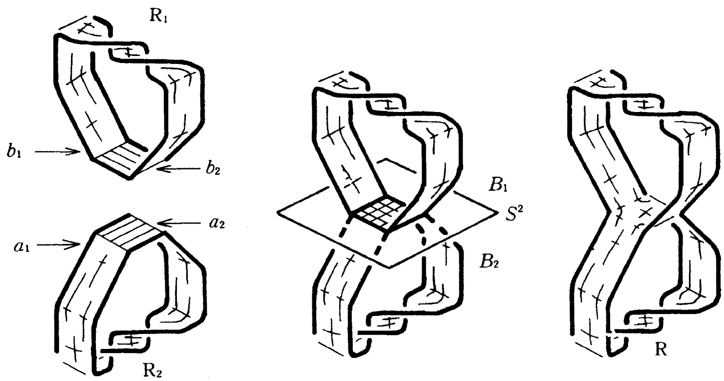


Fig. 4.2.2

The 2-Murasugi sum is nothing but a connected sum of links. The 4-Murasugi sum is also called a *plumbing*. Considering that R, R_1 and R_2 in Definition 4.2.1 are Seifert surfaces for links in S^3, we have the following facts:

Theorem 4.2.2 *If both R_1 and R_2 are incompressible Seifert surfaces, then R is an incompressible Seifert surface.*

Theorem 4.2.3 *R is a minimal genus Seifert surface if and only if both R_1 and R_2 are minimal genus Seifert surfaces.*

We refer to [Gabai 1983] for these proofs. In general, the converse of Theorem 4.2.2 is not true (see figure 4.2.3).

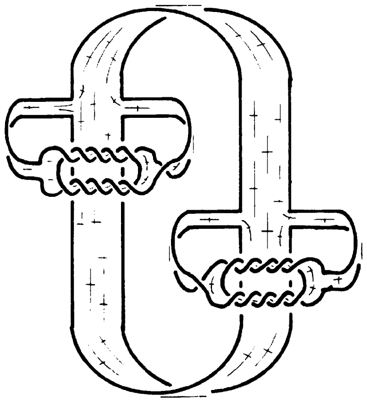


Fig. 4.2.3

Theorem 4.2.4 *R is a fiber surface if and only if both R_1 and R_2 are fiber surfaces.*

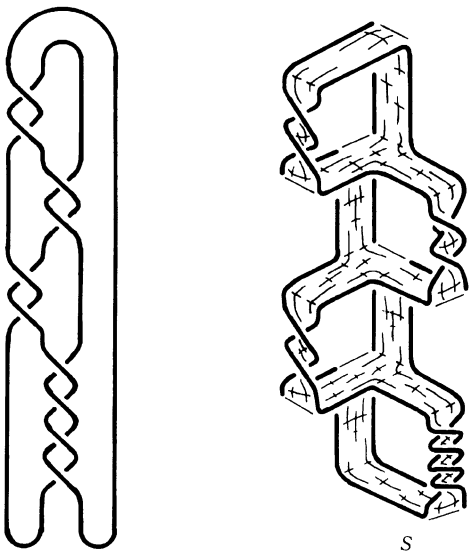


Fig. 4.2.4

The proof of Theorem 4.2.4 will be given in Section 4.3. In the remainder of this section, we suggest some applications of the above theorems as exercises. Let $C(2b_1, 2b_2, \ldots, 2b_{2n})$ be a 2-bridge knot in Conway's normal form (cf. 2.1), where $b_i (i = 1, 2, \ldots, 2n)$ are non-zero integers. This knot bounds a canonical Seifert surface obtained by plumbing together $2n$ unknotted annuli with $b_1, b_2, \ldots, b_{2n}$ full twists, as shown in figure 4.2.4 .

Exercise 4.2.5 Show that S is a minimal genus Seifert surface. [*Hint*: See Exercise 4.1.13 and Theorem 4.2.3.]

Exercise 4.2.6 Show that $C(2b_1, 2b_2, \dots, 2b_{2n})$ is a fibered knot if and only if $|b_i| = 1$ for each i. [*Hint*: See Theorem 4.1.10, Exercises 4.1.12, 4.1.13 and Theorem 4.2.4.]

Exercise 4.2.7 Show that the Whitehead link is a fibered link and that the Seifert surface shown in figure 4.2.5 is a fiber surface.

Exercise 4.2.8 Show that the surface in Exercise 4.1.6 is a fiber surface by using Theorem 4.2.4.

Gabai proposed the following conjecture in [Gabai 1986']:

Conjecture (Gabai) The fiber surface of any non-trivial atoroidal (cf. Definition 3.2.7) fibered knot can be decomposed by a non-trivial Murasugi sum.

Remark This conjecture is not true for a general fibered link (see figure 4.2.6).

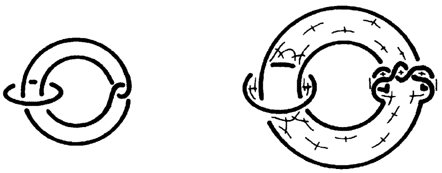


Fig. 4.2.5

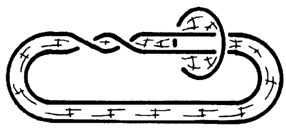


Fig. 4.2.6

4.3 Sutured manifolds

In this section, we introduce the notion of a sutured manifold, which is necessary to prove the theorems stated in Section 4.2. Let M be a (possibly closed) compact oriented 3-manifold. Let N be $\emptyset$ or a compact 2-manifold in ∂M. A proper 2-manifold S in M is said to be *proper* in (M, N) if $\partial S \subset N$. For a *connected* oriented proper 2-manifold S in (M, N), we define $\chi_-(S)$ by

$$\chi_-(S) = \max\{0, -\chi(S)\}.$$

If S is a disconnected oriented proper 2-manifold in (M, N), then we define $\chi_-(S)$ by

$$\chi_-(S) = \sum_{i=1}^{n} \chi_-(S_i)$$

where $S_1, \ldots, S_n$ are the connected components of S. With this notation, we define a non-negative integer-valued function x on $H_2(M, N)$ by

$$x(a) = \min\{\chi_-(S) \mid S \text{ is proper in } (M, N) \text{ with } a = [S] \in H_2(M, N)\}.$$

A proper 2-manifold S in (M, N) is called a *norm-minimizing 2-manifold* for (M, N) if $\chi_-(S) = x([S])$.

We have the following:

Lemma 4.3.1

(1) $x(ka) = |k|x(a)$ *for any* $k \in \mathbf{Z}$ *and* $a \in H_2(M, N)$.
(2) $x(a + b) \leq x(a) + x(b)$ *for any* $a, b \in H_2(M, N)$.

Proof. By definition, we see that $x(ka) \leq |k|x(a)$. We shall show that $x(ka) \geq |k|x(a)$. Since $\chi(a) = \chi(-a)$, we may assume that $k > 1$. Let S be a proper 2-manifold in (M, N) representing ka. For a point $y_0 \in M - S$, we define a function φ from $M - S$ to the cyclic additive group $\mathbf{Z}_k = \{0, 1, \ldots, k-1\}$ of order k by $\varphi(y) = \mathrm{Int}(c, S) \pmod k$ where c is a path in M from y_0 to $y \in M - S$ which intersects S transversely. Here, we note that this definition is well-defined, namely, the number $\mathrm{Int}(c, S) \pmod k$ does not depend on a choice of a path c, since S represents $ka \in H_2(M, N)$. We can also see that φ is constant on any connected component of $M - S$. For each $i\,(i = 0, 1, \ldots, k-1)$ let S_i be the union of components S' of S such that $\varphi(M'_+) = i + 1$ for the component M'_+ of $M - S$ on the $(+)$ side of S' and $\varphi(M'_-) = i$ for the component M'_- of $M - S$ on the $(-)$ side of S'. Then S is the disjoint union of $S_i (i = 0, 1, \ldots, k-1)$. Since $\mathrm{Int}(ka, b) = k\mathrm{Int}(a, b) \equiv 0 \pmod k$ for any $b \in H_1(M, \mathrm{cl}(\partial M - N))$, it follows that there is a number $i_0 \in \mathbf{Z}_k$ such that $\mathrm{cl}(\partial M - N) \subset \varphi^{-1}(i_0)$. Then for each $i \in \mathbf{Z}_k$, $ka = [S] = \sum_{j=0}^{k-1}[S_j] = k[S_i]$ in $H_2(M, N)$. Since $H_2(M, N)$ is a free abelian group, we have $a = [S_i]$, so that $\chi_-(S_i) \geq x(a)$ and $\chi_-(S) = \sum_{i=0}^{k-1} \chi_-(S_i) \geq |k|x(a)$. Thus, we have conclusion (1).

In order to show (2), let S and T be norm-minimizing 2-manifolds which represent a and b respectively and meet transversely. If there is a loop component of $S \cap T$ which bounds a disk D in S or T, say in S, then we take an innermost disk $D' \subset D$ bounded by a loop component of $S \cap T$ in $D \subset S$ and we do surgery on T by a 2-handle along D', namely, replace T with a 2-manifold $\mathrm{cl}(T - c(\partial D' \times [-1, 1])) \cup c(D' \times \{-1, 1\})$ for a *bi-collar* $c : (D', \partial D') \times [-1, 1] \to (M, T)$ of D' with $c(x, 0) = x$ for all $x \in D'$ (cf. 5.1). This modification of T reduces the number of components of $S \cap T$, but does not alter the homology class $[T] \in H_1(M, N)$ or the value of $\chi_-(T)$. Continuing this process, we can assume that there is no loop component of $S \cap T$ which is the boundary of a disk in S or T. Similarly, we can also assume that there is no arc component of $S \cap T$ which is the boundary of a disk in S or T. Let $S + T$ be a 2-manifold obtained from S and T by the orientation-preserving cut-and-paste operation as shown in figure 4.3.1. Then $S + T$ represents $a + b$ and we can see that $\chi_-(S + T) = \chi_-(S) + \chi_-(T)$. Hence, we have

$$x(a + b) \leq \chi_-(S + T) = \chi_-(S) + \chi_-(T) = x(a) + x(b). \qquad \square$$

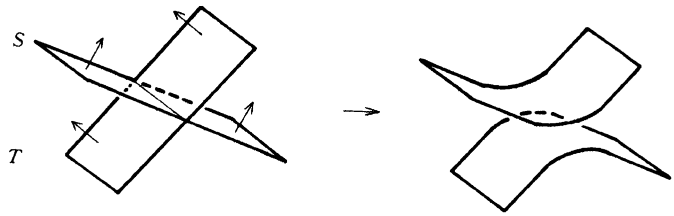


Fig. 4.3.1

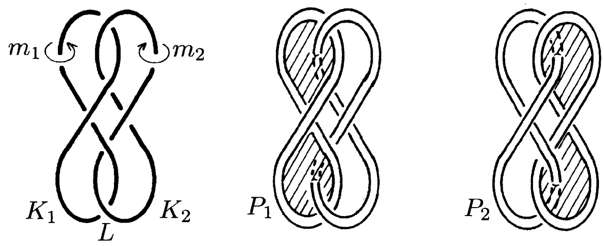


Fig. 4.3.2

Let $\mathbf{R}_+$ denote the non-negative real numbers. Thurston showed the following theorem in [Thurston 1986']:

Theorem 4.3.2 *Let x be a non-negative integral valued function on $H_2(M, N)$. Then x extends to a continuous function $x : H_2(M, N; \mathbf{R}) \to \mathbf{R}_+$ which is linear on any ray from the origin. In particular, we have $x(a + b) \leq x(a) + x(b)$ for all $a, b \in H_2(M, N; \mathbf{R})$.*

By Theorem 4.3.2, we can see that x is a semi-norm, that is, it satisfies all the conditions of a norm $\|\cdot\|$ except "$\|a\| = 0 \Leftrightarrow a = 0$ ".

Example 4.3.3. Let M be the exterior $E(L)$ of the Whitehead link $L = K_1 \cup K_2$. Then $H_2(M, \partial M; \mathbf{R}) \cong H^1(M; \mathbf{R}) \cong \mathbf{R}^2$. We see that the proper surfaces P_1 and P_2 in M shown in figure 4.3.2 form a linear basis $[P_1], [P_2]$ for $H_2(M, \partial M; \mathbf{R})$, which are dual to the meridianal basis m_1, m_2 of $H_1(M; \mathbf{R})$ with respect to the intersection pairing (see figure 4.3.2).

Claim *P_1 and P_2 are norm-minimizing 2-manifolds for $(M, \partial M)$.*

Proof. We shall show it for the case of P_1. If P_1 is not norm-minimizing, there is a proper 2-manifold Q without closed surface components such that $[Q] = [P_1]$, any component of ∂Q is not null-homotopic in ∂M, and $\chi_-(Q) = 0$, since $\chi_-(P_1) = 1$ and the natural homomorphism $H_2(M) \to H_2(M, \partial M)$ is trivial. If there is a disk component in Q, then K_1 spans a disk disjoint from K_2. This contradicts the fact that L is a non-split link. Hence we may suppose that all the components of Q are annuli. Since $\mathrm{Int}(Q, m_1) = \pm 1$ and $\mathrm{Int}(Q, m_2) = 0$, there is a component

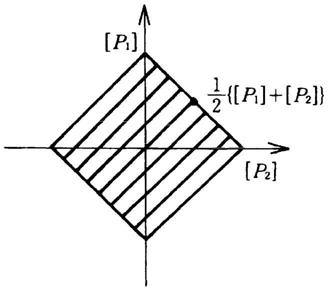


Fig. 4.3.3

Q' in Q which bounds a longitude of K_1 and a meridian of K_2, so we have that $\mathrm{Link}(K_1, K_2) = \pm 1$, contradicting the fact that $\mathrm{Link}(K_1, K_2) = 0$. □

Next, we consider a surface $P_1 + P_2$ obtained from P_1 and P_2 by the orientation-preserving cut-and-paste operation as shown in figure 4.3.1. Observe that $P_1 + P_2$ is a torus with two open disks removed and that $P_1 + P_2$ is a Seifert surface of L. Since L is a genus 1 fibered link (Exercise 4.2.7), $P_1 + P_2$ is norm-minimizing by Theorem 4.1.10, that is, $x([P_1] + [P_2]) = \chi_-(P_1 + P_2) = 2$. Thus, the unit sphere $S_1 = \{a \in H_2(M, \partial M; \mathbf{R}) \mid x(a) = 1\}$ on x in $\mathbf{R}^2$ contains 3 points $[P_1], [P_2]$ and $([P_1] + [P_2])/2$. By the same argument, S_1 further contains $-[P_1], -[P_2], ([P_1] - [P_2])/2, ([P_2] - [P_1])/2$ and $-([P_1] + [P_2])/2$. Since x is a semi-norm, we see that the unit ball $B_1 = \{a \in H_1(M, \partial M; \mathbf{R}) \mid x(a) \leq 1\}$ is a convex set in $\mathbf{R}^2$. Since it contains the above eight points, the figure of this convex set becomes a diamond as shown in figure 4.3.3.

Now, we define a sutured manifold. Let M be a compact oriented 3-manifold with boundary and γ a compact 2-manifold in ∂M.

Definition 4.3.4 The manifold pair (M, γ) is a *sutured manifold* if we have the following conditions (1),(2) and (3):

(1) γ is the union of mutually disjoint annuli and tori. We denote the union of annuli by $A(\gamma)$ and the union of tori by $T(\gamma)$.
(2) Each component of $A(\gamma)$ contains an oriented core loop, called a *suture*. We denote the set of sutures by $s(\gamma)$.
(3) $R(\gamma) = \mathrm{cl}(\partial M - \gamma)$ is oriented so that each component of $\partial R(\gamma)$ is homologous to a component of $s(\gamma)$ in γ.

We denote by $R_+(\gamma)$ the union of those components of $R(\gamma)$ whose positive normal vectors point out of M, and by $R_-(\gamma)$ the union of those components of $R(\gamma)$ whose positive normal vectors point into M.

Example 4.3.5 (Product sutured manifold). Let S be a compact oriented surface such that $\partial S \neq \emptyset$. Set $M = S \times [0, 1], \gamma = \partial S \times [0, 1], R_+(\gamma) = S \times 1$ and $R_-(\gamma) = S \times 0$, then (M, γ) is a sutured manifold, called a *product sutured manifold.*

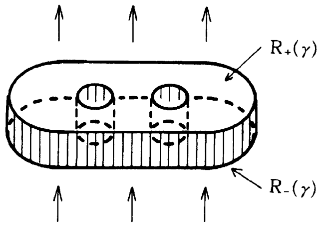


Fig. 4.3.4

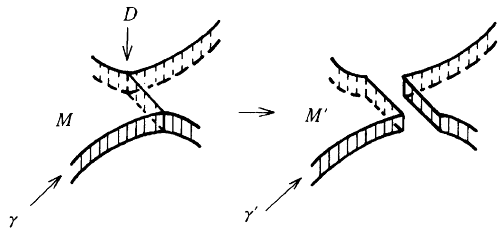


Fig. 4.3.5

For a sutured manifold (M, γ), a proper disk D in M is called a *product disk* if there is an embedding

$$(I \times I; I \times \{0\}, I \times \{1\}, \partial I \times I) \to (M; R_+(\gamma), R_-(\gamma), A(\gamma))$$

under some identification $D \cong I \times I$ for $I = [0, 1]$. We say that (M', γ') is obtained from (M, γ) by a *product decomposition* if (M', γ') is obtained from (M, γ) by cutting along a product disk D, as shown in figure 4.3.5.

Exercise 4.3.6 Show that (M, γ) is a product sutured manifold if and only if (M', γ') is a product sutured manifold.

Example 4.3.7 (Complementary sutured manifold). Let S be a Seifert surface for a link L in S^3. For the exterior $E(= E(L))$ of L, set $S_E = S \cap E$. Then a regular neighborhood pair $(N, \delta) = (N(S_E, E), N(\partial S_E, \partial E))$ admits the structure of a product sutured manifold naturally. We say that this (N, δ) is a *sutured manifold obtained from* S. Let $N^c = \mathrm{cl}(E(L) - N)$ and $\delta^c = \mathrm{cl}(\partial E(L) - \delta)$. We put a sutured manifold structure on (N^c, δ^c) so that $R_\pm(\delta^c) = R_\mp(\delta)$. We say that this sutured manifold (N^c, δ^c) is the *complementary sutured manifold* for S (see figure 4.3.6).

Exercise 4.3.8 Let S be a surface in S^3. Show that the following two conditions are equivalent:

(1) S is a fiber surface.
(2) The complementary sutured manifold for S is a product sutured manifold.

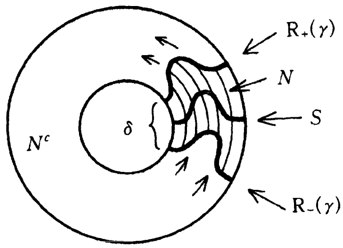


Fig. 4.3.6

Definition 4.3.9 A sutured manifold (M, γ) is a *taut sutured manifold* if M is an irreducible 3-manifold and $R(\gamma)$ is a norm minimizing surface for (M, γ).

The following theorem is important to our argument:

Theorem 4.3.10 (cf. Exercise 4.3.8) *Let S be a Seifert surface for a non-split link in S^3. Then the following two conditions are equivalent:*

(1) *S is a minimal genus Seifert surface.*
(2) *The complementary sutured manifold for S is a taut sutured manifold.*

The proof of Theorem 4.3.10 is much more difficult than the proof of Exercise 4.3.8. We omit it here. See Gabai's paper [Gabai 1983'] or [Scharlemann 1989] for it. For the remainder of this section, we shall prove Theorem 4.2.4.

Proof of Theorem 4.2.4. At first, we prepare some notation and terminology. We take $R, R_1, R_2, D, a_1, \dots, a_n, b_1, \dots, b_n, S^2$ as in Definition 4.2.1. Set $F = \mathrm{cl}(S^2 - D)$ and $T = (R - D) \cup F$. We can see that T is a Seifert surface for a link $L = \partial R$. We may suppose that R is disjoint from T in $E = E(L)$ by a small ambient isotopy (see figure 4.3.7). Let (M, γ) be the complementary sutured manifold for R, (N, δ) be the complementary sutured manifold for T, and (N_i, δ_i) be the complementary sutured manifold for $R_i (i = 1, 2)$. Note that $R \cup T$ decomposes E into two sutured manifolds. Let (H_1, ϵ_1) be the sutured manifold whose "thick part" is in the upper side and (H_2, ϵ_2) the sutured manifold whose "thick part" is in the lower side. Here, we can take product disks $\alpha_1, \dots, \alpha_n$ in (H_1, ϵ_1) corresponding to $a_1, \dots, a_n$. By figure 4.3.8, we can decompose (H_1, ϵ_1) into the sutured manifold homeomorphic to (N_1, δ_1) and the product sutured manifold homeomorphic to $(\mathrm{cl}(R - R_1) \times I, \partial(\mathrm{cl}(R - R_1)) \times I)$ by the product decomposition along $\alpha_1 \cup \cdots \cup \alpha_n$. Suppose that both R_1 and R_2 are fiber surfaces. Then we have that (N_1, δ_1) is a product sutured manifold by Exercises 4.3.6 and 4.3.8. Hence, (H_1, ϵ_1) is also a product sutured manifold. By the same argument, we can see that (H_2, ϵ_2) is a product sutured manifold. Then $(M, \gamma) = (H_1 \cup_T H_2, \epsilon_1 \cup_T \epsilon_2)$ is a product sutured manifold. By Exercise 4.3.8 again, R is a fiber surface.

Conversely, suppose that R is a fiber surface. Then we can see that (H_1, ϵ_1) and (H_2, ϵ_2) are product sutured manifolds. Following the above argument conversely, we have also that (N_1, δ_1) and (N_2, δ_2) are product sutured manifolds. By Exercise 4.3.8, both R_1 and R_2 are fiber surfaces. □

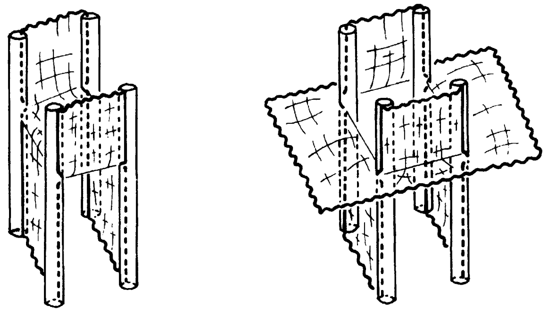


Fig. 4.3.7

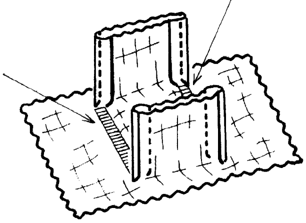


Fig. 4.3.8

Supplementary notes for Chapter 4

The Murasugi sum of Seifert surfaces was introduced originally in [Murasugi 1958, 1958',1958"] in order to estimate the degree of the Alexander polynomial of alternating links. After that, J. Stallings showed in [Stallings 1978] that a Seifert surface obtained by a Murasugi sum of fiber surfaces is a fiber surface. D. Gabai gave the proofs of Theorems 4.2.2 and 4.2.3 in [Gabai 1983] and extended the notion of the Murasugi sum to one in a general 3-manifold to obtain a result similar to Theorem 4.2.4 in [Gabai 1986']. It is also shown in [Kobayashi 1989] that the Murasugi sum is an effective means to decide whether minimal genus Seifert surfaces of a link are unique or not. By extending this method, O. Kakimizu classified the incompressible Seifert surfaces for prime knots of ≤ 10 crossings [Kakimizu *] and M. Sakuma classified the minimal genus Seifert surfaces for special arborescent links [Sakuma 1994]. D. Gabai investigates the properties of Seifert surfaces by looking directly at the complementary sutured manifolds in [Gabai 1986",1987,1987',1987"] and the theory of sutured manifolds is reported synthetically in [Scharlemann 1989]. A knot K has *property P* if for any label $f \neq \infty$ on K, the fundamental group of the Dehn surgery manifold $\chi(S^3;(K,f))$ is not trivial (cf. Appendix D for the notion of Dehn surgery). A knot K has *property Q* if there is a closed surface F in S^3 with $F \supset K$ and $F - K$ connected such that $H_1(V_i, F-K) \neq 0$ for the manifolds $V_i (i = 1,2)$ obtained from S^3 by splitting it along F (cf. [Simon 1970]). A knot K has *property R* if $\chi(S^3;(K,0))$ is not homeomorphic to $S^1 \times S^2$. The sutured manifold theory is discussed in connection with these properties of a knot. The *property P conjecture* is that every non-trivial knot has property P. It is not yet settled, but important progress has been made in [Gordon-Luecke 1989]. Namely, it is proved that $\chi(S^3;(K,f))$ is not homeomorphic to S^3 for any non-trivial knot K and any label $f \neq \infty$. The following theorem is a related general result, given in [Culler-Gordon-Luecke-Shalen 1987]:

Cyclic surgery theorem *Let M be a compact connected Haken 3-manifold such that ∂M is a torus. Assume that $\pi_1(M \cup_\phi D^2)$ and $\pi_1(M \cup_{\phi'} D^2)$ are cyclic groups for two embeddings $\phi, \phi' : S^1 \to \partial M$. Then the intersection number of $\phi(S^1)$ and $\phi'(S^1)$ in ∂M is* 0 *or* ± 1.

For example, we see from this theorem and [Bleiler-Scharlemann 1988] that any non-trivial knots with non-trivial symmetry group have property P. We see also that $\pi_1(\chi(S^3;(K,f)))$ is not trivial for any non-trivial knot K and any label $f \neq \pm 1, \infty$. The *property R conjecture* is that every non-trivial knot has property R, which has been settled in [Gabai 1987'] where it is proved that $\chi(S^3(K,0))$ is a Haken manifold for any non-trivial knot K.

The notion of the genus of a knot is defined in 4.1.5. We remark here that there are two other similar notions, called the *canonical genus* and the *free genus* of a knot. A *canonical Seifert surface* for a knot K is a Seifert surface F_c for K obtained by applying Seifert's algorithm to a diagram D for K. Then the canonical genus of K, denoted by $g_c(K)$, is the minimal genus of all such canonical Seifert surfaces F_c. A *free Seifert surface* for a knot K is a Seifert surface F_f for K such that the fundamental group $\pi_1(S^3 - F_f)$ is a free group, and the free genus of K, denoted by $g_f(K)$, is the minimal genus of all such free Seifert surfaces F_f. Since any canonical Seifert surface is a free Seifert surface, we have the inequality $g_c(K) \geq g_f(K) \geq g(K)$ for all knots K. For example, when K is an alternating knot (cf. 8.4.11), it is known that this inequality is replaced by the equality (cf. [Murasugi 1960]). On the other hand, when K is an untwisted or twisted double of a trefoil knot, we have $g(K) = 1$ and $g_f(K) = 2$ (except the case of ± 6-full twists with $g_f(K) = 1$) by [Kobayashi,M.-Kobayashi,T. 1996] and $g_c(K) = 3$ by [Kawauchi 1994].

Chapter 5
Seifert surfaces II: an algebraic approach

In this chapter, we discuss the Seifert matrix, which is derived from a connected Seifert surface of a link, and related link invariants such as the signature, the nullity, the Arf invariant and the one-variable Alexander polynomial.

5.1 The Seifert matrix

We consider a *connected* Seifert surface F for a link K in S^3. An embedding $c : F \times [-1,1] \to S^3$ is called a *bi-collar* of F in S^3 if $c(F \times 0) = F$ and $c(F \times 1)$ is in the positive normal direction of F (see figure 5.1.1).

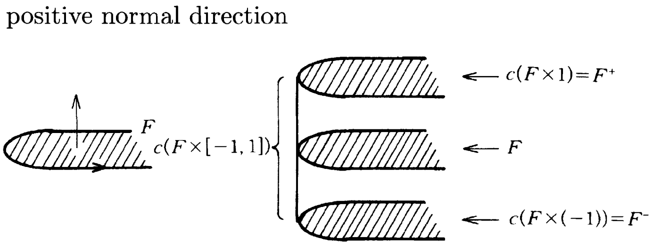


Fig. 5.1.1

For any other bi-collar $c' : F \times [-1,1] \to S^3$ of F, it is shown that there is an orientation-preserving homeomorphism $h : S^3 \to S^3$ such that $h\,|_F = \mathrm{id}$ and $c' = hc$.

Exercise 5.1.1 Show this by considering a general technique of PL topology.

For a bi-collar $c : F \times [-1,1] \to S^3$, we let $F^+ = c(F \times 1)$ and $F^- = c(F \times (-1))$. We orient $F^\pm$ so that $F^\pm$ and F represent the same generator of $H_2(c(F \times [-1,1]), c((\partial F) \times [-1,1]) \cong \mathbf{Z}$. For an element $x = \{c_x\}$ of $H_1(F)$ with c_x a cycle, let $x^\pm = \{c_x^\pm\}$ be the corresponding element of $H_1(F^\pm)$. Using that $F^+ \cap F^- = \emptyset$, we can define the *linking number* $\mathrm{Link}(c_1^+, c_2^-) \in \mathbf{Z}$ in S^3 of a 1-cycle c_1^+ in F^+ and a 1-cycle c_2^- in F^- (which is a generalization of the definition of the linking number in 1.1). In other words, $\mathrm{Link}(c_1^+, c_2^-)$ is the intersection number $\mathrm{Int}(d_1^+, c_2^-)$ of any 2-chain d_1^+ in S^3 and c_2^- such that $\partial d_1^+ = c_1^+$, and each 2-simplex Δ^2 appearing in d_1^+ and each 1-simplex Δ^1 appearing in c_2^- either do not meet, or meet at one point in the interior of each simplex.

Comments. According to whether Δ^2 and Δ^1 do not meet, meet in a positive orientation, or meet in a negative orientation, we take $\epsilon(\Delta^2, \Delta^1)$ to be $0, 1$ or -1, respectively. Then $\mathrm{Int}(d_1^+, c_2^-)$ is the sum of $\epsilon(\Delta^2, \Delta^1)$ on all of such Δ^2 and Δ^1 appearing in d_1^+ and c_2^-.

One can show that $\mathrm{Link}(c_1^+, c_2^-)$ depends only on the homology classes $[c_1^+] \in H_1(F^+)$ and $[c_2^-] \in H_1(F^-)$.

Definition 5.1.2 A *Seifert form* on the link L (associated with a connected Seifert surface F) is a map

$$\phi : H_1(F) \times H_1(F) \to \mathbf{Z}$$

defined by $\phi(x, y) = \mathrm{Link}(c_x^+, c_y^-)$.

Lemma 5.1.3

(1) $\phi(x_1 + x_2, y) = \phi(x_1, y) + \phi(x_2, y), \phi(x, y_1 + y_2) = \phi(x, y_1) + \phi(x, y_2)$.

(2) $\phi(x, y) - \phi(y, x) = I(x, y)$, *where* $I(x, y)$ *denotes the intersection number of* x *and* y *on the surface* F.

Proof. (1) is clear from the definition. We show (2).

$$\begin{aligned}\phi(x, y) - \phi(y, x) &= \mathrm{Link}(c_x^+, c_y^-) - \mathrm{Link}(c_y^+, c_x^-)\\ &= \mathrm{Link}(c_x^+, c_y) - \mathrm{Link}(c_x^-, c_y)\\ &= \mathrm{Link}(c_x^+ - c_x^-, c_y)\\ &= \mathrm{Int}(c(c_x \times [-1,1]), c_y)\\ &= I(x, y),\end{aligned}$$

where $c(c_x \times [-1, 1])$ denotes a 2-cycle constructed from a suitable triangulation of $F \times [-1, 1]$. □

$H_1(F)$ is a free abelian group of finite rank, say n. A square matrix $V = (v_{ij})$ of size n with $v_{ij} = \phi(x_i, x_j)$ for a basis $x_1, x_2, \ldots, x_n$ of $H_1(F)$ is called a *Seifert matrix* of the link L. This matrix of course depends on choice of F and a basis $x_1, x_2, \ldots, x_n$ of $H_1(F)$. Thus, it is not a link invariant, but as we shall show in the next section, the S-equivalence class of it is an invariant of the link L. In this section, we give a characterization of a Seifert matrix. A *unimodular* matrix is an integral square matrix whose determinant is ± 1. Two integral square matrices V and W are *unimodular-congruent* if there is a unimodular matrix P such that $W = PVP'$.

Theorem 5.1.4 *An integral square matrix* V *of size* n *is a Seifert matrix of an* r*-component link* L *if and only if* $m = (n - r + 1)/2$ *is a non-negative integer and* $V - V'$ *is unimodular-congruent to the block sum* T *of* m *copies of* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and the zero matrix* O^{r-1} *of size* $r - 1$.

Exercise 5.1.5 Show that $V - V'$ and T are unimodular-congruent if and only if there are unimodular matrices $P_i, i = 1, 2$, such that $P_1(V - V')P_2$ is the block sum of the unit matrix E^{2m} and O^{r-1}. (*Hint*: Use that $V - V'$ is skew-symmetric.)

Corollary 5.1.6 *An integral square matrix V is a Seifert matrix of a knot if and only if* $\det(V - V') = 1$.

Proof of Theorem 5.1.4. Let V be a Seifert matrix of L. By Lemma 5.1.3, $V - V'$ is an intersection matrix of a connected Seifert surface F of L. With a suitable basis of $H_1(F)$, the intersection form $I : H_1(F) \times H_1(F) \to \mathbf{Z}$ represents the matrix T, so that $V - V'$ and T are unimodular-congruent. To prove sufficiency, we assume that there is a unimodular matrix P with $P(V-V')P' = T$. Clearly, any matrix unimodular-congruent to a Seifert matrix is also a Seifert matrix by a base change. Hence it is sufficient to prove that V is a Seifert matrix when we assume that $P = E^n$. Now we consider F as a genus m closed surface with r open disks removed. F is constructed from a disk D by attaching n bands $B_i (i = 1, 2, \ldots, n)$ to ∂D in order that

$$a_1^+, a_2^+, a_1^-, a_2^-, \ldots, a_{2m-1}^+, a_{2m}^+, a_{2m-1}^-, a_{2m}^-, a_{2m+1}^+, a_{2m+1}^-, \ldots, a_n^+, a_n^-,$$

where a_i^+ and a_i^- are the end arcs of the band B_i. Then $D \cup B_i$ is an annulus for each i. Denoting the homology class in $H_1(F)$ represented by a suitably oriented core circle of the annulus by x_i, we see that $x_1, x_2, \ldots, x_n$ form a basis of $H_1(F)$ and the intersection matrix associated with it is T. We shall embed F into S^3 so that the Seifert matrix associated with the basis $x_1, x_2, \ldots, x_n$ is V. Let $F_i = D \cup B_1 \cup \cdots \cup B_i$. Let $V = (v_{ij})$. First, embed D standardly. Embed B_1 so that $\mathrm{Link}(x_1^+, x_1^-) = v_{11}$ on the Seifert surface F_1. Next, embed B_2 so that $\mathrm{Link}(x_2^+, x_1^-) = v_{21}$, $\mathrm{Link}(x_2^+, x_2^-) = v_{22}$ on the Seifert surface F_2. Continuing this process, we obtain an embedding of $F_n = F$ into S^3 whose Seifert matrix on $x_1, x_2, \ldots, x_n$ is V. □

Exercise 5.1.7 Construct a link L with a Seifert matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

5.2 S-equivalence

For integral square matrices V and W, we say that W is a *row enlargement* of V or V is a *row reduction* of W if

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 1 & x & u \\ 0 & v' & V \end{pmatrix},$$

where x is an integer and u is a row vector and v' is a column vector. If W' is a row enlargement of V', then we say that W is a *column enlargement* of V or V is a *column reduction* of W. We note that if a matrix $\tilde{W}$ is obtained from W by replacing the integral row vector (xu) with any other integral row vector $(\tilde{x}\tilde{u})$, then $\tilde{W}$ is unimodular-congruent to W.

Definition 5.2.1 *S-equivalence* is an equivalence relation which is generated by the following relations: unimodular-congruence, row-enlargement, row-reduction, column-enlargement and column-reduction.

Exercise 5.2.2 Show that a matrix that is S-equivalent to a Seifert matrix is a Seifert matrix.

The following theorem is the main theorem of this section:

Theorem 5.2.3 *Any two Seifert matrices of a link L are S-equivalent.*

We consider a (possibly disconnected) oriented 2-manifold F and a 3-ball B^3 in $\mathbf{R}^3$ such that $F \cap B^3 = (\mathrm{int}F) \cap \partial B^3$ is a disjoint union of two disks D_1, D_2 and the 2-manifold

$$F' = \mathrm{cl}(F - (D_1 \cup D_2)) \cup \mathrm{cl}(\partial B^3 - (D_1 \cup D_2))$$

is orientable. Let F' have the orientation inherited from $F - (D_1 \cup D_2)$. Then we say that F' is a *1-handle enlargement* of F or F is a *1-handle reduction* of F', and this 3-ball B^3 is a *1-handle* on F with attaching disks D_1, D_2. Before proving Theorem 5.2.3, we show the following lemma:

Lemma 5.2.4 *Any two connected Seifert surfaces F_1, F_2 of a link L in $\mathbf{R}^3$ are ambient isotopic after modifying them by a finite sequence of 1-handle enlargements.*

Proof. Let $K_i (i = 1, 2, \ldots, r)$ be the components of L. By an isotopic deformation of F_1 in $\mathbf{R}^3$ keeping L fixed, we may assume that $\mathrm{int}F_1$ and $\mathrm{int}F_2$ meet transversely with $\mathrm{int}F_1 \cap \mathrm{int}F_2$ being a closed 1-manifold (that is, a disjoint union of simple loops).

Exercise 5.2.5 Show this last statement.
[*Hint*: Use that the linking number $\mathrm{Link}(K_i, K_i')$ for a loop $K_i' \subset \mathrm{int}F_1$ parallel to K_i equals $-\mathrm{Link}(K_i, L - K_i)$.]

Let m be the genus of F_1. We take mutually disjoint bands $B_i (i = 1, 2, \ldots, 2m + r - 1)$ in F_1 so that (1) the end arcs of the band B_i belong to K_1 when $i \le 2m$ and belong to K_1 and K_{i-2m+1} when $2m + 1 \le i \le 2m + r - 1$, and (2) $D_1 = \mathrm{cl}(F_1 - \cup_{i=1}^{2m+r-1} B_i)$ is a disk. Let b_i be a proper arc in the band B_i joining the end arcs and meeting $\mathrm{int}F_2$ transversely. For $i \le 2m$, ∂b_i splits K_1 into two arcs. Let b_i' be one of them. Let ℓ_i^+ be the loop in F_1^+ corresponding to the loop $\ell_i = b_i \cup b_i' \subset F_1$. Since $F_1 \cup -F_2$ is a 2-cycle in S^3 and $H_2(S^3) = 0$, we have $\mathrm{int}(\ell_i^+, F_1 \cup -F_2) = 0$. However, $\ell_i^+ \cap F_1 = \emptyset$. This means that b_i meets $\mathrm{int}F_2$ with intersection number 0. When $2m+1 \le i \le 2m+r-1$, we can assume that b_i meets $\mathrm{int}F_2$ with intersection number 0 by considering suitable ambient isotopic deformations of F_1 keeping L fixed as they are shown in figure 5.2.1. Taking the 1-handle enlargements of F_2 along $b_i (i = 1, 2, \ldots, 2m + r - 1)$, we can conclude that the resulting surface F_2 meets $\mathrm{int}F_1$ within the disk D_1. Let ℓ_0 be an innermost loop in D_1 with respect to the simple loops in $(\mathrm{int}F_1) \cap F_2$ and let $D_0 \subset D_1$ be the disk bounded by ℓ_0. We take a bi-collar $c(D_0 \times [-1,1])$ of D_0 with $c(D_0 \times [-1,1]) \cap F_2 = c(\ell_0 \times [-1,1])$. The 2-manifold $F_2' = (F_2 - c(\ell_0 \times (-1,1)) \cup c(D_0 \times \{-1,1\})$ is a 1-handle reduction of F_2. In case F_2' has a closed surface component F_0, then F_0 bounds a compact connected orientable 3-manifold M in S^3 such that $M \cap F_1 = \emptyset$, because F_1 is connected. Using the manifold M, we can eliminate the loop ℓ_0 by a finite sequence of 1-handle enlargements and 1-handle reductions on F_2 (see Exercise 5.2.6 later). By continuing this process, we finally obtain a 2-manifold F_2^* from F_2 such that $F_2^* \cap \mathrm{int} F_1 = \emptyset$, and F_2^* does not have a closed surface component (though it may be disconnected). Since the Seifert surface F_1 is connected, the closed 2-manifold $F_2^* \cup -F_1$ is a closed connected surface. Using the connected 3-manifold bounded by it in S^3, we obtain F_1 from F_2^* by a finite sequence of 1-handle enlargements and 1-handle reductions on F_2^* and ambient isotopies of S^3 (cf. Exercise 5.2.6). Then a handle slide argument shows that we can obtain ambient isotopic surfaces after modifying them by a finite sequence of 1-handle enlargements on F_1 and F_2. □

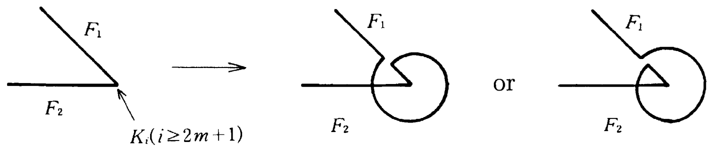


Fig. 5.2.1

Exercise 5.2.6 Let M be a compact connected orientable 3-manifold. Let A and B be compact (possibly disconnected) 2-manifolds such that $A \cup B = \partial M$ and $\partial A = A \cap B = \partial B$. Then show that a surface *parallel* (cf. C.5) to B is obtained by a finite sequence of 1-handle enlargements and 1-handle reductions on A.

Proof of Theorem 5.2.3. Any Seifert matrices of ambient isotopic connected Seifert surfaces are unimodular-congruent. Let F' be a 1-handle enlargement of a connected Seifert surface F. We have a Seifert matrix of F' which is a row or column enlargement of a Seifert matrix of F. Hence by Lemma 5.2.4, Theorem 5.2.3 is proved. □

Exercise 5.2.7 Let A be a Seifert matrix of a connected Seifert surface F of a link L. Show that for any row or column enlargement A^+ of A, there is a 1-handle enlargement F^+ of F with Seifert matrix A^+.

5.3 Number-theoretic invariants

We consider a symmetric bilinear form $b : G \times G \to \mathbf{Z}$ on a free abelian group G of finite rank. This form is said to be *even* if $b(x,x)$ is an even integer for all $x \in G$. Otherwise, it is said to be *odd*. Two such forms $(G,b), (G',b')$ are said to be *isomorphic* if there is an isomorphism $f : G \cong G'$ such that $b(x,y) = b'(f(x), f(y))$ for all $x, y \in G$. The form (G,b) is said to be *non-singular* (or *non-degenerate*, respectively) if the homomorphism $G \to \mathrm{Hom}(G, \mathbf{Z})$ sending x to $b(x, \ )$ is an isomorphism (or a monomorphism, respectively).

Exercise 5.3.1 Show that there is exactly one non-singular symmetric even bilinear form $b : G \times G \to \mathbf{Z}$ with $G \cong \mathbf{Z} \oplus \mathbf{Z}$. (We call this form the *hyperbolic plane form.*)

Two symmetric bilinear forms $(G, b), (G', b')$ are said to be *stably isomorphic* if we obtain isomorphic forms after adding some copies of the hyperbolic plane form to them as orthogonal summands. A *symmetric bilinear form associated with* an integral square matrix V is a symmetric bilinear form $b : G \times G \to \mathbf{Z}$ representing the matrix $V + V'$ with respect to a suitable basis of G. This is clearly an even form. The following theorem is obtained directly from Theorem 5.2.3:

Theorem 5.3.2 *The stable isomorphism class of a symmetric bilinear form b : $G \times G \to \mathbf{Z}$ associated with V is an invariant of the link type of L.*

In particular, the signature and the nullity of b are invariants of the link type of L, called the *signature* and the *nullity* of L and denoted by $\sigma(L)$ and $n(L)$, respectively.

Exercise 5.3.3 For an r-component link L, show that

$$n(L) \leq r - 1, \ \sigma(L) + n(L) \equiv r - 1 \pmod 2.$$

Exercise 5.3.4

(1) For any Seifert matrices $V_i, i = 1, 2$, of two links $L_i, i = 1, 2$, show that the block sum $V_1 \oplus V_2$ is a Seifert matrix of any connected sum $L_1 \sharp L_2$. In particular, show that $\sigma(L_1 \sharp L_2) = \sigma(L_1) + \sigma(L_2)$.

(2) For any Seifert matrix V of a link L, show that V', $-V'$ and $-V$ are Seifert matrices of the inverse $-L$, the mirror image L^* and the inverted mirror image $-L^*$, respectively. In particular, show that $\sigma(\pm L^*) = -\sigma(L)$ for any link L, so that $\sigma(L) = 0$ for any $(+)$amphicheiral or $(-)$amphicheiral link L.

To derive another invariant of a symmetric even bilinear form b, we consider the non-degenerate form $\hat{b} : \hat{G} \times \hat{G} \to \mathbf{Z}$ with $\hat{G} = G/\{x \in G \mid b(x, G) = 0\}$ naturally induced from b. We extend this form $\hat{b}$ to the non-degenerate (= non-singular) $\mathbf{Q}$-form $\hat{b}_{\mathbf{Q}} : \hat{G}_{\mathbf{Q}} \times \hat{G}_{\mathbf{Q}} \to \mathbf{Q}$. Let $\hat{G}^+ = \{u \in \hat{G}_{\mathbf{Q}} \mid \hat{b}_{\mathbf{Q}}(u, \hat{G}) \in \mathbf{Z}\}$ and $T = \hat{G}^+/\hat{G}$. Then we see that T is a finite abelian group and $\hat{b}_{\mathbf{Q}}$ induces a non-degenerate(= non-singular) symmetric bilinear form

$$\ell : T \times T \to \mathbf{Q}/\mathbf{Z},$$

which we call a *non-singular linking form.* Using the fact that $\hat{b}$ is an even form, we have a well-defined function

$$q : T \to \mathbf{Q}/\mathbf{Z}$$

by letting $q(v) = \hat{b}_{\mathbf{Q}}(u, u)/2 \pmod 1$ for $v = \{u\} \in T (u \in \hat{G}_{\mathbf{Q}})$.

In general, a map $q' : T' \to \mathbf{Q}/\mathbf{Z}$ with T' a finite abelian group is called a *non-singular quadratic function* if the map $\ell' : T' \times T' \to \mathbf{Q}/\mathbf{Z}$ defined by $\ell'(v, v') = q'(v + v') - q'(v) - q'(v')$ for all $v, v' \in T'$ is a non-singular linking form. The non-singular quadratic function q' is also called the *quadratic function inducing the non-singular linking form* ℓ'.

Since we have the identity

$$q(v + v') - q(v) - q(v') = \ell(v, v')$$

for all $v, v' \in T$, the map $q : T \to \mathbf{Q}/\mathbf{Z}$ is a non-singular quadratic function. Two non-singular linking forms $\ell : T \times T \to \mathbf{Q}/\mathbf{Z}$ and $\ell' : T' \times T' \to \mathbf{Q}/\mathbf{Z}$ are *isomorphic* if there is an isomorphism $f : T \to T'$ with $b(v, v') = b'(f(v), f'(v'))$ for all $v, v' \in T$. Similarly, two non-singular quadratic functions $q : T \to \mathbf{Q}/\mathbf{Z}$ and $q' : T' \to \mathbf{Q}/\mathbf{Z}$ are *isomorphic* if there is an isomorphism $f : T \cong T'$ with $q(v) = q'(f(v))$ for all $v, v' \in T$.

The following theorem is also obtained directly from Theorem 5.2.3:

Theorem 5.3.5 *The non-singular linking form $\ell : T \times T \to \mathbf{Q}/\mathbf{Z}$ and the non-singular quadratic function $q : T \to \mathbf{Q}/\mathbf{Z}$ which are induced from the even symmetric bilinear form $b : G \times G \to \mathbf{Z}$ associated with a Seifert matrix V of a link L are invariants of the link type of L up to isomorphisms.*

By this theorem, we call ℓ and q the *linking form of* L and the *quadratic function of* L, respectively. Next, we consider a non-singular linking form $\ell : T_2 \times T_2 \to \mathbf{Q}/\mathbf{Z}$ such that T_2 is a direct sum of copies of $\mathbf{Z}_2$ and $\ell(v, v) = 0$ for all $v \in T_2$, and a quadratic function $q : T_2 \to \mathbf{Q}/\mathbf{Z}$ inducing the non-singular linking form ℓ. Since $\ell(T_2 \times T_2) = q(T_2) = \{0, 1/2\} \subset \mathbf{Q}/\mathbf{Z}$, we can regard ℓ and q as $\ell : T_2 \times T_2 \to \mathbf{Z}_2$ and $q : T_2 \to \mathbf{Z}_2$, respectively. Since $\ell(v, v) = 0$ for all $v \in T_2$, there is a basis $x_1, y_1, \dots, x_m, y_m$ of T_2 such that $\ell(x_i, x_j) = \ell(y_i, y_j) = 0$ and $\ell(x_i, y_j) = \delta_{i,j}$ for all i, j. This basis is called a *symplectic basis* (with respect to ℓ). We consider the Gauss sum

$$GS(q) = \sum_{x \in T_2} (-1)^{q(x)}.$$

This value is of course an invariant of the isomorphism class of the quadratic function $q : T_2 \to \mathbf{Z}_2$. Let T_2^i be the direct summand of T_2 generated by x_i and y_i. Since $q(x_i + y_i) - q(x_i) - q(y_i) = 1$, we see that

$$\sum_{x \in T_2^i} (-1)^{q(x)} = (-1)^{q(x_i)q(y_i)} 2,$$

so that

$$GS(q) = \prod_{i=1}^{m} \Big(\sum_{x \in T_2^i} (-1)^{q(x)} \Big) = (-1)^a 2^m,$$

where $a = \sum_{i=1}^m q(x_i)q(y_i)$. This means that $a \in \mathbf{Z}_2$ is an invariant of the isomorphism of q. This value a is called the *Arf invariant* of q and denoted by $\mathrm{Arf}(q)$.

Exercise 5.3.6 Show that there is a symplectic basis $x_1, y_1, \ldots, x_m, y_m$ of T_2 as follows: In case $\mathrm{Arf}(q) = 0$, then $q(x_i + y_i) = 1, q(x_i) = q(y_i) = 0$ for all i. In case $\mathrm{Arf}(q) = 1$, then $q(x_i + y_i) = 1$ for all i, $q(x_1) = q(y_1) = 1$, and $q(x_i) = q(y_i) = 0$ for all i with $i \geq 2$.

A link L with components $K_i (i = 1, 2, \ldots, r)$ is said to be *proper* if $\mathrm{Link}(K_i, L - K_i) \equiv 0 \,(\mathrm{mod}\, 2)$ for all i. In particular, any knot is proper. Let $\phi_2 : H_1(F; \mathbf{Z}_2) \times H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$ be the $\mathbf{Z}_2$-reduction of the Seifert form $\phi : H_1(F) \times H_1(F) \to \mathbf{Z}$ on a connected Seifert surface F of a link L. We define a map $q : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$ by $q(x) = \phi_2(x, x)$. By Lemma 5.1.3, we have the identity

$$q(x + y) - q(x) - q(y) = I_2(x, y)$$

for all $x, y \in H_1(F; \mathbf{Z}_2)$, where I_2 is the $\mathbf{Z}_2$-intersection form on $H_1(F; \mathbf{Z}_2)$. Since F is orientable, we see that $I_2(x, x) = 0$ for all $x \in H_1(F; \mathbf{Z}_2)$. By $\hat{I}_2 : \hat{H}_1(F; \mathbf{Z}_2) \times \hat{H}_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$, we denote the non-singular linking form induced from I_2.

Proposition 5.3.7 *The map $q : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$ defines a quadratic function $\hat{q} : \hat{H}_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$ inducing a non-singular linking form $\hat{I}_2 : \hat{H}_1(F; \mathbf{Z}_2) \times \hat{H}_1(F : \mathbf{Z}_2) \to \mathbf{Z}_2$ if and only if the link L is proper.*

Proof. Note that q induces a non-singular quadratic function $\hat{q}$ if and only if $q(x_i) = 0$ for the element $x_i \in H_1(F; \mathbf{Z}_2)$ representing each component K_i of $\partial F = L$. Since we have $q(x_i) = \mathrm{Link}(K_i, L - K_i) \,(\mathrm{mod}\, 2)$, we have the conclusion. □

By this proposition, $\mathrm{Arf}(\hat{q})$ is defined for a connected Seifert surface F of a proper link L. This value depends only on the S-equivalence class of a Seifert matrix on F.

Exercise 5.3.8 Confirm this assertion.

Accordingly, by Theorem 5.2.3, $\mathrm{Arf}(\hat{q})$ is an invariant of the type of the proper link L, called the *Arf invariant* of L and denoted by $\mathrm{Arf}(L)$. When L is a knot, this invariant is independent of the orientation of L, but in general it depends on the orientation of L (see figure 5.3.1).

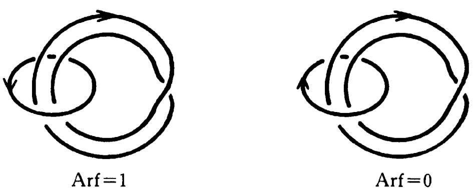


Fig. 5.3.1

5.4 The reduced link module

By Λ, we denote the group ring $\mathbf{Z}\langle t\rangle$ of the infinite cyclic group $\langle t\rangle$ with a generator t, which is seen to be a Noetherian ring. For a Seifert matrix V of size n of a link L, let $\psi_V(t) : \Lambda^n \to \Lambda^n$ be the Λ-homomorphism representing the matrix $tV' - V$ with respect to the standard basis of Λ^n. Then by Theorem 5.2.3, we see that the Λ-module $\Lambda^n/\mathrm{Im}\psi_V(t)$ is an invariant of the type of L up to Λ-isomorphisms, which is called the *reduced link module* of the link L. Let $E = E(L, S^3)$. Let $\gamma : \pi = \pi_1(E, x) \to \langle t\rangle$ be the epimorphism sending each meridian of L to t. The homology group $H_1(E_\infty)$ of the infinite cyclic covering space E_∞ over E corresponding to the subgroup $\mathrm{Ker}\gamma$ of $\pi_1(E, x)$ naturally forms a Λ-module.

Proposition 5.4.1 *The Λ-module $H_1(E_\infty)$ is Λ-isomorphic to the reduced link module of L.*

Proof. For a connected Seifert surface F of L, let E' be the manifold obtained from E by splitting it along $F_E = F \cap E(\cong F)$. Let F^+ and F^- be the two copies of F_E occurring in E'. Then E_∞ is constructed from the infinite copies $(E'_i, F_i^+, F_i^-)(i = 0, \pm 1, \pm 2, \dots)$ of (E', F^+, F^-) as follows: In the topological sum $\coprod_{i=-\infty}^{+\infty} E'_i$, we identify F_i^- with F_{i+1}^+ for all i (see figure 5.4.1).

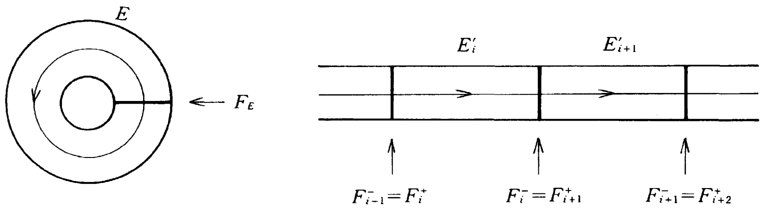


Fig. 5.4.1

Further, the covering transformation $t : E_\infty \to E_\infty$ can be taken to be the translation of the copy E'_i into the copy E'_{i+1} for all i. By the Mayer-Vietoris exact sequence, we obtain an exact sequence

$$H_1(F_E) \otimes_{\mathbf{Z}} \Lambda \xrightarrow{tj_*^+ - j_*^-} H_1(E') \otimes_{\mathbf{Z}} \Lambda \to H_1(E_\infty) \to 0$$

of Λ-modules, where $j^+ : F_E \cong F^+ \subset E'$ and $j^- : F_E \cong F^- \subset E'$ denote the natural injections. Let V be a Seifert matrix associated with a basis $e_1, e_2, \dots, e_n$ of $H_1(F_E)$. Then $H_1(E')$ is a free abelian group and we can have a basis $e'_i (i = 1, 2, \dots, n)$ for it such that $\mathrm{Link}(c'_i, c_j) = \delta_{ij}$ in S^3 for a cycle c'_i in E' with $e'_i = \{c'_i\}$ and a cycle c_j in F_E with $e_j = \{c_j\}$. Then $tj_*^+ - j_*^-$ represents the matrix $tV' - V$ with respect to these bases. This means that $H_1(E_\infty)$ is a reduced link module. $\square$

For a Seifert matrix V of a link L, we take

$$\Delta_L(t) = \Delta(L; t) = \det(tV' - V).$$

By Theorem 5.2.3, $\Delta_L(t)$ is an invariant of the type of L up to multiplication by $t^m (m = 0, \pm 1, \pm 2, \dots)$. This Laurent polynomial is called the *one-variable Alexander polynomial* of L.

Remark Because of a convenient relation to covering space theory, $\Delta_L(t)$ is usually considered to be an element of Λ up to multiplication by the units $\pm t^m$.

Exercise 5.4.2 For an r-component link L, show that $(t-1)^{r-1}$ divides $\Delta_L(t)$, the quotient $\hat{\Delta}_L(t) = \Delta_L(t)/(t-1)^{r-1}$ is a Laurent polynomial of even degree, and the multiplicity of $(t-1)$ in $\hat{\Delta}_L(t)$ is even (this Laurent polynomial will be called the *Hosokawa polynomial* after Theorem 7.3.16). Also, show that when $r = 1$, $\Delta_L(1) = 1$.

Exercise 5.4.3 If a link L bounds a connected Seifert surface of genus g, then show that the Laurent polynomial degree $\deg\hat{\Delta}_L(t)$ has $\deg\hat{\Delta}_L(t) \leq 2g$.

Exercise 5.4.4 If the reduced link module of a link L is finitely generated as an abelian group, then show that any Seifert surface of L is connected. Also, show that any fibered link satisfies this assumption.

5.5 The homology of a branched cyclic covering manifold

For the exterior E of a link L, let $\gamma : \pi_1(E) \to \langle t \rangle$ be the epimorphism sending each meridian of L to t and $\gamma_n : \pi_1(E) \to \langle t \mid t^n = 1 \rangle$ the composition of γ and the natural quotient map $\langle t \rangle \to \langle t \mid t^n = 1 \rangle$. Let $p_n : E_n \to E$ be the covering corresponding to the kernel of γ_n. We consider a branched covering $\hat{p}_n : M_n \to S^3$ with branch set a link L which is a completion of the covering $p_n : E_n \to E$. We note that the infinite cyclic covering $p : E_\infty \to E$ (corresponding to the kernel of γ) is the composite of the covering $p^n : E_\infty \to E_\infty/\langle t^n \rangle = E_n$ and the covering $p_n : E_n \to E$. We denote by $q^n : E_\infty \to M_n$ the composition of $p^n : E_\infty \to E_n$ and the inclusion $j_n : E_n \subset M_n$. The cyclic group $\langle t \mid t^n = 1 \rangle$ acts on M_n so that the map q^n is t-invariant, i.e., $q^n t = t q^n$. Letting $\rho_n(t) = (1-t^n)/(1-t) = 1 + t + \cdots + t^{n-1}$, we have the following:

Theorem 5.5.1 *The map $q^n : E_\infty \to M_n$ induces a Λ-epimorphism*

$$q^n_* : H_1(E_\infty) \to H_1(M_n)$$

whose kernel is $\rho_n(t) H_1(E_\infty)$. Accordingly, q^n_* induces a Λ-isomorphism

$$H_1(E_\infty)/\rho_n(t) H_1(E_\infty) \cong H_1(M_n).$$

We denote the minimal number of Λ-generators of $H_1(E_\infty)$ by $e(L)$ and the minimal number of abelian generators of $H_1(E_n)$ by $e_n(L)$. The following is obtained from Theorem 5.5.1:

Corollary 5.5.2 $e(L) \geq e_n(L)/(n-1)$.

Exercise 5.5.3 Derive this corollary from Theorem 5.5.1.

Proof of Theorem 5.5.1. We consider simplicial triangulations of M_n and S^3 so that $\hat{p}_n$ and t are simplicial maps between them. Let c be a 1-cycle in M_n with this triangulation. Then $\hat{p}_{n\sharp}(c)$ in S^3 is also a 1-cycle and we have a 2-chain c' in S^3 with $\partial c' = \hat{p}_{n\sharp}(c)$ since $H_1(S^3) = 0$. Let c'_n be the preimage of c' under the chain map $\hat{p}_{n\sharp} : C_2(M_n) \to C_2(S^3)$. Then we have $\partial c'_n = \rho_n(t)c$. This means that $\rho_n(t) = 0$ in $H_1(M_n) = 0$. Hence $\rho_n(t)H_1(E_\infty) \subset \mathrm{Ker} q^n_*$, for q^n_* is a Λ-homomorphism. By a general position argument, we see that for any element x in $H_1(M_n)$, there is an element x' in $H_1(E_n)$ with $j_{n*}(x') = x$. Let $\gamma_* : H_1(E) \to \langle t \rangle$ be the epimorphism induced from γ. Let $\gamma^n_* : H_1(E_n) \to \langle t^n \rangle$ be the epimorphism obtained from the composition $\gamma_* p_{n*} : H_1(E_n) \to \langle t \rangle$ by restricting the co-domain to its image which is the subgroup $\langle t^n \rangle \subset \langle t \rangle$. Let $K_i (i = 1, 2, \ldots, r)$ be the components of L. Let $m_i^{(n)}$ be the lift by p_n of a meridian $m_i \subset E$ of K_i in S^3. If $\gamma^n_*(x') = t^{an}$, then the element $x'' = x' - a\{m_1^{(n)}\} \in H_1(E_n)$ has $j_{n*}(x'') = x$ and $\gamma^n_*(x'') = 1$. Note that the covering $p^n : E_\infty \to E_n$ is associated with the kernel of the composite epimorphism $\gamma^n : \pi_1(E_n) \to \langle t^n \rangle$ of the Hurewicz epimorphism $\pi_1(E_n) \to H_1(E_n)$ and γ^n_*. For a simple loop c'' in E_n representing x'', the restriction $p^n |_{(p^n)^{-1}(c'')}$: $(p^n)^{-1}(c'') \to c''$ is a trivial covering. Let c''_0 be any component of $(p^n)^{-1}(c'')$. Then $q^n_*\{c''_0\} = j_{n*}\{c''\} = j_{n*}(x'') = x$. Hence q^n_* is surjective. Next, assume that $q^n_*(y) = 0$ for an element $y \in H_1(E_\infty)$. Then $p^n_*(y) = \sum_{i=1}^r a_i\{m_i\}$ and $\sum_{i=1}^r a_i = 0$. For the epimorphism $\gamma_* : H_1(E) \to \langle t \rangle$ induced from γ, the element $z = \sum_{i=1}^r a_i\{m_i\} \in H_1(E_n)$ has $\gamma_*(z) = 1$. Hence there is an element $\tilde{z} \in H_1(E_\infty)$ such that $p_{n*}p^n_*(\tilde{z}) = p_*(\tilde{z}) = z$. Letting $y' = y - \rho_n(t)\tilde{z}$, we have $q^n_*(y') = 0$ and $p_{n*}p^n_*(y') = 0$. This means that $p^n_*(y') = 0$. We note the following exact sequence

$$\cdots \to H_2(E_n) \xrightarrow{\partial_*} H_1(E_\infty) \xrightarrow{t^n - 1} H_1(E_\infty) \xrightarrow{p^n_*} H_1(E_n) \xrightarrow{\partial_*} H_0(E_\infty) \to \cdots$$

which is induced from this short exact sequence on chain complexes:

$$0 \to C_\sharp(E_\infty) \xrightarrow{t^n - 1} C_\sharp(E_\infty) \xrightarrow{p^n_\sharp} C_\sharp(E_n) \to 0.$$

Then we see that there is an element $y'' \in H_1(E_\infty)$ with $y' = (t^n - 1)y''$, so that $y = \rho_n(t)((t-1)y'' + \tilde{z}) \in \rho_n(t)H_1(E_\infty)$ and $\mathrm{Ker} q^n_* = \rho_n(t)H_1(E_\infty)$. □

Corollary 5.5.4 *The first Betti number $\beta_1(M_2)$ of M_2 is equal to $n(L)$.*

Proof. Since $\rho_2(t) = 1 + t$, we have $H_1(M_2) \cong H_1(E_\infty)/(1+t)H_1(E_\infty)$. By Proposition 5.4.1, $H_1(M_2)$ is isomorphic to the cokernel $\mathbf{Z}^n/\mathrm{Im}\psi_V(-1)$ of the homomorphism $\psi_V(-1) : \mathbf{Z}^n \to \mathbf{Z}^n$ representing the matrix $-(V + V')$ for a Seifert matrix V of L. Hence $\beta_1(M_2) = n(L)$. □

Exercise 5.5.5 When $H_1(M_n)$ is a finite abelian group, show that the order of $H_1(M_n)$ is given by the absolute value $|\prod_{k=1}^n \Delta_L(\omega^k)|$ where $\Delta_L(t)$ is the Alexander polynomial of L and ω is an n-th primitive root of unity (cf. Lemma 7.2.8).

By Proposition 5.4.1, $H_1(E_\infty)$ has a square Λ-presentation matrix. The minimal size of such matrices is denoted by $m(L)$ and is called *Nakanishi's index* (by convention, $m(L) = 0$ when $H_1(E_\infty) = 0$). Clearly, $m(L) \geq e(L)$, but actually we have the following (cf. [Kawauchi 1987]):

Theorem 5.5.6 $m(L) = e(L)$.

Supplementary notes for Chapter 5

The Seifert matrix was introduced by [Seifert 1934] in the case of a knot. The notion of S-equivalence was introduced by [Trotter 1962] and [Murasugi 1965] (see [Trotter 1973]). The proof of Theorem 5.2.3 is similar to that of [Rice 1971] in the case of a knot and included in [Kawauchi *] where an analogous result is established in general dimension. We note that in [Murasugi 1965] $n(L)+1$ (instead of $n(L)$) is called the nullity of L. The stable isomorphism class of the symmetric bilinear even form $b : G \times G \to \mathbf{Z}$ appearing in Theorem 5.3.2 is known to be completely determined by the signature, the nullity and the isomorphism classes of the linking form $\ell : T \times T \to \mathbf{Q}/\mathbf{Z}$ and the quadratic form $q : T \to \mathbf{Q}/\mathbf{Z}$ which are induced from b (see [Wall 1972], [Hirzebruch-Neumann-Koh 1971]). Further, it is shown in [Wall 1972] that the isomorphism class pair of the linking form ℓ and the quadratic form q corresponds bijectively to the isomorphism class of a certain linking form $\ell^* : T^* \times T^* \to \mathbf{Q}/\mathbf{Z}$. Accordingly, the number theoretic invariants of the linking form ℓ^* stated in [Kawauchi-Kojima 1980] (which are computed from q) are a complete invariant of the isomorphism class pair of ℓ and q. The Arf invariant of a proper link was introduced by [Robertello 1965] (see [Kawauchi 1984] for the effect of the link orientation). The reduced link module has been discussed in detail in the case of a knot (see, for example, [Pizer 1985]), but is not well understood in the case of the general link (cf. [Kawauchi 1987], [Pizer 1987]). Theorem 5.5.1 on the homology of a branched cyclic covering manifold was given by [Sakuma 1979] in a more general setting emphasizing the naturality of the isomorphism. The present proof is in [Kawauchi 1994].

Finally, we note that all of the results in this chapter continue to hold when we consider a homology 3-sphere instead of the 3-sphere S^3.

Chapter 6
The fundamental group

In this chapter, we discuss various properties of the fundamental group of a link exterior.

6.1 Link groups and link group systems

Here, we discuss how a certain topological property of a link exterior is related to the fundamental group.

Definition 6.1.1 The *link group* of a link L in S^3 (or $\mathbf{R}^3$), which we denote by π or $\pi(L)$, is the fundamental group $\pi_1(E(L), b)$ of the exterior $E(L)$, where b denotes a base point.

When L is a knot, π is called the *knot group*. Clearly, for any two equivalent links L, L', we have an isomorphism $\pi(L) \cong \pi(L')$. However, the converse is not true in general.

Exercise 6.1.1 Show that the links L_1 and L_2 in figure 6.1.1 are not equivalent, but their exteriors $E(L_1)$ and $E(L_2)$ are homeomorphic (so that there is an isomorphism $\pi(L_1) \cong \pi(L_2)$).

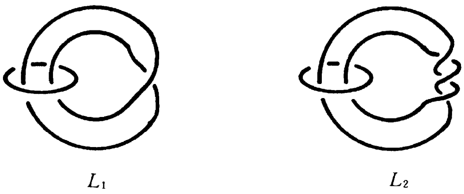


Fig. 6.1.1

Exercise 6.1.3 Show that if a link L is a split union of two links L_1, L_2, then $\pi(L)$ is isomorphic to the free product $\pi(L_1) * \pi(L_2)$.

Theorem 6.1.4 *For a link L in S^3, the following are equivalent:*

(1) *L is non-splittable.*
(2) *L is a trivial knot or $E(L)$ is a Haken manifold with incompressible boundary.*
(3) *The group $\pi(L)$ is indecomposable (with respect to the free product).*

Proof. It follows from the loop theorem (cf. C.2.2) and the sphere theorem (cf. C.2.3) that (1) $\to$ (2), and from the Kneser conjecture (Theorem C.4.4) that (2) $\to$ (3). (3) $\to$ (1) is clear. □

Since a subgroup of a free group is free, we obtain from Theorem 6.1.4 the following:

Corollary 6.1.5 *If $\pi(L)$ is a free group of rank r, then L is an r-component trivial link. (The converse also clearly holds.)*

Let $K_i (i = 1, 2, \ldots, r)$ be the components of a link L. Let T_i be the torus component of $\partial E(L)$ around K_i. Let (m_i, ℓ_i) be a meridian-longitude pair of K_i in S^3 with $m_i \cup \ell_i \subset T_i$. Note that the orientations of m_i and ℓ_i are uniquely specified by the orientations of S^3 and K_i. We consider m_i and ℓ_i as elements of $\pi_1(E(L), p)$ by choosing a path ω_i in $E(L)$ from the base point p to the point $m_i \cap \ell_i$ for each i. Then the subgroup of $\pi = \pi_1(E(L), p)$ generated by m_i and ℓ_i is independent of a choice of ω_i up to conjugation. This subgroup is called a *meridian-longitude subgroup* of π on K_i and denoted by $\langle m_i, \ell_i \rangle^\pi$. By the loop theorem (cf. Appendix C), $\langle m_i, \ell_i \rangle^\pi$ is isomorphic to $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$, and if it is isomorphic to $\mathbf{Z}$, then ℓ_i is the trivial element and L is a split union of the trivial knot K_i and the sublink $L - K_i$. A system $(G; G_i, i = 1, 2, \ldots, r)$ of a group G and its subgroups $G_i (i = 1, 2, \ldots, r)$ is called a *group system.*

Definition 6.1.6 A *group system* of an r-component link L is the group system $(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \ldots, r)$ for some meridian-longitude subgroups $\langle m_i, \ell_i \rangle^\pi$ $(i = 1, 2, \ldots, r)$ of the components of L.

By an *isomorphism* φ from a group system $(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \ldots, r)$ of an r-component link L onto a group system $(\pi(L'); \langle m'_i, \ell'_i \rangle^\pi, i = 1, 2, \ldots, r)$ of an r-component link L', we mean an isomorphism $\varphi : \pi(L) \cong \pi(L')$ such that $\varphi(m_i) = m'_i$ and $\varphi(\ell_i) = \ell'_i$ for all i. The following theorem shows that the link type is determined by the isomorphism classes of group systems of the link.

Theorem 6.1.7 *Two r-component links L and L' in S^3 belong to the same type if and only if there is an isomorphism φ from a group system $(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \ldots, r)$ of L onto a group system $(\pi(L'); \langle m'_i, \ell'_i \rangle^\pi, i = 1, 2, \ldots, r)$ of L'.*

Proof. Since the "only if" part is clear, it suffices to prove the "if"part. We assume that there is an isomorphism φ from $(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \ldots, r)$ onto $(\pi(L'); \langle m'_i, \ell'_i \rangle^\pi, i = 1, 2, \ldots, r)$. Let $L_j (j = 1, 2, \ldots, u)$ and $L'_k (k = 1, 2, \ldots, v)$ be the non-splittable sublinks of L and L', respectively. Then $\pi(L)$ and $\pi(L')$ are free products of the $\pi(L_j)$'s and the $\pi(L'_k)$'s respectively. By Theorem 6.1.4, $\pi(L_j)$ and $\pi(L'_k)$ are indecomposable groups for all j, k. If some L_j is a trivial knot, then the component of L' corresponding to L_j by φ is a trivial knot which is split from the other components of L'. Hence we can assume that neither L_j nor L'_k is a trivial knot, so that none of $\pi(L_j)$, $\pi(L'_k)$ is isomorphic to $\mathbf{Z}$. Then by the Kurosh subgroup theorem (cf. [Lyndon-Schupp 1977], [Magnus-Karass-Solitar 1966]), we have that $u = v$, and $\varphi(\pi(L_j))$ and $\pi(L'_j)$ are conjugate in $\pi(L')$ for each j by permuting the indices of $\pi(L'_j)$, if necessary. Hence φ induces an isomorphism φ_j from a group system $(\pi(L_j); \langle m_{j_i}, \ell_{j_i} \rangle^\pi, i = 1, 2, \ldots, r_j)$ onto a group system $(\pi(L'_j); \langle m'_{j_i}, \ell'_{j_i} \rangle^\pi, i = 1, 2, \ldots, r_j)$ for each j. By Waldhausen's theorem (Theorem C.4.1), there is a homeomorphism from $E(L_j)$ to $E(L'_j)$ preserving the (oriented) meridian-longitude systems of L_j and L'_j for each j. This homeomorphism extends to an auto-homeomorphism of S^3 sending L_j to L'_j and preserving the orientations of S^3, L_j and L'_j for each j, so that L_j and L'_j belong to the same type. □

For a group system $(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \dots, r)$ of a link L, some group systems of the links $-L, L^*$ and $-L^*$ are respectively isomorphic to the group systems $(\pi(L); \langle m_i^{-1}, \ell_i^{-1} \rangle^\pi, i = 1, 2, \dots, r), (\pi(L); \langle m_i^{-1}, \ell_i \rangle^\pi, i = 1, 2, \dots, r)$ and $(\pi(L); \langle m_i, \ell_i^{-1} \rangle^\pi, i = 1, 2, \dots, r)$. Then we obtain from Theorem 6.1.7 the following corollary:

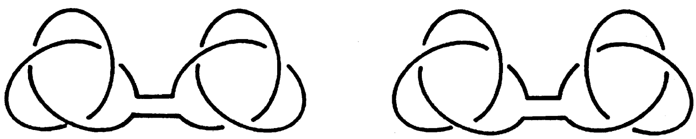
Corollary 6.1.8 *(1) L is invertible if and only if*
$$(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \dots, r) \cong (\pi(L); \langle m_i^{-1}, \ell_i^{-1} \rangle^\pi, i = 1, 2, \dots, r).$$
(2) L is (+)amphicheiral if and only if
$$(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \dots, r) \cong (\pi(L); \langle m_i^{-1}, \ell_i \rangle^\pi, i = 1, 2, \dots, r).$$
(3) L is (−)amphicheiral if and only if
$$(\pi(L); \langle m_i, \ell_i \rangle^\pi, i = 1, 2, \dots, r) \cong (\pi(L); \langle m_i, \ell_i^{-1} \rangle^\pi, i = 1, 2, \dots, r).$$

The following theorem follows from the Seifert-van Kampen theorem (cf. Appendix B):

Theorem 6.1.9 *For any group systems $(\pi(K_i); \langle m_i, \ell_i \rangle^\pi)(i = 1, 2)$ of knots $K_i (i = 1, 2)$ and any group system $(\pi(K); \langle m, \ell \rangle^\pi)$ of the connected sum $K = K_1 \sharp K_2$, the group $\pi(K)$ is isomorphic to the group obtained from the free product $\pi(K_1) * \pi(K_2)$ by adding the relation $m_1 = m_2$. Under this isomorphism, the meridian-longitude subgroup $\langle m, \ell \rangle^\pi$ on K corresponds to the subgroup generated by m_1 and $\ell_1 \ell_2$ up to conjugation.*

In particular, we obtain the following corollary from the observation preceding Corollary 6.1.8 and Theorem 6.1.9:

Corollary 6.1.10
For any knots K_0, K, we have an isomorphism $\pi(K_0 \sharp K) \cong \pi(K_0 \sharp - K^*)$.



$K \# K$ $K \# K^*$

Fig. 6.1.2

Example 6.1.11. For a (left-handed or right-handed) trefoil knot K, we call the connected sum $K \sharp K$ a *granny knot*, and noting that K is invertible, we call the connected sum $K \sharp K^* \cong K \sharp - K^*$ a *square knot* (see figure 6.1.2). By Corollary 6.1.10, we have $\pi(K \sharp K) \cong \pi(K \sharp K^*)$. However, $K \sharp K$ and $K \sharp K^*$ are not equivalent. For if they are equivalent, then their signatures must satisfy $\sigma(K \sharp K) = \pm\sigma(K \sharp K^*)$. But $\sigma(K \sharp K) = \pm 4$ and $\sigma(K \sharp K^*) = 0$ by Exercise 5.3.4, since $\sigma(K) = \pm 2$.

As this example shows, even for a knot, the group is not a complete invariant for knot equivalence. However, for prime knots, it is known to be a complete invariant.

Theorem 6.1.12 *Prime knots K_1, K_2 in S^3 are equivalent if and only if $\pi(K_1) \cong \pi(K_2)$.*

This is essentially a corollary of the following two results:

(1) ([Whitten 1987]) For prime knots K_1, K_2 in S^3, if $\pi(K_1) \cong \pi(K_2)$, then there is a homeomorphism $E(K_1) \cong E(K_2)$.

(2) ([Gordon-Luecke 1989]) For any non-trivial knots K_1, K_2 in S^3, any homeomorphism $E(K_1) \cong E(K_2)$ can be extended to an auto-homeomorphism of S^3 sending K_1 to K_2 setwise.

The following theorem and Thurston's hyperbolization theorem (cf. C.7.2) justify the definition of a "hyperbolic" link in Definition 3.2.11.

Theorem 6.1.13 *A link L in S^3 is simple and anannular if and only if the group $\pi(L)$ is non-abelian and indecomposable (with respect to free product) and any subgroup of $\pi(L)$ isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ is conjugate to a subgroup of a meridian-longitude subgroup $\langle m_i, \ell_i \rangle^\pi$ of some component K_i of L.*

This theorem follows from Theorem 6.1.4, the annulus theorem (cf. C.6.1), some properties of special Seifert manifolds (cf. C.5), and the two well-known facts: that $E(L)$ is not homeomorphic to the twisted I-bundle over the Klein bottle, and that $\pi_1(E(L))$ is abelian if and only if L is a Hopf link or a trivial knot (cf. Theorem 6.3.1).

Exercise 6.1.14 Complete the proof of Theorem 6.1.13.

We obtain the following from Thurston's hyperbolization theorem:

Theorem 6.1.15 *A link L in S^3 is simple and anannular if and only if the group $\pi(L)$ is non-abelian, indecomposable (with respect to the free product) and isomorphic to a discrete subgroup of $PSL_2(\mathbf{C})$.*

Proof. By Theorem 6.1.13 and Thurston's hyperbolization theorem, L is simple and unannular if and only if $E(L)$ is a hyperbolic manifold with finite volume. Hence the 'only if' part is obtained. We show the 'if' part. For any link L, the group $\pi(L)$ is torsion-free (i.e., $x^n = 1$ implies $x = 1$ for any element $x \in \pi(L)$ and any non-zero integer n). Let π be a discrete subgroup of $PSL_2(\mathbf{C})$ isomorphic to $\pi(L)$. Because π is discrete and torsion-free, it is known that π acts properly-discontinuously (cf. B.4) and orientation-preservingly on the hyperbolic 3-space H^3, so that the projection $H^3 \to H^3/\pi = M$ is a covering projection with π the covering transformation group. Since π is a finitely generated group, we see from the Scott Theorem (cf. [Hempel 1976]) that there is a compact orientable 3-submanifold E of M with the natural isomorphism $\pi_1(E) \cong \pi_1(M) = \pi$. Using that the universal covering space of M is $H^3 \cong \mathbf{R}^3$, we can take E to be irreducible, so that E is homotopy equivalent to $E(L)$ which is a Haken manifold with incompressible boundary by Theorem 6.1.4. Since the Euler characteristic $\chi(E) = \chi(E(L)) = 0$, the boundary ∂E of E has only torus components. Then by the loop theorem, E is a Haken manifold with ∂E incompressible in E, for $\pi_1(E)$ is non-abelian and indecomposable. We see that the inclusion $\partial E \subset M - \mathrm{int}E$ is a homotopy equivalence and by [Thurston *], that any subgroup of $\pi_1(M)$ isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ is conjugate to a subgroup of the fundamental group of a component of $M - \mathrm{int}E$ (that is, an end of M). Hence E is algebraically simple. By Thurston's hyperbolization theorem, E is a hyperbolic manifold and any isomorphism $\pi(L) \cong \pi_1(E)$ preserves the peripheral structures. By Waldhausen's theorem, $E(L)$ is homeomorphic to E, so that L is simple and unannular. $\square$

Proposition 6.1.16 *The group of the torus knot $T(p,q)$ of type (p,q) has the group presentation $\langle a, b \mid a^p = b^q \rangle$.*

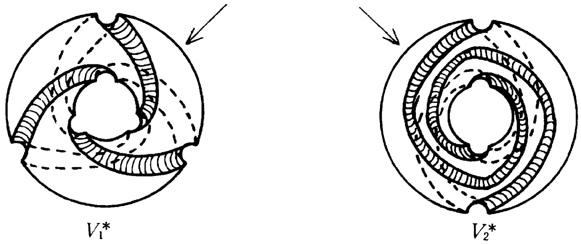


Fig. 6.1.3

Proof. We take the knot $T(p,q)$ on the standard torus T in S^3. Let $V_i, i = 1, 2$, be the solid tori obtained by splitting S^3 along T. Then $E(T(p,q)) = S^3 - \mathrm{int}N(T(p,q))$ is the union of two solid tori $V_i^* = \mathrm{cl}(V_i - N(T(p,q))), i = 1, 2$, pasted along the annulus $A = E(T(p,q)) \cap T$. The central loop C of the annulus A is isotopic to $T(p,q)$ in the torus T. For a meridian-longitude pair (m, ℓ) on V_1 (in S^3), C represents $m^p\ell^q$ in $\pi_1(T)$. Note that T is ambient isotopic to both ∂V_1^* and ∂V_2^* in S^3. Let $(m_i^*, \ell_i^*), i = 1, 2$, be the meridian-longitude pair on V_i^* corresponding to (m, ℓ) by this ambient isotopy. Then m_1^* and ℓ_1^* represent the trivial element and a generator of $\pi_1(V_1^*) \cong \mathbf{Z}$, respectively, so that $[C] = [\ell_1^*]^q$. Also, m_2^* and ℓ_2^* represent a generator and the trivial element of $\pi_1(V_2^*) \cong \mathbf{Z}$, respectively, so that $[C] = [m_2^*]^p$. By the Seifert-van Kampen Theorem, we obtain a group presentation of $\pi(T(p,q))$ as $\langle a, b \mid a^p = b^q \rangle$ with $a = [m_2^*], b = [\ell_1^*]$. $\square$

Using Proposition 6.1.16, we give here a group-theoretic proof of Theorem 2.2.2 (except the proof of non-amphicheirality which is given in 12.2.11).

6.1.17. Group-theoretic Proof of Theorem 2.2.2 except non-amphicheirality. For $p = \pm 1$ or $q = \pm 1$, the knot $T(p,q)$ is trivial. Assume that $p \neq \pm 1$, $q \neq \pm 1$. The center $\xi(\pi)$ of the group π of $T(p,q)$ includes the subgroup A generated by a^p, and the quotient group π/A is the free product $\mathbf{Z}_p * \mathbf{Z}_q$. Hence π is not abelian and we have (1). Further, since the center $\xi(\mathbf{Z}_p * \mathbf{Z}_q)$ is the trivial group, we have $\xi(\pi) = A$ and the group $\mathbf{Z}_p * \mathbf{Z}_q$ is uniquely determined by the group π. We see from the Kurosh subgroup theorem that $\mathbf{Z}_p * \mathbf{Z}_q \cong \mathbf{Z}_{p'} * \mathbf{Z}_{q'}$ if and only if $\{|p|, |q|\} = \{|p'|, |q'|\}$. Hence, we have (2) when we know that $T(p,q)$ is non-amphicheiral. □

Exercise 6.1.18 Find a group presentation of a torus link by using an argument similar to Proposition 6.1.16.

Exercise 6.1.19 Show that the group $\langle a, b \mid a^p = b^q \rangle$ (where p and q are coprime integers not equal to ± 1) is not isomorphic to any subgroup of $PSL_2(\mathbf{C})$. [*Hint*: Use the fact that every non-trivial element x of $PSL_2(\mathbf{C})$ is conjugate to a matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$ according to whether or not the trace $\mathrm{tr}(x) = \pm 2$.]

6.2 Presentations of a link group

We shall give here a presentation of $\pi(L) = \pi_1(\mathbf{R}^3 - L)$. Note that there are natural isomorphisms $\pi_1(E(L, \mathbf{R}^3)) \cong \pi_1(\mathbf{R}^3 - L) \cong \pi_1(E(L, S^3))$ by regarding S^3 as the one-point compactification of $\mathbf{R}^3$. Let $\mathbf{R}^3$ have the orientation of the right-hand rule. For the orthogonal projection $p : \mathbf{R}^3 \to \mathbf{R}^2$ whose image $p(L)$ gives a regular presentation of L, we consider that $\mathbf{R}^2 = \mathbf{R}^2 \times 0$ and p sends each $(x, y, z) \in \mathbf{R}^3$ to $(x, y, 0) \in \mathbf{R}^2 \times 0 = \mathbf{R}^2$. We assume that the regular presentation $p(L)$ is connected and has at least one double point. Let $v_j (j = 1, 2, \ldots, s)$ be the double points of $p(L)$. Then we assume that an open arc neighborhood of the overcrossing point of $p^{-1}(v_j)$ in L is in the upper-half space $\mathbf{R}^3_+$ and the remaining part of L is in $\mathbf{R}^2$ for each j. In this case, we say that the link L is in an *over-normal position.* (As a dual concept, when an open arc neighborhood of the undercrossing point of $p^{-1}(v_j)$ in L is in the lower-half space $\mathbf{R}^3_-$ and the remaining part of L is in $\mathbf{R}^2$ for each j, we say that the link L is in an *under-normal position*).
[Step 1] We take a point a under the plane $\mathbf{R}^2$. Let F be a 2-dimensional polyhedron consisting of the cone $p(L) * a$ and the line segments joining x with $p(x)$ for all $x \in L$ (see figure 6.2.1).

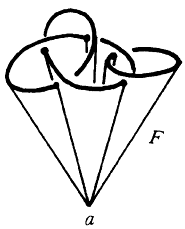


Fig. 6.2.1

[Step 2] We consider $p(L)$ as a graph whose vertices are $v_j (j = 1, 2, \ldots, s)$. Let $e_i (i = 1, 2, \ldots, m)$ be the edges of the graph $p(L)$. We orient each e_i by the orientation induced from that of L. We take a cell decomposition of L so that any element of $\cup_{j=1}^{s} p^{-1}(v_j)$ is the 0-cell and the arc $e_i^{\wedge}$ of L corresponding to e_i for each i is the 1-cell (see figure 6.2.2).

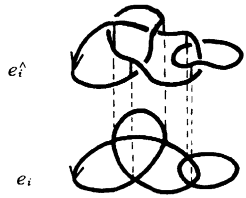


Fig. 6.2.2

[Step 3] We consider F as the polyhedron of a cell complex K given as follows: Namely, the 0-cells of K consist of the 0-cells of L (including $v_j (j = 1, 2, \ldots, s)$ by our assumption) and a. The 1-cells of K consist of the line segment with end point set $p^{-1}(v_j)$ and the line segment with end point set $\{v_j, a\}$ for all j. Denoting the union of these 1-cells by F^1, we take as a 2-cell of K the closure of each component of $F - F^1$ in F. Then the boundary of each 2-cell of K contains just one of the 1-cells $e_i^{\wedge}$ $(i = 1, 2, \ldots, m)$. Let D_i be the 2-cell of K with $\partial D_i \supset e_i^{\wedge}$. We orient D_i by the orientation induced from that of $e_i^{\wedge}$.

[Step 4] $F^c = \mathbf{R}^3 - F$ is simply connected. Let H be the union of F^c and the open 2-cells $\mathrm{int}D_i (i = 1, 2, \ldots, m)$. Then $\pi_1(H)$ is a free group $\langle x_1, x_2, \ldots, x_m \rangle$ of rank m, where the base point b of $\pi_1(H)$ is taken above the link L and the generator x_i is represented by a path intersecting $\mathrm{int}D_i$ in just one point with intersection number $+1$ (see figure 6.2.3).

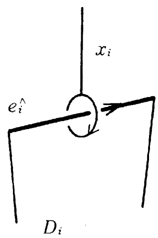


Fig. 6.2.3

[Step 5] Let H' be obtained from H by adjoining all of the open 1-cells in K belonging to $\mathbf{R}^3 - L$. The group $\pi_1(H')$ is obtained from $\pi_1(H) = \langle x_1, x_2, \ldots, x_m \rangle$ by adding one relation for each attached open 1-cell, which is shown as follows:

(1) Add the relation $x_i x_h^{-1} = 1$ for each 1-cell J with $p(J)$ a double point of $p(L)$ as shown in figure 6.2.4.

(2) Add the relation $x_i x_k^{-\epsilon} x_h^{-1} x_s^{\epsilon} = 1$ for each 1-cell J with a as an end point as shown in figure 6.2.5, where $\epsilon = \pm 1$ is the sign of the crossing point of $p(L)$ (see 1.1 and figure 6.2.6).

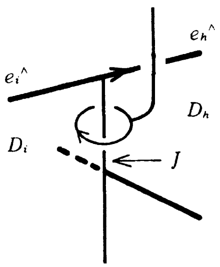


Fig. 6.2.4

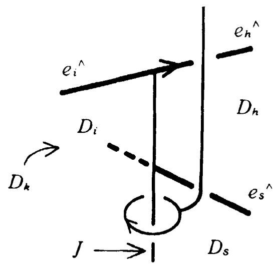


Fig. 6.2.5

Then the path around each 1-cell J in (2) is a path which turns around each 1-cell in (2) except for J once. Hence each relation in (2) is a consequence of the other relations in (2). Finally, add the point a to H' to obtain $\mathbf{R}^3 - L$. No new relation occurs from this addition. Thus, we obtain the following:

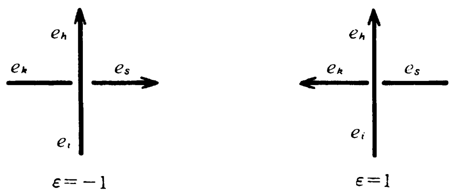


Fig. 6.2.6

Theorem 6.2.1 *For a connected regular presentation $p(L)$ with at least one double point of a link L, we have a presentation of the group $\pi(L)$ whose generators are the words $x_1, x_2, \ldots, x_m$ corresponding to the edges $e_1, e_2, \ldots, e_m$ of $p(L)$ and whose relations consist of $x_i = x_h$ and $x_i x_k^{-\epsilon} x_h^{-1} x_s^{\epsilon} = 1$ for each double point of $p(L)$, shown in figure 6.2.6. Further, we can remove any one relation $x_i x_k^{-\epsilon} x_h^{-1} x_s^{\epsilon} = 1$ from the relations of the presentation without changing the resulting group.*

The presentation of $\pi(L)$ given in Theorem 6.2.1 is called an *over presentation* of $\pi(L)$. When L is in under-normal position, the mirror image L^* of L in $\mathbf{R}^2$ is in over-normal position. By Theorem 6.2.1, we have an over presentation of the group $\pi(L^*)$. When we apply the remark preceding Corollary 6.1.8, we obtain the following new presentation of the group $\pi(L)$ which we call an *under presentation* of the group $\pi(L)$:

Theorem 6.2.2 *For a connected regular presentation $p(L)$ of a link L with at least one double point, we have a presentation of the group $\pi(L)$ whose generators* *are the words $x_1, x_2, \ldots, x_m$ corresponding to the edges $e_1, e_2, \ldots, e_m$ of $p(L)$ and whose relations consist of $x_k = x_s$ and $x_i^{-1}x_k^{\epsilon}x_h x_s^{-\epsilon} = 1$ for each double point of $p(L)$, shown in figure 6.2.6. Further, we can remove any one relation $x_i^{-1}x_k^{\epsilon}x_h x_s^{-\epsilon} = 1$ from the relations of the presentation without altering the group.*

Remark 6.2.3 The generator x_i in Theorem 6.2.1 and the generator x_i in Theorem 6.2.2 induce the same element in the abelianized group of $\pi(L) = \pi_1(\mathbf{R}^3 - L)$ (that is, in $H_1(\mathbf{R}^3 - L)$).

Remark 6.2.4 When a regular presentation $p(L)$ of a link L is disconnected, a presentation of the group $\pi(L)$ is obtained as a free product of group presentations for all connected components of $p(L)$ which are obtained from Theorem 6.2.1 (or Theorem 6.2.2), where we take $\langle x \rangle$ as a group presentation of the connected component without double points.

Example 6.2.5. The over and under presentations of the group of the Hopf link shown in figure 6.2.7 are respectively

$\langle x_1, x_2, y_1, y_2 \mid x_1 = x_2,\ y_1 = y_2, y_2 x_1 y_1^{-1} x_2^{-1} = 1 \rangle$ and

$\langle x_1, x_2, y_1, y_2 \mid x_1 = x_2,\ y_1 = y_2,\ y_2^{-1} x_1^{-1} y_1 x_2 = 1 \rangle$.

Each group is isomorphic to $\langle x, y \mid xy = yx \rangle \cong \mathbf{Z} \oplus \mathbf{Z}$.

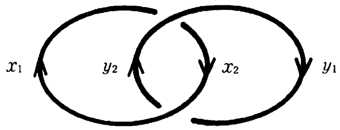


Fig. 6.2.7

Example 6.2.6. The over and under presentations of the group of the trefoil knot shown in figure 6.2.8 are respectively

$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1 = x_2, x_3 = x_4, x_5 = x_6, x_1 x_5^{-1} x_2^{-1} x_4 = 1, x_5 x_3^{-1} x_6^{-1} x_2 = 1 \rangle$ and

$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_2 = x_3, x_4 = x_5, x_6 = x_1, x_1^{-1} x_5 x_2 x_4^{-1} = 1, x_5^{-1} x_3 x_6 x_2^{-1} = 1 \rangle$. Each group is isomorphic to $\langle x, y \mid xyx^{-1} = y^{-1}xy \rangle$.

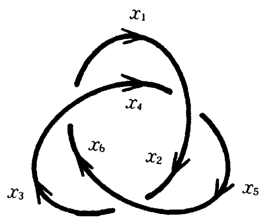


Fig. 6.2.8

Let $p(L)$ be a connected regular presentation of a link L with at least one double point and with n bridges. Let $Y_i (i = 1, 2, \ldots, n)$ be the overbridges. In the over presentation of L on $p(L)$, the generators corresponding to the edges of $p(L)$ forming Y_i can be reduced to one generator, say y_i, so that we have a presentation $\langle y_1, y_2, \ldots, y_n \mid r_1, r_2, \ldots, r_{n-1} \rangle$ of the group $\pi(L)$ where r_i is a relation as shown in figure 6.2.9. (In this figure, r_i is given as $y_h w y_k^{-1} w^{-1}$ with $w = y_a^{\epsilon(a)} y_b^{\epsilon(b)} \ldots y_c^{\epsilon(c)}$ where $\epsilon(a), \epsilon(b), \ldots, \epsilon(c)$ are respectively the signs (see 1.1 or figure 6.2.6) of the crossing points occurring at the intersections of the overbridges $y_a, y_b, \ldots, y_c$ and the underbridge between y_h and y_k. From this argument, we obtain the following:

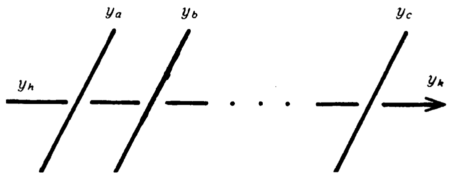


Fig. 6.2.9

Theorem 6.2.7 *The group of an n-bridge link has a Wirtinger presentation with n generators represented by meridians and with deficiency one.*

Example 6.2.8. The group of the Borromean rings in a 3-bridge presentation shown in figure 6.2.10 has the presentation
$\langle a, b, c \mid a = (cb^{-1}c^{-1}b)a(cb^{-1}c^{-1}b)^{-1}, b = (ac^{-1}a^{-1}c)b(ac^{-1}a^{-1}c)^{-1} \rangle$.

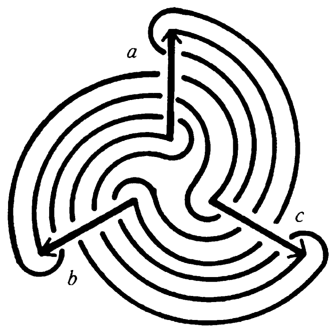


Fig. 6.2.10

Exercise 6.2.9 Find a Wirtinger presentation with 2 generators and one relation of the 2-bridge link $S(\alpha, \beta)$. [*Hint*: Think of Schubert's normal form.]

Exercise 6.2.10 Show the inequality $1 \leq \operatorname{def} \pi(L) \leq r$ for the deficiency $\operatorname{def} \pi(L)$ of the group $\pi(L)$ of an r-component link L.

The following theorem is obtained by using a braid presentation of a link:

Theorem 6.2.11 (Alexander-Artin theorem) *Any link group has a presentation of the following Wirtinger type:* $\langle x_1, x_2, \dots, x_n \mid x_i = w_i x_{p(i)} w_i^{-1}, i = 1, 2, \dots, n\rangle$ *where* $p(1), p(2), \dots, p(n)$ *are a permutation of* $1, 2, \dots, n$ *and* $w_i (i = 1, 2, \dots, n)$ *are words in* $x_1, x_2, \dots, x_n$ *which satisfy the identity* $\prod_{i=1}^n x_i = \prod_{i=1}^n w_i x_{p(i)} w_i^{-1}$ *in the free group* $\langle x_1, x_2, \dots, x_n\rangle$. *Conversely, any group with such a Wirtinger presentation is realized by a link group.*

Proof. As shown in Chapter 1, any link L is obtained from an n-braid $b \in B_n$ by closing it in the vertical direction (Alexander's theorem). We take n points $A_i = (i/(n+1), 1/2)(i = 1, 2, \dots, n)$ in the square $I^2 = \{(x, y) \mid 0 \le x, y \le 1\}$. Then the braid b gives an ambient isotopy from an auto-homeomorphism f_b of I^2 to the identity fixing $\{A_1, A_2, \dots, A_n\}$ setwise. This auto-homeomorphism f_b determines an automorphism φ_b of the free group $\langle x_1, x_2, \dots, x_n\rangle$ of rank n which is the fundamental group of $I^2 - \{A_1, A_2, \dots, A_n\}$ where x_i is represented by a meridian of A_i in I^2. Let $f_b(A_i) = A_{p(i)}$. Then we have $\varphi_b(x_i) = w_i x_{p(i)} w_i^{-1}$. By the Seifert-van Kampen theorem, we obtain the first half of the theorem. Conversely, for any $p(i)$ and w_i stated in the theorem, there is a braid b with $\varphi_b(x_i) = w_i x_{p(i)} w_i^{-1}(i = 1, 2, \dots, n)$. (See Artin's theorem, cf. [Burde-Zieschang 1985].) Hence we obtain the second half of the theorem. □

Exercise 6.2.12 Find a group presentation of the pretzel link $P(q_1, q_2, \dots, q_m)$.

6.3 Subgroups and quotient groups of a link group

Here, we observe miscellaneous properties of a link group. As a preliminary tool, we first explain the homology group of a group. For any group G and any integer $q > 1$, we can construct a path-connected topological space X such that $\pi_1(X, b) = G$ and $\pi_i(X, b) = 0$ $(1 < i < q+1)$, where b is a base point (cf. [Spanier 1966]). Then the *homology group* $H_q(G)$ *of the group* G is defined to be the homology group $H_q(X)$ which is known to be independent of choice of X up to isomorphism. The following theorem follows essentially from a classification argument of the abelian fundamental groups of 3-manifolds (cf. [Hempel 1976]):

Theorem 6.3.1 *A non-trivial abelian subgroup of the group* $\pi(L)$ *of a link* L *is isomorphic to* $\mathbf{Z}$ *or* $\mathbf{Z} \oplus \mathbf{Z}$.

Proof. Let A be a non-trivial abelian subgroup of $\pi = \pi(L)$. If π is not indecomposable, then A is isomorphic to $\mathbf{Z}$ or conjugate to a subgroup of an indecomposable component of π by the Kurosh subgroup theorem. Hence we can assume that π is an indecomposable group. Since it is obvious when $\pi \cong \mathbf{Z}$, we may assume by Theorem 6.1.4 that $E = E(L)$ is a Haken manifold with incompressible boundary. Let $\tilde{E}$ be the covering space over E corresponding to the subgroup A of π. Since $\pi_i(\tilde{E}) = 0$ for all $i \ge 2$, we have $H_q(A) \cong H_q(\tilde{E})$ for all q. In particular, $H_3(A) = 0$. To complete the proof, it suffices to show that the following two cases cannot occur: (1) $A = \mathbf{Z}_m$ with $|m| \ge 2$, (2) $A = \mathbf{Z}^s$ with $s \ge 3$. In the case (1) we have $H_3(A) \cong \mathbf{Z}_m$ and in the case (2) $H_3(A) \cong \mathbf{Z}^u$ for $u = s!/(s-3)!3!$ (cf. [Hempel 1976], p.75). These contradict $H_3(A) = 0$. □

The following theorem is a special case of the Stallings fibration theorem for a compact 3-manifold in [Stallings 1962]:

Theorem 6.3.2 *Consider a (unique) epimorphism γ from the group $\pi(L)$ of a link L to the infinite cyclic group $\langle t\rangle$ sending each meridian of L to t. Then L is a fibered link if and only if the kernel $\mathrm{Ker}\gamma$ of γ is finitely generated.*

Proof. If L is a fibered link, then $\mathrm{Ker}\gamma$ is isomorphic to the fundamental group of the fiber surface, which is a free group of finite rank. Conversely, suppose that $\mathrm{Ker}\gamma$ is finitely generated. By Exercise 5.4.4, a minimal genus Seifert surface F of L is connected. Let $E = E(L)$ and $F_E = E \cap F(\cong F)$. Let E' be the 3-manifold obtained from E by splitting along F_E, and $F_i (i = 1, 2)$ the resulting copies of F_E in E'. The natural homomorphism $(j_i)_\sharp : \pi_1(F_i) \to \pi_1(E')$ is injective for each i. Let E_∞ be the infinite cyclic covering space over E corresponding to the kernel of γ, which is constructed by pasting together the $\mathbf{Z}$-indexed copies of E' (cf. 5.4). If $(j_i)_\sharp$ is not surjective for some i, then we must have that $\pi_1(E_\infty) \cong \mathrm{Ker}\gamma$ is not finitely generated (cf. [Neuwirth 1965]), which is a contradiction. Hence $(j_i)_\sharp$ is an isomorphism. Then using that E' is irreducible, we can show that there is a homeomorphism $(E'; F_1, F_2) \cong (F \times [0,1], F \times 0, F \times 1)$ (cf. [Hempel 1976]). Hence L is a fibered link. □

Exercise 6.3.3 If $\pi(L) \cong \mathbf{Z} \oplus \mathbf{Z}$, then show that the link L is equivalent to a Hopf link.

Link groups with non-trivial center are completely determined in [Burde-Murasugi 1970]. Here we observe only several standard properties.

Theorem 6.3.4 *For a link L in S^3, the center of the group $\pi(L)$ is non-trivial if and only if $E(L)$ is a Seifert manifold.*

Proof. Since the center of $\pi(L)$ is non-trivial, then $\pi(L)$ is an indecomposable group, so that $\pi(L) \cong \mathbf{Z}$ (in this case, $E(L) \cong S^1 \times D^2$) or $E(L)$ is a Haken manifold such that $\partial E(L)$ is incompressible by Theorem 6.1.5. It is clear in the former case and seen from [Waldhausen 1967] in the latter case that $E(L)$ is a Seifert manifold. Conversely, suppose that $E(L)$ is a Seifert manifold. Let B be the base space of this fibration. Since the natural homomorphism $H_1(\partial E(L)) \to H_1(E(L))$ is surjective, it follows that the natural homomorphism $H_1(\partial B) \to H_1(B)$ is surjective. Hence B is homeomorphic to the planar surface obtained from S^2 by removing from it r open disks with r the number of components of L. Since B and $E(L)$ are orientable, a regular fiber of the fibration represents a non-trivial element in the center of $\pi(L)$. □

Proposition 6.3.5 *Let $E(L)$ be a Seifert manifold. Then either there is a Seifert fibered structure on S^3 extending the Seifert fibered structure of $E(L)$ or the link L has a component O which is a trivial knot in S^3 such that the Seifert fibered structure of $E(L)$ extends to a Seifert fibered structure on $E(O) \cong S^1 \times D^2$.*

Proof. Let K be a component of L. Let $N(K)$ be a component of $N(L)$. Let L' be a sublink of L such that a regular fiber of $\partial N(K)(\subset \partial E(L))$ is a meridian of

K for all components K of L'. If $L' = \emptyset$, then we can extend the Seifert structure of $E(L)$ to a Seifert fibered structure on S^3. Let $L' \neq \emptyset$. Then $E(L')$ is a Seifert manifold with a Seifert fibered structure extending the Seifert fibered structure of $E(L)$, and the group obtained from $\pi(L')$ by adding to it a relation $[h'] = 1$ for a regular fiber h' of $E(L')$ is a trivial group. This means that the base space B' of $E(L')$ is a disk with at most one singular point, so that L' is a trivial knot. □

We see easily that any fiber of any Seifert structure on S^3 is a torus knot. Hence we obtain the following from Theorem 6.3.4 and Proposition 6.3.5 (cf. [Burde-Zieschang 1966]):

Corollary 6.3.6 *For any knot K in S^3, the center of $\pi(K)$ is non-trivial if and only if K is a torus knot.*

Exercise 6.3.7 For the link L in S^3 shown in figure 6.3.1, show the following:

(1) The center of $\pi(L)$ is non-trivial.
(2) There is no Seifert fibered structure on S^3 with the link L belonging to the fibers.

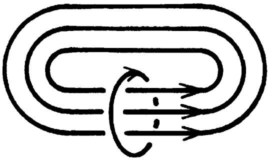


Fig. 6.3.1

A group π is said to be *locally indicable* if every non-trivial finitely generated subgroup of π admits an epimorphism to $\mathbf{Z}$. The following theorem is observed in [Howie-Short 1985]:

Theorem 6.3.8 *Every link group is locally indicable.*

A group π is said to be *residually finite* if for every non-trivial element $x \in \pi$, there is a homomorphism φ from π to a finite group H with $\varphi(x) \neq 1$.

Theorem 6.3.9 *Every link group is residually finite.*

Proof. The free product of residually finite groups is residually finite and $\mathbf{Z}$ is residually finite. Hence it suffices to show it for $\pi(L)$ when $E(L)$ is a Haken manifold such that ∂E is incompressible. In case $E(L)$ is a Seifert manifold, $E(L)$ is a fiber bundle over S^1 with a fiber a compact surface F (cf. [Orlik 1972]). In particular, there is a short exact sequence

$$1 \to \pi_1(F) \to \pi(L) \to \pi_1(S^1) \to 1.$$

Since $\pi_1(F)$ is a free group, $\pi(L)$ is residually finite by a method which is described in [Neuwirth 1965(pp. 63–64)]. If $E(L)$ is a hyperbolic manifold with finite volume, then $\pi(L)$ is a subgroup of $PSL_2(\mathbf{C})$ by Theorem 6.1.5. It is also residually finite by [Lyndon-Schupp 1977], Proposition 7.11. In general, $E(L)$ is a Seifert manifold or a hyperbolic manifold with finite volume or a torus sum of such link exteriors by the torus decomposition theorem (Theorem C.6.3). Then we obtain the proof in the general case. □

Exercise 6.3.10 Complete the proof of Theorem 6.3.9 in the general case. (Cf. [Hempel 1987]).

The following theorem is observed in [González-Acuña 1975]:

Theorem 6.3.11 *For any finitely generated group G, choose elements $x_1, x_2, \ldots, x_n$ in G with $\langle\langle x_1, x_2, \ldots, x_n\rangle\rangle^G = G$. Then there is an epimorphism φ from the group $\pi(L)$ of some n-component link L to G sending a meridian system of L to the elements $x_1, x_2, \ldots, x_n$.*

Supplementary notes for Chapter 6

For a link L with components $K_i(i = 1, 2, \ldots, r)$ we denote the kernel of the natural epimorphism $\pi(L) \to \pi(L - K_i)$ by A_i. Then the product $D(L) = [A_1, A_1]\,[A_2, A_2] \ldots [A_r, A_r]$ of the commutator subgroups $[A_i, A_i]$ of A_i is a normal subgroup of $\pi(L)$. The quotient group $\pi(L)/D(L)$ is invariant under a *link-homotopy* of L (that is to say, a homotopic deformation of L permitting only self-intersections of each component of L). Using this concept, *Milnor's μ^*-invariant*, which is a numerical invariant of a link-homotopy, is defined (cf. [Milnor 1954]). For the lower central series

$$\pi(L) = \pi(L)_0 \supset \pi(L)_1 \supset \pi(L)_2 \supset \cdots, \qquad \pi(L)_{k+1} = [\pi(L), \pi(L)_k],$$

of a link group $\pi(L)$, the quotient group $\pi(L)/\pi(L)_k$ is invariant under a topological *I-equivalence* (that is to say, a link cobordism (= a link concordance), as described in 12. 3, except that the cobordism annuli are only required to be topologically embedded). From this definition, *Milnor's $\bar{\mu}$-invariant*, which is a numerical link invariant of a topological I-equivalence, is defined (cf. [Milnor 1957]).

Chapter 7
Multi-variable Alexander polynomials

In this chapter, we define the Alexander module and the link module of a link and show how to calculate them by Fox's free differential calculus. Then we define the (multi-variable) graded Alexander polynomials to be the characteristic polynomials of these modules and explain the Torres conditions, which the (0-th) Alexander polynomial satisfies.

7.1 The Alexander module

Let $L = K_1 \cup K_2 \cup \cdots \cup K_r$ be a link in S^3 and E be the exterior of L. Let $\pi = \pi_1(E)$ be the group of L. Let t_i be the homology class in $H_1(E) \cong H_1(S^3 - L)$ represented by a meridian of K_i $(1 \leq i \leq r)$. Then $H_1(E)$ is a free abelian group of rank r generated by $t_1, \ldots, t_r$. Let $\gamma : \pi = \pi_1(E) \to H_1(E)$ be the Hurewicz epimorphism. The covering space over E corresponding to the subgroup $\mathrm{Ker}(\gamma) = [\pi, \pi]$ of π is called the *universal abelian covering space* of E and denoted by E_γ. Since $H_1(E)$ acts on E_γ as the covering transformation group, $H_1(E_\gamma)$ is regarded as a module over the integral group ring $\mathbf{Z}H_1(E)$ of $H_1(E)$. By regarding $H_1(E)$ as the multiplicative free abelian group $\prod_{i=1}^r \langle t_i \rangle$ with basis $t_1, t_2, \ldots, t_r$, we identify $\mathbf{Z}H_1(E)$ with the Laurent polynomial ring Λ in the variables $t_1, \ldots, t_r$, so that we can regard $H_1(E_\gamma)$ as a Λ-module. Let $p : E_\gamma \to E$ be the covering projection and b a point in E. Then $H_1(E_\gamma, p^{-1}(b))$ can also be regarded as a Λ-module.

Definition 7.1.1 (1) The *link module* of L is the Λ-module $H_1(E_\gamma)$.
(2) The *Alexander module* of L, denoted by $A(L)$, is the Λ-module $H_1(E_\gamma, p^{-1}(b))$.

Since E_γ is a connected, non-compact 3-manifold, we have $H_0(E_\gamma) \cong \mathbf{Z}$ and $H_i(E_\gamma) \cong H_i(E_\gamma, p^{-1}(b)) \cong 0\,(i \geq 3)$. Here t_i acts on $H_0(E_\gamma)$ as the identity map. Let $\varepsilon : \Lambda \to \mathbf{Z}$ be the Λ-homomorphism defined by $\varepsilon(t_i) = 1$ $(1 \leq i \leq r)$. The kernel $\mathrm{Ker}(\varepsilon)$ is an ideal of Λ generated by $\{t_i - 1 \mid 1 \leq i \leq r\}$ which is denoted by $\varepsilon(\Lambda)$ and called the *augmentation ideal* of Λ. Then $H_0(E_\gamma) \cong \Lambda/\varepsilon(\Lambda)$.

Proposition 7.1.2 *We have the following two Λ-exact sequences (for a suitable positive integer n):*

$$0 \to H_1(E_\gamma) \to A(L) \to \varepsilon(\Lambda) \to 0. \tag{1}$$

$$0 \to H_2(E_\gamma) \to \Lambda^{n-1} \to \Lambda^n \to A(L) \to 0. \tag{2}$$

Proof. (1) follows from the homology exact sequence of the pair $(E_\gamma,\, p^{-1}(b))$. (2) Since E is a compact connected 3-manifold with boundary, there is a deformation retract from E to a connected compact 2-dimensional cell complex W with only one 0-cell b. Let n be the number of 2-cells of W. Then the number of 1-cells of W is equal to $n - 1$ since $\chi(W) = \chi(E) = 0$. Let $p : W_\gamma \to W$ be the universal abelian covering of W, and let $C_\sharp(W_\gamma)$ be the chain complex associated with W_γ. Then the desired result is obtained from the following exact sequence:

$$0 \to H_2(W_\gamma) \to C_2(W_\gamma) \xrightarrow{\partial} C_1(W_\gamma) \to H_1(W_\gamma, p^{-1}(b)) \to 0,$$

where ∂ denotes the boundary homomorphism. □

Exercise 7.1.3 (1) Show that the exterior E of the Hopf link is homeomorphic to $S^1 \times S^1 \times [0,1]$. Further, by using this result, show that
$H_i(E_\gamma) = 0$ $(i \geq 1)$ and $A_\gamma(L) \cong \varepsilon(\Lambda)$.
(2) Show that the exterior E of the 2-component trivial link is homotopy equivalent to a bouquet $S^1 \vee S^1 \vee S^2$. Further, using this result, show that
$H_1(E_\gamma) \cong H_2(E_\gamma) \cong \Lambda$ and $A(L) \cong \Lambda \oplus \Lambda$.

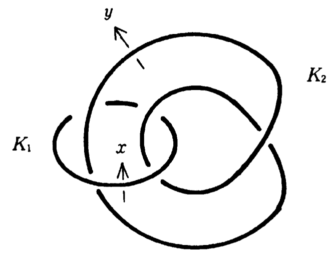


Fig. 7.1.1

We describe a method to calculate the Alexander module and the link module of a given link, for example, the link L shown in figure 7.1.1. The group π of this link L has the presentation: $\langle x, y \mid yxyxy^{-1}x^{-1}y^{-1}x^{-1}\rangle$. Let W be a 2-dimensional cell complex associated with this presentation, i.e., W has one 0-cell b, two 1-cells x^* and y^*, and one 2-cell r^*, where r^* is attached to the 1-skeleton according to the relation in the group presentation. Then $\pi_1(W, b)$ is identified with π. Let $p : W_\gamma \to W$ be the universal abelian covering. Then there are natural isomorphisms $H_1(W_\gamma) \cong H_1(E_\gamma)$ and $H_1(W_\gamma, p^{-1}(b)) \cong A(L)$. To calculate these modules, we describe the chain complex $C_\sharp(W_\gamma)$ explicitly. Choose a lift $\hat{b}$ of b to E_γ and let $\hat{x}^*$,$\hat{y}^*$, and $\hat{r}^*$ be the lifts of x^*, y^*, and r^*, respectively, with base point $\hat{b}$. Then the chain complex $C_\sharp(W_\gamma)$ is given as follows:

$$\begin{array}{ccccccc}
C_2(W_\gamma) & \xrightarrow{\partial_2} & C_1(W_\gamma) & \xrightarrow{\partial_1} & C_0(W_\gamma) & \xrightarrow{\partial_0} & 0, \\
\| & & \| & & \| & & \\
\Lambda[\hat{r}^*] & & \Lambda[\hat{x}^*, \hat{y}^*] & & \Lambda[\hat{e}] & &
\end{array}$$

where $\Lambda[\cdots]$ denotes the free Λ-module with basis $\cdots$ in the parenthesis [ ].

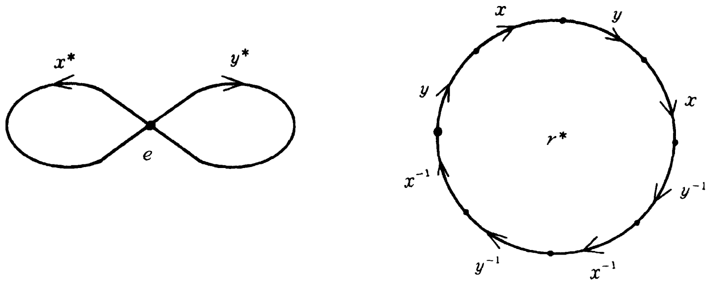


Fig. 7.1.2

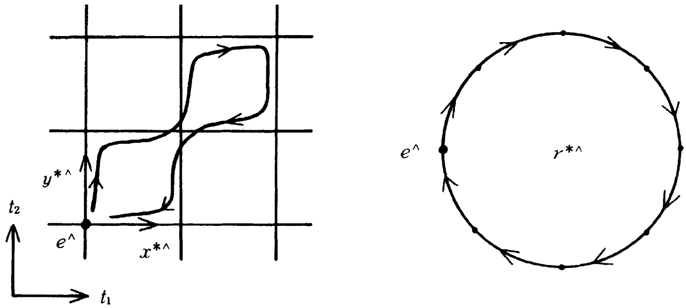


Fig. 7.1.3

We see that $\partial_1(\hat{x}^*) = (t_1 - 1)\hat{e}^*$ and $\partial_1(\hat{y}^*) = (t_2 - 1)\hat{e}^*$ and that $\partial_2(\hat{r}^*)$ is represented by the lift with base point $\hat{b}$ of a loop representing the word $yxyxy^{-1}x^{-1}y^{-1}x^{-1}$. We construct the lift successively from the initial letter of this word and obtain the following:

$$\begin{aligned}\partial_2(\hat{r}^*) &= \hat{y}^* + t_2\hat{x}^* + t_1t_2\hat{y}^* + t_1t_2^2\hat{x}^* - t_1^2t_2\hat{y}^* - t_1t_2\hat{x}^* - t_1y^* - \hat{x}^* \\ &= (-1 + t_2)(1 + t_1t_2)\hat{x}^* + (1 - t_1)(1 + t_1t_2)\hat{y}^*.\end{aligned}$$

The calculation is performed using the following facts:
(1) Let u_1 and u_2 be words of x and y (more precisely, loops in W with base point b representing the words), and $\hat{u}_1$ and $\hat{u}_2$ the lifts of u_1 and u_2 with base point $\hat{b}$, respectively. Then as a 1-chain, we have $\widehat{(u_1u_2)} = \hat{u}_1 + \gamma(u_1)\hat{u}_2$ (see figure 7.1.4), where γ denotes the Hurewicz epimorphism $\pi_1(W, b) \to H_1(W) = \langle t_1\rangle \times \langle t_2\rangle$ and we identify $H_1(W)$ with the covering transformation group of W_γ.
(2) Let $\widehat{(u^{-1})}$ be the lift of u^{-1} with base point $\hat{b}$. Then we have $\widehat{(u^{-1})} = -\gamma(u^{-1})\hat{u}$ (see figure 7.1.5).
Put $z = (1 - t_1)\hat{y}^* - (1 - t_2)\hat{x}^* \in C_1(W_\gamma)$. Then, from figure 7.1.3, we see that $\mathrm{Ker}(\partial_1)$ is the free Λ-module $\Lambda[z]$ with base z. On the other hand, we see $\partial_2(\hat{r}^*) =$

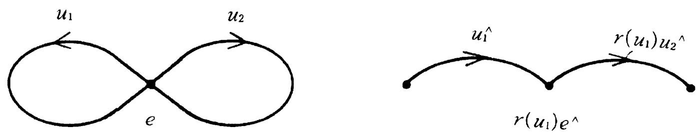


Fig. 7.1.4

$(1+t_1t_2)z$. Hence we obtain the following:

$$A(L) \cong C_1(W_\gamma)/\mathrm{Im}(\partial_2) \cong \Lambda[x^*, y^*]/((1+t_1t_2)z).$$
$$H_1(E_\gamma) \cong \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) \cong \Lambda/(1+t_1t_2).$$

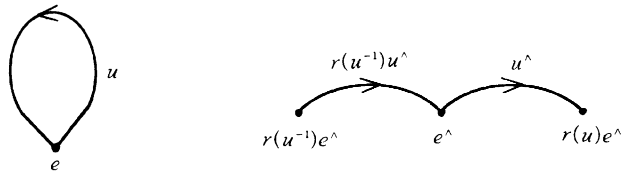


Fig. 7.1.5

The above calculation can be described explicitly using a method called *Fox's free differential calculus* which we shall now explain. Let F_n be the free group with basis $x_1, \ldots, x_n$. For each k $(1 \le k \le n)$, there is a unique map $\partial/\partial x_k : F_n \to \mathbf{Z}F_n$ determined by the following conditions (see [Crowell-Fox, 1963]):

(1) $\partial x_i/\partial x_k = \delta_{ik}$.
(2) $\partial(uv)/\partial x_k = \partial u/\partial x_k + u\partial v/\partial x_k$.

Exercise 7.1.4 Show the following identities:

$$\partial 1/\partial x_k = 0, \quad \partial u^{-1}/\partial x_k = -u^{-1}(\partial u/\partial x_k).$$

The $\mathbf{Z}$-homomorphism extension $\mathbf{Z}F_n \to \mathbf{Z}F_n$ of the map $\partial/\partial x_k$ is also denoted by the same symbol $\partial/\partial x_k$, and we call it the *free derivative* with respect to x_k. The calculation of the example stated above is generalized as follows:

Theorem 7.1.5 *Let $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a presentation of the group π of a link L with r components. Let γ be the homomorphism between the group rings induced from the composite homomorphism $\langle x_1, \ldots, x_n \rangle \to \pi \to \prod_{i=1}^r \langle t_i \rangle$. Then we obtain a chain complex*

$$\Lambda^m \xrightarrow{\partial_2} \Lambda^n \xrightarrow{\partial_1} \Lambda \to 0,$$

where ∂_2 is represented by the (m,n) matrix $(\gamma(\partial r_i/\partial x_k))_{1\le i\le m, 1\le k\le n}$ and ∂_1 is represented by the $(n,1)$ matrix $(\gamma(x_k)-1)_{1\le k\le n}$ such that the Alexander module and the link module of L are obtained as follows:

$$A(L) \cong \mathit{Coker}(\partial_2), \quad H_1(E_\gamma) \cong \mathit{Ker}(\partial_1)/\mathit{Im}(\partial_2).$$

We now know how to derive the Alexander module and the link module from a presentation of a link group, but for applications of these Λ-modules, we must know invariants derived from them, which will be described in the following section.

7.2 Invariants of a Λ-module

In this section, a general theory for deriving some invariants from Λ-modules is discussed. First, we note that the Laurent polynomial ring Λ has the following properties (cf. [Lang 1965]):

(1) Λ is a unique factorization domain.
(2) Λ is Noetherian, i.e., every submodule of a finitely generated Λ-module is finitely generated.

Let H be a finitely generated Λ-module. Then there is an epimorphism from a finitely generated free Λ-module Λ^n to H whose kernel is finitely generated by property (2). So, we obtain an exact sequence $\Lambda^m \to \Lambda^n \to H \to 0$. Let P be the (m, n) matrix representing the homomorphism $\Lambda^m \to \Lambda^n$. We call P a *presentation matrix* of H. Though presentation matrices of H are not unique, we have the following result (see [Zassenhaus 1958(pp. 117–120)]).

Lemma 7.2.1 *Two presentation matrices of H are related by a finite sequence of the following operations.*

(1) *Interchange two rows or two columns.*
(2) *Multiply a row or column by a unit of Λ.*
(3) *Add to any row a Λ-linear combination of other rows or to any column a Λ-linear combination of other columns.*
(4) $P \leftrightarrow \begin{pmatrix} P \\ * \end{pmatrix}$, *where $*$ is a Λ-linear combination of rows of P.*
(5) $P \leftrightarrow \begin{pmatrix} P & 0 \\ * & 1 \end{pmatrix}$, *where $*$ is an arbitrary row.*

By (4) and (5), we can note that for any positive integer d, H has a presentation matrix of size (m, n) with $n > d$ and $m \geq n - d$. Let $\mathbf{Q}(\Lambda)$ be the quotient field of Λ. We define two kinds of invariants of a Λ-module H as follows:

Definition 7.2.2 The *rank* of H, denoted by $\mathrm{rank}_\Lambda H$ or simply $\mathrm{rank} H$ is the dimension of the $\mathbf{Q}(\Lambda)$-vector space $H \otimes \mathbf{Q}(\Lambda)$, where $\otimes$ denotes the tensor product over Λ.

Definition 7.2.3 For each non-negative integer d, the *d-th elementary ideal* of H, denoted by $E_d(H)$, is the ideal of Λ generated by the $(n - d)$-minors of a presentation matrix P of H of size (m, n) with $n > d$ and $m \geq n - d$. The *d-th characteristic polynomial* of H, denoted by $\Delta_d(H)$, is the greatest common divisor of the elements of $E_d(H)$.

Exercise 7.2.4 Show that $E_d(H)$ and $\Delta_d(H)$ (up to multiplication by a unit of Λ) does not depend on the choice of a presentation matrix P of H of size (m,n) with $n > d$ and $m \geq n-d$ by using Lemma 7.2.1.

From Lemma7.2.1(4),(5), we see that $E_d(H) = \Lambda$ if H has a presentation matrix P of size (m,n) with $d \geq n$ and $E_d(H) = \{0\}$ if H has a presentation matrix P of size (m,n) with $n-d > m$.

Example 7.2.5. Let $H = \Lambda^s \oplus \Lambda/(\lambda_1) \oplus \cdots \oplus \Lambda/(\lambda_n)$, where $\lambda_i (i = 1, 2, \ldots, n)$ are non-zero elements of Λ such that $\lambda_{i+1}|\lambda_i$ for each i. Then we have rank$H = s$, and $\Delta_d(H)$ is 0, $\lambda_{d-s+1} \cdots \lambda_n$, or 1, according to whether $0 \leq d \leq s-1$, $s \leq d \leq s+n-1$, or $s+n \leq d$.

Definition 7.2.6 The *torsion submodule* of H is the Λ-submodule $TH = \{x \in H \mid \lambda x = 0 \text{ for some non-zero } \lambda \in \Lambda\}$ and H is *torsion free* if $TH = 0$.

Lemma 7.2.7 *For a Λ-exact sequence*

$$0 \to H_1 \to H \to H_2 \to 0$$

of finitely generated Λ-modules, we have the following:

(1) *rank$H$ = rankH_1 + rankH_2.*

(2) $\Delta_0(H) \doteq \Delta_0(H_1)\Delta_0(H_2)$, *where $\doteq$ denotes the equality up to multiplication by a unit of Λ.*

(3) *If $TH_2 = 0$ and rank$H_2 = r$, then $\Delta_d(H) \doteq \Delta_{d-r}(H_1)$ or 0 according to whether $r \leq d$ or $0 \leq d \leq r-1$.*

Proof. (1) follows from the fact that $\mathbf{Q}(\Lambda)$ is flat over Λ: i.e., the operation $\otimes\mathbf{Q}(\Lambda)$ preserves exactness of a sequence. (2) First, we consider the case where H_1 and H_2 have square presentation matrices P_1 and P_2 respectively. Then H has a presentation matrix of the form $\begin{pmatrix} P_1 & 0 \\ * & P_2 \end{pmatrix}$, and we can easily prove the desired result. The assertion in the general case can be proved by using the concept of localization (see [Lang 1965]). Let p be a prime element of Λ, and suppose $\Delta_0(H) \doteq p^a q$, $\Delta_0(H_1) \doteq p^b q'$, and $\Delta_0(H) \doteq p^c q''$. Here a, b, and c are non-negative integers, and q , q' , and q'' are elements of Λ relatively prime to p. Let $\Lambda_{(p)}$ be the ring obtained from Λ by localizing at the prime ideal (p), i.e., $\Lambda_{(p)} = \{\lambda_1/\lambda_2 \in \mathbf{Q}(\Lambda) \mid p \nmid \lambda_2\}$. This ring is a principal ideal domain where $p\Lambda_{(p)}$ is the only prime ideal. Hence, the $\Lambda_{(p)}$-modules $H \otimes \Lambda_{(p)}$, $H_1 \otimes \Lambda_{(p)}$, and $H_2 \otimes \Lambda_{(p)}$ have square presentation matrices, and their 0-th characteristic polynomials are p^a, p^b, and p^c respectively. Since $\Lambda_{(p)}$ is flat over Λ, these three $\Lambda_{(p)}$-modules are connected by a short exact sequence. Hence, by the previous argument, we see $p^a \doteq p^b p^c$ and therefore $a = b + c$. By applying this argument to each prime element of Λ, we obtain (2). (3) can also be proved using the idea of localization. [*Hint*: $H \otimes \Lambda_{(p)} \cong (H_1 \otimes \Lambda_{(p)}) \oplus \Lambda^r_{(p)}$.] □

The same results also hold for modules over a ring R which has the two conditions stated in the beginning of this section. The following lemma is useful for the homological study of a covering space over S^3 with a link as the branch set and for the study of the Alexander polynomials of symmetric knots (see [Hillman 1981] or [Sakuma 1979]):

Lemma 7.2.8 *Let R be a Noetherian unique factorization domain, and let $R\langle t\rangle$ be the group ring over R of the infinite cyclic group $\langle t\rangle$. Let H be a finitely generated $R\langle t\rangle$-module, and let $\Delta(t)$ be the 0-th characteristic polynomial of H. Let $f(t) = a_0 + a_1 t + \cdots + a_n t^n$ ($a_i \in R$) be an element of $R\langle t\rangle$ such that a_0 and a_n are units of R. Then the 0-th characteristic polynomial of the R-module $H/f(t)H$ is given by $\Pi_{i=1}^n \Delta(\omega_i)$. Here, $\{w_i \mid 1 \le i \le n\}$ are the roots of $f(t)$ (in the splitting field of $f(t)$).*

7.3 Graded Alexander polynomials

In this section, we define the graded Alexander polynomials of a link and state some properties of them.

Definition 7.3.1 (1) For each non-negative integer d, the *d-th Alexander polynomial* of a link L, denoted by $\Delta_L^{(d)} = \Delta_L^{(d)}(t_1, \ldots, t_r)$, is the $(d+1)$-th characteristic polynomial $\Delta_{d+1}(A(L))$ of the Alexander module $A(L)$.
(2) The *Alexander nullity* of L, denoted by $\beta(L)$, is $\operatorname{rank} A(L) - 1$.

In particular, the 0-th Alexander polynomial $\Delta_L^{(0)}$ is called the *Alexander polynomial* of L and denoted by Δ_L. We describe a method of calculating these invariants. Let $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_{n-1}\rangle$ be a Wirtinger presentation of the group π of an r-component link L obtained from a connected link diagram of L. Note that the deficiency of this presentation is 1 and that the image $\gamma(x_k)$ of x_k by the Hurewicz epimorphism γ is equal to some t_h ($1 \le h \le r$). Let P be the presentation matrix of $A(L)$ obtained from this group presentation by the method given by Theorem 7.1.5. Let $X_1, \ldots, X_n$ be the column vectors of P, i.e., $P = (X_1, \ldots, X_n)$ with $X_k = (\gamma(\partial r_i/\partial x_k))_{1\le i\le n-1}$. Let P_k be the square matrix obtained from P by deleting X_k.

Lemma 7.3.2 *If $r = 1$, then* $\det P_k = \Delta_L^{(0)}$. *If $r \ge 2$, then*

$$\det P_k = (\gamma(x_k) - 1)\Delta_L^{(0)}.$$

Proof. By the relation $\partial_1\partial_2 = 0$, we obtain $\sum_{k=1}^n (\gamma(x_k) - 1) X_k = 0$. Hence for each pair i and k with $i \ne k$, we see

$$\begin{aligned}(\gamma(x_i) - 1)\det P_k &= \det(X_1, \ldots, (\gamma(x_i) - 1)X_i, \ldots, \hat{X}_k, \ldots, X_n) \\ &= \det(X_1, \ldots, -\Sigma_{j\ne i}(\gamma(x_j) - 1)X_j, \ldots, \hat{X}_k, \ldots, X_n) \\ &= \pm(\gamma(x_k) - 1)\det P_i\end{aligned}$$

If $r = 1$, we obtain the desired result immediately. If $r \ge 2$, we obtain the desired result by using the fact that g.c.d.$\{t_1 - 1, \ldots, t_r - 1\} = 1$. □

Corollary 7.3.3 *$E_1(A(L))$ is equal to $(\Delta_L^{(0)})$ or $\varepsilon(\Lambda)(\Delta_L^{(0)})$ according to whether $r=1$ or $r \geq 2$.*

The invariants of L given in Definition 7.3.1 are obtained from the link module $H_1(E_\gamma)$ as follows :

Proposition 7.3.4

(1) $\beta(L) = \mathrm{rank}H_1(E_\gamma) = \mathrm{rank}H_2(E_\gamma)$.
(2) $\Delta_L^{(d)} = \Delta_d(H_1(E_\gamma))$.

Proof. These follow from Proposition 7.1.2 and Lemma 7.2.7. □

Exercise 7.3.5 Show that the following three conditions are equivalent:

(1) $\beta_1(L) = 0$.
(2) $\Delta_L^{(0)} \neq 0$.
(3) $H_2(E_\gamma) = 0$.

Although the interpretation of the Alexander polynomial in terms of the link module $H_1(E_\gamma)$ given by the above proposition is theoretically important, the presentation matrix of $H_1(E_\gamma)$ is more complicated than that of $A(L)$ in case the number of components r of L is large. This is caused by the following fact:

Proposition 7.3.6 *The link module of an r-component trivial link O^r is given as follows:*

(1) *If $r=1$, $H_1(E_\gamma) = 0$.*
(2) *If $r=2$, $H_1(E_\gamma) \cong \Lambda$.*
(3) *If $r \geq 3$, $H_1(E_\gamma)$ has a presentation with $\binom{r}{2}$ generators $\hat{e}_{ik}$ $(1 \leq i < k \leq r)$ and $\binom{r}{3}$ relations $(t_i - 1)\hat{e}_{hk} - (t_h - 1)\hat{e}_{ik} + (t_k - 1)\hat{e}_{ih} = 0$ $(1 \leq i < h < k \leq r)$.*

Proof. The link group of O^r has a group presentation $\langle x_1, \ldots, x_r \mid - \rangle$, and the cell complex W associated with this presentation is a bouquet of r circles. We construct an r-dimensional cell complex X whose 1-skeleton is equal to W as follows. First, consider a cell decomposition of S^1 with one 0-cell e^0 and one 1-cell e^1. Consider r circles $S_i^1 = e_i^0 \cup e_i^1$ of this cell complex, and let X be the cell complex which is obtained as the product $S_1^1 \times \cdots \times S_r^1$. Then the number of m-cells of X is $\binom{r}{m}$. Let X_γ be the universal abelian covering of X, and let $C_\sharp(X_\gamma)$ be its chain complex. Since $\pi_1(X)$ is naturally identified with $\mathbf{Z}^r \cong \prod_{i=1}^r \langle t_i \rangle$, $C_\sharp(X_\gamma)$ has a structure of a Λ-chain complex. Since W is the 1-skeleton of X, $C_\sharp(W_\gamma)$ is a sub-chain complex of $C_\sharp(X_\gamma)$. To be precise, $C_i(W_\gamma) = C_i(X_\gamma)$ or 0 according to whether $0 \leq i \leq 1$ or $i \geq 2$. On the other hand, we have $H_i(X_\gamma) = 0$ $(i \geq 1)$ since $X_\gamma \cong \mathbf{R}^r$. Hence we see that

$$\mathrm{Ker}(\partial_1) = \mathrm{Im}(\partial_2) \cong C_2(X_\gamma)/\mathrm{Ker}(\partial_2) = C_2(X_\gamma)/\mathrm{Im}(\partial_3).$$

$C_2(X_\gamma)$ is a free Λ-module with basis $\hat{e}_{ih}$ $(1 \leq i < h \leq r)$ where $\hat{e}_{ih}$ is a lifting 2-cell of $e_{ih} = e_1^0 \times \cdots \times e_i^1 \times \cdots \times e_h^1 \times \cdots \times e_r^0$. $C_3(X_\gamma)$ is generated by $\hat{e}_{ihk}$ $(1 \leq i < h <$

$k \leq r)$ where $\hat{e}_{ihk}$ is a lifting 3-cell of $\hat{e}_{ihk} = e_1^0 \times \cdots \times e_i^1 \times \cdots \times e_h^1 \times \cdots \times e_k^1 \times \cdots \times e_r^0$. As shown in figure 7.3.1, we have

$$\partial_3(\hat{e}_{ihk}) = (t_i - 1)\hat{e}_{hk} - (t_h - 1)\hat{e}_{ik} + (t_k - 1)\hat{e}_{ih}.$$

Hence, we obtain the proposition. □

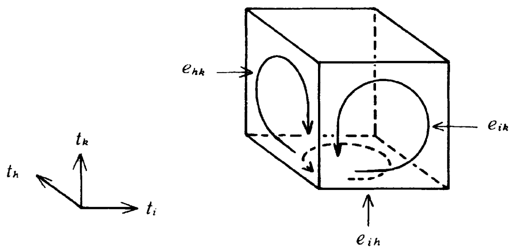


Fig. 7.3.1

Problem 7.3.7 For an r-component trivial link, show the following:
(1) $\mathrm{rank} H_1(E_\gamma) = r - 1$.
(2) $H_2(E_\gamma) \cong \Lambda^{r-1}$.
(3) If $r \geq 3$, then $H_1(E_\gamma) \not\cong \Lambda^{r-1}$. [*Hint*: Consider the elementary ideals.]

Exercise 7.3.8
(1) Show that the link group of an r-component link L has a presentation $\pi = \langle x_1, \ldots, x_r, a_1, \ldots, a_{n-r} \mid r_1, \ldots, r_{n-1} \rangle$ such that $\gamma(x_i) = t_i$ $(1 \leq i \leq r)$ and $\gamma(a_h) = 1$ $(1 \leq h \leq n - r)$.
(2) Show that $H_1(E_\gamma)$ has a presentation with $n - r + \binom{r}{2}$ generators and $n - 1 + \binom{r}{3}$ relations, by using the 2-dimensional cell complex W associated with the group presentation given in (1).

Using Exercise 7.3.8, we can have the following proposition (see [Crowell-Strauss 1969] for the proof):

Proposition 7.3.9
(1) *If $r \leq 3$, then $H_1(E_\gamma)$ has a square presentation matrix. In particular, $E_0(H_1(E_\gamma)) = (\Delta_L)$.*
(2) *If $r \geq 4$, then $E_0(H_1(E_\gamma)) = \varepsilon(\Lambda)^s(\Delta_L)$, where $s = \binom{r-1}{2}$. In particular, $H_1(E_\gamma)$ cannot have a square presentation matrix if $\Delta_L \neq 0$.*

At the end of this section, we observe a homological relationship of the universal abelian covering space of E to the other free abelian covering spaces over E. Let ν be an epimorphism from $H_1(E)$ to a free abelian group J, and let E_ν be the covering space over E corresponding to the kernel of the composite homomorphism $\nu\gamma : \pi_1(E) \to H_1(E) \to J$. Then $H_*(E_\nu)$ has a structure of a $\mathbf{Z}J$-module.

Proposition 7.3.10

(1) *If* $r \geq 2$ *and* $J = \langle a \rangle \cong \mathbf{Z}$, *then* $\Delta_0(H_1(E_\nu)) \doteq (a-1)\Delta_L(\nu(t_1), \ldots, \nu(t_r))$.
(2) *If* $\mathrm{rank} J \geq 2$, *then* $\Delta_0(H_1(E_\nu)) \doteq \Delta_L(\nu(t_1), \ldots, \nu(t_r))$.

Exercise 7.3.11 Prove the above proposition by the method indicated in the following:

(1) If we replace γ with $\nu\gamma$ in Theorem 7.1.5, we obtain a method to calculate $H_1(E_\nu)$.
(2) For $H_1(E_\nu)$, results analogous to those in Proposition 7.3.4 hold.
(3) Apply Lemma 7.3.2 by noting that the g.c.d. of the $\nu\gamma(x_k) - 1$'s ($k = 1, 2, \ldots, n$) is equal to $a - 1$ or 1 according to whether $\mathrm{rank} J$ is 1 or greater than 1.

By comparing $C_\sharp(E_\gamma)$ with $C_\sharp(E_\nu)$, we obtain the following (see [Kawauchi 1978]):

Proposition 7.3.12

$$\mathrm{rank}_{\mathbf{Z}H} H_2(E_\nu) \geq \mathrm{rank}_\Lambda H_2(E_\gamma)$$
$$\mathrm{rank}_{\mathbf{Z}H} H_1(E_\nu) \geq \mathrm{rank}_\Lambda H_1(E_\gamma).$$

In the above proposition, the inequalities hold even when $J = \{1\}$. Hence by combining the first inequality with 7.1.2, we have the following corollary:

Corollary 7.3.13 $0 \leq \beta(L) \leq r - 1$.

We consider the special case that $J = \langle t \rangle$ and ν is given by $\nu(t_i) = t$ for all i. Then E_ν is constructed using a Seifert surface (cf. 5.4) and has a special meaning. In the following proposition, which is proved in [Sakuma 1979], we denote this E_ν by E_∞ and the covering projection $E_\gamma \to E_\infty$ by q:

Proposition 7.3.14 *For* $r \geq 2$, *we have the* $\mathbf{Z}\langle t \rangle$*-exact sequence*

$$0 \to q_* H_1(E_\gamma) \to H_1(E_\infty) \to (\mathbf{Z}\langle t \rangle/(t-1))^{r-1} \to 0.$$

In particular, we have

$$\Delta_0(q_* H_1(E_\gamma)) \doteq \Delta_0(H_1(E_\infty))/(t-1)^{r-1} \doteq \Delta_L(t, \ldots, t)/(t-1)^{r-2}.$$

Definition 7.3.15 The *Hosokawa polynomial* of a link L, denoted by $\hat{\Delta}_L(t)$ is the polynomial $\Delta_L(t, \ldots, t)/(t-1)^{r-2}$. (cf. Exercise 5.4.2.)

The following theorem was proved by [Hosokawa 1958]:

Theorem 7.3.16

(1) $\hat{\Delta}(t)$ *satisfies the following three conditions: (1-1)* $\hat{\Delta}_L(t) \doteq \hat{\Delta}_L(t^{-1})$ *and the multiplicity of the factor* $(t-1)$ *in* $\Delta_L(t)$ *is even (possibly zero), (1-2) the degree (as a Laurent polynomial)* $\deg \hat{\Delta}_L(t)$ *is even. (1-3) when we*

set $\lambda_{ih} = \mathrm{Link}(K_i, K_h)$ *or* $-\sum_{1\le k(\neq i)\le r}\mathrm{Link}(K_i, K_k)$ *according to whether* $i \neq h$ *or* $i = h$, *any* $(r-1)$ *minor of the square matrix* (λ_{ih}) *of size* r *is equal to* $\pm\hat{\Delta}_L(1)$.

(2) *For any* $r \ge 2$ *and for any integral Laurent polynomial* $\hat{\Delta}(t)$ *which satisfies the conditions (1-1) and (1-2), there is an* r*-component link whose Hosokawa polynomial is equal to* $\hat{\Delta}(t)$.

7.4 Torres conditions

Here we describe some well-known conditions, called the *Torres conditions*, which the (0-th) Alexander polynomial satisfies.

Theorem 7.4.1 (Torres conditions) *The Alexander polynomial* $\Delta_L(t_1, \ldots, t_r)$ *of an* r*-component link* $L = K_1 \cup \cdots \cup K_r$ *has the following properties:*

(1) $\Delta_L(t_1,\ldots,t_r) \doteq \Delta_L(t_1^{-1},\ldots,t_r^{-1})$.

(2)

$$\Delta_L(t_1,\ldots,t_{r-1},1) \doteq \begin{cases} \{(t_1^{\lambda_1}-1)/(t_1-1)\}\Delta_{L'}(t_1) & if r = 2 \\ (t_1^{\lambda_1}\ldots t_{r-1}^{\lambda_{r-1}}-1)\Delta_{L'}(t_1,\ldots,t_{r-1}) & if r \ge 3, \end{cases}$$

where $L' = K_1 \cup \cdots \cup K_{r-1}$ *and* $\lambda_i = \mathrm{Link}(K_i, K_r)$.

Proof. For (1), we show the following strengthened result:

$$\Delta_L^{(\beta)}(t_1,\ldots,t_r) \doteq \Delta_L^{(\beta)}(t_1^{-1},\ldots,t_r^{-1})$$

for $\beta = \beta(L)$ by the Blanchfield duality of Appendix E. (cf. Lemma 7.2.7 and Proposition 7.3.4(2).) In fact, by Theorem E.2 applied to the localization at each prime factor of $\Delta_L^{(\beta)}(t_1,\ldots,t_r)$, we see that the β-th characteristic polynomial Δ' of the Λ-module $H_1(E_\gamma, \partial E_\gamma)$ is equal to $\Delta_L^{(\beta)}(t_1^{-1},\ldots,t_r^{-1})$ up to multiplication by a unit of Λ. On the other hand, we see that the following sequence induced from the homology exact sequence of the pair $(E_\gamma, \partial E_\gamma)$

$$TH_1(\partial E_\gamma) \to TH_1(E_\gamma) \to TH_1(E_\gamma, \partial E_\gamma) \to TH_0(\partial E_\gamma)$$

is exact, because $TH_j(\partial E_\gamma) = H_j(\partial E_\gamma)$ for all j. Since there is a Λ-epimorphism $\oplus_{i=1}^r \Lambda/(t_i - 1) \to TH_j(\partial E_\gamma)$, we see from Lemma 7.2.7(2) that $\Delta_L^{(\beta)}(t_1,\ldots,t_r)$ and $\Delta_L^{(\beta)}(t_1^{-1},\ldots,t_r^{-1})$ are equal up to multiplication by a unit of Λ and a factor of $\prod_{i=1}^r(t_i-1)$. Then the desired result follows. For (2), let E' be the exterior of L'. Then the exterior E of L is obtained as $E = E' - \mathrm{int}N(K_r)$. Let ν be the natural composite epimorphism $\pi_1(E) \to \pi_1(E') \to H_1(E') \cong \langle t_1,\ldots,t_{r-1} \mid [t_i, t_h] = 1\rangle$, and let E_ν be the covering space over E corresponding to the kernel of ν. Let $p : E'_\gamma \to E'$ be the universal abelian covering of E'. Then E_ν is identified with $p^{-1}(E)$. By the excision isomorphism, we have $H_*(E'_\gamma, E_\nu) \cong$

$H_*(p^{-1}(N(K_r)), p^{-1}(\partial N(K_r)))$, and hence $H_1(E'_\gamma, E_\nu) = 0$. Let Λ' be the Laurent polynomial ring on $(r-1)$ variables $t_1, \ldots, t_{r-1}$. Then the above isomorphism is a Λ'-isomorphism, and we obtain $H_2(E'_\gamma, E_\nu) \cong \Lambda'/(t_1^{\lambda_1} \ldots t_{r-1}^{\lambda_{r-1}} - 1)$. By the homology exact sequence of the pair (E'_γ, E_ν), we have the exact sequence

$$H_2(E'_\gamma) \to H_2(E'_\gamma, E_\nu) \to H_1(E_\nu) \to H_1(E'_\gamma) \to H_1(E'_\gamma, E_\nu) = 0.$$

Noting that $H_2(E'_\gamma) = 0$ if $\Delta(t_1, \ldots, t_{r-1}) \neq 0$ (Exercise 7.3.5), we obtain the desired result from Proposition 7.3.10 and Lemma 7.2.7 (2). □

Theorem 7.4.1(1) can also be proved by using either dual presentations of a link group (cf. [Torres-Fox 1954]) or the duality of Reidemeister torsion (cf. [Milnor 1962]). Further, in Theorem 7.4.1, by using (2), (1) can be refined as follows:

Theorem 7.4.2 $\Delta_L(t_1, \ldots, t_r) = (-1)^r t_1^{b_1} \ldots t_r^{b_r} \Delta_L(t_1^{-1}, \ldots, t_r^{-1})$, *where* $b_i \equiv 1 - \sum_{1 \leq h (\neq i) \leq r} \mathrm{Link}(K_i, K_h) \pmod 2$.

Exercise 7.4.3 Prove the theorem above using the following method:

(1) In case there is a component K_i such that $\mathrm{Link}(K_i, K_k) \neq 0$ for every $k \neq i$, use induction on the number r of the components of L.

(2) Otherwise, adjoin a new component K_0 such that $\mathrm{Link}(K_0, K_i) \neq 0$ $(1 \leq i \leq r)$. Then the Alexander polynomial of $K_0 \cup L$ satisfies the desired result by (1). Applying Theorem 7.4.1(2), the theorem is proved.

Exercise 7.4.4 Show that the Alexander polynomial $\Delta_{T(p,q)}$ of the (p,q)-torus knot $T(p,q)$ is given by

$$\Delta_{T(p,q)}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}$$

by using the knot group presentation given in Proposition 6.1.16. Also, show that the Hosokawa polynomial $\hat{\Delta}_{T(np,nq)}(t)$ of the n-component torus link $T(np,\ nq)$ is given by

$$\hat{\Delta}_{T(np,nq)}(t) = \frac{\rho_{npq}(t)^n}{\rho_{np}(t)\rho_{nq}(t)},$$

where we let $\rho_m(t) = (1-t^m)/(1-t)$. [*Hint:* By noting that $T(np,nq)$ is a satellite of $T(p,q)$, consider a suitable Mayer-Vietoris exact sequence for $H_*(E(T(np,nq)_\infty)$.]

Supplementary notes for Chapter 7

In [Rolfsen 1976], we can find a nice explanation of link modules using the surgery description of links. [Bailey 1977] and [Nakanishi 1980] gave a characterization of presentation matrices of link modules of 2-component links using this method. From this characterization, it is shown in [Hillman 1981'] that the Torres conditions are insufficient to characterize the Alexander polynomial, i.e., there is a 2-variable polynomial which satisfies the Torres conditions but is not the Alexander polynomial of a link. See [Hillman 1981] for a detailed study of Alexander ideals. For further relations between Alexander invariants and the homology of finite abelian coverings, see [Mayberry-Murasugi 1982] and [Sakuma 1995].

Chapter 8
Jones type polynomials I: a topological approach

In this chapter, we discuss the following polynomial invariants of a link: the Conway polynomial, the Jones polynomial, the skein polynomial, the Q polynomial and the Kauffman polynomial.

8.1 The Jones polynomial

For a link diagram D, let $|D|$ denote the unoriented diagram. A *state* for $|D|$ is a diagram obtained by replacing each crossing ╳ of $|D|$ with)(or $\asymp$; the result is a union of mutually disjoint simple loops. One state of the diagram shown in figure 8.1.1a is illustrated in figure 8.1.1b.

a: a diagram　　　　b: a state

Fig. 8.1.1

The number of states of a diagram with n crossings is 2^n. For example, all the states of the link diagram in figure 8.1.1a are illustrated in figure 8.1.2. Given a state S for $|D|$, we assign an independent variable A or B, which we call a *weight* for S, to each crossing ╳ of $|D|$, according to whether the crossing changes into)(or $\asymp$ in S. In other words, assigning A and B to a neogborhood of a crossing of $|D|$ as shown in figure 8.1.3, we choose A or B of the regions which connect in S as the weight of the vertex for S. Let $\langle |D|/S\rangle$ denote the product of all weights for a state S. For example, for the diagram $|D|$ of figure 8.1.1a and the first state S of figure 8.1.2, we have $\langle |D|/S\rangle = BBAB = AB^3$. Let $|S|$ be the number of components of S. We define the *bracket polynomial* of $|D|$ by the following identity:

$$\langle |D|\rangle = \sum_S \langle |D|/S\rangle \delta^{|S|},$$

where the summation is taken over all states for $|D|$ and δ is a variable independent of A and B.

Exercise 8.1.1 Compute the bracket polynomial $\langle |D|\rangle$ for the diagram $|D|$ of figure 8.1.1a. Find a diagram which presents the same knot as $|D|$ but does not have the same bracket polynomial.

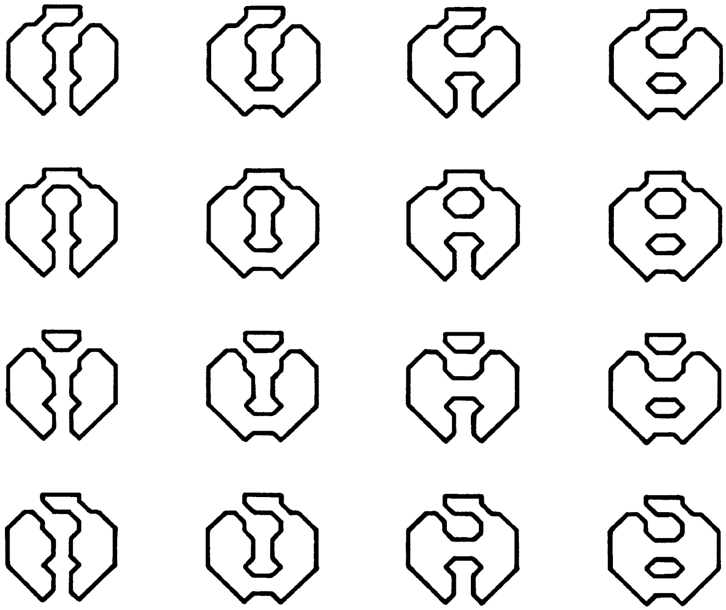


Fig. 8.1.2

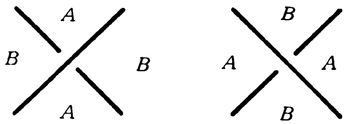


Fig. 8.1.3

By this exercise, the bracket polynomial is not a link invariant, but we can modify it to become a link invariant, as we shall explain. Let O^n denote an n-component trivial link diagram without crossings and let $O^1 = O$.

Proposition 8.1.2 *The bracket polynomial has the following properties:*

(0) $\langle |O^n| \rangle = \delta^n$.

(1) $\langle \times \rangle = A\langle)(\rangle + B\langle \asymp \rangle$.

(2) $\langle \times \rangle = B\langle)(\rangle + A\langle \asymp \rangle$.

Here, $\times$, $)($ *and* $\asymp$ *denote diagrams which are identical except inside the depicted regions.*

We note that (2) differs from (1) by a $90°$ rotation.

Remark 8.1.3 We can employ Proposition 8.1.2 as the definition of the bracket polynomial.

Exercise 8.1.4 Prove Proposition 8.1.2 and confirm Remark 8.1.3.

Let $D \circ D'$ be a link diagram obtained as a split union of two link diagrams D and D' so as not to cause extra crossings.

Exercise 8.1.5 Prove $\langle |D \circ D'| \rangle = \langle |D| \rangle \langle |D'| \rangle$.

To construct a link invariant from the bracket polynomial, we must arrange that it is invariant under the three types of Reidemeister moves. For this purpose, we investigate how the bracket polynomial behaves under the Reidemeister moves.

Lemma 8.1.6

$$\langle \; \rangle = (A + B\delta)\langle \; \rangle.$$
$$\langle \; \rangle = (B + A\delta)\langle \; \rangle.$$

Proof. By Proposition 8.1.2, we have $\langle \; \rangle = A\langle \; \rangle + B\langle \; \rangle$. By Exercise 8.1.5, we have $\langle \; \rangle = \langle |O| \rangle \langle \; \rangle = \delta\langle \; \rangle$. Thus we have the first identity. The second one follows similarly. □

In particular, this lemma means that if the bracket polynomial is to be invariant under the type I move, then we must have $A = B$ or $\delta = 1$ in which case it becomes a trivial invariant.

Exercise 8.1.7 Characterize the bracket polynomial when $A = B$ or $\delta = 1$. Explain the reason why we are not interested in the bracket polynomial in this case.

Setting aside the type I move for a while, we consider only the moves of types II and III.

Definition 8.1.8 Two (unoriented) diagrams of links, $|D|$ and $|D'|$, are *regularly isotopic* if they differ by a finite sequence of the Reidemeister moves of types II, III and IV (and ambient isotopies of $\mathbf{R}^2$). (Cf. 1.1.)

The notion of regular isotopy naturally occurs when we deform a knot formed from a rubber band with the twist of the band taken into consideration. First of all, we modify the bracket polynomial to be a regular isotopy invariant.

Lemma 8.1.9 $\langle \; \rangle = AB\langle \; \rangle + (A^2 + AB\delta + B^2)\langle \; \rangle$.

Proof. By Lemma 8.1.6, we have

$$\begin{aligned} \langle \; \rangle &= A\langle \; \rangle + B\langle \; \rangle \\ &= A\{B\langle \; \rangle + A\langle \; \rangle\} + B(B + A\delta)\langle \; \rangle. \end{aligned}$$

Hence the desired identity is obtained. □

Thus, when we take $AB = 1$ and $A^2 + AB\delta + B^2 = 0$, the bracket polynomial is invariant under the move of type II. Then we can see from similar calculations that these relations also make the bracket polynomial invariant under the moves of types III and IV.

Exercise 8.1.10 Verify this assertion.

Thus, by taking $B = A^{-1}, \delta = -(A^2 + A^{-2})$, the bracket polynomial becomes a regular isotopy invariant. From now on, we assume that A, B and δ are chosen in this way. The observation above is summarized as follows:

Theorem 8.1.11 *Let $|D|$ be an unoriented link diagram. Let $\langle |D| \rangle$ be the Laurent polynomial (in A) defined by the rules:*

(0) $\langle |O^n| \rangle = \{-(A^2 + A^{-2})\}^n$.
(1) $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$.
(2) $\langle \times \rangle = A^{-1}\langle)(\rangle + A\langle \asymp \rangle$.

Then $\langle \quad \rangle$ is a regular isotopy invariant.

We return to the move of type I. If a knot is formed from a rubber band as stated after Definition 8.1.8, then the Reidemeister move of type I corresponds to adding a full twist to the band. This fact that the move of type I changes the twist shows a crucial difference between the move of type I and the other moves. Recall the writhe $w(D)$ of an oriented diagram D, defined in 1.1. For any orientation, we have $w(\gamma) = w(\smile) - 1$ and $w(\gamma) = w(\smile) + 1$. Using this observation, we have the following theorem:

Theorem 8.1.12 *The Laurent polynomial $V(D; A)$ (in A) defined by the identity*

$$V(D; A) = (-A^3)^{-w(D)}\langle |D| \rangle / \{-(A^2 + A^{-2})\}$$

for an (oriented) diagram D of a link L is an invariant of the link type of L.

Remark 8.1.13 The reason we divide the bracket polynomial by $\{-(A^2 + A^{-2})\}$ is not serious. It is only done so that we have the identity $V(O; A) = 1$.

Proof. The invariance under the moves of types II and III follows from Theorem 8.1.11 and the fact that the writhe does not change under them. We have

$$\begin{aligned} V(\gamma; A) &= (-A^3)^{-w(\gamma)}\langle \gamma \rangle / \{-(A^2 + A^{-2})\} \\ &= (-A^3)^{-w(\smile)+1}(-A^{-3})\langle \smile \rangle / \{-(A^2 + A^{-2})\} \\ &= (-A^3)^{-w(\smile)}\langle \smile \rangle / \{-(A^2 + A^{-2})\} \\ &= V(\smile; A). \end{aligned}$$

Similarly, we have $V(\gamma; A) = V(\smile; A)$. Thus, it is also an invariant of the type I move. □

We denote $V(D; A)$ by $V(L; A)$ and call it the *Jones polynomial* of L.

Exercise 8.1.14 Compute the Jones polynomial of a suitable link and, in particular, for a knot with trivial Alexander polynomial.

As seen in this exercise, it turns out that the Jones polynomial is a rather strong invariant. The Jones polynomial was first introduced in [Jones 1985] and the definition given here is due to [Kauffman 1987']. In 8.4, we shall observe to what extent the Jones polynomial is strong (that is, how many links have the same Jones polynomial). Here, we should emphasize that we have obtained a link invariant by a very simple method.

8.2 The skein polynomial

We first state a characterization of the Jones polynomial defined in 8.1.

Theorem 8.2.1 *For link diagrams, we have the following identities:*

$(\mathrm{J_{I_A}})$ $V(O; A) = 1$.

$(\mathrm{J_{II_A}})$ $A^4V(\times_+; A) - A^{-4}V(\times_-; A) = (A^2 - A^{-2})V()(; A)$.

Proof. (JI_A) is clear. Forgetting the orientation, we obtain from Theorem 8.1.11 the following identity:

$$A\langle \times \rangle - A^{-1}\langle \times \rangle = (A^2 - A^{-2})\langle)(\rangle.$$

Recalling the orientation, we obtain from the identities $w(\times_+) = w()() + 1$ and $w(\times_-) = w()() - 1$ the following identity:

$$\begin{aligned}
&A^4V(\times_+; A) - A^{-4}V(\times_-; A)\\
&= A^4(-A^3)^{-w()()-1}\langle \times \rangle - A^{-4}(-A^3)^{-w()()+1}\langle \times \rangle\\
&= (-A^3)^{-w()()}\{A^{-1}\langle \times \rangle - A\langle \times \rangle\}\\
&= (-A^3)^{-w()()}(A^{-2} - A^2)\langle)(\rangle\\
&= (A^{-2} - A^2)V()(; A).\square
\end{aligned}$$

By taking $t^{1/2} = A^{-2}$, $(\mathrm{J_{I_A}})$ and $(\mathrm{J_{II_A}})$ change into the following identities:

$(\mathrm{J_I})$ $V(O; t) = 1$.

$(\mathrm{J_{II}})$ $t^{-1}V(\times_+; t) - tV(\times_-; t) = (t^{1/2} - t^{-1/2})V()(; t)$.

From now on, we assume that the variable of the Jones polynomial is as above. Let us compute the Jones polynomial for a few examples using $(\mathrm{J_I})$ and $(\mathrm{J_{II}})$.

Example 8.2.2.

(1) $t^{-1}V(\bigcirc\!\!\bigcirc) - tV(\bigcirc\!\!\bigcirc) = (t^{1/2} - t^{-1/2})V(\bigcirc\bigcirc)$. Hence, $V(O^2) = -t^{1/2} - t^{-1/2}$.

(2) $t^{-1}V(\bigcirc\!\!\bigcirc) - tV(\bigcirc\!\!\bigcirc) = (t^{1/2} - t^{-1/2})V(\bigcirc)$. Hence, $V(\bigcirc\!\!\bigcirc) = -t^{5/2} - t^{1/2}$.

(3) $t^{-1}V(\bigcirc) - tV(\bigcirc) = (t^{1/2} - t^{-1/2})V(\bigcirc\!\!\bigcirc)$. Hence, $V(\bigcirc) = -t^4 + t^3 + t$.

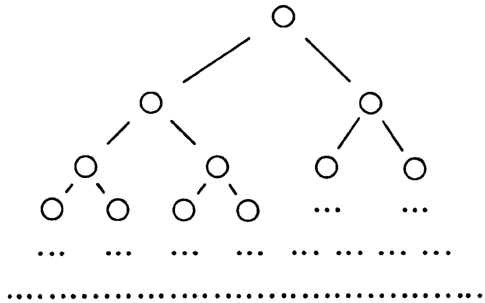


Fig. 8.2.1

As seen from these examples, the Jones polynomial of any link can be calculated by using only (J_{I}) and (J_{II}). It is convenient for our calculation to consider a binary resolution tree, shown in figure 8.2.1.

Here, the central, left and right circles in each stand for either , and , or , and . Further, each bottom circle corresponds to a trivial link. By a calculation similar to Example 8.2.2(1), we have

$$V(O^n) = (-t^{1/2} - t^{-1/2})^{n-1}.$$

Thus, the Jones polynomial of every link can in principle be calculated by tracing back a binary resolution tree from the bottom trivial links. Two links with the same binary resolution tree are said to be *skein equivalent.* (A skein is a thread wound in a loose, elongated coil.) To give the rigorous definition, we call a link triple (L_+, L_-, L_0) a *skein triple* if there is a link diagram triple (D_+, D_-, D_0) of it whose component diagrams are mutually identical except in a neighborhood triple where it is consistent with (, ,).

Definition 8.2.3 The *Skein equivalence* is an equivalence relation " $\sim$ " on the set of all links generated by the following:

(0) if L and L' belong to the same type, then $L \sim L'$,
(1) $L_+ \sim L'_+$ and $L_0 \sim L'_0$ imply $L_- \sim L'_-$,
(2) $L_- \sim L'_-$ and $L_0 \sim L'_0$ imply $L_+ \sim L'_+$,

for skein triples (L_+, L_-, L_0) and (L'_+, L'_-, L'_0).

As seen from the calculation method above, the Jones polynomial is invariant under skein equivalence. Moreover, we see that we may adopt (J_{I}) and (J_{II}) as a definition of the Jones polynomial. (In this case, we must note that Theorem 8.1.12 is needed to see that it is well-defined. Though it can be directly proved by Theorem 8.2.6, it would be much harder than Theorem 8.1.12.) The following theorem shows that the one-variable Alexander polynomial $\Delta(L;t) = \Delta_L(t)$ (see 5.4) is also invariant under skein equivalence.

Theorem 8.2.4 *The one-variable Alexander polynomial $\Delta(L;t)$ (after suitable multiplication by $\pm t^{n/2}$) satisfies the following identities:*
(A_I) $\Delta(O;t) = 1$.
(A_{II}) $\Delta(L_+;t) - \Delta(L_-;t) = (t^{1/2} - t^{-1/2})\Delta(L_0;t)$.

Proof. We consider Seifert surfaces for L_+, L_- and L_0 as shown in figure 8.2.2. Taking certain bases of the first homology groups, we can assume that L_+, L_- and L_0 have Seifert matrices M_+, M_- and M_0 such that

$$M_+ = \begin{pmatrix} a & u \\ v' & M_0 \end{pmatrix}, \qquad M_- = \begin{pmatrix} a+1 & u \\ v' & M_0 \end{pmatrix},$$

where u is a row vector and v' is a column vector. When we define the one-variable Alexander polynomial $\Delta(L;t)$ of a link L to be

$$(-1)^n \det(t^{1/2}M' - t^{-1/2}M)$$

where M is a Seifert matrix of L and n is the size of M, we obtain the desired result by an easy calculation. □

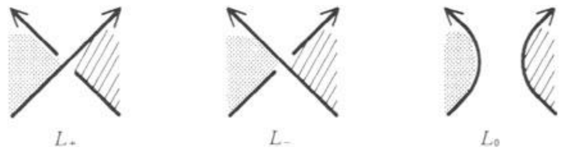


Fig. 8.2.2

To avoid confusion, we use the notation $\Delta^*(L;t)$ for $\Delta(L;t)$ satisfying the identities in Theorem 8.2.4. This polynomial is uniquely determined (without ambiguity concerning multiplication of a unit $\pm t^m$) and is called *Conway's normalized one-variable Alexander polynomial.* Moreover, when we replace $t^{1/2} - t^{-1/2}$ with $z, \Delta^*(L;t)$ changes into a polynomial in z (without a negative power term) which we call the *Conway polynomial* and denote by $\nabla(L;z)$.

Exercise 8.2.5 Compute the Conway polynomial for several links.

An analogy between (A_I), (A_{II}) and (J_I),(J_{II}) leads us to the existence of the following Laurent polynomial invariant:

Theorem 8.2.6 *There is a Laurent polynomial invariant $P(L;a,z) \in \mathbf{Z}[a,a^{-1}, z,z^{-1}]$ of the type of a link L which is determined uniquely by the following identities:*
(P_I) $P(O) = 1$.
(P_{II}) $a^{-1}P(\ ) - aP(\ ) = zP(\ )$.

The invariant $P(L;a,z)$ is called the *skein polynomial* of L. (It is also called the twisted Alexander, HOMFLY, two-variable Jones, FLYPMOTH or LYMPHTOFU polynomial). The skein polynomial was introduced by [Freyd-Yetter-Hoste-Lickorish-Millett-Ocneanu 1985] and its existence can be proved directly using a binary resolution tree (see [Lickorish-Millett 1987]). We shall discuss it from the viewpoint of state models in 8.5 and from the viewpoint of representation theory in Chapter 9.

Exercise 8.2.7 Compute $P(L; a, z)$ for several links L. Also, sketch the proof of Theorem 8.2.6.

The one-variable Alexander polynomial (Conway polynomial) and the Jones polynomial can be derived from the skein polynomial as follows:

Proposition 8.2.8

$$\Delta^*(L;t) = P(L;1,t^{1/2} - t^{-1/2}).$$
$$V(L;t) = P(L;t,t^{1/2} - t^{-1/2}).$$

We see now that the skein polynomial can be re-defined via an invariant of regular isotopy.

Definition 8.2.9 The *R-polynomial* of a link diagram D is a regular isotopy invariant $R(D; a, z) \in \mathbf{Z}[a, a^{-1}, z, z^{-1}]$ of D such that

($\mathrm{R_I}$) $R(O) = 1$.

($\mathrm{R_{II}}$) $R(\;) - R(\;) = zR(\;)$.

($\mathrm{R_{III}}$) $R(\;) = aR(\;)$, $R(\;) = a^{-1}R(\;)$.

Exercise 8.2.10 Show that we can omit one of the relations in ($\mathrm{R_{III}}$). Also show that the R-polynomial of each link diagram is uniquely determined by Definition 8.2.9.

The skein polynomial can be defined from the R-polynomial as follows (and conversely the R-polynomial is defined from the skein polynomial in this way):

Theorem 8.2.11 *For any diagram D of a link L, we have*

$$P(L; a^{-1}, z) = a^{-w(D)}R(D; a, z).$$

This suggests that the variable a measures a twisting information on a link. If we say that the one-variable Alexander polynomial is an invariant ignoring this twisting information, then what can we say about the Jones polynomial or more generally about the skein polynomial? What topological meaning do they potentially have? There remain many questions to be answered.

8.3 The Q and Kauffman polynomials

We sometimes consider a quadruple

$$(\;,\;,\;,\;)$$

instead of a skein triple $(\;,\;,\;)$ associated with a link diagram D. It is also denoted by $(D_+,\ D_-,\ D_0,\ D_\infty)$ for D or $(L_+,\ L_-,\ L_0,\ L_\infty)$ for the link L presented by D. The quadruple $(|D_+|, |D_-|, |D_0|, |D_\infty|)$ of unoriented link diagrams is presented as $(\;,\;,\;,\;)$. We denote by $|L|$ the link obtained from a link L by forgetting the orientation.

Theorem 8.3.1 *There is a Laurent polynomial invariant $Q(|L|;x) \in \mathbf{Z}[x,x^{-1}]$ of an unoriented link $|L|$ which is invariant under positive-equivalent links (cf. 0.3.1) and uniquely determined by the following identities:*

(Q_{I}) $Q(|O|) = 1$.

(Q_{II}) $Q(\times) + Q(\times) = x\{Q()() + Q(\asymp)\}$.

This theorem can be proved directly using unoriented link diagrams (cf. [Brandt-Lickorish-Millett 1986]). An algebraic proof of this theorem as well as of Theorem 8.2.6 is known.

Exercise 8.3.2 Compute the Q-polynomial for several links and give a sketch of the proof of Theorem 8.3.1.

Considering the Q-polynomial instead of the Conway polynomial, we may obtain a new link type invariant in place of the skein polynomial by an argument similar to that used for the establishment of Theorem 8.2.11 (cf. [Kauffman 1990]).

Theorem 8.3.3 *For an unoriented link diagram $|D|$, there is a regular isotopy invariant $\Lambda(|D|;a,x) \in \mathbf{Z}[a,a^{-1},x,x^{-1}]$ which satisfies the following identities:*

(Λ_{I}) $\Lambda(|O|) = 1$.

(Λ_{II}) $\Lambda(\times) + \Lambda(\times) = x\{\Lambda()() + \Lambda(\asymp)\}$.

(Λ_{III}) $\Lambda(\gamma) = a\Lambda(\smile)$, $\Lambda(\gamma) = a^{-1}\Lambda(\smile)$.

Further, if we take

$$F(L;a,x) = a^{-w(D)}\Lambda(|D|;a,x)$$

for a link L represented by $D, F(L;a,x)$ is an invariant of the type of L.

This invariant $F(L;a,x)$ of L is called the *Kauffman polynomial.* Clearly, we have $F(L;1,x) = Q(|L|;x)$.

Exercise 8.3.4 Compute the Kauffman polynomial for several links and give a sketch of the proof of Theorem 8.3.3.

Exercise 8.3.5 For a quadruple $(D_+, D_-, D_0, D_\infty)$ associated with a diagram D of a link L, show that $w(D_+) = w(D_\infty) + 2\nu + 1$ for some integer ν. Using this integer ν, show that

$$aF(L_+;a,x) + a^{-1}F(L_-;a,x) = x\{F(L_0;a,x) + a^{-2\nu}F(L_\infty;a,z)\}.$$

8.4 Properties of the polynomial invariants

We describe some properties of the Jones, skein and Kauffman polynomials, defined in 8.1, 8.2 and 8.3. First, we prepare some notation used in this section. For a link L, $\sharp L$ denotes the number of components of L. We denote a split union of two links L_1 and L_2 by $L_1 + L_2$. The n-fold cyclic branched covering space over S^3 with branch set L is denoted by $M_n(L)$. We can directly obtain the following properties from the recursive definition of the skein polynomial, given in Theorem 8.2.6 (cf. [Lickorish-Millett 1987]):

Theorem 8.4.1

(1) $P(-L; a, z) = P(L; a, z)$.
(2) $P(L^*; a, z) = P(L; -a^{-1}, z)$.
(3) *For any connected sum* $L_1 \sharp L_2$ *of* L_1 *and* L_2,
$P(L_1 \sharp L_2; a, z) = P(L_1; a, z)P(L_2; a, z)$.
(4) $P(L_1 + L_2; a, z) = \{(a^{-1} - a)/z\}P(L_1; a, z)P(L_2; a, z)$.
(5) *For any link* L, $P(L; a, a^{-1} - a) = 1$.
(6) $P(L; -a, -z) = P(L; a, z)$,
$P(L; a, -z) = P(L; -a, z) = (-1)^{\sharp L - 1}P(L; a, z)$.

Some topological interpretations are well-known for the evaluation of the one-variable Alexander polynomial at a root of unity (see Exercise 5.5.5). Likewise, the evaluations of the Jones polynomial at some roots of unity have topological meanings.

Theorem 8.4.2

(1) $V(L; 1) = (-2)^{\sharp L - 1}$.
(2) $V(L; -1) = \Delta^*(L; -1)$.
(3) $V(L; e^{2\pi\sqrt{-1}/3}) = 1$.
(4) $V(L; \sqrt{-1}) = \begin{cases} \sqrt{2}^{\sharp L - 1}(-1)^{\mathrm{Arf}(L)} & \textit{(if } L \textit{ is proper),} \\ 0 & \textit{(otherwise).} \end{cases}$
(5) $V(L; e^{2\pi\sqrt{-1}/6}) = \delta(L)(\sqrt{-1})^{\sharp L - 1}(\sqrt{-3})^{\mathrm{rank} H_1(M_2(L); \mathbf{Z}_3)}$, *where* $\delta(L) = \pm 1$.

For the proof, see [Lickorish-Millett 1986'], [Lipson 1986] and [Murakami, H. 1986, 1986']. (In particular, the sign of $\delta(L)$ in (5) is determined in [Lipson 1986].) As an evaluation of the skein polynomial, the following is known (cf. [Lickorish-Millett 1986']):

Theorem 8.4.3 $P(L; \sqrt{-1}, \sqrt{-1}) = (\sqrt{-2})^{\mathrm{rank} H_1(M_3(L); \mathbf{Z}_2)}$.

As we have seen in Chapter 1, any link diagram can be changed into a set of disjoint simple closed curves, called the Seifert circles, by smoothing all the crossings. Moreover, using this fact, we can deform any link into a closed braid. The following theorem, given in [Murakami,H. 1987], is related to the definition of the skein polynomial via braid groups discussed in the next chapter:

Theorem 8.4.4 *For a diagram* D *of a link* L *with* n *Seifert circles and mutually independent variables* $a_i (i = 0, 1, \ldots, n)$, *we have the identity:*

$$a_0^{-w(D)}P(L; a_0, z) = \sum_{i=1}^{m} a_i^{-w(D)}P(L; a_i, z)F_i(a_0, \ldots, a_n),$$

where

$$F_i(a_0, \ldots, a_n) = \prod_{1 \leq j \leq n, j \neq i} \frac{(a_0 a_j^{-1} - a_0^{-1} a_j)}{(a_i a_j^{-1} - a_i^{-1} a_j)}.$$

Using the fact that choosing a_i, z appropriately makes the skein polynomial change into the one-variable Alexander polynomial or the Jones polynomial, we obtain from Theorem 8.4.4 the following corollary (cf. [Murakami,H. 1987]):

Corollary 8.4.5 *(1) If a link L has a diagram D with three Seifert circles, then $P(L; a, t^{1/2} - t^{-1/2})$ is expressible by either $\Delta^*(L;t)$ and $w(D)$ or $V(L;t)$ and $w(D)$.*
(2) If a link L has a diagram D with four Seifert circles, then $P(L; a, t^{1/2} - t^{-1/2})$ is expressible by either $\Delta^*(L;t), V(L;t)$ and $w(D)$ or $V(L;t), V(L;t^{-1})$ and $w(D)$. Moreover, $\Delta^*(L;t)$ can be expressed by $V(L;t), V(L;t^{-1})$ and $w(D)$.
(3) If a link L has a diagram D with 5 Seifert circles, then $P(L; a, t^{1/2} - t^{-1/2})$ is expressible by $\Delta^*(L;t), V(L;t), V(L;t^{-1})$ and $w(D)$.

From the proof of Theorem 8.4.4 in [Murakami,H. 1987], we also obtain the following estimate on the braid index which is given in [Morton 1986] and [Franks-Williams 1987]:

Theorem 8.4.6 *For a diagram D with n Seifert circles, we have the following inequality:*

$$w(D) - (n-1) \le l\text{--}\deg_a P(L; a, z) \le h\text{--}\deg_a P(L; a, z) \le w(D) + (n-1),$$

where by $l\text{--}\deg_a$ and $h\text{--}\deg_a$ we denote the minimal and maximal degrees of $P(L; a, z)$ regarded as a polynomial in a, respectively.

The following theorem, which is given in [Lickorish-Millett 1986] and [Morton 1986'], shows a big difference between the Jones polynomial and the one-variable Alexander polynomial :

Theorem 8.4.7 *Suppose that L' is obtained from a link L by reversing the orientation of one component K of L. Then we have $V(L';t) = t^{-3\lambda}V(L;t)$, where $\lambda = \mathrm{Link}(K, L-K)$.*

Some evaluations of the Q-polynomial are summarized here (cf. [Brandt-Lickorish-Millett 1986]).

Theorem 8.4.8
(1) $Q(|L|; 1) = 1$.
(2) $Q(|L|; -1) = (-3)^{\mathrm{rank} H_1(M_2(L); \mathbf{Z}_3)}$ *(compare 8.4.2(5)).*
(3) $Q(|L|; 2) = |\Delta^*(L; -1)|^2$.
(4) $Q(|L|; -2) = (-2)^{\sharp L - 1}$.

The following theorem shows that the Kauffman polynomial and the skein polynomial are related via the Jones polynomial (cf. [Lickorish 1986]):

Theorem 8.4.9 $F(L; -t^{3/4}, t^{1/4} + t^{-1/4}) = V(L;t)$.

The following theorem shows that these polynomial invariants are not complete invariants (cf. [Kanenobu 1986,1989']):

Theorem 8.4.10 *There exist infinitely many knots with the same skein polynomial (and hence with the same Jones and the same Alexander polynomials). There exists a pair of knots with the same Kauffman polynomial.*

We conclude this section with an application of the Jones polynomial to alternating links, which is one of the most interesting applications of the Jones polynomial. Compare [Murasugi 1987,1987'] for the detailed argument.

Definition 8.4.11 A link diagram is *alternating* if an over-crossing and an under-crossing appear alternately as one goes along each component. A link is *alternating* if it possesses an alternating diagram.

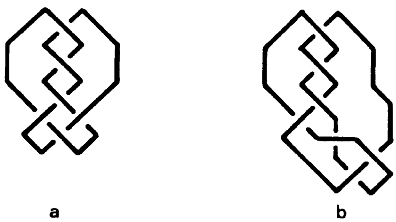


Fig. 8.4.1

For example, figure 8.4.1a is alternating, but figure 8.4.1b is not alternating, though they represent the same link type.

Definition 8.4.12 A link diagram is *reduced* if there is no crossing as shown in figure 8.4.2 when we consider that the diagram is on S^2, where each square means a diagram of a tangle.

Fig. 8.4.2

We have the following theorem:

Theorem 8.4.13 *For any connected diagram D, we have the following inequality:*

$$h\text{-}\deg_A V(D;A) - l\text{-}\deg_A V(D;A) \leq 4c(D),$$

where $c(D)$ is the crossing number of D. Further, if D is alternating and reduced, then the inequality can be replaced by equality.

8.5 The skein polynomial via a state model

We show here the existence of the skein polynomial using a certain state model on a link diagram whose proof is due to [Turaev 1988] and [Jones 1989]. All proofs, however, are omitted. Let D be a link diagram. Fix a positive integer n.

Definition 8.5.1 An *n-state* on D is a function which sends each segment of D to an element of the set $\{-n+1, -n+3, \dots, n-3, n-1\}$.

Let S be an n-state on D. We define a weight $g(D, S; v)$ of S at a crossing v as in figure 8.5.1, where the numbers i and j denote the values of the segments given by S and q denotes a variable. We take $g(D, S; v) = 0$ for any state S and any crossing v except the above cases. We define $G(D, S) = \prod g(D, S; v)$, where the product is taken over all crossings of D. We take $G(D, S) = 1$ if D does not have any crossing.

We consider a smooth (possibly non-simple) planar loop C defined by a function $a(t)$ $(0 \le t \le 1)$ such that $|a'(t)| = 1$ and $a(0) = a(1)$. Let $e_1(t) = a'(t)$. Let $e_2(t)$ be a vector obtained by turning the vector $e_1(t)$ $90°$ anticlockwise. Then the *curvature* of C is a a real-valued function $\kappa(t)$ defined by $e_1'(t) = \kappa(t)e_2(t)$. The *rotation number* $\mathrm{rot}(C)$ of C is defined by $\frac{1}{2\pi}\int_0^1 \kappa(t)dt$. The rotation number of a component of a link diagram is defined to be the rotation number of a smooth loop obtained by rounding each corner off. Thus, for a simple planar loop C, we have that $\mathrm{rot}(C)$ is $+1$ or -1 according to whether it is oriented counterclockwise or clockwise.

Now we consider a link diagram D and an n-state S on D. Let $d(D, S)$ and $s(D, S)$ be a diagram and a state obtained from D and S by changing each crossing of the type ${}_i^i\!\times\!{}_j^j$ into ${}^i)({}^j$. Then we see that if $G(D, S) \neq 0$, then $s(D, S)$ is constant on each component of $d(D, S)$. In this case, we define $\int(D, S) = q^{\sum_C \mathrm{rot}(C)s(C)}$, where the sum is taken over all components C of $d(D, S)$ and $s(C)$ is the value of $s(D, S)$ on C.

Definition 8.5.2 For a link diagram D, we define

$$p_n(D) = (-q^n)^{w(D)} \sum_S G(D, S) \int (D, S),$$

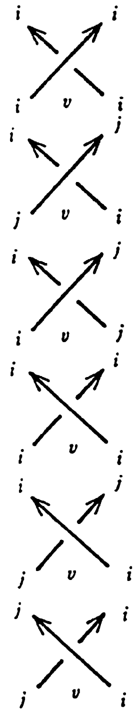
where the sum is taken over all n-states S on D.

Theorem 8.5.3 *$p_n(D)$ is an invariant of the type of a link presented by D and we have the following identities:*

($\mathrm{p_I}$) $p_n(O) = \frac{q^n - q^{-n}}{q - q^{-1}}$.

($\mathrm{p_{II}}$) $q^{-n}p_n(\times) - q^n p_n(\times) + (q - q^{-1})p_n()() = 0$.

The proof is complicated, but elementary. Taking $n = 1, 2, 3, \dots,$ we can deduce from this theorem the existence of the skein polynomial stated in 8.2.



$: \quad g(D,S;v) = -q^{-1}.$

$(i \neq j): \quad g(D,S;v) = 1.$

$(i < j): \quad g(D,S;v) = q - q^{-1}.$

$: \quad g(D,S;v) = -q.$

$(i \neq j): \quad g(D,S;v) = 1.$

$(i < j): \quad g(D,S;v) = q^{-1} - q.$

Fig. 8.5.1

Supplementary notes for Chapter 8

It is known that there exists a pair of 2-bridge knots with the same skein polynomial but with distinct Kauffman polynomials and, conversely, a pair of 2-bridge knots with the same Kauffman polynomial but with distinct skein polynomials. (cf. [Kanenobu 1989'].) In this sense, the skein polynomial and the Kauffman polynomial are distinct invariants. On the other hand, in [Kauffman 1991] the following fact (due to F. Jaeger) is observed: the Kauffman polynomial can be obtained as a certain weighted sum of the skein polynomials of links associated with a given link. In [Kawauchi 1992] we showed how to construct a link whose skein polynomial is "close" to that of a given link. For example, for any link L and any positive integers M, N, we can find a link L^*, not equivalent to L, such that the difference $P(L^*; a, z) - P(L; a, z)$ can be written as a sum of finitely many terms in the following form:

$$f(a)(a^{2N} - 1)z^m$$

with $f(a) \in \mathbf{Z}[a, a^{-1}] = \mathbf{Z}\langle a \rangle$ and $m \geq M$. Then if we take a to satisfy $a^{2N} = 1$, we have that $P(L^*; a, z) = P(L; a, z)$.

Chapter 9
Jones type polynomials II: an algebraic approach

In this chapter, we discuss some algebras related to link polynomials such as the skein polynomial in order to explain how the polynomials arise from the representation theory of algebras.

9.1 Preliminaries from representation theory

Let R be a commutative ring with unit.

Definition 9.1.1 An R-module $\mathcal{A}$ is an *R-algebra* if there is an R-bilinear map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(a, b) \to ab$, which has the following properties:

(1) For any element $a, b, c \in \mathcal{A}$, $(ab)c = a(bc)$.
(2) There is an element $1 \in \mathcal{A}$ such that $a1 = 1a = a$ for all elements $a \in \mathcal{A}$.

A map ρ from an R-algebra $\mathcal{A}$ to an R-algebra $\mathcal{A}'$ is called an *algebra homomorphism* if $\rho(a + b) = \rho(a) + \rho(b)$, $\rho(ab) = \rho(a)\rho(b)$ and$\rho(\lambda a) = \lambda\rho(a)$ for all $a, b \in A$ and all $\lambda \in R$. Let $M_n(R)$ be the matrix R-algebra consisting of all (n, n) matrices over R.

Definition 9.1.2 A *matrix representation* of an R-algebra $\mathcal{A}$ is an algebra homomorphism $\rho : \mathcal{A} \to M_n(R)$ for some n such that $\rho(1) = E^n$.

In this definition, n is called the *degree* of the matrix representation ρ, and we denote it by $\deg(\rho) = n$. For two matrix representations ρ and τ of an R-algebra $\mathcal{A}$ of degree n, ρ is said to be *equivalent* to τ if there is an R-isomorphism Φ : $R^n \to R^n$ such that $\Phi(\rho(a)(\mathbf{x})) = \tau(a)(\Phi(\mathbf{x}))$ for each $a \in \mathcal{A}$ and each $\mathbf{x} \in R^n$. This equivalence gives an equivalence relation among the matrix representations of $\mathcal{A}$ of degree n. We say that a matrix representation ρ of $\mathcal{A}$ of degree n is *irreducible*[1] if it is not equivalent to the direct sum

$$\rho_1 + \rho_2 : \mathcal{A} \to M_{n_1}(R) \oplus M_{n_2}(R) \subset M_{n_1+n_2}(R) = M_n(R)$$

of any two matrix representations $\rho_i : \mathcal{A} \to M_{n_i}(R)$ $(i = 1, 2)$ such that $0 < n_i < n$ and $n = n_1 + n_2$, where $M_{n_1}(R) \oplus M_{n_2}(R)$ is a matrix R-algebra consisting of all the block sums $X_1 \oplus X_2$ for all $X_1 \in M_{n_1}(R)$ and all $X_2 \in M_{n_2}(R)$.

From now on, we take R to be $\mathbf{C}$ and we call a finitely generated $\mathbf{C}$-algebra simply an algebra.

[1] The usual term for this is *indecomposable*. We use *irreducible* here only for "semi-simple" algebras, for which it coincides with the concept of "irreducible" in the usual sense.

Example 9.1.3 (Group algebra). The *symmetric group algebra of degree n* is the group algebra $\mathbf{C}[\mathbf{S}_n]$ of the symmetric group $\mathbf{S}_n$ of degree n over $\mathbf{C}$. We note that $\mathbf{S}_n$ has the following presentation:

generators $s_1, \ldots, s_{n-1}$.
relations (S1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (i = 1, \ldots, n-2)$.
(S2) $s_i s_j = s_j s_i \quad (|i-j| \geq 2)$.
(S3) $s_i^2 = 1 \quad (i = 1, \ldots, n-1)$.

The *braid group algebra of degree n* is the group algebra $\mathbf{C}[B_n]$ of the braid group B_n of degree n over $\mathbf{C}$. We see from the braid group presentation of B_n given in 1.2.1 that there is an algebra epimorphism $\mathbf{C}[B_n] \to \mathbf{C}[\mathbf{S}_n]$ sending σ_i to s_i for all i.

A sequence of integers $n_1, \ldots, n_m$ such that $n = n_1 + \cdots + n_m$ and $n_1 \geq \cdots \geq n_m > 0$ is called a *partition* of n and denoted by $(n_1, \ldots, n_m)$. Then by $\Lambda(n)$ we denote the set of all partitions of n. Then we have the following theorem (see [Iwahori 1978]):

Theorem 9.1.4 *The number of the equivalence classes of irreducible matrix representations of $\mathbf{C}[\mathbf{S}_n]$ is equal to the number of the elements of $\Lambda(n)$.*

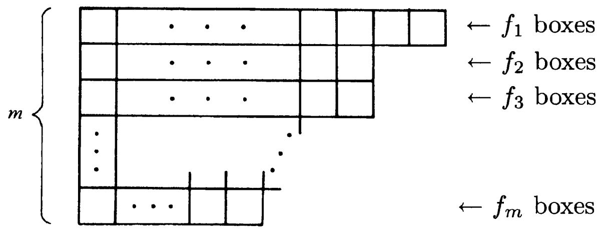


Fig. 9.1.1

Let $\lambda = (n_1, \ldots, n_m)$ be a partition of n. Then we can match λ with a diagram as in figure 9.1.1, which is called a *Young diagram* $Y(\lambda)$ and we can concretely construct the irreducible matrix representation of $\mathbf{C}[\mathbf{S}_n]$ corresponding to λ. We discuss below another example of an algebra (see [Iwahori 1964], [Bourbaki 1968]).

Example 9.1.5(Iwahori-Hecke algebra). Let q be a complex number. The *Iwahori-Hecke algebra of degree n* is an algebra $H_n(q)$ defined by the following generators and relations:

generators $1, g_1, \ldots, g_{n-1}$.
relations (H1) $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (i = 1, \ldots, n-2)$.
(H2) $g_i g_j = g_j g_i \quad (|i-j| \geq 2)$.
(H3) $g_i^2 = (q-1) g_i + q \quad (i = 1, \ldots, n-1)$.

Thus, $H_n(1)$ and $\mathbf{C}[\mathbf{S}_n]$ are naturally isomorphic by the correspondence $g_i \to s_i$ for all i. More generally, it is known that $H_n(q)$ and $\mathbf{C}[\mathbf{S}_n]$ are isomorphic for a generic complex number q. That is, we have the following theorem (see [Gyoja-Uno 1989]):

Theorem 9.1.6 *$H_n(q)$ and $\mathbf{C}[\mathbf{S}_n]$ are isomorphic if and only if q is not 0 or an m-th root of 1 except 1 for any m with $2 \leq m \leq n$.*

Hence, combining this theorem with Theorem 9.1.4, we see that the equivalence classes of irreducible matrix representations of $H_n(q)$ correspond bijectively with $\Lambda(n)$ for a generic complex number q (cf. [Bourbaki 1968]).

9.2 Link invariants of trace type

We denote by B the disjoint union of the braid groups B_n ($n = 1, 2, \dots$) and by $\sim$ the Markov equivalence which is an equivalence relation on B. Then we may identify $B/_\sim$ with the set of link types (cf. Theorem 1.2.5). Throughout this section, we adopt this identification. Let $\varphi : B/_\sim \to \mathbf{C}$ be a function which we call here a *link invariant.* Let $k : B \to \mathbf{C}$ be the composition of the quotient map $B \to B/_\sim$ and the link invariant φ. We denote by k_n the restriction of k to B_n. We assume that k_n is a linear combination of the traces of finitely many matrix representations of $\mathbf{C}[B_n]$ for each n. That is, there are matrix representations $\rho_j^{(n)} : \mathbf{C}[B_n] \to M_{n_j}(\mathbf{C})$ and $a_{n,j} \in \mathbf{C}$ ($j = 1, 2, \dots, r_n$) such that

$$k_n(b) = \sum_{j=1}^{r_n} a_{n,j} \mathrm{tr} \rho_j^{(n)}(b)$$

for each element $b \in B_n$ and each n. Then we denote by $\mathcal{A}_n(\varphi)$ the algebra $\oplus_{j=1}^{r_n} \rho_j^{(n)}(\mathbf{C}[B_n])$ and call it the *algebra belonging to* the link invariant φ. Let

$$s_n = \sum_{j=1}^{r_n} a_{n,j} \rho_j^{(n)} : B_n \to \mathcal{A}_n(\varphi).$$

Then we say that the link invariant φ is a *link invariant of trace type* if there exists an algebra-homomorphism $j_n : \mathcal{A}_n(\varphi) \to \mathcal{A}_{n+1}(\varphi)$ which makes the following diagram commutative for all n:

$$\begin{tikzcd} B_n \arrow[r, "i_n"] \arrow[d, "s_n"'] & B_{n+1} \arrow[d, "s_{n+1}"'] \\ \mathcal{A}_n(\varphi) \arrow[r, "j_n"] & \mathcal{A}_{n+1}(\varphi), \end{tikzcd}$$

where $i_n : B_n \to B_{n+1}$ denotes the homomorphism defined by $i_n(\sigma_i) = \sigma_i$ for all i ($i = 1, 2, \dots, n-1$).

Example 9.2.1. The skein polynomial P is a link invariant of trace type such that $\mathcal{A}_n(P)$ is the Iwahori-Hecke algebra $H_n(q)$ (cf. [Jones 1987]).

Example 9.2.2. The Kauffman polynomial F is a link invariant of trace type such that $\mathcal{A}_n(F)$ is a q-analogue of Brauer algebra (cf. [Birman-Wenzl 1989], [Murakami, J. 1987]).

9.3 The skein polynomial as a link invariant of trace type

The skein polynomial P of Example 9.2.1 can be defined by induction on n from a trace defined on the Iwahori-Hecke algebra $H_n(q)$, so that P is a natural trace type invariant. In this section, we give a method of describing a presentation of this trace type invariant concretely. For this purpose, we give a matrix representation of $\mathbf{C}[B_n]$, denoted by ρ_λ, for each $\lambda \in \Lambda(n)$. We use another presentation of the Iwahori-Hecke algebra $H_n(q)$ by changing the parameter q.

generators $1, g_1, \ldots, g_{n-1}$.
relations (H1)$'$ $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad (i = 1, \ldots, n-2)$.
(H2)$'$ $g_i g_j = g_j g_i \quad (|i-j| \geq 2)$.
(H3)$'$ $g_i^2 - (q - q^{-1}) g_i - 1 = 0 \quad (i = 1, 2, \ldots, n-1)$.

Exercise 9.3.1 Explain the correspondence of the parameter q in Example 9.1.4 to the parameter q in this presentation.

Hereafter, we adopt this presentation for $H_n(q)$. As we claimed in section 9.1, there is a one-to-one correspondence between the set of irreducible matrix representations of $H_n(q)$ and the set $\Lambda(n)$ of partitions of n for a generic complex number q. Then we denote by π_λ the irreducible matrix representation of $H_n(q)$ corresponding to $\lambda \in \Lambda(n)$. For any complex number $\alpha \in \mathbf{C} - \{0\}$, we define an algebra homomorphism

$$p_n(\alpha) : \mathbf{C}[B_n] \to H_n(q)$$

by $p_n(\alpha)(\sigma_i) = \alpha^{-1} g_i$ for each i $(i = 1, 2, \ldots, n-1)$. Now we define $\rho_\lambda = \pi_\lambda p_n(\alpha)$. Let the skein polynomial P be such that

$$P(\hat{b}; \alpha, q) = \sum_{\lambda \in \Lambda(n)} a_\lambda \mathrm{tr} \rho_\lambda(b).$$

Then we give a concrete presentation of a_λ. Let x be the box lying in the i-th row and the j-th column of the Young diagram $Y(\lambda)$ of the partition λ of n, shown in figure 9.3.1.

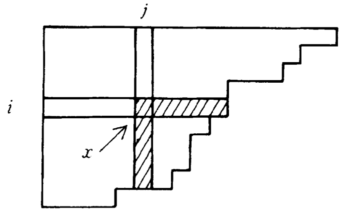


Fig. 9.3.1

Definition 9.3.2 The *hook* of x is the union of x and the boxes lying below x and the boxes lying in the right hand side of x. The *hook length* of x, denoted by $h(x)$, is the number of the boxes in the hook of x.

In the Young diagram $Y(\lambda)$, we put 0 in each diagonal box and put -1 on the boxes adjoining the boxes with the number 0 in the right side and put -2 on the boxes adjoining the boxes with the number -1 on the right hand side. We continue this numbering procedure on the boxes on the right hand side. Next, put 1 on the boxes adjoining below the boxes with the number 0 and put 2 on the boxes adjoining below the boxes with the number 1. We continue this numbering procedure. (cf. figure 9.3.2.) Thus, we can assign an integer to each box x of the Young diagram $Y(\lambda)$ which we denote by $\ell(x)$.

0	-1	-2	-3
1	0	-1	
2			

Fig. 9.3.2

Notation For an integer i, let $[i;q] = (q^i - q^{-i})/(q - q^{-1})$ and $[0;q] = 0$. Then we put

$$p(n;q) = [n;q][n-1;q]\ldots[1;q]$$

for each $n \geq 1$ and

$$D_\lambda(q) = \prod_{x \in Y(\lambda)} [h(x);q].$$

for each $\lambda \in \Lambda(n)$.

Then we have the following presentation of a_λ:

Theorem 9.3.3

$$a_\lambda = (-1)^{n+1}\frac{p(n;q)}{D_\lambda(q)} \prod_{x \in Y(\lambda)} \left(\frac{\alpha q^{\ell(x)} - \alpha^{-1}q^{-\ell(x)}}{q - q^{-1}}\right)\left(\frac{\alpha - \alpha^{-1}}{q - q^{-1}}\right)^{-1}.$$

For the proof of this theorem, we refer to [Gyoja *] and [Jones 1987].

9.4 The Temperley-Lieb algebra

We take the points $P_i = (i/(n+1), 1)$ and $Q_i = (i/(n+1), 0)$ $(i = 1, 2, \ldots, n)$ in the boundary of the square $I^2 = \{(x,y) \mid 0 \leq x, y \leq 1\}$. An (n,n)-*tangle diagram* in I^2 is a diagram d in I^2 which is given by $d = |D| \cap I^2$ for an unoriented link diagram $|D|$ in $\mathbf{R}^2$ such that $|D| \cap \partial I^2$ is equal to the set $\{P_i, Q_i \mid i = 1, 2, \ldots, n\}$ and this set is disjoint from the vertices of $|D|$. A typical example is an n-string braid diagram discussed in 1.2. Two (n,n)-tangle diagrams d and d' in I^2 are said to be *regularly isotopic* if they can be transformed into each other by a regular isotopy keeping ∂I^2 fixed (i.e., by a finite sequence of the Reidemeister moves of type II, type III and type IV in intI^2). We denote by D_n^n the set of the regular isotopy classes of (n,n)-tangles in I^2. By a method similar to forming the n-braid group B_n, we have that D_n^n forms a semi-group with identity 1 which is represented by a trivial n-braid diagram. Let TL_n be the algebra $\mathbf{C}[D_n^n]$ over $\mathbf{C}$ associated with the semi-group D_n^n. Then the Kauffman bracket $\langle\ \ \rangle$ defined by the following (1)-(3) (cf. Theorem 8.1.11) operates on the algebra TL_n:

(1) $\langle |O| \circ d \rangle = \delta \langle d \rangle$ for a split diagram $|O| \circ d$ where $|O|$ is an unoriented trivial knot diagram without crossing and d is an (n,n)-tangle diagram.

(2) $\langle \times \rangle = A \langle)(\rangle + B \langle \asymp \rangle$.

(3) $\langle \times \rangle = B \langle)(\rangle + A \langle \asymp \rangle$.

Here we take A to be a fixed non-zero complex number and $\delta = -(A^2 + A^{-2})$ and $B = A^{-1}$.

By using (1), (2) and (3), the algebra TL_n is easily seen to be generated over $\mathbf{C}$ by (n,n)-tangles without crossing and loop. Let $e_i (i = 1, 2, \ldots, n-1)$ be the elements of D_n^n shown in figure 9.4.1. Then we obtain the following theorem (cf. [Kauffman 1987']):

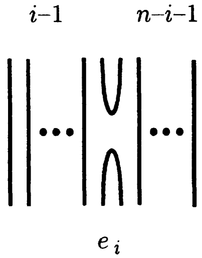


Fig. 9.4.1

Theorem 9.4.1 *The algebra TL_n over $\mathbf{C}$ has the following presentation:*

generators	$1, e_1, \ldots, e_{n-1}$.	
relations	(TL1)	$e_i e_{i\pm 1} e_i = e_i$ *for all possible* i.
	(TL2)	$e_i e_j = e_j e_i$ *for all* i, j *with* $\lvert i - j \rvert \geq 2$.
	(TL3)	$e_i^2 = \delta e_i$ *for all* i.

The algebra over $\mathbf{C}$ determined by these generators and relations is called the *n-th Temperley-Lieb algebra* (cf. [Jones 1983]). For the n-string braid group B_n, we note that there is a natural algebra epimorphism

$$\rho : \mathbf{C}[B_n] \to TL_n$$

sending each primitive generator σ_i of B_n (in figure 1.2.2) to $A^{-1}1 + Ae_i$.

Theorem 9.4.2 *Let $A \in \mathbf{C}$ have $A^{4m} \neq 1$ for any integer m with $1 \leq m \leq n$. Then there exists an element $f_n \in TL_n$ such that*

(1) $e_i f_n = f_n e_i = 0$ *for all* i $(i = 1, 2, \ldots, n-1)$, *and*
(2) $f_n = 1 + w_n$ *for an element* w_n *of the subalgebra of* TL_n *generated by* $e_i (i = 1, 2, \ldots, n-1)$.

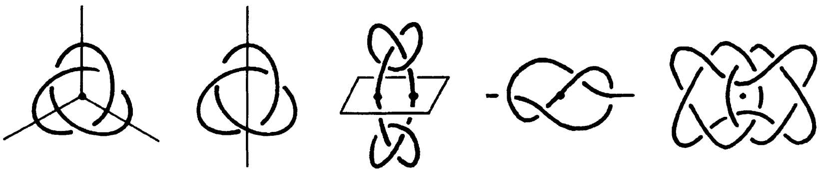
Many proofs of this theorem are known (cf. [Jones 1983], [Wenzl 1987], [Lickorish 1991], [Yamada 1992], [Turaev 1994]). The uniqueness of the element f_n follows from (1) and (2). Also, we have $f_n^2 = f_n$ and $xf_n = f_n x$ for all $x \in TL_n$. This element f_n is called the *Jones-Wenzl idempotent* or the *magic element* of the n-th Temperley-Lieb algebra TL_n. This element plays an important role in many arguments concerning the Temperley-Lieb algebra.

Supplementary notes for Chapter 9

The origin of the results of this chapter was the discovery of the Jones polynomial by V.F.R. Jones in [Jones 1985]. V.F.R. Jones discovered an algebra with a trace, called the *Jones algebra*, in his study of index theory for subfactors of the hyperfinite II_1 factor. Then he observed that there is a representation from the braid group to the Jones algebra; he successfully obtained a link invariant by combining this representation with the trace on the Jones algebra. This is what is called the Jones polynomial. Although it had been known that the Jones algebra is a quotient of the Iwahori-Hecke algebra by a certain ideal, A. Ocneanu showed in [Ocneanu *] that there exists a certain trace on the Iwahori-Hecke algebra which he used to generalize the Jones polynomial to the skein polynomial. After the discovery of the Jones polynomial, Kauffman discovered a link polynomial invariant — the Kauffman polynomial. Then it was shown in [Murakami, J. 1987] and [Birman-Wenzl 1989] that the Kauffman polynomial can be also expressed in terms of a trace of a certain algebra. In [Murakami, J. 1989], parallel versions of these polynomial link invariants are studied from the viewpoint of representation theory.

Chapter 10
Symmetries

As shown in figure 10.0.1, there are various kinds of symmetries on knots. In the first half of this chapter, we study some relationships between symmetries and the polynomial invariants. As an application, we explain the proof of [Kawauchi 1979] on the non-invertibility of 8_{17} (see figure 10.0.2). In the latter half of this chapter, we study the symmetry group of a knot, which essentially controls the symmetries of a knot. We explain a (still unpublished) theory of F. Bonahon and L. Siebenmann (cf. [Bonahon-Siebenmann *]) for a canonical decomposition of a knot, which gives us good insight into the knot and enables us to determine the symmetry groups of algebraic knots including 8_{17} and the Kinoshita-Terasaka knot K_{KT} (see figure 3.8.1a).



a b c d e

Fig. 10.0.1

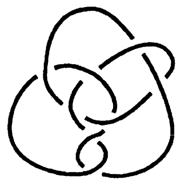


Fig. 10.0.2

10.1 Periodic knots

As a generalization of the symmetry shown in figure 10.0.1a, we have the concept of periodic knots.

Definition 10.1.1 A knot $K \subset \mathbf{R}^3$ is a *periodic knot* of period n if there is a periodic map f of $(\mathbf{R}^3, K)$ which satisfies the following conditions:

(1) f is $2\pi/n$-rotation about a line F in $\mathbf{R}^3$.
(2) $F \cap K = \emptyset$.

Let $p : \mathbf{R}^3 \to \mathbf{R}^3/f$ be the projection. Put $F_* = p(F)$ and $K_* = p(K)$. Then F_* and K_* are 1-dimensional proper submanifolds in $\mathbf{R}^3/f \cong \mathbf{R}^3$. We define the *linking number*, $\text{Link}(F_*, K_*)$, of F_* and K_* to be the linking number of the knots $O_* = F_* \cup \{\infty\}$ and K_* in the 3-sphere $S^3 = \mathbf{R}^3 \cup \{\infty\}$.

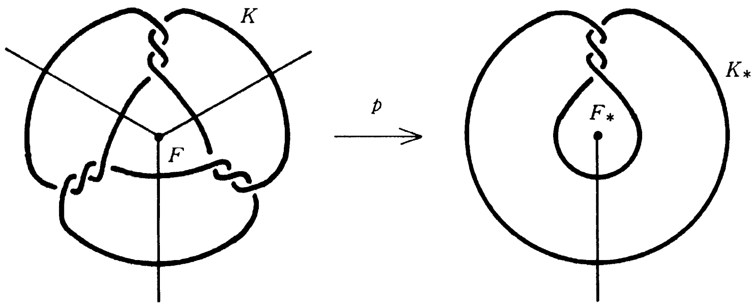


Fig. 10.1.1

Exercise 10.1.2 Let $d = |\text{Link}(F_*, K_*)|$. Then show that d and the period n are relatively prime.

Theorem 10.1.3 *If a knot K has a prime power period $n = p^r$ where p is prime and $r \geq 1$, then we have the following congruence:*

$$\Delta(K;t) \equiv \rho_d(t)^{n-1}\Delta(K_*;t)^n \pmod{p}.$$

Here $\rho_d(t) = 1 + t + \cdots + t^{d-1}$ and $\equiv \pmod{p}$ denotes the congruence modulo p up to multiplication by $\pm t^i$.

Proof. First, we show that the above result holds for the torus knot $T(n,d)$ of type (n,d), which is the "simplest" periodic knot of period n. As shown in figure 10.1.2, we see that the quotient knot $K_* = T(n,d)_*$ is a trivial knot and $\text{Link}(F_*, K_*) = d$. Thus, we obtain the desired result from the following congruence:

$$\begin{aligned}\Delta(K(n,d);t) &= \{(t^{nd}-1)(t-1)\}/\{(t^n-1)(t^d-1)\} \\ &= \rho_d(t^n)/\rho_d(t) \equiv (\rho_d(t))^{n-1} \pmod{p}.\end{aligned}$$

To prove the theorem for general periodic knots, we use the Conway polynomial. Let $\nabla_{n,d}(z)$ be the Conway polynomial of $T(n,d)$. Then the desired formula is equivalent to the following:

$$\nabla(K;z) \equiv \nabla_{n,d}(z)^{n-1}\nabla(K_*;z)^n \pmod{p}.$$

We may assume that f is the $2\pi/n$-rotation around the z-axis of $\mathbf{R}^3$. Let K also denote the diagram of the knot K obtained by projection to the (x,y)-plane, and let K_* also denote the diagram of K_* obtained as the quotient of the diagram K

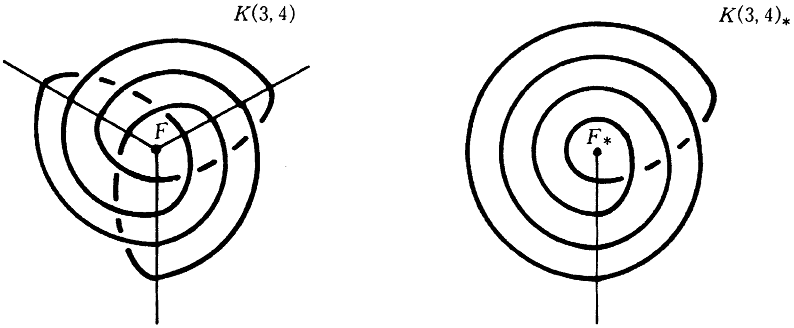


Fig. 10.1.2

(see figure 10.1.1). Let c be a crossing of the diagram K and c_* be the corresponding crossing of the diagram K_*. Let $K_+(\mathbf{Z}_n c)$, $K_-(\mathbf{Z}_n c)$ and $K_0(\mathbf{Z}_n c)$ be the f-invariant diagrams obtained from the diagram K by replacing the crossings $\{f^i(c) \mid 0 \leq i \leq n-1\}$ by $\times$, $\times$, $)($, respectively. Then we obtain the following:

Lemma 10.1.4

$$\nabla(K_+(\mathbf{Z}_n c); z) \equiv \nabla(K_-(\mathbf{Z}_n c); z) + z^n \nabla(K_0(\mathbf{Z}_n c); z) \pmod{p}.$$

Proof of Lemma 10.1.4. Construct the binary resolution tree T of $K_+(\mathbf{Z}_n c)$ obtained from the skein triples on the crossings $c, f(c), \ldots, f^{n-1}(c)$ (see figure 10.1.3). Then this expresses $\nabla(K_+(\mathbf{Z}_n c); z)$ as the sum of 2^n terms corresponding to the branches of T. The periodic map f induces a $\mathbf{Z}_n$-action on the set of branches of T. This action fixes the branches corresponding to $K_-(\mathbf{Z}_n c)$ and $K_0(\mathbf{Z}_n c)$, which contribute to $\nabla(K_+(\mathbf{Z}_n c); z)$ by $\nabla(K_-(\mathbf{Z}_n c); z)$ and $z^n \nabla(K_0(\mathbf{Z}_n c); z)$ respectively. Each of the remaining $\mathbf{Z}_n$-orbits consists of p^s branches for some s $(1 \leq s \leq r)$, all branches of which give the same contribution to $\nabla(K_+(\mathbf{Z}_n c); z)$. Hence the total contribution of each of the remaining $\mathbf{Z}_n$-orbits is 0 modulo p. Lemma 10.1.4 follows. □

Proof of Theorem 10.1.3 (continued). Construct a binary resolution tree of (a diagram of) the quotient knot K_* so that each end is in one of the following two forms (see figure 10.1.4):

(1) A diagram which is separable by the boundary of a regular neighborhood of F_*.
(2) $K(d,n)_*$, i.e., $(d,1)$-cable of a meridian loop of F_* for some integer d.

Let l be the number of the trivalent vertices in the resolution tree above. We prove the theorem by induction on l. To perform this induction, we extend our consideration to periodic links and show that the conclusion of the theorem holds for periodic links. Suppose $l = 0$. Then K_* is of the form (1) or (2) and hence K is either a split link or a torus knot, so we obtain the desired congruence. Next,

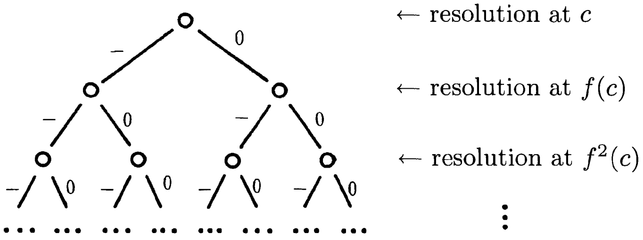


Fig. 10.1.3

we consider the general case. Let c_* be the crossing of K_* which appears in the first step of the resolution tree of K_*. Let c be a crossing of K which projects to c_*. Without loss of generality, we may assume that c is a positive crossing. Then we have $K = K_+(\mathbf{Z}_n c)$ and for the periodic links $K_-(\mathbf{Z}_n c)$ and $K_0(\mathbf{Z}_n c)$, the congruence holds by the inductive hypothesis. Hence, by Lemma 10.1.4, we have the following congruence modulo p:

$$\begin{aligned}\nabla(K; z) &\equiv \nabla(K_-(\mathbf{Z}_n c); z) + z^n \nabla(K_0(\mathbf{Z}_n c); z) \\ &\equiv \nabla_{n,d}(z)^{n-1}\{\nabla(K_{*-}; z) + z\nabla(K_{*0}; z)\}^n \\ &\equiv \nabla_{n,d}(z)^{n-1}\nabla(K_*; z)^n,\end{aligned}$$

where K_{*-} and K_{*0} denote the diagrams such that (K_*, K_{*-}, K_{*0}) is the skein triple (cf. 8.2) at the crossing c_*. This completes the proof of Theorem 10.1.3. □

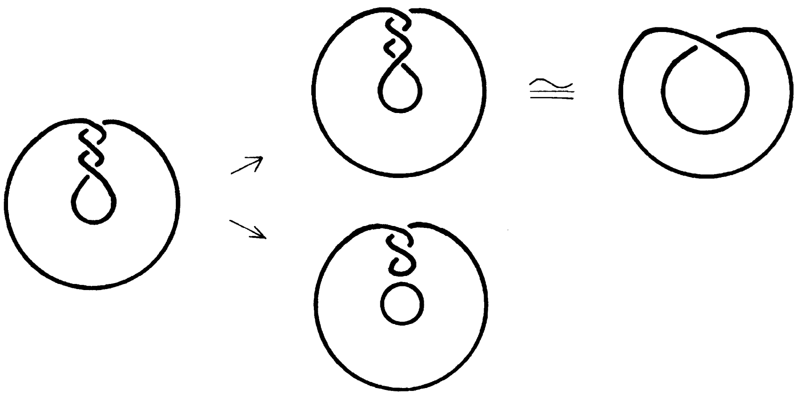


Fig. 10.1.4

Exercise 10.1.5 Show that 2 and 3 are the only periods of the trefoil knot and that 8_{17} does not have any periods.

Exercise 10.1.6 For a knot K as in Theorem 10.1.3, prove the following results by methods similar to the proof of Theorem 10.1.3:

(1) $V(K;t) \equiv V(K_*;t)^{n-1} \pmod{p, \xi_p(t)}$, where $\xi_p(t) = (-t^{1/2} - t^{-1/2})^{p-1} - 1$ (see [Murasugi 1988]).
(2) $V(K;t) \equiv V(K;t^{-1}) \pmod{p, t^p - 1}$ (see [Traczyk 1990], [Yokota 1991]).
(3) Find congruences on the skein and Kauffman polynomials similar to those in (1) and (2) (see [Przytycki 1989], [Yokota 1991', 1993]).

Originally, Theorem 10.1.3 was obtained as a corollary of the following explicit formula for the Alexander polynomials in [Murasugi 1971]:

Theorem 10.1.7 $\Delta(K;t) = \Delta(K_*;t)\Pi_{i=0}^{n-1}\Delta(O_* \cup K_*;\omega^i,t)$, *where* ω *is a primitive* n*-th root of* 1.

For the proof, see also [Burde-Zieschang 1985] and [Hillman 1986]. In [Davis-Livingston 1991], the realization problem of the above formula is investigated.

Exercise 10.1.8 Prove Theorem 10.1.3 by using Theorem 10.1.7.

Exercise 10.1.9 Show that the only periods of the torus knot of type (p,q) are the divisors of p and q.

Finally, we note that the definition of a periodic knot is equivalent to the following.

Definition 10.1.10 A knot $K \subset S^3$ is a periodic knot of period n if there is a periodic map f of (S^3, K) of period n such that $\mathrm{Fix}(f) \cong S^1$ and $\mathrm{Fix}(f) \cap K = \emptyset$.

The fact that the two definitions of periodic knots are equivalent is implied by the positive solution of the Smith Conjecture, which was obtained by a synthesis of deep theories involving the geometry and topology of 3-manifolds (cf. [Morgan-Bass 1984]).

Smith Conjecture 10.1.11 For any orientation preserving periodic map f of S^3 with $\mathrm{Fix}(f) \neq \emptyset$, the fixed point set $\mathrm{Fix}(f)$ is a trivial knot in S^3.

Remark f must be a smooth (or PL) periodic map. Otherwise, there is a counterexample (cf. [Bing 1964]).

10.2 Freely periodic knots

In the previous section, we treated a knot which is invariant under a periodic map of S^3 with non-empty fixed point set. In this section, we study a knot which is invariant under a free cyclic action of S^3.

Definition 10.2.1 A knot $K \subset S^3$ is a *freely periodic knot* with free period n if there is a periodic map f of (S^3, K) of period n such that $\mathrm{Fix}(f^i) = \emptyset$ $(1 \leq i \leq n-1)$.

Exercise 10.2.2 (1) In the definition above, show that f preserves the orientations of S^3 and K.
(2) If (S^3, K) admits a periodic map ($\neq$ id) which preserves the orientations of S^3 and K, then show that K has a period or a free period.
(3) For a torus knot of type (p, q), show that any natural number coprime to both p and q is a free period of the knot.
(4) Let K_* be a knot in a lens space $L(p, q)$ representing a generator of $H_1(L(p, q))$. Then show that the lift of K_* to the universal covering space S^3 over $L(p, q)$ is a freely periodic knot with free period p.

We obtain the following result similar to Theorem 10.1.3 by considering the quotient knot K/f in the quotient manifold S^3/f (cf. [Hartley 1981]):

Theorem 10.2.3 *If K has free period n, then there is an integral Laurent polynomial $\Delta_*(t)$ in t which has the following properties:*

(1) $\Delta(K; t^n) \doteq \prod_{i=0}^{n-1} \Delta_*(\omega^i t)$, *where ω is a primitive n-th root of* 1.
(2) $\Delta_*(t) \doteq \Delta_*(t^{-1}), \Delta_*(1) = \pm 1$.

To prove this theorem, let $p : S^3 \to S^3/f$ be the projection, and put $S_* = S^3/f$, $K_* = p(K)$ and $E_* = S_* - \mathrm{int}N(K_*)$.

Lemma 10.2.4 $H_1(E_*) \cong \mathbf{Z}$.

Proof. Since $p : S^3 \to S_*$ is a $\mathbf{Z}_n$-covering, we see that $H_1(S_*) \cong \pi_1(S_*) \cong \mathbf{Z}_n$ and that it is generated by the homology class of K_*. On the other hand,

$$\begin{aligned} H_1(E_*) &\cong H^2(E_*, \partial E_*) \quad \text{(by Poincaré duality)} \\ &\cong H^2(S_*, K_*) \quad \text{(by the excision isomorphism)} \\ &\cong \mathrm{Tor} H_1(S_*, K_*) \oplus \mathrm{Hom}(H_2(S_*, K_*), \mathbf{Z}) \\ &\qquad \text{(by the universal coefficient theorem).} \end{aligned}$$

Consider the following part of the homology exact sequence of the pair (S_*, K_*):

$$H_2(S_*) \to H_2(S_*, K_*) \to H_1(K_*) \to H_1(S^3) \to H_1(S_*, K_*) \to 0.$$

Since $H_2(S_*) = H_1(S^3) = 0$, we have $H_2(S_*, K_*) \cong \mathbf{Z}$ and $H_1(S_*, K_*) = 0$. These imply the desired result. □

Proof of Theorem 10.2.3. Let E_∞ and $E_{*\infty}$ be the infinite cyclic connected covering spaces of $E = S^3 - \mathrm{int}N(K)$ and E_*, respectively. Let τ and t be the generators of the covering transformation groups of E_∞ and $E_{*\infty}$, respectively. Since $p\,|_E \colon E \to E_*$ is a $\mathbf{Z}_n$-covering, we obtain natural identifications $E_\infty = E_{*\infty}$ and $\tau = t^n$. The $\mathbf{Z}\langle t \rangle$-module $H_1(E_{*\infty})$ has a square presentation matrix and the 0-th characteristic polynomial $\Delta_*(t)$ satisfies (2) of the theorem. In fact, this follows from 5.4.1 since E_* is the exterior of a knot in a homology 3-sphere whose Alexander polynomial is $\Delta_*(t)$ (cf. Supplementary notes to Chapter 5). On the other hand, the 0-th characteristic polynomial of the $\mathbf{Z}\langle \tau \rangle$-module $H_1(E_\infty)$ is equal to the Alexander polynomial $\Delta(K; \tau)$. Since $\Delta(K; 1) \neq 0$, we obtain the desired result from the following lemma. □

Lemma 10.2.5 *Let H be a $\mathbf{Z}\langle\tau\rangle$-module and $\Delta_*(t)$ the 0-th characteristic polynomial. Set $\tau = t^n$. Let $\Delta(\tau)$ be the 0-th characteristic polynomial of the $\mathbf{Z}\langle\tau\rangle$-module H. Suppose $\Delta(1) \neq 0$. Then we have $\Delta(t^n) \doteq \prod_{i=0}^{n-1} \Delta_*(\omega^i t)$, where ω is a primitive n-th root of 1.*

Proof. Note that $\Delta_*(t) \neq 0$. Put $d = \deg\Delta_*(t)$, then $H \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}^d$. Let A be a square matrix of size d over $\mathbf{Q}$ which represents the action of t on $H \otimes_{\mathbf{Z}} \mathbf{Q}$. Then the $\mathbf{Q}\langle t\rangle$-module $H \otimes_{\mathbf{Z}} \mathbf{Q}$ is represented by the matrix $tE - A$ and the $\mathbf{Q}\langle\tau\rangle$-module $H \otimes_{\mathbf{Z}} \mathbf{Q}$ is represented by $\tau E - A^n$. Hence, up to multiplication by an element of $\mathbf{Q} - \{0\}$, we have the following identity:

$$\Delta(t^n) \equiv \det(t^n E - A^n) \equiv \det(\textstyle\prod_{i=0}^{n-1} \omega^i tE - A) \equiv \prod_{i=0}^{n-1} \Delta_*(\omega^i t).$$

On the other hand, by Lemma 7.2.8,

$$|\Delta(1)| = |H/(\tau - 1)H| = |H/(t^n - 1)H| = |\textstyle\prod_{i=0}^{n-1} \Delta_*(\omega^i)| \neq 0.$$

The result follows. □

Exercise 10.2.6 Show that the roots of $\Delta(K;t)$ are the n-th powers of the roots of $\Delta_*(t)$. Using this fact, show that the only periods of the (p,q)-torus knot are the positive integers relatively prime to pq.

Exercise 10.2.7 Show that 8_{17} does not have any free period.

10.3 Invertible knots

In the two previous sections, we considered a knot K invariant under a periodic map which preserves the orientations of S^3 and K. In this section, we treat a knot K as shown in figure 10.0.1b, which is invariant under a periodic map which preserves the orientation of S^3 but reverses the orientation of K.

Exercise 10.3.1 Using the Smith Conjecture, show that such a periodic map is an involution (i.e., a map of period 2) unless K is a trivial knot.

Definition 10.3.2 A knot K in S^3 is *strongly invertible* if there is an involution of (S^3, K) which preserves the orientation of S^3 and reverses the orientation of K.

If K is strongly invertible, then K is invertible (cf. 0.3), but not vice versa (see [Hartley 1980'], [Whitten 1981]). However, for hyperbolic knots (cf. 3.2.11), we have the following:

Proposition 10.3.3 *An invertible knot is strongly invertible if it is a hyperbolic knot.*

Contrary to the results of the previous sections, it is known that there is no restriction on the Alexander polynomials of strongly invertible knots (cf. [Sakai 1983']).

Theorem 10.3.4 *Any Alexander polynomial of a knot is realized as the Alexander polynomial of a strongly invertible knot.*

Although the existence of non-invertible knots had been known by [Trotter 1964], it was a difficult problem to determine whether a given knot is invertible or not, as suggested by the above theorem. For example, although 8_{17} had been known as a candidate for the non-invertible knot of minimal crossing number since the early 60's (cf. [Fox 1962"]), it wasn't until the end of the 70's that the non-invertibility of 8_{17} was proved (see 10.5). Since then [Hartley 1983'] found an effective computer-aided method to solve the invertibility problem of the prime knots with up to 10 crossings. A further development will be discussed in 10.5. At any rate, we note here that the invertibility problem still remains to be solved by a tractable algebraic method (in particular, for non-amphicheiral knots).

10.4 Amphicheiral knots

In the previous sections, we treated knots which are invariant under orientation-preserving periodic maps of S^3. In this section, we study amphicheiral knots, i.e., knots which are preserved by an orientation-reversing auto-homeomorphism of S^3, as shown in figure 10.0.1c-e (see 0.3).

Definition 10.4.1 A knot $K \subset S^3$ is *strongly* $(+)$*amphicheiral* or *periodically* $(+)$*amphicheiral*, if there is an involution or a periodic map respectively of (S^3, K) which reverses the orientation of S^3 and preserves the orientation of K. Similarly, K is *strongly* $(-)$*amphicheiral* or *periodically* $(-)$*amphicheiral*, if there is an involution or a periodic map respectively of (S^3, K) which reverses the orientations of both S^3 and K.

Exercise 10.4.2 Prove the following:

(1) The figure eight knot and 8_{17} is strongly $(-)$amphicheiral.
(2) For each knot K and its mirror image K^*, the connected sum $K\sharp K^*$ is strongly $(+)$amphicheiral and the connected sum $K\sharp -K^*$ is strongly $(-)$amphicheiral.
(3) The figure eight knot is periodically $(+)$amphicheiral with period 4 [*Hint*: figure 4.2.2].
(4) A knot is strongly $(-)$amphicheiral if it is periodically $(-)$amphicheiral.

It is known that there is an amphicheiral knot which is not periodically amphicheiral (cf. [Hartley 1980']). However, for hyperbolic knots, we have the following (see 10.5):

Proposition 10.4.3 *An amphicheiral knot is periodically amphicheiral if it is a hyperbolic knot.*

The following result is easily obtained from the definitions (see 5.3.4 for the knot signature and 8.4.1 for the skein polynomial):

Proposition 10.4.4 *If a knot K is amphicheiral, then we have the following:*

(1) $\sigma(K) = 0$.
(2) $V(K;t) = V(K;t^{-1})$.
$P(K;a,z) = P(K;a^{-1},z)$.
$F(K;a,z) = F(K;a^{-1},z)$.

Exercise 10.4.5 Prove that the Kinoshita-Terasaka knot and Conway's knot are not amphicheiral.

For the Conway polynomial $\nabla(K;z)$, it also follows that $\nabla(K;z) = \nabla(K;-z)$ if K is amphicheiral; however, this identity holds for any (not necessarily amphicheiral) knot. However, for the Alexander polynomial, which is equivalent to the Conway polynomial, we have the following result (see [Hartley-Kawauchi 1979]):

Theorem 10.4.6 (1) *If K is strongly $(-)$amphicheiral, then there is an integral Laurent polynomial $\Delta_*(t)$ with $\Delta_*(t^{-1}) \doteq \Delta_*(-t)$ and $\Delta_*(1) = \pm 1$ such that $\Delta(K;t^2) \doteq \Delta_*(t)\Delta_*(-t)$.*
(2) *If K is strongly $(+)$amphicheiral, then there is an integral Laurent polynomial $\Delta_*(t)$ with $\Delta_*(t^{-1}) \doteq \Delta_*(t)$ and $\Delta_*(1) = \pm 1$ such that $\Delta(K;t) \doteq \Delta_*(t)^2$.*

Outline of the proof. For (1), let f be an involution of (S^3, K) which realizes the strong $(-)$amphicheirality. By Smith theory (see [Bredon 1972]), $\mathrm{Fix}(f)$ is either S^0 or S^2.
Case 1: $\mathrm{Fix}(f) = S^2$. Then we see $K = K_* \sharp - (K_*)^*$ for some knot K_* and hence $\Delta(K;t) = \Delta(K_*, t)^2$. By putting $\Delta_*(t) = \Delta(K_*;t^2)$, we obtain the desired result.
Case 2: $\mathrm{Fix}(f) = S^0$. Then $\mathrm{Fix}(f) \subset K$ and hence $f\,|_{E(K)}$ is a free involution. Putting $E_* = E(K)/f$, we obtain the following (cf. Lemma 10.2.6): $H_1(E_*) \cong \mathbf{Z}$, $H_1(E_*, \partial E_*) = 0$, and ∂E_* is a Klein bottle. This means that E_* is a non-orientable "homology circle" and the "Alexander polynomial" $\Delta_*(t)$ has $\Delta_*(t^{-1}) \doteq \Delta_*(-t)$ and $\Delta_*(1) = \pm 1$ (cf. [Kawauchi 1975]). Further, by Lemma 7.2.8, $\Delta_*(t)$ has $\Delta(K;t^2) \doteq \Delta_*(t)\Delta_*(-t)$. For (2), let f be an involution of (S^3, K) which realizes the strong $(+)$amphicheirality. Since $f\,|_{E(K)}$ has a fixed point, it lifts to an involution f_∞ of the infinite cyclic covering space E_∞ of E. Let $L : H_1(E_\infty;\mathbf{Q}) \times H_1(E_\infty;\mathbf{Q}) \to \mathbf{Q}(t)/\mathbf{Q}\langle t\rangle$ be the Blanchfield pairing (see Appendix D). Then we have $L(f_{\infty *}(x), f_{\infty *}(y)) = -L(x,y)$. From this fact, we can see that the $\mathbf{Q}\langle t\rangle$-module $H_1(E_\infty;\mathbf{Q})$ is a direct double $B \oplus B$. Then we obtain the desired result. □

Exercise 10.4.7 (1) Show that 8_{17} actually satisfies the condition in Theorem 10.4.6(1).
(2) Show that 8_{17} is not strongly $(+)$amphicheiral.

Theorem 10.4.6 originated from a conjecture in [Kawauchi 1975'] for general $(-)$-amphicheiral knots (cf. [Buskirk 1983]). The affirmative solution of this conjecture and its $(+)$amphicheiral knot version have been given by R. Hartley [Hartley 1980"], who reduced the general case to the case of Theorem 10.4.6 using the torus decomposition theorem (C.6.3), Thurston's hyperbolization theorem (C.7.2), and Mostow's rigidity theorem (C.7.3).

10.5 Symmetries of a hyperbolic knot

The definition of a hyperbolic link is given in 3.2.11. The knot case can be simply stated as follows:

Theorem 10.5.1 *A knot $K \subset S^3$ is hyperbolic if and only if K is non-trivial and is neither a torus knot nor a satellite of a non-trivial knot.*

Proof. By 3.2.11, K is hyperbolic if and only if the exterior $E(K)$ has the following properties:

(1) $E(K)$ does not admit an essential torus.
(2) $E(K)$ is not a Seifert manifold.

(1) is equivalent to the condition that K is not a satellite knot of a non-trivial knot and (2) is equivalent to the condition that K is non-trivial and is not a torus knot (cf. 6.3.6). □

Corollary 10.5.2 *If a prime knot K bridge index ≤ 3 and is not a torus knot, then K is hyperbolic.*

Proof. Suppose that K is not hyperbolic. Then K is a satellite of a nontrivial knot K_0. Let V be a regular neighborhood of K_0 containing K. Let s be the minimal number of transverse intersections of K and a meridian disk of V. Then by [Schubert 1954], we have $b(K) \geq sb(K_0)$, where b denotes bridge index. Since $b(K) \leq 3$ and $b(K_0) \leq 2$, we have $s = 1$. This contradicts the assumption that K is prime. □

Since the bridge indices of the prime knots up to 10 crossings are at most 3, all such knots except the torus knots 3_1, 5_1, 7_1, 8_{19}, 9_1, 10_{124}, are hyperbolic. Let $\Psi(E(K))$ be the mapping class group of the exterior $E(K)$ of a knot K and $\mathrm{Out}(\pi(K))$ the outer automorphism group of the knot group $\pi(K)$ (see C.4). For a hyperbolic knot K, let $\mathrm{Isom}E(K)$ be the isometry group of the complete hyperbolic manifold $\mathrm{int}E(K)$ of finite volume (see Remark following 3.2.11). Then we have the following result:

Theorem 10.5.3 *For any hyperbolic knot K, there are isomorphisms:*

$$\Psi(E(K)) \cong \mathrm{Out}(\pi(K)) \cong \mathrm{Isom}E(K).$$

Further, this group is isomorphic to a finite cyclic group $\mathbf{Z}_n$ or a dihedral group D_n.

Proof. The first half follows from Waldhausen's theorem C.4.1 and Mostow's rigidity theorem C.7.3. To prove the latter half, we observe the following:

Lemma 10.5.4
The action of $\mathrm{Isom}E(K)$ *on* $\mathrm{int}E(K)$ *extends to an action of* (S^3, K).

Proof. Since $E(K)$ can be identified with the exterior of an open cusp in the hyperbolic manifold $\mathrm{int}E(K)$ (cf. [Thurston 1982]), the isometry group $\mathrm{Isom}E(K)$ acts on $E(K)$. By [Gordon-Luecke 1989] (cf. 6.1.12), the action of $\mathrm{Isom}E(K)$ on $\partial E(K) = \partial N(K)$ extends to an action on $(N(K), K)$. □

Proof of Theorem 10.5.3(continued). By Smith theory (cf. [Bredon 1972], the fixed point set of an orientation-reversing periodic map is $\emptyset$, S^0 or S^2. Hence by this remark and the Smith Conjecture (10.1.11) the restriction of the action of Isom$E(K)$ on (S^3, K) to K is faithful (in other words, only the identity map acts trivially) unless K is a trivial knot. Since K is hyperbolic, the action of Isom$E(K)$ on K is faithful. Since a finite group which can act on $K(\cong S^1)$ faithfully is a cyclic group or a dihedral group, the theorem is proved. □

Exercise 10.5.5 Prove Propositions 10.3.3 and 10.4.3 using the above argument.

As an application of the results stated so far, we prove the non-invertibility of 8_{17} following [Kawauchi 1979]. First, recall that 8_{17} is hyperbolic (Corollary 10.5.2) and $(-)$amphicheiral (Exercise 10.4.2). Suppose that 8_{17} is invertible. Then it is $(+)$amphicheiral and hence periodically $(+)$amphicheiral (Proposition 10.4.3). Let f be an orientation-reversing periodic map of S^3 realizing the periodic $(+)$amphicheirality of K. We may assume that the period of f is 2^r $(r \geq 1)$ by considering an odd power of f, if necessary. If $r \geq 2$, the map $\varphi = f^n$ with $n = 2^{r-1}$ is an involution of (S^3, K) which preserves the orientations of S^3 and K. Hence 8_{17} must have the period 2 or the free period 2. However, this is impossible by Problems 10.1.5 and 10.2.7. Hence we have $r = 1$ and therefore 8_{17} is strongly $(+)$amphicheiral. However, this is impossible by Exercise 10.4.7. Hence, we can conclude that 8_{17} is non-invertible.

10.6 The symmetry group

Let $PL(S^3, K)$ be the group of the PL automorphisms of (S^3, K). Let $PL_0(S^3, K)$ be the normal subgroup of $PL(S^3, K)$ consisting of those elements which are pairwise ambient isotopic to the identity.

Definition 10.6.1 The *symmetry group* of a knot K, denoteed by Sym(S^3, K), is the group $PL(S^3, K)/PL_0(S^3, K)$.

This symmetry group is intimately related to the mapping class group $\Psi(E(K))$, the peripheral-structure-preserving outer automorphism group $\mathrm{Out}(\pi(K), \partial)$, and the outer automorphism group $\mathrm{Out}(\pi(K))$ (cf. C.4).

Theorem 10.6.2

(1) *For each nontrivial knot K, we have the following natural isomorphisms:*

$$\mathrm{Sym}(S^3, K) \cong \Psi(E(K)) \cong \mathrm{Out}(\pi(K), \partial).$$

(2) *For each (non-trivial) prime knot K, we have the following natural isomorphisms:*

$$\mathrm{Sym}(S^3, K) \cong \Psi(E(K)) \cong \mathrm{Out}(\pi(K)).$$

Proof. The isomorphism on the right in (1) is a direct consequence of Waldhausen's theorem C.4.3. Noting that there is a deformation retraction from $S^3 - K$ to $E(K)$, we see that there is a natural homomorphism $\varphi : \mathrm{Sym}(S^3, K) \to \mathrm{Out}(\pi(K), \partial)$. Let f be an element of $PL(S^3, K)$ such that $[f] \in \mathrm{Ker}\varphi$. By the uniqueness of regular neighborhoods, f is pairwise ambient isotopic to an element $f' \in PL(S^3, K)$ such that $f'(E(K)) = E(K)$. By the right-hand isomorphism, $f' |_{E(K)}$ is ambient isotopic to the identity. From this fact, we see that f' is pairwise ambient isotopic to the identity, and hence $[f] = 1$. Hence φ is injective. By [Gordon-Luecke 1989], any auto-homeomorphism f of $E(K)$ extends to an element of $PL(S^3, K)$ (cf. 6.1.12). Hence the surjectivity of φ is obtained and we have (1). (2) follows from the argument of [Whitten 1987] based on the cyclic surgery theorem [Culler et al. 1987]. (See 6.1.12 and [Tsau 1988]). □

By Theorem 10.5.3, the symmetry groups of hyperbolic knots are finite. However, this does not hold for non-hyperbolic knots. In fact, we have the following (cf. [Sakuma 1989]):

Proposition 10.6.3 *A knot K has a finite symmetry group if and only if K is a hyperbolic knot, a torus knot or a cable of a torus knot.*

Exercise 10.6.4 ([Schreier 1924]) Let $K = T(p, q)$ be a (non-trivial) torus knot of type (p, q) and $G = \pi(K)$. Show that

$$\mathrm{Sym}(S^3, K) \cong \mathrm{Out}(G) \cong \mathbf{Z}_2$$

by the method indicated in the following:
(1) In the group presentation $G = \langle a, b \mid a^p = b^q \rangle$, the center $\xi(G)$ of G is the infinite cyclic group $\langle a^p \rangle = \langle b^q \rangle$.
(2) Any automorphism of $G/\xi(G) \cong \langle a, b \mid a^p = b^q = 1 \rangle$ is of the following form: $a \to ca^r c^{-1}$, $b \to cb^s c^{-1}$ for some $c \in G/\xi(G)$ where r and s are integers coprime to p and q, respectively.
(3) Any automorphism of G is of the following form: $a \to ca^\varepsilon c^{-1}$, $b \to cb^\varepsilon c^{-1}$ for some $c \in G$, where $\varepsilon = \pm 1$.

Now we consider a relationship between the symmetry groups and the "rigid" symmetries discussed in 10.1 $\sim$ 10.4. Suppose that a finite group G acts faithfully on (S^3, K) (i.e., $G \subset PL(S^3, K)$). We consider the restriction of the natural homomorphism $\pi : PL(S^3, K) \to \mathrm{Sym}(S^3, K)$ to G.

Example 10.6.5. Let $K = T(p, q)$ be a non-trivial torus knot.

(1) Cyclic group actions on (S^3, K) realizing periods or free periods of K are embedded in a S^1-action on (S^3, K). Hence the images of these groups in $\mathrm{Sym}(S^3, K)$ are trivial.
(2) An involution on (S^3, K) which realizes the strong invertibility of K generates $\mathrm{Sym}(S^3, K) \cong \mathbf{Z}_2$.

Since the torus knot admits a circle action, there are finite group actions which are not detected by the symmetry groups. However, except for the torus knots, the symmetry group of a knot K basically controls the finite group actions on (S^3, K). In fact, the following result is proved using Borel's theorem(cf. [Conner-Raymond 1972]) and the results of [Takeuchi 1991] and [Zimmermann 1982]:

Theorem 10.6.6 *Let K be a non-trivial knot which is not a torus knot.*

(1) *The image of a finite subgroup $G \subset PL(S^3, K)$ in* $\mathrm{Sym}(S^3, K)$ *is isomorphic to G.*

(2) *Any finite subgroup G of* $\mathrm{Sym}(S^3, K)$ *is realized as a finite group action on (S^3, K), i.e., there is a monomorphism $G \to PL(S^3, K)$ which makes the following diagram commute:*

$$\begin{array}{ccc} PL(S^3, K) & \xrightarrow{\pi} & Sym(S^3, K) \\ \| & & \uparrow \cup \\ PL(S^3, K) & \longleftarrow & G. \end{array}$$

(3) *Let G be a finite subgroup of the group $Sym_+(S^3, K) \subset Sym(S^3, K)$ generated by the elements of $PL(S^3, K)$ that preserve the orientation of S^3. Then the realization of G into $PL(S^3, K)$ assured by (2) is unique modulo conjugation by elements of $PL_0(S^3, K)$.*

At the end of this section, we present the results of [Boileau-Zimmermann 1989] which relate the symmetry groups to the outer automorphism groups of the "π-orbifold groups".

Definition 10.6.7 The π-*orbifold group* $O(K)$ of K is the quotient group

$$\pi(K)/\langle\langle m^2 \rangle\rangle^{\pi(K)}$$

for a meridian element $m \in \pi(K)$.

The geometric meaning of this group is as follows: Let $M(K)$ be the double covering space over S^3 with branch set K, and τ the non-trivial covering transformation. Then $O(K)$ is isomorphic to the group of lifts of τ and id to the universal covering space of $M(K)$, so that we have the following exact sequence:

$$1 \to \pi_1(M(K)) \to O(K) \to \mathbf{Z}_2 \to 1.$$

$O(K)$ is also called the *fundamental group* of the orbifold $(M(K), \mathrm{Fix}(\tau))/\tau$ (cf. [Thurston *], [Scott 1983]).

Exercise 10.6.8 Show that the π-orbifold group of the 2-bridge knot of type (p, q) is isomorphic to the dihedral group of order $2p$.

A knot K is said to be *sufficiently complicated* if K is prime and $O(K)$ is infinite. This condition is equivalent to the condition that $M(K)$ is aspherical. The following theorem is proved by using Thurston's orbifold uniformization theorem [Thurston **].

Theorem 10.6.9

(1) *Let K and K' be sufficiently complicated knots. Then K and K' are equivalent if and only if $O(K)$ and $O(K')$ are isomorphic.*

(2) *If K is sufficiently complicated, then there is a natural isomorphism*

$$\mathrm{Sym}(S^3, K) \cong \mathrm{Out}(O(K)).$$

Exercise 10.6.10 Show that the conclusion of the above theorem does not hold for knots which are not sufficiently complicated.

10.7 Canonical decompositions and symmetry

In [Bonahon-Siebenmann *], Bonahon and Siebenmann established a method to decompose a knot into simpler pieces in a canonical way. This decomposition consists of two steps. The first step is just the torus decomposition (see C.6.3) of the knot exterior. The second step is a decomposition by "Conway spheres" which is regarded as an equivariant torus decomposition of the double covering space over S^3 branch set a knot. Let L be a link in S^3 which is non-splittable and non-trivial. Then $E(L)$ is a Haken manifold with incompressible boundary. By the torus decomposition theorem (C.6.3), there is a characteristic torus family $\mathcal{T}$ of (possibly empty) essential tori which cut $E(L)$ into *simple pieces* and *Seifert pieces*. It should be noted that each piece is homeomorphic to the exterior of some link $L' \subset S^3$ by the following exercise:

Exercise 10.7.1 Let M be a compact 3-manifold which is embedded in S^3 and whose boundary consists of tori. Then show that there is a link L' in S^3 such that $M \cong E(L')$. [*Hint*: Use the fact that each torus in S^3 bounds a solid torus in S^3.]

By the results stated above, the classification problem and the symmetry problem for all links in S^3 are reduced to those of links whose exteriors are either Seifert or simple. Those links with Seifert exteriors were classified by [Burde-Murasugi 1970] and the mapping class groups of these Seifert manifolds were calculated by [Johannson 1979]. Thus we may concentrate on the study of simple links.

The second step of the canonical decomposition is the decomposition of a simple link. To state the result, we need to introduce some notation. By a (3,1)-*manifold pair*, we mean a pair (M, L) of a compact oriented 3-manifold M and a proper 1-submanifold L of M. A *Conway sphere* in (M, L) is a 2-sphere F in int$M$ or ∂M which meets L transversally in 4 points. A Conway sphere F is said to be *compressible* if there is a disk D in $M - L$ such that $D \cap F = \partial D$ and ∂D does not bound a disk in $F - L$. Otherwise, F is said to be *incompressible.* A Conway sphere F in int$M$ is said to be *∂-parallel* if F splits M into two parts N and N' such that for one of which, say N, we have a homeomorphism $(N, N \cap L) \cong (F, F \cap L) \times [0,1]$. A (3,1)-manifold pair (M, L) is said to be *Conway-simple* if there is no incompressible, non-∂-parallel Conway sphere F in int$M$ for (M, L). A *Montesinos pair* is a (3,1)-manifold pair which is built from the pair in figure

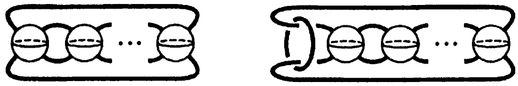


Fig. 10.7.1

10.7.1a or 10.7.1b by plugging some of the holes with rational tangles of finite slopes (see 3.3).

It should be noted that for a Montesinos pair (M, L) the double branched covering space over M with branch set L is a Seifert manifold (see [Montesinos 1975]) whose base space is orientable or non-orientable according to whether the Montesinos pair is obtained from a pair in figure 10.7.1a or 10.7.1b.

Theorem 10.7.2 *Given a simple link $L \subset S^3$, there is a 2-manifold $F \subset S^3$ which is unique up to ambient isotopy of (S^3, L) and has the following properties:*
(1) *The components of F are incompressible Conway spheres, any two of which are not ambient isotopic in (S^3, L).*
(2) *Each component N of the 3-manifold obtained from S^3 by splitting along F gives a (3,1)-manifold pair $(N, L \cap N)$ that is either Conway-simple or a Montesinos pair.*
(3) *When any component is omitted from F, property (2) fails.*

The decomposition given by the above theorem is called the *characteristic decomposition* of a simple link L. The union of the Montesinos pairs in the characteristic splitting of L is called the *algebraic part* of L. We say that a link L is *algebraic* if (S^3, K) consists of only the algebraic part.

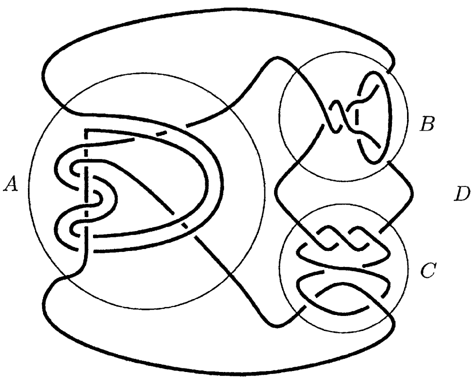


Fig. 10.7.2

Example 10.7.3. (1) The characteristic decomposition of the knot in figure 10.7.2 is given by the two Conway spheres ∂A and ∂B. A is the non-algebraic part while B and $C \cup D$ form the algebraic part. (See Example 10.7.4 and Theorem 10.7.6.)

(2) The characteristic decomposition of the Kinoshita-Terasaka knot and the Conway knot are given in figure 3.8.1. The characteristic splitting of 8_{17} is given in figure 10.7.3 (see [Menasco 1984(p.43)]). Note that all of them are algebraic knots.

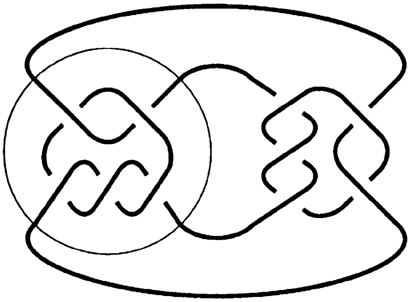
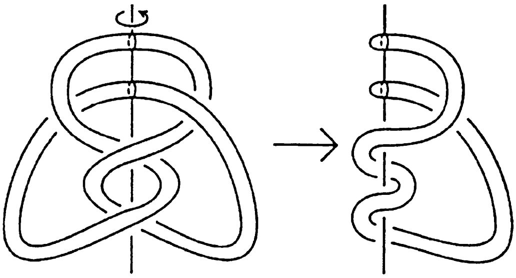


Fig. 10.7.3

The characteristic decomposition provides satisfactory understanding of Montesinos pairs (and hence of algebraic parts). In particular the algebraic knots are completely classified and their symmetry groups are determined (cf. [Boileau-Zimmermann 1987], [Sakuma 1990]). Roughly speaking, it is proved that there are only "canonical" homeomorphisms between two algebraic knots, so it follows that the algebraic knots have only "canonical" symmetries. For example, the symmetry group of 8_{17} is the cyclic group of order 2 generated by an auto-homeomorphism realizing the strong negative amphicheirality while the symmetry group of the Kinosita-Terasaka knot is trivial. Thus we obtain an intuitive proof of the non-invertibility of these two knots.



a b

Fig. 10.7.4

The classification problem and the symmetry problem for an arbitrary link in S^3 are now reduced to those of non-algebraic Conway-simple (3,1)-manifold pairs in S^3. To treat these (3,1)-manifold pairs, we introduce some terminology.

Let (M, L) be a (3,1)-manifold pair in S^3 whose boundary consists of (possibly empty) Conway spheres. (M, L) is said to be *π-hyperbolic* if the interior of the double covering space over M with branch set L admits a complete hyperbolic structure of finite volume such that the covering transformation is an isometry. In other words, (M, L) is π-hyperbolic if M admits the structure of a complete hyperbolic orbifold with singular set L of cone angle π (see [Thurston *], [Scott 1983]).

Example 10.7.4. The (3,1)-manifold pair A in figure 10.7.2 is π-hyperbolic. In fact, it is homeomorphic to the (3,1)-manifold pair in figure 10.7.4b; the double branched covering space is the figure eight knot exterior whose non-trivial covering transformation h is given by a strong inversion as shown in figure 10.7.4a. According to [Thurston *], the (open) figure eight knot complement can be identified with the union of two regular ideal hyperbolic tetrahedra, and hence admits a complete hyperbolic structure for which we can see that h is an isometry which interchanges the two ideal tetrahedra.

Definition 10.7.5 A *Conway graph* is a connected quadrivalent graph Γ in S^2 which is distinct from the graphs in figure 10.7.5 and has the property that every simple loop ℓ in S^2 meeting Γ transversely in 4 points which are not vertex points of Γ bounds a disk D such that $D \cap \Gamma$ is a trivial 2-string tangle in D or is homeomorphic to a cone over the 4-point set $\ell \cap \Gamma$.

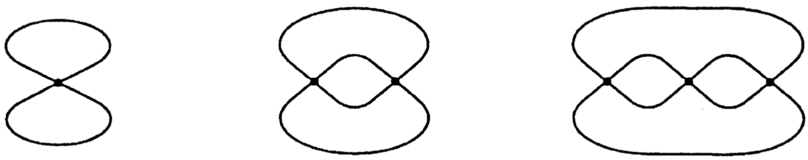
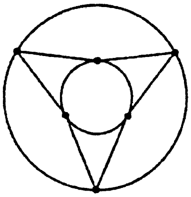


Fig. 10.7.5

Figure 10.7.6 lists the Conway graphs with at most 10 vertices. Let N be a regular neighborhood of the vertices of Γ in S^3 by regarding that $S^2 \subset S^3$. Let $M = S^3 - \mathrm{int}N$. Then Andreev's theorem (see [Thurston *]) implies that the (3,1)-manifold pair $(M, M \cap \Gamma)$ is π-hyperbolic. Thurston's orbifold uniformization theorem [Thurston **] together with the classification of the non-hyperbolic geometric orbifolds due to [Dunber 1988] implies the following theorem:

Theorem 10.7.6 *Let (M, L) be a (3,1)-manifold pair in S^3 whose boundary consists of (possibly empty) incompressible Conway spheres. Then (M, L) admits a π-hyperbolic structure if and only if it has the following properties:*
(1) (M, L) is non-splittable, i.e., the exterior $E(L; M) = M - \mathrm{int}N(L)$ is irreducible.
(2) (M, L) is atoroidal, i.e., the exterior $E(L; M)$ is simple.
(3) (M, L) is Conway-simple.



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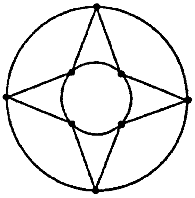

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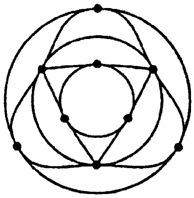

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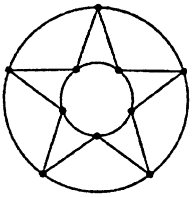

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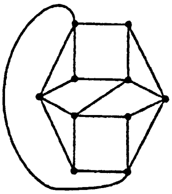

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Fig. 10.7.6

(4) *The exterior $E(L; M)$ is not a Seifert fibered manifold.*
(5) *(M, L) is not a 2-bridge link.*

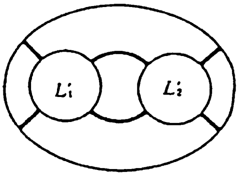


Fig. 10.7.7

It should be noted that if $\partial M \neq \emptyset$, then the theorem above follows from Thurston's hyperbolization theorem for Haken manifolds. Next, we give a quick explanation of Conway's notation for knots in [Conway 1970], which ties together nicely with the theory of Bonahon and Siebenmann. In the following argument, we use the term *tangle* for a diagram of a two-string tangle with four fixed end points in a disk D. Given two tangles L_1 and L_2 in D, we construct a new tangle as shown in figure 10.7.7 with L_1' and L_2' the copies of L_1 and L_2 made smaller, respectively. Note that any rational tangle (cf. 3.3) is obtained from the rational tangle of slope 1 by repeating this construction. An *algebraic tangle* is a tangle obtained from a rational tangle by repeating this construction. Each algebraic tangle is described by a finite sequence of rational numbers and certain symbols which represent the above process. Given a knot diagram Γ, we consider small disks with centers the crossings of Γ. Then these determine the rational tangles of slopes ± 1. By regarding these disks as vertices, we obtain a quadrivalent graph in S^2. If there is a bigon in this plane graph, we amalgamate the corresponding pair of vertices to obtain a new quadrivalent graph in S^2 with fewer vertices. Repeat this process until we get a (quadrivalent) graph in S^2 which does not have any bigons. Such a graph in S^2 is called a *basic polyhedron*. Then the original knot diagram is obtained from the basic polyhedron by substituting some algebraic tangles for the vertices. Conway's notation represents this process. For example, 10_{100} is represented by $6^*3:2:2$ or simply $3:2:2$. See [Conway 1970] for more details. This means that 10_{100} is obtained from the basic polyhedron 6^* (see figure 10.7.6) according to the substitution data $3:2:2$ as shown in figure 10.7.8. A knot with basic polyhedron having only one vertex, denoted by 1^*, is algebraic. A knot with basic polyhedron 6^{**} (whose picture can be drawn from the data of Appendix F) can be seen to be algebraic (cf. [Conway 1970]). The Conway graphs form an important subclass of the basic polyhedra. The following theorem, which is obtained from Andreev's theorem and the hyperbolic Dehn surgery theorem (see [Thurston *]), shows that Conway's notation is not only a convenient method to represent a knot diagram but also has topological meaning in and of itself.

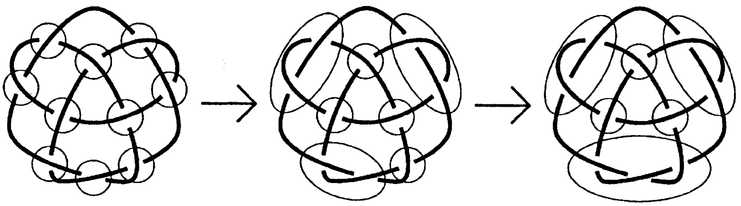


Fig. 10.7.8

Theorem 10.7.7 *Let Γ be a Conway graph. Then for each vertex v of Γ, there is a finite subset $E_v(\Gamma)$ of $\mathbf{Q} \cup \{\infty\}$ with the following properties: consider the set $\mathcal{L}(\Gamma)$ of links which is obtained from Γ by substituting for each vertex v a rational tangle whose slope is not in $E_v(\Gamma)$. Then the only homeomorphisms between links in $\mathcal{L}(\Gamma)$ are the obvious ones. In particular, the symmetry group of a link in $\mathcal{L}(\Gamma)$ is isomorphic to the group of the graph automorphisms of (S^2, Γ) which respect the substitution data.*

In addition, the recent positive solution of the Tait flyping conjecture established by [Menasco-Thistlethwaite 1993] suggests that Conway's notation is a topological invariant for alternating links which are Conway-simple. However, it seems that no practical estimate, except this positive result, is known for the subsets $E_v(\Gamma)$. It would be a challenging problem to find such an estimate.

Supplementary notes to Chapter 10

For each cusped hyperbolic 3-manifold M, there is a canonical way of decomposing M into ideal cells, which is called the *canonical cell decomposition* of M

(see [Epstein-Penner 1988], [Weeks 1993]). By virtue of the Mostow rigidity theorem, two such 3-manifolds are homeomorphic if and only if their canonical cell decompositions have the same combinatorial structure and their mapping class groups are isomorphic to this combinatorial automorphism group. The computer program SnapPea developed by J. Weeks determines the canonical cell decompositions of cusped hyperbolic 3-manifolds and whether two such manifolds are isometric, as well as computes the isometry groups (cf. [Weeks 1993]). For example, the mapping class groups of the exteriors of the hyperbolic knots with up to 10 crossings are determined by using SnapPea (cf. [Adams-Hildebrandt-Weeks 1991], [Henry-Weeks 1992], [Kodama-Sakuma 1992]). Thus, SnapPea is a magic wand for practical knot theorists. However, there remain the following problems (cf. [Sakuma-Weeks 1995']):

(1) Although SnapPea gives us the combinatorial data of the canonical cell decomposition, we cannot *see* it. How can we see and understand it?

(2) SnapPea does not tell us explicit information on a family of infinitely many knots, e.g., the family of 2-bridge knots. How can we know the canonical cell decompositions of their complements simultaneously?

Chapter 11
Local transformations

In this chapter, we discuss several patterns of local transformations on link diagrams, the major theme of which is unknotting operations on knots.

11.1 Unknotting operations

Here, we define several patterns of local transformations on knot diagrams, some of which are shown in figure 11.1.1. The Reidemeister moves I, II and III are also examples of such patterns of local transformations.

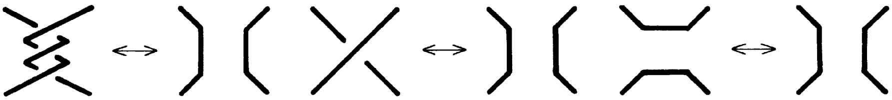


Fig. 11.1.1

Definition 11.1.1 A pattern of local transformation from a knot diagram into a knot diagram is an *unknotting operation* if any knot diagram can be transformed into a trivial knot diagram by a finite sequence of this pattern and Reidemeister moves I, II and III.

The pattern of local transformation shown in figure 11.1.2 is the most well-known unknotting operation (see [Wendt 1937]), which we call here the X-*move*, although it is usually referred to as *the unknotting operation.* The X-move may be considered on unoriented knot diagrams since is has the same effect even if we consider it on oriented knot diagrams as shown in figure 11.1.3.

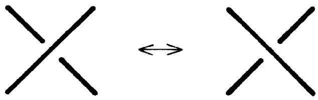


Fig. 11.1.2

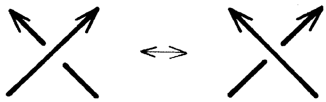


Fig. 11.1.3

Theorem 11.1.2 *The X-move is an unknotting operation.*

Proof. Let K be an oriented knot. We take a diagram of K and choose a base point p on K that is not a crossing. We trace K from p in the direction of the orientation of K and transform the diagram by the X-move so that the first time we reach each crossing we form an over-crossing (see figure 11.1.4). The result is a diagram of a trivial knot. □

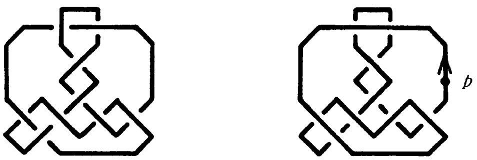


Fig. 11.1.4

Exercise 11.1.3 Prove that the transformation described in the proof above actually produces a diagram of a trivial knot.

Next, we consider the pattern of local transformation of a 3-string tangle (in a knot diagram) shown in figure 11.1.5. This pattern is called the Δ-*move* (see [Murakami-Nakanishi 1989]).

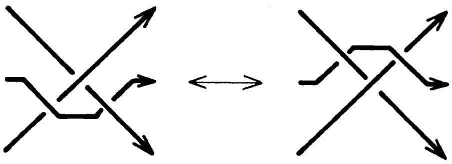


Fig. 11.1.5

Exercise 11.1.4 Show that each of two transformations shown in figure 11.1.6 is obtained by using the pattern of figure 11.1.5 once.

As a consequence of this exercise, we may define the Δ-move to be the pattern of local transformations of unoriented knot diagrams shown in figure 11.1.7.

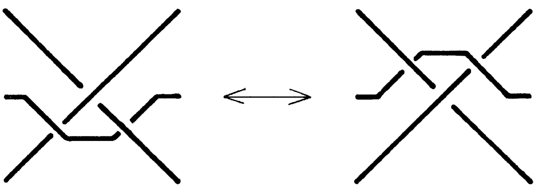
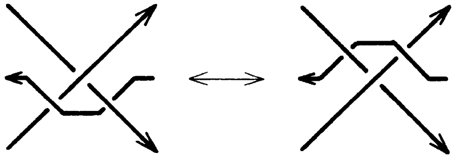
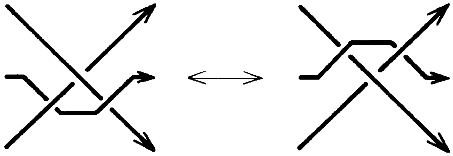


Fig. 11.1.7



Change the orientation of one string



Reverse the upper-lower relations at all crossings

Fig. 11.1.6

Theorem 11.1.5 *The Δ-move is an unknotting operation.*

Proof. By a finite number of Δ-moves, we can arrange the following manouver:

Claim. *A clasp can leap over a hurdle.*

The meaning and the proof are clear from figure 11.1.8. We use this claim to prove that the X-move at any crossing can be accomplished by a sequence of Δ-moves. The proof will be completed by Theorem 11.1.2. Notice that the X-move can remove a clasp as shown in figure 11.1.9. Using the claim repeatedly, we can slide the clasp along the knot until the clasp reaches the root of the clasp as shown in figure 11.1.10. Since we can cancel the twist of the clasp by leaping over a hurdle, we can accomplish the X-move at any crossing. This completes the proof. □

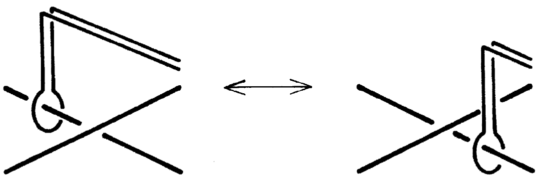


Fig. 11.1.8

Next, we study the pattern of local transformation shown in figure 11.1.11. This pattern is called the *$\sharp$-move* (see [Murakami,H. 1985]).

Theorem 11.1.6 *The $\sharp$-move is an unknotting operation.*

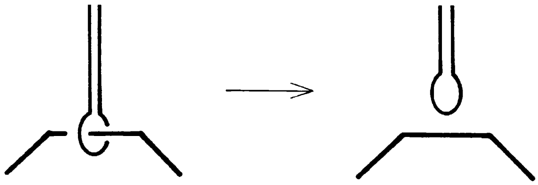


Fig. 11.1.9

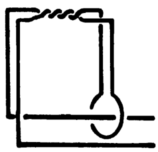


Fig. 11.1.10

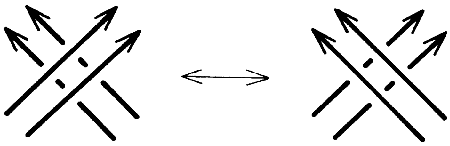


Fig. 11.1.11

Proof. Let F be a non-orientable surface in S^3 with $\partial F = K$. Then F can be represented as a disk with bands shown in figure 11.1.12. Since F is non-orientable, we may assume that these bands are non-orientable and that any two bands cross locally as shown in figure 11.1.13. By $\sharp$-moves, we can eliminate these band-crossings. Further, we can make the band cores unknotted arcs by the $\sharp$-moves which are induced from the X-moves transforming the band cores into unknotted arcs. The resulting bands must have an odd number of half-twists since K is a knot. Since one $\sharp$-move contributes 4 half-twists to each band (see figure 11.1.14), each band can be transformed into one with a positive or negative half-twist by $\sharp$-moves. Since the boundary of a disk with only half-twisted bands attached trivially is a trivial knot, the result follows. □

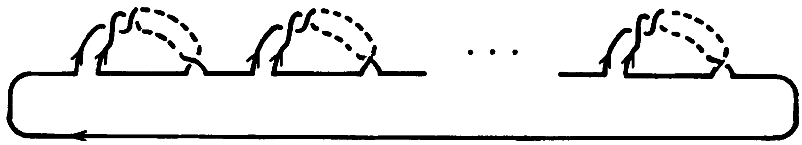


Fig. 11.1.12

For the $\sharp$-move, the orientation shown in figure 11.1.11 is essential. For example, the local transformation shown in figure 11.1.15 cannot be accomplished by one $\sharp$-

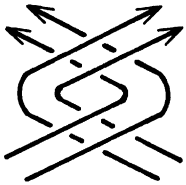


Fig. 11.1.13

Fig. 11.1.14

move (Exercise 11.6.11). Hence, the pattern of local transformation on unoriented knot diagrams shown in figure 11.1.16 is different from the $\sharp$-move although it is still an unknotting operation. Given an unknotting operation, we can define the distance between two knots.

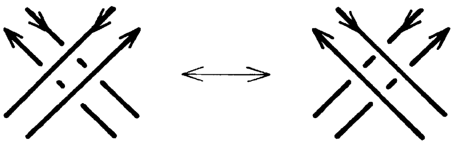


Fig. 11.1.15

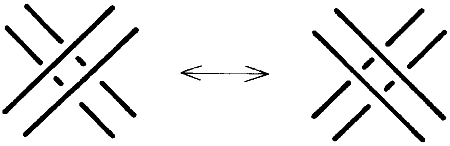


Fig. 11.1.16

Lemma 11.1.7 *Given an unknotting operation and any two knots K and K', then a diagram of K can be transformed into a diagram of K' by a finite sequence of applications of this unknotting operation.*

Exercise 11.1.8 Prove Lemma 11.1.7.

Definition 11.1.9 For two knots K and K', the *distance from K to K' for an unknotting operation* is the minimum of times this unknotting operation needs to be used to transform a diagram of K into a diagram of K', where the minimum is taken over all diagrams of K and K'.

Exercise 11.1.10 Show that the distance in this definition gives a distance function on the set of knot types.

The distances from K to K' defined by the X-move, the Δ-move and the $\sharp$-move are denoted by $d_G^X(K,K')$, $d_G^\Delta(K,K')$ and $d_G^\sharp(K,K')$ respectively, and called the *X-Gordian distance*, the *Δ-Gordian distance* and the *$\sharp$-Gordian distance* from K to K' respectively. The *X-unknotting number* $u^X(K)$, the *Δ-unknotting number* $u^\Delta(K)$ and the *$\sharp$-unknotting number* $u^\sharp(K)$ of a knot K are defined to be $d_G^X(K,O)$, $d_G^\Delta(K,O)$ and $d_G^\sharp(K,O)$ respectively, where O is a trivial knot. The X-unknotting number $u^X(K)$ is usually referred to as *the unknotting number* and denoted by $u(K)$. The X-unknotting number will be examined in 11.5. The X-Gordian distance, the Δ-Gordian distance, and the $\sharp$-Gordian distance are discussed in 11.2, 11.3, and 11.4, respectively. We give here some other unknotting operations as an exercise.

Exercise 11.1.11 Show that each pattern of local transformation shown in figure 11.1.17 is an unknotting operation.

Fig. 11.1.17

11.2 Properties of X-Gordian distance

Here, we discuss some properties of the X-Gordian distance given in [Murakami, H. 1985].

Theorem 11.2.1 *For any two knots K and K',*

$$d_G^X(K,K') \geq |\sigma(K) - \sigma(K')|/2.$$

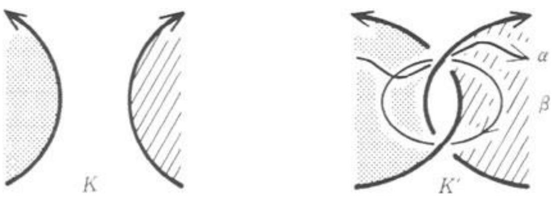


Fig. 11.2.1

Proof. We construct Seifert surfaces F and F' of K and K' as shown in figure 11.2.1. We choose a basis $\mathcal{S}$ for $H_1(F)$ and let $V(K)$ be the Seifert matrix of K defined using it. We add generators α and β to $\mathcal{S}$ as shown in figure 11.2.1 so that $\{\alpha, \beta\} \cup \mathcal{S}$ is a basis for $H_1(F')$. Then the Seifert matrix $V(K')$ of K' defined on this basis has the form

$$V(K') = \begin{pmatrix} \epsilon & \delta & \mathbf{0} \\ 0 & a & N' \\ \mathbf{0} & N & V(K) \end{pmatrix},$$

where ϵ and δ are ± 1 and N' is the transpose matrix of N and $\mathbf{0}$ is the zero (row or column) vector. Since

$$V(K') + V(K')' = \begin{pmatrix} 2\epsilon & \delta & \mathbf{0} \\ \delta & 2a & N + N' \\ \mathbf{0} & N + N' & V(K) + V(K)' \end{pmatrix},$$

we have $|\sigma(K) - \sigma(K')| \leq 2$. This gives the result (see also 12.3.6 for a 4-dimensional proof). $\square$

Exercise 11.2.2 Show that for the $(p, 2)$-torus knot $T_{(p,2)}$,

$$d_G^X(T_{(p,2)}, T_{(q,2)}) = \begin{cases} |p - q|/2 & (pq > 0) \\ |p - q|/2 - 1 & (pq < 0). \end{cases}$$

Next, we consider the linking forms (defined in 5.3) of the knots K and K' with $d_G^X(K, K') = 1$. Let $M(K)$ be the double branched covering space over S^3 with branch set a knot K. The linking form ℓ of the knot K is a form on $H_1(M(K))$ (cf. 5.5.4). Set $D(K) = |H_1(M(K))| (< \infty)$. Then we have the following:

Theorem 11.2.3 *Let K and K' be knots and $\ell : H_1(M(K)) \times H_1((M(K)) \to \mathbf{Q}/\mathbf{Z}$ and $\ell' : H_1(M(K')) \times H_1((M(K')) \to \mathbf{Q}/\mathbf{Z}$ be the linking forms of K and K', respectively. If $d_G^X(K, K') = 1$, then there exist elements $a \in H_1(M(K))$ and $a' \in H_1(M(K))$ such that $\ell(a, a) \equiv \pm n/D(K) \pmod 1$ and $\ell'(a', a') \equiv \pm n/D(K')$ (mod 1), where $n = |D(K) - D(K')|/2$.*

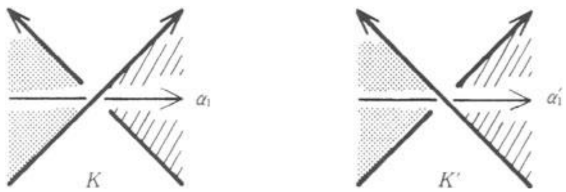


Fig. 11.2.2

Proof. We construct Seifert surfaces F and F' of K and K' as shown in figure 11.2.2 and choose a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}$ for $H_1(F)$ such that α_1 is the loop shown in figure 11.2.2. Taking the loop α_1' on F' shown in figure 11.2.2 instead of α_1, we obtain a basis $\{\alpha_1', \alpha_2, \ldots, \alpha_{2g}\}$ for $H_1(F')$. Let $V(K)$ and $V(K')$ be the Seifert matrices of F and F' defined by these bases, respectively. Then

$$V(F) + V(F)' = \begin{pmatrix} c & * \\ * & V \end{pmatrix} \quad \text{and} \quad V(F') + V(F')' = \begin{pmatrix} c \pm 2 & * \\ * & V \end{pmatrix}.$$

Let $a = [\alpha_1] \in H_1(M(K))$ and $a' = [\alpha_1'] \in H_1(M(K'))$. Then we have

$$\ell(a,a) = -\det V / \det(V(F) + V(F)') \quad \text{and}$$
$$\ell'(a',a') = -\det V / \det(V(F') + V(F')').\square$$

Corollary 11.2.4 *Let K be an X-unknotting number one knot. Then there exists $a \in H_1(M(K))$ such that $\ell(a,a) \equiv \pm 2/D(K) \pmod 1$.*

Exercise 11.2.5 Prove Corollary 11.2.4 and show that $a \in H_1(M(K))$ is a generator of $H_1(M(K))$.

Exercise 11.2.6 Show $d_G^X(3_1, 4_1) = 2$.

11.3 Properties of Δ-Gordian distance

Here, we discuss some properties of the Δ-Gordian distance, given in [Murakami-Nakanishi 1989]. Since the Δ-move is accomplished by two X-moves, the following theorem follows immediately.

Theorem 11.3.1 $d_G^\Delta(K,K') \geq d_G^X(K,K')/2$.

The following theorem is obtained in the same manner as Theorem 11.2.1:

Theorem 11.3.2 $d_G^\Delta(K,K') \geq |\sigma(K) - \sigma(K')|/2$.

Exercise 11.3.3 Prove Theorem 11.3.2. [*Hint*: Use suitable Seifert surfaces to make an argument analogous to that of Theorem 11.2.1. See also 12.3.6 for a 4-dimensional proof.]

In contrast to the X-Gordian distance, we can easily decide parity of the Δ-Gordian distance by the following theorem:

Theorem 11.3.4 $d_G^\Delta(K,K') \equiv \mathrm{Arf}(K) - \mathrm{Arf}(K') \pmod 2$.

Proof. We have only to prove that the Δ-move changes the Arf invariant, that is, the Arf invariant of a knot K is different from that of a knot obtained from K by a single Δ-move. The Δ-move is obtained as the result of a fusion with the Borromean rings as shown in figure 11.3.1 (see 13.1.1 for a general concept of fusion). Since the Arf invariant of the Borromean rings (with any orientation) is 1, we have the result (cf. 12.3.8). $\square$

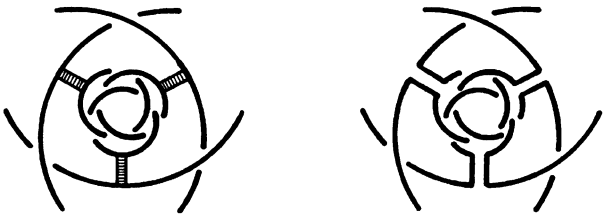


Fig. 11.3.1

Exercise 11.3.5 Show that the Arf invariant of the Borromean rings (with any orientation) is 1.

From Theorems 11.3.1 and 11.5.1 (stated later), we have the following theorem (where $m(K)$ is Nakanishi's index of a knot K defined in 5.5):

Corollary 11.3.6 $u^{\Delta}(K) \geq m(K)/2$.

Exercise 11.3.7 Determine the Δ-unknotting numbers of the knots with less than 7 crossings and the Δ-Gordian distance between them.

11.4 Properties of ♯-Gordian distance

Here, we discuss some properties of the ♯-Gordian distance, given in [Murakami,H. 1985]. From Theorems 11.2.1 and 11.3.2 we have the following estimate:

Theorem 11.4.1 *For any two knots K and K' with $d_G^{\sharp}(K,K') = 1$, $|\sigma(K) - \sigma(K')|$ is 2, 4 or 6.*

Exercise 11.4.2 Prove Theorem 11.4.1 (see [Murakami,H. 1985]).

Corollary 11.4.3 $d_G^{\sharp}(K,K') \geq |\sigma(K) - \sigma(K')|/6$.

The following result is similar to Theorem 11.3.4:

Theorem 11.4.4 $d_G^{\sharp}(K,K') \equiv \mathrm{Arf}(K) - \mathrm{Arf}(K') \pmod 2$.

Exercise 11.4.5 Prove Theorem 11.4.4. [*Hint*: Use that the ♯-move is obtained as the result of fusion with the link shown in figure 11.4.1.]

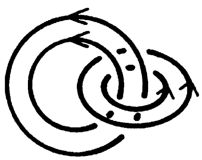


Fig. 11.4.1

Exercise 11.4.6 Determine the ♯-unknotting numbers of the knots with less than 7 crossings and the ♯-Gordian distances between them.

11.5 Estimation of the X-unknotting number

Here, we discuss several topics related to the X-unknotting number of a knot. The following relation between the unknotting number and Nakanishi's index is useful for many knots (cf. [Nakanishi 1981], [Kawauchi 1987]):

Theorem 11.5.1 $u(K) \geq m(K)$.

Proof. We prove this theorem by induction on the unknotting number. If $u(K) = 0$, that is, if K is a trivial knot, then $m(K) = 0$ and it is true. Assume that it is true for any knot K with $u(K) < n$. Let K' be a knot with $u(K') = n$. Let K be a knot with $u(K) = n-1$ which is obtained from K' by a X-move. We may assume that Seifert surfaces of K and K' are given as in figure 11.2.1. We use the Seifert matrices $V(K)$ and $V(K')$ for K and K' shown in the proof of Theorem 11.2.1. Then the following matrices are presentation matrices for the Alexander modules of K and K':

$$A(K) = tV(K) - V(K)',$$

$$A(K') = \begin{pmatrix} \epsilon(t-1) & \delta t & \mathbf{0} \\ -\delta & a(t-1) & (t-1)N' \\ \mathbf{0} & (t-1)N & A(K) \end{pmatrix}.$$

Multiplying $A(K')$ on the left by an invertible matrix over Λ, we can transform $A(K')$ into a matrix

$$B(K') = \begin{pmatrix} 1 & 0 & \mathbf{0} \\ 0 & x & * \\ \mathbf{0} & * & A(K) \end{pmatrix}.$$

Since $m(K) \leq n-1$, we can transform $A(K)$ into an $(n-1, n-1)$ presentation matrix by a finite sequence of the operations (1)-(5) of 7.2.1. Hence $B(K')$ can be transformed into an (n, n) presentation matrix; it follows that $m(K') \leq n$. $\square$

From this theorem and Corollary 5.5.2, we have the following:

Theorem 11.5.2 *Let $M_n(K)$ be the n-fold cyclic covering space over S^3 with branch set a knot K and $e_n(K)$ the minimum number of generators of $H_1(M_n(K))$. Then $u(K) \geq e_n(K)/(n-1)$.*

In contrast with the two theorems above, the following theorem shows that the Alexander polynomial of a knot K does not detect the unknotting number of K.

Theorem 11.5.3 *Let $f(t)$ be any integral Laurent polynomial in t such that $f(1) = 1$ and $f(t^{-1}) \doteq f(t)$. Then there exists an unknotting number one knot K whose Alexander polynomial is $f(t)$.*

The proof is omitted here (see [Kondo 1979] and [Sakai 1977]). The following proposition obtained from Theorems 11.5.2, 8.4.3 and 8.4.8(2) shows that we cannot realize the skein polynomial or the Kauffman polynomial of some knots by an X-unknotting number one knot.

Proposition 11.5.4

(1) $u(K) \geq \log_2 |P(K; \sqrt{-1}, \sqrt{-1})|$.

(2) $u(K) \geq \log_3 |Q(K; -1)|$.

It is also known that there exists a pair of knots which are skein equivalent but whose X-unknotting numbers are different. For example, the knots 8_8 and the mirror image of 10_{129} are skein equivalent but $u(8_8) = 2$ and $u(10_{129}) = 1$ (see [Kanenobu 1986']). Thus the X-unknotting number is not a skein invariant. The following theorem was conjectured by J. W. Milnor in [Milnor 1968] and proved by F. B. Kronheimer and T. S. Mrowka in [Kronheimer-Mrowka 1993] using gauge theory:

Theorem 11.5.5 *For a torus knot $T(p,q)$ of type (p,q), we have $u(T(p,q)) = (|p|-1)(|q|-1)/2$.*

In the remainder of this section, we note a few properties of unknotting-number-one knots without proofs.

Theorem 11.5.6 *The unknotting-number-one knots are prime.*

For proof, see [Scharlemann 1985]. The following result is given in [Kanenobu-Murakami 1985]:

Theorem 11.5.7 *Let K be a nontrivial 2-bridge knot. Then the following conditions are equivalent:*

(1) $u(K) = 1$.

(2) *There exist an odd integer $p(> 1)$ and coprime positive integers m and n with $2mn = p \pm 1$ such that K is equivalent to $S(p, 2n^2)$.*

(3) *K can be expressed as $C(a, a_1, a_2, \ldots, a_k, \pm 2, -a_k, \ldots, -a_2, -a_1)$.*

Exercise 11.5.8 Show that (2) and (3) are equivalent.

The following theorem is given in [Scharlemann-Thompson 1988] and [Kobayashi, T. 1989'''].

Theorem 11.5.9 *Any unknotting-number-one, genus one knot is a double of some knot.*

Exercise 11.5.10 Check that the table of X-unknotting numbers is correct by using the results mentioned above.

11.6 Local transformations of links

Here, we shall apply the X-move, the Δ-move and the $\sharp$-move defined in 11.1 to link diagrams and study whether they can transform any link into a trivial link.

Definition 11.6.1 Two links are *equivalent modulo a pattern of local transformation* if their link diagrams can be transformed into each other by a finite sequence of such transformations and Reidemeister moves I, II and III. In particular, two links are X-, Δ- or $\sharp$-*equivalent* if they are equivalent modulo X-, Δ- or $\sharp$-moves, respectively.

It is easy to see the following result:

Theorem 11.6.2 *Two links L and L' are X-equivalent if and only if they are links with the same number of components.*

Exercise 11.6.3 Prove Theorem 11.6.2.

Next, for study of the Δ-equivalence, we need the following concept:

Definition 11.6.4 Two ordered links $L = K_1 \cup K_2 \cup \cdots \cup K_n$ and $L' = K_1' \cup K_2' \cup \cdots \cup K_m'$ are *link-homologous* if $n = m$ and $\mathrm{Link}(K_i, K_j) = \mathrm{Link}(K_i{}', K_j{}')$ for all i and j with $1 \leq i < j \leq n$.

Let L and L' be ordered links. If L can be transformed into L' by a Δ-move shown in figure 11.1.5, L and L' are link-homologous by a suitable change of the order components of L (cf. figure 11.3.1). The converse is also true and we have the following:

Theorem 11.6.5 *Two links L and L' are Δ-equivalent if and only if they are link-homologous after ordering their components suitably.*

Exercise 11.6.6 Prove Theorem 11.6.5.

Next, we consider $\sharp$-equivalence. In general the $\sharp$-move changes the linking numbers between the components, but it does not change the parity of the linking number of any component and the other components. Using this, we obtain the following:

Theorem 11.6.7 *Two links $L = K_1 \cup K_2 \cup \cdots \cup K_n$ and $L' = K_1{}' \cup K_2{}' \cup \cdots \cup K_n{}'$ are $\sharp$-equivalent if and only if the number of i such that* $\mathrm{Link}(K_i, L - K_i) \equiv 1 \pmod 2$ *is equal to the number of i such that* $\mathrm{Link}(K_i', L' - K_i') \equiv 1 \pmod 2$.

Exercise 11.6.8 Prove Theorem 11.6.7.

Finally, we consider an interesting pattern of local transformation which is not an unknotting operation. The pattern of local transformation shown in figure 11.1.15 is called the *pass move*. It differs from the $\sharp$-move by orientation. Two links are said to be *pass-equivalent* if they are equivalent modulo the pass-move. Note that any link which is pass-equivalent to a proper link is a proper link. The following theorem is due to [Kauffman-Banchoff 1977], [Yamasaki 1977] and [Murakami-Nakanishi 1989]:

Theorem 11.6.9 *Two links L and L' are pass-equivalent if and only if they are $\sharp$-equivalent improper links or $\sharp$-equivalent proper links with* $\mathrm{Arf}(L) = \mathrm{Arf}(L')$. *In particular, two knots K and K' are pass-equivalent if and only if* $\mathrm{Arf}(K) = \mathrm{Arf}(K')$.

Proof. First we prove the "only if" part. It is clear that the pass-move does not change the number of i such that $\mathrm{Link}(K_i, L - K_i) \equiv 1 \pmod 2$. Hence we have

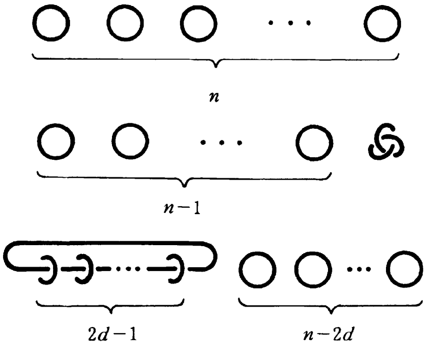


Fig. 11.6.1

only to prove that the pass-move on proper links does not change the Arf invariant. By Theorem 8.4.2 (4), it suffices to show that the pass-move does not change the value of the Jones polynomial at $t = \sqrt{-1}$. To prove the "if" part, we have only to show that we can choose representatives as in figure 11.6.1. This is proved in the same way as Theorem 11.1.6 except we use a Seifert surface instead of a non-orientable surface. □

Exercise 11.6.10 Prove Theorem 11.6.9 in detail.

Exercise 11.6.11 Show that the pass-move cannot be accomplished by one $\sharp$-move.

Supplementary notes for Chapter 11

We first note that there are many other unknotting operations on knots which are not discussed here (see for example [Hoste-Nakanishi-Taniyama 1990], [Aida 1992]). For a knot K with $u(K) = 1$, there is a question concerning the number of places at which the X-move transforms K into a trivial knot. It is shown in [Kobayashi,T 1989'] and [Scharlemann-Thompson 1989] that there is just one such place (up to equivalence) in any non-trivial doubled knot. In [Taniyama 1991] it is shown that there are at most two inequivalent such places in any 2-bridge knot K with $u(K) = 1$. In [Kawauchi 1993'], it is shown that for any integer $n > 1$, there is a knot K with $u(K) = 1$ which admits at least n mutually inequivalent such places. It is an open question whether or not there is a knot K with $u(K) = 1$ which admits infinitely many mutually inequivalent such places. For the Δ-move, Y. Uchida showed in [Uchida 1993] that any knot K with $u^{\Delta}(K) = 1$ admits infinitely many mutually inequivalent places at which the Δ-move transforms K into a trivial knot.

Chapter 12
Cobordisms

In this chapter we discuss the concept of knot cobordism, which is a 4-dimensional property of a knot. In the latter half, this concept is generalized to links.

12.1 The knot cobordism group

A knot K in S^3 is said to be a *slice knot* if K is the boundary of a locally flat proper disk D in the 4-ball B^4. This disk D is called a *slice disk.*

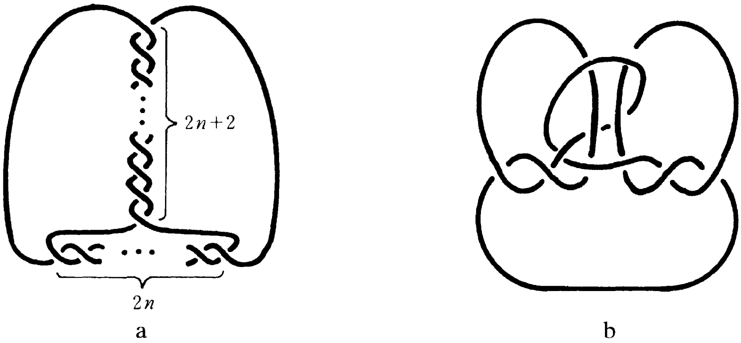


Fig. 12.1.1

Example 12.1.1. The knots K_n for $n \geq 1$ with $4n+2$ crossing points shown in figure 12.1.1a are slice knots. K_1 is called the *stevedore knot* and listed as 6_1 in the knot table. The Kinoshita-Terasaka knot K_{KT} shown in figure 12.1.1b is also a slice knot. The reason they are slice is that K_n and K_{KT} can be transformed into 2-component trivial links by the hyperbolic transformations (see Definition 13.1.1) along the band B shown in figure 12.1.2.

Lemma 12.1.2 (1) *Any knot equivalent to a slice knot is a slice knot.*
(2) *For any knot K, the knot sum $(-K^*)\sharp K$ is a slice knot.*
(3) *If any two knots are slice among three knots K_i $(i = 1, 2, 3)$ such that $K_3 = K_1 \sharp K_2$, then the remaining knot is also a slice knot.*

Proof. If a knot K is a slice knot, then the mirror image K^* and the inverse mirror image $-K^*$ of K are also slice knots. Hence (1) follows from Appendix A. To see (2), we choose a 3-ball $B \subset S^3$ with $K \cap B$ a trivial arc in B. Then $(\mathrm{cl}(S^3 - B), \mathrm{cl}(K - K \cap B)) \times [0,1]$ is a pair consisting of a 4-ball and a locally flat proper disk whose boundary pair shows that the knot sum $(-K^*)\sharp K$ is a slice knot. To see (3), first assume that K_1 and K_2 are slice knots with slice disks $D_1 \subset B^4$ and $D_2 \subset B^4$, respectively. Then the boundary disk sum $(B^4, D_1)\natural(B^4, D_2)$ shows

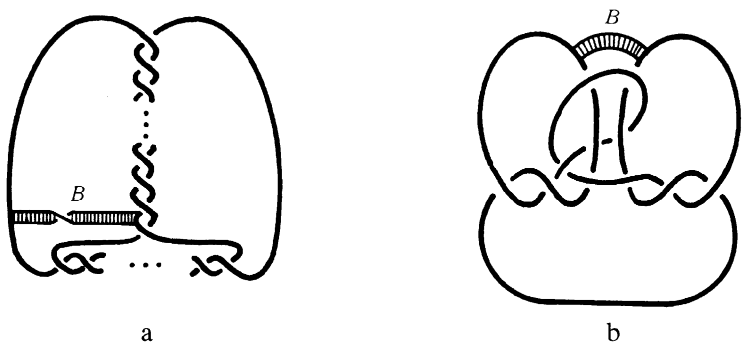


Fig. 12.1.2

that the knot $K_3 = K_1 \sharp K_2$ is a slice knot. Next, assume that K_1 and K_3 are slice knots with slice disks $D_1 \subset B^4$ and $D_3 \subset B^4$, respectively. We choose 3-balls B_1 and B_3 in S^3 so that $\mathrm{cl}(K_1 - B_1 \cap K_1)$ is a trivial arc in the 3-ball $\mathrm{cl}(S^3 - B_1)$ and $(B_3, B_3 \cap K_3) = (B_1, B_1 \cap K_1)$. (We see that ∂B_3 is a sphere defining the knot sum $K_3 = K_1 \sharp K_2$.) The pair consisting of a 4-ball and a locally flat proper disk obtained from the topological sum $(B^4, D_1) + (B^4, D_3)$ by identifying $(B_1, B_1 \cap K_1)$ in the first summand with $(B_3, B_3 \cap K_3)$ in the second summand shows that the knot K_2 is a slice knot. $\square$

By our convention, the product $S^3 \times [0,1]$ is oriented so that the natural projections $p_1 : S^3 \times 1 \cong S^3$ and $p_0 : S^3 \times 0 \cong S^3$ are, respectively, orientation-preserving and orientation-reversing. Further, when we are given a link L in S^3, the orientations of $L \times 1 (\subset S^3 \times 1)$ and $L \times 0 (\subset S^3 \times 0)$ are specified so that $p_1 |_{L \times 1}: L \times 1 \cong L$ and $p_0 |_{L \times 0}: L \cong 0 \to L$ are orientation-preserving and orientation-reversing, respectively. Thus, the links $(S^3 \times 1, L \times 1)$ and $(S^3 \times 0, L \times 0)$ are identified with (S^3, L) and $(S^3, -L^*)$, respectively. Two knots K_0 and K_1 are *(knot) cobordant* (or *concordant*), if there is a locally flat, oriented, proper annulus C with $C \cap S^3 \times 0 = K \times 0$ and $C \cap S^3 \times 1 = K_1 \times 1$. Then K_0 and K_1 are cobordant if and only if the knot sum $(-K_0^*) \sharp K_1$ is a slice knot.

Exercise 12.1.3 Prove the last statement.

Cobordism gives an equivalence relation on the set of knot types. The set of equivalence classes is denoted by C^1. Writing an element of C^1 as $[K]$ with K a knot, the set C^1 forms an abelian group under the sum $[K_1] + [K_2] = [K_1 \sharp K_2]$. The zero element is $[O]$ with O a trivial knot and the inverse element of $[K]$ is $[-K^*]$.

Exercise 12.1.4 Using Lemma 12.1.2, prove these assertions about C^1.

12.2 The matrix cobordism group

We denote by Θ the set of Seifert matrices of knots (namely the set of integral square matrices V with $\det(V - V') = 1$, by Corollary 5.1.6). A matrix $V \in \Theta$ is said to be *null-cobordant* if V is unimodular-congruent to a matrix of the following type

$$\begin{pmatrix} O & V_{21} \\ V_{12} & V_{22} \end{pmatrix},$$

where V_{ij} is a square matrix for all i, j and O is the zero matrix. Null-cobordant matrices are related to slice knots.

Proposition 12.2.1 *Any null-cobordant matrix $V \in \Theta$ is a Seifert matrix of a slice knot.*

Proof. Let the size of V be $2m$. By a unimodular congruence, we can assume that V has the following form:

$$V = \begin{pmatrix} O & V_{21} \\ V_{12} & V_{22} \end{pmatrix} \text{ and } V - V' = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where I denotes a square matrix of size m with the $(i, m-i)$-component equal to 1 for all $i (0 < i < m)$ and the other components equal to 0. For a disk D in S^3, we attach $2m$ bands $B_i (i = 1, 2, \ldots, 2m)$ to D by the method stated in Theorem 5.1.3 so that $F = D \cup B_1 \cup \cdots \cup B_{2m}$ has V as a Seifert matrix, where we can take the first m bands $B_i (i = 1, 2, \ldots, m)$ to be standardly attached to D. Then $K = \partial F$ is a slice knot. In fact, we consider a disk D_i in S^3 which bounds the central loop of the annulus $D \cup B_i$ for each i with $1 \leq i \leq m$. We may assume that $D_i \cap D_j = \emptyset$ for all i, j with $i \neq j$. Then we do 2-handle surgery on F along proper disks obtained from $D_i (1 \leq i \leq m)$ by pushing $\text{int}D_i$ into $\text{int}B^4$ to obtain a disk D_F. Since $\partial D_F = K$, we see that K is a slice knot by pushing $\text{int}D_F$ into $\text{int}B^4$. □

A knot with a null-cobordant Seifert matrix is called an *algebraic slice knot*. Here, we give a characterization of algebraic slice knots. Let M be a compact oriented 3-manifold with ∂M a surface. We consider a compact surface $S \subset \partial M$ such that $F = \text{cl}(\partial M - S)$ is also a compact surface. Let $K(S \subset M)$ be the kernel of the natural homomorphism $i_* : H_1(S) \to H_1(M)$, which is a free abelian group. The surface S is *isotropic* in M if the intersection form Int_S on $H_1(S)$ is non-singular and the rank of $K(S \subset M)$ is equal to the genus $g(S)$ of S. Using that Int_S is non-singular, we can see that the boundary of S is a circle, and that $\text{rank}K(S \subset M) \geq g(S)$ if and only if $\text{rank}K(S \subset M) = g(S)$.

Exercise 12.2.2 Prove this assertion. [*Hint for the second half*: Use the identity $\text{Int}_S(\partial \bar{x}, y) = \text{Int}_M(\bar{x}, i_*(y))$ for $\bar{x} \in H_2(M, S)$ and $y \in H_1(S)$, where Int_M denotes the intersection form $\text{Int}_M : H_2(M, S) \times H_1(M) \to \mathbf{Z}$.]

A locally flat compact oriented proper surface S in B^4 with $K = \partial S$ a knot in $\partial B^4 = S^3$ is said to be *isotropic* if there is a locally flat compact oriented 3-submanifold M in B^4 with $S \subset \partial M$ such that $F = \text{cl}(\partial M - S)$ is a Seifert surface for K in S^3 and S is isotropic in M. We show the following:

Theorem 12.2.3 *A knot K in S^3 is an algebraic slice knot if and only if K bounds an isotropic surface S in B^4. In particular, any slice knot is an algebraic slice knot.*

The following duality is useful for our proof:

Lemma 12.2.4 *A compact oriented surface S with ∂S a circle is isotropic in M if and only if the complementary surface $F = cl((\partial M - S)$ is isotropic in M.*

Proof. We note that the intersection form $\mathrm{Int}_{\partial M}$ on $H_1(\partial M)$ is non-singular and is an orthogonal sum of Int_S and Int_F. It suffices to show that if $\mathrm{rank}K(S \subset M) = g(S)$, then $\mathrm{rank}K(F \subset M) = g(F)$. By Poincaré duality and the hint for Exercise 12.2.2, we have $\mathrm{rank}K(\partial M \subset M) = g$ for $g = g(\partial M) = g(S) + g(F)$. We take $\mathbf{Z}$-linearly independent elements $x_i (i = 1, 2, \ldots, g)$ in $K(\partial M \subset M)$ such that the first $g(S)$ elements $x_i (i = 1, 2, \ldots, g(S))$ belong to $K(S \subset M)$. For each i with $g(S) + 1 \leq i \leq g$, we can write x_i as the sum of an element $x_i^F \in H_1(F)$ and an element $x_i^S \in H_1(S)$. By the hint of Exercise 12.2.2, $\mathrm{Int}_{\partial M}(x, x_i) = 0$ for all $x \in K(S \subset M)$. Hence $\mathrm{Int}_S(x, x_i^S) = 0$ for all $x \in K(S \subset M)$. This means that there is a positive integer n_i such that $n_i x_i^S$ is a $\mathbf{Z}$-linear combination of $x_i (i = 1, 2, \ldots, g(S))$. Then the elements $n_i x_i^F = n_i x_i - n_i x_i^S$ $(g(S) + 1 \leq i \leq g)$ belong to $K(F \subset M)$ and are $\mathbf{Z}$-linearly independent. Hence we have $\mathrm{rank}K(F \subset M) = g(F)$. □

Proof of Theorem 12.2.3. Assume that a knot K in S^3 bounds an isotropic surface S in B^4. Then there is a locally flat compact oriented 3-submanifold M of B^4 such that S is isotropic in M and $F = \partial M - \mathrm{int}S$ is a Seifert surface for K in S^3. By Lemma 12.2.4, F is isotropic in M. Using a collar of M in B^4, we have $\varphi(x, y) = 0$ for all $x, y \in K(F \subset M)$, where φ is the Seifert form on $H_1(F)$. $K(F \subset M)$ is a finite index subgroup of a subgroup $N \subset H_1(F)$ such that $H_1(F)/N$ is a free abelian group of rank $g(F)$. Then N is a direct summand of $H_1(F)$ and we have $\varphi(x, y) = 0$ for all $x, y \in N$. This means that a Seifert matrix associated with φ is null-cobordant. Conversely, assume that a Seifert matrix associated with a Seifert form $\varphi : H_1(F) \times H_1(F) \to \mathbf{Z}$ is null-cobordant. Then we have a basis $x_i, y_i (i = 1, 2, \ldots, g(F))$ of $H_1(F)$ with

$$\varphi(x_i, x_j) = \mathrm{Int}_F(x_i, x_j) = \mathrm{Int}_F(y_i, y_j) = 0,$$

and $\mathrm{Int}_F(x_i, y_j) = \delta_{ij}$ for all i, j, so that there are mutually disjoint simple loops $K_i (i = 1, 2, \ldots, g(F))$ on $\mathrm{int}F$ representing $x_i (i = 1, 2, \ldots, g(F))$. Let F_i be a Seifert surface for K_i in S^3 such that $F_i \cap K_j = F_i \cap K = \emptyset$ for all i, j with $i \neq j$. This is guaranteed to exist because $\mathrm{Link}(K_i, K_j) = \mathrm{Link}(K_i, K) = 0$ for all i, j with $i \neq j$. We push $\mathrm{int}F_i$ into $\mathrm{int}B^4$ so that the resulting proper surfaces $\hat{F}_i (i = 1, 2, \ldots, g(F))$ are mutually disjoint. Since $\varphi(x_i, x_i) = 0$, we can extend a collar of K_i in F to a normal I-bundle N_i of $\hat{F}_i$ in B^4. A collar of F in B^4 and $N_i (i = 1, 2, \ldots, g(F))$ constitute a 3-submanifold $M \subset B^4$ with $F = M \cap S^3$ such that F is isotropic in M. By Lemma 12.2.4, $S = \partial M - \mathrm{int}F$ is isotropic in M. □

Theorem 12.2.5 *Assume that a knot K' is cobordant to an algebraic slice knot K. Then any Seifert matrix of any Seifert surface F' for K' is null-cobordant. In particular, K' is an algebraic slice knot.*

To prove this theorem, we use the following fact:

Lemma 12.2.6 *Let F be a Seifert surface for a link L in S^3. Let S be a locally flat proper oriented 2-manifold in B^4 with $\partial S = -L$. Then there is a locally flat compact oriented 3-manifold M in B^4 with $\partial M = F \cup S$.*

Proof. Let $E = \mathrm{cl}(B^4 - N(S))$ for a tubular neighborhood $N(S)$ of S in B^4 and $E_0 = \mathrm{cl}(S^3 - N(S) \cap S^3)$. Let $F_0 = F \cap E_0 \cong F$. We can choose a homeomorphism $h : (S \times D^2, S \times 0) \cong (N(S), S)$ so that the composition $S \times p \to \mathrm{Fr}N(S) \to E$ for a point $p \in S^1 = \partial D^2$ is trivial in the first homology, where $\mathrm{Fr}N(S)$ denotes the frontier of $N(S)$ in B^4 and the first map is the restriction of h and the second map is the inclusion map. Then the composite map of the homeomorphism $\mathrm{Fr}N(S) \cong S \times S^1$ defined by h^{-1} and the projection $S \times S^1 \to S^1$ extends to a map $f : E \to S^1$. After a homotopy of f, we can further assume that $(f|_{E_0})^{-1}(p) = F_0$ and $M_p = f^{-1}(p)$ is a bi-collared 3-manifold with boundary $F_0 \cup h(S \times p)$ for a non-vertex point p of S^1. The desired 3-manifold M is obtained from M_p by adding a collar. □

Proof of Theorem 12.2.5. Let C be a cobordism annulus in $S^3 \times [0,1]$ with $C \cap (S^3 \times 0) = K \times 0$ and $C \cap (S^3 \times 1) = K' \times 1$. By Theorem 12.2.3, there is a 3-submanifold $M \subset B^4$ such that $F = M \cap S^3$ is a Seifert surface for K and $S = \partial M - \mathrm{int}F$ is isotropic for M. By Lemma 12.2.6, we have a locally flat compact oriented 3-manifold M' in $S^3 \times [0,1]$ with $\partial M' = F \times 0 \cup C \cup F' \times 1$. Identifying $S^3 \times 0$ with S^3 naturally, we obtain a 3-manifold $M' \cup M$ in the 4-ball $S^3 \times [0,1] \cup B^4$ in which the surface $C \cup F (\cong F)$ is isotropic and hence by Lemma 12.2.4 in which $F' \times 1$ is isotropic. By Theorem 12.2.3, any Seifert matrix on F' is null-cobordant. □

Corollary 12.2.7 (1) *Any matrix S-equivalent to a null-cobordant matrix in Θ is null-cobordant.*
(2) *For any three matrices $V_i \in \Theta, i = 1,2,3$, with $V_3 = V_1 \oplus V_2$, if any two of them are null-cobordant, then the remaining one is also null-cobordant.*
(3) *For any matrix $V \in \Theta$, $(-V) \oplus V$ is null-cobordant.*

Proof. For (1), let V^+ be a row or column enlargement of a Seifert matrix V of a Seifert surface for a knot K. By Exercise 5.2.7, V^+ is also a Seifert matrix of a Seifert surface for the same knot K. By Theorem 12.2.5, V^+ is null-cobordant if and only if V is null-cobordant. By induction, the conclusion of (1) holds. For (2), since the block sum of two null-cobordant matrices is null-cobordant, it suffices to show that if V_1 and V_3 are null-cobordant, then V_2 is also null-cobordant. By Theorem 5.1.4, we can realize $V_i, i = 1,2$ as Seifert matrices of knots $K_i, i = 1,2$. Then $K_3 = K_1 \sharp K_2$ has the Seifert matrix $V_3 = V_1 \oplus V_2$. Then since $[K_2] = [K_3] - [K_1] = [K_3 \sharp (-K_1^*)]$ in C^1, K_2 is cobordant to $K_3 \sharp (-K_1^*)$, which has the null-cobordant Seifert matrix $V_3 \oplus (-V_1)$, where we note that $-V_1$ is a Seifert matrix for $-K_1^*$.

By Theorem 12.2.5, K_2 is an algebraic slice knot, so V_2 is null-cobordant. For (3), let V be a Seifert matrix of a knot K. Then $(-V) \oplus V$ is a Seifert matrix of the slice knot $(-K^*)\sharp K$. By Theorem 12.2.5, $V \oplus (-V)$ is null-cobordant. □

We say that two matrices $V_i \in \Theta, i = 1, 2$, are *cobordant* if $(-V_1) \oplus V_2$ is null-cobordant. By Corollary 12.2.7, cobordism gives an equivalence relation on Θ and the set G_- of the equivalence classes $[V], V \in \Theta$, forms an abelian group under the sum $[V_1] + [V_2] = [V_1 \oplus V_2]$. The inverse of $[V]$ is given by $[-V]$.

Theorem 12.2.8 *There is an epimorphism $\psi : C^1 \to G_-$ sending the class of any knot to the class of its Seifert matrix.*

Proof. The well-definedness of ψ follows from Theorem 12.2.5. By Exercise 5.3.4, ψ is a homomorphism. By Theorem 5.1.4, ψ is onto. □

For any knot K, the orders of $[K]$ and $\psi[K]$ in C^1 and G_- are denoted by $o(K)$ and $a(K)$, respectively. The known orders of knots with up to ten crossings are listed in the table in Appendix F. Concerning matrix cobordism, we introduce the *quadratic form* of a Seifert matrix $V \in \Theta$. We say that a matrix $V \in \Theta$ is *non-degenerate* if $\det V \neq 0$. Let $\Delta(V;t) = \det(tV' - V)$, which is an invariant of the S-equivalence class of V up to multiplication by units of Λ.

Theorem 12.2.9 *Every Seifert matrix $V \in \Theta$ is S-equivalent to the empty matrix or a non-degenerate matrix V_* according to whether $\Delta(V;t)$ is a unit of Λ or not. Further, for any two S-equivalent non-degenerate Seifert matrices V_* and W_*, there is a rational non-singular matrix Q with $QV_*Q' = W_*$.*

Proof. For the first half, it suffices to show that any matrix $V \in \Theta$ with $\det W = 0$ is unimodular-congruent to a row-enlargement of a matrix $V_1 \in \Theta$ (or the empty matrix). Let $\varphi : G \times G \to \mathbf{Z}$ be the Seifert form induced from V. Define a skew symmetric form $I : G \times G \to \mathbf{Z}$ by $I(x,y) = \varphi(x,y) - \varphi(y,x)$. Using that $\det V = 0$, we can find an indivisible element $e \in G$ such that $\varphi(e, G) = 0$. Since $\det(V - V') = \pm 1$, I is non-singular, so there is an element $e^* \in G$ with $I(e, e^*) = -1$. I is given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on the subgroup G' with basis e, e^*. Let G'' be the orthogonal complement of G' in G with respect to I. With respect to a basis for G consisting of e, e^* and any basis of G'', we have a unimodular matrix P such that φ and I are represented by the matrices $PVP' = \begin{pmatrix} 0 \\ * \end{pmatrix}$ and

$$PVP' - PV'P' = P(V - V')P' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & * \end{pmatrix}.$$

Thus, we see that PVP' is a row-enlargement of some matrix in Θ (or the empty matrix). To show the later half, we use a method of H. F. Trotter in [Trotter 1962, 1973]. Let V be a Seifert matrix of size n. Let $b_\Lambda : \Lambda^n \times \Lambda^n \to \mathbf{Q}(\Lambda)$ be the Λ-Hermitian pairing representing $(t-1)(tV'-V)^{-1}$ under the standard Λ-basis of Λ^n. Let H be the reduced knot module associated with V defined in 5.4. Then the pairing b_Λ induces a pairing $b_H : H \times H \to \mathbf{Q}(\Lambda)/\Lambda$, where $\mathbf{Q}(\Lambda)$ is the quotient field of Λ. This pairing has the nice property that it depends only on the S-equivalence class of V. Let $b_{H_\mathbf{Q}} : H_\mathbf{Q} \times H_\mathbf{Q} \to \mathbf{Q}(\Lambda)/\Lambda_\mathbf{Q}$ be the coefficient extension to $\Lambda_\mathbf{Q} = \Lambda \otimes_\mathbf{Z} \mathbf{Q}$. Let $\chi : \mathbf{Q}(\Lambda) \to \mathbf{Q}$ be the $\mathbf{Q}$-linear map defined by $\chi(f) = 0$ for $f \in \Lambda_Q$ and $\chi(f) = f'(1)$ for any proper fraction f in $\mathbf{Q}(\Lambda)$, where we note that every element of $\mathbf{Q}(\Lambda)$ has a unique expression as the sum of an element of $\Lambda_\mathbf{Q}$ and a proper fraction in $\mathbf{Q}(\Lambda)$. χ defines a $\mathbf{Q}$-linear map $\mathbf{Q}(\Lambda)/\Lambda_\mathbf{Q} \to \mathbf{Q}$ also denoted by χ, for $\chi(\Lambda_\mathbf{Q}) = 0$. Let V_* be a non-degenerate Seifert matrix of size n^* which is S-equivalent to V. Then there is a $\mathbf{Q}$-basis $e_1, e_2, \dots, e_{n^*}$ for $H_\mathbf{Q}$ such that $t(e_1 e_2 \dots e_{n^*}) = (e_1 e_2 \dots e_{n^*}) V_*'(V_*)^{-1}$ and the composite $\chi b_{H_\mathbf{Q}}$ represents $(V_*' - V_*)^{-1}$ under this basis. If W_* is S-equivalent to V_*, then there is a rational non-singular matrix Q such that $W_*'(W_*)^{-1} = QV_*'(V_*)^{-1}Q^{-1}$ and $(W_*' - W_*)^{-1} = Q(V_*' - V_*)^{-1}Q^{-1}$. Since $V_* = (V_*'V_*^{-1} - E)^{-1}(V_*' - V_*)$, we see that $W_* = QV_*Q^{-1}$. □

For a non-degenerate Seifert matrix $V_* \in \Theta$, we consider a symmetric bilinear form $b : \mathbf{Q}^n \times \mathbf{Q}^n \to \mathbf{Q}$ and a linear automorphism $t : \mathbf{Q}^n \to \mathbf{Q}^n$ which represent the matrices $V_* + V_*'$ and $(V_*')^{-1}V_*$ under the standard basis of $\mathbf{Q}^n$, respectively. Then t satisfies the identity $b(tx, ty) = b(x, y)$ for all $x, y \in \mathbf{Q}^n$, and is called an *isometry* of the form b. Since $\Delta(V_*; 1) = 1$, we see that $\Delta(V_*; -1) = \pm\det(V_* + V_*')$ is an odd integer, so that the form b is non-singular. When V_* is the empty matrix, we understand that b is the trivial form on $\mathbf{Q}^n = 0$ and $t = \mathrm{id}$.

Definition 12.2.10 The *quadratic form* of a Seifert matrix $V \in \Theta$ is the pair (b, t) associated to a non-degenerate or empty matrix V_* which is S-equivalent to V.

The quadratic forms $(b_i, t_i)(i = 1, 2)$ of Seifert matrices $V_i \in \Theta(i = 1, 2)$ are said to be *isomorphic* if there is a $\mathbf{Q}$-linear isomorphism $f : \mathbf{Q}^n \to \mathbf{Q}^n$ with $t_2 = f t_1 f^{-1}$ and $b_2(f(x), f(y)) = b_1(x, y)$ for all $x, y \in \mathbf{Q}^n$. By Theorem 12.2.9, the quadratic form of a Seifert matrix $V \in \Theta$ is uniquely determined up to isomorphism by the S-equivalence class of V. If V is a Seifert matrix on a knot K, the quadratic form of V is also called the *quadratic form of a knot K*.

Corollary 12.2.11 *A Seifert matrix $V \in \Theta$ is null-cobordant if and only if there is a half-dimensional $\mathbf{Q}$-subspace $N_\mathbf{Q} \subset \mathbf{Q}^n$ such that $b(N_\mathbf{Q}, N_\mathbf{Q}) = 0$ and $tN_\mathbf{Q} = N_\mathbf{Q}$ for the quadratic form (b, t) of V.*

Theorem 12.2.12 $G_- \cong \mathbf{Z}^\infty \oplus \mathbf{Z}_4^\infty \oplus \mathbf{Z}_2^\infty$.

The proof of this theorem is omitted (see [Levine 1969']), but the main idea is to consider the quadratic form (b, t) of a Seifert matrix V on the basis of Corollary 12.2.11 and then to extend it to the form over a (Archimedean or non-Archimedean) completion of $\mathbf{Q}$ in order to use the classification result of [Milnor 1969]. We consider the real extension $(b_\mathbf{R}, t)$ of the quadratic form (b, t) of a Seifert matrix $V \in \Theta$, where $b_\mathbf{R} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$. Since $\mathbf{R}^n$ is a $\Lambda_\mathbf{R}$-torsion module for $\Lambda_{\mathbf{R}} = \mathbf{R}\langle t\rangle$, we can uniquely split $\mathbf{R}^n$ into the $p(t)$-components $C_{p(t)}$ with $p(t)$ a (non-unit) real irreducible polynomial in t. Note that $C_{p(t)} \neq 0$ if and only if $p(t)$ is a real factor of $\Delta(V;t)$. In particular, $C_{t\pm 1} = 0$. If $p(t)$ and $p(t^{-1})$ are not equal up to units of $\Lambda_{\mathbf{R}}$, then we can show that the restriction of $b_{\mathbf{R}}$ to $C_{p(t)} \times C_{p(t)}$ is the zero form, so that the signature of the restriction $b'_{\mathbf{R}}$ of $b_{\mathbf{R}}$ to $C_{p(t)} \oplus C_{p(t^{-1})} \times C_{p(t)} \oplus C_{p(t^{-1})}$ (which is a non-singular form) is 0. If $p(t)$ and $p(t^{-1})$ are equal up to units of $\Lambda_{\mathbf{R}}$, then we may write $p(t)$ as $p_a(t) = t^2 - 2at + 1$ for a real number $a \in (-1,1)$. When V is a Seifert matrix of a knot K, the signature of the restriction $b^a_{\mathbf{R}}$ of $b_{\mathbf{R}}$ to $C_{p_a(t)} \times C_{p_a(t)}$ (which is a non-singular form) is called the *local signature* of K at $a \in (-1,1)$ and denoted by $\sigma_a(K)$. It is an invariant of the S-equivalence class of V and hence the knot type of K. Further, we have $\sigma_a(-K) = \sigma_a(K)$ and $\sigma_a(\pm K^*) = -\sigma_a(K)$. Since the form $b_{\mathbf{R}}$ is an orthogonal sum of such forms $b'_{\mathbf{R}}, b^a_{\mathbf{R}}$, it follows that

$$\sigma(K) = \sum_{a\in(-1,1)} \sigma_a(K).$$

By the definition of a null-cobordant matrix and Corollary 12.2.11, we obtain the following:

Corollary 12.2.13 *If K is an algebraic slice knot, then $\Delta(K;t) \doteq F(t)F(t^{-1})$ for some $F(t) \in \Lambda$ and $\sigma(K) = \sigma_a(K) = 0$ for all $a \in (-1,1)$.*

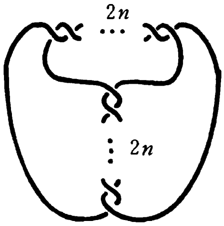


Fig. 12.2.1

Example 12.2.14. The knot $K(n)(n \geq 1)$ shown in figure 12.2.1 (where $K(1)$ denotes the figure eight knot 4_1) is invertible and amphicheiral. Hence $2[K(n)] = 2\psi[K(n)] = 0$. Let $K = K(n_1)\sharp K(n_2)\sharp \ldots \sharp K(n_r)$ for some integers n_i ($i = 1,2,\ldots,r$) with $1 \leq n_1 < n_2 < \cdots < n_r$. Since $\Delta(K;t) = \prod_{i=1}^r (n_i^2 t^2 - (2n_i^2+1)t + n_i^2)$, we do not have any $F(t) \in \Lambda$ with $\Delta(K;t) \doteq F(t)F(t^{-1})$. By Corollary 12.2.13, we have that $o(K) = a(K) = 2$. By $t_2(C^1)$ and $t_2(G_-)$, respectively, we denote the subgroups of C^1 and G_- consisting of all elements of order 2. Thus, $t_2(C^1) \cong \mathbf{Z}_2^\infty$ and $t_2(G_-) \cong \mathbf{Z}_2^\infty$.

Example 12.2.15. Let $T(p,q)$ be a torus knot, which is normalized so that p and q are coprime positive integers up to equivalence. By Exercise 7.4.4, we have that

$$\begin{aligned}\Delta(T(p,q);t) &= (t^{pq}-1)(t-1)/(t^p-1)(t^q-1)\\ &= p_{a_1}(t)p_{a_2}(t)\ldots p_{a_r}(t),\end{aligned}$$

where $r = (p-1)(q-1)/2$ and $p_{a_i}(t) = t^2 - 2a_it + 1$ for some $a_i (i = 1, 2, \ldots, r)$ with $-1 < a_1 < a_2 < \cdots < a_r < 1$. Let $T(p, q)$ be non-trivial, i.e., $r \geq 1$. Then it can be directly checked that for each i, the $p_{a_i}(t)$-component $C_{p_{a_i}(t)}$ is $\Lambda_{\mathbf{R}}$-isomorphic to $\Lambda_{\mathbf{R}}/(p_{a_i}(t))$ and the form $b_{\mathbf{R}}^{a_i}$ is a positive-(or negative-)definite form, so that $\sigma_{a_i}(T(p,q)) = \pm 2 (i = 1, 2, \ldots, r)$. This means that every non-trivial torus knot is non-slice and non-amphicheiral. (In particular, this completes the proof of Theorem 2.2.2 together with 6.1.17.) Denoting the above r by $r(p, q)$, we also see that $\psi[T(p_i, q_i)] (i = 1, 2, \ldots, m)$ are $\mathbf{Z}$-linearly independent in G_- for any torus knots $T(p_i, q_i) (i = 1, 2, \ldots, m)$ with mutually distinct $r(p_i, q_i) (i = 1, 2, \ldots, m)$, so that both C^1 and G_- have infinite ranks.

Open Problem 12.2.16 Is each cobordism class in $t_2(C^1)$ represented by a $(-)$amphicheiral knot?

Open Problem 12.2.17 Does there exist a knot K with $2 < o(K) < +\infty$?

12.3 Link cobordism

Here, the concept of knot cobordism is generalized to a concept for a link. Recall the orientation conventions on $S^3 \times [0,1], L \times 0$ and $L \times 1$ for a link L in S^3 which were discussed in 12.1. Then the links $(S^3 \times 1, L \times 1)$ and $(S^3 \times 0, L \times 0)$ are identified with (S^3, L) and $(S^3, -L^*)$, respectively.

Theorem 12.3.1 *For two links $L, L' \subset S^3$, we consider a compact oriented proper locally flat 2-manifold F in $S^3 \times [0,1]$ such that $\partial F = L \times 0 \cup L \times 1$ and each component of F meets both $S^3 \times 0$ and $S^3 \times 1$. Then we have*

$$|\sigma(L) - \sigma(L')| + |n(L) - n(L')| \leq \beta_1(F, L \times 0) = \beta_1(F, L \times 1),$$

where β_1 denotes the first Betti number.

When each component of F is an annulus, we say that L and L' are *link cobordant* (or *link concordant*).

Corollary 12.3.2 *If L and L' are link cobordant, then we have $\sigma(L) = \sigma(L')$ and $n(L) = n(L')$.*

For the proof of Theorem 12.3.1, we need some preliminaries. For a link L in S^3 (or $\mathbf{R}^3$), we consider mutually disjoint oriented embedded bands $B_i (i = 1, 2, \ldots, r)$ which span L with coherent orientations (see figure 12.3.1).

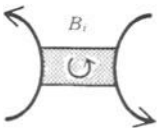


Fig. 12.3.1

Definition 12.3.3 The link $L' = L \cup (\cup_{i=1}^r \partial B_i) - \cup_{i=1}^r \mathrm{int}(L \cap \partial B_i)$ is obtained from L by *hyperbolic transformation* along the bands $B_i (i = 1, 2, \ldots, r)$.

Lemma 12.3.4 (Murasugi's lemma) *Assume that a link L' is obtained from a link L by hyperbolic transformation along a band B. Then we have*

$$|\sigma(L) - \sigma(L')| + |n(L) - n(L')| = 1.$$

Proof. We may assume that $\sharp L' = \sharp L + 1$, if necessary, by exchanging the roles of L and L'. We can take a connected Seifert surface F for L such that $F \cap B = L \cap B$. Then $F' = F \cup B$ is a connected Seifert surface for L'. The Seifert form $\varphi : H_1(F) \times H_1(F) \to \mathbf{Z}$ is the restriction of the Seifert form $\varphi' : H_1(F') \times H_1(F') \to \mathbf{Z}$. Let $b' : H_1(F') \times H_1(F') \to \mathbf{Z}$ be the symmetric form defined by $b'(x, y) = \varphi'(x, y) + \varphi'(y, x)$, and $b : H_1(F) \times H_1(F) \to \mathbf{Z}$ be the restricted form. Note that the the signatures $\sigma(b), \sigma(b')$ and the nullities $n(b), n(b')$ are respectively equal to $\sigma(L), \sigma(L')$ and $n(L), n(L')$. Let $b_{\mathbf{Q}} : G_{\mathbf{Q}} \times G_{\mathbf{Q}} \to \mathbf{Q}$ and $b'_{\mathbf{Q}} : G'_{\mathbf{Q}} \times G'_{\mathbf{Q}} \to \mathbf{Q}$ be the $\mathbf{Q}$-extensions of b and b', respectively. Choose a maximal $\mathbf{Q}$-subspace E of $G_{\mathbf{Q}}$ such that $b_{\mathbf{Q}} |_{E \times E}$ is a non-singular form. Let E_0 and E'_0 be the orthogonal complements of E in $G_{\mathbf{Q}}$ and $G'_{\mathbf{Q}}$ with respect to $b_{\mathbf{Q}}$ and $b'_{\mathbf{Q}}$, respectively. Since $b_{\mathbf{Q}} |_{E_0 \times E_0} = 0$, $E_0 \subset E'_0$ and $\dim_{\mathbf{Q}} E'_0 = \dim_{\mathbf{Q}} E_0 + 1$, we have either $\sigma(b') = \sigma(b)$ and $n(b') = n(b) \pm 1$ or $\sigma(b') = \sigma(b) \pm 1$ and $n(b') = n(b)$. □

Lemma 12.3.5 *Let L' be a link obtained from a split union L^+ of a link L and an r-component trivial link O^r by hyperbolic transformation along r bands $B_i (i = 1, 2, \ldots, r)$ connecting each component of O^r to L. Then we have $\sigma(L') = \sigma(L)$ and $n(L') = n(L)$.*

Proof. We construct a link cobordism 2-manifold $F \subset S^3 \times [0, 1]$ between the links $L, L' \subset S^3$ as follows:

$$F \cap S^3 \times t = \begin{cases} L' \times t & \text{for } 2/3 < t \leq 1, \\ (L^+ \cup B_1 \cup \cdots \cup B_r) \times t & \text{for } t = 2/3, \\ L^+ \times t & \text{for } 1/3 < t < 2/3, \\ (L \cup D(r)) \times t & \text{for } t = 1/3, \\ L \times t & \text{for } 0 \leq t < 1/3, \end{cases}$$

where $D(r)$ denotes a union of mutually disjoint disks bounded by O^r in $S^3 - L$. Let $W = S^3 \times [0, 1] - F$, $E = S^3 \times 0 - L \times 0$, and $E' = S^3 \times 1 - L' \times 1$. We consider the infinite cyclic covering $p : (W_\infty; E_\infty, E'_\infty) \to (W; E, E')$ corresponding to the kernel of the epimorphism $\gamma : \pi_1(W) \to \langle t \rangle$ sending each meridian of F in $S^3 \times [0, 1]$ to t. Considering the homology exact sequence associated with the following short exact sequence of chain complexes:

$$0 \to C_\sharp(W_\infty, E_\infty; \mathbf{Z}_2) \xrightarrow{t-1} C_\sharp(W_\infty, E_\infty; \mathbf{Z}_2) \xrightarrow{p_\sharp} C_\sharp(W, E; \mathbf{Z}_2) \to 0,$$

we have an isomorphism $t-1 : H_*(W_\infty, E_\infty; \mathbf{Z}_2) \cong H_*(W_\infty, E_\infty; \mathbf{Z}_2)$, since $H_*(W, E;\ \mathbf{Z}_2) = 0$. Hence

$$t^2 - 1 = (t-1)^2 : H_*(W_\infty, E_\infty; \mathbf{Z}_2) \cong H_*(W_\infty, E_\infty; \mathbf{Z}_2).$$

Let $(W_2; E_2, E_2') = (W_\infty; E_\infty, E_\infty')/\langle t^2\rangle$. Then p is the composition

$$(W_\infty; E_\infty, E_\infty') \xrightarrow{p^2} (W_2; E_2, E_2') \xrightarrow{p_2} (W; E, E'),$$

where p^2 is an infinite cyclic covering and p_2 is a double covering. By the homology exact sequence associated with the following short exact sequence on chain complexes:

$$0 \to C_\sharp(W_\infty, E_\infty; \mathbf{Z}_2) \xrightarrow{t^2-1} C_\sharp(W_\infty, E_\infty; \mathbf{Z}_2)$$
$$\xrightarrow{(p^2)_\sharp} C_\sharp(W, E; \mathbf{Z}_2) \to 0,$$

we have that $H_*(W_2, E_2; \mathbf{Z}_2) = 0$. Let $(X_2; M_2, M_2')$ be the double branched covering of $(S^3 \times [0,1]; S^3 \times 0, S^3 \times 1)$ branched along F. Then $H_*(X_2, M_2;\ \mathbf{Z}_2) = H_*(W_2, E_2; \mathbf{Z}_2) = 0$. Hence $H_*(X_2, M_2; \mathbf{Q}) = 0$. In particular, $\beta_1(M_2) = \beta_1(X_2)$. Similarly, $\beta_1(M_2') = \beta_1(X_2)$, so that $\beta_1(M_2) = \beta_1(M_2')$. By Corollary 5.5.4, this means that $n(L) = n(L')$. Since $n(L^+) = n(L) + r$ and $\sigma(L^+) = \sigma(L)$, we see from Murasugi's lemma that $\sigma(L') = \sigma(L)$. □

Proof of Theorem 12.3.1. Let $m = \beta_1(F, L \times 0)$. Then deforming F to be in normal form by an ambient isotopy of $S^3 \times [0,1]$, we have links L^* and L^{**} in S^3 as follows:

(1) L^* is obtained from a split union of L and a trivial link O^r by hyperbolic transformation along r bands connecting each component of O^r to L.
(2) L^{**} is obtained from L^* by hyperbolic transformation along m bands.
(3) L^{**} is obtained from a split union of L' and a trivial link O^s by hyperbolic transformation along s bands connecting each component of O^s to L'.

See [Kawauchi-Shibuya-Suzuki 1982] for the entire proof, though the idea of the proof can be found in Theorem 13.1.8. By Murasugi's Lemma, $|\sigma(L^*) - \sigma(L^{**})| + |n(L^*) - n(L^{**})| \le m$. By Lemma 12.3.5, $n(L^*) = n(L)$, $n(L^{**}) = n(L)$, $\sigma(L^*) = \sigma(L)$ and $\sigma(L^{**}) = \sigma(L')$. The result follows. □

Exercise 12.3.6 Show that Theorems 11.2.1 and 11.3.2 are corollaries of Theorem 12.3.1.

The 4-*dimensional genus* of a link L in S^3 is the minimal genus of a locally flat proper oriented surface in B^4 bounded by L. We denote it by $g^*(L)$.

Corollary 12.3.7 *For any link L, $|\sigma(L)| + n(L) \le 2g^*(L) + \sharp L - 1$.*

Theorem 12.3.8 *If L is a proper link in S^3 and $g^*(L) = 0$, then* $\mathrm{Arf}(L) = 0$.

Corollary 12.3.9 *For two proper links $L, L' \subset S^3$, assume that there is a locally flat compact oriented surface F of genus 0 in $S^3 \times [0,1]$ such that $\partial F = L \times 0 \cup L \times 1$. Then* $\mathrm{Arf}(L) = \mathrm{Arf}(L')$.

Proof of Theorem 12.3.8. Let $F^* \subset B^4$ be a genus 0 surface with $\partial F^* = L$. For a connected Seifert surface F for the link L, the closed oriented surface $S = F \cup -F^*$ bounds a compact oriented 3-submanifold M of B^4 with a collar $c : (M, F) \times [-1,1] \to (B^4, S^3)$ which is an embedding for which $c(x, 0) = x$ for all $x \in M$. Note that any simple loop s in S bounds an immersed disk C in B^4 with $C \cap c(s \times [-1,1]) = s$, for $B^4 - s$ is simply connected. Let $C_+ = C \cup c(s \times [0,1])$ and $C_- = C \cup c(s \times [-1,0])$. Then we can define the $\mathbf{Z}_2$-intersection number $I_2(C_+, C_-) \in \mathbf{Z}_2$ in B^4. Let $x = \{s\} \in H_1(S; \mathbf{Z}_2)$.

Exercise 12.3.10 Show that this value is independent of the choice of C and the choice of s representing the $\mathbf{Z}_2$-homology class x.

Denote this value by $q^*(x)$. Because every element of $H_1(S; \mathbf{Z}_2)$ is represented by a simple loop, we have a function $q^* : H_1(S; \mathbf{Z}_2) \to \mathbf{Z}_2$. q^* is a quadratic function belonging to the $\mathbf{Z}_2$-intersection form $I_2 : H_1(S; \mathbf{Z}_2) \times H_1(S; \mathbf{Z}_2) \to \mathbf{Z}_2$. The composition

$$q : H_1(F; \mathbf{Z}_2) \xrightarrow{i_*} H_1(S; \mathbf{Z}_2) \xrightarrow{q^*} \mathbf{Z}_2,$$

with i_* the natural homomorphism, is given by $q(x) = \varphi_2(x, x)$ for the $\mathbf{Z}_2$-reduced Seifert form $\varphi_2 : H_1(F; \mathbf{Z}_2) \times H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$. Using that F^* is connected and the genus of F^* is 0, we see that i_* is a monomorphism and we have a symplectic basis $x_1, y_1, \ldots, x_n, y_n$ for $H_1(S; \mathbf{Z}_2)$ such that the preimages of $x_1, y_1, \ldots, x_m, y_m, x_{m+1}, \ldots, x_n$ by i_* for some $m \le n$ form a $\mathbf{Z}_2$-basis for $H_1(F; \mathbf{Z}_2)$. Since L is a proper link, we have $q^*(x_i) = q(i_*^{-1} x_i) = 0$ for all $i \geq m$. Hence

$$\mathrm{Arf}(q^*) = \sum_{i=1}^{n} q^*(x_i) \cdot q^*(y_i) = \sum_{i=1}^{m} q(i_*^{-1} x_i) \cdot q^*(i_*^{-1} y_i) = \mathrm{Arf}(L).$$

On the other hand, since S is the boundary of a 3-manifold M, we see from Poincaré duality that there is a symplectic basis $x'_1, y'_1, \ldots, x'_n, y'_n$ for $H_1(S; \mathbf{Z}_2)$ such that $x'_1, \ldots, x'_n$ are in the kernel of the natural homomorphism $H_1(S; \mathbf{Z}_2) \to H_1(M; \mathbf{Z})$. Using the collar c, we have that $q^*(x'_i) = 0 (i = 1, 2, \ldots, n)$, so that $\mathrm{Arf}(q^*) = 0$. Hence $\mathrm{Arf}(L) = 0$. □

Next, we investigate how the Alexander polynomial is affected by link cobordism. Let L be a link in S^3 with components $L_i (i = 1, 2, \ldots, n)$. Let $G = \prod_{i=1}^{n} \langle t_i \rangle$ be the free abelian group with basis $t_i (i = 1, 2, \ldots, n)$, and Λ the integral group ring of G. Let $E = E(L)$. We consider the universal abelian covering $E_\gamma \to E$, which is the covering corresponding to the kernel of the epimorphism $\gamma : \pi_1(E) \to G$ sending a meridian of L_i to t_i for each i. Since Λ is a Noetherian ring, $H_*(E_\gamma)$ is a finitely generated Λ-module. We denote the Λ-torsion part of a Λ-module H by TH. The 0-th characteristic polynomial of $TH_1(E_\gamma)$ is the β-th Alexander polynomial $\Delta^{(\beta)}(L) = \Delta^{(\beta)}(L; t_1, \ldots, t_n)$ of L with $\beta = \beta(L)$ (cf. Lemma 7.2.7). For $f = f(t_1, \ldots, t_n) \in \Lambda$, we let $f^* = f(t_1^{-1}, \ldots, t_n^{-1}) \in \Lambda$.

Theorem 12.3.12 *If two n-component links L and L' in S^3 are link cobordant, then:*

$$1)\qquad \beta(L) = \beta(L'),$$
$$2)\qquad \Delta^{(\beta)}(L)ff^* = \Delta^{(\beta)}(L')gg^*$$

for $\beta = \beta(L)$ and some $f, g \in \Lambda$ with $|f(1,\dots,1)| = |g(1,\dots,1)| = 1$.

In this theorem, we understand that the variables $t_i (i = 1, 2, \dots, n)$ of $\Delta^{(\beta)}(L)$ and $\Delta^{(\beta)}(L')$ correspond through the cobordism annuli.

Proof. Let $F \subset S^3 \times [0,1]$ be a link cobordism 2-manifold with $F \cap S^3 \times 0 = L \times 0$ and $F \cap S^3 \times 1 = L' \times 1$, so that $H_*(F, L \times 0) = H_*(F, L' \times 1) = 0$. For a tubular neighborhood $N(\cong F \times D^2)$ of F in $S^3 \times [0,1]$, we let $X = \mathrm{cl}(S^3 \times [0,1] - N)$ and $M = X \cap S^3 \times 0(\cong E(L))$ and $M' = X \cap S^3 \times 1(\cong E(L'))$. Note that $H_*(X, M) = H_*(X, M') = 0$. Let $(X_\gamma; M_\gamma, M'_\gamma)$ be a covering of $(X; M, M')$ which extends the covering $E_\gamma \to E$. We use the following lemma:

Lemma 12.3.13 *Let $(\tilde{X}, \tilde{X}')$ be a regular covering over a polyhedral pair (X, X') with free abelian covering transformation group G. If $H_q(X, X') = 0$, then the 0-th characteristic polynomial Δ_0 of the Λ-module $H_q(\tilde{X}, \tilde{X}')$ has $|\Delta_0(1,\dots,1)| = 1$. In particular, we have that Λ-rank$H_q(\tilde{X}, \tilde{X}') = 0$.*

The proof of this lemma is elementary, but omitted (see [Kawauchi 1978(Lemma 2.1)]). Consider the following natural exact sequence:

$$H_2(X_\gamma, M_\gamma) \to H_1(M_\gamma) \to H_1(X_\gamma) \to H_1(X_\gamma, M_\gamma).$$

By Lemma 12.3.13, we have a $\mathbf{Q}(\Lambda)$-isomorphism

$$H_1(M_\gamma) \otimes_\Lambda \mathbf{Q}(\Lambda) \cong H_1(X_\gamma) \otimes_\Lambda \mathbf{Q}(\Lambda),$$

where $\mathbf{Q}(\Lambda)$ denotes the quotient field of Λ. Thus, $\beta(L) = \Lambda$-rank$H_1(X_\gamma)$. Similarly, $\beta(L') = \Lambda$-rank$H_1(X_\gamma)$ and hence $\beta(L) = \beta(L')$. By Lemma 12.3.13, the exact sequence above induces the following exact sequence:

$$TH_2(X_\gamma, M_\gamma) \to TH_1(M_\gamma) \to TH_1(X_\gamma) \to TH_1(X_\gamma, M_\gamma).$$

We denote the 0-th characteristic polynomials of these modules and $TH_1(\partial X_\gamma)$ by $\Delta_2, \Delta_M, \Delta_X, \Delta_1$ and $\Delta_{\partial X}$, respectively. Then $\Delta_M \doteq \Delta^{(\beta)}(L)$ for $\beta = \beta(L)$. By lemma 12.3.13, $|\Delta_1(1,\dots,1)| = |\Delta_2(1,\dots,1)| = 1$. Then using Lemma 7.2.7, we have that $\Delta^{(\beta)}(L)g_1 \doteq \Delta_X g_2$ for some $g_i \in \Lambda$ with $|g_i(1,\dots,1)| = 1 (i = 1,2)$. Similarly, $\Delta^{(\beta)}(L')g'_1 \doteq \Delta_X g'_2$ for some $g'_i \in \Lambda$ with $|g'_i(1,\dots,1)| = 1 (i = 1,2)$. By using that Λ is a unique factorization domain, we can split $\Delta^{(\beta)}(L) \doteq u(L)v(L)$ and $\Delta^{(\beta)}(L') \doteq u(L')v(L')$ and $\Delta_{\partial X} = u_{\partial X}v_{\partial X}$ uniquely (up to units of Λ) so that $v(L), v(L'), v_{\partial X}$ consist of all irreducible factors $f \in \Lambda$ with $|f(1,\ldots,1)| \neq 1$ in $\Delta^{(\beta)}(L), \Delta^{(\beta)}(L')$, $\Delta_{\partial X}$, respectively. It follows that:

$$(12.3.12.1) \qquad v(L) \doteq v(L').$$

Consider the exact sequence obtained from the Mayer-Vietoris sequence on $(\partial X_\gamma; M_\gamma, M'_\gamma)$:

$$TH_1(\partial M_\gamma) \to TH_1(M_\gamma) \oplus TH_1(M'_\gamma) \to TH_1(\partial X_\gamma) \to TH_0(\partial M_\gamma).$$

Since there are Λ-epimorphisms from $\oplus_{i=1}^n \Lambda/(t_i - 1)$ to $TH_1(\partial M_\gamma)$ and $TH_0(\partial M_\gamma)$, it follows that $\Delta_{\partial X}\lambda \doteq \Delta^{(\beta)}(L)\Delta^{(\beta)}(L')\lambda'$ for some factors λ, λ' of $(t_1 - 1)\ldots(t_n - 1)$ (cf. Lemma 7.2.7). Using that $\Delta^{(\beta)}(L) \doteq \Delta^{(\beta)}(L)^*$, we have the following result:

$$(12.3.12.2) \qquad u_{\partial X} \doteq u(L)u(L') \doteq u(L)^* u(L').$$

For a Λ-module H, let

$$DH = \{x \in H \mid \exists \text{coprime}\, \lambda_1, \ldots, \lambda_m \in \Lambda\, (m \geq 2)\, \text{with}\, \lambda_1 x = \cdots = \lambda_m x = 0\}.$$

Note that the 0-th characteristic polynomial of DH is a unit of Λ. Let $T_D H = TH/DH$. Let x_D be the image of an element $x \in TH$ in $T_D H$. By the Blanchfield duality (cf. Theorem E.3), there are non-degenerate Λ-sesquilinear forms

$$L_D : T_D H_1(\partial X_\gamma) \times T_D H_1(\partial X_\gamma) \to \mathbf{Q}(\Lambda)/\Lambda$$
$$L'_D : T_D H_2(X_\gamma, \partial X_\gamma) \times T_D H_1(X_\gamma) \to \mathbf{Q}(\Lambda)/\Lambda.$$

Let $i'_* : TH_1(\partial X_\gamma) \to TH_1(X_\gamma)$ and $\partial' : TH_2(X_\gamma, \partial X_\gamma) \to TH_1(\partial X_\gamma)$ be the restrictions of the natural homomorphisms $i_* : H_1(\partial X_\gamma) \to H_1(X_\gamma)$ and $\partial : H_2(X_\gamma, \partial X_\gamma) \to H_1(\partial X_\gamma)$, respectively. Using that $\Lambda\text{-rank} H_2(X_\gamma, M_\gamma) = 0$, we see that $\mathrm{Im}\partial' = \mathrm{Ker} i'_*$. Since $L'_D(x_D, i'_*(y)_D) = L_D(\partial'(x)_D, y_D)$ for all $x \in TH_2(X_\gamma, \partial X_\gamma)$, $y \in TH_1(\partial X_\gamma)$, and $TH_1(\partial X_\gamma)/(i'_*)^{-1}(D(\mathrm{Im} i'_*)) \cong T_D \mathrm{Im} i'_*$, it follows that L_D induces a non-degenerate Λ-sesquilinear form

$$T_D \mathrm{Im}\partial' \times T_D \mathrm{Im} i'_* \to \mathbf{Q}(\Lambda)/\Lambda.$$

Let h be the 0-th characteristic polynomial of $\mathrm{Im} i'_*$. By Lemma 7.2.7 and the existence of this form, the 0-th characteristic polynomial of $\mathrm{Im}\partial'$ is equal to h^*. Hence $\Delta_{\partial X} \doteq hh^*$. In particular, we see that:

$$(12.3.12.3) \qquad u_{\partial X} \doteq ff^* \text{for some } f \in \Lambda \text{ with } |f(1,\ldots,1)| = 1.$$

The results in (12.3.12.1), (12.3.12.2) and (12.3.12.3) imply that

$$\Delta^{(\beta)}(L)ff^* \doteq u(L)u_{\partial X}v(L) \doteq u(L)u(L)^*u(L')v(L') \doteq \Delta^{(\beta)}(L')gg^*$$

for $g = u(L)$. □

A link is called a *strongly slice link* if it is link cobordant to a trivial link. For an n-component trivial link O^n, we have $\beta(O^n) = n - 1$ and $\Delta^{(n-1)}(O^n) \doteq 1$. Hence Theorem 12.3.12 yields the following result:

Corollary 12.3.14 *For an n-component strongly slice link L, $\beta(L) = n - 1$ and $\Delta^{(n-1)}(L) \doteq ff^*$ for some $f \in \Lambda$ with $|f(1,\ldots,1)| = 1$.*

This is a generalization of the slice knot case stated in Corollary 12.2.13.

Exercise 12.3.15 Show that the 2-component link L shown in figure 12.3.2 which is called *Milnor's link* is a strongly slice link. Further, show that $\Delta^{(1)}(L) \doteq (1 - t_1 + t_1t_2)(1 - t_1^{-1} + t_1^{-1}t_2^{-1})$.

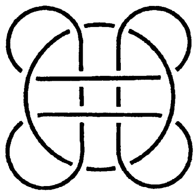


Fig. 12.3.2

Supplementary notes for Chapter 12

The concept of knot cobordism was first introduced by [Fox-Milnor 1966]. The existence of the natural epimorphism $\psi : C^1 \to G_-$ was shown by [Levine 1969]. It was shown by [Casson-Gordon 1978] that Kerψ is not 0. The form b_Λ appearing in the proof of Theorem 12.2.9 is closely related to the Blanchfield duality in Appendix E and discussed in [Trotter 1962, 1973]. The quadratic form of a knot is defined in [Milnor 1968'] from a cohomological viewpoint. Its relation to a Seifert matrix is given in [Erle 1969']. A natural generalization of the quadratic form to a link is given in [Kawauchi 1977], but its relation to a Seifert matrix is not yet completely clear (cf. [Kawauchi 1985]). For a knot K with trivial Alexander polynomial, it was shown by [Freedman 1982] that K is necessarily a slice knot in TOP category. On the other hand, in the PL category which we are discussing, it is known by a result of [Donaldson 1983] in gauge theory that there are many knots K with trivial Alexander polynomial and with $o(K) = \infty$ (cf. [Gompf 1986]).

Chapter 13
Two-knots I: a topological approach

In this chapter, we discuss a normal form for 2-knots, how to construct 2-knots, and some properties of ribbon 2-knots. Most results are described without proof, but the reader can consult the papers and books that are cited there or in the supplementary notes of this chapter.

13.1 A normal form

We first introduce a method called the *moving picture method* to describe a 2-knot. This method was first introduced in an unpublished paper of R.H. Fox and J.W. Milnor and re-introduced in [Fox 1962]. Here we introduce a notion of normal form for a 2-knot in $\mathbf{R}^4$ and discuss how to deform a given 2-knot by an ambient isotopy of $\mathbf{R}^4$ into one in normal form. This is proved in [Kawauchi-Shibuya-Suzuki 1982], although sketchy proofs are given in the unpublished paper of R.H. Fox and J.W. Milnor and [Suzuki 1976]. For this purpose, the notion of hyperbolic transformations of a link, defined in 12.3.3 is important. Let L' be an m-component oriented link, obtained from an n-component oriented link L in $\mathbf{R}^3$ by hyperbolic transformations along a disjoint family of p bands.

Definition 13.1.1 The link L' is a *p-fission* of L if $m = n + p$, and a *p-fusion* of L if $n = m + p$.

The bands used for fission or fusion are called the *fission bands* or *fusion bands*, respectively. Note that L' is obtained from L by p-fusion if and only if L is obtained from L' by p-fission.

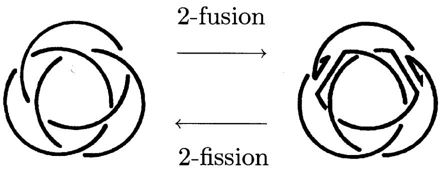


Fig. 13.1.1

Notation 13.1.2 When A is any interval in $\mathbf{R}$ (including a "one-point interval" $[t]$, $t \in \mathbf{R}$), we denote the subspace $\mathbf{R}^3 \times A$ of $\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}$ by $\mathbf{R}^3 A$.

For a subspace X of $\mathbf{R}^4$, the intersection $X \cap \mathbf{R}^3[t]$ is called the *cross-section* of X at the level t.

Definition 13.1.3 For a 2-knot K, the cross-section $K \cap \mathbf{R}^3[t]$ is *regular* if it is empty or a link in $\mathbf{R}^3[t]$. Otherwise, it is *singular*.

In the PL category which we are discussing, the number of singular cross sections of any 2-knot is finite.

Definition 13.1.4 Let $K \cap \mathbf{R}^3[t]$ be a singular cross section. If x is a point of $K \cap \mathbf{R}^3[t]$ whose regular neighborhood in $K \cap \mathbf{R}^3[t]$ is not an arc, the point x is a *critical point* of K and the level t is a *critical level.*

Definition 13.1.5 Let x be a critical point of K in a critical level t. If $N(x;K) \cap \mathbf{R}^3[t+\epsilon, t-\epsilon]$ is one of the figures shown in figures 13.1.2a-c for a regular neighborhood $N(x;K)$ and a small number $\epsilon > 0$, the point x is a *maximum*, *minimum*, or *saddle point*, respectively.

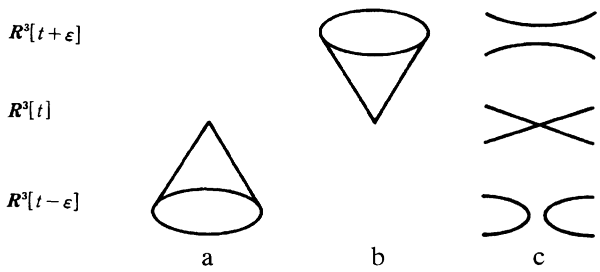


Fig. 13.1.2

In Definition 13.1.5, we can deform the figures of $N(x;K)$ in figure 13.1.2a-c into the figures in figure 13.1.3a-c, by an ambient isotopy of $\mathbf{R}^4$ keeping $\mathrm{cl}(K - N(x;K))$ fixed.

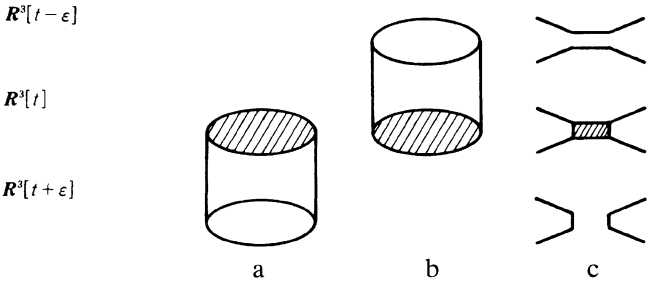


Fig. 13.1.3

Definition 13.1.6 The disks in figure 13.1.3a-c are a *maximal band*, a *minimal band* and a *saddle band*, respectively.

We also call these bands *critical bands*. Let K have exactly one saddle band B in a critical level s. Then for a small number $\epsilon > 0, K \cap \mathbf{R}^3[t]$ is a regular cross section for any t with $0 < |t-s| \leq \epsilon$. Let L_+ and L_- be the links $K \cap \mathbf{R}^3[s+\epsilon]$ and $K \cap \mathbf{R}^3[s-\epsilon]$, respectively. Let $m_\pm$ be the number of components of the link $L_\pm$. Then we see that $|m_+ - m_-| = 1$. If $m_+ = m_- + 1$, then L_+ is a 1-fission of L_- (or equivalently, L_- is a 1-fusion of L_+) along the band B.

Definition 13.1.7 A 2-knot K is in *normal form* if it satisfies the following five conditions for some numbers a and b with $0 < a < b$:

(1) All critical points of K are in critical bands.
(2) K has a regular cross section in any level except $\pm a, \pm b$, and is a knot in the level 0.
(3) All saddle bands of K are in the level $\pm a$.
(4) All maximal bands of K are in the level b.
(5) All minimal bands of K are in the level $-b$.

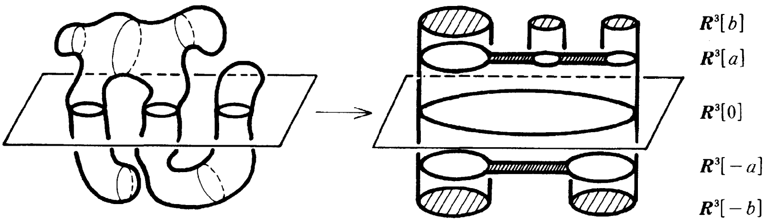


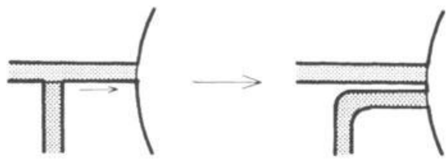
Fig. 13.1.4

The following is the main result of this section:

Theorem 13.1.8 *Any 2-knot in $\mathbf{R}^4$ can be deformed into normal form by an ambient isotopy of $\mathbf{R}^4$.*

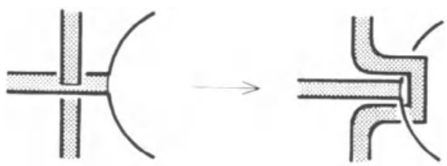
Proof. A detailed proof is given in [Kawauchi-Shibuya-Suzuki 1982]. Here, we give an outline of the proof. Without loss of generality, we can assume that the 2-knot K is contained in $\mathbf{R}^3[-2, 2]$ and that any critical point of K is a maximum, minimum, or saddle point. For a maximum p of K, we can choose a point q in $\mathbf{R}^3[3]$ so that the line segment $\overline{pq}$ intersects K only in p, after a small deformation if necessary. Then we pull the point p up along the segment $\overline{pq}$ to reach q by an ambient isotopy of $\mathbf{R}^4$ keeping the outside of a neighborhood of $\overline{pq}$ in $\mathbf{R}^4$ fixed. We do the same procedure for all other maxima. Thus, we can assume that all maxima of K are in $\mathbf{R}^3[3]$. By a similar modification for the minima of K, we may assume that K is in $\mathbf{R}^3[-3, 3]$ and all of the maxima of K are in $\mathbf{R}^3[3]$ and all of the minima of K are in $\mathbf{R}^3[-3]$. By taking a small ambient isotopy of $\mathbf{R}^4$, we can also assume that all saddle points of K are in $\mathbf{R}^3(-2, 2)$ and that the levels are mutually distinct. Let $p_1, \dots ,p_r$ be the saddle points of K, whose critical levels we denote by $t_1, \dots ,t_r$, where $2 = t_0 > t_1 > \cdots > t_r > t_{r+1} = -2$. In a regular neighborhood in $\mathbf{R}^4$ of p_i, we deform K by an ambient isotopy so that the saddle point p_i changes into a saddle band B_i. Do this procedure for each saddle point. The resulting 2-knot is also denoted by K. Let L be the link that is a cross section of K in the level 2. Then $(\mathbf{R}^3(t_1, 2), K \cap \mathbf{R}^3(t_1, 2))$ can be deformed into $(\mathbf{R}^3(t_1, 2), L(t_1, 2))$ by a level-preserving ambient isotopy of $\mathbf{R}^4$, because K has no saddle points in any level between t_1 and 2. We perform a similar modification on $K \cap \mathbf{R}^3(t_i, t_{i+1})$ for each

i. The resulting 2-knot is also denoted by K. To complete this outline of a proof, it suffices to show that the levels of saddle bands are interchangeable. We move the saddle band B_i in the level t_i down along the 4-th coordinate into the level t_{i+1}. Then there are two saddle bands B_i and B_{i+1} in the level t_{i+1} and there is not any saddle band in the level t_i. If $B_i \cap B_{i+1} = \emptyset$, then the band B_{i+1} can be moved up into the level t_i and this shows that we can exchange two saddle bands B_i and B_{i+1}. If $B_i \cap B_{i+1} \neq \emptyset$, then only the two cases shown in figures 13.1.5a and 13.1.6a can occur, after deforming the bands if necessary. Namely, the case that an end arc of one band attaches to the boundary of the other band (figure 13.1.5a) and the case that one band intersects the other band transversely (figure 13.1.6a). In the former case, we slide the attaching arc along the boundary of the other band until these bands become disjoint (figure 13.1.5b). In the latter case, we pass the band including the intersection arc as a proper arc in the other band until these bands become disjoint (figure 13.1.6b). In both cases, the modifications are realized by an ambient isotopy of $\mathbf{R}^4$ and this means that we can exchange levels of saddle bands arbitrarily keeping the type of the 2-knot K unchanged. Then we can arrange the saddle bands so that any saddle band is in level 2 or -2 and the cross section of K in level 0 appears as a knot. □



a b

Fig. 13.1.5



a b

Fig. 13.1.6

Definition 13.1.9 A knot in $\mathbf{R}^3$ is a *ribbon knot* if it is the boundary of an immersed (i.e., a locally embedded) disk in $\mathbf{R}^3$ such that all of the self-intersections are of *ribbon type*, as shown in figure 13.1.7.

For example, the knot in figure 13.1.8 is a ribbon knot. We see that a knot k in $\mathbf{R}^3$ is a ribbon knot if and only if there exists a locally flat disk D in $\mathbf{R}^3[0, \infty)$ such

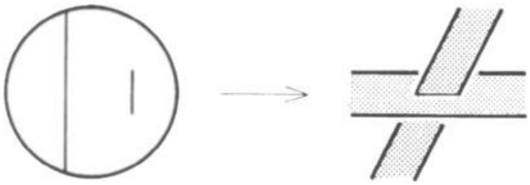


Fig. 13.1.7

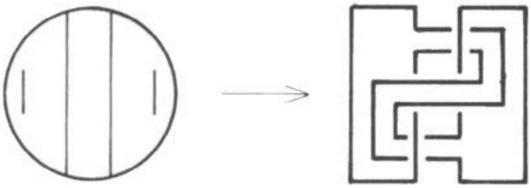


Fig. 13.1.8

that $\partial D = k$ and D has no minimum in $\mathbf{R}^3[0, \infty)$ (cf. [Kawauchi-Shibuya-Suzuki 1983]). That is, we have the following:

Corollary 13.1.10 *For a 2-knot K in normal form, the knot $K \cap \mathbf{R}^3[0]$ is a ribbon knot.*

Exercise 13.1.11 Prove the converse of this corollary. Namely, for any ribbon knot k, there exists a 2-knot in normal form whose level 0 cross-section is k.

A ribbon knot is clearly a slice knot (cf. 12.1), but the converse is an unsolved problem called the *slice-ribbon problem* (cf. [Fox 1962"(Problem 25)]).

Slice-Ribbon Problem Is a slice knot a ribbon knot?

For example, the slice knots which are in figure 12.1.1 or constructed in Lemma 12.2.1 are known to be ribbon knots.

By a *surface-link* in $\mathbf{R}^4$ (or S^4), we mean a locally flat closed 2-manifold M in $\mathbf{R}^4$ (or S^4). When M is a connected 2-manifold (i.e., a surface), it is called a *surface-knot.* In the case when M is an oriented 2-manifold, it is called an *oriented surface-link.* Two surface-links M and M' in $\mathbf{R}^4$ (or S^4) are said to be *equivalent* if there is an auto-homeomorphism h of $\mathbf{R}^4$ (or S^4) such that $h(M) = M'$. More precisely, they are *positive-equivelent* or *negative-equivalent* according to whether h is orientation-preserving or orientation-reversing. They are also said to *belong to the same type* if they are oriented and h and $h|_M\colon M \cong M'$ are orientation-preserving. The proof of Theorem 13.1.8 can be generalized to a surface-link M in $\mathbf{R}^4$ (cf. [Kawauchi-Shibuya-Suzuki 1982], [Kamada 1989]). In particular, M is ambient isotopic to a surface-link M' in $\mathbf{R}^4$ whose critical points are only maxima, minima and saddle points in the levels 1, -1 and 0, respectively. Then we can consider a diagram D in $\mathbf{R}^2$ (like a link diagram) of the graph $\Gamma = M' \cap \mathbf{R}^3[0]$

in $\mathbf{R}^3[0]$ so that D is a quadrivalent planar graph in $\mathbf{R}^2$ whose vertices consist of the vertices of Γ (=the saddle points of M') and the crossing points between the edges of Γ. Let $\mathrm{ch}(D)$ be the number of the vertices of D. The ch-*index* $\mathrm{ch}(M)$ of M in $\mathbf{R}^4$ is defined to be the minimal $\mathrm{ch}(D)$ for all M' and all D. $\mathrm{ch}(M)$ is clearly an invariant of M in $\mathbf{R}^4$ up to equivalence. The ch-index $\mathrm{ch}(M)$ was introduced in [Yoshikawa 1994] to enumerate surface-links in $\mathbf{R}^4$. It turns out that there exist (up to equivalence) just 23 surface-links in $\mathbf{R}^4$ with ch-indices less than or equal to ten; they are listed in Appendix F. The non-trivial 2-knot with the smallest ch-index is the spun knot of the trefoil knot (see 13.2.1(2)).

Exercise 13.1.12 By an argument similar to the proof of Theorem 13.1.8, confirm the following fact:

Any oriented surface-knot M in $\mathbf{R}^4$ is ambient isotopic to a surface-knot M' which has all the following properties:

(1) All critical points of M' are in critical bands.
(2) All maximal bands are in $\mathbf{R}^3[4]$.
(3) All minimal bands are in $\mathbf{R}^3[-4]$.
(4) All saddle bands are in $\mathbf{R}^3[\pm 3] \cup \mathbf{R}^3[\pm 1]$.
(5) $M' \cap \mathbf{R}^3[2]$ and $M \cap \mathbf{R}^3[-2]$ are knots.
(6) $M' \cap \mathbf{R}^3[0]$ is a link of $g+1$ components where g is the genus of M'.
(7) $M' \cap \mathbf{R}^3[0,\infty)$ and $M' \cap \mathbf{R}^3(-\infty, 0]$ are disks with g open disks removed.

We call this oriented surface-knot M' an *oriented surface-knot in normal form.*

Exercise 13.1.13 Assume that a 2-knot in normal form has one maximum and one minimum and has a trivial knot in level 0. Prove that this 2-knot bounds a 3-ball in $\mathbf{R}^4$.

Example 13.1.14. Assume that a 2-knot in normal form has a trivial knot in level 0 and has two maxima in $\mathbf{R}^3[0,\infty)$ and two minima in $\mathbf{R}^3(-\infty, 0]$. Then this 2-knot is known to be trivial (cf. [Scharlemann 1985], [Howie-Short 1985], [Thompson 1987]).

Example 13.1.15. The 2-knot defined by the cross sections shown in figure 13.1.9 has a non-trivial prime knot in the level 0, but it is known to be a trivial 2-knot (cf. [Terasaka-Hosokawa 1961]).

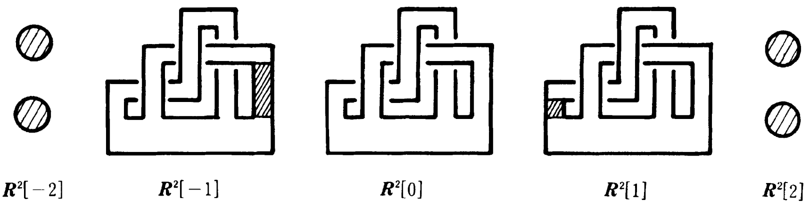


Fig. 13.1.9

Example 13.1.16. The 2-knot defined by the cross sections shown in figure 13.1.10 is not a trivial knot. (See Theorem 13.4.11 for an intuitive proof and figure 14.1.1 for an algebraic proof.)

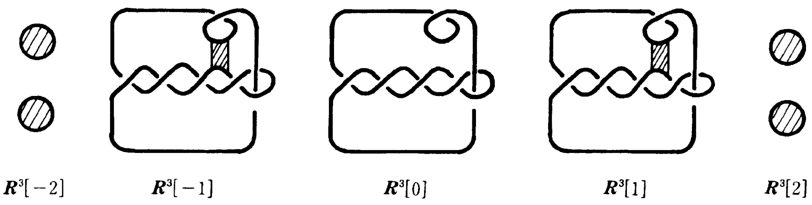


Fig. 13.1.10

13.2 Constructing 2-knots

In this section, we discuss how to construct 2-knots. The following construction was first made by [Artin 1925] and then generalized to the present form by [Zeeman 1965], [Litherland 1979] and many others:

Construction 13.2.1

(1) Let $g : (B^3, \beta) \to (B^3, \beta)$ be an orientation preserving homeomorphism with $g\,|_{\partial B^3}$=id, where (B^3, β) is a (3,1)-ball pair. Identifying $(x, 0)$ and $(g(x), 1)$ in $(B^3, \beta) \times [0, 1]$, we obtain a manifold pair $(B^3, \beta) \times_g S^1$. Then $(\partial B^3, \partial\beta) \times D^2 \cup (B^3, \beta) \times_g S^1$ forms a (4,2)-sphere pair, which is called the *g-spun knot* of (B^3, β).

(2) When we take g to be the identity map, the g-spun knot is called the *spun knot.*

(3) Assume that the knotted arc β in B^3 joins the north pole and the south pole and g is the rotation n times around the axis through these poles. Then the g-spun knot is called the *n-twist spun knot.*

In (1), if we take β to be an r-string tangle in B^3, then we obtain an r-component link in S^4, called the *g-spun link* of (B^3, β). This construction can be generalized to a construction in higher dimensions without essential changes.

Exercise 13.2.2 If two homeomorphisms g and h are isotopic keeping ∂B^3 fixed, then show that the g-spun knot and the h-spun knot belong to the same knot type.

Exercise 13.2.3

(1) Take a suitable example of (B^3, β), and draw normal forms of the spun knot and n-twist spun knot.

(2) Write group presentations of the spun knot and the n-twist spun knot by using a Wirtinger presentation of $\pi_1(B^3 - \beta)$.

We can study a 2-knot in $\mathbf{R}^4$ by projecting it into $\mathbf{R}^3$, which is analogous to the regular projection of a knot. The following construction was first made by [Yajima 1964] from this point of view (see also [Yanagawa 1969]):

Construction 13.2.4 Let $\{B_i^3 \mid i = 0, 1, \ldots, n\}$ be the family of mutually disjoint 3-balls in S^4 (or $\mathbf{R}^4$) and $S_i^2 = \partial B_i^3$. Let $I = [0, 1]$. Assume that a family of embeddings with mutually disjoint images $f_i : B^2 \times I \to S^4$ $(i = 1, 2, \ldots, n)$ has the following property:

$$f_i(B^2 \times I) \cap S_k^2 = \begin{cases} f_i(B^2, 0) & \text{if } k = i - 1, \\ f_i(B^2, 1) & \text{if } k = i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then the embedded 2-sphere

$$(\bigcup_{i=0}^{n} S_i^2) \cup (\bigcup_{i=1}^{n} f_i(\partial B^2 \times I)) - \bigcup_{i=1}^{n} f_i(\mathrm{int} B^2 \times \partial I)$$

is called a *ribbon 2-knot* of *n-fusion* along the *bands* $f_i(B^2 \times I)$ $(i = 1, 2, \ldots, n)$. Further, if we have additional embeddings (with mutually disjoint images) $f_i : B^2 \times I \to S^4$ $(i = n+1, n+2, \ldots, s)$ such that $f_i(B^2 \times I) \cap S_k^2$ is $f_i(B^2 \times \partial I)$ for $k = 0$ and $\emptyset$ for $k \geq 1$ and the surface $S_0^2 \cup f_i(\partial B^2 \times I) - f_i(\mathrm{int} B^2 \times \partial I)$ is orientable, then we have a closed orientable surface F of genus $s - n$ in S^4 (instead of a 2-sphere), which is called a *ribbon surface* in S^4 of *s-fusion* along the *bands* $f_i(B^2 \times I)$ $(i = 1, 2, \ldots, s)$.

Ribbon surfaces are discussed in [Kawauchi-Shibuya-Suzuki 1983] as a generalization of ribbon 2-knots. This construction can be also generalized to a construction in higher dimension without essential changes.

Exercise 13.2.5 Show that any ribbon 2-knot K has a normal form which is invariant under reflection of $\mathbf{R}^4$ with respect to $\mathbf{R}^3[0]$. Using this normal form of K, show that the group $\pi_1(S^4 - K)$ has a Wirtinger presentation of deficiency one.

Let M be an m-manifold with non-empty boundary. Let $\alpha : \partial B^p \times D^{m-p} \to \partial M$ be an embedding, where B^p and D^{m-p} denote the p- and $(m-p)$-balls, respectively. Then we can obtain from M and $B^p \times D^{m-p}$ a new manifold $M \cup_\alpha B^p \times D^{m-p}$ by pasting along α. This $B^p \times D^{n-p}$ with embedding α is called a *p-handle* on M with *attaching map* α and denoted by h^p or $h^p(\alpha)$. An *attaching sphere* of this p-handle on M is the $(p-1)$-sphere $\alpha(\partial B^p \times x)$ for a point $x \in \mathrm{int} D^{m-p}$. We sometimes consider a $(p+1)$-handle $h^{p+1}(\beta)$ on $M \cup h^p(\alpha)$ such that $\alpha(x \times D^{m-p}) \cap \beta(\partial B^{p+1} \times y)$ consists of exactly one point for some points $x \in \partial B^p$ and $y \in D^{m-p-1}$:

Definition 13.2.6 Such a p-handle h^p on M is a *trivial* p-handle on M and the $(p+1)$-handle h^{p+1} on $M \cup h^p$ is a *complementary handle* of h^p.

In this case, we have that $M \cup h^p(\alpha) \cup h^{p+1}(\beta)$ is homeomorphic to M (cf. [Rourke-Sanderson 1972]).

The following construction is a special version due to [Marumoto 1987] of a well-known surgery construction on high-dimensional knots (cf. [Kervaire 1965, 1965'], [Levine 1975]):

Construction 13.2.7 Let p be 1 or 2. Let $\{h_i^p \mid i = 1, 2, \ldots, n\}$ be a disjoint family of trivial p-handles on the 5-ball B^5. Let h_i^{p+1} be a complementary handle of h_i^p with $h_i^p \cap h_k^{p+1} = \emptyset$ for $i \neq k$. Let B^3 be a 3-ball in ∂B^5 such that $B^3 \cap h_i^p = \emptyset$ and $\partial B^3 \cap h_i^{p+1} = \emptyset$ for all i. Then since $B^5 \cup (\bigcup_{i=1}^n h_i^p) \cup (\bigcup_{i=1}^n h_i^{p+1})$ is a 5-ball, say D^5, we see that ∂B^3 is an embedded 2-sphere in the 4-sphere ∂D^5. This 2-sphere is called a *2-knot of type p*.

The following theorem is due to [Marumoto 1987]:

Theorem 13.2.8 *A 2-knot is a ribbon 2-knot if and only if it is a 2-knot of type* 1. *Every 2-knot is of type* 2.

Exercise 13.2.9 Show that the spun knot is a ribbon 2-knot.

Definition 13.2.10 A 2-knot K in $\mathbf{R}^4$ is *simply knotted* if, after an ambient isotopy of K, the image $p(K)$ under an orthogonal projection $p : \mathbf{R}^4 \to \mathbf{R}^3$ is an immersed (i.e., a locally embedded) sphere whose singular set consists of only double points.

The following theorem is due to [Yajima 1964]:

Theorem 13.12.11 *A 2-knot is a ribbon 2-knot if and only if it is simply knotted.*

Exercise 13.2.12 For any 2-knot K, show that $p(K)$ is an immersed 2-sphere in $\mathbf{R}^3$ after an ambient isotopy of K.

There remain many research problems related to $p(K)$. For example, how are the topological properties of K reflected in $p(K)$ (cf. [Giller 1982])? Can one develop a 2-knot theory similar to that in Chapter 8? In [Carter-Saito 1993'], certain moves for 2-knots corresponding to the Reidemeister moves for knots are studied (cf. [Carter 1993]). We also mention here that another description for 2-knots, called the *braid presentation*, is studied in [Kamada 1992].

13.3. Seifert hypersurfaces

Here, we discuss compact connected oriented 3-manifolds in $\mathbf{R}^4$ (or S^4) which are bounded by 2-knots.

Definition 13.3.1 A *Seifert hypersurface* for a 2-knot K in S^4 (or $\mathbf{R}^4$) is a locally flat compact connected oriented 3-manifold V with $\partial V = K$.

The proof of the following theorem is similar to that of Lemma 12.2.6 and is omitted:

Theorem 13.3.2 *Every locally flat closed oriented 2-manifold in S^4 bounds a locally-flat oriented 3-manifold in S^4. In particular, every 2-knot has a Seifert hypersurface.*

It is also known that a locally flat closed non-orientable surface in S^4 bounds a 3-manifold in S^4 if and only if the normal Euler number is 0 (cf. [Kamada 1989]).

Exercise 13.3.3 Using the normal form of a 2-knot K, construct a Seifert hypersurface of K (cf. [Gluck 1962], [Kawauchi-Shibuya-Suzuki 1983]).

Next, we study Seifert hypersurfaces of ribbon 2-knots.

Definition 13.3.4 A Seifert hypersurface M of a 2-knot in S^4 is *semi-unknotted* if there exist finitely many disjoint 2-spheres $S_1^2, \ldots, S_n^2$ in int$M$ such that
(1) S_i^2 $(i = 1, 2, \ldots, n)$ bound mutually disjoint 3-balls B_i^3 $(i = 1, 2, \ldots, n)$ in S^4, such that the restriction to S_i^2 of a bicollar of B_i^3 in S^4 is a regular neighborhood N_i of S_i^2 in M, and
(2) $M - (\cup_i \mathrm{int} N_i)$ is a 3-ball with $2n$ open 3-balls removed.

The 2-sphere family $S_1^2, \ldots, S_n^2$ is called a *trivial system* for the semi-unknotted Seifert hypersurface M. Note that a 3-ball in S^4 is semi-unknotted. The following theorem is given in [Yanagawa 1969]:

Theorem 13.3.5 *A 2-knot is a ribbon 2-knot if and only if it has a semi-unknotted Seifert hypersurface.*

It is shown in [Cochran 1983] that there exists a non-ribbon 2-knot with a Seifert hypersurface homeomorphic to $S^1 \times S^2$ with an open 3-ball removed.

13.4. Exteriors of 2-knots

In this section, we discuss when 2-knots are determined by their exteriors. The results in the first half of this section is given in [Gluck 1962]. We consider the mapping class group $\Psi(S^2 \times S^1)$ of $S^2 \times S^1$. Let f be a representative of an element of $\Psi(S^2 \times S^1)$. Then f induces automorphisms on $H_1(S^2 \times S^1)$ and $H_2(S^2 \times S^1)$, which show that the homomorphism $\phi : \Psi(S^2 \times S^1) \to \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is onto. Taking the antipodal map $r : S^2 \to S^2$ and a map $s : S^1 \to S^1$ defined by $s(x_1, x_2) = (x_1, -x_2)$, where $S^1 = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$, we have $\phi([r \times \mathrm{id}]) = (0, 1)$ and $\phi([\mathrm{id} \times s]) = (1, 0)$. Let ρ_t be the $2\pi t$-rotation of $S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ around the x_3-axis. We define a map $T : S^2 \times S^1 \to S^2 \times S^1$ by $T(x, t) = (\rho_t(x), t)$, where we identify $S^1 = \mathbf{R}/\mathbf{Z}$. One can verify that $\phi([T]) = (1, 1)$ and prove that $\mathrm{Ker}\phi = \{[\mathrm{id}], [T]\}$, which is isomorphic to $\mathbf{Z}_2$. This gives the following theorem:

Theorem 13.4.1 $\Psi(S^2 \times S^1) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

The following two lemmas follow directly:

Lemma 13.4.2 *If* $[h] \in \Psi(S^2 \times S^1)$ *is one of* $[\mathrm{id} \times \mathrm{id}]$, $[r \times \mathrm{id}]$ *or* $[\mathrm{id} \times s]$, *then* h *can be extended to a homeomorphism of* $S^2 \times D^2$ *onto itself keeping* $S^2 \times 0$ *fixed.*

Lemma 13.4.3 *Re-gluing a regular neighborhood of a 2-knot in* S^4 *to the exterior using the auto-homeomorphism* T, *we obtain from the 4-sphere* S^4 *a homotopy 4-sphere.*

Exercise 13.4.4 *Prove Lemmas 13.4.1 and 13.4.2.*

The following theorem is given in [Gluck 1962]:

Theorem 13.4.5 *The number of 2-knots with the same exterior is at most two up to equivalence.*

The following theorem is given in [Gordon 1976] and [Plotnick-Suciu 1985] (see [Cappell-Shaneson 1976] for higher dimensional knots):

Theorem 13.4.6 *There exist inequivalent 2-knots with the same exterior.*

For ribbon 2-knots, we can see the uniqueness of a 2-knot with a given exterior.

Lemma 13.4.7 *For $V = \sharp S^2 \times S^1 - \mathrm{int}B^3$, there exists an ambient isotopy $\{\rho_t^*\}$: $V \to V$ such that $\rho_t^* |_{\partial V} = \rho_t$ and $\rho_0^* = \rho_1^* = \mathrm{id}$.*

Theorem 13.4.8 *A ribbon 2-knot is uniquely determined by the exterior up to positive-equivalence.*

Proof. Let K be a ribbon 2-knot. Let V be a semi-unknotted Seifert hypersurface V of K. Take a regular neighborhood $S^2 \times D^2$ of K in S^4 so that $K = S^2 \times 0$. Let $E = \mathrm{cl}(S^4 - S^2 \times D^2)$. Apply Lemma 13.4.7 to a collar $V \times I$ of V in S^4. Then we can see that T, which is defined on $\partial E = S^2 \times S^1$, can be extendable to an auto-homeomorphism of E. This gives a homeomorphism from S^4 onto $E \cup_r S^2 \times D^2$ sending K to $S^2 \times 0$. We can assume that this homeomorphism is orientation-preserving due to 13.2.5. □

Exercise 13.4.9 Complete the proof above explicitly.

The exterior of a trivial n-knot is $B^{n+1} \times S^1$, which is homotopy-equivalent to S^1.

The unknotting theorem An n-knot with $n \neq 2$ is trivial if and only if the exterior is homotopy-equivalent to S^1.

This is proved for $n = 1$ by [Papakyriakopoulos 1957] and [Homma 1957] and for $n \geq 3$ by [Levine 1965"], [Shaneson 1968] and [Wall 1965]. For $n = 2$, it is observed in [Kawauchi 1974] that the 2-knot exterior E is homotopy equivalent to S^1 if and only if $\pi_1(E) \cong \mathbf{Z}$. Thus, we have the following conjecture:

The unknotting conjecture for 2-knots A 2-knot is trivial if and only if the fundamental group of the 2-knot exterior is infinite cyclic.

This was proved by [Freedman 1983] in TOP category, although it is still unsolved in the PL (or smooth) category. The unknotting result for general oriented surface-knots was claimed in [Hillman-Kawauchi 1995] (also in the TOP category), but unfortunately the proof used the splitting result for closed oriented 4-manifolds with infinite cyclic fundamental group of [Kawauchi 1994"'] which contains an error in its proof and is seen to be false in general by [Hambleton-Teichner *]. For non-orientable surface-knots, the unknotting result is known only in a few cases (also in the TOP category), see [Kreck 1990] and [Lawson 1984]. Thus, regarding the unknotting of surface-knots, there remain many important research problems to be solved.

By a *simple-ribbon* 2-knot, we mean a 2-knot constructed as in Construction 13.2.7 in the case $p = n = 1$. The following theorem is obtained from [Marumoto 1977, 1984]:

Theorem 13.4.11 *For the class of simple-ribbon* 2*-knots, the unknotting conjecture is true.*

Proof. We use the notations in 13.2.7. Let $V = B^5 \cup h^1$. We note that the group $\pi_1(\partial V - \partial B^3)$ is a rank two free group with a basis t, a such that t is a meridian element of ∂B^3 in ∂V and a comes from a generator of $\pi_1(\partial V - B^3) \cong \mathbf{Z}$ naturally related to the 1-handle attachment of h^1 to B^5. Then the group $\pi = \pi_1(\partial D^5 - \partial B^3)$ has the presentation $\langle a, t \mid w = 1\rangle$ where w is a word in a and t which is represented by an attaching sphere of the 2-handle h^2 in $\partial V - \partial B^3$. Since h^2 is a complementary handle for h^1, the exponent sum of a in w is ± 1. Then we can arrange the handles so that

(13.4.11.1) The exponent sum of t in w is zero

(cf. [Marumoto 1984]). If π is infinite cyclic, then we have that

$$\langle a, t \mid w = 1\rangle = \langle a, t \mid a = 1\rangle.$$

By the conjugacy theorem for one-relator groups (see [Magnus-Karrass-Solitar 1966]), we have that $w = a$ or $w = a^{-1}$; this implies that ∂B^3 is a trivial 2-knot in ∂D^5. □

The following corollary is proved in [Marumoto 1977] and is generalized to a fibering theorem for a certain class of high-dimensional knots by [Yoshikawa 1981]:

Corollary 13.4.12 *If the exterior of a ribbon* 2*-knot of* 1*-fusion has infinite cyclic fundamental group, then the* 2*-knot is trivial.*

If a 2-knot K in normal form has a trivial knot in level 0, then we can see that the fundamental group of the exterior of K is infinite cyclic. Example 13.1.13 gives a partial answer to the following conjecture:

Conjecture If a 2-knot in normal form has a trivial knot in level 0, then the 2-knot is trivial.

13.5. Cyclic covering spaces

In this section, for simplicity, we restrict our attention to simple-ribbon 2-knots. That is, we consider the situation in which we are given a trivial 1-handle h^1 on a 5-ball B^5, a complementary handle h^2 for h^1, and a 3-ball B^3 in ∂B^5 such that $B^3 \cap h^1 = \emptyset$ and $\partial B^3 \cap h^2 = \emptyset$. Let $D^5 = B^5 \cup h^1 \cup h^2$, which is a 5-ball, and $(S^4, K) = (\partial D^5, \partial B^3)$. We also assume that the attaching sphere of the 2-handle h^2 has property (13.4.11.1). Let W_∞ be the infinite cyclic connected covering space over $\partial V - \partial B^3$. Then W_∞ can be extended to an infinite cyclic covering space $\widehat{W_\infty}$ over $V - \widehat{B}$ where $\widehat{B}$ is a proper 3-ball in V obtained from B^3 by pushing $\mathrm{int}B^3$ into $\mathrm{int}V$. Let $\{h_i^1\}$ be the lifts of h^1 in $\widehat{W_\infty}$. Since an attaching sphere of the 2-handle h^2 can be lifted to W_∞, the lifts $\{h_i^2\}$ of h^2 are 2-handles on W_∞ so that $W_\infty \cup \partial(\bigcup_i h^2) - \mathrm{int}(W \cap (\bigcup_i h_i^2))$ is the infinite cyclic connected covering space X_∞ over $X = S^4 - K$. Using this construction, we can calculate the first homology group of X_∞ as follows:

Theorem 13.5.1 *The Λ-module $H_1(X_\infty)$ is isomorphic to $\Lambda/(f(t))$, where $f(t)$ is a Laurent polynomial obtained from w by the Fox derivative on a.*

Clearly, $\Delta \doteq f(t)$ for the Alexander polynomial Δ of the 2-knot group $\pi = \pi_1(S^4 - K)$ defined in 14.1. As noted there, $\Delta(1) = \pm 1$ for any 2-knot group.

Exercise 13.5.2

(1) Give a complete proof of Theorem 13.5.1 and confirm that $f(1) = \pm 1$.
(2) Generalize Theorem 13.5.1 to a general 2-knot of type 1 (that is a general ribbon knot).

The following theorem, given first in [Kinoshita 1961], characterizes the Alexander polynomials of 2-knot groups:

Theorem 13.5.3 *Let $f(t)$ be a Laurent polynomial on t with $f(1) = \pm 1$. Then there exists a simple-ribbon 2-knot with $H_1(X_\infty) = \Lambda/(f(t))$.*

Proof. Assume $f(t) = \sum_{i=0}^{n} c_i t^i$. Using the notation in the proof of Theorem 13.4.11, we set $w = a^{c_0} t a^{c_1} t a^{c_2} t \dots t a^{c_n} t^{-n}$. Then Theorem 13.5.1 implies the required result. □

As for the infinite cyclic covering space above, we can develop a theory for a finite cyclic covering space over S^4 with branch set a ribbon 2-knot. The following theorem due to [Sumners 1975] shows that the generalized Smith conjecture does not hold for high-dimensional knots (cf. [Giffen 1966], [Gordon 1974], [Kanenobu 1985] and [Teragaito 1989]):

Theorem 13.5.4 *There exists a $\mathbf{Z}_p$-action on S^4 whose fixed point set is a non-trivial simple-ribbon 2-knot.*

Proof. We set $w = a^2 t^p a^{-1} t^{-p}$ according to the notation in the proof of Theorem 13.4.11. Then the p-fold cyclic covering space over S^4 with branch set K is seen to be S^4. □

Exercise 13.5.5 In Theorem 13.5.4, let $\widetilde{K}$ be the lift of K in the p-fold branched covering space. Is $\widetilde{K}$ a ribbon 2-knot? Study the relation between K and $\widetilde{K}$.

13.6. The k-invariant

We discuss here the second homotopy group $\pi_2(X)$ of a 2-knot exterior X. Computations of the higher homotopy groups as abstract groups for high-dimensional knots and links are made, for example, in [Andrews-Curtis 1959], [Epstein 1960] and [Gordon 1973]. Here, we consider $\pi_2(X)$ as a left $\mathbf{Z}[\pi]$-module with $\pi = \pi_1(X)$ (cf. [Hu 1959]). Let X be the exterior of a ribbon 2-knot of 1-fusion. Then we can write that $\pi = \langle a, t \mid w \rangle$. By considering the universal covering space of X, we obtain the following (cf. [Lomonaco 1981], [Suciu 1985]):

Theorem 13.6.1 *There is an isomorphism*

$$\pi_2(X) \cong \mathbf{Z}[\pi]/(\partial w/\partial a)^*$$

as left $\mathbf{Z}[\pi]$*-modules, where* $(\sum n_g g)^* = \sum n_g g^{-1}$ *for* $n_g \in \mathbf{Z}$ *and* $g \in \pi$*, and* $\partial/\partial a$ *denotes the Fox derivative.*

Example 13.6.2. Let X be the exterior of the ribbon 2-knot K of 1-fusion shown in figure 13.6.1. Then $\pi(X) = \langle a, t \mid tat^{-1}a^{-2}\rangle$ and $\pi_2(X) \cong \mathbf{Z}[\pi]/(1 + a^{-1} - t^{-1})$. Note that K is the spun knot of a trefoil knot.

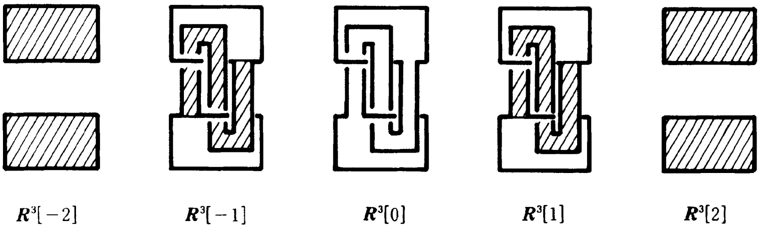


Fig. 13.6.1

In [Suciu 1985], the following theorem is obtained from Theorem 13.6.1:

Theorem 13.6.3 *There exist infinitely many ribbon 2-knots with the same* π *but with different* π_2 *as left* $\mathbf{Z}[\pi]$*-modules.*

Next, we study 2-knots with the same π and with the same π_2 as left $\mathbf{Z}\pi$-modules. To do it, let X be a polyhedron of a connected cell complex and $\pi = \pi_1(X)$. We construct a path-connected topological space X^+ from X with a cell structure (called a *CW complex*) by attaching 3- and 4-cells so that the inclusion map $X \subset X^+$ induces an isomorphism $\pi_1(X) \cong \pi_1(X^+)$ and $\pi_i(X^+) = 0$ for $i = 2, 3$. (Note that the base point is omitted here.) Let $\overline{X}$ be the universal covering space of X, and $\overline{X}^+$ the union of $\overline{X}$ and the lifting 3- and 4-cells. The following part of the cellular chain complex $C_\sharp(\overline{X}^+)$ is exact as left $\mathbf{Z}[\pi]$-modules:

$$C_4(\overline{X}^+) \to C_3^+ \oplus C_3(\overline{X}) \xrightarrow{\partial^+ + \partial_3} C_2(\overline{X}) \xrightarrow{\partial_2} C_1(\overline{X}) \to C_0(\overline{X}) \to \mathbf{Z} \to 0,$$

where we write $C_3(\overline{X}^+) = C_3^+ \oplus C_3(\overline{X})$. Then we can regard ∂^+ as a left $\mathbf{Z}[\pi]$-homomorphism $C_3^+ \to \mathrm{Ker}\partial_2$. For the projection $p : \mathrm{Ker}\partial_2 \to H_2(\overline{X}) = \pi_2(\overline{X}) = \pi_2(X)$, the composite left $\mathbf{Z}[\pi]$-homomorphism $p\partial^+ : C_3^+ \to \pi_2(X)$ defines a unique 3-cocycle of the cochain complex $\mathrm{Hom}(C_\sharp(X^+), \pi_2(X))$ consisting of left $\mathbf{Z}[\pi]$-homomorphisms. By taking the cohomology, we obtain an element in $H^3(\pi_1(X); \pi_2(X))$, which is seen to be a homotopy type invariant of X.

Definition 13.6.4 The *k-invariant* (or the *Eilenberg-Postonikov invariant*) of X is this element.

The following result is obtained in [Plotonick-Suciu 1985] using the k-invariant:

Theorem 13.6.5 *There exist distinct 2-knots (up to equivalence) with the same* π *and with the same* π_2 *as left* $\mathbf{Z}[\pi]$*-modules.*

Open question Are there distinct ribbon 2-knots (up to equivalence) with the same π and with the same π_2 as left $\mathbf{Z}[\pi]$-modules? Does the homotopy type of the exterior determine a ribbon 2-knot up to equivalence?

In [Lomonaco 1981], a notion called the *quasi-asphericity* of a 2-knot is introduced, and it is proved that the homotopy type of the exterior of a quasi-aspherical 2-knot is determined by π and π_2 as a left $\mathbf{Z}[\pi]$-module.

13.7 Ribbon presentations

In Construction 13.2.4, a ribbon 2-knot K of n-fusion is constructed from a family $\mathcal{O}$ of disjoint 3-balls B_i^3 $(i = 0, \ldots, n)$ in S^4 and a family $\mathcal{B}$ of $n + 1$ bands by piping the components of $\partial\mathcal{O}$ along the bands.

Definition 13.7.1 This pair $(\mathcal{O}, \mathcal{B})$ is a *ribbon presentation* for the ribbon 2-knot K.

We also call $\mathcal{O}$ the *base* of the ribbon presentation.

Definition 13.7.2 (1) Two ribbon presentations $(\mathcal{O}_1, \mathcal{B}_1)$ and $(\mathcal{O}_2, \mathcal{B}_2)$ are *simply equivalent* if there exists an orientation-preserving homeomorphism f of S^4 onto itself sending $\partial\mathcal{O}_1$ onto $\partial\mathcal{O}_2$ orientation-preservingly, and $\mathcal{B}_1$ onto $\mathcal{B}_2$.
(2) Two ribbon presentations are *stably equivalent* if one can be deformed into the other by a finite sequence of the following three types of moves: *trivial addition/deletion*, *band slide* and *band pass*.

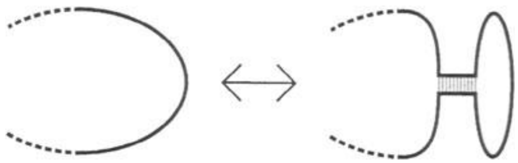


Fig. 13.7.1: trivial addition/deletion

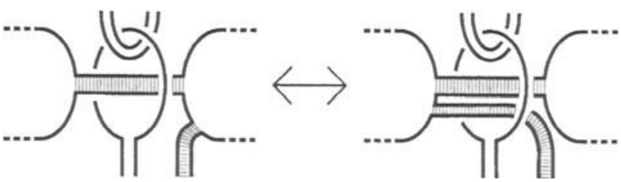


Fig. 13.7.2: band slide

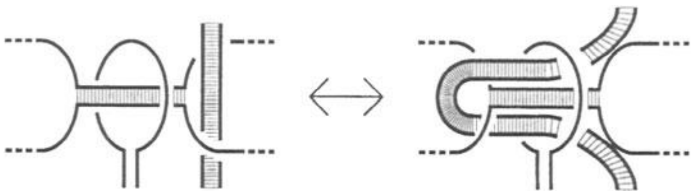


Fig. 13.7.3: band pass

Let β be a proper arc in the half-space $R^3_+ = \{(x,y,z) \in \mathbf{R}^3 \mid z \geq 0\}$. Let $\mathbf{R}^2 = \{(x,y,z) \in \mathbf{R}^3 \mid z = 0\}$. Then we can consider that the image of β under the orthogonal projection $\mathbf{R}^3_+ \to \mathbf{R}^2$ with respect to the z-axis is a diagram with overbridges $a_1, \ldots, a_m$ and underbridges $c_1, \ldots, c_{m-1}$. For an arc l in R^3_+, let

$$l^* = \{(x,y,z) \mid \exists (x,y,z') \in l, 0 \leq z \leq z'\}.$$

We call l^* the *shadow* of l. Regard a 3-ball B^3 as the one-point compactification of R^3_+. When we construct the spun knot of (B^3, β) by Construction 13.2.1, we obtain 3-balls B^3_i $(i = 1, 2, \ldots, m)$ and Δ_i $(i = 1, 2, \ldots, m-1)$ where B^3_i and Δ_i are obtained from the shadows a^*_i and c^*_i respectively of the overbridges and underbridges a_i and c_i by the spinning process. Taking a suitable embedding $f_i : B^2 \times I \to S^4$ with $f_i(B^2 \times I) = \Delta_i$, we obtain the following:

Theorem 13.7.3 *$(\{\Delta_i\}, \{f_i(B^2 \times I)\})$ is a ribbon presentation of the spun knot of (B^3, β).*

Exercise 13.7.4 Prove Theorem 13.7.3 (cf. [Marumoto 1992]).

This ribbon presentation strongly depends on the choice of diagram of the proper arc β. However, this difference is not serious, as is stated below (cf. [Marumoto 1992]):

Theorem 13.7.5 *The ribbon presentations in Theorem 13.7.3 of the spun knot of (B^3, β) that are constructed from any two diagrams of β are stably equivalent.*

The following theorem is proved by [Yasuda 1992]:

Theorem 13.7.6 *There exists a ribbon 2-knot with at least two ribbon presentations of 1-fusion which are not simply equivalent.*

The corresponding result for knots in S^3 is obtained in [Nakanishi-Nakagawa 1982] (see also [Kawauchi 1993'] for another example).

Open Question

(1) Are any two ribbon presentations of a 2-knot stably equivalent?
(2) Do there exist infinitely many ribbon presentations for a ribbon 2-knot of 1-fusion?

Supplementary notes for Chapter 13

Several results on high-dimensional knots which are not discussed here or in the next chapter are found in [Suzuki 1976], [Kervaire-Weber 1978] and [Hillman 1989]. A certain "normal form" for high-dimensional knot exteriors is discussed in [Kearton 1975], which gives an effective method to classify high-dimensional knots algebraically. Example 13.1.14 is related to the properties P and Q on knots (see Supplementary notes for Chapter 4). Combining it with a result of [Marumoto 1977], we see that strongly invertible knots have property Q (see [Gabai 1987] for a more general result). In [Kato 1969], an $(n+2)$-dimensional "knot manifold" (obtained from S^{n+2} by a 2-handle surgery along an n-dimensional knot) is studied.

Chapter 14
Two-knots II: an algebraic approach

By an *n-knot group*, we mean the fundamental group $\pi_1(S^{n+2} - K^n, b)$ for an n-knot K^n in S^{n+2}. Similarly, by a *surface-knot group*, we mean the fundamental group $\pi_1(S^4 - F, b)$ for an oriented surface-knot F in S^4. In this chapter, we discuss some properties of 2-knot groups in comparison with those of the other dimensional knot groups and surface-knot groups.

14.1 High-dimensional knot groups

Here, we discuss some general properties of the n-knot groups for each n. A necessary condition for a group to be an n-knot group is known (see 6.3 for the homology of a group):

Theorem 14.1.1 *An n-knot group π $(n \geq 1)$ has the following properties:*

(1) *π is finitely presented.*
(2) *π/π' is an infinite cyclic group, where π' denotes the commutator subgroup of π.*
(3) *$H_2(\pi) = 0$.*
(4) *There is an element x (which we call a meridian) of π such that $\langle\langle x \rangle\rangle^\pi = \pi$.*

Proof. Let $E(K^n)$ be the exterior of an n-knot K^n in S^{n+2} with $\pi \cong \pi_1(E(K^n), b)$. (1) is trivial. By the Alexander duality theorem, $\pi/\pi' \cong H_1(S^{n+2} - K^n) \cong H^n(K^n) \cong \mathbf{Z}$, which proves (2). It is known that for the Hurewicz homomorphism $\rho : \pi_2(E(K^n)) \to H_2(E(K^n))$, $H_2(\pi) \cong H_2(E(K^n))/\rho(\pi_2(E(K^n))$ (cf.[Hopf 1941]). Since $H_2(E(K^n)) = 0$, we have $H_2(\pi) = 0$, which proves (3). Now we prove (4). Let $a : (S^1, b') \to (S^{n+2} - K^n, b)$ be an embedding such that the image $a(S^1)$ is a meridian loop of K^n in S^{n+2}. We assume that $\pi = \pi_1(S^{n+2} - K^n, b)$. We show that $\ll x \gg^\pi = \pi$ for the element x of π represented by the loop $a : (S^1, b') \to (S^{n+2} - K^n, b)$. Let y be an element of π and $f : (S^1, b') \to (S^{n+2} - K^n, b)$ be a loop representing y. Since S^{n+2} is simply connected, we have an extension $F : D^2 \to S^{n+2}$ of f which intersects K^n transversely. Then $F^{-1}(K^n)$ consists of a finite number of points $u_1, \ldots, u_r$ of $\mathrm{int}D^2$. Let $D_1, D_2, \ldots, D_r$ be mutually disjoint disks in $\mathrm{int}D^2$ around the points $u_1, u_2, \ldots, u_r$, respectively. Since these are meridian disks of K^n in S^{n+2} up to orientations, we can deform each loop $F(\partial D_i)$ into $a(S^1)$ up to orientations, which represents an element x^{ϵ_i} $(\epsilon_i = \pm 1)$ of π. Let b'_i be the point of ∂D_i with $F(b'_i) = b$, and ω_i be a path in $\mathrm{cl}(D^2 - \cup_{i=1}^r D_i)$ from b'_i to b' such that $\omega_1, \ldots, \omega_r$ are mutually disjoint except at b'. Let β_i be the element of π represented by the loop $F(\omega_i)$. Then $y = \Pi_{i=1}^r \beta_i^{-1} x^{\epsilon_i} \beta_i$ (after a suitable permutation of the indices). Hence $y \in \ll x \gg^\pi$. □

It is also known that this necessary condition is sufficient for $n \geq 3$ (cf. [Kervaire 1965,1965']):

Theorem 14.1.2 *If a group π has the properties (1)–(4) of Theorem 14.1.1, then π is an n-knot group for each $n \geq 3$.*

Artin's spinning construction in [Artin 1925] (cf. 13.2.1(2)) implies the following:

Theorem 14.1.3 *For each $n \geq 1$, any n-knot group is an $(n+1)$-knot group.*

The next theorem was proved by [Yajima 1970] using a group theoretic method:

Theorem 14.1.4 *A group π has a Wirtinger presentation if it has properties (1)–(4) of Theorem 14.1.1.*

For $n = 1$ and 2, this can be also seen from Theorems 6.2.1 and 13.1.8 (cf. Theorem 14.2.1). Using Theorems 14.1.3 and 14.1.4, we can give a relatively simple proof of Theorem 14.1.2.

Proof of Theorem 14.1.2. By Theorem 14.1.3, it is sufficient to prove that π is a 3-knot group. By Theorem 14.1.4, we can find a Wirtinger presentation $\langle x_0, x_1, \ldots, x_m \mid r_1, r_2, \ldots, r_s \rangle$ for π such that $s \geq m$, $r_i = x_0^{-1} w_i x_i w_i^{-1}$ for $i = 1, 2, \ldots, m$, and $r_i = x_0^{-1} w_i x_0 w_i^{-1}$ for $i = m+1, \ldots, s$, where w_i denotes a word in $x_0, x_1, \ldots, x_m$. Let M_0 be an m-fold connected sum of $S^1 \times S^4$. Let K_0 be a trivial 3-knot in a 5-ball in M_0. We identify the group $\pi_1(M_0 - K_0, b)$ with the free group $\langle x_0, x_1, \ldots, x_m \rangle$ so that x_0 is represented by a meridian of K_0 and $x_i (i \geq 1)$ is represented by a loop homotopic to a factor $S^1 \times p (p \in S^4)$ of the i-th connected summand of $S^1 \times S^4$ in M_0. We take mutually disjoint simple loops $\ell_j, j = 1, 2, \ldots, s$ in $M_0 - K_0$ which are homotopic to loops representing the words $r_j, j = 1, 2, \ldots, s$. We note that ℓ_j is isotopic in M_0 to the j-th factor $S^1 \times p$ for each $j \leq m$ and to a trivial loop for each $j \geq m+1$ in M_0. We do surgeries on M_0 along these loops $\ell_j, j = 1, 2, \ldots, s$. Choosing framings of the ℓ_j's carefully, we obtain from the pair (M_0, K_0) a pair (M_1, K_1) such that M_1 is S^5 for $s = m$ or the $(s-m)$-fold connected sum of $S^2 \times S^3$ for $s > m$ and K_1 is a 3-knot in M_1 with $\pi_1(M_1 - K_1, b) \cong \pi$. Let $s > m$. By (2), we have a natural isomorphism $H_2(M_1 - K_1) \cong H_2(M_1)$. By (3) and [Hopf 1941], the Hurewicz homomorphism $\pi_2(M_1 - K_1, b) \to H_2(M_1 - K_1)$ is onto. Then by general position, we can represent a basis for the free abelian group $H_2(M_1)$ by mutually disjoint embedded 2-spheres in $M_1 - K_1$. Since the normal bundles of these 2-spheres in M_1 are trivial, we can do 3-handle surgeries on M_1 along these 2-spheres. Then we obtain from the pair (M_1, K_1) a pair (M_2, K_2) such that M_2 is a homotopy 5-sphere and K_2 is a 3-knot in M_2 with $\pi_1(M_2 - K_2) \cong \pi$. Since $M_2 \sharp - M_2$ is h-cobordant (and hence homeomorphic) to S^5 (cf. [Smale 1962]), we see that π is a 3-knot group. □

For an n-knot group π, the first homology group $H_1(\pi')$ is a finitely generated torsion module over $\Lambda = \mathbf{Z}\langle t \rangle$. The 0-th characteristic polynomial Δ is called the *Alexander polynomial* of π. Since $t-1$ is an automorphism of $H_1(\pi')$, we see that $\Delta(1) = \pm 1$.

Theorem 14.1.5 *There exists a 2-knot group which is not a 1-knot group.*

Proof. The Alexander polynomial of the group of Fox's 2-knot K_F shown in figure 14.1.1 is $\Delta(t) = 2t - 1$, which is not symmetric up to units $\pm t^m (m \in \mathbf{Z})$ of Λ. Since the Alexander polynomials of 1-knots are symmetric up to the units of Λ, the group of K_F is not a 1-knot group. $\square$

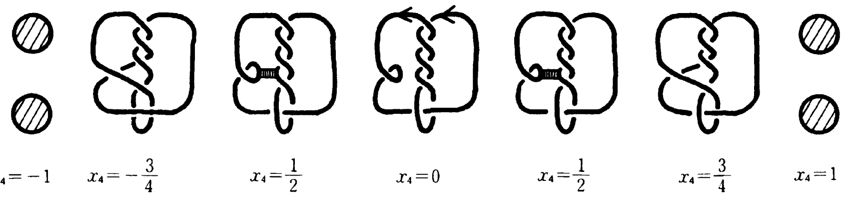


Fig. 14.1.1

Let X be the exterior $E(K^n)$ of an n-knot K^n in S^{n+2}. Let $\pi = \pi_1(X)$ be the group of K^n. The homology group $H_q(X_\infty)$ of the universal abelian (= infinite cyclic) covering space X_∞ is a finitely generated torsion Λ-module of which $t-1$ is an automorphism. Let $\mathrm{t}H_q(X_\infty)$ be the $\mathbf{Z}$-torsion part of $H_q(X_\infty)$, which is a Λ-submodule of $H_q(X_\infty)$. Then we can see that $\mathrm{t}H_q(X_\infty)$ is a finite abelian group (cf. [Kervaire 1965], [Kawauchi 1986']). For a prime number p, $(\mathrm{t}H_q(X_\infty))_p = \mathrm{t}H_q(X_\infty) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is naturally assigned a Λ_p-module structure, where $\Lambda_p = \Lambda \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Since Λ_p is a principal ideal domain, the elementary divisors $\lambda^q_{p,i}$ of $(\mathrm{t}H_q(X_\infty))_p$ are defined. (We assume $\lambda^q_{p,i+1} | \lambda^q_{p,i}$.) We call $\{\lambda^q_{p,i}\}_i$ the q-th *p-local Alexander polynomial* of K^n. The following theorem is given in [Gutiérrez 1972']:

Theorem 14.1.6 *For the q-th p-local Alexander polynomial of any n-knot, we have the following:*

(a) $\lambda^q_{p,i+1} \mid \lambda^q_{p,i}$.
(b) $\lambda^q_{p,i}(1) \doteq 1$ *(as an element of* $\mathbf{Z}_p$*).*
(c) $\lambda^q_{p,i}(t) \doteq \lambda^{n-q}_{p,i}(t^{-1})$ *(as an element of* Λ_p*).*

By this lemma, the following result is obtained (see also [Farber 1975], [Hosokawa-Kawauchi 1979], [Levine 1978], [Hillman 1989]):

Theorem 14.1.7 *There exists a 3-knot group which is not a 2-knot group.*

Proof. Let π be a group with the following presentation
(14.1.7.1) $\quad \langle a, x \mid x^{-1}ax = a^2, x^{-3}ax^3 = a \rangle$.
It is a 3-knot group, for it is isomorphic to the group of the 3-knot obtained from Fox's 2-knot K_F by 3-twist spinning. (One can also directly check that π has properties (1)–(4) of Theorem 14.1.1 in order to show that π is a 3-knot group.) We show that π is not a 2-knot group. Since $a = x^{-3}ax^3 = x^{-2}a^2x^2 = x^{-1}a^4x = a^8$, we have $a^7 = 1$ and $\pi = \langle a, x \mid x^{-1}ax = a^2, x^{-3}ax^3 = a, a^7 = 1 \rangle = \langle a, x \mid x^{-1}ax = a^2, a^7 = 1 \rangle$. Hence the commutator subgroup π' of π is a cyclic group of order 7. When we assign x to a generator t of π/π', we have

(14.1.7.2) $\quad H_1(\pi') \cong \Lambda/(2t-1,7) \cong \Lambda_7/(2t-1)$.

Suppose that π is the group of a 2-knot K^2. Since $H_1(X_\infty) \cong H_1(\pi')$, (14.1.7.2) implies that the first 7-local Alexander polynomial $\lambda^1_{7,1}$ of K^2 is $2t-1$. In Λ_7, we have that $2t-1 \neq 2t^{-1}-1$, which contradicts Theorem 14.1.6. Hence π is not a 2-knot group. □

Exercise 14.1.8 Let π be a group with a presentation

$\langle a, x \mid x^{-1}ax = a^2, a^5 = 1\rangle$.

Prove that π is a 3-knot group but not a 2-knot group.

Theorem 14.1.6 can be proved using the existence of the Farber-Levine pairing

$$\ell : \mathrm{t}H_q(X_\infty) \times \mathrm{t}H_{n-q}(X_\infty) \to \mathbf{Q}/\mathbf{Z}$$

which is a t-isometric $(-1)^{q(n-q)+1}$-symmetric non-singular pairing (cf. [Farber 1977], [Levine 1977]). This pairing is generalized to a pairing on any infinite cyclic covering space over any compact oriented manifold in [Kawauchi 1986']. This generalized pairing is applied to the exterior of a surface-knot F in S^4 in [Kawauchi 1990] in order to estimate the genus $g(F)$ of F by the group $\pi_1(S^4-F,b)$. This will be explained in Theorem 14.1.9.

Let $\mathcal{K}^n$ be the family of n-knot groups and $\mathcal{F}_g$ the family of surface-knot groups of surfaces of genus g. We note that for each $g \geq 1$, there exists a group $\pi \in \mathcal{F}_g$ such that $\pi \notin \mathcal{F}_{g-1}$, because in [Litherland 1981] it is shown that there is a group $\pi \in \mathcal{F}_g$ with $H_2(\pi) \cong A$ for any abelian group A generated by $2g$ elements. We consider the family $\mathcal{K}^2_g = \mathcal{K}^3 \cap \mathcal{F}_g$. Then the following theorem holds:

Theorem 14.1.9

$$\mathcal{K}^1 \subsetneqq \mathcal{K}^2 = \mathcal{K}^2_0 \subsetneqq \mathcal{K}^2_1 \subsetneqq \mathcal{K}^2_2 \subsetneqq \mathcal{K}^2_3 \subsetneqq \cdots \subsetneqq \mathcal{K}^3 = \mathcal{K}^4 = \mathcal{K}^5 = \cdots, \qquad \mathcal{K}^3 = \bigcup_{g=0}^{+\infty} \mathcal{K}^2_g.$$

Proof. In Theorem 14.2.1, we shall show that every group π with a Wirtinger presentation and with $\pi/\pi' \cong \mathbf{Z}$ is a surface-knot group. Then by Theorem 14.1.4, every group in $\mathcal{K}^3$ is a surface-knot group and hence $\mathcal{K}^3 = \bigcup_{g=0}^{+\infty} \mathcal{K}^2_g$. By Theorems 14.1.1,14.1.2,14.1.3, 14.1.5 and 14.1.7, it remains only to prove that $\mathcal{K}^2_g \subsetneqq \mathcal{K}^2_{g+1}$ for each $g \geq 0$. It is clear that $\mathcal{K}^2_g \subset \mathcal{K}^2_{g+1}$. We show that there is a group $\pi \in \mathcal{K}^2_{g+1}$ such that $\pi \notin \mathcal{K}^2_g$. We consider the group π of Exercise 14.1.8. It is shown in [Hosokawa-Kawauchi 1979] that π is a surface-knot group of a genus one surface F_1 in S^4. For $n > 1$, let $F_n \subset S^4$ be the surface-knot obtained by taking the n-fold connected sum of $F_1 \subset S^4$ and $\pi_{(n)}$ the surface-knot group. We consider a surface-knot F with minimal genus whose surface-knot group $\pi_1(S^4-F)$ is isomorphic to $\pi_{(n)}$. Let X be the exterior $\mathrm{cl}(S^4 - N(F))$. We have

$$H_1(\pi'_{(n)}) = H_1(X_\infty) \cong H_1(X_\infty, \partial X_\infty) \cong (\Lambda_5/(2t-1))^n$$

for the universal abelian covering space X_∞ of X. Since any non-trivial submodule of this finite Λ-module does not have a symmetric divisor in Λ_5, the pairing defined on a characteristic finite submodule of $H_1(X_\infty)$ in [Kawauchi 1986'] must be trivial. This means that there is a t-anti isomorphism

$$H_1(X_\infty) \cong \mathrm{Ext}^1_\Lambda(BH_2, \Lambda),$$

where $BH_2(X_\infty)$ denotes the quotient module of $H_2(X_\infty)$ by the Λ-torsion part. Since $\mathrm{Ext}^2_\Lambda(\Lambda_5/(2t-1), \Lambda) \cong \Lambda_5/(2t-1)$ (cf. [Levine 1977], [Kawauchi 1986']), we have

$$\mathrm{Ext}^2_\Lambda(\mathrm{Ext}^1_\Lambda(BH_2, \Lambda), \Lambda) \cong (\Lambda_5/(2t^{-1}-1))^n,$$

and the minimal number of Λ-generators of this module is n. Note that there is a natural Λ-epimorphism

$$\mathrm{Ext}^0_\Lambda(\mathrm{Ext}^0_\Lambda(BH_2, \Lambda), \Lambda) \to \mathrm{Ext}^2_\Lambda(\mathrm{Ext}^1_\Lambda(BH_2, \Lambda), \Lambda)$$

and that the domain of this epimorphism is a free Λ-module of rank $2g(F)$ (cf. [Kawauchi 1986',1990]). Thus, we see that $2g(F) \geq n$. Noting that the group $\pi_{(n)}$ is also a 3-knot group for all n, we can find a positive integer n_g such that $\pi_{(n_g)} \in \mathcal{K}^2_{g+1}$ but $\pi_{(n_g)} \notin \mathcal{K}^2_g$, for each $g \geq 0$. □

As observed before, for $n \geq 3$, $\mathcal{K}^n$ is characterized by properties (1)–(4) of Theorem 14.1.1. $\mathcal{K}^1$ is characterized in terms of group presentations by the Alexander-Artin theorem (Theorem 6.2.11). $\mathcal{K}^2_g$ for each g was also characterized in terms of group presentations and group homology by [González-Acuña *] and [Kamada 1994] using a 2-dimensional braid argument analogous to the Alexander-Artin theorem.

14.2 Ribbon 2-knot groups

In this section, we discuss some properties of ribbon 2-knot groups. First, we observe the fact that the surface-knot groups can be characterized by Wirtinger presentations.

Theorem 14.2.1 *Every surface-knot group has a Wirtinger presentation. Conversely, every group π which has a Wirtinger presentation and has the property that $\pi/\pi' \cong \mathbf{Z}$ is realized as the group of a ribbon surface in $\mathbf{R}^4$.*

Proof. Let π be the surface-knot group of a surface F in $\mathbf{R}^4$. By Exercise 13.1.12, we may assume that the surface F is in normal form. Take a connected diagram D of the link $L = \mathbf{R}^3[0] \cap F$ in a plane $\mathbf{R}^2$ so that all the saddle bands of F are projected to $\mathbf{R}^2$ as mutually disjoint bands $B_k (k = 1, 2, \ldots, s)$ meeting D only in the overpaths of D. Let $\pi(D)$ be the over presentation of the group of L associated with D, which is a Wirtinger presentation (cf. Theorem 6.2.1). By the Seifert-van Kampen theorem (cf. Appendix B), the group π is obtained from $\pi(D)$ by adding the relations $x_{i_k} = x_{j_k} (k = 1, 2, \ldots, s)$ where x_{i_k} and x_{j_k} denote the generators of $\pi(D)$ connected by the band B_k. Thus, π has a Wirtinger presentation. Conversely, let π have a Wirtinger presentation $\langle x_0, x_1, \ldots, x_n \mid r_1, \ldots, r_s \rangle$ with $r_k = x_{i_k}^{-1} w_k x_{j_k} w_k^{-1}$. Since $\pi/\pi' \cong \mathbf{Z}$, we see that any x_i with $i \geq 1$ is conjugate to x_0 and $s - n \geq 0$. Without loss of generality, we can assume that $x_{i_k} = x_k$ and $x_{j_k} = x_{k-1}$ for $k = 1, 2, \ldots, n$ and $x_{i_k} = x_{j_k} = x_0$ for $k = n+1, \ldots, s$. We consider a trivial 2-link L in S^4 of $n+1$ components $S_i^2 (i = 0, 1, \ldots, n)$ and bands $f_k(B^3 \times I)(k = 1, 2, \ldots, s)$ as stated in Construction 13.2.4. Let $\langle x_0, x_1, \ldots, x_n \rangle$ be the free group identified with $\pi_1(S^4 - L, b)$ so that x_i represents a meridian of S_i^2. Then T. Yajima showed in [Yajima 1969] that if we choose the arc $f_k(p \times I)$ for a point $p \in B^2$ to represent the word w_i^{-1}, then the surface-knot group of the ribbon surface F of s-fusion along the bands $f_i(B^2 \times I)$ $(i = 1, 2, \ldots, s)$ has the presentation $\langle x_0, x_1, \ldots, x_n \mid r_1, \ldots, r_s \rangle$. □

Since the genus of the ribbon surface F constructed in the proof above is $s - n$, we have the following corollary:

Corollary 14.2.2([Yajima 1969]) *A group π is a ribbon 2-knot group if and only if (1) π/π' is an infinite cyclic group and (2) π has a Wirtinger presentation of deficiency one.*

The 1-knot groups have the properties (1) and (2) of this corollary and hence they are ribbon 2-knot groups. It is an open question whether a group π is a ribbon 2-knot group if it has a finite presentation of deficiency one, the property $\pi/\pi' \cong \mathbf{Z}$, and a meridian element. It is known that such a group π is the group of a 2-knot in a *homotopy* 4-*sphere* (cf. [Kervaire 1965,1965']).

Some properties of 1-knot groups can be extended to ribbon 2-knot groups. The following can be obtained from the Fox calculus (cf. Theorem 7.1.5):

Lemma 14.2.3 *For any group π which has a finite presentation of deficiency one and $\pi/\pi' \cong \mathbf{Z}$, the Λ-module $H_1(\pi') = \pi'/\pi''$ has a square Λ-presentation matrix and the first elementary ideal is principal.*

It is known that no 1-knot group has a non-trivial torsion element (cf. 6.3.1). In [Kawauchi-Shibuya-Suzuki 1983], it is asked whether any ribbon 2-knot does, but this question is still open.

Theorem 14.2.4 *If π is a group which has a finite presentation of deficiency one and for which $\pi/\pi' \cong \mathbf{Z}$, then the quotient group π/π'' of π is torsion-free.*

Proof. Since $\pi/\pi' \cong \mathbf{Z}$ is torsion-free, it suffices to show that $A = H_1(\pi') = \pi'/\pi''$ is torsion-free. By Lemma 14.2.3, A has a square Λ-presentation matrix P of size, say, n. Since A is a Λ-torsion module, we have that $\det P \neq 0$; hence there is a Λ-exact sequence

$$0 \to \Lambda^n \xrightarrow{P} \Lambda^n \to A \to 0.$$

This means that $\mathrm{Ext}^2_\Lambda(A, \Lambda) = 0$. Let $\mathrm{t}A$ be the $\mathbf{Z}$-torsion part of A. Since $t - 1$ is an automorphism of A, we see from [Kervaire 1965], [Kawauchi 1986'] that $\mathrm{t}A$ is a finite abelian group. By [Levine 1977], [Kawauchi 1986'], we have that

$$\mathrm{Ext}^2_\Lambda(A,\Lambda) \cong \mathrm{Ext}^2_\Lambda(\mathrm{t}A,\Lambda) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathrm{t}A,\mathbf{Q}/\mathbf{Z}).$$

Thus, $\mathrm{t}A = 0$, i.e., A is torsion-free. □

A group π is called *Hopfian* if π is not isomorphic to π/N for any proper normal subgroup N of π. The following is well-known (cf. [Lyndon-Schupp 1977]):

Lemma 14.2.5 *A finitely generated residually finite group is Hopfian.*

Proof. We note that a group is Hopfian if and only if any surjective endmorphism of it is an automorphism. Let π be a finitely generated residually finite group, and let $\varphi : \pi \to \pi$ be a surjective endomorphism of π. Suppose $\mathrm{Ker}\varphi \neq \{1\}$ and take an element $g \neq 1$ of $\mathrm{Ker}\varphi$. Since π is residually finite, there is a normal subgroup N of π such that $g \notin N$ and $n = [\pi : N] < \infty$. Since π is finitely generated, by the Hall theorem ([Lyndon-Schupp 1977(p196)]), the number of subgroups of π with index n is finite. Let $N_1, \ldots, N_k$ be such subgroups. ($N_1 = N$ and $N_i \neq N_h$ $(i \neq h)$.) For each i $(i = 1, \ldots, k)$, let $L_i = \varphi^{-1}(N_i)$, which has also index n. Since $L_1, \ldots, L_k$ are all distinct, $\{L_1, \ldots, L_k\} = \{N_1, \ldots, N_k\}$. $\mathrm{Ker}\varphi$ is contained in each L_i $(i = 1, \ldots, k)$ and hence in each N_i. In particular, $\mathrm{Ker}\varphi \subset N_1 = N$, which contradicts that $g \notin N$. Hence $\mathrm{Ker}\varphi = \{1\}$. □

It is known that any 1-knot group is residually finite (cf. Theorem 6.3.9) and, by Lemma 14.2.5, it is Hopfian.

Theorem 14.2.6 *There is a ribbon 2-knot group which is neither residually finite nor Hopfian.*

Proof. It is sufficient to show the existence of a ribbon 2-knot group which is not Hopfian. Let π be a group with presentation $\langle x, y \mid y = (x^{-1}y)^2 x (x^{-1}y)^{-2}\rangle$. It has properties (1) and (2) of Corollary 14.2.2 and hence is a ribbon 2-knot group. We show that it is not Hopfian. By introducing $a = x^{-1}y$ and deleting y, we have $\pi = \langle a, x \mid xa = a^2xa^{-2}\rangle = \langle a, x \mid x^{-1}a^2x = a^3\rangle$. Let N be the normal subgroup of π generated by $a(a^{-1}x^{-1}ax)^{-2}$. The quotient group π/N has a presentation $\langle a, x \mid x^{-1}a^2x = a^3, a(a^{-1}x^{-1}ax)^{-2} = 1\rangle$. Putting $b = a^{-1}x^{-1}ax$, we have $\pi/N = \langle a, b, x \mid x^{-1}a^2x = a^3, ab^{-2} = 1, b = a^{-1}x^{-1}ax\rangle$. Using the second relation, we can delete a and we obtain $\pi/N = \langle b, x \mid x^{-1}b^4x = b^6, b = b^{-2}x^{-1}b^2x\rangle = \langle b, x \mid x^{-1}b^2x = b^3\rangle$. Hence we see that $\pi/N \cong \pi$. The remaining task is to show that $N \neq \{1\}$, in particular, that $g = a(a^{-1}x^{-1}ax)^{-2}$ is not a trivial element. The commutator subgroup π' of π is presented by the following amalgamated free product:

$$\cdots *_{H_{-2,-1}} H_{-1} *_{H_{-1,0}} H_0 *_{H_{0,1}} H_1 *_{H_{1,2}} \cdots,$$

where each H_i is the infinite cyclic group generated by $t^{-i}at^i$, which we denote by a_i, and the amalgamation between H_i and H_{i+1} is given by $a_i^3 = a_{i+1}^2$. Since $g = a_0(a_0^{-1}a_1)^{-2}$, it is contained in $H_0 *_{H_{0,1}} H_1 = \langle a_0, a_1 \mid a_0^3 = a_1^2\rangle$. Let φ be the representation of $H_0 *_{H_{0,1}} H_1$ in $\mathbf{S}_3$ defined by $\varphi(a_0) = (123)$ and $\varphi(a_1) = (12)$. Then $\varphi(g) = (123)((123)^{-1}(12))^{-2} = (123) \neq 1$; thus $g \neq 1$. □

Exercise 14.2.7 Let π be a group having a presentation with a single relation such that $\pi/\pi' \cong \mathbf{Z}$. Show that π is a ribbon 2-knot group.

Exercise 14.2.8 Prove that (1) and (2) of Corollary 14.2.2 are necessary and sufficient conditions for a group to be a ribbon n-knot group for each $n \geq 3$.

Exercise 14.2.9 Show by an algebraic argument, as in [Yajima 1970], that properties (1) and (2) of Corollary 14.2.2 imply that $H_2(\pi) = 0$.

14.3 Torsion elements and the deficiency of 2-knot groups

Though there are no non-trivial torsion elements in the 1-knot groups (cf. 6.3.1), it is well-known that there are 2-knot groups with non-trivial torsion elements. For example, the 2-knot group $\langle a, x \mid x^{-1}ax = a^{-1}, a^{2n+1} = 1\rangle$ of the 2-knot shown in figure 14.3.1 (cf.[Fox 1962]) has an element of order $2n+1$ ($n \geq 1$). (In fact, the commutator subgroup is the cyclic group of order $2n+1$ generated by a.) This example also shows that any positive odd integer can be the order of an element of a 2-knot group. Another example is the group of a 2-knot obtained from a trefoil knot by 5-twist spinning, which has elements of order 2, 4, 6 and 10. The following result is due to [Kanenobu 1980]:

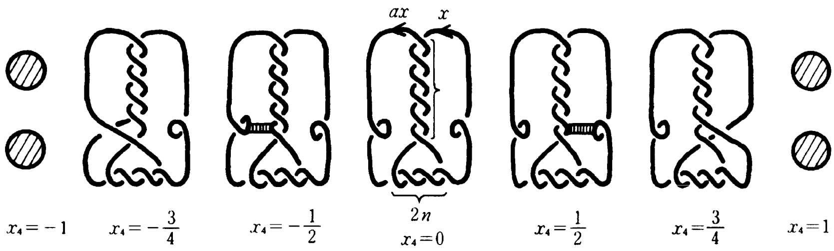


Fig. 14.3.1

Theorem 14.3.1 *For any positive integer n, there exists a 2-knot group with an element of order n.*

Proof. Let $\pi_{(n)}$ be the group of a 2-knot shown in figure 14.3.2, which has the presentation

(14.3.1.1) $\quad \langle a, x \mid xa^{2n} = a^{2n}x, xa^{n+1}x = a^n xa^{-n}xa^n\rangle$.

We prove that a has order $4n$. Since a^{2n} commutes with x, it follows from the second relation that $xa^{n+1}x = a^{-n}xa^nxa^n = a^nxa^nxa^{-n}$. Hence $a^nxa^{n+1}x = xa^nxa^n$ and $xa^{n+1}xa^n = a^nxa^nx$. Using these two relations, we have that

$$(a^nxa^{n+1}x)(a^nxa^nx)^{-1} = (xa^nxa^n)(xa^{n+1}xa^n)^{-1},$$

which implies the following relation:

(14.3.1.2) $\quad a^nxax^{-1}a^{-n} = xa^{-1}x^{-1}$.

By (14.3.1.2), $a^n x a^{2n} x^{-1} a^{-n} = x a^{-2n} x^{-1}$. Since a^{2n} commutes with x, $a^n a^{2n} a^{-1} = a^{-2n}$ and hence $a^{4n} = 1$. It remains to prove that the order of a is exactly $4n$. By letting $y = a^n x$, we delete x from (14.3.1.1). Then we have a presentation $\langle a, y \mid ya^{2n} = a^{2n}y, yay = a^n y^2 a^{-n}\rangle$ of $\pi_{(n)}$. Define a representation φ of $\pi_{(n)}$ in the special linear group $SL(2, \mathbf{C})$ by

$$\text{(14.3.1.3)} \qquad \varphi(a) = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \text{ and } \varphi(y) = \begin{pmatrix} -\alpha & \beta \\ \alpha & \beta \end{pmatrix},$$

where $\omega = \exp(\pi\sqrt{-1}/2n)$, $\alpha = \exp((1 - 1/2n)\pi\sqrt{-1}/2)/\sqrt{2}$, $\beta = \exp((1 + 1/2n)\pi\sqrt{-1}/2)/\sqrt{2}$. It is easily checked that φ is well-defined (Exercise 14.3.8). Since ω is a primitive $4n$-th root of unity, the order of $\varphi(a)$ is $4n$. Hence the order of a is $4n$. □

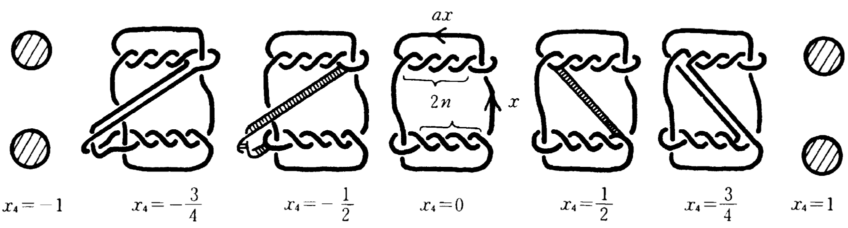


Fig. 14.3.2

Next, we consider the deficiency of a 2-knot group.

Lemma 14.3.2 *The deficiency of an n-knot group is less than or equal to one.*

Proof. Let π be an n-knot group. Since π/π' is an infinite cyclic group, $\operatorname{def}\pi \leq 1$. □

Any ribbon 2-knot group has a Wirtinger presentation of deficiency one, hence its deficiency is one. On the other hand, if an n-knot group has deficiency one, then the first elementary ideal is principal. Since the group $\pi_{(n)}$ of the 2-knot shown in figure 14.3.1 has first elementary ideal $(2t - 1, 2n + 1)$, which is not principal, the deficiency of $\pi_{(n)}$ is less than one. Since $\pi_{(n)}$ has a presentation of deficiency zero, $\operatorname{def}\pi_{(n)} = 0$. In this way, we see that there are 2-knot groups of deficiency one and zero. Moreover, we can prove the following theorem (cf. [Levine 1978]):

Theorem 14.3.3 *For any integer $d \leq 1$, there exists a 2-knot group π of deficiency d.*

Before proving this theorem, we prove the following lemma:

Lemma 14.3.4 *Let A be a finitely generated Λ-module of rank r. If A has an (s_1, s_0)-matrix as a presentation matrix over Λ, then the Λ-module $\operatorname{Ext}^2_\Lambda(A, \Lambda)$ is generated by $r + s_1 - s_0$ elements over Λ.*

Proof. Since Λ has homological dimension two (cf. [MacLane 1963]) and a projective Λ-module is Λ-free (cf. [Seshadri 1958]), there is a Λ-exact sequence

$$\text{(14.3.4.1)} \qquad 0 \to F_2 \xrightarrow{e} F_1 \xrightarrow{d} F_0 \to A \to 0$$

where F_i $(i = 0, 1, 2)$ is a free Λ-module of rank s_i. Since $s_2 - s_1 + s_0 - r = 0$, we have that $s_2 = s_1 - s_0 + r$. Consider the following part of the Λ-dual chain complex of (14.3.4.1)

$$\mathrm{Hom}_\Lambda(F_0, \Lambda) \xrightarrow{d^\sharp} \mathrm{Hom}_\Lambda(F_1, \Lambda) \xrightarrow{e^\sharp} \mathrm{Hom}_\Lambda(F_2, \Lambda) \to 0.$$

Observe that $\mathrm{Ext}^2_\Lambda(A, \Lambda)$ is given by $\mathrm{Coker} e^\sharp$. Since $\mathrm{Hom}_\Lambda(F_2, \Lambda)$ is a free Λ-module of rank s_2, $\mathrm{Ext}^2_\Lambda(A, \Lambda)$ is generated by s_2 $(= r + s_1 - s_0)$ elements over Λ. □

In [Sekine 1989'], it is shown that for any positive integers m and n, there is a finite Λ-module A such that the minimal numbers of Λ-generators of A and $\mathrm{Ext}^2_\Lambda(A, \Lambda)$ are m and n, respectively.

Proof of Theorem 14.3.3. It is enough to prove the case where $d < 1$. Let $K_{(1)}$ be the 2-knot shown in figure 14.3.1 with $n = 1$. Let $\pi_{(m)}$ be the group of the 2-knot which is the m-fold connected sum of $K_{(1)}$. We prove $\mathrm{def}\,\pi_{(m)} = 1 - m$. Since $\mathrm{def}\,\pi_{(1)} = 0$, we have that $\mathrm{def}\,\pi_{(m)} \geq 1 - m$. Then it is sufficient to prove that $\mathrm{def}\,\pi_{(m)} \leq 1 - m$. Let $A_m = H_1(\pi'_{(m)})$. Using $\pi_{(1)} = \langle a, x \mid x^{-1}ax = a^{-1}, a^3 = 1\rangle$, it follows that $A_1 \cong \Lambda/(t+1, 3) \cong \Lambda_3/(t+1)$, so that $A_m \cong (A_1)^m$. Hence the minimal number of Λ-generators of A_m is m. Now suppose that $\mathrm{def}\,\pi_{(m)} > 1 - m$. Then $\pi_{(m)}$ has a group presentation with p generators and q relations such that $p - q > 1 - m$, so that the Λ-module A_m has a Λ-presentation matrix of size $(q, p-1)$. Since the rank of A_m is zero, we see from Lemma 14.3.4 that $\mathrm{Ext}^2_\Lambda(A_m, \Lambda)$ is generated by $0 + q - p + 1$ $(< m)$ elements. On the other hand, $\mathrm{Ext}^2_\Lambda(A_m, \Lambda) \cong A_m$ (cf. [Levine 1977], [Kawauchi 1986']), so that the minimal number of Λ-generators of A_m is less than m, which is a contradiction. Therefore $\mathrm{def}\,\pi_{(m)} \leq 1 - m$ and hence $\mathrm{def}\,\pi_{(m)} = 1 - m$. □

Exercise 14.3.5 Show that the commutator subgroup of the group of the 2-knot shown in figure 14.3.1 is a cyclic group of order $2n + 1$.

Exercise 14.3.6 Let $\pi_{(n)}$ be the group of the 2-knot obtained from a trefoil knot by n-twist spinning. Verify the following:

(1) $\pi_{(1)} \cong \mathbf{Z}$.
(2) $\pi'_{(2)} \cong \mathbf{Z}_3$.
(3) $\pi'_{(3)} \cong Q_8$ (the quaternion group).
(4) $\pi'_{(4)} \cong T^*$ (the binary tetrahedral group).
(5) $\pi'_{(5)} \cong I^*$ (the binary icosahedral group).

Exercise 14.3.7 Show that the group of the 2-knot shown in figure 14.3.2 has the presentation (14.3.1.1).

Exercise 14.3.8 Show that (14.3.1.3) defines a representation
$\varphi : \pi_{(n)} \to SL(2, \mathbf{C})$.

Exercise 14.3.9 Show that the ideal $(2t-1, 2n+1)$ (for $n > 0$) of Λ is not principal.

Supplementary notes for Chapter 14

The commutator subgroup of a 1-knot group is finitely generated if and only if it is a free group (cf. [Neuwirth 1965]). There is more variety for commutator subgroups of 2-knot groups. For instance, the commutator subgroup of a 2-knot shown in figure 14.3.1 is a finite cyclic group. Here, we give some results on finite or abelian commutator subgroups of 2-knot groups.

Theorem ([Hillman 1977']) *If the commutator subgroup of a 2-knot group is a finite group, then it is isomorphic to one of the following (1)-(4):*
(1) $\mathbf{Z}_n$ *where* n *is an odd integer.*
(2) $Q_8 \times \mathbf{Z}_n$ *where* Q_8 *is the quaternion group and* n *is an odd integer.*
(3) $I^* \times \mathbf{Z}_n$ *where* I^* *is a binary icosahedral group and* $gcd(|I^*|, n) = 1$.
(4) $T_{(i)} \times \mathbf{Z}_n$ *where* $i \geq 1$ *and* $T_{(i)}$ *is*

$$\langle x, y, z \mid x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^i} = 1\rangle$$

and $\gcd(|T_{(i)}|, n) = 1$.

Conversely, it is proved in [Yoshikawa 1980] that every group listed above appears as the commutator subgroup of the group of a 2-knot in a homotopy 4-sphere. Further, we can see that each group in (1) is the commutator subgroup of the group of a 2-knot shown in figure 14.3.1. The groups in (2) with $n = 1$ and the groups in (3) and (4) are those of 2-knots obtained from certain 1-knots by twist spinnings. It is also proved in [Kanenobu 1988'] and [Teragaito 1989'] that the groups in (2) with $n = 5, 11, 13, 19$ are the commutator subgroups of certain 2-knot groups. It is still an open question whether the remaining groups in (2) (that is, the groups in (2) for all odd $n > 0$ with $n \neq 1, 5, 11, 13, 19$) are actually 2-knot groups.

Theorem ([Hillman 1989], [Levine 1978]) *If the commutator subgroup* π' *of a 2-knot group* π *is a finitely generated abelian group, then it is isomorphic to either a cyclic group of order* $2n + 1$ *for some* n*, or a free abelian group of rank 3. Furthermore,* π *has a presentation*
(1) $\langle a, x \mid x^{-1}ax = a^{-1}, a^{2n+1} = 1\rangle$ *in the first case, or*
(2) $\langle a_1, a_2, a_3, x \mid x^{-1}a_ix = \varphi(a_i), [a_i, a_k] = 1, 1 \leq i, k \leq 3\rangle$ *in the second case, where* φ *is an automorphism of the free abelian group* $\prod_{i=1}^{3}\langle a_i\rangle$ *which is presented by a matrix* $M(\varphi)$ *such that* $\det M(\varphi) = 1$ *and* $\det(M(\varphi) - E^3) = \pm 1$.

Conversely, the group with presentation (1) is the group of a 2-knot shown in figure 14.3.1. It is also shown in [Cappell-Shaneson 1976] that there is a 2-knot in a homotopy 4-sphere whose group has the presentation (2). It is still unknown whether any group presented by (2) is actually a 2-knot group (cf. [Gompf 1991,1991']).

Theorem ([Yoshikawa 1984], [Hillman 1986"']) *If the commutator subgroup* π' *of a 2-knot group* π *is an abelian group which is not finitely generated, then* π' *is isomorphic to the abelian group* $\mathbf{Z}[1/2]$ *consisting of rational numbers of the form* $m/2^n$*, and* π *is presented by (*)* $\langle a, x \mid x^{-1}ax = a^2\rangle$.

Conversely, the group presented by $(*)$ is the group of a 2-knot shown in figure 14.1.1. Thus, the 2-knot groups with abelian commutator subgroups are almost classified. The commutator subgroups of 3-knot groups with finitely generated abelian commutator subgroups are determined as follows:

Theorem ([Hausmann-Kervaire 1978]) *A finitely generated abelian group H is the commutator subgroup of a 3-knot group if and only if H has the following properties (where $\rho_H(p^m)$ denotes the number of $\mathbf{Z}_{p^m}$ factors in the p-component of H):*
(1)rank $H \neq 1, 2$.
(2)$\rho_H(2^m) \neq 1, 2$ *for any positive integer m.*
(3) *There is at most one positive integer m such that $\rho_H(3^m) = 1$.*

Further, the 3-knot groups with infinitely generated abelian commutator subgroups are determined in [Yoshikawa 1986] in the case of the rank one or two.

Chapter 15
Knot theory of spatial graphs

The topological study of spatial graphs is considered to be a natural extension of knot theory, although it has not been paid much attention until quite recently. In this chapter, we regard two notions on "equivalence" of graphs. The first one is a notion naturally extending positive-equivalence of links and is called *equivalence*. The second one is a notion which is useful when we study the exterior of a spatial graph and is called *neighborhood-equivalence*. Since the importance of the first concept is motivated by recent developments in molecular chemistry, we devote the first section to some comments on the topology of molecules. In 15.2 we discuss some results on the first notion, and in 15.3 some results on the second notion, including an explanation of recent developments on the tunnel number.

15.1 Topology of molecules

The constitutional formula of a molecule is a graph whose vertices correspond to the atoms in the molecule and whose edges express the combination data between the atoms by covalent bonds. This graph is obviously in $\mathbf{R}^3$. In chemistry, geometric invariants of the graph such as the length of an edge and the angle between two edges were originally important. However, since the beginning of this century, some chemists have been interested in synthesizing artificially molecules with constitutional formulae containing knots or links. Then, H. L. Frisch and E. Wasserman succeeded in synthesizing a molecule with constitutional formula containing a link (the Hopf link) (cf. [Frisch-Wasserman 1961], [Wasserman 1962]). This news promptly appeared in [Crowell-Fox 1963], but afterward information on developments in this area was not well communicated to topologists. The attempt to synthesize a molecule with a knot as its constitutional formula has been continued by some chemists, and molecules with topologically interesting constitutional formulae have been synthesized (figure 15.1.1). D. M. Walba succeeded in synthesizing a molecule with the Möbius ladder (the M_3-graph) as its constitutional formula in the process of trying to construct molecule with constitutional formula which is a knot (cf. [Walba 1985]). This news was announced by J. Simon to topologists at the meeting of the American Mathematical Society in Richmond, Virginia in 1984 (cf. [Simon 1986]). It signaled the birth of the study of the topology of molecules. The purpose of this research is to study topological properties of graphs expressing constitutional formulae in $\mathbf{R}^3$, and hence we assume from the start that we may topologically deform the edges of the graphs. In our study of molecules, because topological invariants are geometric invariants, we give priority to the topological study rather than the geometric study of molecules (cf. [Simon 1986], [Walba 1987]). Not all problems on molecules are reduced to topological problems on spatial graphs because we cannot disregard the difference between individual atoms (such as atoms of carbon, oxygen, nitrogen and so on). However, the birth of the study of the topology of molecules gave the topological study of spatial graphs a new guiding principle (cf. [Simon 1986, 1987, 1987']). For example, the notion of neighborhood-equivalence, discussed in 15.3, is not important here and, if necessary, we may impose non-topological conditions on the graph itself. The attempt of Walba and others to synthesize a molecule with a knot, as the constitutional formula, does not appear to have been achieved. On the other hand, molecular biologists have succeeded in making a circular DNA (called a *DNA knot*) artificially. Knot theory has been applied to study the structure and function of circular DNA. Concerning this topic, see [Wasserman-Cozzarelli 1986], [Wasserman-Dungan-Cozzarelli 1985], and [Sumners 1987, 1987', 1988, 1990].

Fig. 15.1.1

15.2 Uses of the notion of equivalence

Throughout this section and the next section, by a *graph*, we mean a graph without vertices of degrees 0 or 1, unless otherwise specified. Among the graphs that are not closed 1-manifolds, the θ_n*-curve* is one of the simplest graphs. It is a graph consisting of two vertices and n edges such that each edge joins the vertices. For $n = 2$, it is homeomorphic to a simple loop. For $n = 3$, it is simply called the θ*-curve*. For any integers $m, n \geq 2$, the (m, n)*-complete bipartite graph*, denoted by $K_{m,n}$, is the graph with $m+n$ vertices and mn edges whose vertex set is divided into a set V_m of m vertices and a set V_n of n vertices so that each vertex in V_m is joined with all the vertices in V_n by n edges. Then the θ_n-curve is homeomorphic to the $(2, n)$-complete bipartite graph $K_{2,n}$. For any integer $n \geq 3$, the *n-complete graph*, denoted by K_n, is the graph with n vertices and $n(n-1)/2$ edges such that any two distinct vertices are joined by just one edge. A graph is called a *planar* graph if it can be embedded into $\mathbf{R}^2$. The following theorem is well-known (see a textbook on graph theory, for example, [Harary 1969]):

Theorem 15.2.1 (Kuratowski's theorem) *A graph G is planar if and only if G does not have a subgraph homeomorphic to the 5-complete graph K_5 or the $(3,3)$-complete bipartite graph $K_{3,3}$.*

We consider a *spatial graph*, that is, a graph in $\mathbf{R}^3$ or S^3.

Definition 15.2.2 Two graphs K and K' in $\mathbf{R}^3$ (or S^3) are *equivalent* if there is an orientation-preserving auto-homeomorphism h of $\mathbf{R}^3$ (or S^3) with $h(K) = K'$.

It is possible to establish a spatial graph version of Theorems A.3 and A.4 on link equivalence by a similar argument (cf. [Yamada 1992(Fig.1)]).

Definition 15.2.3 A graph in $\mathbf{R}^3$ or S^3 is *trivial* if it is equivalent to a graph in a plane in $\mathbf{R}^3$ or a standard 2-sphere in S^3, respectively.

By this definition, any spatial graph of any non-planar graph is a priori non-trivial. The following theorem on the uniqueness of a trivial spatial graph is due to [Mason 1969]:

Theorem 15.2.4 *Two trivial graphs in $\mathbf{R}^3$ are equivalent if they are homeomorphic graphs.*

It is natural to consider first the following concept to determine the knotting property of a spatial graph:

Definition 15.2.5 For a graph K in $\mathbf{R}^3$, a *constituent knot* of K is a simple loop in K which is a knot in $\mathbf{R}^3$. Further, a *constituent link* of K is a union of mutually disjoint constituent knots in K.

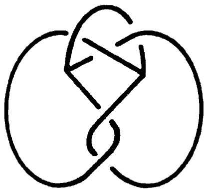


Fig. 15.2.1

For example, since there are three simple loops on the θ-curve, any θ-curve in $\mathbf{R}^3$ has three constituent knots. It is natural to ask whether or not a θ-curve in $\mathbf{R}^3$ must be trivial if these three constituent knots are trivial. The answer is negative by *Kinoshita's θ-curve*, shown in figure 15.2.1 (cf. [Kinoshita 1958", 1972]). This example leads to the following concept:

Definition 15.2.6 A graph K in $\mathbf{R}^3$ (or S^3) is *almost trivial* if K is a non-trivial planar graph and every proper subgraph of K is trivial.

For almost trivial (possibly disconnected) graphs, we have the following theorem:

Theorem 15.2.7

(1) *For every almost trivial graph K in S^3, the fundamental group $\pi_1(S^3 - K)$ is not free.*
(2) *For every planar graph K, there is an embedding of K into S^3 whose image is an almost trivial graph.*

(1) of Theorem 15.2.7 was proved in [Scharlemann-Thompson 1991] and (2) was proved in [Kawauchi 1989']. Both were conjectured in [Simon 1987]. Partial results

were given in [Wolcott 1987] and [Simon-Wolcott 1990]. The first example of (2) was Kinoshita's θ-curve, which has been generalized to an almost trivial θ_n-curve in [Suzuki 1984']. The argument of [Kawauchi 1989'] actually gives a stronger fact that for any given graph K in S^3, there exists a graph K^* in S^3 which has the following properties:

(1) K^* is not equivalent to K.
(2) There is a map $q : (S^3, K^*) \to (S^3, K)$ with $K^* = q^{-1}(K)$ such that $q\,|_{K^*}$: $K^* \to K$ is a homeomorphism and $q\,|_{(S^3, q^{-1}(K'))}$: $(S^3, q^{-1}(K')) \to (S^3, K')$ is homotopic to an orientation-preserving homeomorphism for each proper subgraph K' of K.

We can construct a 2-knot $L(k)$ in S^4 from a knot k in S^3 by Artin's spinning method (cf. Construction 13.2.1). Then by the Seifert-van Kampen theorem we have an isomorphism $\pi_1(S^4 - L(k)) \cong \pi_1(B^3 - k)$. Likewise, by Artin's spinning method, we can construct an r-component 2-link $L(t)$ in S^4 from an r-string tangle t (without loop component) in a 3-ball B^3, and by the Seifert-van Kampen theorem, we have an isomorphism $\pi_1(S^4 - L(t)) \cong \pi_1(B^3 - t)$. Given a connected graph K in S^3, we take a 3-ball B_0 which is a regular neighborhood of a maximal tree of K in S^3. Let $B^3 = \mathrm{cl}(S^3 - B_0)$. Then $t = K \cap B^3$ is a tangle (without loop components) in the 3-ball B^3. We construct a spun 2-link $(S^4, L(t))$ from (B^3, t). Since there are finitely many choices of B_0, we obtain from the connected graph K in S^3 finitely many spun links $(S^4, L(t))$ with $\pi_1(S^4 - L(t)) \cong \pi_1(S^3 - K)$. In this way, several examples of 2-component spun 2-links are constructed from θ-curves in S^3 in [Kinoshita 1973]. When we take any almost trivial graph K in S^3, we see from Theorem 15.2.7 that any spun link $L(t)$ constructed from it is an almost trivial 2-link (that is to say, a non-trivial 2-link such that every proper sublink is trivial). We can also define the connected sum and the prime decomposition of spatial graphs in a way similar to the case of links (cf. [Suzuki 1987]). The following theorem is due to [Conway-Gordon 1983]:

Theorem 15.2.8 *Any embedded image of the* 6*-complete graph* K_6 *into* $\mathbf{R}^3$ *contains a non-trivial constituent link. Any embedded image of the* 7*-complete graph* K_7 *into* $\mathbf{R}^3$ *contains a non-trivial constituent knot.*

We note that both K_6 and K_7 are non-planar graphs by Theorem 15.2.1. A similar result, proved in [Shimabara 1988], is that any embedded image of $K_{4,4}$ in $\mathbf{R}^3$ contains a non-trivial constituent link and that any embedded image of $K_{5,5}$ in $\mathbf{R}^3$ contains a non-trivial constituent knot. On the other hand, when a family of any three knots is given in $\mathbf{R}^3$, then we can construct a θ-curve in $\mathbf{R}^3$ such that the family of constituent knots coincides with this given family up to positive-equivalence. A similar result holds for the θ_n-curves (cf. [Kinoshita 1987]) and the 4-complete graph (cf. [Yamamoto 1990]) and graphs with three vertices (cf. [Harikae-Kinoshita 1991]). A related topic, belonging to abstract graph theory, is the problem of determining how many simple loops a given graph contains; it is studied in [Entringer-Slater 1981].

Definition 15.2.9 A graph K in $\mathbf{R}^3$ or S^3 is *achiral*, if there exists an orientation-reversing homeomorphism h of $\mathbf{R}^3$ or S^3 such that $h(K) = K$, respectively. Otherwise, K is *chiral*.

Achirality is a natural extension of amphicheirality for links. Molecular chemists habitually use the term "achiral" instead of "amphicheiral". An important problem in chemistry is to determine whether or not a given graph in $\mathbf{R}^3$ is chiral. J. Simon and E. Flapan studied in [Simon 1986] and [Flapan 1987, 1988] the chirality of the extended Möbius ladder M_n including the Möbius ladder $M_3 = K_{3,3}$ which is the constitutional formula of a molecule chemically synthesized by [Walba 1985]. As described in Chapters 8 and 9, several new polynomial invariants for links such as the Jones polynomial have been introduced. For tri-valent graphs in $\mathbf{R}^3$, the *Yamada polynomial* was defined in [Yamada 1989', 1992] as an invariant of equivalence. It was generalized to ribbon graphs in [Reshetikhin-Turaev 1990], to more general type invariants of tri-valent graphs in [Murakami, J. 1992], and to any spatial graph in [Yokota 1994, 1996].

15.3 Uses of the notion of neighborhood-equivalence

We first observe some facts about closed surfaces in S^3. The following theorem is well-known [Alexander 1924] (cf. [Lickorish 1989]):

Theorem 15.3.1 (Alexander's theorem)

(1) *Embeddings of S^2 into S^3 are unique up to ambient isotopy (that is, each embedded image of S^2 is the boundary of a 3-ball in S^3).*
(2) *Embeddings of the torus $S^1 \times S^1$ into S^3 arise from knots (that is, each embedding of the torus is ambient isotopic to the boundary of a regular neighborhood of a knot in S^3).*

By the *exterior of a graph* K in S^3, we mean the complement $\mathrm{cl}(S^3 - N(K))$ of a regular neighborhood $N(K)$ of K in S^3, which is unique by the uniqueness of regular neighborhoods. The following theorem is proved in [Fox 1948] using a method of [Alexander 1924]:

Theorem 15.3.2 *A closed surface in S^3 separates S^3 into two compact connected 3-manifolds, each of which is homeomorphic to the exterior of a connected graph in S^3.*

According to this theorem, the topological study of closed surfaces in $\mathbf{R}^3$ can be reduced, in a sense, to the topological study of graphs in $\mathbf{R}^3$, although the graph is not uniquely determined by the surface. For example, the three graphs shown in figure 15.3.1 have mutually ambient isotopic regular neighborhoods.
From this viewpoint, the following definition due to [Suzuki 1970] is reasonable:

Definition 15.3.3 Two graphs K and K' in $\mathbf{R}^3$ (or S^3) are *neighborhood-equivalent* if regular neighborhoods $N(K)$ and $N(K')$ of K and K' in it are ambient isotopic in $\mathbf{R}^3$ (or S^3).

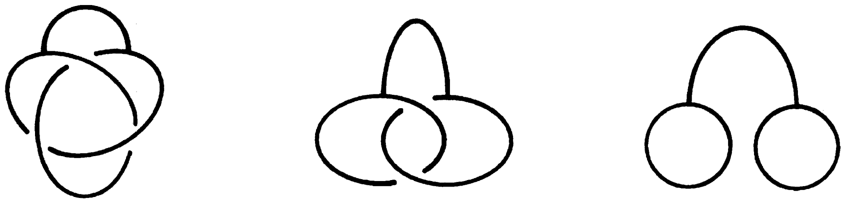


Fig. 15.3.1

The Alexander module of a link (cf. Chapter 7) can be obtained directly from the link group. In this group theoretic version, the Alexander module of a finitely presented group has been studied in detail by R. H. Fox and R. H. Crowell (cf. [Fox 1953, 1954], [Crowell 1961]).

Theorem 15.3.4 *Let π be a group with a finite presentation $\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle$. For a free abelian group H with a fixed basis, we assume that there is an epimorphism $\gamma : \pi \to H$. The group ring extension $\mathbf{Z}[\pi] \to \mathbf{Z}[H] = \Lambda$ of γ is also denoted by γ. Then we have the following Λ-exact sequence (depending only on the epimorphism $\gamma : \pi \to H$):*

$$0 \to \operatorname{Ker}\gamma/(\operatorname{Ker}\gamma)' \to A_\gamma \to \varepsilon(\Lambda) \to 0,$$

where $\varepsilon(\Lambda)$ denotes the augmentation ideal of Λ and the Λ-module A_γ has the Jacobian matrix $(\gamma(\partial r_i/\partial x_k))_{1\le i\le m, 1\le k\le n}$ as a Λ-presentation matrix.

We call A_γ the *Alexander module* of γ. The proof of this theorem is similar to that of Theorem 7.1.5 if we take a compact connected polyhedron X with $\pi_1(X, b) = \pi$. Then we can take $H_1(X_\gamma) = \operatorname{Ker}\gamma/(\operatorname{Ker}\gamma)'$ and $H_1(X_\gamma, p^{-1}(b)) = A_\gamma$ for the covering $p : X_\gamma \to X$ associated with the kernel of γ. This algebraic theory, called *Fox's free differential calculus*, can be applied to the fundamental group $\pi(K)$ of the exterior of a graph K in S^3, since we can obtain a finite group presentation of $\pi(K)$ by a method similar to the case of a link. For any epimorphism $\gamma : \pi(K) \to H$, we have the Λ-module $A_\gamma(K)$ of γ and its derived invariants such as the characteristic polynomials and the elementary ideals, which are calculable from the Jacobian matrix of the group presentation. In this theory, however, there is a subtlety in that we cannot select a canonical epimorphism γ in general (although we can specify γ up to finitely many choices if we are interested in the equivalence of K instead of the neighborhood-equivalence). Nevertheless, we can obtain some information on neighborhood-equivalent graphs (cf. [Kinoshita 1958", 1972, 1973, 1975], [Suzuki 1970, 1984]) and hence on surfaces in S^3 (cf. [Kinoshita 1986], [Suzuki 1984]). For example, if a graph K in S^3 is neighborhood-equivalent to a trivial graph, then the Alexander module $A_\gamma(K)$ of any epimorphism γ must be Λ-free. Hence if $A_\gamma(K)$ is not Λ-free for some γ, then we can conclude that K is not neighborhood-equivalent to a trivial graph. For a connected graph K in S^3, we can define effective polynomial invariants related to the Alexander module $A_\gamma(K)$

as we shall now describe. Considering a regular neighborhood of a tree in K, we can easily find a compact connected planar subsurface P in the boundary ∂X of the exterior X of K in S^3 such that $H_1(X,P) = 0$. Then by Lemma 12.3.13, $H_1(X_\gamma, P_\gamma)$ is a Λ-torsion module for all epimorphisms $\gamma : \pi(K) \to H$ whose 0-th characteristic polynomial $\Delta_\gamma(K;P)$ has $\Delta_\gamma(K;P)(1,\ldots,1) = \pm 1$. When K is a θ_n-curve, this polynomial is studied in [Litherland 1989] by choosing P carefully.

As the final topic of this section, we consider the tunnel number of a graph in S^3. For a graph K, let $K^{(0)}$ be the set of the vertices of degree greater than 2.

Definition 15.3.5 An *unknotting tunnel system* of a graph K in $\mathbf{R}^3$ or S^3 is a collection of mutually disjoint arcs $t_i (i = 1,2,\ldots,s)$ in $\mathbf{R}^3$ or S^3 such that $t_i \cap K = \partial t_i \subset K - K^{(0)}$ and the graph $K \cup (\bigcup_{i=1}^s t_i)$ in $\mathbf{R}^3$ or S^3 is neighborhood-equivalent to a trivial connected graph.

The minimal cardinality of unknotting tunnel systems of K is called the *tunnel number* of K and denoted by $t(K)$. We define $t(K) = 0$ if K is neighborhood-equivalent to a trivial connected graph. By definition, if K has c components, then $c - 1 \leq t(K)$. The following proposition follows directly from the definition:

Proposition 15.3.6 (1) *$t(K) < +\infty$ for any spatial graph K.*
(2) *The fundamental group $\pi(K)$ of the exterior of a connected graph K in S^3 is generated by at most $t(K) + 1 - \chi(K)$ elements, where $\chi(K)$ denotes the Euler characteristic of K.*
(3) *$t(K_1 \sharp K_2) \leq t(K_1) + t(K_2) + 1$ for any connected sum $K_1 \sharp K_2$ of two graphs $K_i, i = 1,2$, in S^3.*

Many results on tunnel numbers are known when K is a knot or link. For Proposition 15.3.6(3), this inequality is shown to be the best possible for knots in [Moriah-Rubinstein *] and [Morimoto-Sakuma-Yokota *] and for links in [Boileau-Lustig-Moriah *]. On the other hand, T. Kobayashi constructed a pair of knots $K_i, i = 1,2$, such that $t(K_1 \sharp K_2) \leq t(K_1) + t(K_2) - N$ for any given integer $N > 0$ (cf. [Kobayashi 1994], [Morimoto *]). The tunnel numbers of prime knots with up to 10 crossings, which are listed in Appendix F, are determined in [Morimoto-Sakuma-Yokota **]. For other topics on tunnel numbers, see, for example, [Boileau-Rost-Zieschang 1988], [Morimoto-Sakuma 1991], [Norwood 1982], [Scharlemann 1984], and [Fujii *]. In the last paper, H. Fujii showed that the Alexander polynomial of every knot is realizable by a tunnel number one knot.

Exercise 15.3.7 Show that $t(K) = 1$ if K is a non-trivial 2-bridge or torus knot.

Exercise 15.3.8 Let K be a connected sum of a trefoil knot and a Hopf link. Then show that $t(K) = 1$.

The *h-genus* of a graph K in S^3 is the minimal genus of a Heegaard surface of S^3 which contains K, which we denote by $h(K)$. The h-genus $h(K)$ is closely related to the tunnel number $t(K)$ as follows:

Proposition 15.3.9 *For any graph K in S^3, we have*

$$t(K) + 1 - c(K) \leq h(K) \leq t(K) + 1 - \chi(K),$$

where $c(K)$ denotes the number of connected components of K.

The proof is similar to that of [Morimoto 1994] in the case when K is a knot.

Supplementary notes for Chapter 15

There is another concept of a spatial graph, called a *rigid-vertex graph.* It is a graph K in $\mathbf{R}^3$ (or S^3) together with a small oriented disk $B(v, K)$ in $\mathbf{R}^3$ (or S^3) for each vertex v of K such that $B(v, K) \cap K$ is a regular neighborhood of v in K. Two such rigid-vertex graphs K and K' in $\mathbf{R}^3$ (or S^3) are said to be *rigid-vertex-equivalent* if there is an orientation-preserving auto-homeomorphism f of $\mathbf{R}^3$ (or S^3) such that $f(K) = K'$ and $f\,|_{B(v,K)}$ defines an orientation-preserving homeomorphism $B(v, K) \cong B(f(v), K')$ for each vertex v of K. It is possible to establish a spatial rigid-vertex graph version of Theorems A.3 and A.4 on link equivalence by a method analogous to Appendix A (see [Kauffman 1989] for the "Reidemeister moves" on rigid-vertex graphs). Finally, we observe that the tunnel number is independent of the unknotting number. In fact, for the torus knot $K = T(2, n)$ for an odd positive integer n, we have that the tunnel number $t(K) = 1$ and the unknotting number $u(K) = (n-1)/2$ (cf. Exercise 11.2.2). On the other hand, we take a twisted or untwisted double K of a knot K^* in S^3 such that the exterior $E(K^*)$ is decomposed into n pieces by the torus decomposition. Then we have $u(K) = 1$ and by [Kobayashi,T. 1987(Corollary 1)], $t(K) \geq (n+1)/3$.

Chapter 16
Vassiliev-Gusarov invariants

In this chapter, we discuss a graded $\mathbf{Q}$-algebra of numerical link invariants which we call the *Vassiliev-Gusarov invariants* (cf. [Vassiliev 1990], [Gusarov 1994, 1994']). An important observation is that this algebra determines the Jones, skein and Kauffman polynomials and their satellite version invariants, as is discussed in 16.2. In 16.3, we discuss Kontsevich's iterated integral invariant, and characterize the Vassiliev-Gusarov algebra in terms of the weight systems on chord diagrams. In 16.4, we discuss numerical link invariants which are not of Vassiliev-Gusarov type.

16.1 Vassiliev-Gusarov algebra

For each positive integer r, we consider the set $\mathcal{L}^r$ of the types of r-component ordered links in $\mathbf{R}^3$. The definition of the type of an ordered link is the same as the ordinary one except that an *order-preserving* homeomorphism instead of a homeomorphism should be used in the definition. For a diagram D of an element of $\mathcal{L}^r$, let $c(D)$ be the set of crossing points in D and $r(D)$ be the number of components of the link presented by D. For a subset $S \subset c(D)$, let D_S be the diagram obtained from D by changing the over-under relation at each crossing point in S. Let $\sharp S$ be cardinality of S. Let R be a commutative ring with unit. Let v be an R-valued invariant of $\mathcal{L}^r$. The following definition is due to [Gusarov 1994] and [Stanford 1994]:

Definition 16.1.1 An invariant v is a *Vassiliev-Gusarov invariant* of order $\leq n$ if v has $\sum_{S \subset X}(-1)^{\sharp S} v(D_S) = 0$ for any D with $\sharp c(D) \geq n+1$ and any subset $X \subset c(D)$ with $\sharp X = n+1$.

In this definition, we note that $r(D_S) = r(D) = r$. We also say that v is of *order* n if it is of order $\leq n$ but not of order $\leq n-1$. If v is a Vassiliev-Gusarov invariant of order 0, then v is a constant function on $\mathcal{L}^r$. For a subset $V \subset c(D)$, we denote by $D_{V(\times)}$ the diagram obtained from D by changing each crossing point in V into a vertex. Let $\mathcal{L}^r_{(\times)}$ be the set of the (ordered) types of quadrivalent rigid-vertex spatial graphs with oriented edges which are associated with the diagrams $D_{V(\times)}$ for all diagrams D in $\mathcal{L}^r$ and all $V \subset c(D)$ (cf. Supplementary notes for Chapter 15). Here we consider that $\mathcal{L}^r \subset \mathcal{L}^r_{(\times)}$. The invariant v on $\mathcal{L}^r$ is inductively extended on $\mathcal{L}^r_{(\times)}$ by the following rule:

$$v(D_{V(\times)}) = \operatorname{sign}(p)\{v(D_{(V-\{p\})(\times)}) - v((D_{\{p\}})_{(V-\{p\})(\times)})\}$$

for any $p \in V$, where $\operatorname{sign}(p)$ denotes the sign ± 1 of p. For a subset $S \subset c(D)$, we denote by $\operatorname{sign}(S)$ the product of the signs of all crossing points p in S. Let $c(D_{V(\times)}) = c(D) - V$. We denote the diagram $D_{V(\times)}$ simply by $\triangle$ unless confusion might occur. Then the notation $\triangle_{S\cup T(\times)}$ is used for $(D_S)_{(V\cup T)(\times)}$ where S, T are disjoint subsets of $c(\triangle)$. For a positive number n, let $\mathcal{X}$ be the family of mutually disjoint non-empty subsets $X_i (i = 1, 2, \ldots, n)$ of $c(\triangle)$. For a subfamily $\mathcal{S} \subset \mathcal{X}$, let $\triangle_{\mathcal{S}}$ be the diagram $\triangle_{\cup\{X_i \in \mathcal{S}\}}$ and $\sharp\mathcal{S}$ the number of X_i belonging to $\mathcal{S}$. The following lemma is easily obtained by induction on n:

Lemma 16.1.2

$$\sum_{\mathcal{S}\subset\mathcal{X}} (-1)^{\sharp\mathcal{S}} v(\triangle_{\mathcal{S}}) = \sum_{\emptyset\neq S_i\subset X_i(1\leq i\leq n)} \operatorname{sign}(\bigcup_{i=1}^{n} S_i) v(\triangle_{\bigcup_{i=1}^{n}[S_i(\times)\cup(X_i - S_i)]}).$$

The following corollary gives an alternate description of Vassiliev-Gusarov invariants which is employed in [Birman-Lin 1993] as the definition:

Corollary 16.1.3 *An invariant v on $\mathcal{L}^r$ is a Vassiliev-Gusarov invariant of order $\leq n$ if and only if $v(D_{X(\times)}) = 0$ for any diagram D on $\mathcal{L}^r$ with $\sharp c(D) \geq n+1$ and any subset $X \subset c(D)$ with $\sharp X = n+1$.*

The components of a tubular neighborhood of an ordered link $L \in \mathcal{L}^r$ are uniquely identified with the n copies $(S^1 \times D^2)_i (i = 1, 2, \ldots, r)$ of $S^1 \times D^2$ by using the longitude of each component of L. Embedding a fixed $r_i (\geq 1)$-component ordered link into $\operatorname{int}(S^1 \times D^2)_i$ for each i, we obtain from L a new ordered link L' with $r' = \sum_{i=1}^{r} r_i$ components (cf. 3.1.7). This defines a map $\psi : \mathcal{L}^r \to \mathcal{L}^{r'}$, called a *satellite map*. The following property of Vassiliev-Gusarov invariants with respect to satellites is known (cf. [Gusarov 1994'], [Stanford 1994], [Bar-Natan 1995]):

Corollary 16.1.4 *For any satellite map $\psi : \mathcal{L}^r \to \mathcal{L}^{r'}$ and any Vassiliev-Gusarov invariant v' of order $\leq n'$ on $\mathcal{L}^{r'}$, the invariant $v' \circ \psi$ on $\mathcal{L}^r$ is a Vassiliev-Gusarov invariant of order $\leq n'$.*

For any invariants v, w, diagram $\triangle$ in $\mathcal{L}^r_{(\times)}$, and subset $X \subset c(\triangle)$, we have the following lemma due to [Kanenobu-Miyazawa *]:

Lemma 16.1.5

$$\sum_{S\subset X} (-1)^{\sharp S} v(\triangle_S) w(\triangle_S) = \sum_{S\subset X} (\sum_{T\subset S} (-1)^{\sharp T} v(\triangle_T))(\sum_{U\subset X-S} (-1)^{\sharp U} w(\triangle_{S\cup U})).$$

The proof is by induction on n. The following corollary is a consequence of Lemma 16.1.5 (cf. [Gusarov 1994'], [Bar-Natan 1995]):

Corollary 16.1.6 *If v, w are Vassiliev-Gusarov invariants on $\mathcal{L}^r$ of orders $\leq p, q$ respectively, then the product vw is a Vassiliev-Gusarov invariant on $\mathcal{L}^r$ of order $\leq p+q$.*

Let V^r_R be the set of all R-valued Vassiliev-Gusarov invariants on $\mathcal{L}^r$. By Corollary 16.1.6, multiplication on R gives V^r_R an algebra structure over R. When $R = \mathbf{Q}$, we call V^r_R the *Vassiliev-Gusarov algebra* on $\mathcal{L}^r$ and denote it by V^r.

16.2 Vassiliev-Gusarov invariants and Jones type polynomials

Let V_n^r be the subset of V^r consisting of all Vassiliev-Gusarov invariants of order $\leq n$, which is a vector space over $\mathbf{Q}$. We discuss here how to derive elements of V_n^r from the skein polynomial $P(L; a, z)$ in 8.2.6 (see [Gusarov 1994], [Bar-Natan 1995], [Birman-Lin 1993] and [Kanenobu-Miyazawa *] for similar arguments). For $L \in \mathcal{L}^r$, we let $P_\sharp(L; a, z) = (a^{-1}z)^{r-1}P(L; a, z)$. Then we have the following identities:

(1) For a trivial knot O, $\quad P_\sharp(O; a, z) = 1$.

(2) For a skein triple (L_+, L_-, L_0),
$$a^{-2}P_\sharp(L_+; a, z) - P_\sharp(L_-; a, z) = (a^{-2}z^2)^\delta P_\sharp(L_0; a, z),$$
where $\delta = (r_+ - r_0 + 1)/2$ (=1 or 0) for the component numbers r_+, r_0 of L_+, L_0, respectively.

Then we see that $P_\sharp(L; a, z)$ can be written as $\Sigma_{n=0}^{+\infty} p_{2n}(L; a)z^{2n}$, where $p_{2n}(L; a)$ is a Laurent polynomial in a^2, which is 0 except for finitely many n. We denote a^{-2} and z^2 by x and y, and $P_\sharp(L; a, z)$ and $p_{2n}(L; a)$ by $C_\sharp(L; x, y)$ and $c_n(L; x)x^{-n}$, respectively. Clearly, $c_n(L; x)$ is a Laurent polynomial in x and we have

$$C_\sharp(L; x, y) = \sum_{n=0}^{+\infty} c_n(L; x)(xy)^n.$$

Taking $c_n(L; x) = 0$ for $n < 0$, we obtain the following alternate description of the skein polynomial:

Theorem 16.2.1 *There is one and only one sequence $\{c_n \mid n \in \mathbf{Z}\}$ of Laurent polynomial link invariants in x with integer coefficients that satisfy the following identities:*

(1) *For a trivial knot O,*
$$c_n(O; x) = \begin{cases} 0 & (n \neq 0) \\ 1 & (n = 0). \end{cases}$$

(2) $xc_n(L_+; x) - c_n(L_-; x) = c_{n-\delta}(L_0; x)$ *for all n, where $\delta = (r_+ - r_0 + 1)/2$ ($= 0$ or 1) for the number of components r_+, r_0 of L_+, L_0, respectively.*

We call c_n the n-th *coefficient polynomial* of the skein polynomial (or of the link). Some calculations of the coefficient polynomials have been made in [Lickorish-Millett 1987] and [Kawauchi 1994]. For example, for any link L with the components $K_i (i = 1, 2, \dots, r)$ and total linking number λ, we have

$$c_0(L; x) = (x-1)^{r-1}x^{-\lambda}c_0(K_1; x)c_0(K_2; x)\dots c_0(K_r; x),$$
$$c_0(K_i; 1) = 1, \quad \frac{d}{dx}c_0(K_i; 1) = 0 \quad (i = 1, 2, \dots, r).$$

We can show that these properties characterize c_0 and that $(x-1)^{r-1-n}$ divides $c_n(L; x)$ for each integer n with $0 \leq n < r-1$ (cf. [Kawauchi 1994]). Since $c_0(L; x)$

determines the component number r of L, we can conclude that the sequence $\{c_n\}$ determines the skein polynomial itself. The Conway polynomial $\nabla(L;z)$ and the Jones polynomial $V(L;t)$ are obtained from $\{c_n\}$ as follows:

$$\nabla(L;z)z^{r-1} = \sum_{n=0}^{+\infty} c_n(L;1)z^{2n},$$

$$V(t^{-1})(t^{1/2} - t^{3/2})^{r-1} = \sum_{n=0}^{+\infty} c_n(L;t^2)t^n(t-1)^{2n}.$$

We show the following:

Theorem 16.2.2 *For any non-negative integers m and n and any link diagram D and any subset $X \subset c(D)$ such that $\sharp X \geq 2n+2-r(D)+m$, we have*

$$\sum_{S \subset X} (-1)^{\sharp S} \frac{d^m}{dx^m} c_n(D_S;1) = 0.$$

Here are three corollaries of Theorem 16.2.2:

Corollary 16.2.3 *The invariant $\frac{d^m}{dx^m}c_n(1)$ on $\mathcal{L}^r$ belongs to V_p^r for $p = \max(0, 2n+1-r+m)$.*

Write the Conway polynomial $\nabla(L;z)$ as the sum $\sum_{n=0}^{+\infty} a_n(L)z^n$ with $a_n(L) \in \mathbf{Z}$. Since $\nabla(L;z)z^{r-1} = \sum_{n=r-1}^{+\infty} c_n(L;1)z^{2n}$, we obtain the following:

Corollary 16.2.4 *The invariant a_n on $\mathcal{L}^r$ belongs to V_n^r.*

Corollary 16.2.5 *For the Jones polynomial $V(t)$, the invariant $\frac{d^m}{dt^m}V(1)$ on $\mathcal{L}^r$ belongs to V_m^r.*

Proof of Theorem 16.2.2. We proceed by induction on $\sharp X$. Let $\sharp X = 0$, i.e., $X = \emptyset$. Then $r(D) - 1 > n + m$. Since $(x-1)^{r(D)-1-n}$ divides $c_n(D;x)$, we see that $(x-1)^{r(D)-1-n-m}$ divides $\frac{d^m}{dx^m}c_n(D;x)$. Hence $\frac{d^m}{dx^m}c_n(D;1) = 0$. Consider a subset $X \subset c(D)$ such that $\sharp X \geq 2n+2-r(D)+m$. By Theorem 16.2.1, we have that

$$\frac{d^m}{dx^m}c_n(D;1) - \frac{d^m}{dx^m}c_n(D_{\{p\}};1) = \frac{d^m}{dx^m}c_{n-\delta}(D_0;1) - m\frac{d^{m-1}}{dx^{m-1}}c_n(D;1)$$

for any positive crossing $p \in c(D)$, where D_0 denotes the diagram obtained from D by smoothing at p, and $\delta = (r(D)-r(D_0)+1)/2$. When p is a negative crossing, we have a similar identity obtained by interchanging D and $D_{\{p\}}$. Let $p \in X$ and $X_1 = X - \{p\}$. When $\mathrm{sign}(p) = +1$,

$$\begin{aligned}
&\sum_{S \subset X} (-1)^{\sharp S} \frac{d^m}{dx^m} c_n(D_S;1) \\
&= \sum_{S \subset X_1} (-1)^{\sharp S} \left(\frac{d^m}{dx^m}c_n(D_S;1) - \frac{d^m}{dx^m}c_n(D_{S\cup\{p\}};1)\right) \\
&= \sum_{S \subset X_1} (-1)^{\sharp S} \frac{d^m}{dx^m}c_{n-\delta}((D_S)_0;1) - m \sum_{S \subset X_1} (-1)^{\sharp S} \frac{d^{m-1}}{dx^{m-1}}c_n(D_S;1).
\end{aligned}$$

Since $\sharp X_1 = \sharp X - 1 \geq 2n + 2 - r(D) + (m-1) = 2(n-\delta) + 2 - r(D_0) + m$, we see that the last two summations are zero by the induction hypothesis. When sign(p)$= -1$, the last two summations are multiplied by -1 and $\frac{d^{m-1}}{dx^{m-1}} c_n(D_S;1)$ is replaced by $\frac{d^{m-1}}{dx^{m-1}} c_n(D_{S\cup\{p\}};1)$. For the same reason, the last two summations are also zero. □

Example 16.2.6. The Vassiliev-Gusarov invariant $v_m = \frac{d^m}{dx^m} c_0(1)$ on $\mathcal{L}^1$ is precisely of order m for each $m \geq 2$. By Corollary 16.1.6, it is of order $\leq m$. Let D_m be a standard $(2m-1)$-crossing diagram of the torus knot $T(2, 2m-1)$ which has only negative crossing points. We have $c_0(D_m; x) = mx^{m-1} - (m-1)x^m$. Take any subset $X \subset c(D_m)$ with $\sharp X = m$. For each non-empty subset $S \subset X$, there is an integer m_S with $0 \leq m_S < m$ such that $c_0((D_m)_S; x) = m_S x^{m_S - 1} - (m_S - 1)x^{m_S}$, so that $\frac{d^m}{dx^m} c_0((D_m)_S; 1) = 0$. Thus, we have

$$\sum_{S\subset X} (-1)^{\sharp S} v_m((D_m)_S; 1) = v_m(D_m) = -(m-1)m! \neq 0$$

and v_m is of order m. We note that any $\mathbf{Q}$-valued Vassiliev-Gusarov invariant on $\mathcal{L}^1$ of order ≤ 1 is of order 0 (cf. [Chmutov-Duzhin 1994]).

Example 16.2.7. Here we assume that $r \geq 2$. For $L \in \mathcal{L}^r$, we set $\tilde{c}_0(L;x) = c_0(L;x)/(x-1)^{r-1}$. Then we have

$$(r-1)! \frac{d^m}{dx^m} \tilde{c}_0(L;1) = \frac{d^{r-1+m}}{dx^{r-1+m}} c_0(L;1).$$

We show that the invariant $\tilde{v}_m = \frac{d^m}{dx^m} \tilde{c}_0(1)$ on $\mathcal{L}^r$ is a Vassiliev-Gusarov invariant of order m. It is of order $\leq m$ by Corollary 16.2.3. Note that $\lambda = -\tilde{v}_1$ for the total linking number λ of L. Hence $\tilde{v}_1$ is precisely of order 1 since λ is not constant on $\mathcal{L}^r$. Let $m \geq 2$. We take L_m to be the split union of an $(r-1)$-component trivial link and the torus knot $T(2, 2m-1)$. Since $\tilde{c}_0(L_m; x) = c_0(T(2, 2m-1); x)$, we see from the argument of Example 16.2.6 that the invariant $\tilde{v}_m$ is precisely of order m. We further show that the m-fold product λ^m is a Vassiliev-Gusarov invariant on $\mathcal{L}^r$ of order m and the exponential invariant $e^\lambda = \sum_{n=0}^{+\infty} \lambda^n/n!$ on $\mathcal{L}^r$ is not a Vassiliev-Gusarov invariant of any order. By Corollary 16.1.6, λ^m is of order $\leq m$. Let D_m be a standard $2m$-crossing diagram of the split sum of a trivial $(r-2)$-component link and the torus link $T^*(2, 2m)$ which has only positive crossing points. For any subset $X \subset c(D_m)$ with $\sharp X = m$, we have $\sum_{S\subset X} (-1)^{\sharp S} \lambda((D_m)_S)^m = m! \neq 0$ by Lemma 16.1.5. Hence λ^m is of order m. Next, by a direct computation, we see that

$$\sum_{S\subset X} (-1)^{\sharp S} e^{\lambda((D_m)_S)} = \sum_{s=0}^{m} (-1)^s \binom{m}{s} e^{m-s},$$

which cannot be a rational number because e is a transcendental number over $\mathbf{Q}$ (cf. [Lang 1965]). Hence e^λ is not a Vassiliev-Gusarov invariant of order $m-1$ for any m.

From these examples, we see that V_n^r is a proper subspace of V_{n+1}^r for each $n \geq 0$, except for the case when $r = 1$ and $n = 0$ (where $V_0^1 = V_1^1$).

For the Kauffman polynomial $F(L; a, x)$, we have the following identity (cf. Exercise 8.3.5):

$$a^2 F(L_+; a, x) + F(L_-; a, x) = ax\{F(L_0; a, x) + a^{-2\nu} F(L_\infty; a, x)\}.$$

Letting $y = -a^2$ and $z = -ax$, we write $F(L; a, x)$ as $G(L; y, z)$. Further, we let $G(L; y, z)_\sharp = G(L; y, z) z^{r-1}$ for the number of components r of L. Then we have

$$yG(L_+; y, z)_\sharp - G(L_-; y, z)_\sharp = G(L_0; y, z)_\sharp z^{2\delta} + (-y)^{-\nu} G(L_\infty; y, z)_\sharp z^{1+\delta}.$$

Since $G(O^r; y, z)_\sharp = (y - 1 - z)^{r-1}$ for an r-component trivial link O^r, we see from induction on $c(D)$ that $G(L; y, z)_\sharp$ is written as the sum $\sum_{n=0}^{\infty} f_n(L; y) z^n$ where $f_n(L; y)$ is a Laurent polynomial in y which is zero except for finitely many n. We note that $f_0(L; y) = c_0(L; y)$.

Exercise 16.2.8 For any non-negative integers m and n, any link diagram D, and any subset $X \subset c(D)$ such that $\sharp X \geq n + 2 - r(D) + m$, show that

$$\sum_{S \subset X} (-1)^{\sharp S} \frac{d^m}{dy^m} f_n(D_S; 1) = 0.$$

16.3 Kontsevich's iterated integral invariant

Let r be a positive integer. Let ℓ^r be the disjoint union of oriented ordered r loops, called the r-string *Wilson loop system.* By a *chord system* γ on ℓ^r, we mean finitely many, mutually disjoint, unoriented dotted arcs spanning ℓ^r. Each member of γ is called a *chord* on ℓ^r. Note that a chord system γ on ℓ^r is uniquely determined by the system of the boundary point pairs of all the chords in γ, denoted by $\partial\gamma$, namely a system of finitely many, mutually disjoint, unordered pairs of points in ℓ^r.

Definition 16.3.1 An r-string *chord diagram* c is a pair $(\ell^r; \gamma)$ of the r-string Wilson loop system ℓ^r and a chord system γ on ℓ^r. The *degree* of the chord diagram $c = (\ell^r; \gamma)$, denoted by $\deg c$ is the number of chords in γ.

We consider two chord diagrams $c = (\ell^r; \gamma)$ and $c' = (\ell^{r'}; \gamma')$ to be the *same* if $r = r'$ and there is an orientation-preserving, order-preserving auto-homeomorphism h of ℓ^r such that $h(\partial\gamma) = \partial\gamma'$. Let C_d^r be the set of r-string chord diagrams of degree d. Let $\mathbf{Q}[C_d^r]$ be the vector space over $\mathbf{Q}$ spanned by C_d^r. Let A_d^r be the quotient space of $\mathbf{Q}[C_d^r]$ by the *4-term relation* shown in figure 16.3.1 and the *framing independence relation* shown in figure 16.3.2. We note that $\dim_{\mathbf{Q}} A_d^r < +\infty$. Let A_*^r be the product vector space $\prod_{d=0}^{+\infty} A_d^r$ over $\mathbf{Q}$. We regard A_d^r as a subspace of A_*^r by the natural injection for each d. For an r-string chord diagram c and 1-string chord diagrams $c_i^1 (i = 1, 2, \ldots, r)$, we take an oriented connected sum of the i-th Wilson loop of c and the Wilson loop of c_i^1 along arcs disjoint from the chords. Then we obtain a new r-string chord diagram c' with $\deg c' = \deg c + \sum_{i=1}^{r} \deg c_i^1$. When we consider c' as an element of $A_{d'}^r$ where $d' = \deg c'$, we see that c' does not depend on the choice of arcs used for the connected sum because of the 4-term relation. Thus, we have an action of the r-fold product $(A_*^1)^r$ of A_*^1 on A_*^r. In particular, taking $r = 1$, we see that by this action A_*^1 forms a $\mathbf{Q}$-algebra with identity 1 represented by the Wilson loop without chords.

We regard 3-space $\mathbf{R}^3$ as $\mathbf{C} \times \mathbf{R}$ whose points are written as (z, t), $z \in \mathbf{C}, t \in \mathbf{R}$. For each $L \in \mathcal{L}^r$ and $t \in \mathbf{R}$, we also assume that $L_t = L \cap \mathbf{C} \times t$ is a finite (or empty) set. Further, for any two distinct points $(z, t), (z', t)$ of L_t which are not maxima or minima with respect to t, the assignment $t \longrightarrow (z, z')$ defines an embedding from a neighborhood of t in $\mathbf{R}$ to $\mathbf{C} \times \mathbf{C}$. The following definition due to [Kontsevich 1993], [Bar-Natan 1995] generalizes the Gauss integral for the linking number (cf. [Gauss 1833], [Rolfsen 1976]):

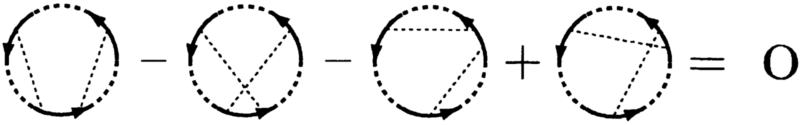


Fig. 16.3.1

Fig. 16.3.2

Definition 16.3.2 The *Kontsevich iterated integral* of a link $L \in \mathcal{L}^r$ is an element $Z(L)$ of A_*^r which is given on A_d^r by

$$\sum_P (-1)^{\sharp P \downarrow} \left(\frac{1}{(2\pi\sqrt{-1})^d} \int_{t_{min} < t_1 < \cdots < t_d < t_{max}} \wedge_{i=1}^{d} \frac{dz_i - dz_i'}{z_i - z_i'}\right) c_P,$$

where we use the following notations: t_{max} and t_{min} denote the maximum and minimum values of t on L, respectively. P denotes the set of the unordered pairs $\{(z_i, t_i), (z_i', t_i)\}$ $(i = 1, 2, \ldots, d)$ such that (z_i, t_i) and (z_i', t_i) are distinct points of L which are neither maxima nor minima and $t_{min} < t_1 < \cdots < t_d < t_{max}$. c_P denotes the chord diagram of degree d determined by L and P, which is regarded as an element of A_d^r. We denote by $\sharp P \downarrow$ the number of the points of the form (z_i, t_i) or (z_i', t_i) at which L is decreasing.

For the definition above, it is proved in [Le-Murakami **] (based on a suggestion by M. Kontsevich) that the coefficient of c_P in $Z(L)$ is rational. Let $\heartsuit$ be a heart-shaped loop embedded in $\mathbf{R} \times \mathbf{R} \subset \mathbf{C} \times \mathbf{R}$. Let $a = Z(\heartsuit) \in A_*^1$. Since a takes 1 on

A_0^1, the formal power series $\sum_{n=0}^{+\infty} b^n$ with $b = 1 - a$ (where we put $b^0 = 1$) defines the multiplicative inverse $a^{-1} \in A_*^1$ of a. Let m_i be the number of maxima of the i-th component of L with respect to t.

Theorem 16.3.3 *The element $\tilde{Z}(L) = (a^{-m_1}, a^{-m_2}, \ldots, a^{-m_r}) \cdot Z(L)$ of A_*^r is an invariant of the type of $L \in \mathcal{L}^r$.*

See [Le-Murakami 1995] for this formulation and [Bar-Natan 1995] for the proof (where only the case $r = 1$ is discussed, but the proof can be naturally extended to the general case). In [Le-Murakami *], a method for the combinatorial calculation of $\tilde{Z}(L)$ is given. Let

$$W_d^r = \{w \in \mathrm{Hom}_{\mathbf{Q}}(A_*^r, \mathbf{Q}) \mid w(A_{d'}^r) = 0, \forall d' \neq d\}.$$

We call each element $w \in W_d^r$ a *weight system* of degree d. Each element of $\mathcal{L}_{(\times)}^r$ is represented by an r-string Wilson loop system immersed in $\mathbf{C} \times \mathbf{R}$ with only isolated double points. A member of $\mathcal{L}_{(\times)}^r$ with just n double points is denoted by $L_{n(\times)}$. A $\mathbf{Q}$-valued invariant v on $\mathcal{L}^r$ can be extended to an invariant on $\mathcal{L}_{(\times)}^r$, as stated in 16.1. The invariant $\tilde{Z}$ on $\mathcal{L}^r$ is similarly extended to an invariant on $\mathcal{L}_{(\times)}^r$. By Corollary 16.1.3, v belongs to V_d^r if and only if for each $L_{d(\times)} \in \mathcal{L}_{(\times)}^r$, the value $v(L_{d(\times)})$ is determined uniquely by the chord diagram c associated with $L_{d(\times)}$. For any $v \in V_d^r$, we define $w(c) = v(L(c)_{d(\times)})$ by realizing $c \in C_d^r$ as an $L(c)_{d(\times)} \in \mathcal{L}_{(\times)}^r$. Then w belongs to W_d^r and we have a natural linear map

$$\zeta : V_d^r \to W_d^r$$

sending v to w. Clearly, $\zeta(V_{d-1}^r) = 0$. The following theorem reveals the importance of the Kontsevich iterated integral invariant $\tilde{Z}(L)$:

Theorem 16.3.4 *The linear map $\zeta : V_d^r \to W_d^r$ is an epimorphism whose kernel is equal to V_{d-1}^r.*

Proof. For $L \in \mathcal{L}^r$ and $w \in W_d^r$, we define $v(L) = w(\tilde{Z}(L))$. Then $v(L_{n(\times)}) = w(\tilde{Z}(L_{n(\times)}))$ for all $L_{n(\times)} \in \mathcal{L}_{(\times)}^r$ and all n. An argument of [Bar-Natan 1995] shows that $\tilde{Z}(L_{n(\times)})$ takes 0 on A_m^r with $m < n$ and takes the chord diagram c associated with $L_{n(\times)}$ on A_n^r. In particular, $v(L_{n(\times)}) = w(\tilde{Z}(L_{n(\times)})) = 0$ for any $n > d$, which implies that $v \in V_d^r$. Let $L(c)_{d(\times)}$ be a realization of $c \in C_d^r$ in $\mathcal{L}_{(\times)}^r$. Then

$$\zeta(v)(c) = v(L(c)_{d(\times)}) = w(\tilde{Z}(L(c)_{d(\times)}) = w(c).$$

Hence $\zeta(v) = w$ and ζ is onto. If $\zeta(v) = 0$ for some $v \in V_d^r$, then we see from the definition of ζ that $v(L_{d(\times)}) = 0$ for any $L_{d(\times)} \in \mathcal{L}_{(\times)}^r$. Hence $v \in V_{d-1}^r$. □

The following corollary follows immediately from Theorem 16.3.4:

Corollary 16.3.5 $\dim_{\mathbf{Q}} V_d^r < +\infty$.

For example, it is shown in [Chmutov-Duzhin 1994] that

$$\dim_{\mathbf{Q}}(V_d^1/V_{d-1}^1) \leq (d-1)!.$$

Although we discussed the Vassiliev-Gusarov invariants over $\mathbf{Q}$, the same arguments work over $\mathbf{C}$ without any essential changes. Then the $\mathbf{C}$-version of Theorem 16.3.4 and the rationality of $\tilde{Z}(L)$ imply the following corollary:

Corollary 16.3.6 *If v is a $\mathbf{C}$-valued Vassiliev-Gusarov invariant of order $\leq d$, then v is a $\mathbf{C}$-linear combination of a $\mathbf{Q}$-basis $v_i (i = 1, 2, \ldots, n)$ of V_d^r.*

16.4 Numerical invariants not of Vassiliev-Gusarov type

We say that two $\mathbf{C}$-valued invariants v and $\tilde{v}$ on $\mathcal{L}^r$ are *equivalent* if the condition $v(L) = v(L')$ on all $L, L' \in \mathcal{L}^r$ is equivalent to the condition $\tilde{v}(L) = \tilde{v}(L')$ on all $L, L' \in \mathcal{L}^r$. We note that in general this equivalence on link invariants does not preserve the order of Vassiliev-Gusarov invariants. For example, the total linking number $\lambda (\in V_1^r - V_0^r)$ is equivalent to the n-fold product invariant $\lambda^n (\in V_n^r - V_{n-1}^r)$ for each $r > 1$ and each odd $n > 1$. It is also equivalent to the exponential invariant e^λ which is not a Vassiliev-Gusarov invariant of any order (cf. Example 16.2.7).

Definition 16.4.1 A $\mathbf{C}$-valued invariant v on $\mathcal{L}^r$ is a *Vassiliev-Gusarov type* invariant if it is equivalent to a Vassiliev-Gusarov invariant.

The *order* of a Vassiliev-Gusarov type invariant v is the minimum of the orders of Vassiliev-Gusarov invariants which are equivalent to v. For a diagram D in $\mathcal{L}^r$, we choose a disk B so that $T = D \cap B$ is a 2-string trivial tangle without crossing in B. For each integer m, let $D(m)$ be a diagram in $\mathcal{L}^r$ obtained from D by replacing T with an m-full twist $T(m)$ of T, so that we have $D(0) = D$ and $c(D(m)) = c(D) + 2m$. The diagram sequence $\{D(m) \mid m = 0, 1, 2, \ldots\}$ or $\{D(m) \mid m = 0, -1, -2, \ldots\}$ is called a *twist diagram sequence* in $\mathcal{L}^r$ with initial diagram D and denoted by $\mathcal{T}(D)$. The following simple criterion is useful to determine whether or not a given link invariant is not of Vassiliev-Gusarov type (cf. [Birman-Lin 1993],[Dean 1994],[Trapp 1994]):

Lemma 16.4.2 *A $\mathbf{C}$-valued invariant v on $\mathcal{L}^r$ is not of Vassiliev-Gusarov type if there is a twist diagram sequence $\mathcal{T}(D)$ in $\mathcal{L}^r$ such that v is a constant function on $\mathcal{T}(D) - \{D\}$ whose value is distinct from $v(D)$.*

Proof. We give the proof for the case that $\mathcal{T}(D) = \{D(m) \mid m = 0, 1, 2, \ldots\}$. The other case is similar. Suppose that v is equivalent to a Vassiliev-Gusarov invariant $\tilde{v}$ of order, say m. Let X be a set of $m+1$ crossing points of $T(m+1) \subset D(m+1)$. Then

$$\sum_{S \subset X} (-1)^{\sharp S} \tilde{v}(D(m+1)_S) = (\sum_{s=0}^{m} (-1)^s \binom{m+1}{s}) \tilde{v}(D(1)) + (-1)^{m+1} \tilde{v}(D) \neq 0,$$

because $\sum_{s=0}^{m+1}(-1)^s\binom{m+1}{s} = 0$ and $\tilde{v}(D(1)) \neq \tilde{v}(D)$. Thus, $\tilde{v}$ is not of order m, which is a contradiction. $\square$

An example of a twist diagram sequence on $\mathcal{L}^1$ is given by the sequence $D(m)(m = 0, 1, 2, \dots)$ such that $D(m)$ is a standard diagram with $c(D(m)) = w(D(m)) = 2m+2$ of a twist knot with m full twists. From this example, we can see that none of the following invariants on $\mathcal{L}^r$ is of Vassiliev-Gusarov type: the genus, the g^*-genus, the unknotting number, the signature and the bridge index. Another example of a twist diagram sequence on $\mathcal{L}^1$ is given by the sequence $D(m)(m = 0, 1, 2, \dots)$ such that $D(m)$ is a standard diagram with $c(D(m)) = w(D(m)) = 2m + 1$ of the torus knot $T(2, 2m + 1)$. From this, we also see that the braid index is also not an invariant of Vassiliev-Gusarov type on $\mathcal{L}^r$.

Definition 16.4.3 An r-component link L is *n-similar* to a link L_0 if there are a link diagram D of L and a family $\mathcal{X}$ of mutually disjoint non-empty subsets $X_i \subset c(D)(i = 1, 2, \dots, n)$ such that $D_{\mathcal{S}}$ is a diagram of L_0 for any non-empty subfamily $\mathcal{S} \subset \mathcal{X}$.

We have the following lemma due to [Ohyama 1995] as a corollary of Lemma 16.1.2:

Lemma 16.4.4 *If v is a Vassiliev-Gusarov type invariant of order d on $\mathcal{L}^r$, then we have $v(L) = v(L_0)$ for any link $L \in \mathcal{L}^r$ which is n-similar to a given link $L_0 \in \mathcal{L}^r$ with $n > d$.*

Proof. Let D be a diagram of L and $\mathcal{X}$ a family of subsets $X_i(i = 1, 2, \dots, n)$ of $c(D)$ which are used for the n-similarity of L to L_0. Let $\tilde{v}$ be a Vassiliev-Gusarov invariant of order d which is equivalent to v. For $n \geq d + 1$, we see from Lemma 16.1.2 and Corollary 16.1.3 that

$$\sum_{\mathcal{S} \subset \mathcal{X}} (-1)^{\sharp\mathcal{S}} \tilde{v}(D_{\mathcal{S}}) = 0.$$

By definition, $\tilde{v}(D_{\mathcal{S}}) = \tilde{v}(L_0)$ for any non-empty subfamily $\mathcal{S}$ of $\mathcal{X}$. Thus, the equality above reduces to the following identity:

$$\tilde{v}(D) + \sum_{i=1}^{n}(-1)^i\binom{n}{i}\tilde{v}(L_0) = 0.$$

Since $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$, we have $\tilde{v}(D) - \tilde{v}(L_0) = 0$, implying that $v(L) = v(L_0)$. $\square$

For any $L \in \mathcal{L}^r$, we consider the connected sum decomposition of the exterior $E(L)$ of L in the 3-sphere $S^3 = \mathbf{R}^3 \cup \{\infty\}$ and then consider the torus decomposition of each connected summand (cf. Appendix C). Let $M_i(i = 1, 2, \dots, r)$ be the resulting hyperbolic pieces. We define $\mathrm{Vol}(L)$ to be the sum of the hyperbolic volumes $\mathrm{Vol}(M_i)(i = 1, 2, \dots, r)$. $\mathrm{Vol}(L)$ is an $\mathbf{R}$-valued link invariant by the uniqueness of connected sum and torus decompositions and Mostow rigidity (cf. Appendix C). It is known that Vol(L) is equivalent to the Gromov invariant of $E(L)$ (cf. [Thurston *], [Soma 1981]). As an application of Lemma 16.4.4, we have the following corollary:

Corollary 16.4.5 *The invariant* Vol *on* $\mathcal{L}^r$ *is not of Vassiliev-Gusarov type.*

Proof. Suppose that $v = \mathrm{Vol}$ is equivalent to a Vassiliev-Gusarov invariant $\tilde{v}$ of order, say d. Let L be a split union of an $(r-1)$-component trivial link and the d-iterated untwisted double of the figure eight knot 4_1. Then we can directly see that L is $(d+2)$-similar to an r-component trivial link O^r. Hence $\tilde{v}(L) = \tilde{v}(O^r)$, so that $v(L) = v(O^r)$. On the other hand, we have

$$v(L) = v(4_1) + dv(\text{Whitehead link}) > v(O^r) = 0,$$

which is a contradiction. □

From this proof, we see also that the crossing number is not a Vassiliev-Gusarov type invariant on $\mathcal{L}^r$. It is shown in [Yamamoto 1990] that every knot is 2-similar to a trivial knot O and in [Taniyama 1992] that the untwisted double of a knot which is n-similar to O is $(n+1)$-similar to O (cf. [Lin 1994]). See [Stanford **], [Ohyama 1995] and [Kawauchi 1992'] for further examples on n-similarity. The following question due to [Vassiliev 1990] and [Birman-Lin 1993] remains an open question:

Question 16.4.6 For any **C**-valued invariant v on $\mathcal{L}^r$, does there exist a sequence $\{v_n\}$ such that v_n is a **C**-valued Vassiliev-Gusarov invariant of some order on $\mathcal{L}^r$ for all n and $\lim_{n\to+\infty} v_n(L) = v(L)$ for each $L \in \mathcal{L}^r$?

It is observed in [Trapp 1994] that if $\{v_n\}$ converges uniformly to v, then v is also a Vassiliev-Gusarov invariant of some order.

Supplementary notes for Chapter 16

Vassiliev-Gusarov invariants were originally introduced in [Vassiliev 1990] as knot invariants coming from the 0-th cohomology of the space obtained from the function space of all smooth maps $S^1 \to \mathbf{R}^3$ by removing the subspace of all non-embeddings. It is not yet known whether or not the Vassiliev-Gusarov algebra V^1 determines the knot types. In particular, does V^1 determine the invertibility of 8_{17}?

Appendix A
The equivalence of several notions of "link equivalence"

In this appendix, we show the equivalence of several conditions for two links to belong to the same type. The results are more or less known as classical results and are discussed in [Burde-Zieschang 1985]. Here, by M, we mean S^3 or $\mathbf{R}^3$. We first note that the condition for two links to belong to the same type is stated in Chapter 0.

Definition A.1 Two links L and L' in M belong to *the same topological type* if there is an orientation-preserving auto-homeomorphism h (not necessarily PL) of M such that $h(L) = L'$ and $h\,|_L\colon L \cong L'$ is orientation-preserving.

For a link L and a disk D in M such that $L \cap D = L \cap \partial D$ which is an arc, the new link $L' = \mathrm{cl}(L - L \cap D) \cup (\partial D - L \cap D)$ is said to be obtained from L by a *disk move*. Here, the link L' is oriented so that the orientations on $L' - D$ and $L - D$ coincide.

Definition A.2 Two links L and L' in M belong to *the same combinatorial type* if there is a sequence of links $L_i (i = 0, 1, \ldots, s+1)$ with $L_0 = L$ and $L_{s+1} = L'$ such that L_{i+1} is obtained from L_i by a disk move for each i.

Two links L and L' in M are said to be *ambient isotopic* if there is an ambient isotopy $h_t (0 \leq t \leq 1)$ of M such that $h_0 = \mathrm{id}$ and h_1 gives the same type $L \cong L'$ (defined in Chapter 0). In case $M = \mathbf{R}^3$, they are said to be *ambient isotopic with a compact support* if, in addition, there is a compact subset $X \subset \mathbf{R}^3$ such that $h_t(x) = x$ for all $x \in \mathbf{R}^3 - X$ and all $t \in [0, 1]$.

Theorem A.3 *For links L and L' in S^3, the following are mutually equivalent:*

(1) *L and L' belong to the same topological type.*
(2) *L and L' belong to the same type.*
(3) *L and L' belong to the same combinatorial type.*
(4) *L and L' are ambient isotopic.*

Theorem A.4 *For two polygonal links L and L' in $\mathbf{R}^3$ which are in a regular position with respect to the plane $\mathbf{R}^2$, the following are mutually equivalent:*

(1) *L and L' belong to the same topological type.*
(2) *L and L' belong to the same type.*
(3) *L and L' belong to the same combinatorial type.*
(4) *The regular projections of L and L' in $\mathbf{R}^2$ are R-isotopic, namely they are mutually related by a finite sequence of Reidemeister moves I,II,III (cf. Chapter 1).*
(5) *L and L' are ambient isotopic with a compact support.*

Proof of Theorem A.3. The proof of the assertion that (1) $\to$ (2) follows from the theory of PL approximation of a topological homeomorphism by [Bing 1954] and [Moise 1952]. A relatively modern proof proceeds as follows (cf. [Moise 1954]): Let f be a topological homeomorphism $(S^3, L) \to (S^3, L')$ whose existence is implied by (1). By a PL approximation, we can assume that $f|_P \colon P \to S^3$ is a PL embedding for any compact PL subspace P of $S^3 - L$. Let $N(L)$ be a regular neighborhood of L in S^3. Let (m, ℓ) be a meridian-longitude system of L in $\partial N(L)$. Since $f|_{\partial N(L)}$ is a PL embedding, we see that $f(N(L))$ is a PL submanifold of S^3. Take a regular neighborhood $N'(L')$ of L' in $f(N(L))$. Then each component of $f(N(L)) - \mathrm{int}N'(L')$ is PL homeomorphic to $S^1 \times S^1 \times [0,1]$ since its fundamental group is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ (cf. [Hempel 1976]). Hence we see that $f(N(L))$ is a regular neighborhood of L' in S^3. Using that $(f(m), f(\ell))$ is a meridian-longitude system of L' in $\partial f(N(L))$, we can extend the PL homeomorphism $f|_{S^3 - \mathrm{int}N(L)} \colon S^3 - \mathrm{int}N(L) \to S^3 - \mathrm{int}f(N(L))$ to a PL homeomorphism $(S^3, L) \to (S^3, L')$, showing that (1) $\to$ (2). To show that (2) $\to$ (3), we first show that there is a link L^* in B' with the same combinatorial type as L for any 3-ball $B' \subset S^3$ so that $B' \cap L = \emptyset$. For this purpose, we further choose 3-balls B_0, B so that

$$B_0 \subset \mathrm{int}B' \subset B' \cup L \subset \mathrm{int}B \subset S^3.$$

Let $S = \partial B$. Note that there is a homeomorphism

$$d : B - \mathrm{int}B_0 \to S \times [1,3]$$

such that $d(B' - \mathrm{int}B_0) = S \times [1,2]$. Let $p_S : S \times [1,3] \to S$ and $p_I : S \times [1,3] \to [1,3]$ be the projections from $S \times [1,3]$ to the first factor S and the second factor $[1,3]$, respectively. Let $e_S = p_S d : B - \mathrm{int}B_0 \to S$ and $e_I = p_I d : B - \mathrm{int}B_0 \to [1,3]$. We assume that $e_S|_L \colon L \to S$ is an immersion (i.e., a local homeomorphism), if necessary, by composing d with an auto-homeomorphism h of $S \times [1,3]$ such that $h(S \times [1,2]) = S \times [1,2]$. Noting that $e_I(L) \subset (2,3)$, we can define a map $F : L \times [0,1] \to S \times [1,3]$ by $F(x,t) = (e_S(x), e_I(x) - t)$ for $x \in L$ and $t \in [0,1]$. Strictly speaking, this map is not a PL map, although the image of every PL subspace of $L \times [0,1]$ is a PL subspace of $S \times [1,3]$. For each $t \in [0,1]$, there is a closed interval neighborhood $N(t)$ of t in $[0,1]$ with the following property:

($*$) $F|_{L \times N(t)}$ is injective and $F(L \times N(t))$ is a PL 2-submanifold of $S \times [1,3]$.

Hence there are division-points $0 = t_0 < t_1 < \cdots < t_r = 1$ of $[0,1]$ such that $[t_i, t_{i+1}]$ for each i is included in $N(t)$ for some $t \in [0,1]$. Since $F(L \times t_i)$ is deformed into $F(L \times t_{i+1})$ by a finite number of disk moves for each i, it follows that $L = d^{-1}F(L \times 0)$ and $L^* = d^{-1}F(L \times 1) \subset B'$ belong to the same combinatorial type. To complete the proof of the assertion that (2) $\to$ (3), we choose a homeomorphism $h : (S^3, L) \to (S^3, L')$ guaranteed by (2) such that there is a 3-ball $B' \subset S^3 - L - L'$ with $h(x) = x$ for all $x \in B'$. By the above argument, L has the same combinatorial type as a link L^* in B'. Hence as the images of h, the links $L' = h(L)$ and $L^* = h(L^*)$ belong to the same combinatorial type. Hence L and L' belong to the same combinatorial type. The assertion that (3) $\to$ (4) is easily shown. The assertion that (4) $\to$ (1) is clear. $\square$

Proof of Theorem A.4. By regarding S^3 as $\mathbf{R}^3 \cup \{\infty\}$, L and L' are links in S^3. Let $f^+ : (S^3, L) \to (S^3, L')$ be a homeomorphism obtained by a one-point compactification of a homeomorphism $f : (\mathbf{R}^3, L) \to (\mathbf{R}^3, L')$ provided by (1). By Theorem A.3, the assertion that (1) $\to$ (2) $\to$ (3) is easily obtained. We show that (3) $\to$ (4). When L is deformed into L' by a finite sequence of disk moves, we triangulate all the disks that we are using. Then L can be deformed into L' by a finite sequence of disk moves on the 2-simplices. If we slightly incline the plane $\mathbf{R}^2$ in $\mathbf{R}^3$, then the regular diagrams of L, L' with respect to this new plane $\mathbf{R}^2$ are identical with the original ones and the link resulting from the disk move of each 2-simplex is regularly projected into this new plane $\mathbf{R}^2$. Now it is easy to check that the regular diagrams of L and L' are R-isotopic. The assertion that (4) $\to$ (5) $\to$ (1) is clear. $\square$

Appendix B
Covering spaces

Here, we discuss the theory of unbranched and branched covering spaces in the PL category. In this category, spaces are necessarily paracompact, Hausdorff, locally path-connected and semilocally 1-connected. In particular, connected spaces are path-connected. Thus, most statements on covering spaces are much simpler than those in the TOP category (cf. [Spanier 1966]).

B.1 The Fundamental group

A map from $I = [0,1] = \{t \mid 0 \leq t \leq 1\}$ to a space X is called a *path* which joins the *initial point* $\alpha(0)$ to the *terminal point* $\alpha(1)$. If β is also a path in X and the terminal point of α is the initial point of β, then we have a path $\alpha * \beta$ defined by the following identity:

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & (0 \leq t \leq 1/2) \\ \beta(2t-1) & (1/2 \leq t \leq 1). \end{cases}$$

A path such that $\alpha(0) = \alpha(1)$ is called a *closed path* with *base point* $\alpha(0)$. Two closed paths α and β with base point x_0 in a space X are *homotopic* (*relative to the base point* x_0), if there exists a map F from $[0,1] \times [0,1]$ to X such that $F(t,0) = \alpha(t), F(t,1) = \beta(t)\, (0 \leq t \leq 1)$ and $F(0,s) = F(1,s) = x_0\, (0 \leq s \leq 1)$. That is, $\alpha(t)$ can be deformed into $\beta(t)$ by a family of closed paths, $\alpha_s(t) = F(t,s)$ $(0 \leq s \leq 1)$ (see figure B.1.1). For any closed path α, let $[\alpha]$ denote the homotopy class. Let $\pi_1(X, x_0)$ be the set of all homotopy classes of closed paths in X with base point x_0. If we define the product by $[\alpha] \cdot [\beta] = [\alpha * \beta]$ for $[\alpha]$ and $[\beta]$ in $\pi_1(X, x_0)$, then $\pi_1(X, x_0)$ forms a group with this operation. This group is called the *fundamental group* of X with the base point $x_0 \in X$. Moreover, $[\alpha]^{-1} = [\alpha^{-1}]$, where $\alpha^{-1}(t) = \alpha(1-t)$, and the identity is the homotopy class of the constant path $\alpha(t) = x_0$ $(0 \leq t \leq 1)$. Let x_0 and x_1 be two points in X. Let γ be a path with initial point x_0 and terminal point x_1. By sending $[\alpha] \in \pi_1(X, x_0)$ to $[(\gamma^{-1} * \alpha) * \gamma] \in \pi_1(X, x_1)$, we can define an isomorphism $\pi_1(X, x_0) \cong \pi_1(X, x_1)$, which we call *conjugation.* For a map $f : (X, x_0) \longrightarrow (Y, y_0)$, we define a homomorphism $f_\sharp : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ by the correspondence $[\alpha] \longrightarrow [f\alpha]$. We call this map the *induced homomorphism* of f. For $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$, we have $(gf)_\sharp = g_\sharp f_\sharp$. Two maps f, $g : X \longrightarrow Y$ are said to be *homotopic relative to* A, where A is a subspace of X, if there is a map $F : X \times [0,1] \longrightarrow Y$ (called a *homotopy relative to* A from f to g) such that $F(x,0) = f(x),\ F(x,1) = g(x) (x \in X)$ and $F(x,s) = f(x) = g(x)\, (0 \leq s \leq 1, x \in A)$. If f is homotopic to g relatively to the base point, then $f_\sharp = g_\sharp$. To calculate the fundamental group $\pi_1(X, x_0)$, the well-known Seifert-van Kampen theorem is useful.

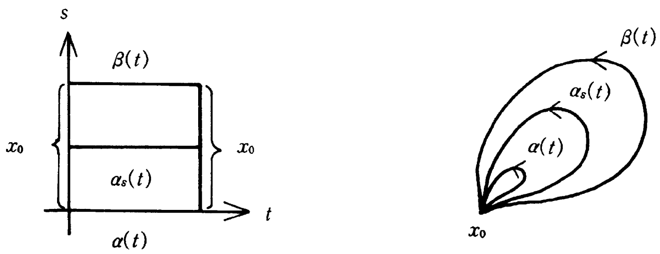


Fig. B.1.1

Theorem B.1.1 (Seifert-van Kampen theorem) *Let A, B and $A \cap B$ be connected subspaces of X. Let $x_0 \in A \cap B$. We assume that group presentations of $\pi_1(A, x_0)$,$\pi_1(B, x_0)$, $\pi_1(A \cup B, x_0)$ are given as follows:*

$$\pi_1(A, x_0) \cong \langle a_1, \cdots \mid r_1, \ldots \rangle$$
$$\pi_1(B, x_0) \cong \langle b_1, \cdots \mid s_1, \ldots \rangle$$
$$\pi_1(A \cap B, x_0) \cong \langle c_1, \cdots \mid t_1, \ldots \rangle.$$

Then

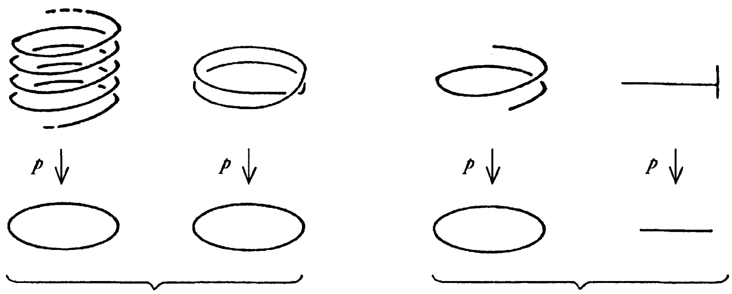
$$\pi_1(X, x_0) \cong \langle a_1, b_1, \cdots \mid r_1, \ldots, s_1, \ldots, i_\sharp(c_1) = j_\sharp(c_1), \ldots \rangle,$$

where i and j denote the inclusion maps $i : A \cap B \to A$ and $j : A \cap B \to B$.

We can also define higher homotopy groups in a similar way: Namely, for a space X and a point $x_0 \in X$ and each integer $q > 1$, we define $\pi_q(X, x_0)$ to be the set of homotopy classes of maps $(I^q, \partial I^q) \to (X, x_0)$, which forms an abelian group called the *q-th homotopy group* of X with base point x_0. For a map $f : (X, x_0) \to (Y, y_0)$, we have also the *induced homomorphism* $f_\sharp : \pi_q(X, x_0) \to \pi_q(Y, y_0)$ of f. We note that fundamental groups and higher homotopy groups are usually defined in the TOP category. The statement of the Seifert-van Kampen theorem in the TOP category needs some additional assumptions on A, B and $A \cap B$ (cf. [Crowell-Fox 1963]).

B.2 Definitions and properties of covering spaces

A map p from a space $\tilde{X}$ onto a space X is a *covering projection* if there exists an open neighborhood U for each point x in X, such that for each connected component $\tilde{U}_\lambda$ of $p^{-1}(U)$, the restricted map $p \mid_{\tilde{U}_\lambda} : \tilde{U}_\lambda \longrightarrow U$ is a homeomorphism. Then $\tilde{X}$ is called a *covering space* over X by p (see figure B.2.1). Since the covering space is a concept consisting of $\tilde{X}$, p and X, we denote it by $(\tilde{X}, p, X)$, but unless it will cause confusion, we also denote it by $(\tilde{X}, p)$ or $\tilde{X}$. For a covering space $(\tilde{X}, p, X)$, X is called the *base space* and $p^{-1}(x)$ $(x \in X)$ is called a *fiber* over x. The fiber is a discrete set in $\tilde{X}$. Let f be a map from a space Y to X. Let $\tilde{f}$ be a map from a space Y to $\tilde{X}$ such that $p\tilde{f} = f$. Then $\tilde{f}$ is called a *lift* of f.



covering spaces non-covering spaces

Fig. B.2.1

Theorem B.2.1 (Unique lifting property) *Let $(\tilde{X}, p, X)$ be a covering space. Let f be a map from a connected space Y into X. If two lifts $\tilde{f}$, $\tilde{g} : Y \longrightarrow \tilde{X}$ of f have $\tilde{f}(y_0) = \tilde{g}(y_0)$ for some point y_0 in Y, then $\tilde{f} = \tilde{g}$.*

Theorem B.2.2 (Path lifting property) *Let $(\tilde{X}, p, X)$ be a covering space. Let $\alpha : I \longrightarrow X$ be a path in X such that $\alpha(0) = x_0$. Then for any point $\tilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = \tilde{x}_0$.*

Theorem B.2.3 (Homotopy lifting property) *Let $(\tilde{X}, p, X)$ be a covering space. For a space Y, let F be a map from $Y \times [0, 1]$ to X which is a homotopy from $f_0 = F(y, 0)$ to $f_1 = F(y, 1)$. Let $\tilde{f}_0 : Y \longrightarrow \tilde{X}$ be a lift of f, then there exists a lift $\tilde{F} : Y \times I \longrightarrow \tilde{X}$ of F such that $\tilde{F}(y, 0) = \tilde{f}_0$.*

Corollary B.2.4 *Let $(\tilde{X}, p, X)$ be a covering space over a connected space X. Then the fiber $p^{-1}(x_1)$ is in one-to-one correspondence with the fiber $p^{-1}(x_2)$ for any points x_1, $x_2 \in X$.*

The cardinality of $p^{-1}(x)$ $(x \in X)$ is called the *degree of the covering*. If the degree of the covering is finite, say n, then the covering space is called an *n-fold covering space.*

Corollary B.2.5 *Let $(\tilde{X}, p, X)$ be a covering space. Then the induced homomorphism $p_\sharp : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is a monomorphism.*

Theorem B.2.6 (Monodromy principle) *Let $(\tilde{X}, p, X)$ be a covering space. Let α and β be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifting paths of α and β which have the same initial point $\tilde{x}_0 \in \tilde{X}$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ if and only if $[\alpha * \beta^{-1}] \in p_\sharp\pi_1(\tilde{X}, \tilde{x}_0)$.*

Theorem B.2.7 (Lifting theorem)
For a covering space $(\tilde{X}, p, X)$, a map $f : (Y, y_0) \longrightarrow (X, x_0)$ lifts to a map $\tilde{f} : Y \to \tilde{X}$ with $\tilde{f}(y_0) = \tilde{x}_0 \in p^{-1}(x_0)$ if and only if $f_\sharp\pi_1(Y, y_0) \subset p_\sharp\pi_1(\tilde{X}, \tilde{x}_0)$.

If such a lift $\tilde{f}$ of f exists, then $\tilde{f}$ is unique by Theorem B.2.1.

Corollary B.2.8 *Let $(\tilde{X}, p, X)$ be a covering space. For each integer $q > 1$, the induced homomorphism $p_\sharp : \pi_q(\tilde{X}, \tilde{x}_0) \to \pi_q(X, x_0)$ is an isomorphism for any $\tilde{x}_0 \in p^{-1}(x_0)$.*

B.3 The classification of covering spaces

Let $(\tilde{X}_1, p_1)$ $(\tilde{X}_2, p_2)$ be two connected covering spaces over a connected space X. A map $f : \tilde{X}_1 \to \tilde{X}_2$ with $p_2 f = p_1$ is called a *morphism* from $\tilde{X}_1$ to $\tilde{X}_2$. Moreover, if f is a homeomorphism, then we say that $(\tilde{X}, p_1)$ and $(\tilde{X}_2, p_2)$ are *equivalent* and f is an equivalence. We see that any morphism is itself a covering projection.

Theorem B.3.1 *Let $(\tilde{X}, p, X)$ be a connected covering space. Then we have the following:*
(1) The subgroups $p_\sharp \pi_1(\tilde{X}, \tilde{x}_1)$ and $p_\sharp \pi_1(\tilde{X}, \tilde{x}_2)$ are conjugate in $\pi_1(X, x_0)$ for any $\tilde{x}_1$, $\tilde{x}_2 \in p^{-1}(x_0)$.
(2) Conversely, for any subgroup H of $\pi_1(X, x_0)$ which is conjugate to $p_\sharp \pi_1(\tilde{X}, \tilde{x}_1)$, there exists $\tilde{x}_2 \in p^{-1}(x_0)$ such that $p_\sharp \pi_1(\tilde{X}, \tilde{x}_2) = H$.

If $p_{1\sharp} \pi_1(\tilde{X}, \tilde{x}_1)$ and $p_{2\sharp} \pi_1(\tilde{X}_2, \tilde{x}_2)$ are conjugate in $\pi_1(X, x_0)$ for two connected covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ over X, then there exists $\tilde{x}_2' \in \tilde{X}_2$ such that $p_{2\sharp} \pi_1(\tilde{X}_2, \tilde{x}_2') = p_{1\sharp} \pi_1(\tilde{X}, \tilde{x}_1)$. By Theorem B.2.7, there exists an equivalence $f : \tilde{X}_1 \longrightarrow \tilde{X}_2$. Thus we obtain the following theorem:

Theorem B.3.2 *Two connected covering spaces $(\tilde{X}_1, p_1)$, $(\tilde{X}_2, p_2)$ over X are equivalent if and only if $p_{1\sharp} \pi_1(\tilde{X}_1, \tilde{x}_1)$ and $p_{2\sharp} \pi_1(\tilde{X}_2, \tilde{x}_2)$ are conjugate in $\pi_1(X, x_0)$.*

Theorem B.3.3 (Existence theorem) *Let X be a connected space. For any subgroup H of $\pi_1(X, x_0)$, there exists a connected covering space $(\tilde{X}, p, X)$ such that $p_\sharp \pi_1(\tilde{X}, \tilde{x}_0) = H$.*

Theorems B.3.2 and B.3.3 imply the following theorem:

Theorem B.3.4 (Classification theorem) *The equivalence classes of connected covering spaces over a connected space X correspond bijectively to the conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

A connected covering space $(\bar{X}, p, X)$ is said to be *universal* if it is a covering space over X corresponding to the trivial subgroup of $\pi_1(X, x_0)$. Then $\pi_1(\bar{X}, \bar{x}_0) = \{1\}$ and any universal covering space of X is uniquely determined up to equivalence. The universal covering space of X is a covering space of any covering space over X (cf. Theorem B.4.7). For this reason, this covering space is said to be "universal". A covering space over X corresponding to the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$ of $\pi_1(X, x_0)$ is called a *universal abelian covering space* of X which is uniquely determined up to equivalence.

B.4 Covering transformations and the monodromy map

Throughout this section, we consider only a connected covering space over a connected space. Let $(\tilde{X}, p, X)$ be such a covering. A self-equivalence of $\tilde{X}$ is called a *covering transformation* on $\tilde{X}$. The set of covering transformations on $\tilde{X}$, denoted by $A(\tilde{X}, p, X)$ forms a group under composition. We call it the *covering transformation group* of $(\tilde{X}, p, X)$. Let $x \in X$. For any point $\tilde{x} \in p^{-1}(x)$ and any $\alpha \in \pi_1(X, x)$, we define $\tilde{x} \cdot \alpha$ to be the terminal point of the lift of α whose initial point is $\tilde{x}$. Then by the correspondence $\tilde{x} \in p^{-1}(x) \longrightarrow \tilde{x} \cdot \alpha \in p^{-1}(x)$, we have a bijection $\sigma(\alpha) : p^{-1}(x) \to p^{-1}(x)$, which is called the *monodromy map* of $p^{-1}(x)$ induced by α. The correspondence from α to the monodromy map $\sigma(\alpha)$ of $p^{-1}(x)$ induces a homomorphism σ from $\pi_1(X, x)$ to the symmetric group on the set $p^{-1}(x)$, called the *monodromy representation.* The group of monodromy maps of $p^{-1}(x)$, namely the image of σ, is called the *monodromy group* and denoted by $M_x(\tilde{X})$. For any points $\tilde{x}_1$, $\tilde{x}_2 \in p^{-1}(x)$, we have an $\alpha \in \pi_1(X, x)$ with $\sigma(\alpha)(\tilde{x}_1) = \tilde{x}_2$. Further, we have that $\{\alpha \mid \tilde{x} \cdot \alpha = \tilde{x}\} = p_\sharp \pi_1(\tilde{X}, \tilde{x})$. This implies the following:

Theorem B.4.1 $[\pi_1(X, x) : p_\sharp \pi_1(\tilde{X}, \tilde{x})]$ *equals the covering degree.*

Since $\tilde{x} \cdot \alpha = \tilde{x}$ for any $\tilde{x} \in p^{-1}(x)$ if and only if α belongs to $\bigcap_{\tilde{x} \in p^{-1}(x)} p_\sharp \pi_1(\tilde{X}, \tilde{x})$, we have the following result:

Theorem B.4.2 $M_x(\tilde{X}) \cong \pi_1(X, x) / \bigcap_{\tilde{x} \in p^{-1}(x)} p_\sharp \pi_1(\tilde{X}, \tilde{x})$.

Let X be a connected space. We say that a group G *acts on* X (*from the right*) if there is a map $\varphi : X \times G \longrightarrow X$, denoted by $\varphi(x, g) = x \cdot g$, with the following properties:

(1) For any $x \in X$ and the trivial element 1 of G, we have $x \cdot 1 = x$.
(2) For any non-trivial $g \in G$, the correspondence $x \longrightarrow x \cdot g$ is an auto-homeomorphism of X which is not the identity.
(3) For any $x \in X$, and any g_1, $g_2 \in G$, we have $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$.

Further, this action is said to be *transitive* if for any two x_1, $x_2 \in X$ there exists a $g \in G$ with $x_1 \cdot g = x_2$. According to these definitions, we can say that $\pi_1(X, x)$ acts transitively on $p^{-1}(x)$ from the right. We note that $f(\tilde{x} \cdot \alpha) = f(\tilde{x}) \cdot \alpha$ for any $f \in A(\tilde{X}, p, X)$, any $\alpha \in \pi_1(X, x)$, and any $\tilde{x} \in p^{-1}(x)$. Then we see the following:

Theorem B.4.3 *Let N be the normalizer $\{g \in \pi_1(X, x) \mid gH = Hg\}$ of $H = p_\sharp \pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$. Then the covering transformation group $A(\tilde{X}, p, X)$ is isomorphic to the quotient group N/H.*

We have the following inequality:

$$\text{the order of } M_x(\tilde{X}) \geq \text{the covering degree} \geq \text{the order of } A(\tilde{X}, p, X).$$

If $p_\sharp\pi_1(\tilde{X},\tilde{x})$ is a normal subgroup of $\pi_1(X,x)$, then

$$N = p_\sharp\pi_1(\tilde{X},\tilde{x}) = \bigcap_{\tilde{x}\in p^{-1}(x)} p_\sharp\pi_1(\tilde{X},\tilde{x}), \quad \text{and}$$
$$\text{the monodromy group} \cong \text{the covering transformation group} \cong \pi_1(X,x)/p_\sharp\pi_1(\tilde{X},\tilde{x}).$$

In this case, $(\tilde{X},p,X)$ is called a *regular covering space.*

Theorem B.4.4 *$(\tilde{X},p,X)$ is a regular covering space if and only if there exists an $f \in A(\tilde{X},p,X)$ such that $f(\tilde{x}_1) = \tilde{x}_2$ for any two points $\tilde{x}_1$, $\tilde{x}_2 \in p^{-1}(x)$.*

We say that a group G *acts properly discontinuously* on X if for any $x_0 \in X$ there exists a neighborhood $U(x_0)$ of x_0 such that $U(x_0) \cap U(x_0)\cdot g = \phi$ for any $g \in G$ $(g \neq 1)$, where $U(x_0)\cdot g = \{x\cdot g \mid x \in U(x_0)\}$.

Theorem B.4.5 *If G acts properly discontinuously on X, then $p : X \longrightarrow X/G$ is a regular covering space with $A(X,p,X/G) \cong G$, where X/G denotes the orbit space of X by G, i.e., X/G is the quotient space of X by the equivalence relation: $x \sim y \longleftrightarrow x\cdot g = y$ for some $g \in G$ and p is the quotient map from X to X/G.*

Theorem B.4.6 *$A(\tilde{X},p,X)$ acts properly discontinuously on $\tilde{X}$.*

Let $\bar{X}$ be the universal covering space of X. Since $A(\bar{X},p,X)$ is isomorphic to $\pi_1(X,x)$, the subgroup H of $\pi_1(X,x)$ acts properly discontinuously on $\bar{X}$ and $(\bar{X},p',\bar{X}/H)$ is a regular covering space. The induced projection $p'' : \bar{X}/H \longrightarrow X$ defines also a covering $(\bar{X}/H,p'',X)$, which is equivalent to the covering space over X corresponding to H by Theorem B.3.4. Any covering space over X is isomorphic to $\bar{X}/H$ for a subgroup H of $\pi_1(X,x) \cong A(\bar{X},p,X)$ and $\bar{X}$ is a covering space of $\bar{X}/H$. Thus we obtain the following theorem:

Theorem B.4.7 *The universal covering space of X is a covering space over any covering space of X.*

B.5 Branched covering spaces

Let p_m be a map from the disk $D^2 = \{z \mid |z| \leq 1, z \in \mathbf{C}\}$ onto itself defined by $p_m(z) = z^m$ $(m = 1,2,3,\dots)$. Then p_m defines a covering projection of D^2 onto itself except near the central point $z = 0$. For the $(n-2)$-ball D^{n-2}, we define a map $q_m : D^2 \times D^{n-2} \longrightarrow D^2 \times D^{n-2}$ by $q_m = p_m \times \mathrm{id}$. Then q_m defines a covering projection from the part of $D^2 \times D^{n-2}$ away from $0 \times D^{n-2}$ onto itself. Using this map q_m, we can define the notion of a branched covering space over a general manifold. Let $\tilde{M}$, M be connected $n(\geq 2)$-manifolds. Let $p : \tilde{M} \longrightarrow M$ be a surjective map with $\partial\tilde{M} = p^{-1}(\partial M)$. A point in $\tilde{M}$ is called a *singular point* of the map p if p is not a local homeomorphism at this point. Let S_p be the set of singular points of p. Let $B_p = p(S_p)$ and $\tilde{B}_p = p^{-1}(B_p)$. Clearly, we have $\tilde{B}_p \supset S_p$.

Definition B.5.1 A surjective map $p : \tilde{M} \to M$ is a branched covering projection if p has the following properties:

(1) The restriction $p_0 : \tilde{M} - \tilde{B}_p \longrightarrow M - B_p$ of p is a covering projection.

(2) For any $\tilde{x} \in \tilde{B}_p$, there exist a neighborhood $\tilde{N}$ of $\tilde{x}$ in $\tilde{M}$, a homeomorphism $\tilde{h} : \tilde{N} \longrightarrow D^2 \times D^{n-2}$, a neighborhood N of $p(\tilde{x}) = x$ in M, a homeomorphism $h : N \longrightarrow D^2 \times D^{n-2}$, and a positive integer m such that the following diagram is commutative:

$$\begin{array}{ccc} (\tilde{N}, \tilde{N} \cap \tilde{B}_p) & \xrightarrow{\tilde{h}} & (D^2 \times D^{n-2}, 0 \times D^{n-2}) \\ {\scriptstyle p|_{\tilde{N}}}\downarrow & & \downarrow{\scriptstyle q_m} \\ (N, N \cap B_p) & \xrightarrow{h} & (D^2 \times D^{n-2}, 0 \times D^{n-2}). \end{array}$$

In this definition, $(\tilde{M}, p, M)$ or simply $\tilde{M}$ is called a *(connected) branched covering space (or manifold)* over M and the integer m is called the *branching index* of $\tilde{x}$. The branching index is constant on each connected component of $\tilde{B}_p$. We call B_p or $\tilde{B}_p$ the *branch set* of M or $\tilde{M}$, respectively. Each point of B_p or $\tilde{B}_p$ is also called a *branch point* of B_p or $\tilde{B}_p$, respectively. The covering space $(\tilde{M} - \tilde{B}_p, p_0, M - B_p)$ is called the *unbranched covering space associated with* the branched covering space $(\tilde{M}, p, M)$. A branched covering space is called a *finite*, *k-fold*, or *regular* branched covering manifold, respectively, if the associated unbranched covering space is a finite, k-fold, or regular covering space, respectively. By condition (2), B_p and $\tilde{B}_p$ are locally flat, proper $(n-2)$-submanifolds of M and $\tilde{M}$, respectively.

Remark B.5.2 We have defined here only a connected branched covering manifold over a connected manifold with branch set a locally flat proper submanifold. This concept can be generalized to a branched covering space over a locally connected Hausdorff space (cf. [Fox 1957]).

Two branched covering spaces $(\tilde{M}_i, p_i, M)$ $(i = 1, 2)$ over M are said to be *equivalent*, if there exists a homeomorphism $h : \tilde{M}_1 \to \tilde{M}_2$ with $p_2 h = p_1$. A representation σ of a group G into the symmetric group $\mathbf{S}_k$ of order k is said to be *transitive* if the image $\sigma(G)$ acts transitively on the set of k letters used to define $\mathbf{S}_k$. Then we have the following classification theorem for finite branched covering manifolds:

Theorem B.5.3 *Let M be a connected n-manifold. Let B be a locally flat proper $(n-2)$-submanifold of M. The equivalence classes of k-fold connected branched covering manifolds over M with branch set B correspond bijectively to the conjugacy classes of transitive representations of $G = \pi_1(M - B, x_0)$ into $\mathbf{S}_k$ sending each meridian of B in M to a non-trivial element of $\mathbf{S}_k$.*

In this theorem, a transitive representation $\sigma : G \longrightarrow \mathbf{S}_k$ is called the *monodromy representation* of the associated branched covering space. The following theorem is well-known(cf. [Hilden 1976], [Montesinos 1976']):

Theorem B.5.4 *Every closed connected oriented 3-manifold is a 3-fold branched covering space over S^3 with branch set a knot.*

Appendix C
Canonical decompositions of 3-manifolds

In this appendix, we explain standard concepts and results on canonical decompositions of compact connected 3-manifolds.

C.1 The connected sum

Let $M_i (i = 1, 2)$ be connected 3-manifolds. When they are orientable, we assume that they are oriented. For each i, choose a 3-ball B_i in $\mathrm{int}M_i$. Let $S_i = \partial B_i$, which is a sphere. We take a homeomorphism $f : S_1 \to S_2$. When $M_i (i = 1, 2)$ are oriented, we assume that f reverses the orientations of $S_i (i = 1, 2)$ induced from those of $B_i (i = 1, 2)$. Then we can construct a connected 3-manifold from $M_1 - \mathrm{int}B_1$ and $M_2 - \mathrm{int}B_2$ by identifying S_1 with S_2 by f. This new 3-manifold is determined only by M_1 and M_2 up to homeomorphism (or oriented homeomorphism when $M_i (i = 1, 2)$ are oriented). It is denoted by $M_1 \sharp M_2$ and called the *connected sum* of M_1 and M_2. A 3-manifold M is *prime* provided that M is connected and M homeomorphic to $M_1 \sharp M_2$ implies that one of $M_i (i = 1, 2)$ is homeomorphic to S^3. Then we have the following theorems (cf. for example [Hempel 1976] for the proofs):

Theorem C.1.1 (Existence of prime decompositions) *Every compact connected 3-manifold is homeomorphic to a connected sum of finitely many compact prime 3-manifolds.*

Theorem C.1.2 (Uniqueness of prime decompositions)
Assume that a compact connected oriented 3-manifold M is homeomorphic to $M_1 \sharp M_2 \sharp \dots \sharp M_r$ and $N_1 \sharp N_2 \sharp \dots \sharp N_s$ by orientation preserving homeomorphisms where M_i and N_i are compact prime 3-manifolds, not homeomorphic to S^3. Then $r = s$ and (after permuting the indices suitably) there is an orientation-preserving homeomorphism from M_i to N_i for each $i (i = 1, 2, \dots, r)$.

Exercise C.1.3 Let $S^2 \times_\tau S^1$ be a non-orientable handle or, in other words, a non-orientable S^2-bundle over S^1 (which is homeomorphic to the double of the solid Klein bottle). Show that $S^2 \times_\tau S^1 \sharp S^2 \times S^1 \cong S^2 \times_\tau S^1 \sharp S^2 \times_\tau S^1$.

A sphere S in a 3-manifold M is said to be *essential* if S is not the boundary of a 3-ball in M. A 3-manifold M is *irreducible* if M is connected and there are no essential spheres in M. By Alexander's theorem (Theorem 15.3.1), S^3 is irreducible. By definition, any irreducible 3-manifold is prime. However, the converse does not hold.

Exercise C.1.4 Show that if a compact connected oriented 3-manifold M is prime but not irreducible, then M is homeomorphic to $S^2 \times S^1$.

By this exercise and the existence and the uniqueness theorems for connected sum decompositions, we can say that irreducible 3-manifolds are fundamental objects in the study of 3-manifold topology.

C.2 Dehn's lemma and the loop and sphere theorems

Here we describe three results due to C. D. Papakyriakopoulos. The first one is called *Dehn's lemma* after M. Dehn, who proved it in 1910 (although his proof had a serious gap). The correct proof was given independently by [Papakyriakopoulos 1957] and [Homma 1957]. See [Hempel 1976] for the generalized versions and proofs of the following three theorems:

Theorem C.2.1 (Dehn's lemma) *Let $f : D \to M$ be a map from a disk D to a connected 3-manifold M. If there exists a neighborhood N of ∂D in D such that $f|_N : N \to M$ is an embedding and $f^{-1}(f(N)) = N$, then there exists an (embedded) disk D' in M such that $\partial D' = f(\partial D)$.*

We say that a loop in a surface is *essential* if it is not null-homotopic in the surface. Let F be a surface in a connected 3-manifold M. A disk D in M is called a *compressing disk* for F in M if $D \cap F = \partial D$ and ∂D is essential in F.

Theorem C.2.2 (Loop theorem) *Let M be a connected 3-manifold with non-empty boundary. Let F be a surface in ∂M. If the homomorphism $i_\sharp : \pi_1(F) \to \pi_1(M)$ induced from the inclusion map is not injective, then there exists a compressing disk for F in M.*

Theorem C.2.3 (Sphere theorem) *Let M be a connected oriented 3-manifold. If $\pi_2(M)$ is non-trivial, then there exists an essential sphere in M.*

By the sphere theorem, any irreducible oriented 3-manifold M has $\pi_2(M) = 0$. The question asking whether the converse holds for all oriented 3-manifolds is equivalent to the Poincaré conjecture, which is not yet settled (cf. [Poincaré 1904], [Homma 1985]):

Poincaré conjecture Every closed connected 3-manifold M with $\pi_1(M, x) = \{1\}$ is homeomorphic to S^3.

We can regard Dehn's lemma as a special case of the loop theorem.

Exercise C.2.4 Derive Dehn's lemma from the loop theorem.

C.3 The equivariant loop and sphere theorems

Let G be a group acting on a connected 3-manifold M. A subset N of M is G-*equivariant* if $g(N) = N$ or $g(N) \cap N = \emptyset$ for all elements g of G. In this section, we describe the equivariant versions of the loop and sphere theorems for a finite group action. For more details and the proofs, see [Meeks-Yau 1980, 1982], [Meeks-Simon-Yau 1982] and [Plotnick 1984'].

Theorem C.3.1 (Equivariant loop theorem) *We assume that a finite group G acts on a connected 3-manifold M with non-empty boundary. Let F be a G-equivariant surface in ∂M. If the homomorphism $i_\sharp : \pi_1(F) \to \pi_1(M)$ induced from the inclusion map is not injective, then there exists a G-equivariant compressing disk for F in M.*

Theorem C.3.2 (Equivariant sphere theorem) *We assume that a finite group G acts on a connected oriented 3-manifold M. If $\pi_2(M)$ is non-trivial, then there exists a G-equivariant essential sphere in M.*

Exercise C.3.3 By the equivariant sphere theorem, show the following: a connected oriented 3-manifold M is irreducible if and only if any finite connected covering 3-manifold $\tilde{M}$ over M is irreducible.

C.4 Haken manifolds

Let F be a compact proper surface in a compact connected 3-manifold M. For F not a sphere or disk, we say that F is *compressible* in M if there exists a compressing disk for F in M. Otherwise, we say that F is *incompressible* in M. In the case that F is a sphere, we say that F is *incompressible* in M if F is essential in M. Otherwise, we say that F is *compressible* in M. When F is a disk, we say that F is *compressible* in M if there exists a disk D in ∂M with $D \cap F = \partial D = \partial F$ such that $D \cup F$ is the boundary of a 3-ball in M. Otherwise, we say that F is *incompressible* in M. If a regular neighborhood of F in M is homeomorphic to $F \times I$, where $I = [0, 1]$, then we call F a *two-sided surface* in M. Otherwise, we call F a *one-sided surface* in M. For a compact surface S in ∂M which is not a disk, we say that S is *compressible* or *incompressible* in M according to whether a proper surface S' obtained from S by pushing $\mathrm{int}S$ into $\mathrm{int}M$ is compressible or incompressible in M. (We also understand that such a surface S is a 2-sided surface since S' is 2-sided.) A compact oriented 3-manifold is called a *Haken manifold* if it is either a 3-ball, or an irreducible 3-manifold which contains a two-sided incompressible surface. Let M and N be connected 3-manifolds whose boundaries are $\emptyset$ or consist of incompressible surfaces. We say that a homomorphism $\psi : \pi_1(M) \to \pi_1(N)$ preserves the *peripheral structure* if for each connected component F of ∂M, there exists a connected component G of ∂N such that $\psi(i_\sharp(\pi_1(F)))$ is conjugate to a subgroup of $i_\sharp(\pi_1(G))$ in $\pi_1(N)$.

Waldhausen's theorem ([Waldhausen 1968]), which follows, implies that the homeomorphism type of a Haken manifold is determined by the fundamental group and the peripheral structure. See [Hempel 1976] for details and the proof.

Theorem C.4.1 *Let M and N be Haken manifolds whose boundaries are $\emptyset$ or consist of incompressible surfaces. For any isomorphism $\psi : \pi_1(M) \to \pi_1(N)$ which preserves the peripheral structure, there exists a homeomorphism $f : M \to N$ such that $f_\sharp = \psi$.*

We say that a 3-manifold is an I-bundle over a surface if it is a fiber bundle such that the base space is homeomorphic to a surface and the fiber is homeomorphic to the unit interval I.

Theorem C.4.2 *Let M be a Haken manifold whose boundary is $\emptyset$ or consists of incompressible surfaces. Assume that M is not homeomorphic to an I-bundle over a surface. Suppose that $f : (M, \partial M) \to (M, \partial M)$ is an auto-homeomorphism which is homotopic to the identity map. Then f is ambient isotopic to the identity map.*

Let $PL(M)$ be the group of auto-homeomorphisms of a connected 3-manifold M. Let $PL_0(M)$ be the subgroup of $PL(M)$ consisting of auto-homeomorphisms which are ambient isotopic to the identity map. Since $PL_0(M)$ is a normal subgroup of $PL(M)$, we can consider the quotient group $PL(M)/PL_0(M)$ which we call the *mapping class group* of M and denote by $\Psi(M)$. The *outer automorphism group* $\mathrm{Out}(\pi_1(M))$ of $\pi_1(M)$ is the quotient group of the automorphism group $A(\pi_1(M))$ of $\pi_1(M)$ by the inner automorphism group $\mathrm{Inn}(\pi_1(M))$ of $\pi_1(M)$ which is a normal subgroup of $A(\pi_1(M))$. For a connected 3-manifold M whose boundary is $\emptyset$ or consists of incompressible surfaces, let $A(\pi_1(M), \partial)$ be the subgroup of $A(\pi_1(M))$ consisting of automorphisms which preserve the peripheral structure. Then $\mathrm{Inn}(\pi_1(M))$ is a normal subgroup of $A(\pi_1(M), \partial)$ and we call the quotient group $A(\pi_1(M), \partial)/\mathrm{Inn}(\pi_1(M))$ the *peripheral structure preserving outer automorphism group* of $\pi_1(M)$ and denote it by $\mathrm{Out}(\pi_1(M), \partial)$. Then we have the following corollary:

Corollary C.4.3 *Let M be a Haken manifold whose boundary is $\emptyset$ or consists of incompressible surfaces. Assume that M is not homeomorphic to an I-bundle over a surface. Then the natural homomorphism $\Psi(M) \to Out(\pi_1(M), \partial)$ induces an isomorphism.*

A group G is *indecomposable* provided that if G is isomorphic to a free product $G_1 * G_2$, then one of G_1 and G_2 is a trivial group. See [Hempel 1976] for the details of the following theorem:

Theorem C.4.4 (Kneser's conjecture) *Let M be a compact connected 3-manifold whose boundary is $\emptyset$ or consists of incompressible surfaces. If $\pi_1(M) \cong G_1 * G_2$, then there is a connected sum decomposition $M \cong M_1 \sharp M_2$ with $\pi_1(M_i) \cong G_i (i = 1, 2)$.*

This theorem means that the fundamental group of a Haken manifold whose boundary is $\emptyset$ or consists of incompressible surfaces is an indecomposable group.

C.5 Seifert manifolds

We consider D^2 as $\{re^{\theta\sqrt{-1}} \mid 0 \le r \le 1, 0 \le \theta < 2\pi\}$. We simply denote it by $D^2 = \{(r, \theta)\}$. Let $I = [0, 1]$. For a coprime integer pair (p, q) $(p > 0)$, we define a homeomorphism $f : D^2 \times 0 \to D^2 \times 1$ by $f((r, \theta), 0) = ((r, \theta + 2\pi q/p), 1)$. Let $V_{p,q}$ be a 3-manifold obtained from $D^2 \times I$ by identifying $D^2 \times 0$ with $D^2 \times 1$ by

f. Then $V_{p,q}$ is a solid torus. We consider $D^2 \times I$ as the union of the segments $x \times I$'s for all $x \in D^2$. For the central point $\mathbf{0}$ of D^2, the image of $\mathbf{0} \times I$ in $V_{p,q}$ is a single simple loop. Otherwise, p segments around $\mathbf{0} \times I$ form a simple loop. Hence we can regard $V_{p,q}$ as the union of these simple loops. The solid torus $V_{p,q}$ with this structure is called the *fibered solid torus* of *type* (p, q). Each simple loop is called a *fiber*. A compact oriented 3-manifold M is called a 3-manifold with a *Seifert structure* or simply a *Seifert manifold* if M is connected and satisfies the following two conditions:

(1) M is the union of a family $\mathcal{F}$ of mutually disjoint simple loops in M called *fibers*.
(2) For each fiber C of M, there is a regular neighborhood N of C in M which is the union of fibers and is homeomorphic to the fibered solid torus of some type by a homeomorphism preserving the fiber structures.

A fiber C of the Seifert manifold M is called a *regular fiber* if the type of the fibered solid torus in condition (2) is given by $(1, q)$ for some q. Otherwise, C is called a *singular fiber*. By the compactness of M, the number of singular fibers is finite. By the definition, each fiber in ∂M is a regular fiber. Regarding each fiber of M as a point, we have a natural quotient map $p : M \to B$ with B a compact surface. This surface B is called the *base space* of the Seifert manifold M. A point $x \in B$ is called a *regular* point if $p^{-1}(x)$ is a regular fiber. Otherwise, $x \in B$ is called a *singular* point. See [Orlik 1972], [Scott 1983] and [Seifert-Threlfall 1980] for the details of Seifert manifolds. The following theorem was proved in [Casson-Jungreis 1994] and [Gabai 1992]:

Theorem C.5.1 *If M is a compact oriented irreducible 3-manifold such that $\pi_1(M)$ has an infinite cyclic normal subgroup, then M is a Seifert manifold.*

A *special Seifert manifold* is a Seifert manifold whose base space is one of the following:

(1) A disk with at most two singular points.
(2) An annulus with at most one singular point.
(3) A surface which is obtained from a disk without singular points by removing two open disks.
(4) A Möbius band without singular points.
(5) A sphere with at most three singular points.
(6) A projective plane with at most one singular points.

Let F be a proper surface in a connected 3-manifold M. If there is an embedding f from $F \times I$ to M such that $f(F \times 0) = F$ and $\partial M \cap f(F \times I) = f(\partial F \times I)$ or $f(\partial F \times I \cup F \times 1)$, then we say that F is *parallel* to $f(F \times 1)$. In particular, in the latter case, we say that F is *∂-parallel*. In the case that F is not a sphere or a disk, F is said to be *essential* in M if F is incompressible and not ∂-parallel. In the case that F is a disk, F is *essential* if F is incompressible. A compact oriented 3-manifold is said to be *simple* if it is connected and there are no essential tori in it. Then we have the following proposition (cf. [Jaco 1980]):

Proposition C.5.2 *A Seifert manifold is simple if and only if it is homeomorphic to a special Seifert manifold.*

C.6 The annulus and torus theorems and the torus decomposition theorem

For a connected 3-manifold M, we consider a map $f : (S^1 \times I, S^1 \times \partial I) \to (M, \partial M)$. If $f_\sharp : \pi_1(S^1 \times I) \to \pi_1(M)$ is injective and f is not homotopic to a map $g : (S^1 \times I, S^1 \times \partial I) \to (M, \partial M)$ such that $g(S^1 \times I) \subset \partial M$, then we say that f is a *non-degenerate* map. Likewise, a map $f : S^1 \times S^1 \to M$ is said to be a *non-degenerate* map if $f_\sharp : \pi_1(S^1 \times S^1) \to \pi_1(M)$ is injective and f is not homotopic to a map $g : S^1 \times S^1 \to M$ such that $g(S^1 \times S^1) \subset \partial M$.

Theorem C.6.1 (Annulus theorem) *Let M be a Haken manifold whose boundary consists of incompressible surfaces. If a map $f : (S^1 \times I, S^1 \times \partial I) \to (M, \partial M)$ is non-degenerate, then there exists a proper essential annulus A in M. In particular, if $f\,|_{S^1 \times \partial I}$ is an embedding, then we can find an annulus A with $\partial A = f(S^1 \times \partial I)$.*

Theorem C.6.2 (Torus theorem) *Let M be a Haken manifold whose boundary is $\emptyset$ or consists of incompressible surfaces. If a map $f : S^1 \times S^1 \to M$ is non-degenerate, then there exists an essential torus in M or M is homeomorphic to a special Seifert manifold.*

Next, we consider a Haken manifold M whose boundary is empty or consists of incompressible tori. A finite family $\mathcal{T}$ of mutually disjoint, mutually non-parallel (embedded) tori in M is called a *characteristic torus family* of M if $\mathcal{T}$ has the following properties:

(1) Each torus in $\mathcal{T}$ is essential in M.
(2) Each component of the manifolds obtained from M by splitting it along $\mathcal{T}$ is a Seifert manifold or a simple manifold.
(3) For any finite family $\mathcal{T}'$ of mutually disjoint, mutually non-parallel tori in M which satisfies conditions (1) and (2), $\mathcal{T}$ is ambient isotopic to a subfamily of $\mathcal{T}'$ in M.

When M is a Seifert manifold or a simple manifold, we have $\mathcal{T} = \emptyset$. By definition, a characteristic torus family $\mathcal{T}$ of M is unique up to ambient isotopy of M if it exists. We call the pair $(M, \mathcal{T})$ a *torus decomposition* of M.

Theorem C.6.3 (Torus decomposition theorem) *Let M be a Haken manifold whose boundary is $\emptyset$ or consists of incompressible tori. Then M admits a torus decomposition $(M, \mathcal{T})$.*

This theorem was proved by [Jaco-Shalen 1979] and [Johannson 1979] independently (see [Jaco 1980] for the details). In a manner analogous to the equivariant loop and sphere theorems, it is also possible to establish an equivariant version of the torus decomposition theorem (cf. [Freedman-Hass-Scott 1983]). A generalization of the torus decomposition theorem to Haken manifolds whose boundary may contain high genus incompressible surfaces is also known. In this case, the family $\mathcal{T}$ is taken to be a family of annuli and/or tori.

C.7 Hyperbolic 3-manifolds

The upper half space $\{(x, y, z) \in \mathbf{R}^3 \mid z > 0\}$ of $\mathbf{R}^3$ with Riemannian metric $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$ is called *hyperbolic 3-space* and denoted by H^3. H^3 is characterized up to isometry as a simply connected 3-manifold which admits a complete Riemannian metric such that the sectional curvature at each point is -1. We regard $\mathbf{R}^3$ as $\mathbf{C} \times \mathbf{R}$. Then $\mathbf{C} \times 0$ is not contained in H^3, but we may consider it to be at infinity with respect to the metric of H^3. The sphere obtained from $\mathbf{C} \times 0$ by one point compactification is denoted by ∂H^3 and called the *sphere at infinity* of H^3. The group which consists of all orientation preserving isometries of H^3 is denoted by $\mathrm{Isom}_+(H^3)$. It is known that each element of $\mathrm{Isom}_+(H^3)$ extends to an auto-homeomorphism of ∂H^3 naturally, and when we regard ∂H^3 as the Riemannian sphere, this auto-homeomorphism can be written as a linear fractional transformation $z \to (az + b)/(cz + d)$ for some $a, b, c, d \in \mathbf{C}$ with $ad - bc = 1$. Conversely, it is also known that any linear fractional transformation on ∂H^3, which we may take to be defined by an element of

$$SL_2(\mathbf{C}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbf{C}\},$$

can be extended to an isometry of H^3 uniquely. Therefore, we have a homomorphism from $SL_2(\mathbf{C})$ to $\mathrm{Isom}_+(H^3)$. Note here that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ induce the same linear fractional transformation. The kernel of this homomorphism is $\{\pm E^2\}$, which is the center of $SL_2(\mathbf{C})$. Thus, we have the following:

Proposition C.7.1 *The quotient group $PSL_2(\mathbf{C})$ of $SL_2(\mathbf{C})$ by $\{\pm E^2\}$ is naturally isomorphic to* $\mathrm{Isom}_+(H^3)$.

Consider a subgroup G of $\mathrm{Isom}_+(H^3)$ which acts on H^3 properly discontinuously in the sense of Appendix B. Equivalently, by using Proposition C.7.1, G is a torsion-free discrete subgroup of $PSL_2(\mathbf{C})$. We say that a 3-manifold M is *hyperbolic* if $\mathrm{int}M$ is homeomorphic to the orbit manifold of H^3 by a properly discontinuous action of a subgroup G of $\mathrm{Isom}_+(H^3)$. In other words, a hyperbolic 3-manifold is an oriented 3-manifold whose interior admits a complete Riemannian metric such that the sectional curvature at each point is -1, which we simply call a *hyperbolic structure*. The hyperbolic 3-manifold M is said to have *finite volume* if the volume on the hyperbolic structure of $\mathrm{int}M$ is finite. By a general argument on covering spaces (cf. Appendix B), the fundamental group of the hyperbolic manifold M is isomorphic to G. A serious problem is to determine which 3-manifolds can admit a hyperbolic structure. For Haken manifolds, W. P. Thurston gave a complete solution as we shall state in the following theorem. For the proof, see [Morgan-Bass 1984] and [Thurston 1982, 1986] (though the entire proof is not yet published). (Incidentally, geometric structures on surfaces and 3-manifolds are summarized in [Scott 1983].) We say that a compact oriented 3-manifold is *algebraically simple* if it is connected and there are no non-degenerate maps from a torus to it. Let KI be a twisted I-bundle on a Klein bottle and T a torus.

Theorem C.7.2 (Thurston's hyperbolization theorem) *A Haken manifold M is a hyperbolic 3-manifold if and only if M is algebraically simple and not homeomorphic to KI. Moreover, M has finite volume if and only if ∂M is empty or consists of tori, and M is homeomorphic to neither $T \times I$ nor a solid torus.*

The following theorem shows that the interior of a hyperbolic 3-manifold of finite volume admits only one hyperbolic structure. See [Mostow 1973], [Prasad 1973], [Thurston *] for the proof.

Theorem C.7.3 (Mostow's rigidity theorem)
Let M_1, M_2 be hyperbolic 3-manifolds of finite volume. For any isomorphism $\psi : \pi_1(M_1) \to \pi_1(M_2)$, there exists a unique isometry $f : \mathrm{int}(M_1) \to \mathrm{int}(M_2)$ such that $f_\sharp$ is conjugate to ψ.

Appendix D
Heegaard splittings and Dehn surgery descriptions

In this appendix we explain Heegaard splittings of 3-manifolds and Dehn surgery manifolds of S^3 along links.

D.1 Heegaard splittings

An oriented 3-manifold which is homeomorphic to the product $S^1 \times D^2$ is called a *solid torus*. By a *handlebody*, we mean a bounded connected oriented 3-manifold V which admits mutually disjoint proper disks such that they split V into solid tori. Let M be a closed connected oriented 3-manifold. A *Heegaard handlebody* of M is a handlebody V in M such that $V' = \mathrm{cl}(M - V)$ is also a handlebody. The surface $F = \partial V$ and the triad $(V, V'; F)$ are called a *Heegaard surface* and a *Heegaard splitting* of M, respectively. Every closed connected oriented 3-manifold M admits a Heegaard handlebody. In fact, any regular neighborhood of the 1-skeleton of any triangulation of M is a Heegaard handlebody of M. The minimum of the genera of Heegaard surfaces of M is a topological invariant of M which we call the *genus* of M and denote by $g(M)$. Taking a minimal genus Heegaard splitting of M, we see from the Seifert-van Kampen theorem (Theorem B.1.1) that the fundamental group $\pi_1(M)$ is generated by at most $g(M)$ elements. $g(M) = 0$ if and only if $M \cong S^3$. The 3-manifolds M with $g(M) \leq 1$ are completely classified, as we now describe. Let V be a solid torus. By a *meridian-longitude system* (m, ℓ) on V, we mean the image of the pair $(S^1 \times x, y \times \partial D^2)$ for $x \in \partial D^2$, $y \in S^1$ by an orientation-preserving homeomorphism $S^1 \times D^2 \to V$. Given a meridian-longitude system (m, ℓ) on V and an essential simple loop c in ∂V, there is a pair of coprime integers p, q such that $[c] = p[\ell] + q[m]$ in $H_1(\partial V)$. Conversely, for any pair of coprime integers p, q, there is an essential simple loop c in ∂V such that $[c] = p[\ell] + q[m]$ in $H_1(\partial V)$. Let $V_i (i = 1, 2)$ be solid tori with meridian-longitude systems $(m_i, \ell_i)(i = 1, 2)$. Let $f : \partial V_2 \to \partial V_1$ be an orientation-reversing homeomorphism. Since $f(m_2)$ is an essential simple loop on ∂V_1, there is a pair of coprime integers p, q such that $[f(m_2)] = p[\ell_1] + q[m_1]$ in $H_1(\partial V_1)$. Then we see that the oriented 3-manifold obtained from V_1 and V_2 by identifying ∂V_1 with ∂V_2 by f is determined uniquely (up to oriented homeomorphisms) by the homotopy class of $f(m_2)$ in ∂V_1, i.e., by the pair (p, q). Thus, this manifold is denoted by $L(p, q)$ and called the *lens space* of *type* (p, q). Note that $L(1, 0) = S^3$ and $L(0, 1) = S^2 \times S^1$. Usually, S^3 and $S^2 \times S^1$ are not regarded as lens spaces, but for convenience we include them as lens spaces. Using the Seifert-van Kampen theorem, we see that the fundamental group of $L(p, q)$ is isomorphic to the cyclic group $\mathbf{Z}_p$ (where $\mathbf{Z}_{\pm 1} = 0$ and $\mathbf{Z}_0 = \mathbf{Z}$).

Exercise D.1.1 For any coprime integers p, q, show the following:
(1) $-L(p,q) \cong L(-p,q) \cong L(p,-q)$, $L(p,q) \cong L(-p,-q)$.
(2) For any integer n, $L(p,q) \cong L(p,q+np)$.

Here $-L(p,q)$ denotes the same manifold as $L(p,q)$ but with orientation reversed, and $\cong$ denotes an orientation-preserving homeomorphism.

By (1) of this exercise, we can assume that p and q are non-negative integers. Then the lens spaces are classified as follows:

Theorem D.1.2 *Let (p,q) and (p',q') be two pairs of coprime non-negative integers. Then $L(p,q)$ and $L(p',q')$ are orientation-preservingly homeomorphic if and only if $p = p'$ and we have $q \equiv q' \pmod{p}$ or $qq' \equiv 1 \pmod{p}$.*

Theorem D.1.3 *Let (p,q) and (p',q') be two pairs of coprime non-negative integers. Then $L(p,q)$ and $L(p',q')$ are orientation-preservingly homotopy equivalent if and only if $p = p'$ and there exists an integer m such that $qq' \equiv m^2 \pmod{p}$.*

For example, we see from these theorems that $L(7,1)$ and $L(7,2)$ are orientation-preservingly homotopy equivalent but not homeomorphic. For the proofs of these theorems, see for example [Brody 1960], [Cohen 1973] and [Hempel 1976]. For the 3-manifolds M with $g(M) \leq 2$, we have the following theorem due to [Birman-Hilden 1975] and [Viro 1972]:

Theorem D.1.4 *Every closed connected oriented 3-manifold M with $g(M) \leq 2$ is a double branched covering space over S^3 with branch set a link L with bridge number $b(L) \leq 3$.*

The following theorem is given in [Jaco 1980] by an argument based on Haken's normal surface theory:

Theorem D.1.5 *For any two closed connected oriented 3-manifolds $M_i (i = 1, 2)$, we have $g(M_1 \sharp M_2) = g(M_1) + g(M_2)$.*

In a similar way, a relationship between the characteristic torus family and the genus of a Haken manifold is given in [Kobayashi,T. 1987]. Given a Heegaard splitting $(V, V'; F)$ of M, the natural homomorphisms $\iota : \pi_1(F, b) \to \pi_1(V, b)$ and $\iota' : \pi_1(F, b) \to \pi_1(V', b)$ are onto. We call the homomorphism

$$\iota \times \iota' : \pi_1(F, b) \to \pi_1(V, b) \times \pi_1(V', b)$$

the *splitting homomorphism associated with the Heegaard splitting* $(V, V'; F)$ of M. In case $g(F) \geq 2$, we can see from Dehn's lemma that if a non-trivial element of $\mathrm{Ker}(\iota \times \iota')$ is represented by a simple loop in F, then M is a connected sum of two 3-manifolds $M_i (i = 1, 2)$ such that M_i admits a Heegaard splitting $(V_i, V_i'; F_i)$ with $g(F_i) \geq 1$ and $g(F) = g(F_1) + g(F_2)$ (cf. [Hempel 1976]). It is proved in [Jaco 1969] that if we replace ι and ι' by arbitrary epimorphisms (instead of the natural epimorphisms), then the homomorphism $\iota \times \iota'$ is realizable as the splitting homomorphism associated with a Heegaard splitting $(V, V'; F)$ of some new closed connected oriented 3-manifold M'. A proper 1-manifold L in a handlebody V is called a *trivial tangle* in V if there is a 3-ball V_0 cut out of V by a proper disk which contains L as a trivial tangle. For a Heegaard handlebody V of M and a trivial r-string tangle t in $V' = \mathrm{cl}(M - V)$, the union $V \cup N(t)$ with $N(t)$ a tubular neighborhood of t in V' also gives a Heegaard handlebody of M, which is said to be *obtained from the Heegaard handlebody V by stabilization.* We have the following theorem (cf. [Reidemeister 1933, 1938], [Singer 1933], [Siebenmann **]):

Theorem D.1.6 (Reidemeister-Singer theorem) *Two Heegaard handlebodies of a closed connected oriented 3-manifold are ambient isotopic after some stabilizations.*

For the 3-sphere S^3, it is shown in [Waldhausen 1968'] that any two Heegaard handlebodies of the same genus of S^3 are ambient isotopic. On the other hand, it is known by [Casson-Gordon 1987] and [Kobayashi, T. 1992] that there are 3-manifolds admitting two or more Heegaard handlebodies of the same genus which cannot be transformed into each other by any auto-homeomorphism of M.

D.2 Dehn surgery descriptions

A *labeled knot* is a pair (K, f) of a knot K in S^3 and an unoriented simple loop f on $\partial N(K)$, called a *label* of K, where $N(K)$ denotes a tubular neighborhood of K in S^3. Using a meridian-longitude pair (m, ℓ) of K on $\partial N(K)$, we can write $[f] = a[m] + b[\ell]$ in $H_1(\partial N(K))$ for some coprime integers $a, b \in \mathbf{Z}$ including the critical cases $(a, b) = (0, \pm 1), (\pm 1, 0)$. Then we see that the fraction a/b which is a rational number (when $b \neq 0$) or the infinity ∞ (when $b = 0$) determines the label f of K. We often identify this label with this rational number or ∞. A *labeled link* (L, f) is the union of labeled knots $(K_i, f_i)(i = 1, 2, \ldots, n)$ such that the tubular neighborhoods $N(K_i)(i = 1, 2, \ldots, n)$ are mutually disjoint in S^3. Conversely, (K_i, f_i) is called a *component* of the labeled link (L, f). The *Dehn surgery manifold of S^3 along the labeled link* (L, f) is the 3-manifold obtained from S^3 by splitting S^3 into tubular neighborhoods $N(K_i)$ of $K_i(i = 1, 2, \ldots, n)$ and the exterior $E(L)$ and then by re-gluing the $N(K_i)$ along their boundaries so that the meridian m_i of $N(K_i)$ is identified with f_i for each i. We denote this manifold by $\chi(S^3; (K_i, f_i), i = 1, 2, \ldots, n)$ or $\chi(S^3; (L, f))$. We say that this notation gives a *Dehn surgery description* of a 3-manifold M if M is orientation-preserving homeomorphic to it. By a *Dehn twist* of an annulus A, we mean an auto-homeomorphism f_A of A defined by the mapping

$$(e^{2\theta\pi\sqrt{-1}}, t) \to (e^{(2\theta+t+1)\pi\sqrt{-1}}, t)$$

under the identification $A \cong S^1 \times I$ where $S^1 = \{e^{2\pi\theta\sqrt{-1}} \mid 0 \leq \theta \leq 1\}$ and $I = [-1, 1]$. We note that $f_A\,|_{\partial A} = \mathrm{id}$. When A is on an orientable surface F, the auto-homeomorphism f_A of A is uniquely extended to an auto-homeomorphism of F by taking the identity on $F - A$, which we call a *Dehn twist on the surface F*. The following theorem is widely known (cf. [Dehn 1938], [Lickorish 1962, 1964], [Birman 1974]):

Theorem D.2.1 *Every orientation-preserving auto-homeomorphism of a closed orientable surface F is ambient isotopic to the composition of a finite sequence of Dehn twists on F.*

For any closed connected oriented 3-manifold M, we take a Heegaard handlebody V in M. Then M is homeomorphic to the 3-manifold $V \cup_j V$ obtained from the two copies of V by attaching them along an orientation-reversing auto-homeomorphism j of ∂V. We choose an orientation-reversing auto-homeomorphism d_0 of ∂V so that $V \cup_{d_0} V \cong S^3$. Then by Theorem D.2.1 $(d_0)^{-1}j$ is ambient isotopic to the composition of a finite sequence $d_i (i = 1, 2, \ldots, n)$ of Dehn twists on ∂V. Hence $M \cong V \cup_{d_0 d_1 \ldots d_n} V$. Let $M_i = V \cup_{d_0 d_1 \ldots d_i} V (i = 0, 1, \ldots, n)$. Then $M_0 \cong S^3$ and each $M_i (i = 1, 2, \ldots, n)$ is obtained from M_{i-1} by a surgery attaching a 2-handle to a solid torus $V_i \subset M_{i-1}$ which is a collar in M_{i-1} of the annulus A_i used for the Dehn twist d_i. We can arrange that the solid tori $V_i (i = 1, 2, \ldots, n)$ are disjointedly embedded in $M_0 \cong S^3$. Thus, we obtain the following theorem due to [Lickorish 1962] and [Wallace 1960]:

Theorem D.2.2 *Every closed connected oriented 3-manifold is homeomorphic to the Dehn surgery manifold $\chi(S^3; (L, f))$ of S^3 along a labeled link (L, f) each of whose labels is ± 1.*

We assume that each label of a labeled link (L, f) is an integer. Using the Reidemeister move of type I, we can regard the labeled link (L, f) as a diagram D of L on S^2 together with the planar framing (in other words, the blackboard framing). Then we may write $\chi(S^3; (L, f))$ as $\chi(S^3; D)$. In [Kirby 1978], some local moves called the *Kirby moves* are given so that two Dehn surgery manifolds $\chi(S^3; (L, f))$ and $\chi(S^3; (L', f'))$ are orientation-preservingly homeomorphic if and only if (L, f) and (L', f') are transformed into each other by a finite sequence of such moves. The following diagrammatic version is due to [Fenn-Rourke 1979]:

Theorem D.2.3 $\chi(S^3; D)$ *and* $\chi(S^3; D')$ *are orientation-preservingly homeomorphic if and only if D can be transformed into D' by a finite sequence of Reidemeister moves of type II,III and IV and the moves T_{+1} and T_{-1} shown in figure D.2.1.*

T_{+1}

T_{-1}

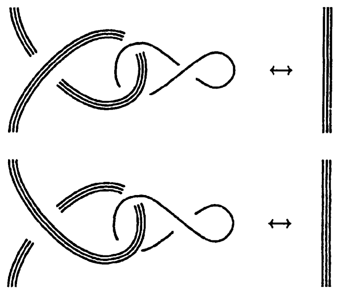


Fig. D.2.1

It is sometimes useful to know the local moves on labeled links which do not change the oriented homeomorphism type of the Dehn surgery manifolds. Figure D.2.1 tells us how to change the longitude of each component of a labeled link by one twist. The local moves stated in the following exercise are well-known:

Exercise D.2.4 Let $(K_i, f_i)(i = 1, 2, \ldots, n)$ be the components of a labeled link (L, f) with $f_i \neq \infty$ for all i. Then show that there is an orientation-preserving homeomorphism $\chi(S^3; (L, f)) \cong \chi(S^3; (L', f'))$ if the labeled link (L', f') conforms to one of the following three cases:
(1) We assume that the label f_n is an integer a. Let ℓ'_n be a longitude of $N(K_n)$ such that $\mathrm{Link}(\ell'_n, K_n) = a$. Let K'_{n-1} be a knot obtained by fusion of K_{n-1} and ℓ'_n along a band not meeting $\mathrm{int}N(K_n) \cup (L - K_{n-1} - K_n)$. We take a label f'_{n-1} of K'_{n-1} to be $f_{n-1} + a \pm 2\ \mathrm{Link}(K_{n-1}, K_n)$. Let (L', f') be the labeled link obtained from (L, f) by replacing (K_{n-1}, f_{n-1}) with (K'_{n-1}, f'_{n-1}). (See [Kirby 1978].)
(2) We assume that K_n is a trivial knot. Let $(L - K_n)_\varepsilon$ be the link obtained from $L - K_n$ by the move T_ε on K_n as shown in figure D.2.1, where $\varepsilon = \pm 1$. Let $L' = (L - K_n)_\varepsilon \cup K_n$. Let $f'_i = f_i + \varepsilon a_i^2$ with $a_i = \mathrm{Link}(K_i, K_n)$ for $i < n$ and $f'_n = 1/[(1/f_n) + \varepsilon]$ (when $f_n = 0$, we understand $f'_n = 0$). Let (L', f') be the resulting labeled link. (See [Rolfsen 1976].)
(3) We assume that K_n is a meridian of K_{n-1} and that $f_{n-1} = 0$. Let (L', f') be a labeled link consisting of the components $(K_i, f_i)(i = 1, 2, \ldots, n-2)$ and (K_{n-1}, f'_{n-1}) with $f'_{n-1} = -1/f_n$ (when $f_n = 0$, we understand $f'_{n-1} = \infty$). (See [Cochran-Gompf 1988].)

When $\chi(S^3; (L, f))$ is a surgery description of a 3-manifold M such that L is a link with two components or more, we note that there are infinitely many Dehn surgery descriptions $\chi(S^3; (L^*, f))$ of M such that there is a map $q : (S^3, L^*) \to (S^3, L)$ with a homeomorphism $q : N(L^*) \cong N(L)$ and $q(E(L^*)) = E(L)$, but L^* is not equivalent to L (cf. [Kawauchi 1989']).

Theorem D.2.3 has been used in [Reshetikhin-Turaev 1991] (and then in many other authors' papers) to define some numerical invariants of 3-manifolds, which were foreseen in [Witten 1989] and are called *the quantum invariants*. For this topic, we refer to the survey article [Morton 1993] and the book [Turaev 1994]. We can define the Dehn surgery manifold of a homology 3-sphere M (instead of S^3) along a labeled link in M without essential changes. Another 3-manifold invariant recently defined is the *Floer homology* (cf. [Floer 1988, 1990]). It is a $\mathbf{Z}_8$-graded homology theory defined for homology 3-spheres and homology handles using gauge theory. The Floer homology admits an exact triangle connecting the Floer homology groups of a homology 3-sphere M and the Dehn surgery manifolds $\chi(M; (K, 0))$ and $\chi(M; (K, 1))$ for a knot K in M (cf. [Braam-Donaldson 1995]).

Appendix E
The Blanchfield duality theorem

In this appendix, we are mainly concerned with a non-compact (PL) pair, obtained by taking a free abelian covering. By (X^*, A^*) we denote a triangulation of a space pair (X, A). By $C_\sharp(X^*, A^*)$ and $C^\sharp(X^*, A^*)$, we denote the ordinary simplicial chain and cochain complexes, respectively. Let $X_n^*(n = 0, 1, 2, \dots)$ be finite subcomplexes of X^* such that $X_n^* \subset X_{n+1}^*$ and $X^* = \cup_{n=0}^{\infty} X_n^*$. Let $\mathrm{cl}(X^* - X_n^*)$ be the subcomplex of X^* consisting of all simplexes in $X^* - X_n^*$ and their faces. Let

$$C_c^\sharp(X^*, A^*) = \lim_{n \to +\infty} C^\sharp(X^*, A^* \cup \mathrm{cl}(X^* - X_n^*))$$

be a cochain complex which is a natural subcomplex of the cochain complex $C^\sharp(X^*, A^*)$. This cohomology is called the *cohomology of* (X, A) *with compact support* and denoted by $H_c^*(X, A)$. It is well-known that this cohomology is independent of the choice of triangulation (X^*, A^*) and the subcomplexes $X_n^*(n = 0, 1, 2, \dots)$ and hence is a topological invariant of (X, A) (cf. [Spanier 1966]). Let X be a non-compact oriented PL m-manifold. By a *splitting* A, B of ∂X, we mean that A and B are $(m-1)$-submanifolds or the empty subset of ∂X with $A = \mathrm{cl}(\partial X - B)$ and $B = \mathrm{cl}(\partial X - A)$. For any splitting A, B of ∂X and any integers p, q with $p + q = m$, we have the following:

Poincaré duality theorem (the non-compact version) *There is an isomorphism* $\rho : H_c^p(X, A) \to H_q(X, B)$ *with identity* $\rho h^* = (h_*)^{-1}\rho$ *for any auto-homeomorphism* h *of* $(X; A, B)$ *preserving the orientation of* X.

Proof. This duality can be obtained from the well-known Poincaré duality theorem in the compact case. We take compact PL m-submanifolds $X_n(n = 0, 1, 2, \dots)$ of X so that $X = \cup_{n=0}^{\infty} X_n$, $X_n \subset X_{n+1}$, and $X_n \cap \partial X, X_n \cap A$, and $X_n \cap B$ are compact PL $(m-1)$-submanifolds or the empty subset of ∂X. Let $\mathrm{Fr}X_n = \mathrm{cl}(\partial X_n - \partial X_n \cap \partial X)$. Then we have isomorphisms

$$H^p(X, A \cup \mathrm{cl}(X - X_n)) \cong H^p(X_n, (A \cap X_n) \cup \mathrm{Fr}X_n) \cong H_q(X_n, B \cap X_n),$$

where the former isomorphism is the excision isomorphism and the latter isomorphism is the Poincaé duality isomorphism in the compact case. Denoting this composite isomorphism by ρ_n, we see that the following diagram with vertical homomorphisms induced from the inclusions is commutative:

$$\begin{tikzcd}
H^p(X, A \cup \mathrm{cl}(X - X_n)) \arrow[r, "\rho_n"] \arrow[d] & H_q(X_n, B \cap X_n) \arrow[d] \\
H^p(X, A \cup \mathrm{cl}(X - X_{n+1})) \arrow[r, "\rho_{n+1}"] & H_q(X_{n+1}, B \cap X_{n+1}).
\end{tikzcd}$$

The desired isomorphism ρ is obtained from it by letting $n \to \infty$. By a property of cap products, we have the identity $\rho h^* = (h_*)^{-1}\rho$ for any auto-homeomorphism h of $(X; A, B)$ preserving the orientation of X. $\square$

Let Λ be the integral group ring $\mathbf{Z}G$ of a free abelian group G of rank n. Let $\bar{\lambda} = \sum_{g\in G} n_g g^{-1}$ for any element $\lambda = \sum_{g\in G} n_g g (n_g \in \mathbf{Z})$ of Λ. Let $\bar{q} = \bar{\lambda}_1/\bar{\lambda}_2$ for any element $q = \lambda_1/\lambda_2$ of the quotient field $\mathbf{Q}(\Lambda)$ of Λ. We consider a compact connected oriented m-manifold M with an epimorphism $\gamma : \pi_1(M, p) \to G$. Let A, B be a splitting of ∂M. Let $(M_\gamma; A_\gamma, B_\gamma)$ be the covering space triad over the triad $(M; A, B)$ corresponding to the kernel of γ. Then $H_c^*(M_\gamma, A_\gamma)$ and $H_*(M_\gamma, B_\gamma)$ are natural Λ-modules and the Poincaré duality isomorphism $\rho : H_c^p(M_\gamma, A_\gamma) \to H_q(M_\gamma, B_\gamma)$ is an anti-Λ-map,i.e., it satisfies $\rho(\lambda x) = \bar{\lambda}\rho(x)$ for $\lambda \in \Lambda$ and $x \in H_c^p(M_\gamma, A_\gamma)$. Let $(M_\gamma^*; A_\gamma^*, B_\gamma^*)$ be a triangulation of $(M_\gamma; A_\gamma, B_\gamma)$ induced from a triangulation $(M^*; A^*, B^*)$ of $(M; A, B)$ by the covering $M_\gamma \to M$. Let $\Delta_i (i = 1, 2, \ldots, r)$ be the mutually distinct p-simplices of $M^* - A^*$. For each i, we choose a p-simplex in M_γ^* lifting Δ_i and denote it by Δ_i'. Then we have

$$C_p(M_\gamma^*, A_\gamma^*) = \Lambda\Delta_1' \oplus \cdots \oplus \Lambda\Delta_r'.$$

The map

$$\varphi : C_c^*(M_\gamma^*, A_\gamma^*) \to \mathrm{Hom}(C_\sharp(M_\gamma^*, A_\gamma^*), \Lambda)$$

defined by $\varphi(f)(c) = \sum_{g\in G} f(gc)g^{-1}$ for $c \in C_\sharp(M_\gamma^*, A_\gamma^*)$ and $f \in C_c^\sharp(M_\gamma, A_\gamma)$ is seen to be a Λ-isomorphism of cochain complexes over Λ, where Hom denotes the collection of all Λ-homomorphisms. The cohomology of the co-domain of φ is also independent of a choice of the triangulation (M^*, A^*) and denoted by $H^*(M_\gamma, A_\gamma; \Lambda)$ (cf.[Kawauchi 1986]). We define a Λ-homomorphism

$$h : H^p(M_\gamma, A_\gamma; \Lambda) \to \mathrm{Hom}(H_p(M_\gamma, A_\gamma), \Lambda)$$

by $h(\{f\})(\{c\}) = f(c)$ for $\{f\} \in H^p(M_\gamma, A_\gamma; \Lambda)$ and $\{c\} \in H_p(M_\gamma, A_\gamma)$. For a Λ-module H, let TH be the Λ-torsion part of H and $BH = H/TH$ and

$$\begin{aligned} DH = \{x \in H \mid \lambda_1 x = \lambda_2 x = \cdots = \lambda_s x = 0 \\ \text{for some coprime}\, \lambda_1, \lambda_2, \ldots, \lambda_s \in \Lambda (s \geq 2)\} \end{aligned}$$

and $T_D H = TH/DH$. We define a Λ-homomorphism

$$k : TH^p(M_\gamma, A_\gamma; \Lambda) \to \mathrm{Hom}(TH_{p-1}(M_\gamma, A_\gamma), \mathbf{Q}(\Lambda)/\Lambda)$$

as follows: For any element $\{f\} \in TH^p(M_\gamma, A_\gamma; \Lambda)$, there are a non-zero $\lambda \in \Lambda$ and a Λ-homomorphism $f^+ : C_{p-1}(M_\gamma^*, A_\gamma^*) \to \Lambda$ such that $\lambda f = \delta(f^+)$. Then we set $k(\{f\})(\{c\}) = f^+(c)/\lambda$. We can see that this is well-defined. Using h, k and the anti-Λ-isomorphism $\rho^* = \rho(\varphi^*)^{-1} : H^p(M_\gamma, A_\gamma; \Lambda) \cong H_q(M_\gamma, B_\gamma)$, we define Λ-sesquilinear forms

$$\begin{aligned} S &: H_p(M_\gamma, A_\gamma) \times H_q(M_\gamma, B_\gamma) \to \Lambda, \\ L &: TH_{p-1}(M_\gamma, A_\gamma) \times TH_q(M_\gamma, B_\gamma) \to \mathbf{Q}(\Lambda)/\Lambda \end{aligned}$$

by $S(x,y) = (h(\rho^*)^{-1}(y))(x)$ and $L(x,y) = (-1)^{q+1}(k(\rho^*)^{-1}(y))(x)$. It is easily checked that

$$\begin{aligned} S(\lambda x, y) &= \lambda S(x,y) = S(x, \bar{\lambda} y), \\ L(\lambda x, y) &= \lambda L(x,y) = L(x, \bar{\lambda}) \end{aligned}$$

for any $\lambda \in \Lambda$. Similarly, we have the following Λ-sesquilinear forms

$$\begin{aligned} S' &: H_q(M_\gamma, B_\gamma) \times H_p(M_\gamma, A_\gamma) \to \Lambda, \\ L' &: TH_q(M_\gamma, B_\gamma) \times TH_{p-1}(M_\gamma, A_\gamma) \to \mathbf{Q}(\Lambda)/\Lambda. \end{aligned}$$

Lemma E.1

(1) $S(x,y) = (-1)^{pq}\overline{S'(y,x)}$.
(2) $L(x,y) = (-1)^{(p-1)q+1}\overline{L'(y,x)}$.

Proof. For $x = \{c_x\} \in H_p(M_\gamma, A_\gamma), y = \{c_y\} \in H_q(M_\gamma, B_\gamma)$ and $\rho^{-1}(y) = \{f_y\}$, we have

$$\begin{aligned} S(x,y) &= h\varphi^* \rho^{-1}(y)(x) \\ &= \sum_{g \in G} f_y(gc_x)g^{-1} \\ &= \sum_{g \in G} \mathrm{Int}(c_y, gc_x)g^{-1}. \end{aligned}$$

Then we obtain (1), because

$$\begin{aligned} \mathrm{Int}(c_y, gc_x) &= (-1)^{pq}\mathrm{Int}(gc_x, c_y) \\ &= (-1)^{pq}\mathrm{Int}(c_x, g^{-1}c_y). \end{aligned}$$

Next, we take $x = \{c_x\} \in TH_{p-1}(M_\gamma, A_\gamma)$ and $y = \{c_y\} \in TH_q(M_\gamma, B_\gamma)$ and $\{f_y\} = \rho^{-1}(y)$. We note that the underlying polyhedra of c_x and c_y can be taken to be disjoint by a general position argument. We take non-zero $\lambda_x, \lambda_y \in \Lambda$ and $c_x^+ \in C_p(M_\gamma^*, A_\gamma^*), c_y^+ \in C_{q+1}(M_\gamma^*, B_\gamma^*)$ so that $\partial c_x^+ = \lambda_x c_x, \partial c_y^+ = \lambda_y c_y$. Since there is an $f_y^+ \in C_c^{p-1}(M_\gamma, A_\gamma)$ with $\bar{\lambda}_y f_y = \delta f_y^+$, we have that

$$\begin{aligned} L(x,y) &= (-1)^{q+1} \sum_{g\in G} f_y^+(gc_x)g^{-1}/\bar{\lambda}_y \\ &= (-1)^{q+1} \sum_{g\in G} f_y^+(g\partial c_x^+)g^{-1}/\lambda_x\bar{\lambda}_y \\ &= (-1)^{q+1} \sum_{g\in G} f_y(gc_x^+)g^{-1}/\lambda_x \\ &= (-1)^{q+1} \sum_{g\in G} \mathrm{Int}(c_y, gc_x^+)g^{-1}/\lambda_x. \end{aligned}$$

Then we obtain (2), because

$$\begin{aligned} &(-1)^{q+1}\mathrm{Int}(c_y, gc_x^+) = (-1)^{q+1}\mathrm{Int}(\partial c_y^+, gc_x^+) = \mathrm{Int}(c_y^+, g\partial c_x^+) \\ &= (-1)^{(p-1)(q+1)}\mathrm{Int}(g\partial c_x^+, c_y^+) = (-1)^{(p-1)q+1} \cdot (-1)^p \mathrm{Int}(c_x, g^{-1}c_y^+). \square \end{aligned}$$

By $\Lambda_{(\alpha)}$, we denote the local ring $\{\lambda_1/\lambda_2 \in \mathbf{Q}(\Lambda) \mid \lambda_1, \lambda_2 \in \Lambda, (\lambda_2, \alpha) = 1\}$ of Λ at a prime (or unit) element $\alpha \in \Lambda$. For example, for the unit $1 \in \Lambda$, we have $\Lambda_{(1)} = \mathbf{Q}(\Lambda)$. For a Λ-module H, the tensor product $H \otimes \Lambda_{(\alpha)}$ over Λ is simply written as $H_{(\alpha)}$. Since $\Lambda_{(\alpha)}$ is a principal ideal domain and there is a $\Lambda_{(\alpha)}$-isomorphism

$$H^p(M_\gamma, A_\gamma; \Lambda)_{(\alpha)} \cong H^p(\mathrm{Hom}_{(\alpha)}(C_\sharp(M_\gamma, A_\gamma)_{(\alpha)}, \Lambda_{(\alpha)})),$$

where $\mathrm{Hom}_{(\alpha)}$ denotes the set of $\Lambda_{(\alpha)}$-homomorphisms, we obtain from the usual universal coefficient theorem(cf.[Spanier 1966]) the following natural exact sequence:

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\Lambda_{(\alpha)}}(H_{p-1}(M_\gamma, A_\gamma)_{(\alpha)}, \Lambda_{(\alpha)}) \to H^p(M_\gamma, A_\gamma; \Lambda)_{(\alpha)} \to \\ \mathrm{Hom}_{(\alpha)}(H_p(M_\gamma, A_\gamma)_{(\alpha)}, \Lambda_{(\alpha)}) \to 0. \end{aligned}$$

The anti-Λ-isomorphism $\rho^* : H^p(M_\gamma, A_\gamma; \Lambda) \cong H_q(M_\gamma, B_\gamma)$ induces an anti-$\Lambda_{(\alpha)}$-isomorphism

$$\rho^*_{(\alpha)} : H^p(M_\gamma, A_\gamma; \Lambda)_{(\alpha)} \cong H_q(M_\gamma, B_\gamma)_{(\bar{\alpha})}.$$

Further, we have

$$BH^p(M_\gamma, A_\gamma; \Lambda)_{(\alpha)} \cong \mathrm{Hom}_{(\alpha)}(BH_p(M_\gamma, A_\gamma)_{(\alpha)}, \Lambda_{(\alpha)})$$

and

$$\begin{aligned} TH^p(M_\gamma, A_\gamma; \Lambda)_{(\alpha)} &\cong \mathrm{Ext}^1_{\Lambda_{(\alpha)}}(H_{p-1}(M_\gamma, A_\gamma)_{(\alpha)}, \Lambda_{(\alpha)}) \\ &\cong \mathrm{Ext}^1_{\Lambda_{(\alpha)}}(TH_{p-1}(M_\gamma, A_\gamma)_{(\alpha)}, \Lambda_{(\alpha)}) \\ &\cong \mathrm{Hom}_{(\alpha)}(TH_{p-1}(M_\gamma, A_\gamma)_{(\alpha)}, \mathbf{Q}(\Lambda)/\Lambda_{(\alpha)}). \end{aligned}$$

Thus, the Λ-homomorphisms h and k induce $\Lambda_{(\alpha)}$-isomorphisms

$$h_{(\alpha)} : BH^p(M_\gamma, A_\gamma; \Lambda)_{(\alpha)} \to \mathrm{Hom}_{(\alpha)}(BH_p(M_\gamma, A_\gamma)_{(\alpha)}, \Lambda_{(\alpha)}) \text{ and}$$
$$k_{(\alpha)} : TH^p(M_\gamma, A_\gamma; \Lambda)_{(\alpha)} \to \mathrm{Hom}_{(\alpha)}(TH_{p-1}(M_\gamma, A_\gamma)_{(\alpha)}, \mathbf{Q}(\Lambda)/\Lambda_{(\alpha)}).$$

By the anti-$\Lambda_{(\alpha)}$-isomorphism $\rho^*_{(\alpha)}$, the $\Lambda_{(\alpha)}$-isomorphisms $h_{(\alpha)}, k_{(\alpha)}$, and Lemma E.1, the following theorem is obtained:

Theorem E.2 *For any integers p, q with $p+q=n$ and any prime (or unit) element $\alpha \in \Lambda$, the Λ-sesquilinear forms S and L induce the following non-singular $\Lambda_{(\alpha)}$-sesquilinear forms:*

$$S_{(\alpha)} : BH_p(M_\gamma, A_\gamma)_{(\alpha)} \times BH_q(M_\gamma, B_\gamma)_{(\bar{\alpha})} \to \Lambda_{(\alpha)},$$
$$L_{(\alpha)} : TH_{p-1}(M_\gamma, A_\gamma)_{(\alpha)} \times TH_q(M_\gamma, B_\gamma)_{(\bar{\alpha})} \to \mathbf{Q}(\Lambda)/\Lambda_{(\alpha)}.$$

From Theorem E.2, we also obtain the following:

Theorem E.3 *For any integers p, q with $p+q=n$, the Λ-sesquilinear forms S and L induce the following non-degenerate Λ-sesquilinear forms:*

$$S_T : BH_p(M_\gamma, A_\gamma) \times BH_q(M_\gamma, B_\gamma) \to \Lambda,$$
$$L_D : T_D H_{p-1}(M_\gamma, A_\gamma) \times T_D H_q(M_\gamma, B_\gamma) \to \mathbf{Q}(\Lambda)/\Lambda.$$

Proof. It is easy to see that the Λ-sesquilinear forms S and L induce the Λ-sesquilinear forms S_T and L_D. $BH_p(M_\gamma, A_\gamma)$ and $BH_q(M_\gamma, B_\gamma)$ are naturally embedded in $BH_p(M_\gamma, A_\gamma)_{(1)}$ and $BH_q(M_\gamma, B_\gamma)_{(1)}$, respectively. Then the non-singularity of $S_{(1)}$ implies the non-degeneracy of S_T. Next, we take any element $x \in TH_{p-1}(M_\gamma, A_\gamma)$ which is not in $DH_{p-1}(M_\gamma, A_\gamma)$. For the ideal $I_x = \{\lambda \in \Lambda \mid \lambda x = 0\}$, there is a non-unit prime element $\alpha \in \Lambda$ with $I_x \subset \alpha\Lambda$. Since $\Lambda_{(\alpha)}$ is flat over Λ, the homomorphism $(\Lambda/I_x)_{(\alpha)} \to TH_{p-1}(M_\gamma, A_\gamma)_{(\alpha)}$ sending 1 to x is injective. Since $(\Lambda/I_x)_{(\alpha)} \neq 0$, the element x is not zero in $TH_{p-1}(M_\gamma, A_\gamma)_{(\alpha)} = T_D H_{p-1}(M_\gamma, A_\gamma)_{(\alpha)}$. By the non-degeneracy of $L_{(\alpha)}$, there is an element $y \in TH_q(M_\gamma, B_\gamma)_{(\bar{\alpha})} = T_D H_q(M_\gamma, B_\gamma)_{(\bar{\alpha})}$ such that $L_{(\alpha)}(x, y) \neq 0$ in $\mathbf{Q}(\Lambda)/\Lambda_{(\alpha)}$. We can take y in $T_D H_q(M_\gamma, B_\gamma)$ and the left non-degeneracy of L_D is demonstrated. By Lemma E.1, the right non-degeneracy of L_D is also obtained. $\square$

Theorems E.1 and E.2 are called the *Blanchfield duality theorem*, although the original expression is slightly different from ours (cf. [Blanchfield 1957]). When $G \cong \mathbf{Z}$, it is known that L_D is always non-singular (cf. [Kawauchi 1986]).

Appendix F
Tables of data

F.0 Comments on the data

In Tables 0-5, we list many data concerning the prime knots with up to 10 crossings. The numbering of the knots follows that of [Rolfsen 1976], which is taken from [Alexander-Briggs 1927] for knots with up to 9 crossings and from [Conway 1970] for knots with 10 crossings. The following points should be noted:

(1) It was pointed out in [Perko 1974'] that 10_{161} and 10_{162} are identical.
(2) The figures 10_{83} and 10_{86} in [Rolfsen 1976] should be interchanged.

In Table 6, we list some diagrams of the surfaces in $\mathbf{R}^4$ with up to 9 ch-indices (cf. 13.1).

Table 0 This table exhibits diagrams of the prime knots with up to 10 crossings which are drawn by Y. Nakanishi, together with their Conway notation (cf. 10.7 and Table 1).

Table 1 This table contains the types and the symmetries of knots. In the type-column, the symbols $T(\cdots), S(\cdots)$ and $M(\cdots)$ represent a torus knot, a 2-bridge knot and a Montesinos knot, respectively (cf. Chapter 2). The other symbols in the type-column represent Conway's notation (cf. 10.7 and Table 1). The Symbols r, i, f, and n come from the words; *reversible*, *involutory*, *full* and *none*, respectively, and have the following meaning:

	amphicheiral	non-amphicheiral
invertible	f	r
non-invertible	i	n

All the prime knots of type i with up to 10 crossings are $(-)$amphicheiral. P_C and P_F denote periods and free periods, respectively. The sign "$-$" in these columns means that the knot does not have the corresponding symmetry. In the P_F-column, for example, $(p, 6) = 1$ denotes that any number p relatively prime to 6 is a free period of the knot. Sym denotes the symmetry group $\mathrm{Sym}(S^3, K)$, which is presented so as to pay attention to its action on $K \cong S^1$: Thus, D_1 or $\mathbf{Z}_2$ means that $\mathrm{Sym}(S^3, K)$ is the cyclic group of order 2 generated by an auto-homeomorphism which reverses or preserves the orientation of K, respectively. Except for the following list, all the symmetries can be "naturally" seen from the information in the type-column:

(1) The period of 9_{47} comes from its representation as the closed braid $(\sigma_1\sigma_2\sigma_3^{-1})^2$.

(2) The free periods of 10_{155} and 10_{157} can be seen from their representations as closed braids $\Delta^2\sigma_1^{-3}\sigma_2^{-3}\sigma_2$ and $\Delta^2(\sigma_1^{-1}\sigma_2)^4$ respectively, where $\Delta = \sigma_1\sigma_2\sigma_1$.
(3) The invertibility of the knots $10_{156}, 10_{159}, 10_{165}, 10_{166}$ can be seen from the diagrams in [Rolfsen 1976].

Table 2 This table contains the Alexander polynomial, Nakanishi's index m, the half absolute value of the signature $\sigma' = |\sigma|/2$, the unknotting number u, the 4-dimensional genus g^*, the order in the algebraic knot cobordism group a, the order in the knot cobordism group o, the Δ-unknotting number u_Δ and the tunnel number t. However, the genus is omitted from the table, since it is equal to half of the degree of the Alexander polynomial. The symbol $[a_0 + a_1 + \cdots + a_n$ in the column of the Alexander polynomial represents the polynomial $a_0 + a_1(t^{-1} + t) + \cdots + a_n(t^{-n} + t^n)$. The symbols A, B, M, N, X, Z have the following meanings:

A:	1 or 2.	M:	2 or 4.	X:	2 or 3.
B:	1, 2, or 3.	N:	1 or 3.	Z:	3 or 4.

Table 3 This table is a list of presentation matrices for the knot modules obtained by omitting the knot modules with Nakanishi's index = 1 from the knot modules of prime knots with up to 10 crossings. This list is taken from [Nakanishi 1980"]. The question mark (?) represents that it is not confirmed that Nakanishi's index is actually 2.

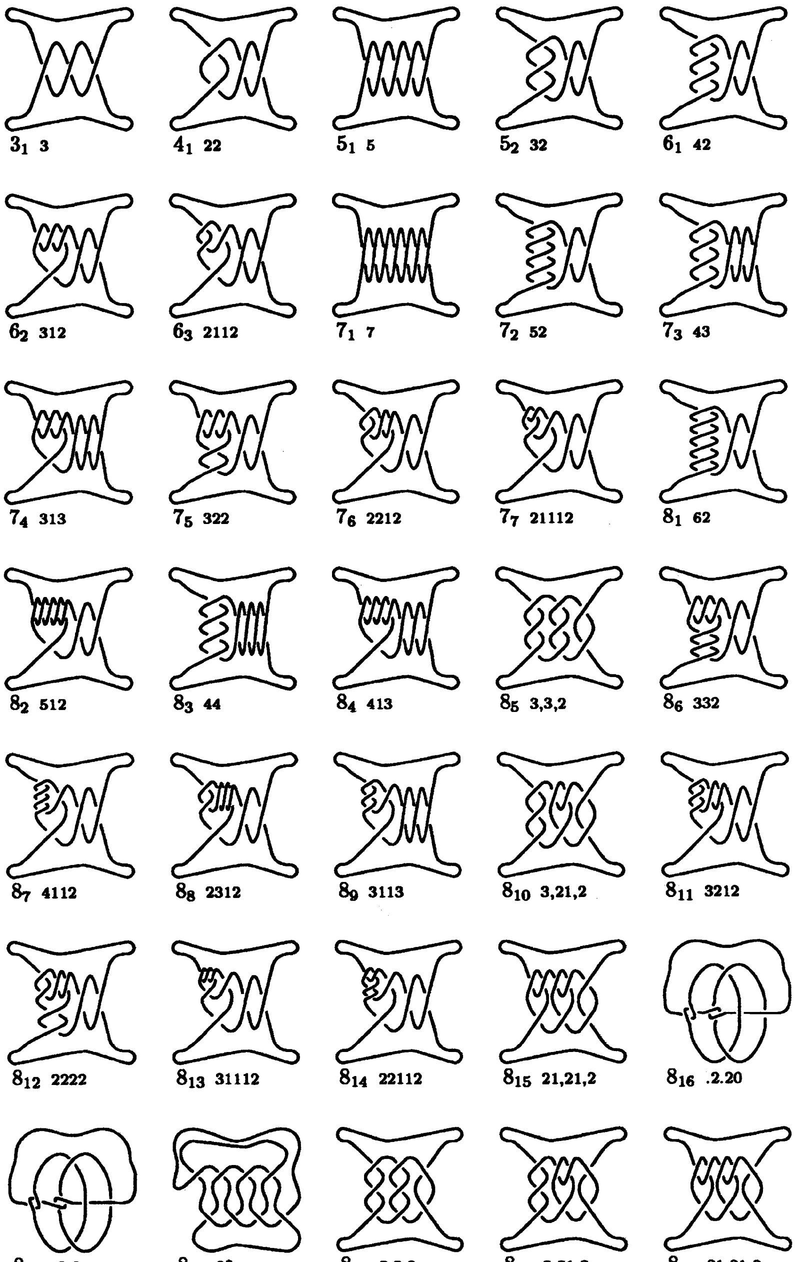
Table 4 This table is a list of ribbon presentations of ribbon prime knots with up to 10 crossings which was made by Y. Nakanishi. We note that each slice knot in the table is a ribbon knot and each ribbon knot can have non-equivalent ribbon presentations.

Table 5 This table enumerates the skein polynomials $P(K) = P(K; a, z)$ and the Kauffman polynomials $F(K) = F(K; a, x)$ of the prime knots K with up to 10 crossings under the conventions for the variables that we used in Chapter 8. These calculations were made by K. Kodama by using his NEC-PC98 computer program "KODAMA SOFT".

Table 6 This table enumerates the diagrams of surface-links in $\mathbf{R}^4$ with up to 10 ch-indices (cf. 13.1). This list is taken from [Yoshikawa 1994]. In the diagrams, the saddle point shown in figure 13.1.2c is denoted by $\times$. The notation $I_k^{g_1,\dots,g_c}$ means the k-th surface-link with ch-index I and with c components of genera $|g_1|, \dots, |g_c|$. If $g_i < 0$, then we mean that the i-th component is non-orientable. I_k^0 is simply written as I_k.

In compiling these tables many studies listed in the references were useful, among them:
[Adams-Hildebrand-Weeks 1991], [Burde-Zieschang 1985], [Kanenobu-Murakami 1986], [Kobayashi 1989'], [Kodama-Sakuma 1992], [Lickorish 1985], [Gordon-Litherland-Murasugi 1981], [Miyazawa 1994], [Morita, T. 1988], [Murakami-Sugishita 1984], [Nakamura 1995], [Nakanishi 1981], [Okada 1990].

F.1 knot diagram



3_1 3 | 4_1 22 | 5_1 5 | 5_2 32 | 6_1 42

6_2 312 | 6_3 2112 | 7_1 7 | 7_2 52 | 7_3 43

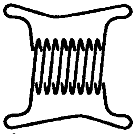
7_4 313 | 7_5 322 | 7_6 2212 | 7_7 21112 | 8_1 62

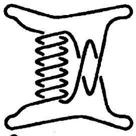
8_2 512 | 8_3 44 | 8_4 413 | 8_5 3,3,2 | 8_6 332

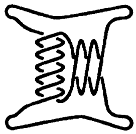
8_7 4112 | 8_8 2312 | 8_9 3113 | 8_{10} 3,21,2 | 8_{11} 3212

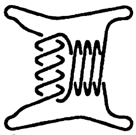
8_{12} 2222 | 8_{13} 31112 | 8_{14} 22112 | 8_{15} 21,21,2 | 8_{16} .2.20

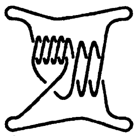
8_{17} .2.2 | 8_{18} 8* | 8_{19} 3,3,2– | 8_{20} 3,21,2– | 8_{21} 21,21,2–

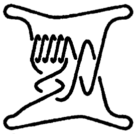

9_{1} 9

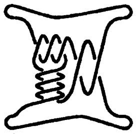

9_{2} 72

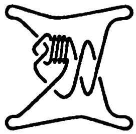

9_{3} 63

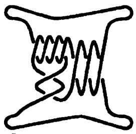

9_{4} 54

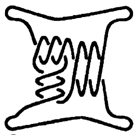

9_{5} 513

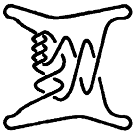

9_{6} 522

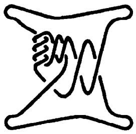

9_{7} 342

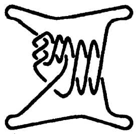

9_{8} 2412

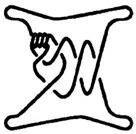

9_{9} 423

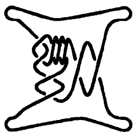

9_{10} 333


9_{11} 4122


9_{12} 4212


9_{13} 3213


9_{14} 41112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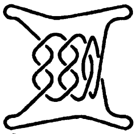
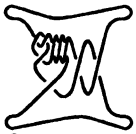

9_{16} 3,3,2+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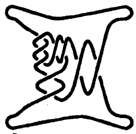
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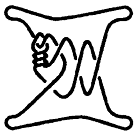
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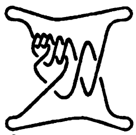
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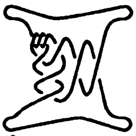
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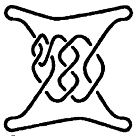
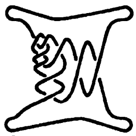

9_{22} 211,3,2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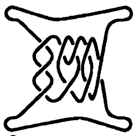
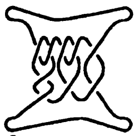

9_{24} 3,21,2+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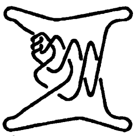
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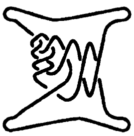
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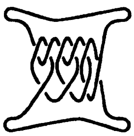
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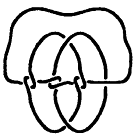
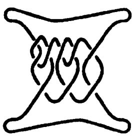

9_{29} .2.20.2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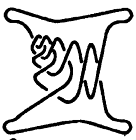
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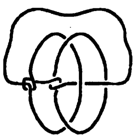
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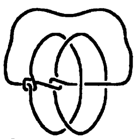
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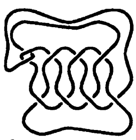
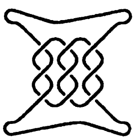

9_{34} 8*20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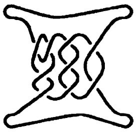
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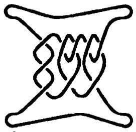
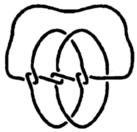
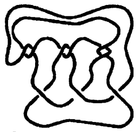
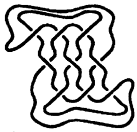

9_{37} 3,21,21


9_{38} .2.2.2


9_{39} 2:2:20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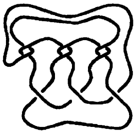
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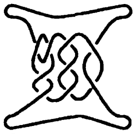
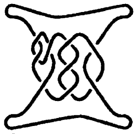
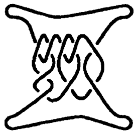
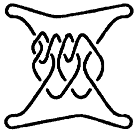
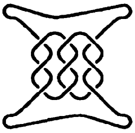
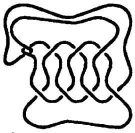

9_{42} 22,3,2−

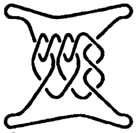

9_{43} 211,3,2−

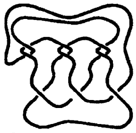

9_{44} 22,21,2−

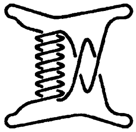

9_{45} 211,21,2−

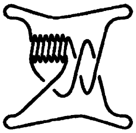

9_{46} 3,3,21−

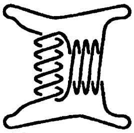

9_{47} 8*−20

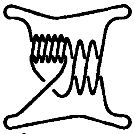

9_{48} 21,21,21−


9_{49} −20:−20:−20


10_{1} 82


10_{2} 712


10_{3} 64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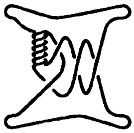
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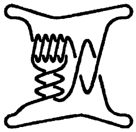
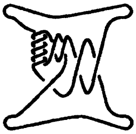
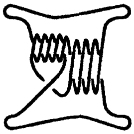
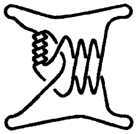

10_{6} 532


10_{7} 5212


10_{8} 514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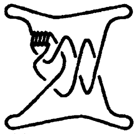
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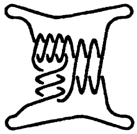
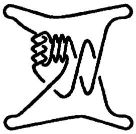

10_{11} 433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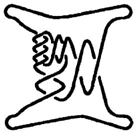
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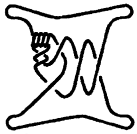
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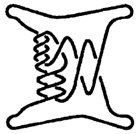
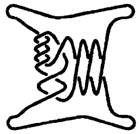

10_{15} 4132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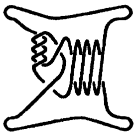
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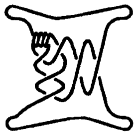
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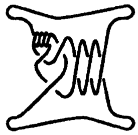
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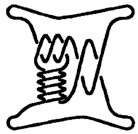
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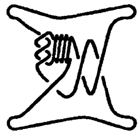
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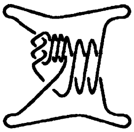
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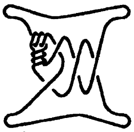
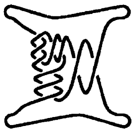

10_{23} 33112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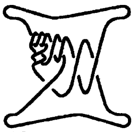
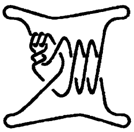

10_{25} 32212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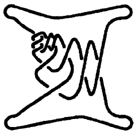
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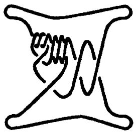
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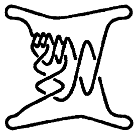
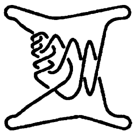

10_{29} 31222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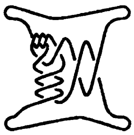
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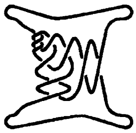
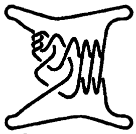
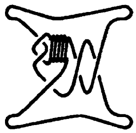

10_{32} 311122


10_{33} 311113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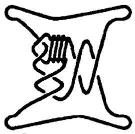
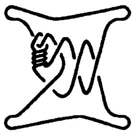

10_{35} 2422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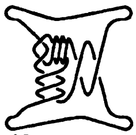
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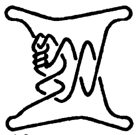
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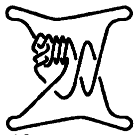
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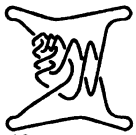
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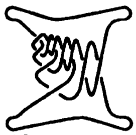
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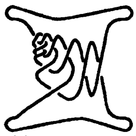
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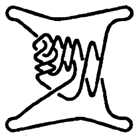
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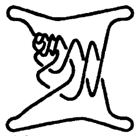
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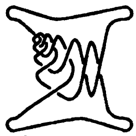
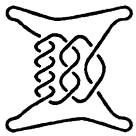
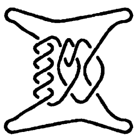

10_{45} 21111112


10_{46} 5,3,2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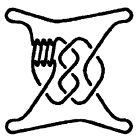
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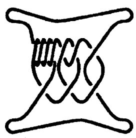
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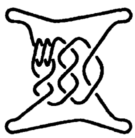
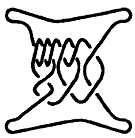
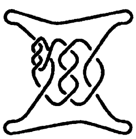
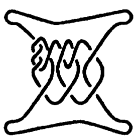
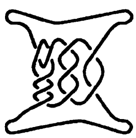

10_{50} 32,3,2


10_{51} 32,21,2


10_{52} 311,3,2


10_{53} 311,21,2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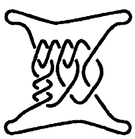
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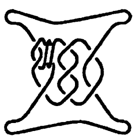
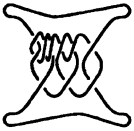
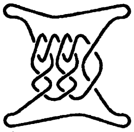
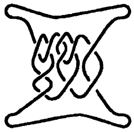
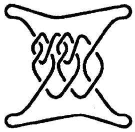
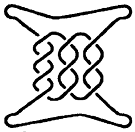

10_{56} 221,3,2

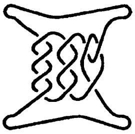

10_{57} 221,21,2

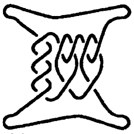

10_{58} 22,22,2

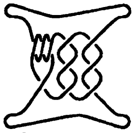

10_{59} 22,211,2

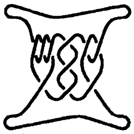

10_{60} 211,211,2

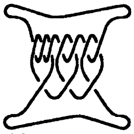

10_{61} 4,3,3

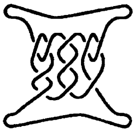

10_{62} 4,3,21

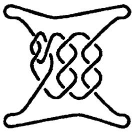

10_{63} 4,21,21

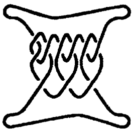

10_{64} 31,3,3

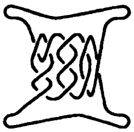

10_{65} 31,3,21

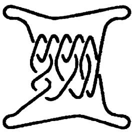

10_{66} 31,21,21

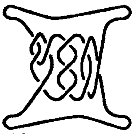

10_{67} 22,3,21

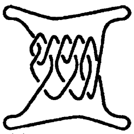

10_{68} 211,3,3

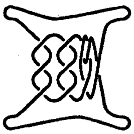

10_{69} 211,21,21

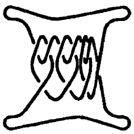

10_{70} 22,3,2+

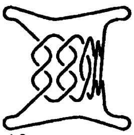

10_{71} 22,21,2+

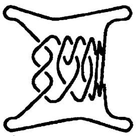

10_{72} 211,3,2+

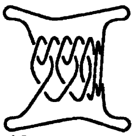

10_{73} 211,21,2+

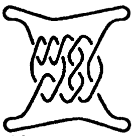

10_{74} 3,3,21+

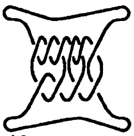

10_{75} 21,21,21+

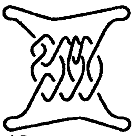

10_{76} 3,3,2++

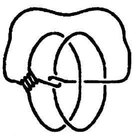

10_{77} 3,21,2++

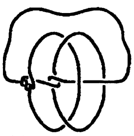

10_{78} 21,21,2++

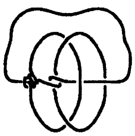

10_{79} (3,2)(3,2)

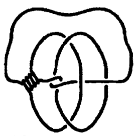

10_{80} (3,2)(21,2)

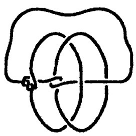

10_{81} (21,2)(21,2)

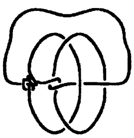

10_{82} .4.2

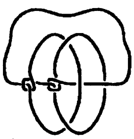

10_{83} .31.2

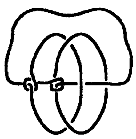

10_{84} .22.2

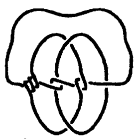

10_{85} .4.20

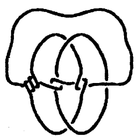

10_{86} .31.20

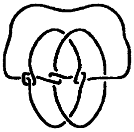

10_{87} .22.20

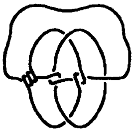

10_{88} .21.21

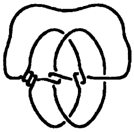

10_{89} .21.210

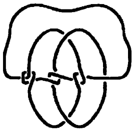

10_{90} .3.2.2

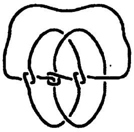

10_{91} .3.2.20

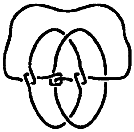

10_{92} .21.2.20

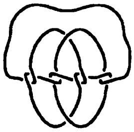

10_{93} .3.20.2

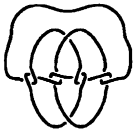

10_{94} .30.2.2

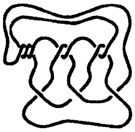

10_{95} .210.2.2

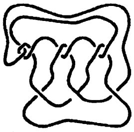

10_{96} .2.21.2

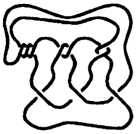

10_{97} .2.210.2

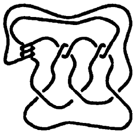

10_{98} .2.2.2.20

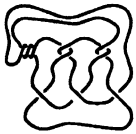

10_{99} .2.2.20.20

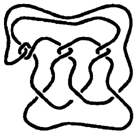

10_{100} 3:2:2

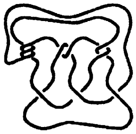

10_{101} 21:2:2

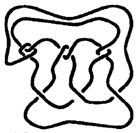

10_{102} 3:2:20

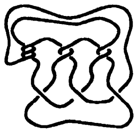

10_{103} 30:2:2

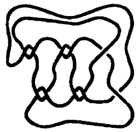

10_{104} 3:20:20

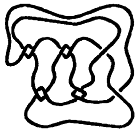

10_{105} 21:20:20

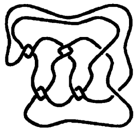

10_{106} 30:2:20

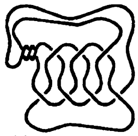

10_{107} 210:2:20

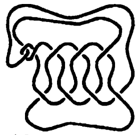

10_{108} 30:20:20

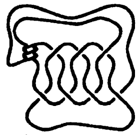

10_{109} 2.2.2.2

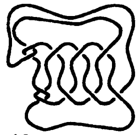

10_{110} 2.2.2.20

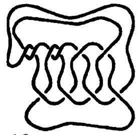

10_{111} 2.2.20.2

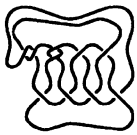

10_{112} 8*3

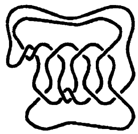

10_{113} 8*21

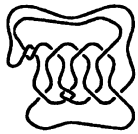

10_{114} 8*30

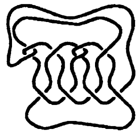

10_{115} 8*20.20

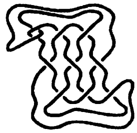

10_{116} 8*2:2

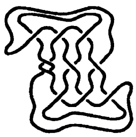

10_{117} 8*2:20

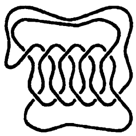

10_{118} 8*2:.2

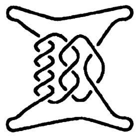

10_{119} 8*2:.20

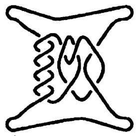

10_{120} 8*20::20

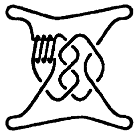

10_{121} 9*20

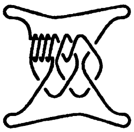

10_{122} 9*.20

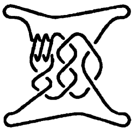

10_{123} 10*

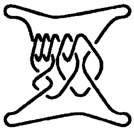

10_{124} 5,3,2–

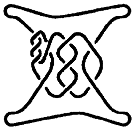

10_{125} 5,21,2–

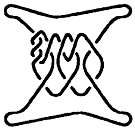

10_{126} 41,3,2–

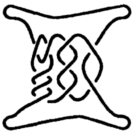

10_{127} 41,21,2−

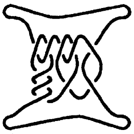

10_{128} 32,3,2−

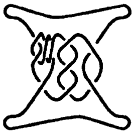

10_{129} 32,21,2−

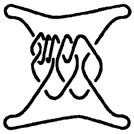

10_{130} 311,3,2−

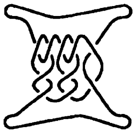

10_{131} 311,21,2−

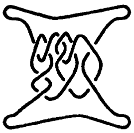

10_{132} 23,3,2−

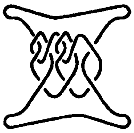

10_{133} 23,21,2−

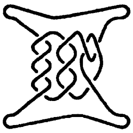

10_{134} 221,3,2−

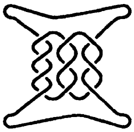

10_{135} 221,21,2−

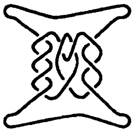

10_{136} 22,22,2−

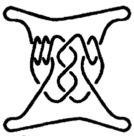

10_{137} 22,211,2−

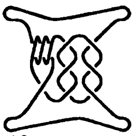

10_{138} 211,211,2−

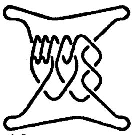

10_{139} 4,3,3−

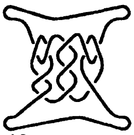

10_{140} 4,3,21−

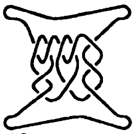

10_{141} 4,21,21−

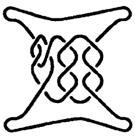

10_{142} 31,3,3−

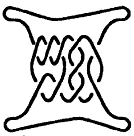

10_{143} 31,3,21−

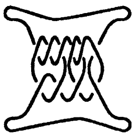

10_{144} 31,21,21−

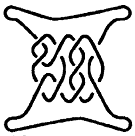

10_{145} 22,3,3−

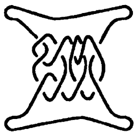

10_{146} 22,21,21−

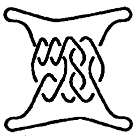

10_{147} 211,3,21−

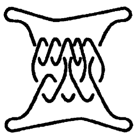

10_{148} (3,2)(3,2−)

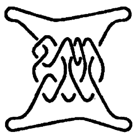

10_{149} (3,2)(21,2−)

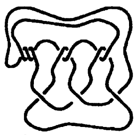

10_{150} (21,2)(3,2−)

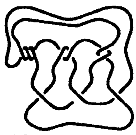

10_{151} (21,2)(21,2−

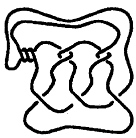

10_{152} (3,2)−(3,2)

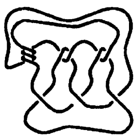

10_{153} (3,2)−(21,2)

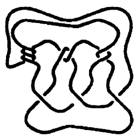

10_{154} (21,2)−(21,2)

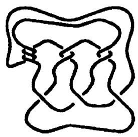

10_{155} −3:2:2

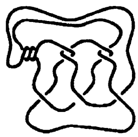

10_{156} −3:2:20

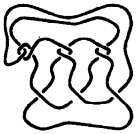

10_{157} −3:20:20


10_{158} −30:2:2

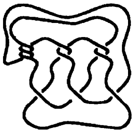

10_{159} −30:2:20


10_{160} −30:20:20

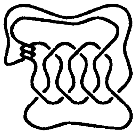

10_{161} 3:−20:−20



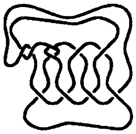
10_{162} 21:−20:−20



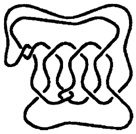
10_{163} −30:−20:−20



10_{164} 8*−30



10_{165} 8*2:−20



10_{166} 8*2:.−20

F.2 Type and symmetry

K	type	S_T	P_C	P_F	S_{YM}
3_1	T $(2,3)$	r	2, 3	(p, 6)=1	D_1
4_1	S $(5,2)$	f	2	—	D_4
5_1	T $(2,5)$	r	2, 5	(p, 10)=1	D_1
5_2	S $(7,3)$	r	2	—	D_2
6_1	S $(9,4)$	r	2	—	D_2
6_2	S $(11,4)$	r	2	—	D_2
6_3	S $(13,5)$	f	2	—	D_4
7_1	T $(2,7)$	r	2, 7	(p, 14)=1	D_1
7_2	S $(11,5)$	r	2	—	D_2
7_3	S $(13,4)$	r	2	—	D_2
7_4	S $(15,4)$	r	2	—	D_4
7_5	S $(17,7)$	r	2	—	D_2
7_6	S $(19,7)$	r	2	—	D_2
7_7	S $(21,8)$	r	2	—	D_4
8_1	S $(13,6)$	r	2	—	D_2
8_2	S $(17,6)$	r	2	—	D_2
8_3	S $(17,4)$	f	2	—	D_4
8_4	S $(19,5)$	r	2	—	D_2
8_5	M $(0;(2,1),(3,1),(3,1))$	r	2	—	D_2
8_6	S $(23,10)$	r	2	—	D_2
8_7	S $(23,9)$	r	2	—	D_2
8_8	S $(25,9)$	r	2	—	D_2
8_9	S $(25,7)$	f	2	—	D_4
8_{10}	M $(0;(2,1),(3,1),(3,2))$	r	—	—	D_1
8_{11}	S $(27,10)$	r	2	—	D_2
8_{12}	S $(29,12)$	f	2	—	D_4
8_{13}	S $(29,11)$	r	2	—	D_2
8_{14}	S $(31,12)$	r	2	—	D_2
8_{15}	M $(0;(2,1),(3,2),(3,2))$	r	2	—	D_2
8_{16}	6**2.20	r	—	—	D_1
8_{17}	6**2.2	i	—	—	D_1
8_{18}	8*	f	2, 4	—	D_8
8_{19}	T $(3,4)$	r	2, 3, 4	(p, 12)=1	D_1
8_{20}	M $(1;(2,1),(3,1),(3,2))$	r	—	—	D_1
8_{21}	M $(1;(2,1),(3,2),(3,2))$	r	2	—	D_2

K	type	S_T	P_C	P_F	S_{YM}
9_1	T $(2,9)$	r	2, 3, 9	(p, 18)=1	D_1
9_2	S $(15,7)$	r	2	—	D_2
9_3	S $(19,6)$	r	2	—	D_2
9_4	S $(21,5)$	r	2	—	D_2
9_5	S $(23,6)$	r	2	—	D_2
9_6	S $(27,5)$	r	2	—	D_2
9_7	S $(29,13)$	r	2	—	D_2
9_8	S $(31,11)$	r	2	—	D_2
9_9	S $(31,9)$	r	2	—	D_2
9_{10}	S $(33,10)$	r	2	—	D_4
9_{11}	S $(33,14)$	r	2	—	D_2
9_{12}	S $(35,13)$	r	2	—	D_2
9_{13}	S $(37,10)$	r	2	—	D_2
9_{14}	S $(37,14)$	r	2	—	D_2
9_{15}	S $(39,16)$	r	2	—	D_2
9_{16}	M $(-1;(2,1),(3,1),(3,1))$	r	2	—	D_2
9_{17}	S $(39,14)$	r	2	—	D_4
9_{18}	S $(41,17)$	r	2	—	D_2
9_{19}	S $(41,16)$	r	2	—	D_2
9_{20}	S $(41,15)$	r	2	—	D_2
9_{21}	S $(43,18)$	r	2	—	D_2
9_{22}	M $(0;(2,1),(3,1),(5,3))$	r	—	—	D_1
9_{23}	S $(45,19)$	r	2	—	D_4
9_{24}	M $(-1;(2,1),(3,1),(3,2))$	r	—	--	D_1
9_{25}	M $(0;(2,1),(3,2),(5,2))$	r	—	—	D_1
9_{26}	S $(47,18)$	r	2	—	D_2
9_{27}	S $(49,19)$	r	2	—	D_2
9_{28}	M $(-1;(2,1),(3,2),(3,2))$	r	2	—	D_2
9_{29}	6**2.20.2	r	—	—	D_1
9_{30}	M $(0;(2,1),(3,2),(5,3))$	r	—	—	D_1
9_{31}	S $(55,21)$	r	2	—	D_4
9_{32}	6**21.20	n	—	—	1
9_{33}	6**21.2	n	—	—	1
9_{34}	8*20	r	—	—	D_1

K	type	S_T	P_C	P_F	S_{YM}
9_{35}	M $(0;(3,1),(3,1),(3,1))$	r	2, 3	- -	D_6
9_{36}	M $(0;(2,1),(3,1),(5,2))$	r	—	—	D_1
9_{37}	M $(0;(3,1),(3,2),(3,2))$	r	2	—	D_2
9_{38}	6**2.2.2	r	—	—	D_1
9_{39}	6*2 : 2 : 20	r	—	—	D_1
9_{40}	9*	r	2, 3	—	D_6
9_{41}	6*20 : 20 : 20	r	3	—	D_3
9_{42}	M $(1;(2,1),(3,1),(5,2))$	r	—	—	D_1
9_{43}	M $(1;(2,1),(3,1),(5,3))$	r	—	—	D_1
9_{44}	M $(1;(2,1),(3,2),(5,2))$	r	—	—	D_1
9_{45}	M $(1;(2,1),(3,2),(5,3))$	r	—	—	D_1
9_{46}	M $(1;(3,1),(3,1),(3,2))$	r	2	—	D_2
9_{47}	$8^* - 20$	r	3	—	D_3
9_{48}	M $(1;(3,2),(3,2),(3,2))$	r	2	3	D_6
9_{49}	$6^* - 20 : -20 : -20$	r	3	—	D_3
10_1	S $(17,8)$	r	2	—	D_2
10_2	S $(23,8)$	r	2	—	D_2
10_3	S $(25,6)$	r	2	—	D_2
10_4	S $(27,7)$	r	2	—	D_2
10_5	S $(33,13)$	r	2	—	D_2
10_6	S $(37,16)$	r	2	—	D_2
10_7	S $(43,16)$	r	2	—	D_2
10_8	S $(29,6)$	r	2	—	D_2
10_9	S $(39,11)$	r	2	—	D_2
10_{10}	S $(45,17)$	r	2	—	D_2
10_{11}	S $(43,13)$	r	2	—	D_2
10_{12}	S $(47,17)$	r	2	—	D_2
10_{13}	S $(53,22)$	r	2	—	D_2
10_{14}	S $(57,22)$	r	2	—	D_2
10_{15}	S $(43,19)$	r	2	—	D_2
10_{16}	S $(47,14)$	r	2	—	D_2
10_{17}	S $(41,9)$	f	2	—	D_4
10_{18}	S $(55,23)$	r	2	—	D_2
10_{19}	S $(51,14)$	r	2	—	D_2
10_{20}	S $(35,16)$	r	2	—	D_2
10_{21}	S $(45,16)$	r	2	—	D_2

K	type	S_T	P_C	P_F	S_{YM}
10_{22}	S $(49,13)$	r	2	—	D_2
10_{23}	S $(59,23)$	r	2	—	D_2
10_{24}	S $(55,24)$	r	2	—	D_2
10_{25}	S $(65,24)$	r	2	—	D_2
10_{26}	S $(61,17)$	r	2	—	D_2
10_{27}	S $(71,27)$	r	2	—	D_2
10_{28}	S $(53,19)$	r	2	—	D_2
10_{29}	S $(63,26)$	r	2	—	D_2
10_{30}	S $(67,26)$	r	2	—	D_2
10_{31}	S $(57,25)$	r	2	—	D_2
10_{32}	S $(69,29)$	r	2	—	D_2
10_{33}	S $(65,18)$	f	2	—	D_4
10_{34}	S $(37,13)$	r	2	—	D_2
10_{35}	S $(49,20)$	r	2	—	D_2
10_{36}	S $(51,20)$	r	2	—	D_2
10_{37}	S $(53,23)$	f	2	—	D_4
10_{38}	S $(59,25)$	r	2	—	D_2
10_{39}	S $(61,22)$	r	2	—	D_2
10_{40}	S $(75,29)$	r	2	—	D_2
10_{41}	S $(71,26)$	r	2	—	D_2
10_{42}	S $(81,31)$	r	2	—	D_2
10_{43}	S $(73,27)$	f	2	—	D_4
10_{44}	S $(79,30)$	r	2	—	D_2
10_{45}	S $(89,34)$	f	2	—	D_4
10_{46}	M $(0;(2,1),(3,1),(5,1))$	r	—	—	D_1
10_{47}	M $(0;(2,1),(3,2),(5,1))$	r	—	—	D_1
10_{48}	M $(0;(2,1),(3,1),(5,4))$	r	—	—	D_1
10_{49}	M $(0;(2,1),(3,2),(5,4))$	r	—	—	D_1
10_{50}	M $(0;(2,1),(3,1),(7,3))$	r	—	—	D_1
10_{51}	M $(0;(2,1),(3,2),(7,3))$	r	—	—	D_1
10_{52}	M $(0;(2,1),(3,1),(7,4))$	r	—	—	D_1
10_{53}	M $(0;(2,1),(3,2),(7,4))$	r	—	—	D_1
10_{54}	M $(0;(2,1),(3,1),(7,2))$	r	—	—	D_1
10_{55}	M $(0;(2,1),(3,2),(7,2))$	r	—	—	D_1
10_{56}	M $(0;(2,1),(3,1),(7,5))$	r	—	—	D_1
10_{57}	M $(0;(2,1),(3,2),(7,5))$	r	—	—	D_1

K	t y p e	S_T	P_C	P_F	S_{YM}
10_{58}	M $(0;(2,1),(5,2),(5,2))$	r	2	—	D_2
10_{59}	M $(0;(2,1),(5,2),(5,3))$	r	—	—	D_1
10_{60}	M $(0;(2,1),(5,3),(5,3))$	r	2	—	D_2
10_{61}	M $(0;(3,1),(3,1),(4,1))$	r	2	—	D_2
10_{62}	M $(0;(3,1),(3,2),(4,1))$	r	—	—	D_1
10_{63}	M $(0;(3,2),(3,2),(4,1))$	r	2	—	D_2
10_{64}	M $(0;(3,1),(3,1),(4,3))$	r	2	—	D_2
10_{65}	M $(0;(3,1),(3,2),(4,3))$	r	—	—	D_1
10_{66}	M $(0;(3,2),(3,2),(4,3))$	r	2	—	D_2
10_{67}	M $(0;(3,1),(3,2),(5,2))$	n	2	—	Z_2
10_{68}	M $(0;(3,1),(3,1),(5,3))$	r	2	—	D_2
10_{69}	M $(0;(3,2),(3,2),(5,3))$	r	2	—	D_2
10_{70}	M $(-1;(2,1),(3,1),(5,2))$	r	—	—	D_1
10_{71}	M $(-1;(2,1),(3,2),(5,2))$	r	—	—	D_1
10_{72}	M $(-1;(2,1),(3,1),(5,3))$	r	—	—	D_1
10_{73}	M $(-1;(2,1),(3,2),(5,3))$	r	—	—	D_1
10_{74}	M $(-1;(3,1),(3,1),(3,2))$	r	2	—	D_2
10_{75}	M $(-1;(3,2),(3,2),(3,2))$	r	2	3	D_6
10_{76}	M $(-2;(2,1),(3,1),(3,1))$	r	2	—	D_2
10_{77}	M $(-2;(2,1),(3,1),(3,2))$	r	—	—	D_1
10_{78}	M $(-2;(2,1),(3,2),(3,2))$	r	2	—	D_2
10_{79}	$1^*(3,2)(3,2)$	i	—	—	D_1
10_{80}	$1^*(3,2)(21,2)$	n	—	—	1
10_{81}	$1^*(21,2)(21,2)$	i	—	—	D_1
10_{82}	6**4.2	n	—	—	1
10_{83}	6**31.2	n	—	—	1
10_{84}	6**22.2	n	—	—	1
10_{85}	6**4.20	n	—	—	1
10_{86}	6**31.2	n	—	—	1
10_{87}	6**22.20	n	—	—	1
10_{88}	6**21.21	i	—	—	D_1
10_{89}	6**21.210	r	—	—	D_1
10_{90}	6**3.2.2	n	—	—	1
10_{91}	6**3.2.20	n	—	—	1
10_{92}	6**21.2.20	n	—	—	1
10_{93}	6**3.20.2	n	—	—	1

K	type	S_T	P_C	P_F	S_{YM}
10_{94}	6**30.2.2	n	—	—	1
10_{95}	6**210.2.2	n	—	—	1
10_{96}	6**2.21.2	r	—	—	D_1
10_{97}	6**2.210.2	r	—	—	D_1
10_{98}	6**2.2.2.20	n	2	—	Z_2
10_{99}	6**2.2.20.20	f	—	—	D_2
10_{100}	6*3 : 2 : 2	r	—	—	D_1
10_{101}	6*21 : 2 : 2	r	—	—	D_1
10_{102}	6*3 : 2 : 20	n	—	—	1
10_{103}	6*30 : 2 : 2	r	—	—	D_1
10_{104}	6*3 : 20 : 20	r	—	—	D_1
10_{105}	6*21 : 20 : 20	r	—	—	D_1
10_{106}	6*30 : 2 : 20	n	—	—	1
10_{107}	6*210 : 2 : 20	n	—	—	1
10_{108}	6*30 : 20 : 20	r	—	—	D_1
10_{109}	6*2.2.2.2	i	—	—	D_1
10_{110}	6*2.2.2.20	n	—	—	1
10_{111}	6*2.2.20.2	r	—	—	D_1
10_{112}	8*3	r	—	—	D_1
10_{113}	8*21	r	—	—	D_1
10_{114}	8*30	r	—	—	D_1
10_{115}	8*20.20	i	—	—	D_1
10_{116}	8*2 : 2	r	—	—	D_1
10_{117}	8*2 : 20	n	—	—	1
10_{118}	8*2 : .2	i	—	—	D_1
10_{119}	8*2 : .20	n	—	—	1
10_{120}	8*20 :: 20	r	2	—	D_2
10_{121}	9*20	r	—	—	D_1
10_{122}	9*.20	r	2	—	D_2
10_{123}	10*	f	5	—	D_{10}
10_{124}	T $(3,5)$	r	3, 5	(p, 15)=1	D_1
10_{125}	M $(1;(2,1),(3,2),(5,1))$	r	—	—	D_1
10_{126}	M $(1;(2,1),(3,1),(5,4))$	r	—	—	D_1
10_{127}	M $(1;(2,1),(3,2),(5,4))$	r	—	—	D_1
10_{128}	M $(1;(2,1),(3,1),(7,3))$	r	—	—	D_1
10_{129}	M $(1;(2,1),(3,2),(7,3))$	r	—	—	D_1

K	type	S_T	P_C	P_F	S_{YM}
10_{130}	M $(1;(2,1),(3,1),(7,4))$	r	—	—	D_1
10_{131}	M $(1;(2,1),(3,2),(7,4))$	r	—	—	D_1
10_{132}	M $(1;(2,1),(3,1),(7,2))$	r	—	—	D_1
10_{133}	M $(1;(2,1),(3,2),(7,2))$	r	—	—	D_1
10_{134}	M $(1;(2,1),(3,1),(7,5))$	r	—	—	D_1
10_{135}	M $(1;(2,1),(3,2),(7,5))$	r	—	—	D_1
10_{136}	M $(1;(2,1),(5,2),(5,2))$	r	2	—	D_2
10_{137}	M $(1;(2,1),(5,2),(5,3))$	r	—	—	D_1
10_{138}	M $(1;(2,1),(5,3),(5,3))$	r	2	—	D_2
10_{139}	M $(1;(3,1),(3,1),(4,1))$	r	2	—	D_2
10_{140}	M $(1;(3,1),(3,2),(4,1))$	r	—	—	D_1
10_{141}	M $(1;(3,2),(3,2),(4,1))$	r	2	—	D_2
10_{142}	M $(1;(3,1),(3,1),(4,3))$	r	2	—	D_2
10_{143}	M $(1;(3,1),(3,2),(4,3))$	r	—	—	D_1
10_{144}	M $(1;(3,2),(3,2),(4,3))$	r	2	—	D_2
10_{145}	M $(1;(3,1),(3,1),(5,2))$	r	2	—	D_2
10_{146}	M $(1;(3,2),(3,2),(5,2))$	r	2	—	D_2
10_{147}	M $(1;(3,1),(3,2),(5,3))$	n	2	—	Z_2
10_{148}	$1^*(3,2)(3,2-)$	n	—	—	1
10_{149}	$1^*(3,2)(21,2-)$	n	—	—	1
10_{150}	$1^*(21,2)(3,2-)$	n	—	—	1
10_{151}	$1^*(21,2)(21,2-)$	n	—	—	1
10_{152}	$1^*(3,2)-(3,2)$	r	—	—	D_1
10_{153}	$1^*(3,2)-(21,2)$	n	—	—	1
10_{154}	$1^*(21,2)-(21,2)$	r	—	—	D_1
10_{155}	$6^*-3:2:2$	r	—	2	D_2
10_{156}	$6^*-3:2:20$	r	—	—	D_1
10_{157}	$6^*-3:20:20$	r	—	2, 4	D_4
10_{158}	$6^*-30:2:2$	r	—	—	D_1
10_{159}	$6^*-30:2:20$	r	—	—	D_1
10_{160}	$6^*-30:20:20$	r	—	—	D_1
10_{161}	$6^*3:-20:-20$	r	—	—	D_1
10_{162}	$6^*21:-20:-20$	r	—	—	D_1
10_{163}	$6^*-30:-20:-20$	r	—	—	D_1
10_{164}	8^*-30	r	—	—	D_1
10_{165}	$8^*2:-20$	r	—	—	D_1
10_{166}	$8^*2:.-20$	r	—	—	D_1

F.3 Knot invariants

K	$A-polynomial$	m	σ'	u	g^*	a	o	t	Δ
3_1	$[1-1$	1	1	1	1	∞	∞	1	1
4_1	$[3-1$	1	0	1	1	2	2	1	1
5_1	$[1-1+1$	1	2	2	2	∞	∞	1	3
5_2	$[3-2$	1	1	1	1	∞	∞	1	2
6_1	$[5-2$	1	0	1	0	1	1	1	2
6_2	$[3-3+1$	1	1	1	1	∞	∞	1	1
6_3	$[5-3+1$	1	0	1	1	2	2	1	1
7_1	$[1-1+1-1$	1	3	3	3	∞	∞	1	6
7_2	$[5-3$	1	1	1	1	∞	∞	1	3
7_3	$[3-3+2$	1	2	2	2	∞	∞	1	5
7_4	$[7-4$	1	1	2	1	∞	∞	1	4
7_5	$[5-4+2$	1	2	2	2	∞	∞	1	4
7_6	$[7-5+1$	1	1	1	1	∞	∞	1	1
7_7	$[9-5+1$	1	0	1	1	4	?	1	1
8_1	$[7-3$	1	0	1	1	2	?	1	3
8_2	$[3-3+3-1$	1	2	2	2	∞	∞	1	2
8_3	$[9-4$	1	0	2	1	2	2	1	4
8_4	$[5-5+2$	1	1	2	1	∞	∞	1	3
8_5	$[5-4+3-1$	1	2	2	2	∞	∞	1	3
8_6	$[7-6+2$	1	1	2	1	∞	∞	1	2
8_7	$[5-5+3-1$	1	1	1	1	∞	∞	1	2
8_8	$[9-6+2$	1	0	2	0	1	1	1	2
8_9	$[7-5+3-1$	1	0	1	0	1	1	1	2
8_{10}	$[7-6+3-1$	1	1	A	1	∞	∞	1	3
8_{11}	$[9-7+2$	1	1	1	1	∞	∞	1	1
8_{12}	$[13-7+1$	1	0	2	1	2	2	1	3
8_{13}	$[11-7+2$	1	0	1	1	2	?	1	1
8_{14}	$[11-8+2$	1	1	1	1	∞	∞	1	2
8_{15}	$[11-8+3$	1	2	2	2	∞	∞	1	4
8_{16}	$[9-8+4-1$	1	1	A	A	∞	∞	2	1
8_{17}	$[11-8+4-1$	1	0	1	1	2	2	1	1
8_{18}	$[13-10+5-1$	2	0	2	A	2	2	2	1
8_{19}	$[1+0-1+1$	1	3	3	3	∞	∞	1	5
8_{20}	$[3-2+1$	1	0	1	0	1	1	1	2
8_{21}	$[5-4+1$	1	1	1	1	∞	∞	1	2
9_1	$[1-1+1-1+1$	1	4	4	4	∞	∞	1	10

K	$A-polynomial$	m	σ'	u	g^*	a	o	t	Δ
9_2	$[7-4$	1	1	1	1	∞	∞	1	4
9_3	$[3-3+3-2$	1	3	3	3	∞	∞	1	9
9_4	$[5-5+3$	1	2	2	2	∞	∞	1	7
9_5	$[11-6$	1	1	2	1	∞	∞	1	6
9_6	$[5-5+4-2$	1	3	3	3	∞	∞	1	7
9_7	$[9-7+3$	1	2	2	2	∞	∞	1	5
9_8	$[11-8+2$	1	1	2	1	∞	∞	1	2
9_9	$[7-6+4-2$	1	3	3	3	∞	∞	1	8
9_{10}	$[9-8+4$	1	2	X	2	∞	∞	1	8
9_{11}	$[7-7+5-1$	1	2	2	2	∞	∞	1	4
9_{12}	$[13-9+2$	1	1	1	1	∞	∞	1	1
9_{13}	$[11-9+4$	1	2	X	2	∞	∞	1	7
9_{14}	$[15-9+2$	1	0	1	1	2	?	1	1
9_{15}	$[15-10+2$	1	1	2	1	∞	∞	1	2
9_{16}	$[9-8+5-2$	1	3	3	3	∞	∞	1	6
9_{17}	$[9-9+5-1$	1	1	A	A	∞	∞	1	2
9_{18}	$[13-10+4$	1	2	2	2	∞	∞	1	6
9_{19}	$[17-10+2$	1	0	1	1	2	?	1	2
9_{20}	$[11-9+5-1$	1	2	2	2	∞	∞	1	2
9_{21}	$[17-11+2$	1	1	1	1	∞	∞	1	3
9_{22}	$[11-10+5-1$	1	1	1	1	∞	∞	1	1
9_{23}	$[15-11+4$	1	2	2	2	∞	∞	1	5
9_{24}	$[13-10+5-1$	1	0	1	1	2	2	1	1
9_{25}	$[17-12+3$	1	1	2	1	∞	∞	1	2
9_{26}	$[13-11+5-1$	1	1	1	1	∞	∞	1	2
9_{27}	$[15-11+5-1$	1	0	1	0	1	1	1	2
9_{28}	$[15-12+5-1$	1	1	1	1	∞	∞	1	1
9_{29}	$[15-12+5-1$	1	1	A	1	∞	∞	2	1
9_{30}	$[17-12+5-1$	1	0	1	1	2	?	1	1
9_{31}	$[17-13+5-1$	1	1	2	A	∞	∞	1	2
9_{32}	$[17-14+6-1$	1	1	A	A	∞	∞	2	1
9_{33}	$[19-14+6-1$	1	0	1	1	2	?	1	1
9_{34}	$[23-16+6-1$	1	0	1	1	4	?	1	1
9_{35}	$[13-7$	2	1	X	1	∞	∞	2	7
9_{36}	$[9-8+5-1$	1	2	2	2	∞	∞	1	3
9_{37}	$[19-11+2$	2	0	2	1	2	2	2	3

K	$A - polynomial$	m	σ'	u	g^*	a	o	t	Δ
9_{38}	$[19 - 14 + 5$	A	2	X	2	∞	∞	1	6
9_{39}	$[21 - 14 + 3$	1	1	1	1	∞	∞	1	2
9_{40}	$[23 - 18 + 7 - 1$	2	1	2	A	∞	∞	1	1
9_{41}	$[19 - 12 + 3$	2	0	2	0	1	1	2	2
9_{42}	$[1 - 2 + 1$	1	1	1	1	∞	∞	1	2
9_{43}	$[1 - 2 + 3 - 1$	1	2	2	2	∞	∞	1	3
9_{44}	$[7 - 4 + 1$	1	0	1	1	2	?	1	2
9_{45}	$[9 - 6 + 1$	1	1	1	1	∞	∞	1	2
9_{46}	$[5 - 2$	2	0	2	0	1	1	2	2
9_{47}	$[5 - 6 + 4 - 1$	2	1	2	A	∞	∞	1	1
9_{48}	$[11 - 7 + 1$	2	1	2	A	∞	∞	1	3
9_{49}	$[7 - 6 + 3$	2	2	X	2	∞	∞	2	6
10_{1}	$[9 - 4$	1	0	1	1	2	?	1	4
10_{2}	$[3 - 3 + 3 - 3 + 1$	1	3	3	3	∞	∞	1	4
10_{3}	$[13 - 6$	1	0	2	0	1	1	1	6
10_{4}	$[7 - 7 + 3$	1	1	2	1	∞	∞	1	5
10_{5}	$[5 - 5 + 5 - 3 + 1$	1	2	2	2	∞	∞	1	4
10_{6}	$[7 - 7 + 6 - 2$	1	2	3	2	∞	∞	1	3
10_{7}	$[15 - 11 + 3$	1	1	1	1	∞	∞	1	1
10_{8}	$[5 - 5 + 5 - 2$	1	2	2	2	∞	∞	1	3
10_{9}	$[7 - 7 + 5 - 3 + 1$	1	1	1	1	∞	∞	1	2
10_{10}	$[17 - 11 + 3$	1	0	1	1	2	?	1	1
10_{11}	$[13 - 11 + 4$	1	1	X	1	∞	∞	1	5
10_{12}	$[11 - 10 + 6 - 2$	1	1	2	1	∞	∞	1	4
10_{13}	$[23 - 13 + 2$	1	0	2	1	2	?	1	5
10_{14}	$[13 - 12 + 8 - 2$	1	2	2	2	∞	∞	1	M
10_{15}	$[9 - 9 + 6 - 2$	1	1	2	1	∞	∞	1	3
10_{16}	$[15 - 12 + 4$	1	1	2	1	∞	∞	1	4
10_{17}	$[9 - 7 + 5 - 3 + 1$	1	0	1	1	2	2	1	2
10_{18}	$[19 - 14 + 4$	1	1	1	1	∞	∞	1	2
10_{19}	$[11 - 11 + 7 - 2$	1	1	2	1	∞	∞	1	1
10_{20}	$[11 - 9 + 3$	1	1	2	1	∞	∞	1	3
10_{21}	$[9 - 9 + 7 - 2$	1	2	2	2	∞	∞	1	3
10_{22}	$[13 - 10 + 6 - 2$	1	0	2	0	1	1	1	4
10_{23}	$[15 - 13 + 7 - 2$	1	1	1	1	∞	∞	1	3
10_{24}	$[19 - 14 + 4$	1	1	2	1	∞	∞	1	2

K	$A-polynomial$	m	σ'	u	g^*	a	o	t	Δ
10_{25}	[17 − 14 + 8 − 2	1	2	2	2	∞	∞	1	2
10_{26}	[17 − 13 + 7 − 2	1	0	1	1	2	?	1	3
10_{27}	[19 − 16 + 8 − 2	1	1	1	1	∞	∞	1	2
10_{28}	[19 − 13 + 4	1	0	2	1	2	?	1	3
10_{29}	[17 − 15 + 7 − 1	1	1	2	1	∞	∞	1	4
10_{30}	[25 − 17 + 4	1	1	1	1	∞	∞	1	N
10_{31}	[21 − 14 + 4	1	0	1	1	4	?	1	2
10_{32}	[19 − 15 + 8 − 2	1	0	1	1	∞	∞	1	1
10_{33}	[25 − 16 + 4	1	0	1	1	2	2	1	2
10_{34}	[13 − 9 + 3	1	0	2	1	2	?	1	3
10_{35}	[21 − 12 + 2	1	0	2	0	1	1	1	4
10_{36}	[19 − 13 + 3	1	1	2	1	∞	∞	1	N
10_{37}	[19 − 13 + 4	1	0	2	1	2	2	1	3
10_{38}	[21 − 15 + 4	1	1	2	1	∞	∞	1	N
10_{39}	[15 − 13 + 8 − 2	1	2	2	2	∞	∞	1	3
10_{40}	[21 − 17 + 8 − 2	1	1	2	1	∞	∞	1	3
10_{41}	[21 − 17 + 7 − 1	1	1	2	1	∞	∞	1	2
10_{42}	[27 − 19 + 7 − 1	1	0	1	0	1	1	1	2
10_{43}	[23 − 17 + 7 − 1	1	0	2	1	2	2	1	2
10_{44}	[25 − 19 + 7 − 1	1	1	1	1	∞	∞	1	2
10_{45}	[31 − 21 + 7 − 1	1	0	2	1	2	2	1	2
10_{46}	[5 − 5 + 4 − 3 + 1	1	3	3	3	∞	∞	1	4
10_{47}	[7 − 7 + 6 − 3 + 1	1	2	X	2	∞	∞	1	6
10_{48}	[11 − 9 + 6 − 3 + 1	1	0	A	0	1	1	1	4
10_{49}	[13 − 12 + 8 − 3	1	3	3	3	∞	∞	1	7
10_{50}	[13 − 11 + 7 − 2	1	2	2	2	∞	∞	1	3
10_{51}	[19 − 15 + 7 − 2	1	1	B	A	∞	∞	1	5
10_{52}	[15 − 13 + 7 − 2	1	1	A	1	∞	∞	1	3
10_{53}	[25 − 18 + 6	1	2	X	2	∞	∞	1	6
10_{54}	[11 − 10 + 6 − 2	1	1	B	A	∞	∞	1	4
10_{55}	[21 − 15 + 5	1	2	2	2	∞	∞	1	5
10_{56}	[17 − 14 + 8 − 2	1	2	2	2	∞	∞	1	M
10_{57}	[23 − 18 + 8 − 2	1	1	A	1	∞	∞	1	4
10_{58}	[27 − 16 + 3	1	0	A	1	2	?	1	4
10_{59}	[23 − 18 + 7 − 1	1	1	1	1	∞	∞	1	1
10_{60}	[29 − 20 + 7 − 1	1	0	1	1	2	?	1	1

K	$A-polynomial$	m	σ'	u	g^*	a	o	t	Δ
10_{61}	$[7-6+5-2$	2	2	X	2	∞	∞	2	4
10_{62}	$[9-8+6-3+1$	1	2	2	2	∞	∞	1	5
10_{63}	$[19-14+5$	2	2	2	2	∞	∞	1	6
10_{64}	$[11-10+6-3+1$	1	1	A	1	∞	∞	1	3
10_{65}	$[17-14+7-2$	1	1	2	1	∞	∞	1	4
10_{66}	$[19-16+9-3$	1	3	3	3	∞	∞	1	7
10_{67}	$[23-16+4$	1	1	A	1	∞	∞	1	2
10_{68}	$[21-14+4$	1	0	A	1	4	?	1	2
10_{69}	$[29-21+7-1$	A	1	2	1	∞	∞	2	2
10_{70}	$[19-16+7-1$	1	1	A	A	∞	∞	1	3
10_{71}	$[25-18+7-1$	1	0	1	1	4	?	1	1
10_{72}	$[19-16+9-2$	1	2	2	2	∞	∞	1	M
10_{73}	$[27-20+7-1$	1	1	1	1	∞	∞	1	1
10_{74}	$[23-16+4$	2	1	2	1	∞	∞	2	2
10_{75}	$[27-19+7-1$	2	0	2	0	1	1	2	2
10_{76}	$[15-12+7-2$	1	2	X	2	∞	∞	1	M
10_{77}	$[17-14+7-2$	1	1	B	1	∞	∞	1	4
10_{78}	$[21-16+7-1$	1	2	2	2	∞	∞	1	3
10_{79}	$[15-12+7-3+1$	1	0	B	1	2	2	2	5
10_{80}	$[17-15+9-3$	1	3	3	3	∞	∞	1	6
10_{81}	$[27-20+8-1$	1	0	A	1	2	2	1	3
10_{82}	$[13-12+8-4+1$	1	1	1	1	∞	∞	1	2
10_{83}	$[25-19+9-2$	1	1	A	1	∞	∞	1	1
10_{84}	$[25-20+9-2$	1	1	1	1	∞	∞	1	2
10_{85}	$[11-10+8-4+1$	1	2	2	2	∞	∞	1	2
10_{86}	$[23-19+9-2$	1	0	A	1	4	?	1	1
10_{87}	$[23-18+9-2$	1	0	A	0	1	1	1	2
10_{88}	$[35-24+8-1$	1	0	1	1	2	2	1	1
10_{89}	$[33-24+8-1$	1	1	2	1	∞	∞	1	1
10_{90}	$[23-17+8-2$	1	0	A	1	4	?	1	3
10_{91}	$[17-14+9-4+1$	1	0	1	1	2	?	1	2
10_{92}	$[25-20+10-2$	1	2	2	2	∞	∞	1	M
10_{93}	$[17-15+8-2$	1	1	A	1	∞	∞	1	N
10_{94}	$[15-14+9-4+1$	1	1	A	1	∞	∞	1	2
10_{95}	$[27-21+9-2$	1	1	1	1	∞	∞	1	3
10_{96}	$[33-22+7-1$	1	0	A	1	4	?	1	3

K	$A-polynomial$	m	σ'	u	g^*	a	o	t	Δ
10_{97}	$[33-22+5$	1	1	2	A	∞	∞	1	M
10_{98}	$[23-18+9-2$	2	2	2	2	∞	∞	1	M
10_{99}	$[19-16+10-4+1$	2	0	2	0	1	1	1	4
10_{100}	$[13-12+9-4+1$	1	2	X	2	∞	∞	1	4
10_{101}	$[29-21+7$	A	2	X	2	∞	∞	1	7
10_{102}	$[21-16+8-2$	1	0	1	1	2	?	1	2
10_{103}	$[21-17+8-2$	2	1	X	1	∞	∞	1	3
10_{104}	$[19-15+9-4+1$	1	0	1	1	4	?	1	1
10_{105}	$[29-22+8-1$	1	1	A	1	∞	∞	1	1
10_{106}	$[17-15+9-4+1$	1	1	A	1	∞	∞	1	1
10_{107}	$[31-22+8-1$	1	0	1	1	4	?	1	1
10_{108}	$[15-14+8-2$	1	1	2	1	∞	∞	1	2
10_{109}	$[21-17+10-4+1$	1	0	A	1	2	2	1	3
10_{110}	$[25-20+8-1$	1	1	A	1	∞	∞	1	3
10_{111}	$[21-17+9-2$	1	2	2	2	∞	∞	1	3
10_{112}	$[19-17+11-5+1$	1	1	A	1	∞	∞	1	2
10_{113}	$[33-26+11-2$	1	1	1	1	∞	∞	1	2
10_{114}	$[27-21+10-2$	1	0	1	1	∞	∞	1	1
10_{115}	$[37-26+9-1$	1	0	2	1	2	2	1	1
10_{116}	$[21-19+12-5+1$	1	1	A	1	∞	∞	1	2
10_{117}	$[31-24+10-1$	1	1	A	A	∞	∞	1	2
10_{118}	$[23-19+12-5+1$	1	0	1	1	2	2	1	2
10_{119}	$[31-23+10-2$	1	0	1	1	2	?	1	1
10_{120}	$[37-26+8$	1	2	X	2	∞	∞	1	6
10_{121}	$[35-27+11-2$	1	1	A	1	∞	∞	1	1
10_{122}	$[31-24+11-2$	2	0	2	1	∞	∞	1	2
10_{123}	$[29-24+15-6+1$	2	0	2	0	1	1	2	2
10_{124}	$[1-1+0+1-1$	1	4	4	4	∞	∞	1	8
10_{125}	$[1-2+2-1$	1	1	A	1	∞	∞	1	3
10_{126}	$[5-4+2-1$	1	1	A	1	∞	∞	1	5
10_{127}	$[7-6+4-1$	1	2	2	2	∞	∞	1	3
10_{128}	$[1+1-3+2$	1	3	3	3	∞	∞	1	7
10_{129}	$[9-6+2$	1	0	1	0	1	1	1	2
10_{130}	$[5-4+2$	1	0	A	1	∞	∞	1	4
10_{131}	$[11-8+2$	1	1	A	1	∞	∞	1	2
10_{132}	$[1-1+1$	1	0	1	1	∞	∞	1	3

K	$A - polynomial$	m	σ'	u	g^*	a	o	t	Δ
10_{133}	$[7-5+1$	1	1	1	1	∞	∞	1	N
10_{134}	$[3-4+4-2$	1	3	3	3	∞	∞	1	6
10_{135}	$[13-9+3$	1	0	A	1	2	?	1	3
10_{136}	$[5-4+1$	1	1	1	1	∞	∞	1	2
10_{137}	$[11-6+1$	1	0	1	0	1	1	1	2
10_{138}	$[7-8+5-1$	1	1	A	1	∞	∞	1	3
10_{139}	$[3-2+0+1-1$	1	3	Z	Z	∞	∞	1	9
10_{140}	$[3-2+1$	2	0	2	0	1	1	2	2
10_{141}	$[5-4+3-1$	1	0	1	1	∞	∞	1	1
10_{142}	$[1-2+3-2$	2	3	3	3	∞	∞	1	8
10_{143}	$[7-6+3-1$	1	1	1	1	∞	∞	1	3
10_{144}	$[13-10+3$	2	1	2	A	∞	∞	2	2
10_{145}	$[3-1-1$	1	1	A	A	∞	∞	1	5
10_{146}	$[13-8+2$	1	0	1	1	4	?	2	2
10_{147}	$[9-7+2$	1	1	1	1	∞	∞	1	1
10_{148}	$[9-7+3-1$	1	1	A	A	∞	∞	1	4
10_{149}	$[11-9+5-1$	1	2	2	2	∞	∞	1	M
10_{150}	$[7-6+4-1$	1	2	2	2	∞	∞	1	3
10_{151}	$[13-10+4-1$	1	1	A	A	∞	∞	1	3
10_{152}	$[5-4+1+1-1$	1	3	Z	Z	∞	∞	1	7
10_{153}	$[3-1-1+1$	1	0	A	0	1	1	1	4
10_{154}	$[7-4+0+1$	1	2	X	X	∞	∞	1	5
10_{155}	$[7-5+3-1$	2	0	2	0	1	1	1	2
10_{156}	$[9-8+4-1$	1	1	1	1	∞	∞	1	1
10_{157}	$[13-11+6-1$	2	2	2	2	∞	∞	1	4
10_{158}	$[15-10+4-1$	1	0	A	1	2	?	1	3
10_{159}	$[11-9+4-1$	1	1	1	1	∞	∞	1	2
10_{160}	$[3-4+4-1$	2	2	2	2	∞	∞	2	3
10_{161}	$[3-2+0+1$	1	2	X	X	∞	∞	1	7
10_{162}	$[3-2+0+1$	1	2	X	X	∞	∞	1	7
10_{163}	$[11-9+3$	1	1	A	1	∞	∞	2	3
10_{164}	$[15-12+5-1$	1	1	2	1	∞	∞	1	1
10_{165}	$[17-11+3$	1	0	1	1	2	?	1	1
10_{166}	$[15-10+2$	1	1	2	1	∞	∞	2	M

F.4 Presentation matrix

$$8_{18} \quad \begin{pmatrix} t^2-t+1 & 0 \\ 0 & t^4-4t^3+5t^2-4t+1 \end{pmatrix}$$

$$9_{35} \quad \begin{pmatrix} 3t-3 & -2t+1 \\ -t+2 & 3t-3 \end{pmatrix}$$

$$9_{37} \quad \begin{pmatrix} 2t-1 & 0 \\ 0 & t^3-5t^2+7t-2 \end{pmatrix}$$

$$9_{38} \quad \begin{pmatrix} t^2-t+1 & t+1 \\ 0 & 5t^2-9t+5 \end{pmatrix}?$$

$$9_{40} \quad \begin{pmatrix} t^2-3t+1 & 0 \\ 0 & t^4-4t^3+5t^2-4t+1 \end{pmatrix}$$

$$9_{41} \quad \begin{pmatrix} 3t^2-3t+1 & t^2-1 \\ 0 & t^2-3t+3 \end{pmatrix}$$

$$9_{46} \quad \begin{pmatrix} t-2 & 0 \\ 0 & 2t-1 \end{pmatrix}$$

$$9_{47} \quad \begin{pmatrix} t^2-1 & 5t-4 \\ t^2-2t & t^4-4t^3+7t^2-4t-1 \end{pmatrix}$$

$$9_{48} \quad \begin{pmatrix} 3t^2-4t+2 & t^2-t+1 \\ t^2-1 & 2t-1 \end{pmatrix}$$

$$9_{49} \quad \begin{pmatrix} t^2-2t+2 & t^2-1 \\ t^2-1 & -2t^2+2t-1 \end{pmatrix}$$

$$10_{61} \quad \begin{pmatrix} t^2-t+1 & 2 \\ 0 & 2t^4-3t^3+t^2-3t+2 \end{pmatrix}$$

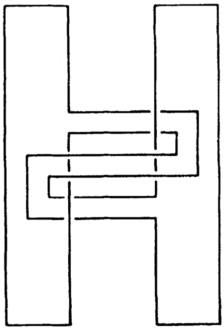
$$10_{63} \quad \begin{pmatrix} t^2-t+1 & 2 \\ 0 & 5t^2-9t+5 \end{pmatrix}$$

$$10_{69} \quad \begin{pmatrix} t^2+t-1 & 3t^3-9t^2+5t-1 \\ 2t^2-2t+1 & t^4-2t^3 \end{pmatrix}?$$

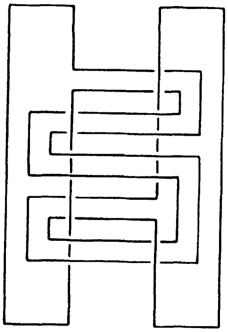
$$10_{74} \quad \begin{pmatrix} t-2 & 0 \\ 0 & 4t^3-8t^2+7t-2 \end{pmatrix}$$

10_{75}	$\begin{pmatrix} t^3-3t^2+4t-1 & 3 \\ 0 & t^3-4t^2+3t-1 \end{pmatrix}$
10_{98}	$\begin{pmatrix} t^2-t+1 & 0 \\ 0 & 2t^4-7t^3+9t^2-7t+2 \end{pmatrix}$
10_{99}	$\begin{pmatrix} t^4-2t^3+3t^2-2t+1 & 0 \\ 0 & t^4-2t^3+3t^2-2t+1 \end{pmatrix}$
10_{101}	$\begin{pmatrix} t^2-t+1 & t-3 \\ -2t+9 & 7t^2-14t+20 \end{pmatrix}?$
10_{103}	$\begin{pmatrix} t^2-2t+2 & 0 \\ 0 & 2t^4-4t^3+5t^2-3t+1 \end{pmatrix}$
10_{115}	$\begin{pmatrix} t^2-t+1 & 2t \\ 2t-2 & t^4-8t^3+17t^2-12t+1 \end{pmatrix}$
10_{122}	$\begin{pmatrix} t^2-3t+1 & 3 \\ 0 & 2t^4-5t^3+7t^2-5t+2 \end{pmatrix}$
10_{123}	$\begin{pmatrix} t^4-3t^3+3t^2-3t+1 & 0 \\ 0 & t^4-3t^3+3t^2-3t+1 \end{pmatrix}$
10_{140}	$\begin{pmatrix} t^2-t+1 & 2 \\ 0 & t^2-t+1 \end{pmatrix}$
10_{142}	$\begin{pmatrix} t^2-t+1 & 2 \\ 0 & 2t^4-t^3-t^2-t+2 \end{pmatrix}$
10_{144}	$\begin{pmatrix} t^2-t+1 & 2 \\ 0 & 3t^2-7t+3 \end{pmatrix}$
10_{155}	$\begin{pmatrix} t^3-2t^2+t-1 & 0 \\ 0 & t^3-t^2+2t-1 \end{pmatrix}$
10_{157}	$\begin{pmatrix} t^3-2t^2+3t-1 & t^4-3t^3+3t^2 \\ t^2-1 & 3t^2-3t+1 \end{pmatrix}$
10_{160}	$\begin{pmatrix} t^3+t^2-1 & t^3+t \\ 3t^2-3t+1 & t^3-2t^2+3t-1 \end{pmatrix}$
10_{164}	$\begin{pmatrix} t^2-t+1 & 0 \\ 0 & t^4-4t^3+7t^2-4t+1 \end{pmatrix}$

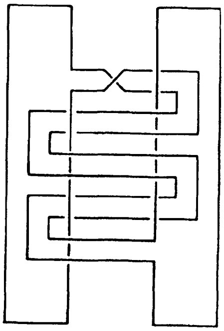
F.5 Ribbon presentation



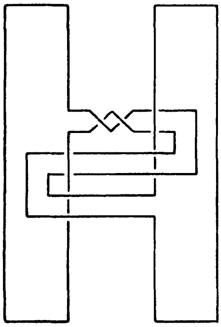
6_1



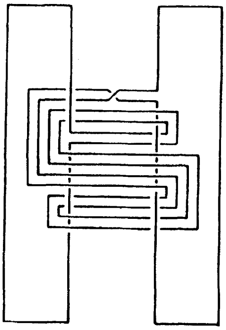
8_8



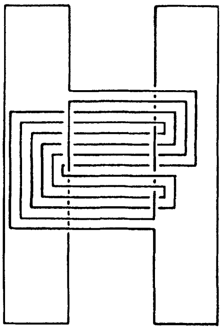
8_9



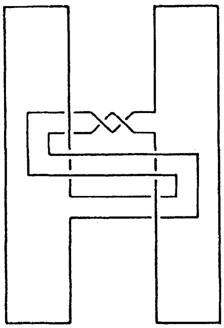
8_{20}



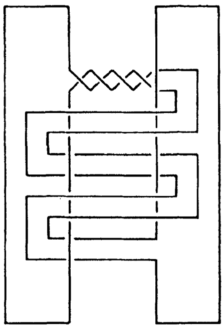
9_{27}



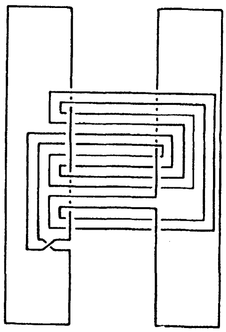
9_{41}



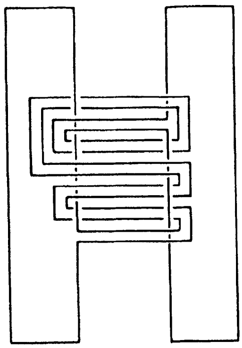
9_{46}



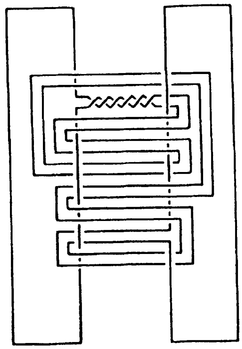
10_3



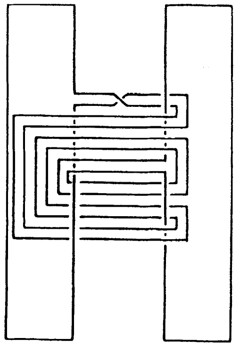
10_{22}



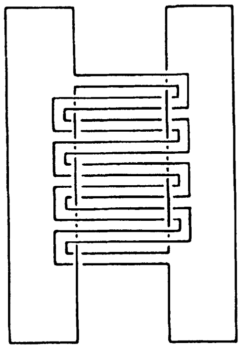
10_{35}



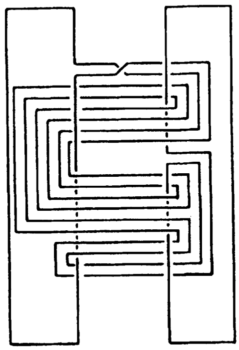
10_{42}



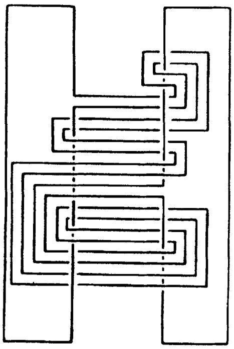
10_{48}



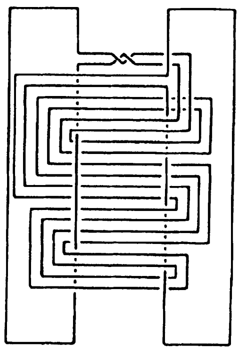
10_{75}



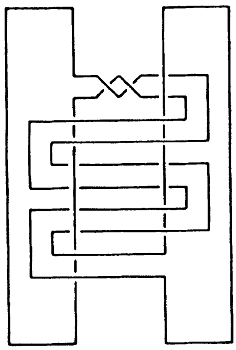
10_{87}



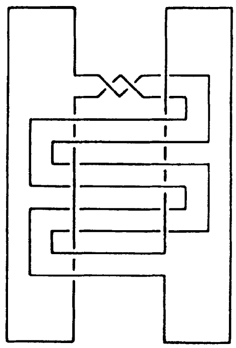
10_{99}



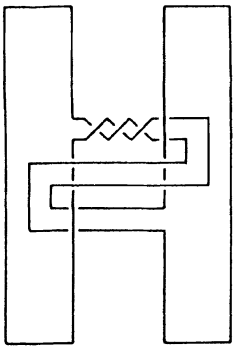
10_{123}



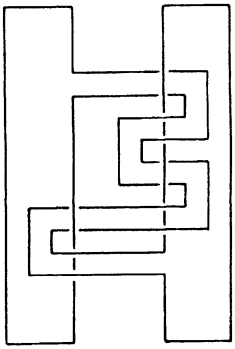
10_{129}



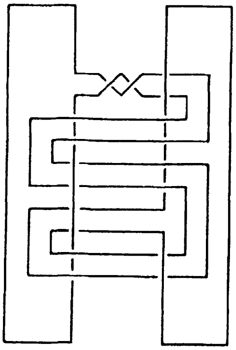
10_{137}



10_{140}



10_{153}



10_{155}

F.6 Skein and Kauffman polynomials

$\mathbf{P}(3_1) = (-a^{-4} + 2a^{-2}) + a^{-2}z^2$
$\mathbf{P}(4_1) = (a^{-2} - 1 + a^2) - z^2$
$\mathbf{P}(5_1) = (-2a^{-6} + 3a^{-4}) + z^2(-a^{-6} + 4a^{-4}) + a^{-4}z^4$
$\mathbf{P}(5_2) = (-a^{-6} + a^{-4} + a^{-2}) + z^2(a^{-4} + a^{-2})$
$\mathbf{P}(6_1) = (a^{-4} - a^{-2} + a^2) + z^2(-a^{-2} - 1)$
$\mathbf{P}(6_2) = (a^{-4} - 2a^{-2} + 2) + z^2(a^{-4} - 3a^{-2} + 1) - a^{-2}z^4$
$\mathbf{P}(6_3) = (-a^{-2} + 3 - a^2) + z^2(-a^{-2} + 3 - a^2) + z^4$
$\mathbf{P}(7_1) = (-3a^{-8} + 4a^{-6}) + z^2(-4a^{-8} + 10a^{-6}) + z^4(-a^{-8} + 6a^{-6}) + a^{-6}z^6$
$\mathbf{P}(7_2) = (-a^{-8} + a^{-6} + a^{-2}) + z^2(a^{-6} + a^{-4} + a^{-2})$
$\mathbf{P}(7_3) = (a^4 + 2a^6 - 2a^8) + z^2(3a^4 + 3a^6 - a^8) + z^4(a^4 + a^6)$
$\mathbf{P}(7_4) = (2a^4 - a^8) + z^2(a^2 + 2a^4 + a^6)$
$\mathbf{P}(7_5) = (-a^{-8} + 2a^{-4}) + z^2(-a^{-8} + 2a^{-6} + 3a^{-4}) + z^4(a^{-6} + a^{-4})$
$\mathbf{P}(7_6) = (-a^{-6} + 2a^{-4} - a^{-2} + 1) + z^2(2a^{-4} - 2a^{-2} + 1) - a^{-2}z^4$
$\mathbf{P}(7_7) = (2 - 2a^2 + a^4) + z^2(-a^{-2} + 2 - 2a^2) + z^4$
$\mathbf{P}(8_1) = (a^{-6} - a^{-4} + a^2) + z^2(-a^{-4} - a^{-2} - 1)$
$\mathbf{P}(8_2) = (a^{-6} - 3a^{-4} + 3a^{-2}) + z^2(3a^{-6} - 7a^{-4} + 4a^{-2}) + z^4(a^{-6} - 5a^{-4} + a^{-2}) - a^{-4}z^6$
$\mathbf{P}(8_3) = (a^{-4} - 1 + a^4) + z^2(-a^{-2} - 2 - a^2)$
$\mathbf{P}(8_4) = (a^{-4} - 2 + 2a^2) + z^2(a^{-4} - 2a^{-2} - 3 + a^2) + z^4(-a^{-2} - 1)$
$\mathbf{P}(8_5) = (4a^2 - 5a^4 + 2a^6) + z^2(4a^2 - 8a^4 + 3a^6) + z^4(a^2 - 5a^4 + a^6) - a^4z^6$
$\mathbf{P}(8_6) = (a^{-6} - a^{-4} - a^{-2} + 2) + z^2(a^{-6} - 2a^{-4} - 2a^{-2} + 1) + z^4(-a^{-4} - a^{-2})$
$\mathbf{P}(8_7) = (-1 + 4a^2 - 2a^4) + z^2(-3 + 8a^2 - 3a^4) + z^4(-1 + 5a^2 - a^4) + a^2z^6$
$\mathbf{P}(8_8) = (-a^{-2} + 2 + a^2 - a^4) + z^2(-a^{-2} + 2 + 2a^2 - a^4) + z^4(1 + a^2)$
$\mathbf{P}(8_9) = (2a^{-2} - 3 + 2a^2) + z^2(3a^{-2} - 8 + 3a^2) + z^4(a^{-2} - 5 + a^2) - z^6$
$\mathbf{P}(8_{10}) = (-2 + 6a^2 - 3a^4) + z^2(-3 + 9a^2 - 3a^4) + z^4(-1 + 5a^2 - a^4) + a^2z^6$
$\mathbf{P}(8_{11}) = (a^{-6} - 2a^{-4} + a^{-2} + 1) + z^2(a^{-6} - 2a^{-4} - a^{-2} + 1) + z^4(-a^{-4} - a^{-2})$
$\mathbf{P}(8_{12}) = (a^{-4} - a^{-2} + 1 - a^2 + a^4) + z^2(-2a^{-2} + 1 - 2a^2) + z^4$
$\mathbf{P}(8_{13}) = (2a^2 - a^4) + z^2(-a^{-2} + 1 + 2a^2 - a^4) + z^4(1 + a^2)$
$\mathbf{P}(8_{14}) = 1 + z^2(a^{-6} - a^{-4} - a^{-2} + 1) + z^4(-a^{-4} - a^{-2})$
$\mathbf{P}(8_{15}) = (a^{-10} - 4a^{-8} + 3a^{-6} + a^{-4}) + z^2(-3a^{-8} + 5a^{-6} + 2a^{-4}) + z^4(2a^{-6} + a^{-4})$
$\mathbf{P}(8_{16}) = (-a^{-4} + 2a^{-2}) + z^2(-2a^{-4} + 5a^{-2} - 2) + z^4(-a^{-4} + 4a^{-2} - 1) + a^{-2}z^6$
$\mathbf{P}(8_{17}) = (a^{-2} - 1 + a^2) + z^2(2a^{-2} - 5 + 2a^2) + z^4(a^{-2} - 4 + a^2) - z^6$
$\mathbf{P}(8_{18}) = (-a^{-2} + 3 - a^2) + z^2(a^{-2} - 1 + a^2) + z^4(a^{-2} - 3 + a^2) - z^6$
$\mathbf{P}(8_{19}) = (5a^6 - 5a^8 + a^{10}) + z^2(10a^6 - 5a^8) + z^4(6a^6 - a^8) + a^6z^6$
$\mathbf{P}(8_{20}) = (-2a^{-4} + 4a^{-2} - 1) + z^2(-a^{-4} + 4a^{-2} - 1) + a^{-2}z^4$
$\mathbf{P}(8_{21}) = (a^{-6} - 3a^{-4} + 3a^{-2}) + z^2(a^{-6} - 3a^{-4} + 2a^{-2}) - a^{-4}z^4$
$\mathbf{P}(9_1) = (-4a^{-10} + 5a^{-8}) + z^2(-10a^{-10} + 20a^{-8}) + z^4(-6a^{-10} + 21a^{-8}) + z^6(-a^{-10} + 8a^{-8}) + a^{-8}z^8$
$\mathbf{P}(9_2) = (-a^{-10} + a^{-8} + a^{-2}) + z^2(a^{-8} + a^{-6} + a^{-4} + a^{-2})$
$\mathbf{P}(9_3) = (a^6 + 3a^8 - 3a^{10}) + z^2(6a^6 + 7a^8 - 4a^{10}) + z^4(5a^6 + 5a^8 - a^{10}) + z^6(a^6 + a^8)$
$\mathbf{P}(9_4) = (-2a^{-10} + 2a^{-8} + a^{-4}) + z^2(-a^{-10} + 3a^{-8} + 2a^{-6} + 3a^{-4}) + z^4(a^{-8} + a^{-6} + a^{-4})$
$\mathbf{P}(9_5) = (a^4 + a^6 - a^{10}) + z^2(a^2 + 2a^4 + 2a^6 + a^8)$
$\mathbf{P}(9_6) = (-a^{-10} - a^{-8} + 3a^{-6}) + z^2(-3a^{-10} + 3a^{-8} + 7a^{-6}) + z^4(-a^{-10} + 4a^{-8} + 5a^{-6}) + z^6(a^{-8} + a^{-6})$
$\mathbf{P}(9_7) = (-a^{-10} + a^{-8} - a^{-6} + 2a^{-4}) + z^2(-a^{-10} + 2a^{-8} + a^{-6} + 3a^{-4}) + z^4(a^{-8} + a^{-6} + a^{-4})$
$\mathbf{P}(9_8) = (-a^{-6} + 2a^{-4} - 1 + a^2) + z^2(2a^{-4} - a^{-2} - 2 + a^2) + z^4(-a^{-2} - 1)$
$\mathbf{P}(9_9) = (-2a^{-10} + a^{-8} + 2a^{-6}) + z^2(-3a^{-10} + 4a^{-8} + 7a^{-6}) + z^4(-a^{-10} + 4a^{-8} + 5a^{-6}) + z^6(a^{-8} + a^{-6})$
$\mathbf{P}(9_{10}) = (2a^6 + a^8 - 2a^{10}) + z^2(2a^4 + 5a^6 + 2a^8 - a^{10}) + z^4(a^4 + 2a^6 + a^8)$
$\mathbf{P}(9_{11}) = (a^2 - a^4 + 3a^6 - 2a^8) + z^2(3a^2 - 4a^4 + 6a^6 - a^8) + z^4(a^2 - 4a^4 + 2a^6) - a^4z^6$
$\mathbf{P}(9_{12}) = (-a^{-8} + 2a^{-6} - a^{-4} + 1) + z^2(2a^{-6} - a^{-4} - a^{-2} + 1) + z^4(-a^{-4} - a^{-2})$

$\mathbf{P}(9_{13}) = (3a^6 - a^8 - a^{10}) + z^2(2a^4 + 5a^6 + a^8 - a^{10}) + z^4(a^4 + 2a^6 + a^8)$
$\mathbf{P}(9_{14}) = (1 + a^2 - 2a^4 + a^6) + z^2(-a^{-2} + 1 + a^2 - 2a^4) + z^4(1 + a^2)$
$\mathbf{P}(9_{15}) = (1 - a^2 + a^4 + a^6 - a^8) + z^2(1 - a^2 + 2a^6) + z^4(-a^2 - a^4)$
$\mathbf{P}(9_{16}) = (4a^6 - 3a^8) + z^2(8a^6 - 2a^{10}) + z^4(5a^6 + 3a^8 - a^{10}) + z^6(a^6 + a^8)$
$\mathbf{P}(9_{17}) = (2a^{-2} - 3 + 2a^2) + z^2(-2a^{-4} + 5a^{-2} - 6 + a^2) + z^4(-a^{-4} + 4a^{-2} - 2) + a^{-2}z^6$
$\mathbf{P}(9_{18}) = (-a^{-10} + a^{-6} + a^{-4}) + z^2(-a^{-10} + a^{-8} + 4a^{-6} + 2a^{-4}) + z^4(a^{-8} + 2a^{-6} + a^{-4})$
$\mathbf{P}(9_{19}) = (a^{-2} - a^2 + a^4) + z^2(-a^{-4} + a^{-2} - 2a^2) + z^4(a^{-2} + 1)$
$\mathbf{P}(9_{20}) = (-a^{-8} + 2a^{-6} - 2a^{-4} + 2a^{-2}) + z^2(-a^{-8} + 5a^{-6} - 5a^{-4} + 3a^{-2}) + z^4(2a^{-6} - 4a^{-4} + a^{-2}) - a^{-4}z^6$
$\mathbf{P}(9_{21}) = (a^2 + a^6 - a^8) + z^2(1 + 2a^6) + z^4(-a^2 - a^4)$
$\mathbf{P}(9_{22}) = (2a^{-2} - 4 + 4a^2 - a^4) + z^2(a^{-2} - 6 + 6a^2 - 2a^4) + z^4(-2 + 4a^2 - a^4) + a^2z^6$
$\mathbf{P}(9_{23}) = (-2a^{-8} + 2a^{-6} + a^{-4}) + z^2(-a^{-10} + 4a^{-6} + 2a^{-4}) + z^4(a^{-8} + 2a^{-6} + a^{-4})$
$\mathbf{P}(9_{24}) = (-2a^{-4} + 5a^{-2} - 3 + a^2) + z^2(-a^{-4} + 6a^{-2} - 6 + 2a^2) + z^4(2a^{-2} - 4 + a^2) - z^6$
$\mathbf{P}(9_{25}) = (-a^{-8} + 3a^{-6} - 3a^{-4} + a^{-2} + 1) + z^2(3a^{-6} - 4a^{-4} + 1) + z^4(-2a^{-4} - a^{-2})$
$\mathbf{P}(9_{26}) = (3a^2 - 3a^4 + a^6) + z^2(-2 + 6a^2 - 5a^4 + a^6) + z^4(-1 + 4a^2 - 2a^4) + a^2z^6$
$\mathbf{P}(9_{27}) = (-a^{-4} + 3a^{-2} - 2 + a^2) + z^2(-a^{-4} + 5a^{-2} - 6 + 2a^2) + z^4(2a^{-2} - 4 + a^2) - z^6$
$\mathbf{P}(9_{28}) = (a^{-6} - 4a^{-4} + 5a^{-2} - 1) + z^2(a^{-6} - 5a^{-4} + 7a^{-2} - 2) + z^4(-2a^{-4} + 4a^{-2} - 1) + a^{-2}z^6$
$\mathbf{P}(9_{29}) = (-2a^{-4} + 5a^{-2} - 3 + a^2) + z^2(-2a^{-4} + 7a^{-2} - 5 + a^2) + z^4(-a^{-4} + 4a^{-2} - 2) + a^{-2}z^6$
$\mathbf{P}(9_{30}) = (-a^{-4} + 4a^{-2} - 4 + 2a^2) + z^2(-a^{-4} + 5a^{-2} - 7 + 2a^2) + z^4(2a^{-2} - 4 + a^2) - z^6$
$\mathbf{P}(9_{31}) = (-2a^{-4} + 4a^{-2} - 1) + z^2(a^{-6} - 4a^{-4} + 7a^{-2} - 2) + z^4(-2a^{-4} + 4a^{-2} - 1) + a^{-2}z^6$
$\mathbf{P}(9_{32}) = (1 + a^2 - 2a^4 + a^6) + z^2(-1 + 3a^2 - 4a^4 + a^6) + z^4(-1 + 3a^2 - 2a^4) + a^2z^6$
$\mathbf{P}(9_{33}) = (-a^{-4} + 2a^{-2}) + z^2(-a^{-4} + 4a^{-2} - 3 + a^2) + z^4(2a^{-2} - 3 + a^2) - z^6$
$\mathbf{P}(9_{34}) = (a^{-2} - 1 + a^2) + z^2(-a^{-4} + 3a^{-2} - 4 + a^2) + z^4(2a^{-2} - 3 + a^2) - z^6$
$\mathbf{P}(9_{35}) = (-a^{-10} - a^{-8} + 3a^{-6}) + z^2(a^{-8} + 3a^{-6} + 2a^{-4} + a^{-2})$
$\mathbf{P}(9_{36}) = (2a^2 - 3a^4 + 4a^6 - 2a^8) + z^2(3a^2 - 5a^4 + 6a^6 - a^8) + z^4(a^2 - 4a^4 + 2a^6) - a^4z^6$
$\mathbf{P}(9_{37}) = (2a^{-2} - 2 + a^4) + z^2(-a^{-4} + a^{-2} - 1 - 2a^2) + z^4(a^{-2} + 1)$
$\mathbf{P}(9_{38}) = (-3a^{-8} + 4a^{-6}) + z^2(-a^{-10} - a^{-8} + 7a^{-6} + a^{-4}) + z^4(a^{-8} + 3a^{-6} + a^{-4})$
$\mathbf{P}(9_{39}) = (2a^2 - 2a^4 + 2a^6 - a^8) + z^2(1 + a^2 - 3a^4 + 3a^6) + z^4(-a^2 - 2a^4)$
$\mathbf{P}(9_{40}) = (a^{-4} - 2a^{-2} + 2) + z^2(a^{-6} - 2a^{-4}) + z^4(-2a^{-4} + 2a^{-2} - 1) + a^{-2}z^6$
$\mathbf{P}(9_{41}) = (a^{-6} - 3a^{-4} + 3a^{-2}) + z^2(-3a^{-4} + 4a^{-2} - a^2) + z^4(2a^{-2} + 1)$
$\mathbf{P}(9_{42}) = (2a^{-2} - 3 + 2a^2) + z^2(a^{-2} - 4 + a^2) - z^4$
$\mathbf{P}(9_{43}) = (3a^2 - 4a^4 + 3a^6 - a^8) + z^2(4a^2 - 7a^4 + 4a^6) + z^4(a^2 - 5a^4 + a^6) - a^4z^6$
$\mathbf{P}(9_{44}) = (-a^{-4} + 3a^{-2} - 2 + a^2) + z^2(-a^{-4} + 3a^{-2} - 2) + a^{-2}z^4$
$\mathbf{P}(9_{45}) = (-a^{-8} + 2a^{-6} - 2a^{-4} + 2a^{-2}) + z^2(2a^{-6} - 2a^{-4} + 2a^{-2}) - a^{-4}z^4$
$\mathbf{P}(9_{46}) = (a^{-6} - a^{-4} - a^{-2} + 2) + z^2(-a^{-4} - a^{-2})$
$\mathbf{P}(9_{47}) = (1 + a^2 - 2a^4 + a^6) + z^2(-2 + 4a^2 - 3a^4) + z^4(-1 + 4a^2 - a^4) + a^2z^6$
$\mathbf{P}(9_{48}) = (3a^4 - 2a^6) + z^2(1 - a^2 + 3a^4) - a^2z^4$
$\mathbf{P}(9_{49}) = (4a^6 - 3a^8) + z^2(2a^4 + 6a^6 - 2a^8) + z^4(a^4 + 2a^6)$
$\mathbf{P}(10_1) = (a^{-8} - a^{-6} + a^2) + z^2(-a^{-6} - a^{-4} - a^{-2} - 1)$
$\mathbf{P}(10_2) = (a^{-8} - 4a^{-6} + 4a^{-4}) + z^2(6a^{-8} - 14a^{-6} + 10a^{-4}) + z^4(5a^{-8} - 16a^{-6} + 6a^{-4}) + z^6(a^{-8} - 7a^{-6} + a^{-4}) - a^{-6}z^8$
$\mathbf{P}(10_3) = (a^{-6} - a^{-2} + a^4) + z^2(-a^{-4} - 2a^{-2} - 2 - a^2)$
$\mathbf{P}(10_4) = (a^{-4} - 2a^2 + 2a^4) + z^2(a^{-4} - 2a^{-2} - 2 - 3a^2 + a^4) + z^4(-a^{-2} - 1 - a^2)$
$\mathbf{P}(10_5) = (-a^2 + 5a^4 - 3a^6) + z^2(-6a^2 + 17a^4 - 7a^6) + z^4(-5a^2 + 17a^4 - 5a^6) + z^6(-a^2 + 7a^4 - a^6) + a^4z^8$
$\mathbf{P}(10_6) = (a^{-8} - a^{-6} - 2a^{-4} + 3a^{-2}) + z^2(3a^{-8} - 4a^{-6} - 4a^{-4} + 4a^{-2}) + z^4(a^{-8} - 4a^{-6} - 4a^{-4} + a^{-2}) + z^6(-a^{-6} - a^{-4})$
$\mathbf{P}(10_7) = (a^{-8} - 2a^{-6} + a^{-4} + 1) + z^2(a^{-8} - 2a^{-6} - a^{-2} + 1) + z^4(-a^{-6} - a^{-4} - a^{-2})$
$\mathbf{P}(10_8) = (a^{-6} - 3a^{-2} + 3) + z^2(3a^{-6} - 3a^{-4} - 7a^{-2} + 4) + z^4(a^{-6} - 4a^{-4} - 5a^{-2} + 1) + z^6(-a^{-4} - a^{-2})$
$\mathbf{P}(10_9) = (3 - 4a^2 + 2a^4) + z^2(7 - 16a^2 + 7a^4) + z^4(5 - 17a^2 + 5a^4) + z^6(1 - 7a^2 + a^4) - a^2z^8$
$\mathbf{P}(10_{10}) = (1 - a^2 + 2a^4 - a^6) + z^2(-a^{-2} + 1 + 2a^4 - a^6) + z^4(1 + a^2 + a^4)$

$\mathbf{P}(10_{11}) = (a^{-6} - a^{-2} - 1 + 2a^2) + z^2(a^{-6} - a^{-4} - 4a^{-2} - 2 + a^2) + z^4(-a^{-4} - 2a^{-2} - 1)$
$\mathbf{P}(10_{12}) = (-1+2a^2+2a^4-2a^6)+z^2(-3+5a^2+5a^4-3a^6)+z^4(-1+4a^2+4a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{13}) = (a^{-6} - a^{-4} + a^{-2} - 1 + a^4) + z^2(-2a^{-4} - 1 - 2a^2) + z^4(a^{-2} + 1)$
$\mathbf{P}(10_{14}) = (-a^{-6} + a^{-4} + a^{-2}) + z^2(2a^{-8} - 2a^{-6} - a^{-4} + 3a^{-2}) + z^4(a^{-8} - 3a^{-6} - 3a^{-4} + a^{-2}) + z^6(-a^{-6} - a^{-4})$
$\mathbf{P}(10_{15}) = (-a^{-2}+1+3a^2-2a^4)+z^2(-3a^{-2}+4+5a^2-3a^4)+z^4(-a^{-2}+4+4a^2-a^4)+z^6(1+a^2)$
$\mathbf{P}(10_{16}) = (a^{-2} + 1 - 2a^2 + a^6) + z^2(a^{-2} - 1 - 4a^2 - a^4 + a^6) + z^4(-1 - 2a^2 - a^4)$
$\mathbf{P}(10_{17}) = (-2a^{-2}+5-2a^2)+z^2(-7a^{-2}+16-7a^2)+z^4(-5a^{-2}+17-5a^2)+z^6(-a^{-2}+7-a^2)+z^8$
$\mathbf{P}(10_{18}) = (a^{-4} - a^{-2} + a^2) + z^2(a^{-6} - 3a^{-2} - 1 + a^2) + z^4(-a^{-4} - 2a^{-2} - 1)$
$\mathbf{P}(10_{19}) = (-a^{-2}+3-a^2)+z^2(-2a^{-4}+a^{-2}+5-3a^2)+z^4(-a^{-4}+3a^{-2}+4-a^2)+z^6(a^{-2}+1)$
$\mathbf{P}(10_{20}) = (a^{-8} - a^{-6} - a^{-2} + 2) + z^2(a^{-8} - 2a^{-6} - a^{-4} - 2a^{-2} + 1) + z^4(-a^{-6} - a^{-4} - a^{-2})$
$\mathbf{P}(10_{21}) = (a^{-8} - 3a^{-6} + 2a^{-4} + a^{-2}) + z^2(3a^{-8} - 5a^{-6} + 3a^{-2}) + z^4(a^{-8} - 4a^{-6} - 3a^{-4} + a^{-2}) + z^6(-a^{-6} - a^{-4})$
$\mathbf{P}(10_{22}) = (2a^{-2}-1-2a^2+2a^4)+z^2(3a^{-2}-5-5a^2+3a^4)+z^4(a^{-2}-4-4a^2+a^4)+z^6(-1-a^2)$
$\mathbf{P}(10_{23}) = (3a^4 - 2a^6) + z^2(-2 + 2a^2 + 6a^4 - 3a^6) + z^4(-1 + 3a^2 + 4a^4 - a^6) + z^6(a^2 + a^4)$
$\mathbf{P}(10_{24}) = (a^{-8} - a^{-6} - a^{-4} + a^{-2} + 1) + z^2(a^{-8} - a^{-6} - 3a^{-4} + 1) + z^4(-a^{-6} - 2a^{-4} - a^{-2})$
$\mathbf{P}(10_{25}) = (a^{-8} - 2a^{-6} + 2a^{-2}) + z^2(2a^{-8} - 3a^{-6} - 2a^{-4} + 3a^{-2}) + z^4(a^{-8} - 3a^{-6} - 3a^{-4} + a^{-2}) + z^6(-a^{-6} - a^{-4})$
$\mathbf{P}(10_{26}) = (a^{-2}+1-3a^2+2a^4)+z^2(2a^{-2}-2-6a^2+3a^4)+z^4(a^{-2}-3-4a^2+a^4)+z^6(-1-a^2)$
$\mathbf{P}(10_{27}) = (-a^{-6} + a^{-4} + a^{-2}) + z^2(-2a^{-6} + 3a^{-4} + 3a^{-2} - 2) + z^4(-a^{-6} + 3a^{-4} + 3a^{-2} - 1) + z^6(a^{-4} + a^{-2})$
$\mathbf{P}(10_{28}) = (-1 + 3a^2 - a^6) + z^2(-a^{-2} + 4a^2 + a^4 - a^6) + z^4(1 + 2a^2 + a^4)$
$\mathbf{P}(10_{29}) = (a^{-6} - a^{-4} + a^{-2} - 2 + 2a^2) + z^2(a^{-6} - 4a^{-4} + 3a^{-2} - 5 + a^2) + z^4(-2a^{-4} + 3a^{-2} - 2) + a^{-2}z^6$
$\mathbf{P}(10_{30}) = (-a^{-4} + 2a^{-2}) + z^2(a^{-8} - 2a^{-4} + a^{-2} + 1) + z^4(-a^{-6} - 2a^{-4} - a^{-2})$
$\mathbf{P}(10_{31}) = (-a^{-2} + 2 + a^2 - a^4) + z^2(-a^{-4} + 3 + a^2 - a^4) + z^4(a^{-2} + 2 + a^2)$
$\mathbf{P}(10_{32}) = (a^{-2} - 1 + a^2) + z^2(2a^{-4} - 2a^{-2} - 3 + 2a^2) + z^4(a^{-4} - 3a^{-2} - 3 + a^2) + z^6(-a^{-2} - 1)$
$\mathbf{P}(10_{33}) = 1 + z^2(-a^{-4} + 2 - a^4) + z^4(a^{-2} + 2 + a^2)$
$\mathbf{P}(10_{34}) = (-a^{-2} + 2 + a^4 - a^6) + z^2(-a^{-2} + 2 + a^2 + 2a^4 - a^6) + z^4(1 + a^2 + a^4)$
$\mathbf{P}(10_{35}) = (a^{-4} - a^{-2} + 1 - a^4 + a^6) + z^2(-2a^{-2} - 2a^4) + z^4(1 + a^2)$
$\mathbf{P}(10_{36}) = (-a^{-6} + 2a^{-4} - a^{-2} + 1) + z^2(a^{-8} - a^{-6} + a^{-4} - a^{-2} + 1) + z^4(-a^{-6} - a^{-4} - a^{-2})$
$\mathbf{P}(10_{37}) = (-a^{-4} + a^{-2} + 1 + a^2 - a^4) + z^2(-a^{-4} + a^{-2} + 3 + a^2 - a^4) + z^4(a^{-2} + 2 + a^2)$
$\mathbf{P}(10_{38}) = (a^{-6} - 2a^{-4} + a^{-2} + 1) + z^2(a^{-8} - 3a^{-4} + 1) + z^4(-a^{-6} - 2a^{-4} - a^{-2})$
$\mathbf{P}(10_{39}) = (-a^{-4} + 2a^{-2}) + z^2(2a^{-8} - 2a^{-6} - 2a^{-4} + 3a^{-2}) + z^4(a^{-8} - 3a^{-6} - 3a^{-4} + a^{-2}) + z^6(-a^{-6} - a^{-4})$
$\mathbf{P}(10_{40}) = (-1 + 3a^2 - a^6) + z^2(-2 + 4a^2 + 3a^4 - 2a^6) + z^4(-1 + 3a^2 + 3a^4 - a^6) + z^6(a^2 + a^4)$
$\mathbf{P}(10_{41}) = (a^{-6} - 2a^{-4} + 2a^{-2} - 1 + a^2) + z^2(a^{-6} - 4a^{-4} + 4a^{-2} - 4 + a^2) + z^4(-2a^{-4} + 3a^{-2} - 2) + a^{-2}z^6$
$\mathbf{P}(10_{42}) = (a^{-2} - 2 + 3a^2 - a^4) + z^2(-a^{-4} + 3a^{-2} - 5 + 4a^2 - a^4) + z^4(2a^{-2} - 3 + 2a^2) - z^6$
$\mathbf{P}(10_{43}) = (-a^{-4}+2a^{-2}-1+2a^2-a^4)+z^2(-a^{-4}+4a^{-2}-4+4a^2-a^4)+z^4(2a^{-2}-3+2a^2)-z^6$
$\mathbf{P}(10_{44}) = (-a^{-4}+3a^{-2}-2+a^2)+z^2(a^{-6}-3a^{-4}+5a^{-2}-4+a^2)+z^4(-2a^{-4}+3a^{-2}-2)+a^{-2}z^6$
$\mathbf{P}(10_{45}) = (2a^{-2} - 3 + 2a^2) + z^2(-a^{-4} + 3a^{-2} - 6 + 3a^2 - a^4) + z^4(2a^{-2} - 3 + 2a^2) - z^6$
$\mathbf{P}(10_{46}) = (6a^4-8a^6+3a^8)+z^2(11a^4-18a^6+7a^8)+z^4(6a^4-17a^6+5a^8)+z^6(a^4-7a^6+a^8)-a^6z^8$
$\mathbf{P}(10_{47}) = (-3a^2 + 9a^4 - 5a^6) + z^2(-7a^2 + 21a^4 - 8a^6) + z^4(-5a^2 + 18a^4 - 5a^6) + z^6(-a^2 + 7a^4 - a^6) + a^4z^8$
$\mathbf{P}(10_{48}) = (-4a^{-2}+9-4a^2)+z^2(-8a^{-2}+20-8a^2)+z^4(-5a^{-2}+18-5a^2)+z^6(-a^{-2}+7-a^2)+z^8$
$\mathbf{P}(10_{49}) = (2a^{-12} - 7a^{-10} + 5a^{-8} + a^{-6}) + z^2(a^{-12} - 10a^{-10} + 12a^{-8} + 4a^{-6}) + z^4(-3a^{-10} + 9a^{-8} + 4a^{-6}) + z^6(2a^{-8} + a^{-6})$
$\mathbf{P}(10_{50}) = (2a^2+a^4-4a^6+2a^8)+z^2(3a^2-a^4-6a^6+3a^8)+z^4(a^2-3a^4-4a^6+a^8)+z^6(-a^4-a^6)$
$\mathbf{P}(10_{51}) = (-1+a^2+4a^4-3a^6)+z^2(-2+3a^2+7a^4-3a^6)+z^4(-1+3a^2+4a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{52}) = (-2a^{-2}+4-a^4)+z^2(-3a^{-2}+6+2a^2-2a^4)+z^4(-a^{-2}+4+3a^2-a^4)+z^6(1+a^2)$
$\mathbf{P}(10_{53}) = (a^{-12} - 3a^{-10} + 3a^{-6}) + z^2(-3a^{-10} + 2a^{-8} + 6a^{-6} + a^{-4}) + z^4(2a^{-8} + 3a^{-6} + a^{-4})$

$\mathbf{P}(10_{54}) = (-2a^{-2}+3+2a^2-2a^4)+z^2(-3a^{-2}+5+5a^2-3a^4)+z^4(-a^{-2}+4+4a^2-a^4)+z^6(1+a^2)$
$\mathbf{P}(10_{55}) = (a^{-12}-3a^{-10}+a^{-8}+a^{-6}+a^{-4})+z^2(-3a^{-10}+3a^{-8}+3a^{-6}+2a^{-4})+z^4(2a^{-8}+2a^{-6}+a^{-4})$
$\mathbf{P}(10_{56}) = (2a^2-2a^6+a^8)+z^2(3a^2-2a^4-3a^6+2a^8)+z^4(a^2-3a^4-3a^6+a^8)+z^6(-a^4-a^6)$
$\mathbf{P}(10_{57}) = (-1+2a^2+2a^4-2a^6)+z^2(-2+4a^2+4a^4-2a^6)+z^4(-1+3a^2+3a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{58}) = (a^{-6}-2a^{-4}+3a^{-2}-2+a^4)+z^2(-3a^{-4}+3a^{-2}-2-2a^2)+z^4(2a^{-2}+1)$
$\mathbf{P}(10_{59}) = (a^{-2}-2+4a^2-3a^4+a^6)+z^2(a^{-2}-4+5a^2-4a^4+a^6)+z^4(-2+3a^2-2a^4)+a^2z^6$
$\mathbf{P}(10_{60}) = (a^{-6}-3a^{-4}+4a^{-2}-2+a^2)+z^2(-3a^{-4}+6a^{-2}-5+a^2)+z^4(3a^{-2}-3+a^2)-z^6$
$\mathbf{P}(10_{61}) = (4-5a^2+a^4+a^6)+z^2(4-8a^2-3a^4+3a^6)+z^4(1-5a^2-4a^4+a^6)+z^6(-a^2-a^4)$
$\mathbf{P}(10_{62}) = (-2a^2+7a^4-4a^6)+z^2(-7a^2+20a^4-8a^6)+z^4(-5a^2+18a^4-5a^6)+z^6(-a^2+7a^4-a^6)+a^4z^8$
$\mathbf{P}(10_{63}) = (a^{-12}-4a^{-10}+3a^{-8}+a^{-4})+z^2(-3a^{-10}+4a^{-8}+3a^{-6}+2a^{-4})+z^4(2a^{-8}+2a^{-6}+a^{-4})$
$\mathbf{P}(10_{64}) = (4-6a^2+3a^4)+z^2(8-19a^2+8a^4)+z^4(5-18a^2+5a^4)+z^6(1-7a^2+a^4)-a^2z^8$
$\mathbf{P}(10_{65}) = (-a^2+5a^4-3a^6)+z^2(-2+2a^2+7a^4-3a^6)+z^4(-1+3a^2+4a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{66}) = (a^{-12}-4a^{-10}+2a^{-8}+2a^{-6})+z^2(a^{-12}-8a^{-10}+9a^{-8}+5a^{-6})+z^4(-3a^{-10}+8a^{-8}+4a^{-6})+z^6(2a^{-8}+a^{-6})$
$\mathbf{P}(10_{67}) = 1+z^2(a^{-8}-2a^{-4}+1)+z^4(-a^{-6}-2a^{-4}-a^{-2})$
$\mathbf{P}(10_{68}) = (-a^{-6}+a^{-4}+a^{-2})+z^2(-a^{-6}+a^{-4}+3a^{-2}-a^2)+z^4(a^{-4}+2a^{-2}+1)$
$\mathbf{P}(10_{69}) = (2a^2-2a^4+2a^6-a^8)+z^2(-1+5a^2-5a^4+3a^6)+z^4(-1+3a^2-3a^4)+a^2z^6$
$\mathbf{P}(10_{70}) = (2a^{-2}-3+3a^2-2a^4+a^6)+z^2(a^{-2}-5+4a^2-4a^4+a^6)+z^4(-2+3a^2-2a^4)+a^2z^6$
$\mathbf{P}(10_{71}) = (-a^{-4}+3a^{-2}-3+3a^2-a^4)+z^2(-a^{-4}+4a^{-2}-5+4a^2-a^4)+z^4(2a^{-2}-3+2a^2)-z^6$
$\mathbf{P}(10_{72}) = (2a^2-2a^4+2a^6-a^8)+z^2(3a^2-3a^4+a^6+a^8)+z^4(a^2-3a^4-2a^6+a^8)+z^6(-a^4-a^6)$
$\mathbf{P}(10_{73}) = (-a^{-8}+3a^{-6}-4a^{-4}+3a^{-2})+z^2(3a^{-6}-6a^{-4}+5a^{-2}-1)+z^4(-3a^{-4}+3a^{-2}-1)+a^{-2}z^6$
$\mathbf{P}(10_{74}) = (a^{-8}-2a^{-6}+2a^{-2})+z^2(a^{-8}-a^{-6}-2a^{-4}+a^{-2}+1)+z^4(-a^{-6}-2a^{-4}-a^{-2})$
$\mathbf{P}(10_{75}) = (3a^2-3a^4+a^6)+z^2(a^{-2}-4+6a^2-3a^4)+z^4(a^{-2}-3+3a^2)-z^6$
$\mathbf{P}(10_{76}) = (4a^2-4a^4+a^8)+z^2(4a^2-6a^4-2a^6+2a^8)+z^4(a^2-4a^4-3a^6+a^8)+z^6(-a^4-a^6)$
$\mathbf{P}(10_{77}) = (-2+5a^2-a^4-a^6)+z^2(-3+7a^2+2a^4-2a^6)+z^4(-1+4a^2+3a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{78}) = (a^{-10}-4a^{-8}+4a^{-6}-a^{-4}+a^{-2})+z^2(-3a^{-8}+7a^{-6}-3a^{-4}+2a^{-2})+z^4(3a^{-6}-3a^{-4}+a^{-2})-a^{-4}z^6$
$\mathbf{P}(10_{79}) = (-5a^{-2}+11-5a^2)+z^2(-9a^{-2}+23-9a^2)+z^4(-5a^{-2}+19-5a^2)+z^6(-a^{-2}+7-a^2)+z^8$
$\mathbf{P}(10_{80}) = (2a^{-12}-6a^{-10}+3a^{-8}+2a^{-6})+z^2(a^{-12}-9a^{-10}+9a^{-8}+5a^{-6})+z^4(-3a^{-10}+8a^{-8}+4a^{-6})+z^6(2a^{-8}+a^{-6})$
$\mathbf{P}(10_{81}) = (-a^{-4}+a^{-2}+1+a^2-a^4)+z^2(-a^{-4}+3a^{-2}-1+3a^2-a^4)+z^4(2a^{-2}-2+2a^2)-z^6$
$\mathbf{P}(10_{82}) = 1+z^2(4a^{-4}-8a^{-2}+4)+z^4(4a^{-4}-12a^{-2}+4)+z^6(a^{-4}-6a^{-2}+1)-a^{-2}z^8$
$\mathbf{P}(10_{83}) = (2-2a^2+a^4)+z^2(a^{-2}-4a^2+2a^4)+z^4(a^{-2}-2-3a^2+a^4)+z^6(-1-a^2)$
$\mathbf{P}(10_{84}) = (-1+4a^2-2a^4)+z^2(-2+5a^2-a^6)+z^4(-1+3a^2+2a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{85}) = (-a^{-6}+a^{-4}+a^{-2})+z^2(-4a^{-6}+9a^{-4}-3a^{-2})+z^4(-4a^{-6}+12a^{-4}-4a^{-2})+z^6(-a^{-6}+6a^{-4}-a^{-2})+a^{-4}z^8$
$\mathbf{P}(10_{86}) = (1-a^2+2a^4-a^6)+z^2(-1+4a^4-2a^6)+z^4(-1+2a^2+3a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{87}) = (a^{-2}-2+3a^2-a^4)+z^2(2a^{-2}-4+a^2+a^4)+z^4(a^{-2}-3-2a^2+a^4)+z^6(-1-a^2)$
$\mathbf{P}(10_{88}) = (a^{-2}-1+a^2)+z^2(-a^{-4}+2a^{-2}-3+2a^2-a^4)+z^4(2a^{-2}-2+2a^2)-z^6$
$\mathbf{P}(10_{89}) = (-a^{-8}+2a^{-6}-a^{-4}+1)+z^2(3a^{-6}-4a^{-4}+2a^{-2})+z^4(-3a^{-4}+2a^{-2}-1)+a^{-2}z^6$
$\mathbf{P}(10_{90}) = (2a^{-2}-2+a^4)+z^2(2a^{-2}-4-3a^2+2a^4)+z^4(a^{-2}-3-3a^2+a^4)+z^6(-1-a^2)$
$\mathbf{P}(10_{91}) = (-2a^{-2}+5-2a^2)+z^2(-5a^{-2}+12-5a^2)+z^4(-4a^{-2}+13-4a^2)+z^6(-a^{-2}+6-a^2)+z^8$
$\mathbf{P}(10_{92}) = (a^2+a^4-a^6)+z^2(2a^2-a^6+a^8)+z^4(a^2-2a^4-2a^6+a^8)+z^6(-a^4-a^6)$
$\mathbf{P}(10_{93}) = (-a^{-4}+2a^{-2})+z^2(-2a^{-4}+3a^{-2}+2-2a^2)+z^4(-a^{-4}+3a^{-2}+3-a^2)+z^6(a^{-2}+1)$
$\mathbf{P}(10_{94}) = (3-4a^2+2a^4)+z^2(5-12a^2+5a^4)+z^4(4-13a^2+4a^4)+z^6(1-6a^2+a^4)-a^2z^8$
$\mathbf{P}(10_{95}) = (3a^4-2a^6)+z^2(-1+a^2+5a^4-2a^6)+z^4(-1+2a^2+3a^4-a^6)+z^6(a^2+a^4)$
$\mathbf{P}(10_{96}) = (2a^{-2}-3+3a^2-2a^4+a^6)+z^2(a^{-2}-6+5a^2-3a^4)+z^4(a^{-2}-3+3a^2)-z^6$
$\mathbf{P}(10_{97}) = (2a^2-2a^4+2a^6-a^8)+z^2(1+2a^2-4a^4+2a^6+a^8)+z^4(-a^2-3a^4-a^6)$

$\mathbf{P}(10_{98}) = (2a^{-8} - 5a^{-6} + 3a^{-4} + a^{-2}) + z^2(2a^{-8} - 5a^{-6} + a^{-4} + 2a^{-2}) + z^4(a^{-8} - 3a^{-6} - 2a^{-4} + a^{-2}) + z^6(-a^{-6} - a^{-4})$
$\mathbf{P}(10_{99}) = (-4a^{-2}+9-4a^2)+z^2(-6a^{-2}+16-6a^2)+z^4(-4a^{-2}+14-4a^2)+z^6(-a^{-2}+6-a^2)+z^8$
$\mathbf{P}(10_{100}) = (-3a^{-6} + 5a^{-4} - a^{-2}) + z^2(-5a^{-6} + 13a^{-4} - 4a^{-2}) + z^4(-4a^{-6} + 13a^{-4} - 4a^{-2}) + z^6(-a^{-6} + 6a^{-4} - a^{-2}) + a^{-4}z^8$
$\mathbf{P}(10_{101}) = (2a^6 + 2a^8 - 4a^{10} + a^{12}) + z^2(a^4 + 5a^6 + 5a^8 - 4a^{10}) + z^4(a^4 + 3a^6 + 3a^8)$
$\mathbf{P}(10_{102}) = (a^{-2} - a^2 + a^4) + z^2(2a^{-2} - 3 - 3a^2 + 2a^4) + z^4(a^{-2} - 3 - 3a^2 + a^4) + z^6(-1 - a^2)$
$\mathbf{P}(10_{103}) = (-a^{-6} + 3a^{-2} - 1) + z^2(-2a^{-6} + 3a^{-4} + 4a^{-2} - 2) + z^4(-a^{-6} + 3a^{-4} + 3a^{-2} - 1) + z^6(a^{-4} + a^{-2})$
$\mathbf{P}(10_{104}) = (-a^{-2}+3-a^2)+z^2(-5a^{-2}+11-5a^2)+z^4(-4a^{-2}+13-4a^2)+z^6(-a^{-2}+6-a^2)+z^8$
$\mathbf{P}(10_{105}) = (a^{-2} - 1 + a^2) + z^2(a^{-2} - 3 + 2a^2 - 2a^4 + a^6) + z^4(-2 + 2a^2 - 2a^4) + a^2z^6$
$\mathbf{P}(10_{106}) = (2 - 2a^2 + a^4) + z^2(5 - 11a^2 + 5a^4) + z^4(4 - 13a^2 + 4a^4) + z^6(1 - 6a^2 + a^4) - a^2z^8$
$\mathbf{P}(10_{107}) = (2a^2 - a^4) + z^2(-a^{-4} + 2a^{-2} - 2 + 3a^2 - a^4) + z^4(2a^{-2} - 2 + 2a^2) - z^6$
$\mathbf{P}(10_{108}) = 1 + z^2(-2a^{-2} + 2 + 2a^2 - 2a^4) + z^4(-a^{-2} + 3 + 3a^2 - a^4) + z^6(1 + a^2)$
$\mathbf{P}(10_{109}) = (-3a^{-2}+7-3a^2)+z^2(-6a^{-2}+15-6a^2)+z^4(-4a^{-2}+14-4a^2)+z^6(-a^{-2}+6-a^2)+z^8$
$\mathbf{P}(10_{110}) = (a^{-6} - a^{-4} + a^2) + z^2(a^{-6} - 3a^{-4} + a^{-2} - 3 + a^2) + z^4(-2a^{-4} + 2a^{-2} - 2) + a^{-2}z^6$
$\mathbf{P}(10_{111}) = (a^2+2a^4-3a^6+a^8)+z^2(2a^2+a^4-4a^6+2a^8)+z^4(a^2-2a^4-3a^6+a^8)+z^6(-a^4-a^6)$
$\mathbf{P}(10_{112}) = (-2a^{-4}+4a^{-2}-1)+z^2(a^{-4}+1)+z^4(3a^{-4}-7a^{-2}+3)+z^6(a^{-4}-5a^{-2}+1)-a^{-2}z^8$
$\mathbf{P}(10_{113}) = (3a^2 - 3a^4 + a^6) + z^2(-1 + 3a^2 - 2a^4) + z^4(-1 + 2a^2 + a^4 - a^6) + z^6(a^2 + a^4)$
$\mathbf{P}(10_{114}) = (-a^{-4} + 2a^{-2}) + z^2(a^{-4} - 1 + a^2) + z^4(a^{-4} - 2a^{-2} - 2 + a^2) + z^6(-a^{-2} - 1)$
$\mathbf{P}(10_{115}) = (-a^{-2} + 3 - a^2) + z^2(-a^{-4} + a^{-2} + 1 + a^2 - a^4) + z^4(2a^{-2} - 1 + 2a^2) - z^6$
$\mathbf{P}(10_{116}) = 1 + z^2(2a^{-4} - 4a^{-2} + 2) + z^4(3a^{-4} - 8a^{-2} + 3) + z^6(a^{-4} - 5a^{-2} + 1) - a^{-2}z^8$
$\mathbf{P}(10_{117}) = (a^2 + a^4 - a^6) + z^2(-1 + 2a^2 + 2a^4 - a^6) + z^4(-1 + 2a^2 + 2a^4 - a^6) + z^6(a^2 + a^4)$
$\mathbf{P}(10_{118}) = 1 + z^2(-2a^{-2} + 4 - 2a^2) + z^4(-3a^{-2} + 8 - 3a^2) + z^6(-a^{-2} + 5 - a^2) + z^8$
$\mathbf{P}(10_{119}) = (a^{-2} - 1 + a^2) + z^2(a^{-2} - 2 - a^2 + a^4) + z^4(a^{-2} - 2 - 2a^2 + a^4) + z^6(-1 - a^2)$
$\mathbf{P}(10_{120}) = (a^{-12} - 3a^{-10} + 3a^{-6}) + z^2(-4a^{-10} + 3a^{-8} + 7a^{-6}) + z^4(3a^{-8} + 4a^{-6} + a^{-4})$
$\mathbf{P}(10_{121}) = (-a^{-6} + 2a^{-4} - a^{-2} + 1) + z^2(-a^{-6} + 3a^{-4} - a^{-2}) + z^4(-a^{-6} + 2a^{-4} + a^{-2} - 1) + z^6(a^{-4} + a^{-2})$
$\mathbf{P}(10_{122}) = (-1 + 4a^2 - 2a^4) + z^2(a^{-2} - 2 + 3a^2) + z^4(a^{-2} - 2 - a^2 + a^4) + z^6(-1 - a^2)$
$\mathbf{P}(10_{123}) = (2a^{-2} - 3 + 2a^2) + z^2(a^{-2} - 4 + a^2) + z^4(-2a^{-2} + 3 - 2a^2) + z^6(-a^{-2} + 4 - a^2) + z^8$
$\mathbf{P}(10_{124}) = (7a^8 - 8a^{10} + 2a^{12}) + z^2(21a^8 - 14a^{10} + a^{12}) + z^4(21a^8 - 7a^{10}) + z^6(8a^8 - a^{10}) + a^8z^8$
$\mathbf{P}(10_{125}) = (-3a^{-2} + 7 - 3a^2) + z^2(-4a^{-2} + 11 - 4a^2) + z^4(-a^{-2} + 6 - a^2) + z^6$
$\mathbf{P}(10_{126}) = (-4a^{-6}+7a^{-4}-2a^{-2})+z^2(-4a^{-6}+12a^{-4}-3a^{-2})+z^4(-a^{-6}+6a^{-4}-a^{-2})+a^{-4}z^6$
$\mathbf{P}(10_{127}) = (2a^{-8} - 6a^{-6} + 5a^{-4}) + z^2(3a^{-8} - 9a^{-6} + 7a^{-4}) + z^4(a^{-8} - 5a^{-6} + 2a^{-4}) - a^{-6}z^6$
$\mathbf{P}(10_{128}) = (2a^6 + 2a^8 - 4a^{10} + a^{12}) + z^2(6a^6 + 6a^8 - 5a^{10}) + z^4(5a^6 + 5a^8 - a^{10}) + z^6(a^6 + a^8)$
$\mathbf{P}(10_{129}) = (-a^{-4} + a^{-2} + 2 - a^2) + z^2(-a^{-4} + 2a^{-2} + 2 - a^2) + z^4(a^{-2} + 1)$
$\mathbf{P}(10_{130}) = (-2a^{-6} + 2a^{-4} + 2a^{-2} - 1) + z^2(-a^{-6} + 3a^{-4} + 3a^{-2} - 1) + z^4(a^{-4} + a^{-2})$
$\mathbf{P}(10_{131}) = (a^{-8} - 2a^{-6} + 2a^{-2}) + z^2(a^{-8} - 2a^{-6} - a^{-4} + 2a^{-2}) + z^4(-a^{-6} - a^{-4})$
$\mathbf{P}(10_{132}) = (-2a^{-6} + 3a^{-4}) + z^2(-a^{-6} + 4a^{-4}) + a^{-4}z^4$
$\mathbf{P}(10_{133}) = (a^{-8} - 3a^{-6} + 2a^{-4} + a^{-2}) + z^2(a^{-8} - 3a^{-6} + 2a^{-4} + a^{-2}) - a^{-6}z^4$
$\mathbf{P}(10_{134}) = (3a^6 - 3a^{10} + a^{12}) + z^2(7a^6 + 3a^8 - 4a^{10}) + z^4(5a^6 + 4a^8 - a^{10}) + z^6(a^6 + a^8)$
$\mathbf{P}(10_{135}) = (-a^{-4} + 4 - 2a^2) + z^2(-a^{-4} + a^{-2} + 5 - 2a^2) + z^4(a^{-2} + 2)$
$\mathbf{P}(10_{136}) = (a^{-2} - 2 + 3a^2 - a^4) + z^2(a^{-2} - 3 + 2a^2) - z^4$
$\mathbf{P}(10_{137}) = (a^{-6} - 2a^{-4} + 2a^{-2} - 1 + a^2) + z^2(-2a^{-4} + 2a^{-2} - 2) + a^{-2}z^4$
$\mathbf{P}(10_{138}) = (2a^{-2} - 3 + 3a^2 - 2a^4 + a^6) + z^2(a^{-2} - 6 + 5a^2 - 3a^4) + z^4(-2 + 4a^2 - a^4) + a^2z^6$
$\mathbf{P}(10_{139}) = (6a^8 - 6a^{10} + a^{12}) + z^2(21a^8 - 13a^{10} + a^{12}) + z^4(21a^8 - 7a^{10}) + z^6(8a^8 - a^{10}) + a^8z^8$
$\mathbf{P}(10_{140}) = (-2a^{-6} + 4a^{-4} - 2a^{-2} + 1) + z^2(-a^{-6} + 4a^{-4} - a^{-2}) + a^{-4}z^4$
$\mathbf{P}(10_{141}) = (a^{-4} - 2a^{-2} + 2) + z^2(3a^{-4} - 7a^{-2} + 3) + z^4(a^{-4} - 5a^{-2} + 1) - a^{-2}z^6$
$\mathbf{P}(10_{142}) = (a^6 + 4a^8 - 5a^{10} + a^{12}) + z^2(6a^6 + 7a^8 - 5a^{10}) + z^4(5a^6 + 5a^8 - a^{10}) + z^6(a^6 + a^8)$
$\mathbf{P}(10_{143}) = (-2a^{-6} + 3a^{-4}) + z^2(-3a^{-6} + 8a^{-4} - 2a^{-2}) + z^4(-a^{-6} + 5a^{-4} - a^{-2}) + a^{-4}z^6$
$\mathbf{P}(10_{144}) = (2a^{-4} - 4a^{-2} + 3) + z^2(a^{-6} - 5a^{-2} + 2) + z^4(-a^{-4} - 2a^{-2})$
$\mathbf{P}(10_{145}) = (-a^{-10} + a^{-8} - a^{-6} + 2a^{-4}) + z^2(a^{-8} + 4a^{-4}) + a^{-4}z^4$

$\mathbf{P}(10_{146}) = 1 + z^2(-a^{-4} + a^{-2} + 1 - a^2) + z^4(a^{-2} + 1)$
$\mathbf{P}(10_{147}) = (a^{-2} - 1 + a^2) + z^2(a^{-2} - 2 - a^2 + a^4) + z^4(-1 - a^2)$
$\mathbf{P}(10_{148}) = (-3a^{-6} + 5a^{-4} - a^{-2}) + z^2(-3a^{-6} + 9a^{-4} - 2a^{-2}) + z^4(-a^{-6} + 5a^{-4} - a^{-2}) + a^{-4}z^6$
$\mathbf{P}(10_{149}) = (a^{-8} - 4a^{-6} + 4a^{-4}) + z^2(2a^{-8} - 6a^{-6} + 6a^{-4}) + z^4(a^{-8} - 4a^{-6} + 2a^{-4}) - a^{-6}z^6$
$\mathbf{P}(10_{150}) = (2a^2 - a^4) + z^2(3a^2 - 4a^4 + 2a^6) + z^4(a^2 - 4a^4 + a^6) - a^4z^6$
$\mathbf{P}(10_{151}) = (-1 + 3a^2 - a^6) + z^2(-2 + 6a^2 - a^4) + z^4(-1 + 4a^2 - a^4) + a^2z^6$
$\mathbf{P}(10_{152}) = (3a^{-12} - 10a^{-10} + 8a^{-8}) + z^2(2a^{-12} - 17a^{-10} + 22a^{-8}) + z^4(-8a^{-10} + 21a^{-8}) + z^6(-a^{-10} + 8a^{-8}) + a^{-8}z^8$
$\mathbf{P}(10_{153}) = (-a^{-4} - a^{-2} + 6 - 3a^2) + z^2(-a^{-4} - a^{-2} + 10 - 4a^2) + z^4(6 - a^2) + z^6$
$\mathbf{P}(10_{154}) = (4a^6 - 2a^8 - 2a^{10} + a^{12}) + z^2(9a^6 - 2a^8 - 2a^{10}) + 6a^6z^4 + a^6z^6$
$\mathbf{P}(10_{155}) = (3 - 4a^2 + 2a^4) + z^2(3 - 8a^2 + 3a^4) + z^4(1 - 5a^2 + a^4) - a^2z^6$
$\mathbf{P}(10_{156}) = (-a^{-4} + 2a^{-2}) + z^2(-2a^{-4} + 5a^{-2} - 2) + z^4(-a^{-4} + 4a^{-2} - 1) + a^{-2}z^6$
$\mathbf{P}(10_{157}) = (2a^4 - a^8) + z^2(5a^4 - 2a^6 + a^8) + z^4(2a^4 - 3a^6 + a^8) - a^6z^6$
$\mathbf{P}(10_{158}) = (2a^{-2} - 2 + a^4) + z^2(2a^{-2} - 6 + a^2) + z^4(a^{-2} - 4 + a^2) - z^6$
$\mathbf{P}(10_{159}) = (-a^{-6} + a^{-4} + a^{-2}) + z^2(-2a^{-6} + 5a^{-4} - a^{-2}) + z^4(-a^{-6} + 4a^{-4} - a^{-2}) + a^{-4}z^6$
$\mathbf{P}(10_{160}) = (a^2 + a^6 - a^8) + z^2(3a^2 - 3a^4 + 3a^6) + z^4(a^2 - 4a^4 + a^6) - a^4z^6$
$\mathbf{P}(10_{161}) = (-a^{-10} - a^{-8} + 3a^{-6}) + z^2(-a^{-10} - a^{-8} + 9a^{-6}) + 6a^{-6}z^4 + a^{-6}z^6$
$\mathbf{P}(10_{162}) = (3a^6 - a^8 - a^{10}) + z^2(9a^6 - a^8 - a^{10}) + 6a^6z^4 + a^6z^6$
$\mathbf{P}(10_{163}) = (a^{-6} - 3a^{-2} + 3) + z^2(a^{-6} - a^{-4} - 5a^{-2} + 2) + z^4(-a^{-4} - 2a^{-2})$
$\mathbf{P}(10_{164}) = (1 - a^2 + 2a^4 - a^6) + z^2(-1 + 2a^2) + z^4(-1 + 3a^2 - a^4) + a^2z^6$
$\mathbf{P}(10_{165}) = (-a^{-2} + 3 - a^2) + z^2(-a^{-4} + 4 - 2a^2) + z^4(a^{-2} + 2)$
$\mathbf{P}(10_{166}) = (a^2 + a^4 - a^6) + z^2(2a^2 - a^6 + a^8) + z^4(-a^4 - a^6)$
$\mathbf{F}(3_1) = (-2a^{-1} - a) + x(1 + a^2) + x^2(a^{-1} + a)$
$\mathbf{F}(4_1) = (-a^{-2} - 1 - a^2) + x(-a^{-1} - a) + x^2(a^{-2} + 2 + a^2) + x^3(a^{-1} + a)$
$\mathbf{F}(5_1) = (3a^{-1} + 2a) + x(-2 - a^2 + a^4) + x^2(-4a^{-1} - 3a + a^3) + x^3(1 + a^2) + x^4(a^{-1} + a)$
$\mathbf{F}(5_2) = (-a^{-3} + a^{-1} + a) + x(-2 - 2a^2) + x^2(a^{-3} - a^{-1} - 2a) + x^3(a^{-2} + 2 + a^2) + x^4(a^{-1} + a)$
$\mathbf{F}(6_1) = (-a^{-4} + 1 + a^2) + x(2a^{-1} + 2a) + x^2(a^{-4} - 4 - 3a^2) + x^3(a^{-3} - 2a^{-1} - 3a) + x^4(a^{-2} + 2 + a^2) + x^5(a^{-1} + a)$
$\mathbf{F}(6_2) = (2a^{-2} + 2 + a^2) + x(-a - a^3) + x^2(-3a^{-2} - 6 - 2a^2 + a^4) + x^3(-2a^{-1} + 2a^3) + x^4(a^{-2} + 3 + 2a^2) + x^5(a^{-1} + a)$
$\mathbf{F}(6_3) = (a^{-2} + 3 + a^2) + x(-a^{-3} - 2a^{-1} - 2a - a^3) + x^2(-3a^{-2} - 6 - 3a^2) + x^3(a^{-3} + a^{-1} + a + a^3) + x^4(2a^{-2} + 4 + 2a^2) + x^5(a^{-1} + a)$
$\mathbf{F}(7_1) = (-4a^{-1} - 3a) + x(3 + a^2 - a^4 + a^6) + x^2(10a^{-1} + 7a - 2a^3 + a^5) + x^3(-4 - 3a^2 + a^4) + x^4(-6a^{-1} - 5a + a^3) + x^5(1 + a^2) + x^6(a^{-1} + a)$
$\mathbf{F}(7_2) = (-a^{-5} - a^{-1} - a) + x(3 + 3a^2) + x^2(a^{-5} + 3a^{-1} + 4a) + x^3(a^{-4} - a^{-2} - 6 - 4a^2) + x^4(a^{-3} - 3a^{-1} - 4a) + x^5(a^{-2} + 2 + a^2) + x^6(a^{-1} + a)$
$\mathbf{F}(7_3) = (-2a^{-1} - 2a + a^3) + x(-2a^{-4} + a^{-2} + 3) + x^2(-a^{-3} + 6a^{-1} + 4a - 3a^3) + x^3(a^{-4} - a^{-2} - 4 - 2a^2) + x^4(a^{-3} - 3a^{-1} - 3a + a^3) + x^5(a^{-2} + 2 + a^2) + x^6(a^{-1} + a)$
$\mathbf{F}(7_4) = (-a^{-1} + 2a^3) + x(4a^{-2} + 4) + x^2(2a^{-1} - 3a - 4a^3 + a^5) + x^3(-4a^{-2} - 8 - 2a^2 + 2a^4) + x^4(-3a^{-1} + 3a^3) + x^5(a^{-2} + 3 + 2a^2) + x^6(a^{-1} + a)$
$\mathbf{F}(7_5) = (2a^{-3} - a) + x(-a^{-2} + 1 + a^2 - a^4) + x^2(-3a^{-3} + a - 2a^3) + x^3(-a^{-2} - 4 - 2a^2 + a^4) + x^4(a^{-3} - a^{-1} + 2a^3) + x^5(a^{-2} + 3 + 2a^2) + x^6(a^{-1} + a)$
$\mathbf{F}(7_6) = (a^{-3} + a^{-1} + 2a + a^3) + x(a^{-2} + 2 - a^4) + x^2(-2a^{-3} - 4a^{-1} - 4a - 2a^3) + x^3(-4a^{-2} - 6 - a^2 + a^4) + x^4(a^{-3} + a^{-1} + 2a + 2a^3) + x^5(2a^{-2} + 4 + 2a^2) + x^6(a^{-1} + a)$
$\mathbf{F}(7_7) = (a^{-3} + 2a^{-1} + 2a) + x(2a^{-2} + 3 + a^2) + x^2(-2a^{-3} - 6a^{-1} - 7a - 3a^3) + x^3(-4a^{-2} - 8 - 3a^2 + a^4) + x^4(a^{-3} + 2a^{-1} + 4a + 3a^3) + x^5(2a^{-2} + 5 + 3a^2) + x^6(a^{-1} + a)$
$\mathbf{F}(8_1) = (-a^{-6} - 1 - a^2) + x(-3a^{-1} - 3a) + x^2(a^{-6} + 7 + 6a^2) + x^3(a^{-5} - a^{-3} + 5a^{-1} + 7a) + x^4(a^{-4} - 2a^{-2} - 8 - 5a^2) + x^5(a^{-3} - 4a^{-1} - 5a) + x^6(a^{-2} + 2 + a^2) + x^7(a^{-1} + a)$
$\mathbf{F}(8_2) = (-3a^{-2} - 3 - a^2) + x(a^{-1} + a - a^3 - a^5) + x^2(7a^{-2} + 12 + 3a^2 - a^4 + a^6) + x^3(3a^{-1} - a - 2a^3 + 2a^5) + x^4(-5a^{-2} - 12 - 5a^2 + 2a^4) + x^5(-4a^{-1} - 2a + 2a^3) + x^6(a^{-2} + 3 + 2a^2) + x^7(a^{-1} + a)$
$\mathbf{F}(8_3) = (a^{-4} - 1 + a^4) + x(-4a^{-1} - 4a) + x^2(-3a^{-4} + a^{-2} + 8 + a^2 - 3a^4) + x^3(-2a^{-3} + 8a^{-1} + 8a - 2a^3) + x^4(a^{-4} - 2a^{-2} - 6 - 2a^2 + a^4) + x^5(a^{-3} - 4a^{-1} - 4a + a^3) + x^6(a^{-2} + 2 + a^2) + x^7(a^{-1} + a)$

$\mathbf{F}(8_4) = (-2a^{-2}-2+a^4)+x(-a^{-1}+a+2a^3)+x^2(7a^{-2}+10-a^2-3a^4+a^6)+x^3(4a^{-1}-3a-5a^3+2a^5)+x^4(-5a^{-2}-11-3a^2+3a^4)+x^5(-4a^{-1}-a+3a^3)+x^6(a^{-2}+3+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_5) = (-2a^{-2}-5-4a^2)+x(4a^{-3}+7a^{-1}+3a)+x^2(a^{-6}-2a^{-4}+4a^{-2}+15+8a^2)+x^3(2a^{-5}-8a^{-3}-10a^{-1})+x^4(3a^{-4}-7a^{-2}-15-5a^2)+x^5(4a^{-3}+a^{-1}-3a)+x^6(3a^{-2}+4+a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_6) = (2a^{-4}+a^{-2}-1-a^2)+x(-a^{-3}-3a^{-1}-a+a^3)+x^2(-3a^{-4}-2a^{-2}+6+3a^2-2a^4)+x^3(-a^{-3}+5a^{-1}+2a-4a^3)+x^4(a^{-4}-6-4a^2+a^4)+x^5(a^{-3}-2a^{-1}-a+2a^3)+x^6(a^{-2}+3+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_7) = (-2a^{-2}-4-a^2)+x(-a^{-5}+2a^{-1}+2a+a^3)+x^2(-2a^{-4}+4a^{-2}+12+6a^2)+x^3(a^{-5}-a^{-3}-2a^{-1}-3a-3a^3)+x^4(2a^{-4}-3a^{-2}-12-7a^2)+x^5(2a^{-3}-a+a^3)+x^6(2a^{-2}+4+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_8) = (-a^{-2}-1+2a^2+a^4)+x(2a^{-3}+3a^{-1}+a-a^3-a^5)+x^2(4a^{-2}+5-a^2-2a^4)+x^3(-3a^{-3}-5a^{-1}-3a+a^5)+x^4(-6a^{-2}-9-a^2+2a^4)+x^5(a^{-3}+a+2a^3)+x^6(2a^{-2}+4+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_9) = (-2a^{-2}-3-2a^2)+x(a^{-3}+a^{-1}+a+a^3)+x^2(-2a^{-4}+4a^{-2}+12+4a^2-2a^4)+x^3(-4a^{-3}-a^{-1}-a-4a^3)+x^4(a^{-4}-4a^{-2}-10-4a^2+a^4)+x^5(2a^{-3}+2a^3)+x^6(2a^{-2}+4+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_{10}) = (-3a^{-2}-6-2a^2)+x(-a^{-5}+2a^{-3}+6a^{-1}+5a+2a^3)+x^2(-a^{-4}+6a^{-2}+12+5a^2)+x^3(a^{-5}-3a^{-3}-9a^{-1}-8a-3a^3)+x^4(2a^{-4}-5a^{-2}-13-6a^2)+x^5(3a^{-3}+3a^{-1}+a+a^3)+x^6(3a^{-2}+5+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_{11}) = (a^{-4}-a^{-2}-2-a^2)+x(a^{-1}+3a+2a^3)+x^2(-2a^{-4}+6+2a^2-2a^4)+x^3(-3a^{-3}-2a^{-1}-3a-4a^3)+x^4(a^{-4}-2a^{-2}-7-3a^2+a^4)+x^5(2a^{-3}+a^{-1}+a+2a^3)+x^6(2a^{-2}+4+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_{12}) = (a^{-4}+a^{-2}+1+a^2+a^4)+x(a^{-3}+a^3)+x^2(-2a^{-4}-2a^{-2}-2a^2-2a^4)+x^3(-3a^{-3}-3a^{-1}-3a-3a^3)+x^4(a^{-4}-a^{-2}-4-a^2+a^4)+x^5(2a^{-3}+2a^{-1}+2a+2a^3)+x^6(2a^{-2}+4+2a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_{13}) = (-a^{-2}-2)+x(2a^{-3}+4a^{-1}+3a+a^3)+x^2(5a^{-2}+7-2a^4)+x^3(-3a^{-3}-7a^{-1}-9a-4a^3+a^5)+x^4(-6a^{-2}-11-2a^2+3a^4)+x^5(a^{-3}+a^{-1}+4a+4a^3)+x^6(2a^{-2}+5+3a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_{14}) = a^{-4}+x(a^{-3}+3a^{-1}+3a+a^3)+x^2(-2a^{-4}-a^{-2}+3+a^2-a^4)+x^3(-3a^{-3}-6a^{-1}-8a-5a^3)+x^4(a^{-4}-a^{-2}-7-4a^2+a^4)+x^5(2a^{-3}+3a^{-1}+4a+3a^3)+x^6(2a^{-2}+5+3a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_{15}) = (a^{-4}-3a^{-2}-4-a^2)+x(6a^{-1}+8a+2a^3)+x^2(-2a^{-4}+5a^{-2}+8-a^4)+x^3(-2a^{-3}-11a^{-1}-14a-5a^3)+x^4(a^{-4}-5a^{-2}-10-3a^2+a^4)+x^5(2a^{-3}+5a^{-1}+6a+3a^3)+x^6(3a^{-2}+6+3a^2)+x^7(a^{-1}+a)$

$\mathbf{F}(8_{16}) = (-2-a^2)+x(a^{-3}+3a^{-1}+4a+2a^3)+x^2(5a^{-2}+10+4a^2-a^4)+x^3(-2a^{-3}-6a^{-1}-10a-5a^3+a^5)+x^4(-8a^{-2}-18-7a^2+3a^4)+x^5(a^{-3}-a^{-1}+3a+5a^3)+x^6(3a^{-2}+8+5a^2)+x^7(2a^{-1}+2a)$

$\mathbf{F}(8_{17}) = (-a^{-2}-1-a^2)+x(a^{-3}+2a^{-1}+2a+a^3)+x^2(-a^{-4}+3a^{-2}+8+3a^2-a^4)+x^3(-4a^{-3}-6a^{-1}-6a-4a^3)+x^4(a^{-4}-6a^{-2}-14-6a^2+a^4)+x^5(3a^{-3}+2a^{-1}+2a+3a^3)+x^6(4a^{-2}+8+4a^2)+x^7(2a^{-1}+2a)$

$\mathbf{F}(8_{18}) = (a^{-2}+3+a^2)+x(a^{-1}+a)+x^2(3a^{-2}+6+3a^2)+x^3(-4a^{-3}-9a^{-1}-9a-4a^3)+x^4(a^{-4}-9a^{-2}-20-9a^2+a^4)+x^5(4a^{-3}+3a^{-1}+3a+4a^3)+x^6(6a^{-2}+12+6a^2)+x^7(3a^{-1}+3a)$

$\mathbf{F}(8_{19}) = (-a^{-2}-5-5a^2)+x(5a^{-1}+5a)+x^2(10+10a^2)+x^3(-5a^{-1}-5a)+x^4(-6-6a^2)+x^5(a^{-1}+a)+x^6(1+a^2)$

$\mathbf{F}(8_{20}) = (-a^{-2}-4-2a^2)+x(a^{-3}+3a^{-1}+5a+3a^3)+x^2(2a^{-2}+6+4a^2)+x^3(-3a^{-1}-7a-4a^3)+x^4(-4-4a^2)+x^5(a^{-1}+2a+a^3)+x^6(1+a^2)$

$\mathbf{F}(8_{21}) = (-3a^{-2}-3-a^2)+x(2a^{-1}+4a+2a^3)+x^2(3a^{-2}+5-2a^4)+x^3(-a^{-1}-6a-5a^3)+x^4(-2-a^2+a^4)+x^5(a^{-1}+3a+2a^3)+x^6(1+a^2)$

$\mathbf{F}(9_1) = (5a^{-1}+4a)+x(-4-a^2+a^4-a^6+a^8)+x^2(-20a^{-1}-14a+3a^3-2a^5+a^7)+x^3(10+6a^2-3a^4+a^6)+x^4(21a^{-1}+16a-4a^3+a^5)+x^5(-6-5a^2+a^4)+x^6(-8a^{-1}-7a+a^3)+x^7(1+a^2)+x^8(a^{-1}+a)$

$\mathbf{F}(9_2) = (-a^{-7}+a^{-1}+a)+x(-4-4a^2)+x^2(a^{-7}-6a^{-1}-7a)+x^3(a^{-6}-a^{-4}+a^{-2}+13+10a^2)+x^4(a^{-5}-2a^{-3}+8a^{-1}+11a)+x^5(a^{-4}-3a^{-2}-10-6a^2)+x^6(a^{-3}-5a^{-1}-6a)+x^7(a^{-2}+2+a^2)+x^8(a^{-1}+a)$

$\mathbf{F}(9_3) = (3a^{-1}+3a-a^3)+x(-2a^{-6}+a^{-4}-a^{-2}-4)+x^2(-a^{-5}+3a^{-3}-11a^{-1}-9a+6a^3)+x^3(a^{-6}-a^{-4}+4a^{-2}+9+3a^2)+x^4(a^{-5}-2a^{-3}+11a^{-1}+9a-5a^3)+x^5(a^{-4}-3a^{-2}-8-4a^2)+x^6(a^{-3}-5a^{-1}-5a+a^3)+x^7(a^{-2}+2+a^2)+x^8(a^{-1}+a)$

$\mathbf{F}(9_4) = (a^{-5}+2a^{-1}+2a)+x(-4-a^2+3a^4)+x^2(-3a^{-5}+a^{-3}-7a^{-1}-10a+a^3)+x^3(-2a^{-4}+4a^{-2}+12+2a^2-4a^4)+x^4(a^{-5}-2a^{-3}+11a^{-1}+11a-3a^3)+x^5(a^{-4}-3a^{-2}-8-3a^2+a^4)+x^6(a^{-3}-5a^{-1}-5a+a^3)+x^7(a^{-2}+2+a^2)+x^8(a^{-1}+a)$

$\mathbf{F}(9_5) = (a^{-1} - a^3 + a^5) + x(-6a^{-2} - 6) + x^2(-3a^{-1} + 4a + 3a^3 - 3a^5 + a^7) + x^3(11a^{-2} + 18 + a^2 - 4a^4 + 2a^6) + x^4(7a^{-1} - 3a - 7a^3 + 3a^5) + x^5(-6a^{-2} - 14 - 5a^2 + 3a^4) + x^6(-5a^{-1} - 2a + 3a^3) + x^7(a^{-2} + 3 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_6) = (-3a^{-3} - a^{-1} + a) + x(2a^{-2} - 1 - 2a^2 - a^6) + x^2(7a^{-3} + a^{-1} - 3a + a^3 - 2a^5) + x^3(8 + 6a^2 - a^4 + a^6) + x^4(-5a^{-3} + a^{-1} + 2a - 2a^3 + 2a^5) + x^5(-3a^{-2} - 10 - 5a^2 + 2a^4) + x^6(a^{-3} - 3a^{-1} - 2a + 2a^3) + x^7(a^{-2} + 3 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_7) = (2a^{-5} + a^{-3} + a^{-1} + a) + x(-a^{-4} - a^{-2} - 3 - 2a^2 + a^4) + x^2(-3a^{-5} - 2a^{-3} - 4a^{-1} - 2a + 3a^3) + x^3(-a^{-4} + 2a^{-2} + 11 + 5a^2 - 3a^4) + x^4(a^{-5} + 7a^{-1} + 2a - 6a^3) + x^5(a^{-4} - a^{-2} - 9 - 6a^2 + a^4) + x^6(a^{-3} - 3a^{-1} - 2a + 2a^3) + x^7(a^{-2} + 3 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_8) = (-a^{-3} - a^{-1} + 2a^3 + a^5) + x(-2a^{-2} - 3 - a^2 - a^4 - a^6) + x^2(4a^{-3} + 7a^{-1} + 2a - 3a^3 - 2a^5) + x^3(8a^{-2} + 11 + 2a^2 + a^6) + x^4(-4a^{-3} - 6a^{-1} - 4a + 2a^5) + x^5(-8a^{-2} - 13 - 3a^2 + 2a^4) + x^6(a^{-3} - a^{-1} + 2a^3) + x^7(2a^{-2} + 4 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_9) = (-2a^{-3} + a^{-1} + 2a) + x(a^{-2} - 2 + 2a^4 - a^6) + x^2(7a^{-3} - 3a^{-1} - 6a + 3a^3 - a^5) + x^3(a^{-2} + 5 - 3a^4 + a^6) + x^4(-5a^{-3} + 3a^{-1} + 2a - 4a^3 + 2a^5) + x^5(-3a^{-2} - 8 - 2a^2 + 3a^4) + x^6(a^{-3} - 3a^{-1} - a + 3a^3) + x^7(a^{-2} + 3 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{10}) = (2a^{-1} + a - 2a^3) + x(4a^{-4} - 4) + x^2(-11a^{-1} - 2a + 7a^3 - 2a^5) + x^3(-4a^{-4} - a^{-2} + 9 + 3a^2 - 3a^4) + x^4(-2a^{-3} + 9a^{-1} + 3a - 7a^3 + a^5) + x^5(a^{-4} - a^{-2} - 7 - 3a^2 + 2a^4) + x^6(a^{-3} - 3a^{-1} - a + 3a^3) + x^7(a^{-2} + 3 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{11}) = (-2a^{-3} - 3a^{-1} - a - a^3) + x(-a^{-6} + 2a^{-4} + 2a^{-2} - 2 - a^2) + x^2(-a^{-5} + 4a^{-3} + 6a^{-1} + 5a + 4a^3) + x^3(a^{-6} - 3a^{-4} - 3a^{-2} + 9 + 8a^2) + x^4(2a^{-5} - 4a^{-3} - 7a^{-1} - 5a - 4a^3) + x^5(3a^{-4} - a^{-2} - 12 - 8a^2) + x^6(3a^{-3} + a^{-1} - a + a^3) + x^7(2a^{-2} + 4 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{12}) = (a^{-5} - a^{-1} - 2a - a^3) + x(-2a^{-2} - 4 - a^2 + a^4) + x^2(-2a^{-5} - 2a^{-3} + 3a^{-1} + 7a + 4a^3) + x^3(-3a^{-4} + 4a^{-2} + 13 + 3a^2 - 3a^4) + x^4(a^{-5} - a^{-3} - a^{-1} - 5a - 6a^3) + x^5(2a^{-4} - 3a^{-2} - 11 - 5a^2 + a^4) + x^6(2a^{-3} + 2a^3) + x^7(2a^{-2} + 4 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{13}) = (a^{-1} - a - 3a^3) + x(2a^{-4} - 2a^{-2} - 3 + a^2) + x^2(2a^{-3} - 2a^{-1} + 6a + 8a^3 - 2a^5) + x^3(-3a^{-4} + 2a^{-2} + 9 + a^2 - 3a^4) + x^4(-5a^{-3} - a^{-1} - 4a - 7a^3 + a^5) + x^5(a^{-4} - 4a^{-2} - 9 - 2a^2 + 2a^4) + x^6(2a^{-3} + a + 3a^3) + x^7(2a^{-2} + 4 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{14}) = (-a^{-3} - 2a^{-1} - a + a^3) + x(-3a^{-2} - 5 - 2a^2) + x^2(4a^{-3} + 10a^{-1} + 8a - 2a^5) + x^3(9a^{-2} + 15 + 2a^2 - 3a^4 + a^6) + x^4(-4a^{-3} - 9a^{-1} - 12a - 4a^3 + 3a^5) + x^5(-8a^{-2} - 16 - 4a^2 + 4a^4) + x^6(a^{-3} + 3a + 4a^3) + x^7(2a^{-2} + 5 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{15}) = (-a^{-3} - a^{-1} + a + a^3 + a^5) + x(2a^{-4} + a^{-2} - 1 + a^2 + a^4) + x^2(3a^{-3} + 2a^{-1} - 2a - 3a^3 - 2a^5) + x^3(-3a^{-4} - a^{-2} + 5 - 3a^4) + x^4(-5a^{-3} - 4a^{-1} + a^5) + x^5(a^{-4} - 3a^{-2} - 7 - a^2 + 2a^4) + x^6(2a^{-3} + a^{-1} + a + 2a^3) + x^7(2a^{-2} + 4 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{16}) = (-3a - 4a^3) + x(2a^{-4} + 2a^{-2} + 4 + 4a^2) + x^2(-a^{-5} + 2a^{-3} + a^{-1} + 6a + 8a^3) + x^3(a^{-6} - 5a^{-4} - 5a^{-2} - 1 - 2a^2) + x^4(3a^{-5} - 6a^{-3} - 8a^{-1} - 4a - 5a^3) + x^5(5a^{-4} - a^{-2} - 8 - 2a^2) + x^6(5a^{-3} + 3a^{-1} - a + a^3) + x^7(3a^{-2} + 4 + a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{17}) = (-2a^{-3} - 3a^{-1} - 2a) + x(-a^{-2} + 1 + 3a^2 + a^4) + x^2(5a^{-3} + 13a^{-1} + 9a - a^3 - 2a^5) + x^3(6a^{-2} + 6 - 4a^2 - 3a^4 + a^6) + x^4(-4a^{-3} - 12a^{-1} - 14a - 3a^3 + 3a^5) + x^5(-7a^{-2} - 13 - 2a^2 + 4a^4) + x^6(a^{-3} + a^{-1} + 4a + 4a^3) + x^7(2a^{-2} + 5 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{18}) = (a^{-5} - a^{-3} + a) + x(2a^{-2} + 2a^4) + x^2(-2a^{-5} + 3a^{-3} - 2a + 3a^3) + x^3(-2a^{-4} - 4a^{-2} + 1 - 3a^4) + x^4(a^{-5} - 4a^{-3} - 2a^{-1} - 2a - 5a^3) + x^5(2a^{-4} + a^{-2} - 5 - 3a^2 + a^4) + x^6(3a^{-3} + 2a^{-1} + a + 2a^3) + x^7(2a^{-2} + 4 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{19}) = (a^{-5} + a^{-3} - a) + x(a^{-4} - a^{-2} - 3 - a^2) + x^2(-2a^{-5} - 3a^{-3} + 3a^{-1} + 8a + 4a^3) + x^3(-3a^{-4} + a^{-2} + 10 + 4a^2 - 2a^4) + x^4(a^{-5} - 4a^{-1} - 11a - 8a^3) + x^5(2a^{-4} - a^{-2} - 11 - 7a^2 + a^4) + x^6(2a^{-3} + 2a^{-1} + 3a + 3a^3) + x^7(2a^{-2} + 5 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{20}) = (-2a^{-3} - 2a^{-1} - 2a - a^3) + x(2a^2 + 2a^4) + x^2(5a^{-3} + 11a^{-1} + 10a + 3a^3 - a^5) + x^3(6a^{-2} + 5 - 7a^2 - 5a^4 + a^6) + x^4(-4a^{-3} - 11a^{-1} - 16a - 6a^3 + 3a^5) + x^5(-7a^{-2} - 12 + 5a^4) + x^6(a^{-3} + a^{-1} + 5a + 5a^3) + x^7(2a^{-2} + 5 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{21}) = (-a^{-3} - a^{-1} - a^3) + x(2a^{-4} - 3 - a^2) + x^2(3a^{-3} + 5a^{-1} + 6a + 3a^3 - a^5) + x^3(-3a^{-4} + 9 + 2a^2 - 4a^4) + x^4(-5a^{-3} - 7a^{-1} - 9a - 6a^3 + a^5) + x^5(a^{-4} - 3a^{-2} - 10 - 3a^2 + 3a^4) + x^6(2a^{-3} + 2a^{-1} + 4a + 4a^3) + x^7(2a^{-2} + 5 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{22}) = (-a^{-3} - 4a^{-1} - 4a - 2a^3) + x(a^{-4} + a^{-2} - 2 - 2a^2) + x^2(-a^{-5} + 5a^{-3} + 17a^{-1} + 16a + 5a^3) + x^3(a^{-6} - 4a^{-4} - 2a^{-2} + 10 + 7a^2) + x^4(3a^{-5} - 9a^{-3} - 23a^{-1} - 15a - 4a^3) + x^5(5a^{-4} - 4a^{-2} - 16 - 7a^2) + x^6(6a^{-3} + 7a^{-1} + 2a + a^3) + x^7(4a^{-2} + 6 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{23}) = (a^{-5} - 2a^{-3} - 2a^{-1}) + x(4a^{-2} + 4 + a^2 + a^4) + x^2(-2a^{-5} + 4a^{-3} + 6a^{-1} + 3a + 3a^3) + x^3(-2a^{-4} - 6a^{-2} - 2 - 2a^4) + x^4(a^{-5} - 4a^{-3} - 8a^{-1} - 10a - 7a^3) + x^5(2a^{-4} + 2a^{-2} - 6 - 5a^2 + a^4) + x^6(3a^{-3} + 4a^{-1} + 4a + 3a^3) + x^7(2a^{-2} + 5 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{24}) = (-a^{-3} - 3a^{-1} - 5a - 2a^3) + x(a^{-4} + 2a^{-2} + 2 + 3a^2 + 2a^4) + x^2(-a^{-5} + 2a^{-3} + 9a^{-1} + 10a + 4a^3) + x^3(-4a^{-4} - 3a^{-2} + 1 - 3a^2 - 3a^4) + x^4(a^{-5} - 5a^{-3} - 11a^{-1} - 10a - 5a^3) + x^5(3a^{-4} - a^{-2} - 7 - 2a^2 + a^4) + x^6(4a^{-3} + 5a^{-1} + 3a + 2a^3) + x^7(3a^{-2} + 5 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{25}) = (a^{-5} - a^{-3} - 3a^{-1} - 3a - a^3) + x(-a^{-2} - 1 + a^2 + a^4) + x^2(-2a^{-5} + 2a^{-3} + 13a^{-1} + 13a + 4a^3) + x^3(-2a^{-4} + 3a^{-2} + 5 - 2a^2 - 2a^4) + x^4(a^{-5} - 3a^{-3} - 15a^{-1} - 18a - 7a^3) + x^5(2a^{-4} - 3a^{-2} - 10 - 4a^2 + a^4) + x^6(3a^{-3} + 6a^{-1} + 6a + 3a^3) + x^7(3a^{-2} + 6 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{26}) = (-a^{-3} - 3a^{-1} - 3a) + x(a^{-4} + a^{-2} - 1 - a^2) + x^2(-a^{-5} + 2a^{-3} + 11a^{-1} + 13a + 5a^3) + x^3(-4a^{-4} - 2a^{-2} + 7 + 3a^2 - 2a^4) + x^4(a^{-5} - 5a^{-3} - 14a^{-1} - 16a - 8a^3) + x^5(3a^{-4} - a^{-2} - 11 - 6a^2 + a^4) + x^6(4a^{-3} + 6a^{-1} + 5a + 3a^3) + x^7(3a^{-2} + 6 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{27}) = (-a^{-3} - 2a^{-1} - 3a - a^3) + x(a^{-4} + 2a^{-2} + 2 + 2a^2 + a^4) + x^2(-a^{-5} + 3a^{-3} + 12a^{-1} + 12a + 4a^3) + x^3(-4a^{-4} - 4a^{-2} - 2a^2 - 2a^4) + x^4(a^{-5} - 5a^{-3} - 16a^{-1} - 17a - 7a^3) + x^5(3a^{-4} - 8 - 4a^2 + a^4) + x^6(4a^{-3} + 7a^{-1} + 6a + 3a^3) + x^7(3a^{-2} + 6 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{28}) = (-a^{-3} - 5a^{-1} - 4a - a^3) + x(a^{-4} + 3a^{-2} + 6 + 6a^2 + 2a^4) + x^2(5a^{-3} + 14a^{-1} + 12a + 2a^3 - a^5) + x^3(-2a^{-4} - 4a^{-2} - 7 - 9a^2 - 4a^4) + x^4(-7a^{-3} - 19a^{-1} - 17a - 4a^3 + a^5) + x^5(a^{-4} - 3a^{-2} - 5 + 2a^2 + 3a^4) + x^6(3a^{-3} + 7a^{-1} + 8a + 4a^3) + x^7(3a^{-2} + 6 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{29}) = (-a^{-3} - 3a^{-1} - 5a - 2a^3) + x(-a^{-2} - 1 + 2a^2 + 2a^4) + x^2(3a^{-3} + 12a^{-1} + 17a + 8a^3) + x^3(9a^{-2} + 14 - a^2 - 5a^4 + a^6) + x^4(-3a^{-3} - 11a^{-1} - 24a - 13a^3 + 3a^5) + x^5(-10a^{-2} - 24 - 8a^2 + 6a^4) + x^6(a^{-3} - a^{-1} + 6a + 8a^3) + x^7(3a^{-2} + 9 + 6a^2) + x^8(2a^{-1} + 2a)$

$\mathbf{F}(9_{30}) = (-2a^{-3} - 4a^{-1} - 4a - a^3) + x(a^{-4} + a^{-2} + 1 + 2a^2 + a^4) + x^2(-a^{-5} + 5a^{-3} + 17a^{-1} + 16a + 5a^3) + x^3(-3a^{-4} - 2a^{-2} - 3a^2 - 2a^4) + x^4(a^{-5} - 7a^{-3} - 23a^{-1} - 22a - 7a^3) + x^5(3a^{-4} - 2a^{-2} - 9 - 3a^2 + a^4) + x^6(5a^{-3} + 10a^{-1} + 8a + 3a^3) + x^7(4a^{-2} + 7 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{31}) = (-a^{-3} - 4a^{-1} - 2a) + x(a^{-4} + 3a^{-2} + 5 + 3a^2) + x^2(5a^{-3} + 15a^{-1} + 13a + 3a^3) + x^3(-2a^{-4} - 3a^{-2} - 5 - 8a^2 - 4a^4) + x^4(-7a^{-3} - 21a^{-1} - 23a - 8a^3 + a^5) + x^5(a^{-4} - 3a^{-2} - 7 + a^2 + 4a^4) + x^6(3a^{-3} + 8a^{-1} + 11a + 6a^3) + x^7(3a^{-2} + 7 + 4a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{32}) = (-a^{-3} - 2a^{-1} - a + a^3) + x(a^{-4} - 2 - a^2) + x^2(-a^{-5} + 4a^{-3} + 12a^{-1} + 10a + 3a^3) + x^3(-3a^{-4} + 2a^{-2} + 9 + 3a^2 - a^4) + x^4(a^{-5} - 6a^{-3} - 18a^{-1} - 19a - 8a^3) + x^5(3a^{-4} - 5a^{-2} - 18 - 9a^2 + a^4) + x^6(5a^{-3} + 7a^{-1} + 6a + 4a^3) + x^7(5a^{-2} + 10 + 5a^2) + x^8(2a^{-1} + 2a)$

$\mathbf{F}(9_{33}) = (-2a - a^3) + x(a^2 + a^4) + x^2(3a^{-3} + 9a^{-1} + 10a + 4a^3) + x^3(-3a^{-4} - a^{-2} + 5 + a^2 - 2a^4) + x^4(a^{-5} - 9a^{-3} - 20a^{-1} - 16a - 6a^3) + x^5(4a^{-4} - 5a^{-2} - 16 - 6a^2 + a^4) + x^6(7a^{-3} + 9a^{-1} + 5a + 3a^3) + x^7(6a^{-2} + 10 + 4a^2) + x^8(2a^{-1} + 2a)$

$\mathbf{F}(9_{34}) = (-a^{-3} - a^{-1} - a) + x(-a^{-2} - 1) + x^2(4a^{-3} + 11a^{-1} + 10a + 3a^3) + x^3(-2a^{-4} + 4a^{-2} + 12 + 5a^2 - a^4) + x^4(a^{-5} - 10a^{-3} - 23a^{-1} - 19a - 7a^3) + x^5(4a^{-4} - 10a^{-2} - 26 - 11a^2 + a^4) + x^6(8a^{-3} + 9a^{-1} + 5a + 4a^3) + x^7(8a^{-2} + 14 + 6a^2) + x^8(3a^{-1} + 3a)$

$\mathbf{F}(9_{35}) = (-3a^{-3} - a^{-1} + a) + x(-a^{-2} - 9 - 8a^2) + x^2(a^{-7} - 2a^{-5} + 12a^{-3} + 16a^{-1} + a) + x^3(2a^{-6} - 6a^{-4} + 3a^{-2} + 23 + 12a^2) + x^4(3a^{-5} - 15a^{-3} - 15a^{-1} + 3a) + x^5(4a^{-4} - 8a^{-2} - 18 - 6a^2) + x^6(5a^{-3} + a^{-1} - 4a) + x^7(3a^{-2} + 4 + a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{36}) = (-2a^{-3} - 4a^{-1} - 3a - 2a^3) + x(-a^{-6} + a^{-4} + a^{-2} - 2 - a^2) + x^2(-a^{-5} + 7a^{-3} + 15a^{-1} + 12a + 5a^3) + x^3(a^{-6} - 2a^{-4} + 9 + 6a^2) + x^4(2a^{-5} - 7a^{-3} - 17a^{-1} - 12a - 4a^3) + x^5(3a^{-4} - 4a^{-2} - 14 - 7a^2) + x^6(4a^{-3} + 4a^{-1} + a + a^3) + x^7(3a^{-2} + 5 + 2a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{37}) = (a^{-5} - 2a^{-1} - 2a) + x(-5a^{-2} - 7 - 2a^2) + x^2(-2a^{-5} + a^{-3} + 12a^{-1} + 14a + 5a^3) + x^3(-2a^{-4} + 6a^{-2} + 13 + 3a^2 - 2a^4) + x^4(a^{-5} - 3a^{-3} - 13a^{-1} - 17a - 8a^3) + x^5(2a^{-4} - 4a^{-2} - 13 - 6a^2 + a^4) + x^6(3a^{-3} + 5a^{-1} + 5a + 3a^3) + x^7(3a^{-2} + 6 + 3a^2) + x^8(a^{-1} + a)$

$\mathbf{F}(9_{38}) = (-4a^{-3} - 3a^{-1}) + x(3a^{-2} + 1 - a^2 + a^4) + x^2(-a^{-5} + 9a^{-3} + 10a^{-1} + 3a + 3a^3) + x^3(-2a^{-4} - 2a^{-2} + 5 + 3a^2 - 2a^4) + x^4(a^{-5} - 10a^{-3} - 15a^{-1} - 10a - 6a^3) + x^5(3a^{-4} - 4a^{-2} - 15 - 7a^2 + a^4) + x^6(6a^{-3} + 6a^{-1} + 3a + 3a^3) + x^7(5a^{-2} + 9 + 4a^2) + x^8(2a^{-1} + 2a)$

$\mathbf{F}(9_{39}) = (-a^{-3} - 2a^{-1} - 2a - 2a^3) + x(a^{-4} - a^{-2} - 3 - a^2) + x^2(3a^{-3} + 9a^{-1} + 12a + 5a^3 - a^5) + x^3(-2a^{-4} + 2a^{-2} + 12 + 5a^2 - 3a^4) + x^4(-6a^{-3} - 13a^{-1} - 15a - 7a^3 + a^5) + x^5(a^{-4} - 7a^{-2} - 18 - 7a^2 + 3a^4) + x^6(3a^{-3} + 3a^{-1} + 5a + 5a^3) + x^7(4a^{-2} + 9 + 5a^2) + x^8(2a^{-1} + 2a)$
$\mathbf{F}(9_{40}) = (2a^{-3} + 2a^{-1} + a) + x(-1 - a^2) + x^2(3a^{-1} + 7a + 4a^3) + x^3(6a^{-2} + 14 + 6a^2 - 2a^4) + x^4(-7a^{-3} - 17a^{-1} - 20a - 9a^3 + a^5) + x^5(a^{-4} - 15a^{-2} - 32 - 12a^2 + 4a^4) + x^6(5a^{-3} + 4a^{-1} + 7a + 8a^3) + x^7(8a^{-2} + 17 + 9a^2) + x^8(4a^{-1} + 4a)$
$\mathbf{F}(9_{41}) = (-3a^{-1} - 3a - a^3) + x(-2a^{-2} - 4 - 2a^2) + x^2(-a^{-5} + 6a^{-3} + 17a^{-1} + 13a + 3a^3) + x^3(a^{-6} - 3a^{-4} + 6a^{-2} + 19 + 9a^2) + x^4(3a^{-5} - 11a^{-3} - 23a^{-1} - 12a - 3a^3) + x^5(5a^{-4} - 11a^{-2} - 26 - 10a^2) + x^6(7a^{-3} + 5a^{-1} - a + a^3) + x^7(6a^{-2} + 9 + 3a^2) + x^8(2a^{-1} + 2a)$
$\mathbf{F}(9_{42}) = (-2a^{-1} - 3a - 2a^3) + x(-2 - 2a^2) + x^2(6a^{-1} + 12a + 6a^3) + x^3(6 + 6a^2) + x^4(-5a^{-1} - 10a - 5a^3) + x^5(-5 - 5a^2) + x^6(a^{-1} + 2a + a^3) + x^7(1 + a^2)$
$\mathbf{F}(9_{43}) = (-a^{-3} - 3a^{-1} - 4a - 3a^3) + x(a^{-4} + a^{-2}) + x^2(2a^{-3} + 9a^{-1} + 14a + 7a^3) + x^3(-2a^{-2} + 1 + 3a^2) + x^4(-8a^{-1} - 13a - 5a^3) + x^5(a^{-2} - 3 - 4a^2) + x^6(2a^{-1} + 3a + a^3) + x^7(1 + a^2)$
$\mathbf{F}(9_{44}) = (-a^{-3} - 2a^{-1} - 3a - a^3) + x(-a^{-2} - 1 + a^2 + a^4) + x^2(a^{-3} + 6a^{-1} + 10a + 5a^3) + x^3(2a^{-2} + 4 - a^2 - 3a^4) + x^4(-3a^{-1} - 10a - 7a^3) + x^5(-3 - 2a^2 + a^4) + x^6(a^{-1} + 3a + 2a^3) + x^7(1 + a^2)$
$\mathbf{F}(9_{45}) = (-2a^{-3} - 2a^{-1} - 2a - a^3) + x(2a^2 + 2a^4) + x^2(3a^{-3} + 6a^{-1} + 7a + 4a^3) + x^3(a^{-2} - 1 - 5a^2 - 3a^4) + x^4(-4a^{-1} - 10a - 6a^3) + x^5(a^{-2} + a^4) + x^6(2a^{-1} + 4a + 2a^3) + x^7(1 + a^2)$
$\mathbf{F}(9_{46}) = (2a^{-3} + a^{-1} - a - a^3) + x(-2a^{-2} - 6 - 4a^2) + x^2(3a^{-1} + 9a + 6a^3) + x^3(a^{-2} + 8 + 7a^2) + x^4(-4a^{-1} - 9a - 5a^3) + x^5(-5 - 5a^2) + x^6(a^{-1} + 2a + a^3) + x^7(1 + a^2)$
$\mathbf{F}(9_{47}) = (-a^{-3} - 2a^{-1} - a + a^3) + x(-3a^{-2} - 5 - 2a^2) + x^2(3a^{-3} + 9a^{-1} + 11a + 5a^3) + x^3(3a^{-2} + 6 + a^2 - 2a^4) + x^4(-7a^{-1} - 16a - 9a^3) + x^5(a^{-2} - 4 - 4a^2 + a^4) + x^6(3a^{-1} + 6a + 3a^3) + x^7(2 + 2a^2)$
$\mathbf{F}(9_{48}) = (2a^{-1} + 3a) + x(-4a^{-2} - 5 - a^2) + x^2(-a^{-1} + 2a + 2a^3 - a^5) + x^3(3a^{-2} + 5 - 3a^2 - 5a^4) + x^4(-6a - 5a^3 + a^5) + x^5(-1 + 2a^2 + 3a^4) + x^6(a^{-1} + 4a + 3a^3) + x^7(1 + a^2)$
$\mathbf{F}(9_{49}) = (-3a - 4a^3) + x(-4a^{-2} - 2 + 2a^2) + x^2(-a^{-1} + 10a + 9a^3 - 2a^5) + x^3(3a^{-2} + 3 - 3a^2 - 3a^4) + x^4(-9a - 8a^3 + a^5) + x^5(-1 + a^2 + 2a^4) + x^6(a^{-1} + 4a + 3a^3) + x^7(1 + a^2)$
$\mathbf{F}(10_1) = (-a^{-8} + 1 + a^2) + x(4a^{-1} + 4a) + x^2(a^{-8} - 11 - 10a^2) + x^3(a^{-7} - a^{-5} + a^{-3} - 11a^{-1} - 14a) + x^4(a^{-6} - 2a^{-4} + 3a^{-2} + 21 + 15a^2) + x^5(a^{-5} - 3a^{-3} + 12a^{-1} + 16a) + x^6(a^{-4} - 4a^{-2} - 12 - 7a^2) + x^7(a^{-3} - 6a^{-1} - 7a) + x^8(a^{-2} + 2 + a^2) + x^9(a^{-1} + a)$
$\mathbf{F}(10_2) = (4a^{-2} + 4 + a^2) + x(-2a^{-1} - a + a^3 - a^5 - a^7) + x^2(-14a^{-2} - 21 - 5a^2 - a^6 + a^8) + x^3(-3a^{-1} + 3a + 2a^3 - 2a^5 + 2a^7) + x^4(16a^{-2} + 33 + 11a^2 - 4a^4 + 2a^6) + x^5(10a^{-1} + 2a - 6a^3 + 2a^5) + x^6(-7a^{-2} - 18 - 9a^2 + 2a^4) + x^7(-6a^{-1} - 4a + 2a^3) + x^8(a^{-2} + 3 + 2a^2) + x^9(a^{-1} + a)$
$\mathbf{F}(10_3) = (a^{-6} + 1 - a^4) + x(6a^{-1} + 6a) + x^2(-3a^{-6} + a^{-4} - 12 - 2a^2 + 6a^4) + x^3(-2a^{-5} + 4a^{-3} - 15a^{-1} - 18a + 3a^3) + x^4(a^{-6} - 2a^{-4} + 6a^{-2} + 18 + 4a^2 - 5a^4) + x^5(a^{-5} - 3a^{-3} + 15a^{-1} + 15a - 4a^3) + x^6(a^{-4} - 4a^{-2} - 10 - 4a^2 + a^4) + x^7(a^{-3} - 6a^{-1} - 6a + a^3) + x^8(a^{-2} + 2 + a^2) + x^9(a^{-1} + a)$
$\mathbf{F}(10_4) = (2a^{-2} + 2 + a^6) + x(2a^{-1} - a - 3a^3) + x^2(-13a^{-2} - 16 + a^2 - 3a^6 + a^8) + x^3(-7a^{-1} + 7a + 8a^3 - 4a^5 + 2a^7) + x^4(16a^{-2} + 29 + 4a^2 - 6a^4 + 3a^6) + x^5(11a^{-1} - 2a - 10a^3 + 3a^5) + x^6(-7a^{-2} - 17 - 7a^2 + 3a^4) + x^7(-6a^{-1} - 3a + 3a^3) + x^8(a^{-2} + 3 + 2a^2) + x^9(a^{-1} + a)$
$\mathbf{F}(10_5) = (3a^{-2} + 5 + a^2) + x(-a^{-7} - a^{-3} - 3a^{-1} - 2a - a^3) + x^2(-2a^{-6} + a^{-4} - 9a^{-2} - 22 - 10a^2) + x^3(a^{-7} - a^{-5} + 3a^{-3} + 6a^{-1} + 7a + 6a^3) + x^4(2a^{-6} - 2a^{-4} + 10a^{-2} + 32 + 18a^2) + x^5(2a^{-5} - 4a^{-3} - 3a^{-1} - 2a - 5a^3) + x^6(2a^{-4} - 7a^{-2} - 20 - 11a^2) + x^7(2a^{-3} - 2a^{-1} - 3a + a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$
$\mathbf{F}(10_6) = (-3a^{-4} - 2a^{-2} + 1 + a^2) + x(2a^{-3} + 3a^{-1} + a^5) + x^2(7a^{-4} + 5a^{-2} - 10 - 5a^2 + a^4 - 2a^6) + x^3(-10a^{-1} - 2a + 4a^3 - 4a^5) + x^4(-5a^{-4} - 3a^{-2} + 18 + 12a^2 - 3a^4 + a^6) + x^5(-3a^{-3} + 8a^{-1} + 5a - 4a^3 + 2a^5) + x^6(a^{-4} - 2a^{-2} - 12 - 7a^2 + 2a^4) + x^7(a^{-3} - 4a^{-1} - 3a + 2a^3) + x^8(a^{-2} + 3 + 2a^2) + x^9(a^{-1} + a)$
$\mathbf{F}(10_7) = (a^{-6} + a^{-2} + 2 + a^2) + x(-2a^{-1} - 5a - 3a^3) + x^2(-2a^{-6} - 2a^{-4} - 4a^{-2} - 10 - 3a^2 + 3a^4) + x^3(-3a^{-5} + a^{-3} + 6a^{-1} + 10a + 8a^3) + x^4(a^{-6} - a^{-4} + 8a^{-2} + 20 + 6a^2 - 4a^4) + x^5(2a^{-5} - 2a^{-3} - 2a^{-1} - 6a - 8a^3) + x^6(2a^{-4} - 5a^{-2} - 15 - 7a^2 + a^4) + x^7(2a^{-3} - a^{-1} - a + 2a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$
$\mathbf{F}(10_8) = (3a^{-2} + 3 - a^4) + x(-a + a^3 + 2a^5) + x^2(-13a^{-2} - 18 + 3a^2 + 5a^4 - 2a^6 + a^8) + x^3(-6a^{-1} + 5a + 2a^3 - 7a^5 + 2a^7) + x^4(16a^{-2} + 30 + a^2 - 10a^4 + 3a^6) + x^5(11a^{-1} - a - 8a^3 + 4a^5) + x^6(-7a^{-2} - 17 - 6a^2 + 4a^4) + x^7(-6a^{-1} - 3a + 3a^3) + x^8(a^{-2} + 3 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_9) = (2a^{-2} + 4 + 3a^2) + x(a^{-5} - 2a^{-1} - 2a - a^3) + x^2(-2a^{-6} + a^{-4} - 8a^{-2} - 22 - 8a^2 + 3a^4) + x^3(-4a^{-5} + 4a^{-3} + 5a^{-1} + 4a + 7a^3) + x^4(a^{-6} - 3a^{-4} + 13a^{-2} + 31 + 10a^2 - 4a^4) + x^5(2a^{-5} - 4a^{-3} - 2a - 8a^3) + x^6(2a^{-4} - 7a^{-2} - 18 - 8a^2 + a^4) + x^7(2a^{-3} - 2a^{-1} - 2a + 2a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{10}) = (a^{-2} + 2 + a^2 + a^4) + x(-3a^{-3} - 6a^{-1} - 4a - a^3) + x^2(-8a^{-2} - 12 - 4a^2 - 2a^4 - 2a^6) + x^3(7a^{-3} + 17a^{-1} + 17a + 3a^3 - 3a^5 + a^7) + x^4(15a^{-2} + 26 + 5a^2 - 3a^4 + 3a^6) + x^5(-5a^{-3} - 10a^{-1} - 16a - 7a^3 + 4a^5) + x^6(-10a^{-2} - 21 - 7a^2 + 4a^4) + x^7(a^{-3} - a^{-1} + 2a + 4a^3) + x^8(2a^{-2} + 5 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{11}) = (-2a^{-4} - a^{-2} + 1 - a^4) + x(a^{-3} + 5a^{-1} + 2a - 2a^3) + x^2(7a^{-4} + 2a^{-2} - 12 + 5a^4 - 2a^6) + x^3(a^{-3} - 16a^{-1} - 5a + 9a^3 - 3a^5) + x^4(-5a^{-4} - a^{-2} + 16 + 5a^2 - 6a^4 + a^6) + x^5(-3a^{-3} + 11a^{-1} + 5a - 7a^3 + 2a^5) + x^6(a^{-4} - 2a^{-2} - 10 - 4a^2 + 3a^4) + x^7(a^{-3} - 4a^{-1} - 2a + 3a^3) + x^8(a^{-2} + 3 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{12}) = (2a^{-2} + 2 - 2a^2 - a^4) + x(2a^{-5} - 3a^{-1} - a + a^3 + a^5) + x^2(2a^{-4} - 8a^{-2} - 12 + 2a^2 + 4a^4) + x^3(-3a^{-5} - a^{-3} + 4a^{-1} + 5a - 3a^5) + x^4(-5a^{-4} + 8a^{-2} + 23 + 4a^2 - 6a^4) + x^5(a^{-5} - 3a^{-3} - a - 4a^3 + a^5) + x^6(2a^{-4} - 5a^{-2} - 14 - 5a^2 + 2a^4) + x^7(2a^{-3} - a^{-1} - a + 2a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{13}) = (a^{-6} - a^{-2} - 1 - a^2 - a^4) + x(-2a^{-3} + a - a^3) + x^2(-2a^{-6} + a^{-4} + 4a^{-2} - 1 + 2a^2 + 4a^4) + x^3(-2a^{-5} + 3a^{-3} + a + 6a^3) + x^4(a^{-6} - 3a^{-4} - 3a^{-2} + 6 + a^2 - 4a^4) + x^5(2a^{-5} - 3a^{-3} - 2a^{-1} - 4a - 7a^3) + x^6(3a^{-4} - 9 - 5a^2 + a^4) + x^7(3a^{-3} + a^{-1} + 2a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{14}) = (-a^{-4} + a^{-2} + 1) + x(-a^{-3} - 4a^{-1} - 2a + 2a^3 + a^5) + x^2(4a^{-4} - a^{-2} - 9 - 3a^2 - a^6) + x^3(6a^{-3} + 10a^{-1} + 8a - 4a^5) + x^4(-4a^{-4} + 2a^{-2} + 16 + 5a^2 - 4a^4 + a^6) + x^5(-7a^{-3} - 9a^{-1} - 9a - 4a^3 + 3a^5) + x^6(a^{-4} - 5a^{-2} - 14 - 4a^2 + 4a^4) + x^7(2a^{-3} + a^{-1} + 3a + 4a^3) + x^8(2a^{-2} + 5 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{15}) = (-2a^{-4} - 3a^{-2} + 1 + a^2) + x(-a^{-7} + a^{-5} + 3a^{-3} - 3a - 2a^3) + x^2(-a^{-6} + 7a^{-4} + 8a^{-2} - 7 - 7a^2) + x^3(a^{-7} - 2a^{-5} - 3a^{-3} + a^{-1} + 8a + 7a^3) + x^4(2a^{-6} - 7a^{-4} - 8a^{-2} + 16 + 15a^2) + x^5(3a^{-5} - 3a^{-3} - 5a^{-1} - 4a - 5a^3) + x^6(4a^{-4} - a^{-2} - 15 - 10a^2) + x^7(3a^{-3} - 2a + a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{16}) = (-a^{-4} + 2 + a^2 - a^4) + x(-4a^{-3} - 4a^{-1}) + x^2(-2a^{-6} + 5a^{-4} + 2a^{-2} - 11 - 2a^2 + 4a^4) + x^3(-3a^{-5} + 10a^{-3} + 8a^{-1} + 5a^3) + x^4(a^{-6} - 6a^{-4} + 2a^{-2} + 17 + 4a^2 - 4a^4) + x^5(2a^{-5} - 7a^{-3} - 4a^{-1} - 2a - 7a^3) + x^6(3a^{-4} - 3a^{-2} - 13 - 6a^2 + a^4) + x^7(3a^{-3} - a + 2a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{17}) = (2a^{-2} + 5 + 2a^2) + x(a^{-5} - 3a^{-1} - 3a + a^5) + x^2(3a^{-4} - 8a^{-2} - 22 - 8a^2 + 3a^4) + x^3(-3a^{-5} + 2a^{-3} + 6a^{-1} + 6a + 2a^3 - 3a^5) + x^4(-6a^{-4} + 11a^{-2} + 34 + 11a^2 - 6a^4) + x^5(a^{-5} - 5a^{-3} - 5a^3 + a^5) + x^6(2a^{-4} - 7a^{-2} - 18 - 7a^2 + 2a^4) + x^7(2a^{-3} - 2a^{-1} - 2a + 2a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{18}) = (-a^{-4} + 1 + a^2) + x(-2a^{-3} - 4a^{-1} - 4a - 2a^3) + x^2(4a^{-4} + a^{-2} - 8 - 3a^2 + a^4 - a^6) + x^3(6a^{-3} + 11a^{-1} + 14a + 5a^3 - 4a^5) + x^4(-4a^{-4} + a^{-2} + 17 + 6a^2 - 5a^4 + a^6) + x^5(-7a^{-3} - 10a^{-1} - 12a - 6a^3 + 3a^5) + x^6(a^{-4} - 5a^{-2} - 15 - 5a^2 + 4a^4) + x^7(2a^{-3} + a^{-1} + 3a + 4a^3) + x^8(2a^{-2} + 5 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{19}) = (a^{-2} + 3 + a^2) + x(-2a^{-3} - 4a^{-1} - 2a + a^3 + a^5) + x^2(-9a^{-2} - 13 + 3a^4 - a^6) + x^3(7a^{-3} + 13a^{-1} + 11a - 4a^5 + a^7) + x^4(16a^{-2} + 23 - 4a^2 - 8a^4 + 3a^6) + x^5(-5a^{-3} - 8a^{-1} - 15a - 7a^3 + 5a^5) + x^6(-10a^{-2} - 19 - 3a^2 + 6a^4) + x^7(a^{-3} - a^{-1} + 3a + 5a^3) + x^8(2a^{-2} + 5 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{20}) = (2a^{-6} + a^{-4} + 1 + a^2) + x(-a^{-5} - a^{-3} + 3a^{-1} + 2a - a^3) + x^2(-3a^{-6} - 2a^{-4} - 9 - 5a^2 + 3a^4) + x^3(-a^{-5} + 2a^{-3} - 8a^{-1} - 4a + 7a^3) + x^4(a^{-6} + 3a^{-2} + 17 + 9a^2 - 4a^4) + x^5(a^{-5} - a^{-3} + 9a^{-1} + 3a - 8a^3) + x^6(a^{-4} - 2a^{-2} - 12 - 8a^2 + a^4) + x^7(a^{-3} - 4a^{-1} - 3a + 2a^3) + x^8(a^{-2} + 3 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{21}) = (-a^{-4} + 2a^{-2} + 3 + a^2) + x(-2a^{-1} - a + 3a^3 + 2a^5) + x^2(4a^{-4} - 3a^{-2} - 14 - 5a^2 - 2a^6) + x^3(5a^{-3} + 3a^{-1} + 2a - 4a^5) + x^4(-4a^{-4} + 4a^{-2} + 20 + 9a^2 - 2a^4 + a^6) + x^5(-7a^{-3} - 3a^{-1} - 2a^3 + 2a^5) + x^6(a^{-4} - 6a^{-2} - 14 - 5a^2 + 2a^4) + x^7(2a^{-3} - a^{-1} - a + 2a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{22}) = (2a^{-2} + 2 - a^2 - 2a^4) + x(-a^{-3} + a^{-1} + a - a^3) + x^2(4a^{-4} - 6a^{-2} - 12 + 6a^2 + 6a^4 - 2a^6) + x^3(6a^{-3} - 4a^{-1} + 7a^3 - 3a^5) + x^4(-4a^{-4} + 6a^{-2} + 16 - a^2 - 6a^4 + a^6) + x^5(-7a^{-3} - a - 6a^3 + 2a^5) + x^6(a^{-4} - 6a^{-2} - 12 - 2a^2 + 3a^4) + x^7(2a^{-3} - a^{-1} + 3a^3) + x^8(2a^{-2} + 4 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{23}) = (2a^{-2} + 3) + x(2a^{-5} + a^{-3} - 2a^{-1} - 2a - a^3) + x^2(3a^{-4} - 6a^{-2} - 13 - a^2 + 3a^4) + x^3(-3a^{-5} - 2a^{-3} + 3a^{-1} + 9a + 5a^3 - 2a^5) + x^4(-5a^{-4} + 5a^{-2} + 20 + 3a^2 - 7a^4) + x^5(a^{-5} - 2a^{-3} - 2a^{-1} - 9a - 9a^3 + a^5) + x^6(2a^{-4} - 3a^{-2} - 13 - 5a^2 + 3a^4) + x^7(2a^{-3} + a^{-1} + 3a + 4a^3) + x^8(2a^{-2} + 5 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{24}) = (a^{-6} - a^{-4} - a^{-2} + 1 + a^2) + x(2a^{-3} + 4a^{-1} - 2a^3) + x^2(-2a^{-6} + 2a^{-4} + 5a^{-2} - 5 - 2a^2 +$

$4a^4)+x^3(-2a^{-5}-7a^{-1}-2a+7a^3)+x^4(a^{-6}-3a^{-4}-5a^{-2}+6+3a^2-4a^4)+x^5(2a^{-5}-2a^{-3}+a^{-1}-2a-7a^3)+x^6(3a^{-4}+a^{-2}-8-5a^2+a^4)+x^7(3a^{-3}+a^{-1}+2a^3)+x^8(2a^{-2}+4+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{25}) = (-2a^{-4}+2+a^2)+x(a^{-3}-2a+a^5)+x^2(5a^{-4}+4a^{-2}-4+a^2+3a^4-a^6)+x^3(4a^{-3}+2a^{-1}+3a+2a^3-3a^5)+x^4(-4a^{-4}-3a^{-2}+3-5a^2-6a^4+a^6)+x^5(-6a^{-3}-7a^{-1}-9a-5a^3+3a^5)+x^6(a^{-4}-3a^{-2}-8+a^2+5a^4)+x^7(2a^{-3}+2a^{-1}+5a+5a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{26}) = (2a^{-2}+3+a^2-a^4)+x(-2a^{-3}-2a^{-1}-a-a^3)+x^2(4a^{-4}-4a^{-2}-12+a^2+4a^4-a^6)+x^3(7a^{-3}+4a^{-1}+5a+5a^3-3a^5)+x^4(-4a^{-4}+4a^{-2}+14-2a^2-7a^4+a^6)+x^5(-7a^{-3}-6a^{-1}-9a-7a^3+3a^5)+x^6(a^{-4}-5a^{-2}-12-a^2+5a^4)+x^7(2a^{-3}+a^{-1}+4a+5a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{27}) = (-a^{-2}+1+a^2)+x(-a^{-3}-2a^{-1}-2a+a^5)+x^2(4a^{-4}+4a^{-2}-4-a^2+3a^4)+x^3(-2a^{-5}+5a^{-3}+11a^{-1}+7a+a^3-2a^5)+x^4(-7a^{-4}-3a^{-2}+7-3a^2-6a^4)+x^5(a^{-5}-8a^{-3}-14a^{-1}-12a-6a^3+a^5)+x^6(3a^{-4}-2a^{-2}-9-a^2+3a^4)+x^7(4a^{-3}+6a^{-1}+6a+4a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{28}) = (a^{-2}-3a^2-a^4)+x(-4a^{-3}-6a^{-1}-2a+a^3+a^5)+x^2(-5a^{-2}+10a^2+4a^4-a^6)+x^3(8a^{-3}+18a^{-1}+13a-2a^3-4a^5+a^7)+x^4(12a^{-2}+11-12a^2-8a^4+3a^6)+x^5(-5a^{-3}-12a^{-1}-18a-6a^3+5a^5)+x^6(-9a^{-2}-16-a^2+6a^4)+x^7(a^{-3}+4a+5a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{29}) = (-2a^{-4}-2a^{-2}-1-a^2-a^4)+x(2a^{-1}-2a^3)+x^2(5a^{-4}+6a^{-2}+4a^2+4a^4-a^6)+x^3(4a^{-3}+2a^{-1}+7a+6a^3-3a^5)+x^4(-4a^{-4}-4a^{-2}+3-5a^2-7a^4+a^6)+x^5(-6a^{-3}-8a^{-1}-12a-7a^3+3a^5)+x^6(a^{-4}-3a^{-2}-9+5a^4)+x^7(2a^{-3}+2a^{-1}+5a+5a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{30}) = (-2a^{-4}-a^{-2})+x(-a^{-3}-5a^{-1}-6a-2a^3)+x^2(-a^{-6}+5a^{-4}+9a^{-2}+2+a^2+2a^4)+x^3(-3a^{-5}+4a^{-3}+16a^{-1}+18a+9a^3)+x^4(a^{-6}-7a^{-4}-11a^{-2}+2+2a^2-3a^4)+x^5(3a^{-5}-6a^{-3}-19a^{-1}-20a-10a^3)+x^6(5a^{-4}+2a^{-2}-11-7a^2+a^4)+x^7(5a^{-3}+7a^{-1}+5a+3a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{31}) = (-a^{-4}-a^{-2}+2+a^2)+x(2a^{-5}+2a^{-3}-2a^{-1}-4a-2a^3)+x^2(3a^{-4}-2a^{-2}-10-3a^2+2a^4)+x^3(-3a^{-5}-3a^{-3}+6a^{-1}+15a+7a^3-2a^5)+x^4(-5a^{-4}+3a^{-2}+20+5a^2-7a^4)+x^5(a^{-5}-2a^{-3}-4a^{-1}-12a-10a^3+a^5)+x^6(2a^{-4}-3a^{-2}-14-6a^2+3a^4)+x^7(2a^{-3}+a^{-1}+3a+4a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{32}) = (-a^{-4}-a^{-2}-1)+x(a^{-5}+a^{-3}-a^{-1}-2a-a^3)+x^2(-a^{-6}+4a^{-4}+7a^{-2}+2a^4)+x^3(-3a^{-5}+7a^{-1}+13a+9a^3)+x^4(a^{-6}-6a^{-4}-11a^{-2}+2+3a^2-3a^4)+x^5(3a^{-5}-4a^{-3}-15a^{-1}-18a-10a^3)+x^6(5a^{-4}+3a^{-2}-10-7a^2+a^4)+x^7(5a^{-3}+7a^{-1}+5a+3a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{33}) = 1+x(-2a^{-3}-6a^{-1}-6a-2a^3)+x^2(3a^{-4}-6+3a^4)+x^3(-2a^{-5}+6a^{-3}+18a^{-1}+18a+6a^3-2a^5)+x^4(-7a^{-4}+a^{-2}+16+a^2-7a^4)+x^5(a^{-5}-9a^{-3}-16a^{-1}-16a-9a^3+a^5)+x^6(3a^{-4}-4a^{-2}-14-4a^2+3a^4)+x^7(4a^{-3}+5a^{-1}+5a+4a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{34}) = (a^{-2}+1+2a^4+a^6)+x(-3a^{-3}-4a^{-1}-a-a^5-a^7)+x^2(-6a^{-2}-8-3a^2-3a^4-2a^6)+x^3(7a^{-3}+12a^{-1}+5a-a^3+a^7)+x^4(14a^{-2}+20+4a^2+2a^6)+x^5(-5a^{-3}-6a^{-1}-5a-2a^3+2a^5)+x^6(-10a^{-2}-17-5a^2+2a^4)+x^7(a^{-3}-2a^{-1}-a+2a^3)+x^8(2a^{-2}+4+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{35}) = (-a^{-4}-a^{-2}+a^2+a^4+a^6)+x(-2a^{-3}-a^{-1}+a+a^3+a^5)+x^2(4a^{-4}+3a^{-2}-3-3a^2-3a^4-2a^6)+x^3(6a^{-3}+5a^{-1}-2a^3-3a^5)+x^4(-4a^{-4}+10+5a^2+a^6)+x^5(-7a^{-3}-6a^{-1}-a+2a^5)+x^6(a^{-4}-5a^{-2}-11-3a^2+2a^4)+x^7(2a^{-3}+2a^3)+x^8(2a^{-2}+4+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{36}) = (a^{-6}+a^{-4}+2a^{-2}+1)+x(a^{-5}+a^{-3}-3a^{-1}-4a-a^3)+x^2(-2a^{-6}-3a^{-4}-6a^{-2}-8-2a^2+a^4)+x^3(-3a^{-5}-2a^{-3}+8a^{-1}+16a+9a^3)+x^4(a^{-6}+6a^{-2}+18+8a^2-3a^4)+x^5(2a^{-5}-6a^{-1}-15a-11a^3)+x^6(2a^{-4}-3a^{-2}-16-10a^2+a^4)+x^7(2a^{-3}+a^{-1}+2a+3a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{37}) = (-a^{-4}-a^{-2}+1-a^2-a^4)+x(2a^{-5}+2a^{-3}-a^{-1}-a+2a^3+2a^5)+x^2(3a^{-4}-6+3a^4)+x^3(-3a^{-5}-3a^{-3}+a^{-1}+a-3a^3-3a^5)+x^4(-5a^{-4}+2a^{-2}+14+2a^2-5a^4)+x^5(a^{-5}-2a^{-3}-2a^3+a^5)+x^6(2a^{-4}-3a^{-2}-10-3a^2+2a^4)+x^7(2a^{-3}+2a^3)+x^8(2a^{-2}+4+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{38}) = (a^{-6}-a^{-4}-2a^{-2}-1)+x(-a-a^3)+x^2(-2a^{-6}+2a^{-4}+8a^{-2}+2+2a^4)+x^3(-2a^{-5}+a^{-3}+3a^{-1}+8a+8a^3)+x^4(a^{-6}-3a^{-4}-8a^{-2}+3+4a^2-3a^4)+x^5(2a^{-5}-2a^{-3}-7a^{-1}-13a-10a^3)+x^6(3a^{-4}+2a^{-2}-10-8a^2+a^4)+x^7(3a^{-3}+3a^{-1}+3a+3a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{39}) = (-2a^{-4}-a^{-2})+x(-a^{-1}+2a^3+a^5)+x^2(5a^{-4}+5a^{-2}-1+a^2+a^4-a^6)+x^3(4a^{-3}+$

$5a^{-1}+4a-a^3-4a^5)+x^4(-4a^{-4}-4a^{-2}+5-4a^4+a^6)+x^5(-6a^{-3}-9a^{-1}-9a-3a^3+3a^5)+x^6(a^{-4}-3a^{-2}-10-2a^2+4a^4)+x^7(2a^{-3}+2a^{-1}+4a+4a^3)+x^8(2a^{-2}+5+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{40})=(a^{-2}-3a^2-a^4)+x(a^{-5}+2a+2a^3+a^5)+x^2(3a^{-4}+a^{-2}+1+7a^2+4a^4)+x^3(-2a^{-5}+2a^{-3}+6a^{-1}+3a-a^3-2a^5)+x^4(-6a^{-4}-5a^{-2}-2-9a^2-6a^4)+x^5(a^{-5}-6a^{-3}-13a^{-1}-12a-5a^3+a^5)+x^6(3a^{-4}-5+a^2+3a^4)+x^7(4a^{-3}+7a^{-1}+7a+4a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{41})=(-a^{-4}-a^{-2}-2-2a^2-a^4)+x(-a^{-3}-2a^{-1}-2a+a^5)+x^2(3a^{-4}+7a^{-2}+9+10a^2+4a^4-a^6)+x^3(7a^{-3}+13a^{-1}+10a+a^3-3a^5)+x^4(-3a^{-4}-4a^{-2}-8-14a^2-6a^4+a^6)+x^5(-9a^{-3}-20a^{-1}-18a-4a^3+3a^5)+x^6(a^{-4}-5a^{-2}-7+4a^2+5a^4)+x^7(3a^{-3}+6a^{-1}+8a+5a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{42})=(-a^{-4}-3a^{-2}-2-a^2)+x(a^{-5}+a^{-3}-a^{-1}-a)+x^2(4a^{-4}+9a^{-2}+9+6a^2+2a^4)+x^3(-2a^{-5}+10a^{-1}+14a+5a^3-a^5)+x^4(-6a^{-4}-11a^{-2}-8-10a^2-7a^4)+x^5(a^{-5}-5a^{-3}-18a^{-1}-24a-11a^3+a^5)+x^6(3a^{-4}+2a^{-2}-5+4a^4)+x^7(4a^{-3}+9a^{-1}+11a+6a^3)+x^8(3a^{-2}+7+4a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{43})=(-a^{-4}-2a^{-2}-1-2a^2-a^4)+x(a^{-5}-3a^{-1}-3a+a^5)+x^2(3a^{-4}+7a^{-2}+8+7a^2+3a^4)+x^3(-2a^{-5}+a^{-3}+12a^{-1}+12a+a^3-2a^5)+x^4(-6a^{-4}-8a^{-2}-4-8a^2-6a^4)+x^5(a^{-5}-6a^{-3}-16a^{-1}-16a-6a^3+a^5)+x^6(3a^{-4}-6+3a^4)+x^7(4a^{-3}+7a^{-1}+7a+4a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{44})=(-a^{-4}-2a^{-2}-3-a^2)+x(-2a^{-3}-4a^{-1}-2a)+x^2(3a^{-4}+9a^{-2}+13+10a^2+3a^4)+x^3(8a^{-3}+20a^{-1}+15a-3a^5)+x^4(-3a^{-4}-6a^{-2}-12-18a^2-8a^4+a^6)+x^5(-9a^{-3}-26a^{-1}-27a-6a^3+4a^5)+x^6(a^{-4}-4a^{-2}-7+5a^2+7a^4)+x^7(3a^{-3}+8a^{-1}+12a+7a^3)+x^8(3a^{-2}+7+4a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{45})=(-2a^{-2}-3-2a^2)+x(-a^{-3}-5a^{-1}-5a-a^3)+x^2(3a^{-4}+12a^{-2}+18+12a^2+3a^4)+x^3(-a^{-5}+5a^{-3}+21a^{-1}+21a+5a^3-a^5)+x^4(-7a^{-4}-17a^{-2}-20-17a^2-7a^4)+x^5(a^{-5}-10a^{-3}-31a^{-1}-31a-10a^3+a^5)+x^6(4a^{-4}+3a^{-2}-2+3a^2+4a^4)+x^7(6a^{-3}+14a^{-1}+14a+6a^3)+x^8(4a^{-2}+8+4a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{46})=(3a^{-2}+8+6a^2)+x(2a^{-5}-2a^{-3}-10a^{-1}-6a)+x^2(a^{-8}-2a^{-6}+2a^{-4}-7a^{-2}-29-17a^2)+x^3(2a^{-7}-7a^{-5}+9a^{-3}+23a^{-1}+5a)+x^4(3a^{-6}-9a^{-4}+13a^{-2}+42+17a^2)+x^5(4a^{-5}-13a^{-3}-12a^{-1}+5a)+x^6(4a^{-4}-12a^{-2}-23-7a^2)+x^7(4a^{-3}-a^{-1}-5a)+x^8(3a^{-2}+4+a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{47})=(5a^{-2}+9+3a^2)+x(-a^{-7}+2a^{-5}-a^{-3}-9a^{-1}-8a-3a^3)+x^2(-a^{-6}+a^{-4}-15a^{-2}-26-9a^2)+x^3(a^{-7}-3a^{-5}+2a^{-3}+19a^{-1}+20a+7a^3)+x^4(2a^{-6}-3a^{-4}+15a^{-2}+35+15a^2)+x^5(3a^{-5}-5a^{-3}-14a^{-1}-11a-5a^3)+x^6(3a^{-4}-10a^{-2}-23-10a^2)+x^7(3a^{-3}+a^{-1}-a+a^3)+x^8(3a^{-2}+5+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{48})=(4a^{-2}+9+4a^2)+x(a^{-5}-3a^{-3}-9a^{-1}-7a+2a^5)+x^2(a^{-4}-13a^{-2}-27-11a^2+2a^4)+x^3(-3a^{-5}+8a^{-3}+21a^{-1}+12a-a^3-3a^5)+x^4(-5a^{-4}+18a^{-2}+37+9a^2-5a^4)+x^5(a^{-5}-9a^{-3}-11a^{-1}-5a-3a^3+a^5)+x^6(2a^{-4}-11a^{-2}-20-5a^2+2a^4)+x^7(3a^{-3}+a^{-1}+2a^3)+x^8(3a^{-2}+5+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{49})=(-a^{-4}+5a^{-2}+7+2a^2)+x(-9a^{-1}-10a+a^5)+x^2(4a^{-4}-13a^{-2}-20-2a^2-a^6)+x^3(3a^{-3}+22a^{-1}+24a+a^3-4a^5)+x^4(-4a^{-4}+15a^{-2}+26+2a^2-4a^4+a^6)+x^5(-6a^{-3}-18a^{-1}-19a-4a^3+3a^5)+x^6(a^{-4}-11a^{-2}-19-3a^2+4a^4)+x^7(2a^{-3}+3a^{-1}+5a+4a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{50})=(2a^{-2}+4+a^2-2a^4)+x(-6a^{-3}-10a^{-1}-3a+a^3)+x^2(-2a^{-6}+3a^{-4}-3a^{-2}-13+5a^4)+x^3(-3a^{-5}+16a^{-3}+22a^{-1}+6a+3a^3)+x^4(a^{-6}-5a^{-4}+9a^{-2}+18-a^2-4a^4)+x^5(2a^{-5}-11a^{-3}-15a^{-1}-8a-6a^3)+x^6(3a^{-4}-7a^{-2}-15-4a^2+a^4)+x^7(4a^{-3}+3a^{-1}+a+2a^3)+x^8(3a^{-2}+5+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{51})=(3a^{-2}+4-a^2-a^4)+x(2a^{-5}-3a^{-3}-9a^{-1}-5a+a^5)+x^2(a^{-4}-8a^{-2}-8+4a^2+3a^4)+x^3(-3a^{-5}+5a^{-3}+21a^{-1}+15a-2a^5)+x^4(-4a^{-4}+9a^{-2}+13-6a^2-6a^4)+x^5(a^{-5}-6a^{-3}-16a^{-1}-16a-6a^3+a^5)+x^6(2a^{-4}-6a^{-2}-12-a^2+3a^4)+x^7(3a^{-3}+5a^{-1}+6a+4a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{52})=(-a^{-4}+4+2a^2)+x(2a^{-5}-7a^{-1}-9a-4a^3)+x^2(6a^{-4}+4a^{-2}-9-7a^2)+x^3(a^{-7}-5a^{-5}+2a^{-3}+24a^{-1}+24a+8a^3)+x^4(3a^{-6}-12a^{-4}-9a^{-2}+19+13a^2)+x^5(6a^{-5}-11a^{-3}-28a^{-1}-16a-5a^3)+x^6(8a^{-4}-3a^{-2}-20-9a^2)+x^7(7a^{-3}+7a^{-1}+a+a^3)+x^8(4a^{-2}+6+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{53}) = (-3a^{-4}+3+a^2)+x(a^{-3}-7a^{-1}-11a-3a^3)+x^2(-a^{-6}+8a^{-4}+4a^{-2}-5+2a^2+2a^4)+x^3(-2a^{-5}+a^{-3}+21a^{-1}+28a+10a^3)+x^4(a^{-6}-9a^{-4}-7a^{-2}+6-3a^4)+x^5(3a^{-5}-6a^{-3}-26a^{-1}-27a-10a^3)+x^6(6a^{-4}-13-6a^2+a^4)+x^7(6a^{-3}+10a^{-1}+7a+3a^3)+x^8(4a^{-2}+7+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{54}) = (-2a^{-4}-2a^{-2}+3+2a^2)+x(-a^{-7}+a^{-5}+a^{-3}-5a^{-1}-8a-4a^3)+x^2(-a^{-6}+5a^{-4}+5a^{-2}-7-6a^2)+x^3(a^{-7}-2a^{-5}+2a^{-3}+17a^{-1}+20a+8a^3)+x^4(2a^{-6}-6a^{-4}-3a^{-2}+17+12a^2)+x^5(3a^{-5}-7a^{-3}-18a^{-1}-13a-5a^3)+x^6(4a^{-4}-5a^{-2}-18-9a^2)+x^7(4a^{-3}+3a^{-1}+a^3)+x^8(3a^{-2}+5+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{55}) = (a^{-6}-a^{-4}+a^{-2}+3+a^2)+x(2a^{-3}-4a^{-1}-9a-3a^3)+x^2(-2a^{-6}+2a^{-4}-3a^{-2}-8+a^2+2a^4)+x^3(-2a^{-5}-2a^{-3}+15a^{-1}+24a+9a^3)+x^4(a^{-6}-3a^{-4}+5a^{-2}+13+a^2-3a^4)+x^5(2a^{-5}-a^{-3}-16a^{-1}-23a-10a^3)+x^6(3a^{-4}-4a^{-2}-15-7a^2+a^4)+x^7(3a^{-3}+5a^{-1}+5a+3a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{56}) = (a^{-2}+2-2a^4)+x(-4a^{-3}-8a^{-1}-4a)+x^2(-a^{-6}+2a^{-4}-2a^{-2}-7+3a^2+5a^4)+x^3(-3a^{-5}+11a^{-3}+21a^{-1}+11a+4a^3)+x^4(a^{-6}-6a^{-4}+4a^{-2}+12-3a^2-4a^4)+x^5(3a^{-5}-11a^{-3}-21a^{-1}-13a-6a^3)+x^6(5a^{-4}-5a^{-2}-14-3a^2+a^4)+x^7(6a^{-3}+7a^{-1}+3a+2a^3)+x^8(4a^{-2}+6+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{57}) = (2a^{-2}+2-2a^2-a^4)+x(a^{-5}-3a^{-3}-6a^{-1}-2a+a^3+a^5)+x^2(2a^{-4}-2a^{-2}+8a^2+4a^4)+x^3(-2a^{-5}+6a^{-3}+18a^{-1}+12a-2a^5)+x^4(-5a^{-4}-a^{-2}-1-11a^2-6a^4)+x^5(a^{-5}-9a^{-3}-23a^{-1}-19a-5a^3+a^5)+x^6(3a^{-4}-3a^{-2}-7+2a^2+3a^4)+x^7(5a^{-3}+10a^{-1}+9a+4a^3)+x^8(4a^{-2}+7+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{58}) = (a^{-6}-2a^{-2}-3-2a^2-a^4)+x(-4a^{-3}-6a^{-1}-4a-2a^3)+x^2(-2a^{-6}+8a^{-2}+10+7a^2+3a^4)+x^3(-2a^{-5}+8a^{-3}+21a^{-1}+18a+7a^3)+x^4(a^{-6}-2a^{-4}-5a^{-2}-4-5a^2-3a^4)+x^5(2a^{-5}-6a^{-3}-22a^{-1}-23a-9a^3)+x^6(3a^{-4}-a^{-2}-10-5a^2+a^4)+x^7(4a^{-3}+7a^{-1}+6a+3a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{59}) = (-a^{-4}-3a^{-2}-4-2a^2-a^4)+x(a^{-5}-4a^{-1}-5a-2a^3)+x^2(-a^{-6}+3a^{-4}+10a^{-2}+11+8a^2+3a^4)+x^3(-3a^{-5}+4a^{-3}+20a^{-1}+21a+8a^3)+x^4(a^{-6}-5a^{-4}-11a^{-2}-8-6a^2-3a^4)+x^5(3a^{-5}-7a^{-3}-28a^{-1}-27a-9a^3)+x^6(5a^{-4}+a^{-2}-9-4a^2+a^4)+x^7(6a^{-3}+11a^{-1}+8a+3a^3)+x^8(4a^{-2}+7+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{60}) = (-a^{-4}-2a^{-2}-4-3a^2-a^4)+x(-2a^{-3}-6a^{-1}-7a-3a^3)+x^2(4a^{-4}+14a^{-2}+18+11a^2+3a^4)+x^3(-2a^{-5}+5a^{-3}+25a^{-1}+27a+9a^3)+x^4(a^{-6}-9a^{-4}-22a^{-2}-17-8a^2-3a^4)+x^5(4a^{-5}-11a^{-3}-38a^{-1}-32a-9a^3)+x^6(8a^{-4}+5a^{-2}-7-3a^2+a^4)+x^7(9a^{-3}+16a^{-1}+10a+3a^3)+x^8(5a^{-2}+8+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{61}) = (-a^{-4}+a^{-2}+5+4a^2)+x(-6a^{-3}-8a^{-1}-2a)+x^2(a^{-8}-2a^{-6}+6a^{-4}+a^{-2}-24-16a^2)+x^3(2a^{-7}-6a^{-5}+17a^{-3}+26a^{-1}+a)+x^4(3a^{-6}-13a^{-4}+5a^{-2}+38+17a^2)+x^5(4a^{-5}-18a^{-3}-16a^{-1}+6a)+x^6(5a^{-4}-10a^{-2}-22-7a^2)+x^7(5a^{-3}-5a)+x^8(3a^{-2}+4+a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{62}) = (4a^{-2}+7+2a^2)+x(-a^{-7}+a^{-5}-a^{-3}-6a^{-1}-5a-2a^3)+x^2(-a^{-6}+4a^{-4}-8a^{-2}-23-10a^2)+x^3(a^{-7}-2a^{-5}+5a^{-3}+16a^{-1}+15a+7a^3)+x^4(2a^{-6}-6a^{-4}+6a^{-2}+30+16a^2)+x^5(3a^{-5}-8a^{-3}-15a^{-1}-9a-5a^3)+x^6(4a^{-4}-7a^{-2}-21-10a^2)+x^7(4a^{-3}+2a^{-1}-a+a^3)+x^8(3a^{-2}+5+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{63}) = (a^{-6}+3a^{-2}+4+a^2)+x(-8a^{-1}-10a-2a^3)+x^2(-2a^{-6}+a^{-4}-10a^{-2}-16-2a^2+a^4)+x^3(-2a^{-5}+20a^{-1}+28a+10a^3)+x^4(a^{-6}-3a^{-4}+11a^{-2}+24+6a^2-3a^4)+x^5(2a^{-5}-2a^{-3}-16a^{-1}-23a-11a^3)+x^6(3a^{-4}-6a^{-2}-19-9a^2+a^4)+x^7(3a^{-3}+4a^{-1}+4a+3a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{64}) = (3a^{-2}+6+4a^2)+x(-4a^{-3}-6a^{-1}-3a-a^3)+x^2(-2a^{-6}+3a^{-4}-8a^{-2}-26-9a^2+4a^4)+x^3(-3a^{-5}+15a^{-3}+16a^{-1}+4a+6a^3)+x^4(a^{-6}-5a^{-4}+13a^{-2}+30+7a^2-4a^4)+x^5(2a^{-5}-11a^{-3}-11a^{-1}-5a-7a^3)+x^6(3a^{-4}-8a^{-2}-18-6a^2+a^4)+x^7(4a^{-3}+2a^{-1}+2a^3)+x^8(3a^{-2}+5+2a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{65}) = (3a^{-2}+5+a^2)+x(2a^{-5}-2a^{-3}-8a^{-1}-6a-2a^3)+x^2(a^{-4}-12a^{-2}-17-a^2+3a^4)+x^3(-3a^{-5}+4a^{-3}+19a^{-1}+20a+6a^3-2a^5)+x^4(-4a^{-4}+12a^{-2}+24+a^2-7a^4)+x^5(a^{-5}-6a^{-3}-14a^{-1}-17a-9a^3+a^5)+x^6(2a^{-4}-7a^{-2}-16-4a^2+3a^4)+x^7(3a^{-3}+4a^{-1}+5a+4a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$

$\mathbf{F}(10_{66}) = (-2a^{-4}+2a^{-2}+4+a^2)+x(a^{-3}-5a^{-1}-6a)+x^2(5a^{-4}-6a^{-2}-8+5a^2+2a^4)+x^3(2a^{-3}+20a^{-1}+22a+a^3-3a^5)+x^4(-4a^{-4}+8a^{-2}+8-13a^2-8a^4+a^6)+x^5(-5a^{-3}-$

$22a^{-1} - 28a - 7a^3 + 4a^5) + x^6(a^{-4} - 8a^{-2} - 13 + 3a^2 + 7a^4) + x^7(2a^{-3} + 6a^{-1} + 11a + 7a^3) + x^8(3a^{-2} + 7 + 4a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{67}) = a^{-6} + x(-2a^{-3} - 6a^{-1} - 6a - 2a^3) + x^2(-2a^{-6} + 2a^{-2} - 2 + 2a^4) + x^3(-2a^{-5} + 7a^{-3} + 19a^{-1} + 19a + 9a^3) + x^4(a^{-6} - 2a^{-4} - a^{-2} + 7 + 2a^2 - 3a^4) + x^5(2a^{-5} - 6a^{-3} - 19a^{-1} - 21a - 10a^3) + x^6(3a^{-4} - 2a^{-2} - 13 - 7a^2 + a^4) + x^7(4a^{-3} + 6a^{-1} + 5a + 3a^3) + x^8(3a^{-2} + 6 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{68}) = (-a^{-2} + 1 + a^2) + x(-2a^{-3} - 6a^{-1} - 8a - 4a^3) + x^2(-a^{-6} + 4a^{-4} + 7a^{-2} - 5 - 7a^2) + x^3(a^{-7} - 3a^{-5} + 8a^{-3} + 27a^{-1} + 23a + 8a^3) + x^4(3a^{-6} - 10a^{-4} - 9a^{-2} + 17 + 13a^2) + x^5(5a^{-5} - 14a^{-3} - 30a^{-1} - 16a - 5a^3) + x^6(7a^{-4} - 4a^{-2} - 20 - 9a^2) + x^7(7a^{-3} + 7a^{-1} + a + a^3) + x^8(4a^{-2} + 6 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{69}) = (-a^{-4} - 2a^{-2} - 2 - 2a^2) + x(a^{-5} - 2a^{-3} - 6a^{-1} - 4a - a^3) + x^2(3a^{-4} + 7a^{-2} + 12 + 11a^2 + 3a^4) + x^3(-2a^{-5} + 5a^{-3} + 23a^{-1} + 22a + 5a^3 - a^5) + x^4(-5a^{-4} - 9a^{-2} - 14 - 17a^2 - 7a^4) + x^5(a^{-5} - 8a^{-3} - 30a^{-1} - 32a - 10a^3 + a^5) + x^6(3a^{-4} - 4 + 3a^2 + 4a^4) + x^7(5a^{-3} + 13a^{-1} + 14a + 6a^3) + x^8(4a^{-2} + 8 + 4a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{70}) = (-a^{-4} - 2a^{-2} - 3 - 3a^2 - 2a^4) + x(a^{-5} + a^{-3} + a^{-1} - a^3) + x^2(-a^{-6} + 4a^{-4} + 9a^{-2} + 9 + 10a^2 + 5a^4) + x^3(-3a^{-5} + 2a^{-1} + 4a + 5a^3) + x^4(a^{-6} - 6a^{-4} - 12a^{-2} - 8 - 7a^2 - 4a^4) + x^5(3a^{-5} - 4a^{-3} - 11a^{-1} - 10a - 6a^3) + x^6(5a^{-4} + 3a^{-2} - 5 - 2a^2 + a^4) + x^7(5a^{-3} + 6a^{-1} + 3a + 2a^3) + x^8(3a^{-2} + 5 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{71}) = (-a^{-4} - 3a^{-2} - 3 - 3a^2 - a^4) + x(a^{-5} + a^{-3} - a^{-1} - a + a^3 + a^5) + x^2(4a^{-4} + 10a^{-2} + 12 + 10a^2 + 4a^4) + x^3(-2a^{-5} + 7a^{-1} + 7a - 2a^5) + x^4(-6a^{-4} - 12a^{-2} - 12 - 12a^2 - 6a^4) + x^5(a^{-5} - 5a^{-3} - 15a^{-1} - 15a - 5a^3 + a^5) + x^6(3a^{-4} + 2a^{-2} - 2 + 2a^2 + 3a^4) + x^7(4a^{-3} + 8a^{-1} + 8a + 4a^3) + x^8(3a^{-2} + 6 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{72}) = (-a^{-2} - 2 - 2a^2 - 2a^4) + x(a^{-3} - a^{-1} - 3a - a^3) + x^2(2a^{-4} + 6a^{-2} + 7 + 8a^2 + 5a^4) + x^3(-3a^{-5} + 9a^{-1} + 11a + 5a^3) + x^4(a^{-6} - 8a^{-4} - 11a^{-2} - 4 - 6a^2 - 4a^4) + x^5(4a^{-5} - 7a^{-3} - 19a^{-1} - 14a - 6a^3) + x^6(7a^{-4} + 2a^{-2} - 8 - 2a^2 + a^4) + x^7(7a^{-3} + 9a^{-1} + 4a + 2a^3) + x^8(4a^{-2} + 6 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{73}) = (-3a^{-2} - 4 - 3a^2 - a^4) + x(-a^{-3} - 3a^{-1} - 3a + a^5) + x^2(3a^{-4} + 12a^{-2} + 17 + 12a^2 + 4a^4) + x^3(-a^{-5} + 4a^{-3} + 16a^{-1} + 14a + a^3 - 2a^5) + x^4(-7a^{-4} - 16a^{-2} - 17 - 14a^2 - 6a^4) + x^5(a^{-5} - 10a^{-3} - 26a^{-1} - 21a - 5a^3 + a^5) + x^6(4a^{-4} + 2a^{-2} - 2 + 3a^2 + 3a^4) + x^7(6a^{-3} + 12a^{-1} + 10a + 4a^3) + x^8(4a^{-2} + 7 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{74}) = (-2a^{-4} + 2 + a^2) + x(-4a^{-1} - 8a - 4a^3) + x^2(-a^{-6} + 5a^{-4} + 8a^{-2} - 1 + a^2 + 4a^4) + x^3(-3a^{-5} + 3a^{-3} + 9a^{-1} + 11a + 8a^3) + x^4(a^{-6} - 7a^{-4} - 9a^{-2} + 3 - 4a^4) + x^5(3a^{-5} - 6a^{-3} - 12a^{-1} - 10a - 7a^3) + x^6(5a^{-4} + a^{-2} - 9 - 4a^2 + a^4) + x^7(5a^{-3} + 5a^{-1} + 2a + 2a^3) + x^8(3a^{-2} + 5 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{75}) = (-a^{-4} - 3a^{-2} - 3) + x(-3a^{-3} - 7a^{-1} - 5a - a^3) + x^2(3a^{-4} + 12a^{-2} + 20 + 15a^2 + 4a^4) + x^3(9a^{-3} + 24a^{-1} + 17a - a^3 - 3a^5) + x^4(-3a^{-4} - 9a^{-2} - 21 - 24a^2 - 8a^4 + a^6) + x^5(-9a^{-3} - 29a^{-1} - 29a - 5a^3 + 4a^5) + x^6(a^{-4} - 3a^{-2} - 4 + 7a^2 + 7a^4) + x^7(3a^{-3} + 9a^{-1} + 13a + 7a^3) + x^8(3a^{-2} + 7 + 4a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{76}) = (a^{-2} - 4a^2 - 4a^4) + x(-2a^{-3} + 2a^{-1} + 8a + 4a^3) + x^2(-a^{-6} + 3a^{-4} - 4a^{-2} - 7 + 9a^2 + 8a^4) + x^3(-3a^{-5} + 7a^{-3} - 3a^{-1} - 15a - 2a^3) + x^4(a^{-6} - 7a^{-4} + 4a^{-2} + 10 - 7a^2 - 5a^4) + x^5(3a^{-5} - 8a^{-3} - 2a^{-1} + 7a - 2a^3) + x^6(5a^{-4} - 3a^{-2} - 9 + a^4) + x^7(5a^{-3} + 2a^{-1} - 2a + a^3) + x^8(3a^{-2} + 4 + a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{77}) = (a^{-2} - 1 - 5a^2 - 2a^4) + x(a^{-5} - a^{-3} - a^{-1} + 3a + 4a^3 + 2a^5) + x^2(2a^{-4} - 2a^{-2} - 1 + 7a^2 + 4a^4) + x^3(-2a^{-5} + 2a^{-3} + 6a^{-1} - 5a^3 - 3a^5) + x^4(-6a^{-4} + 8 - 3a^2 - 5a^4) + x^5(a^{-5} - 7a^{-3} - 9a^{-1} - 3a - a^3 + a^5) + x^6(3a^{-4} - 3a^{-2} - 9 - a^2 + 2a^4) + x^7(4a^{-3} + 4a^{-1} + 2a + 2a^3) + x^8(3a^{-2} + 5 + 2a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{78}) = (-a^{-4} - a^{-2} - 4 - 4a^2 - a^4) + x(-a^{-3} - 3a^{-1} + 2a + 6a^3 + 2a^5) + x^2(3a^{-4} + 6a^{-2} + 11 + 10a^2 + a^4 - a^6) + x^3(7a^{-3} + 15a^{-1} + 5a - 7a^3 - 4a^5) + x^4(-3a^{-4} - 4a^{-2} - 7 - 10a^2 - 3a^4 + a^6) + x^5(-9a^{-3} - 21a^{-1} - 15a + 3a^5) + x^6(a^{-4} - 5a^{-2} - 8 + 2a^2 + 4a^4) + x^7(3a^{-3} + 6a^{-1} + 7a + 4a^3) + x^8(3a^{-2} + 6 + 3a^2) + x^9(a^{-1} + a)$

$\mathbf{F}(10_{79}) = (5a^{-2} + 11 + 5a^2) + x(2a^{-5} - 2a^{-3} - 11a^{-1} - 11a - 2a^3 + 2a^5) + x^2(a^{-4} - 13a^{-2} - 28 - 13a^2 + a^4) + x^3(-3a^{-5} + 4a^{-3} + 22a^{-1} + 22a + 4a^3 - 3a^5) + x^4(-4a^{-4} + 12a^{-2} + 32 + 12a^2 - 4a^4) + x^5(a^{-5} - 6a^{-3} - 15a^{-1} - 15a - 6a^3 + a^5) + x^6(2a^{-4} - 7a^{-2} - 18 - 7a^2 + 2a^4) +$

$x^7(3a^{-3}+4a^{-1}+4a+3a^3)+x^8(3a^{-2}+6+3a^2)+x^9(a^{-1}+a)$
$\mathbf{F}(10_{80})=(-2a^{-4}+3a^{-2}+6+2a^2)+x(a^{-3}-8a^{-1}-12a-2a^3+a^5)+x^2(5a^{-4}-7a^{-2}-13+2a^2+2a^4-a^6)+x^3(2a^{-3}+22a^{-1}+29a+6a^3-3a^5)+x^4(-4a^{-4}+8a^{-2}+13-5a^2-5a^4+a^6)+x^5(-5a^{-3}-23a^{-1}-29a-8a^3+3a^5)+x^6(a^{-4}-8a^{-2}-15-a^2+5a^4)+x^7(2a^{-3}+6a^{-1}+10a+6a^3)+x^8(3a^{-2}+7+4a^2)+x^9(a^{-1}+a)$
$\mathbf{F}(10_{81})=(-a^{-4}-a^{-2}+1-a^2-a^4)+x(a^{-5}-2a^{-3}-8a^{-1}-8a-2a^3+a^5)+x^2(3a^{-4}+6a^{-2}+6+6a^2+3a^4)+x^3(-2a^{-5}+5a^{-3}+25a^{-1}+25a+5a^3-2a^5)+x^4(-5a^{-4}-9a^{-2}-8-9a^2-5a^4)+x^5(a^{-5}-8a^{-3}-31a^{-1}-31a-8a^3+a^5)+x^6(3a^{-4}-6+3a^4)+x^7(5a^{-3}+13a^{-1}+13a+5a^3)+x^8(4a^{-2}+8+4a^2)+x^9(a^{-1}+a)$
$\mathbf{F}(10_{82})=a^{-2}+x(-a^{-3}-2a^{-1}+2a^3+a^5)+x^2(a^{-4}-6a^{-2}-13-5a^2-a^6)+x^3(7a^{-3}+10a^{-1}+5a-2a^3-4a^5)+x^4(-3a^{-4}+14a^{-2}+32+10a^2-4a^4+a^6)+x^5(-10a^{-3}-8a^{-1}-4a-3a^3+3a^5)+x^6(a^{-4}-14a^{-2}-27-8a^2+4a^4)+x^7(3a^{-3}-2a^{-1}-a+4a^3)+x^8(4a^{-2}+8+4a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{83})=(a^{-2}+2+2a^2)+x(-2a^{-3}-4a^{-1}-3a-a^3)+x^2(2a^{-4}-2a^{-2}-6+a^2+3a^4)+x^3(8a^{-3}+15a^{-1}+13a+4a^3-2a^5)+x^4(-3a^{-4}+7a^{-2}+13-7a^2-9a^4+a^6)+x^5(-9a^{-3}-17a^{-1}-23a-11a^3+4a^5)+x^6(a^{-4}-10a^{-2}-21-2a^2+8a^4)+x^7(3a^{-3}+3a^{-1}+9a+9a^3)+x^8(4a^{-2}+10+6a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{84})=(-2-4a^2-a^4)+x(-a^{-3}+2a+2a^3+a^5)+x^2(a^{-4}-a^{-2}+1+7a^2+4a^4)+x^3(-a^{-5}+6a^{-3}+11a^{-1}+4a-2a^3-2a^5)+x^4(-6a^{-4}+2a^{-2}+9-5a^2-6a^4)+x^5(a^{-5}-13a^{-3}-20a^{-1}-11a-4a^3+a^5)+x^6(4a^{-4}-8a^{-2}-17-2a^2+3a^4)+x^7(7a^{-3}+8a^{-1}+5a+4a^3)+x^8(6a^{-2}+10+4a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{85})=(-a^{-2}+1+a^2)+x(-a^{-3}-2a^{-1}-2a+a^5)+x^2(-7a^{-2}-14-5a^2+a^4-a^6)+x^3(4a^{-3}+11a^{-1}+14a+2a^3-4a^5+a^7)+x^4(19a^{-2}+37+8a^2-7a^4+3a^6)+x^5(-4a^{-3}-4a^{-1}-15a-10a^3+5a^5)+x^6(-14a^{-2}-32-12a^2+6a^4)+x^7(a^{-3}-5a^{-1}+6a^3)+x^8(3a^{-2}+8+5a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{86})=(a^{-2}+2+a^2+a^4)+x(-3a^{-3}-6a^{-1}-4a-a^3)+x^2(2a^{-4}-4a^{-2}-10-2a^2+2a^4)+x^3(-2a^{-5}+9a^{-3}+20a^{-1}+13a+3a^3-a^5)+x^4(-6a^{-4}+10a^{-2}+22-a^2-7a^4)+x^5(a^{-5}-11a^{-3}-18a^{-1}-17a-10a^3+a^5)+x^6(3a^{-4}-10a^{-2}-22-5a^2+4a^4)+x^7(5a^{-3}+5a^{-1}+6a+6a^3)+x^8(5a^{-2}+10+5a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{87})=(-a^{-2}-3-2a^2-a^4)+x(-a^{-1}-a+a^3+a^5)+x^2(a^{-4}+a^{-2}+3+7a^2+3a^4-a^6)+x^3(7a^{-3}+15a^{-1}+13a+2a^3-3a^5)+x^4(-2a^{-4}+5a^{-2}+8-5a^2-5a^4+a^6)+x^5(-11a^{-3}-23a^{-1}-21a-6a^3+3a^5)+x^6(a^{-4}-12a^{-2}-21-3a^2+5a^4)+x^7(4a^{-3}+5a^{-1}+7a+6a^3)+x^8(5a^{-2}+10+5a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{88})=(-a^{-2}-1-a^2)+x(-a^{-3}-4a^{-1}-4a-a^3)+x^2(3a^{-4}+7a^{-2}+8+7a^2+3a^4)+x^3(-a^{-5}+6a^{-3}+19a^{-1}+19a+6a^3-a^5)+x^4(-6a^{-4}-10a^{-2}-8-10a^2-6a^4)+x^5(a^{-5}-11a^{-3}-32a^{-1}-32a-11a^3+a^5)+x^6(4a^{-4}-2a^{-2}-12-2a^2+4a^4)+x^7(7a^{-3}+14a^{-1}+14a+7a^3)+x^8(6a^{-2}+12+6a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{89})=(a^{-4}-1-2a^2-a^4)+x(-2a^{-1}-4a-a^3+a^5)+x^2(3a^{-2}+6+6a^2+3a^4)+x^3(5a^{-3}+19a^{-1}+20a+4a^3-2a^5)+x^4(-6a^{-4}-9a^{-2}-2-4a^2-5a^4)+x^5(a^{-5}-15a^{-3}-35a^{-1}-27a-7a^3+a^5)+x^6(5a^{-4}-4a^{-2}-15-3a^2+3a^4)+x^7(9a^{-3}+15a^{-1}+11a+5a^3)+x^8(7a^{-2}+12+5a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{90})=(a^{-2}-2a^2-2a^4)+x(-a^{-3}-a^{-1}-2a-2a^3)+x^2(2a^{-4}-4a^{-2}-5+8a^2+6a^4-a^6)+x^3(7a^{-3}+7a^{-1}+9a+7a^3-2a^5)+x^4(-3a^{-4}+9a^{-2}+15-6a^2-8a^4+a^6)+x^5(-9a^{-3}-11a^{-1}-15a-10a^3+3a^5)+x^6(a^{-4}-11a^{-2}-21-3a^2+6a^4)+x^7(3a^{-3}+a^{-1}+5a+7a^3)+x^8(4a^{-2}+9+5a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{91})=(2a^{-2}+5+2a^2)+x(-3a^{-3}-6a^{-1}-4a+a^5)+x^2(a^{-4}-9a^{-2}-19-7a^2+2a^4)+x^3(-2a^{-5}+9a^{-3}+18a^{-1}+9a-2a^5)+x^4(-6a^{-4}+16a^{-2}+35+7a^2-6a^4)+x^5(a^{-5}-12a^{-3}-13a^{-1}-7a-6a^3+a^5)+x^6(3a^{-4}-13a^{-2}-26-7a^2+3a^4)+x^7(5a^{-3}+2a^{-1}+a+4a^3)+x^8(5a^{-2}+9+4a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{92})=(1+a^2-a^4)+x(-a^{-3}-5a^{-1}-5a-a^3)+x^2(2a^{-4}+2a^{-2}-2+a^2+3a^4)+x^3(-2a^{-5}+7a^{-3}+21a^{-1}+18a+6a^3)+x^4(a^{-6}-8a^{-4}-4a^{-2}+10+2a^2-3a^4)+x^5(4a^{-5}-14a^{-3}-32a^{-1}-22a-8a^3)+x^6(8a^{-4}-5a^{-2}-22-8a^2+a^4)+x^7(10a^{-3}+12a^{-1}+5a+3a^3)+x^8(7a^{-2}+11+4a^2)+x^9(2a^{-1}+2a)$
$\mathbf{F}(10_{93})=(-2a^2-a^4)+x(-2a^{-3}-6a^{-1}-6a-a^3+a^5)+x^2(-6a^{-2}-6+7a^2+7a^4)+x^3(5a^{-3}+$

$18a^{-1}+25a+7a^3-4a^5+a^7)+x^4(17a^{-2}+28-6a^2-14a^4+3a^6)+x^5(-4a^{-3}-10a^{-1}-29a-17a^3+6a^5)+x^6(-13a^{-2}-31-9a^2+9a^4)+x^7(a^{-3}-3a^{-1}+5a+9a^3)+x^8(3a^{-2}+9+6a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{94}) = (2a^{-2}+4+3a^2)+x(-3a^{-3}-5a^{-1}-3a-a^3)+x^2(-a^{-6}+2a^{-4}-6a^{-2}-18-7a^2+2a^4)+x^3(-3a^{-5}+9a^{-3}+16a^{-1}+10a+6a^3)+x^4(a^{-6}-6a^{-4}+10a^{-2}+31+11a^2-3a^4)+x^5(3a^{-5}-10a^{-3}-15a^{-1}-11a-9a^3)+x^6(5a^{-4}-9a^{-2}-27-12a^2+a^4)+x^7(6a^{-3}+3a^{-1}+3a^3)+x^8(5a^{-2}+9+4a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{95}) = (2a^{-2}+3)+x(a^{-5}-2a^{-3}-5a^{-1}-3a-a^3)+x^2(2a^{-4}-4a^{-2}-7+a^2+2a^4)+x^3(-2a^{-5}+5a^{-3}+17a^{-1}+16a+5a^3-a^5)+x^4(-5a^{-4}+4a^{-2}+13-2a^2-6a^4)+x^5(a^{-5}-8a^{-3}-21a^{-1}-25a-12a^3+a^5)+x^6(3a^{-4}-6a^{-2}-19-6a^2+4a^4)+x^7(5a^{-3}+8a^{-1}+10a+7a^3)+x^8(5a^{-2}+11+6a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{96}) = (-a^{-4}-2a^{-2}-3-3a^2-2a^4)+x(-2a^{-3}-2a^{-1}-a-a^3)+x^2(3a^{-4}+6a^{-2}+10+12a^2+5a^4)+x^3(7a^{-3}+16a^{-1}+17a+7a^3-a^5)+x^4(-3a^{-4}-a^{-2}-4-17a^2-10a^4+a^6)+x^5(-8a^{-3}-23a^{-1}-34a-15a^3+4a^5)+x^6(a^{-4}-7a^{-2}-17+9a^4)+x^7(3a^{-3}+6a^{-1}+14a+11a^3)+x^8(4a^{-2}+11+7a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{97}) = (-a^{-2}-2-2a^2-2a^4)+x(-4a^{-1}-6a-2a^3)+x^2(a^{-4}+a^{-2}+3+10a^2+6a^4-a^6)+x^3(8a^{-3}+20a^{-1}+24a+10a^3-2a^5)+x^4(-2a^{-4}+4a^{-2}+5-9a^2-7a^4+a^6)+x^5(-11a^{-3}-28a^{-1}-32a-12a^3+3a^5)+x^6(a^{-4}-11a^{-2}-21-3a^2+6a^4)+x^7(4a^{-3}+7a^{-1}+11a+8a^3)+x^8(5a^{-2}+11+6a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{98}) = (-a^{-4}+3a^{-2}+5+2a^2)+x(-6a^{-1}-12a-6a^3)+x^2(3a^{-4}-2a^{-2}-10+4a^4-a^6)+x^3(5a^{-3}+14a^{-1}+25a+14a^3-2a^5)+x^4(-3a^{-4}+4a^{-2}+17+2a^2-7a^4+a^6)+x^5(-8a^{-3}-17a^{-1}-26a-14a^3+3a^5)+x^6(a^{-4}-9a^{-2}-23-7a^2+6a^4)+x^7(3a^{-3}+3a^{-1}+8a+8a^3)+x^8(4a^{-2}+10+6a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{99}) = (4a^{-2}+9+4a^2)+x(a^{-5}-3a^{-3}-10a^{-1}-10a-3a^3+a^5)+x^2(a^{-4}-8a^{-2}-18-8a^2+a^4)+x^3(-2a^{-5}+5a^{-3}+21a^{-1}+21a+5a^3-2a^5)+x^4(-5a^{-4}+9a^{-2}+28+9a^2-5a^4)+x^5(a^{-5}-9a^{-3}-18a^{-1}-18a-9a^3+a^5)+x^6(3a^{-4}-9a^{-2}-24-9a^2+3a^4)+x^7(5a^{-3}+5a^{-1}+5a+5a^3)+x^8(5a^{-2}+10+5a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{100}) = (a^{-2}+5+3a^2)+x(-2a^{-3}-6a^{-1}-8a-2a^3+2a^5)+x^2(-7a^{-2}-17-6a^2+4a^4)+x^3(5a^{-3}+20a^{-1}+26a+5a^3-5a^5+a^7)+x^4(17a^{-2}+36+5a^2-11a^4+3a^6)+x^5(-4a^{-3}-11a^{-1}-27a-14a^3+6a^5)+x^6(-13a^{-2}-33-12a^2+8a^4)+x^7(a^{-3}-3a^{-1}+4a+8a^3)+x^8(3a^{-2}+9+6a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{101}) = (a^{-2}+4+2a^2-2a^4)+x(-a^{-3}-9a^{-1}-8a)+x^2(a^{-4}+a^{-2}-9-a^2+7a^4-a^6)+x^3(8a^{-3}+28a^{-1}+26a+4a^3-2a^5)+x^4(-2a^{-4}+3a^{-2}+15+a^2-8a^4+a^6)+x^5(-11a^{-3}-31a^{-1}-31a-8a^3+3a^5)+x^6(a^{-4}-11a^{-2}-24-6a^2+6a^4)+x^7(4a^{-3}+7a^{-1}+10a+7a^3)+x^8(5a^{-2}+11+6a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{102}) = (a^{-2}+1-a^4)+x(-2a^{-3}-4a^{-1}-4a-2a^3)+x^2(2a^{-4}-4a^{-2}-8+2a^2+3a^4-a^6)+x^3(7a^{-3}+13a^{-1}+16a+7a^3-3a^5)+x^4(-3a^{-4}+8a^{-2}+21+3a^2-6a^4+a^6)+x^5(-9a^{-3}-14a^{-1}-17a-9a^3+3a^5)+x^6(a^{-4}-11a^{-2}-24-7a^2+5a^4)+x^7(3a^{-3}+a^{-1}+4a+6a^3)+x^8(4a^{-2}+9+5a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{103}) = (-a^{-4}-3a^{-2}+a^2)+x(a^{-5}+a^{-3}-2a^{-1}-6a-4a^3)+x^2(3a^{-4}+2a^{-2}-8-6a^2+a^4)+x^3(-2a^{-5}-2a^{-3}+9a^{-1}+21a+10a^3-2a^5)+x^4(-6a^{-4}+25+13a^2-6a^4)+x^5(a^{-5}-5a^{-3}-9a^{-1}-16a-12a^3+a^5)+x^6(3a^{-4}-5a^{-2}-23-12a^2+3a^4)+x^7(4a^{-3}+2a^{-1}+3a+5a^3)+x^8(4a^{-2}+9+5a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{104}) = (a^{-2}+3+a^2)+x(-2a^{-3}-4a^{-1}-2a+a^3+a^5)+x^2(2a^{-4}-6a^{-2}-15-4a^2+3a^4)+x^3(-2a^{-5}+8a^{-3}+13a^{-1}+4a-a^3-2a^5)+x^4(-6a^{-4}+12a^{-2}+27+3a^2-6a^4)+x^5(a^{-5}-11a^{-3}-12a^{-1}-6a-5a^3+a^5)+x^6(3a^{-4}-11a^{-2}-22-5a^2+3a^4)+x^7(5a^{-3}+3a^{-1}+2a+4a^3)+x^8(5a^{-2}+9+4a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{105}) = (-1-a^2-a^4)+x(-a^{-3}-3a^{-1}-4a-2a^3)+x^2(3a^{-4}+5a^{-2}+4+5a^2+3a^4)+x^3(-2a^{-5}+6a^{-3}+19a^{-1}+18a+7a^3)+x^4(a^{-6}-8a^{-4}-9a^{-2}+2-a^2-3a^4)+x^5(4a^{-5}-13a^{-3}-33a^{-1}-24a-8a^3)+x^6(8a^{-4}-3a^{-2}-19-7a^2+a^4)+x^7(10a^{-3}+13a^{-1}+6a+3a^3)+x^8(7a^{-2}+11+4a^2)+x^9(2a^{-1}+2a)$

$\mathbf{F}(10_{106}) = (a^{-2}+2+2a^2)+x(a^{-5}+a^{-3}-a^{-1}-2a-a^3)+x^2(-a^{-6}+2a^{-4}-3a^{-2}-13-5a^2+2a^4)+x^3(-3a^{-5}+3a^{-3}+8a^{-1}+9a+7a^3)+x^4(a^{-6}-5a^{-4}+4a^{-2}+22+9a^2-3a^4)+$

$x^5(3a^{-5} - 7a^{-3} - 13a^{-1} - 12a - 9a^3) + x^6(5a^{-4} - 6a^{-2} - 23 - 11a^2 + a^4) + x^7(6a^{-3} + 4a^{-1} + a + 3a^3) + x^8(5a^{-2} + 9 + 4a^2) + x^9(2a^{-1} + 2a)$

$\mathbf{F}(10_{107}) = (-a^{-4} - 2a^{-2}) + x(a^{-5} - 3a^{-1} - 3a - a^3) + x^2(3a^{-4} + 3a^{-2} + 2a^2 + 2a^4) + x^3(-2a^{-5} + 3a^{-3} + 15a^{-1} + 17a + 6a^3 - a^5) + x^4(-5a^{-4} - 2a^{-2} + 5 - 4a^2 - 6a^4) + x^5(a^{-5} - 7a^{-3} - 22a^{-1} - 27a - 12a^3 + a^5) + x^6(3a^{-4} - 4a^{-2} - 16 - 5a^2 + 4a^4) + x^7(5a^{-3} + 9a^{-1} + 11a + 7a^3) + x^8(5a^{-2} + 11 + 6a^2) + x^9(2a^{-1} + 2a)$

$\mathbf{F}(10_{108}) = 1 + x(-2a^{-3} - 6a^{-1} - 6a - 2a^3) + x^2(-a^{-6} + 2a^{-4} - 10 - 7a^2) + x^3(a^{-7} - 3a^{-5} + 10a^{-3} + 28a^{-1} + 19a + 5a^3) + x^4(3a^{-6} - 9a^{-4} + 4a^{-2} + 33 + 17a^2) + x^5(5a^{-5} - 17a^{-3} - 29a^{-1} - 11a - 4a^3) + x^6(7a^{-4} - 13a^{-2} - 33 - 13a^2) + x^7(8a^{-3} + 4a^{-1} - 3a + a^3) + x^8(6a^{-2} + 9 + 3a^2) + x^9(2a^{-1} + 2a)$

$\mathbf{F}(10_{109}) = (3a^{-2} + 7 + 3a^2) + x(a^{-5} - a^{-3} - 5a^{-1} - 5a - a^3 + a^5) + x^2(2a^{-4} - 7a^{-2} - 18 - 7a^2 + 2a^4) + x^3(-2a^{-5} + 4a^{-3} + 13a^{-1} + 13a + 4a^3 - 2a^5) + x^4(-5a^{-4} + 6a^{-2} + 22 + 6a^2 - 5a^4) + x^5(a^{-5} - 8a^{-3} - 16a^{-1} - 16a - 8a^3 + a^5) + x^6(3a^{-4} - 7a^{-2} - 20 - 7a^2 + 3a^4) + x^7(5a^{-3} + 6a^{-1} + 6a + 5a^3) + x^8(5a^{-2} + 10 + 5a^2) + x^9(2a^{-1} + 2a)$

$\mathbf{F}(10_{110}) = (-a^{-4} - a^2 - a^4) + x(-a^{-3} - 3a^{-1} - 6a - 4a^3) + x^2(3a^{-4} + 2a^{-2} - 1 + 6a^2 + 5a^4 - a^6) + x^3(6a^{-3} + 13a^{-1} + 21a + 12a^3 - 2a^5) + x^4(-3a^{-4} + a^{-2} + 8 - 4a^2 - 7a^4 + a^6) + x^5(-8a^{-3} - 19a^{-1} - 27a - 13a^3 + 3a^5) + x^6(a^{-4} - 8a^{-2} - 20 - 5a^2 + 6a^4) + x^7(3a^{-3} + 4a^{-1} + 9a + 8a^3) + x^8(4a^{-2} + 10 + 6a^2) + x^9(2a^{-1} + 2a)$

$\mathbf{F}(10_{111}) = (a^{-2} + 3 + 2a^2 - a^4) + x(-4a^{-3} - 10a^{-1} - 7a - a^3) + x^2(-a^{-6} + a^{-4} - 3a^{-2} - 10 - 2a^2 + 3a^4) + x^3(-3a^{-5} + 13a^{-3} + 30a^{-1} + 19a + 5a^3) + x^4(a^{-6} - 5a^{-4} + 10a^{-2} + 22 + 3a^2 - 3a^4) + x^5(3a^{-5} - 13a^{-3} - 28a^{-1} - 20a - 8a^3) + x^6(5a^{-4} - 11a^{-2} - 26 - 9a^2 + a^4) + x^7(7a^{-3} + 7a^{-1} + 3a + 3a^3) + x^8(6a^{-2} + 10 + 4a^2) + x^9(2a^{-1} + 2a)$

$\mathbf{F}(10_{112}) = (-a^{-2} - 4 - 2a^2) + x(2a + 2a^3) + x^2(-3a^{-2} - 3 + a^2 + a^4) + x^3(6a^{-3} + 13a^{-1} + 9a - a^3 - 3a^5) + x^4(-2a^{-4} + 15a^{-2} + 28 + 3a^2 - 7a^4 + a^6) + x^5(-11a^{-3} - 16a^{-1} - 17a - 8a^3 + 4a^5) + x^6(a^{-4} - 18a^{-2} - 35 - 9a^2 + 7a^4) + x^7(4a^{-3} + 4a + 8a^3) + x^8(6a^{-2} + 13 + 7a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{113}) = (-a^{-2} - 3 - 3a^2) + x(a^{-3} + a^{-1} - a - a^3) + x^2(3a^{-2} + 8 + 8a^2 + 3a^4) + x^3(5a^{-3} + 16a^{-1} + 17a + 5a^3 - a^5) + x^4(-5a^{-4} - 4a^{-2} + 1 - 6a^2 - 6a^4) + x^5(a^{-5} - 16a^{-3} - 36a^{-1} - 30a - 10a^3 + a^5) + x^6(5a^{-4} - 9a^{-2} - 23 - 5a^2 + 4a^4) + x^7(10a^{-3} + 15a^{-1} + 12a + 7a^3) + x^8(9a^{-2} + 16 + 7a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{114}) = (-2 - a^2) + x(-a^{-3} - 3a^{-1} - 2a) + x^2(2a^{-4} - 5 - 3a^2) + x^3(-2a^{-5} + 5a^{-3} + 18a^{-1} + 18a + 7a^3) + x^4(a^{-6} - 8a^{-4} + a^{-2} + 26 + 14a^2 - 2a^4) + x^5(4a^{-5} - 13a^{-3} - 27a^{-1} - 21a - 11a^3) + x^6(8a^{-4} - 9a^{-2} - 35 - 17a^2 + a^4) + x^7(10a^{-3} + 8a^{-1} + 2a + 4a^3) + x^8(8a^{-2} + 14 + 6a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{115}) = (a^{-2} + 3 + a^2) + x(-2a^{-3} - 5a^{-1} - 5a - 2a^3) + x^2(2a^{-4} - a^{-2} - 6 - a^2 + 2a^4) + x^3(-a^{-5} + 8a^{-3} + 22a^{-1} + 22a + 8a^3 - a^5) + x^4(-5a^{-4} + a^{-2} + 12 + a^2 - 5a^4) + x^5(a^{-5} - 13a^{-3} - 34a^{-1} - 34a - 13a^3 + a^5) + x^6(4a^{-4} - 9a^{-2} - 26 - 9a^2 + 4a^4) + x^7(8a^{-3} + 13a^{-1} + 13a + 8a^3) + x^8(8a^{-2} + 16 + 8a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{116}) = a^{-2} + x(-a^{-3} - 3a^{-1} - 3a - a^3) + x^2(a^{-4} - a^{-2} - 3 + a^2 + 2a^4) + x^3(6a^{-3} + 17a^{-1} + 19a + 6a^3 - 2a^5) + x^4(-2a^{-4} + 9a^{-2} + 19 - a^2 - 8a^4 + a^6) + x^5(-10a^{-3} - 22a^{-1} - 29a - 13a^3 + 4a^5) + x^6(a^{-4} - 15a^{-2} - 32 - 8a^2 + 8a^4) + x^7(4a^{-3} + 3a^{-1} + 9a + 10a^3) + x^8(6a^{-2} + 14 + 8a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{117}) = (a^{-2} + 1 - a^2) + x(-3a^{-3} - 5a^{-1} - 3a - a^3) + x^2(a^{-4} - 3a^{-2} - 4 + 2a^2 + 2a^4) + x^3(-a^{-5} + 8a^{-3} + 21a^{-1} + 18a + 5a^3 - a^5) + x^4(-5a^{-4} + 6a^{-2} + 17 - 6a^4) + x^5(a^{-5} - 14a^{-3} - 29a^{-1} - 26a - 11a^3 + a^5) + x^6(4a^{-4} - 12a^{-2} - 28 - 8a^2 + 4a^4) + x^7(8a^{-3} + 10a^{-1} + 9a + 7a^3) + x^8(8a^{-2} + 15 + 7a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{118}) = 1 + x(-a^{-3} - 3a^{-1} - 3a - a^3) + x^2(a^{-4} - 2a^{-2} - 6 - 2a^2 + a^4) + x^3(-a^{-5} + 5a^{-3} + 15a^{-1} + 15a + 5a^3 - a^5) + x^4(-6a^{-4} + 6a^{-2} + 24 + 6a^2 - 6a^4) + x^5(a^{-5} - 12a^{-3} - 20a^{-1} - 20a - 12a^3 + a^5) + x^6(4a^{-4} - 11a^{-2} - 30 - 11a^2 + 4a^4) + x^7(7a^{-3} + 6a^{-1} + 6a + 7a^3) + x^8(7a^{-2} + 14 + 7a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{119}) = (-1 - a^2 - a^4) + x(-a^{-3} - 3a^{-1} - 4a - 2a^3) + x^2(a^{-4} + 1 + 6a^2 + 4a^4) + x^3(7a^{-3} + 19a^{-1} + 22a + 9a^3 - a^5) + x^4(-2a^{-4} + 8a^{-2} + 13 - 7a^2 - 9a^4 + a^6) + x^5(-10a^{-3} - 26a^{-1} - 37a - 17a^3 + 4a^5) + x^6(a^{-4} - 14a^{-2} - 31 - 7a^2 + 9a^4) + x^7(4a^{-3} + 5a^{-1} + 13a + 12a^3) + x^8(6a^{-2} + 15 + 9a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{120}) = (-3a^{-4} + 3 + a^2) + x(2a^{-3} - 4a^{-1} - 8a - 2a^3) + x^2(7a^{-4} - 7 + a^2 + a^4) + x^3(5a^{-3} + 26a^{-1} + 29a + 8a^3) + x^4(a^{-6} - 11a^{-4} - 3a^{-2} + 17 + 6a^2 - 2a^4) + x^5(4a^{-5} - 17a^{-3} - 44a^{-1} - 33a - 10a^3) + x^6(10a^{-4} - 9a^{-2} - 33 - 13a^2 + a^4) + x^7(13a^{-3} + 16a^{-1} + 7a + 4a^3) + x^8(10a^{-2} + 16 + 6a^2) + x^9(3a^{-1} + 3a)$

$\mathbf{F}(10_{121}) = (a^{-4}+a^{-2}+2+a^2)+x(-a^{-1}-3a-2a^3)+x^2(-3a^{-2}-7-3a^2+a^4)+x^3(4a^{-3}+14a^{-1}+19a+8a^3-a^5)+x^4(-5a^{-4}+3a^{-2}+22+9a^2-5a^4)+x^5(a^{-5}-15a^{-3}-30a^{-1}-28a-13a^3+a^5)+x^6(5a^{-4}-13a^{-2}-36-14a^2+4a^4)+x^7(10a^{-3}+11a^{-1}+9a+8a^3)+x^8(10a^{-2}+19+9a^2)+x^9(4a^{-1}+4a)$

$\mathbf{F}(10_{122}) = (-2a^{-2}-4-a^2)+x(2a^{-3}+2a^{-1})+x^2(2a^2+2a^4)+x^3(4a^{-3}+14a^{-1}+18a+6a^3-2a^5)+x^4(-a^{-4}+12a^{-2}+24+3a^2-7a^4+a^6)+x^5(-11a^{-3}-25a^{-1}-32a-14a^3+4a^5)+x^6(a^{-4}-20a^{-2}-42-13a^2+8a^4)+x^7(5a^{-3}+3a^{-1}+9a+11a^3)+x^8(8a^{-2}+18+10a^2)+x^9(4a^{-1}+4a)$

$\mathbf{F}(10_{123}) = (-2a^{-2}-3-2a^2)+x(-2a^{-1}-2a)+x^2(6a^{-2}+12+6a^2)+x^3(5a^{-3}+21a^{-1}+21a+5a^3)+x^4(-5a^{-4}-3a^{-2}+4-3a^2-5a^4)+x^5(a^{-5}-15a^{-3}-38a^{-1}-38a-15a^3+a^5)+x^6(5a^{-4}-11a^{-2}-32-11a^2+5a^4)+x^7(10a^{-3}+14a^{-1}+14a+10a^3)+x^8(10a^{-2}+20+10a^2)+x^9(4a^{-1}+4a)$

$\mathbf{F}(10_{124}) = (2a^{-2}+8+7a^2)+x(-8a^{-1}-8a)+x^2(-a^{-2}-22-21a^2)+x^3(14a^{-1}+14a)+x^4(21+21a^2)+x^5(-7a^{-1}-7a)+x^6(-8-8a^2)+x^7(a^{-1}+a)+x^8(1+a^2)$

$\mathbf{F}(10_{125}) = (3a^{-2}+7+3a^2)+x(a^{-5}-a^{-3}-6a^{-1}-8a-4a^3)+x^2(a^{-4}-6a^{-2}-15-8a^2)+x^3(a^{-3}+8a^{-1}+17a+10a^3)+x^4(2a^{-2}+13+11a^2)+x^5(-5a^{-1}-11a-6a^3)+x^6(-6-6a^2)+x^7(a^{-1}+2a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{126}) = (2a^{-2}+7+4a^2)+x(-2a^{-3}-6a^{-1}-8a-a^3+3a^5)+x^2(-4a^{-2}-16-11a^2+a^4)+x^3(a^{-3}+11a^{-1}+16a+2a^3-4a^5)+x^4(2a^{-2}+16+11a^2-3a^4)+x^5(-5a^{-1}-9a-3a^3+a^5)+x^6(-6-5a^2+a^4)+x^7(a^{-1}+2a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{127}) = (5a^{-2}+6+2a^2)+x(-5a^{-1}-8a-2a^3+a^5)+x^2(-9a^{-2}-14-2a^2+a^4-2a^6)+x^3(5a^{-1}+16a+7a^3-4a^5)+x^4(3a^{-2}+11+4a^2-3a^4+a^6)+x^5(-3a^{-1}-10a-5a^3+2a^5)+x^6(-4-2a^2+2a^4)+x^7(a^{-1}+3a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{128}) = (a^{-2}+4+2a^2-2a^4)+x(-6a^{-1}-5a+a^3)+x^2(-11-5a^2+6a^4)+x^3(11a^{-1}+13a+2a^3)+x^4(12+7a^2-5a^4)+x^5(-6a^{-1}-10a-4a^3)+x^6(-6-5a^2+a^4)+x^7(a^{-1}+2a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{129}) = (a^{-2}+2-a^2-a^4)+x(-2a^{-3}-5a^{-1}-5a-a^3+a^5)+x^2(-3a^{-2}-4+2a^2+3a^4)+x^3(a^{-3}+9a^{-1}+15a+4a^3-3a^5)+x^4(2a^{-2}+8-6a^4)+x^5(-4a^{-1}-11a-6a^3+a^5)+x^6(-4-2a^2+2a^4)+x^7(a^{-1}+3a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{130}) = (-a^{-4}-2a^{-2}+2+2a^2)+x(a^{-5}+a^{-3}-3a^{-1}-9a-6a^3)+x^2(2a^{-4}+6a^{-2}-4a^2)+x^3(-2a^{-3}+8a^{-1}+21a+11a^3)+x^4(-7a^{-2}+7a^2)+x^5(a^{-3}-8a^{-1}-15a-6a^3)+x^6(2a^{-2}-3-5a^2)+x^7(2a^{-1}+3a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{131}) = (-2a^{-4}+2+a^2)+x(a^{-3}-a^{-1}-5a-3a^3)+x^2(3a^{-4}+2a^{-2}-3+2a^2+4a^4)+x^3(a^{-3}+2a^{-1}+10a+9a^3)+x^4(-2a^{-2}-2-4a^2-4a^4)+x^5(a^{-3}-3a^{-1}-12a-8a^3)+x^6(2a^{-2}-a^2+a^4)+x^7(2a^{-1}+4a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{132}) = (3+2a^2)+x(-a^{-3}-4a^{-1}-8a-5a^3)+x^2(-a^{-2}-7-6a^2)+x^3(9a^{-1}+19a+10a^3)+x^4(10+10a^2)+x^5(-6a^{-1}-12a-6a^3)+x^6(-6-6a^2)+x^7(a^{-1}+2a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{133}) = (-a^{-4}+2a^{-2}+3+a^2)+x(-4a^{-1}-7a-3a^3)+x^2(a^{-4}-3a^{-2}-6+a^2+3a^4)+x^3(a^{-3}+7a^{-1}+16a+10a^3)+x^4(2a^{-2}+6-4a^4)+x^5(-4a^{-1}-13a-9a^3)+x^6(-4-3a^2+a^4)+x^7(a^{-1}+3a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{134}) = (a^{-2}+3-3a^4)+x(-2a^{-3}-8a^{-1}-4a+2a^3)+x^2(a^{-4}+a^{-2}-7+7a^4)+x^3(3a^{-3}+14a^{-1}+11a)+x^4(-a^{-2}+5+a^2-5a^4)+x^5(-8a^{-1}-11a-3a^3)+x^6(a^{-2}-3-3a^2+a^4)+x^7(2a^{-1}+3a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{135}) = (2a^{-2}+4-a^4)+x(-3a^{-3}-6a^{-1}-4a+a^3+2a^5)+x^2(-4a^{-2}-6+a^2+3a^4)+x^3(3a^{-3}+9a^{-1}+8a-a^3-3a^5)+x^4(2a^{-2}+3-4a^2-5a^4)+x^5(-4a^{-1}-8a-3a^3+a^5)+x^6(a^{-2}+a^2+2a^4)+x^7(2a^{-1}+4a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{136}) = (-a^{-2}-3-2a^2-a^4)+x(-2a^{-1}-4a-2a^3)+x^2(a^{-2}+4+6a^2+3a^4)+x^3(7a^{-1}+16a+9a^3)+x^4(2-2a^2-4a^4)+x^5(-5a^{-1}-14a-9a^3)+x^6(-4-3a^2+a^4)+x^7(a^{-1}+3a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{137}) = (-a^{-4}-a^{-2}-2-2a^2-a^4)+x(-a^{-3}-3a^{-1}-5a-3a^3)+x^2(a^{-4}+4a^{-2}+7+8a^2+4a^4)+x^3(2a^{-3}+9a^{-1}+15a+8a^3)+x^4(-2a^{-2}-5-7a^2-4a^4)+x^5(-7a^{-1}-15a-8a^3)+x^6(a^{-2}-1-a^2+a^4)+x^7(2a^{-1}+4a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{138}) = (-a^{-4}-2a^{-2}-3-3a^2-2a^4)+x(-2a^{-3}-2a^{-1}-a-a^3)+x^2(3a^{-4}+6a^{-2}+10+12a^2+5a^4)+x^3(3a^{-3}+5a^{-1}+8a+6a^3)+x^4(-5a^{-2}-13-12a^2-4a^4)+x^5(a^{-3}-6a^{-1}-14a-7a^3)+x^6(3a^{-2}+3+a^2+a^4)+x^7(3a^{-1}+5a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{139}) = (a^{-2}+6+6a^2)+x(-2a^{-5}-a^{-3}-5a^{-1}-6a)+x^2(-2a^{-4}-19-21a^2)+x^3(a^{-5}+a^{-3}+13a^{-1}+13a)+x^4(a^{-4}+20+21a^2)+x^5(-7a^{-1}-7a)+x^6(-8-8a^2)+x^7(a^{-1}+a)+x^8(1+a^2)$

$\mathbf{F}(10_{140}) = (a^{-4}+2a^{-2}+4+2a^2)+x(-2a^{-1}-6a-4a^3)+x^2(-4a^{-2}-12-8a^2)+x^3(6a^{-1}+16a+10a^3)+x^4(a^{-2}+12+11a^2)+x^5(-5a^{-1}-11a-6a^3)+x^6(-6-6a^2)+x^7(a^{-1}+2a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{141}) = (2a^{-2}+2+a^2)+x(-a^{-3}-3a^{-1}-4a-2a^3)+x^2(a^{-4}-4a^{-2}-9-a^2+3a^4)+x^3(2a^{-3}+5a^{-1}+13a+10a^3)+x^4(3a^{-2}+8+a^2-4a^4)+x^5(-3a^{-1}-12a-9a^3)+x^6(-4-3a^2+a^4)+x^7(a^{-1}+3a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{142}) = (a^{-2}+5+4a^2-a^4)+x(2a^{-3}-4a^{-1}-6a)+x^2(-a^{-2}-17-10a^2+6a^4)+x^3(9a^{-1}+12a+3a^3)+x^4(a^{-2}+15+9a^2-5a^4)+x^5(-5a^{-1}-9a-4a^3)+x^6(-6-5a^2+a^4)+x^7(a^{-1}+2a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{143}) = (3+2a^2)+x(-a^{-3}-3a^{-1}-5a-2a^3+a^5)+x^2(-4a^{-2}-10-3a^2+3a^4)+x^3(a^{-3}+7a^{-1}+14a+5a^3-3a^5)+x^4(3a^{-2}+11+2a^2-6a^4)+x^5(-3a^{-1}-10a-6a^3+a^5)+x^6(-4-2a^2+2a^4)+x^7(a^{-1}+3a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{144}) = (3a^{-2}+4+2a^2)+x(-2a-2a^3)+x^2(-7a^{-2}-12-2a^2+2a^4-a^6)+x^3(8a+4a^3-4a^5)+x^4(3a^{-2}+8-2a^2-6a^4+a^6)+x^5(-a^{-1}-8a-4a^3+3a^5)+x^6(-2+2a^2+4a^4)+x^7(a^{-1}+4a+3a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{145}) = (2a^{-4}+a^{-2}+1+a^2)+x(-a^{-3}-2a^{-1}-6a-5a^3)+x^2(-4a^{-4}-2a^{-2}-4-6a^2)+x^3(8a^{-1}+18a+10a^3)+x^4(a^{-4}+9+10a^2)+x^5(-6a^{-1}-12a-6a^3)+x^6(-6-6a^2)+x^7(a^{-1}+2a+a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{146}) = 1+x(-a^{-3}-3a^{-1}-3a-a^3)+x^2(-3a^{-2}-3+3a^2+3a^4)+x^3(a^{-3}+6a^{-1}+12a+5a^3-2a^5)+x^4(3a^{-2}+5-6a^2-8a^4)+x^5(-2a^{-1}-11a-8a^3+a^5)+x^6(-2+a^2+3a^4)+x^7(a^{-1}+4a+3a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{147}) = (-1-a^2-a^4)+x(-a^{-3}-3a^{-1}-4a-2a^3)+x^2(a^{-4}+1+6a^2+4a^4)+x^3(3a^{-3}+8a^{-1}+13a+8a^3)+x^4(-2-6a^2-4a^4)+x^5(-6a^{-1}-14a-8a^3)+x^6(a^{-2}-1-a^2+a^4)+x^7(2a^{-1}+4a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{148}) = (a^{-2}+5+3a^2)+x(-a^{-3}-3a^{-1}-5a-a^3+2a^5)+x^2(-3a^{-2}-11-6a^2+2a^4)+x^3(a^{-3}+6a^{-1}+9a+a^3-3a^5)+x^4(3a^{-2}+10+2a^2-5a^4)+x^5(-2a^{-1}-7a-4a^3+a^5)+x^6(-3-a^2+2a^4)+x^7(a^{-1}+3a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{149}) = (4a^{-2}+4+a^2)+x(-3a^{-1}-3a+a^3+a^5)+x^2(-7a^{-2}-9+a^4-a^6)+x^3(2a^{-1}+5a-a^3-4a^5)+x^4(3a^{-2}+5-4a^2-5a^4+a^6)+x^5(-a^{-1}-6a-2a^3+3a^5)+x^6(-1+3a^2+4a^4)+x^7(a^{-1}+4a+3a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{150}) = (-a^2-2a^4)+x(-a^{-3}-3a^{-1}-2a)+x^2(a^{-4}+a^{-2}+3+8a^2+5a^4)+x^3(3a^{-3}+6a^{-1}+8a+5a^3)+x^4(-5-9a^2-4a^4)+x^5(-5a^{-1}-12a-7a^3)+x^6(a^{-2}+a^4)+x^7(2a^{-1}+4a+2a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{151}) = (a^{-2}-3a^2-a^4)+x(-3a^{-3}-3a^{-1}+a+2a^3+a^5)+x^2(-2a^{-2}+4+10a^2+4a^4)+x^3(3a^{-3}+5a^{-1}+a-3a^3-2a^5)+x^4(2a^{-2}-6-15a^2-7a^4)+x^5(-2a^{-1}-7a-4a^3+a^5)+x^6(a^{-2}+3+5a^2+3a^4)+x^7(2a^{-1}+5a+3a^3)+x^8(1+a^2)$

$\mathbf{F}(10_{152}) = (8a^{-2}+10+3a^2)+x(-10a^{-1}-11a+a^3+2a^5)+x^2(-22a^{-2}-26-3a^2-a^4-2a^6)+x^3(17a^{-1}+19a-3a^3-5a^5)+x^4(21a^{-2}+25+2a^2-a^4+a^6)+x^5(-8a^{-1}-8a+2a^3+2a^5)+x^6(-8a^{-2}-9+a^4)+x^7(a^{-1}+a)+x^8(a^{-2}+1)$

$\mathbf{F}(10_{153}) = (3a^{-2}+6+a^2-a^4)+x(-5a^{-3}-10a^{-1}-6a+2a^3+3a^5)+x^2(-7a^{-2}-12-2a^2+3a^4)+x^3(10a^{-3}+22a^{-1}+12a-4a^3-4a^5)+x^4(10a^{-2}+14-4a^4)+x^5(-6a^{-3}-13a^{-1}-7a+a^3+a^5)+x^6(-6a^{-2}-7+a^4)+x^7(a^{-3}+2a^{-1}+a)+x^8(a^{-2}+1)$

$\mathbf{F}(10_{154}) = (a^{-2}+2-2a^2-4a^4)+x(-4a^{-3}-10a^{-1}-3a+3a^3)+x^2(3a^{-4}+2a^{-2}-5+5a^2+9a^4)+x^3(10a^{-3}+21a^{-1}+9a-2a^3)+x^4(-4a^{-4}-a^{-2}+7-2a^2-6a^4)+x^5(-9a^{-3}-15a^{-1}-6a)+x^6(a^{-4}-3a^{-2}-5+a^4)+x^7(2a^{-3}+3a^{-1}+a)+x^8(a^{-2}+1)$

$\mathbf{F}(10_{155}) = (2a^{-2}+4+3a^2)+x(-2a^{-3}-2a^{-1})+x^2(4a^{-4}-a^{-2}-11-5a^2+a^4)+x^3(8a^{-3}+6a^{-1}+2a^3)+x^4(-4a^{-4}-a^{-2}+7+4a^2)+x^5(-8a^{-3}-9a^{-1}-a)+x^6(a^{-4}-2a^{-2}-3)+x^7(2a^{-3}+3a^{-1}+a)+x^8(a^{-2}+1)$

$\mathbf{F}(10_{156}) = (-2a^{-2}-1)+x(-a^{-3}-2a^{-1}-2a-a^3)+x^2(4a^{-4}+7a^{-2}+1-2a^2)+x^3(-2a^{-5}+3a^{-3}+8a^{-1}+4a+a^3)+x^4(-8a^{-4}-9a^{-2}+2+3a^2)+x^5(a^{-5}-7a^{-3}-9a^{-1}-a)+x^6(3a^{-4}+2a^{-2}-1)+x^7(3a^{-3}+4a^{-1}+a)+x^8(a^{-2}+1)$

$\mathbf{F}(10_{157}) = (-a^{-2} + 2a^2) + x(4a^{-3} + 4a^{-1}) + x^2(2a^{-4} + 7a^{-2} - 5a^2) + x^3(-4a^{-5} - 8a^{-3} - 6a^{-1} - 2a) + x^4(a^{-6} - 8a^{-4} - 15a^{-2} - 3 + 3a^2) + x^5(4a^{-5} - 3a^{-1} + a) + x^6(6a^{-4} + 8a^{-2} + 2) + x^7(4a^{-3} + 5a^{-1} + a) + x^8(a^{-2} + 1)$

$\mathbf{F}(10_{158}) = (a^{-2} - 2a^2 - 2a^4) + x(2a^{-1} + a - a^3) + x^2(-5a^{-2} - 2 + 9a^2 + 5a^4 - a^6) + x^3(-4a^{-1} + 2a + 3a^3 - 3a^5) + x^4(3a^{-2} - 1 - 13a^2 - 8a^4 + a^6) + x^5(a^{-1} - 7a - 5a^3 + 3a^5) + x^6(1 + 6a^2 + 5a^4) + x^7(a^{-1} + 5a + 4a^3) + x^8(1 + a^2)$

$\mathbf{F}(10_{159}) = (-a^{-2} + 1 + a^2) + x(a^{-1} + a + a^3 + a^5) + x^2(-2a^{-2} - 4 + a^2 + 3a^4) + x^3(a^{-3} - a^3 - 2a^5) + x^4(4a^{-2} + 3 - 8a^2 - 7a^4) + x^5(a^{-1} - 5a - 5a^3 + a^5) + x^6(3a^2 + 3a^4) + x^7(a^{-1} + 4a + 3a^3) + x^8(1 + a^2)$

$\mathbf{F}(10_{160}) = (-a^{-2} - 1 - a^4) + x(2a^{-3} - 3a - a^3) + x^2(a^{-2} + 3a^2 + 4a^4) + x^3(3a^{-1} + 10a + 7a^3) + x^4(a^{-2} + 2 - 3a^2 - 4a^4) + x^5(-3a^{-1} - 11a - 8a^3) + x^6(-3 - 2a^2 + a^4) + x^7(a^{-1} + 3a + 2a^3) + x^8(1 + a^2)$

$\mathbf{F}(10_{161}) = (-3a^{-2} - 1 + a^2) + x(2a^{-1} + a^3 + 3a^5) + x^2(9a^{-2} + 3 - 3a^2 + 3a^4) + x^3(-a^{-1} - 3a^3 - 4a^5) + x^4(-6a^{-2} - 1 + a^2 - 4a^4) + x^5(a^3 + a^5) + x^6(a^{-2} + a^4)$

$\mathbf{F}(10_{162}) = (1 - a^2 - 3a^4) + x(3a^{-3} + a^{-1} + 2a^3) + x^2(3a^{-2} - 3 + 3a^2 + 9a^4) + x^3(-4a^{-3} - 3a^{-1} - a^3) + x^4(-4a^{-2} + 1 - a^2 - 6a^4) + x^5(a^{-3} + a^{-1}) + x^6(a^{-2} + a^4)$

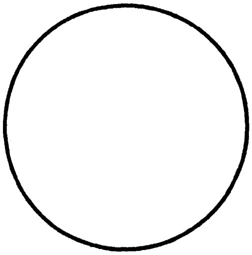
$\mathbf{F}(10_{163}) = (3a^{-2} + 3 - a^4) + x(-2a^{-1} - 7a - 5a^3) + x^2(-7a^{-2} - 9 + 5a^2 + 5a^4 - 2a^6) + x^3(15a + 12a^3 - 3a^5) + x^4(3a^{-2} + 6 - 4a^2 - 6a^4 + a^6) + x^5(-a^{-1} - 11a - 8a^3 + 2a^5) + x^6(-2 + a^2 + 3a^4) + x^7(a^{-1} + 4a + 3a^3) + x^8(1 + a^2)$

$\mathbf{F}(10_{164}) = (a^{-2} + 2 + a^2 + a^4) + x(-2a^{-3} - 3a^{-1} - a) + x^2(-4a^{-2} - 4 + 2a^2 + 2a^4) + x^3(3a^{-3} + 7a^{-1} + 8a + 3a^3 - a^5) + x^4(4a^{-2} + 1 - 11a^2 - 8a^4) + x^5(-4a^{-1} - 15a - 10a^3 + a^5) + x^6(a^{-2} + 3a^2 + 4a^4) + x^7(3a^{-1} + 8a + 5a^3) + x^8(2 + 2a^2)$

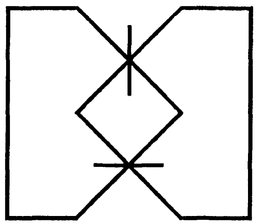
$\mathbf{F}(10_{165}) = (a^{-2} + 3 + a^2) + x(-2a^{-3} - 5a^{-1} - 5a - 2a^3) + x^2(-6a^{-2} - 9 + 3a^4) + x^3(3a^{-3} + 10a^{-1} + 16a + 7a^3 - 2a^5) + x^4(4a^{-2} + 8 - 3a^2 - 7a^4) + x^5(-6a^{-1} - 17a - 10a^3 + a^5) + x^6(a^{-2} - 3 - a^2 + 3a^4) + x^7(3a^{-1} + 7a + 4a^3) + x^8(2 + 2a^2)$

$\mathbf{F}(10_{166}) = (1 + a^2 - a^4) + x(-a^{-3} - 5a^{-1} - 5a - a^3) + x^2(2a^{-4} + 2a^{-2} - 2 + a^2 + 3a^4) + x^3(10a^{-3} + 18a^{-1} + 11a + 3a^3) + x^4(-3a^{-4} - 2a^{-2} - 2 - 3a^2) + x^5(-11a^{-3} - 22a^{-1} - 10a + a^3) + x^6(a^{-4} - 4a^{-2} - 2 + 3a^2) + x^7(3a^{-3} + 7a^{-1} + 4a) + x^8(2a^{-2} + 2)$

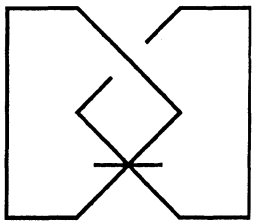
F.7 Surface-link diagram



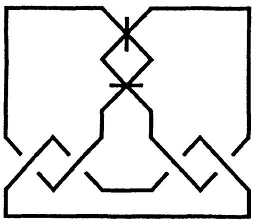
0_1



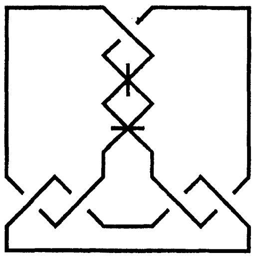
2_1^1



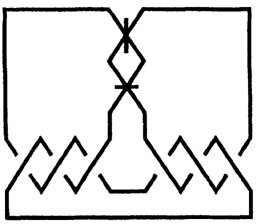
2_1^{-1}



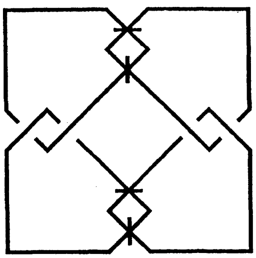
$6_1^{0,1}$



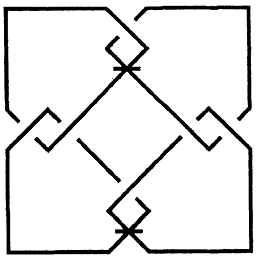
$7_1^{0,-2}$



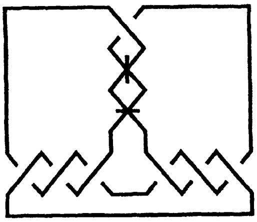
8_1



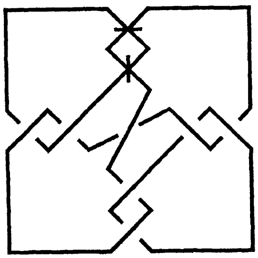
$8_1^{1,1}$



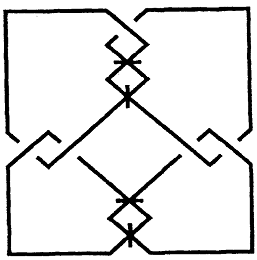
$8_1^{-1,-1}$



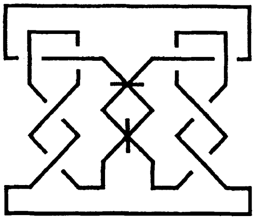
9^1



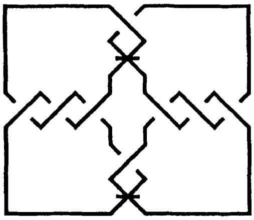
$9_1^{0,1}$



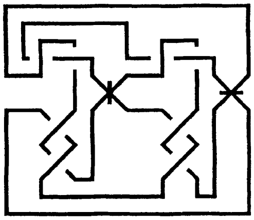
$9_1^{1,-2}$



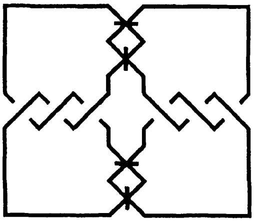
10_1



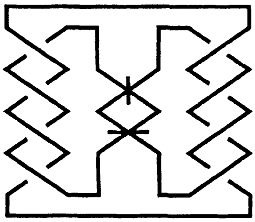
10_2



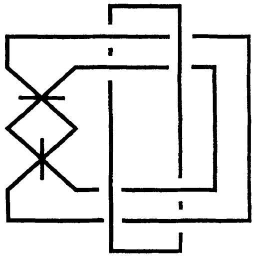
10_3



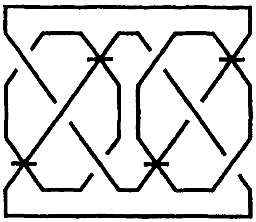
10_1^1



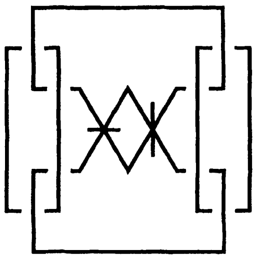
$10_1^{0,1}$



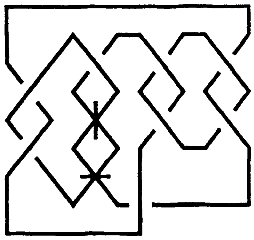
$10_2^{0,1}$



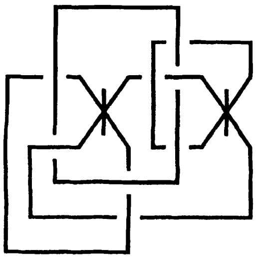
$10_1^{1,1}$



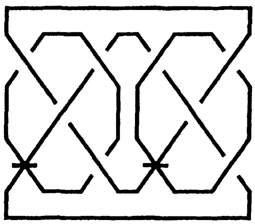
$10_1^{0,0,1}$



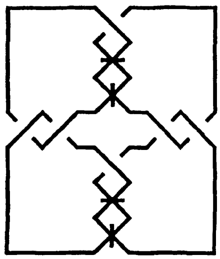
$10_1^{0,-2}$



$10_2^{0,-2}$



$10_1^{-1,-1}$



$10_1^{-2,-2}$

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MATHEMATICS WITH BIRKHÄUSER

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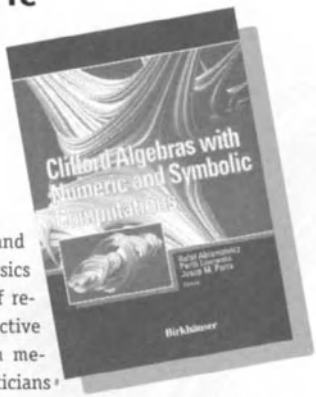
Clifford Algebras with Numeric and Symbolic Computations



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