EVALUATION AND DECISION MODELS WITH MULTIPLE CRITERIA

Stepping stones for the analyst

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Denis Bouyssou
CNRS – LAMSADE

Thierry Marchant
Universiteit Gent

Marc Pirlot
Faculté Polytechnique de Mons

Alexis Tsoukiàs
CNRS – LAMSADE

Philippe Vincke
Université Libre de Bruxelles

Springer

Denis Bouyssou
LAMSADE - CNRS
Paris, France

Thierry Marchant
Universiteit Gent
Belgium

Marc Pirlot
Polytechnique de Mons
Belgium

Alexis Tsoukias
LAMSADE - CNRS
Paris, France

Philippe Vincke
Universite Libre de Bruxelles
Belgium

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INTRODUCTION AND PREVIEW

1.1 Motivations

Deciding is a very complex and difficult task. Some people even argue that our ability to make decisions in complex situations is the main feature that distinguishes us from animals (it is also common to say that laughing is the main difference). Nevertheless, when the task is too complex or the interests at stake are too important, we quite often do not know or are not sure what to decide and, in many instances, we resort to a decision support technique: an informal one—we toss a coin, we ask an oracle, we visit an astrologer, we consult an expert—or a formal one. Although informal decision support techniques can be of interest, in this book, we will focus on formal ones. Among the latter, we find some well-known decision support techniques: cost-benefit analysis, multiple criteria decision analysis, decision trees, ... But there are many others, some not presented as decision support techniques, that help making decisions. Let us give a few examples.

- When a school director has to decide whether a given student will pass or fail, he usually asks each teacher to assess the student's merits by means of a grade. The director then sums the grades and compares the result to a threshold.
- When a bank has to decide whether a given client will obtain a credit, a technique, called credit scoring, is often used.
- When the mayor of a city decides to temporarily forbid car traffic in a city because of air pollution, he probably takes the value of some indicators, e.g. the air quality index, into account.
- Groups or committees also make decisions. In order to do so, they often use voting procedures.

All these formal techniques are what we call (formal) *decision and evaluation models*, i.e. a set of explicit and well-defined rules to collect, assess and process information in order to be able to make recommendations in decision and/or evaluation processes. They are so widespread that almost no one can pretend not to have used or suffered the consequences of one of them. These models—probably due to their formal character—inspire respect and trust: they seem scientific. But are they really well founded? Do they perform as well as we want them to? Can we safely rely on them when we have to make important decisions? It is crucial to answer these questions because formal models are so widespread in many domains of human activity. This is why we chose seven popular evaluation or decision models and thoroughly analysed them, revealing their weaknesses and how things can go wrong, in a previous volume published by the same authors and Patrice Perny (Evaluation and decision models: A critical perspective, Bouyssou, Marchant, Pirlot, Perny, Tsoukiàs, and Vincke, 2000). We also claimed that the difficulties encountered are not specific to these seven models but common to all evaluation and decision models: a perfect or not even a best formal model do not exist. Actually, defining a 'perfect model' would be a difficult, if not impossible, task. You might then ask why bother with formal decision models if they raise so many problems. The answer given in the first volume provided three arguments in favour of formal models and showing that, besides their weaknesses, they also have advantages.

1. First, it should not be forgotten that formal tools lend themselves more easily to criticism and close examination than other types of tools. However, whenever "intuition" or "expertise" was subjected to close scrutiny, it was more or less always shown that such types of judgments are based on heuristics that are likely to neglect important aspects of the situation and/or are affected by many biases (see the syntheses of Bazerman, 1990; Hogarth, 1987; Kahneman, Slovic, and Tversky, 1981; Poulton, 1994; Rivett, 1994; Russo and Schoemaker, 1989; Thaler, 1991)

2. Second, formal methods have a number of advantages that often prove crucial in complex organisational and/or social processes:

   - they promote *communication* between the actors of a decision or evaluation process by offering them a *common language*;
   - they require the building of models of certain aspects of "reality"; this implies concentrating efforts on crucial matters. Thus, formal methods are often indispensable *structuring* instruments.
   - they easily lend themselves to "what-if" types of questions. These *exploration* capabilities are crucial in order to devise *robust recommendations.*

   Although these advantages may have little weight in terms of effort involved, money and time consumed in some situations (e.g. a very simple decision / evaluation process involving a single actor) they appear fundamental to us in most social or organisational processes.

3. Third, casual observation suggests that there is an increasing demand for such tools in various fields (going from executive information systems, decision support systems and expert systems to standardised evaluation tests and impact studies). It is our belief that the introduction of such tools can

have quite a beneficial impact in many areas in which they are not commonly used. Although many companies use tools such as graphology and/or astrology to choose between applicants for a given position, we believe that the use of more formal methods could improve such selection processes in a significant way (if only issues such as fairness and equity). Similarly, the introduction of more formal evaluation tools in the evaluation of public policies, laws and regulations (e.g. policy against crime and drugs, gun control policy, fiscal policy, the establishment of environmental standards, etc.), an area in which they are strikingly absent in many countries, would surely contribute to a more transparent and effective governance.

So, where are we now? In the first volume, we heavily criticised formal models but we also argued that they can be useful. It is now time to make a proposal. Unfortunately, we have no miraculous solution but we can propose something: a kind of guide, a way of reasoning aimed at helping the analyst to choose a model and use it consistently. In this volume, we will systematically analyse many formal models (often using an axiomatic approach). We will try to find their most characteristic properties and show what makes them different from other models. As they are different and thus cannot be used in the same way, our analysis will therefore naturally lead us to determine a consistent way to use each of them. We will also see in which context a given property seems useful, desirable or undesirable.

Let us use a metaphor to clarify our purpose. Suppose you run a small low-tech company which has four employees not including yourself. The company has no computers and you are computer-illiterate but you believe that things have changed and that it is now time to make a move. After looking at a few catalogues presenting hundreds of different models, you feel lost. You therefore go to the nearest computer store and ask the salesman what he has for you. He shows you one model—a desktop—and tells you it is the best one; it was shown on TV and it ranks number one in sales. You decide to buy five of them. The salesman congratulates you: this is your best buy and you return home, happy. It is possible, if you are lucky, that these computers will allow you to run your business more efficiently and that you will be satisfied in the long term. But you may also discover after some time that the computers are not as good as the advertisements claim and that the ones you bought cannot do what you expected of them.

Let us now imagine another scenario. When you go to the computer store, the salesman tells you that computers are not perfect. They sometimes crash and, if a hard disk crashes, you loose all of your data. They can become infested by viruses. Some models are more reliable but their price is higher. Furthermore, no computer is ideal for all types of applications. Some are bad for graphic applications, others are not compatible with most other computers or are not user-friendly. You then leave the store without buying a single computer, very frustrated because you still think that you need computers, but you do not know what to buy.

In a third scenario, after telling you that no computer is perfect and that you therefore need to know more about computers, the salesman explains how a computer works, what the main functionalities are, how you can partially protect yourself against hard disk failures by making backups or against viruses. He also sketches a classification of computers. There are mainly three types of computers: CP, Pear and Plurax. CP's are bad for graphics and not very reliable but they are cheap and compatible with most computers. They are quite user-friendly and there are a lot of business applications designed for CP's. Pears are very reliable, good for graphics and very user-friendly, but they are expensive and not compatible with most computers although there are solutions to improve the compatibility. Finally, Plurax's are probably the most reliable and virus-proof computers. This explains their high cost. They are not user-friendly and are therefore better suited for computer specialists. The salesman then asks what your needs are and eventually helps you to formulate them. He finally helps you design a complete solution, i.e. not just five identical computers but, for example, several different computers—for different uses—with the adequate software and, perhaps, linked by a network.

It is clear that the third scenario is the best one and this book is meant for those wishing they met the third salesman rather than the first two.

Let us return to evaluation and decision models. A naive decision maker consulting an analyst that always uses the same decision aiding method (because he only knows that one or because it is the one he developed and he wants to sell it) is like our business man in the first scenario. This is something we cannot, of course, recommend.

After reading our first volume, a reader may feel very frustrated like the business man in the second scenario, because we criticised so many different models without proposing alternatives or a way to cope with the problems. If we (the authors) stopped after the first volume, we would be like the second salesman, but with this second volume, we hope to be like the third salesman, using criticism as a stimulus, a springboard for going beyond the surface and analysing the situation in depth.

In the next section, before shortly presenting the content of this book, we will summarise what we learned in the first volume. Note that the first book is more a companion volume than one 'to-be-read-before-the-second', but because it appeared first and for the ease of reference, we call it first.

1.2 What have we learned in the first volume?

Let us summarise the conclusions of the first volume in a few points.

Objective and scope of formal decision / evaluation models

- Formal decision and evaluation models are implemented in complex decision / evaluation processes. Using them rarely amounts to solving a well-defined mathematical problem. Their usefulness not only depends on their intrinsic formal qualities, but also on the quality of their implementation (structuring of the problem, communication with actors involved in the process, transparency of the model, etc.). Having a sound theoretical basis is therefore a necessary but insufficient condition of their usefulness (see first volume, chapter 9).

- The objective of these models may not be to recommend the choice of a "best" course of action. More complex recommendations, e.g. ranking the possible courses of action or comparing them to standards, are also frequently needed (see first volume, chapters 3, 4, 6 and 7). Moreover, the usefulness of such models is not limited to the elaboration of several types of recommendations. When properly used, they may provide support at all steps of a decision process (see first volume, chapter 9).

Collecting data

- All models imply collecting and assessing "data" of various types and qualities and manipulating these data in order to derive conclusions that will hopefully be useful in a decision or evaluation process. This more or less inevitably implies building "evaluation models" trying to capture aspects of "reality" that are sometimes difficult to define with great precision (see first volume, chapters 3, 4, 6 and 9).

- The numbers resulting from such "evaluation models" often appear as constructs that are the result of multiple options. The choice between these possible options is only partly guided by "scientific considerations". These numbers should not be confused with numbers resulting from classical measurement operations in Physics. They are measured on scales that are difficult to characterise properly. Furthermore, they are often plagued with imprecision, ambiguity and/or uncertainty. Therefore, more often than not, these numbers seem, at best, to give an order of magnitude of what is intended to be captured (see first volume, chapters 3, 4, 6 and 8).

- The properties of the numbers manipulated in such models should be examined with care; using "numbers" may only be a matter of convenience and does not imply that any operation can be meaningfully performed on them (see first volume, chapters 3, 4, 6 and 7).

- The use of evaluation models greatly contributes to shaping and transforming the "reality" that we would like to "measure". Implementing a decision / evaluation model only rarely implies capturing aspects of reality that can be considered as independent of the model (see first volume, chapters 4, 6 and 9).

Aggregating evaluations

- Aggregating the results of complex "evaluation models" is far from being an easy task. Although many aggregation models amount to summarising these numbers into a single one, this is not the only possible aggregation strategy (see first volume, chapters 3, 4, 5 and 6).

- The pervasive use of simple tools such as weighted averages can lead to disappointing and/or unwanted results. The use of weighted averages should in fact be restricted to rather specific situations that are seldom met in practice (see first volume, chapters 3, 4 and 6).

- Devising an aggregation technique is not an easy task. Apparently reasonable principles can lead to a model with poor properties. A formal analysis of such models may therefore prove of utmost importance (see first volume, chapters 2, 4 and 6).

- Aggregation techniques often call for the introduction of "preference information". The type of aggregation model that is used greatly contributes to shaping this information. Assessment techniques, therefore, not only collect but shape and/or create preference information (see first volume, chapter 6).

- Many different tools can be envisaged to model the preferences of an actor in a decision/evaluation process (see first volume, chapters 2 and 6).

- Intuitive preference information, e.g. concerning the relative importance of several points of view, can be difficult to interpret within a well-defined aggregation model (see first volume, chapter 6).

Dealing with imprecision, ambiguity and uncertainty

- In order to allow the analyst to derive convincing recommendations, the model should explicitly deal with imprecision, uncertainty and inaccurate determination. Modelling all these elements into the classical framework of Decision Theory using probabilities may not always lead to an adequate model. It is not easy to create an alternative framework in which problems such as dynamic consistency or respect of (first order) stochastic dominance are dealt with satisfactorily (see first volume, chapters 6 and 8).

- Deriving robust conclusions on the basis of such aggregation models requires a lot of work and care. The search for robust conclusions may imply analyses much more complex than simple sensitivity analyses varying one parameter at a time to test the stability of a solution (see first volume, chapters 6 and 8).

1.3 Stepping stones for the analyst

As we said above, we do not have solutions to all of the problems encountered and all of the questions raised in the first volume. We do not have a unique and well-defined methodology that one could follow step-by-step from the beginning to the end of a decision aiding process. What we can do, is simply propose, here and there, a sound analysis of techniques aimed at supporting a part of the decision aiding process. These are what we call the 'stepping stones for the analyst'. They do not form a single and continuous path to cross the river but can help.

We will use a special formatting—as shown in this paragraph—at different places in this book, to draw the reader's attention to a 'stepping stone', to the summary of a section, to something we consider important or of much practical interest or when we present the conclusion of a long development.

1.3.1 Structure

The focus in this book—compared to its companion volume—is on *multicriteria* evaluation and decision models: three chapters are devoted to the analysis of aggregation methods (chapters 4–6). In chapter 5, we analyse different aggregation methods in the light of Social Choice Theory, while, in chapter 6, we use the framework of conjoint measurement theory in order to study many aggregation methods (often the same ones as in chapter 5). Chapter 4 serves as an introduction to chapters 5 and 6.

It is well-known that some aggregation methods (for instance the outranking methods) yield relations that are not always transitive. It is therefore necessary, after the aggregation, to use an exploitation technique the purpose of which is to help make a recommendation to the decision maker. In chapter 7, we show that an exploitation is often necessary, not only after an aggregation using an outranking method, but in many other cases, even, in some cases, with a multi-attribute additive model. We then analyse several exploitation techniques. Another topic addressed in chapter 7 is uncertainty. Uncertainty is present in many decision problems and decision aiding processes. A very common and reasonable attitude in presence of uncertainty, is to try to model it in order to take it into account in the decision aiding process. Many different models of uncertainty are available in the literature: probabilities, possibilities, belief functions, upper probabilities, fuzzy sets, etc. There are also a lot of models incorporating these representations of uncertainty in decision models (for instance, Subjective Expected Utility). Some of them even cope with multiple attributes. All these models deserve great attention and the literature devoted to them is vast but we do not discuss them for two reasons: we do not feel competent and the subject is much too vast for this volume. Nevertheless, we discuss the important case, seldom treated in the literature, where nothing is known about the uncertainty distribution or where the hypotheses underlying some models are not met. In such cases, even if it is not possible to define a best or rational decision, we can try to draw robust conclusions or to make robust recommendations, i.e. conclusions that are true or approximately true or recommendations that lead to good—even if not optimal—outcomes under all possible scenarios or states of the nature.

In many evaluation and decision models, and in particular in many aggregation and exploitation methods, we use numbers. Sometimes, we use them to represent preferences but in other cases we derive preferences from them. Sometimes they are factual evaluations of an alternative on an attribute; sometimes, these evaluations result from a more or less subjective process. In other cases, they are the result of the aggregation (as in MAVT) and/or exploitation method. The pervasiveness of these numbers and the variety of their roles makes it necessary to analyse their meaning and what we can meaningfully do with them. As they play an important role from the beginning of the decision aiding process, the chapter on numbers and preferences (chapter 3) comes before those on aggregation and exploitation methods.

We have not presented the second chapter yet. Its status is quite different from that of the other chapters. In chapters 3–7, we analyse some formal techniques aimed at supporting one part of the decision aiding process but, in chapter 2, the focus is on the whole decision aiding process, on the different ways of conducting such a process and of introducing rationality into it, on a formalisation of the different elements of the decision aiding process: the client, the analyst, the stakes, the problem formulation, the model, the recommendation, etc. Because this chapter provides a general framework for describing all parts of the decision aiding process, it comes just after this introduction. Figure 1.1 presents the logical dependencies amongst the chapters. Readers interested mostly by theoretical

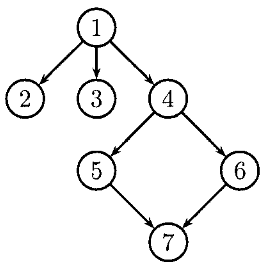

Figure 1.1: Reading schema.

aspects can concentrate on chapters 4–7 while practitioners will find it interesting to first read chapters 2–3.

Although most questions raised in the first volume are addressed in this volume, there is no chapter-by-chapter correspondence between both volumes. In the first volume, we presented several applications of evaluation and decision models. Most of them raised questions that are discussed in different chapters of this volume. We discussed, for example, the problem of grading students. This problem raises questions related to measurement, aggregation, uncertainty, robustness and so on, which are discussed in various parts of the present book.

1.3.2 Outline of the chapters

1.3.2.1 Chapter 2: "Problem formulation and structuring: the decision aiding process"

We introduce two basic subjects in chapter 2. The first is a presentation of what we call a "decision aiding approach": a perspective of how "rationality" (a key concept for formal decision aiding) enters into a decision and/or evaluation model. One of our principal claims is that decision aiding approaches are NOT characterised by any method used in providing decision support, but by how such methods are used.

However, the use of a combinatorial optimisation algorithm within the evaluation model does not preclude that the whole decision aiding process was conducted using a constructive approach. It simply shows that the precise decision maker's problem can be formulated using combinatorial optimisation.

Example 1.1
Consider the following (simplified) situation. A client is planning to open a number of shops in a town structured in districts. He might start by formulating the problem of "covering" the whole town with the minimum number of shops (under the hypothesis that shops opened in one district also "cover" the adjacent ones). This is a typical combinatorial optimisation problem. A solution to this "problem" (let's say a minimum of 3 shops are required) could lead the client to believe that this is too expensive. The "problem" will now be reformulated as maximising coverage under a budget constraint (a new issue for the client). Again this is a well-known combinatorial optimisation problem. The new results, which do not cover the whole town, could lead to considering that coverage could be "weighted" (the districts having different commercial importance), thus slightly modifying the previous formulation. At this point, the client and the analyst could go one step further and consider a bi-objective combinatorial optimisation problem: maximising weighted coverage and minimising costs. The sequence previously described is typically constructive (different problem formulations, alternative evaluation models, different recommendations), since the client constructed the final model without any ex-ante hypothesis about the problem situation and his preferences. Nevertheless, the methods and algorithms are coming from optimisation.

On the other hand, the use of a preference aggregation procedure based on the concordance-discordance principle could be seen as the result of a normative approach if the analyst imposes the axioms of such a model as "*the* model" of rationality.

Example 1.2
The Italian law concerning the call for tender for the allocation of public works contracts (L. 109/1994) imposes, among others, that all tenders should include an assessment of the environmental impact of the work to be undertaken in their offer. Regulation DPR 554/1999 published as the application code of the above law explicitly names the ELECTRE, AHP and TOPSIS methods in its annexes A and B as the ones to be used in order to perform such an assessment. It is interesting to note here that methods which have been explicitly conceived within a constructive approach, become norms for this law. This is due to the fact that in this context the decision rules have to be announced before the decision process itself begins. ◇

The second subject presented in chapter 2 is the "decision aiding process": the interactions between a client (a decision maker) and an analyst, aiming to aid the client within a decision process. Decision aiding cannot be seen as just the construction of a formal decision model. It is a complex activity (a process), which can be described and characterised by its outcomes, summarised as follows:

- a representation of the problem situation;

- a problem formulation;
- an evaluation model;
- a final recommendation.

A large part of chapter 2 is dedicated to discussing how such outcomes are constructed within a decision aiding process and in presenting practical recommendations (stepping stones) about conducting this process. The subsequent chapters go through a more thorough analysis of the more technical and formal among the above outcomes: the evaluation model. The elements of this are discussed in detail level and the interested reader will also find several stepping stones enabling an analyst and his client to establish meaningful and useful "evaluation models".

1.3.2.2 Chapter 3: "Numbers and preferences"

For most people evaluating implies using numbers. It is only after some second thought that we realise we can also evaluate objects by assigning labels such as "good" or "bad" to characterise the way they perform for a given feature. Yet, when using numbers, it is not always obvious to interpret the numbers attached to objects in terms of achieving some level of performance; it is even less obvious to see how they can be interpreted as reflecting the decision maker's preference. You may like having your coffee/tea hot. If this is the case, you probably prefer it when it is served at a temperature of 40 ℃ rather than 30 ℃ and at 50 ℃ rather than 40 ℃. But do you prefer a cup of coffee/tea served at 80 ℃ to a cup at 70 ℃. Coffee/tea can be too hot and you might prefer "not warm enough" to "too hot". More basically, without looking at preferences, but just in terms of warmth, a cup of tea/coffee served at 80 ℃ is clearly hotter than a cup at 40 ℃; can we say that the former is twice as hot as the latter? (if the temperature was measured in degrees Fahrenheit—80 ℃ is equal to 176 °F and 40 ℃ to 104 °F—the former would not be twice as hot as the latter).

Chapter 3 is devoted to examining what numbers mean and, also, how they may relate to preference. To start with, we discuss the former issue, leading, in an informal manner, to the notion of measurement scale.

We then contrast measurement and preference. Even when numbers really measure a dimension of an object (for instance its cost, provided the latter is precisely known) it is often the case that what we can say about the cost does not directly transpose in terms of preference. If I want to buy a car, I may for instance feel—in terms of preference—that paying 11 000 € instead of 10 000 € is more painful than paying 15 000 € instead of 14 000 €; in other words, my appreciation of a cost difference may differ depending on where it is located on the cost scale.

Preference is modelled as one or several relations. For instance, we say that alternative a is preferred to alternative b and note it $a\ P\ b$; alternatives may be indifferent; there may be degrees in the preference. Many types of relations or families of relations can be used to model preference and we try to link these with numbers. We do this in both ways. Starting with numbers assigned to alternatives, we list a number of likely interpretations of these numbers in terms of preference relations. We go from purely ordinal to more "quantitative" interpretations of the numbers. We also deal with the case in which an interval is attached to each alternative rather than a number, thus aiming at taking imprecision into account, in a certain—non probabilistic—way. This interpretation of numbers in terms of preference is what could be called "preference modelling".

In the last part of the chapter, we study how some structures of preference relations can be represented by numbers.

In the whole chapter, we consider that the objects are described on a single dimension; the presence of several dimensions or criteria will be dealt with in the subsequent chapters. At the end, we suggest at the end how these single dimensional considerations can be related to multi-criteria evaluation models.

1.3.2.3 Chapter 4: "Aggregation—Overture"

When dealing with objects that can only be described and compared using several characteristics, aggregation is a major issue: it aims at operating a synthesis of the, usually contradictory, features of the objects, in view of achieving a goal such as choosing among the objects, rank ordering them, sorting them into categories and so on. There are at least two ways of looking at the operation that we call "aggregation".

One way, is to approach aggregation as a mechanism that transforms the assessments or description of the alternatives on the various dimensions into a ranking (or some other structure). Similar mechanisms were studied from a theoretical point of view in the framework of Social Choice Theory.

There is another way of looking at aggregation that changes the point of view on the subject significantly. This theory is usually called "Conjoint Measurement". In this approach, we consider the result of the aggregation, not the process itself. For example, we consider a relation on the set of alternatives, which is one of the possible outputs when applying an aggregation mechanism.

Aggregation procedures are studied in some depth in chapter 5, while conjoint measurement models are described in chapter 6. In chapter 4, we propose an introduction to both chapters: we present an example of an axiomatic characterisation obtained in Social Choice Theory (the Borda method) and in conjoint measurement (the additive value model), and we try to show why and how these characterisations can be useful to the analyst. Chapter 4 also contains a section on parameters. Most aggregation methods use parameters: weights, importance coefficients, preference thresholds, value functions, etc. We believe that the best way to elicit these parameters is to ask the decision maker to compare some alternatives (as is often done, for instance, with the additive value model) or to make some statements about alternatives, but not about the parameters themselves. We motivate this view and we present a general approach to the elicitation of parameters that can be adapted to most—if not all—aggregation methods. Another section in this chapter should help the reader interested in a specific method to find the most relevant results in chapters 5 and 6.

1.3.2.4 Chapter 5: "Aggregation procedures"

Suppose that you have gathered all the information that you need to assess the alternatives in a decision problem (buying a computer or a house, hiring an employee for a particular job, choosing a spot where to spend a holiday, ...); assume that the assessments are provided on a numerical scale. There are several paths you may be tempted to follow to obtain a ranking of the alternatives. One could be: compute a weighted sum of the assessments and rank the alternatives according to the value of the sum. Although this is the most common way to proceed, there are many other possible procedures. You could consider all pairs of alternatives in turn and determine which alternative is to be preferred to the other for each pair; this can be done using a form of majority rule, for instance, based on the number of attributes for which one alternative is better than the other. How can we choose one procedure among all those available? We try to answer this question by presenting—in an informal and hopefully intuitive manner—axiomatic characterisations of a number of these procedures. Our credo is that knowing the characteristic properties of the procedures helps to perceive their "spirit", supports some particular interpretations of its parameters and dismisses others, and consequently helps to understand for which situations (nature, quality, availability of information; time pressure, goal of the process, ...) they are best suited [1].

Chapter 5, dealing with the characterisation of procedures, is subdivided according to the type of input needed and the type of output provided by the procedure. The input can possibly be a set of preference relations that are aggregated into a global preference relation; the corresponding section of the chapter is thus concerned with the case in which the preferential information for the alternatives with respect to the various dimensions was modelled as relations, usually rankings. Among the procedures of this type, we characterise the Borda rule, as well as various types of majority rules and the lexicographic procedure. The output is a relation interpreted as a global preference on the set of alternatives and operating a synthesis of the partial preferences for the various viewpoints. In this setting, we come close to methods that are actually used in multi-criteria decision analysis, such as ELECTRE I, ELECTRE II or TACTIC. Arrow's Theorem is also presented in this section and its implications for decision aiding are discussed at length.

The decision maker's preference on each dimension cannot always be expressed as a relation; in the next two sections, we consider the cases in which these preferences are respectively formulated as fuzzy or valued relations and performance tables. A fuzzy or valued relation occurs as input, for example, when a value can be assigned to each pair of alternatives, reflecting the way or the intensity with which one is preferred to the other; a performance table is essentially a function that assigns a number to each alternative on each dimension; this number positions the alternative on the preference scale associated with the dimension. For both types of inputs, the fuzzy (valued) relation and the performance table, we consider that the output is a relation on the set of the alternatives. If the input is

[1] Note that there may be several equivalent characterisations for a single procedure; in such a case, a procedure can be interpreted in different ways.

a fuzzy (or valued) relation, we characterise generalisations of the Borda and majority rules, that can easily be adapted to deal with fuzzy relations. Here, we pay special attention to the construction of the fuzzy relations and to the consistency between the nature of the fuzziness, its representation and the aggregation technique. The procedures using fuzzy relations as input shed some light on methods like PROMETHEE II and ELECTRE III. We then turn to some results obtained in the framework of cardinal Social Choice Theory, i.e. when the information to be aggregated into one relation does not consist of one preference relation per criteria but of a number (a performance, an evaluation, a utility, ...) for each alternative on each criterion. The case in which the input is a "performance tableau" gives us the opportunity of characterising the minimum, the weighted sum as well as the leximin and leximax procedures, that are commonly used in practice. Here again, we insist on the necessary consistency between the meaning of the numbers to be aggregated and the aggregation method.

Performance tableaus do not always contain numbers; evaluations often are expressed on qualitative scales, using verbal labels or statements. Even on numerical scales, the significance of the numbers may only be ordinal. We briefly address the question of procedures using this type of information as input, which we refer to as "linguistic performance tables".

The output of a procedure is not always a relation. Another case of interest is that of procedures yielding a set as output, this set being usually interpreted as a choice set, i.e. a subset of alternatives, possibly a single one, that would be proposed to the decision maker as the best candidates. We show that the characterisations of the procedures leading to a global preference relation can easily be adapted to procedures for which the output is a choice set. It is, of course, pleasant that the interpretation of the input data supported by the characterisation of procedures leading to a ranking of the alternatives (or another type of a preference relation) can also be used when dealing with a choice problem.

The last section covers some aggregation techniques with characterisations that are not usually presented as similar to what is done in 'Social Choice Theory': the so-called aggregation operators that are very popular in statistics and in the fuzzy literature (the various means, the order statistics, the Choquet and Sugeno integrals, ...). Unlike most aggregation methods in Social Choice Theory, aggregation operators use numbers as input (as in Cardinal Social Choice Theory), but yield one number per alternative and not a relation on the set of alternatives as output. Despite this difference, we present results about aggregation operators in this chapter because these results are very similar to those obtained in Social Choice Theory: they impose conditions on the method transforming the input into an output and not on the preference relations that can be represented by a given model (as is the case in conjoint measurement).

1.3.2.5 Chapter 6: "Multi-dimensional preference models"

While chapter 5 views aggregation as a mechanism that transforms a multi-dimensional input into a more synthetic single-dimensional output, the main primitive of Conjoint Measurement Theory is a preference relation on a set of alternatives. Conjoint measurement examines conditions on the relation under which can be represented in a model linking the preference to the description of the alternatives along the various relevant dimensions. The archetypical relations in this theory are those that can be described in the *additive value model*, studied in the first section of chapter 6. A preference $\succsim$ fulfills the additive value model if one can decide that an alternative a, described by its evaluations $a_1, \ldots, a_n$ on n dimensions, is preferred to an alternative b, described in terms of its evaluations $b_1, \ldots, b_n$, by comparing the values $u(a)$ and $u(b)$ of a function u; the peculiarity of the latter is that its value, for an alternative a described by $a_1, \ldots, a_n$, is computed as a sum of partial value functions $u_i(a_i)$ that only depend on the evaluation of a on dimension i. In the first section of chapter 6, we do not only describe the characteristic properties of the relations that can be represented by an additive value function; more importantly, the analysis of the model draws attention to the central concept of marginal preferences, i.e. the preferences induced by the global one on the scales associated with the various dimensions. Marginal preferences are the building blocks that can be combined to give the global preference. In other words, the analysis of the model suggests ways of eliciting the preference by asking well-chosen questions to the decision maker and these questions rely on marginal preferences in an essential way. It is the main goal of the first section of chapter 6 to stress these features of conjoint measurement theory in the particular case of the additive value model.

The rest of the chapter enlarges the scope of the conjoint measurement models that we consider. Why is this needed? Because not all preferences fulfil the conditions under which they can be represented by an additive value function. For instance, preference relations obtained through applying some sort of a majority rule while comparing each pair of alternatives in turn on all relevant dimensions, typically lack the transitivity property (alternative a may be preferred to b and b to c, while a is not preferred to c). Another example is seen when comparing objects measured on a single dimension, using a measurement device. If objects are only slightly different with respect to their measured characteristic, the measure is usually repeated a certain number of times, to control the uncertainty on the measure. Each object is thus associated with but a vector recording a sample of noisy measures of this object and not with a single measure. Comparing two objects is then done through a statistical test of comparison of means, namely the means of the measures performed on each object. Such a comparison can lead to an intransitive relation on the set of objects ; more precisely, it can occur that an object a can be undistinguishable from b, which is undistinguishable from c, while a and c can indeed be distinguished (a may be significantly greater than c or the opposite). This case suggests that a comparison relation, and by extension, a preference relation, may not be transitive. Moreover, the marginal relations of this comparison relation may be quite rough in the sense that they do not differentiate the values on each dimension sharply; hence the marginal preferences of non-transitive preferences may not convey all the information needed to construct the global preference relation.

The need to deal with preferences that are not necessarily transitive and preferences for which marginal analysis is not sufficient, leads us to propose two more general frameworks in the conjoint measurement spirit, both encompassing the additive value model.

In the first of these frameworks, the marginal preferences are substituted with more subtle relations conveying all the information induced by the global preference on the scales of the various dimensions; they are called the marginal traces. We describe a variety of very general models that encompass not only the additive value model but also a more general one called the decomposable model. An important feature of a large particular class of these models is that they respect dominance with respect to the marginal traces; we provide a characterisation of this class of models.

Another way of generalising the additive value model is particularly suited to describe preferences that can be established on the basis of pairwise comparisons of alternatives, as is the case when using majority rules. The building blocks for constructing preferences in our second framework are relations called traces on differences. By means of these relations, it is possible to express that the difference between two levels a_i and b_i on scale i is at least as large as that between two levels c_i and d_i on the same scale. Within this framework, we analyse the preferences obtained through well-known procedures that are mostly variants of the majority rule, possibly with vetoes. These procedures are shown to correspond to very rough differentiation of preference differences between levels on a scale: essentially, a preference difference can be positive, negative or equal to zero; this can reasonably arise when the scales are purely ordinal. Introducing vetoes in variants of the majority rule amounts to distinguishing five classes of preference differences instead of three.

Finally, a third general framework is explored, that obtained by combining the two previous ones; in the corresponding models, marginal preferences on differences can be expressed in terms of marginal traces. The use of these refinements allows us to further investigate the models based on majority rules, with or without vetoes and more generally, the models for preferences distinguishing few levels of preference differences. At the other extreme, this framework encompasses the additive difference model, which can finely distinguish preference differences.

After a brief section devoted to valued (or fuzzy) preference models (in which we look at the measurement of global preference differences), we close chapter 6 with a rejoinder, stressing the links that exist between the two different approaches to aggregation, described in chapters 5 and 6.

1.3.2.6 Chapter 7: "Making recommendation"

The ultimate aim of a decision aiding study is to build *recommendations* that will hopefully be considered as useful by the participants in the decision process. Such recommendations, at least in our approach to decision aiding, are based on formal preference models. Many different tasks are required in order to obtain a recommendation from such models. Some of them are rather informal, involving, e.g., a good strategy of communication with the actors in the decision process, the need for transparency in the decision aiding process, a sound management of multiple stakeholders, etc. The last chapter of this volume discusses the formal tasks that are involved in the elaboration of a recommendation.

The analyst's task at this stage is clearly dependent upon:

- the nature of the recommendation that is sought, which, in turn, is linked to the problem formulation that has been adopted. This chapter will concentrate on the three problem formulations that are most frequently encountered in practice. The first two (i.e. choosing and ranking) involve a relative evaluation of the alternatives, while the last one (sorting) is concerned with absolute evaluation.

- the nature of the preference models that have been built. We distinguish two main types of preference models: the ones based on value functions leading to well-behaved preference relations and the ones tolerating incomparability and/or intransitivity.

We first deal with the, relatively easy, case of preference models based on a value function. We then tackle the much more difficult case of preference models tolerating incompleteness and/or intransitivity. We also envisage the situation in which the recommendation is based on *several* preference models, a situation that frequently arises in practice. The main difficulty here will be to reach conclusions that will hold with all possible preference models, i.e. *robust conclusions*.

The chapter concludes with a more general perspective on robustness, an important emerging theme in the field of decision aiding. Indeed, all scientists who have dealt with real decision problems know that the numerical values used in their models are often questionable. This is the case for information describing the decision situation, traditionally called the "data". They are often values built by the analyst according to the model he wants to use; they result from assumptions about the context of the problem, from estimations of badly known or random values, from the forecasting of future events. Therefore, it often occurs that several plausible "sets of data", possibly very different from each other, can constitute a good representation of the situation. This is also the case for the parameters that have to be (more or less arbitrarily) chosen by the analyst using a formal decision aiding tool (e.g. value functions, weights, thresholds, etc.).

In such a context, working with a unique (e.g. the "most plausible") set of values can be very risky. What the decision maker generally wants is a recommendation that makes sense with all (or almost all) of the plausible sets of data. This is the basis of the concept of robustness.

1.4 Intended audience

Most of us are confronted with formal evaluation and decision models. Very often, we use them without even thinking about it. This book is intended for the aware or enlightened practitioner, for anyone who uses decision or evaluation models—for research or for applications—and is willing to question his practice, to have a deeper understanding of what he does. We have tried to keep mathematics at a minimum, so that, hopefully, most of the material will be accessible to the not mathematically-inclined readers. We do not use sophisticated mathematical tools such as differential equations, abstract algebra or calculus and we do not prove the theorems we present. Nevertheless, in order to make our definitions precise and to be able to meaningfully manipulate formal concepts, we need to use a formal language. That is why, compared to the first volume, this book requires more mathematical maturity, even if, sometimes, we have privileged intuition and accessibility over mathematical correctness. A rich bibliography will allow the interested reader to locate the more technical literature easily.

This book can certainly be used for teaching purposes, but not for introductory classes because it assumes a basic knowledge of multicriteria decision and evaluation models. For example, we only give a very short presentation of the main aggregation methods. For an introduction, we suggest Bouyssou et al. (2000), Vincke (1992b) or Belton and Stewart (2001).

1.5 Who are the authors?

The authors of this book are European academics working in four different universities and research institutions, in France and in Belgium. They teach in engineering, mathematics, computer science and psychology schools. Their background is quite varied: mathematics, economics, engineering, law and geology, but they are all active in decision support and more particularly in multiple criteria decision support. Preference modelling, fuzzy logic, aggregation techniques, social choice theory, artificial intelligence, problem structuring, measurement theory, Operational Research, ... are among their special interests. Besides their interest in multiple criteria decision support, they share a common view on this field. Four of the five authors of the present volume, together with Patrice Perny, presented their thoughts on the past and the objectives of future research in multiple criteria decision support in the *Manifesto of the new MCDA era* (Bouyssou, Perny, Pirlot, Tsoukiàs, and Vincke, 1993). In 2000, the five authors of this book, once again with Patrice Perny, published a book entitled "Evaluation and decision models: A critical perspective" (Bouyssou et al., 2000).

The authors are active in theoretical research on the foundations of decision aiding, mainly from an axiomatic point of view, but have been involved in a variety of applications ranging from software evaluation to location of a nuclear repository, through the rehabilitation of a sewer network or the location of high-voltage lines.

In spite of the large number of co-authors, this book is not a collection of papers. It is a joint work.

1.6 Conventions

To refer to a decision maker, a voter or an individual whose sex is not determined, we decided not to use the politically correct "he/she" but just "he" in order to make the text easy to read. The fact that all of the authors are male has nothing to do with this choice. The same applies for "his/her".

None of the authors is a native English speaker. Therefore, even if we did our best to write in correct English, the reader should not be surprised to find some mistakes and inelegant expressions. We beg the reader's leniency for any incorrectness that might remain. Throughout, we have tried to stick to the spelling used in the U.K.

1.7 Acknowledgements

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2

PROBLEM FORMULATION AND STRUCTURING: THE DECISION AIDING PROCESS

Consider the following situations:

1. A family discovers that their daughter systematically refuses to eat any type of food claiming that eating for her is "disgusting" (a typical symptom of "anorexia mentalis"). It is reasonable to expect that the family will contact a psychotherapist in order to conceive appropriate therapies to face this (possibly extremely dangerous) situation.

2. A lady becomes pregnant. Soon after she gradually becomes physically upset. Again we can expect that she will consult a physician in order to establish an appropriate treatment.

3. A large company providing mobile communication services is facing the possibility that the European Union will introduce a new directive concerning ownership of networks across Europe, thus seriously affecting its business. We can expect that this company will contact a primary legal adviser in order to appropriately redesign the company's structure.

4. A manager has to reconsider the company's supply chain management in order to improve productivity and delivery time to the customer performance. It is reasonable to believe that he will contact a supply chain management specialist in order to study different policies and establish one.

These situations all share a common characteristic: there is "a problem", for which "a client" (the family, the lady, the company, the manager) asks the advice of "an analyst" (the psychologist, the physician, the lawyer, the supply chain management specialist) in order to "find a solution".

There is, however, an important difference when we compare the advice of the psychologist, the physician, the lawyer to that of the supply chain management specialist: the language (for more details on this issue the reader is referred to Ackoff, 1962; Bevan, 1976; Capurso and Tsoukiàs, 2003). Although all of these advisers might use a "scientific approach" to help their clients, the psychologist, the physician and the lawyer will use a human natural language (naturally ambiguous) and a terminology depending on their specific domain, while the supply chain specialist will be likely to use a formal language (like mathematics) which reduces (if does not exclude) ambiguity and is independent of the field of supply chain management. He will use what we call a "decision support language", thus introducing a "model of rationality" in his decision aiding activity.

Does it make sense to use such a language in any context and at all times? Obviously not. The use of a "decision support language" presents several disadvantages:

- it is much less effective with respect to human communication;
- it has a cost (not necessarily monetary);
- reducing ambiguity might not be desirable;
- it imposes a limiting framework on people's intuition and creativity.

Nevertheless, such a language also presents several advantages, which in some circumstances can be interesting (see also Bouyssou et al., 2000):

- it allows the participants in a decision process to speak the same language, a fact that improves the transparency of the process and possibly increases participation (for an example see Bana e Costa, Nunes da Silva, and Vansnick, 2001);
- it allows the identification of the underlying structure of a decision problem (if there is any) and therefore allows the re-use of procedures and models (for interesting examples see any textbook of Operational Research, e.g., Williams, 1990);
- it is not affected by the biases of human reasoning that are due to education or tradition (for examples see Rivett, 1994);
- it may help to avoid the common errors that are due to an informal use of formal methods; a typical case being the use of averages as a universal grading procedure (see Bouyssou et al., 2000, chapter 3, for a critical discussion of this issue).

In this chapter we will focus on a number of this language's concepts and terms. In our first volume (Bouyssou et al., 2000) we have shown that within such a language we make choices about models, procedures, numerical representations and logics, which are not neutral with respect to the final result of the interaction between the client and the analyst. Furthermore we have shown that a problem situation is not perceived and modelled in a unique and objective way, but there exist several different problem formulations. The use of a formal, domain-independent language forces us to be more precise when terms such as problem, objective, solution etc. are adopted (see, e.g., Belton and Stewart, 2001; Checkland, 1981; Rosenhead, 1989; Roy, 1996; Roy and Bouyssou, 1993).

The aim of this chapter is to introduce the reader to the concept of "decision aiding process", the activities occurring between a client (somebody looking for decision support) and an analyst (somebody providing decision support). Although each such process has a unique history (once accomplished), we claim that there are a number of invariants within it and that these can be used in order to provide useful recommendations on how such a process can be conducted. In other words: conducting a decision aiding process is a combination of personal skills (in human communication, group conduction, listening etc.) and of formal skills characterised by the establishment of precise cognitive artefacts which are used by the client and the analyst in order to represent the problem and its solution(s). This chapter as well as the whole book is dedicated to analysing such steps, providing concepts, tools and methods to appropriately follow them.

In order to better understand our point of view, in the first section of this chapter we discuss four different decision aiding approaches: normative, descriptive, prescriptive and constructive (Bell, Raiffa, and Tversky, 1988; Dias and Tsoukiàs, 2004). Under our perspective, the decision support language makes sense within a particular context: the interactions between the client and the analyst. Such a stream of interactions is denoted as "decision aiding process" and is viewed as a particular type of decision process. For this purpose, we briefly discuss the concept of decision process in section 2.2 as well as the differences between "deciding" and "aiding to decide". In section 2.3, we then introduce a formal model of the decision aiding process. Such a model is based on the cognitive artefacts, the "products" of the process: a problem situation, one or more problem formulations, one or more evaluation models, a final recommendation. In section 2.4, we focus on the construction of such cognitive artefacts. Large part of the book will be dedicated to a deeper analysis of the problems identified in section 2.4. A final section concludes showing the research directions opened by such an approach.

2.1 Decision Aiding Approaches

In order to help someone to "make" a decision we normally elaborate preferences. *"Preferences are rational desires"* (Aristotle, 1990). Practically what we usually know is what a decision maker or a client [1] desires. Where does rationality come from?

Suppose a client faces a health problem. He has a set of more or less sure diagnoses and a number of possible treatments of more or less uncertain results. A manual of decision theory will suggest to consider each possible treatment as an alternative action and each possible diagnosis as a possible state of the world to which a probability might be associated. For each treatment we thus obtain the consequences of its application for each diagnosis. Such consequences allow to establish a utility function. Maximising such an utility (function), will provide the client with the best solution to his problem. The existence of such a utility function is guaranteed through a number of axioms (Savage, 1954) which are supposed to express the idea of rationality in a formal way. Such axioms are independent of the client. Preferences among the potential consequences should be transitive and this is imposed because it is considered essential in order to be rational, otherwise

[1] Hereafter we will substitute the term "decision maker" with that of "client". The reason will become clear later in the text. A "client" is someone who seeks advice for a decision issue. From such a perspective, he is a potential decision maker, but not necessarily.

the client should be ready to pay an infinitely increasing amount of money for the same solution (see the "money pump" discussion in Raiffa, 1970, p. 78). Similarly, preferences about consequences ought to be "independent" (the fact that we prefer a certain consequence to another should not depend on the likelihood that any of the two will occur) (see Fishburn, 1970, p. 107). Rationality here is established independently from the client. We should also note that, although we allow for uncertainties in the diagnosis, there is no uncertainty in the model itself. Diagnoses are all the possible diagnoses, the treatments are all the possible ones and it is clear that the problem is to choose the best one for this specific client (who only has to express his preferences) who is supposed to be "rational". If he is not, then he should modify his preferences in order to become so. Which is what we call a normative approach.

Since von Neumann and Morgenstern (1947) and Savage (1954) this is the dominant paradigm in decision analysis and decision support with or without uncertainty, in the presence or not of multiple evaluation dimensions. Traditional Operational Research techniques fit the same idea: maximise an economic function in the presence of feasibility constraints (usually all expressed in terms of linear functions). Rationality is imposed through a number of hypotheses and axioms which exist independently from the client and his problem.

Returning to our client, we can argue whether his behaviour is effectively "rational" (in the sense of the axioms of economic rationality). Indeed, since Allais (1953) (see also Kahneman and Tversky, 1979), it has been shown that real decision makers in real decision situations behave in a way that violates the axioms of economic rationality. For instance "negative" outcomes may be considered in a totally different way with respect to "positive" outcomes such that the axioms are violated. Moreover, to explain observed patterns of behaviour of decision makers it is often necessary to adopt "distorted probabilities" (Kahneman and Tversky, 1979) in order to take into account the perception of uncertainty that the decision makers have. What should we do? One way could be to use any of the so called "decision heuristics" derived through direct observation of real decision makers. Consider the following frequent decision situation for instance. A decision maker has to choose among candidates using a number of criteria. He may first rank the criteria from the most important to the less important one. He then uses the most important one in order to extract a subset of candidates who are the best on that criterion. He then uses the second most important criterion in order to extract a further reduced subset of candidates from the previously established one. He then uses the third criterion in the same way until he (possibly) ends with a single candidate. This is a lexicographic procedure (extensively studied in Fishburn, 1974). Another example is "dominance structuring" where the decision maker, having once identified a "promising alternative" (intuitively or through another decision procedure) will try to consolidate his opinion by looking whether it is possible to construct a dominance relation between this alternative and the rest. This might be possible by the de-emphasising of certain criteria (up to eliminating them), bolstering the positive features of the "promising alternative" or modifying the criteria set (Montgomery, 1983; Montgomery and Svenson, 1976). Clearly this procedure aims at establishing a justification rather than making a choice.

It should be noted that in the above approach, although we do not impose a normative model of rationality we do impose one, but on empirical grounds. Its validity derives from the fact that several "other" decision makers do behave following a precise model. It should also be noted that we again consider the model as sure. Diagnoses, treatments and probabilities are given and the client has to choose one. The difference is that the model of rationality adopted is derived from analysing the cognitive effort of other decision makers (Svenson, 1996). We call such an approach descriptive since it is based on *descriptive models* of human behaviour when decision situations are faced.

Both approaches presented impose a model of rationality to the client. The question one could introduce is what happens if such a model of rationality cannot be imposed. What happens if the client expresses preferences which do not fit any model of rationality be it normative or descriptive. It might be the case that the client has preferences which are neither transitive nor complete. He might not be able to tell whether one alternative is preferable to another or he might not be willing to do so. He might have a perception of the uncertainty associated with the potential states of the world, but he might not be able to consider them within a model of probability. It is also probable that, although he understands the necessity to better shape his preferences, he has neither sufficient resources nor the time to do it. At the same time, something has to be done and the analyst has to be able to produce a recommendation.

In such cases we may adopt an approach which tries to construct a model as coherent as possible with the information provided by the client, while trying to satisfy minimum requirements of meaningfulness in manipulating such information. In other words we are not going to ask the client to adapt himself to a model of rationality, but try to adaptively model the available information and derive a reasonable recommendation. Considering the health example, we will try to identify a prescription which fits best with the client's preferences (even if these do not obey a model of rationality) and his personal perception of the uncertainty. Nevertheless, we are not going to accept any type of information manipulation, but only those which respect the "nature" of the data (Bouyssou et al., 2000; Roberts, 1979).

It should be noted that, while in the normative and the descriptive approach we consider models of rationality defined "from outside" the client's decision situation, in this case we try to model the precise rationality the client exhibits at the moment. Therefore, rationality is defined "within" the decision situation and not "from outside". We call such an approach prescriptive since it is aimed to "prescribe" to the client the action which appears hic et nunc as the most preferred.

However, again the problem is not discussed. We always consider that the diagnoses, the treatments, the uncertainties are given and that we are looking for the best therapy to follow. In other words, the problem is well established and our main concern is the potentially "irrational" information the client may provide or the fact that such information is difficult to represent under usual quantitative measures. Is it always the case? Are we really sure that all possible diagnoses have been obtained? What if there were other experimental treatments we are not aware of at this moment? Are we sure that the problem is to find a treatment? In several real decision situations neither the client nor the analyst are really aware of what the problem exactly is. What often happens is that, while these two actors try to model a problem, they also shape what the decision situation is about, thus ending up formulating a completely new problem and so on. In our example, although the client claims that he is looking for the best treatment, he might well end up understanding that his problem is to take a long holiday (possibly together with the analyst).

In other words, looking for the solution of a well established problem is always possible, but could be the wrong thing to do, since it might not be the right problem to solve. The problem is that neither the client nor the analyst know what the problem is a priori. Therefore, a decision support activity should also contain the structuring of the problem situation in which the client claims to be and the construction of several different problem formulations. Moreover, representing the client's preferences is not merely an elicitation process in which the analyst helps the client to state his values or to discuss them. It is a dialogue aimed at strengthening the conviction of the client that he actually does prefer "x" to "y", establishing the reasons for supporting such a conviction or the opposite one. Within such an approach we do not limit ourselves to using the most appropriate method for well established problem formulation, but we try to support the whole decision process in which the client is engaged. From such a perspective, nothing can be considered as "given" (if not the client's demand for help), while everything has to be constructed. Furthermore, within such an approach there is a fundamental learning dimension, since both the analyst and the client have to learn about the client's problem. We call such an approach *constructive* in the sense that the problem and its solution are constructed, while in all other approaches the problem is given and the solution is more or less discovered.

We can summarise the above presentation as follows.

Normative approaches

Normative approaches derive rationality models from a priori established norms . Such norms are postulated as necessary for rational behaviour. Deviations from these norms reflect mistakes or shortcomings of the client who should be aided in learning to decide in a rational way. These models are intended to be universal, in that they should apply to all decision makers who want to behave rationally. We may consider ethical norms, laws and religious norms as analogies (for more detail, the reader is referred to the following classics: Fishburn, 1970, 1982; Luce and Raiffa, 1957; Raiffa, 1970; Savage, 1954; von Neumann and Morgenstern, 1947; Wakker, 1989).

Descriptive approaches

Descriptive approaches derive rationality models from observing how decision makers make decisions. Such models are general, in that they should apply to a wide range of decision makers facing similar decision problems. We may consider scientists trying to derive laws from observed phenomena as an analogy (for more details, the reader can refer to: Allais, 1979; Barthélemy and Mullet, 1992; Gigerenzer and Todd, 1999; Humphreys, Svenson, and Vári, 1983; Kahneman et al., 1981; Kahneman and Tversky, 1979; Montgomery, 1983; Montgomery and Svenson, 1976; Poulton, 1994; Svenson, 1996; Tversky, 1969, 1972; von Winterfeldt and Edwards, 1986).

Prescriptive approaches

Prescriptive approaches *discover* rationality models for a given client from his/her answers to preference-related questions. Modelling consists in discovering the model of the person being aided to decide, i.e. unveiling his/her system of values. Therefore, they do not intend to be general, but only to be suitable for the given client in a particular context. Indeed the client can run into some difficulties trying to reply to the analyst's questions and/or be unable to provide a complete description of the problem situation and his/her values. Nevertheless, a prescriptive approach aims being in a position to provide an answer best fitting the decision maker's information *here and now*.Here, we may consider a physician asking questions to a patient, in order to discover his illness and prescribe a treatment as an analogy (for more details, the reader is referred to: Belton and Stewart, 2001; Brown, 1989; Keeney, 1992; Larichev and Moskovich, 1995; Roy, 1996; Tversky, 1977; Vanderpooten, 2002; Vincke, 1992b; Weber and Çoskunoğlu, 1990).

Constructive approaches

Constructive approaches *build* rationality models for a given client from his/her answers to preference-related questions. However, the "discussion" between the client and the analyst is not "neutral" in such an approach. Actually such a discussion is part of the decision aiding process since it constructs the representation of the client's problem and anticipates, to some extent, its solution. If, while talking about what to do tonight, we ask the question *"where should we go tonight?"* we implicitly do not consider all options implying staying at home. If we ask *"Who should we meet?"* we implicitly do not consider all options involving staying alone. In such an approach, structuring and formulating a problem becomes as important as trying to "solve" it. Recent real world applications (see, e.g., Bana e Costa, Ensslin, Corrêa, and Vansnick, 1999; Belton, Ackermann, and Shepherd, 1997; Paschetta and Tsoukiàs, 2000; Stamelos and Tsoukiàs, 2003) do emphasise the importance of supporting the whole decision aiding process and not just the construction of the evaluation model. Modelling using this approach consists in aiding a client to construct his own model, suitable for that contingency and particular context. Indeed, we can adopt the term of "co-modelling" (co-construction of the model). Here, we may consider a designer or an engineer tentatively developing a new car as an analogy (for details, the reader is referred to: Checkland, 1981; Genard and Pirlot, 2002; Habermas, 1990; Landry, Banville, and Oral, 1996; Landry, Malouin, and Oral, 1983a; Landry, Pascot, and Briolat, 1983b; Rosenhead, 1989; Roy, 1996; Schaffer, 1988; Watzlawick, Beavin, and Jackson, 1967).

Approach	Characteristics	Process to obtain the model
Normative	Exogenous rationality, ideal economic behaviour	To postulate
Descriptive	Exogenous rationality, empirical behaviour models	To observe
Prescriptive	Endogenous rationality, coherence with the decision situation	To unveil
Constructive	Learning process, coherence with the decision process	To reach a consensus

Table 2.1: Differences between approaches.

Theoretical differences...

Table 2.1 summarises the differences between the approaches. We may start by dividing these in two groups. On the one hand, normative and descriptive approaches use general models of rationality, established independently from the client and the decision process, intended to model the rationality of decision makers in general. On the other hand, prescriptive and constructive approaches derive a model for the rationality of the contingent client, and only that particular client.

The difference between normative and descriptive models mostly lies in the process of obtaining the model. Normative models are grounded on abstract economic considerations (rationality corresponds to the behaviour of an abstract "homo economicus"), whereas descriptive models are grounded on empirical observation. The former focus on how decision makers ought to decide, whereas the latter focus on how decision makers actually make decisions.

The difference between prescriptive and constructive models also lies to a great extent in how the model is obtained. Prescriptive models intend to unveil a system of values that exists before the decision aiding process starts, hidden somewhere inside the client's mind. Constructive models do not assume that preferences pre-exist, but let the client construct his/her system of values while the model is being constructed, recognising that one construction cannot be isolated from the other. Indeed, the final model is expected to be validated through a consensus reached between the client and the analyst. Such a "consensual" model is expected to satisfy both the client's perception of his/her problem and the analyst's methodological requirements of meaningfulness and formal coherence (on this point see Genard and Pirlot, 2002; Landry et al., 1996, 1983a,b).

...and practical issues

It should be noted that it often (usually in practice) does not happen that an analyst follows any of the above approaches as if he was following a decision theory manual. Normative approaches might be used with weaker versions of their axiomatics (see, e.g., Dubois and Prade, 1995; Dubois, Prade, and Sabbadin, 2001c; Wakker, 1989) knowing that this is empirically grounded. At the same time, someone adopting a prescriptive or a constructive approach might decide to introduce and fix a dimension of rationality in order to ease the dialogue with the client and "force him" to accept a certain point of view. Such interactions between the approaches can be better understood when decision support tools come into practice (see also Belton and Stewart, 2001).

The number of decision support tools and methods available today in literature and more or less applied is incredibly high (see Bouyssou et al., 2000). They range from optimisation techniques to cognitive approaches, from artificial intelligence tools to multiple criteria decision analysis methods, from extremely sophisticated tools (such as logic argumentation and ordered sets) to "soft", natural language-oriented and user-friendly ones. We are not going to present these tools here. Each of such tools however, has been created with a more or less precise "philosophical" background (see Genard and Pirlot, 2002) and with a more or less precise decision aiding approach in mind.

It is clear for instance that traditional Operational Research techniques such as linear programming, combinatorial optimisation and queuing theory reflect a normative idea of rationality as well as expected utility theory and game theory (see the discussion in Moscarola, 1984). On the other hand, several decision heuristics as well as some early artificial intelligence knowledge representation techniques reflect a descriptive approach: capture the way in which decision makers and/or experts do it and generalise it. Much cognitive analysis can be associated to such an approach.

At the same time, several multiple criteria decision support methods were developed under a prescriptive approach and several artificial intelligence tools make explicitly or implicitly reference to such an approach. Note for instance the common argumentation concerning intransitive preferences in decision analysis and non monotonic reasoning in logic (see, e.g., Doyle and Wellman, 1991; Tsoukiàs, 1991). It should also be noted that the seminal work of Simon (1954, 1979) on the concept of bounded rationality can be viewed as the background of both of several decision support methods (developed under a descriptive or a prescriptive approach) and of several artificial intelligence achievements.

Finally, several "soft" OR methods implicitly and several MCDA methods explicitly refer to a constructive approach. Indeed Roy (1996) explicitly claims that the philosophical justification for the methods developed by himself and his group is "constructivism", while the description of the Soft Systems Methodology (Checkland, 1981) clearly focuses on the decision aiding process and the structuring issue although it does not explicitly mention a constructive approach.

However, despite the fact that more or less each decision support method can be associated to a decision aiding approach, we claim that such an association is misleading since it reduces such approaches to a mere collection of methods (on this, note the examples used in chapter 1 of this book).

Our thesis is that decision aiding approaches do not imply the use of an exclusive set of methods and that at the same time, the use of a precise method does not imply the adoption of a decision aiding approach. In the extreme: we consider it possible to use a constructive approach and adopt at a certain point a combinatorial optimisation technique as well as using an outranking based preference aggregation procedure within a normative approach. The difference really is observable in the conducting of the decision aiding process. This is the reason why we dedicate a chapter to discussing how such a process can be structured and conducted.

In the following we are going to explore the constructive approach in more detail. This book however, and the one we have already published (Bouyssou et al., 2000) can be used in order to build models within any approach.

2.2 Decision Processes and Decision Aiding Processes

The concept of decision process is due to Simon (1947). As early as in 1947, Simon observed decision processes occurring within real organisations and concluded that the behaviour of real decision makers is far from the postulates of decision theory, at least as this theory was formulated at that time. During the '50s, Simon (1954, 1956, 1957) developed his "bounded rationality" theory, which states that a decision maker facing a choice behaves on the basis of a local satisfaction criterion, in the sense that he will choose the first solution that he subjectively considers as satisfactory without trying to attain an unrealistic (and useless) optimal solution. Actually Simon considers decision theory to be based on three implicit hypotheses (see the discussion in Moscarola, 1984):

- decision makers always know their problems well;
- such problems can always be formulated as an effectiveness (or efficiency) problem;
- the information and the resources necessary to find a solution are always available.

According to Simon, any of these hypotheses is not true in reality:

- decision makers never have a very precise idea of their problem;
- often their problems can be formulated as the search for a compromise;
- solving a problem is always constrained by the available resources and time.

The innovation introduced by Simon is radical. Decision theory as had been developed up to that moment always considered the rationality model as existing independently from the decision maker and his decision process. Simon put the decision process (the mental activities of a decision maker) and postulated that a rationality model has to be found within such a process at the centre of his reflection and not outside it. Most of the literature around this concept is based on the hypothesis that such cognitive activities are scientifically observable (either empirically or in experimental settings) and that "patterns" of "decision behaviour" can be established (see Humphreys et al., 1983; Kahneman and Tversky, 1979; Montgomery, 1983; Montgomery and Svenson, 1976; Slovic and Lichtentstein, 1983; Slovic and Tversky, 1974; Svenson, 1996; Vári and Vescenyi, 1983). The use of this concept in decision theory introduced two major innovations:

- rationality is expected to be linked to the process and not to the final decision; coherence is expected along the process, but such coherence is not necessarily reducible to the classic economic rationality;
- rationality is bounded in time, space and the cognitive capacity of the decision maker, therefore is subjectively defined and only locally valid.

The concept of decision process was later associated to organisational studies and more precisely to the study of how organisations and other collective bodies face decision situations (see Cyert and March, 1963; Emerson, 1962; March and Simon, 1958). These works showed that the behaviour of an organisation (assumed to be composed of rational decision makers) does not correspond to the rational behaviour as described by decision theory (the reader can see an extreme model in Cohen, March, and Olson, 1972, which describes the famous garbage can model, in which organisations are seen precisely as garbage cans). The problem, already observed by Weber (1922) in his studies during the 20's on bureaucracies, is that within an organisation different forms of rationality may co-exist (see Simon, 1976). Later on, related research was condensed in Mintzberg's work (see Mintzberg, 1979, 1983; Mintzberg, Raisinghani, and Théoret, 1976) as well as by other authors (see Benson, 1975; Dean and Sharfman, 1996; Huber, 1991; Ilgen, Major, and Tower, 1994; Mackenzie, 1986; Masser, 1983; Mélèse, 1978; Norese and Ostanello, 1984, 1989; Nutt, 1984, 1993, 1999; Ostanello, 1990; Ostanello and Tsoukiàs, 1993).

The observation of organisational decision processes leads to at least the following remarks:

- multiple rationalities that can be associated to different individuals and/or organisations coexist within organisational decision processes;
- such different rationalities rarely aggregate into a unique rationality characterising a process; an organised collection (a system) of rational individuals does not constitute a rational entity.

We are not going to further discuss the issue of the decision process and its models. Indeed, our aim is not just to propose another model of how decisions are made, but to show how analysts can help their clients when they act as "decision makers" either in individual or in organisational decision processes. Of course accepting an hypothesis on how decision processes are structured might influence the adopted decision aiding approach, but this is only one dimension among others in conducting a decision aiding process. The following section considers a model of decision process, but our choice is essentially operational.

2.2.1 A descriptive model of the decision process

In this section we will use a descriptive model of the decision process, introduced by Ostanello and Tsoukiàs (1993). This precise model originated to describe inter-organisational decision processes, but is sufficiently general to be used in more abstract contexts.

A decision process is characterised by the appearance of an "interaction space", an informal abstract space in which actors introduce and share a set of concerns (named "objects"). The awareness of the existence of such an interaction space is due to the existence of a "meta-object" (a concern which only exists in order to allow the actors to justify their presence in the interaction space projecting their concerns on such meta-object).

A temporal instance of a decision process (a state of the process) is characterised by: the participating actors, their concerns (the objects) and the resources committed by each actor to each object. Different levels of commitment and the number of actors interested in the same object characterise the structure of such a temporal instance, anticipating the dynamics under which such a state can be reached. In Ostanello and Tsoukiàs (1993), the following characteristic states were suggested:

- controlled expansion;
- uncontrolled expansion;
- controlled reduction;
- stalemate;
- dissolution;
- institutionalisation

in order to show the different directions towards which the state of the process can evolve (for more details, the reader can refer to Ostanello and Tsoukiàs, 1993).Recognising the present state and fixing a state one wishes to reach can help in understanding the strategy to follow within the decision process.

Example 2.1
Consider the construction of a new highway expected to improve the accessibility of two towns and going through a certain region.

There are a number of participating actors: the potential constructors of the highway, the local, regional and national institutions (including the "National Road Agency"), which have to authorise the construction besides as well as be concerned by the use of the highway and the consequences of its construction, the population affected by the highway and its construction, the social, political and economic groups etc.

Each of these actors has specific concerns about:

- the highway construction;
- the environmental impact;
- the socio-economic impact;
- the transformation of the land use;
- the transportation policy;
- the environmental policy;

which are all evoked by the "meta-object": the idea of a highway between A and B. Each participant commits and demands resources: for instance the potential constructors commit money and demand knowledge and authorisation, the regional authority commits authorisation and political legitimation and demands infrastructures and political legitimation, etc. Different decision problems can be identified such as:

- build the highway or not?
- freeway or toll-highway?
- which route?
- what the procedure to approve the route should be?

and each of them will be treated differently by the different actors depending on the concerns they have.

An external observer could identify the interaction space in which the concerned actors "meet" and can also recognise how the process reached its present "state". However, there are several different ways to conduct such a process (in a more or less authoritarian or participatory way) and for each of these, different types of decision, support can be demanded by different participants. It is not possible to identify a unique decision support. Decision aiding always refers to a participant and his concerns. ◇

As already discussed in the previous section, we are interested in decision aiding. From such a perspective the introduction of the above model of the decision process is functional to our purpose to describe the decision aiding process. Intuitively, in decision aiding we also make decisions (what, why and how to model and support). *Decision aiding is also a decision process but of a particular nature.*

Our claim is that in decision aiding contexts an interaction space (for at least two actors: the client and the analyst) appears, characterised by a meta-object which is the "consensual construction of a client's concern representation" through the use of the technical and methodological skills of the analyst and the domain knowledge of the client. Such a hypothesis implies that the two actors engage themselves in a decision process, that is, the decision aiding process is a special type of decision process.

2.2.2 Decision Making and Decision Aiding

The difference between these two concepts has already been discussed in Roy (1993) (see also Brown, 1989; Brown and Vári, 1992). However, Roy considers these as two different approaches and not, as we do, as different situations. In a decision making context we consider a decision maker who, having a concern, might use a decision theory tool in order to establish potential actions to undertake (although in more general terms decision making can be decision theory free). From such a perspective, the reader will often find the term "decision making" in this text. With this term, we will indicate the activities of an individual who develops some information in order to establish a "decision" to carry on within a decision process.[2] In such a setting, decision theory is directly used by the decision maker. There is no distinction between an analyst and a client. The decision maker is at the same time someone looking for support in his decision process and someone endowed with the appropriate knowledge to give himself this support. If there is an analyst, his presence is justified either because he acts as a tutor or because he is a "clone" of the decision maker (somebody who represents the decision maker, but who shares the same information, knowledge and values). It should also be clear that in such a setting we consider the decision maker as endowed with decision power and therefore also responsible for the decision to make.

On the other hand, a decision aiding context implies the existence of at least two distinctive actors: the client and the analyst, both playing different "roles" with respect to the concern of the client. More actors may exist in such a setting, the client not necessarily being a decision maker (he might not have decision power and be for instance in turn the analyst for another client). For simplicity, we only consider the simpler setting with only these two actors present and use with no further distinctions the concepts of decision maker and client.

A decision aiding context only makes sense with respect to one or more decision processes, the ones in which the client's concerns originate. In this chapter we focus our attention on the set of activities occurring within such a setting. We will call such a set of activities a "decision aiding process". The ultimate objective of this process is to attain a consensus between the client and the analyst. On the one hand, the client has a domain knowledge concerning the decision process. On the other hand, the analyst has a methodological knowledge, which is more or less domain independent. The task can be summarised as: given the client's domain knowledge and the analyst's methodological knowledge (and the associated formal and abstract language), interpret the client's concerns and knowledge so that he can improve his perceived position with respect to the reference decision process. Such an interpretation ought be "consensual": the client should consider it as his own interpretation, while the analyst should consider it correct and meaningful. However, the coherence sought by the actors does not refer to a given situation, information or knowledge, but to the cognitive artefacts they produce working together. From this point of view, the decision aiding process is an autopoietic

[2] This is not in contradiction with our emphasis on decision aiding. Indeed the activity of supporting a decision maker can be considered as the support to a decision making process.

system (a self reference system which maintains its organisation constant, but not a closed system since the environment is part of the system's organisation, see Maturana and Varela, 1984). Using a stakeholder approach (see Banville, Landry, Martel, and Boulaire, 1998)decision aiding sees the emergence of a new stakeholder in the decision process, which is the couple "client-analyst". The decision aiding process represents the cognitive efforts undertaken by this couple in order to "positively" influence the decision process in which they are involved.

Example 2.2
Consider again the previous section's highway example. If decision aiding is requested by any of the participating actors, this will concern "an object" among those evoked by the decision process (and its meta-object: the new highway).

Providing some decision aiding in this context raises questions of the type:

- what is the precise issue concerning the client and why (money, authority, natural resources, power, etc.)?
- how can we formulate such an issue in a decision support language, in terms of a decision problem (do we have to convince, to justify, to choose, to analyse, etc.)?
- how exactly will the decision support be designed (which alternatives do we consider, is there any uncertainty, are there several scenarios etc.)?
- what will effectively be done (negotiate with the other actors, impose a precise policy, expand the interaction space, etc

In a constructive decision aiding approach the answers to the above questions are not unique and have to be provided by both the client and the analyst who are now perceived as a unique stakeholder within the process. ◇

Within a decision process, several specific decision processes are structured. A particular type of decision process occurs when an individual (or more), acting as a client, asks another individual (or more), acting as an analyst, some advice concerning an object of the client's concern within another decision process.

We denote such a process as a "decision aiding process", where we can recognise:

- at least two actors, the client and the analyst;
- at least two objects, the client's concern and the analyst's (economic, scientific or other) interest (economic, scientific or other) to contribute;
- a set of resources including the client's domain knowledge, the analyst's methodological knowledge, money (or whatever the analyst asks), time;
- the meta-object being the construction of a shared representation of the client's object and concern.

Example 2.3
Consider an airline company. The sales department (the client) considers that, in order to face tough competition (the decision process), it needs to diversify the offer of seats on each route with respect to the season and the prices to apply, possibly adapting the offer dynamically as the demand evolves (the client's concern). They contact the company's Operational Research department (the analyst) asking for support. The Operational Research department replies positively since this is its job, but also because this is a good opportunity to show to the CEO that they are useful (the analyst's concern). The two actors (which in this case are units of an organisation and not individuals) will share the knowledge of the sales department (structure of the demand, structure of the supply, constraints of the commercial policy, competitors policy etc.), the analyst's knowledge (models and methods for yield management), the company's investment (time, money, resources) as well as the "award" in the case of success. The Operational Research department will possibly convince the sales department that their problem fits the well know "yield management problem" (thus creating the meta-object of the decision aiding process). However, we can expect that the result of the decision aiding process will not just be the construction of a yield management model (and possibly its successful implementation), but more generally an improvement of the company's commercial policy through the adoption of further actions conceived while discussing the yield management problem. ◇

2.3 A model of the Decision Aiding Process

A decision aiding process is a process of distributed cognition (Massey and Wallace, 1996; Vygotsky, 1978). With this term we indicate any process in which different agents endowed with cognitive capabilities have to share some information and knowledge in order to establish some shared representation of the process object. We call such shared representations *shared cognitive artefacts*. For example, consider two persons observing a painting at an exposition, discussing the interpretation to give to the artist's effort.

Within a decision aiding process we have at least two such "cognitive agents" (the client and the analyst) who share information and knowledge with the perspective of producing a set of shared cognitive artefacts, replying to questions such as:

- who has which problem?
- what could a solution to that problem be?
- why such a solution could be successful? etc.

However, our analysis of the decision aiding process will not be cognitive (describe and analyse the mental activities of the actors involved), but operational (how to conduct the process?). Actually, we are not going to analyse how such a distributed cognition occurs and how it works (although analysing how the two agents interact can be extremely interesting). Our basic hypothesis is that since we are looking for formal models of decision support, there is a basic agreement between the client and the analyst that they are looking for such a model and that they are going to use a formal representation language (this may possibly reduce the cognitive effort). There is no loss of generality with such a hypothesis. If such an agreement does not exist in reality, it is always possible to consider that the analyst will spend some of his time to convincing his client of the opportunity to follow a formal approach. The operational question we therefore have to ask is the following: what are precisely the cognitive artefacts that we expect from a decision aiding process?

In other words, we model the decision aiding process through its main products, the ones we consider mandatory in order to obtain "a consensual representation of the client's concern". At the same time, we can see such products as the deliverables honouring the contract with the client.

We introduce four cognitive artefacts as products of the decision aiding process:

- a representation of the problem situation;
- a problem formulation;
- an evaluation model;
- a final recommendation.

In the following section we intend to discuss such artefacts in the form of "checklists" to follow during the interaction with the client. We are aware that a real decision aiding process rarely follows such a checklist, but we have adopted such a rationalisation for the following two reasons.

1. It may help a novice decision analyst in structuring his interaction with his client in order to better conduct their discussion.

2. It may allow any experienced analyst going through a validation of his work to verify if the key issues and deliverables have been appropriately considered and how.

We understand that there is a risk of reducing decision aiding to "applying a manual", but we are confident that the reader also understands that a real decision aiding process is far more complex and that these are suggestions for support.

2.3.1 The Problem Situation

The first deliverable consists in offering a representation of the problem situation for which the client has asked the analyst to intervene. The main idea is to enable the analyst to answer questions of the type:

- who has a problem?
- why is this a problem?

- who decides on this problem (who is responsible)?
- who pays for the job?
- what is really important for the client?
- how is the client committed in this situation?

Such an analysis might also be useful for the client since it could help him to better situate himself with respect to the decision process for which he asked the analyst's advice.

A representation of the problem situation can be conceived as a triplet

$$\mathcal{P} = \langle \mathcal{A}, \mathcal{O}, \mathcal{S} \rangle$$

where:

- $\mathcal{A}$ are the actors involved in the process (as described by the client and perceived by the analyst);
- $\mathcal{O}$ are the objects (stakes) of the different actors;
- $\mathcal{S}$ are the resources committed by each actor on each object of his concern.

The reader should remember that a decision aiding process always refers to a decision process in which the client is involved. Decision support is always requested with respect to a decision process. Representing a problem situation corresponds to taking a picture of the decision process at the moment the decision support is requested. In this picture, the analyst and the client should recognise who participates (the actors), why they participate and what their concerns (the objects) are and what their level of commitment (the resources) is. Several different representations of the problem situation can be constructed during a decision aiding process. This is due both to the natural evolution of the decision process in which the client is involved (the pictures will be different) and to the decision aiding process itself which might modify the perception of the decision process for the client and the analyst (they might observe the same picture differently).

Example 2.4 (Selection of a Billing System)
A new mobile telecommunications operator has been established in a small, but highly competitive European market. One of the basic operational tools of such companies is their billing system (*BS*). This system allows both a structured accountancy of the traffic and a flexible policy towards the existing and potential clients (enabling for instance a variety of services beyond the basic ones, the creation of packages of services oriented to specific market targets, the monitoring of each subscriber's traffic).

Some years after the establishment of the company, the necessity to upgrade or to substitute the existing billing system became evident to the management. A decision process was therefore triggered, and we were asked to provide decision support (for details, see Stamelos and Tsoukiàs, 2003). An analysis of the problem situation showed that:

- The actors $\mathcal{A}$ involved were:
  - the acquisition manager;
  - the information systems manager (IS);
  - the marketing and sales manager;
  - the software suppliers;
  - the IS consultants.
- The objects $\mathcal{O}$ involved in the process were:
  - the market share of the company;
  - the policy towards the suppliers;
  - the company's internal organisation;
  - the billing system itself.
- The resources $\mathcal{S}$ implied in the process included the necessary funds for the billing system, the knowledge about billing systems and the relations with the software suppliers. The available time was very short, since all decisions had to be made in the least possible time due to the extremely competitive environment.
- The problem situation $\mathcal{P}$ results from the explicit representation of the sets described above.

The client in this study was the IS manager. The identification of the actors, their concerns and the resources were exploited in order to establish a set of problem formulations (see next section) that were meaningful for the client and his concerns within this situation. ◇

2.3.2 Problem Formulation

Given a representation of the problem situation, the analyst may provide the client with one or more problem formulations. This is a crucial point of the decision aiding process. While the first deliverable has mainly a descriptive (possibly explicative) nature, the construction of a problem formulation goes further towards formalising the interaction between the client and the analyst and introduces the use of the decision support language. The result is by definition reductive with respect to the reality of the decision process.

The idea is that a problem formulation translates the client's concern, using the decision support language, into a formal "problem" (a problem to which decision support techniques and methods apply). For instance, the client may claim that he has a problem to "buy a new bus in order to improve service to the clients". This may result in different problem formulations such as:

- choose one among the potential suppliers of buses;
- choose one among the set of offers submitted by the suppliers;

- choose one among the set of all combinations of two offers.

The above problem formulations are not similar and are not neutral with respect to the possible final recommendation. Indeed we, want to emphasise that adopting a problem formulation implies adopting a precise "strategy" towards the problem situation. Each such strategy will lead the decision aiding process to different recommendations. It is necessary to establish which strategy is going to be pursued with the client. Returning to the bus acquisition example, the first problem formulation focuses the attention on the suppliers and not on the offers they may make. The second problem formulation implicitly assumes that only one type of bus will be bought, while the third one allows to buy combinations of two different offers. It is clear that the choice of one of the above problem formulations will greatly influence the evaluation of the alternatives and the final solution.

A problem formulation can be conceived as a triplet:

$$\Gamma = \langle \mathbb{A}, V, \Pi \rangle$$

where:

- $\mathbb{A}$: is a set of potential actions that can be undertaken by the client with respect to the problem situation $\mathcal{P}$;
- V: is a set of points of view from which the potential actions are observed, analysed, evaluated, compared, etc.;
- Π: is a problem statement which anticipates what is expected to be done with the elements of $\mathbb{A}$. The reader will find more details on this point in Bana e Costa (1996), Ostanello (1990) and Roy and Bouyssou (1993) (see also section 2.4.3).

The use of problem formulations aims to anticipate the possible conclusions of the decision aiding process. The awareness of such possible conclusions allows the client to check whether these are compatible with his expectations. Moreover, if the effective conclusions are unsatisfactory to the client, he has the possibility of revising the problem formulation opening new modelling possibilities. The analyst's second deliverable consists in submitting a number of problem formulations to the client . The client validates them and chooses the ones with which the analysis might continue. Hereunder, we continue with the real case study (Stamelos and Tsoukiàs, 2003) concerning the selection of a billing system.

Example 2.5 (Selection of a Billing System BS)
The strategic decision with which the management was faced consisted in choosing one among the following options: upgrade the existing BS, buy and customise an existing BS, buy a BS created ad-hoc for the company by an external supplier (*bespoke system*), develop an ad-hoc BS in collaboration with an external supplier. However, the management was not able to choose an option without analysing what the billing system would eventually be in all such options. We therefore provided three problem formulations (the fourth option being the upgrade of the existing BS, was considered familiar) which we will call:

- B: buy (and customise an existing BS);
- M: make (externally a new ad-hoc BS);
- D: develop (a new ad-hoc BS in collaboration with a supplier).

In all three cases, a call for tenders was provided. The three problem formulations become:

1. $\Gamma_B = \langle \mathbb{A}_B, V_B, \Pi_B \rangle$ where:

   $\mathbb{A}_B$: offers proposed by specific suppliers of existing BS accompanied by a proposal for the customisation phase.

   V_B: points of view of the evaluation:

   - costs (including training, insurance fees and payment conditions);
   - quality (based on ISO9126 and benchmarks on the proposed product);
   - timing (of delivery, test and installation);
   - installed base of the proposed BS (including performance reports on already installed BS of the same type).

   Π_B: ranking of the offers in order to enable further negotiations on the price.

2. $\Gamma_M = \langle \mathbb{A}_M, V_M, \Pi_M \rangle$ where:

   $\mathbb{A}_M$: offers proposed by specific software developers with different degrees of experience in BS development.

   V_M: points of view of the evaluation:

   - costs (including training, insurance fees and payment conditions);
   - requirements satisfaction (client driven requirements);
   - timing (of delivery, test and installation);
   - type of supplier-developer (taking into account the company's supplying policy);
   - consequences for the company's internal organisation (including project management).

   Π_M: selection of a supplier - developer with whom to establish a supplying process (consisting of benchmarks, tests, training and delivery).

3. $\Gamma_D = \langle \mathbb{A}_D, V_D, \Pi_D \rangle$ where:

   $\mathbb{A}_D$: set of suppliers with whom it could be possible to co-develop a new BS.

   V_D: points of view of the evaluation:

   - costs (distinguishing internal and external costs);
   - requirements analysis and satisfaction;
   - timing (including the time in which the product could be ready for the market);

- type of supplier-developer (including company's supplying policy);
- consequences for the company's internal organisation (including project management);
- benefits to the company by entering the market of billing systems as a supplier itself.

Π_D: selection of a co-developer to establish a co-makership policy and therefore a long-term collaboration.

The client finally chose the first problem formulation, implicitly accepting a pure buying policy with respect to the basic strategic choice. We are not going to explain this choice. We would however, like to emphasise two observations:

1. From a general point of view, each problem formulation may generate quite a different evaluation model. The set of potential actions is different (existing BS in Γ_B, offers of non existing software in Γ_M, co-developing suppliers in Γ_D). The set of criteria may also be quite different (it is sufficient to note that the "make" and the "development" option requires to consider as a criterion the implication of the information systems department in the development process, a fact that may alter the distribution of resources and responsibilities in the company's organisation or that the development option requires to evaluate the eventual benefits of "selling" the new billing system). The relative importance of the criteria may also be different, while the aggregation procedures in each model have to be adapted to the different problem statements and the different nature of the criteria.

2. Focusing on the problem, the different problem formulations also lead to different models. In the Γ_B case, existing software products must be compared (even if the one chosen will be customised), a fact that allows the use of existing models (as the ISO9126 standard). Benchmark tests must also be performed. On the other hand, in the Γ_M case, the software artefact does not yet exist. The attention of the evaluation will shift to the satisfaction of the requirements during software development, and therefore some of the supplier's quality requirements have to be considered a priori. Finally, in the Γ_D case, the evaluation consists in the comparison of possible partners for software development, implying the comparison of the compliance of the partner's software development process with the company's standards (assuming that they exist).

   Furthermore, the priorities among the different criteria and attributes will change from one problem formulation to another, independently of the uncertainty associated with the available or required information. Finally, in order to aggregate the different software measurements, different necessities arise from one problem formulation to another (e.g., in the Γ_B case, measurements may correspond to observations and therefore a functional aggregation can be allowed, while in the Γ_M and in the Γ_D cases, the measurements are predictions or estimations based on expert opinions, a fact that requires a different treatment). ◇

Obtaining the client's consensus on a problem formulation leads to a gain of insight, since instead of having an "ambiguous" description of the problem we have an abstract and formal problem. Several decision aiding approaches will stop here, considering that formulating (and understanding) a problem is equivalent to solving it, thus limiting decision aiding to helping to formulate problems, the solution being the client's personal issue. Other approaches might consider the problem formulation as given. Within a constructive approach the problem formulation is one among the artefacts of the decision aiding process, the one used in order to construct the evaluation model.

2.3.3 Evaluation Model

For a given problem formulation, the analyst may construct an evaluation model, that is to organise the available information in such a way that it will be possible to obtain a formal answer to a problem statement (defined within Γ).

An evaluation model can be viewed as an 5-tuple:

$$\mathcal{M} = \langle A, \{D, \mathcal{E}\}, H, \mathcal{U}, \mathcal{R} \rangle$$

where:

- A is the set of alternatives to which the model applies. Formally it establishes the universe of discourse (including the domain) of all relations and functions that are going to be used in order to describe the client's problem.

- D is the set of dimensions (attributes) under which the elements of A are observed, described, measured etc. (the set D might be endowed with different structuring properties such as an hierarchy). Formally D is a set of functions such that each element of A is mapped to a co-domain that we denote as X_i.

- $\mathcal{E}$ is the set of X_i associated to each element of D. Each X_i can be considered as a set of "levels" or "degrees" to which a structure such as an "order" is possibly associated. Intuitively we can consider the functions in D as measurements using the X_i as "scales". Issues concerning measurement are discussed in more detail in chapter 3 of this book.

- H is the set of criteria under which each element of A is evaluated in order to take in account the client's preferences. Formally a criterion is a preference relation, that is a binary relation on A (a subset of $A \times A$) or a function representing the relation. The reader will find more details about preference models in chapter 3 of this book.

- $\mathcal{U}$ is a set of uncertainty structures to apply to D and/or H. Formally $\mathcal{U}$ collects all uncertainty distributions that can be associated to the relations and functions applied to A, besides possible scenarios to which uncertainty measures can be associated.

- $\mathcal{R}$ is a set of operators such that the information available on A, through D and H can be synthesised to a more concise evaluation. Formally $\mathcal{R}$ is a set of operators such that it is possible to obtain a comprehensive relation and/or function on A, possibly allowing to infer a final recommendation.

The reader can observe that a large part of the existing decision aiding models and methods can be represented through the above description. It also allows to draw the reader's attention to a number of important points:

1. It is easy to understand why the differences between the approaches do not depend on the adopted method. The fact that we work with only one evaluation dimension, a single criterion, a combinatorial optimisation algorithm can be the result of applying a constructive approach. It is important not to choose the method before the problem has been formulated and the evaluation model constructed, but to show that this is the natural consequence of the decision aiding process as conducted up to that moment.

2. The reader should note the difference between D and H. The former represents the "empirical" knowledge available or collected about A, but says nothing about the preferences of the client. The fact that such knowledge may use a structure such as an order (possibly coded in X_i) does not establish any knowledge about the client's "desires". These are modelled in H where preferences are explicitly represented. In the literature the elements of D are often called "attributes". Chapter 6 will extensively discuss the direct use of such "dimensions" in decision aiding.

3. The technical choices (typology of the measurement scales, different preferences or difference models, different aggregation operators) are not neutral. Even in the case in which the client has was to formulate his problem clearly and he is convinced about it (possibly using one of the techniques aiding in formulating problems presented in section 2.4), the choice of a particular technique, procedure, operator can have important consequences that are not discussed when the problem is formulated (for a critical discussion see Bouyssou et al., 2000). Characterising such techniques, procedures and operators is therefore crucial since it allows to control their applicability to the problem as formulated during the decision aiding process.

4. The evaluation models are subject to validation processes. This includes namely (see Landry et al., 1983a):

    - conceptual validation: verify whether the concepts used within the model in order to describe the client's concerns and problem situation are meaningful for the client, i.e., that he understands them and finds them useful; in other words the client and the analyst have to agree on what each precise concept represents and how this is useful for the client's problem;

    - logical validation: verify whether the concepts and the tools used within the model are logically consistent and meaningful (from a measurement theory perspective); the reader should pay attention to the fact that logical consistency does not necessary imply that the client is consistent in his claims, but that the model handles the information consistently (including possible inconsistencies and ambiguities);

- experimental validation: test the model using experimental data (and examples) in order to show that the model provides the expected results and possibly check formal requirements such as convergence of an algorithm, accuracy of a classification, sensitivity to small variations of the parameters, etc.;
- operational validation: show that the model when confronted with the decision process for which it was conceived acts as expected and that the client can indeed use it within such a process; further unforeseen consequences of using the model can be observed at this point.

It should be noted that validating the model is a crucial activity to establish the necessary consensus between the client and the analyst, consensus which (at least partially) legitimates the model to be used within the decision process for which it was conceived.

Example 2.6
Let us again consider the example of buying a bus. Suppose that the problem formulation adopted was the second one (choose one among the offers from suppliers). Suppose also that in reply to a call for tenders a number of offers are available. An evaluation model for this problem formulation could be (we use subscript 2 in order to denote that is the second problem formulation considered):

- A_2: set of offers received, legally acceptable;
- D_2: economic dimension (costs, maintenance, payment conditions), technical dimension (technical characteristics), quality characteristics (comfort, luggage capacity etc.); it should be mentioned that the set of dimensions in this case has an hierarchical nature (each of the above dimensions being further decomposable);
- $\mathcal{E}_2$: we are not going to show the whole set of scales, but we can mention that for instance maintenance is measured in "estimated numbers of man-hours per month", that one of the technical characteristics is the brakes capacity measured in "metres to stop the bus at max speed and full charge", that the comfort is a qualitative measure provided by an external expert on a scale of the type "good", "acceptable", "unacceptable";
- H_2: again we are not going to give the whole set of criteria; a generalised cost criterion putting together all different costs and the number of buses to buy is considered, while several technical and quality criteria have to be constructed such that the client's preferences can be represented; for instance a safety criterion is established (offer x is preferred to offer y iff the "brake's capacity of x" is at least 20 metres less than the "brake's capacity of y"); again an hierarchy of criteria has to be defined;

- $\mathcal{U}_2$ will be considered empty, all measures and preferences being considered by the client as "sure" and "precise";

- $\mathcal{R}_2$ is a set of aggregation procedures including the necessary parameters; it should be noted that the presence of an hierarchical structure on the criteria could be seen as the creation of a number of evaluation models one for each node of the hierarchy excluding the leaves. A precise aggregation procedure can be associated to each such evaluation model for instance, the quality criterion is obtained using a sorting (ordered classification) procedure by which each offer is classified in one among a set of merit classes (very good, good, acceptable, unacceptable) based on the values of the offers on the different quality criteria (comfort, luggage capacity, number of seats) (on such ordered classification procedures, the reader can be referred to: Belacel, 2000; Bouyssou and Marchant, 2005a,b; Bouyssou et al., 2000; Henriet, 2000; Massaglia and Ostanello, 1991; Mousseau, Słowiński, and Zielniewicz, 2000; Paschetta and Tsoukiàs, 2000; Perny, 1998; Yu, 1992b). Of course each aggregation procedure requires a number of parameters (importance of the criteria, thresholds, etc.). In our example the final aggregation was expected to compute a value for each offer and a multi-attribute value function was constructed. Therefore, tradeoffs between the three criteria (cost, technical, quality) had to be established (on such procedures, the reader may refer to Bouyssou et al., 2000, and chapter 6 of this book).

It is worth noting that had the third problem formulation been adopted, the evaluation model would have been quite different. The set of alternatives would be the set of all combinations of two offers. Furthermore, the reason for which such a problem formulation was considered derives from the observation that two different buses might better fit the variety of client the company serves (one for child transportation and the other for medium range tourism services). At least a criterion such as "fitting the market variety" should be added, while an uncertainty could now be considered (unknown behaviour of the market). ◇

2.3.4 Final Recommendation

The evaluation model will provide an output (denoted by Φ) which is still expressed in terms of the decision support language. The final recommendation is the final deliverable which translates Φ into the client's language.

It should be possible to check whether this final recommendation:

1. is technically sound (no incorrect or meaningless manipulations should be undertaken). Since the output Φ is the result of a number of manipulations on the available information (representing consequences, modelling preferences and uncertainties, aggregating measures, preferences and uncertainties etc.), it is important that such operations fulfil basic requirements of meaningfulness (for definitions, see Roberts, 1979). The number of situations in which intuitive reasoning leads us to undertake meaningless operations is incredibly high (for examples and further discussion see Bouyssou et al., 2000).

Care should be taken to verify whether the evaluation model is free of such biases;

2. is operationally complete (the client understands the recommendation and is able to apply it). The fact that the output is technically sound does not necessarily mean that this is useful for the client's problem. An arithmetic average of three measures of length is technically correct, but useless in case the client is looking for an aggregate measure of a volume (where a geometric average will fit perfectly). The final recommendation should be able to give an operational reply to the client's concerns (as these were established in the problem formulation) and enable him to undertake some deliberation and/or action (including doing nothing, provided this is deliberated);

3. is legitimated with respect to the decision process for which it was conceived. We should always remember that the advice requested by the client refers to some decision process in which he is involved. A technically sound and operationally complete recommendation is not sufficient in order to be incisive within the decision process. The reality of such processes includes organisational, cultural, ethical and interpersonal dimensions which are not necessarily (and rarely are) considered within the construction of the evaluation model and the establishment of the output Φ. When we return to the reality of the decision process we should take care to present the final recommendation in such a way that this can be inserted in the process.

In other words, the final recommendation should be able to translate the conclusions of the decision aiding process into a format that can be used within the client's decision process and/or organisation process in which the client is involved. In order to do that, the model, should not only be convincing for the client (which should be the case if a consensus was reached between the client and the analyst), but also should be able to convince the other actors participating in the process in which it is going to be used. Theoretical soundness, operational completeness and legitimation are the essential features the final recommendation should satisfy.

In the following sections we are going to focus our attention on how the previously introduced cognitive artefacts can be established, with particular emphasis on the definition of a problem formulation and the construction of an evaluation model. We try to outline a number of recommendations on how the decision aiding process should be conducted as well as a number of technical issues to which the analyst should pay attention. In this chapter, we do not provide the precise theory concerning the items of the evaluation model. These are discussed in a structured way and with much more detail in the following chapters. More precisely, the use of D, $\mathcal{E}$ and the construction of elements of H are mainly discussed in chapter 3. $\mathcal{R}$ is thoroughly discussed in chapters 4, 5 and 6 since it represents a crucial component in multiple criteria decision and evaluation models. Chapter 7 is dedicated to several technical aspects of the final recommendation construction and the treatment of robustness.

2.4 Problem structuring

There is a lot of literature on problem structuring (Abualsamh, Carlin, and McDaniel, 1990; Belton and Stewart, 2001; Binbasioğlu, 2000; Buchanan, Henig, and Henig, 1998; Corner, Buchanan, and Henig, 2001; Courtney and Paradice, 1993; Eden, 1988, 1994; Eden, Jones, and Sims, 1983; Keller and Ho, 1988; Landry, 1995; Lehaney, Martin, and Clarke, 1997; Massey and Wallace, 1996; McGregor, Lichtenstein, Baron, and Bossuyt, 1991; Mingers and Rosenhead, 2004; Norese, 1996; Pidd, 1988; Smith, 1988, 1989; Sycara, 1991; Woolley and Pidd, 1981). A common characteristic of this literature is the emphasis on the claim that supporting decisions should not be limited to solving well established decision models, but should help in facing more "soft", "ill-structured" decision situations that need to be "structured". The idea is that trying to fit a decision situation to a given decision model may result in solving the wrong problem correctly. It is therefore necessary to have methods and tools enabling to establish a problem formulation *before* any choice concerning the decision and/or evaluation model. The issue is (simplifying): *first set what the problem is and only then consider how to solve it.* This may appear to be common sense, but several authors cited above have shown that decision theory traditionally focuses its attention on how to solve the problem and not on how to formulate it.

Our claim is that our model of the decision aiding process can be used as a problem structuring method. Before showing how this can occur in detail, we discuss some of the best known methods found in the literature.

2.4.1 Problem Structuring Methods

Problem structuring methodologies aim to help decision makers to better understand their concerns (Checkland, 1981; Landry, 1995; Landry et al., 1983b; Rosenhead, 1989), better justify and legitimate their conclusions (Landry et al., 1996) and ease the validation process (Landry et al., 1983a; Ostanello, 1997).

Several among the problem structuring methodologies consider that decision aiding *is* problem structuring (see, e.g., Checkland, 1981; Friend and Hickling, 1987; Rosenhead, 1989). In other words, the quantitative aspects on which evaluation models usually rely are considered irrelevant, neglected or not at all considered under the not unrealistic claim that once the decision maker has a definitely clear idea of what the problem is, he also knows how to solve it.

2.4.1.1 Cognitive Mapping

Particularly "cognitive mapping" (see Eden, 1988, 1994; Eden et al., 1983) aims to give a representation of how a person (the client) "thinks" about a set of issues. The basic tool is simple: a network in which nodes represent the issues concerning the client(s) for whom the map is constructed and arrows represent the way in which one issue may lead to or have an implication on another. Issues are represented as sentences calling for "action" or "problem solving" and arrows show how one such action (or possible solution) will influence the outcome of another.

What is important in this method however, is not the tool itself, but the conducting of the interview which will lead to the establishment of the cognitive map. Indeed, the existing software implementing the method (Decision Explorer™) [3] is just a support for the discussion rather than a decision support tool. In the construction of a cognitive map a key role is played by the "facilitator" (the analyst in our terminology). He is expected to conduct the discussion and practically to design the cognitive map using the client's replies as well as the discussion developed during a cognitive mapping session. Actually, such sessions are carefully prepared and precise rules on how the discussion has to be conducted by the facilitator are established (see Rosenhead, 1989, ch. 3).

Cognitive mapping seems extremely useful when the client(s) consist in a group of people involved in organisational decision processes in which the emergence of consensus on different issues is extremely difficult and remains subject to power manipulations. In such a situation, it can also be very useful in giving a "sense" to discussions occurring within an (formal or informal) organisation.

The scope of a cognitive mapping session (possibly more than one session might be necessary) is to provide the client(s) with a representation of how they perceive their "problems" and how they expect to act on them. This a clearer representation and the structuring of the problem situation should enable the emergence of a consensus among the participants on how to act further and which actions it might be necessary to undertake.

2.4.1.2 Strategic Choice

Another well known problem structuring method is "strategic choice" (see Friend and Hickling, 1987; Friend and Jessop, 1969). Such a method is expected to handle the complexity of interconnected decision problems. The basic idea is that these complex problem situations are characterised by large uncertainties requiring strategic management. The authors claim that the basic philosophy of their method is "managing uncertainty in a strategic way". Within such a method three principal sources of uncertainty are identified:

- uncertainties about guiding values;
- uncertainties about the working environment;
- uncertainties about choices and related agendas.

The dynamics of a "strategic choice process" distinguish four "modes" of decision making:

- the "shaping mode" where the decision maker(s) are add concerns about the structure of the set of decision problems they are facing;
- the "designing mode" where the decision maker(s) are concerned about which actions are feasible with respect to their view of the problem;

[3] Decision Explorer is a product of Banxia Software, see http://www.banxia.com.

- the "comparing mode" where the decision maker(s) look for the different dimensions under which different actions could be compared;
- the "choosing mode" where the decision maker(s) look for arguments and commitment to pursue actions over time.

Strategic Choice can be seen as a toolbox of procedures aimed to support the four different "modes" previously introduced. However, such a toolbox (see also the software STRAD2^{TM})[4] is expected to be used within a precise approach in which the decision makers are seen as "stakeholders" of the final decision. It is mainly based on conducting workshops facilitating communication among the participants through the use of graphical tools manipulated by a facilitator who also conducts the workshop. The different modes of decision making are seen as interchangeable loops. This implies that within a workshop it is also important to register the dynamics of the interactions and of the outcomes. Indeed, the result of the method should not only be the deliverables (argued actions and policies), but also new ways of pursuing the organisational decision process.

2.4.1.3 Soft Systems Methodology

Soft Systems Methodology was developed by Checkland (1981) as an alternative to classic systems engineering (see Hall, 1962) seen mainly as a problem solving process (in which traditional OR techniques could apply).

"SSM is a learning system. The learning is about a complex problematical human situation, and leads to finding accommodations and taking purposeful action in the situation aimed at improvement, action which seems sensible to those concerned. SSM articulates a process of enquiry which leads to the action, but that is not an end point unless you choose to make it one (in Rosenhead, 1989, p. 67, ch. 4).

Although its presentation has evolved in recent years, we are going to present SSM in its original form, as a series of stages taking place in two worlds: the real world and an "abstract world" created through "systems thinking" on the real world:

1. enter situation considered problematic (real world);
2. express the problem situation (real world);
3. formulate root definitions of related systems of purposeful activity (abstract world);
4. build conceptual models of the systems used in the root definitions (abstract world);
5. compare models with the real world actions (real world);
6. define possible changes which are both feasible and desirable (real world);

[4] STRAD2 is a product of Stradspan, see `http://www.btinternet.com/~stradspan/products.htm`.

7. take action to improve the problem situation.

From a modelling point of view what is important is how "root definitions" are formulated. Under such a perspective SSM suggests a checklist of the following demands:

Context: who would be victim or beneficiary of the purposeful activity?

Actor: who would carry out the activities?

Transformation process: what is the purposeful activity expressed as "input-transformation-output"?

Weltanschauung: what view of the world makes this definition meaningful?

Owner: who could stop this activity?

Environment constraints: what constraints in its environment does this system take as given?

The second important modelling step is to build conceptual models of the system used in the root definitions. In order to do this, on the one hand it should be possible to consider actions on the systems and on the other hand, it should be possible to monitor and control them. This generates the following three basic modelling criteria:

Effectiveness: is this the right thing to be doing?

Efficacy: do the means work?

Efficiency: is a minimum of resources used?

The reader should pay attention to the fact that the above recommendations have to be seen within the whole process of understanding the problem situation and the different purposeful activities that can be undertaken. SSM is not just a simple checklist of modelling acts to follow in order to establish a deliverable for the client.

Practically SSM is applied through extensive interviews with the problem owners and large workshops including the stakeholders of the problem situation. The aim is that the modelling process suggested by SSM will allow such stakeholders to converge to a shared representation of both the problem situation (what is the problem?) and the actions to undertake (what to do?). Again the reader should consider that SSM has to be viewed as a "methodology" and not just a method, thus remaining situation driven and not method driven (the reader is referred to Checkland and Scholes, 1990, for more details concerning the use of SSM).

2.4.1.4 Valued Focussed Thinking

In his challenging book, Keeney (1992), suggests that usually decision making methods focus their attention on evaluating alternatives *after* such alternatives have been established or given. Instead, focus should be given to how such alternatives are or can be established and the author's suggestion is: *thinking about values and objectives.*

The idea is that as soon as the client has been able to structure his objectives (with respect to a given problem situation) he is also able not only to compare ready-made alternatives, but also to consider alternatives that were not there at the beginning of the process, but appear desirable and feasible within the objectives and values structure. For instance, it might be that only after understanding the importance of CO reduction in car engines for future sales, that CO absorption devices could be considered as components of such car engines.

Structuring objectives implies establishing an hierarchy of values starting from what Keeney calls "fundamental objectives". These should be (see table 3.2, in Keeney, 1992, page 82):

essential: indicate consequences in terms of the fundamental reasons for interest in the decision situation;

controllable: address consequences that are influenced only by the choice of alternatives in the decision context;

complete: include all fundamental aspects of the consequences of the decision alternatives;

measurable: define objectives precisely and specify the degrees to which objectives may be achieved;

operational: make the collection of the information required for an analysis reasonable, considering the time and effort available;

decomposable: allow the separate treatment of different objectives in the analysis;

non redundant: avoid double counting of possible consequences;

concise: reduce the number of objectives needed for the analysis of a decision;

understandable: facilitate generation and communication of insights for guiding the decision making process.

Fundamental objectives are then structured in attributes for which value functions (or utility functions in the case uncertainty has to be considered) can be constructed in order to "measure" the desirability of the outcomes and achievements for each objective. Such attributes result in "decomposing" the fundamental objectives into "sub-objectives", dimensions that contribute to defining the client's values. For instance, while looking to buy a car, a fundamental objective could be "safety". Such an objective can be decomposed into two attributes: "brakes efficiency" and "steering efficiency" which can be appropriately measured and for which the client could express preferences. The resulting structure of objectives (and attributes) allows the decision maker to have insight into the problem situation and, more importantly, to have an organised insight. Indeed he might be able to concentrate his attention on high-valued alternatives or make use of generic alternatives, to expand the decision context or even to consider any of his concerns as decision opportunities rather than as decision problems, thus allowing new unforeseeable paths of action to be taken into account. Keeney considers his approach as a path to creative decision making, claiming that structuring the client's values enables to expand the set of feasible actions through structured desirability. From this perspective his suggestion can be considered as a problem structuring approach, although, in this case, the use of quantitative methods is essential (in order to build the value and/or utility functions to be associated to attributes).

2.4.1.5 Integrating Approaches

In their book, Belton and Stewart (2001), advocate the necessity of integrating different approaches of multiple criteria decision analysis. In doing this they base their argumentation on their model of the process of decision analysis in which the following stages are distinguished:

- identification of the problem issue;
- problem structuring;
- model building;
- using the model to inform and challenge thinking;
- development of an action plan.

In discussing the problem structuring part of the MCDA process, the authors suggest a checklist of issues to analyse in order to be able to establish a model:

- criteria;
- alternatives;
- uncertainties;
- stakeholders;
- environmental facts and constraints.

However, since the author's proposal is essentially a way through which to integrate different approaches, the idea is to consider within a MCDA process the use of different techniques, driven by the problem situation and not by a particular method, an idea shared by several scholars in this field (see Bana e Costa et al., 1999; Belton et al., 1997; Norese, 1988, 1996). Multi-methodological approaches have been considered in a wider perspective in the literature (for a presentation see Rosenhead, 1989, ch. 13).

2.4.1.6 Discussion

All the approaches introduced above are basically prescriptive in nature. They suggest how an analyst should conduct the interaction with his client in order to lead him (the client) in a reasonably structured representation of his problem. However, they are either based on empirical grounds (we tried this several times and it works) or they represent a consistent theoretical conjecture. In all cases they have never been based on a descriptive model of the decision aiding activities, fixing the cognitive artefacts of the process, thus allowing the client and the analyst to control the process in a formal way. The result is that either they have to neglect the evaluation model aspect (ignoring situations when the problem formulated still does not allow to find intuitively dominant solutions or underestimates the cognitive biases that affect the decision maker's behaviour) or they have to fix a priori some of the artefacts by adopting a precise shape for the evaluation model (using value functions) thus limiting the applicability of the approach or they underestimate the influence that the analyst can have on his client, influencing his behaviour. Moreover, all such approaches do not explicitly take the process dimension of the decision aiding activities into account. Such a dimension is essential in order to be able to revise and update the outcomes of the decision aiding as the decision process evolves and the client learns.

The model of the decision aiding process previously suggested aims to fill such a gap. It is a descriptive model (showing how the decision aiding process gets structured) and at the same time is constructive since it suggests a path for the process concerning both the client and the analyst. Moreover, it allows to control the conducting of the process since it fixes the cognitive artefacts that are expected to be constructed during the process. This allows to control the process itself since each such artefact is precisely defined. In the next section, we are going to present how such artefacts can be constructed in more detail, suggesting empirical procedures for conducting the interaction with the client.

2.4.2 Representing the problem situation

We consider as given the interest of the client to work with the analyst. This interest is expected to be due to one or more concerns for which the client seeks advice due to his (possibly justifiable) conviction that he is unable to do this alone.

The construction of such a representation begins by establishing a list of actors potentially affected by the interaction between the client and the analyst (see also the so-called stakeholders approaches in decision aiding Banville et al., 1998; De Marchi, Funtowicz, Lo Cascio, and Munda, 2000; Shakun, 1991). We try to answer the question "Who else could be concerned by the client's concern?". A particular issue to explore here is whether the client is the (only) "owner" of this particular concern. It is often the case that the client himself is involved in a decision aiding process as an analyst or that this concern originates within a particular organisational structure. Actually he might not necessarily be a decision maker. For instance, the advice could be asked:

- for a (a priori or a posteriori) justification purpose;

- in order to understand a problem, but where no immediate action is expected to be undertaken;
- because the client has to report to somebody within the organisational structure.

This leads to the following questions: why could the other actors be concerned and what other concerns could they associate to the client's concerns? Intuitively we trace a map associating actors to concerns. Two questions arise at this point:

- are there any links among the concerns?
- how important are such concerns to the different actors?

In order to reply to the first question we can make use of a "projection" relation (see Ostanello and Tsoukiàs, 1993) showing how a concern projects to another one (usually from simple very specific concerns to more general and abstract ones). Usually such a relation results in a tree in which the leaves represent the simple (not further "decomposable") concerns and the root represents the meta-object characterising the decision process for which the decision aiding was requested.

Example 2.7
Imagine an artificial lake, created by the construction of a dam required to operate a hydroelectric power station, but also used for recreational activities (fishing, sailing etc.). The concern of "fish availability" (associated to the local fishermen) as well as the concern of "hydrogeological stability" (associated to the local electricity company) both project to the concern "lake management" (associated to the local authority: the local province). ◇

In order to reply to the second question we can associate the resources committed or requested by each actor for each of his concerns to each object. The client's commitment is in particular a key issue for two reasons:

- it will influence the content of the problem formulation and the evaluation model;
- it will play a specific role as far as the timing of the decision aiding process is concerned.

Establishing a representation of the problem situation enables the two actors (the client and the analyst) to "situate" themselves with respect to the decision process for which the aid was requested. This is important for at least two reasons:

- it offers the basic information to formulate the decision aiding problem and the associated evaluation models;
- it allows the two actors, in case of unsatisfactory conclusions, to come back and re-interpret the problem situation or to update it in order to take the evolution of the decision process into account.

2.4.3 Formulating a problem

As already introduced, formulating a problem is the first effort to translate the client's concern into a formal problem. The first question to ask here is: "what are we going to decide about"? We might call this set decision variables or alternatives or potential decisions. At this stage, it is important to establish with sufficient clarity what the set $\mathbb{A}$ does represent (e.g., suppliers or bids or combinations of bids etc.) and how (are they quantities, alternatives, combination of actions etc.).

Where does such information come from? One source is of course the client who might be able to provide at least part of the set $\mathbb{A}$ directly (for the cognitive problems associated to this activity see Newstead, Thompson, and Handley, 2002). The actors and their concerns as identified in the problem situation representation can also be sources. However, quite often the elements of set $\mathbb{A}$ have to be "designed" (see Hatchuel, 2001), in the sense that such a set does not already exist somewhere (and we just have to find it), but has to be constructed from existing or yet to be expressed information (the reader can see examples of such process in Keeney, 1992, a couple starting comparing one week holiday packages in national tourist resorts and ending up considering a one month holiday in the Pacific islands). A way to do this can be to work on the client's structure of values and expectations (as Keeney, 1992, suggested by) or using an "expandable rationality" (see Hatchuel, 2001) allowing to make the set of alternatives evolve. Another way is through an analysis of the structure of concerns in the problem situation. The client typically presents himself with a concern that remains somewhere at an intermediate level of the tree of concerns. Going up and down such a tree enables to identify different sets of potential actions (considering the resources the client may commit for each such concern).

Example 2.8
Using the holiday example, the concern of an ordinary holiday may project on a more general one which is the well being of the couple, for which further resources could be committed and thus allow to consider a concern of a special holiday. ◇

The final shape of set $\mathbb{A}$ will only be fixed when the evaluation model is established, but the effort of constructing set $\mathbb{A}$ during the problem formulation will pay during the whole decision aiding process: *half of a problem is deciding what to decide.*

The analysis of the different concerns (and how and why these associate to the different actors) leads to the establishment of the points of view to be considered in the decision aiding process. These represent the different dimensions under which we observe, analyse, describe, evaluate, compare the objects in $\mathbb{A}$. At this stage, the elements of V do not have any formal properties and do not necessarily define a structure (such as a hierarchy). They simply represent what the client knows or wishes to know about set $\mathbb{A}$. The key question here is: "among all this knowledge, what is relevant for the decision situation under analysis?" Again the representation of the problem situation can be useful here, since certain concerns can be of a descriptive nature (thus resulting in points of view), while the identification of the different resources to be committed to the concern may reveal other points of view. A more structured approach for this particular problem can be the use of cognitive maps (Eden, 1988, 1994) or Checkland's soft systems methodology (see Checkland and Scholes, 1990).

Last, but not least, we have to establish a problem statement Π. Do we optimise or do we look for a compromise? Do we just try to provide a formal description of the problem? Do we evaluate or do we design alternatives? Establishing a problem formulation implies announcing what we expect to do with set $\mathbb{A}$. We can first distinguish three basic attitudes:

- the first is constructing a set of feasible and realistic alternative actions without any necessary further evaluation purpose (as, for instance, in the "constraint satisfaction" case, see Brailsford, Potts, and Smith, 1999);
- the second is describing a set of actions under a set of precise instances of the points of view established in V;
- the third one, which we will call "purposeful" (also named operational, see Roy, 1996), consists in partitioning set $\mathbb{A}$.

Let us focus on this third attitude. Partitioning the set $\mathbb{A}$ implies establishing a set of categories to which each element of $\mathbb{A}$ is univocally associated (the "good" elements and the "rest", the "better", the "second best", etc., the "type X", the "type Y", the "type Z", etc.). In all cases and under all approaches, a purposeful problem statement results from the replies to the following questions:

- are the categories predefined or do they result from the comparison of the elements of $\mathbb{A}$ among themselves?
- are the categories ordered (at least partially) or not?
- how many such categories can exist (if they are not predefined)? Just two complementary ones or more than two?

A purposeful problem statement is a combination of answers to the above questions and establishes a precise form of partition of set $\mathbb{A}$:

1. in predefined, not ordered categories (a typical example being a diagnosis problem: patient x has appendicitis, patient y has a simple abdominal pain, etc.);
2. in predefined, ordered categories (as in the "sorting" procedures: tender x is "acceptable", tender y is "good", etc.);
3. in not predefined, not ordered categories (as in the clustering and more generally classification case: cluster the students of a class on the basis of their height);
4. in two, not predefined, ordered categories (for instance, the chosen or rejected objects and the rest or the optimal solutions and the rest: the outcome of all mathematical programming algorithms result in such a partition);

5. in more than two, not predefined, ordered categories (as in ranking procedures: rank the students on the basis of their performances in the different classes they followed in a year).

Up to now we have presented seven possible problem statements, the five purposeful ones previously described, and the two "non purposeful" ones which we call "design" and "description". All such statements can be further characterised by the possibility of looking to "robust" decision aiding. We will not further discuss this issue which already attracted the interest of several researchers (see Chu, Moskowitz, and Wong, 1988; Kouvelis and Yu, 1997; Ríos-Insua and Martin, 1994; Roy, 1998; Vincke, 1999a,b; Wong and Rosenhead, 2000). Further discussion can be found in chapter 7.

Operational Research and Decision Theory usually focus their attention on optimisation and more generally on "choice" problem statements in which one alternative or vector of decision variables is expected to be established as a solution (thus introducing the use of only two categories of solutions: the chosen ones and the rest). However, decision aiding is also provided when we rank-order the alternatives, when we classify them in categories (ordered or not, pre-existing or not) through internal (relative) or external (absolute) comparison. Establishing the problem statement with the client enables to focus on the appropriate methods and procedures to be used and avoids wasting time trying to force the information in irrelevant ones. Nevertheless, the establishment of Π is an anticipation of the final solution and as such it is rare that the client is able to provide it through simple questioning. The work of the analyst here is to show (through examples) the different possible problem statements and the different outcomes to which they lead.

As already mentioned, the establishment of a problem formulation is a key issue in the decision aiding process. It represents a tentative start to foreseeing and anticipating the conclusions of the process and as such has a "strategic" character (de facto establishing a strategy with respect to the decision process). From this perspective, revising the problem formulation represents a revision of "strategy".

2.4.4 Constructing the Evaluation Model

This is the typical task in which the analyst applies his methodological knowledge to the information provided by the client in order to produce a model which can be elaborated through a Decision Analysis method.

Again the first step is to fix the set of potential decisions or alternatives A. At this stage set A should have precise formal properties such as:

- being a compact (in a topological sense) or a discrete subset of an n-dimensional space;
- being a list of objects or an enumeration of options;
- having a combinatorial structure.

The existence of feasibility (or acceptability) constraints should apply here either directly (limiting the enumeration of A) or indirectly (limiting the space where A can be defined). Set $\mathbb{A}$, established in the problem formulation, is the starting point of this process, but new elements may be added (such as dump alternatives or ideal solutions) or eliminated. Within an evaluation model we consider the set A as stable across time and if it has a combinatorial structure, we have to fix whether we are going to focus on the elementary components or on a list of combinations. For instance, in evaluating investment portfolios, we could either consider each single investment (and then possibly try to find an optimal combination) or lists of ready-made combinations proposed by an investment company (and then possibly try to choose one of these).

Set A is described through a set of dimensions D. These represent the relevant knowledge we have about A. Some of these dimensions might have already been introduced in the form of constraints (used in order to fix set A), but other dimensions might be necessary for evaluation purposes, that is they should allow to evaluate the performance of each element of A under certain characteristics. Again the establishment of D requires fixing some formal properties. Each element of D is considered as a form of measurement, therefore the precise structure (X_i) of such a measure should be established ("a measurement scale"). Several types of measurement scales are possible and might co-exist within an evaluation model such as nominal, ordinal, etc (for more details see chapter 3 in this book). Furthermore, set D may have a structure such as a hierarchy. Set D cannot be empty. At least one dimension (the nominal description of A) exists. Usually set D is constructed using set V as a starting point. Typically the construction of D involves structuring V (if necessary) and associating a measurement structure to each element thus defined.

In the case in where a purposeful problem statement has been adopted (such as an optimisation or a ranking one), we then have to construct the set of criteria H to be used for such a purpose. The key issues here are the client's preferences. We define as a criterion any *dimension to which it is possible to associate a preference model, even a partial one, such that the client should be able to make a choice along this single dimension.* The construction of the set of criteria is a central activity in the decision aiding process. Dimensions expressed under "nominal measurement" (dimensions where we only know "labels" of the alternatives, but we are unable to provide any ordering among them) definitely require the establishment of a preference model. Dimensions using X_i endowed with some ordering structure can be transformed directly into criteria using an ordering as a preference structure, but this is rather exceptional. Usually the preference model is an interpretation of the available ordering (consider for example the use of a semi-order as a preference structure for a dimension endowed with a ratio scale) and therefore requires careful elaboration. The reader will find more details in chapter 3. Furthermore, it should be clear that if we are looking for a "rich" (in information) final result (such as an optimal solution), then the preference information ought to be "rich" itself. It should also be noted that the construction of H can be either the result of a direct process (creating criteria from dimensions through direct questioning of the client) or of an indirect process (establishing criteria "explaining" global preferences expressed by the client on examples or already known cases). When several criteria are considered, the first approach is described in more detail in chapter 5, while the second approach is described in chapter 6.

Last but not least, set H has to fulfil a number of conditions depending on the type of procedure that is foreseen to be used in order to elaborate the solution. A basic requirement is separability of the criteria: each criterion alone should be able to discriminate the alternatives, regardless of how these behave under the other criteria. A more complex requirement is the establishment of a consistent family of criteria: a set which contains the strictly necessary criteria and only these (see also chapter 4 in this book). Further conditions can apply, such as independence in the sense of the preferences (when an additive composition of the criteria is foreseen), etc. (for more details, the reader is referred to Keeney and Raiffa, 1976; Roy and Bouyssou, 1993; Vincke, 1992b).

At this point an element which has to be added to the model is the presence of any uncertainty structure $\mathcal{U}$. Uncertainty can be exogenous or endogenous with respect to the model. Typical cases of exogenous uncertainty include the presence of different scenarios or states of the nature under which the evaluation has to be pursued, poor or missing information as far as certain dimensions or criteria are concerned, hesitation or inconsistency of the client in establishing his preference on one or more criteria. Typical cases of endogenous uncertainty include the difficulty to discriminate alternatives in a dimension or criterion due to its ambiguous definition or linguistic nature, the appearance of inconsistencies due to conflicting information in different parts of the model, the impoverishment of the information due to the aggregation of dimensions or criteria. In all such cases the model must contain the appropriate structure for each particular type of uncertainty (if any). It should be noted that choosing a particular representation for a given uncertainty is not neutral with respect to the final result and that the client should be aware of the different results to which such a choice may lead.

The last element to be established within the evaluation model is the precise method $\mathcal{R}$ to be used in order to elaborate a solution to the model. Such a choice is not neutral, since different methods can result in completely different conclusions. Classic decision theory usually neglects this issue since it always considers as given the method (an optimisation procedure).This is however, not generally the case. The choice of $\mathcal{R}$ depends on the problem statement Π adopted in the problem formulation and should depend on two criteria:

- theoretical meaningfulness (in the sense of measurement theory): the method should be sound with respect to the information used. Typical errors in this case include the use of averaging operators on ordinal information, the use of a conventional optimisation algorithm when the cost coefficients are only ordinal, the underestimation of the importance of the independence of criteria when an additive value function is used.

- operational meaningfulness (in the sense that the client should be able to understand and use the result within the decision process). It should be noted that theoretical meaningfulness does not prevent the problem of establishing a useless result (an arithmetic mean of lengths is theoretically sound, but useless if the client is looking for a volume). Typical errors here include the underestimation of the quantity of information required by the client (a simple ranking of the alternatives can be insufficient for the client's concerns) or the aggregation of criteria without verifying their coherence.

A critical aspect in establishing $\mathcal{R}$ is the set of properties each such method fulfils. Each method may satisfy some useful properties, but may also not satisfy some other useful ones. It may present undesired side effects (see Bouyssou et al., 2000)such as non monotonicity, dependence on circuits, different forms of manipulability etc. The analyst should establish a set of properties that the method should fulfil (not necessarily of normative nature, but simply prescriptive ones) and make the client aware of the possible side-effects of the use of a potential method. From this perspective, the axiomatic study of the methods is a key knowledge for the analyst since it allows to have a precise map of the properties each method satisfies (see the discussion in chapter 4).

Furthermore, each method $\mathcal{R}$ requires the use of a number of parameters: some of these directly representing preferential information to be obtained from the client and his/her knowledge, others more or less arbitrary interpretations of such knowledge and depending on $\mathcal{R}$ itself.

The best known example concerns the use of coefficients of importance when several criteria have to be considered simultaneously. Here the client can have an "intuition" on "how important" certain criteria are with respect to others, but the precise formalisation of this concept strictly depends on how $\mathcal{R}$ works (see Borcherding, Eppel, and von Winterfeldt, 1991; Mousseau, 1997). If, for instance, $\mathcal{R}$ is based on the construction of a value function, then such parameters are tradeoffs among the criteria and have to be established together with the value function associated to each criterion. If on the other hand, $\mathcal{R}$ is a majority procedure then these parameters are "power indices" to be associated to potential coalitions of criteria. It is clear that, depending on what $\mathcal{R}$ is and on the available information, the establishment of these parameters requires precise procedures and interaction protocols with the client (see Mousseau, 1995; Mousseau, Dias, Figueira, Gomes, and Clímaco, 2003; von Winterfeldt and Edwards, 1986; Weber and Borcherding, 1993).

The same reasoning applies to other parameters that could be necessary for a given $\mathcal{R}$, such as discrimination thresholds, cutting levels for valued preference relations, cost coefficients and right hand side terms in mathematical programmes, boundaries of categories in classification procedures etc. Most of these parameters are an interpretation of what the client considers relevant for the problem and such an interpretation depends on how $\mathcal{R}$ is defined. Not all interpretations might be consistent with the client's information and knowledge and different consistent interpretations might lead to completely different results. The reader will find further details in section 4.4 of chapter 4.

Although constructing the evaluation model can be seen as a traditional decision aiding activity, on which the analyst's decision aiding knowledge usually focuses, it remains a crucial activity to which major attention has to be dedicated. Several technical choices have to be made here and not all of them are either straightforward or neutral with respect to the final recommendation. The accurate selection and justification of such choices enables on the one hand to guarantee meaningfulness of this artefact and on the other hand to identify the precise reasons why this specific final recommendation has been obtained. from this perspective, a sound construction of the evaluation model is crucial for easy revision and update, as well as enabling a clear justification of its adoption.

2.4.5 Constructing the final recommendation

The output of the evaluation model is essentially a result which is consistent with the model itself. This does not guarantee that this result is consistent with the client's concern and even less with the decision process for which the aid has been requested. As the client and the analyst return to reality they should take at least three precautions before they formulate the final recommendation (to be noted that due to the expected consensus between client and analyst, we consider that the outcome is also considered as "owned" by the client).

Sensitivity analysis. How will the suggested solution vary when the parameters of the model are perturbed? What is the range of values of such parameters for which the solution will remain, at least structurally, the same? A solution that appears to be sensitive to very small perturbations of some technical parameters implies that the solution strongly depends on this particular instance of the parameters and less on the preferential information. Since such an instance can be quite an arbitrary interpretation, a thorough investigation of the model should be conducted.

Robustness. We have already seen that robustness can be conceived as a dimension of the problem statement within a problem formulation. How good will the solution (or the method) be under different scenarios and combinations of the parameters? Being able to show that a particular solution will remain "good" (although perhaps not the best one) under the worst conditions that may occur should be considered as an advantage. Depending on the particular type of robustness considered, it is reasonable to verify whether such a feature holds or not. On the other hand a typical error in robustness analysis consists in testing different methods in order to find out if a certain solution will remain "the best". This is meaningless, since each method provides qualitatively different results that cannot be compared.

Legitimation. How legitimated is the foreseeable recommendation with respect to the organisational context of the decision process (David, 2001; Hatchuel and Molet, 1986; Landry et al., 1996)? As already mentioned, each decision aiding process refers to a decision process that usually occurs within a certain organisation (possibly of informal nature). Coming up with a recommendation that could be in conflict with such an organisation implies assuming risks. Either the client and the analyst explicitly pursue this conflict or they risk wasting time and resources. It should be noted that in considering legitimation, besides its precise contents, we have to take into account how a recommendation is presented, implemented and perceived by the other actors. From this perspective, a valid representation of the problem situation helps in verifying the legitimation.

Establishing the final recommendation implies the return to the reality of the decision process for the client and the analyst. A successful return is not only guaranteed by the scientific legitimation of the final recommendation (theoretical and operational meaningfulness), but also by the capacity of the two actors to take the dynamics of the decision process as well as its organisational complexity into account.

2.5 Update and Revision: an open problem

Conducting a decision aiding process is not a linear process in which the four cognitive artefacts are established one after the other. Since a decision aiding process always refers to a decision process which has a time and space extension, it is natural that the outcomes of the decision aiding process remain *defeasible cognitive artefacts*. Usually the process will encounter situations in which any of the above artefacts:

- may be in conflict with the evolution of the client's expectations, preferences and knowledge;
- may be in conflict with the updated state of the decision process and the new information available.

It is therefore necessary to adapt the contents of such artefacts as the decision aiding process evolves in time and space. see example 2.9 below.

Example 2.9
Consider again the case of the bus acquisition. A client looking for decision support within a problem situation described as: "the client's bus company is looking for a bus". He presents a set of offers received from several suppliers, each offer concerning a precise type of bus (thus a supplier may introduce several offers). The analyst will establish a problem formulation in which:

- $\mathbb{A}$ is the list of offers received;
- V is the list of points of view that are customary in such cases, (e.g., retrieved from past decisions) let's say cost, quality and transportation capacity;
- Π is a choice problem statement (an offer has to be chosen).

It is possible to construct an evaluation model with such information in which:

- A are the feasible offers;
- D are the dimensions under which the offers are analysed: price and management costs (for the cost point of view), technical features (for the quality point of view), loading capacity (for the transportation capacity point of view), etc.;
- H are the criteria that the client agrees to use in order to represent his preferences (the cheapest the better, the more loading capacity the better, better quality resulting from better performances on technical features, etc.);
- there is no uncertainty;
- $\mathcal{R}$ could be a multi-attribute value function provided the client is able to establish the marginal value function on each criterion.

When this model is presented to the client his reaction could be: *"in reality we can buy more than one bus and there is no reason that we should buy two identical buses, since these could be used for different purposes such as long range leisure travels or urban school transport"*. With such information, it is now possible to establish a new evaluation model in which:

- A are all pairs of feasible offers;
- D are the dimensions under which the offers are analysed (price, management costs, technical features, loading capacity etc.), but now concerning pairs of offers plus a classification of the buses in categories (luxury liner, mass transit etc.);
- H are the same criteria as previously plus a criterion about "fitting the demand" since two different types of buses may fit the demand better;
- uncertainty is now associated to the different scenarios of bus use;
- $\mathcal{R}$ could be a multi-attribute utility function provided the client is able to establish the marginal value function on each criterion.

A possible reaction to this suggestion could be the following: *"meanwhile we had a strategic discussion and the company considers that in reality the issue is to find a supplier with whom to establish a strategic partnership considering the expansion of our activities"*. Clearly, not only does the evaluation model makes no sense, but the problem formulation also has to be revised. We now have:

- $\mathbb{A}$ are potential suppliers;
- $\mathcal{V}$ concern the suppliers reliability, market share, availability to strategic partnerships, quality record, etc.;
- Π will now become a classification problem statement, the issue being to find out whether each supplier fits the company's strategy.

A new evaluation model has to be built now in such a way that:

- A are potential suppliers;
- D are the dimensions under which the suppliers are analysed (market share, quality certification, history of past supplies, management structure etc.);
- H are the criteria the client agrees to use in order to represent his preferences;
- there is no uncertainty;
- $\mathcal{R}$ could be a multiple criteria classification procedure.

The process may continue revising models and problem formulations until the client is satisfied. ◇

The above example shows that during a decision aiding process several different versions of the cognitive artefacts may be established. However, such different versions are strongly related to each other since they carry essentially the same information and only a small part of the model has to be revised. The problem is: is it possible to give a formal representation of how such an evolution occurs? In other words: is it possible to show how a set of alternatives or some preferential information may change while shifting from one model to another? It is out of the scope of this volume to find an answer to this question which requires further theory on the dynamics of the decision aiding process. We will just mention that the descriptive model of the decision aiding process turns out to be useful since it allows to establish a set of possible problem formulations and evaluation models to be used in different contexts, thus preventing the necessity of re-starting the modelling process from the beginning each time.

2.6 Conclusion

This is a book aiming at helping decision makers, analysts, practitioners and researchers to appropriately use tools and methods of decision support. However, such tools and methods are not independent algorithms and models which we just have to apply to some information to obtain the conclusion. They are used within a stream of interactions structured around a decision process in which an actor involved (the client) asks for advice and support from another actor who becomes involved (the analyst). In other words they are used within a *decision aiding process*. It is therefore necessary to analyse them from the perspective of such a process. Talking about the correct use of such tools, about their meaningfulness, about their legitimation and the usefulness of their results only makes sense with respect to such a decision aiding process.

In this chapter we tried to introduce a general description of what such a decision aiding process is and how it can be conducted in order to pursue meaningful, useful and legitimated recommendations. In order to do so, we first had to show that aiding someone involved in a decision process cannot just be limited to solving a well established formal problem. It concerns a wide set of issues including the understanding of the problem situation in which the client is involved as well as formulating a number of formal problems to choose from. Such concerns are independent of the formal model that is going to be used to elaborate the client's problem. In practice, such concerns are always considered. However, different decision aiding approaches can be characterised by the fact that such concerns are explicitly or implicitly considered as outcomes of the decision aiding approach.

In the chapter we basically introduce two contributions.

1. A model of the decision aiding process based on the description of the cognitive artefacts such a process produces. Indeed, our point of view is that decision aiding is a process in which the actors engaged have to establish a set of shared representations of issues such as:
   - a representation of the problem situation within which the client (and consequently the analyst) are engaged;
   - one or more problem formulations, a formal anticipation of the model to construct, in which the client's concerns are expressed in a "decision support language";
   - one or more evaluation models enabling to elaborate the problem formulation(s) and to establish a conclusion;
   - a final recommendation in which the conclusions of the decision aiding process are summarised, expressed in natural language and prepared to be confronted with the real world (the client's decision process).
2. A number of recommendations on how the above cognitive artefacts can be constructed through interaction with the client. Such recommendations are expected to be helpful in order to:
   - guarantee the theoretical soundness of the result (meaningfulness);
   - guarantee the operational completeness of the result (usefulness);
   - guarantee the legitimation of the results within the client's decision process.

In the following chapters the reader will see how the construction of the evaluation model can be pursued following the above requirements in further detail. More precisely, chapter 3 will discuss how it is possible to establish models of preferences (on a single criterion) and how to use numerical representations of measures and preferences correctly. Chapter 4 gives a general introduction to the problem of aggregating preferences expressed on several criteria or performances established on several dimensions (attributes). Chapter 5 will focus on the use of "procedures" allowing to undertake such an aggregation (and will therefore study the properties of such procedures), while chapter 6 will focus on the use of "models" representing a global preference and how these influence the preferences on single criteria and their aggregation (and will therefore study the properties of such models). Finally, chapter 7 will discuss the problem of constructing the final recommendation, mainly when the result of the aggregation is not directly usable and issues concerning the robustness of such a recommendation.

3

NUMBERS AND PREFERENCES

3.1 Introduction

This book is devoted to the use of formal models in evaluation and decision aiding models. Most of the formal models presented in the literature and used in practice are based on two fundamental mathematical concepts: numbers and relations. They are also present on each page of this book. In this chapter, we will focus on these two basic tools.

Sections 3.1 to 3.3 are devoted to the use of numbers for representing various aspects of the observed reality and the adequateness of performing certain calculations with respect to this reality. The rest of the chapter deals with the connections between the language of preference relations (evaluation and decision aiding is impossible in the absence of preferences) and the language of numbers. This chapter does not aim at being exhaustive: it is to be seen as a collection of questions that naturally arise in the course of using formal models for evaluation and decision aiding, either to build preferences on the basis of numerical information, or to build numerical models of preferences. It should be noted that, contrary to the next chapters, this one does not provide operational tools for decision aiding, but it points out fundamental aspects which will be (sometimes implicitly) present everywhere in the book.

3.2 Numbers

To our knowledge, there is no culture, even very primitive, which does not use numbers or, more generally, mathematics. "Everything is numbering" said Pythagoras and many stories and legends but also very concrete political decisions are based on what could be called the mysticism of numbers. Martzloff (1981) writes: "Without them, it is impossible to understand the measure of the sky and of the earth, to manage the taxes and finances, to pitch military camps or to arrange bodies of soldiers, to govern the city".

Galileo (published 1966) translated many experimental observations about the physical world into mathematics and said that (our translation):

> Philosophy is written in this very vast book that is eternally open in

> front of our eyes—I mean the universe—but one cannot read it before having learned the language and before having become familiar with the characters in which it is written. It is written in mathematical language and its letters are triangles, circles and other geometric shapes, means without which it is humanly impossible to understand a single word, without which we vainly roam in a dark labyrinth [1]

It is clear that the success of mathematics in the description and the explanation of the solar system (for example forecasting the return of the Halley's Comet) was crucial in the development of the role of mathematics in the explanation of other natural phenomena: capillarity, electromagnetism, classification of crystals, heat propagation, ... Extending the domain of application of mathematics, Condorcet introduced what is now called "social mathematics" (see Condorcet, 1785). He was convinced that this discipline would contribute to the welfare and the progress of humanity.

Today, mathematics are used in all the fields of human activity, not only as a tool to make calculations, but also in the education, the methodology and everyone's way of thinking: we all reason in terms of measures, percentages, ratios, logical deductions, statistics, ... In fact, numbers are present everywhere. Most of the people consider that "natural numbers" (positive integers) exist independently of any mathematics or, even, of any human intervention. However, our intuitive perception is limited to very small numbers, associated to the counting of objects (in some primitive tribes people only count up to five or have no specific words for the numbers; very small numbers are considered to be particular characteristics of the counted objects and are treated as attributive adjectives). The constitution of a system of numbers is already a mathematical theory, with many rules, conventions or axioms. These rules can be different depending on what these numbers represent.

A first use of numbers is of course numbering (first, second, ...), i.e. giving a list in a certain order (ordinal aspect of numbers). A second use of natural numbers (positive integers) is to count objects (cardinal aspect); in this perspective, some basic operations can be introduced, such as addition and subtraction. However, the main use of numbers resides in one of the most natural activities of humans: measuring. Measuring allows to quantify phenomena, to make calculations in order to understand, to foresee and to manage our environment. Measuring weights, lengths or volumes is necessary in commercial transactions. Measuring heat, duration or flow is useful to describe physical phenomena. Measuring wealth, unemployment or production allows to analyse economy. Measuring pollution, noise or vegetation density is necessary in environmental management. Numbers are used to measure many other things (as illustrated in Bouyssou et al., 2000, ch. 4): speed, age, density, score, social impact, economic index, probability, possibility, credibility, preference intensity, latitude, date, earthquake intensity, popularity,

[1] "La filosofia é scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, e conoscer i caratteri, ne' quali é scritto. Egli é scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi é impossibile a intenderne umanamente parola; senza questi é un aggirarsi vanamente per un oscuro laberinto".

political success, financial ratio, friction coefficient, coordinates, radioactivity, electric power, angle magnitude, percentage, severity of a tumour, cash-flow, exchange rate, productivity, landscape harmony, ... A number can indicate the presence or not of a specific property (boolean variable). It can be a tool for recognising an object (i.e. a label). Sometimes, it is possible to define a standard and to express the measure of every object in function of this standard thanks to physical instruments, as is the case for lengths or weights. Sometimes, it is necessary to define reference states on an "arbitrary" scale, as is the case for temperatures or dates (of course, this can depend on the state of knowledge: thermodynamics allowed to define an "absolute" zero for temperature and cosmology will perhaps allow to do so for dates).

Manipulating "numbers" in social sciences, as most of the decision aiding tools try to do, raises the question of measuring human or social characteristics, such as satisfaction, risk aversion, preference, group cohesion, etc. However, contrary to what happens for the characteristics measured in the natural sciences (length, weight, duration, etc.), there is no real consensus on what measuring means in social sciences. Does the way of measuring depend on the goal of the process (description of reality, construction of models or laws, decision support)? Does the duplication of social objects make sense (two apples having the same weight versus two individuals having the same preference)? How to aggregate measures (the weight of a package of apples versus the preference of a group of individuals)?

Some of these questions were raised in Bouyssou et al. (2000): remember, in particular, the role of numbers in voting systems (chapter 2), in evaluating students (chapter 3), in characterising the development of a country, the quality of air or the performance of a decathlete (chapter 4), in assessing competing projects (chapter 5) or in automatic decision making (chapter 7).

It seems clear that the numbers representing measures cannot always be treated in the same way because the underlying information can be completely different from one context to another. This chapter certainly does not give a definitive answer to this fundamental and difficult problem. Its purpose is to try to clarify the various types of numerical scales that are used, especially in the field of decision aiding. We will first present four basic examples in order to introduce the main types of scales that are usually discussed in measurement theory (see Krantz, Luce, Suppes, and Tversky, 1971; Narens and Luce, 1986; Roberts, 1979, 1994). The rest of the chapter is a study of the connections between numbers and relations in preference modelling.

3.3 Four basic examples

3.3.1 The race

The arrival order in a race is the following: Alfred, Bill, Carl, David, Ernest, Franz, Gilles, Henry, Isidore and John. Alfred, David, Franz and John form team a, the others form team b. The duration of the race has been registered, in seconds, yielding for each runner, giving the numbers in table 3.1. The purpose is to compare these two teams and, if possible, to decide which team is the best. On

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

Table 3.1: Race example: times in seconds.
Team $a = \{A, D, F, J\}$, team $b = \{B, C, E, G, H, I\}$.

the basis of these numbers, the following assertions can be verified:

(i) the mean time of team $b = \{B, C, E, G, H, I\}$, is higher than the mean time of team $a = \{A, D, F, J\}$;

(ii) the second best (lowest) time in team b is lower than the second best time in team a;

(iii) the mean time computed on the basis of all the runner' results, is beaten by three runners of team a and three runners of team b;

(iv) the median[2] time, calculated on the basis of all of the runners' results, is exceeded by two runners of team a and three runners of team b;

(v) the third best time in team a is lower than the times of three runners of team b;

(vi) the worst time in team b is more than 1.2 times the best time in team a;

(vii) the difference between the worst time in team a and the worst time in team b is 12.5 times the difference between the best time in team a and the best time in team b;

(viii) the sum of the two best times in team a is higher than the sum of the two best times in team b;

(ix) the difference between the two best times in team a is triple the difference between the two best times in team b;

(x) if we consider the three best times, team b is more often represented than team a;

(xi) the sum of the three best times in team a is higher than the sum of the two best times in team b;

(xii) the mean time of team b is 1.015 times the mean time of team a;

(xiii) the ratio between the worst and the best times is higher in team a than in team b;

(xiv) in team a, the square of the worst time is 1.6 times the square of the best time;

[2] The median of a set of numbers is a value x such that there as many numbers greater than x than number smaller than x (some specific conventions exist to avoid ambiguity in the even case).

(xv) the difference between the best and the worst times in team a is equal to 11.5.

Now, if we convert all the times into minutes, we see that all the assertions remain valid, except (xiv) and (xv). More generally, as a duration is completely defined by the choice of a unit (the origin being "natural"), every multiplication by a positive constant should be possible without changing the conclusions. This shows that some assertions that use numbers resulting from measurement, even in cases where measurement is of "high quality" should be considered with care. The way in which numbers are obtained is crucial for their interpretation. This will be all the more true with numbers obtained by measurement operations of "decreasing quality".

3.3.2 The weather

Temperatures were measured at noon in two European countries, during respectively 10 and 8 consecutive days. The results, in Celsius degrees, are presented in table 3.2. On the basis of these observations, how could we help a tourist choose a country for his holidays? As in the previous example, many assertions can be pro-

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	–	–

Table 3.2: Temperatures in two countries (Celsius degrees).

posed for the comparison of the countries a and b, on the basis of these numbers. Here are some examples of such valid assertions:

(i) the mean temperature in country a is higher than the mean temperature in country b;

(ii) the second highest temperature in country a is higher than the highest temperature in country b;

(iii) the mean temperature calculated on the basis of all the measures in both countries, is exceeded seven times in country a and three times in country b;

(iv) the median value, calculated on the basis of all the measures in both countries, is exceeded four times in country a and once in country b;

(v) the fourth highest temperature in country a is higher than the temperatures in country b during 5 days;

(vi) the highest temperature in country a is more than 1.5 times the lowest temperature in country b;

(vii) the difference between the highest temperature in country a and the highest temperature in country b is four times the difference between the lowest temperature in country a and the lowest temperature in country b;

(viii) the sum of the two highest temperatures in country a is larger than the sum of the two highest temperatures in country b;

(ix) the difference between the two highest temperatures in country a is two times the difference between the two highest temperatures in country b;

(x) if we consider the five highest temperatures in table 3.2, country a is more often represented than country b;

(xi) the sum of the three highest temperatures in country a is larger than the sum of the four lowest temperatures in country b;

(xii) the mean temperature in country a is 1.1 times the mean temperature in country b;

(xiii) the ratio between the highest and the lowest temperatures is larger in country a than in country b;

(xiv) in country a, the square of the highest temperature is 2.37 times the square of the lowest temperature;

(xv) the difference between the highest and the smallest temperatures in country a is equal to 7.

The temperatures in table 3.2 are expressed in Celsius degrees, but they could be expressed on another temperature scale. In table 3.3, they have been converted into Fahrenheit degrees (in order to limit the number of decimals, we have simply multiplied by 1.8 and added 32). On the basis of these new numbers, we see

	1	2	3	4	5	6	7	8	9	10
a	68	60.8	59	57.2	57.2	59	55.4	59	60.8	64.4
b	57.2	53.6	55.4	59	57.2	55.4	59	60.8	–	–

Table 3.3: Temperatures in two countries (Fahrenheit degrees).

that some assertions remain valid and other do not. As a temperature scale is completely defined when the origin and the unit are fixed, every transformation of the form

$$\alpha x + \beta \text{ (with } \alpha > 0),$$

should be possible without changing the conclusions. The reader can verify that this is the case for all the assertions except (vi), (xi), (xii), (xiii), (xiv) and (xv). It should be noted that what is verified is not the veracity or not of an assertion, but the fact that its veracity (resp. falsity) is unchanged for an admissible change of scale. For example, let us verify that the following assertion is not invariant for the transformation $\alpha x + \beta$: "this temperature is the double of that one". Numerically, this assertion can be written

$$x_1 = 2x_2.$$

As this equality does not imply that

$$\alpha x_1 + \beta = 2(\alpha x_2 + \beta), \quad \forall \alpha > 0, \forall \beta,$$

the veracity (resp. falsity) of the assertion can change for an admissible change of scale.

On the contrary, the assertion "this difference of temperature is the double of that one" remains true (resp. false) when an admissible change of scale is applied. Indeed,

$$x_1 - x_2 = 2(x_3 - x_4)$$

implies, $\forall \alpha > 0, \forall \beta$:

$$(\alpha x_1 + \beta) - (\alpha x_2 + \beta) = 2\Big[(\alpha x_3 + \beta) - (\alpha x_4 + \beta)\Big].$$

3.3.3 The race again

Let us take again example 3.3.1. Suppose that the only available information is the ranking of the runners and that numbers have been associated to them in decreasing order of the arrivals, as in table 3.4. On the basis of these numbers,

A	B	C	D	E	F	G	H	I	J
10	9	8	7	6	5	4	3	2	1

Table 3.4: Race example: numbers associated to the runners. Team $a = \{A, D, F, J\}$. Team $b = \{B, C, E, G, H, I\}$.

many assertions can be proposed for the comparison of the teams a and b; here are some examples of such valid assertions:

(i) the mean of team a is greater than the mean of team b;

(ii) the second highest number in team b is bigger than the second highest number in team a;

(iii) two runners of team a and three runners of team b have a number that is bigger than the mean of the whole set of runners;

(iv) two runners of team a and three runners of team b have a number which is bigger than the median of the whole set of runners;

(v) the third highest number in team a is greater than the numbers of three runners of team b;

(vi) the greatest number in team a is less than two times the number of the third runner in team b;

(vii) the difference between the numbers of the best runners of teams a and b is equal to the difference between the numbers of the last runners of these teams;

(viii) the sum of the two highest numbers in team a is equal to the sum of the two highest numbers in team b;

(ix) the difference between the numbers of the first and the second runners of team a is triple the difference between the numbers of the first and the second runners of team b;

(x) team b has more runners among the three highest numbers than team a, and also among the five highest;

(xi) the highest number in team a is larger than the sum of the three lowest numbers in team b but smaller than the sum of the four lowest numbers in team b;

(xii) the mean of team a is 1.17 times the mean of team b;

(xiii) the ratio between the second and the third highest numbers is larger in team a than in team b;

(xiv) in team b, the square of the highest number is 20.25 times the square of the smallest one;

(xv) the difference between the greatest and the smallest number in team a is equal to 9.

In fact, in this example, the only relevant information is the ranking of the runners, and there is no reason to privilege one numerical representation over another (unless very specific assumptions are added). Consider, for instance, the numerical representation given in table 3.5. On the basis of these new numbers, we see that

A	B	C	D	E	F	G	H	I	J
100	90	80	10	9	8	7	6	5	0

Table 3.5: Race example: other possible values for the runners.

assertions (ii), (iv), (v) and (x) remain verified but not the others. This means that the other assertions cannot be considered as reliable information: their truth or falsity depends on the particular numerical representation which is chosen. As only the ranking of the runners is known, every strictly increasing transformation of the numbers should be possible without changing the conclusions derived from these numbers. This is clearly the case only for assertions (ii), (iv), (v) and (x). Note also that all these considerations do not allow to definitely decide which team is the best.

3.3.4 The expert's advice

Suppose that an expert evaluated social projects in a city by assigning numbers to them in function of what he considers as their chance of success and their global interest for the city. The scale is $[0, 20]$ and the higher the evaluation, the higher the quality of the project. What kind of information can we deduce from these evaluations? As in the previous examples, many assertions can be proposed on

A	B	C	D	E	F	G	H	I	J
17	16	14	12	10	10	9	5	3	2

Table 3.6: Evaluations by the expert on a scale from 0 to 20.

the basis of these numbers; here are some examples of valid assertions:

(i) project A is the best;

(ii) project E is two times better than project H;

(iii) the difference between projects A and B is less than that between D and E;

(iv) the differences between B and C and between C and D are equal;

(v) four projects are "below the mean" (which is equal to 10);

(vi) if two projects can be chosen, the pair $\{B, C\}$ is better than $\{A, D\}$ (as the sum of their evaluations is higher).

In this example, the numbers are associated to subjective evaluations (by the expert) and not to some "objective facts" such as times, temperatures or ranking, as was the case in the previous examples. This means that the reliability of a conclusion based on these numbers depends on the type of information they really support. This can be the subject of additional assumptions or can be obtained by a dialogue with the expert on how he has built his evaluations. Such a dialog could reveal, for example, that his evaluations of "bad" projects were only very roughly made (so that the difference between H and I has no meaning at all), or that he really hesitated to consider that A is better than B, while he was sure that C is much better than D. Moreover, if this expertise has to be merged with other information, the decision maker may want to take into account the inevitable imprecisions of such subjective evaluations by considering that a difference of 1 point between two projects can be ignored. In this case, table 3.7 of evaluations could be considered as equivalent to table 3.6 for the purpose of comparing projects. With

A	B	C	D	E	F	G	H	I	J
17	17	15	12	10	10	10	0	0	0

Table 3.7: "Equivalent" possible evaluations.

these new evaluations, we see that some of the assertions proposed before are no longer true. Finally, the only reliable information could be the following:

- A and B are the best projects and are very similar,
- C is strictly better than D, which is better than E, F and G,
- There is no significant difference between E, F and G,
- H, I, J are the only bad projects (evaluations less than 7),

and every set of numbers supporting this information could be accepted (and not only the strictly increasing transformations as in the example of section 3.3.3). For example, every set of numbers satisfying the following conditions could be considered as an acceptable numerical representation of the information (in the following expressions, $h(x)$ is the numerical evaluation of project x):

$$\begin{cases} |h(A) - h(B)| \leq 1, \\ h(A) > h(x) + 1, \forall x \neq A, B, \\ h(B) > h(x) + 1, \forall x \neq A, B, \\ h(C) > h(x) + 1, \forall x \neq A, B, C, \\ h(D) > h(x) + 1, \forall x \neq A, B, C, D, \\ |h(E) - h(F)| \leq 1, \\ |h(E) - h(G)| \leq 1, \\ |h(F) - h(G)| \leq 1, \\ h(x) \leq 7 \text{ iff } x \in \{H, I, J\}. \end{cases} \tag{3.1}$$

Of course, the solution of this system is not unique. Moreover, the threshold, equal to 1 here, could be variable along the scale.

3.4 Evaluation and meaningfulness

3.4.1 Definitions

Let us consider a completely ordered set (i.e. a set with elements ranked from the first to the last, without ties). As a completely ordered set can generally be mapped to the real numbers (see Fishburn, 1970), we limit ourselves to numerical ordered sets, i.e. subsets of real numbers, and we call them "numerical scales". "Evaluating" an object consists in associating an element of a numerical scale to it, according to some conventions as, for example, the choice of a measurement instrument (sometimes, the element of the numerical scale that is associated to an object is not unique, because of imprecision or uncertainty, but we put these situations aside for the moment). The evaluation of an object along a numerical scale is supposed to characterise or to represent a particular information about certain aspects of this object (weight, temperature, age, number of votes, development of a country, air quality, performance of a sportsman, etc. (see Bouyssou et al., 2000, ch. 2 and 4). Changing the conventions leads to changing the evaluations of the objects. An important question is to know whether changing the conventions leads to a modification of the underlying information about the objects (in terms, for instance, of comparisons between the objects). The examples in section 3.3 show that, depending on the context, some assertions remain true or remain false when the evaluations of the objects are transformed, while some assertions do not. Remember that what is being verified is not the veracity or the falsity of an assertion, but the fact that its veracity (falsity) is unchanged when the conventions used for evaluation are modified.

Different numerical scales are considered as being "equivalent" if they support (represent) the same information about the considered objects: we will call them "info-equivalent". Moreover, it is sometimes possible to characterise the mathematical transformations between info-equivalent numerical scales. This observation lead Stevens (1946) to define three important types of scale respectively called ordinal, interval and ratio scales (for more details, see Krantz et al., 1971; Narens and Luce, 1986; Roberts, 1979, 1994). In the following definitions, the expression "admissible transformations" means "transformations into info-equivalent numerical scales". A scale is *ordinal* if its admissible transformations are all strictly increasing transformations; it is an *interval* scale if its admissible transformations are all positive affine transformations of the form $\varphi(x) = \alpha x + \beta$, with $\alpha > 0$ (in this case, the scale is univocally determined by the choice of an origin and a unit); it is a *ratio* scale if its admissible transformations are the positive homothetic transformations of the form $\varphi(x) = \alpha x$, with $\alpha > 0$ (in this case, the scale is univocally determined by the choice of a unit, the origin being "naturally fixed"). Let us also mention the *absolute* scale which does not accept any admissible transformation (except the identity), as a counting or a probability scale. Other, more or less sophisticated, scale types can be defined (see Roberts, 1979) by their sets of admissible transformations, but will not be developed here. It is also important to note that, in many cases, it is not possible to characterise the transformations between info-equivalent numerical scales in an analytical way (this is the case in example 3.3.4).

In classical measurement theory, an assertion is declared to be meaningful if its truth value is unchanged when admissible transformations are applied to the scales used in the assertion. More generally (when the admissible transformations are not identifiable), we will say that an assertion is meaningful if its truth value is unchanged when the numerical scales used in the assertion are replaced by info-equivalent scales (see the concept of "technical sound" introduced in chapter 2, section 2.3.4). For instance, if we consider that a numerical scale used for evaluating durations is a ratio scale, then all the assertions in the basic example 3.3.1 are meaningful, except (xiv) and (xv), because their veracity (or falsity) is unchanged by any positive homothetic transformation of the scale. If we accept that a scale used for measuring temperatures is an interval scale, then all the assertions in the basic example 3.3.2 are meaningful, except (vi), (xi), (xii), (xiii), (xiv) and (xv), because their veracity (or falsity) is unchanged by any positive affine transformation of the scale. If we consider that a scale used for representing a ranking is an ordinal scale, then only the assertions (ii), (iv), (v), and (x) are meaningful, in the basic example 3.3.3, because their veracity or falsity resists to any strictly increasing transformation of the scale. In example 3.3.4, only the assertions that remain true (or false) for all the sets of all numerical values verifying the constraints system are meaningful. As we see, depending on the scale type (i.e. depending on the information supported by the scale), some caution is necessary in the manipulation and the interpretation of the numbers if we want to obtain meaningful conclusions based on these numbers. A conclusion that is true using a given scale but that is meaningless (not meaningful) for this type of scale is completely dependent of the particular scale which is considered, has no character of generality and is thus, probably, of very limited interest. It can even be dangerous because of the tendency of humans to generalise ideas without sufficient precautions. The analysis of scale types allows to detect manipulations (mathematical operations) which can lead to meaningless conclusions. In this case, we can speak of meaningless operations or procedures. In this perspective, the analysis of scale types is a useful tool for scientists.

3.4.2 Comments

Identifying the type of scale of a given set of evaluations is not always an easy task. Besides the scales used for measuring physical phenomena (length, weight, volume, force, time, energy, power, etc.), most of which are ratio or interval scales, many situations lead to the use of scales of a type that does not belong to the classical ones and is often the result of an empirical judgement, as in example 3.3.4 (see also Knapp, 1990). This is the reason why the concept of meaningfulness has some limits and was the subject of some criticisms. It should also be noted that a "meaningless" manipulation of some numbers (because of the scale type considered) can sometimes yield pertinent information. If you respectively assign the numbers 0, 1 and 2 to people having brown, blue and green eyes, the fact that the arithmetic mean of these numbers, computed in a given population, is 1.2, yields the certainty that there are more green than brown eyes, although the arithmetic mean is generally considered as meaningless in this context (being a so-called nominal scale). Another example is the minimum spanning tree problem where the sum can be applied to the numerical values of the edges of the given graph in order to find the optimal solution, even if the numerical scale is ordinal (so that, theoretically, the sum is meaningless). Conversely, it is possible that an assertion is meaningful but without any interest for solving the problem. An over-enthusiastic application of this theory may lead to the fanatic attitude where "meaningless" is synonym of senseless. As we will see in the following sections, it often happens, particularly in decision aiding, that the scales are an intermediary between the classical types defined in section 3.4.1. In these cases, a punctilious application of meaningfulness theory generally leads to an impoverishment of the data, due to the important gap between ordinal and interval scales.

Defining admissible transformations or info-equivalent scales implies knowing what kind of information we want to represent by the scale. The knowledge of the nature of the data is not enough to determine the scale type or the info-equivalent scales, especially in decision aiding. The context, the perception by the decision maker and its purpose play an important role in the interpretation of the numbers and the scale type and therefore in the conditions for meaningfulness and in the acceptable manipulations. A price, for instance, is "naturally" a ratio scale, so that, on this basis, it is possible to give a sense to the assertion "this object is k times better than that one with regards to the price". However, it may happen that the decision maker refuses this kind of assertion (due to his perception of the prices in terms of preferences), but only accepts to compare prices in an ordinal way. In this case, the price should be considered as an ordinal scale instead of a ratio scale. Conversely, a decision maker may decide to give a sense to the comparison of intervals on a scale that is "naturally" ordinal. To conclude, we consider that scale types are not "naturally given" in decision aiding, even for physical measures, and that every use of numbers must be accompanied by some precisions on the information they are supposed to support. Despite the limitations of meaningfulness theory, we consider it an important tool for the analysts in order to avoid the development of completely arbitrary decision aiding procedures.

3.5 Stepping stones for this chapter

Here are the main ideas that we want to put forward in this chapter.

1. Numbers are present everywhere; however their origins can be very dissimilar and the information supported by these numbers can be very different from one situation to another. One consequence is that not all mathematical operations are justified for all these numbers (see section 3.3).

2. In evaluation and decision problems (which constitute the subject of this book), the analyst is often confronted with two types of numbers: "data", which can be considered as pre-existing to the intervention of the analyst (the maximum speed of this car) and "parameters", which are introduced by the analyst in the decision aiding process (see the example in Bouyssou et al., 2000, ch. 6). This distinction will be illustrated in the next sections of this chapter (comparison versus representation problems).

3. In evaluation and decision problems, the nature of the numbers used is partially in the hands of the analyst: it mainly depends on the purpose of the decision aiding process and on the future steps of the process (is it really useful to build a ratio scale if the next step only exploits the ordinal properties of the numbers?). The role of the analyst is to be sure that all the operations are compatible with his choice, from the assessment of the numbers to their interpretation, including the mathematical manipulations of these numbers. This essential aspect was widely illustrated in Bouyssou et al. (2000); the more theoretical aspects were introduced in section 3.4.

4. As mentioned in chapter 1, we are interested in formal models. It is important to point out that a formal model does not necessarily imply the presence of numbers. Many other concepts can be used in formal models (sets, relations, geometrical figures, logic languages, ...). Even if the numbers are useful, their presence in a "model" does not guarantee

that it is a formal model. In a sense, the ease of use of the numbers may be a pitfall since it can lead to instrumental bias.

5. Another confusion is often made between the term "qualitative" and the absence of numerical information. The colour of an object is typically qualitative but can be represented by a number (the wave length). On the contrary, the expression "a small number of students" does not contain any number but is certainly not qualitative. It represents a quantity.

The next sections of this chapter will illustrate these stepping stones through the study of the connection between numbers and preferences.

3.6 Numbers and preference relations

A fundamental step in decision aiding is the modelling and representation of the decision maker's preferences over the set A of alternatives. Two main situations can be distinguished in this framework and will be developed in this chapter. First, the alternatives can be evaluated according to one or several dimensions (cost, acceleration, pick-up, brakes and road-holding of cars, as in Bouyssou et al. (2000, ch. 6); see also chapter 7, section 7.3.5 of this book). An interesting question is to find out what kind of preference relation can be deduced from these evaluations. Of course, many variants can be considered, depending on the nature of the dimensions, the way the evaluations are expressed and the interpretation that the decision maker wants to give to these evaluations. Second, the alternatives can be compared pairwise according to one or several dimensions, in terms of preferences. A problem is then to try to model this information by assigning numbers to the alternatives. This problem is extensively studied in the literature under the denomination "preference modelling" or "measurement". These two main situations are illustrated in figure 3.1 and respectively called the comparison problem and the numerical representation problem. Note that the comparison and the numer-

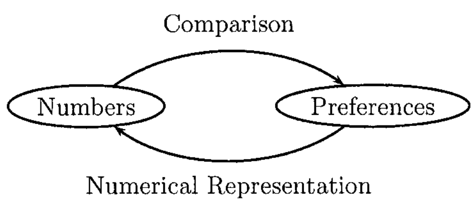


Figure 3.1: Numbers and preference relations.

ical representation problems do not only concern the preferences of the decision maker when an analyst tries to build an evaluation or a decision model. Other concepts such as "the likelihood of events" or "the importance of dimensions" lead to the same kind of questions. We only consider "preferences" here because this concept is probably the most intuitive and can be apprehended with rather simple questions (easily understandable by the decision maker).

3.6.1 The comparison problem

In the comparison problem, the alternatives are evaluated according to a set of n dimensions (with n eventually equals to 1). Each dimension i ($i = 1, 2, \cdots, n$) is represented by a set of "states" X_i, called attribute, which can be expressed by symbols, linguistic expressions or numbers. In our context of decision aiding, we assume that X_i is completely ordered. If not, no preference relation can be established between the alternatives and none of the decision aiding procedures presented in the following chapters is applicable. This means that, under certain assumptions, X_i can be considered as a numerical scale (see section 3.4) and that the elements of X_i are real numbers. However, this basic structure can be completed by additional information about how these numbers (the elements of X_i) must be compared: presence of thresholds, comparisons of differences, ... This additional information comes from the context of the decision problem and from the meaning that the decision maker wants to give to the elements of X_i. Here the notion of scale type and of meaningfulness that we discussed in section 3.4 comes into the picture. In the comparison problem, we have to make the scale type of the X_i's precise in order to be able to infer meaningful preference assertions. As we have seen, the scale type is not necessarily one of the three main types presented in section 3.4. Moreover, it is not given naturally but depends on the meaning given by the decision maker to that particular scale (see section 3.5, point (4)). Remember also that the scale type cannot always be characterised in a simple and concise way. Finally, given X_i and additional information on how its elements must be compared, the evaluation of an alternative according to i may just be an element of X_i, a probability distribution on X_i, a fuzzy subset of X_i, ..., expressing the fact that the evaluation of an alternative according to a dimension can be imprecise, uncertain or undetermined. To conclude, the comparison problem consists in building preference relations over the set of alternatives, based on their evaluations on the X_i's and on the information we have on the nature of the scales. This is a common situation in multiple criteria decision aiding.

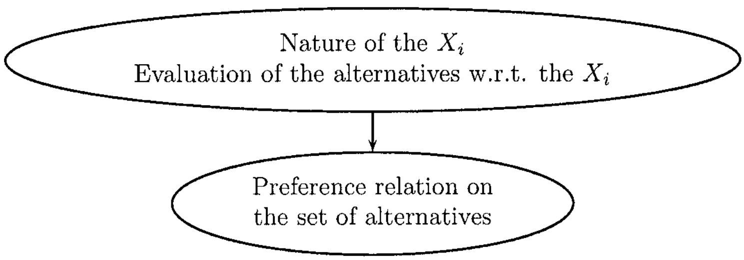


Figure 3.2: The comparison problem.

3.6.2 The numerical representation problem

In the numerical representation problem, alternatives are compared according to one or several points of view. The comparisons can generally be expressed by a binary relation S defined on the set A, where $a \ S \ b$ means "a is at least as good as b". This relation can be completed by additional information (more or less strong preference, preference with a certain credibility or probability, ...). If there are several independent dimensions, the comparison of the alternatives can also be expressed by n binary relations S_i (one for each dimension), with possible additional information for each of them. The numerical representation problem consists in associating numbers to the alternatives in such a way that the pairwise comparison of these numbers is a good model of the pairwise comparison of the alternatives. In other words, the numerical representation problem consists in building numerical scales on the set of alternatives and in making the meaning of the obtained numbers in connection with the given preferences clear. Measurement theory and preference modelling typically apply to this type of situation.

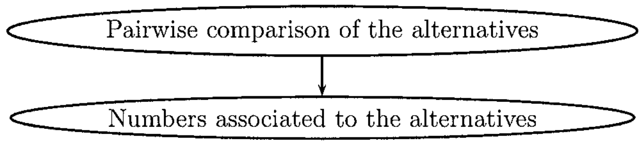


Figure 3.3: The numerical representation problem.

3.6.3 Content of the following sections

Section 3.7 is devoted to the *comparison problem* in the particular case in which $n = 1$; the alternatives are evaluated according to a single dimension which is a subset X of $\mathbb{R}$ (the case in which several dimensions have to be taken into account will be dealt with in chapters 4, 5 and 6). In each subsection, we describe the basic nature of X, the additional information on how to compare the elements of X and the resulting evaluation of the alternatives. We then propose some preference relations which can be deduced from this information and illustrate the case.

Section 3.8 is devoted to the *numerical representation problem* in the particular case in which the alternatives are compared according to a single dimension, giving rise to a relation S (the case of several dimensions in covered in chapter 6. In each subsection, we consider a set of properties for the relation S and we propose a numerical representation of this relation. We then discuss the meaningfulness aspects connected to the numbers obtained. The reader will find the basic definitions about the properties of relations in section 3.10.

3.7 The comparison problem

3.7.1 Pointwise evaluations on an ordinal scale

In this section, each alternative $a \in A$ is evaluated by a single element $x(a)$ of an attribute $X \subseteq \mathbb{R}$. This attribute is considered by the decision maker as an ordinal scale (see section 3.4). This basic structure can eventually be completed by additional information about how the elements of X must be compared (the variants on this additional information correspond to the diverse subsections).

3.7.1.1 Pure ordinal scale

Without any additional information, the relation $>$ on the set X of numbers naturally induces a preference and an indifference relations on A defined by:

$$\forall a, b \in A, \begin{cases} a \; P \; b & \Leftrightarrow \quad x(a) > x(b) \\ a \; I \; b & \Leftrightarrow \quad x(a) = x(b), \end{cases}$$

where $a \; P \; b$ means "a is preferred to b" and $a \; I \; b$ means "a is indifferent to b" (or, more precisely, the decision maker is indifferent between a and b).

It should be noted that P and I are invariant for any strictly increasing transformation of the scale of X (leading to an info-equivalent scale). Every assertion based on these relations can thus be considered as "meaningful".

Of course, any "poorer" conclusion can also be considered (retaining only the best or the worst, identifying the ties, ...), but any richer conclusion would imply that X is not a pure ordinal scale. The example in section 3.3.3 is typically a situation where the numbers that are associated to the alternatives are elements of a pure ordinal scale: the only information they are supposed to support is the ranking of the runners. Other examples were presented in "Thierry's choice problem" presented in Bouyssou et al. (2000, section 6.1). When the evaluations are purely ordinal, it is probably better, from a practical viewpoint, to introduce a non-numerical coding to express them, in order to avoid any attempt to compare differences or to make meaningless calculations. The reader will easily verify that the obtained preference and indifference relations satisfy the following properties:

(1) it is impossible to have $a \; P \; b$ and $b \; P \; a$ simultaneously (one says that P is asymmetric),

(2) if a is preferred to b and b is preferred to c, then a is preferred to c (P is transitive),

(3) if a is not preferred to b and b is not preferred to c, then a is not preferred to c (P is negatively transitive); this is due to the fact that, in the case considered here, a is not preferred to b iff $x(a) \leq x(b)$,

(4) one always has $a \; I \; b$ and $b \; I \; a$ simultaneously (I is symmetric),

(5) for each alternative a, $a \; I \; a$ (I is reflexive),

(6) if a is indifferent to b and b is indifferent to c, then a is indifferent to c (I is transitive).

Consequently, P is a strict weak order (it satisfies (1) and (3), hence (2)) and I is an equivalence relation (it satisfies (4), (5) and (6)). Note also that, given two alternatives a and b, only three situations are possible: $a\ P\ b$ or $b\ P\ a$ or $a\ I\ b$, and they are also mutually exclusive. In other words, I can be seen as an absence of P so that I is completely determined by the knowledge of P.

Consider the following evaluations given by an expert for assessing the "aesthetic" of objects as a numerical example:

a	b	c	d	e	f
13	12	8	5	4	2

where the higher the evaluation the more beautiful the object. The comparison of the objects leads to the comparisons presented in table 3.8. Every strictly

$\hookrightarrow$	a	b	c	d	e	f
a	I	P	P	P	P	P
b		I	P	P	P	P
c			I	P	P	P
d				I	P	P
e					I	P
f						I

Table 3.8: Comparison of objects: linear order.

increasing transformation of the numerical scale would lead to exactly the same result (this is the case, e.g., if 13 becomes 1 000, 12 becomes 397, 8 becomes 200, 5 becomes 80, 4 becomes 10 and 2 becomes 0).

Note also that our example is very particular because of the fact that all the alternatives have different evaluations. This means that the indifference relation is restricted to the identical pairs and that the preference relation is weakly complete (given two distinct alternatives a and b, only two situations are possible: $a\ P\ b$ or $b\ P\ a$). Relation P is then called a strict linear order (asymmetric, transitive and weakly complete or, equivalently, asymmetric, negatively transitive and weakly complete).

If the example given above is modified as follows:

a	b	c	d	e	f	g	h
13	13	11	9	9	9	7	7

the classes of indifferent alternatives are not restricted to singletons and the relation P is not weakly complete. Comparing the objects leads to the relations presented in table 3.9. The "step type" matrix obtained, the "noses" of which are on the diagonal, generalises the previous one. We can also describe the model introduced in this section by using the relation S, where $a\ S\ b$ means "a is preferred or indifferent to b" or "a is at least as good as b". It is obtained from the

$\hookrightarrow$	a	b	c	d	e	f	g	h
a	I	I	P	P	P	P	P	P
b	I	I	P	P	P	P	P	P
c			I	P	P	P	P	P
d				I	I	I	P	P
e				I	I	I	P	P
f				I	I	I	P	P
g							I	I
h							I	I

Table 3.9: Comparison of objects: weak order.

numerical evaluations by defining, $\forall a, b \in A$,

$$a \; S \; b \Leftrightarrow x(a) \geq x(b).$$

This relation is reflexive, transitive and complete and is called *a* weak order (see section 3.10 for definitions). It is of course the union of P and I and, conversely, given S, the relations P and I are obtained by

$$a \; P \; b \Leftrightarrow a \; S \; b \text{ and } Not[\, b \; S \; a\,]$$
$$a \; I \; b \Leftrightarrow a \; S \; b \text{ and } b \; S \; a.$$

Instead of $Not[\, b \; S \; a\,]$, we shall sometimes write $b \neg S \; a$. A weak order is nothing but a ranking with possible ties. If there is no tie, it is called a complete or linear order. In conclusion, comparing alternatives that are evaluated by elements of a pure ordinal scale leads to a strict preference relation that is a strict weak order and to an indifference relation which is an equivalence relation or, equivalently, to an "at least as good as" relation which is a weak order.

3.7.1.2 Ordinal scale with a threshold

Consider the case in which, besides the natural relation $>$ on the set X of numbers, the decision maker considers that there is a threshold q such that he does not want to make a distinction between two numbers x and y such that $|x - y| \leq q$ (for some comments about thresholding, see Bouyssou et al., 2000, page 142). This information induces a preference and an indifference relation on A given by:

$$\forall a, b \in A, \begin{cases} a \; P \; b \Leftrightarrow x(a) > x(b) + q, \\ a \; I \; b \Leftrightarrow |x(a) - x(b)| \leq q, \end{cases}$$

and every assertion based on these relations will be meaningful. As a numerical example, consider again the numerical evaluations given by an expert for assessing the aesthetic of objects:

a	b	c	d	e	f
13	12	8	5	4	2

Suppose now that the decision maker considers, in agreement with the analyst, that a difference of 2 is not very significant. The resulting comparisons are presented in table 3.10. In this step-type matrix, generalising the previous ones, the "noses" are no longer on the diagonal. The reader can easily verify that, as in the previous

$\hookrightarrow$	a	b	c	d	e	f
a	I	I	P	P	P	P
b	I	I	P	P	P	P
c			I	P	P	P
d				I	I	P
e				I	I	I
f					I	I

Table 3.10: Comparison of objects: semiorder.

section, P is still asymmetric (property (1)) and transitive (property (2)) and I is still symmetric (property (4)) and reflexive (property (5)). However, P is no longer negatively transitive (property (3)) and I is no longer transitive (property (6), as illustrated by the triplet d, e, f), so that P is no longer a strict weak order and I is no longer an equivalence relation. There are always three possible and mutually exclusive situations: $a\ P\ b$ or $b\ P\ a$ or $a\ I\ b$, so that the information about the pairwise comparison of the alternatives is entirely determined by P (I being an absence of P), the properties of which are the following (besides asymmetry and transitivity):

(7) if a is preferred to b, b indifferent to c and c preferred to d, then a is preferred to d (note that this property implies the transitivity of P, because of the reflexivity of I),

(8) if a is preferred to b, b preferred to c and c indifferent to d, then a is preferred to d (it also implies the transitivity of P, because of the reflexivity of I).

The relation P is called a strict semiorder (see Pirlot and Vincke, 1997). Of course a strict weak order (section 3.7.1.1) is a particular strict semiorder, corresponding to the case in which the threshold is set as being equal to 0.

It is important to note that in this case, no special meaning should be attached to the numerical value of q. This implies that the additions and subtractions in the formulae written at the beginning of this section, do not induce any particular algebraic structure on X; they are only convenient ways to express the fact that each number cannot be distinguished from some other numbers in its neighbourhood. An equivalent manner of describing the situation, which points out the ordinal character of the scale and avoids an eventual misunderstanding about the interpretations of the numbers, is the following. Besides the relation $>$ on X, the decision maker considers that, an element $x' > x$ is associated with each $x \in X$ and that every z such that $x' \geq z \geq x$ is not distinguished from x; moreover, for every $x, y \in X$, $x > y$ implies $x' > y'$. This information induces a preference and an indifference relation on A given by:

$$\forall a, b \in A, \begin{cases} a\ P\ b \Leftrightarrow x(a) > x'(b), \\ a\ I\ b \Leftrightarrow x'(b) \geq x(a) \text{ and } x'(a) \geq x(b), \end{cases}$$

which are the same as before ($x + q = x'$).

This presentation also shows that q is not necessarily a "constant", as its intrinsic value has no meaning. The only important property, which seems to be rather unsurprising in the present context, is the fact that $x > y$ implies $x' > y'$ (which could be written, with the idea of threshold, $x > y$ implies $x + q(x) > y + q(y)$, the threshold becoming a function). In other words, if $z > x > y$ and if z cannot be distinguished from y, then it cannot be distinguished from x. The case in which this assumption is not satisfied is not treated here (in our opinion, it is not realistic) but will be evoked in section 3.7.3).

So, the following numerical evaluations would lead to the same preference structure as before, provided that the thresholds are chosen as indicated:

	a	b	c	d	e	f
Values	180	140	80	30	10	0
Thresholds	55	50	40	35	25	15

Again, this illustrates the fact that it is more prudent to work with the relations P and I than with the numbers, because the temptation to make calculations is great (for instance, the difference between a and b is greater than the difference between d and f but a is indifferent to b while d is preferred to f).

Finally, the presence of a threshold seems to introduce an idea of "distance" or "difference" between the evaluations. However, this does not mean that all comparisons can be made between differences of evaluations. Let us denote by $[x, y]$ the preference difference between the evaluations x and y, where $y > x$. Imbedded preference differences can be completely ranked on the basis of the relation $>$ on X. If two preference differences are not imbedded, one of them can be declared "bigger" than the other only if the first one corresponds to a strict preference situation and the second to an indifference situation. If two preference differences are not imbedded and both correspond to strict preference situations, then neither can be declared "bigger" than the other. The situation is similar if they both correspond to indifference situations. In the numerical example introduced in this section, the relation "bigger than" in the set of preference differences is given in figure 3.4 (the edges obtained through transitivity are not represented).

Formally, if $\succ$ is the relation "bigger than" in the set of preference differences, we have:

$$[x, y] \succ [z, t] \text{ if } \begin{cases} y \geq t > z > x, \text{ or} \\ y > t > z \geq x, \text{ or} \\ y > x' \text{ and } z' \geq t \end{cases}$$

3.7.1.3 Ordinal scale with two thresholds

It may happen that, besides the relation $>$ on X, the decision maker considers that two elements x' and x'' of X are associated with each $x \in X$ in such a way

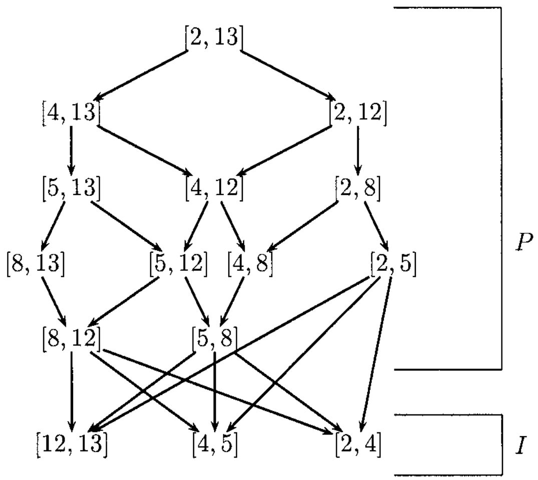


Figure 3.4: Ordinal scale with a threshold: comparison of preference differences.

that $x'' > x' > x$ and

- every z verifying $x' \geq z \geq x$ must not be distinguished from x,
- every z verifying $x'' \geq z > x'$ is weakly preferred to x in the sense that there is a hesitation between indifference and strong preference,
- every z verifying $z > x''$ is strongly preferred to x.

Another presentation of the same situation is to denote $x' = x + q(x)$ and $x'' = x + p(x)$ and to present $q(x)$ and $p(x)$ as thresholds associated to x. However, the remark made in section 3.7.1.2 about the ordinal character of the information also applies here, so we prefer to avoid the introduction of arithmetic operations in the description of the situation. As in section 3.7.1.2 it seems natural to accept the assumption stating that $x > y$ implies $x' > y'$ and $x'' > y''$ (the reader interested in other situations is referred to Vincke, 1988). This information induces three relations on A: I (indifference), Q (weak preference) and P (strict preference) defined by:

$$\forall a, b \in A, \begin{cases} a\ P\ b \Leftrightarrow x(a) > x''(b), \\ a\ Q\ b \Leftrightarrow x''(b) \geq x(a) > x'(b), \\ a\ I\ b \Leftrightarrow x'(b) \geq x(a) \text{ and } x'(a) \geq x(b), \end{cases}$$

and these three relations constitute a so-called pseudo-order (see Roy and Vincke, 1987). As in the previous sections, every assertion based on these relations will be meaningful.

↪	a	b	c	d	e	f
a	I	I	P	P	P	P
b	I	I	P	P	P	P
c			I	Q	P	P
d				I	I	Q
e				I	I	I
f					I	I

Table 3.11: Comparison of objects: pseudo-order.

Let us consider again the data given in section 3.7.1.1 as a numerical example. Suppose that the decision maker considers a difference smaller or equal to 2 as not being significant and that a difference strictly greater than 3 is necessary to justify a strict preference. These rules lead to the double step type matrix presented in table 3.11. This matrix characterises a pseudo-order and generalises the previous ones. In this case, the relation "bigger than" in the set of preference differences is given in figure 3.5 (the edges obtained by transitivity are not represented) and is obtained by the same reasoning as in section 3.7.1.2).

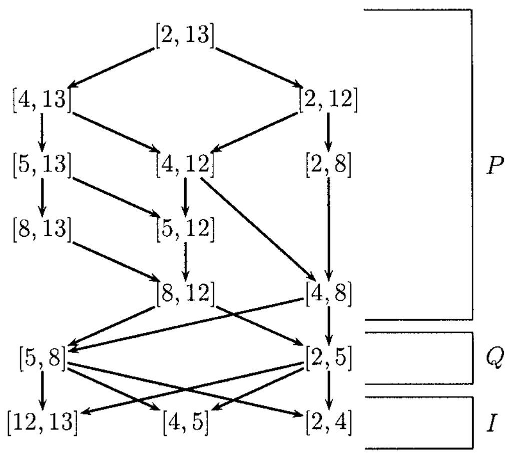


Figure 3.5: Ordinal scale with two thresholds: comparison of preference differences.

3.7.1.4 Ordinal scale with k thresholds

The situation described in section 3.7.1.3 can be generalised by associating a set of elements $\{x^{(1)}, x^{(2)}, \cdots, x^{(k)}\}$ with each element x of X such that $x^{(k)} > x^{(k-1)} > \cdots > x^{(1)}$ and delimiting zones of more and more strong preferences over

x. Assuming that $x > y$ implies $x^{(j)} > y^{(j)}, \forall j$, this information induces a set $\{I, P_1, P_2, \ldots, P_k\}$ of relations on A defined by:

$$\forall a, b \in A \begin{cases} a\ P_k\ b & \Leftrightarrow \quad x(a) > x^{(k)}(b), \\ a\ P_j\ b & \Leftrightarrow \quad x^{(j+1)}(b) \geq x(a) > x^{(j)}(b), \forall j < k, \\ a\ I\ b & \Leftrightarrow \quad x^{(1)}(b) \geq x(a) \text{ and } x^{(1)}(a) \geq x(b). \end{cases}$$

These relations are a so-called "homogeneous family of semiorders", (see Doignon, Monjardet, Roubens, and Vincke, 1986; Roubens and Vincke, 1985). Taking the example of section 3.7.1.2 again and introducing "thresholds" equal to 2, 3, 5 and 10, we obtain the following set of relations presented in table 3.12, where P_1 to P_4 can be interpreted as preferences that are more and more strong. The relation

$\hookrightarrow$	a	b	c	d	e	f
a	I	I	P_2	P_3	P_3	P_4
b	I	I	P_2	P_3	P_3	P_3
c			I	P_1	P_2	P_3
d				I	I	P_1
e				I	I	I
f					I	I

Table 3.12: Comparison of objects: homogeneous family of semiorders.

"bigger than" in the set of preference differences is then given by figure 3.6. Remember that the edges obtained through transitivity are not represented; the other missing edges correspond to pairs of preference differences for the relation "bigger than" that cannot be compared. As in the previous sections, every assertion based on the relations $\{I, P_1, \cdots, P_k\}$ will be meaningful.

3.7.1.5 Ordinal scale with a degree of preference

Consider the case in which, besides the relation $>$ on X, the decision maker is able to associate a "degree" $d(x, y)$ of preference of x over y (increasing with x and decreasing with y) with every pair (x, y) of elements of X such that $x > y$. As X is supposed to be an ordinal scale, this degree must also be an element of an ordinal scale: any richer structure on the degree would imply a richer structure on X. In other words, we are in a situation where the decision maker is able to rank the pairs (x, y) of elements of X in function of the strength of preference of x over y (with eventual ties). This situation is similar to the previous one (with k thresholds), the number of thresholds being equal to the number of different values of the degree of preference, so that it also induces a homogeneous family of semiorders on A, with the same remark about meaningfulness. Another presentation of the same situation consists in defining a valued preference relation S on A as follows: $\forall a, b \in A$,

$$S(a, b) = \begin{cases} d(x(a), x(b)) & \text{if } x(a) > x(b), \\ 0 & \text{otherwise,} \end{cases}$$

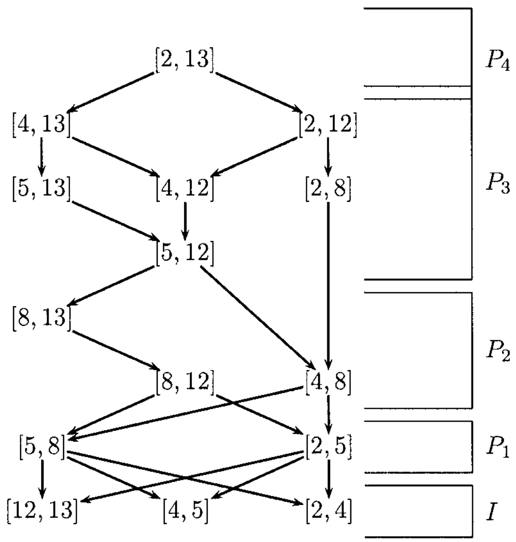


Figure 3.6: Ordinal scale with several thresholds (2, 3, 5, 10): comparison of preference differences.

which will be a so-called semiordered valued relation (see Pirlot and Vincke, 1997; Roubens and Vincke, 1985). This approach is used in methods such as ELECTRE III or PROMETHEE (see section 4.5 of chapter 4). Note also that, in this situation, the relation "bigger than" in the set of preference differences will be defined by:

$$[x, y] > [z, t] \text{ iff } d(x, y) > d(z, t).$$

It will be a strict weak order as each preference difference is associated to an element of an ordinal scale (same situation as in section 3.7.1.1 where the elements of A are now the preference differences). Considering again the example given in section 3.7.1.2, assume that the degrees of preference between evaluations are those given in table 3.13. We then obtain the set of relations presented in table 3.14 and the comparison of preference differences presented in figure 3.7.

$\hookrightarrow$	13	12	8	5	4	2
13		1	7	16	18	20
12			5	8	10	16
8				4	6	9
5					2	3
4						2
2						

Table 3.13: Degrees of preference between evaluations.

$\hookrightarrow$	a	b	c	d	e	f
a	I	P_1	P_7	P_{11}	P_{12}	P_{13}
b		I	P_5	P_8	P_{10}	P_{11}
c			I	P_4	P_6	P_9
d				I	P_2	P_3
e					I	P_2
f						I

Table 3.14: Comparison of objects: preference structure with degrees of preference.

3.7.2 Pointwise evaluations on an interval scale

In this section, each alternative a of A is evaluated by a single element $x(a)$ of an attribute $X \subseteq \mathbb{R}$ which is considered as an interval scale (see section 3.4). The fact that the attribute is an interval scale can be established on the basis of the information we have on how the decision maker compares the elements of X. It can also happen that the attribute is "naturally" an interval scale (as a temperature for instance). In this last case, the basic structure of X can eventually be completed with additional information about how the elements of X must be compared. The variants on this additional information correspond to the various subsections that follow.

3.7.2.1 Pure interval scale

Without any additional information, the structure of X induces the following relations on A and on the set of ordered pairs of elements of A:

$$\forall a,b,c,d \in A \begin{cases} a\ P\ b \Leftrightarrow x(a) > x(b), \\ a\ I\ b \Leftrightarrow x(a) = x(b), \\ (a,b)\ P^*\ (c,d) \Leftrightarrow x(a) - x(b) > x(c) - x(d), \\ (a,b)\ I^*\ (c,d) \Leftrightarrow x(a) - x(b) = x(c) - x(d). \end{cases}$$

It is tempting to interpret "$(a,b)\ P^*\ (c,d)$ " as "the preference of a over b is stronger than that of c over d" and "$(a,b)\ I^*\ (c,d)$ " means "the preference of a over b is as strong as that of c over d". As shown in section 4.3.9 of chapter 4, this interpretation is not always justified however.

With such a definition, the relations P and P^* are strict weak orders, whatever the interpretation of P^* (see section 3.7.1.1). Furthermore, these two relations satisfy many additional conditions that have been studied in the theory of the measurement of differences (see,e.g., Krantz et al., 1971, ch. 4).It is clear that these two strict weak orders are invariant for any positive affine transformation of X (leading to an info-equivalent scale), so that assertions solely based on them are meaningful.

Consider the following table, giving the temperature (in Celsius degrees) in eight different cities, as a numerical example:

a	b	c	d	e	f	g	h
12	5	14	11	7	11	18	15

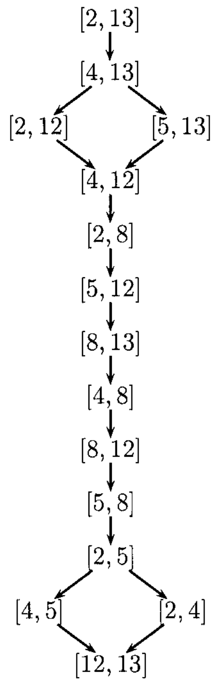


Figure 3.7: Ordinal scale with a degree of preference: comparison of preference differences.

Assuming that the preferences are completely determined by the temperatures, these data lead to the preference structure presented in table 3.15. It will be the same for any scale of temperatures (we assume that the preference increases with the temperature). Moreover, differences in temperatures induce a complete ordering of the ordered pairs of cities as given in figure 3.8. In this figure, arrows denote the presence of the relation P^*. The resulting comparisons are the same for any scale of temperatures. More generally, any assertion which resists to a positive affine transformation (i.e. a transformation of the type $\alpha x + \beta$ where α is positive) will be meaningful, as for example:

- I prefer the temperature of c to the mean temperature of the other cities,
- the difference of preference between g and a is twice larger than the difference of preference between c and d.

Note that these kinds of assertions were not allowed in section 3.7.1, even in the case where a degree of preference was given (see section 3.7.1.5).

$\hookrightarrow$	g	h	c	a	d	f	e	b
g	I	P	P	P	P	P	P	P
h		I	P	P	P	P	P	P
c			I	P	P	P	P	P
a				I	P	P	P	P
d					I	I	P	P
f					I	I	P	P
e							I	P
b								I

Table 3.15: Comparison of objects: linear order.

3.7.2.2 Interval scale with a threshold

While X is still considered as an interval scale, suppose that the decision maker considers that there is a threshold q such that he does not want to make a distinction between two numbers x and y when $|x - y| \leq q$. Note that here, contrary to what happened in section 3.7.1.2, the numerical value of q can be seen as a gap which can be compared to the distances between elements of X. This information allows first to define, as in section 3.7.1.2, a semiorder on A given by:

$$a \; P \; b \Leftrightarrow x(a) > x(b) + q,$$
$$a \; I \; b \Leftrightarrow |x(a) - x(b)| \leq q.$$

Moreover (and contrary to section 3.7.1.2), the interval scale structure of X allows to compare all the differences of evaluations through the strict weak order $\succ^*$ ("bigger than") and the equivalence relation $\sim^*$ ("equal to") defined by:

$$[x, y] \succ^* [z, t] \Leftrightarrow y - x > t - z,$$
$$[x, y] \sim^* [z, t] \Leftrightarrow y - x = t - z.$$

However, due to the presence of q, this strict weak order cannot be used as such to compare differences of evaluation: because of the existence of a threshold q, "small" differences should be considered as non-significant.

That is why we suggest to define the relations P^* and I^* as follows:

$$(a, b) \; P^* \; (c, d) \Leftrightarrow x(a) - x(b) > max[q, x(c) - x(d)],$$
$$(a, b) \; I^* \; (c, d) \Leftrightarrow [Not[(a, b) \; P^* \; (c, d)] \text{ and } Not[(c, d) \; P^* \; (a, b)]].$$

With this definition, the reader can verify that P^* is a weak order satisfying the following desirable property:

$$a \; P \; b \Leftrightarrow (a, b) \; P^* \; (a, a)$$

(as was the case in section 3.7.2.1, where P was also a strict weak order while it is a strict semiorder here). Let us illustrate this case with the numerical example of section 3.7.2.1 where we introduce an indifference threshold equal to 2. We

$$\begin{array}{c}
(g,b) \\ \downarrow \\ (g,e) \\ \downarrow \\ (h,b) \\ \downarrow \\ (c,b) \\ \downarrow \\ (h,e) \\ \downarrow \\ (g,d),(g,f),(c,e),(a,b) \\ \downarrow \\ (g,a),(d,b),(f,b) \\ \downarrow \\ (a,e) \\ \downarrow \\ (g,c),(h,d),(d,e),(f,e) \\ \downarrow \\ (g,h),(h,a),(c,d),(c,f) \\ \downarrow \\ (c,a),(e,b) \\ \downarrow \\ (h,c),(a,d),(a,f)
\end{array}$$

Figure 3.8: Representation of the relation P^*.
(for positive differences of temperatures only)

obtain the preference structure presented in table 3.16. The comparison of pairs of alternatives will give the same figure as in figure 3.8, except that the last two classes are now merged into the single class $\{(c,a),(e,b),(h,c),(a,d),(a,f)\}$.

Insofar as meaningfulness is concerned, it is clear that the previous relations are invariant for any positive affine transformation of X, provided that q be submitted to the same transformation (in order to obtain an info-equivalent scale).

To generalise the results of measurement theory to this situation remains an open problem. It would be interesting, in particular, to establish the properties of P, I, P^* and I^* characterising the previous model. Such structures have received little attention in the literature up to now.

3.7.3 Pointwise evaluations on a ratio scale

In this section, each alternative a of A is evaluated by a single element $x(a)$ of an attribute $X \subseteq \mathbb{R}$ which is a ratio scale. The fact that the attribute is a ratio scale can be established on the basis of the information we have on how the decision maker compares the elements of X. It can also happen that the attribute is "naturally" a ratio scale (e.g., a length or a weight or a price). In the latter case, the basic structure of X can eventually be completed with additional information about how the elements of X must be compared. Various variants on

$\hookrightarrow$	g	h	c	a	d	f	e	b
g	I	P	P	P	P	P	P	P
h		I	I	P	P	P	P	P
c		I	I	I	P	P	P	P
a			I	I	I	I	P	P
f				I	I	I	P	P
f				I	I	I	P	P
e							I	I
b							I	I

Table 3.16: Comparison of objects: semiorder.

this additional information are studied in the following subsections.

3.7.3.1 Pure ratio scale

In the absence of any additional information, the structure of X induces the following relations on A and on the set of ordered pairs of elements of A:

$$\forall a,b,c,d \in A \begin{cases} a\ P\ b \Leftrightarrow x(a) > x(b), \\ a\ I\ b \Leftrightarrow x(a) = x(b), \\ (a,b)\ P^*\ (c,d) \Leftrightarrow x(a) - x(b) > x(c) - x(d), \\ (a,b)\ I^*\ (c,d) \Leftrightarrow x(a) - x(b) = x(c) - x(d), \\ (a,b)\ P^{**}\ (c,d) \Leftrightarrow x(a)/x(b) > x(c)/x(d), \\ (a,b)\ I^{**}\ (c,d) \Leftrightarrow x(a)/x(b) = x(c)/x(d), \end{cases}$$

where the interpretation of P^* and I^* is similar to the one presented in section 3.7.2.1. A possible interpretation for "$(a,b)\ P^{**}\ (c,d)$" is that "the preference ratio between a and b is higher than between c and d", the relation I^{**} being interpreted similarly. The relations between P^* and P^{**} have been analysed in Krantz et al. (1971, section 4.4.3, page 152).

As an illustration, consider a set A of eight possible decisions that have been evaluated according to the gains (expressed in thousands of euros) they are supposed to bring:

a	b	c	d	e	f	g	h
18	15	12	12	11	9	8	5

The preference structure will be given by the step type matrix (P is a strict weak order) given in table 3.17. For positive differences, the relation P^* is the following strict weak order:

$$(a,h)\ P^*\ [(a,g),(b,h)]\ P^*\ (a,f)\ P^*\ [(a,e),(b,g),(c,h),(d,h)]$$
$$P^*\ [(a,c)(a,d),(b,f),(e,h)]\ P^*\ [(b,e),(c,g),(d,g),(f,h)]$$
$$P^*\ [(a,b)(b,c)(b,d),(c,f),(d,f),(e,g)(g,h)]\ P^*\ (e,f)$$
$$P^*\ [(c,e),(d,e),(f,g)].$$

$\hookrightarrow$	a	b	c	d	e	f	g	h
a	I	P	P	P	P	P	P	P
b		I	P	P	P	P	P	P
c			I	I	P	P	P	P
d			I	I	P	P	P	P
e					I	P	P	P
f						I	P	P
g							I	P
h								I

Table 3.17: Comparison of objects: weak order.

For ratios greater than 1, the relation P^{**} is the following strict weak order:

$$(a,h)\ P^{**}\ (b,h)\ P^{**}\ [(c,h),(d,h)]\ P^{**}\ (a,g)\ P^{**}\ (e,h)\ P^{**}\ (a,f)\ P^{**}$$
$$(b,g)\ P^{**}\ (f,h)\ P^{**}\ (b,f)\ P^{**}\ (a,e)\ P^{**}\ (g,h)\ P^{**}$$
$$[(a,c),(a,d),(c,g),(d,g)]\ P^{**}\ (e,g)\ P^{**}\ (b,e)\ P^{**}\ [(c,f),(d,f)]\ P^{**}$$
$$[(b,c),(b,d)]\ P^{**}\ (e,f)\ P^{**}\ (a,b)\ P^{**}\ (f,g)\ P^{**}\ [(c,e),(d,e)].$$

The strict weak orders obtained are invariant for any positive homothetic transformation of X, so that the assertions based on them are meaningful. When P^{**} is interpreted in terms of "ratio of preference", the assertion "a is k times better than b" is here meaningful, contrary to the previous cases.

3.7.4 Interval evaluations on an ordinal scale

Sections 3.7.1, 3.7.2 and 3.7.3 were devoted to comparison problems in a set of alternatives that are evaluated by elements of numerical scales. In other words, the evaluation of each alternative is considered as precise and certain. It often happens in practice that the context of the problem does not allow the obtention of such evaluations. Imprecisions, uncertainties, vagueness have to be taken into account and many tools were developed in the literature to cope with these phenomena. In Bouyssou et al. (2000, ch. 8), we analysed the very traditional tool offered by probability theory. We will consider here the simplest way to introduce lack of precision in the evaluation and decision models: it consists in assuming that the evaluations of the alternatives are defined by intervals (on numerical scales). Moreover, we assume that there is no dependence between these intervals, in the sense that each alternative can have any value in its interval independently of the values of the other alternatives.

In this section, each alternative is evaluated by an interval $I(a) = [\underline{x}(a), \overline{x}(a)]$ of an attribute $X \subseteq \mathbb{R}$. This attribute is considered by the decision maker as an ordinal scale. This basic structure can eventually be completed with additional information about how the elements of X must be compared (the variants of this additional information correspond to the following subsections).

3.7.4.1 Pure ordinal scale

In the absence of additional information, the relation $>$ on X can induce different preference structures on A: we present and illustrate three of them here. A first possibility is to define a preference and an indifference relation in the following way:

$$\forall a, b \in A, \begin{cases} a\ P\ b \Leftrightarrow \underline{x}(a) > \overline{x}(b), \\ a\ I\ b \Leftrightarrow I(a) \cap I(b) \neq \varnothing, \end{cases}$$

expressing the fact that strict preference of a over b only occurs when the "worst" evaluation of a is higher than the "best" evaluation of b. In this case, P is a strict interval order (i.e. an asymmetric relation satisfying property (7) presented in section 3.7.1.2; (see also Fishburn, 1985)).

Note the difference between the situation studied in section 3.7.1.2 (pointwise evaluations on an ordinal scale with a threshold) and the present situation (interval evaluations on an ordinal scale). Here, it can happen that an interval is included in another, while in section 3.7.1.2, we made the (reasonable) assumption that a threshold could not be included in another threshold (represented by the fact that $x > y$ implies $x' > y'$). In the particular case in which the interval evaluations present the property that no interval is included in another, the preference structure is a strict semiorder, exactly as in section 3.7.1.2 (see Pirlot and Vincke, 1997).

A second possibility is to consider that there is a strict preference for a over b as soon as the interval evaluation of a is "more on the right" than the interval evaluation of b, as follows:

$$\forall a, b \in A \begin{cases} a\ P\ b \Leftrightarrow \overline{x}(a) > \overline{x}(b) \text{ and } \underline{x}(a) > \underline{x}(b), \\ a\ I\ b \Leftrightarrow I(a) \subseteq I(b) \text{ or } I(b) \subseteq I(a). \end{cases}$$

The obtained relation P is a strict partial order, i.e. an asymmetric and transitive relation. It can be seen as the intersection of the two strict linear orders (see section 3.7.1.1) L_1 and L_2 defined by

$$a\ L_1\ b \Leftrightarrow \overline{x}(a) > \overline{x}(b$$
$$a\ L_2\ b \Leftrightarrow \underline{x}(a) > \underline{x}(b).$$

That is why one says that it is a strict partial order of dimension two, representable in a plane (see the example below).

A third possibility is to introduce a distinction between a strict preference P and a weak preference Q and to consider that:

$$\forall a, b \in A \begin{cases} a\ P\ b \Leftrightarrow \underline{x}(a) > \overline{x}(b), \\ a\ Q\ b \Leftrightarrow \overline{x}(a) > \overline{x}(b) > \underline{x}(a) > \underline{x}(b), \\ a\ I\ b \Leftrightarrow I(a) \subseteq I(b) \text{ or } I(b) \subseteq (a). \end{cases}$$

This model leads to the so-called (P, Q, I)-interval order or (P, Q, I)-semiorder that were studied in Tsoukiàs and Vincke (2003). All the relations obtained are invariant for any strictly increasing transformation of X, so that every assertion based on these relations is meaningful.

$\hookrightarrow$	a	b	c	d	e	f
a	I	I	P^{-1}	I	P^{-1}	P^{-1}
b	I	I	P^{-1}	I	P^{-1}	P^{-1}
c	P	P	I	P	I	I
d	I	I	P^{-1}	I	P^{-1}	I
e	P	P	I	P	I	I
f	P	P	I	I	I	I

$\hookrightarrow$	a	b	c	d	e	f
a	I	I	P^{-1}	P^{-1}	P^{-1}	P^{-1}
b	I	I	P^{-1}	I	P^{-1}	P^{-1}
c	P	P	I	P	I	I
d	P	I	P^{-1}	I	P^{-1}	P^{-1}
e	P	P	I	P	I	P
f	P	P	I	P	P^{-1}	I

$\hookrightarrow$	a	b	c	d	e	f
a	I	I	P^{-1}	Q^{-1}	P^{-1}	P^{-1}
b	I	I	P^{-1}	I	P^{-1}	P^{-1}
c	P	P	I	P	I	I
d	Q	I	P^{-1}	I	P^{-1}	Q^{-1}
e	P	P	I	P	I	Q
f	P	P	I	Q	Q^{-1}	I

Table 3.18: Preference structure: interval order, partial order of dimension 2 and (P, Q, I) structure.

As a numerical example, consider the following interval evaluations given by an expert for assessing the comfort of different transportation systems, on the scale $X = \{1, 2, 3, 4, 5, 6, 7\}$ where the elements of X respectively correspond to very bad, bad, medium, acceptable, good, very good and excellent,

a	b	c	d	e	f
[1,3]	[2,3]	[5,6]	[2,4]	[5,7]	[4,6]

as illustrated in figure 3.9. Table 3.18, presents the three preference structures

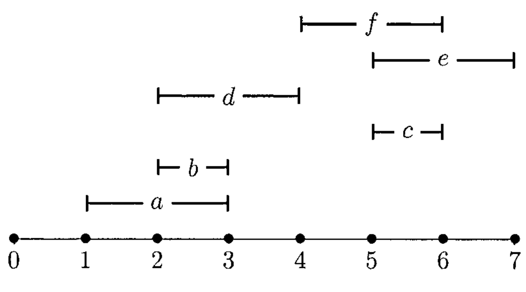


Figure 3.9: Interval evaluations on an ordinal scale.

corresponding to the three previous models (the notation P^{-1}, in the case (a, c) for instance, means that $c\ P\ a$). Every strictly increasing transformation of X would yield the same preference structures (e.g., considering an increasing transformation ϕ such that $\phi(1) = 0$, $\phi(2) = 5$, $\phi(3) = 10$, $\phi(4) = 12$, $\phi(5) = 14$, $\phi(6) = 16$ and $\phi(7) = 18$).

However, no meaning can be given to the differences between these numbers. The only meaningful information contained in the data, in terms of preferences, are the preference structures described above. There is no objective argument allowing to choose between the three solutions, but the third one is of course more discriminating than the other two. As mentioned before, the second preference structure is a strict partial order of dimension 2 and can be represented in a plane where an alternative is strictly better than another if both its coordinates are strictly larger as depicted in figure 3.10.

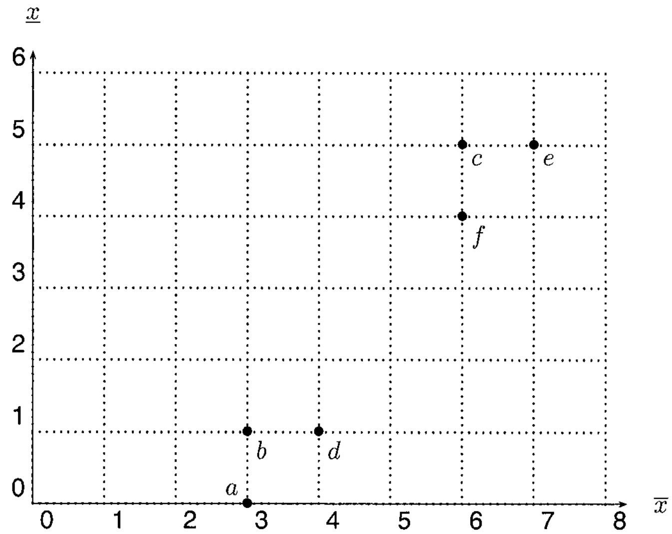


Figure 3.10: Geometrical representation of the strict partial order of dimension 2.

3.7.4.2 Ordinal scale with a threshold

Consider the case where, besides the strict linear order $>$ on X, the decision maker considers that there is a threshold q such that he does not want to make a distinction between two numbers x and y such that $|x - y| \leq q$. Remember that the scale is ordinal and that the numerical value of q has no particular meaning: we could thus adopt a different presentation from the one made in section 3.7.1.2. As the evaluations of the alternatives are intervals, we cannot simply transpose the structure of X (as we did in section 3.7.1.2) to A. In fact, many possibilities exist for defining a preference structure on A on the basis of the given information. A first possibility is to consider that the comparison between two alternatives a and b could be made through the comparison of the intervals $[\underline{x}(a) - q, \overline{x}(a) + q]$ and $[\underline{x}(b) - q, \overline{x}(b) + q]$ and apply one of the models described in section 3.7.4.1, but this would mean that no distinction is made between the "imprecision" of the evaluation of the alternative (leading to interval evaluations) and the perception of the elements of the attribute by the decision maker (leading to the introduction of an indifference threshold), although the two phenomena are of very different natures. Another possibility, presented in the literature, consists in applying the extension principle used in fuzzy logic for defining a fuzzy preference relation on A, as explained below (see Perny and Roubens, 1998, and the example below). One associates to each element a of A the fuzzy number Π_a defined on X by:

$$\forall x \in X, \Pi_a(x) = \begin{cases} 1 & \text{if } x \in [\underline{x}(a), \overline{x}(a)] \\ \alpha & \text{if } x \in [\overline{x}(a), \overline{x}(a)+q] \text{ or } x \in [\underline{x}(a)-q, \underline{x}(a)] \\ 0 & \text{elsewhere,} \end{cases}$$

where $\alpha \in [0, 1)$. Moreover, we define, on $X \times X$, the function θ by:

$$\theta(x,y) = \begin{cases} 1 & \text{if } x \geq y - q \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\theta(x, y) = 1$ iff x is not worse than y.

Two indices are then introduced to compare every pair $\{a, b\}$ of elements of A:

$$R^+(a,b) = \sup_{x,y} \min\{\theta(x,y), \Pi_a(x), \Pi_b(y)\}$$

$$R^-(a,b) = \inf_{x,y} \max\{\theta(x,y), 1-\Pi_a(x), 1-\Pi_b(y)\}.$$

As illustrated in the example below, $R^+(a, b)$ is maximum ($= 1$) when there exist two elements x and y of X such that x is not worse than y ($x \geq y - q$), x belongs to $I(a)$ and y belongs to $I(b)$; it is equal to α if it is not maximum but there exist two elements x and y of X such that x is not worse than y, x is "close" to $I(a)$ and/or y is "close" to $I(b)$; it is minimum ($= 0$) if, for every x such that $\Pi_a(x) \neq 0$ and every y such that $\Pi_b(y) \neq 0$, x is worse than y.

The number $R^-(a, b)$ is maximum ($= 1$) if, for every pair of elements x, y of X such that x is worse than y, either $\Pi_a(x) = 0$ or $\Pi_b(y) = 0$; it is minimum ($= 0$) if there exist x and y such that $x \in I(a), y \in I(b)$ and x is worse than y; it is equal to $(1 - \alpha)$ if it is not minimum but there exist x and y such that x is worse than y, x is close to $I(\alpha)$ and y is close to $I(b)$.

So, we see that $R^+(a, b)$ can be considered as an optimistic indicator of the preference of a over b while $R^-(a, b)$ is a pessimistic indicator of the preference of a over b. Combining them (for example using a convex combination) leads to the definition of a fuzzy preference relation on A. The reader will find interesting results about this construction in Perny (1992).

It is easy to see that no strictly increasing transformation of X can change the values of R^+ or R^-, so that the assertions based on these indicators are meaningful (of course, the threshold must also be transformed in order to maintain the same structure on X).

To illustrate the previous construction, consider the following example: students are evaluated on the ordinal scale $\{10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0\}$, where a difference of one is considered as not significant. Four students a, b, c, d have been evaluated on this scale, respectively yielding the following intervals: $[6, 8]$, $[5, 6]$, $[5, 7]$ and $[3, 5]$. Table 3.19 gives the values obtained for R^+ and R^-: we see that, for R^+ all the students are pairwise indifferent except a and d while for R^-, they are ranked in the order a, c, b, d at the level $1 - \alpha$. To conclude this section, let us

R^+	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	1	1	1	1
d	α	1	1	1

R^-	a	b	c	d
a	1	$1-\alpha$	$1-\alpha$	1
b	0	1	0	$1-\alpha$
c	0	$1-\alpha$	1	$1-\alpha$
d	0	0	0	1

Table 3.19: Comparison of objects: relations R^- and R^+.

consider the previous model in the particular case where $\alpha = 0$, so that, $\forall a \in A$,

$$\Pi_a(x) = \begin{cases} 1 & \text{if } x \in [\underline{x}(a), \overline{x}(a)], \\ 0 & \text{elsewhere.} \end{cases}$$

In this case, it is easy to see that the preferences between the alternatives are defined on the basis of the relative positions of the intervals $[\underline{x} - \frac{q}{2}, \overline{x} + \frac{q}{2}]$ for the optimistic indicator and of the intervals $[\underline{x} + \frac{q}{2}, \overline{x} - \frac{q}{2}]$ for the pessimistic indicator. More precisely, we obtain $R^+ \in \{0, 1\}$, $R^- \in \{0, 1\}$ and

$$R^+(a, b) = 0 \Leftrightarrow \underline{x}(b) - \frac{q}{2} > \overline{x}(a) + \frac{q}{2},$$
$$R^-(a, b) = 0 \Leftrightarrow \overline{x}(b) - \frac{q}{2} > \underline{x}(a) + \frac{q}{2},$$

leading to interval orders. In the case where $\alpha \neq 0$, the indicators R^+ and R^- have three possible values so that a connection could perhaps be established with some well-known (P, Q, I)-structures (see Tsoukiàs and Vincke, 2003; Vincke, 1988): to our knowledge, this connection has not been studied to date.

3.7.5 Interval evaluations on an interval scale

In this section, each alternative is evaluated by an interval $I(a) = [\underline{x}(a), \overline{x}(a)]$ of an attribute $X \subseteq \mathbb{R}$, which is an interval scale. As in section 3.7.2, in the case where the attribute is "naturally" an interval scale, some additional information can be given about how the elements of X must be compared.

3.7.5.1 Pure interval scale

Many preference structures can be proposed, based on the models presented in sections 3.7.2.1 and 3.7.4.1. For example, we could define P and P^* as follows:

$$\forall a, b, c, d \in A \begin{cases} a\ P\ b \Leftrightarrow \underline{x}(a) > \overline{x}(b), \\ a\ I\ b \Leftrightarrow I(a) \cap I(b) \neq \varnothing, \\ (a, b)\ P^*\ (c, d) \Leftrightarrow \underline{x}(a) - \overline{x}(b) > \underline{x}(c) - \overline{x}(d), \\ (a, b)\ I^*\ (c, d) \Leftrightarrow \underline{x}(a) - \overline{x}(b) = \underline{x}(c) - \overline{x}(d), \end{cases}$$

leading to an interval order P and a strict weak order P^* (the additional properties of these two relations have never been studied in the literature, to our knowledge).

The important point is to define preference relations which are invariant for positive affine transformations of X, in order to ensure the meaningfulness of the assertions based on these relations.

3.7.5.2 Interval scale with a threshold

As in section 3.7.4.2, fuzzy logic can again be used here, with the remark that the operators which are used for defining R^+ and R^- can be different from min and max.

3.7.6 Summary of the comparison problem

The situations that were analysed in section 3.7 are only a small part of the large number of possibilities that can be of interest. In each case, the purpose was to show that, given numerical evaluations of alternatives, different meaningful preference structures can be built depending on the types of evaluations (pointwise versus interval evaluations), depending on the nature of the scale on which the evaluations are defined (ordinal, interval or ratio scales) and depending on the complementary information given on the way the elements of the scale can be compared (thresholds).

Table 3.20 summarises the results presented. Much work still needs to be done to analyse other situations and propose rigourous ways of treating comparison problems to the analyst.

Situation	*section, page*	*preference structure*
Pointwise evaluations		
pure ordinal scale	3.7.1.1, p. 83	strict weak order (strict linear order, weak order)
ordinal scale with a threshold	3.7.1.2, p. 85	strict semiorder
ordinal scale with 2 thresholds	3.7.1.3, p. 87	pseudo-order
ordinal scale with k thresholds	3.7.1.4, p. 89	homogeneous family of semiorders
ordinal scale with a degree of preference	3.7.1.5, p. 90	homogeneous family of semiorders + weak order on preference differences
pure interval scale	3.7.2.1, p. 92	strict weak orders on A and $A \times A$ + properties
interval scale with a threshold	3.7.2.2, p. 94	strict semiorder on A + strict weak order on $A \times A$ + properties (open problem)
pure ratio scale	3.7.3.1, p. 96	three strict weak orders + properties
Interval evaluations		
pure ordinal scale	3.7.4.1, p. 98	strict interval order or strict partial order or (P, Q, I)-structure
ordinal scale with a threshold	3.7.4.2, p. 100	fuzzy preference relation
pure interval scale	3.7.5.1, p. 102	strict interval order on A + strict weak order on $A \times A$ + properties (open problem)
interval scale with a threshold	3.7.5.2, p. 103	fuzzy preference relation

Table 3.20: Summary of the comparison problem.

3.8 The numerical representation problem

In the following subsections, we consider situations where the alternatives are compared pairwise according to a single dimension. Our purpose is to study the numerical representation of the obtained relation and to analyse the admissible transformations of these models. Note that, as a particular case, we find out how to determine the type of an attribute X on the basis of pairwise comparisons of its elements.

3.8.1 Weak order

If the pairwise comparisons of the alternatives lead to a strict preference relation P which is a strict weak order (see section 3.7.1.1) and consequently to an indifference relation I which is an equivalence relation or, equivalently, to a weak order $S{=}P \cup I$ (complete and transitive relation), assuming that A is finite or countable it has been proved (see Krantz et al., 1971) that it is always possible to build a real valued function g on A such that:

$$\forall a, b \in A \begin{cases} a \; P \; b \Leftrightarrow g(a) > g(b), \\ a \; I \; b \Leftrightarrow g(a) = g(b), \end{cases}$$

or equivalently:

$$a \; S \; b \Leftrightarrow g(a) \geq g(b).$$

When A is not countable (as in econometric models, for instance, where A is a continuous subset of a real space), an order-density condition must be added (see Krantz et al., 1971, ch. 2).

Of course, g is not unique: every strictly increasing monotonic transformation of g provides another admissible numerical representation and every admissible numerical representation is a strictly increasing monotonic transformation of g. According to the definitions in section 3.7.1, the numerical scale obtained is an ordinal scale, and the meaningful assertions based on this scale are those whose truth value is unchanged by any strictly increasing monotonic transformation of the scale.

For example, suppose that the pairwise comparisons of the elements of $A = \{a, b, c, d, e, f\}$ have led to the following preference structure:

$\hookrightarrow$	a	b	c	d	e	f
a	I	P	P^-	P^-	P	P
b	P^-	I	P^-	P^-	P	I
c	P	P	I	I	P	P
d	P	P	I	I	P	P
e	P^-	P^-	P^-	P^-	I	P^-
f	P^-	I	P^-	P^-	P	I

An easy way to verify that it is a weak order is to reorder the alternatives in the decreasing order of the number of Ps in the associated lines, as shown below:

$\hookrightarrow$	d	c	a	b	f	e
d	I	I	P	P	P	P
c	I	I	P	P	P	P
a	P^-	P^-	I	P	P	P
b	P^-	P^-	P^-	I	I	P
f	P^-	P^-	P^-	I	I	P
e	P^-	P^-	P^-	P^-	P^-	I

If all the Ps are grouped above the diagonal of the matrix and separated from the Is by a step-type line the "noses" of which are on the diagonal and if all the P^-s are exactly in the symmetric part under the diagonal (so that the Is are grouped in several disjoint squares along the diagonal), then the relation P is a strict weak order (consequence of the definition) and the number of Ps in the row associated to each alternative can be taken as the numerical value of this alternative, giving in this case: $g(d) = g(c) = 4$, $g(a) = 3$, $g(b) = g(f) = 1$ and $g(e) = 0$.

Of course, every strictly increasing monotonic transformation of g provides another admissible numerical representation of the weak order, as for example: $g'(d) = g'(c) = 1\,000$, $g'(a) = 800$, $g'(b) = g'(f) = 100$, $g'(e) = 10$.

Another way of verifying that the given preference structure has the requested properties is to check that the relation $S{=}P \cup I$ is a weak order (complete and transitive relation). Replace first all the Ps and Is by 1 and all the P^-s by 0 , in the initial matrix, as follows:

$\hookrightarrow$	a	b	c	d	e	f
a	1	1	0	0	1	1
b	0	1	0	0	1	1
c	1	1	1	1	1	1
d	1	1	1	1	1	1
e	0	0	0	0	1	0
f	0	1	0	0	1	1

To verify completeness, sum this matrix with its transpose (i.e. the matrix obtained by permuting the rows and the columns): the relation S is complete iff the resulting matrix does not contain any 0 (immediate consequence of the definition).

To verify transitivity, compute the product of the above matrix with itself: relation S is transitive iff for each 1 in the obtained matrix, there is a 1 in the initial matrix (immediate consequence of the definition). Note that the previous operations can easily be implemented on a computer, in case A is large. Having checked that the preference structure is a weak order, a numerical representation is obtained by associating, to each alternative, the number of 1s in its row. Here, this yields $g(c) = g(d) = 6, g(a) = 4, g(b) = g(f) = 3$ and $g(e) = 1$. If all the values are different from each other, this means that P is a strict linear order.

As an exercise, the reader can verify that the following preference structure is not a weak order:

$\hookrightarrow$	a	b	c	d	e	f
a	I	P^-	P^-	I	P	P^-
b	P	I	P	P	P	I
c	P	P^-	I	I	P	I
d	I	P^-	I	I	P	P^-
e	P^-	P^-	P^-	P^-	I	P^-
f	P	I	I	P	P	I

Indeed, the matrix of the relation $S = P \cup I$ is the following:

$\hookrightarrow$	a	b	c	d	e	f
a	1	0	0	1	1	0
b	1	1	1	1	1	1
c	1	0	1	1	1	1
d	1	0	1	1	1	0
e	0	0	0	0	1	0
f	1	1	1	1	1	1

Computing the product of this matrix with itself, we obtain a matrix in which the cells $(a, c), (c, b)$ and (d, f) are nonzero, proving that S is not transitive. If we associate, to each alternative, the number of 1s of its row, we obtain: $g(b) = g(f) = 6$, $g(c) = 5$, $g(d) = 4$, $g(a) = 3$ and $g(e) = 1$, but this numerical representation is not acceptable because, for example, $g(f) > g(c)$ while $f\ I\ c$. In fact, as $b\ I\ f$ and $f\ I\ c$, we should have a numerical representation where $g(b) = g(f)$ and $g(f) = g(c)$, implying $g(b) = g(c)$ which is incompatible with the fact that $b\ P\ c$.

So, it is not possible to obtain a numerical representation of the given preference structure by associating an element of a pure ordinal scale to each alternative. We will see in the next section that a numerical representation is possible if we introduce a threshold on the scale.

3.8.2 Semiorder

If the pairwise comparisons of the alternatives lead to a strict preference relation P which is a strict semiorder (see section 3.7.1.2) then, assuming that A is finite, it has been proven (see Scott and Suppes, 1958) that it is always possible to choose a positive threshold q and to build a real valued function g on A such that:

$$\forall a, b \in A \begin{cases} a\ P\ b \Leftrightarrow g(a) > g(b) + q, \\ a\ I\ b \Leftrightarrow |g(a) - g(b)| \leq q. \end{cases}$$

As an example, suppose that the pairwise comparisons of the elements of $A = \{a, b, c, d, e, f\}$ have led to the following preference structure (it is the same as in the last example of the previous section):

$\hookrightarrow$	a	b	c	d	e	f
a	I	P^-	P^-	I	P	P^-
b	P	I	P	P	P	I
c	P	P^-	I	I	P	I
d	I	P^-	I	I	P	P^-
e	P^-	P^-	P^-	P^-	I	P^-
f	P	I	I	P	P	I

As before, let us reorder the alternatives in the decreasing order of the number of Ps in the associated rows. We see again that P is not a strict weak order because the "noses" are not on the diagonal (see previous section):

$\hookrightarrow$	b	f	c	d	a	e
b	I	I	P	P	P	P
f	I	I	I	P	P	P
c	P^-	I	I	I	P	P
d	P^-	P^-	I	I	I	P
a	P^-	P^-	P^-	I	I	P
e	P^-	P^-	P^-	P^-	P^-	I

However, as all the Ps are grouped above the diagonal of the matrix and separated from the Is by a step-type line and as all the P^-s are exactly in the symmetric part under the diagonal, then the relation P is a strict semiorder (see Pirlot and Vincke, 1997). Choosing a threshold q and an arbitrary value for the "worst" alternative (e in our example), we can attribute increasing values from e to b in such a way that the difference of values between two alternatives is larger than q when one alternative is preferred to the other and less than q when the alternatives are indifferent. In our example, taking $q = 3$, we can define successively: $g(e) = 0$, $g(a) = 4$, $g(d) = 6$, $g(c) = 8$, $g(f) = 10$ and $g(b) = 12$.

If the set of alternatives is too large, checking that the preference structure is a semiorder can be done using operations on matrices, as in the previous section. As mentioned in section 3.7.1.2, P is a strict semiorder if it is asymmetric and if the following two properties are satisfied (see properties (7) and (8) in section 3.7.1.2):

- if a is preferred to b, b indifferent to c and c preferred to d, then a must be preferred to d;
- if a is preferred to b, b preferred to c and c indifferent to d, then a must be preferred to d.

In order to check these properties using operations on matrices, let us build the matrix M^P, obtained from the initial matrix by replacing all the Ps by 1 and all the Is and P^-s by 0, and the matrix M^I, obtained from the initial matrix by replacing all the I by 1 and all the P and P^- by 0. The asymmetry of P will be verified if the sum of the matrix M^P with its transpose does not contain any element strictly greater than 1 (immediate consequence of the definition). The other two conditions will be satisfied if, for each 0 in matrix M^P, there is a 0 in the matrix defined by the product $M^P\ M^I\ M^P$ and in the matrix defined by product $M^P\ M^P\ M^I$ (consequence of the definition).

In our example, here are the matrices M^P, M^I, $M^I\,M^I\,M^P$ and $M^P\,M^P\,M^I$ (where the elements larger than 1 have been replaced by 1):

M^P	a	b	c	d	e	f
a	0	0	0	0	1	0
b	1	0	1	1	1	0
c	1	0	0	0	1	0
d	0	0	0	0	1	0
e	0	0	0	0	0	0
f	1	0	0	1	1	0

M^I	a	b	c	d	e	f
a	1	0	0	1	0	0
b	0	1	0	0	0	1
c	0	0	1	1	0	1
d	1	0	1	1	0	0
e	0	0	0	0	1	0
f	0	1	1	0	0	1

$M^P\,M^I\,M^P$	a	b	c	d	e	f
a	0	0	0	0	0	0
b	1	0	0	1	1	0
c	0	0	0	0	1	0
d	0	0	0	0	0	0
e	0	0	0	0	0	0
f	1	0	0	0	1	0

$M^P\,M^P\,M^I$	a	b	c	d	e	f
a	0	0	0	0	0	0
b	1	0	0	1	1	0
c	0	0	0	0	1	0
d	0	0	0	0	0	0
e	0	0	0	0	0	0
f	0	0	0	0	1	0

Once it has been checked that the preference structure is a strict semiorder, the numerical representation is obtained by ordering the alternatives in the decreasing order of the number of Ps in their rows, by choosing a constant threshold q and in giving numerical values to the alternatives, from the "worst" to the "best", in "the good way" (as we did above).

Of course the constant q and the obtained numerical representation are not unique: every strictly increasing transformation of the set of values $\{g(a), g(a)+q, a \in A\}$ provides another acceptable numerical representation. However, not all the acceptable numerical representations are obtained in this way. As an example, consider $A = \{a, b, c\}$, with $a\ I\ b, b\ I\ c$ and $a\ P\ c$. Here are two numerical representations of this semiorder, without any strictly increasing transformation between the two sets of values $\{g(a), g(a)+q, a \in A\}$ and $\{g'(a), g'(a)+q', a \in A\}$:

	a	b	c	
g	4	2	1	$q = 2$
g'	3	2	1	$q' = 2$

When there is no pair of equivalent alternatives for the semiorder (two alternatives are equivalent if they are indifferent between themselves and if they are compared to the other alternatives in exactly the same way), then every admissible transformation of g must be strictly increasing. However, any strictly increasing transformation of g is not admissible because of the presence of the threshold. For example, consider $A = \{a, b, c, d\}$, with $a\ I\ b, a\ P\ c, a\ P\ d, b\ P\ c, b\ P\ d, c\ I\ d$. Taking $q = 2$, a numerical representation of this semiorder is given by $g(a) = 5, g(b) = 4, g(c) = 1$ and $g(d) = 0$. Now, taking $g'(a) = 5, g'(b) = 4, g'(c) = 3$ and $g'(d) = 0$ (which can be seen as the result of a strictly increasing transformation of g), there is no threshold q' allowing to represent the given semiorder since we must necessarily have $q' \geq g'(c) - g'(d) = 3$ and $q' < g'(b) - g'(c) = 1$.

As a last example, consider $A = \{a, b, c\}$ with $a\ P\ b, a\ P\ c$ and $b\ I\ c$. As a weak order, it has a numerical representation which is unique up to a strictly increasing transformation. However, as a semiorder (with two equivalent elements b and c) it has several numerical representations with no strictly increasing transformation between them, as illustrated below:

	a	b	c	
g	5	2	1	$q = 2$
g'	5	1	2	$q = 2$

Let us also mention the fact that a sort of "canonical" representation of a semiorder is given by the concept of minimal representation (see Pirlot and Vincke, 1997, for precise definitions and properties. This representation has the advantage of being unique).

Finally, an assertion based on a numerical representation of a semiorder is meaningful if its truth value is unchanged when another numerical representation of this semiorder (i.e. an info-equivalent scale) is used (unfortunately, as we have seen above, there is no simple analytic expression of the admissible transformations for the numerical representations of a given semiorder).

Note that we can also decide to numerically represent the semiorder with intervals on an ordinal scale, with the property that no interval is included in any other. In this case, we can give arbitrary values to the alternatives, in the increasing order of the number of Ps in their rows: these values will be the left end points of the intervals. The right end points are then fixed in the same increasing order in such a way that, for each alternative a, the right-end point of its interval is:

- smaller than the left-end point of any alternative b such that $b\ P\ a$,
- larger than the left-end point of any alternative c such that $c\ I\ a$.

For the example introduced at the beginning of this section, here is a possible result:

alternatives	left-end points	right-end points
e	0	0.5
a	1	2.5
d	2	3.5
c	3	4.5
f	4	5.5
b	5	6

As an exercise, let us verify that the following preference structure is not a semiorder:

$\hookrightarrow$	a	b	c	d	e	f
a	I	I	I	P^-	P	P^-
b	I	I	I	P^-	I	P^-
c	I	I	I	I	I	P^-
d	P	P	I	I	P	I
e	P^-	I	I	P^-	I	P^-
f	P	P	P	I	P	I

Ordering the alternatives in the decreasing order of the number of Ps in the lines gives the following matrix (if two alternatives have the same number of Ps in their rows, we order them in the increasing order of the number of P^-s in their rows), which is not characteristic of a semiorder:

$\hookrightarrow$	f	d	a	c	b	e
f	I	I	P	P	P	P
d	I	I	P	I	P	P
a	P^-	P^-	I	I	I	P
c	P^-	I	I	I	I	I
b	P^-	P^-	I	I	I	I
e	P^-	P^-	P^-	I	I	I

Moreover, computing the matrices M^P, M^I, $M^P M^I M^P$ and $M^P M^P M^I$ gives the following results:

M^P	a	b	c	d	e	f
a	0	0	0	0	1	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	1	1	0	0	1	0
e	0	0	0	0	0	0
f	1	1	1	0	1	0

M^I	a	b	c	d	e	f
a	1	1	1	0	0	0
b	1	1	1	0	1	0
c	1	1	1	1	1	0
d	0	0	1	1	0	1
e	0	1	1	0	1	0
f	0	0	0	1	0	1

$M^P M^I M^P$	a	b	c	d	e	f
a	0	0	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	1	0
e	0	0	0	0	0	0
f	1	1	0	0	1	0

$M^P M^P M^I$	a	b	c	d	e	f
a	0	0	0	0	0	0
b	0	0	0	0	0	0
c	0	0	0	0	0	0
d	0	1	1	0	1	0
e	0	0	0	0	0	0
f	0	1	1	0	1	0

As we can see, P is asymmetric and satisfies the $(M^P M^I M^P)$-condition, but the $(M^P M^P M^I)$-condition is not satisfied as there is a 0 in the cell (d, c) of matrix M^P but not in the same cell of matrix $M^P M^P M^I$. The reason is that d is preferred to a which is preferred to e which is indifferent to c, but d is not preferred to c.

3.8.3 Interval order

If the pairwise comparisons of the alternatives lead to a strict preference relation P which is a strict interval order (see section 3.7.4.1), then, assuming that A is finite or countable it has been proven (see Fishburn, 1985) that it is always possible to build two real valued functions g and $q(\geq 0)$ such that:

$$\forall a,b \in A \begin{cases} a\ P\ b \Leftrightarrow g(a) > g(b) + q(g(b)), \\ a\ I\ b \Leftrightarrow \begin{cases} g(a) \leq g(b) + q(g(b)), \\ g(b) \leq g(a) + q(g(a)). \end{cases} \end{cases}$$

Equivalently, if P is a strict interval order, it is always possible to associate an interval $G(a) = [\underline{g}(a), \overline{g}(a)]$ to each alternative $a \in A$ in such a way that:

$$\begin{cases} a\ P\ b \Leftrightarrow \underline{g}(a) > \overline{g}(b), \\ a\ I\ b \Leftrightarrow G(a) \cap G(b) \neq \varnothing. \end{cases}$$

Taking $g = \underline{g}$ and $g + q = \overline{g}$, we obtain the representation given above.

For example, suppose that the pairwise comparisons of the elements of $A = \{a, b, c, d, e, f\}$ have led to the following preference structure (the same as in the last example of the previous section):

$\hookrightarrow$	a	b	c	d	e	f
a	I	I	I	P^-	P	P^-
b	I	I	I	P^-	I	P^-
c	I	I	I	I	I	P^-
d	P	P	I	I	P	I
e	P^-	I	I	P^-	I	P^-
f	P	P	P	I	P	I

We have seen in the previous section that ordering the alternatives in the decreasing order of the number of Ps in their rows did not lead to a configuration characterising a semiorder. So, we will build two different rankings of the alternatives: the first one will be defined by the decreasing number of Ps in the rows (in case of ties, put the alternative with the smallest number of P^-s in its row first) and the second one will be defined by the increasing number of Ps in the columns (in case of ties, put the alternative with the largest number of P^-s in its column first). The first one is (f, d, a, c, b, e) and the second one is (f, d, c, a, b, e). Using these two rankings to respectively reorder the rows and the columns of the initial matrix, we obtain:

$\hookrightarrow$	f	d	c	a	b	e
f	I	I	P	P	P	P
d	I	I	I	P	P	P
a	P^-	P^-	I	I	I	P
c	P^-	I	I	I	I	I
b	P^-	P^-	I	I	I	I
e	P^-	P^-	I	P^-	I	I

As P is asymmetric (this was verified at the end of section 3.8.2) and as all the Ps are grouped above the diagonal and separated from the Is by a step-type line, the relation P is a strict interval order. The values of the function g (or of the left end points $\underline{g}$ of the intervals) are chosen arbitrarily in the increasing order from the bottom to the top of the rows. The values of the function $g+q$ (or of the right end points $\overline{g}$ of the intervals) are chosen in the increasing order from the right to the left of the columns in such a way that, for each alternative a:

$$(g+q)(a) < g(b) \text{ when } b\ P\ a,$$
$$(g+q)(a) > g(b) \text{ when } b\ I\ a.$$

For the preference structure treated here, we obtain for example:

$$g(e) = 0 \quad g(b) = 5 \quad g(c) = 10$$
$$g(a) = 15 \quad g(d) = 20 \quad g(f) = 25$$

$$(g+q)(e) = 12 \quad (g+q)(b) = 17 \quad (g+q)(a) = 19$$
$$(g+q)(c) = 23 \quad (g+q)(d) = 28 \quad (g+q)(f) = 30$$

If the set of alternatives is too large, checking that the preference structure is a

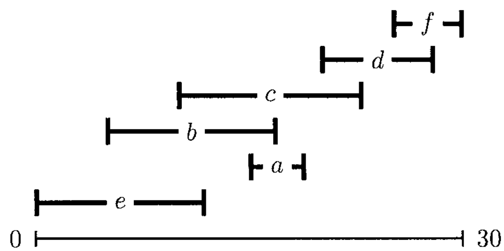


Figure 3.11: Representation by intervals.

strict interval order can be carried out through operations on matrices, as in the previous sections. We already know that P is an interval order if it is asymmetric and if it satisfies the following property (see property (7) in section 3.7.1.2):

- if a is preferred to b, b indifferent to c and c preferred to d, then a must be preferred to d.

Checking these properties through operations on matrices was explained in section 3.8.2, using the matrices M^P and M^I. We concluded, at the end of section 3.8.2 that the preference structure treated here satisfied the asymmetry of P and the $(M^P\,M^I\,M^P)$-condition, proving that P is a strict interval order. Of course, g and q are not unique: every strictly increasing transformation of the set of values $\{g(a), g(a)+q(g(a)), a \in A\}$ provides another acceptable numerical representation. However, not all the acceptable numerical representations are obtained in this way. For example, consider $A = \{a, b, c, d\}$, with $a\ P\ b\ P\ c, d\ I\ a, d\ I\ b$ and $d\ I\ c$. Here are two numerical representations of this interval order where the two sets of values $\{g(a), g(a) + q(g(a)), a \in A\}$ and $\{g'(a), g'(a) + q'(g'(a)), a \in A\}$ are not ordered in the same way:

	a	b	c	d
g	6	4	1	2
$g+q$	8	5	3	7
g'	6	4	2	1
$g'+q'$	7	5	3	8

Note that even the two sets of values $\{g(a), a \in A\}$ and $\{g'(a), a \in A\}$ are not ordered in the same way, showing that non increasing transformations of g can be admissible here, even when there is no pair of equivalent alternatives (which was not the case for the semiorders). Also note that here, contrary to the semiorders again, every strictly increasing transformation of g is admissible because it is always possible to adapt the variable threshold in order to have a numerical representation of the interval order. This is due to the fact that, given g, the only constraints that must be satisfied by $g(a) + q(g(a))$, for a certain $a \in A$, are:

$$\begin{cases} g(a) + q(g(a)) < g(b), \forall b : b\ P\ a, \\ g(a) + q(g(a)) \geq g(c), \forall c : c\ I\ a. \end{cases}$$

Finally, as for semiorders, we can conclude that an assertion based on a numerical representation of an interval order is meaningful if its truth value is unchanged when another numerical representation of this interval order is used (without having the possibility of giving an analytic expression of the admissible transformations for the numerical representations of a given interval order).

3.8.4 (P, Q, I)-structure

If the pairwise comparisons of the alternatives lead to a strict preference relation P, a weak preference relation Q and an indifference relation I, then, in function of the properties of these relations, numerical representations with two thresholds or representations by intervals are possible. We refer the reader to Vincke (1988) and Tsoukiàs and Vincke (2003) for some examples of results which were proved.

Generally speaking, these representations are not unique: when the thresholds are not constant, every strictly increasing transformation of g is admissible but all the admissible transformations are not of this type. If one (or both) threshold(s) must be constant, not all the strictly increasing transformations of g are admissible. As in the previous cases, an assertion based on a numerical representation of a (P, Q, I)-structure is meaningful if its truth value is unchanged when another numerical representation of this (P, Q, I)-structure is used.

3.8.5 Valued preference relation

Different situations can lead to the necessity of working with valued (or fuzzy) relations. Let us point out two of them, which are very frequently encountered. The first one is the case where the data associated to the alternatives are precisely known and the relation used to compare them is vague (example: the relation "much smaller" in a set of individuals whose heights are precisely measured). The second one is the case where the relation used to compare the alternatives is precise but the data are not (example: the relation "smaller" in a set of individuals whose heights are imprecise). Of course, the two types of imprecisions can also be combined.

In these cases, the pairwise comparisons of the alternatives can lead to a valued relation in A, a "degree" of preference being associated with each ordered pair of elements of A. This "degree" of preference can reflect the imprecisions mentioned above, but it can also express the result of a voting procedure, the probability of an external event, a credibility index built in a decision aiding procedure (as in Bouyssou et al., 2000, ch. 6), an intensity of preference, etc.

In most cases, it is an element of a numerical scale, so that the acceptable numerical representations and the meaningfulness of the assertions depend on the nature of the numerical scale on which the "degree" of preference is defined.

If the "degree" of preference is an element of an ordinal scale, the valued relation is strictly equivalent to an embedded family of preference relations and one may be interested in the numerical representation of this information by a function g and a family of thresholds (see sections 3.7.1.4 and 3.7.1.5). This situation was studied by Doignon et al. (1986) and conditions were established for the existence of this type of numerical representation. Such a model is used, for example, in MACBETH (see section 7.3.1.3.1).

If the "degree" of preference of a over b is the number (or the proportion) of people who prefer a to b in a jury, one may want to take some cardinal aspects of this degree into account. For example, the assertion "the degree of a over b is worth twice the degree of c over d" is meaningful.

There is also a very abundant literature on so-called stochastic relations, where the values associated with the pairs of alternatives are probabilities, with the property that, $\forall a, b \in A$,

$$p(a,b) + p(b,a) = 1.$$

The interested reader is referred to Fishburn (1973a) and Roubens and Vincke (1985).

If the valued relation is additive, in the sense that, $\forall a, b, c \in A$,

$$v(a,c) = v(a,b) + v(b,c),$$

then the "degree" of preference can be interpreted as an intensity of preference and one may want to look for a numerical representation such that, $\forall a, b \in A$,

$$g(a) - g(b) = v(a,b).$$

Measurement theory (see Krantz et al., 1971) provides many results in this context.

3.9 Conclusion

As mentioned in the introduction, the purpose of this chapter was to show the connections between two languages that are naturally used in evaluations and decision aiding problems: the language of numbers and the language of preference relations. We first pointed out the great diversity of information that can be supported by numbers and the necessity of being very cautious in their use and manipulation. The rest of the chapter gave some guidelines:

- to build preference relations on the basis of numerical evaluations of a set of alternatives,
- to build numerical models of preferences expressed on a set of alternatives.

These two situations are permanently present in the decision aiding processes which will be developed in the next chapter. More precisely, let us give some examples of the relevance of the above considerations for the evaluation phase of the decision aiding process:

- not all aggregation methods require an explicit modelling of the preference of the decision maker on each dimension of evaluation. Some of them do the job implicitly, in the process of aggregating the various dimensions. After the latter is completed, one may observe the resulting preference structures on these dimensions and they may be related to some of the interpretations of numerical scales proposed in this chapter (see for instance, sections 5.4 or 6.2.9).
- some aggregation methods require a description of the alternatives on the various dimensions, not by means of performance assessments, but by means of preference relations (see sections 5.2 and 6.2.6). If the information available on these alternatives are performance measurements (possibly only on a subset of the dimensions), "converting" them into preference relations is directly related to section 3.7.6
- conversely, when the single dimensional information is ordinal (e.g. rankings) some aggregation procedures (for instance the Borda rule, introduced in 4.2.2) use numerical representations of these relations as an intermediary step in the aggregation process.

3.10 Appendix: binary relations and ordered sets

The purpose of this appendix is to recall some basic definitions about binary relations and their properties. Let A denote a finite set of elements $a, b, c, \ldots$ and $|A|$, its number of elements. A *binary relation* S on the set A is a subset of the Cartesian product $A \times A$, that is, a set of ordered pairs (a, b) such that a and b belong to A, i.e., $S \subseteq A \times A$. If the ordered pair (a, b) is in S, we write $(a, b) \in S$ or $a\ S\ b$. Otherwise, we write $(a, b) \notin S$ or $Not[a\ S\ b]$ or $a \neg S\ b$.

Let S and T be two relations on the same set A. The following notations will be used.

- $S \subseteq T$ iff $a\ S\ b \Rightarrow a\ T\ b, \forall a, b \in A$ (inclusion),
- $a\ (S \cup T)\ b$ iff $a\ S\ b$ or (inclusive) $a\ T\ b$ (union),

- $a\ (S \cap T)\ b$ iff $a\ S\ b$ and $a\ T\ b$ (intersection),
- $a\ ST\ b$ iff $\exists c \in A : a\ S\ c$ and $c\ T\ b$ (product),
- $a\ S^2\ b$ iff $\exists c \in A : a\ S\ c$ and $c\ S\ b$.

A binary relation S on the set A is:

- *reflexive* iff $a\ S\ a, \forall a \in A$,
- *irreflexive* iff $a\ \neg S\ a,\ \forall a \in A$,
- *symmetric* iff $a\ S\ b \Rightarrow b\ S\ a, \forall a, b \in A$,
- *antisymmetric* iff $a\ S\ b \Rightarrow b\ \neg S\ a, \forall a, b \in A$ such that $a \neq b$,
- *asymmetric* iff $a\ S\ b \Rightarrow b\ \neg S\ a, \forall a, b \in A$,
- *complete* iff $a\ S\ b$ or $b\ S\ a, \forall a, b \in A$
- *weakly complete* iff $a\ S\ b$ or $b\ S\ a, \forall a, b \in A$ such that $a \neq b$,
- *transitive* iff $a\ S\ b, b\ S\ c \Rightarrow a\ S\ c, \forall a, b, c \in A$,
- *negatively transitive* iff $a\ \neg S\ b,\ b\ \neg S\ c \Rightarrow a\ \neg S\ c, \forall a, b, c \in A$,
- an *equivalence relation* iff it is reflexive, symmetric and transitive,
- a *strict partial order* iff it is asymmetric and transitive,
- a *partial order* iff it is reflexive, antisymmetric and transitive,
- a *partial preorder* or simply *preorder* iff it is reflexive and transitive,
- a *strict linear* order iff it is asymmetric, transitive and weakly complete,
- a *strict weak order* iff it is asymmetric and negatively transitive,
- a *weak order* iff it is complete and transitive,
- a *linear order* iff it is complete, transitive and antisymmetric.

Given a binary relation S on a set A, we respectively denote by P_S and I_S the asymmetric and the symmetric parts of S;

$$a\ P_S\ b \Leftrightarrow a\ S\ b \text{ and } b\ \neg S\ a,$$
$$a\ I_S\ b \Leftrightarrow a\ S\ b \text{ and } b\ S\ a.$$

It is clear that $S = P_S \cup I_S$. When no confusion is possible, P_S and I_S will be replaced by P and I. Given a binary relation $S = P \cup I$, the relation E defined by

$$a\ E\ b \Leftrightarrow \forall c \in A, \begin{cases} a\ S\ c \Leftrightarrow b\ S\ c, \\ c\ S\ a \Leftrightarrow c\ S\ b. \end{cases}$$

is clearly an equivalence relation. Also note that:

- S is a linear order iff P is a strict linear order,
- S is a weak order iff P is a strict weak order,

when I is defined as the absence of P.

4

AGGREGATION—OVERTURE

4.1 Introduction

In this chapter and the next two, we concentrate on aggregation, an important point in the building of an evaluation model, itself a step of the decision aiding process as in section 2.3 of chapter 2. This point is both crucial and highly controversial since a profusion of methods have been—and are still—proposed to overcome it. Let us first state exactly where we stand in the decision aiding process; we recall the scheme described mainly in section 2.3.3. The analyst and his client (to keep it simple) have determined—possibly after major efforts—a *problem formulation* (a triplet $\langle \mathbb{A}, V, \Pi \rangle$, in the language of chapter 2) relative to a *problem situation* $\mathcal{P}$. And they have started to build an evaluation model $\langle A, \{D, \mathcal{E}\}, H, \mathcal{U}, \mathcal{R} \rangle$ (see section 2.3.3). That is, a set A of alternatives has been eventually settled and each alternative a in A has been assigned an element $g_i(a)$ on the scale X_i associated with dimension i, this for all dimensions that have been determined relevant in the problem situation $\mathcal{P}$. The "level" $g_i(a)$ describes, measures, characterises the alternative x on dimension i. It may happen that the analyst and his client, have gone one step further, incorporating the client's *a priori preference* on each dimension; this, as we have seen in chapter 3, may result for instance in a binary relation on A for each dimension or in a function h_i that usually associates a number $h_i(a)$ to alternative a on each dimension. Most of the time, the number $h_i(a)$ can be viewed as a function of $g_i(a)$ (which, we insist, is not necessarily a number; X_i may well be an unordered set of labels, for instance).

Let us take the example of buying a sports car, which was discussed in chapter 6 of Bouyssou et al. (2000). The client, Thierry, who is an engineering student, earns little money and participates in car races, wants to buy a sportive second hand car. Here, rather exceptionally, the client also plays the role of the analyst. Thierry selects a set A of 14 cars in the middle range segment, but with powerful engines. Three points of view are of importance to him for assessing these cars: cost, performance of the engine and safety; Thierry is not concerned at all with such issues as comfort or aesthetics. He constructs an evaluation model taking these three viewpoints into account. Cost is a single dimension, since Thierry manages to estimate the yearly expenses g_1 that each car would generate for him. The numbers labelling the scale X_1 associated to the cost represent amounts of money. For assessing the engine performance, Thierry uses two dimensions, namely, acceleration and pick up; the measures g_2 and g_3 on the corresponding scales X_2 and X_3 are expressed in seconds of time. The safety viewpoint is associated two dimensions, one evaluating the cars brakes while the other evaluates roadholding. Cars are assessed on these dimensions using aggregates of several indicators assessed by experts and found in specialised magazines; this yields two functions g_4 and g_5 ranging respectively in the numerical scales X_4 and X_5. Due to their mode of computation, the significance of these numbers is rather unclear, but Thierry believes that they correctly reflect his feelings about the safety of the cars; he is ready to use them to compare cars, saying, for instance, that a car rated "2" on the "brakes scale" X_4 is better than a car rated 1.67 on the same scale. The evaluation model built so far has specified the set of alternatives A, the dimensions and scales, D and $\mathcal{E}$; the preferences of Thierry have not been incorporated into the evaluations. This is quite clear for cost, acceleration and pick up, the assessment of which being measures expressed in physical units (€, seconds). This is also largely true regarding the latter two dimensions, although one may consider that Thierry's preferences are reflected in the way he interprets the numbers g_4 and g_5; one might argue that h_4 and h_5, the preference-coloured information on dimensions X_4 and X_5, are in fact relations ordering of the cars according to their value g_4 and g_5, respectively (many other interpretations of the numbers g_4 and g_5 could be made, as was shown in chapter 3. In this decision problem, uncertainty (that should be described in the $\mathcal{U}$ structure) has not been explicitly modelled, although there are many elements of uncertainty, for instance, in the assessment of the cost. At this point, the set $\mathcal{R}$ of aggregation procedures to be used is still undetermined.

The crux of the evaluation process—and the central topic of this chapter and the two next ones—is to select, build or elicit the link between the description D of the alternatives—or the preference-coloured description H—and the output of the evaluation process. The output may be, for instance, a relation on A or a real-valued function on A, that synthesises the multi-dimensional description of the alternatives, which incorporates the client's preferences. The output of the evaluation process is intended to allow the analyst to derive a recommendation for the client (this is dealt with in chapter 7). The link between D and H, on the one hand, and the output of the evaluation process, on the other, is symbolised by $\mathcal{R}$ in the model described in chapter 2; this is also what we call "aggregation". In the buying a sports car example, Thierry has to combine the evaluations of the cars on the various dimensions with his personal priorities to derive synthetic global statements about the cars, that should help him make a decision.

This chapter and the following two (chapters 4–6) try to deal in a general way with the operation of aggregating descriptions on various dimensions into a global object, called *preference*, which summarises all relevant features of the alternatives and incorporates the client's preference in a given problem situation $\mathcal{P}$. These chapters are built as a piece of music for two voices. These voices develop the theme of aggregation in rather different ways.

The first voice views aggregation as an operator that transforms single-dimensional information on the alternatives (sets of relations or vectors of numbers, see chapter 2) into a global preference. It takes its inspiration from the tradition of *social choice theory.* It characterises a number of *mechanisms* that can transform a certain type of input information related to the evaluation of the alternatives on several dimensions into a synthetic output, most of the time a relation. The characterisations are expressed as properties of the mechanism.

The other voice follows the tradition of *conjoint measurement theory.* It provides us with families of models that decompose a global preference relation into elements related to the description of the alternatives on the various dimensions. The characterisations are of the following type: if a global preference relation satisfies some conditions, then it admits a description within a particular model. In this approach, one does not investigate the properties of mechanisms but those of preference relations. Characterising a model amounts to finding the properties of all the preference relations that fit the model.

4.1.1 How can this help the analyst?

With both voices, we focus on characterisations either of mechanisms or of models (i.e. of subsets of preference relations) by groups of properties that we shall call *axioms.* What can the benefits of having characterisations of a number of mechanisms or models be in practice? Axioms usually have an intuitive content (which we have tried to make as explicit as possible in the presentation that follows) as they express:

- in the first approach, how an aggregation mechanism behaves, i.e. how the output changes in response to particular changes in the input information

- in the second approach, how the preference behaves in various configurations, i.e. on selected subsets (often pairs) of alternatives.

This offers the analyst an opportunity to test (at least partly) whether a set of properties is likely to be verified in a particular decisional context. How? By asking the client how he feels the mechanism should behave or how the preference behaves in the situations evoked in the axioms (or some of these situations). So, ideally, one might expect that the analyst who knows about the various mechanisms or models and their characterisations is helped in his choice of a particular mechanism or model in a given decisional context. The client's answers to some well-chosen questions may suggest that the analyst eliminate some methods and drive him towards others or, at best, point him to a single particular method.

The two approaches we follow are not exclusive; the same methods commonly used in practice for constructing preference relations (additive value model, ELECTRE or PROMETHEE) can be understood using the tools and concepts of both approaches. The interesting feature is that they can be analysed from two different perspectives and using different concepts. This should help the analyst diversify the "languages" in which he can talk with the client to better understand the decision problem and elicit the client's preferences in a reliable way.

4.1.2 Organisation of chapters 4–6

The chapters on aggregation are organised in the following way. The concert starts with a brief presentation, in chapter 4 sections 4.2 and 4.3, of the main themes of the two voices; they are illustrated with well known situations. They should help the reader understand the specificity of each approach more precisely, how they contrast from one another and also on which points they converge. We then have a few bars with three themes common to both voices.

- Since, in both approaches, systems of axioms fail most of the time to determine a single mechanism or a single preference but rather select a family of aggregation procedures or a model for a family of preferences, there usually remain "parameters" (e.g. weights, value functions, thresholds) to be determined. The axioms generally offer clues on how to determine these parameters. This question will be discussed in section 4.4 in a general way; more practical issues on how to determine the parameters in the context of a particular procedure or model will be addressed in chapters 5 and 6.

- The reader may sometimes be interested in a particular aggregation procedure and not be willing to read the three chapters on aggregation before finding the information he is looking for. Section 4.5 was written for this reader; it is a kind of commented index of some popular aggregation methods, we give a list of all sections of chapters 4–6 that are relevant and we briefly explain why.

- Our analysis of aggregation procedures is often axiomatic. We believe that this has a lot of advantages but it also suffers some limitations. These are discussed in section 4.6.

We come back to the main themes of the first and second voice and develop them thoroughly in chapters 5 and 6. In chapter 5, the characteristic properties of a variety of mechanisms (called *procedures*) are described. The rationale for grouping the procedures is the type of input information needed and the type of output that is desired. For example, section 5.2 deals with the aggregation of a profile of binary relations into a binary relation, as the expression of the global preference; section 5.4 accepts a performance table as input (each row represents the description of an alternative on a dimension) and associates a binary relation to any such table.

Chapter 6 mainly analyses two types of models. The first is, the comparison of two alternatives resulting from the comparison of the description of each of them on the different dimensions. In the second type of models, the preference difference between alternatives is assessed for each pair of alternatives and each dimension. The model then balances all these preference differences in order to determine which of the two alternatives is the preferred one. Each type of model has its own logic and suggests a corresponding strategy of elicitation.

4.2 Theoretical results inspired by social choice theory: introduction

In social choice theory, and more particularly in voting theory, a society needs to choose a candidate from a set of candidates. The choice of the candidate is, in most cases, based on the preferences of the voters. This problem bears a striking similarity to the multiple criteria decision support problem in which a client needs to choose an alternative, based on preferences on different dimensions. In multiple criteria decision support, the client plays the role of society, criteria play the role of the voters, and alternatives, the role of the candidates [1].

Social choice theory was already an active research field in the eighteenth century with people like M. J. A. N. Caritat, marquis de Condorcet and J.-Ch. de Borda, but it grew dramatically since the 1950s, thanks to the celebrated works of K. J. Arrow and D. Black (see, among others, Arrow, 1963; Black, 1958). Since the 1980s, some concepts and theorems originally developed in the framework of social choice theory have been adapted to the problem of multiple criteria decision support (see Arrow and Raynaud, 1986; Bouyssou and Perny, 1992; Marchant, 1996; Nurmi and Meskanen, 2000; Pérez and Barba-Romero, 1995). Some other results have been completely developed in the framework of multiple criteria decision support, but using an approach that is typical of voting theory.

In this section as well as in chapter 5, we present some of these results and we try to show how they can be used to help the client and the analyst. We introduce some concepts, an example illustrates why an axiomatic characterisation can be useful and we explain why the theoretical results inspired by social choice theory are fundamentally different from those obtained using measurement theory that are presented in section 4.3 and chapter 6.

4.2.1 Aggregation functions

Suppose we have a set of alternatives $A = \{a, b, c, \ldots\}$ and a set of dimensions $N = \{1, 2, \ldots, n\}$. We have some ordinal information about the alternatives along each dimension. For example,

- linguistic assessments (excellent, good, average, bad or beautiful, average, ugly or ...),
- numbers the meaning of which is only ordinal (expert evaluations on a ten point scale),
- ranks (1 for the best alternative, 2 for the second, ...).

This ordinal information can be modelled or represented by a binary preference relation. So, for each dimension i, we have, a preference relation $\succsim_i$ defined on A. We call p the n-tuple $(\succsim_1, \succsim_2, \ldots, \succsim_n)$. Such a vector is called a *profile*. Lastly, suppose that we would like to construct a global preference relation $\succsim$ on A and

[1] For a more thorough discussion of the analogy between social choice theory and multiple criteria decision support, see Bouyssou et al. (2000, ch. 2).

that we want this global preference relation to be a weak order (a complete ranking, possibly with ties).

To illustrate these first definitions, let us use an example. Let $A = \{a, b, c, d\}$ be the set of the alternatives and $N = \{1, 2, 3\}$ the set of the dimensions. The profile p thus contains three preference relations. We suppose here that these relations are linear orders (complete rankings without ties):

1. $a \succ_1 b \succ_1 d \succ_1 c$,
2. $a \succ_2 b \succ_2 c \succ_2 d$,
3. $b \succ_3 d \succ_3 c \succ_3 a$.

The notation $x \succ_i y \succ_i z$ means that x is strictly better than y on dimension i and that y is strictly better than z on the same dimension. Because we assumed that the relations are linear orders, we also have, by transitivity, x strictly better than z.

Our goal is now to construct a global preference relation $\succsim$ on A, taking the preferences on each dimension into account. In other words, our goal is to aggregate the n preference relations $\succsim_i$ into one global preference relation $\succsim$. And we want the global preference relation $\succsim$ to be a weak order (as decided above). A possible way to do this is to adopt the majority principle. Alternative a is the best one for a majority of criteria (2/3); therefore, it is the best alternative. Then we see that b is better than d and c for a majority of criteria (3/3) and, finally, d is better than c for a majority of criteria (2/3). It happens that, for this example, the result is a complete ranking[2]:

$$a \succ b \succ d \succ c.$$

One could possibly argue that, even if a beats b on two criteria, there is a criterion for which a is the worst alternative. Therefore, a should not be considered as the best alternative. Instead, an alternative should be penalised for each bad position. A possible way to construct the preference relation $\succsim$ is then the following: an alternative gets one point for each first rank (best position), two points for each second rank, three points for each third rank, and so on. These points can be considered as penalties. The worse the position, the higher the penalty.

In our example, a obtains 6 points $(1 + 1 + 4)$, b obtains 5 points (2+2+1), c obtains 10 points $(4 + 3 + 3)$ and d obtains 9 points $(3 + 4 + 2)$. Hence, b is the best alternative because it obtained the lowest penalty. Similarly, we find:

$$b \succ a \succ d \succ c.$$

Note that the ranking was different using the majority principle. So, starting from the same data (the profile), there are different ways, different aggregation procedures to construct a global preference relation. And these different aggregation procedures do not yield the same result, the same global preferences.

[2]As is well known, the majority rule may give rise to global preference relations that are not rankings and may have cycles; see section 5.2.1.3 in chapter 5

Once an aggregation procedure has been chosen, the global preference relation $\succsim$ is of course a function of the profile, of the n preference relations $\succsim_i, i = 1 \ldots n$. In other words, $\succsim = \succsim(\succsim_1, \succsim_2, \ldots, \succsim_n) = \succsim(p)$. We call $\succsim$ an aggregation function and each aggregation procedure corresponds to a different aggregation function. Selecting an aggregation function amounts to setting the set $\mathcal{R}$ of "operators" that appears in the description of an evaluation model (section 2.3.2).

When the problem is formulated in these terms, the task of the analyst is then to choose a function $\succsim$. In this process, he can be helped by some theoretical results, by characterisations. These characterisations tell us what the fundamental properties or characteristics of an aggregation function $\succsim$ are. In the next paragraphs, we briefly show, in the light of an example, how these characterisations can be helpful.

4.2.2 An example: the Borda method

The method we presented above, using penalties, is called the Borda method (de Borda, 1784). It can be used in a number of different contexts, but we will consider it in this section only for the aggregation of linear orders. This method has a number of interesting properties of which we now present five.

4.2.2.1 Axioms and characterisation

- *Weak Order.* The global preference relation is always a weak order (a ranking, possibly with ties).

- *Faithfulness.* If we have only one dimension, i.e. $N = \{1\}$, then $\succsim(p)$ is equal to $\succsim_1$, i.e. the global preference is identical to the preference relation on the unique dimension.

- *Cancellation.* If, for every pair of alternatives, there are as many criteria in favour of the first alternative as in favour of the second , then all alternatives are tied.

- *Neutrality.* The result of the aggregation does not depend on the labels of the alternatives but only on their positions in the n preference relations $\succsim_i$.

- *Consistency.* Suppose that, for some reason, you divide your n dimensions in two subsets $N_1 = \{1, 2, \ldots, k\}$ and $N_2 = \{k+1, k+2, \ldots, n\}$ (for example, costs / benefits or financial / non-financial). This also corresponds to two profiles, p_1 and p_2. Then, taking only the dimensions in N_1 into account, you use an aggregation function to construct a global preference relation. Suppose this yields $a \succsim(p_1)\ b$ (a is not worse than b). Then, taking only the dimensions in N_2 into account, you use the same aggregation function in order to construct a global preference relation. Suppose this also yields $a \succsim(p_2)\ b$ (or even $a \succ(p_2)\ b$). Now, if you take all the dimensions in N into account, you probably expect that the aggregation function will tell you $a \succsim(p)\ b$ (or $a \succ(p)\ b$). If it does, then we say that the aggregation function satisfies Consistency.

Using these five properties, Debord (1987) proved the following theorem [3].

Theorem 4.1
Suppose we want to aggregate profiles of linear orders. The only aggregation function satisfying Weak Order, Faithfulness, Cancellation, Neutrality and Consistency is the Borda method.

In other words, if you want to use a procedure that satisfies the five above-mentioned properties, you must use the Borda method. Conversely, if you use the Borda method, these five properties are necessarily satisfied. Many other properties are also satisfied, but only the Borda method satisfies these five. Since the five properties completely characterise the Borda method, this theorem is called a characterisation and the five properties are called axioms.

Note that it might be possible to find other conditions that also characterise the Borda method. They would necessarily be logically equivalent to those of theorem 4.1 because they are necessary and sufficient. The reason why we use these particular conditions is that we think they are more or less intuitively interpretable. So, they can help the analyst or the client to better understand the procedure he uses. We have also chosen the conditions in such a way that they are independent, i.e. none of them is implied by the other ones. In other words, you cannot drop one of them in the statement of the theorem.

Note also that Theorem 4.1 does not apply if we want to aggregate linear orders and, simultaneously, take some additional information into account. For example the fact that the preference between a and b on dimension 1 is much stronger than the preference between c and d. Or the fact that b is definitely not a good alternative. An aggregation function associates a linear order to each profile, without considering any other information than the profiles. While this is quite natural in Social Choice Theory, it is sometimes a limitation in multiple criteria decision aiding.

Remark 4.2.1
For more on the Borda method, see, among others, Chamberlin and Courant (1983), Debord (1992), Dummett (1998), McLean and Urken (1995), Marchant (1996, 1998, 2000, 2001), Nitzan and Rubinstein (1981), Pattanaik (2002), Regenwetter and Grofman (1998), Saari (1990, 1994), Smith (1973) and Van Newenhizen (1992) •

4.2.2.2 Usefulness of the characterisation

We believe that such a theorem can be useful for the client and the analyst because, if the analyst is able to explain the intuitive content of the axioms to the client and if the client finds them appealing or at least acceptable, then he should probably use the Borda method—no other method satisfies the same axioms. On the contrary, if he dislikes one or more axioms, then he should probably not use the Borda method.

[3] The first characterisation of the Borda method was presented by Young (1974), but in a somewhat different context.

In order to make the axioms intuitively understandable to the client, the analyst can state them in natural language, however he also does need to show why they could be desirable or why not. He can also speak of the axioms that are not satisfied by a method. This should help the client make up his mind. In the previous section, while we presented the axioms, we also showed why they could eventually be considered as sensible. In the following paragraphs, we show some reasons to eventually reject them.

Consider Cancellation for example: it might seem reasonable in some applications but probably is not in most of them. Suppose a client faces a problem with two dimensions and somehow finds that a criterion, say 1, is much more important than the other criterion, say 2, he then will probably not find Cancellation attractive. Indeed, for him, not only the number of criteria in favour of a against b is relevant when comparing a and b but also the importance of each criterion. And it is often the case that not all criteria play the same role.

Another reason why Cancellation might not be adequate is the following: suppose that there are only two alternatives a and b (this makes the presentation simpler but doesn't change the reasoning) and that there are as many criteria in favour of a as in favour of b. Suppose also that a client considers all criteria as equally important. Then, according to Cancellation, a and b should be tied. But suppose finally that the client considers that a is not only better than b on dimension 1 but much better than b. Then, this large advantage of a on dimension 1 combined with the advantages on the other dimensions in favour of a might be too large to be compensated by the advantages of b on the rest of the criteria. It would therefore be reasonable to consider a strictly better than b. Of course, if we strictly respect the setting in which we presented the Borda method (aggregation of linear orders), this cannot happen. We have only ordinal information and no information about the size of some advantages or differences. But, in practice, it is not always clear whether the information we have about the alternatives is purely ordinal or not.

Another axiom that might not seem attractive in some cases is Consistency. Suppose that four high school students take four exams and are ranked as follows.

Physics $a \succ_p b \succ_p c \succ_p d$,

Maths $c \succ_m a \succ_m d \succ_m b$,

Economics $d \succ_e b \succ_e c \succ_e a$,

Law $c \succ_l a \succ_l d \succ_l b$.

They apply for scholarships and we want to give the best scholarships to the best students. We therefore need to rank them. If they apply for a scholarship in Physics, we might only look at the rankings in Physics and Maths. Because Maths and Physics are very important in a cursus in Physics, we might also consider that both dimensions play the same role. A reasonable ranking of the candidates is then $a \succ(p)\ c \succ(p)\ b \succ(p)\ d$ (according to our opinion, intuitively).

Suppose then that the four students apply for a scholarship in Economics. For similar reasons, we look only at the rankings in Economics and Law. Observe that

c and d have symmetric positions in the two rankings. Hence, c and d should be considered as equivalent. The same applies to a and b. Observe also that c and d have ranks 1 and 3 in Economics and Law while a and b have ranks 2 and 4. Therefore, the most plausible rankings of the four candidates is $[c \sim(p)\ d] \succ(p)$ $[a \sim(p)\ b]$.

Suppose now that the four students apply for the best student award in their high school. A first and a second prize will be awarded. We therefore need to rank the candidates. Because c is ranked before b in the rankings for both scholarships, then, using Consistency, we might conclude that c should be ranked before b in the award contest. But if we look at the profile—at the four dimensions—we see that b is perhaps better than c. Student b is better than c in two rankings. His only bad grades are in Maths and Law. But, because he is good in Physics and in Economics, it is hard to believe that he is really bad in Maths. So, in this case, the use of Consistency seems to yield an unsatisfactory result.

The problem illustrated in this example is typical of an interaction between two or more criteria. Here, the interaction is positive, between Physics and Economics. The impact of a good rank simultaneously in Economics and in Physics is larger than the impact of a good rank in Physics "plus" the impact of a good rank in Economics. The interested reader will find more about interaction and ways to handle it in Grabisch, Labreuche, and Vansnick (2003), Marichal (2004) and Marichal and Roubens (2000), among others.

On the contrary, Neutrality and Faithfulness seem to be two conditions an aggregation function should satisfy in any context.

Weak Order has a different status. In our opinion, any client that wants to construct a global preference relation, wants it to be a ranking, possibly with ties; not a partial order or a cyclical relation—these are not easy to interpret. But, for some reasons that we will present in section 5.2, p. 174, it is sometimes difficult to obtain a ranking. So, in some cases, a client might be satisfied with a partial order or even a preference relation with some cycles and eventually decide to use an exploitation procedure (see chapter 7, section 7.4) later in the process. In such a case an analyst could be interested by an aggregation function that doesn't satisfy the Weak Order property.

In chapter 5, we will present various results similar to theorem 4.1 and show how they can be used to help the client and the analyst. We now turn to some problems and limits of this approach.

4.2.3 Specificity of this approach

In section 4.2.2, we presented a characterisation of the Borda method as an example of the results (and their usefulness) that can be obtained in a framework inspired by social choice theory. The Borda method, like many other procedures that have been characterised in social choice theory, aggregates ordinal information: the information on each dimension is ordinal, it is a binary relation.

Until recently, all results of social choice theory applied to multiple criteria decision support were characterisations of ordinal aggregation procedures. The aggregation procedures that are not ordinal (for example MAVT, Multi-Attribute

Value Theory) have always been studied in a different framework, in conjoint measurement (see section 4.3 and chapter 6). It would be misleading to think that social choice theory is devoted to the problem of ordinal aggregation and conjoint measurement to cardinal aggregation. In fact, there is a part of social choice theory called cardinal social choice theory. It studies procedures for aggregating cardinal information into a weak order or a choice set. We will develop this in section 5.4. Besides, conjoint measurement can also be used to study the problem of ordinal aggregation (Bouyssou and Pirlot, 2002a).

What is then the difference between the two approaches? All characterisations that will be presented in chapter 5 are to some extent similar to theorem 4.1. They are often inspired by social choice theory (ordinal or cardinal). They characterise aggregation procedures, i.e. procedures that transform an input—a profile or a performance table—into an output—a weak order, most of the time. Characterisations tell us which properties make a given procedure unique.

Roughly speaking, in conjoint measurement, the input is the global preference relation and the set of dimensions. One then tries to represent the global preference relation by means of a model, the parameters of which must be estimated. In conjoint measurement, a typical theorem tells us under which conditions a global preference relation can be represented by a given model. Note that the conditions are imposed on the global preference relation, not on the model.

Some researchers have used the results of conjoint measurement in multiple criteria decision support. For them, the model used in conjoint measurement becomes the aggregation procedure. A typical result of conjoint measurement applied to multiple criteria decision support therefore tells us under which conditions (imposed on the global preferences) a given aggregation procedure can be used. It also suggests a way to set the parameters. This is completely different from social choice theorems where most of the conditions are imposed on the aggregation procedure.

In the next section as well as in chapter 6, we will present some conjoint measurement results applied to multiple criteria decision support.

4.3 Conjoint measurement theory interpreted in MCDA

Measurement theory aims towards examining the conditions and the meaning of *measurement*, which consists in representing "some attributes of objects, substances, and events" (Krantz et al., 1971, p. xvii) numerically. In problems in which a relevant description of an object requires several dimensions, a major question arises: is there an "aggregated measure" or "aggregated descriptor" that allows us to compare these objects even when the measurements on the various dimensions are expressed on incommensurable scales (e.g. mass and length)? *Conjoint measurement* theory examines the conditions under which a relation on a set of objects described by a vector of evaluations is determined by a sort of synthetic measurement that takes the relevant attributes of the objects into account in an appropriate manner. This theory was first developed in Economics (Debreu, 1960) and in Psychology (Luce and Tukey, 1964). It did not take long before people working in decision analysis realised that it could also be used to represent preferences (Edwards, 1971; Raiffa, 1969). In a decision context, the aggregated measure does not reflect an intrinsic property of the objects, which would be independent of the particular evaluation model; it is usually related to a client's subjective preference, which is assumed to be, in some way, related to several objective characteristics of the objects.

4.3.1 The additive value function model

To be more concrete, suppose that within a certain problem formulation, we have started to build an evaluation model: we have determined a set of alternatives A and n dimensions that can describe all the aspects relevant to the decision problem at hand. Suppose that a descriptor for assessing the alternatives on each of the n aspects settled on has been constructed; let $g_i : \mathcal{A} \to X_i$ be the descriptor used for dimension i, with X_i the set of levels of the associated scale. Referring to the evaluation model concept described in chapter 2, section 2.3.3, A precisely denotes the set of alternatives to which the evaluation model applies; it may be larger than the set of alternatives that can be actually chosen by the client; it may contain ideal alternatives that could help in the elicitation of the evaluation model. In this section and more generally in the framework of conjoint measurement, we shall assume that the set of functions g_i used to describe the alternatives on each dimension is exhaustive, so that any alternative a can be identified with the vector $(g_1(a), \ldots, g_i(a), \ldots, g_n(a))$. We may then work with the set of vectors representing the alternatives instead of the alternatives themselves. These vectors form a subset $\{(g_1(a), \ldots, g_n(a)),\ a \in A\}$ of the Cartesian product $X = X_1 \times X_2 \times \ldots X_i \times \ldots X_n$ of the various scales. We assume further that each vector of X corresponds to an alternative and that the client's preferences, denoted by $\succsim$, is a relation on the whole[4] set X. Conjoint measurement theory studies the links that may exist—depending on the properties of $\succsim$—between any pair (x, y) of vectors of X and the fact that this pair is or is not in the preference relation ($x \succsim y$ or $Not[\, x \succsim y\,]$).

In the most popular model of this theory, it can be determined that x is preferred to y by comparing the values that a function u, defined on X, assigns to x and y; u is called a *multi-attribute value function* (MAV function). A very particular case for u, but also by far the most frequent in practice, is when u decomposes into a sum of n functions u_i each of a single variable, i.e. $u(x) = u(x_1, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i)$. The main model of conjoint measurement—

[4] This postulates the extension to all the Cartesian product X of the preference relation that is perceived on $\overline{g}(A) = \{(g_1(a), \ldots, g_n(a)),\ a \in A\}$. In practice, such an extension could force the client to compare alternatives that appear artificial or unrealistic to him. Monotonicity considerations should help to position such unrealistic alternatives with respect to the real ones; for instance, the fictitious cheap car with high performance on all dimensions would clearly be ranked at the top; it is true however that meaningless comparison between unrealistic alternatives could entail contradictions with groups of actual alternatives. Despite possible unwanted practical consequences and provided that the range X_i is not unrealistic, we consider that the extension of $\succsim$ to X is not an outrageous assumption.

called *additive value function model*—thus deals with preferences on X such that for all $x, y \in X$:

$$x \succsim y \Leftrightarrow u(x) = \sum_{i=1}^{n} u_i(x_i) \geq u(y) = \sum_{i=1}^{n} u_i(y_i), \tag{4.1}$$

where u_i is a function mapping X_i into $\mathbb{R}$ for all i. In this representation, the relative importance of the criteria is reflected in the magnitude of the functions u_i. There is an alternative way of representing the same model, which makes the importance of the criteria more explicit.

4.3.2 An alternative formulation showing tradeoffs

Let us start by normalising the values of u_i to fit in the $[0, 1]$ interval (this is always possible if the set X_i is finite or, more generally, if $u_i(X_i)$ is a bounded set of $\mathbb{R}$); denoting by $\underline{u}_i$ (resp. $\overline{u}_i$) the minimal (resp. the maximal) value taken by $u_i(x_i)$ when x_i varies in the set X_i, we define the normalised value v_i of u_i as

$$v_i(x_i) = \frac{u_i(x_i) - \underline{u}_i}{\overline{u}_i - \underline{u}_i}. \tag{4.2}$$

Expressing u_i as a function of v_i yields

$$u_i(x_i) = (\overline{u}_i - \underline{u}_i)v_i(x_i) + \underline{u}_i; \tag{4.3}$$

substituting this expression in equation (4.1) yields:

$$x \succsim y \Leftrightarrow u(x) = \sum_{i=1}^{n}(\overline{u}_i - \underline{u}_i)v_i(x_i) + \sum_{i=1}^{n}\underline{u}_i \geq u(y) = \sum_{i=1}^{n}(\overline{u}_i - \underline{u}_i)v_i(y_i) + \sum_{i=1}^{n}\underline{u}_i.$$

Subtracting $\sum_{i=1}^{n}\underline{u}_i$ from both sides of the inequality and dividing by $\sum_{j=1}^{n}(\overline{u}_j - \underline{u}_j)$ yields another additive value function v that represents the same relation $\succsim$; we have:

$$x \succsim y \Leftrightarrow v(x) = \sum_{i=1}^{n} \frac{\overline{u}_i - \underline{u}_i}{\sum_{j=1}^{n}(\overline{u}_j - \underline{u}_j)} v_i(x_i) \geq v(y) = \sum_{i=1}^{n} \frac{\overline{u}_i - \underline{u}_i}{\sum_{j=1}^{n}(\overline{u}_j - \underline{u}_j)} v_i(y_i).$$

The transformed value function v is thus defined by:

$$v(x) = \frac{1}{\sum_{j=1}^{n}(\overline{u}_j - \underline{u}_j)} u(x) - \sum_{i=1}^{n}\underline{u}_i.$$

Defining coefficients k_i as:

$$k_i = \frac{\overline{u}_i - \underline{u}_i}{\sum_{j=1}^{n}(\overline{u}_j - \underline{u}_j)}, \tag{4.4}$$

we have the following representation of the preference $\succsim$:

$$x \succsim y \Leftrightarrow v(x) = \sum_{i=1}^{n} k_i v_i(x_i) \geq v(y) = \sum_{i=1}^{n} k_i v_i(y_i), \tag{4.5}$$

in which k_i are nonnegative "weighting factors" adding up to 1; in this representation the maximal value of v_i is 1 and the minimal is 0.

In the sequel we assume that $u_i(X_i)$ is a bounded set of $\mathbb{R}$. Starting with any representation of a preference in model (4.1), we can derive a representation in model (4.5), as we have just shown. Conversely, from a representation in model (4.5), we immediately derive a representation in model (4.1); letting $u'_i = k_i v_i$, we get: $x \succsim y \Leftrightarrow v(x) = \sum_{i=1}^{n} u'_i(x_i) \geq v(y) = \sum_{i=1}^{n} u'_i(y_i)$. Hence, models (4.5) and (4.1) are equivalent in the sense that all preferences that can be represented by one of them can be represented by the other.

Depending on the context, one or another formulation of the model may offer an advantage. From equation (4.4), we infer that k_i can be computed as the length of the range of variation of function u_i relatively to the sum of all ranges; the value of k_i remains invariant when we apply a positive affine transformation to u_i. In section 4.3.8, we shall see how the "weights" k_i can be interpreted as *tradeoffs*.

4.3.3 Additive value function and conjoint measurement

The model described above, in either of its forms (4.1) or (4.5), will be referred to as the *additive value function model*; u is called an additive MAV function. Conjoint measurement theory is concerned with establishing conditions on $\succsim$ under which a representation according to model (4.1) (or (4.5)) exists. The uniqueness of the representation is also studied.

Why is this interesting? Clearly, if we have reasons to believe that a preference might obey model (4.1), we can try to determine the preference—which is usually not known explicitly—by constructing the functions u_i; alternatively, for eliciting model (4.5), we should construct the functions v_i and assess the coefficients k_i. Each model suggests a strategy (or several) for eliciting preferences that are representable in the model. Of course, not all preferences satisfy model (4.1); we shall not specify the necessary and sufficient conditions here but just mention the following two important and obvious requirements for the preference:

- $\succsim$ must be a weak order (see chapter 3, section 3.10), i.e. a transitive and complete preference, in other words a complete ranking, possibly with tied alternatives. This is clearly a necessary requirement since model (4.1) exactly says that the order $\succsim$ on X is obtained by transporting the natural order of $\mathbb{R}$ onto X using the function u.

- $\succsim$ must satisfy (strong) preference independence. The decomposition of u into a sum of functions each of a single variable reveals that if $x \succsim y$ while x and y have received the same assessment on dimension i, then, if we change that common level into another common level, the transformed x and y will compare in the same way as before. More formally, let x and y be such that $x_i = y_i = a_i$; let x' be equal to x except that $x'_i = b_i \neq x_i$ and let y' be equal to y except that $y'_i = b_i \neq y_i$, then:

$$x \succsim y \Leftrightarrow x' \succsim y'$$

since

$$u_i(a_i) + \sum_{j \neq i} u_j(x_j) \geq u_i(a_i) + \sum_{j \neq i} u_j(y_j) \Leftrightarrow$$
$$u_i(b_i) + \sum_{j \neq i} u_j(x_j) \geq u_i(b_i) + \sum_{j \neq i} u_j(y_j)$$

The independence property of the preference has far-reaching consequences; it allows in particular for ceteris paribus reasoning, i.e. comparing alternatives the evaluations of which differ only on a few attributes without specifying the common level of their evaluations on the remaining attributes; the independence property guarantees that the result of such a comparison is not altered when changing the common level on the attributes that do not discriminate between the alternatives. We shall further discuss this property in section 4.3.5.

The two conditions stated above are not sufficient for ensuring that $\succsim$ satisfies model (4.1). If the evaluation space X is infinite, various sets of sufficient conditions are provided in the literature; they are often categorised into two branches, the algebraic and the topological theories, respectively (see e.g. Fishburn, 1970, ch. 5). We give a schematic outline of the algebraic approach in section 6.1.2 of chapter 6, including an intuitive presentation of the additional conditions that are necessary for the additive model. If the set of possible levels X_i on each dimension is finite, the situation is rather unpleasant since necessary and sufficient conditions are not generic: using mathematical tools (mainly the theorem of the alternative for systems of linear equations and inequalities, see Fishburn (1970), p. 46), one can write a system of compatibility conditions for each particular set X (Fishburn, 1970, ch. 4) that guarantees the existence of a representation of $\succsim$ according to model (4.1); we outline the theory for the finite case in chapter 6, section 6.1.3. So, without explaining the formulation of necessary and sufficient conditions here, we just bear the two necessary conditions cited above in mind.

4.3.4 Uniqueness issues

If the model is to be used to elicit preferences through the construction of functions u_i, it may also be important to know whether these u_i are uniquely determined. Actually, the u_i's are not unique. For a preference $\succsim$ that fits in the additive value model, there is a family of value functions u that both

- decompose additively as $u(x) = \sum_{i=1}^{n} u_i(x_i)$ and
- represent the preference, i.e., satisfy $x \succsim y \Leftrightarrow u(x) \geq u(y)$.

Suppose indeed that we start with a particular representation of $\succsim$, $u(x) = \sum_{i=1}^{n} u_i(x_i)$ and transform u_i into u'_i by using a *positive affine transformation*

$$u'_i = \alpha u_i + \beta_i, \tag{4.6}$$

with $\alpha > 0$ and β_i a real number (that may vary with i). By using u'_i instead of u_i in the additive model, we obtain:

$$u'(x) = \sum_{i=1}^{n} u'_i(x_i) = \alpha \sum_{i=1}^{n} u_i(x_i) + \sum_{i=1}^{n} \beta_i = \alpha u(x) + \sum_{i=1}^{n} \beta_i.$$

Clearly, u' is an alternative representation of the preference $\succsim$ since $x \succsim y \Leftrightarrow u(x) \geq u(y) \Leftrightarrow u'(x) \geq u'(y)$. So, the u_i's to be used in an additive representation are at best determined up to a positive affine transformation.

If X is infinite, a number of systems of conditions that guarantee the existence of an additive representation according to model (4.1) are known; this additive representation is unique up to a positive affine transformation of the u_i's according to equation (4.6) (the positive coefficient α is the same for all i but β_i may depend on i). These conditions involve structural assumptions that are sufficient but not necessary; however they may be reasonable in practical situations. For instance, in the algebraic theories mentioned in the last paragraph of the previous section, one postulates that the set of levels X_i on each attribute are "sufficiently rich" so that some "solvability conditions" are fulfilled; roughly speaking, it is required that it always be possible to find a level x_i such that an alternative involving x_i is indifferent to a specified alternative (see section 6.1.2 for more details). Richness is not a necessary assumption, but it corresponds to our intuition related to the measurement of length, for instance. In the finite case, provided the representation of a preference by the additive model exists, it is generally not unique (even up to a positive affine transformation).

Remark 4.3.1 (Normalisation)
If the representation in the additive model is unique up to a positive affine transformation of the u_i's, it is not difficult to impose additional constraints to the function u in order to fix the degrees of freedom left for the determination of the u_i's. One may, for instance, scale u in order for its minimal value on X to be 0 and its maximal value to be 1. If such a requirement is imposed and the functions u_i are constrained to be nonnegative, then they are exactly determined. Indeed, consider any additive representation u of the preference, with $u(x) = \sum_{i=1}^{n} u_i(x_i)$; due to the uniqueness hypothesis of the u_i's up to a positive affine transformation, all other additive representations are of the form

$$u'(x) = \sum_{i=1}^{n} u'_i(x_i) = \sum_{i=1}^{n} (\alpha u_i(x_i) + \beta_i),$$

with $u'_i(x_i) = \alpha u_i(x_i) + \beta_i$. If we impose that u' is scaled as said above and using the notations introduced page 129, we must have

$$0 = \sum_{i=1}^{n} (\alpha \underline{u}_i + \beta_i)$$

$$1 = \sum_{i=1}^{n} (\alpha \overline{u}_i + \beta_i).$$

Imposing that the u_i''s are nonnegative, implies that $\alpha \underline{u}_i + \beta_i \geq 0$; this, combined with the first equation, forces $\beta_i = -\alpha \underline{u}_i$; the second equation entails

$$\alpha = \frac{1}{\sum_{i=1}^{n}(\overline{u}_i - \underline{u}_i)}.$$

Hence there is no degree of freedom left on α and the β_i's. •

Assuming that the u_i's are determined up to a positive affine transformation, we shall briefly explain in section 4.3.7 how we can take advantage of this to construct an additive representation of the preference.

4.3.5 Relevance of conjoint measurement results for MCDA

It may seem disturbing at first glance that conjoint measurement results require the verification of properties of a preference that will only be known at the end of the MCDA process. To use these results, the client is asked to answer questions that refer to his intuitive perception of his own preferences. For instance, the preference independence hypothesis that is crucial for model (4.1) can be at least partially tested by asking the client questions like: "Do you prefer a meal with fish and red wine or a meal with fish and white wine?" "Do you prefer a meal with meat and red wine or a meal with meat and white wine?". If the client is consistent in preferring the same type of wine with both meat and fish, then there is no clue that his preference might not satisfy preference independence, with main course and wine as attributes. In the opposite case, we know that his preferences cannot be represented by model (4.1). Thus, the characterisation of conjoint measurement models has the advantage of allowing to test whether the model is likely to be able to fit the preference. Of course the possibility of testing such hypotheses is often theoretical: some axioms may have little intuitive content; even if it is not the case, most of the time it is only possible to "falsify" a model by exhibiting a situation where an axiom fails to be satisfied (like, potentially, in our question about fish, meat and wine) while it is seldom possible to positively establish that a preference will fit with the model.

4.3.6 Marginal preferences within the additive value model

The type of function u associated to model (4.1) suggests a stepping stone for its elicitation. Under the hypothesis that $\succsim$ fits with model (4.1), the model suggests that functions u_i could be elicited. Going one step further, it is readily seen that $u_i(x_i)$ must be compatible with the marginal preference relation $\succsim_i$ defined as:

$$x_i \succsim_i y_i \Leftrightarrow \forall\ a_{-i} \in X_i, (x_i, a_{-i}) \succsim (y_i, a_{-i}). \tag{4.7}$$

Consider two alternatives (x_i, a_{-i}) and (y_i, a_{-i}) that may only differ on attribute i; they have common evaluations a_j on all attributes j except for $j = i$. If the client says that he likes (x_i, a_{-i}) at least as much as (y_i, a_{-i}), this means, in terms of the marginal preference relation $\succsim_i$, that $x_i \succsim_i y_i$ and it translates in model

(4.1) into:

$$u_i(x_i) + \sum_{j \neq i} u_j(a_j) \geq u_i(y_i) + \sum_{j \neq i} u_j(a_j),$$

from which we deduce $u_i(x_i) \geq u_i(y_i)$. Thus, whenever $x_i \succsim_i y_i$, we have $u_i(x_i) \geq u_i(y_i)$ and it is easily seen that the converse is also true; for all levels x_i, y_i in X_i, we have $x_i \succsim_i y_i$ iff $u_i(x_i) \geq u_i(y_i)$. Therefore, in model (4.1), the function u_i can be interpreted as a numerical representation of the marginal preference $\succsim_i$, which is a weak order (a ranking of the alternatives, possibly with ties).

The fact that the marginal preference is a weak order has strong links with the independence property of preference $\succsim$ (this will be analysed much more in depth in section 6.2.9). This is also of significant practical importance. However, a difficulty remains; the u_i functions that we need to use in the additive representation of the preference are not just *any* numerical representation of the marginal preference relations $\succsim_i$. A weak order like $\succsim_i$, has many different numerical representations since *any increasing function* of a representation is in turn a representation; the numerical representation of a weak order is determined up to an increasing transformation. Among the whole set of possible representations of the weak order $\succsim_i$, we have to select the right one (determined up to a positive affine transformation), the one that is needed for a representation of the global preference in the additive model.

Example 4.1 (Buying a sports car)
We consider the example briefly described in section 4.1 (see also in chapter 6 of Bouyssou et al. (2000)). Thierry, a student who is passionate about sports cars but earns little money, assesses fourteen cars among which he considers buying one, based on the five dimensions that are of importance to him, namely cost, acceleration, pick up, brakes and road holding. Assume that his preference fits with the additive value model (4.1) and let us help Thierry build a value function u that represents his preference in accordance with the additive model.

We first settle the ranges X_i in which the attributes will reasonably vary (in view of the evaluation of the fourteen selected cars). These ranges are shown in table 4.1. The evaluations on the first three attributes are expressed in "physical" units (thousands of €, and twice in seconds, respectively); the last two belong to a qualitative scale. On the first three attribute scales, less is better, while on the last two, more is better. What is the relationship between the evaluations and the

Attribute	i	X_i	unit	to be
Cost	1	[13; 21]	1 000 €	minimised
Acceleration	2	[28; 31]	second	minimised
Pick up	3	[34; 42]	second	minimised
Brakes	4	[1; 3]	qualitative	maximised
Road holding	5	[1; 4]	qualitative	maximised

Table 4.1: Ranges of the five dimensions in the "Buying a sports car example".

value function u? There are two main features that we want to emphasise:

- the information contained in the evaluations is transferred to the value function through the marginal preferences;
- the marginal preferences, which are weak orders in the additive model (4.1), cannot be considered as identical to the natural ordering of the evaluations although these weak orders are not unrelated.

Take for example the cost attribute. Clearly, a car, say x, that costs 15 000 € is not preferred to a car y that costs 14 000 € if both cars are tied for all other dimensions. And the conclusion will be the same when comparing the first car with any other car that costs less and has the same evaluation on all other attributes. More formally, car x can be described by the vector $(15, a_2, a_3, a_4, a_5)$ and y by $(14, a_2, a_3, a_4, a_5)$; the first dimension of these vectors represents the cost (in thousands of €) and a_i, for $i = 2, \ldots, 5$, designates any level on the other attributes. Car y is certainly at least as preferred as x ($y \succsim x$) since y is cheaper than x and all other evaluations are identical for both cars. This is a typical case in which "ceteris paribus" reasoning applies; the property of the preference we use here is *weak preference independence* (see page 239, definition 6.3); it is implied by strong preference independence which is a necessary condition for a preference being represented by the additive value model (4.1).

The fact that car y is preferred to car x, independently of the value of a_j, can be translated into a statement involving the marginal preference $\succsim_1$ on the Cost attribute, namely $14 \succsim_1 15$. For all pairs of costs x_1, y_1 in the range $[13; 21]$, we would similarly have $y_1 \succsim_1 x_1$ as soon as the cost x_1 is higher than the cost y_1. ◇

Remark 4.3.2
This does not mean, however, that the marginal preference $\succsim_1$ is necessarily the reversed natural order for the costs in the $[13; 21]$ interval. The marginal preference $\succsim_1$ might indeed not discriminate between x_1 and y_1 when the difference $|x_1 - y_1|$ is small enough. The client could feel that, due to the imprecision of the evaluation of the costs, he cannot distinguish, in terms of preference, between costs that round up to the same nearest thousand of Euros. In such a case, the marginal preference relation $\succsim_1$ would be less discriminating than the reversed natural order on the real numbers. A numerical representation u_1 of the weak order $\succsim_1$ is graphed in figure 4.1.

4.3.7 Leaning on the additive value model to elicit preferences

The additive value model suggests a general strategy for the elicitation of a preference that fits with the model. We assume here that the conditions of uniqueness of the additive representation are fulfilled (see section 4.3.4; i.e., that the functions u_i, which intervene in the sum are determined up to a positive affine transformation (see (4.6)). The strategy consists in eliciting the functions u_i, relying on the fact, observed in the previous section, that the u_i's are numerical representations of the marginal preferences. The main problem is to find among the many representations of the marginal preferences, the essentially unique ones that can be

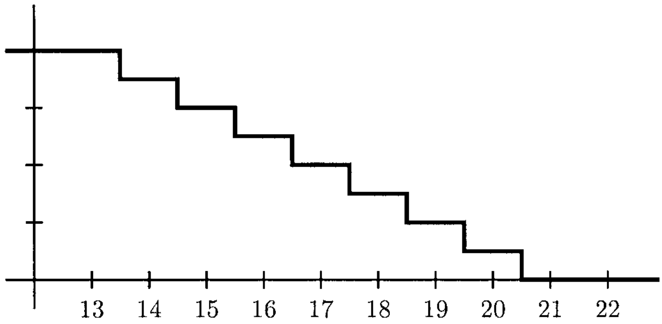


Figure 4.1: Numerical representation of the marginal preference for Cost in a case where it is a nonincreasing function of the cost.

summed up and yield an additive representation u of the preference. This can be done in many different ways, which have been thoroughly studied (see, e.g., Fishburn, 1967; Keeney and Raiffa, 1976; von Winterfeldt and Edwards, 1986). We briefly illustrate the method of *standard sequences* using the example of ranking sports cars outlined in the previous section; we refer the reader to Bouyssou et al. (2000, ch. 6) for more details and for the illustration of other elicitation methods applied to the same example.

We limit ourselves here to the elicitation of the marginal value function u_2, corresponding to the "Acceleration" attribute, by means of indifference judgements requested from the client. We start by considering two hypothetical cars that differ only on the cost and acceleration attributes, their performance levels on the other dimensions being tied (this is again "ceteris paribus" reasoning but with only three common levels, instead of four in the previous section). We assume that the two cars differ in cost by a noticeable amount, say for instance 1 000 €; we locate an interval of cost of that amplitude in the middle of the cost range, say for example [16 500; 17 500] €. We then fix a value for the acceleration, also in the middle of the acceleration range, say, 29.5. We ask the client to consider a car costing 16 500 € and accelerating in 29.5 seconds, the evaluations on the other dimensions being fixed at an arbitrary (say mid-range) value. We ask the client to assess a value x_2 of the acceleration such that he would be indifferent between the cars $(16.5; 29.5)$ and $(17.5; x_2)$ (the cars are sufficiently specified by a pair of levels, on cost and acceleration attributes, since we assume that their evaluations on the remaining dimensions are identical and that the preference is independent, i.e. that ceteris paribus reasoning makes sense). This question amounts to determining which improvement of the performance on the acceleration attribute (starting from a value of 29.5 seconds) would be worth a cost increase of 1 000 € (starting from 16 500 €), all other performance levels remaining constant.

Since the client is assumed to be fond of sports cars, he could say for instance that $x_2 = 29.2$ seconds, which would result in the following indifference judgement: $(16.5; 29.5) \sim (17.5; 29.2)$. In view of the hypothesis that the client's preference fits into the additive value model, this indifference judgement can be translated into the following equality:

$$u_1(16.5) + u_2(29.5) + \sum_{j=3}^{5} u_j(x_j) = u_1(17.5) + u_2(29.2) + \sum_{j=3}^{5} u_j(x_j) \tag{4.8}$$

Since the performance of both cars on attributes $j = 3, 4, 5$ are equal, the corresponding terms of the sum cancel and we are left with $u_1(16.5) + u_2(29.5) = u_1(17.5) + u_2(29.2)$ or:

$$u_1(16.5) - u_1(17.5) = u_2(29.2) - u_2(29.5), \tag{4.9}$$

which translates as an equality between differences of marginal values on attributes 1 and 2.

The second question to the client uses his answer to the first question; we ask him to assess the value x_2 of the acceleration that would leave him indifferent between the two cars $(16.5; 29.2)$ and $(17.5; x_2)$. Suppose the answer is $x_2 = 28.9$; we would then infer that:

$$u_1(16.5) - u_1(17.5) = u_2(28.9) - u_2(29.2). \tag{4.10}$$

Note that the left-hand side has remained unchanged: we always ask for acceleration intervals that are considered as equivalent to the same cost interval.

The next question asks for a value x_2 such that $(16.5; 28.9) \sim (17.5; x_2)$ and so on. Let us imagine that the sequence of answers is e.g.: 29.5; 29.2; 28.9; 28.7; 28.5; 28.3; 28.1. In view of (4.9), this amounts to saying that this sequence of levels on the marginal value scale of the acceleration attribute are equally spaced and that all differences of value between consecutive pairs of levels in the list are worth the same difference in cost, namely a difference of 1 000 € placed between 16 500 and 17 500 €. In other words, the client values 1 000 € as an improvement of

0.3 seconds	w.r.t. a performance level of 29.5s or 29.2s
0.2 seconds	w.r.t. a performance level of 28.9s, 28.7s, 28.5s or 28.3s

on the acceleration attribute. He thus values improvements in the lower range of the scale more. Similar questions are asked for the upper half of the range of the acceleration attribute, i.e., from 29.5 to 31 seconds. We ask the client to assess x_2 such that he would be indifferent between $(16.5; x_2)$ and $(17.5; 29.5)$. Assume the client's answer is $x_2 = 30.0$. Then we go on asking for x_2 such that $(16.5; x_2) \sim (17.5; 30.0)$ and suppose we get $x_2 = 31$. From all these answers, one understands that the client values a gain in acceleration performance of 1 second between 31 and 30 and a gain of 0.2 second between e.g. between 28.9 and 28.7 in the same way, a ratio of 5 to 1.

What can we do with this piece of information? We can build a piecewise linear approximation of the function u_2 (defined on the range going from 28 to 31 seconds). Using an arbitrary unit of length on the vertical axis (the unit of length represents 1 000 € or more precisely the difference $u_1(16.5) - u_1(17.5)$), we obtain the function u_2 represented in figure 4.2; this is in fact a linear interpolation of nine points the first coordinates of which correspond to the answers given by the client to seven indifference judgments; the second coordinates of these points have just to be equally spaced (by one unit of length). The position of the origin is arbitrary. We have extrapolated the line from 28.1 to 28 (thinner piece of line). Note that function u_2 is decreasing, since smaller is better with the measure chosen for evaluating the acceleration.

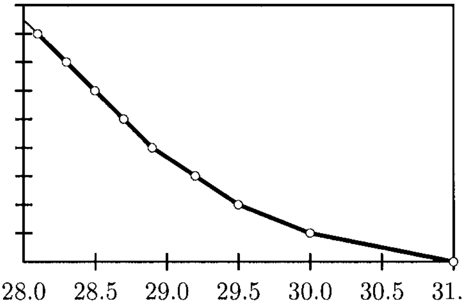


Figure 4.2: Piecewise linear interpolation of the marginal value function u_2 on the acceleration attribute.

To determine u_3, u_4 and u_5, we search successively, in the same way as for acceleration, for intervals on the pick up, brakes and road holding scales that would compensate exactly the cost interval (16.5; 17.5) in terms of preference.

Finally, we have to do the same recoding for the cost itself. We fix an interval for instance on the acceleration scale, say [29.2; 29.5]. We already know the answer to one question: (17.5; 29.2) is indifferent to $(x_1, 29.5)$ when $x_1 = 16.5$. We then ask the client, which level x_1 on the cost scale would leave him indifferent between (16.5; 29.2) and $(x_1, 29.5)$. A cost lower than 16 500 € is expected and we use this in the next question, and so on. We might end up, for instance, with the curve shown in figure 4.3. Looking at this curve indicates that the client is inclined to pay more for the same improvement on the acceleration attribute for a car priced in the lower part of the cost range than for one priced in the higher part. Plausibly, with the limited budget of a student, Thierry can reasonably spend up to 17 500 € on buying a car; paying more would imply restrictions on other expenses. Suppose we have built piecewise linear approximations of u_1 to u_5 in this way. If we have chosen the same unit to represent intervals equivalent to $u_1(16.5) - u_1(17.5)$ on all vertical axes, all that remains is to add up these functions to obtain a piecewise linear approximation of u; ranking in turn the alternatives according to their decreasing value of u (formula (4.1)) yields the preference $\succsim$ (or an approximation of it). For the sake of illustration, we show the additive value function [5] computed

[5] In fact, these values have been determined by means of another elicitation method; details are provided in Bouyssou et al. (2000, ch. 6).

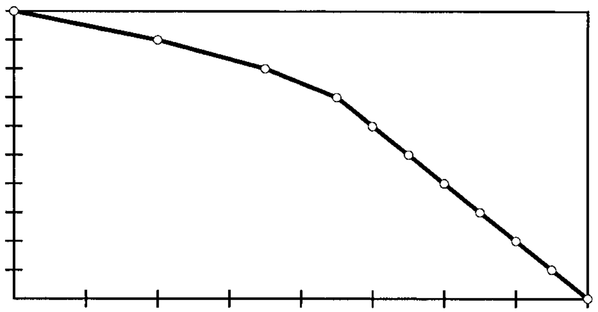


Figure 4.3: Piecewise linear interpolation of the marginal value function u_2 on the cost attribute.

for each of the 14 cars selected as alternatives by Thierry in table 4.2. Ranking the cars in decreasing order of the value function yields Thierry's preference relation $\succsim$ on the set of alternatives. This preference is a weak order; its equivalence classes are labelled by their rank in the table. If we admit that the precision of the indifference judgments made by the client is absolute, there are no ties in this ranking.

Cars	Value u	Rank
Peugeot 309/16	0.85	1
Nissan Sunny	0.75	2
Honda Civic	0.66	3
Peugeot 309	0.65	4
Renault 19	0.61	5
Opel Astra	0.55	6
Mitsubishi Colt	0.54	7
Mazda 323	0.53	8
Fiat Tipo	0.51	9
Toyota Corolla	0.50	10
Mitsubishi Galant	0.48	11
Alfa 33	0.47	12
Ford Escort	0.32	13
R 21	0.16	14

Table 4.2: Ranking of the cars in decreasing order of the value function u.

Remark 4.3.3

As just outlined, the construction of an additive representation incorporates the client's preference in the u_i's; one can thus interpret the u_i's as the set of criteria H of the evaluation model $\langle A, \{D, \mathcal{E}\}, H, \mathcal{U}, \mathcal{R}\rangle$ (see section 2.3.3). The synthesis of the various criteria into global preferential information, here the preference relation $\succsim$, decomposes in two steps that can be interpreted as constituting the set $\mathcal{R}$ of "operators" in the evaluation model. First the criteria (i.e., the u_i functions) are summed up, yielding a numerical representation u of the preference; then this representation is used according to model (4.1) to determine the preference relation $\succsim$. Note that the sole significance of the function u is to be a numerical representation of the preference relation; larger or smaller differences $u(x) - u(y)$ may not, in principle, be interpreted as reflecting larger or smaller differences in preference intensities between the alternatives x and y. We will come back to this point in section 4.3.9 below.

The construction of an additive representation shows the following important features:

- the representations u_i of the marginal preferences $\succsim_i$ are built jointly, using one of the dimensions (here the cost) as reference dimension;

- the elicitation process transforms the dimensions (attributes) into criteria, incorporating the client's preference into the model;

- comparing differences $u(x) - u(y)$ is meaningless; in the additive value function model, these differences do not, in general, model preference intensity.

4.3.8 Tradeoffs or substitution rates

An interesting feature of the form (4.5) of the additive model is that it allows us to give a precise meaning to the intuitively appealing notion of "importance of the criteria"; in this model, this notion can be represented by the "weights" k_i, provided that the v_i's are normalised in such a way that their maximum is 1 and their minimum is 0, as was assumed in section 4.3.2. The "weights" k_i in model (4.5), can be interpreted as *substitution rates* or *tradeoffs*. Consider two alternatives x and y that share all levels except those on two dimensions i and j, i.e., $x = (x_i, x_j, a_{-\{i,j\}})$ and $y = (y_i, y_j, a_{-\{i,j\}})$, where $a_{-\{i,j\}}$ denotes a vector of dimension $n - 2$, the coordinates of which are those of alternative a except for dimensions i and j; suppose that these alternatives are indifferent, implying that $u(x) = u(y)$; using the form (4.5), after having cancelled the terms $k_l v_l(a_l)$ for $l \neq i, j$ that appear on both sides of the equality we obtain:

$$k_i v_i(x_i) + k_j v_j(x_j) = k_i v_i(y_i) + k_j v_j(y_j),$$

from which we get:

$$\frac{k_i}{k_j} = \frac{v_j(y_j) - v_j(x_j)}{v_i(x_i) - v_i(y_i)}. \tag{4.11}$$

In other words, indifference between x and y means that the "difference of preference" between the levels x_i and y_i on attribute i is exactly balanced by the "difference of preference" between the levels y_j and x_j on attribute j, the alternatives being tied on all other attributes (*ceteris paribus* reasoning again!). If we know that $v_i(x_i)-v_i(y_i)$ correctly represents the preference differences on attribute i, i.e. that the difference of preference between the levels x_i and y_i is at least as large as the difference of preference between any two levels z_i and k_i if and only if $v_i(x_i) - v_i(y_i) \geq v_i(z_i) - v_i(k_i)$, and if the same is true for v_j on attribute j, then the coefficients k_i and k_j allow us to compare "inter-attribute" preference differences. Equation (4.11) tells us that the difference $v_j(y_j) - v_j(x_j)$ can be balanced by

$$\frac{k_i}{k_j} \times (v_i(x_i) - v_i(y_i)) \tag{4.12}$$

the ratio k_i/k_j being the substitution rate between the differences in marginal values.

Let us assume that the conditions of uniqueness of the u_i's up to a positive affine transformation are fulfilled. If model (4.1) is considered instead of model (4.5), we may obtain a representation in the latter, as shown in section 4.3.2, by applying the transformation (4.2), i.e., computing $v_i(x_i) = (u_i(x_i) - \underline{u}_i)/(\overline{u}_i - \underline{u}_i)$. Substituting v_i in equation (4.12) yields

$$\frac{k_i}{k_j} = \frac{u_j(y_j) - u_j(x_j)}{u_i(x_i) - u_i(y_i)} \times \frac{\overline{u}_i - \underline{u}_i}{\overline{u}_j - \underline{u}_j}. \tag{4.13}$$

When using model (4.1), the ratio k_i/k_j can still be computed on the basis of a ratio of differences (here involving u_i and u_j), except that the differences have to be normalised by the range of the corresponding function (u_i or u_j).

Example 4.2

Consider for instance the "Buying a sports car" example (described on page 134) and suppose that the u_i's are unique up to a positive affine transformation. If we accept that the curves in figures 4.3 and 4.2 correctly represent the marginal value functions u_1 and u_2 on cost and acceleration, respectively, then we may estimate the ratio k_1/k_2 by substituting u_i by $k_i v_i$, for $i = 1, 2$, in the following equation (see (4.9)):

$$u_1(16.5) + u_2(29.2) = u_1(17.5) + u_2(28.9).$$

Using (4.13), we obtain:

$$\frac{k_1}{k_2} = \frac{u_2(28.9) - u_2(29.2)}{u_1(16.5) - u_1(17.5)} \times \frac{\overline{u}_1 - \underline{u}_1}{\overline{u}_2 - \underline{u}_2};$$

In figure 4.3, we see that $\overline{u}_1 = 10$ units and $\underline{u}_1 = 0$; in figure 4.2, we obtain $\overline{u}_2 = 8$ units and $\underline{u}_2 = 0$; remember that the units are the same on both attributes (due to the elicitation procedure) and we have determined that they are equal to the differences $u_2(28.9) - u_2(29.2) = u_1(16.5) - u_1(17.5)$. Hence,

$$\frac{k_1}{k_2} = 1 \times \frac{10}{8} = 1.2.$$

What does this mean? If we normalise the ranges of variation of the marginal value functions on the two dimensions in order for the normalised values to vary between

0 and 1, then the substitution rate between one unit on the normalised range of the cost criterion is worth 1.2 units on the normalised range of the acceleration criterion. ◇

4.3.9 The measurement of global preference differences

Consider a preference $\succsim$ for which there exists a unique additive representation (up to a positive affine transformation of the u_i's) within model (4.1). Through an elicitation procedure—such as, for instance, the standard sequence method outlined in section 4.3.7—one obtains a value function u that represents $\succsim$. Let us assume that Thierry's preference in the case outlined above fulfils the hypotheses of model (4.1) and that a value function u representing Thierry's preference in the model has been correctly elicited. Note that this function represents Thierry's preference not only for the cars in the set of selected alternatives, but also for the whole Cartesian product X determined by the ranges of the attributes (see table 4.1). In other words, we know the u-value of any "car"—real or fictitious—described by a vector $(x_1, x_2, x_3, x_4, x_5)$, with x_i varying in the ranges specified in table 4.1.

Under the above conditions, as we have seen in remark 4.3.1, the normalised additive representation is uniquely determined. In other words, if we set the value u of the ideal[6] car $\bar{x}$ at 1 and the value of the anti-ideal car $\underline{x}$ at 0, then there is only one additive value function representing $\succsim$. The u-function in table 4.2 has been set using these constraints.

It is a common mistake to interpret the uniqueness of the additive value function representing $\succsim$ as implying that the size of the difference $u(x) - u(y)$ can be interpreted as measuring a *preference difference* and that such differences can be compared meaningfully. In the "Buying a sports car" example, the difference in the values of the two top-ranked cars, the Peugeot 309/16 and the Nissan Sunny, is $0.85 - 0.75 = 0.10$; the difference in the values associated to the cars ranked in 8th and 9th positions, the Mazda 323 and the Fiat Tipo, is equal to $0.53 - 0.51 = 0.02$. Comparing these differences does not make any sense because we did not ask the client any information on global preference differences. One cannot meaningfully say something like "the difference (of preference) between the former two cars is five times the difference between the latter two cars". It cannot even be said that the preference of the Peugeot to the Nissan is stronger than the preference of the Mazda to the Fiat. Differences in u-values, although u is numeric and unique, may not be meaningfully related with "strength of preference" or any analogous concept.

We emphasise here that the only legitimate interpretation of u is *ordinal*. The only conclusion we can meaningfully draw from the fact that the u-value attached to the Peugeot is 0.85 and that attached to the Nissan is 0.75, is that the Peugeot is preferred to the Nissan. And that's all! The uniqueness result discussed above only concerns *additive* representations of $\succsim$. There are clearly many other value

[6] The ideal car is the fictitious car that realises the best performance level on all attributes, in the range specified in table 4.1: it is a car that costs 13 000 €, accelerates in 28 seconds, etc. The anti-ideal car is defined symmetrically.

functions—assigning 1 to the ideal car and 0 to the anti-ideal—that can represent the preference $\succsim$ *equally well*, but the only one that is normalised and decomposes into a sum of marginal value functions u_i is u. In table 4.3, apart from the additive value function we already know, we give an equivalent representation of the preference using a value function v. We see that the "difference" between the Peugeot and the Nissan is 0.01 according to v, while it is 0.10 between Mazda and Fiat. Any increasing transformation of the $[0, 1]$ interval into itself provides an alternative representation of the preference $\succsim$ when applied to u. Does the above

Cars	Value u	Value v
Peugeot 309/16	0.85	0.60
Nissan Sunny	0.75	0.59
Mazda 323	0.53	0.58
Fiat Tipo	0.51	0.48
Ideal	1.00	1.00
Anti-ideal	0.00	0.00

Table 4.3: Two equivalent representations of the preference; u is the additive one.

analysis imply that comparing preference differences is meaningless? By no means! But the model one uses must be *specifically* designed for that purpose. Difference—or strength—of preference is a different notion from that of preference. Formally, it is a relation, that we shall denote by $\succsim^*$, defined on the pairs of alternatives, i.e. on X^2. It enables to compare one pair of alternatives (x, y) to another pair (z, w). There are various ways of interpreting the relation resulting from such a comparison. One reads $(x, y) \succsim^* (z, w)$ as "the preference difference between x and y is larger (or not smaller) than the preference difference between z and w". Another way of expressing the same idea is in terms of *strength of preference*: the preference of x to y is at least as strong as the preference of z to w.

In the comparison of the four cars discussed above, if we want to make sense when comparing preference differences or talking in terms of strength of preference, we need a value function, say v, that meets the following *two* requirements: for all alternatives $x, y, z, w \in X$,

$$x \succsim y \Leftrightarrow v(x) \geq v(y) \tag{4.14}$$

and

$$(x, y) \succsim^* (z, w) \Leftrightarrow v(x) - v(y) \geq v(z) - v(w). \tag{4.15}$$

Representations satisfying these two conditions were studied in the literature; conditions have been provided, in particular, for the existence (and uniqueness) of a value function v that satisfies (4.14) and (4.15) (see Krantz et al., 1971, ch. 4 and Fishburn, 1970, ch. 6).

In these models, the value function does not, in general, decompose additively. The conditions to be imposed on the pair of primitive relations $(\succsim, \succsim^*)$ so that they admit a representation as described above, with a function v that is also an additive value function, are of course more restrictive than those just guaranteeing a representation of the sole relation $\succsim$ with an additive value function. To be more specific, assume that the client not only has a preference on the set X of alternatives, but is also able to compare preference differences between all pairs of alternatives in X, yielding a relation on X^2. If the pair $(\succsim, \succsim^*)$ satisfies the axioms that guarantee the existence of an additive value function v representing $\succsim$, and at the same time guarantee that differences in the values of v can be used to represent $\succsim^*$ according with (4.15), then building v as proposed for instance in section 4.3.7 will also yield a representation of $\succsim^*$. On the contrary, it may happen, if $\succsim$ and $\succsim^*$ do not satisfy all the axioms of additive difference of preference measurement, that the preference $\succsim$ has a unique additive representation v according to (4.1) but that the relation $\succsim^*$ comparing preference differences cannot be represented in accordance with (4.15) by differences of this function v. In this case, no other function v could satisfy the latter condition on the representation of preference differences and at the same time, constitute an additive value model for $\succsim$ since the latter is unique. The additive model of preference differences is a very constrained one; axiomatic characterisations of this model have been obtained; the interested reader is referred to Dyer and Sarin (1979) and von Winterfeldt and Edwards (1986, chapter 9) on this issue.

4.3.10 Insufficiency of classical conjoint measurement

We now come back to the additive value model (4.1) and describe several examples showing that there are preferences that are both reasonable and do not satisfy the hypotheses for an additive representation.

4.3.10.1 Example 1: Flexible CSP

A solution to a Flexible Constraint Satisfaction Problem is assessed by a vector of n numbers that represent the degree to which each of the n constraints are satisfied; the degree of satisfaction is usually modelled as a number between 0 and 1. For instance, in certain scheduling problems (Dubois, Fargier, and Prade, 1995; Dubois and Fortemps, 1999), there may be an ideal range of time between the end of some tasks and the start of other; if more (or less) time elapses, the schedule is then less satisfactory; for each constraint of this type, the degree of satisfaction is equal to 1 if the corresponding slack time lies within the ideal range; it decreases outside this range; and outside a larger interval corresponding to the admissible delays between the end of a task and the beginning of another, the degree of satisfaction reaches 0. Usually, one considers that the scale on which the satisfaction degrees are assessed is ordinal (see chapter 3, section 3.4) and the same goes for all constraints: one may meaningfully compare degrees of satisfaction (saying for instance that one is higher than the other), but the difference between two degrees cannot be compared meaningfully to another difference; moreover, the degrees of satisfactions of two different constraints are commensurate: it is meaningful to say that a constraint is satisfied to a higher level than another one. A solution to such a scheduling problem is an assignment of a starting time to each task; comparing two solutions amounts to comparing their associated vectors of degrees of satisfaction. Usually in practice, a solution is evaluated using its weakest aspect, i.e. the lowest degree of satisfaction it attains on the set of constraints. Clearly, the relation comparing the vectors of satisfaction degrees can be viewed as a relation $\succsim$ on the product set $X = [0,1]^n$. In other words, vectors of satisfaction can be compared using the "min-score"; for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, where x_i and y_i respectively denote the degrees of satisfaction of constraint i for the two alternatives to be compared, we have:

$$x \succsim y \Leftrightarrow \min(x_1, \ldots, x_n) \geq \min(y_1, \ldots, y_n) \tag{4.16}$$

Clearly, the relation comparing the vectors of degrees of satisfaction can be viewed as a relation $\succsim$ on the product set $X = [0,1]^n$. It is defined by means of the "min"-score instead of an additive value function as in model (4.1). Of course, it may occur that a preference relation can be defined using several different scores and one can not exclude a priori that the relation defined by (4.16) could also be represented in model (4.1). This is however not the case, since this relation does not satisfy one of the necessary conditions stated above, namely the strong independence property: we can indeed transform an indifference into a strict preference by changing the common level of satisfaction that is achieved by two alternatives for the same constraint. This is shown with the following example. Suppose there are two constraints ($n = 2$) and $x = (0.6, 0.5)$, $y = (0.6, 0.7)$; one has $y \succ x$, but lowering to 0.3, for instance, the common satisfaction level yields $x' \sim y'$ (with $x' = (0.3, 0.5)$ and $y' = (0.3, 0.7)$). It should be clear from this example that there are simple and well-motivated procedures the additive value function model is not able to encompass.

4.3.10.2 Example 2: Non-transitive preferences

In the previous example, we described a procedure leading to a preference that lacks the strong independence property. The other necessary condition for model (4.1), namely transitivity, may also fail to be satisfied by some reasonable preferences.

Let us just recall R. D. Luce's famous example (Luce, 1956) of the sugar in the cup of coffee: a person who likes to drink coffee is indifferent between two cups of coffee that differ by the adjunction of one grain of sugar; he normally would not be indifferent between a cup with no sugar and a cup containing one thousand grains of sugar; according to whether he likes drinking sugared coffee or not, he would definitely prefer the latter or the former. A long sequence of indifferent alternatives may thus result in a preference, contrary to the hypothesis of the additive value model, in which preferences are weak orders, hence transitive[7].

4.3.10.3 Example 3: PROMETHEE II and the additive value function model

There are preferences that can be represented within the additive value function model but:

[7] For further discussion of the transitivity of preference issue, mainly in the context of decision under risk, the reader is referred to Fishburn (1991b). For counter-arguments against considering intransitive preferences, see (Luce, 2000, section 2.2).

- the model is more specific than (4.1): the u_i functions have special characteristics;
- using model (4.1) offers no clues for eliciting the u_i's and constructing the global preference.

Consider for example the PROMETHEE II method (Brans and Vincke, 1985). It is usually presented as a method that relies on pairwise comparisons of alternatives, as in the Condorcet method. It is nevertheless amenable to a representation within the additive value model.

PROMETHEE II starts by comparing alternatives, in a pairwise manner, with respect to each attribute i. Consider two alternatives x (resp. y) characterised by their description $(x_1, \ldots, x_i, \ldots, x_n)$ (resp. $(y_1, \ldots, y_i, \ldots, y_n)$) on the n attributes; we assume that x_i, y_i are numbers, usually obtained as evaluations $g_i(x)$, $g_i(y)$ of the alternatives on attribute i. The intensity $S_i(x, y)$ of the preference of x to y on attribute i is a nondecreasing function P_i of the difference $x_i - y_i$:

$$S_i(x, y) = P_i(x_i - y_i). \tag{4.17}$$

When the difference $x_i - y_i$ is negative, it is assumed that $S_i(x, y) = 0$: the intensity of the preference of x over y on attribute i is zero. The global intensity of the preference of x to y is described using a weighted sum of the S_i functions:

$$S(x, y) = \sum_{i=1}^{n} w_i S_i(x, y), \tag{4.18}$$

where w_i is the weight associated to attribute i.

One can view S as a valued relation assigning the value $S(x, y)$ to the pair (x, y) for all $x, y \in \mathcal{A}$. In a further step, the alternatives are evaluated using their score, computed as the "net flow" Φ at each node, i.e., for alternative x, $\Phi(x)$ is the difference of the sum of the values of all arcs emanating from x minus the sum of the values of all arcs entering x:

$$\Phi(x) = \sum_{y \in \mathcal{A}} S(x, y) - S(y, x). \tag{4.19}$$

This score is then used to determine that x is preferred to y if $\Phi(x) \geq \Phi(y)$. This is the customary presentation of PROMETHEE II (see, e.g., Vincke, 1992b, page 74).

By using equations (4.19), it is easy to rewrite $\Phi(x)$ as follows:

$$\Phi(x) = \sum_{i=1}^{n} w_i \sum_{y \in \mathcal{A}} [S_i(x, y) - S_i(y, x)]. \tag{4.20}$$

The latter formula can be seen as defining an additive value model in which the marginal value functions u_i have the following particular form:

$$u_i(x_i) = \sum_{y \in \mathcal{A}} [S_i(x, y) - S_i(y, x)]. \tag{4.21}$$

The computation of function u_i that models the influence of criterion i depends on the other alternatives (as in the Borda method; see section 4.2.2 and, below, section 5.2.1.1 for a discussion of a property called "(in)dependence of irrelevant alternatives"). Equation (4.21) suggests that the preference can be constructed through modelling the value of any echelon x_i as the sum of its "advantages" and "disadvantages" for each dimension , respectively coded by $S_i(x, y)$ and $S_i(y, x)$ (remember that $S_i(x, y) = 0$ whenever $y_i \geq x_i$). Model (4.1) makes no mention of intuitively interpretable concepts that would suggest that u_i could be viewed as a superposition (using a sum) of more elementary elements. The basic notion emerging from model (4.1) is the marginal preference $\succsim_i$ defined by (4.7); the basic hint provided by the model for building the preference, is that one may construct u_i as a numerical representation of $\succsim_i$. In section 6.6.2, it will be shown that the valued version of a family of models studied in section 6.4 offers better insight into the process of constructing a preference according to the PROMETHEE II model.

This example suggests that one of the virtues of a formal model could be pedantically called its "hermeneutic power", i.e. the fact that it facilitates the elicitation process; of course this power depends on the context of the problem situation, including the cultural and intellectual background of the client.

4.3.11 Conclusion

We hope to have shown:

- that the additive value function model is not appropriate for all possible evaluation problems;
- that one virtue of the models that provide a preference representation (i.e. models in which preference can be described using a condition of the type $x \succsim y \Leftrightarrow \ldots$) is to support the process of constructing of the preference by:
  - implying intuitively interpretable concepts (such as that of marginal preference),
  - establishing a link between these concepts and elements of the representation built in the model (such as the link between the marginal preferences and the marginal value functions u_i).

In chapter 6 we present more general conjoint measurement models (which provide more general representations of the preference); the models proposed all induce concepts—usually different from marginal preferences—that can support the construction or elicitation process.

4.4 General comment on the status of the parameters

Many aggregation methods require some parameters: weights, importance coefficients, indifference thresholds, concordance thresholds, veto thresholds, and so on.

These parameters are very important; they allow us to adapt or modulate to some extent a rigid mathematical model, taking the values or preferences of the client into account. Thanks to the parameters, we can hope that the outcome of the aggregation procedure will make sense to the client. Indeed, using the weighted sum, ELECTRE, PROMETHEE II, AHP, ... without weights (or all weights being equal) is bound to fail. There is very little chance that a client will trust the outcome of such an aggregation procedure because it is not faithful to his values, preferences or beliefs. It does not reflect his subjective perception of the situation. In this section, we will examine a few popular methods used to set parameters. We will then present a general approach that can help us set the parameters in a meaningful way. We will often speak of weights although not all parameters are weights. But almost everything we will say about weights can be transposed to other types of parameters.

4.4.1 Direct rating

In many applications, the analyst just asks the client to give numerical values to the weights. These numbers are then eventually normalised, in order for them to add up to one, and they are used in an aggregation procedure. What we would like to show now, is that such weights should probably not be used in an aggregation procedure. Not because the client gives wrong answers to the question raised by the analyst or because the weights given by the client are only approximations of the "true" weights, but because the analyst's question is very ambiguous. We know that weights (sometimes called importance coefficients) do not play the same role in different aggregation procedures (Bouyssou et al., 2000; Roy and Mousseau, 1996). Furthermore, in most aggregation procedures, the role of the weights is not well understood. So, how can we hope that the weights given by the client can be adapted to the aggregation procedure to be used thereafter?

Besides, even if we use an aggregation procedure in which the weights have a simple and well-understood role (say the weighted sum), the weights do not have any intrinsic numerical value in a given application. It is well known that the value of the weights (using the weighted sum) must depend on the units used for the different criteria. For example, if we already have the weights, if a dimension is measured in metres and we change it to centimetres, then the weight of that criterion needs to be divided by 100. Suppose now that we do not know the weights and that we ask the client for them. Will he give a weight 100 times smaller if we express the evaluations in centimetres instead of metres? Probably not.

Consider now an aggregation procedure where the units of measurement play no role: absolute majority. Each criterion has a weight and the weights add up to 1. An alternative a is globally at least as good as b if the total weight of the criteria with $a \succsim_i b$ is not smaller than 1/2. Suppose we decided to use three criteria in a given problem formulation and we ask the client for the weights he wishes to use. Feeling that criterion 1 is slightly more important than criterion 2 but much more important than criterion 3, he gives respective weights 0.45, 0.40 and 0.15. Note that no criterion is strong enough to attain the threshold 0.5 on its own. Note also that any coalition of two criteria is strong enough to attain the threshold. So, the three criteria play exactly the same role; the real weight or importance (game theorists speak of power index) of the three criteria are $1/3, 1/3$ and $1/3$. They have the same power despite the very different weights given by the client. When the number of criteria increases, computing the power of a criterion becomes very difficult and we cannot expect a client to assign weights such that the powers reflect his beliefs.

Besides, if we used simple majority—a is globally at least as good as b if the total weight of the criteria with $a \succsim_i b$ is not smaller than the total weight of the criteria with $b \succsim_i a$—the result would be very different, using the same weights. Criterion 1 would have more power than criterion 2 and criterion 2 would have more power than criterion 3. Yet the odds are small that a client assign different weights if we use simple instead of absolute majority.

In ELECTRE I, the weights are independent of the units of measurement, but are not independent of the other parameters of the methods (the various thresholds). We cannot expect that the client to assign weights that are consistent with the other parameters, the aggregation procedure and his preferences.

So, even if this has not been empirically proven, it seems extremely plausible that the weights spontaneously given by a client are not reliable, and it is not a matter of precision. This cannot be solved by a sensitivity analysis. The weights can differ by several orders of magnitude from weights that we would obtain through a sound procedure (an example of such a procedure is given in section 4.4.5).

4.4.2 Simos' cards method

In the method proposed by J. Simos (see Roy and Figueira, 2002; Simos, 1990), the client receives n cards; The name of one dimension is written on each one. The first task is for the client to rank these cards from the least to the most important criterion. Ties are allowed. The client can then insert one or more white cards between the previously ranked cards. The number of white cards between two criteria indicates the difference in importance between these criteria. The more white cards, the larger the difference. Simos then suggests a simple algorithm that computes weights based on the cards ranking given by the client.

In this method, the analyst does not ask the client for the numerical values of the weights but he nevertheless asks him to reason about the weights, to make statements about the weights. We are convinced that this does not make sense. This would make sense only if the client had even some vague or imprecise knowledge about the weights to be used with a particular procedure. It would help us to set precise numerical values without asking for these values. But even just asking the client to rank the criteria by importance is in fact too much. With the weighted sum, there is no such thing as an intrinsic ranking of the criteria: it depends on the units. Suppose we have two criteria and the performances on those criteria are expressed in metres and Euros. Suppose also we have found weights that perfectly reflect the client's preferences: 0.3 and 0.7. It seems, therefore, that the second criterion is more important than the first. If we now express the performances on the first dimension in kilometres, we must multiply the weight of the first criterion by 1 000. The weights are then 300 and 0.7. If we want to normalise them, we find 0.998 and 0.002. It now seems that the first criterion is much more important than the second . All this is of course spurious.

With ELECTRE, the value of the weights is not independent of the other parameters (concordance, discordance, indifference and preference thresholds) and the role of the weights is not completely clear. With some variants of ELECTRE (such as ELECTRE III) or when an exploitation procedure is used, the role of the weights can even become obscure (see also Bouyssou et al., 2000, ch. 6). We illustrate this with an example using a simplified version of ELECTRE, namely qualified majority. Suppose we have three criteria and we want to use qualified majority, i.e. absolute majority with a threshold possibly different to 1/2. The client thinks that criterion 1 is more important than criterion 2, which itself is more important than criterion 3. We can represent this using the weights 0.45, 0.40 and 0.15. If we use a threshold equal to 1/2, we have seen in section 4.4.1 that the three criteria have the same power, namely 1/3. But if we use a threshold of 0.6, we see that criterion 1 has more power than criteria 2 and 3 because the coalition of criteria 2 and 3 is not strong enough to attain the threshold, while the other coalitions of two criteria are strong enough. If we use a threshold of 0.7, we see that criteria 1 and 2 have the same power but more power than criterion 3 because the only strong enough coalition of two criteria is the coalition of 1 and 2. In these three examples, we see that although the client uses the same ranking of the criteria, we obtain three different rankings of the power of the criteria. And, if we were to use other weights such as 0.45, 0.30 and 0.25 (reflecting the same ranking of the criteria), we would obtain other orderings of the power of the criteria. The client can certainly not anticipate this and, so, we cannot expect him to provide us with the correct ranking of the criteria. Note that if we consider the possibility of using other kinds of majorities (like simple majority), the situation worsens.

4.4.3 Ranking the criteria

The goal of the designers of MELCHIOR (Leclercq, 1984), ORESTE (Roubens, 1982) and QUALIFLEX (Paelinck, 1978) was to avoid asking the client for numerical weights because it is too difficult. So, they decided to just ask for a ranking of the criteria, from the most to the least important. But, in fact, this is not easier.

No method is used for eliciting this ranking. The analyst just asks the client to provide a ranking of the criteria. For the reasons presented in the previous section on Simos' method, we think that a ranking of the criteria given by a client is no more reliable than numerical values, because, we insist, it is not a problem of precision. Several methods can be thought of that only use a ranking of the criteria but that lead to different results. So, the meaning of the relation "more important than" can vary from one aggregation procedure to another. It has no meaning *per se* and, even if the client has the impression that he understands it, we can never be sure that his concept of the relation "more important than" coincides with the one to be used with a particular aggregation method. So, if no *absolute* ranking exists, how can we expect the client to provide us with the right one, adapted to his problem, to the aggregation procedure to be used, to the units and scales of the different criteria and to the other parameters used in the chosen method?

4.4.4 Analytic Hierarchy Process (AHP)

The technique used in AHP (Saaty, 1980), to set the value of the weights is very sophisticated (see section 4.5.1). It also avoids asking the client for numerical values, but it fails for the same reason as the previous methods. It asks the client to compare the importance of the criteria. But the concept of importance, even in its relative form (more important than), is so ill-defined that the answers given by the client and used with a particular aggregation procedure cannot reliably reflect his value system. See Belton and Gear (1983) and Dyer (1990).

4.4.5 A classical technique in MAVT

In this subsection, we present an interesting technique, which is classical in MAVT, but that can easily be adapted to other parameters in other aggregation procedures. The reader should therefore not understand this subsection as an argument in favour of MAVT and against other methods. Instead, it should be a source of inspiration for a sound elicitation of parameters using other methods (this will be developed in the next section).

Suppose a client and an analyst have decided to use an additive model, i.e., given the performances $g_1(a), \ldots, g_n(a)$ and $g_1(b), \ldots, g_n(b)$ of two alternatives a and b, they will consider a, globally at least as good as b if and only if

$$u(a) = \sum_{i=1}^{n} k_i v_i(g_i(a)) \geq \sum_{i=1}^{n} k_i v_i(g_i(b)) = u(b), \qquad (4.22)$$

where k_i is the weight associated to dimension i and v_i is the value function corresponding to dimension i (note the similarity with equation (4.5); this will be discussed at the end of this subsection, on p.153). Two kinds of parameters thus need be determined: the marginal value functions and the weights. Suppose they used the *midvalue splitting technique* (see Keeney and Raiffa, 1976, section 3.4.7) to elicit the n value functions. We will not say much about this technique (though it is quite interesting), but we will focus on the next step—the elicitation of the weights or scaling constants—because these parameters are in some way comparable to those discussed in the previous sections.

After the midvalue splitting technique, the client has n value functions such that $v_i(\underline{g}_i) = 0$ and $v_i(\overline{g}_i) = 1$, where $\underline{g}_i$ is the worst performance and $\overline{g}_i$ is the best performance on dimension i. The range of each value function is thus $[0, 1]$. If we then want to additively combine these values, we must use some weights, as in (4.22). Indeed, even if the numerical difference in value between $\underline{g}_i$ and $\overline{g}_i$ is the same as between $\underline{g}_j$ and $\overline{g}_j$ (it is equal to one), these differences may represent very different things for the client. The difference between $\underline{g}_i$ and $\overline{g}_i$ is perhaps perceived as much bigger than the difference between $\underline{g}_j$ and $\overline{g}_j$. So, we need weights or scaling constants or substitution rates to make these differences comparable.

Keeney and Raiffa (1976) suggest to use the following technique. For the sake of clarity, suppose there are only three dimensions. Present the following three fictitious alternatives to the client: $a = (\overline{g}_1, \underline{g}_2, \underline{g}_3)$, $b = (\underline{g}_1, \overline{g}_2, \underline{g}_3)$ and $c = (\underline{g}_1, \underline{g}_2, \overline{g}_3)$ where (x, y, z) stands for an alternative with performances x, y and z on criteria 1, 2 and 3. Then ask the client to rank them from best to worst. Suppose his answer is $b \succ c \succ a$. From this, we can conclude that

$$u(b) > u(c) > u(a)$$

or, using (4.22),

$$k_2 > k_3 > k_1.$$

We present then the following pair of fictitious alternatives to the client: $d = (\underline{g}_1, \underline{g}_2, \overline{g}_3)$ and $e = (\underline{g}_1, x_2, \underline{g}_3)$ where x_2 stands for an unspecified performance on criterion 2, with the constraint that $\underline{g}_2 \leq x_2 \leq \overline{g}_2$. The client must then say for which value of x_2 he is indifferent between d and e. We then again write (4.22):

$$u(d) = k_1 0 + k_2 0 + k_3 1 k_1 0 + k_2 v_2(x_2) + k_3 0 = u(e),$$

that is, $k_3 = k_2 v_2(x_2)$. Because the value functions v_i have been previously determined by the midvalue splitting technique, we know the value of $v_2(x_2)$. Let us call it μ. So, $k_3 = k_2\mu$. This means that a difference of one unit on criterion 3 is worth a difference of μ units on criterion 2.

If we now present the pair $d' = (\overline{g}_1, \underline{g}_2, \underline{g}_3)$ and $e' = (\underline{g}_1, x_2', \underline{g}_3)$ where x_2' stands for an unspecified performance on criterion 2, we can find a value ν such that $k_1 = k_2\nu$ in a similar way. This means that one unit of value on criterion 1 is worth ν units on criterion 2. We can then choose any value for k_2 and the other weights are automatically determined by the relations $k_3 = k_2\mu$ and $k_1 = k_2\nu$. In particular, we can arrange to have $\sum_{i=1}^{n} k_i = 1$ but this is not necessary. If there are more criteria, the same technique can be generalised.

The interest of this technique is that all questions we ask to the client are formulated in his language and are directly related to his problem, not to a model: we only ask him to compare alternatives. We never ask for the value of a parameter. We do not even mention parameters. Besides, we are sure that the parameters we obtain are to some extent (we do not believe in a perfect model) compatible with the client's preferences: if we use (4.22) with the obtained parameters and the alternatives a, b, c, d, e, d' or e', we necessarily obtain the preferences that were previously stated by the client. If we do the same with other alternatives, we are no longer sure that the obtained preferences will coincide with those of the client. It will only be so if the chosen aggregation method is well-adapted to the client's preferences. This is not due to an incorrect choice of the weights.

But if we use (4.22) with the parameters elicited through a direct rating procedure, we are not even certain that the resulting preferences will coincide with the client's preference for even one pair of alternatives.

In section 4.3, we presented the additive model (4.1) and showed that it had the equivalent alternative formulation (4.5). The latter form corresponds exactly to model (4.22) above, provided we identify $g_i(a)$ and x_i. Despite the fact that models (4.1) and (4.5) are equivalent, their different formulations suggest different elicitation procedures for their parameters. This is why we presented another elicitation procedure in section 4.3. Ideally, both procedures should lead to the same result, i.e. to the same preference relation.

4.4.6 General approach

The technique used in the previous section to elicit the substitution rates or weights can be adapted to many different parameters and aggregation procedures. For example, Mousseau, Figueira, and Naux (2001) and Mousseau and Dias (2004) have proposed a similar technique for eliciting the parameters of a variant of ELECTRE I devoted to the problem of sorting. The same reasoning also lies at the heart of the aggregation-disaggregation approach (see Jacquet-Lagrèze and Siskos, 2001, which is the editorial of a special issue on preference disaggregation) and in particular the UTA method implemented in PREFCALC (see Jacquet-Lagrèze and Siskos, 1982 and also, to some extent Mousseau et al., 2003). Köksalan and Ulu (2003) use this approach for setting the parameters of a linear utility model in a sorting problem. It is also possible to elicit preference thresholds (for example in ELECTRE) by asking the client to compare some pairs of alternatives instead of asking for the threshold directly, this whether one criterion or several are considered (see chapter 3).

For any aggregation procedure involving parameters, if we present a pair of alternatives to the client and if he tells us which one he prefers, we can always draw some conclusion about the parameters (all of them, not just the weights) of the aggregation method we want to use. This conclusion is generally under the form of a constraint. When we repeat this process and present more and more pairs of alternatives, we obtain more and more constraints. By combining these constraints, we can eventually isolate a set (hopefully not too large) containing the suitable parameters. Note that the representation theorems obtained in the framework of measurement theory can help us determine the questions we must ask to arrive at unique parameters or, more realistically, at a small set of parameters as fast as possible (i.e. with a minimal number of questions).

This can sometimes be difficult. It may require the use of complex algorithms. And these algorithms are yet to be developed for some aggregation procedures. But we are convinced that it is the best way to arrive at parameters that make sense with respect to a particular aggregation procedure.

The difficulty is not only computational. Sometimes, the client will be able to compare only a few pairs of alternatives. He will be undecided about the other pairs. So, we might have too few constraints and not be able to set, even approximately, the value of the parameters.

Sometimes, we will need to ask hundreds of questions in order to have enough constraints, but this will not be possible due to lack of time. We will therefore need to cope with poor information. Some hypotheses will eventually help narrow the set of possible parameters. We could, for example, invoke Laplace's principle of insufficient reason or, more generally, any means for setting default parameter values. A sensitivity analysis or a robustness approach (see chapter 7, section 7.5) might prove helpful in these cases.

Sometimes, in order to gain more information with less questions, we might be tempted to present pairs of fictitious alternatives, with a particular structure (as in the previous section). But the client must then compare alternatives that he does not know and, possibly, alternatives that are not realistic. So, his answers become less reliable. We must therefore balance the need for specially structured alternatives that bring a lot of information and the need for reliable answers.

In the following two chapters, we discuss many aggregation procedures involving parameters. For some of them, we mention a technique that can be used for eliciting these parameters, which always follows the approach presented in this section. For the other aggregation procedures (not presented in this book or for which we do not present an elicitation technique), the same approach can and should always be used. It does not yield the "right" or "correct" parameters (which we cannot define) but it guarantees parameters that make sense.

4.4.7 Laplace's principle of insufficient reason and other principles

Suppose we are in the middle of a decision aiding process. We have been through the formulation phase, we have constructed an evaluation model (see chapter 2, section 2.3.3) and we have decided to use a particular aggregation method involving weights, but we have no idea what the weights should be. It is then tempting to invoke Laplace's principle of insufficient reason and take all weights as being equal. In a constructive approach (see chapter 2, p. 26), we can justify this by saying that the client's preferences do not exist a priori, that they are constructed during the decision aiding process, through the interaction, the discourse between the client and the analyst. So, if the client and the analyst agree on the relevance of Laplace's principle, then it is fine.

But we must not forget that in many cases, the client has some a priori preferences. They are of course incomplete and some are not stable (they might change during the decision aiding process). But often, there are probably some elements of preference that exist and that are stable. If we do not respect these preferences, we should wonder what the client's role is in such a process and we should not be surprised if the client does not accept our recommendations. So, in a constructive approach, we have to build the preferences around some elements that already exist.

Let us come back to our example. Taking all weights as being equal amounts precisely to not take the client's preferences into account. This will yield a preference relation that is completely (and in some sense artificially) constructed. By chance, it might contain the pairs that existed in the preferences of the client. The client might then accept the outcome of the aggregation and the recommendation based on it. But the outcome can also contain none or only few of the pairs that existed in the preferences of the client. There is then a risk that he accepts a recommendation based on a constructed preference relation that has nothing to do with *his* preferences.

We therefore think that the use of Laplace's principle is not to be recommended in our example and in many other cases. If we do not know which weights to use, we may not guess or toss a coin. Our duty, as analysts, is to take one of the following two routes:

- to honestly acknowledge our ignorance and use only the available information. For example, to use the dominance relation or a robustness approach (see chapter 7, section 7.5).
- to work harder and search for the information that can help us set the weights. There are techniques for that. We presented some of them in connection with some aggregation procedures. We also presented a general approach (section 4.4) that can be used with any aggregation procedure.

This discussion is not limited to the use of Laplace's principle for setting weights. It holds for all parameters (thresholds, importance of criteria, value functions, ...). It also holds for other, somehow related, principles that can help set parameters, like Jaynes' maximum entropy or minimum information principle[8], Wald's maximin criterion or Savage's minimax regret criterion[9] (see Luce and Raiffa, 1957). This also holds for the principle lying at the heart of the Regime method (Hinloopen, Nijkkamp, and Rietveld, 1983). In this method, when the performances of the alternatives on a criterion are ordinal, it is assumed that they can be represented by a value function but, because this value function is unknown, all value functions compatible with the ordering of the performances are used in the computation of some indexes. Yet, if the client has some preferences—and we believe that this is often the case—there are value functions that better represent his preferences than others and there are techniques to construct such functions.

Now, let us be more pragmatic. We know, that in almost all decision aiding processes, it will not be possible to find a unique value for the weights or the other parameters, even if we use the best techniques and devote a lot of time: these techniques will probably give us intervals for the weights or at least narrow the range of the possible values. What can we do then to set the weights within the limits of our techniques? Toss a coin? This is hardly recommendable. The best solution is probably to try to find robust alternatives, i.e. alternatives that are good, even if not the best, under all possible scenarios (see chapter 7, section 7.5). But this might take too long or be difficult in some circumstances. We could

[8] See Jaynes (2003). Fine (1973, chapter 6) shows that the use of information-theoretic principles like maximum entropy is an attempt to "enlarge the domain of classical probability to include unequal probability assignments"; it thus clearly shows the filiation with Laplace.

[9] These principles are mainly concerned, at least originally, with the assessment of probabilities or statistical decision theory; they have links with Bayesian statistics; the minimal specificity principle (Benferhat, Dubois, and Prade, 1997) used in the possibilistic approach to default or nonmonotonic reasoning, is a similar idea proposed in the field of artificial intelligence.

then invoke Laplace's principle or another similar principle. But that is precisely what we strongly criticised in the previous paragraphs! What is the difference between the situation here and the situation in the first paragraph of this section? Unfortunately, there is no fundamental difference, it is only a matter of degree. In both situations, we try to use a principle as a remedy to our ignorance. But it is our conviction that we should do this only after we have used all available techniques that are feasible in the decision aiding context.

Let us finally mention a situation where Laplace's principle of insufficient reason or another similar principle might be used to set the weights or some other parameters. Suppose we have decided to use an aggregation procedure involving weights but we do not know these weights. We therefore ask the client to compare some alternatives, according to the general strategy presented in section 4.4.6, in order to find constraints on the weights. Unfortunately, the client finds the task difficult and can compare almost no alternatives, so we have almost no constraints. We might then decide to arbitrarily choose some weights (according to Laplace's principle or to the throw of a dice), to use the aggregation procedure with these weights and to present the resulting ranking to the client. The ranking should be—unless we are extremely lucky—very different from what the client expects, at least for some pairs. It can therefore be used as a provocation, as a support for an interaction or a dialogue between the analyst and the client. It should force the client to react and say, for example, 'it is not possible that a is better than b; I am sure b is better than a.' So, it can help us find constraints on the weights.

In such a case, the weights obtained by the application of Laplace's principle are in no way meant to be sensible, correct or even approximately correct weights. We use them only for their maieutic virtues.

Note that a situation where the client does not answer any of our questions or where he answers only a few of them is problematic; not only for the determination of the parameters but also for the whole decision aiding process. The client must use some of his resources (time, money, ...) to interact with the analyst and he must commit himself. Otherwise, we can hardly speak of a decision aiding process.

4.5 I am using the *XYZ* method. Which results are useful for me?

In this section, we have selected a few popular aggregation methods or models of preference in MCDA: AHP, ELECTRE I, ELECTRE III, MAVT, PROMETHEE and TACTIC (in alphabetical order). We will list the relevant sections of chapters 4–6 for each of them. We also give a short presentation and a few references to important publications for each one. The reader interested in a more extensive presentation of the different methods is referred to Belton and Stewart (2001) and Vincke (1992b). There are of course, many other interesting methods but we do not mention them here because chapters 5 and 6 do not contain any material pertaining to them or because they are not popular.

4.5.1 AHP

References: Saaty (1980) and Harker and Vargas (1987)

4.5.1.1 The method

The Analytic Hierarchy Process (AHP) is a method for building an evaluation model. Its main characteristics are the following:

- the evaluation model is structured in a hierarchical way;
- the same assessment technique is used at each node of the hierarchy;
- the assessment of the "children" nodes of a common "parent" node is based on pairwise comparisons.

In the simplest case, the hierarchy has three levels. The node at the top level represents the client's global objective and is analysed as resulting from the aggregation of n dimensions (or criteria) represented by the second level nodes; each dimension is split into as many nodes as there are alternatives (which are represented as bottom nodes and duplicated as many times as there are dimensions). In more complex cases, there may be more levels, corresponding to splitting dimensions in sub-dimensions.

The assessment technique, used at each node (except for the bottom nodes), assigns a weight or score to each of the "children" nodes of a "parent" node. For instance, the procedure for assessing the n dimensions in terms of their contribution to the client's global objective runs as follows:

- the client is asked to compare the dimensions (or criteria)
  - in a pairwise manner,
  - in terms of their relative importance,
  - using a conventional "semantic" scale with five levels (these levels being labelled "equally important", "weakly more important", "strongly more important", "very strongly more important", "extremely more important") with possibly 4 intermediate levels;
- the qualitative assessments made by the client are given a quantitative interpretation; the five levels of the semantic scale are respectively coded as $1, 3, 5, 7$ and 9; this process results in a $n \times n$ pairwise comparison matrix; for instance, when the client considers that dimension i is "weakly more important" than dimension j, 3 is written in row i column j of the matrix and $1/3$ in row j column i;
- from the (numerically coded) pairwise comparison matrix, one computes a score or weight w_i attached to each dimension i; the scores are computed as the eigenvector corresponding to the maximum eigenvalue of the matrix and are normalised to add up to 1.

The reasons for applying such a procedure are complex. Briefly, the judgments made by the client when comparing two dimensions i and j are assumed to be strongly related to the ratio w_i/w_j of the scores of the corresponding dimensions, as they are computed by the procedure.

The same procedure (differing only in the labelling of the levels on the semantic scale), is applied to compare the alternatives on each dimension. This yields a score $\alpha_i(x)$ attached to each alternative x on each dimension i.

The global score of each alternative w.r.t. the global objective is finally computed as:

$$\sum_{i=1}^{n} w_i \alpha_i(x)$$

and the alternatives are ranked accordingly.

4.5.1.2 Some pointers

A very distinctive feature of AHP is the derivation of the value of each alternative on each dimension by means of pairwise comparisons and the eigenvector technique. But a very standard aggregation procedure lies at the heart of AHP: the weighted sum. The values obtained through the eigenvalue technique are aggregated by nothing else than a weighted sum. Section 5.4.4 about the weighted sum thus gives us some insight into AHP and section 5.4.4.3 is particularly relevant since it presents two meaningful techniques for the elicitation of weights to be used in a weighted sum. Other techniques exist but are variants of these two (e.g. von Winterfeldt and Edwards, 1986, ch. 7). The technique generally used with AHP for the elicitation of the weights is not a variant of the techniques we present. It yields weights that are independent of the aggregation procedure (the weighted sum) and of the values of the alternatives on the dimensions. For reasons presented in section 4.4, it should therefore never be used in the way advocated in the orthodox AHP method. See also (Belton, 1986; Belton and Gear, 1983; Bouyssou et al., 2000; Dyer, 1990, ch. 6).

Although some proponents of AHP sustain that AHP is not a value function method, the theory of additive value functions (MAVT) has at least some relevance w.r.t. AHP. It cannot be doubted that the preference $\succsim$ induced by the global score yielded by AHP satisfies the hypotheses of the additive value model since alternatives, say x and y are ranked according to the rule:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} w_i \alpha_i(x) \geq \sum_{i=1}^{n} w_i \alpha_i(y).$$

Comparing this expression with model (4.1), p. 129, one concludes that $w_i \alpha_i(x)$ can play the role of the marginal value function $u_i(x_i)$, when AHP is applied to alternatives described by a performance vector. Can we infer from this that $w_i \alpha_i(x)$ are the marginal value functions? In some cases we do! In section 4.3.4, we discussed uniqueness issues related to the representation in the additive value model. If the conditions for uniqueness are fulfilled, there is only one representation of the client's preference in the additive model, up to positive affine transformations of the u_i's. In other words, if AHP gives us an additive representation of the preference, it is the only one and $u_i(x_i) = w_i \alpha_i(x)$. So, in case of uniqueness of the additive representation, AHP should be considered as a method for eliciting an additive value function and could (should) be compared to the other elicitation methods on empirical grounds. We are not aware of empirical tests on this particular issue. But in view of the considerable experience accumulated on the elicitation of the additive model, it can hardly be doubted that the elicitation of an additive model using AHP would be biased. Indeed, with AHP, the questions asked for eliciting, e.g., the "weights" do not refer to the scales of the associated dimensions or to the additive model that will be used for the aggregation. We therefore have the same problem as with the direct rating technique, discussed in section 4.4.1, p. 148.

4.5.2 ELECTRE I

References: Roy (1968, 1971) and Maystre, Pictet, and Simos (1994)

4.5.2.1 The method

ELECTRE I is aimed at the aggregation of a performance table into a choice set. It is often presented as a three-step procedure (preference modelling, aggregation and exploitation) although the first step is almost trivial. But using this presentation in three steps allows a unified presentation of ELECTRE I and ELECTRE III. Several variants of ELECTRE I have been proposed and, because the original version of ELECTRE I is almost never used, we present the most common variant here.

Preference modelling We define two binary relations S_i and V_i for each dimension by

$$a\ S_i\ b \Leftrightarrow g_i(a) \geq g_i(b)$$

and

$$a\ V_i\ b \Leftrightarrow g_i(a) > g_i(b) + \tau_i,$$

where τ_i is positive[10]. The first relation simply expresses the fact that a is at least as good as b on a given dimension. The second one expresses the fact that a is much better than b on a given dimension, because the difference between their performances exceeds a threshold that the client considers as very large.

Aggregation A binary relation, called outranking relation, is constructed on A. We will consider that a outranks b iff the coalition of criteria such that a is better than b is sufficiently large and if b is not much better than a on a dimension. In order to define large coalitions of criteria, an importance coefficient w_i is associated to each criterion and the large coalitions are those

[10] In section 5.4.6, the notation for the thresholds is more cumbersome because we consider the case where there are several thresholds per criterion.

for which the sum of the importance coefficients is larger than a threshold c, called concordance threshold. Formally, the outranking relation depends on performance table $\mathbf{g}$ but also on the n-tuples $w = (w_1, \ldots, w_n)$ and $\bar{\tau} = (\tau_1, \ldots, \tau_n)$ and on c. So, we write

$$a \succsim(\mathbf{g}, w, \bar{\tau}, c)\ b \Leftrightarrow \sum_{i:aS_ib} w_i \geq c \text{ and } \nexists i : b\ V_i\ a.$$

Exploitation The outranking relation built during the aggregation is usually not a weak order. It is therefore not easy to see which alternatives are the best ones and a complementary analysis is often necessary. The author of the method recommends using the kernel (see section 7.4.3.1, p. 367) of the relation $\succsim$. The kernel $\mathcal{K}$ is a subset of alternatives such that

- each alternative not in the kernel is outranked by at least one alternative in the kernel and
- no alternative in the kernel outranks any other alternative in the kernel.

In other words, $\forall b \in A \setminus \mathcal{K}, \exists a \in \mathcal{K} : a \succsim(\mathbf{g}, w, \bar{\tau}, c)\ b$ and $\forall b \in \mathcal{K}, \nexists a \in \mathcal{K} : a \succsim(\mathbf{g}, w, \bar{\tau}, c)\ b$. The kernel is not necessarily unique and does not always exist. In such cases, several solutions have been proposed in the literature. The kernel will not necessarily contain the best alternatives but a set of promising alternatives that must be further analysed.

4.5.2.2 Some pointers

Until now, ELECTRE I, with its concordance thresholds, discordance thresholds and weights, has not been characterised as an aggregation procedure but different special cases (simplified versions) have been. The concordance relation in ELECTRE I is nothing but a kind of majority relation. It is therefore not surprising that relevant results can be found in section 5.2.3 about the qualified majority. These results do not take weights and vetoes into account.

Simple majority is another kind of majority and its weighted variant is characterised. Due to the similarity between qualified and simple majority, the analysis of weighted simple majority can be fruitful for our understanding of ELECTRE I. In particular, we present a technique that can be used for setting the weights in section 5.2.2. It can easily be adapted for using it with ELECTRE I.

One last result inspired by Social Choice Theory and relevant for understanding ELECTRE I is theorem 5.13 in section 5.4.6. It does not take weights into account, even though not all coalitions are assumed to be equally important. Contrary to the other results related to ELECTRE I, these two address the aggregation of performances and not of preference relations.

Other kinds of results have been found in the framework of conjoint measurement (see section 6.5). In ELECTRE I, we add the weights of the criteria supporting an alternative against another one. If we replace the sum of the weights by a more general or abstract operation, we obtain a general model (or a family of models) that is characterised. This general model contains ELECTRE I as a special case (see sections 6.3.5.3, 6.3.4, 6.4.3). The analysis of this model thus tells us a lot about ELECTRE I, in particular that the traces on differences (see (6.28), p. 275) are a fundamental elicitation tool. A slightly more general version helps to clarify the very special way in which vetoes intervene in the global preference relation; this issue is addressed in section 6.3.6.

4.5.3 ELECTRE III

References: Roy (1978), Vincke (1992b) and Roy and Bouyssou (1993, ch. 5 and 6)

4.5.3.1 The method

ELECTRE III is aimed at the aggregation of a performance table into a ranking (partial weak order). The main difference with respect to ELECTRE I lies in the preference modelling and the exploitation. With ELECTRE I, we say that a is preferred to b with respect to dimension i as soon as the performance of a is at least as good as that of b on dimension i. The change from non-preference to preference is therefore very discontinuous. Here, we will try to make the change more continuous.

Preference modelling We define two valued binary relations S_i and V_i for each dimension by

$$S_i(a,b) = \begin{cases} 1 & \text{if} \quad g_i(a) + \tau_{i,1} \geq g_i(b) \\ 0 & \text{if} \quad g_i(a) + \tau_{i,2} \leq g_i(b) \\ \text{linear in between} \end{cases}$$

and

$$V_i(a,b) = \begin{cases} 0 & \text{if} \quad g_i(a) \leq g_i(b) + \tau_{i,2} \\ 1 & \text{if} \quad g_i(a) \geq g_i(b) + \tau_{i,3} \\ \text{linear in between} \end{cases}$$

where $\tau_{i,1} < \tau_{i,2} < \tau_{i,3}$ are positive. The first relation simply expresses the fact that a is at least as good as b on a given dimension. The second one expresses the fact that a is much better than b on a given dimension.

Aggregation A binary relation S, called outranking relation, is constructed on A. With ELECTRE III, the outranking relation is valued between 0 en 1. In order to build S, we first compute a concordance index CI, for each pair, by

$$CI(a,b) = \sum_{i=1}^{n} w_i S_i(a,b).$$

Then,

$$S(a,b) = \begin{cases} CI(a,b) & \text{if } V_i(b,a) \leq CI(a,b),\ \forall i \\ CI(a,b) \prod_{i:V_i(b,a) > CI(a,b)} \frac{1-V_i(b,a)}{1-CI(a,b)} & \text{otherwise} \end{cases}$$

Exploitation The outranking relation constructed during the aggregation is valued and, furthermore, often lacks nice properties. It is therefore not easy to see which alternatives are the best ones and an exploitation is often necessary. The author of the method recommends to use the so-called distillation procedure. We do not present it here, because chapters 5 and 6 contain no result pertaining to the exploitation procedure of ELECTRE III.

4.5.3.2 Some pointers

ELECTRE III builds a valued relation based on a concordance-discordance principle then exploits this relation in view of producing rankings. Only the part leading to a valued relation is analysed in chapters 5 and 6. A version without discordance is described as a generalisation of the Condorcet method (section 5.3.3). Conjoint measurement models of valued preferences (section 6.6) offer a framework in which the valued relation produced by ELECTRE III can be fully analysed; some of the main features of ELECTRE III are emphasised by the model, namely the construction of the relation based on the modelling of preference differences. However, no characterisation of the specific ELECTRE III valued relations is provided and it would probably be very difficult to find one.

4.5.4 MAVT

References: Fishburn (1970), Keeney and Raiffa (1976), Wakker (1991b) and von Winterfeldt and Edwards (1986)

4.5.4.1 The method

MultiAttribute Value Theory (MAVT)—also called MultiAttribute Utility Theory (MAUT), but this terminology is better suited to decision under risk (not covered in this volume)—is not the theory of an aggregation procedure, contrary to all the other items in this section. MAVT studies a collection of models of preference relations. Once it has been recognised (or assumed) that the client's preference can be represented in such a model, MAVT usually indicates strategies or procedures for eliciting the model, hence the preference.

MAVT deals with preference relations $\succsim$ that can be represented by a value function u in the following way:

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Such preferences are thus necessarily weak orders (rankings, possibly with ties). The particular form of u that has received the most attention is the additive model. Each alternative x is assumed to be completely described by a performance vector $x = (x_1, x_2, \dots, x_n)$ and each vector, provided its coordinates vary within a specified range, is assigned a value $u(x) = \sum_{i=1}^{n} u_i(x_i)$, that decomposes additively along the n dimensions. Systems of conditions on the preference $\succsim$ are known, guaranteeing that such a preference can be represented in the additive model, i.e. satisfies

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i),$$

for some specification of the u_i functions. Only preferences that are independent weak orders can be represented in this way (additional restrictions on the preference are required). The form of the model suggests a strategy for eliciting the client's preference through the elicitation of the u_i functions; many procedures for doing this have been proposed in the literature (see, e.g., section 4.3.7).

4.5.4.2 Some pointers

It is not easy to approach multiattribute value (or utility) models using the characterisation of aggregation procedures. The characterisation of the weighted sum in section 5.4.4 is the closest in this chapter; what we miss is the possibility of recoding the evaluations using marginal values that reflect single-attribute preferences.

Contrarily, MAVT is at the heart of conjoint measurement theory; in which it has had many different full characterisations. The most relevant section of this chapter is section 4.3. Section 6.1 of chapter 6 is devoted to a relatively detailed presentation of the additive value model. Elicitation issues—mainly through using standard sequences—are dealt with in sections 4.3.7, 6.1.2, 6.1.2.2; section 6.1.2.3 opens to other elicitation methods.

4.5.5 PROMETHEE II

References: Brans and Vincke (1985), Brans and Mareschal (2002)

4.5.5.1 The method

PROMETHEE II is aimed at the aggregation of a performance table into a weak order and is often presented, as ELECTRE III, as a three-step procedure. In the first step, a valued preference relation is built for each criterion. In the second one, these valued relations are aggregated into one global preference relation. In the last step, the global preference relation is exploited using a net flow procedure in order to obtain a weak order. We hereunder detail the three steps.

Preference modelling The first step for the client is to choose a *preference function* P_i for each dimension i. A preference function P_i is a non decreasing function from $\mathbb{R}$ into $[0, 1]$ and such that $P_i(z) = 0$ for all $z < 0$. Six such functions are proposed in the software PROMCALC (see figure 4.4). A fuzzy relation S_i is constructed for each dimension on the basis of these preference functions. It is defined by

$$S_i(a, b) = P_i(g_i(a) - g_i(b)), \tag{4.23}$$

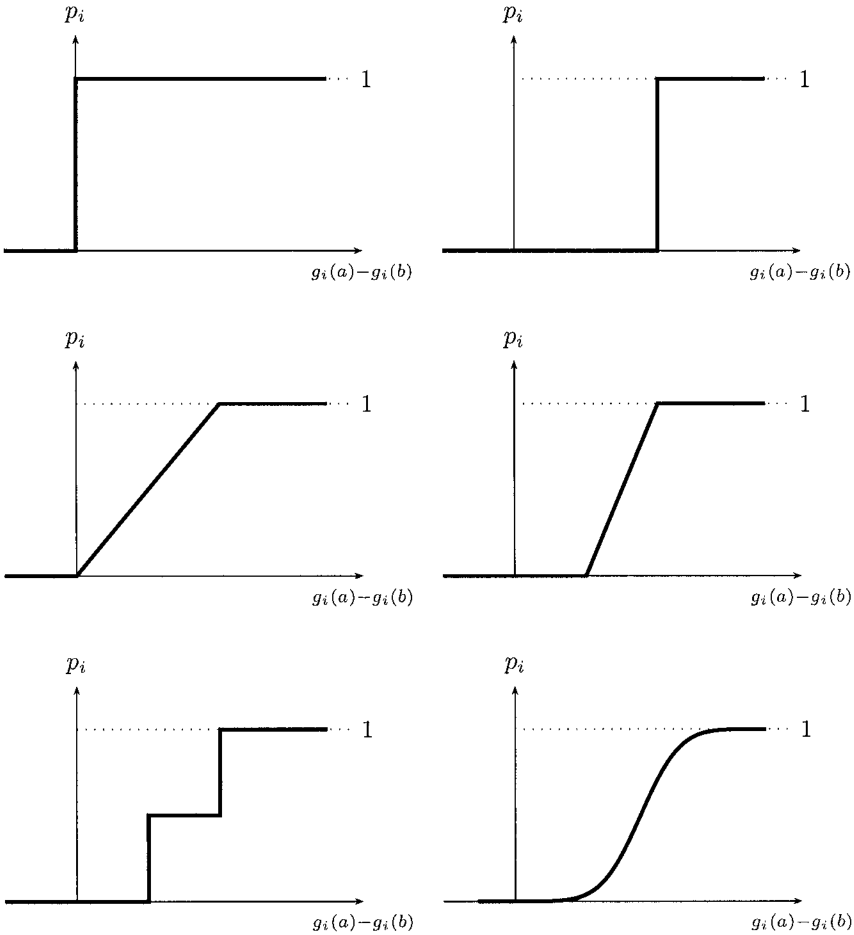


Figure 4.4: The six preference functions proposed in PROMCALC.

where $g_i(a)$ is the performance of alternative a on dimension i. The value $S_i(a,b)$ measures, in some sense, the intensity of the preference for a over b with respect to dimension i.

Aggregation In the second step, the profile of fuzzy relations obtained at the end of step one, is aggregated into one fuzzy preference relation. The aggregation is performed simply by computing a weighted average for each pair of alternatives. Formally, for every a and b in A, we define

$$S(a,b) = \sum_{i=1}^{n} w_i S_i(a,b),$$

where w_i is the weight of criterion i and the sum of the weights is 1. In some sense, the value $S(a,b)$ measures the global intensity of the preference for a over b taking all criteria into account.

Exploitation The fuzzy relation obtained at the end of step 2 is often very difficult to interpret. An exploitation is therefore often necessary and is performed using the net flow. For each alternative we compute

$$\Phi_i(a) = \sum_{b \in A} S(a,b) - \sum_{b \in A} S(b,a). \tag{4.24}$$

The net flow of alternative a, $\Phi_i(a)$, is the sum of the valuations on all arcs leaving a minus the sum on all arcs entering a. The alternatives are then ranked in decreasing order of their net flows. Other exploitation techniques are also proposed by the authors of the method.

4.5.5.2 Some pointers

The last step is analysed in section 7.4.3.2 of chapter 7. If we consider the last two steps as one procedure for the aggregation of valued relations into one weak order, then we also have interesting results in section 5.3.2.

It is interesting to note that, if we consider all three steps together, then PROMETHEE II can be put into a very simple form, which is a particular case of the additive value function model (see section 4.3.10.3). So, it can be described in the framework of conjoint measurement theory. Unfortunately, the additive value function model is much more general, so its analysis doesn't tell us much about PROMETHEE II.

The first two steps, leading to a valued relation, can also be analysed within conjoint measurement; the resulting valued relation can be described in the framework of models $L-D$ (section 6.6.2). Although only describing part of the method, this model is perhaps more in phase with the "philosophy" of PROMETHEE II that, classified within the outranking methods, is based on pairwise comparisons and is close in spirit to ELECTRE III.

Let us finally mention a result in Myerson (1995) that characterises an aggregation procedure in the framework of Social Choice Theory, which is very general but not so far from the three steps of PROMETHEE II.

4.5.6 TACTIC

Reference: Vansnick (1986b)

4.5.6.1 The method

TACTIC is very similar to ELECTRE I but yields a global preference relation instead of a choice set. Like ELECTRE I, it consists of three main steps.

Preference modelling We define two binary relations P_i and V_i for each dimension by

$$a\ P_i\ b \Leftrightarrow g_i(a) - g_i(b) > \tau_{i,1}$$

and

$$a\ V_i\ b \Leftrightarrow g_i(a) \geq g_i(b) + \tau_{i,2},$$

where $0 \leq \tau_{i,1} < \tau_{i,2}$. The first relation simply expresses the fact that a is better than b on a given dimension because the difference between the preferences exceeds some indifference or discrimination threshold. The second one expresses the fact that a is much better than b on a given dimension, because the difference between their performances exceeds a threshold that the client considers as being very large. In the version presented in Vansnick (1986b), instead of using the difference in the performances, one uses the difference in the image of the performances through a value function.

Aggregation A binary relation, sometimes called outranking relation, is constructed on A. We will consider a outranks b iff the coalition of criteria such that a is better than b is sufficiently larger than the coalition of criteria such that b is better than a and if b is not much better than a on at least one dimension. In order to define the importance of a coalition of criteria, an importance coefficient w_i is associated to each criterion and the importance of a coalition is the sum of the importance coefficients of the criteria in the coalition. Formally, the outranking relation depends on the performance table $\mathbf{g}$ but also on the n-tuple $w = (w_1, \ldots, w_n)$, the $2n$-tuple $\bar{\tau} = (\tau_{1,1}, \ldots, \tau_{1,n}, \tau_{2,1}, \ldots, \tau_{2,n})$ and on a threshold ρ. So, we write

$$a \succ(\mathbf{g}, w, \bar{\tau}, \rho)\ b \Leftrightarrow \sum_{i:aP_ib} w_i > \rho \sum_{i:bP_ia} w_i \text{ and } \nexists i : b\ V_i\ a.$$

Note that contrary to ELECTRE I, the global preference relation is asymmetric; it represents a strict preference.

Exploitation The outranking relation constructed during the aggregation is usually not a weak order and, hence, is difficult to interpret. In order to facilitate the interpretation by the client, the simply connected components of the relation are isolated and, within each component, the alternatives are grouped suitably after eliminating possible cycles.

4.5.6.2 Some pointers

TACTIC is very close to the (weighted) Condorcet method. Two particularly interesting sections are therefore sections 5.2.1 and 5.2.2. From a conjoint measurement viewpoint, the most relevant sections are sections 6.3.5.5 and 6.3.6.2 as well as the more general section 6.5.

4.6 Limitations of the axiomatic approach

In this chapter, we suggested that the axiomatic analysis of an aggregation procedure can help the analyst or client to choose one that is well-suited to his problem and to use it in a consistent way. We suggested that the client and the analyst mutually agree on a set of sensible or attractive conditions, hereby reducing the set of available aggregation methods. But it is certainly not always easy to understand an axiom and all of its consequences. This is even more difficult if we consider groups of axioms, because they interact and as a group, can have far-reaching consequences. In the course of a decision aiding process, it can thus happen that a client does not agree with the outcome of the aggregation or that he finds a new axiom attractive whilst this new axiom is not compatible with those he selected at the beginning of the process. Choosing a particular method because the axioms that characterise it seem attractive or relevant does therefore not always lead to an adequate choice.

It might help in the discussion about the choice of the procedure to "test" the axioms instead of asking the client if he agrees with them. We do not mean extensive tests such as, for example, in the experimental assessment of mathematical models in psychology. We simply suggest to present a few well-chosen pairs of alternatives (real or not too fictitious ones) to the client and see if his preferences are compatible with the axiom. This will not guarantee the choice of the "right" aggregation procedure but it might help.

Many axioms force the aggregation procedures to behave consistently when we change the preferences (Positive Responsiveness, Limited Influence of Indifference), the weights (Convexity, Archimedeanness), the number of criteria (consistency) or the set of alternatives (a variant of Independence of Irrelevant Alternatives—not presented here—does this). These conditions often look attractive to the analyst or the client because they guarantee that the outcome will not vary too much or in the wrong direction if the data change a little bit. But these consistency conditions are imposed for all logically possible profiles whilst the client is usually only interested in small changes. Even if the single-dimension preferences, the set of dimensions or the set of alternatives can change during the decision aiding process, we do not expect dramatic changes. So, these consistency conditions, although quite appealing, are perhaps too strong and there might be aggregation procedures that are only locally consistent, but we do not know them. There are two reasons for this:

- It seems technically difficult to characterise an aggregation procedure with a restricted domain.

- It is not at all clear how we should define the restricted domain, i.e. the set of admissible data (alternatives, dimensions, performances, preferences, ...), in a given decision aiding process, how we should define which changes are possible and which are not.

When a theorem states that only one method or no method satisfies a given set of axioms, we must not forget that there are perhaps other methods that *almost* satisfy these axioms. And these other methods may have such nice properties that we might be willing to use them, even if they do not satisfy the axioms we had first chosen. For example, there is an aggregation procedure which almost satisfies the axioms of Arrow's Theorem under some conditions: the Condorcet method. It satisfies Independence of Irrelevant Alternatives, Pareto and Non-Dictatorship in all cases. And, with 3 alternatives, the proportion of profiles for which it yields a weak order varies between 91 and 94%, according to the number of voters (Gehrlein and Fishburn, 1976). Unfortunately, when the number of alternatives grows, the proportion rapidly decreases and tends towards 0.

The axiomatic approach probably suffers more limitations than those listed above but these limitations certainly do not cancel the advantages. Instead, we are convinced that it is a powerful tool for the analysis of aggregation procedures and that it helps to understand a lot of their characteristics.

5

AGGREGATION PROCEDURES

In this chapter we analyse different aggregation procedures with an emphasis on their axiomatic characterisations. Contrary to chapter 6, the viewpoint is that of social choice theory.

5.1 Different kinds of aggregation functions

In section 4.2, we presented an example of an aggregation function: the Borda method. It maps a profile of linear orders on a weak order. But we sometimes want to aggregate other kinds of profiles: profiles of weak orders, of semi-orders, of fuzzy relations, ... (in chapter 3, section 3.6.2, we discuss how these different relations might arise), profiles of real valued mappings (performance table), etc.

Similarly, we do not always want the result of the aggregation to be a weak order. A choice set or an acyclic relation might also be fine. So, we can define other kinds of aggregation functions:

- functions mapping a profile of weak orders on a weak order,
- functions mapping a profile of linear orders on a subset of alternatives containing only the best one,
- functions mapping a profile of linear orders on an acyclic preference relation,
- functions mapping a profile of fuzzy relations on a weak order,
- functions mapping a performance table on a fuzzy relation,
- functions mapping a performance table on a weak order,
- and so on.

The next two sections will be devoted to the problem of aggregating a profile of binary preference relations (fuzzy in section 5.2, or not in section 5.3) into one preference relation. In section 5.4, we will turn to the problem of aggregating a performance table into one preference relation. In section 5.5, we will very shortly discuss the aggregation of linguistic performances into one relation. The outcome of the aggregation in sections 5.2 to 5.5 is always a preference relation. This is well suited when the decision maker wants to rank a set of objects, but there are cases where he wants to choose one object. It is then more adequate to use an aggregation procedure (a choice function) leading to a subset of best alternatives and not to a preference relation. This is discussed in section 5.6. We will conclude this chapter by a presentation of some techniques (e.g. the arithmetic mean) aiming at the aggregation of a vector of performances into one single performance (section 5.7).

5.2 Aggregation of preference relations into one relation

In this section, we present different procedures aimed at aggregating a profile of preference relations into one binary relation. These preference relations (binary relations) can be the outcome of a preference modelling process (see chapter 3, section 3.10) or be formally derived from performances (numerical, linguistic, ...) or they can be directly stated by the decision maker. In the evaluation model presented in chapter 2, p. 41, the preference relations are elements of H. Unless otherwise stated, we do not assume that preference relations have particular properties like reflexivity or transitivity.

We will discuss the following methods:

The simple majority or Condorcet method. We present the method itself, its characterisation and the celebrated Arrow's Theorem that explains why the Condorcet method does not always help us make a decision. Some aspects of the TACTIC method are discussed.

The weighted Condorcet method. This method is a variant of the one stated above. A characterisation is presented as well as some consequences for the TACTIC method.

The qualified majority. This method is to some extent similar to simple majority. A characterisation is presented and a link is established with the ELECTRE-like methods.

The lexicographic method. This simple method, although its use is not very widespread, has some interesting properties and allows us to introduce some important concepts.

Note that we already extensively presented a method aggregating several preference relations into one: the Borda method (see section 4.2.2).

Many other methods for the aggregation of preference relations into one relation can be found in the literature. They are so numerous that it is definitely not possible to present them all here. We present only five of them: we chose these because they help us understand some aggregation methods commonly used in multicriteria decision aiding.

5.2.1 The simple majority or Condorcet method

This procedure, named after the French mathematician and philosopher Condorcet (1743–1794), works as follows [1]. Take any pair (a, b) of alternatives. If the number of criteria such that a defeats b is larger than the number of criteria such that b defeats a, then a is globally preferred to b. If the two numbers are equal, then a is globally indifferent to b. We illustrate this method with a simple example (the notation has been introduced in section 4.2.1, p. 121). Let $A = \{a, b, c\}$, $N = \{1, 2, 3\}$ and p contain the following three weak orders (rankings, possibly with ties).

1. $a \succ_1 c \succ_1 b$,
2. $c \succ_2 a \succ_2 b$,
3. $b \succ_3 [c \sim_3 a]$.

Alternative a defeats b twice. Therefore, a is globally better than b. Alternative c defeats a once and a defeats c also once. Therefore, a and c are globally indifferent. Alternative c defeats b twice. Therefore, c is globally better than b. Finally, we obtain the weak order $[a \sim(p)\ c] \succ(p)\ b$.

5.2.1.1 Axioms and characterisation

- *Anonymity.* All criteria play exactly the same role. In other words,

$$\succsim(\succsim_1, \succsim_2, \ldots, \succsim_n) = \succsim(\succsim_n, \succsim_1, \ldots, \succsim_2) = \succsim(\succsim_2, \succsim_1, \ldots, \succsim_n) = \ldots$$

  whatever the order of the relations in the profile.

- *Completeness.* The global preference relation is always complete, i.e. for any pair (a, b), we have either $a \succsim(p)\ b$ or $b \succsim(p)\ a$ (possibly both). In other words, no pairs of alternatives are incomparable.

- *Neutrality.* See section 4.2.2, p. 123.

- *Positive Responsiveness.* Suppose that, using $\succsim$, we obtain Not $[b \succ(p)\ a]$, i.e. a is strictly preferred to b or they are incomparable or indifferent. Suppose also that p' is identical to p except for one criterion, on which the position of a has improved with respect to b. If $\succsim$ satisfies Positive Responsiveness, then $a \succ(p')\ b$. In other words, if a is globally not worse than b in p and if p' is identical to p except for one criterion where the position of a has improved with respect to b, then a should be globally better than b. By an improvement of a with respect to b, we mean one of the following situations.

  – $b \succ_i a$ in p and $a \succsim'_i b$ in p' or

[1] This procedure was presented very informally in section 4.2.1, as a procedure for aggregating linear orders.

– $a \sim_i b$ in p and $a \succ'_i b$ in p'.

An example will clarify the picture. Let $A = \{a, b, c, d\}$ be a set of projects, $N = \{1, 2, 3\}$ and p contain the following three linear orders.

1. $d \succ_1 b \succ_1 a \succ_1 c$,
2. $c \succ_2 a \succ_2 d \succ_2 b$,
3. $b \succ_3 c \succ_3 a \succ_3 d$.

Suppose that the decision maker uses the aggregation function $\succsim$ and finds $c \sim(p)\, b$. Later, the decision maker improves project c and uses $\succsim$ again with the new profile p' in which the position of c has improved on criterion 3. In this new profile, $c \succ'_3 b \succ'_3 a \succ'_3 d$. Because the position of c has improved and because c was as good as b in p, Positive Responsiveness imposes that c is now better than b in p', i.e. $c \succ(p')\, b$.

- *Independence of Irrelevant Alternatives.* The global preference between a and b depends only on their relative position in p, not on other alternatives. In other words, if p and p' are two profiles such that, for each criterion i, $a \succsim_i b$ in $p \Leftrightarrow a \succsim'_i b$ in p', then $a \succsim(p)\, b \Leftrightarrow a \succsim(p')\, b$.

  An example will help to understand this property. Let $A = \{a, b, c, d\}$, $N = \{1, 2, 3\}$ and p contain the following three linear orders.

  1. $d \succ_1 b \succ_1 a \succ_1 c$,
  2. $c \succ_2 a \succ_2 d \succ_2 b$,
  3. $b \succ_3 c \succ_3 a \succ_3 d$.

  Suppose that the decision maker uses the aggregation function $\succsim$ and finds $c \succsim(p)\, b$. Suppose that he later uses the same aggregation function with a profile p' differing from p only on the first criterion: this time, $b \succ'_1 d \succ'_1 a \succ'_1 c$. In the new profile p', b is still better than c on criteria 1 and 3 and c is better than b on the second criterion, precisely as in p. Because no change occurred in the relative position of c and b, Independence of Irrelevant Alternatives imposes that $c \succsim(p')\, b$, as in p. The positions of d and a have no influence on the way $\succsim$ compares b and c.

The following theorem uses these 5 axioms to characterise the Condorcet method. It is based on a theorem by May (1952).

Theorem 5.1
Suppose we want to aggregate profiles of weak orders into a binary relation. The only aggregation function satisfying Anonymity, Completeness, Neutrality, Positive Responsiveness and Independence of Irrelevant Alternatives is the Condorcet method.

Remark 5.2.1
In his theorem, May only considers aggregation procedures that satisfy Completeness. He therefore uses a version of Positive Responsiveness that works only with

procedures satisfying Completeness. But later in this book (Section 5.2.1.4), we will consider aggregation procedures that do not satisfy Completeness. Hence we need a version of Positive Responsiveness that works well with or without Completeness. That is why the version of Positive Responsiveness that we use is not exactly the same as that of May. Note that our Positive Responsiveness plus Completeness implies the Positive Responsiveness of May. Another difference between May's theorem and theorem 5.1 is that May's theorem is stated for only two alternatives. But, because we impose Independence of Irrelevant Alternatives, we may apply May's theorem to each pair of alternatives and, so, the proof is straightforward. •

5.2.1.2 Discussion

The five axioms characterising the Condorcet method look reasonable to some extent but, nevertheless, deserve discussion.

Neutrality This condition is very compelling under most circumstances: we do not want to favour any alternative a priori. Yet there are circumstances where some alternatives have a different status than others and may be treated in a different way. For instance, when there is a status quo, i.e. an alternative representing the opportunity of doing nothing.

Completeness If two alternatives are indifferent on six criteria, if a is better than b on one criterion and b is better than a on one criterion, then it seems reasonable to conclude that a and b are globally indifferent. But consider now a quite different situation: a is better than b on four criteria and b is better than a on four criteria; then it might be the case that the decision maker concludes that he is indifferent between a and b, but a more likely situation is that he would be unable or unwilling to conclude anything, because of the highly conflicting information he has about a and b. So, imposing Completeness is a strong requirement.

Anonymity In most applications, even if no question about the importance of the criteria is asked, we can expect that the decision maker will consider some criteria more important than others or that they do not play the same role. If this is the case, then the Condorcet method should not be used. If the other axioms of the Condorcet method look attractive or seem adequate, a possible solution is then to use the weighted Condorcet method (see p. 178).

Positive Responsiveness This property may seem desirable in many conditions but it has some consequences that need to be considered. Suppose that, given a profile p, the result of the aggregation is $a \sim(p)\ b$. If the aggregation procedure $\succsim$ satisfies Positive Responsiveness, then any change in favour of a or in favour of b will break the indifference and we will have $a \succ(p')\ b$ or $b \succ(p')\ a$; even if there are many criteria and if the change occurs only on one criterion. Therefore, a situation where two alternatives are indifferent is very unstable: the smallest change can break the indifference.

Independence of Irrelevant Alternatives Let $A = \{a, b, c\}$, $N = \{1, 2\}$ and p contain the following two linear orders.

1. $b \succ_1 c \succ_1 a$,
2. $a \succ_2 b \succ_2 c$,

Note that b compares to a in exactly the same way that c compares to a. Hence, under Independence of Irrelevant Alternatives, if an aggregation function concludes that $b \succ(p)\ a$, then it must also conclude that $c \succ(p)\ a$. But suppose that the decision maker knows or feels or is convinced that the difference between b and a on criterion 1 is much larger than the difference between c and a. On the contrary, on criterion 2, he feels that the difference between a and b is much smaller than between a and c. It might then be very reasonable to conclude that $b \succ(p)\ a$ but that $a \succ(p)\ c$, thereby violating Independence of Irrelevant Alternatives.

This example shows us that, even if there are cases where Independence of Irrelevant Alternatives makes sense, it is no longer a desirable property when the information to be aggregated is not purely ordinal, because it discards any information on preference differences, anything that is not ordinal.

Note that in this example, the available information is very poor, somewhere between ordinal and interval: we have two linear orders (ordinal information) plus the fact that some differences are larger than others. When we have cardinal information (interval, ratio or absolute scales) for each criterion, we then definitely have good reasons to reject Independence of Irrelevant Alternatives.

5.2.1.3 When simple majority fails

In the previous section, we characterised the Condorcet method as a function aggregating a profile of linear orders into a binary relation with Theorem 5.1. The main problem with the Condorcet method is that this binary relation is not always a weak order, as shown in the following example where p contains the following three linear orders.

1. $a \succ_1 b \succ_1 c$,
2. $c \succ_2 a \succ_2 b$,
3. $b \succ_3 c \succ_3 a$.

Alternative a defeats b twice. Therefore, a is globally better than b. Alternative c defeats a twice. Therefore, c is globally better than a. Alternative b defeats c twice. Therefore, b is globally better than c. Unfortunately, this is not a weak order: the global preference relation is cyclical (see figure 5.1). No alternative is better than all of the other ones and it is therefore impossible to make a decision. This situation is often called the Condorcet paradox. If an analyst presents such an outcome to a decision maker, we can safely consider that it is of no help to the decision maker.

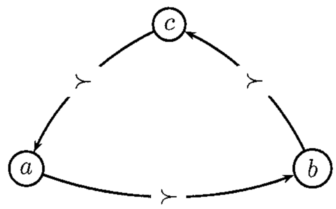


Figure 5.1: The preferences $a \succ b$, $b \succ c$, $c \succ a$.

Arrow (1963) proved a theorem that helps us understand why the Condorcet method does not always work and why it is difficult to avoid this problem. We now present this theorem (slightly modified) and the axioms it uses.

- *Weak Order.* See p. 123.
- *Independence of Irrelevant Alternatives* (see p. 172). In (Arrow, 1963), this condition is stated in terms of social choice and not in terms of social preferences. A proof of Arrow's Theorem with the condition we use can be found in Sen (1986).
- *Non-Dictatorship.* There is no criterion k such that, in any profile p, if $a \succ_k b$, then, necessarily, $a \succ(p)\ b$. In other words, no criterion can impose its strict preference or, very roughly, the global preferences depend on more than one criterion. Ideally, we would like $\succsim(p)$ to depend on all criteria. So, imposing that it depends on more than one criterion is certainly a basic condition.
- *Pareto.* If a is strictly better than b on all criteria ($a \succ_i b$ for all i), then a is globally strictly better than b, i.e. $a \succ(p)\ b$. It would indeed be strange that an alternative b worse than a with respect to all criteria would globally defeat a.

Theorem 5.2 (Arrow's Theorem)
Suppose we want to aggregate profiles of weak orders and there are at least three alternatives. There is no aggregation function satisfying Weak Order, Independence of Irrelevant Alternatives, Non-Dictatorship and Pareto (in fact this theorem also applies to profiles of linear orders, semi-orders,if there are at least four alternatives, and many different kinds of binary relations).

The four properties involved in Arrow's Theorem are not compatible. It is not possible to find a method that satisfies all of them. Therefore, if an aggregation function satisfies Weak Order, Independence of Irrelevant Alternatives and Pareto, it necessarily does not satisfy Non-Dictatorship. In other words, such a method always yields a global preference the asymmetric part of which is the same as the asymmetric part of the preference relation along a given criterion (the same for all profiles). Similarly, if an aggregation function satisfies Weak Order, Non-Dictatorship and Pareto, it cannot satisfy Independence of Irrelevant Alternatives.

Let us now come back to the Condorcet method. This method satisfies Pareto, Non-Dictatorship and Independence of Irrelevant Alternatives. Hence, it cannot satisfy Weak Order: this is why the outcome of the Condorcet method is not always a weak order.

But it also tells us something which is much more important: there is no perfect aggregation function. Or, we should say: no aggregation method can satisfy all our expectations. But this is not a problem of the aggregation method, it is our problem. Our expectations are not reasonable. We should not expect to find an aggregation method satisfying all the axioms of Arrow's Theorem, whether we aggregate ordinal information or we use only the ordinal part of the available information: it is too poor. So, if we want to aggregate a profile of binary relations, we have to abandon one of the four properties proposed by Arrow; we have to use a method that suffers some flaws, some weaknesses or at least some imperfections. We can, for instance, drop transitivity. We then obtain a method that, in some cases, will yield intransitive global preference relations. In order to arrive at a recommendation for the decision maker, some further analysis is then necessary (see section 5.2.1.5 and chapter 7).

Another possibility is to drop Independence of Irrelevant Alternatives. Look for example at the Borda method. It satisfies Weak Order, Non-Dictatorship and Pareto. Therefore, according to Arrow's Theorem, it cannot satisfy Independence or Irrelevant Alternatives. And, indeed, if you apply the Borda method to the example on page 172, you will find that it doesn't.

The task of choosing an aggregation procedure may then be seen as the task of looking for the procedure whose strengths and weaknesses best fit to the decision context. This fit must take into account some more or less objective elements such as the axioms satisfied by the method or the computational tractability but also some more subjective elements such as the ease of use, the confidence put by the decision maker in the procedure, the existence of a software, and so on. In this chapter, we will focus on the axiomatic properties satisfied by the procedure but the other aspects also need to be examined with care.

Remark 5.2.2

As stated above, Arrow's Theorem is limited to the aggregation of weak orders. It tells us nothing about the aggregation of relations that are not weak orders. In fact, it holds for most relations that we encounter in our applications: semi-orders, interval orders, partial orders, some kinds of trees, and so on. This has been proven in different variants or generalisations of Arrow's original Theorem (for example Barthélemy, McMorris, and Powers, 1995; Sen, 1986).

Note also that many researchers have tried to escape from Arrow's Theorem by weakening some of its axioms. For a survey of this literature, see Campbell and Kelly (2002); for for more specific surveys focusing on conditions on profiles (on the domain of aggregation functions), see Gaertner (2002); Weymark (forthcoming).

5.2.1.4 Condorcet and TACTIC

Consider an application of TACTIC (Vansnick, 1986b) without weights (or all weights set to 1), without veto and with a concordance threshold equal to 1. What we obtain then is almost the Condorcet method. The only difference is that TACTIC yields incomparability where the Condorcet method yields indifference. This special version of TACTIC is therefore characterised by almost the same axioms as the Condorcet method. We just have to replace Completeness by Antisymmetry.

No indifference. The global preference relation is always antisymmetric, i.e. for any pair (a, b) of distinct alternatives, we never have $a \sim(p)\ b$. In other words, no pairs of alternatives are indifferent.

Theorem 5.3
Suppose we want to aggregate profiles of weak orders. The only aggregation function satisfying Anonymity, No indifference, Neutrality, Positive Responsiveness and Independence of Irrelevant Alternatives is TACTIC, without weights, without veto and with a concordance threshold equal to 1.

Among the conditions of this theorem, the only one we have not yet discussed is No indifference. In some sense, it poses the same problem as Completeness. Completeness does not allow incomparability. But No indifference does not allow indifference. Yet, in our discussion of Theorem 5.1 (p. 172), we showed that indifference and incomparability can, at least in some cases, be desirable.

If we now consider a version of TACTIC without weights and without veto but with a Concordance threshold different from 1, the axioms of Theorem 5.3 remain valid, except for one: Positive Responsiveness. Indeed, if two alternatives are incomparable in $\succsim(p)$, because the threshold is larger than 1, if the position of b is improved on one of the criteria, the global preference relation might not change.

As the Condorcet method, TACTIC satisfies Pareto, Independence of Irrelevant Alternatives and Non-Dictatorship. Therefore, as proved by Arrow's Theorem (Theorem 5.2), it cannot always yield a weak order. That is why an exploitation phase is sometimes necessary after the aggregation phase (see chapter 7, section 7.4).

5.2.1.5 What do we do with a non-transitive relation?

We showed in the two previous sections that the outcome of an aggregation with simple majority (or a special case of TACTIC) is not always transitive. So, if we decide to use simple majority, we must be prepared to face cases in which the global preference relation is not transitive, i.e. preference relations on which it is not easy to base a recommendation (see section 7.4) for the decision maker. Indeed, if the global preference relation is a weak order (ranking with ties), then there is one or several best alternative(s) and it is easy to see that they are good candidates for a recommendation if the decision maker must choose an alternative. But if the global preference relation is not transitive, then there is not necessarily an alternative (or several) that is better than all of the other ones (see figure 5.1, p. 175) and it is not at all obvious at to decide which one should be recommended.

Hence, a careful analysis of the global preference relation is needed in order to derive a recommendation. This analysis is usually called the *exploitation* of the global preference relation. Many techniques have been developed for the exploitation: they are presented in chapter 7, section 7.4. The analyst performing an exploitation should always bear the technique used for the construction of the global preference relation in mind because not all exploitation techniques are compatible with a specific construction technique. Note that if the exploitation is necessary when the outcome of the aggregation is not transitive, there are also cases in which an exploitation is helpful even if the outcome of the aggregation is transitive. This is also discussed in chapter 7.

5.2.2 The weighted simple majority or weighted Condorcet method

This procedure is a generalisation of the classical Condorcet method. In this procedure, each criterion i is assigned a weight w_i reflecting its importance. The n-tuple $w = (w_1, \dots, w_n)$ is called the weight vector. We define $W_{ab}(p, w)$ as the sum of the weights of the criteria such that a is at least as good as b ($a \succsim_i b$) in the profile p. This number can be interpreted as the strength of the coalition of criteria supporting a against b. We will therefore say that a is globally at least as good as b if $W_{ab}(p, w) \geq W_{ba}(p, w)$. Because the global preferences now depend on the preferences for each criterion and also on the weights, we use the notation $\succsim(p, w)$ for the global preference relation. This explicitly indicates that $\succsim$ is a function of p and w.

We illustrate this method with a simple example. Let $A = \{a, b, c\}$, $N = \{1, 2, 3\}$ and p contain the following three linear orders.

1. $a \succ_1 c \succ_1 b$,
2. $c \succ_2 a \succ_2 b$,
3. $b \succ_3 c \succ_3 a$.

Let the weights of the three criteria respectively be: 3, 1 and 1. We have $W_{ab}(p, w) = 4$ and $W_{ba}(p, w) = 1$. Therefore, $a \succ(p, w)\ b$. Also, $W_{ac}(p, w) = 3$ and $W_{ca}(p, w) = 2$. Therefore, $a \succ(p, w)\ c$. Finally, $W_{cb}(p, w) = 4$ and $W_{bc}(p, w) = 1$, so, Therefore, $c \succ(p, w)\ b$. The final ranking is thus:

$$a \succ(p, w)\ c \succ(p, w)\ b.$$

Note that if we choose all weights equal to 1, we obtain the classical Condorcet method.

5.2.2.1 Axioms and characterisation

Here are the properties that we will use to characterise the weighted Condorcet method.

- *Weighted Anonymity.* All criteria play the same role but their weight makes a difference. Therefore, if you rename the criteria (for example, 3 becomes 2, 2 becomes 3 and 1 doesn't change) and if you accordingly change the weights (w_3 becomes w_2, w_2 becomes w_3 and w_1 doesn't change), the result of the aggregation doesn't change.

Slightly more formally,

$$\begin{aligned}
&\succsim(\succsim_1, \succsim_2, \ldots, \succsim_n, w_1, w_2, \ldots, w_n) \\
&= \succsim(\succsim_n, \succsim_1, \ldots, \succsim_2, w_n, w_1, \ldots, w_2) \\
&= \succsim(\succsim_3, \succsim_n, \ldots, \succsim_1, w_3, w_n, \ldots, w_1) = \ldots.
\end{aligned}$$

- *Convexity.* Let p be a profile. Suppose that, using the weight vector $w = (w_1, \ldots, w_n)$, we obtain $a \succsim(p, w)\ b$. Using another vector $w' = (w'_1, \ldots, w'_n)$, we also obtain $a \succsim(p, w')\ b$. Suppose finally that we use a third weight vector w'' such that each weight w''_i is the average of w_i and w'_i, i.e. $w'' = (\frac{w_1+w'_1}{2}, \ldots, \frac{w_n+w'_n}{2})$. Because $a \succsim(p, w)\ b$ and $a \succsim(p, w')\ b$ and also because w'' is half-way between w and w', we might expect that $a \succsim(p, w'')\ b$. This is precisely what convexity says.

- *Archimedeanness.* Very roughly, Archimedeanness imposes that, if you raise the weight of a criterion high enough, the global preferences will be identical to the preferences for that criterion. A consequence of this is that a good rank on a criterion with a high weight can compensate anything.

- *Neutrality.* See p. 123

- *Positive Responsiveness.* See p. 171.

- *Faithfulness.* See p. 123

- *Independence of Irrelevant Criteria.* A criterion which is assigned a weight equal to zero plays no role. If a criterion has a weight equal to zero, modifying the preferences along that criterion will not affect the global preference relation.

- *Independence of Irrelevant Alternatives.* See p. 172.

The following theorem uses these 8 axioms to characterise the weighted Condorcet method. It is based on proposition 3 in Marchant (2003).

Theorem 5.4
Suppose we want to aggregate profiles of complete binary relations into complete binary relations (not necessarily weak orders). The only aggregation function satisfying Weighted Anonymity, Convexity, Archimedeanness, Neutrality, Positive Responsiveness, Faithfulness, Independence of Irrelevant Criteria and Independence of Irrelevant Alternatives is the weighted Condorcet method.

5.2.2.2 Discussion

Some of the conditions characterising the weighted Condorcet method have already been discussed: Neutrality, Positive Responsiveness, Faithfulness, Independence of Irrelevant Alternatives (see p. 124 and p. 173). Among the other axioms, Weighted Anonymity and Independence of Irrelevant Criteria seem to be unavoidable. But the need for Convexity and Archimedeanness is subject to criticism, at least in some cases.

- In this context, Convexity is, to some extent, similar to Consistency (see p. 123). We can use the same example of four students ranked in physics, maths, economics and law to show that Convexity is not always desirable. If there is some interaction between two or more criteria, Convexity is a drawback.
- Due to Archimedeanness, as said earlier, a good rank on a criterion with a high weight can compensate anything, even the worst ranks on all other criteria. This is clearly not always what a decision maker wants.

5.2.2.3 Cyclical preferences

Because the simple weighted majority method is just a generalisation of the plain simple majority method, it is also possible that cyclical global preferences appear. Hence, an exploitation (see chapter 7) of the global preference relation will often be needed in order to arrive at a final recommendation (see chapter 2).

5.2.2.4 Choosing the weights

Archimedeanness tells us that, if the weight of a criterion is too high, then the other criteria no longer play a role (they cannot even break ties). This criterion becomes overwhelming. It is not hard to show that this happens when a criterion has a weight larger than the sum of the weights of the other criteria. This is definitely not desirable and it therefore puts a constraint on the weights. Let W be the sum of the weights, for all criteria. Then, each weight w_i must be lower than $W/2$. Of course, this constraint is very weak and doesn't help us very much in choosing the weights.

If we want to be more practical, then we might use the following method, consistent with the general approach presented in section 4.4.6. We present a (fictitious or not) profile on two alternatives a and b to the decision maker and we ask him which one he prefers. He is not necessarily able to answer our question; he can hesitate, but if he does answer, then we can use his answer to set the weights. Suppose he says that a is strictly better than b. Then, we know that $W_{ab}(p,w) > W_{ba}(p,w)$. The weight of the coalition in favour of a is strictly larger than the weight of the coalition in favour of b. If he says that a and b are indifferent, then we know that $W_{ab}(p,w) = W_{ba}(p,w)$.

For example, if we present this profile p

1. $a \succ_1 b$,

2. $b \succ_2 a$,

3. $b \sim_3 a$,

and if he says that b is strictly better than a, then we know that $w_2+w_3 > w_1+w_3$ and, so, $w_2 > w_1$.

With another profile p',

1. $a \succ_1 b$,

2. $b \succ_2 a$,

3. $b \succ_3 a$,

if he says that a is strictly better than b, then we know that $w_1 > w_2 + w_3$.

If we then present other profiles $p'', p''', \ldots$ on two alternatives, we can eventually find all inequalities (or equalities in cases of indifferences) involving every pair of coalitions that restrict the set of possible weights. Our task is then to find weights $w_1, w_2, \ldots, w_n$ such that all inequalities are satisfied. There can be many such vectors but they all correspond to the same ordering of the coalitions and, hence, they all yield the same result. So, picking any of them is fine.

But, unfortunately, it is not always possible to find weights satisfying all the inequalities. For example, suppose that using different profiles, we find

$$w_2 + w_3 > w_1 + w_2 > w_1 + w_3 > w_1 > w_2 > w_3.$$

It is not possible to satisfy all these inequalities simultaneously. Indeed, $w_2+w_3 > w_1 + w_2$ implies $w_3 > w_1$ and this is not compatible with $w_1 > w_3$.

If such a problem occurs, we can ask the decision maker if he wants to revise his position. If he does, there is no problem, but if he doesn't, then, strictly speaking, it means that the weighted Condorcet method is not appropriate for this decision maker, in this context. But, of course, if we can find weights such that *almost* all inequalities are simultaneously satisfied, then the decision maker and the analyst might decide to go on and neglect the inconsistencies.

The number of profiles that is needed on two alternatives in order to rank all coalitions can be very high if the number of criteria is not small. For n larger than 5 or 6, this number can be prohibitive. Performing all necessary comparisons would take too much time. Instead of ranking all coalitions, a possible attitude is then to rank only the criteria (coalitions of size 1) or only the coalitions of size 1 or 2. In order to rank the criteria, we present profiles where a and b are indifferent on all criteria but two. On one of these two criteria, a is better than b. On the other one, b is better than a. For example, with 4 criteria, if we want to rank criteria 2 and 3, we use the following profile.

1. $a \sim_1 b$,

2. $a \succ_2 b$,

3. $b \succ_3 a$,

4. $a \sim_4 b$.

If the decision maker says that he prefers a to b, then we know that $w_2 > w_3$. Once more, strictly speaking, it is necessary to compare all $n(n-1)/2$ pairs of criteria in order to rank them and to test the consistency of the comparisons. We illustrate this with an example. Suppose that after comparing a and b in two profiles, we obtain $w_1 > w_2$ and $w_2 > w_3$. We might stop here and consider that $w_1 > w_3$. But if we ask the decision maker to compare a and b in a profile where they differ only on criteria 1 and 3, we would perhaps find $w_1 < w_3$. This would indicate that the decision maker's preferences are not compatible with the weighted Condorcet method.

So, comparing all $n(n-1)/2$ pairs of criteria is time consuming— perhaps sometimes impossible—but it is a good method for checking the compatibility between the decision maker's preferences and the aggregation function. Note that the only way to really check this compatibility is to compare all pairs of coalitions—there are $2^n - 2$ such pairs.

Note also that ranking only the criteria and not all coalitions yields a number of inequalities that is not sufficient for determining the weights. Several weight vectors satisfying the inequalities can yield different results. The choice of a weight vector among those satisfying the inequalities is then, to some extent, arbitrary and leads to an arbitrary ranking of the alternatives. It might then be wise to perform a robustness analysis (see chapter 7, section 7.5) and to draw only robust conclusions, i.e. conclusions that hold with any weight vector satisfying the inequalities.

Choice of the weights. A good way to set the weights with the weighted Condorcet method is to present various profiles on two alternatives to the decision maker. The profiles are constructed in such a way that, if the decision maker can state which alternatives he prefers, then we can derive some inequalities involving the weights. For example, $w_1 + w_3 + w_4 > w_2 + w_5$. If we present $2^n - 2$ carefully constructed profiles and if the decision maker can say which of the two alternatives he prefers for each profile, we then can unambiguously set the weights and we can also fully check the compatibility between the decision maker's preferences and the aggregation procedure. Unfortunately, the decision maker is not always able to compare the two alternatives and we do not always have the time to perform all $2^n - 2$ comparisons. Nevertheless, performing as many comparisons as possible is probably the best way to set the weights and to check the adequacy of the aggregation procedure.

5.2.2.5 TACTIC and Condorcet

On page 176, we already discussed the similarity between a simplified version of TACTIC and the Condorcet method. The same analogy exists between the weighted Condorcet method and TACTIC with weights but without veto and with a concordance threshold equal to 1.

A sensible way to set the weights in TACTIC is the procedure described on p. 180, where indifference must be replaced by incomparability in the global preference relation.

5.2.3 Absolute and qualified majorities

Simple majority (see p. 171) is an operationalisation of the concept of majority. Absolute majority is another one. With absolute majority, an alternative a is globally preferred to b (i.e. $a \succ(p)\ b$) if the number of criteria such that a is better than b is larger than $n/2$. If a is not globally preferred to b and b is not globally preferred to a, then a and b are indifferent. When the preferences on all criteria are rankings *without* ties, simple and absolute majority always yield the same result. But when there are ties, they can yield different results. This is illustrated by the following example. Let $A = \{a, b\}$, $N = \{1, 2, 3, 4\}$ and p contain the following four weak orders:

1. $a \succ_1 b$,
2. $a \succ_2 b$,
3. $b \succ_3 a$.
4. $b \sim_4 a$.

Alternative a defeats b on more criteria than b defeats a (2 against 1). So, $a \succ(p)\ b$ according to simple majority. But, with absolute majority, $a \sim(p)\ b$ because the number of criteria such that a is better than b is not larger than $n/2$ (criteria 1 and 2) and the number of criteria such that b is better than a is smaller than $n/2$ (criterion 3).

Note that absolute majority can also be defined as follows: an alternative a is globally at least as good as b (i.e. $a \succsim(p)\ b$) if and only if the number of criteria such that a is at least as good as b is at least $n/2$. This definition is equivalent to the previous one.

In the simple example given above, absolute majority yields a tie while simple majority does not. This is not a special case: absolute majority will very often yield a tie where simple majority does not. We say that simple majority is more decisive.

Qualified majority is a generalisation of absolute majority. With qualified majority, a is at least as good as b (i.e. $a \succsim(p)\ b$) if and only if the number of criteria such that a is at least as good as b is at least equal to some fixed concordance threshold δ, between 0 and n.

Fishburn (1973b) characterised absolute majority. We do not present his result but a slightly modified version of another one, proved by Marchant (unpublished), characterising qualified majority and, so, including the cases where thresholds different from $n/2$ are used.

5.2.3.1 Axioms and characterisation

We will use the following properties to characterise qualified majority.

- *Anonymity.* See p. 215.
- *Neutrality.* See p. 123.

- *Non-Negative Responsiveness* If p and p' are two identical profiles except that the position of a with respect to b has been improved on one criterion in p', then the position of a w.r.t. b in the global preference relation $\succsim(p')$ cannot be worse than in $\succsim(p)$. In other words, if a is globally at least as good as b in p, then this is still the case in p'. If a is globally better than b in p, then it is also so in p'. This condition is a weak version of Positive responsiveness (see p. 171). With Positive Responsiveness, an improvement of the position of a on one criterion must lead to a global improvement. With Non-Negative Responsiveness, an improvement of the position of a on one criterion cannot lead to a global deterioration.

- *Limited Influence of Indifference* Consider two identical profiles p and p' except that, on one criterion i, $a \succ_i b$ in p and $a \sim_i b$ in p'. Suppose that $a \succsim(p)\ b$. Even if there is less support for the global strict preference of a over b in p' than in p, we might consider that there is not less support for the global weak preference of a over b in p' than in p. It is then reasonable to impose that $a \succsim(p')\ b$ and this is precisely what Limited Influence of Indifference does.

  Suppose now that $a \succ(p)\ b$. Then, because there is less support for the global strict preference of a over b in p' than in p, it might happen that $a \sim(p')\ b$. This is not prevented by Limited Influence of Indifference. So, moving from $a \succ_i b$ in p to $a \sim_i b$ in p' can influence the global preference relation and cause a deterioration of the global position of a, but not in all situations. That is why we speak of "limited influence."

- *Independence of Irrelevant Alternatives.* See p. 172.

- *Pareto.* See p. 175.

The following theorem uses these 6 axioms to characterise qualified majority.

Theorem 5.5
Suppose we want to aggregate profiles of weak orders. The only aggregation function satisfying Anonymity, Neutrality, Non-Negative Responsiveness, Limited Influence of Indifference, Independence of Irrelevant Alternatives and Pareto is a qualified majority, i.e. there is an integer δ $(0 \leq \delta \leq n)$ such that $a \succsim(p)\ b$ iff the number of criteria such that $a \succsim_i b$ is at least δ.

5.2.3.2 Discussion

Only two of the 6 conditions characterising qualified majority have not yet been discussed.

Non-Negative Responsiveness This condition is very natural. It says that the global preference cannot react in the wrong direction when the preferences on one criterion change. Contrary to Positive Responsiveness (see p. 171 and 173), it does not impose that global indifference be broken as soon as an improvement occurs on one criterion. Therefore, it is hard to find situations in which Non-Negative Responsiveness is a problem.

Limited Influence of Indifference This condition is not as natural as Non-Negative Responsiveness but it is at least reasonable. It is a type of prudence condition. It makes the global preference stable or robust. It perhaps makes sense when "$a \sim_i b$" does not mean that a is perfectly equivalent to b on criterion i, but just means that a is approximately equivalent to b on criterion i.

5.2.3.3 Cyclical preferences

Just like with the Condorcet method (see p. 174), it is possible to arrive at cyclical or non-transitive preferences with qualified majority. Consider for example a profile p consisting of the following three linear orders (this is the same profile as in section 5.2.1.3):

1. $a \succ_1 b \succ_1 c$,
2. $c \succ_2 a \succ_2 b$,
3. $b \succ_3 c \succ_3 a$.

Applying qualified majority with a threshold of 0.6, a cycle appears. Hence, an exploitation of the global preference relation will often be needed (see section 7.4).

5.2.3.4 The choice of the concordance threshold δ for the qualified majority

We now present a method to set the value of the concordance threshold δ. It is based on the general method that we have presented in section 4.4.6. Present two alternatives a and b to the decision maker and ask him if he strictly prefers a to b, b to a or if he is indifferent between them. Let $N_{ab}(p)$ be the number of criteria in p such that a is at least as good as b. According to the decision maker's answer, there are four cases.

i) He strictly prefers a to b. If $N_{ab}(p) \geq N_{ba}(p)$, we know that $N_{ba}(p) < \delta \leq N_{ab}(p)$. But if $N_{ab}(p) < N_{ba}(p)$, we are in trouble because this is not compatible with the qualified majority. There are then three possible attitudes: we can ask the decision maker to revise his position, we can decide that qualified majority is not the right model for this decision maker in this context or we can just ignore his answer.

ii) He strictly prefers b to a. Symmetrically, if $N_{ba}(p) \geq N_{ab}(p)$, we know that $N_{ab}(p) < \delta \leq N_{ba}(p)$. But if $N_{ba}(p) < N_{ab}(p)$, we are in trouble as in the previous case.

iii) He is indifferent between a and b. We then know that $\delta \leq N_{ab}(p)$ and $\delta \leq N_{ba}(p)$.

iv) He cannot compare a and b. We then know that $\delta > N_{ab}(p)$ and $\delta > N_{ba}(p)$.

So we have found two constraints on δ. If we then repeat this process with different pairs of alternatives, we obtain more constraints on δ. Some of them take the form $\delta > \ldots$, some the form $\delta \leq \ldots$ Hence, ideally, we can finally identify a value or a range of possible values for δ.

Of course, this procedure does not always work. As always with this way of working, some contradictions may appear between constraints; for example a constraint saying that $\delta \geq 3$ and another saying that $\delta < 2$. Here again, three attitudes are possible. We ask the decision maker to revise his position, we abandon qualified majority or, if the contradictions are not too numerous and too severe, we can try to find a value for δ that is "almost" compatible with the constraints. In our example, 2 would be a reasonable value for δ, but 1 and 3 are also reasonable candidates: they almost satisfy the constraints.

Another problem with this approach is the treatment of incomparability. If we always handle incomparable pairs as in iv), then we will not help the decision maker. Indeed, if his answers are totally compatible with the qualified majority and if we respect all his preferences, indifferences and incomparabilities, then the aggregation procedure will just restate the decision maker's answers. This is definitely not what he needs.

In a decision aiding process, there are almost always pairs that the decision maker cannot compare; this is usually why he uses a decision aiding method. What can we then offer him with our aggregation procedures? First, a structured process that can help him reason about his problem and analyse his values, needs and goals. Second, a formal technique for constructing a global preference relation that obeys two (often contradictory) principles in a consistent way: respecting the information provided by the criteria and by the decision maker, and easy to use and interpret (roughly speaking, as complete and transitive as possible). According to the latter principle, it is necessary that the aggregation procedure changes at least some incomparabilities stated by the decision maker into indifferences or preferences.

The question is then: which ones? We do not have a clear answer, just two suggestions:

- Drop the incomparabilities yielding constraints that are incompatible with the other constraints. These incomparabilities are easy to identify: they correspond to pairs with high values of $N_{ab}(p)$ or $N_{ba}(p)$ or both.

- Ask the decision maker to distinguish between pairs that are incomparable because he does not know how to compare them or because he is convinced that they cannot be compared. Then drop the incomparabilities of the former type.

Finally, if the decision maker is undecided about many pairs and if we do not want to keep all the corresponding constraints, we may have too few constraints. It is then perhaps wise to present pairs of fictitious alternatives in order to try to obtain more constraints.

5.2.3.5 The qualified majority and ELECTRE I

It is easy to see that ELECTRE I (Roy, 1996), without weights and vetoes, is equivalent to a qualified majority with $\delta \geq n/2$. The characterisation of the qualified majority and the discussion about the choice of the threshold can thus help us use ELECTRE I in a consistent way. In fact, we just need one additional condition (Marchant, unpublished) to characterise ELECTRE I, without weights and vetoes.

Restricted Positive Responsiveness. Suppose p and p' are two identical profiles except that the position of a has been strongly improved on one criterion i, in the following sense: $b \succ_i a$ and $a \succ'_i b$. Suppose also that there is no criterion for which a and b are indifferent. Then, if a and b are globally indifferent in p, a should be globally preferred to b in p'.

It is interesting to compare this condition with Positive Responsiveness (p. 171). Both conditions impose that the global preference reacts positively to an improvement of the position of an alternative; but with Restricted Positive Responsiveness, this positive reaction is imposed only in some special cases.

Theorem 5.6
Suppose we want to aggregate profiles of weak orders. The only aggregation function satisfying Anonymity, Neutrality, Non-Negative Responsiveness, Limited Influence of Indifference, Independence of Irrelevant Alternatives, Pareto and Restricted Positive Responsiveness is ELECTRE I without weights and vetoes, i.e. there is an integer δ ($\delta \geq n/2$) such that $a \succsim(p)\, b$ iff the number of criteria i such that $a \succsim_i b$ is at least δ.

We will not devote a lot of time to the discussion of Theorem 5.6 because all the axioms it uses have been discussed previously, except Restricted Positive Responsiveness. This condition might be seen as too restrictive, in some situations, just for the same reason as Positive Responsiveness. But because Restricted Positive Responsiveness is much weaker, the problem (if any) is much less serious. Note finally that, in most cases, ELECTRE I is used with a threshold larger than $n/2$, so, it never happens that there is no criterion on which a and b are indifferent and a and b are globally indifferent. Hence, Restricted Positive Responsiveness is trivially satisfied and does not really help us to understand how ELECTRE I works.

It is therefore more interesting to characterise ELECTRE I with $\delta > n/2$, which is why we introduce a new condition.

Minimal Incomparability. Assume n is even. There is at least one situation where a and b must be considered incomparable: when the conflict is maximal, i.e. when a is strictly better than b on $n/2$ criteria and b is strictly better than a on the other $n/2$ criteria. But, because n can be odd, we must adapt the condition for this case. Alternatives a and b must be incomparable if a is strictly better than b on $(n+1)/2$ criteria and b is strictly better than a on the other $(n-1)/2$ criteria, or the converse.

The meaning of this condition is clear. If you impose it, you adopt a prudent attitude. You avoid that the aggregation function always gives a clear-cut and easily interpretable result, even when it (perhaps) should not. But the cost of imposing this condition is also clear. The global preference relation might be difficult to interpret or use. It is up to the decision maker to choose between prudence and ease of interpretation.

Theorem 5.7
Suppose we want to aggregate profiles of weak orders. The only aggregation function satisfying Anonymity, Neutrality, Non-Negative Responsiveness, Limited Influence of Indifference, Independence of Irrelevant Alternatives, Pareto and Minimal Incomparability is ELECTRE I without weights and vetoes, i.e. there is an integer $\delta > n/2$ such that $a \succsim(p)\, b$ iff the number of criteria i such that $a \succsim_i b$ is at least δ.

There is, to the best of our knowledge, no characterisation of the weighted qualified majority. So, we cannot analyse the role and meaning of the weights in details, although ELECTRE I is almost always used with weights. The method described on p. 180 for setting the weights of the Condorcet or simple majority method can easily be adapted to the qualified majority and ELECTRE I.

5.2.4 The lexicographic method

This very simple method works as follows: first, you need a linear order (a ranking without ties) on the set of criteria N. This linear order is denoted by $>_\ell$. The maximal criterion (with respect to $>_\ell$) is denoted by 1^ℓ, the second one, by 2^ℓ and so on. So, we have $1^\ell >_\ell 2^\ell >_\ell 3^\ell \ldots >_\ell n^\ell$. Then you look at the first criterion in $>_\ell$, i.e. 1^ℓ. If a is strictly better than b on criterion 1^ℓ, then a is declared globally preferred to b without even considering the other criteria. Similarly, if b is strictly better than a on criterion 1^ℓ, then b is considered as globally preferred to a without considering the other criteria. But if a and b are indifferent on criterion 1^ℓ, you look at the second criterion in $>_\ell$, i.e. 2^ℓ. If a or b is strictly better than the other on criterion 2^ℓ, then it is declared globally better than the other one without considering criteria 3^ℓ, 4^ℓ, ... If you still can not make a difference between a and b using criterion 2^ℓ, you proceed with criterion 3^ℓ, then 4^ℓ and so on until you can make a difference or until you have considered all criteria. In that case, a and b are globally tied. Formally,

$$a \succsim(p)\, b \text{ iff } \begin{cases} a \sim_i b \;\text{ for all criteria} \\ \text{or} \\ a \succ_i b \text{ for the first criterion } i, \text{ w.r.t. } >_\ell, \text{ for which } a \not\sim_i b. \end{cases}$$

We illustrate the lexicographic method with a simple example. Suppose there are three voters, three candidates and the profile p is

1. $a \sim_1 b \succ_1 c$,
2. $c \succ_2 b \sim_2 a$,
3. $b \succ_3 a \sim_3 c$.

Suppose also that $2 >_\ell 1 >_\ell 3$. Consider the pair a, b. Begin with criterion 2 on which a and b are indifferent. They are also indifferent on criterion 1 (the next one in $>_\ell$). But b is strictly better than a on criterion 3. Therefore, $b \succ(p)\ a$. Consider now c and b. Alternative c is strictly better than b on criterion 2 (the first one in $>_\ell$). So, $c \succ(p)\ b$. Similarly, $c \succ(p)\ a$. The final result is thus the linear order $c \succ(p)\ b \succ(p)\ a$.

Note that, in this example, the relation $\succsim(p)$ is complete and transitive. But there are cases where this is not so. For example, when the preferences along each criterion are semi-orders, the global preference relation needs not be transitive.

Leaving aside some very particular (degenerated) cases, we can say that $\succ(p)$, the asymmetric part of the global preference relation, is transitive if and only if each relation $\succsim_i$ is transitive. Moreover, $\succsim(p)$ is a weak order iff all relations $\succsim_i$ are weak orders (for the aggregation of semi-orders, see e.g. Pirlot and Vincke, 1992).

The particularity of the lexicographic method is the existence of the order $>_\ell$ and the fact that each criterion is totally or infinitely more important than all other criteria lower in the order $>_\ell$. If a is better than b on criterion 1^ℓ, it will be globally better than b, even if b is better than a on ten or one hundred other criteria. No compensation is possible.

There are not many cases in which a decision maker would say that only one criterion (except in case of a tie) must be taken into account for deciding if $a \succsim(p)\ b$ or the converse. In most cases, a decision maker would be willing to consider all criteria because the difference between a and b on criterion 1^ℓ might not be so large (even if this is not explicit) and could be compensated by opposite differences on other criteria.

But there is at least one context for which this does make sense: screening. Screening can be applied to a wide variety of problems and is very popular in recruitment processes. In a screening process, all applicants take a first test measuring one or several abilities or competencies. The best applicants are then selected while the others are eliminated. The remaining ones then take a second test measuring some other characteristics. A new elimination follows. Then a third test is given, and so on until one or only a few applicants remain.

In such a process, each test can be seen as a criterion. The first test corresponds to criterion 1^ℓ, the second one to criterion 2^ℓ, and so on. The ranking of the applicants after test 1 is $\succsim_{1^\ell}$. The ranking given by test 2 is $\succsim_{2^\ell}$, and so on. If, at each step i, we keep the best candidates according to $\succsim_{i^\ell}$, then the set of applicants that remain at the end of the process is the set of the best applicants according to the lexicographic method.

The motivation for using the lexicographic method in this context is obvious. After the first test, only $\succsim_{1^\ell}$ is known. Furthermore testing all applicants is expensive. One therefore tries to give the second test to as few applicants as possible. This is why a first elimination occurs after the first test, taking into account only $\succsim_{1^\ell}$. For the same reason, a second elimination occurs after test 2, taking into account only $\succsim_{2^\ell}$, and so on. Of course, if the human resources manager knew all relations $\succsim_{1^\ell}, \succsim_{2^\ell}, \ldots, \succsim_{n^\ell}$ from the beginning, he would probably not use the lexicographic method. But the cost of information prevents him from giving all tests to all applicants.

5.2.4.1 Axioms and characterisation

Here are the most characteristic properties of the lexicographic method.

- *Strong Pareto.* If a is strictly better than b on some criteria ($a \succ_i b$ for some i) and a is at least as good as b on all criteria ($a \succsim_i b$ for all i), then a is globally strictly better than b, i.e. $a \succ(p)\ b$. Furthermore, if a is indifferent to b on all criteria ($a \sim_i b$ for all i), then a is globally indifferent to b, i.e. $a \sim(p)\ b$. Note that this condition implies Pareto (see p. 175). It is a kind of unanimity condition.

- *Independence of Irrelevant Alternatives.* See p. 172.

- *Weak Order.* See p. 123.

The following theorem uses these three axioms to characterise the lexicographic method. It can be found in Fishburn (1974).

Theorem 5.8
Suppose we want to aggregate profiles of weak orders and $n \geq 3$. An aggregation function satisfies Weak Order, Independence of Irrelevant Alternatives and Strong Pareto if and only if it is a lexicographic method.

5.2.4.2 Discussion

It is important to notice the similarity between this theorem and Arrow's Theorem 5.2. Non-Dictatorship has been deleted but Pareto has been strengthened to Strong Pareto. The result is now that there is not one dictator but a hierarchy $>_\ell$ of dictators. Let us now discuss some of the axioms used in this characterisation.

Strong Pareto Consider the following case: $a \sim_i b$ for all criteria but the least important one, i.e. n^ℓ, and $a \succ_{n^\ell} b$. According to Strong Pareto (but not to Pareto), we have $a \succ(p)\ b$. At first sight, this seems quite reasonable but it is so only if "$a \sim_i b$" really means that a and b are perfectly equivalent on criterion i. Indeed, if a and b are perfectly equivalent on all criteria except n^ℓ, then a difference on n^ℓ can make a global difference. But suppose now that "$a \sim_i b$" just means that a and b are approximately indifferent on criterion i. Then, the strict preference $a \succ_{n^\ell} b$, on the least important criterion, might not be enough to conclude that $a \succ(p)\ b$.

Independence of Irrelevant alternatives As mentioned earlier (when discussing the Condorcet method, p. 174), Independence of Irrelevant Alternatives is probably not a good property when the information on each criterion is richer (even slightly) than ordinal. It is worth saying a word about this in the context of screening processes. Consider the recruitment example. When the applicants take the first test, the result of the test can be a ranking (purely ordinal information) but, very often, the result of the test is a score. The recruiter then keeps all the applicants with a score above a given threshold or the m best applicants, where m is a predetermined number. The scores are often measured on scales with well-known psychometric and/or statistic properties but with measurement-theoretic characteristics that are not so well understood. Nevertheless, they are very often a bit more than ordinal. Suppose for example that the score is the number of correctly answered items in a multiple choice questionnaire. Some items are easier than others. So, the number of correct answers cannot be considered as a measure of some ability on an interval scale. But suppose there are 20 items; then the difference between a score of 18 and one of 10 is certainly much larger than the difference between a score of 5 and one of 2. Hence, some comparisons of differences make sense: the information is more than ordinal. This indicates that, when the result of a test is a score and not a ranking, then the lexicographic method should not be used because the information is not purely ordinal. This is even more true if the scores are measured on interval or ratio scales.

The lexicographic method should probably be used only in two cases:

- when there are good reasons to consider that one criterion is infinitely more important than the other ones, i.e. a difference on that criterion, no matter how small it is, cannot be compensated by any number of differences , no matter how large they are, on the other criteria. We must also be sure that an indifference between a and b on a criterion really means that a and b are perfectly equivalent regarding that criterion only. The preference relations $\succsim_i$ thus need be very finely grained.
- when the cost of constructing the preference relations $\succsim_i$ is high and one wants to reduce the costs by means of a screening process. But in such a case, it is probably better not to use a pure lexicographic method. The cost of constructing a very fine-grained preference relation $\succsim_i$ might be too high. Furthermore, some kind of compensation is very often desirable. So, instead of keeping only the best applicants, one might want to keep all the good ones, so that, after the last step, when the remaining applicants have taken all tests, the decision maker can use a method (AHP, MAVT, ELECTRE, ...) allowing some compensation. The decision maker then needs to decide in which order the tests will be given. He definitely must take the cost of the tests into account: a cheap one will be given in the first steps, an expensive one later. But he must also consider some preferential aspects of the problem: he does not want to eliminate an applicant early on in the process who might prove globally excellent later.

5.3 Aggregation of fuzzy relations into one relation

So far we only considered the aggregation of classical preference relations, where "classical" is used as opposed to "fuzzy." We also sometimes use the term crisp relations instead of classical relations. In a classical preference relation, when comparing two alternatives a and b, there are four and only four possible cases:

- a is strictly preferred to b ($a \succ_i b$),
- a and b are indifferent ($a \sim_i b$),
- b is strictly preferred to a ($a \prec_i b$) or
- a and b are not comparable.

But the situation is often not so clear-cut and there are many circumstances in which more nuances would be useful. For example, a decision maker is not always sure that he prefers a to b, even when considering only one criterion, because his knowledge of a and b is uncertain or not perfect. Another example is when the decision maker is sure of his preference but would like to make a distinction between different degrees or intensities of preference. A last example is when some criterion i can be decomposed into several sub criteria. When the decision maker must say whether he prefers a to b, taking into account only criterion i, he might hesitate because he does not know how to balance the pros and cons of the different sub criteria. For a more thorough discussion of these questions, see Bouyssou et al. (2000) and Perny and Roubens (1998).

In these cases, a fuzzy preference relation can be used to model the preferences of the decision maker. A fuzzy preference relation is a relation in which the preference between each pair of alternatives a and b is measured by a number between 0 and 1. The value 1 indicates that a is preferred to b with certainty (if we want to model uncertainty) or with maximum intensity (if we want to model intensity). The value 0 indicates that it is certain that a is not preferred to b (uncertainty representation) or that the intensity of the preference is 0 (intensity representation). An intermediate value, say 0.7, indicates that it is fairly certain that a is preferred to b or that the preference intensity between a and b is high but not maximum. Note that, in some cases, a fuzzy preference relation can be used to simultaneously capture uncertainty and intensity. In any case, the analyst should always clearly know what he wants to represent by means of preferences valued between 0 and 1.

Formally a fuzzy preference relation S (sometimes called valued relation) on the set A maps each pair of alternatives (a, b) to a real number $S(a, b)$ in $[0, 1]$. According to the context, $S(a, b)$ will denote the intensity or the certainty of the preference of a over b. The larger $S(a, b)$, the larger the certainty or the intensity of the preference. Note that $S(b, a)$, the certainty or intensity of the preference of b over a, is usually not linked to $S(a, b)$. Yet, in some applications, these two numbers are linked, for example by the relation $S(a, b) + S(b, a) = 1$ (reciprocal relations) or $\max[S(a, b), S(b, a)] = 1$, etc. This link can result from an arbitrary choice by the analyst or from the construction technique. For example, $S(a,b)$ can be the proportion of cases where the decision maker chooses a over b in a forced-choice pairwise presentation. In this case, we necessarily have $S(a,b)+S(b,a)=1$. This is also true if $S(a,b)$ is the proportion of experts in a panel choosing a over b.

Suppose now that the preferences of a decision maker, along each criterion i, are modelled by a fuzzy preference relation S_i. We then have a profile $p = (S_1, S_2, \ldots, S_n)$ of fuzzy preference relations and the following question arises: how can we aggregate or synthesise this profile into one (classical or fuzzy) preference relation. This is almost the same problem as the one considered in section 5.2. The only difference is that our profiles now consist of fuzzy relations and that the global preference relation will in some cases be fuzzy (section 5.3.4).

5.3.1 Construction of fuzzy preference relations

Assume we have a small-sized problem with 6 alternatives and 4 criteria. A profile of fuzzy preference relations is in this case defined by $4 \times 6 \times 6 = 144$ numbers between 0 and 1. Until now, to the best of our knowledge, no method has been proposed in the literature to elicit $S_i(a,b)$, the certainty or intensity of the preference of a over b on criterion i. But suppose we have such a method. We would then need to apply it 144 times to determine the profile p. This shows why profiles of fuzzy preference relations are almost never elicited by questioning the decision maker: the process would be far too long. Instead, fuzzy preference relations are usually obtained by construction, starting from a performance table where the performances are real numbers or fuzzy numbers.

PROMETHEE II is an example of a method where a fuzzy preference relation is constructed for each criterion, starting from real performances (see p. 196). ELECTRE III (Roy, 1996) is another example (for a brief description of these methods, see section 4.5). Note that the numbers $S_i(a,b)$ in PROMETHEE II are usually interpreted as intensities of preference, while they are often considered in ELECTRE III as degrees of credibility of the statement "a is at least as good as b." In our opinion, the construction techniques are so similar in ELECTRE III and PROMETHEE II that such a dramatic difference between the interpretations can hardly be justified. But it is not clear to us which of these interpretations is correct.

We now present a completely different example of a construction technique in a case where the membership degrees reflect some kind of uncertainty or imprecision rather than an intensity. Suppose the performances of the alternatives on criterion i are not perfectly known: they are modelled using fuzzy numbers. Figure 5.2 presents an example with three alternatives. In this figure, the three curves $\mu_{g_i(a)}, \mu_{g_i(b)}$ and $\mu_{g_i(c)}$ are the fuzzy performances of the three alternatives a, b and c. We might choose $S_i(b,a)$ equal to 1 minus the value of $\mu_{g_i(a)}$ and $\mu_{g_i(b)}$ at the intersection of the two curves, as depicted in figure 5.2. This is in fact the *necessity* (see Dubois and Prade, 1983) that b be strictly better than a. In other words, taking only criterion i into account, b is preferred to a with certainty $S_i(b,a)$.

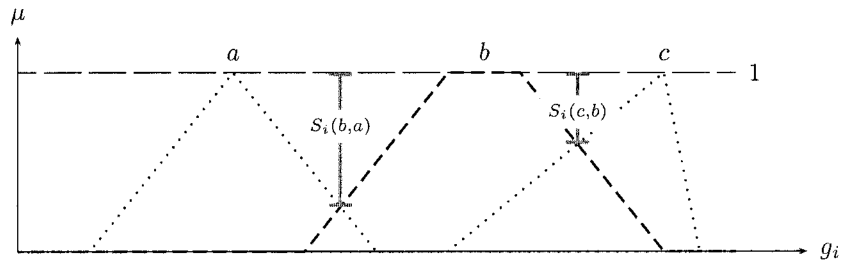


Figure 5.2: A fuzzy preference relation based on fuzzy performances and aiming at modelling uncertainty.

The three construction techniques presented here are just examples. Many others are used and can be considered. But these examples show that the meaning of the numbers $S_i(a, b)$ can vary and is not always clear. This should be kept in mind when aggregating fuzzy preference relations.

We will now present two methods for the aggregation of fuzzy relations into one relation: a generalisation of the Borda method and a generalisation of the Condorcet method. There are of course many other methods, but we present these two because they allow us to shed some light on some aggregation methods that are commonly used in multicriteria decision aiding.

5.3.2 The Generalised Borda method

In section 4.2.2, we presented the Borda method as a method aimed at aggregating a profile of linear orders into a weak order. In fact, the Borda method can be used for aggregating any kind of binary relations and also fuzzy relations. Here is how it works. Suppose we have a profile p of fuzzy relations: $p = (S_1, \ldots, S_n)$. We will not discuss the nature of the valuations for the moment; This will be addressed later. We define $b_a(S_i)$, the single-criterion score of alternative a in S_i, as follows.

$$b_a(S_i) = \sum_{b \in A} S_i(a, b) - \sum_{b \in A} S_i(b, a). \tag{5.1}$$

It is the sum of the valuations on the arcs leaving a minus the sum of the valuations on the arcs entering a. Thus, the larger $b_a(S_i)$, the better a in S_i. The Borda score of alternative a, $B_a(p)$, is then defined as the sum over all criteria of the single-criterion scores.

$$B_a(p) = \sum_{i \in N} b_a(S_i). \tag{5.2}$$

We then say that $a \succsim(p)\ b$ iff $B_a(p) \geq B_b(p)$. In other words, we rank the alternatives in the decreasing order of their Borda scores.

We illustrate the Borda method with the profile presented in figure 5.3. The Borda score of a, $B_a(p)$, is equal to -1.7. Similarly, $B_b(p) = 1.4$ and $B_c(p) = 0.3$. Hence, $b \succ(p)\ c \succ(a)$.

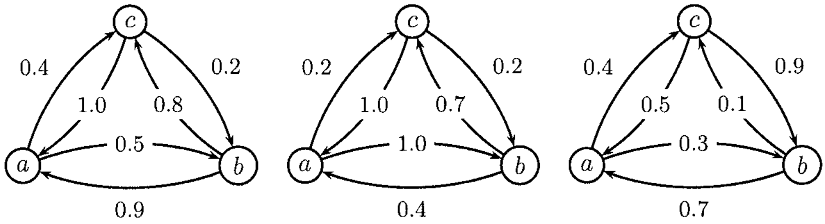


Figure 5.3: A profile of fuzzy preference relations.

Note that if a fuzzy relation happens to be a crisp linear order, then $b_a(S_i)$ is just the number of alternatives beaten by a in S_i minus the number of alternatives beating a in S_i. It can be shown that this number is a negative affine transformation of the rank of a in S_i. Therefore, if the profile p contains only linear orders, ranking the alternatives in increasing order of their sum of ranks (Borda method) or in decreasing order of their sum of single-criterion scores (generalised Borda method) is equivalent and the two methods coincide.

5.3.2.1 Axioms and characterisation

Here are the main characteristics of the generalised Borda method.

- *Weak Order.* See p. 123.
- *Generalised Faithfulness.* If we have only one criterion, i.e. $N = \{1\}$, and if S_i is a linear order, then $\succsim(p)$ is equal to S_1, i.e. the global preference is identical to the preference relation along the unique criterion.
- *Generalised Cancellation.* For any pair (a, b) of alternatives, let r_{ab} be defined by

$$r_{ab}(p) = \sum_{i \in N} S_i(a, b).$$

  If, for all pairs (a, b) of alternatives, $r_{ab}(p) = r_{ba}(p)$, then all alternatives tie. Note that, if all relations in p are linear orders, then $r_{ab}(p)$ is the number of criteria such that a is preferred to b. Generalised Cancellation is thus equivalent to Cancellation.
- *Neutrality.* See p. 123.
- *Consistency.* See p. 123.

Marchant (1996) proved the following theorem.

Theorem 5.9
Let $\mathcal{F}$ be a set of fuzzy relations. Suppose we want to aggregate profiles of fuzzy relations taken in $\mathcal{F}$. The only aggregation function satisfying Weak Order, Generalised Faithfulness, Generalised Cancellation, Neutrality and Consistency is the Generalised Borda method. This theorem holds for almost any $\mathcal{F}$ (see Remark 5.3.1 below).

5.3.2.2 Discussion

This characterisation is very similar to Theorem 4.1. It uses almost the same axioms. It is interesting to note that only one axiom is based on the valuations: Generalised Cancellation. In this axiom we use the sum of some valuations. So, these valuations must be cardinal, they should in principle be measured at least on an interval scale, otherwise we are note sure it makes sense to add them; it would not be meaningful in the sense of meaningfulness theory (see chapter 3, section 3.4). Furthermore, even if we have interval or ratio scales, we must wonder if the sum of the valuations makes sense, if it represents something. In order to illustrate this last point, let us consider density (of mass). Density is measured on a ratio scale, so, a statement like $d_1 + d_2 = d_3 + d_4$, involving the sum of densities, is meaningful, in the sense of meaningfulness theory (with d_i the density of object i). But, even if it is meaningful, it does not mean anything except if we speak of objects of identical volume. Similarly, even if the valuations are on interval or ratio scales, we need to consider if their sum represents something. In principle, measurement theory should be used to answer this question but, very often, the problem is too difficult and the answer is unknown. So, the analyst and the decision maker need to consider if they are willing and if it is sensible to give a meaning to the sum of the valuations.

Hence, looking at this characterisation in order to find out if the Generalised Borda method is appropriate in a given context makes sense only if the valuations of the fuzzy preference relations are cardinal and if their sum represents something. If not, then this characterisation may not be used.

Remark 5.3.1
Note that this theorem is valid in many different cases. Not only for profiles of fuzzy relations without restriction but for virtually all kinds of profiles (for a precise statement, see Marchant, 1996): profiles of t-transitive fuzzy relations (where the t-norm t can be chosen arbitrarily), profiles of fuzzy relations such that $S_i(a,b) + S_i(b,a) = 1$, profiles of semi-orders (they are also fuzzy relations, with $S_i(a,b) \in \{0,1\}$), profiles of interval orders, ...

5.3.2.3 The Generalised Borda method and PROMETHEE II

Some aspects of PROMETHEE II (Brans and Vincke, 1985) have been shortly discussed in section 4.3.10.3, p. 145 (see also section 6.6.2, p. 320). For a short presentation of PROMETHEE, see p. 163.

The net flow of alternative a, denoted by $\Phi(a)$, is defined (see p. 165) by:

$$\Phi_i(a) = \sum_{b \in A} S_i(a,b) - \sum_{b \in A} S_i(b,a),$$

where

$$S(a,b) = \sum_{i=1}^{n} w_i S_i(a,b)$$

and

$$S_i(a,b) = P_i(g_i(a) - g_i(b)). \tag{5.3}$$

It is easy to see that the net flow can also be written as

$$\Phi(a) = \sum_{i \in N} w_i \Phi_i(a), \tag{5.4}$$

where

$$\Phi_i(a) = \sum_{b \in A} S_i(a,b) - \sum_{b \in A} S_i(b,a). \tag{5.5}$$

The net flow $\Phi(a)$ of alternative a can therefore be seen as the sum on all criteria of the single criterion net flow $\Phi_i(a)$. This is very similar to what was shown for the Borda method.

A close look at Equations 5.1 and 5.2 will convince the reader that, once the preference functions P_i have been chosen, the PROMETHEE II method is nothing but a weighted version of the Borda method applied to the valued relations S_i defined by (5.3). Theorem 5.9 thus tells us a lot about the PROMETHEE II method. Roughly speaking, once the decision maker has decided to use preference functions and has agreed on the axioms characterising the Generalised Borda method, he no longer has choice. He must use the PROMETHEE II method. We say "roughly" because a small issue has not yet been addressed: the weights. In the generalised Borda method, there are no weights. In PROMETHEE II, there are weights. But, if the weights are integer, it can be shown that assigning a weight w_i to criterion i amounts to considering a problem without weights and where each criterion i is taken into account w_i times. The number of criteria in this new problem is then no longer n but the sum of the weights. If the weights are not integer but rational, they can be transformed into integers through a multiplication.

The axioms characterising the Generalised Borda method have already been discussed, but one of them deserves a deeper discussion in relation to the PROMETHEE II method: Generalised Cancellation. In this axiom, we add the numbers $S_i(a,b)$, for $i \in N$. These numbers thus need to be at least taken on an interval scale; otherwise the sum and, hence, the condition, do not make sense. But it is not an easy task to decide if this is the case. The analysis of the construction technique (the preference functions) does not tell us much about the scale on which $S_i(a,b)$ is measured. Today, we do not have a clear understanding of what is really modelled by the preference functions and we do not really know how to set the parameters of these functions. The question of the scale type of $S_i(a,b)$ thus remains open. Deciding if Generalised Cancellation is an appropriate condition in a given decision problem is thus problematic because we do not even know if the condition makes sense.

5.3.3 The Generalised Condorcet method and other majorities

One of the most cited arguments against the Borda method (generalised or not) is that it does not satisfy Independence of Irrelevant Alternatives (see p. 172). An obvious way to avoid this problem is to aggregate the preference relations in a pairwise manner, i.e. to consider in turn all pairs of alternatives and to decide for each pair which alternative is preferred to the other, taking only the preferential information about that pair into account. We have already considered this idea when we wanted to aggregate crisp (not fuzzy) relations (section 5.2) with the Condorcet method (p. 171) and qualified majority (p. 183). We might then want to adapt one of these methods for the aggregation of fuzzy preference relations. Let us carry out the exercise with the Condorcet method. In the crisp case, $a \succ(p)\ b$ iff the number of criteria such that a is preferred to b is larger than the number of criteria such that b is preferred to a. If these two numbers are equal, then $a \sim(p)\ b$. More formally,

$$a \succsim(p)\ b \Leftrightarrow |\{i : a \succ_i b\}| \geq |\{i : b \succ_i a\}|$$

or, equivalently,

$$a \succsim(p)\ b \ \Leftrightarrow \ |\{i : a \succsim_i b\}| \geq |\{i : b \succsim_i a\}|. \tag{5.6}$$

For a crisp preference relation $\succsim_i$, let us define $S_i(a,b) = 1$ iff $a \succsim_i b$. Otherwise, $S_i(a,b) = 0$. We can then rewrite (5.6) as

$$a \succsim(p)\ b \ \Leftrightarrow \ \sum_{i \in N} S_i(a,b) \geq \sum_{i \in N} S_i(b,a). \tag{5.7}$$

If we now apply (5.7) with fuzzy relations (where $S_i(a,b)$ can take any value between 0 and 1), we have a generalisation or an extension of the Condorcet method for fuzzy relations. The adaptation is thus very simple and we can follow the same steps to generalise another type of majority method, e.g. the qualified majority. In the next few paragraphs, we make some comments about this generalisation.

5.3.3.1 Does it make sense to add the valuations?

This question is in fact twofold. First, is it meaningful in the sense of meaningfulness theory (see chapter 3) to add the valuations? Second, even if the valuations are measured on interval or ratio scales, does the sum of the valuations represent anything? These two questions have already been discussed in section 5.3.2.2 about the Generalised Borda method.

Note that when $S_i(a,b)$ takes its values in $\{0,1\}$, i.e. when the preference relations are crisp, the sum of the valuations correspond to the number of criteria for which a is better than b and this makes sense.

5.3.3.2 Other possible extensions

Have a look at (5.6). It is the definition of the simple majority for crisp relations. It can also be written as (5.7). But it could also be written as

$$a \succsim(p)\ b \ \Leftrightarrow \ \sum_{i \in N} {S_i}^2(a,b) \geq \sum_{i \in N} {S_i}^2(b,a). \tag{5.8}$$

This expression is perfectly equivalent to (5.6) and (5.7) if we consider only crisp relations, because $S_i(b, a)$ can take only two values: 0 and 1. But using (5.8) for the aggregation of fuzzy relations is not equivalent to (5.7). So, we now have two different extensions of the Condorcet method and it would not be difficult to find many more. This raises a new problem: which extension is the right one? Once more, there is no universal answer. A correct extension in a given context is meaningful, in the sense of meaningfulness theory, and involves arithmetic operations that represent something, that make sense for the decision maker. Each case thus requires a careful analysis but this is often difficult.

5.3.3.3 Transitivity

As seen in section 5.2.1.3 on Arrow's Theorem, an aggregation method satisfying Independence of Irrelevant Alternatives, Non-dictatorship and Pareto (that is the case of the simple majority and of qualified majority) does not always yield a transitive global preference relation. This was in the section about the aggregation of crisp preference relations. But we now want to aggregate fuzzy preference relations. Is it any different? Unfortunately not and the reason is simple: if we want to aggregate fuzzy preference relations into a crisp preference relation, we must be prepared to aggregate all kinds of fuzzy relations, for instance fuzzy relations where $S_i(a, b)$ is 0 or 1 for every pair a, b. But such relations are obviously equivalent to crisp relations and so, Arrow's Theorem applies.

The outcome of an aggregation method (for fuzzy preferences) satisfying Independence of Irrelevant Alternatives, Non-dictatorship and Pareto (like (5.7) and different forms of majority) will therefore not always be transitive. Hence, an exploitation (see chapter 7) of the global preference relation will often be needed in order to reach a final recommendation (see chapter 2).

5.3.3.4 ELECTRE III

Equation (5.7) is a particular generalisation of simple majority. If we omit the weights and vetoes, the aggregation mechanism in ELECTRE III (Roy, 1996) can be seen as the same kind of generalisation of qualified majority. Our comments in the three previous paragraphs (sum of the valuations, other extensions and transitivity) thus also apply to ELECTRE III.

Let us mention here a recent paper by Mousseau and Dias (2004) about the elicitation of the parameters of a variant of ELECTRE III. They propose a disaggregation technique in the spirit of the techniques we recommend in section 4.4.6.

5.3.4 Pairwise aggregation into a fuzzy relation

In this section, we consider the aggregation of a profile of fuzzy preference relations into one fuzzy preference relation, contrary to the previous sections where the outcome of the aggregation was a crisp relation. In order to do this, we aggregate the relations S_i in a pairwise manner into one fuzzy relation S by means of an aggregation operator such as the arithmetic mean. For example, we could define

$$S(a,b) = \frac{1}{n}\sum_{i=1}^{n} S_i(a,b),\ \forall a,b \in A.$$

Of course, instead of the arithmetic mean, we can use the weighted arithmetic mean, the geometric mean, the median, an Ordered Weighted Average (OWA) operator, the min, the max, the leximin, etc. Any averaging or aggregation operator can do the trick (see García-Lapresta and Llamazares, 2000 and Example 4.11 in Perny, 1992). Section 5.7, of this chapter, is devoted to these operators, in a different context but the analysis remains valid. When evaluating the relevance of a condition for his application, the interested reader will thus go to section 5.7, bearing in mind that he is aggregating valuations of arcs from different preference relations (see Fodor and Roubens, 1994, section 5.10).

A key property in the analysis of aggregation operators is commensurability (p. 203–205), this is the fact that valuations on different criteria can be compared. So, a very careful construction of these valuations is necessary in order to guarantee commensurability.

Another important issue for the aggregation of valuations is the scale on which they are measured. As already mentioned in section 5.3.2.3, not much is known today about the scales on which preference intensities, credibilities or certainties are measured. And, as far as we know, to date, no technique has been proposed to construct fuzzy preference relations such that the valuations would be measured on, say, an interval scale. It is therefore prudent to use aggregation procedures that only take the ordering of the valuations into account and not the values themselves, unless there are good reasons to use the values.

5.3.5 General comment on the aggregation into a fuzzy relation

As seen in sections 5.2.1.3 and 5.3.3.3, when we want to aggregate crisp or fuzzy relations into one crisp relation using an aggregation method satisfying Independence of Irrelevant Alternatives, Non-dictatorship and Pareto, the result is not always transitive. But in this section, we want to aggregate profiles of preference relations into a fuzzy relation, not a crisp one. We may thus wonder if it is now possible to always obtain a transitive result. Unfortunately, the answer is negative.

Let us be more explicit: it is not possible to apply Arrow's Theorem in this context because Transitivity, as defined previously, does not make sense when the global preference relation is fuzzy. Transitivity must be redefined for fuzzy relations. A popular definition of transitivity for fuzzy relations is

$$S_i(a,c) \geq \min\{S_i(a,b), S_i(b,c)\}$$

(called min-transitivity). But another definition is

$$S_i(a,c) \geq S_i(a,b)S_i(b,c)$$

(called product transitivity) or

$$S_i(a,c) \geq \max(S_i(a,b) + S_i(b,c) - 1, 0)$$

(called Lukasiewicz transitivity). And there are many more. But there are many results in the literature showing that, for different definitions of transitivity, a variant of Arrow's theorem can be proved, showing that Independence of Irrelevant Alternatives, Transitivity and Pareto are not compatible with Non-dictatorship or with a slightly stronger condition imposing the absence of a coalition of criteria that would play the role of a dictator (see Banerjee, 1994; Barrett, Pattanaik, and Salles, 1986, 1992; Dutta, 1987). There are also few results showing that Independence of Irrelevant Alternatives, Transitivity and Pareto are compatible with a particular definition of transitivity. One such result, due to Ovchinnikov (1991), uses Lukasiewicz's transitivity. But this apparently positive result is not totally positive because Lukasiewicz-transitive relations can be very difficult to interpret as illustrated in figure 5.4. The relation depicted in this figure is Lukasiewicz-

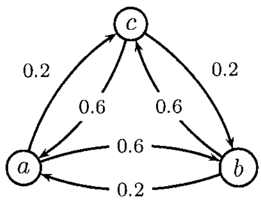


Figure 5.4: A Lukasiewicz-transitive fuzzy relation.

transitive because $0.2 \geq 0.6 + 0.6 - 1 \geq 0$. But, for a decision maker, it is as difficult to interpret as a cyclic crisp relation: it looks like a cycle. Note that if we cut the relation at any level between 0.2 and 0.6, we obtain the cyclic crisp relation $(a,b), (b,c), (c,a)$. If we cut above 0.6, we obtain an empty relation that does not help the decision maker either. Only when cutting under 0.2 do we obtain a weak order, but one where all alternatives are indifferent.

So, the aggregation of fuzzy relations into one fuzzy relation is also difficult.

As an application of these results, note that the methods described in the previous section (5.3.4) do not always yield transitive relations (whatever definition of transitivity you choose), since they satisfy Independence of Irrelevant Alternatives, Pareto and Non-Dictatorship.

Note that, when the outcome of the aggregation is a fuzzy preference relation, an exploitation (see chapter 7, section 7.4.3.2 and 7.4.4.2) is almost always necessary if we want to make a recommendation (see chapter 2). Suppose that the decision makers's problem is to find the best alternative. It is not possible, in general, to identify the best alternative just by looking at a fuzzy relation, even if it is transitive in some sense. This is why exploitation techniques are needed. If in addition, the outcome is not transitive, then the exploitation is even more necessary . . . and difficult.

5.3.6 The difficulty of aggregating fuzzy relations

Let us summarise the steps involved in the process of aiding a decision maker by modelling his preferences by means of fuzzy relations and aggregating them.

- It is almost never the case that fuzzy preference relations exist a priori, so, we must first construct the fuzzy preference relations (1 per criterion). This can be done by directly questioning the decision maker or using a construction technique (see section 5.3.1 and chapter 3). In the first case, we do not know exactly what properties the resulting fuzzy relations have. In particular, we do not know on what kind of scale the valuations lie. In the second case, it is sometimes possible to obtain well-behaved fuzzy relations, but with many techniques, the obtained fuzzy relations are not well understood (with PROMETHEE and ELECTRE III for example).

- We must then aggregate the fuzzy preference relations. Very few aggregation methods yield a weak order (a ranking, possibly with ties). The generalised Borda method does but it requires valuations that can be added and we probably seldom have such valuations. We will thus almost always end up with a non-transitive and eventually fuzzy global preference relation.

- We therefore need an exploitation step, which will not be simple. This is particularly true if the global preference relation is fuzzy because, in many cases, we will not exactly know the properties and the meaning of the valuations obtained at the end of the aggregation.

The number of steps and their complexity is such that we fear that the outcome will seldom be reliable (although there are cases where it is). Given our current knowledge, we think that it is often more sensible or prudent to take a simpler route. For example, instead of constructing fuzzy relations from performances and aggregating these relations, it might be better to directly use the performances in an aggregation method. It is simpler (one step instead of two) and is perhaps better understood so that we can use sound techniques for setting the parameters (if any) of the aggregation method.

5.4 Aggregation of a performance table into one relation

As mentioned in section 4.2.3, social choice theory is not only concerned with the aggregation of ordinal information (preference relations) but also of cardinal information. In this section, we present different characterisations of aggregation procedures that were first formulated in the frame of social choice theory. We will discuss the min, the weighted sum, the leximin and a family of procedures called *outranking procedures*, similar in some sense to the ELECTRE-like methods and PROMETHEE. There are of course many other aggregation procedures but we chose these because they allow us to present some important concepts or because they are close to some aggregation procedures commonly used in multicriteria decision aiding.

Before presenting these characterisations, we introduce some new notations and make some general comments about the nature of the cardinal information.

5.4.1 Notations and definitions

In this section, the descriptor g_i (introduced in section 4.3, p. 128) is assumed to take its values in $\mathbb{R}$. The values $g_i(a), g_i(b), \ldots$ can be interpreted as a more or less factual description of $a, b, \ldots$ on dimension i—g_i is then an element of D and $\mathcal{E}$ could be formalised as $X_1 \times \ldots \times X_n$ where each $X_i = \mathbb{R}$ (see chapter 2, p. 41)—or as the numerical representation of the decision maker's preferences with respect to viewpoint i—g_i is then an element of H (see chapter 2, p. 41).

In the first interpretation, our hypothesis is that all preferences are increasing with g_i, i.e. the larger an evaluation, on any criterion, the better the alternative. If we then face a problem in which the preference on a criterion is decreasing with g_i, it is generally obvious to make the necessary adaptations in the aggregation methods or in the axioms that we will present in this section.

In this context, a profile is a n-tuple of functions $g_1, \ldots, g_n$. We now use the symbol $\mathbf{g}$ for a profile. The symbol p is used only for profiles of preference relations. Note that a profile $\mathbf{g}$ can also be seen as a performance or evaluation table or matrix. It *contains* an evaluation for each alternative on each criterion.

We must now consider the nature of the information provided by the functions g_i. We distinguish several cases (according to d'Aspremont and Gevers, 1977, and Roberts, 1980).

Ordinal non commensurable. In this case, the only meaningful operation we can perform is the comparison of two evaluations on a single criterion. For example, $g_3(b) = 357$ is obviously larger than $g_3(c) = 287$. We cannot say anything about the distance between b and c. Only the order matters. Furthermore it is also impossible to compare evaluations on different criteria. For example, the statement $g_3(c) = 287 > g_1(c) = 36$ has no meaning.

Because the information is purely ordinal, it is perfectly equivalent to use a profile of weak orders, which contains exactly the same information, instead of the functions g_i. Therefore, we will not discuss this case; it has already been treated in section 5.2.

Ordinal commensurable. It can happen that all evaluations, for all criteria, are measured on the same ordinal scale. In such a case, only order matters, as in the previous case, but, in addition, inter-criteria comparisons make sense. For example, we can say that $g_2(c) = 8.6 > g_1(b) = 7.1$. This will make it possible to use the min or leximin, for instance. This hypothesis is frequent made for example in constraint satisfaction problems (see p. 144). There, the alternatives are different solutions to a problem (for example a scheduling problem) and, for $i = 1, .., n$, the function g_i measures the extent to which constraint i is satisfied (between 0 and 1). Because the different criteria (the satisfaction of the constraints) are of a similar nature, it is not unreasonable to consider that the satisfaction degrees can be compared across criteria.

It therefore makes sense to compare evaluations on different criteria when all criteria are measured on the same scale. But, be careful! Same scale is not equivalent to same range. For example, if an expert assesses the alternatives on three criteria (financial, social and environmental costs), using a scale from 0 to 10, it is very unlikely that the scales are the same, even if they have the same range. Indeed, it is hard to say if a 5 on the financial criterion is better, worse or equivalent to a 5 on the environmental criterion.

Interval non commensurable. The scale for each criterion is an interval scale (see chapter 3), i.e. the evaluations can only be transformed through positive affine transformations[2]. Therefore, $g'_i = \alpha_i g_i + \beta_i$ is as good an evaluation function as g_i. Note that the transformations need not be the same for all criteria. We can have all different α_i's and all different β_i's. Therefore, saying that

$$g_2(c) - g_2(a) = 2\ (g_2(b) - g_2(a))$$

is meaningful while saying

$$g_2(c) - g_2(a) = 2\ (g_1(d) - g_1(a))$$

is not. Indeed, if we use two different transformations for criterion 1 and criterion 2, the first statement remains true while the second one becomes false.

Interval with the same unit. The scale for each criterion is an interval scale. The evaluations can only be transformed using affine transformations such that $g'_i = \alpha g_i + \beta_i$. Here, α is the same for all criteria, which means that the same unit is used for all criteria but not necessarily the same origin. In this case, a statement like

$$g_2(c) - g_2(a) = 2\ (g_1(d) - g_1(a))$$

is meaningful because it is independent of the origin. But

$$g_2(c) = 2\ g_1(b)$$

is not meaningful. In other words, comparisons across criteria are not permitted but comparisons of differences make sense. This will make tradeoffs possible.

Ratio non commensurable. The scale for each criterion is a ratio scale. Evaluations can only be transformed by linear transformations such that $g'_i = \alpha_i g_i$. Comparisons across criteria are in general not meaningful but, because there

[2] An affine transformation f is a mapping that can be written in the form $f(u) = \alpha u + \beta$, where α and β are real constants. It is positive if $\alpha > 0$.

is a fixed origin, a special kind of comparison is possible. Suppose that $g_2(c) > 0$ and $g_1(a) < 0$. Then, after any transformation, $g'_2(c) > 0$ and $g'_1(a) < 0$. Therefore, the statement

$$g_2(c) > g_1(a)$$

is meaningful.

Ratio commensurable The scale for each criterion is a ratio scale with the same unit. The evaluations can only be transformed by linear transformations such that $g'_i = \alpha g_i$, with α identical for all criteria. So, all statements that are meaningful with ratio scales are meaningful here, even across criteria.

This case occurs, for example, when all criteria are expressed in monetary units and when we are interested in the amounts of money and not by their value or utility for one or several persons.

We can of course distinguish many other cases. Our list is not exhaustive. For example, interval with the same unit and origin. But this case is not interesting: on an interval scale, we always compare differences; so, the origin doesn't play any role and this case boils down to the case of interval scales with the same unit. Another case is when all criteria are expressed on incommensurable ordinal scales with a common fixed point. This can happen if all criteria are ordinal but, a neutral point is precisely identified on each one; a point such that every evaluation above it is considered as attractive and every evaluation under it is repulsive. An aggregation procedure might take advantage of the existence of this special point (see e.g. Grabisch and Labreuche, 2004). Unfortunately, we do not have much to say about this case.

Note that if some preferences are increasing with g_i and others are decreasing with g_j (see our comment p. 203), then it is very unlikely that some sort of commensurability exists between the criteria.

5.4.2 A comment about commensurability

Commensurability is very rare. It almost never happens that two scales are commensurable, even if we often assume they are, for commodity reasons. Often, when two scales seem commensurable, they are not. For example, suppose some projects must be ranked, taking only their costs in year 1 and 2 into account. The two costs are measured in Euros. So, apparently, the two scales are identical but it could be the case that the decision maker prefers a cost of 10^6 € in the second year to the same cost in the first year because he expects to have more liquidities in the second year (even after discounting). So, even if the consequences, measured in monetary amounts are commensurable, we do not necessarily know how a decision maker compares them. The simple algebraic comparison $g_i(a) \geq g_j(b)$ does not necessarily imply that $g_i(a)$ is at least as good as $g_j(b)$ (preferential comparison).

There are techniques that can help build commensurable scales. These techniques are based on conjoint measurement (MAVT) and are discussed in section 4.3 and chapter 6. But, using these techniques, in the process of constructing the scales, we necessarily also build a global preference relation on A. So, once we have obtained commensurable scales, we no longer need to use an aggregation method, because we already performed the aggregation. These techniques do not help us to construct commensurable scales that we can later use in a weighted sum for example. They yield the scales and the global preference relation simultaneously.

If we do not construct commensurable scales, when do we face such scales? A general answer to this question probably does not exits, but we see at least three classes of problems in which we might have commensurable scales.

- When alternatives have dispersed consequences. An alternative has dispersed consequences (Azibi and Vanderpooten, 2003; Keeney and Raiffa, 1976) when it has consequences of the same nature in different places (the impact of a factory all along a river it pollutes), at different moments (the impact of an investment over the next ten years) or for different persons (the impact of a new community policy on all people in that community) or units (the impact of a policy decided by a bank for all its branches).

  Suppose that, in order to estimate the aesthetical impact of a new freeway on the landscape, twenty points are selected along it. At each point, the maximum distance from which the freeway can be seen is considered as the impact. From a purely algebraic viewpoint, the twenty evaluations are on commensurable ratio scales. Now, from a preferential viewpoint, we probably do not have ratio scales (because the aesthetical impact might not vary linearly with distance) but we might still have commensurable scales (except if one of the twenty points lies in a national park, for example).

  Dispersion in time is more problematic because we seldom give the same importance to yesterday, today, tomorrow, next year and the next millennium. One Euro today often has more value than one Euro tomorrow while one Euro yesterday or in the next millennium has no value at all. But, in some cases, short- or mid-term, commensurability might hold. Suppose for example that, in a production planning problem, you estimate the average delivery time for every month of a year (the mean of the delivery times for all orders received during that month). It is probably reasonable to assume that a given average delivery time for month 3 is equivalent (in terms of preference) to the same average delivery time for month 7.

- In the constraint satisfaction problem (see p. 144).

- In the pairwise aggregation of fuzzy preference relations (see p. 199).

Our remark for the first case also holds for the last two ones: it is not because the ranges of the scales are the same that they are commensurable. So, a careful construction of the scales is necessary but perhaps not always possible. We now turn to the description and analysis of some simple and/or popular aggregation methods.

5.4.3 The min

The min, also called maximin, is a very simple method. The alternatives are ranked in the decreasing order of their minimum performance. Formally,

$$a \succsim(\mathbf{g})\, b \Leftrightarrow \min_{i \in N}[g_i(a)] \geq \min_{i \in N}[g_i(b)].$$

It is a very pessimistic aggregation method because it only takes the worst performance into account. Table 5.1 illustrates how the min 'works'. Note that, using

	g_1	g_2	g_3	g_4
a	10	*6*	8	*6*
b	*4*	9	5	7
c	7	9	6	*4*

Table 5.1: The min: the smallest performance of a (resp. b and c) is 6 (resp. 4 and 4). The ranking is thus $a \succ(\mathbf{g})\, [b \sim(\mathbf{g})\, c]$.

the min, the global preference relation $\succsim(\mathbf{g})$ is always a weak order, i.e. a ranking, possibly with ties.

Of course, a symmetrical or dual aggregation procedure can be defined: the max (or minimax). The alternatives are ranked in the increasing order of their maximal performance. Formally,

$$a \succsim(\mathbf{g})\, b \Leftrightarrow \max_{i \in N}[g_i(a)] \leq \max_{i \in N}[g_i(b)].$$

It is a very optimistic aggregation method because it only takes the best performance into account. Everything we state about the min in the next paragraphs can easily be adapted to the max.

5.4.3.1 Axioms and characterisation

We will use the following condition to characterise the min.

- *Weak Order.* See p. 123.

- *Strong Ordinality.* Suppose that, given $\mathbf{g}$, we change the performances in $\mathbf{g}$ in order to obtain $\mathbf{g}'$, in such a way that we never reverse the order between two performances or break an indifference. In other words, if $g_i(c) > g_j(d)$, then two cases are possible: $g'_i(c) > g'_j(d)$ or $g'_i(c) = g'_j(d)$. The case $g'_i(c) < g'_j(d)$ is not allowed (this is a reversal). If $g_i(c) = g_j(d)$, then only one case is possible: $g'_i(c) = g'_j(d)$. Suppose now that alternative a is globally at least as good as b in $\mathbf{g}$. If $\succsim$ is Strongly Ordinal, then a must still be globally at least as good as b in $\mathbf{g}'$. Very roughly, this means that only the order of the performances is relevant.

  Formally: let g_- be the smallest admissible or possible performance. Let ϕ be a non-decreasing mapping from $[g_-, \infty[$ into $\mathbb{R}$. If $\mathbf{g}'$ is such that $g'_i(a) = \phi(g_i(a))$ for all $i \in N$ and all $a \in A$, then $a \succsim(\mathbf{g})\, b \Rightarrow a \succsim(\mathbf{g}')\, b$.

- *Weak Reversibility.* Suppose that $a \succsim(\mathbf{g})\ b$. Then, by lowering any performance of a sufficiently, it is possible to obtain $b \succsim(\mathbf{g})\ a$.

- *Strong Reversibility.* Suppose that $a \succsim(\mathbf{g})\ b$ and none of the performances of b are equal to g_{-}. Then, by lowering any performance of a sufficiently, it is possible to obtain $b \succ(\mathbf{g})\ a$.

In a different context, Bouyssou and Pirlot (1997) proved the following theorem.

Theorem 5.10
If the smallest possible performance g_{-} can be attained, then the only aggregation function satisfying Weak Order, Strong Ordinality, Weak Reversibility and Strong Reversibility is the min, i.e. the alternatives are ranked in the decreasing order of their minimum performance.

For alternative characterisations, see Bouyssou (1991, 1995); Fortemps and Pirlot (2004); Pirlot (1995).

5.4.3.2 Discussion

Because the only operation we carry out on the performances is comparing them, they do not need to be on a scale that is stronger than ordinal. But, because we compare performances on different criteria, it is necessary that these performances be measured on commensurable scales. So, if we have ordinal commensurable scales and if the four axioms which characterise the min seem appealing to the decision maker, it makes sense to use the min. But this is not the only case. We have seen that the scales must at least be ordinal. So, if the scales are commensurable ratio scales and if the decision maker agrees with the axioms, then he should also use the min. Its use is not restricted to ordinal scales. The important issue is the commensurability of the scales.

If the smallest possible performance cannot be attained—for example if the range for the performances is $\mathbb{R}$—then Theorem 5.10 does not hold. This does not mean that the min should not be used. It just means that there might then exist other aggregation procedures which also satisfy Weak Order, Strong Ordinality, Weak Reversibility and Strong Reversibility. Furthermore one of them might be better suited to the decision maker's need than the min.

When using the min, it is very important to make sure that the performances on all criteria are on the same scales, in a very strong sense: if $g_i(a) \geq g_j(b)$, the decision maker must then agree that $g_i(a)$ is at least as good (or as desirable, as attractive, ...) as $g_j(b)$, for all pairs of criteria i, j and all pairs of alternatives a, b. This is a strong requirement. Here is an example where it is not satisfied: suppose some projects must be ranked, taking only their costs in years 1 and 2 into account. The two costs are measured in €. So, apparently, the two scales are identical but it could be the case that the decision maker prefers a cost of 10^6 € in the second year to the same cost in the first year because he expects to have more liquidities in the second year (even after actualisation).

Later on, we will present a method bearing some similarities with the min—the leximin—but let us first discuss the weighted sum.

5.4.4 The weighted sum

The weighted sum is a very popular and simple aggregation method. For each alternative, we compute a score $s_a(p)$ which is defined as the weighted sum of its evaluations:

$$s_a(\mathbf{g}) = \sum_{i \in N} w_i g_i(a). \tag{5.9}$$

The alternatives are then ranked in the decreasing order of their score. Table 5.2 illustrates how the weighted sum works.

	g_1	g_2	g_3
a	10	6	8
b	4	9	5
c	7	9	6

Table 5.2: The weighted sum: if $\mathbf{w} = (1, 2, 1)$ the score of a (resp. b and c) is 30 (resp. 27 and 31). The ranking is thus $c \succ(\mathbf{g})\ a \succ(\mathbf{g})\ b$.

5.4.4.1 Axioms and characterisation

- *Weak Order.* See p. 123.
- *Cardinal Pareto.* If alternative a is strictly better than b on all criteria, then a is globally preferred to b. Formally,

$$g_i(a) > g_i(b)\ \forall i \in N \ \Rightarrow a \succ(\mathbf{g})\ b.$$

  This condition is in fact almost the same as Pareto (see p. 175). The only difference is that Pareto is formulated in terms of preference relations while Cardinal Pareto is formulated in terms of performance tables.

- *Cardinal Independence of Irrelevant Alternatives.* The global preference between a and b depends only on their evaluations in $\mathbf{g}$ and not on the evaluations of other alternatives. In other words, if $\mathbf{g}$ and $\mathbf{g}'$ are two profiles such that, for every criterion i, $g_i(a) = g'_i(a)$ and $g_i(b) = g'_i(b)$, then $a \succsim(\mathbf{g})\ b \Leftrightarrow a \succsim(\mathbf{g}')\ b$. This condition is almost the same as Independence of Irrelevant Alternatives (see p. 172).
- *Cardinal Neutrality.* The result of the aggregation does not depend on the labels of the alternatives, but only on their evaluations in $\mathbf{g}$. This condition is almost the same as Neutrality (see p. 123).
- *Invariance w.r.t. Independent Translations.* Suppose that, given some $\mathbf{g}$, we change the performances in $\mathbf{g}$ in order to obtain $\mathbf{g}'$, in such a way that

$g'_i(a) = g_i(a) + \beta_i$. Suppose now that alternative a is globally better than b in $\mathbf{g}$. If $\succsim$ is Invariant w.r.t. Independent Translations, then a must remain globally better than b in $\mathbf{g}'$. Very roughly, this means that the performances as such are not really important. What really matters are the differences between performances. We speak here of Independent Translations because the performances on each criterion are translated by a different quantity β_i. We already presented a somewhat similar condition: Strong Ordinality (see p. 207). It could have been called Invariance w.r.t. a Common Non-Decreasing Transformation.

Note that if the performances $g_i(a)$ are utilities that have been previously constructed in such a way that differences of utility are meaningful, then Invariance w.r.t. Independent Translations certainly makes sense. But it might make sense in other circumstances as well.

- *Invariance w.r.t. a Common Multiplication.* Suppose that, given $\mathbf{g}$, we change the performances in $\mathbf{g}$ in order to obtain $\mathbf{g}'$, in such a way that $g'_i(a) = \alpha g_i(a)$. Suppose now that alternative a is globally better than b in $\mathbf{g}$. If $\succsim$ is Invariant w.r.t. a Common Multiplication, then a must still be globally better than b in $\mathbf{g}'$. This, combined with the previous condition, implies that only ratios of differences between performances are important. We speak of a Common Multiplication because the performances on all criteria are multiplied by the same amount α.

K. W. S. Roberts (1980) proved the following theorem.

Theorem 5.11
Suppose that, for each criterion i and each alternative a, the performance $g_i(a)$ can be any real number. Then, the only aggregation function satisfying Weak Order, Cardinal Pareto, Cardinal Independence of Irrelevant Alternatives, Cardinal Neutrality, Invariance w.r.t. Independent Translations and Invariance w.r.t. a Common Multiplication is the weighted sum, i.e. the alternatives are ranked in the decreasing order of their weighted sum.

5.4.4.2 Discussion

In this characterisation, the first four conditions imposed on the aggregation function are extremely reasonable. It is hard to find an example of a decision problem in which one of these conditions is questionable. But the last two deserve a closer examination; combined with the first four, they impose that the scales of the different criteria be interval scales with the same unit. We will show this now.

For the sake of clarity, let us consider a profile with two criteria. Suppose the decision maker is indifferent between a and b. We then have to choose the weights w_1 and w_2 in such a way that $w_1 g_1(a) + w_2 g_2(a) = w_1 g_1(b) + w_2 g_2(b)$ (we will discuss the choice of the weights later). This can be rewritten as

$$w_1[g_1(a) - g_1(b)] = w_2[g_2(b) - g_2(a)]$$

or

$$g_1(a) - g_1(b) = r_{21}[g_2(b) - g_2(a)], \tag{5.10}$$

where $r_{21} = w_2/w_1$. If we now want to compare two other alternatives c and d, we need to compare

$$w_1 g_1(c) + w_2 g_2(c) \quad \text{and} \quad w_1 g_1(d) + w_2 g_2(d)$$

or

$$g_1(c) - g_1(d) \quad \text{and} \quad r_{21}[g_2(c) - g_2(d)].$$

In the last line, it is clear that the quantities we compare are differences of performances. The performances thus need to be measured on interval scales. Furthermore, we compare differences of performances on different criteria. The scales for the criteria must therefore have the same unit.

But how do we know if we have interval scales with the same unit? Or how can we construct our scales in such a way that we are sure that they are interval scales with the same unit? These questions are addressed, to some extent, in chapter 3.

In (5.10), it also clearly appears that any difference on the second criterion is exactly compensated by r_{21} times that difference on the first criterion. A difference of 1 on the second criterion is compensated by a difference of r_{21} on the first one. A difference of 2 on the second criterion is compensated by a difference of $2r_{21}$ on the first one. And so on. The ratio r_{21} is called the substitution rate or tradeoff. It tells us how many units of criterion 1 each unit of criterion 2 is worth. It is important to note that substitution rate r_{21} is independent of the level of the performances. Whether the performances $g_2(a)$ and $g_2(b)$ are both low, both average or both high, their difference is compensated exactly by $r_{21}[g_2(a) - g_2(b)]$. This is a consequence of Invariance w.r.t. Independent Translations.

When there are more than two criteria, a substitution rate can be defined for all pairs of criteria and has the same properties as the substitution rate in a bicriteria problem.

In order to use the weighted sum, it is important that the performances be measured on interval scales with the same unit for all criteria and that the substitution rates be constant for all levels of the criteria. Here is an example (already introduced on p. 208) where it is not constant. Suppose some projects must be ranked, taking only their costs in year 1 and 2 into account. The two costs are measured in Euros. The two scales are thus interval scales with the same unit. Suppose the decision maker expects to have more liquidities in the second year and he is indifferent between a ($1.5\,10^6$ € in year 1, $2\,10^6$ € in year 2) and b ($1\,10^6$ €, $3\,10^6$ €). The substitution rate r_{21} is thus $1/2$. Let us now present two other projects to the decision maker: c ($1.5\,10^6$ €, $50\,10^6$ €) and d ($1\,10^6$ €, $51\,10^6$ €). The difference between the costs in the second year might now appear to be very small (relatively) when compared to the difference between the costs in the first year. The decision maker might then prefer c to d. The substitution rate would then be smaller than $1/2$. In other words, it would have changed.

It is interesting to note that none of the axioms of Theorem 5.11 involve weights. So, even if no weights are given, if the decision maker does not think about weights, the need for weights may appear as a consequence of the axioms or conditions imposed on the aggregation function. This is quite different from what we had in Theorem 5.4 where, in the axioms, the weights were considered as given.

Last remark: for convenience reasons, we often write that a performance $g_i(a)$ is any real number although, for a given criterion i, the range of g_i is often limited. The cost of a piece of equipment can neither be negative nor infinite! So, in some cases, strictly speaking, Theorem 5.11 cannot be applied. But it is probably not terribly wrong to apply it.

5.4.4.3 Choosing the weights

In order to set the weights, we need to ask questions to the decision maker, somehow as with the Weighted Condorcet method (see p. 180).

1. A first strategy is the following. Present a profile on two alternatives to the decision maker and ask him which one he prefers. Using (5.9), we then obtain an inequality involving the weights and the performances of the two alternatives. For example, suppose we present the following profile

	g_1	g_2	g_3	g_4
a	10	6	100	67
b	7	9	88	79

   and the decision maker says he prefers a to b. We then know that

$$10w_1 + 6w_2 + 100w_3 + 67w_4 > 7w_1 + 9w_2 + 88w_3 + 79w_4.$$

   Presenting more profiles will give us more inequalities. Eventually, we will have so many constraints that all weight vectors satisfying them will yield the same global preference relation. We then just have to pick one of these weight vectors and we are done. But it can also happen that no weight vector simultaneously satisfies all constraints. The decision maker might then be willing to modify his judgements. If not, he might be satisfied with a weight vector satisfying almost all constraints. If not, we are in trouble. The weighted sum is an aggregation function that is not well suited to the problem; probably because the scales are not interval scales with the same unit.

2. A second and more direct strategy is to present only profiles on two alternatives such that the performances of both alternatives are equal on all criteria but two. In addition, the performance of one alternative is not fixed on one of these two criteria. An example of this type of profile is:
   We then ask to the decision maker for what value of $g_1(b)$ he would be indifferent between a and b. Suppose he says 8. We then write (5.9):

$$10w_1 + 6w_2 + 100w_3 + 67w_4 = 8w_1 + 9w_2 + 100w_3 + 67w_4.$$

	g_1	g_2	g_3	g_4
a	10	6	100	67
b	$g_1(b)$	9	100	67

Table 5.3: Setting the weights: a profile on two alternatives.

After some simplifications, we find:

$$2w_1 = 3w_2.$$

If we repeat this operation for the pairs of criteria (1,3), (1,4), ... (1,n), we find all weights up to a multiplicative constant. If we follow the convention that the weights add up to 1, then they are completely known; after only $n-1$ questions.

Note that if we ask only the $n-1$ above-mentioned questions, we are sure that all answers will be compatible and will lead to a unique weight vector, up to a multiplicative constant. But if we ask additional questions, we might obtain contradictory information, as with the first strategy.

3. Many other strategies (mostly variants of the first two) can be thought of (see e.g. von Winterfeldt and Edwards, 1986, table 8.3).

The second strategy is faster to implement than the first but it also has some drawbacks. The compared profiles are almost always hypothetical, fictitious and any statement about such profiles is probably not very reliable. Any such statement is about something that the decision maker does not really know and, might even be about something impossible. Furthermore the task to be performed by the decision maker is not familiar to him. Comparing alternatives (as in the first strategy) is something he can (sometimes) do. Finding a performance for a given criterion that makes two alternatives indifferent is something he probably never does. It is therefore not certain he can do it in a reliable way. Nevertheless, this technique can 'force' the decision maker to think about his problem and promote the dialogue with the analyst. It is therefore an interesting maieutic tool.

Choosing the weights. A good way to set the weights with the weighted sum is to present a profile on two alternatives to the decision maker, in which the performances of the alternatives are identical except on two criteria, as in table 5.3. The decision maker needs to find the value $g_1(b)$ that makes a and b indifferent. This is done for the $n-1$ pairs (1,2), (1,3), ... (1,n) and yields $n-1$ equations, each one involving w_1 and one of the other weights. If we then choose a value for w_1, all other weights are fixed. They can eventually be normalised. It is a good idea to present more than the $n-1$ required profiles, in order to check the adequacy of the weighted sum.

In many applications of the weighted sum (see, e.g., Liu, Lai, and Wang, 2000), the criteria are first rescaled or normalised so that the largest performance is 1 and the smallest 0. Suppose we then apply a meaningful technique (like one of those presented above) for the elicitation of the weights. If, later, a new alternative is added and if some of its performances fall outside of the previous range for some criteria, we may be tempted to normalise the criteria (although this is not necessary). But if we do so, the scale of the values that will be added in the weighted sum will change and, hence, the weights must be changed accordingly. It is in fact much simpler not to normalise or renormalise the criteria. Then the weights never need to be changed when a new alternative is added. Furthermore, this normalisation is quite misleading: it gives the impression that a difference of, say, 0.1 on one criterion is worth the same difference on another criterion. This is not true. Even if two differences are numerically equal, they are not necessarily identical in terms of preferences. The normalisation also leads some people to believe that, because the normalised performances are without units, the weights only need to reflect the subjective intrinsic importance of the criteria. This is not true: the weights remain substitution rates or tradeoffs and must be elicited as above.

5.4.5 The leximin and leximax

The leximin is another simple aggregation method. For each alternative a, we define a new vector $\mathbf{g}^{\nearrow}(a)$ which is just a reordering of $\mathbf{g}(a)$ such that the performances in $\mathbf{g}^{\nearrow}(a)$ are increasing or at least not decreasing from left to right. For example, if $\mathbf{g}$ is as in table 5.4, then $\mathbf{g}^{\nearrow}(a) = (4, 5, 10)$, $\mathbf{g}^{\nearrow}(b) = (4, 5, 9)$ and $\mathbf{g}^{\nearrow}(c) = (5, 6, 6)$. In order to determine the ranking between two alternatives, say

	g_1	g_2	g_3
a	10	4	5
b	4	9	5
c	5	6	6

Table 5.4: The leximin. The ranking is $c \succ(\mathbf{g})\ a \succ(\mathbf{g})\ b$.

a and b, we then look at their reordered vector of performances $\mathbf{g}^{\nearrow}(a)$ and $\mathbf{g}^{\nearrow}(b)$. We first focus on the first component $g_1^{\nearrow}(a)$ and $g_1^{\nearrow}(b)$. If $g_1^{\nearrow}(a) > g_1^{\nearrow}(b)$, then $a \succ(\mathbf{g})\ b$. On the contrary, if $g_1^{\nearrow}(b) > g_1^{\nearrow}(a)$, then $b \succ(\mathbf{g})\ a$. If the two smallest performances are equal, i.e. $g_1^{\nearrow}(a) = g_1^{\nearrow}(b)$, then we look at the second component in order to try to make a distinction between a and b. We do this in the same way as for the first component. If we cannot make a distinction between a and b, i.e. $g_2^{\nearrow}(a) = g_2^{\nearrow}(b)$, we then look at the third component, and so on. If two vectors $\mathbf{g}^{\nearrow}(a)$ and $\mathbf{g}^{\nearrow}(b)$ are identical, then $a \sim(\mathbf{g})\ b$.

In other words: the leximin is the lexicographic method applied to $\mathbf{g}^{\nearrow}(a)$ and $\mathbf{g}^{\nearrow}(b)$.

We illustrate this using the example in table 5.4. By looking at the first component of $\mathbf{g}^{\nearrow}(a)$, $\mathbf{g}^{\nearrow}(b)$ and $\mathbf{g}^{\nearrow}(c)$, we find that $c \succ(\mathbf{g})\ a$ and $c \succ(\mathbf{g})\ b$ but we do not know how to rank a and b because their smallest performances are equal: it is 4. We thus look at the second smallest performance of a and b, i.e. at the second component of $\mathbf{g}^{\nearrow}(a)$ and $\mathbf{g}^{\nearrow}(b)$. They are also equal. So, we look at the third component and we find that $a \succ(\mathbf{g})\ b$. The ranking is thus $c \succ(\mathbf{g})\ a \succ(\mathbf{g})\ b$.

The leximax is the same method except that we use the vector $\mathbf{g}^{\searrow}(a)$, where the performances are ordered in decreasing order, instead of $\mathbf{g}^{\nearrow}(a)$.

Very briefly, the leximin focuses on the worst performances, irrespective of the criteria on which they are measured, while the leximax focuses on the best performances.

5.4.5.1 Axioms and characterisation

The following properties are characteristic of the leximin.

- *Weak Order.* See p. 123.
- *Cardinal Pareto.* See p. 209.
- *Cardinal Independence of Irrelevant Alternatives.* See p. 209.
- *Cardinal Neutrality.* See p. 209.
- *Anonymity.* Anonymity is verified when all criteria play exactly the same role. That is, we can permute the components of **g** (the columns of the performance table) without modifying the ranking of the alternatives.
- *Ordinality.* Suppose that, given **g**, we change the performances in **g** in order to obtain **g**′, in such a way that we completely preserve the order between the performances. In other words,

  $$g_i(c) \geq g_j(d) \Leftrightarrow g'_i(c) \geq g'_j(d).$$

  Suppose now that alternative a is globally better than b in **g**. If $\succsim$ is Ordinal, then a must still be globally better than b in **g**′. This just means that only the order of the performances is relevant. Note the difference with Strong Ordinality (see the characterisation of the min, p. 207): with Strong Ordinality, it is admitted that $g_i(c) > g_j(d)$ and $g'_i(c) = g'_j(d)$.
- *Independence.* Suppose we have two alternatives a and b such that the performances of a and b in **g** are identical on some but not all criteria (say the criteria in a set M). Suppose also that $a \succsim(\mathbf{g})\ b$. Consider now a new profile **g**′ identical to **g** except that some of the performances of a and b on the criteria in M have been modified, while keeping them equal (like in table 5.5).

	g_1	g_2	g_3	g_4
a	4	6	5	8
b	4	9	5	6

	g'_1	g'_2	g'_3	g'_4
a	3	6	7	8
b	3	9	7	6

Table 5.5: Two profiles on two alternatives such that Independence applies. The set M consists of criteria 1 and 3.

If Independence is satisfied, then $a \succsim(\mathbf{g}')\ b$. The reason for imposing such a condition is simple: when we compare a and b in profile **g**, we only pay attention to the criteria which are not in M because the criteria in M do not make a difference. And this leads us to considering that $a \succsim(\mathbf{g})\ b$. But when we compare a and b in the profile $\mathbf{g}'$, we also only pay attention to the criteria which are not in M for the same reason. Therefore, it seems logical that we come to the same conclusion, i.e. $a \succsim(\mathbf{g}')\ b$, because $\mathbf{g}$ and $\mathbf{g}'$ are just the same when we consider only the criteria not in M.

K. W. S. Roberts (1980) proved the following theorem.

Theorem 5.12
Suppose that, for each criterion i and each alternative a, the performance $g_i(a)$ can be any real number. Then, if there are at least three criteria, the only aggregation functions satisfying Weak Order, Cardinal Pareto, Cardinal Independence of Irrelevant Alternatives, Cardinal Neutrality, Anonymity, Ordinality and Independence are the leximin and the leximax.

For another characterisation, see Fortemps and Pirlot (2004).

5.4.5.2 Discussion

Anonymity is seldom a desirable condition: in many applications, we do not want the criteria to play identical roles. But here are some cases where it seems a reasonable condition:

- When the performances are evaluations given by different experts, stakeholders or voters and there is no hierarchy among these persons, then it seems a good thing to impose Anonymity.

- When all the performances express the same sort of consequence but for different comparable units, objects or persons (dispersed consequences) and if there is no hierarchy among these units, then Anonymity also seems interesting. For example, consider the choice among different potential locations for a household garbage dump. The garbage dump will cause nauseous smells. The nuisance caused by a smell is measured on an ordinal scale and varies with the nauseous gas concentration. Using a mathematical model, it is possible to predict the gas concentration in each house (the units) within a radius of 10 kilometres around the location. The consequences are thus nuisances caused by smells in each house. If we do not want to favour some house or inhabitant, we will impose Anonymity. Because all other conditions of Theorem 5.12 seem reasonable (at least to us) in this application and because the consequences are measured on the same ordinal scale (see below for a more thorough discussion of this point), the leximin and the leximax are probably "adequate" aggregation functions.)

- The flexible constraint satisfaction problem (flexible CSP) is another kind of problem where Anonymity seems natural (see p. 144).

Ordinality, as stated above, means that only order matters, not the performances themselves. It is very similar to Strong Ordinality (see par. 5.4.3.1, p. 207 and par. 5.4.3.2). Note that, just like Strong Ordinality, Ordinality makes sense only if all scales are commensurable, i.e. a performance on a scale must be comparable with a performance on another scale. This calls for a comment about the example of the garbage dump. It is probably true that the nuisance caused by the odour varies with the gas concentration: the higher the concentration, the stronger the nuisance. But it is certainly not true that the same gas concentration causes the same nuisance to each person. Some individuals are more sensitive to odours than others. Ordinality is therefore questionable in this application and, strictly speaking, it should be rejected. But, we might argue—this is open to discussion—that smell sensitivity does not vary so much between individuals, except for some rare cases, and that Ordinality is rather a reasonable condition. Then, the leximin and leximax are the only possibilities.

On the contrary, if we decide that nuisances are not comparable between individuals, then the available information we have (the performances) is ordinal and not commensurable. This is in fact equivalent to a profile of rankings and we are then back to the problem discussed in section 5.2. The leximin and leximax are no longer available options.

We repeat that, as on p. 208, even if the performances are measured on interval or ratio scales, it might make sense to impose Ordinality. Consider for example the choice between different potential locations for a facility to be accessed by different customers (the units). The consequences are travel times for each customer. If all customers have approximately the same importance, then we do not want to favour any customer and we might impose Anonymity. Contrary to the garbage dump example, the consequences are measured on ratio scales (travel times). Because all performances are measured on the same ratio scale, we might just impose Invariance w.r.t. a Common Multiplication (see p. 210) and not Ordinality (which could be called Invariance w.r.t. a Common Increasing Transformation). This, combined with the other axioms of Theorem 5.12 and some kind of continuity would force us to use the arithmetic mean.[3]

But it is not because we have a ratio scale (travel time) that we must impose Invariance w.r.t. a Common Multiplication. We may think that a gain of 5 minutes in travel time for a 15 minutes journey is more important, has more value than the same gain for a 30 minutes journey. The utility or value of travel time would thus not be proportional to travel time. Then, instead of arbitrarily saying that the utility or value of a travel time is equal to its square root (or logarithm or square or exponential), we might just say that the utility of travel time is measured on an ordinal scale. This brings us back to the problem of the garbage dump. An important question is then to find out whether the utilities are identical for all customers, in other words, if the travel times (or their utility) are commensurable across customers. If the customers are individuals, then we suspect that the answer is negative: not all people perceive and value time the same way. But if the customers are similar companies, then the answer could be

[3] This theorem (Roberts, 1980) is not presented in this book, but another characterisation of the arithmetic mean can easily be obtained by adding Anonymity to the conditions of Theorem 5.11, characterising the weighted sum. It is indeed clear that, if Anonymity is imposed, then all weights must be identical and we obtain the arithmetic mean. Here, we consider the arithmetic mean as an aggregation function and not as an aggregation operator as in section 5.7.

approximately 'yes' because each company consists of different people and that their "average" perception and value for time is perhaps roughly the same.

We now present an application, in robustness analysis, in which Ordinality definitely makes sense. Suppose we want to go from A to B and there are six possible routes: through C, D, E, F, G or H. We would like to choose the fastest route but because of some unpredictable events, the travel times for each route can take two different values. The travel times for these six routes are presented in table 7.9 in chapter 7, where this example is introduced. If we want to find a robust solution, we may then consider a decision problem with two criteria such that the performances on the two criteria are the travel times under the two different scenarios (for a motivation of this approach, see section 7.5.2). If we consider that the utility or value of travel time is measured on an ordinal scale, then Ordinality is a condition that we will definitely impose because the performances on the two criteria are utilities of travel times, for the same person. They are therefore perfectly commensurable. We know for sure that 28 minutes on route AEB is better than 29 minutes on the same route.

Suppose now that both scenarios seem equally possible; we then do not want to favour one of them and we may impose Anonymity. If we then also impose Independence and the other conditions of Theorem 5.12 (these are very reasonable), we find that we must use the leximin or the leximax.

Just as for the min, a crucial issue for the leximin is the commensurability of the scales. We must be able to compare performances on different criteria. If 5 is a performance on criterion i and 6 on criterion j, then it must be that the decision maker considers 6 as better than 5. The main difference with respect to the min is Independence. Take two alternatives that have the same performance on one criterion and lower that performance; if you lower it enough and if you use the min, you are sure that the two alternatives will be indifferent. With the leximin, because of Independence, if you lower two identical performances, the ranking between these alternatives will not be affected and if one is strictly preferred to the other one, they will remain so.

The last condition of Theorem 5.12 we want to discuss is Independence (see p. 215). This condition looks innocuous but is in fact not always acceptable. It is very similar to several other conditions introduced in section 4.3 and chapter 6: namely Weak Separability (p. 258), Weak Preference independence (p. 239) and Strong Preference Independence (p. 239). Because these conditions are discussed at length in section 4.3 and chapter 6, we refer the reader to those sections.

5.4.6 The outranking procedures

The expression *outranking procedure* has often been used informally to designate aggregation methods that, like ELECTRE I, PROMETHEE, TACTIC, etc., produce a global preference relation, sometimes not complete, based on pairwise comparisons of the alternatives. There is no formal definition of an outranking procedure.

In this section, we use the expression *outranking procedure* in a different way, as defined by Pirlot (1997). It is not unrelated to the informal expression but it is not equivalent. Outranking procedures are a large family of aggregation functions. Unlike most families of aggregation functions we have seen so far, outranking procedures are functions not only of a performance table but also of some parameters: the thresholds. For each criterion i, there are t_i thresholds: $\tau_{i,1}, \tau_{i,2}, \ldots, \tau_{i,t_i}$. These thresholds are non-negative real numbers and are ordered, i.e. $0 \leq \tau_{i,1} < \tau_{i,2} < \tau_{i,t_i}$. We use the symbol $\bar{\tau}$ to represent the collection of all thresholds to be used in a particular problem. In this section, an aggregation function is therefore a function $\succsim$ that associates a global preference relation denoted by $\succsim(\mathbf{g}, \bar{\tau})$ to each profile $\mathbf{g}$ and each collection of thresholds $\bar{\tau}$.

Another difference with the aggregation functions presented so far: outranking procedures are not defined by the computations we need to perform in order to use them, but by a list of properties or axioms that they satisfy. We can therefore not present outranking procedures without first presenting the axioms that they all must satisfy (by definition).

- *Cardinal Neutrality.* See p. 209.

- *No Reversal.* Suppose that, given a profile $\mathbf{g}$ and a collection of thresholds $\bar{\tau}$, we have $a \succsim(\mathbf{g}, \bar{\tau})\ b$. Construct a new profile $\mathbf{g}'$ identical to $\mathbf{g}$ except that some performances of a are raised and some performances of b are lowered. Formally, for every criterion i, $g'_i(a) \geq g_i(a)$ and $g'_i(b) \leq g_i(b)$. Because the position of a has improved and that of b has deteriorated and because we had $a \succsim(\mathbf{g}, \bar{\tau})\ b$, No Reversal imposes $a \succsim(\mathbf{g}', \bar{\tau})\ b$.

- *Cardinal Independence of Irrelevant Alternatives with thresholds.* The global preference between a and b depends only on their evaluations in $\mathbf{g}$ and on the thresholds, but not on the evaluations of other alternatives. In other words, if $\mathbf{g}$ and $\mathbf{g}'$ are two profiles with a vector $\bar{\tau}$ such that, for each criterion i, $g_i(a) = g'_i(a)$ and $g_i(b) = g'_i(b)$, then $a \succsim(\mathbf{g}, \bar{\tau})\ b \Leftrightarrow a \succsim(\mathbf{g}', \bar{\tau})\ b$. This condition is almost the same as Independence of Irrelevant Alternatives (see p. 172) and Cardinal Independence of Irrelevant Alternatives (see p. 209). The difference is purely formal.

- *Semi-Pareto.* This axiom is a variant of Cardinal Pareto (p. 209) and gives a meaning to the first threshold $\tau_{i,1}$. For a criterion i, $\tau_{i,1}$ represents the limit between the differences of performances that are considered as negligible or not important and those that are significant or not negligible. The threshold $\tau_{i,1}$ can therefore be seen as an indifference threshold. Taking this into account, we can adapt the Pareto condition as follows: if an alternative a is significantly better than another one (say b) on all criteria, then it cannot be globally worse than b. By 'significantly better', we mean that the difference in performances is larger than $\tau_{i,1}$. Formally, Semi-Pareto is satisfied if

$$[\forall i,\ g_i(a) > g_i(b) + \tau_{i,1}] \ \Rightarrow\ \text{Not}[b \succsim(\mathbf{g}, \bar{\tau})\ a].$$

- *Semi-Ordinality.* This axiom gives a meaning to the other thresholds. As we have seen, the first threshold $\tau_{i,1}$ represents the limit between the differences in performances that are considered as negligible and those that are significant. No distinction is made between negligible differences (thus smaller than $\tau_{i,1}$). The other thresholds will partition the larger differences in performances as follows: the differences in performance lying between $\tau_{i,1}$ and $\tau_{i,2}$ are considered as larger than those smaller than $\tau_{i,1}$ but no distinction is made between them. The differences in performance lying between $\tau_{i,2}$ and $\tau_{i,3}$ are considered as larger than those smaller than $\tau_{i,2}$ but no distinction is made among them. And so on. Hence, if we change some performances but if the differences in performances remain between the same thresholds, for each criterion, then the result of the aggregation should not vary. Formally, suppose we have two profiles $\mathbf{g}$ and $\mathbf{g}'$ such that, for every pair a, b in A, every criterion i and every j between 1 and t_i,

$$g_i(a) \geq g_i(b) + \tau_{i,j} \Leftrightarrow g'_i(a) \geq g'_i(b) + \tau_{i,j}.$$

Semi-Ordinality then imposes that $\succsim(\mathbf{g}, \bar{\tau}) = \succsim(\mathbf{g}', \bar{\tau})$.

Pirlot (1997) defines an outranking procedure as any aggregation function satisfying Cardinal Neutrality, No Reversal, Cardinal Independence of Irrelevant Alternatives with Thresholds, Semi-Pareto and Semi-Ordinality. The reader may now wonder what these outranking procedures look like, but it is very difficult to answer this question because this family is very large. Nevertheless, it is not difficult to see that they are based on pairwise comparisons (because of Cardinal Independence of Irrelevant Alternatives with Thresholds) and on differences in performances for each criterion (because of Semi-Ordinality). They are therefore quite close to the ELECTRE methods, TACTIC, PROMETHEE, etc. This is why they are been called outranking procedures.

By imposing an additional condition, Pirlot (1997) characterises a family of aggregation functions that is very much like ELECTRE I, without veto. This condition is

Componentwise Strong Ordinality. This condition is similar to Strong Ordinality (p. 207) but here we consider independent transformations for all criteria, i.e. we have n mappings ϕ_i instead of one mapping ϕ.
Formally, let ϕ_i be a non-decreasing mapping from $\mathbb{R}$ into $\mathbb{R}$. If $\mathbf{g}'$ is such that $g'_i(a) = \phi_i(g_i(a))$ for all $i \in N$ and all $a \in A$, then $a \succsim(\mathbf{g}, \bar{\tau})\, b \Rightarrow a \succsim(\mathbf{g}', \bar{\tau})\, b$.

Theorem 5.13
An aggregation function satisfies Cardinal Neutrality, No Reversal, Cardinal Independence of Irrelevant Alternatives with Thresholds, Semi-Pareto, Semi-Ordinality and Componentwise Strong Ordinality if and only if there is a set $\mathcal{C}$ of coalitions of criteria (to be interpreted as strong coalitions) such that

$$a \succsim(\mathbf{g}, \bar{\tau})\, b \quad \textit{iff} \quad \{i : g_i(a) \geq g_i(b)\} \in \mathcal{C}.$$

5.4.6.1 Discussion

In general, outranking procedures allow for the existence of an indifference threshold as defined by Semi-Pareto but in Theorem 5.13, there is no indifference threshold (in fact, it is equal to zero). This is a consequence of Componentwise Strong Ordinality. So, we could have used Cardinal Pareto instead of Semi-Pareto in the statement of Theorem 5.13 but because Semi-Pareto is part of the definition of an outranking procedure and because it is weaker than Cardinal Pareto, we prefer the statement with Semi-Pareto.

As already discussed, Cardinal Neutrality is essentially the same condition as Neutrality (see p. 123) and is very compelling except perhaps in situations involving a status quo.

No Reversal is in fact a weak version of Non-Negative Responsiveness (see p. 184). With Non-Negative responsiveness, when the position of a improves on some criteria w.r.t. b, the global position of a w.r.t. b can not deteriorate. With No Reversal, the global position of a w.r.t. b can deteriorate in the sense that a strict preference of a over b can be transformed in an indifference. But a preference (strict or not) of a over b cannot be transformed in a strict preference of b over a. This is an extremely weak condition. In just about all cases, we will want more than No reversal. It is therefore a very compelling condition.

Cardinal Independence of Irrelevant Alternatives with Thresholds is a more disputable condition. Because it is essentially the same condition as Independence of Irrelevant Alternatives, we refer the reader to the discussion on p. 174.

Semi-Pareto is a weaker condition than Pareto. It applies only if a is significantly better than b on all criteria whilst Pareto applies when a is better than b on all criteria. Because Pareto is a very reasonable condition in most (or even all) cases, Semi-Pareto seems very appealing.

Semi-Ordinality is a strange property of outranking procedures. In some sense, it imposes that a method be not too sensitive to small changes. Indeed, if we change some performances and if all performances remain between the same thresholds, the result of the aggregation must be the same. But, at the same time, the thresholds induce some discontinuities: if the difference between two performances is just smaller than a threshold and if we slightly change these performances in such a way that the difference between them becomes just larger than the threshold, anything can happen.

Another salient aspect of outranking procedures is very clear when we look at Semi-Ordinality: the outcome of an outranking procedure does not depend on the magnitude of the performances, but only on their order and their differences. If we add a constant to all performances on a criterion, no difference changes and, hence, the outcome does not change. Suppose now that the indifference threshold for a criterion expressed in Euros is 1 000€. This threshold will play the same role when we compare two alternatives with costs of 99 000€ and 99 500€ or two alternatives with costs of 100€ and 600€ because the differences are the same. But this is not necessarily what we want. If this is a problem in a given context, we can easily avoid it by using variable thresholds. But we then leave the world of outranking procedures (as defined by Pirlot).

The last condition we must discuss is Componentwise Strong Ordinality. It is not a characteristic of all outranking procedures, but only of those characterised by Theorem 5.13. Componentwise Strong Ordinality is a very strong condition. It has two aspects.

First, it imposes that the outcome of the aggregation be insensitive to any non-decreasing transformation of the performances. So, the magnitude of the performances is not relevant (this is also a consequence of Semi-Ordinality) but, in addition, the differences are also not relevant. So, only the order of the performances matters and, hence, all thresholds must be zero. This is probably reasonable if the performances are measured on an ordinal scale, that is, we have no information about distances between the various performances.

Second, it imposes that the outcome be insensitive to different transformations on the different criteria. This is quite different from Strong Ordinality that we presented in section 5.4.3 about the min (see p. 207). There we considered the same transformation for all criteria. Here, the performances on one criterion can be transformed independently of those on another criterion. A consequence of this is that the methods characterised by Theorem 5.13 are noncompensatory. Indeed, suppose we have two criteria (investment and exploitation cost) and two alternatives with the following performances:

	g_1	g_2
a	4	2
b	1	4

Suppose also that $a \succsim(\mathbf{g}, \bar{\tau})\ b$. Then, if we want to use the same procedure in order to compare two other alternatives c and d with the following performances

	g_1	g_2
c	10	0.2
d	1	0.4

we must conclude $c \succsim(\mathbf{g}, \bar{\tau})\ d$ because, on each criterion, the performances of c and d are in the same order as those of a and b. So, even though c is much worse than d on criterion 1 and only slightly better than d on criterion 2, the bad performance of d on criterion 2 cannot be compensated by its very good performance on criterion 1.

A situation where noncompensation is probably desirable or useful is when comparisons across criteria are difficult, as in the following example. Suppose your beloved is in a coma and you must choose between 3 different surgical treatments for him or her. You have a performance table with three criteria (cost in Euros, quality of life after treatment and chances of success). If you use a compensatory technique—say additive utility (see section 4.3)—to choose a treatment, then during the elicitation process, you will have to answer questions like 'Do you prefer a or b' where a and b are characterised by the following performances.

	Cost	QOL	Chances
a	30 000	6	0.7
b	10 000	6	0.6

This means that you will have to decide if increasing the chance of success of your beloved by 0.1 is worth 20 000 €. This is a tough question! Chances are you will not be able or willing to answer it. With a procedure like those characterised by Theorem 5.13, you would not have to answer such a question, where an increase on one criterion is compared to an increase on another one.

Note that, in our example, comparisons across criteria are difficult and, so, perhaps justify the use of a method insensitive to independent transformations of the scales but the performances are not measured on ordinal scales (at least not all of them). Using a method satisfying Componentwise Strong Ordinality is therefore not completely justified.

In Pirlot (1997), another result is presented, similar to Theorem 5.13 but where Componentwise Strong Ordinality is replaced by two other conditions, one of them expressing the fact that the second threshold $\tau_{i,2}$ is a veto threshold. The methods characterised by this theorem are very close to those of Theorem 5.13. The difference is the presence of a veto.

5.4.6.2 The outranking procedures and ELECTRE I

It is clear that ELECTRE I is an outranking procedure, in the sense of Pirlot (1997). If we put aside the vetoes, ELECTRE I is even one of the methods characterised by Theorem 5.13. These are more general than ELECTRE I because the set of strong coalitions is not necessarily defined by the addition of weights. The second main result in Pirlot (1997), that we do not present here, characterises a family even closer to ELECTRE I since it allows for vetoes.

A very particular property of ELECTRE I is its noncompensatory character which, in the absence of vetoes, is formally expressed by Componentwise Strong Ordinality. Before using ELECTRE I within an evaluation model, it is probably a good idea to check if the absence of compensation is desirable. A simple way to do this is to present two alternatives a and b such that the decision maker prefers one of them (say a) and such that the number of criteria for which a is better than b is as small as possible. Then improve significantly the performances of b on all criteria for which b is better than a. If the decision maker still prefers a to b, then ELECTRE I might be an appropriate aggregation method.

We already mentioned ELECTRE I several times in this chapter but we never mentioned noncompensation. The reason is that this is the first time we consider the aggregation of performances. Until now, we always considered the aggregation of preference relations and noncompensation was not really relevant in that context. The condition that makes the outranking procedures noncompensatory is clearly Componentwise Strong Ordinality.

5.5 Aggregation of a linguistic performance table into one relation

When each alternative is evaluated on each criterion by means of linguistic evaluations ("good", "average", "bad" or ' 'very comfortable", "comfortable", "acceptable", "unacceptable" or ...) and we want to construct a preference relation on the set of alternatives, several attitudes are possible.

- We have no other information than the linguistic evaluations and an order on these evaluations. For example, we know that "good" is better than "average" which is better than "bad". For some reason, we do not try or do not succeed in gaining more information. In this case, we are back to the case explored in section 5.2 (the aggregation of several binary relations into one relation).

- If the decision maker has enough time and is willing to spend some energy in such a process, it might be interesting to try to build a numerical representation on each criterion. Using the techniques of preference modelling (see chapter 3), we can arrive at numerical evaluations for each alternative on each criterion on scales that are stronger than ordinal (eventually interval scales). Instead of a numerical representation, we can also build a fuzzy representation, i.e. a model where each alternative is characterised on each criterion by a fuzzy number instead of a number. The next step is then to aggregate these performances into one global preference relation. If the performances are crisp, this brings us back to section 5.4; the case of the aggregation of fuzzy performances into one relation—which is popular in fuzzy control (see Bouyssou et al., 2000, chapter 7 or Nguyen and Kreinovich, 1998)—is not addressed in this book.

  Note that transforming the linguistic evaluations into numerical or fuzzy evaluations is not an easy task. It is not enough to say that we will (re)code "good" by 3, "average" by 2 and "bad" by 1 or to decide that "good" is represented by the trapezoidal[4] fuzzy number (6, 8.5, 10, 10), "average" by (3.5, 4, 6, 8.5) and "bad" by (0, 0, 3.5, 4). The representation must really represent the decision maker's preferences. It is therefore necessary to have a deep interaction with the decision maker to obtain the information that can help us move from an ordinal scale to a richer scale. For a discussion of techniques that may be of some help with fuzzy sets, see Bollmann-Sdorra, Wong, and Yao (1993); Marchant (2004a,b, forthcoming).

- In the previous case, we suggested replacing the linguistic evaluations by numerical or fuzzy evaluations through a preference modelling step. This can take place independently for each criterion. The evaluations we obtain are therefore on incommensurable scales and it is then not at all obvious how to aggregate these evaluations. In section 5.4, all of the aggregation methods we presented require some commensurability.

[4]An example of a trapezoidal fuzzy number is curve b in fig. 5.2.

It might then be wiser (but this might also be more difficult) to model the preferences on all criteria simultaneously in order to obtain commensurable scales. This process is known by the name conjoint measurement. It is familiar to all those who used MAVT, but it is not limited to MAVT. Thanks to the work of Bouyssou and Pirlot (2002a), we know that many different aggregation methods (even those usually grouped under the label of outranking methods) can be described in the framework of conjoint measurement. The reader interested in this approach should go to section 4.3 and chapter 6.

Note that, after the conjoint measurement (or conjoint preference modelling) step, no aggregation is needed because the modelling of the preferences on all criteria simultaneously is necessarily accompanied by the construction of the global preference.

Choosing between the three attitudes described above is not easy. If the decision maker has a lot of time and is willing to cooperate with the analyst, if the decision maker is able to give consistent answers to the analyst, to give him the necessary information, then the last attitude will probably yield the global preference relation with the strongest validity because it will be based on a large amount of consistent and relevant preferential information.

But if time is short or if the decision maker does not really want to cooperate or if he is not able to give consistent answers, then the second or even the first attitude might be equally or even more valid than the third one. If the decision maker does not really understand your questions or if he has no time to think about your questions, the global preference relation might just be the outcome of a stochastic (or chaotic) process and not really reflect the decision maker's preferences. In such a case, it might then be better to ask the decision maker for less information and to replace it by some more or less normative principles (some axioms characterising an aggregation function) that can eventually be discussed with the decision maker. The result would then neither be a better reflection of the decision maker's preferences nor a worse one. But—this is important—it would be consistent and you would know on what it is based.

5.6 Choice functions

Until now, we always considered the problem of aggregating a profile of preference relations (or a performance table) into a global preference relation. If this global preference relation is a weak order, then this problem is often referred to as the ranking problem statement (see section 2.4.3 and Roy, 1996). But, in many cases, the decision maker's problem is not stated in terms of ranking alternatives. There are many other possible problem statements. Formulating a problem statement is a difficult issue and also a very important one if we do not want to find the correct solution of an erroneous problem. We will not develop this point here because it has been partially addressed in chapter 2 and will be further discussed in chapter 7. Suppose our decision maker just needs to choose one alternative: the best one. He is not interested in a ranking. This problem is known as the choice problem statement. There are typically two ways to handle this problem.

- The decision maker first constructs a global preference relation, by aggregating the single-criterion preference relations. He then tries to base his choice on the analysis of the global preference relation. This second step is often called the exploitation of the global preference relation. It is discussed in chapter 7, section 7.4.
- The decision maker directly constructs a choice set (a set with the best alternatives) in one step. Only this approach will be discussed in this section.

Many aggregation procedures can be used for ranking or choosing; in the latter case, they provide the decision maker with a choice set which contains the best alternatives (one or more alternatives). For example, with the Borda method, instead of ranking the alternatives according to their score, you just choose the alternatives with the highest score. With the Condorcet method, the choice set contains the alternatives that are preferred to all other alternatives (this choice set can eventually be empty). For many procedures, switching from ranking to choosing requires only a small and obvious adaptation. But the choice functions that we then obtain are no longer aggregation functions and the characterisations that we presented above no longer hold. We need new characterisations of choice functions.

For some methods, we have both kinds of characterisations (choice and ranking). For example, Young (1974) characterised the Borda method as a choice function. Fortunately, in all cases in which we have the two kinds of characterisations, the axioms are almost the same; they express the same kind of ideas and only small adaptations are necessary. This seems to indicate (but does not prove) that characterisations of ranking and choice methods are essentially the same. Therefore, if we know a characterisation of an aggregation procedure used for ranking, we can quite safely use this characterisation for practical purpose, in order to try to understand the corresponding choice function.

The opposite adaptation, i.e. transforming a choice function into an aggregation function (for ranking) is often possible but less obvious. There can be several ways to convert a choice function into an aggregation function.

In the following subsection, we explain how it is possible to take axioms pertaining to the ranking problem and adapt them to the choice problem. Of course, this is relevant only if the method one wants to analyse comes in two flavours: ranking and choice.

5.6.1 Adapting the axioms to the choice problem

Adapting the axioms is usually an easy job. Let us use the symbol C for the choice set. C is a function of p. $C(p)$ is thus the choice set based on the profile p. We first give three general principles of axioms translation and we then show some examples:

ranking	choice
$a \succ(p)\, b$	if a is in $C(p)$ then b is not in $C(p)$
$a \succsim(p)\, b$	if b is in $C(p)$ then a is also in $C(p)$
$\succsim(p)$ depends only on ...	$C(p)$ depends only on ...

Based on these principles, it is easy to adapt some axioms to the choice problem. We provide some examples below.

Cancellation. If, for any pair of alternatives, there are as many criteria in favour of the first alternative as in favour of the second one, then all alternatives are in the choice set.

Faithfulness (in the case of a profile of linear orders). If there is only one criterion, i.e. $N = \{1\}$, then the choice set contains only the alternative which is ranked first on that criterion.

Neutrality. The axiom presented in section 4.2.2 (p. 123) does not need to be changed.

Anonymity. All criteria play the same role. In other words,

$$C(\succsim_1, \succsim_2, \ldots, \succsim_n) = C(\succsim_n, \succsim_1, \ldots, \succsim_2) = C(\succsim_3, \succsim_n, \ldots, \succsim_1) = \ldots.$$

Positive Responsiveness. Suppose a is globally not worse than b (i.e., if b is in $C(p)$ then a is also in $C(p)$). Suppose also that p' is identical to p except for one criterion where the position of a has improved with respect to b. If $\succsim$ satisfies positive responsiveness, then a is globally strictly better than b (if a is in $C(p')$ then b is not in $C(p')$).

Most axioms can easily be transposed to the choice problem, as illustrated above. The interested reader will thus be able to analyse choice methods by transposing characterisations obtained for ranking methods. Of course, before trying such a transposition, one should check that no characterisation of the choice method exists in the literature.

5.7 Aggregation of a performance vector into one single performance

Suppose we have an object a, characterised by some performances or evaluations (real numbers) on different criteria: $g_1(a)$ on the first criterion, $g_2(a)$ on the second, and so on. These numbers may eventually be the outcome of a preference modelling process, they may be utilities (g_j is then an element of H in the evaluation model. See chapter 2, p. 41). Or they can be just performances or evaluations not reflecting any preferences (g_j is then an element of D in the evaluation model. See chapter 2, p. 41). We might want to aggregate or summarise these performances into one single global performance. We then immediately think of the arithmetic mean, or the weighted arithmetic mean, the median, the min, etc. All these operations that aggregate a vector $\mathbf{g}(a)$ of real numbers into a single real number $G(a)$ are called aggregation operators and this section will be devoted to them.

It is important to make a clear distinction between this section and section 5.4. In that section, we also use the weighted sum and the min (among others), but we then use the global score or performance to derive a ranking: the alternatives are ranked in decreasing (or increasing) order of their global scores. In this section, we are interested in the global performance itself, not in the ranking that we can derive by comparing global scores.

At first sight, this does not make a big difference and, indeed, the calculations are exactly the same. But the conditions or axioms characterising, for example, the weighted sum are not the same if we are interested in the ranking derived from the weighted sum or in the weighted sum itself. In particular, the axioms in this section will usually be stronger than those given in section 5.4 for the following reason: suppose that, using the weighted sum as in section 5.4, we obtain $a \succ(\mathbf{g})\ b$. If we used the weighted sum raised to the power 3, we would get the same result, because

$$x \geq y \Leftrightarrow x^3 \geq y^3.$$

In other words, the ordering between any two numbers is preserved if we raise them to the third power. Similarly, if we used the exponential of the weighted sum, we would also get $a \succ(\mathbf{g})\ b$ because

$$x \geq y \Leftrightarrow e^x \geq e^y.$$

This is in fact true for any strictly increasing function applied to the weighted sum (e.g., the square root, arctan, ...). So, when we only look at the derived ranking, the conditions characterising the weighted sum also characterise the square root of the weighted sum and many others because they are all equivalent. But if we want to characterise the weighted sum itself (not the derived ranking), then we need to impose additional conditions that make a distinction between the weighted sum and all the increasing transformations of the weighted sum (square root of the weighted sum, etc.).

This way of reasoning not only applies to the weighted sum, but to all aggregation operators: the min, the max, the ordered weighted average (see below in this section), and so on.

It is then natural to ask the following question: when are we interested in the global performance of an alternative rather than in its position in a ranking ? We can distinguish at least two situations where this is the case.

- Suppose the decision maker considers a hierarchy of criteria, i.e. some criteria are decomposed into sub criteria which, in turn, can also be decomposed into sub-sub criteria, and so on. It is then sometimes interesting or convenient to perform the aggregation at different levels, i.e. aggregate at the level of the sub criteria and then at the level of the criteria (we suppose here that there are only two levels in the hierarchy). In such a case, after the first aggregation (at sub criteria level), what we are interested in is not a ranking but aggregated performances that we can use in the next aggregation (at criteria level).

- There are also situations in which, after the aggregation, a relative evaluation (a ranking) of the alternatives is not sufficient. An absolute evaluation is needed, imposed or customary. Think of the students' grades, the Dow Jones, the life expectancy, and the many different indexes used in almost all areas of human activity (see Bouyssou et al., 2000, ch. 4).

In these cases, it is however important to remain critical. It is not because an absolute evaluation is customary that we necessarily need it. For example, a bank that evaluates credit applications probably does not need an absolute and numerical evaluation. A very rough ranking or a classification in ordered classes (e.g., very good, good, acceptable, problematic, unacceptable) is probably significant enough.

Another context where the aggregation of a vector of numbers into one number is relevant is the pairwise aggregation of fuzzy preference relations (section 5.3.4). But, here, the numbers are not performances of alternatives, they are valuations of different fuzzy preference relations for a given pair of alternatives.

In this section, we will review some popular aggregation operators, present their characterisation and try to identify contexts in which they are appropriate. As in section 5.4, it is important to consider the nature of the information provided by the numerical performances. Are the scales identical or not? Are the performances measured on an ordinal, interval or ratio scale? The reader will find a discussion of these points in section 5.4, p. 203.

5.7.1 Notation

As in section 5.4, the performance vector of alternative a could be denoted by $\mathbf{g}(a)$. But, in this section, because we are looking at absolute evaluations and not at rankings, most of the time, we will consider only one alternative at a time. We can thus safely drop the name of the alternative and use the simplified notation $\mathbf{g} = (g_1, \ldots, g_n)$ for the vector of performances. The aggregation operator will be denoted by G. Because the number of performances (the number of criteria or dimensions) to be aggregated can vary, we will use the superscript (n) to specify the number of arguments of a given operation aggregator. For example, $G^{(3)}$ is an aggregation operator for vectors of size 3. When we speak of a specific aggregation operator, we indicate this with a subscript. For example, if G is the arithmetic mean, we use the subscript $\overline{x}$ and we write

$$G_{\overline{x}}^{(n)}(g_1, \ldots, g_n) = \frac{g_1 + \ldots + g_n}{n}.$$

Similarly, $G_{min}^{(n)}$ will denote the min, i.e.

$$G_{min}^{(n)}(g_1, \ldots, g_n) = \min(g_1, \ldots, g_n).$$

5.7.2 The arithmetic mean

This well-known operator is defined by:

$$G_{\overline{x}}^{(n)}(g_1, \ldots, g_n) = \frac{g_1 + \ldots + g_n}{n}.$$

5.7.2.1 Axioms and characterisation

The arithmetic mean has, among others, the following properties.

- *Idempotency.* When all performances of an alternative are identical, then the global performance should be equal to the single-criterion performances. In other words,

$$G^{(n)}_{\overline{x}}(g,\ldots,g) = g.$$

  This condition obviously makes sense only if all performances are measured on the same scale and if the global performance also needs to be measured on the same scale. We stress again that "same scale" does not only mean scales with the same range (say, from 0 to 1) but fully commensurable scales (for a more thorough discussion of this question, see stepping stones on p. 208). This can eventually be the case if the performances are not just "raw" performances but utilities or numbers resulting from a preference modelling process.

- *Cardinal Neutrality.* See section 5.4.4.1, p. 209.

- *Continuity.* Continuity ensures that, if a performance g_i changes slightly, then the global performance will not change dramatically. Small changes in the single-criterion performances can only cause small changes in the global performance $G^{(n)}(\mathbf{g})$. Technically, this condition is spelled out as follows: $G^{(n)}$ is a continuous function of $g_1, g_2, \ldots, g_n$.

- *Strict Monotonicity.* Suppose that two alternatives a and b are identical except on criterion i. Suppose also that $g_i(a) > g_i(b)$. Strict monotonicity imposes then that $G^{(n)}(\mathbf{g}(a)) > G^{(n)}(\mathbf{g}(b))$. This condition is of course related to Cardinal Pareto (p. 209) but it is stronger because Cardinal Pareto applies only if a is strictly better than b on *all* criteria.

  Note that we often use a rounded version of the arithmetic mean and that this rounded version does not verify Strict Monotonicity. Indeed, take a and b such that $\mathbf{g}(a) = (3.12, 4.32, 2.71)$ and $\mathbf{g}(b) = (3.12, 4.32, 2.70)$. After rounding, we obtain $G^{(n)}_{\overline{x}}(\mathbf{g}(a)) = 3.38 = G^{(n)}_{\overline{x}}(\mathbf{g}(b))$. So, even if a and b are not the same, they get the same global performance because the small difference between them was lost in the rounding process. But this is usually not a problem because, if we did not use rounding, we would probably consider the difference as negligible anyway. So, this shows that Strict monotonicity is probably not an important or crucial condition.

- *Decomposability.* This property is convenient when we perform the aggregation at different levels. Suppose that we have some criteria that are decomposed into sub criteria, themselves decomposed into sub-sub criteria. We do not want the overall performance to depend on the way this hierarchy of criteria is structured. If we use such a hierarchy, it is because it is convenient and helps us think in a structured way, considering only one small sub problem at a time. Of course, another decomposition of the problem into sub problems might also work. The overall performance should therefore be independent of the decomposition we chose. This is in some sense what Decomposability says. Formally, an aggregation operator G satisfies Decomposability if

$$G^{(n)}(g_1, \ldots, g_k, g_{k+1}, \ldots, g_n) = G^{(n)}(\underbrace{g, \ldots, g}_{k \text{ times}}, g_{k+1}, \ldots, g_n),$$

where $g = G^{(k)}(g_1, \ldots, g_k)$. We may replace k different performances by k times the value corresponding to their aggregation. Note that this condition has some similarities with Consistency (see p. 123).

- *Stability w.r.t. a Common Translation.* Suppose two alternatives a and b are such that the performance of b on each criterion is equal to the performance of a plus a constant β (for example $\mathbf{g}(a) = (3, 4.5, 1.8)$ and $\mathbf{g}(b) = (4, 5.5, 2.8)$). Then the global performance of b is equal to the global performance of a plus the same constant β. Formally, Stability w.r.t. a Common Translation is defined by

$$G^{(n)}(g_1 + \beta, \ldots, g_n + \beta) = G^{(n)}(g_1, \ldots, g_n) + \beta.$$

This condition is often presented as follows.

> If we change the scale of measurement by adding a constant to all performances (like displacing the origin of the time scale from 1 A.D. to 622 A.D. as in the traditional muslim calendar) and we then compute the global performance, we obtain the same result as if we first aggregated the performances and then added the same constant to the global score.

This interpretation is quite attractive but misleading. It implicitly assumes that we want to use the same aggregation operator on both scales (before and after the addition of a constant). But it is not clear at all why we would want to use the same one (see Narens, 2002, for a deep discussion of these issues). When we use a weighted sum, it is clear for everyone that, if we change the unit of measurement of one criterion, we have to accordingly change the weight of that criterion, i.e. we change the aggregation procedure. And this is accepted by everyone. So, why impose that the aggregation operator be independent of the scale of measurement?

We must therefore only keep the first interpretation in mind, which looks at the relation between the global performances of pairs of alternatives with a particular structure, in the absence of a change of scale. The question we must now answer is: when is Stability w.r.t. a Common Translation a sensible condition? Instead of a clear answer, we will just present two examples of cases where Stability w.r.t. a Common Translation is not desirable.

  - Suppose the performances of student a for three different courses are $\mathbf{g}(a) = (3, 2, 3)$, with 0 and 10 indicating respectively the worst and best possible performances. It would make sense to give him a global score equal to 2, thereby penalising his consistently bad scores. Suppose now that the performances of another student b are $\mathbf{g}(b) = (9,8,9)$. A decision maker might be tempted to reward his consistently good scores and give him 9 as global score. Now note that $\mathbf{g}(b) = \mathbf{g}(a) + 6$ and $G^{(n)}(\mathbf{g}(b)) \neq G^{(n)}(\mathbf{g}(a)) + 6$, contradicting Stability w.r.t. a Common Translation. Finally, note that this has nothing to do with the nature of the scales: this discussion does not rely on the hypothesis that the performances lie on some particular scale. It just depends on the opinion or preferences of a decision maker.

- Consider now a set of investments characterised by their rates of return in years 1, 2 and 3 (for example, 0.95, 1.01 and 1.12). If we want to aggregate the three rates into one rate, representing the average rate over three years, it is well-known that we must use the geometric mean, i.e.

$$G^3(g_1, g_2, g_3) = (g_1 g_2 g_3)^{1/3}$$

and not the arithmetic mean. This indicates that Stability w.r.t. a Common Translation is not a desirable condition in this case. Note though that the scale on which the rates are measured is a strong one: it is an absolute scale.

This condition is very similar to Invariance w.r.t. Independent Translations introduced on p. 209, for the weighted sum. There are however two differences:

- Here, we look at the global performance and not at the ranking derived from the global performance. For this reason, we use the name Stability instead of Invariance.
- Here, we consider identical translations on all criteria while the translations could be different on different criteria with the weighted sum (see p. 209).

Finally note that it is necessary that the performances $g_i(a)$ be constructed in such a way that differences of utility are meaningful (see section 4.3.9, p. 142), if we want to impose Stability w.r.t. a Common Translation but it is not sufficient.

- *Stability w.r.t. a Common Multiplication.* This condition is very similar to the previous one: Stability w.r.t. a Common Translation. Here, instead of adding a constant β we multiply by a constant α. Formally, Stability w.r.t. a Common Multiplication is defined by

$$G^{(n)}(\alpha g_1, \ldots, \alpha g_n) = \alpha G^{(n)}(g_1, \ldots, g_n).$$

There is also a misleading presentation of this condition, based on a change of scale. In this case, the change is no longer a change of origin (a translation) but a change of unit (a multiplication). This interpretation should not be used for the same reasons as those presented for Stability w.r.t. a Common Translation (see p. 231).

The following theorem (Kolmogoroff, 1930) uses these seven axioms to characterise the arithmetic mean.

Theorem 5.14
Suppose that, for each dimension i and each alternative a, the performance $g_i(a)$ can be any real number in some interval. The only aggregation operator satisfying Idempotency, Cardinal Neutrality, Continuity, Strict Monotonicity, Decomposability, Stability w.r.t. a Common Translation and Stability w.r.t. a Common Multiplication is the arithmetic mean.

5.7.2.2 Discussion

The long list of axioms characterising the arithmetic mean and the nature of these axioms (see above for a discussion) show that the arithmetic mean, although very natural in statistics, is not so likely to be a "good" aggregation operator in many contexts. Yet such a context may exist. Note also that the range of the scale (we don't use the plural because all criteria must be measured on the same scale) for the performances can take different forms. For example, $[0, 1]$, $[0, \infty[$ and $]-\infty, \infty[$.

To summarise very roughly, if we want to use the arithmetic mean, we need to be sure that all criteria play exactly the same role, that all performances, along all criteria, are measured on the same interval scale. If the performances are measured on a "strong" scale (like mass, length, price, temperature, ...) and if we just want to summarise this factual information, the conditions for the arithmetic mean are probably met. But if the performances are subjective and/or measured on ordinal scales (loudness, risk, aesthetic, reliability, ... although it is sometimes possible to measure these attributes on interval or ratio scales) or if we are not interested by the performances *per se* but by the value or utility attached to these performances by the decision maker, then we should probably not use the arithmetic mean, unless we build the scales very carefully.

5.7.3 Quasi-arithmetic means

This is a family of aggregation operators, generalising the arithmetic mean. We say that an aggregation operator $G^{(n)}$ is a quasi-arithmetic mean if there is a continuous and strictly monotonic function f such that

$$G^{(n)}(g_1, \ldots, g_n) = f^{-1}\left(\frac{f(g_1) + \ldots + f(g_n)}{n}\right).$$

The arithmetic mean is a quasi-arithmetic mean with $f(x) = x$. The geometric and harmonic means are also quasi-arithmetic means with $f(x) = \log x$ and $f(x) = 1/x$ respectively.

The family of quasi-arithmetic means is characterised (Kolmogoroff, 1930) by Idempotency, Neutrality, Continuity, Strict Monotonicity and Decomposability. The arithmetic mean is the only one that also satisfies Stability w.r.t. a Common Translation and a Common Multiplication. Any other quasi-arithmetic mean can be characterised by imposing some kind of Stability condition applied not to the performances themselves but to the performances transformed by f on top of the previous five conditions.

The five axioms characterising the family of quasi-arithmetic means are quite reasonable and tend to make them attractive. But as soon as we want to isolate one of them, we need some kind of stability condition that we cannot easily justify. This makes these quasi-arithmetic means no more attractive than the arithmetic mean.

5.7.4 Min, max and the other order statistics

The min is the operator that maps each vector of performances on the smallest performance.

$$G^{(n)}_{\min}(g_1, \ldots, g_n) = \min(g_1, \ldots, g_n).$$

The max is defined in a similar way. Min and max are particular cases of order statistics. The kth order statistic, denoted by $G^{(n)}_{OS_k}$, is equal to the k-th smallest performance. It is defined for $k = 1 \ldots n$. Obviously, when $k = 1$, $G^{(n)}_{OS_1} = G^{(n)}_{\min}$. When $k = n$, $G^{(n)}_{OS_n} = G^{(n)}_{\max}$. Another particular case is the median: it corresponds to the case $k = (n+1)/2$ if n is odd.

The family of order statistics was characterised by Fodor and Roubens (1995). The axioms in their characterisation implicitly indicate that the performances to be aggregated are measured on an ordinal scale and that the global performance must also be interpreted on an ordinal scale. The use of the aggregation operator $G^{(n)}_{\min}$ in such a context is in fact equivalent to the use of the min as an aggregation function, aggregating a performance table into a binary relation (see section 5.4, p. 207). We therefore do not present the characterisation of Fodor and Roubens (1995); not only because it would be redundant with Theorem 5.10, but also to stress the fact that, in spite of appearances, it sheds some light on the min as an aggregation function, in the spirit of section 5.4, and not as an aggregation operator.

This does not mean that using the min or another order statistic as an aggregation operator with performances on interval or ratio scales is meaningless. But we are not aware of any theoretical result that would help us understand what the distinctive properties of this operator are.

5.7.5 The weighted mean, the weighted sum and the other aggregation operators

It is possible to generalise the weighted mean exactly in the same way as the quasi-arithmetic mean generalises the arithmetic mean. We then obtain the quasi-linear weighted mean, that was characterised by Aczél (1948). If we add a stability condition to Aczél's axioms, it is possible to characterise the weighted mean. We do not present these results here. They can be found, with related results, in Fodor and Roubens (1994) for example. These results always involve Idempotency. But if we want to aggregate performances on different scales, Idempotency does not make sense. So, it is natural to turn to the weighted sum because it can eventually be used even if the scales are not the same for all criteria. Surprisingly, we did not find a characterisation of the weighted sum.

In the case of aggregation functions (see section 5.4), the weighted mean and the weighted sum are equivalent because we look only at the induced ordering. But in this section, we are considering aggregation operators and the two operators are not equivalent.

There are of course many other aggregation operators and several of them have been characterised: the weighted minimum and maximum (Dubois and Prade, 1986), the OWA or ordered weighted average (Fodor, Marichal, and Roubens, 1995; Yager, 1988), the Choquet and Sugeno integrals (Grabisch, Nguyen, and Walker, 1995), ... (see also Marichal, 1998). We will not discuss them here because we do not aim at exhaustivity. We just presented some operators that are often used by analysts or that allow us to introduce some important concepts. We hope that, after reading this section, the reader interested in other operators will be prepared to dive into the relevant literature.

6

MULTI-DIMENSIONAL PREFERENCE MODELS

This chapter may look more formal than the other parts of this book. We acknowledge this, believing that a formal presentation of this material is both unavoidable and worth the effort. Our goal is to offer a picture of the variety of preference models, showing them in a structured way that is the result of recent research. We mainly present three frameworks for describing preferences. Each one is a sort of matryoshka or Russian doll formed of embedded families of models. Each one relies on a clear basic principle for decomposing preferences; they both start with a very general family of models that gradually specialises when further properties are added. The remarkable thing is that most preference models that have been proposed and that are used in practice belong to a family of models in these hierarchies.

There is a price to pay: the—sometimes tiny—differences between families of models in a hierarchy can only be understood by stating precise definitions and theorems. In order to ease the reading and to convince the reader of the relevance of our frameworks, we have illustrated the definitions as often as possible;

- we describe the insertion of as many examples of actually used models as possible in the hierarchies;
- we emphasise the consequences of the progressive structuring of the hierarchies of models on the elicitation process of these models.

In the previous chapter we characterised a number of aggregation *procedures*; we now explore another way of analysing multiple criteria preferences, by characterising preference *relations*. The conjoint measurement approach was introduced in section 4.3 where its main model, the additive value function model, was briefly described. Some limitations of this model were discussed. We first come back to the additive value model in more detail, focusing on the conditions under which a preference relation can be represented in it and how it is possible to elicit the parameters of the model; we then develop three types of extensions of this fundamental model, namely:

- models based on marginal traces (section 6.2)
- models based on traces on differences (section 6.3)

- models based on marginal traces and on traces on differences (section 6.4)

In the light of these extensions, we then discuss models that distinguish a small number of differences in preference (which are useful for understanding ordinal aggregation; see section 6.5). Finally in section 6.6, we introduce valued preferences and the related conjoint measurement models, in connection with the measurement of preference differences.

Throughout this chapter we consider preferences, denoted by $\succsim$, defined on a product set $X = \prod_{i=1}^{n} X_i$. Each of the sets X_i is typically the co-domain of the scale associated to dimension i or the co-domain of a numerical representation of the client's preferences on dimension i (see chapter 2, section 2.3.3). Any alternative will be identified by a vector $x = (x_1, \ldots, x_n)$ of X where $x_1, \ldots, x_i, \ldots, x_n$ denote the evaluations of alternative x on the n dimensions. We shall use the notation x_{-i} to refer to a $(n-1)$ components vector obtained by dropping the ith coordinate of vector x; this allows us to define an "alternative" (x_i, a_{-i}) as the vector that has x_i as its ith component while the other components are those of vector a. Such $n-1$ dimensional vectors form the set $X_{-i} = \prod_{j \neq i} X_i$. We denote by N the set of integers $\{1, 2, \ldots, n\}$. For any subset J of N, X_J is the product set $\prod_{i \in J} X_i$.

6.1 The additive value model

The *additive value function model* was introduced in section 4.3.1 of chapter 4. We recall that a preference $\succsim$ on X can be represented in the additive value function model (or *additive value model* for short) if there are functions u_i from X_i into $\mathbb{R}$ for all i, such that, for all $x, y \in X$:

$$x \succsim y \Leftrightarrow u(x) = \sum_{i=1}^{n} u_i(x_i) \geq u(y) = \sum_{i=1}^{n} u_i(y_i). \tag{6.1}$$

Not all preferences, of course, satisfy such a condition. Before considering the hypotheses under which this is the case, we investigate the notion of marginal preference that is an important one in the process of elicitation of the additive value model.

6.1.1 Independence and marginal preferences

In conjoint measurement, one starts with a preference relation $\succsim$ on X. It is then of vital importance—as anticipated in section 4.3.6—to investigate how this information makes it possible to define preference relations on dimensions or subsets of dimensions.

Let $J \subseteq N$ be a nonempty set of dimensions. We define the *marginal relation* $\succsim_J$ induced on X_J by $\succsim$ letting, for all $x_J, y_J \in X_J$:

$$x_J \succsim_J y_J \Leftrightarrow (x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for all } z_{-J} \in X_{-J},$$

with asymmetric (resp. symmetric) part $\succ_J$ (resp. $\sim_J$). When $J = \{i\}$, we often abuse notation and write $\succsim_i$ instead of $\succsim_{\{i\}}$ (see the definition (4.7) of $\succsim_i$ on p. 133). Note that if $\succsim$ is reflexive (resp. transitive), the same will be true for $\succsim_J$. This is clearly not true for completeness however.

Definition 6.1 (Independence)
Consider a binary relation $\succsim$ on a set $X = \prod_{i=1}^n X_i$ and let $J \subseteq N$ be a nonempty subset of dimensions. We say that $\succsim$ is independent for J if, for all $x_J, y_J \in X_J$,

$$[(x_J, z_{-J}) \succsim (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J.$$

If $\succsim$ is independent for all nonempty subsets of N, we say that $\succsim$ is independent *(or strongly independent). If $\succsim$ is independent for all subsets containing a single dimension, we say that $\succsim$ is* weakly independent.

In view of (6.1), it is clear that the additive value model will require that $\succsim$ is independent. This crucial condition says that common evaluations on some dimensions do not influence preference. Whereas independence implies weak independence, it is well-know that the converse is not true (Wakker, 1989).

Remark 6.1.1
The (strong) independence condition is equivalent to an apparently weaker condition, i.e. independence with respect to all subsets J containing $n - 1$ elements. It is easy to convince oneself that this condition indeed implies independence with respect to all subsets J of X. To contrast this condition with weak independence we state both of them explicitly below. A relation $\succsim$ on X is

- *(strongly) independent* if, for all i and all x_i, y_i, a_{-i}, b_{-i},

$$(x_i, a_{-i}) \succsim (x_i, b_{-i}) \Rightarrow (y_i, a_{-i}) \succsim (y_i, b_{-i}) \tag{6.2}$$

- *weakly independent* if, for all i and all x_i, y_i, a_{-i}, b_{-i},

$$(x_i, a_{-i}) \succsim (y_i, a_{-i}) \Rightarrow (x_i, b_{-i}) \succsim (y_i, b_{-i}) \tag{6.3}$$

In other words, (strong) independence means that once one has $(x_i, a_{-i}) \succsim (x_i, b_{-i})$ for some x_i, then $a_{-i} \succsim_{-i} b_{-i}$; weak independence says that once for some a_{-i}, one has $(x_i, a_{-i}) \succsim (y_i, a_{-i})$, then $x_i \succsim_i y_i$. When dealing with (strong) independence, the alternatives share a common level on a single criterion (x_i is the common level), while in the weak independence property, all levels are common but one. This makes the latter condition less restrictive than the former. Strong independence says that when two alternatives share the same evaluation on a criterion, their relative position in the preference does not change when this common level is changed in any other common one; weak independence says something similar when two alternatives share all their evaluations but one: changing all those common levels into other common levels does not change the way the alternatives compare. •

Remark 6.1.2
Weak independence is referred to as "weak separability" in Wakker (1989); in section 6.2.3, we use "weak separability" (and "separability") with a different meaning. •

Remark 6.1.3
Independence, or at least weak independence, is an almost universally accepted hypothesis in multiple criteria decision making. It cannot be overemphasised that it is easy to find examples in which it is inadequate.

We have already examined the following example of likely non-independent preference in section 4.3.5: if a meal is described by the two dimensions, main course and wine, it is highly likely that most gourmets will violate independence, preferring red wine with beef and white wine with fish. Similarly, in a dynamic decision problem, a preference for variety will often lead to violating independence: you may prefer Pizza to Steak, but your preference for meals today (first dimension) and tomorrow (second dimension) may well be such that (Pizza, Steak) is preferred to (Pizza, Pizza), while (Steak, Pizza) is preferred to (Steak, Steak).

Many authors (Keeney, 1992; Roy, 1996; von Winterfeldt and Edwards, 1986) have argued that such failures of independence were almost always due to a poor structuring of dimensions (e.g. in our choice of the meal example above, preference for variety should be explicitly modelled). •

When $\succsim$ is a weak order (complete transitive relation or a ranking, possibly with ties) and is weakly independent, marginal preferences are well-behaved and combine with the preference $\succsim$ in a *monotonic* manner. For instance, if an alternative is preferred to another on all dimensions, then the former should be globally preferred to the latter. This monotonicity property of the preference with respect to the marginal preferences has strong links with the idea of *dominance* that we shall discuss more in depth later (see section 6.2.8). We put forward some useful properties of independent weak orders in the next proposition.

Proposition 6.1 (Properties of independent weakly ordered preferences)
Let $\succsim$ be a weakly independent weak order on $X = \prod_{i=1}^{n} X_i$. Then $\succsim_i$ is a weak order on X_i and for all $x, y \in X$ and all $z_i, w_i \in X_i$:

1. $[x \succsim y \text{ and } z_i \succsim_i x_i] \Rightarrow (z_i, x_{-i}) \succsim y$,
2. $[x \succsim y \text{ and } y_i \succsim_i w_i] \Rightarrow x \succsim (w_i, y_{-i})$,
3. $[x \succsim y \text{ and } z_i \succ_i x_i] \Rightarrow (z_i, x_{-i}) \succ y$,
4. $[x \succsim y \text{ and } y_i \succ_i w_i] \Rightarrow x \succ (w_i, y_{-i})$.

The latter four properties express the way the preference $\succsim$ responds to marginal improvement or worsening of the alternatives involved: the response is monotonic (or non-negative) and even strictly monotonic (positive) with respect to marginal preferences as we see from the last two properties. Non-negative or positive responsiveness properties of the preference were discussed several times in section 5.2 (see p. 171, 173, 184).

For preferences that are independent weak orders, the marginal preferences are also weak orders and the preference responds monotonically with respect to the marginal preferences. The importance of these properties cannot be overemphasised, since direct procedures for eliciting preferences that are independent weak orders, usually rely on the relationship between marginal preferences and the global preference; this will be the case for the additive value model as shown in the following sub-sections.

It should however be kept in mind that preferences that are not weak orders may show different behaviours. For more general preferences, the marginal preferences may no longer be the adequate tool on which to rely for eliciting the preference. This will be strongly emphasised and analysed in the generalisations of the additive value model discussed in sections 6.2 to 6.5.

6.1.2 The additive value model in the "rich" case

The purpose of the remainder of section 6.1 is to present the conditions under which a preference relation on a product set may be represented by the additive value function model (6.1) and how such a model can be assessed; the presentation of this material follows Bouyssou and Pirlot (2005b). We begin here with the case that most closely resembles the measurement of physical dimensions such as length.

When the structure of X is presumed to be "adequately rich", conjoint measurement is an adaptation of the process that is used for the measurement of physical extensive quantities such as length. The basic idea of this type of measurement (called *extensive measurement*, see Krantz et al., 1971, ch. 3) consists in comparing the object to be measured with a standard object that can be replicated while the length of the chains of replicas is an integer number of times that of the standard "unit" object. The "length" of preference intervals on a dimension will be measured here, using a preference interval on another dimension as a standard. A sequence of "equal length" intervals, called a *standard sequence*, will be built on each dimension; the procedure used to build such a sequence is known as the "standard sequence method" (von Winterfeldt and Edwards, 1986).

6.1.2.1 The case of two dimensions

Consider first the two dimension case, where the relation $\succsim$ is defined on a set $X = X_1 \times X_2$. In section 4.3, p. 130, we already identified necessary conditions for a relation to be representable in the additive value model, namely, we have to assume that $\succsim$ is an *independent weak order*. In such a case, $\succsim_1$ and $\succsim_2$ are weak orders, as stated in proposition 6.1. Consider two levels $x_1^0, x_1^1 \in X_1$ on the first dimension such that $x_1^1 \succ_1 x_1^0$, i.e. x_1^1 is preferable to x_1^0. Note that in order to be able to find such levels, we will have to exclude the case in which all levels on the first dimension are marginally indifferent.

Choose any $x_2^0 \in X_2$. The arbitrarily chosen element $(x_1^0, x_2^0) \in X$ will be our "reference point". The basic idea is to use this reference point and the "unit" on the first dimension given by the reference preference interval $[x_1^0, x_1^1]$ to build a standard sequence on the preference intervals on the second dimension. We are therefore looking for an element $x_2^1 \in X_2$ that would verify:

$$(x_1^0, x_2^1) \sim (x_1^1, x_2^0). \tag{6.4}$$

Clearly, this will require the structure of X_2 to be adequately "rich" to be able to find the level $x_2^1 \in X_2$ such that the reference preference interval on the first dimension $[x_1^0, x_1^1]$ is exactly matched by a preference interval of the same "length" on the second dimension $[x_2^0, x_2^1]$. Technically, this calls for a solvability assumption or, more restrictively, for the assumption that X_2 has a (topological) structure that is close to that of an interval of $\mathbb{R}$ and that $\succsim$ is "somehow" continuous.

If such a level x_2^1 can be found, model (6.1) implies:

$$\begin{aligned} u_1(x_1^0) + u_2(x_2^1) &= u_1(x_1^1) + u_2(x_2^0) \text{ so that} \\ u_2(x_2^1) - u_2(x_2^0) &= u_1(x_1^1) - u_1(x_1^0). \end{aligned} \tag{6.5}$$

Let us set the origin of measurement letting:

$$u_1(x_1^0) = u_2(x_2^0) = 0,$$

and our unit of measurement letting:

$$u_1(x_1^1) = 1 \text{ so that } u_1(x_1^1) - u_1(x_1^0) = 1.$$

Using (6.5), we obtain $u_2(x_2^1) = 1$. We have therefore found an interval between levels on the second dimension ($[x_2^0, x_2^1]$) that exactly matches our reference interval on the first dimension ($[x_1^0, x_1^1]$). We may proceed with building our standard sequence on the second dimension (see figure 6.1) asking for levels $x_2^2, x_2^3, \ldots$ such that:

$$\begin{aligned} (x_1^0, x_2^2) &\sim (x_1^1, x_2^1), \\ (x_1^0, x_2^3) &\sim (x_1^1, x_2^2), \\ &\cdots \\ (x_1^0, x_2^k) &\sim (x_1^1, x_2^{k-1}). \end{aligned}$$

As above, using (6.1) leads to:

$$\begin{aligned} u_2(x_2^2) - u_2(x_2^1) &= u_1(x_1^1) - u_1(x_1^0), \\ u_2(x_2^3) - u_2(x_2^2) &= u_1(x_1^1) - u_1(x_1^0), \\ &\cdots \\ u_2(x_2^k) - u_2(x_2^{k-1}) &= u_1(x_1^1) - u_1(x_1^0), \end{aligned}$$

so that:

$$u_2(x_2^2) = 2, u_2(x_2^3) = 3, \ldots, u_2(x_2^k) = k.$$

This process of building a standard sequence on the second dimension therefore leads to defining u_2 on a number of carefully selected elements of X_2. When

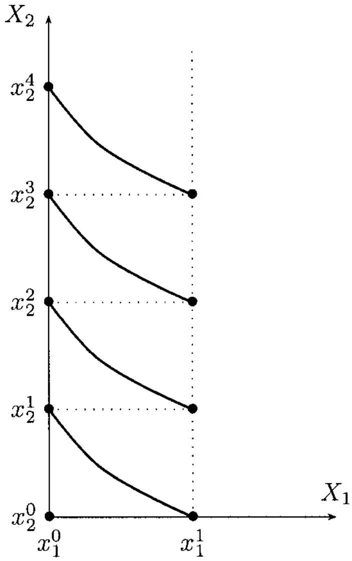


Figure 6.1: Building a standard sequence on X_2.

measuring physical quantities such as length, a key idea is that it is always possible to concatenate copies of a unit rod or ruler and to compare, with respect to length, any object to a composite one obtained by concatenating copies of a unit rod. This is a basic feature of what is technically called *extensive measurement* (Krantz et al., 1971, ch. 3). An implicit hypothesis is that the length of any object can be exceeded by the length of a composite object obtained by concatenating a sufficient number of perfect copies of a standard rod. Such a hypothesis is called *Archimedean* since it mimics the property of the real numbers which says that for any positive real numbers x, y it is true that $nx > y$ for some integer n, i.e. y, no matter how large, may always be exceeded by taking any x, no matter how small, and adding it with itself and repeating the operation a sufficient number of times. Clearly, we will need a similar hypothesis here. Failing this, there might be a level $y_2 \in X_2$ that will never be "reached" by our standard sequence, i.e. such that $y_2 \succ_2 x_2^k$, for $k = 1, 2, \ldots$. For measurement models in which this Archimedean condition is omitted, see Narens (1985) and Skala (1975).

Remark 6.1.4
At this point a good exercise for the reader is to figure out how we may extend the standard sequence to cover levels of X_2 that are "below" the reference level x_2^0. This should not be difficult. •

Now that a standard sequence is built on the second dimension, we may use any part of it to build a standard sequence on the first dimension. This will require

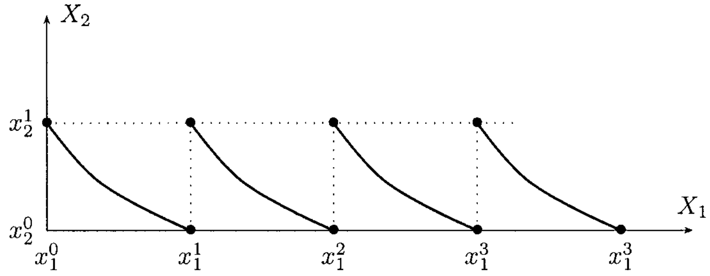


Figure 6.2: Building a standard sequence on X_1.

finding levels $x_1^2, x_1^3, \ldots \in X_1$ such that (see figure 6.2):

$$\begin{aligned}(x_1^2, x_2^0) &\sim (x_1^1, x_2^1),\\ (x_1^3, x_2^0) &\sim (x_1^2, x_2^1),\\ &\ldots\\ (x_1^k, x_2^0) &\sim (x_1^{k-1}, x_2^1).\end{aligned}$$

Using (6.1) leads to:

$$\begin{aligned}u_1(x_1^2) - u_1(x_1^1) &= u_2(x_2^1) - u_2(x_2^0),\\ u_1(x_1^3) - u_1(x_1^2) &= u_2(x_2^1) - u_2(x_2^0),\\ &\ldots\\ u_1(x_1^k) - u_1(x_1^{k-1}) &= u_2(x_2^1) - u_2(x_2^0),\end{aligned}$$

so that:

$$u_1(x_1^2) = 2, u_1(x_1^3) = 3, \ldots, u_1(x_1^k) = k.$$

As was the case for the second dimension, the construction of such a sequence will require the structure of X_1 to be adequately rich, which calls for a solvability assumption. An Archimedean condition will also be needed to ensure that all levels of X_1 can be reached by the sequence.

At this point, we have defined a "grid" in X (see figure 6.3) and we have $u_1(x_1^k) = k$ and $u_2(x_2^k) = k$ for all elements of this grid. Intuitively, such numerical assignments seem to define an adequate additive value function on the grid. We have to prove that this intuition is correct. Let us first verify that, for all integers $\alpha, \beta, \gamma, \delta$:

$$\alpha + \beta = \gamma + \delta = \epsilon \Rightarrow (x_1^\alpha, x_2^\beta) \sim (x_1^\gamma, x_2^\delta). \tag{6.6}$$

When $\epsilon = 1$, (6.6) holds by construction because we have: $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$. When $\epsilon = 2$, we know that $(x_1^0, x_2^2) \sim (x_1^1, x_2^1)$ and $(x_1^2, x_2^0) \sim (x_1^1, x_2^1)$ and the claim is proved using the transitivity of $\sim$.

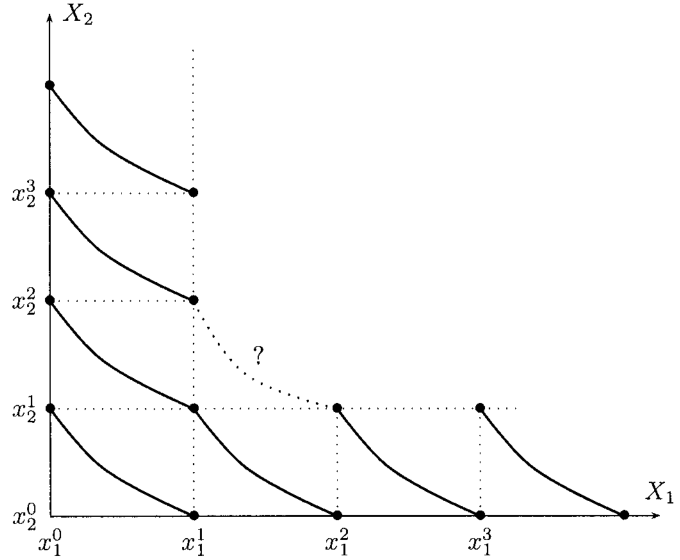


Figure 6.3: The grid.

Consider $\epsilon = 3$. We have $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$ and $(x_1^0, x_2^3) \sim (x_1^1, x_2^2)$. It remains to be shown that $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$ (see the dotted arc in figure 6.3). This does not seem to follow from the previous conditions that we more or less explicitly used: transitivity, independence, "richness", Archimedean property. Indeed, it does not. Hence, we have to suppose that: $(x_1^2, x_2^0) \sim (x_1^0, x_2^2)$ and $(x_1^0, x_2^1) \sim (x_1^1, x_2^0)$ imply $(x_1^2, x_2^1) \sim (x_1^1, x_2^2)$. This condition, called the *Thomsen condition*, is clearly necessary for (6.1). The above reasoning easily extends to all points on the grid, using weak ordering, independence and the Thomsen condition. Hence, (6.6) holds on the grid.

It remains to show that:

$$\epsilon = \alpha + \beta > \epsilon' = \gamma + \delta \Rightarrow (x_1^\alpha, x_2^\beta) \succ (x_1^\gamma, x_2^\delta). \tag{6.7}$$

Using transitivity, it is sufficient to show that (6.7) holds when $\epsilon = \epsilon' + 1$. By construction, we know that $(x_1^1, x_2^0) \succ (x_1^0, x_2^0)$. Using independence, this implies that $(x_1^1, x_2^k) \succ (x_1^0, x_2^k)$. Using (6.6) we have $(x_1^1, x_2^k) \sim (x_1^{k+1}, x_2^0)$ and $(x_1^0, x_2^k) \sim (x_1^k, x_2^0)$. Therefore we have $(x_1^{k+1}, x_2^0) \succ (x_1^k, x_2^0)$, the desired conclusion.

We have thus built an additive value function of a suitably chosen grid (see figure 6.4). The logic of the assessment procedure is then to assess more and more points, somehow considering more finely grained standard sequences. Going to the limit then unambiguously defines the functions u_1 and u_2. Clearly such u_1 and u_2 are quite closely related. Once we have chosen an arbitrary reference point (x_1^0, x_2^0) and a level x_1^1 defining the unit of measurement, the process we just described entirely defines u_1 and u_2. It follows that the only possible transformations that can be applied to u_1 and u_2, is to multiply both by the same positive number α

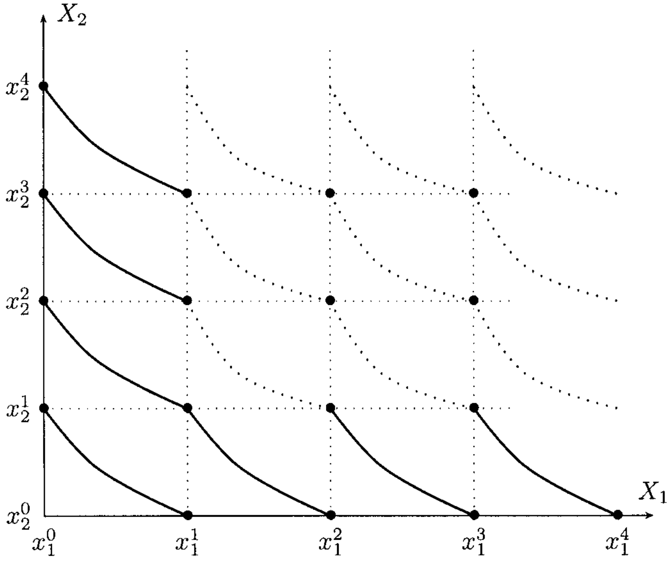


Figure 6.4: The entire grid.

and to add to both a (possibly different) constant. This is usually summarised by saying that u_1 and u_2 define interval scales with a common unit.

The above reasoning is a rough sketch of the proof of the existence of an additive value function when $n = 2$, as well as an outline of how it could be assessed. The careful readers can refer to Fishburn (1970, ch. 5), Krantz et al. (1971, ch. 6) and Wakker (1989, ch. 3).

Remark 6.1.5

The measurement of lengths through standard sequences as described above leads to a scale that is unique once the unit of measurement is chosen. At this point, a good exercise is to find an intuitive explanation to the fact that, when measuring the "length" of preference intervals, the origin of measurement becomes arbitrary. The analogy with the measurement of duration on the one hand and dates, as given in a calendar on the other hand, should help. •

It is worth emphasising that the assessment technique using standard sequences outlined above makes no use of the vague notion of the "importance" of the various dimensions. The "importance" is in fact captured in the lengths of the preference intervals on the various dimensions.

A common but critical mistake is to confuse the additive value function model (6.1) with a weighted average and to try to assess weights asking whether a dimension is "more important" than another. This makes no sense.

6.1.2.2 The case of more than two dimensions

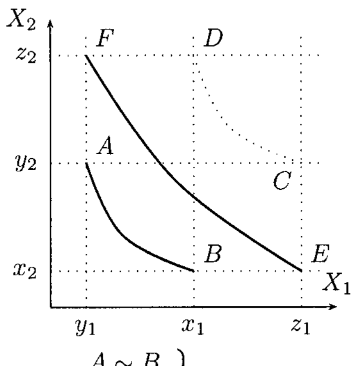
The good news is that the process is exactly the same when there are more than two dimensions. There is one surprise: the Thomsen condition is no longer needed to prove that the standard sequences defined on each dimension lead to an adequate value function on the grid. A heuristic explanation of this strange result is that, when $n = 2$, there is no difference between independence and weak independence. This is no longer true when $n \geq 3$ and assuming independence is much stronger than just assuming weak independence. We use the "algebraic approach" below (Krantz, 1964; Krantz et al., 1971; Luce and Tukey, 1964). A more restrictive approach using a topological structure on X is given in Debreu (1960), Fishburn (1970, ch. 5) and Wakker (1989, ch. 3). We formalise the conditions informally introduced in the previous section below. The reader not interested in the precise statement of the results or, even better, having already written down his own statement, may skip this section.

Definition 6.2 (Thomsen condition)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. It is said to satisfy the Thomsen condition if

$$(x_1, x_2) \sim (y_1, y_2) \text{ and } (y_1, z_2) \sim (z_1, x_2) \Rightarrow (x_1, z_2) \sim (z_1, y_2),$$

for all $x_1, y_1, z_1 \in X_1$ and all $x_2, y_2, z_2 \in X_2$.

Figure 6.5 shows how the Thomsen condition uses two "indifference curves" (i.e. curves linking points that are indifferent) to place a constraint on a third one. This was needed above to prove the existence of an additive value function on our grid. Remember that the Thomsen condition is only needed when $n = 2$; so, we only stated it in this case.



$$\left.\begin{array}{l} A \sim B \\ E \sim F \end{array}\right\} \Rightarrow C \sim D$$

Figure 6.5: The Thomsen condition.

Definition 6.3 (Standard sequences)
A standard sequence on dimension $i \in N$ is a set $\{a_i^k : a_i^k \in X_i, k \in K\}$ where K is a set of consecutive integers (positive or negative, finite or infinite) such that there

are $x_{-i}, y_{-i} \in X_{-i}$ *satisfying* $Not[x_{-i} \sim_{-i} y_{-i}]$ *and* $(a_i^k, x_{-i}) \sim (a_i^{k+1}, y_{-i})$*, for all* $k \in K$.

A standard sequence on dimension $i \in N$ is said to be *strictly bounded* if there are $b_i, c_i \in X_i$ such that $b_i \succ_i a_i^k \succ_i c_i$, for all $k \in K$. It is then clear that, when model (6.1) holds, any strictly bounded standard sequence must be finite.

Definition 6.4 (Archimedean)
For all $i \in N$*, any strictly bounded standard sequence on* $i \in N$ *is finite.*

The following condition rules out the case in which a standard sequence cannot be built because all levels are indifferent.

Definition 6.5 (Essentiality)
Let $\succsim$ *be a binary relation on a set* $X = X_1 \times X_2 \times \cdots \times X_n$*. Dimension* $i \in N$ *is said to be essential if* $(x_i, a_{-i}) \succ (y_i, a_{-i})$*, for some* $x_i, y_i \in X_i$ *and some* $a_{-i} \in X_{-i}$.

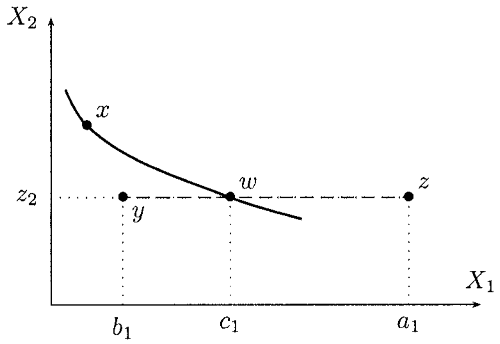
Definition 6.6 (Restricted Solvability)
Let $\succsim$ *be a binary relation on a set* $X = X_1 \times X_2 \times \cdots \times X_n$*. Restricted solvability is said to hold with respect to dimension* $i \in N$ *if, for all* $x \in X$*, all* $z_{-i} \in X_{-i}$ *and all* $a_i, b_i \in X_i$*,* $[(a_i, z_{-i}) \succsim x \succsim (b_i, z_{-i})] \Rightarrow [x \sim (c_i, z_{-i})$*, for some* $c_i \in X_i]$.

Remark 6.1.6
Restricted solvability in the case where $n = 2$ is illustrated in figure 6.6. It states that, given any $x \in X$, if it is possible find two levels $a_i, b_i \in X_i$ such that when combined with a certain level $z_{-i} \in X_{-i}$ on the other dimensions, (a_i, z_{-i}) is preferred to x and x is preferred to (b_i, z_{-i}), it should be possible to find a level c_i, "in between" a_i and b_i, such that (c_i, z_{-i}) is exactly indifferent to x.

A much stronger hypothesis is unrestricted solvability asserting that for all $x \in X$ and all $z_{-i} \in X_{-i}$, $x \sim (c_i, z_{-i})$, for some $c_i \in X_i$. Its use leads to much simpler proofs (Fishburn, 1970; Gonzales, 1996b).

It is easy to imagine situations in which restricted solvability holds while unrestricted solvability fails. Suppose, e.g. that a firm has to choose between several investment projects, two dimensions being the Net Present Value (NPV) of the projects and their impact on the public image of the firm. Consider a project consisting in investing in the software market. It has a reasonable NPV and no adverse consequences on the image of the firm. Consider another project that could have dramatic consequences on the image of the firm, because it leads to investing in the cocaine market. Unrestricted solvability would require that by sufficiently increasing the NPV of the second project it would become indifferent to the more standard project of investing in the software market. This is not required by restricted solvability. •

We are now in a position to state the central results concerning model (6.1). Proofs may be found in Krantz et al. (1971, ch. 6) and Wakker (1991b).



$$\left.\begin{array}{l} z \succ x \\ x \succ y \end{array}\right\} \Rightarrow \text{ there is a } w \text{ such that } x \sim w$$

Figure 6.6: Restricted Solvability on X_1.

Theorem 6.1 (Additive value function when $n = 2$)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2$. If restricted solvability holds on all dimensions and each dimension is essential, then $\succsim$ has a representation in model (6.1) if and only if $\succsim$ is an independent weak order satisfying the Thomsen and Archimedean conditions

Furthermore in this representation, u_1 and u_2 are interval scales with a common unit, i.e. if u_1, u_2 and v_1, v_2 are two pairs of functions satisfying (6.1), there are real numbers α, β_1, β_2 with $\alpha > 0$ such that, for all $x_1 \in X_1$ and all $x_2 \in X_2$

$$u_1(x_1) = \alpha v_1(x_1) + \beta_1 \text{ and } u_2(x_2) = \alpha v_2(x_2) + \beta_2.$$

When $n \geq 3$ and at least three dimensions are essential, the above result simplifies in that the Thomsen condition can now be omitted.

Theorem 6.2 (Additive value function when $n \geq 3$)
Let $\succsim$ be a binary relation on a set $X = X_1 \times X_2 \times \ldots \times X_n$ with $n \geq 3$. If restricted solvability holds on all dimensions and at least 3 dimensions are essential, then $\succsim$ has a representation in model (6.1) if and only if $\succsim$ is an independent weak order satisfying the Archimedean condition.

Furthermore in this representation u_1, $u_2, \ldots, u_n$ are interval scales with a common unit.

Remark 6.1.7
The additive value model is central to several fields in decision theory. It is therefore not surprising that quite a lot of energy has been devoted to analyse variants and refinements of the results given above. Among the most significant ones, let us mention:

- the study of cases in which solvability holds only on some or none of the dimensions (Fishburn, 1992b; Gonzales, 1996a,b, 2000, 2003; Jaffray, 1974a,b; Nakamura, 2002),
- the study of the relation between the "algebraic approach" introduced above and the topological one used in Debreu (1960), see e.g. Karni and Safra (1998), Köbberling (2003), Wakker (1989, ch. 3) and Wakker (1991b).

The results given above are only valid when X is the entire Cartesian product of the sets X_i. Results in which X is a subset of the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ are not easy to obtain, see Chateauneuf and Wakker (1993) and Segal (1994) (the situation is "easier" in the special case of homogeneous product sets, see Wakker, 1991c, 1993)). •

We have shown how additive value functions can be assessed using the standard sequence technique. We pinpoint some of the characteristics of this assessment procedure:

- It requires the set X_i to be *rich* so that it is possible to find a preference interval on X_i that will exactly match a preference interval on another dimension. This excludes using such an assessment procedure when some of the sets X_i are discrete.
- It relies on *indifference* judgements which, a priori, are less firmly established than preference judgements.
- It relies on judgements concerning fictitious alternatives which, a priori, are harder to conceive than judgements concerning real alternatives.
- The various assessments are thoroughly intertwined and, e.g., an imprecision on the assessment of x_2^1, i.e. the endpoint of the first interval in the standard sequence on X_2 (see figure 6.1), will propagate to many assessed values,
- The assessment of tradeoffs may be plagued with cognitive biases (see, e.g., Delquié, 1993; Stillwell, von Winterfeldt, and John, 1987).

6.1.2.3 Implementation: Standard sequences and beyond

The assessment procedure based on standard sequences is, as we have seen, rather demanding; hence, it seems to be seldom used in the practice of decision analysis (Keeney and Raiffa, 1976; von Winterfeldt and Edwards, 1986). The literature on the experimental assessment of additive value functions (see, e.g., Stillwell et al., 1987; von Nitzsch and Weber, 1993; Weber, Eisenfuhr, and von Winterfeldt, 1988) suggests that this assessment is a difficult task that may be affected by several cognitive biases.

Many other simplified assessment procedures that are less firmly grounded in theory have been proposed. In many of them, the assessment of partial value functions u_i relies on *direct* comparison of preference differences without recourse to an interval on another dimension used as a "yard-stick". We refer to Dyer and Sarin (1979) for a theoretical analysis of these techniques.

These procedures include:

- *direct rating* techniques in which values of u_i are directly assessed with reference to two arbitrarily chosen points (Edwards, 1977; Edwards and Hutton Barron, 1994),
- procedures based on *bisection*, the client being asked to assess a point that is "half way" between two reference points in terms of preference(von Winterfeldt and Edwards, 1986),
- procedures that try to build *standard sequences* on each dimension in terms of "preference differences" (see Krantz et al., 1971, ch. 4).

An excellent overview of these techniques may be found in von Winterfeldt and Edwards (1986, ch. 7).

6.1.3 The additive value model in the "finite" case

In this section, we assumed that $\succsim$ is a binary relation on a finite set $X \subseteq X_1 \times X_2 \times \cdots \times X_n$ (contrary to the previous section, dealing with subsets of product sets will raise no difficulty here). The finiteness hypothesis clearly invalidates the standard sequence mechanism used so far. There will only be a finite number of "preference intervals" on each dimension and exact matches between preference intervals will only exceptionally occur, see Wakker (1991a)..

Clearly, as before, independence remains a necessary condition for model (6.1). Given the absence of structure of the set X, it is unlikely that this condition is sufficient to ensure (6.1). The following example shows that this intuition is indeed correct.

Example 6.1
Let $X = X_1 \times X_2$ with $X_1 = \{a, b, c\}$ and $X_2 = \{d, e, f\}$. Consider the weak order on X, such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ af \succ be \succ cd \succ ce \succ bf \succ cf.$$

It is easy to check that $\succsim$ is independent. Indeed, we may for instance check that:

$$ad \succ bd \text{ and } ae \succ be \text{ and } af \succ bf,$$
$$ad \succ ae \text{ and } bd \succ be \text{ and } cd \succ ce.$$

This relation cannot however be represented in model (6.1) since:

$$af \succ be \Rightarrow u_1(a) + u_2(f) > u_1(b) + u_2(e),$$
$$be \succ cd \Rightarrow u_1(b) + u_2(e) > u_1(c) + u_2(d),$$
$$ce \succ bf \Rightarrow u_1(c) + u_2(e) > u_1(b) + u_2(f),$$
$$bd \succ ae \Rightarrow u_1(b) + u_2(d) > u_1(a) + u_2(e).$$

Summing the first two inequalities leads to:

$$u_1(a) + u_2(f) > u_1(c) + u_2(d).$$

Summing the last two inequalities leads to:

$$u_1(c) + u_2(d) > u_1(a) + u_2(f),$$

a contradiction.

Note that, since no indifference is involved, the Thomsen condition is trivially satisfied. Although this is clearly necessary for model (6.1), adding it to independence will therefore not solve the problem. ◇

The conditions allowing to build an additive value model in the finite case were investigated in Adams (1965), Adams and Fagot (1959) and Scott (1964). Although the resulting conditions turn out to be complex, the underlying idea is quite simple. It amounts to finding conditions under which a system of linear inequalities has a solution.

Suppose that $x \succ y$. If model (6.1) holds, we have:

$$\sum_{i=1}^{n} u_i(x_i) > \sum_{i=1}^{n} u_i(y_i). \tag{6.8}$$

Similarly if $x \sim y$, we obtain:

$$\sum_{i=1}^{n} u_i(x_i) = \sum_{i=1}^{n} u_i(y_i). \tag{6.9}$$

The problem is then to find conditions on $\succsim$ such that the system of finitely many equalities and inequalities (6.8–6.9) has a solution. This is a classical problem in Linear Algebra (see, e.g., Gale, 1960).

Definition 6.7 (Relation E^m)
Let m be an integer ≥ 2. Let $x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$. We say that

$$(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m)$$

if, for all $i \in N$, $(x_i^1, x_i^2, \ldots, x_i^m)$ is a permutation of $(y_i^1, y_i^2, \ldots, y_i^m)$.

Suppose that $(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m)$. Then model (6.1) implies that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} u_i(x_i^j) = \sum_{j=1}^{m} \sum_{i=1}^{n} u_i(y_i^j).$$

Therefore if $x^j \succsim y^j$ for $j = 1, 2, \ldots, m-1$, it cannot be true that $x^m \succ y^m$. This condition must hold for all $m = 2, 3, \ldots$.

Definition 6.8 (Condition C^m)
Let m be an integer ≥ 2. We say that condition C^m holds if

$$[x^j \succsim y^j \text{ for } j = 1, 2, \ldots, m-1] \Rightarrow Not[x^m \succ y^m]$$

for all $x^1, x^2, \ldots, x^m, y^1, y^2, \ldots, y^m \in X$ such that

$$(x^1, x^2, \ldots, x^m) E^m (y^1, y^2, \ldots, y^m).$$

Remark 6.1.8
It is not difficult to check that:

- $C^{m+1} \Rightarrow C^m$,
- $C^2 \Rightarrow \succsim$ is independent,
- $C^3 \Rightarrow \succsim$ is transitive. •

We already observed that C^m was implied by the existence of an additive representation. The main result for the finite case states that requiring that $\succsim$ is complete and that C^m holds for $m = 2, 3, \ldots$ is also sufficient. Proofs can be found in Fishburn (1970, ch. 4) and Krantz et al. (1971, ch. 9).

Theorem 6.3 (Additive value function in the finite case)
Let $\succsim$ be a binary relation on a finite set $X \subseteq X_1 \times X_2 \times \cdots \times X_n$. There are real-valued functions u_i on X_i such that (6.1) holds if and only if $\succsim$ is complete and satisfies C^m for $m = 2, 3, \ldots$.

Remark 6.1.9
Contrary to the "rich" case considered in the preceding section, here we have necessary and sufficient conditions for the additive value model (6.1). However, it is important to note that the above result uses a denumerable scheme of conditions. Scott and Suppes (1958) show that this denumerable scheme cannot be truncated: for all $m \geq 2$, there is a relation $\succsim$ on a finite set X such that C^m holds, but violating C^{m+1} (this is studied in more detail in Luce, Krantz, Suppes, and Tversky (1990), Titiev (1972) and Wille (2000)).Therefore, no finite scheme of axioms is sufficient to characterise model (6.1) for all finite sets X.

Given a finite set X *of given cardinality*, it is well-known that the denumerable scheme of conditions can be truncated. The precise relation between the cardinality of X and the number of conditions required raises difficult combinatorial questions that are studied in Fishburn (1996, 1997). •

Remark 6.1.10
It is clear that, if a relation $\succsim$ has a representation in model (6.1) using functions u_i, it also has a representation using functions $v_i' = \alpha u_i + \beta_i$ with $\alpha > 0$. Contrary to the rich case, the uniqueness of the functions u_i is more complex, as shown by the following example.

Example 6.2
Let $X = X_1 \times X_2$ with $X_1 = \{a, b, c\}$ and $X_2 = \{d, e\}$. Consider the weak order on X such that, abusing notation in an obvious way,

$$ad \succ bd \succ ae \succ cd \succ be \succ ce.$$

This relation has a representation in model (6.1) with

$$u_1(a) = 3, u_1(b) = 1, u_1(c) = 0, u_2(d) = 3, u_2(e) = 0.5.$$

An equally valid representation would be given taking $u_1(b) = 2$. Clearly this new representation cannot be deduced from the original one applying a positive affine transformation. ◇

Remark 6.1.11
Theorem 6.3 has been extended to the case of an arbitrary set X in Jaffray (1974a,b) (see also Fishburn, 1992b; Furkhen and Richter, 1991). The resulting conditions are however quite complex. This explains why we spent time on this "rich" case in the previous section. •

Remark 6.1.12
The use of a denumerable scheme of conditions in theorem 6.3 does not ease the interpretation and test of conditions. However it should be noted that, on a given set X, the test of the C^m conditions amounts to finding if a system of a finite number of linear inequalities has a solution. It is well-known that Linear Programming techniques are quite efficient for such a task. In chapter 7, section 7.3.1, we show how to use LP techniques to assess an additive value model (6.1). •

6.2 A first line of generalisation: models based on marginal traces or preferences

Section 4.3 focused on the additive value function model (equation (6.1)); examples were presented showing the need for preference models that cannot be described by means of an additive value function. In this section we discuss a generalisation of the additive value function model, while preserving the possibility of using the fundamental construction tool suggested by the model, namely marginal preferences that are weak orders represented by the functions u_i in (6.1). Interestingly, the generalised model admits a full characterisation through fairly simple and intuitive axioms, which was not the case for model (6.1) as we have just seen.

Since we limit ourselves to evaluation models in which there are a finite number of alternatives, we may restrict the X_i's to be finite sets, but the reader might be interested to know that the theorems below remain valid when the X_i are countably infinite and that for sets of arbitrary cardinality, necessary and sufficient conditions are known (the references provided below deal with the general case).

6.2.1 Decomposable preferences

The so-called decomposable model was introduced in Krantz et al. (1971, ch. 7) as a natural generalisation of model (6.1). The preference $\succsim$ is meant to be a weak order and can thus be represented by a rule of the type

$$x \succsim y \Leftrightarrow u(x) \geq u(y) \tag{6.10}$$

with u, a real-valued function defined on X. Instead of specifying u as a sum of functions u_i of the variables x_i, u is just assumed to be decomposable in the form

$$u(x) = U(u_1(x_1), \ldots, u_n(x_n)) \tag{6.11}$$

where u_i is a function mapping X_i onto $\mathbb{R}$ (the set of real numbers) and U is increasing in all its arguments.

This model encompasses the case in which u is a non-additive function of the u_i's, which is suitable for non-additive utility models, for instance polynomial models (see, e.g., Krantz et al., 1971, ch. 7).

The interesting point with this model is that it admits an intuitively appealing full characterisation. The basic axiom for characterising the decomposable model described above (with increasing function U) is the weak independence condition (see definition 6.1, page 239).

For preferences that are weak orders, we know that the weak independence property is equivalent to the fact that the marginal preferences $\succsim_i$ are weak orders (proposition 6.1). Moreover, it is easy to see that u_i in (6.11) is necessarily a numerical representation of $\succsim_i$, i.e. $x_i \succsim_i y_i$ iff $u_i(x_i) \geq u_i(y_i)$. This is an important result, since it opens the door to the elicitation of the u_i's by questioning in terms of marginal preferences $\succsim_i$ as in the additive utility model.

The following theorem states a simple and important characterisation of the decomposable model. This result was first proved in Krantz et al. (1971, ch. 7).

Theorem 6.4 (Representation in the decomposable model)
A preference relation $\succsim$ on X admits a representation in the decomposable model:

$$x \succsim y \Leftrightarrow U(u_1(x_1), \ldots, u_n(x_n)) \geq U(u_i(y_1), \ldots, u_i(y_n))$$

with U increasing in all its arguments iff $\succsim$ is a weak order and satisfies weak independence.

If one intended to apply this model, one would specify the type of function U, possibly by verifying further conditions on the preference that impose that U belongs to a parameterised family of functions (e.g. polynomials of bounded degree). However, the structure of the model suggests a general elicitation scheme that is quite complex due to the high generality of the model, but could be envisaged for instance, when the number of alternatives is small. Even if this scheme may be of little practical value, it is nevertheless fully compatible with the model and logically valid when the number of alternatives is finite.

6.2.2 A procedure for eliciting the general decomposable model

Once it is recognised that model (6.11) could apply, the first step consists in eliciting the marginal preferences $\succsim_i$. An arbitrary numerical representation of the weak order $\succsim_i$ may then be chosen for each u_i. There are many possible strategies for obtaining (through questioning a client) a function U that assigns a rank to each profile of levels $\overline{u} = (u_1, \ldots, u_n) \in \prod_{i=1}^n u_i(X_i)$ (we abuse notations denoting by u_i a value taken by the function $u_i : X_i \rightarrow \mathcal{R}$). We just suggest one way of doing this here.

On each scale $u_i(X_i)$, select a reference level u_i^0 (this could be the minimum or the maximum on each scale, but it is perhaps better—for cognitive reasons—to start from the "middle" of the scale (see von Winterfeldt and Edwards, 1986, ch. 7)) and form the reference profile $\overline{u}^0 = (u_1^0, \ldots, u_n^0)$. Assign the value 0 to this profile, i.e. set $U(\overline{u}^0) = 0$. Then build a number of "milestone profiles" for instance in the following way: denote by $\mathcal{N}(u_i)$ the level just above u_i on the scale $u_i(X_i)$ or u_i itself if the latter is the highest level on the scale[1]. Conversely, $\mathcal{N}^{-1}(u_i)$ denotes the level just below u_i or u_i itself if there is no level below u_i. $\mathcal{N}$ will be called the "next level" operator and $\mathcal{N}^{-1}$, the "preceding level" operator. Using $\mathcal{N}$, we recursively define the milestone profiles $\overline{u}^1$, $\overline{u}^2$, ... by

$$\overline{u}^1 = (\mathcal{N}(u_1^0), \ldots, \mathcal{N}(u_n^0))$$

and more generally for all $k = 0, 1, 2, \ldots$,

$$\overline{u}^{k+1} = (\mathcal{N}(u_1^k), \ldots, \mathcal{N}(u_n^k));$$

the milestone profiles below $\overline{u}^0$ are defined for $k = 0, -1, -2, \ldots$ by

$$\overline{u}^{k-1} = (\mathcal{N}^{-1}(u_1^k), \ldots, \mathcal{N}^{-1}(u_n^k)).$$

Of course we stop generating milestone profiles as soon as $\overline{u}^{k+1} = \overline{u}^k$ and as soon as $\overline{u}^{k-1} = \overline{u}^k$. Suppose that the generation stops when $k = \overline{k}$ above $\overline{u}^0$ and when $k = \underline{k}$ below $\overline{u}^0$. Due to the properties of the marginal preferences $\succsim_i$ w.r.t. $\succsim$, we have:

$$\overline{u}^{\overline{k}} \succ \ldots \succ \overline{u}^1 \succ \overline{u}^0 \succ \overline{u}^{-1} \succ \ldots \succ \overline{u}^{\underline{k}}. \tag{6.12}$$

We assign the value k to $U(\overline{u}^k)$.

The next step consists in inserting all other profiles in between the appropriate consecutive "milestones". Start for instance with the profiles that differ from $\overline{u}^k$ on a single coordinate and by one level, i.e., for some i, $u_i = \mathcal{N}(u_i^k) \neq u_i^k$. There are at most n such (distinct) profiles that all lie between $\overline{u}^{\underline{k}}$ and $\overline{u}^{\overline{k}}$. One has to situate them in this interval with respect to each other and then give them an arbitrary value of U with the constraint that the assigned values reflect the order of the profiles in the interval. One may then consider profiles at "distance 2" above

[1] Formally, $\mathcal{N}(u_i) \geq u_i$ and for all $u_i \in u_i(X_i)$, if there is $v_i \in u_i(X_i)$ such that $v_i > u_i$, then $\mathcal{N}(u_i) > u_i$.

$\overline{u}^k$, i.e. profiles that differ from the latter either on two coordinates by one level (above) or on a single coordinate by two levels (above). After having inserted all such profiles, one has to consider profiles that are more and more distant, until all profiles have been inserted between those previously assessed. Note that the only constraint to be fulfilled when inserting a new profile, is to respect the dominance relation w.r.t. the previously inserted profiles; this means that if $\overline{u}'$ is to be inserted, the value $U(\overline{u}')$ must be

- larger than the value assigned to any already assigned profile $\overline{u}$ that is dominated by $\overline{u}'$, i.e. to any profile such that

$$u'_i \geq u_i, \text{ for all } i \text{ and for some } i, u'_i > u_i;$$

- smaller than the value assigned to any already assigned profile $\overline{u}$ that dominates $\overline{u}'$

Remark 6.2.1
Our goal in outlining the above procedure was just to suggest a way of eliciting a decomposable model; we do not say that the suggested strategy is the best one possible; it is certainly not the only one. A lot of additional effort would be needed to make it precise and operational. •

Remark 6.2.2
This procedure is also quite complex and it could be envisaged to use it only when the number of different profiles is small. Note however, that the latter number is not directly determined by the number of alternatives, but rather by the number of levels on the scales $u_i(X_i)$, i.e. the number of equivalence classes of the marginal preferences. If the discrimination power of the marginal preferences is weak, the cardinality of the set of profiles $\prod_{i=1}^{n} u_i(X_i)$ is low. •

Remark 6.2.3
Software could help operationalising the above procedure by

- prompting the next profile to be inserted;
- propagating the consequences of the last insertion, i.e. automatically inserting all profiles that can unambiguously be placed due to dominance considerations;
- list the places where a profile could be inserted, taking dominance considerations into account . •

The previous section shows that it is possible, at least in theory, to devise a procedure for faithfully assessing a general decomposable model or, in other words, to elicit a preference that is assumed to be a weakly independent weak order. The procedure mainly relies on the elicitation of marginal preferences. This is in line with the procedures used for eliciting the additive value model (see section 6.1.2)

6.2.3 Non-strict decomposable model

6.2.3.1 The non-strict decomposable model

The decomposable model is fairly general, yet not general enough to encompass a widely used aggregation procedure such as the "min". In the example of Flexible CSP (p. 144), we pointed out that the "min-score" aggregation method (equation 4.16) does not satisfy the strong independence property; the same small numerical example given there shows that it also fails to verify weak independence. However, it is not difficult to convince oneself that changing all the levels that are common to two alternatives into other common levels can only transform strict preference into indifference, excluding strict preference in the opposite direction (the numerical example shows a transformation of strict preference into indifference). This is a motivation to consider a weakened variant of the decomposable model in which U is not assumed to be increasing, but just non-decreasing in all its arguments; we shall refer to such a model as *non-strict decomposable model.* The relevant weakening of weak independence is called *weak separability.* This property was considered for instance in Blackorby, Primont, and Russell (1978). We give its definition below and, for reasons of symmetry, we also define *(strong) separability.*

Definition 6.9 (Separability)
Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$ and $J \subseteq N$ be a nonempty subset of dimensions. We say that $\succsim$ is separable for J if, for all $x_J, y_J \in X_J$,

$$[(x_J, z_{-J}) \succ (y_J, z_{-J}), \text{ for some } z_{-J} \in X_{-J}] \Rightarrow x_J \succsim_J y_J.$$

If $\succsim$ is separable for all nonempty subsets of N, we say that $\succsim$ is separable *(or strongly separable). If $\succsim$ is separable for all subsets containing a single dimension, we say that $\succsim$ is* weakly separable.

Comparing the definition of "separable for J" to that of "independent for J" (see definition 6.1, p. 239), shows that the only difference is the substitution of $\succsim$ with $\succ$ in the premise of the condition. Hence separability is a weaker requirement than independence and weak separability weakens weak independence. The weak separability property tells us that when two alternatives share the same evaluations on all but one criterion, changing all these common levels into any other common level, can neither transform a strict preference into a strict preference in the other "direction", nor into no preference at all. Weak separability and weak independence are intuitively very similar conditions.

Substituting weak independence with weak separability leads to a slightly more general family of models that can be characterised. They admit a numerical representation of type (6.11) with U *nondecreasing* instead of *increasing.* This result is stated below; it was proven (in the special case where $X = \mathbb{R}^n$) by Blackorby et al. (1978) and is revisited in Bouyssou and Pirlot (2004b, Proposition 8).

Theorem 6.5 (Representation in the non-strict decomposable model)
A preference relation $\succsim$ on X admits a representation in the non-strict decompos-

able model

$$x \succsim y \Leftrightarrow U(u_1(x_1), \ldots, u_n(x_n)) \geq U(u_i(y_1), \ldots, u_i(y_n)),$$

with U nondecreasing in all its arguments, iff $\succsim$ is a weak order and satisfies weak separability.

6.2.3.2 Eliciting the non-strict decomposable model

A non-strict decomposable model can be elicited in a very similar way to what we proposed for the strict case. The only difference results from the fact that the preference $\succsim$ is no longer positively responsive to the marginal preferences. It may occur that for some common levels a_{-i}, $(x'_i, a_{-i}) \succ (x_i, a_{-i})$ while for other common levels b_{-i}, $(x'_i, b_{-i}) \sim (x_i, b_{-i})$; in contrast, in the decomposable model, one always has $x'_i \succ_i x_i \Rightarrow (x'_i, a_{-i}) \succ (x_i, a_{-i})$, for all a_{-i}. Consequently, in the process proposed above for eliciting a decomposable model, one can not exclude that $\mathcal{N}(u_i) \neq u_i$ while $(u_1, \ldots, \mathcal{N}(u_i), \ldots, u_n) \sim (u_1, \ldots, u_i, \ldots, u_n)$; in particular, it may occur that in (6.12), $\overline{u}^k \sim \overline{u}^{k-1}$, for some k. As a consequence, the suggested elicitation procedure must be adapted as follows.

For eliciting a non-strict decomposable model, follow the same lines as indicated for the decomposable model. For each profile generated by applying the "next level" operator $\mathcal{N}$ or the "preceding level" operator $\mathcal{N}^{-1}$ to a starting profile, one has to verify whether the generated profile is strictly preferred ($\succ$) or indifferent ($\sim$) to the initial one. For instance, if for some k, $\overline{u}^{k+1} \sim \overline{u}^k$, i.e. if consecutive milestones are indistinguishable, then all profiles in-between also collapse; more precisely, any profile $\overline{u}$ for which $u_i = \overline{u}_i^{k+1}$ or $\overline{u}_i^k$ for all i is indifferent both to $\overline{u}^{k+1}$ and $\overline{u}^k$. Profiles generated by applying the "preceding level" operator should be similarly checked for indifference or strict preference.

6.2.4 Insufficiency of the decomposable model

Decomposable preferences form a large family of preferences though not large enough to encompass all useful cases. A major restriction is that not all preferences may be assumed to be weak orders. The example of the sequence of cups of coffee, each differing from the previous one by an imperceptible added quantity of sugar, is famous; it leads to the notion of semiorder (Luce, 1956), in which indifference is not transitive, while strict preference is. A classic example of such a situation occurs in statistical decision contexts.

Example 6.3 (Testing for equality of means)
Let $X = \mathbb{R}^n$; a vector $x \in X$ can be viewed as a sample of n independent trials drawn from a normal probability distribution. Let a and b be two vectors of X; assuming that they are respectively samples of the variables A and B, both normal and with respective means μ_A and μ_B (and known variance σ^2), one may wish to test for equality of the means μ_A and μ_B. Let us mention as a relevant example, the case in which the length (or mass, or volume, ...) of a collection of objects are measured using an appropriate measuring device; the measure of each object is

repeated n times in order to control errors in measurement; vector a records the n measures performed on a specific object that can be identified with the variable A; the "true" measure of object A is μ_A. If we want to test whether there is evidence that $\mu_A > \mu_B$ or on the contrary, if the data point to accepting that $\mu_A = \mu_B$ (one-sided test), we have to compute $\overline{a} = 1/n \sum_{i=1}^{n} a_i$ and $\overline{b} = 1/n \sum_{i=1}^{n} b_i$. We shall reject the hypothesis that $\mu_A = \mu_B$ and consider that $\mu_A > \mu_B$ if

$$\overline{a} > \overline{b} + k\frac{\sigma}{\sqrt{n}},$$

where k is a positive value determined in order to limit the risk of type I (i.e. the risk of rejecting the hypothesis while it is true). One could decide that $\mu_A > \mu_B$ as soon as $\overline{a} > \overline{b}$ but due to random effects, using this criterion would lead us to frequent errors, especially when μ_B is only slightly larger than μ_A.

This classical statistical test interpreted within our framework, amounts to define a relation $\succ$ on X by

$$a \succ b \text{ iff } \overline{a} > \overline{b} + k\frac{\sigma}{\sqrt{n}}.$$

It models the decision that would be taken, according to standard statistical theory, on the issue of mean equality in all possible cases of two independent samples of normal distributions with known common variance. The relation that one obtains on X is the asymmetric part of a semiorder (see chapter 3). In order to deal with reflexive preferences as we did before, we may consider the semiorder of which the asymmetric part is $\succ$ that is defined as follows:

$$a \succsim b \text{ iff } \overline{a} \geq \overline{b} - k\frac{\sigma}{\sqrt{n}}.$$

The information conveyed in this relation, although it has no classical statistical interpretation, is logically equivalent to its asymmetric part (the latter can be reconstructed from the former without loss of information).

This semiorder cannot be represented within the decomposable model (or in its non-strict variant). Instead, here is another type of a representation for $\succsim$:

$$a \succsim b \text{ iff } \frac{1}{n}\sum_{i=1}^{n} a_i \geq \frac{1}{n}\sum_{i=1}^{n} b_i - k\frac{\sigma}{\sqrt{n}}, \tag{6.13}$$

or, equivalently,

$$a \succsim b \text{ iff } \frac{1}{n}\sum_{i=1}^{n}(a_i - b_i) + k\frac{\sigma}{\sqrt{n}} \geq 0. \tag{6.14}$$

◇

The previous example shows that the decomposable model may prove insufficient for representing relations that occur rather naturally; in particular, the model does not cover preference relations that are no weak orders. One objection to example 6.3 may be that it does not deal with preferences in the true sense. We decided to develop this example because it is familiar to all those who have received at least a basic course in statistics; it also has the advantage of providing a simple and unambiguous mathematical formulation. In addition, we are convinced that the type of behaviour producing the relation $\succsim$ or $\succ$ in example 6.3 is very common when dealing with an additive model for decision. Due to the uncertainty and errors in the assessment of the marginal value functions u_i, one would probably refrain from claiming that $a \succ b$ as soon as $u(a) = 1/n \sum_{i=1}^{n} a_i > u(b) = 1/n \sum_{i=1}^{n} b_i$. Instead, it would be more reasonable to say that $a \succ b$ when $u(a) > u(b) + \varepsilon$, i.e. when $1/n \sum_{i=1}^{n} a_i > 1/n \sum_{i=1}^{n} b_i + \varepsilon$, where ε is a positive constant that offers some guarantee against estimation errors. One would thus have:

$$\begin{aligned} a \sim b &\text{ iff } |u(a) - u(b)| \leq \varepsilon \\ a \succ b &\text{ iff } u(a) > u(b) + \varepsilon \end{aligned}$$

Estimating ε is not an easy task. For example, considering an additive value model involving evaluations on a cost criterion, one can compute the cost equivalent to a unit of u and ask the client which monetary amount he would consider as an indifference threshold, taking his perception of the uncertainty on cost evaluation into account. Of course, this will only reflect the uncertainty as to cost (and not error or imprecision in the elicitation of the functions u_i) but it should provide at least a lower bound on ε, which, most likely, will be positive. The concern for not asserting conclusions that are not well-established because the parameters of a model are not precisely or reliably assessed, is the central topic of section 7.5 in chapter 7.

6.2.5 Insufficiency of marginal analysis: marginal traces

In the decomposable model, the preference may be reconstructed on the basis of the marginal preferences $\succsim_i$ since it is represented by a function of the u_i's, themselves representing $\succsim_i$ (at least in the strict decomposable model).

This is no longer the case when $\succsim$ is not a weak order. Again take example 6.3. We have:

$$\begin{aligned} a_i \succsim_i b_i &\text{ iff } (a_i, c_{-i}) \succsim (b_i, c_{-i}), \forall c_{-i} \\ &\text{ iff } \frac{1}{n}(a_i + \sum_{i \neq j} c_j) \geq \frac{1}{n}(b_i + \sum_{i \neq j} c_j) - k\frac{\sigma}{\sqrt{n}}, \forall c_{-i} \\ &\text{ iff } a_i \geq b_i - k\sigma\sqrt{n} \end{aligned} \quad (6.15)$$

The marginal preference $\succsim_i$ is thus itself a semiorder with a larger threshold $-k\sigma\sqrt{n}$ (larger in absolute value compared to the threshold associated with $\succsim$). This threshold is n times larger than the one associated with the representation of $\succsim$, which means that the relation $\succsim_i$ on X_i is not very discriminating.

Suppose that in example 6.3 the variance σ^2 of both observed normal distributions is equal to 1 and their mean μ are known to range between $-1/2$ and $1/2$. Let $n = 25$, which corresponds to samples of 25 observations. Fix the type I risk to 5%; in our case, this corresponds to a constant k equal to 1.96 in formula (6.15). We thus have:

$$\begin{aligned} a \succsim b &\text{ iff } \overline{a} \geq \overline{b} - \frac{1.96}{\sqrt{n}} = \overline{b} - 0.392 \\ a_i \succsim_i b_i &\text{ iff } a_i \geq b_i - 1.96\sqrt{n} = b_i - 9.8 \end{aligned} \quad (6.16)$$

In view of (6.16), $a_i \sim_i b_i$ iff $|a_i - b_i| \leq 9.8$, which means that $\succsim_i$ does not discriminate between levels on X_i that differ by less than 9.8. If X_i is bound to take values, e.g. in the interval $[-3.5; 3.5]$—which would be reasonable since Gaussian distributions very seldom deviate from their mean by more than 3σ—all levels of X_i are *indifferent* w.r.t the marginal preference.

Is there a relation on X_i that has stronger links with the global preference $\succsim$ than marginal preference $\succsim_i$? The answer is the *marginal trace* $\succsim_i^{\pm}$ that is defined below.

Definition 6.10 (Marginal trace)
The marginal trace $\succsim_i^{\pm}$ of relation $\succsim$ on the product set $X = \prod X_i$ is the relation on X_i defined by:

$$a_i \succsim_i^{\pm} b_i \text{ iff } \begin{cases} \text{for all } c, d \in X, \\ [(b_i, c_{-i}) \succsim d] \Rightarrow [(a_i, c_{-i}) \succsim d] \text{ and} \\ [c \succsim (a_i, d_{-i})] \Rightarrow [c \succsim (b_i, d_{-i})] \end{cases} \tag{6.17}$$

In other words, if a test decides that $\mu_C \geq \mu_D$ (where C and D are normal variables with equal variances) on the basis of a sample of C containing b_i, the same decision would be made if the sample contained a_i instead of b_i; conversely, if a test decides that $\mu_C \geq \mu_D$ on the basis of a sample of D containing a_i, the same decision would be made if the sample contained b_i instead of a_i.

In example 6.3, one has $a_i \succsim_i^{\pm} b_i$ iff $a_i \geq b_i$, which is easily verified. Suppose indeed that $(b_i, c_{-i}) \succsim d$ for some $c_{-i} \in X_{-i}$ and $d \in X$; this means that

$$\frac{1}{n}(b_i + \sum_{i \neq j} c_j) \geq \frac{1}{n}\sum_{j=1}^{n} d_j - k\frac{\sigma}{\sqrt{n}}. \tag{6.18}$$

Substituting b_i by $a_i \geq b_i$ preserves the inequality. Conversely, when $a_i < b_i$, there are situations in which substituting b_i by a_i results in violating the inequality. Choose for instance c_j and d_j in (6.18), such that the inequality becomes an equality; this can be done in many ways since c_j and d_j can take any value in $\mathbb{R}$. In such a case, substituting b_i by a_i breaks the tie in the wrong direction.

In models in which $\succsim$ is not assumed to be a weak order, the information conveyed in the marginal preference relations may be insufficient to reconstruct the preference. As we shall see, marginal traces, provided they are weak orders, always convey sufficient information.

The reason why the insufficiency of marginal preferences did not show up in the decomposable model is a consequence of the following result.

Proposition 6.2 (Marginal preferences and marginal traces)
If a preference relation $\succsim$ on X is reflexive and transitive, its marginal preferences $\succsim_i$ and its marginal traces $\succsim_i^{\pm}$ are identical for all i.

This proposition almost directly results from the definitions of marginal preferences and traces. It implies that there is no need to worry about marginal traces unless $\succsim$ is not transitive. More precisely, as we shall see below, the notion that conveys all the information needed to reconstruct the global preference from relations on each scale X_i, is always the marginal traces; but when $\succsim$ is reflexive and transitive, you may equivalently use marginal preferences. The converse of the proposition is not true however: there are cases where $\succsim$ is not transitive (e.g. when $\succsim$ is a semiorder) and $\succsim_i = \succsim_i^{\pm}$ (see Bouyssou and Pirlot, 2004b, Example 4).

For transitive preferences, the marginal preference contains all the relevant information on the client's preferences in relation to the corresponding dimension. When a preference happens to be nontransitive, the marginal preference may cease to be the central tool for eliciting the preference, because it may not contain all the relevant one-dimensional information. In the case of intransitive preferences, the full one-dimensional information is to be sought in the marginal traces.

Instead of generalising the decomposable model again to encompass preferences that are, for instance, semiorders [2], we propose and study a much more general model. It is so general that it encompasses all relations on X. Considering this model as a framework, we introduce successive specialisations that will bring us back to the decomposable model, but "from above", i.e. in a movement from the general to the particular. In this specialisation process, it is the marginal trace—not the marginal preference—that is the central tool. Our main axioms will have a direct impact on the properties of marginal traces while they will be used to further specify the models.

6.2.6 Generalising decomposable models using marginal traces

Consider the very general representation of a relation $\succsim$ described by:

$$x \succsim y \Leftrightarrow F(u_1(x_1), u_2(x_2), \ldots, u_n(x_n), u_1(y_1), u_2(y_2), \ldots, u_n(y_n)) \geq 0 \qquad (L0)$$

The main difference w.r.t. the decomposable model is that the evaluations of the two alternatives are not dealt with separately.

If no property is imposed on function F, the model is trivial since any relation can be represented within it. It obviously generalises the decomposable model and encompasses as a special case, the representation involving a threshold described in (6.14) (in which the preference is a semiorder).

It is easy to obtain representations that guarantee simple properties of $\succsim$. For instance, we have that:

- $\succsim$ is reflexive iff it has a representation in model $(L0)$ with

$$F([u_i(x_i)]; [u_i(x_i)]) \geq 0;$$

[2] Obviously one could think of generalising the decomposable model to represent some preferences that are semiorders, getting inspiration from the numerical representation with threshold in (6.13).

- $\succsim$ is complete iff it has a representation in model $(L0)$ with

$$F([u_i(x_i)];[u_i(y_i)]) = -F([u_i(y_i)];[u_i(x_i)]).$$

What if we impose monotonicity conditions on F? The natural conditions in view of the decomposable model are:

- F increasing in its first n arguments and decreasing in its last n arguments
- F non-decreasing in its first n arguments and non-increasing in its last n arguments

The following axioms are closely linked to imposing monotonicity properties to F and, as we shall see, with properties of the marginal traces.

Definition 6.11 (Axioms $AC1$, $AC2$, $AC3$, $AC4$)
We say that $\succsim$ satisfies:
$AC1_i$ if

$$\left.\begin{array}{r}(x_i,a_{-i}) \succsim y\\ and\\ (z_i,b_{-i}) \succsim w\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i,a_{-i}) \succsim y\\ or\\ (x_i,b_{-i}) \succsim w\end{array}\right.$$

$AC2_i$ if

$$\left.\begin{array}{r}y \succsim (x_i,a_{-i})\\ and\\ w \succsim (z_i,b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}y \succsim (z_i,a_{-i})\\ or\\ w \succsim (x_i,b_{-i})\end{array}\right.$$

$AC3_i$ if

$$\left.\begin{array}{r}(x_i,a_{-i}) \succsim y\\ and\\ w \succsim (x_i,b_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i,a_{-i}) \succsim y\\ or\\ w \succsim (z_i,b_{-i}),\end{array}\right.$$

for all $x_i, z_i \in X_i$, all $a_{-i}, b_{-i} \in X_{-i}$ and all $y, w \in X$,
$AC4_i$ if $\succsim$ satisfies $AC3_i$ and, whenever one of the conclusions of $AC3_i$ is false, then the other one holds with $\succ$ instead of $\succsim$.

We say that $\succsim$ satisfies $AC1$ (resp. $AC2$, $AC3$, $AC4$) if it satisfies $AC1_i$ (resp. $AC2_i$, $AC3_i$, $AC4_i$) for all $i \in N$. We also use $AC123_i$ (resp. $AC123$) as shorthand for $AC1_i$, $AC2_i$ and $AC3_i$ (resp. $AC1$, $AC2$ and $AC3$).

The intuition behind these axioms is the following. Take axiom $AC1_i$. It suggests that x_i and z_i can be compared: either x_i corresponds to a "level" on a "scale" on X_i that is "above" z_i or the other way around. Suppose indeed that x_i is involved in an alternative that is preferred to another alternative $((x_i,x_{-i}) \succsim y)$; suppose further that substituting z_i to x_i would not allow to preserve the preference $(Not[(z_i,x_{-i}) \succsim y])$. Then $AC1_i$ says that substituting z_i with x_i when z_i is involved in an alternative that is preferred to another $((z_i,z_{-i}) \succsim w)$ will always preserve the preference (i.e. we have: $(x_i,z_{-i}) \succsim w$). One can interpret such a situation by saying that x_i is "above" z_i. The "being above" relation on X_i is what we call the *left marginal trace* of $\succsim$ and we denote it by $\succsim_i^+$; it is defined as follows:

$$x_i \succsim_i^+ z_i \Leftrightarrow [(z_i, z_{-i}) \succsim w \Rightarrow (x_i, z_{-i}) \succsim w]. \tag{6.19}$$

We explained above that $AC1_i$ means that x_i and z_i can always be compared, which, in terms of the left trace, interprets as: "We may not have $Not[x_i \succsim_i^+ z_i]$ and $Not[z_i \succsim_i^+ x_i]$" at the same time. It is easy to see that assuming the latter amounts to having some z_{-i} and some w such that:

$$(z_i, z_{-i}) \succsim w \text{ and } Not[(x_i, z_{-i}) \succsim w]$$

and at the same time, for some x_{-i} and some y,

$$(x_i, x_{-i}) \succsim y \text{ and } Not[(z_i, x_{-i}) \succsim y],$$

which is exactly the negation of $AC1_i$. Axiom $AC1_i$ thus says that the left marginal trace $\succsim_i^+$ is a complete relation; since it is transitive by definition, $AC1_i$ means that $\succsim_i^+$ is a weak order. The natural order induced by $\succsim$ on X_i is $\succsim_i^+$; this order on X_i may be interpreted as a criterion, an element of the set H defined in chapter 2, section 2.3.3. So, in a sense, knowing $\succsim_i^+$ transforms dimension i into a criterion; $\succsim_i^+$ encodes the client's preference as far as dimension i is concerned. Endowed with the order $\succsim_i^+$, X_i can also be seen as an ordinal scale (see chapter 3, section 3.7.1.1).

$AC1_i$ deals with levels involved in alternatives that are preferred to other alternatives, thus in the strong (left hand side) position in the comparison of two alternatives; in contrast, $AC2_i$ rules the behaviour of $\succsim$ when changing levels in alternatives in the weak position (another alternative is preferred to them). Clearly, $AC2_i$ is concerned with a *right marginal trace* $\succsim_i^-$ that is defined as follows:

$$y_i \succsim_i^- w_i \Leftrightarrow [x \succsim (y_i, y_{-i}) \Rightarrow x \succsim (w_i, y_{-i})]. \tag{6.20}$$

The interpretation of $y_i \succsim_i^- w_i$ is clear: when an alternative is beaten showing the level y_i, it would also be beaten if y_i was changed into w_i. In other words, $\succsim_i^-$ compares levels on X_i when these levels are involved in alternatives in the weak (right hand side) position of a preference relation. By reasoning as above, one sees that $AC2_i$ be equivalent to requiring that $\succsim_i^-$ is a complete relation and thus a weak order (since it is transitive by definition).

At this stage, it is natural to ask whether the left marginal trace is related to the right one. The role of $AC3_i$ is to ensure that $\succsim_i^+$ and $\succsim_i^-$ are not incompatible, i.e. that one cannot have $Not[x_i \succsim_i^+ y_i]$ and $Not[y_i \succsim_i^- x_i]$ at the same time. If $\succsim_i^+$ and $\succsim_i^-$ are complete, this means that one cannot have $[y_i \succ_i^+ x_i]$ and $[x_i \succ_i^- y_i]$ (where $\succ_i^+$ and $\succ_i^-$ denote the asymmetric part of $\succsim_i^+$ and $\succsim_i^-$, respectively) or, in other words, that $[x_i \succ_i^+ y_i]$ implies $[x_i \succsim_i^- y_i]$ and $[x_i \succ_i^- y_i]$ implies $[x_i \succsim_i^+ y_i]$. As a consequence of $AC123_i$, the intersection of the (complete) relations $\succsim_i^+$ and $\succsim_i^-$ is a complete relation, that is nothing else than the marginal trace $\succsim_i^\pm$ since definition (6.17) is equivalent to

$$a_i \succsim_i^\pm b_i \Leftrightarrow a_i \succsim_i^+ b_i \text{ and } a_i \succsim_i^- b_i.$$

We summarise our findings in the next proposition (Bouyssou and Pirlot, 2004b, Lemma 3).

Proposition 6.3 (Properties of marginal traces)

1. $\succsim_i^+$ *is a weak order iff* $AC1_i$ *holds,*

2. $\succsim_i^-$ *is a weak order iff* $AC2_i$ *holds,*

3. $\succsim_i^\pm$ *is a weak order iff* $AC1_i, AC2_i$ *and* $AC3_i$ *hold.*

The exact role of $AC4_i$ is less transparent. It is related to the *monotonicity* properties of the preference $\succsim$ with respect to its marginal traces. By definition of the marginal traces, without assuming any of $AC1$, $AC2$ or $AC3$, the preference $\succsim$ responds monotonically w.r.t. $\succsim_i^+$ and $\succsim_i^-$ as follows (Bouyssou and Pirlot, 2004b, Lemma 2).

Proposition 6.4 (Responsiveness w.r.t. marginal traces)
For all $x, y \in X$ *and all* $z_i, w_i \in X_i$,

1. $[x \succsim y$ *and* $z_i \succsim_i^+ x_i] \Rightarrow (z_i, x_{-i}) \succsim y$,

2. $[x \succsim y$ *and* $y_i \succsim_i^- w_i] \Rightarrow x \succsim (w_i, y_{-i})$.

These properties hold a fortiori if $\succsim_i^+$ (resp. $\succsim_i^-$) is substituted by $\succsim_i^\pm$. Contrasting the latter with proposition 6.1 that describes the responsiveness of independent weakly ordered preferences $\succsim$ w.r.t. marginal preferences $\succsim_i$, we note, in the present case, that there is no mention of strict or positive responsiveness (proposition 6.1.3 and 4). The latter property is not true of general preferences and is indeed related to axiom $AC4$. As soon as $\succsim$ is reflexive, one can show that $AC4_i$ implies $AC123_i$ and, moreover, that the preference is strictly responsive to $\succ_i^\pm$, i.e.:

$$[x \succsim y \text{ and } z_i \succ_i^\pm x_i] \Rightarrow (z_i, x_{-i}) \succ y, \tag{6.21}$$

$$[x \succsim y \text{ and } y_i \succ_i^\pm w_i] \Rightarrow x \succ (w_i, y_{-i}). \tag{6.22}$$

Remark 6.2.4 (Positive responsiveness w.r.t. marginal traces)
The property just discussed corresponds to the *positive responsiveness* property introduced in section 5.2 (see pages 171, 173). The two points of view on aggregation developed in this book (characterisation of procedures, in chapter 5 and characterisation of relations in a conjoint measurement framework, in the present chapter) meet here. The positive responsiveness property defined on p. 171 relates profiles of relations to the global preference obtained through an aggregation procedure applied to these profiles. If the relations in a profile (that can be interpreted as a priori modelling the preferences of the client on the various dimensions, i.e. as criteria in the sense of section 2.3.3 of chapter 2) happen to be our marginal traces, then the positive responsiveness defined have the same meaning in both cases. •

The links between the axioms given above and the marginal traces can be directly exploited in the construction of a monotone numerical representation of $\succsim$ in model $(L0)$. We have the following result (Bouyssou and Pirlot, 2004b, Theorem 2).

Proposition 6.5 (Representation in models L)
A preference relation $\succsim$ on X admits a representation in model ($L0$) with F non-decreasing in its first n arguments, and non-increasing in the last n arguments if and only if it satisfies $AC1$, $AC2$ and $AC3$.

To clarify how the marginal traces intervene in the construction of the representation, we describe how a representation can be obtained with F monotone as indicated. Due to the fact that $\succsim$ satisfies $AC123$, we know that the marginal traces $\succsim_i^{\pm}$ are weak orders. Take any numerical representation of the weak order $\succsim_i^{\pm}$ for u_i, i.e. u_i is any real-valued function defined on X_i, such that

$$x_i \succsim_i^{\pm} z_i \text{ iff } u_i(x_i) \geq u_i(z_i).$$

Then define F as follows:

$$F([u_i(x_i)];[u_i(y_i)]) = \begin{cases} +\exp(\sum_{i=1}^{n}(u_i(x_i) - u_i(y_i))) & \text{if } x \succsim y, \\ -\exp(\sum_{i=1}^{n}(u_i(y_i) - u_i(x_i))) & \text{otherwise.} \end{cases} \tag{6.23}$$

It can easily be shown that this representation satisfies the requirements. Clearly, the choice of the exponential function in the definition of F is arbitrary; any other positive and non-decreasing function could have been chosen instead. Again the choice of a representation u_i of the weak orders $\succsim_i^{\pm}$ is highly arbitrary. We are thus far from the uniqueness results that can be obtained for the representation of preferences in the additive utility model (6.1). However, all these representations are however equivalent from the point of view of the description of a preference.

6.2.7 Models using marginal traces

At this point, it might be useful to give a full picture of the models based on marginal traces. We have identified three variants of model ($L0$) above: those corresponding respectively to reflexive or complete preference $\succsim$ or to a preference with complete marginal traces. One can associate particular features of the numerical representation in model ($L0$) to each variant. Systematising the analysis, we may define the variants of model ($L0$) listed in table 6.1. This table also shows a characterisation of the models using the axioms introduced in the previous section.

Remark 6.2.5
Note that requiring that F be strictly monotone instead of monotone makes no difference unless $\succsim$ is complete. This is quite understandable and is due to the fact that, when $\succsim$ is complete, the value of F dedicated to representing indifference is 0. In such a case, if F is strictly monotone, any increase (with respect to marginal traces) of an evaluation of an alternative produces an alternative that is strictly preferred to the original one. Not all preferences show this feature. •

Remark 6.2.6
Model ($L8$) is the closest to the (strict) decomposable model; while model ($L7$) is the closest to the non-strict decomposable model. Each of them generalises the corresponding decomposable model to non-necessarily transitive preference relations. If $\succsim$ is transitive (and complete, hence a weak order) the corresponding decomposable model is the appropriate tool for analysing preferences. •

Table 6.1: Main models using traces on levels and their characterisation.

Models	Definition	Conditions
$(L0)$	$x \succsim y \Leftrightarrow F([u_i(x_i)], [u_i(y_i)]) \geq 0$	$\varnothing$
$(L1)$	$(L0)$ with $F([u_i(x_i)], [u_i(x_i)]) = 0$	refl.
$(L2)$	$(L1)$ with $F([u_i(x_i)]; [u_i(y_i)]) = -F([u_i(y_i)]; [u_i(x_i)])$	cpl.
$(L3)$ $\Updownarrow$ $(L4)$	$(L0)$ with $F(\nearrow, \searrow)$ $(L0)$ with $F(\nearrow\!\!\nearrow, \searrow\!\!\searrow)$	$AC123$
$(L5)$ $\Updownarrow$ $(L6)$	$(L1)$ with $F(\nearrow, \searrow)$ $(L1)$ with $F(\nearrow\!\!\nearrow, \searrow\!\!\searrow)$	refl., $AC123$
$(L7)$	$(L2)$ with $F(\nearrow, \searrow)$	cpl., $AC123$
$(L8)$	$(L2)$ with $F(\nearrow\!\!\nearrow, \searrow\!\!\searrow)$	cpl., $AC4$

$\nearrow$ means nondecreasing, $\searrow$ means nonincreasing
$\nearrow\!\!\nearrow$ means increasing, $\searrow\!\!\searrow$ means decreasing
refl. means reflexive, cpl. means complete

6.2.8 Respect of the dominance relation

Why is the monotonicity of F in proposition 6.5 an appealing property? In conjoint measurement, we do not suppose a priori that there is any preference information on the sets X_i; it is the "observed" global preference on X that reveals how the client values the levels on each viewpoint. In the practice of MCDM, very often, the set of levels on the scales attached to each viewpoint are at least ordered[3]. Suppose that there is an a priori weak order S_i on each set X_i, with $x_i \ S_i \ z_i$ meaning that level x_i is at least as good as level z_i. In other words, S_i orders the levels of X_i from the least desirable to the most desirable. We emphasise that this order a priori has nothing to do with a particular client's preference. For instance, if the alternatives are cars and we consider the point of view of cost, the ordering S_i would correspond to "the cheaper the better". Similarly, if cars are supposed to be assessed on a comfort scale with 5 degrees, these degrees will usually be ranked by increasing order of comfort, independently of the cars to be assessed and one may presume for instance, that no client who considers comfort a relevant

[3] We do not consider the case where the "natural" order on the scale is not compatible with the "natural" preferences of the client here, i.e. for instance when the client's preference initially increases with the performance until a maximum is reached, after which the preference decreases; for an analysis of the requirements for an appropriate system of criteria in multiple criteria decision analysis, see Roy and Bouyssou (1993, chapter 2) or Bouyssou (1990).

criterion for choosing a car, will find degree 3 more desirable than degree 4 (he may possibly be indifferent).

So, if there is such a priori information available on the sets X_i, one may expect that the client's preference $\succsim$ on the set X of alternatives fulfills the following consistency property that we call "respect of dominance". We restrict ourselves to reflexive preference relations in this section.

Definition 6.12 (Respect of dominance)
Let S_i be a weak order on X_i, for all i, and let x, y be alternatives in X. The dominance relation Δ_S on X is defined by

$$x \, \Delta_S \, y \text{ iff } x_i \, S_i \, y_i, \text{ for all } i \in N; \tag{6.24}$$

when this condition is fulfilled, we say that x dominates y. If $\succsim$ is a reflexive preference on X, $\succsim$ is "compatible with the dominance relation Δ_S" if the following condition holds:

$$[z \, \Delta_S \, x, \, x \succsim y \text{ and } y \, \Delta_S \, w] \Rightarrow z \succsim w. \tag{6.25}$$

We say that $\succsim$ is strictly compatible with Δ_S if, in addition, we have $z \succ w$ as soon as at least one of the two dominance pairs corresponds to strict dominance, i.e. $z_i \, P_i \, x_i$ or $y_i \, P_i \, w_i$ for some $i \in N$, where P_i denotes the asymmetric part of S_i.

When $\succsim$ is compatible with the dominance relation Δ_S, we also say that it "respects dominance"; "strict respect of dominance" occurs when $\succsim$ is strictly compatible with Δ_S. This definition requires that combining preference with dominance on both sides of the preference yields a preference; in particular it entails (provided $\succsim$ is reflexive, which we assume here) that dominance implies preference (i.e. $x \, \Delta_S \, y \Rightarrow x \succsim y$).

Remark 6.2.7
Note that the above definition of dominance is the non-strict version of the usual one. Usually (see, e.g., Steuer, 1986, p. 147), dominance is defined as an irreflexive relation: on top of condition (6.24), we must have $x_i \, P_i \, y_i$ for at least one i. The latter relation, that we could refer to a *strict dominance*, is just the asymmetric part of the above-defined Δ or in other words, we include in Δ pairs of alternatives that are indifferent on all dimensions ($x_i \, I_i \, y_i$ for all i, where I_i denotes the symmetric part of S_i). •

The first question that arises in the framework of conjoint measurement is: "Are all preferences compatible with some weak order S_i on each X_i? And if this is not the case, which preferences are?". It is important to understand that, for the moment, we do not presuppose the knowledge of weak orders on X_i; we adopt the typical point of view of conjoint measurement theory, assuming only that we have a preference $\succsim$ on X. We thus investigate the conditions on $\succsim$ under which *there may exist* weak orders S_i on X_i such that $\succsim$ is compatible with the dominance relation Δ_S these weak orders define. These conditions are readily obtained using $AC123$, and $AC4$, for the strict respect of dominance (Bouyssou and Pirlot, 2004b, Theorem 1).

Theorem 6.6 (Compatibility with the dominance relation)
Let $\succsim$ be a reflexive relation on a set X. There are weak orders S_i on X_i for all $i \in N$ such that:

1. *$\succsim$ is compatible with the dominance relation Δ_S iff $\succsim$ satisfies $AC123$;*

2. *$\succsim$ is strictly compatible with the dominance relation Δ_S iff $\succsim$ satisfies $AC4$*

The conditions that guarantee the compatibility of $\succsim$ with a dominance relation are precisely those ensuring that the marginal traces $\succsim_i^{\pm}$ are weak orders and that there is a monotone representation of $\succsim$ in model $(L0)$. So it would be no wonder if the marginal traces and the weak orders S_i had close connections. It is indeed the case that

$$x_i \; S_i \; z_i \Rightarrow x_i \succsim_i^{\pm} z_i,$$

which means that $\succsim_i^{\pm}$ is usually less discriminant than S_i. Thus, to be compatible with Δ_S, $\succsim$ must have marginal traces that never contradict the weak orders S_i, i.e. if $x_i \; S_i \; z_i$, either $x_i \succ_i^{\pm} y_i$ or $x_i \sim_i^{\pm} z_i$, but we never have $z_i \succ_i^{\pm} x_i$. Returning to the comfort criterion in the buying-a-car example alluded to in the beginning of this section, the qualitative levels 3 and 4, with 4 P_i 3, may be considered by the marginal trace as indifferent or distinct (with, in the latter case, $4 \succ_i^{\pm} 3$).

Strict compatibility with a dominance relation is quite a stringent requirement. It imposes that raising the evaluation of an alternative on the scale S_i of any criterion i yields another alternative that is strictly preferred to the original one. And a symmetric behaviour is expected when any evaluation is lowered. Indifference is very "thin" with such a preference relation. This is indeed the case with preferences that can be represented by an additive value function (model 6.1) and also by a decomposable model (6.11), but not with the non-strict decomposable model.

In practice, the client often has a priori preferences on each dimension: cost should be minimised, time needed for accelerating to reach a certain speed should be minimised, etc. This section clarifies how these a priori one-dimensional preferences combine with the global preference. If the latter satisfies some reasonable properties, the marginal traces contain the a priori preferences. However, the a priori preference may distinguish pairs of levels that are indifferent in the marginal trace. This is quite natural; small differences in cost, for instance, will usually not influence the way two costly equipments compare to all other alternatives, provided the characteristics of the former two are tied on all other dimensions.

6.2.9 Properties of marginal preferences in $(L0)$ and variants

We briefly come back to the analysis of marginal preferences in connection with the variants of $(L0)$ characterised above. As stated before (proposition 6.2), we know that for reflexive and transitive preferences, $\succsim_i = \succsim_i^{\pm}$. For reflexive preferences, $x_i \succsim_i^{\pm} z_i$ implies $x_i \succsim_i z_i$.

The incidence of axioms $AC1$, $AC2$, $AC3$ and $AC4$ on marginal preferences is summarised in the next proposition (Bouyssou and Pirlot, 2004b, Proposition 3 and Lemma 4.3).

Proposition 6.6 (Properties of marginal preferences)

1. *If* $\succsim$ *is reflexive and either* $AC1_i$ *or* $AC2_i$ *holds then* $\succsim_i$ *is an interval order.*

2. *If, in addition,* $\succsim$ *satisfies* $AC3_i$ *then* $\succsim_i$ *is a semiorder.*

3. *If* $\succsim$ *is reflexive and* $AC4_i$ *holds then* $\succsim_i$ *is a weak-order and* $\succsim_i = \succsim_i^{\pm}$.

The preference $\succsim$ in example 6.3, page 259 has marginal preferences $\succsim_i$ that are semiorders as is shown by equation (6.15), while marginal traces are the natural weak orders on $\mathbb{R}$. From the latter, applying proposition 6.5 (in its version for sets X of arbitrary cardinality), we deduce that $\succsim$ satisfies $AC123$. Applying the third part of proposition 6.6, we deduce further that $\succsim$ does not satisfy $AC4$.

6.2.9.1 Separability and independence

Conditions $AC1$, $AC2$, $AC3$ and $AC4$ also have an impact on the separability and independence properties of $\succsim$ (Bouyssou and Pirlot, 2004b, Proposition 3.1 and Lemma 4.3).

Proposition 6.7 (Separability and independence)
Let $\succsim$ *be a reflexive relation on* X. *We have:*

1. *If* $\succsim$ *satisfies* $AC1_i$ *or* $AC2_i$ *then* $\succsim$ *is weakly separable for* $i \in N$.

2. *If* $\succsim$ *satisfies* $AC4_i$ *then* $\succsim$ *is independent for* $\{i\}$,

Preference $\succsim$ in the example of the statistical test (example 6.3, p. 259) is weakly separable for all i (since $\succsim$ satisfies $AC123$ and in view of part 1 of proposition 6.7); although $\succsim$ does not satisfy $AC4$, it is easy to see, applying the definition, that $\succsim$ is also independent for all i.

6.2.9.2 The case of weak orders

The case in which $\succsim$ is a weak order is quite particular. We have the following result (Bouyssou and Pirlot, 2004b, Lemma 5 and Lemma 4.3).

Proposition 6.8 (Case of weakly ordered preferences)
Let $\succsim$ *be a weak order on a set* X. *Then:*

1. [$\succsim$ *is weakly separable*] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC1$] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC2$] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC3$],

2. [$\succsim$ *is weakly independent*] $\Leftrightarrow$ [$\succsim$ *satisfies* $AC4$],

3. *If* $\succsim$ *is weakly separable, the marginal preference* $\succsim_i$ *equals the marginal trace* $\succsim_i^{\pm}$, *for all* i, *and these relations are weak orders.*

This result recalls that when analysing weakly separable weak orders, marginal traces can be substituted by marginal preferences (as is classically done); it also shows that weak separability masks $AC123$.

Example 6.4 (Min, LexiMin and DiscriMin)
In section 4.3.10.1, we have shown that comparing vectors of satisfaction degrees associated with a set of constraints could be done by comparing the lowest satisfaction degree in each vector, i.e.

$$x \succsim y \Leftrightarrow \min(x_1, \ldots, x_n) \geq \min(y_1, \ldots, y_n),$$

where x and y are n-tuples of numbers in the $[0, 1]$ interval. This method for comparing vectors is known as the "Min" or "MaxMin" method. Clearly, the preference $\succsim$ that this method yields is a weak order; it is not weakly independent as was shown in section 4.3.10.1, but it is weakly separable since $\succsim_i^{\pm}$ is simply the natural weak order on the interval $[0, 1]$; the relation $\succsim$ thus satisfies $AC123$ but not $AC4$. By theorem 6.8.3, $\succsim_i^{\pm} = \succsim_i$, for all i.

A refinement of the "Min" or "MaxMin" method is the "LexiMin" method that was studied in section 5.4.5; the latter discriminates between alternatives that the former leaves tied. When comparing alternatives x and y, LexiMin ranks x before y if $\min x_i > \min y_i$; when the minimal value of both profiles are equal, LexiMin looks at the second minimum and decides in favour of the alternative with the highest second minimum; if again the second minima are equal, it goes to the third and so on. Alternatives will only be indifferent for LexiMin when they cannot be distinguished when their coordinates are rearranged in non-decreasing order.

The preference yielded by LexiMin is again an independent weak order and $\succsim_i^{\pm} = \succsim_i$, for all i.

There is another interesting procedure that is less commonly used: the "DiscriMin" method. To compare two alternatives, DiscriMin first eliminates the dimensions on which their evaluations are equal; then it ranks as first, the alternative that has the highest minimal value (on the remaining dimensions). The obtained preference is not a weak order because the associated indifference is not transitive (for instance, $(0.1, 0.3) \sim (0.2, 0.1) \sim (0.1, 0.2)$ but $(0.1, 0.3) \succ (0.1, 0.2)$); it is nevertheless weakly separable; its marginal traces and preferences are again the natural order on $[0.1]$. ◇

6.2.10 Eliciting the variants of model $(L0)$

This family of models suggests an elicitation strategy similar to that used for the decomposable model, but based on the marginal traces instead of the marginal preferences. It is not likely, however, that such a general model could serve as a basis for a direct practical elicitation process; instead, we think that it is a framework for conceiving more specific models associated to a method; the additive value function model could be considered in this framework; the DiscriMin method, described above, is another example that doesn't yield a preference that is a weak order. Although it may seem unrealistic to work in such a general framework, Greco, Matarazzo, and Słowiński (1999a) have proposed to do so and elicit preferences using an adapted rough sets approach (indirect approach).

The family of models based on marginal traces constitute a framework that encompasses many common preference models; basic properties that distinguish them (such as independence vs. separability, responsiveness, respect of the dominance relation, etc.) can be understood in this framework. This may help the analyst to select appropriate evaluation models in practical problem situations, for instance by looking for evidence that the preference satisfies some discriminating properties or not.

6.3 Following another path: models using marginal traces on differences

The generalisation of the additive value model was pursued to its most extreme limits, since with model ($L0$) we encompass all possible binary relations on a product set. This generalisation relies on the marginal traces on the sets X_i. These relations were shown to be the stepping stones to lean on to elicit this type of model, for relations that are not transitive. For transitive (and reflexive) relations, marginal traces reduce to the usual marginal preferences.

There is, however, another line of generalisation of the additive value model. Obviously, it cannot be advocated as more general than the models based on marginal traces; it nevertheless sheds another light on the picture, since it is based on an entirely different fundamental notion: *traces on differences.* Instead of comparing performance profiles alternatives such as in the additive value model or the decomposable model or even, in a more implicit form, in model ($L0$), we can see the preference of x over y as resulting from a balance made between advantages and disadvantages of x w.r.t. y on all criteria. While the approach followed in the additive value model could be described as *Aggregate then Compare*, the latter is more relevant to the opposite paradigm *Compare* (on each dimension) *then Aggregate* (Dubois, Fargier, Perny, and Prade, 2003; Perny, 2000). The origins of such a paradigm can perhaps be found in social choice theory and, in particular, the majority rule *à la* Condorcet (see section 5.2.1). If we consider alternatives as candidates and points of view as voters, we may use the majority rule to compare the positions of each pair of candidates x, y in the ranking of each voter and then "aggregate" these comparisons by counting the number of voters that place candidate x ahead of candidate y and conversely. The ELECTRE methods (see p. 187) exploit the same idea in the context of multiple criteria decision analysis.

6.3.1 The additive difference model

This paradigm is not new in conjoint measurement either. It is related to the introduction of the intransitivity of preference. A. Tversky (1969) was one of the first to propose a model generalising the additive value model and able to encompass preferences that lack transitivity. It is known as the *additive difference model* in which,

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} \Phi_i(u_i(x_i) - u_i(y_i)) \geq 0, \tag{6.26}$$

where Φ_i are increasing and odd functions.

Preferences that satisfy (6.26) may be intransitive, but they are complete (due to the postulated oddness of Φ_i). When attention is restricted to the comparison of objects that only differ on one dimension, (6.26) implies that the preference between these objects is independent of their common level on the remaining $n - 1$ dimensions. This amounts to saying that $\succsim$ is independent for all i; the marginal preferences $\succsim_i$, clearly, are complete and transitive (hence weak orders) due to the oddness and the increasingness of the Φ_i. This, in particular, excludes the possibility of any perception threshold on dimensions, which would lead to an intransitive indifference relation on these dimensions. Imposing that Φ_i are nondecreasing instead of being increasing allows for such a possibility. This gives rise to what Bouyssou (1986) called the *weak additive difference model.*

Model (6.26) sums up the differences of preference represented by the functions $\Phi_i(u_i(x_i) - u_i(y_i))$; these differences are themselves obtained by recoding the algebraic difference of partial value functions u_i through the functions Φ_i. Due to the presence of two algebraic operations—the sum of the Φ_i and the difference of the u_i—the difficulties faced when axiomatising (6.26) are of the same order as (or worse than) for the additive value function model. The characterisations obtained in the "rich case" incorporate unnecessary structural assumptions on the set X, either in the topological or the algebraic approach: for $n = 2$, see Bouyssou (1986); Croon (1984); Fishburn (1980); for $n \geq 3$, see Fishburn (1992a).

Dropping the subtractivity requirement in (6.26) (as suggested in Bouyssou, 1986; Fishburn, 1990a,b, 1991a; Vind, 1991) is a partial answer to the limitations of the additive difference model. This leads to *nontransitive additive* conjoint measurement models in which:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} p_i(x_i, y_i) \geq 0, \tag{6.27}$$

where the p_i's are real-valued functions on X_i^2 and may have several additional properties (e.g. $p_i(x_i, x_i) = 0$, for all $i \in \{1, 2, \ldots, n\}$ and all $x_i \in X_i$).

This model is an obvious generalisation of the (weak) additive difference model. It allows for intransitive and incomplete preference relations $\succsim$ as well as for intransitive and incomplete marginal preferences. An interesting specialisation of (6.27) is obtained when the functions p_i are required to be *skew symmetric*, i.e., such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$. This skew symmetric nontransitive additive conjoint measurement model implies the completeness and the independence of $\succsim$. In view of the addition operation involved in the model, the difficulties in obtaining a satisfactory axiomatisation of the model remain essentially as in model (6.26). Fishburn (1990b, 1991a) axiomatises the skew symmetric version of (6.27) both in the finite and the infinite case; Vind (1991) provides axioms for (6.27) with

$p_i(x_i, x_i) = 0$ when $n \geq 4$; Bouyssou (1986) gives necessary and sufficient conditions for (6.27) with and without skew symmetry in the denumerable case, when $n = 2$.

6.3.2 Comparison of preference differences

With the nontransitive additive model (6.27), the notion of "preference difference" becomes more abstract than it seems to be in Tversky's model (6.26); we still refer to p_i as to a representation of preference differences on i even though there is no algebraic difference operation involved.

This prompts the following question: is there any intrinsic way of defining the notion of "difference of preference" by referring only to the preference relation $\succsim$? The answer is pretty much in the spirit of what we discovered in the previous section: differences of preference can be compared in terms of traces, here, of traces on "differences". We define a relation $\succsim_i^*$, that we shall call *marginal trace on differences*, comparing any two pairs of levels (x_i, y_i) and $(z_i, w_i) \in X_i^2$ in the following way.

Definition 6.13 (Marginal trace on differences $\succsim_i^*$)
The marginal trace on differences $\succsim_i^*$ is the relation on the pairs of levels X_i^2 defined by:

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ iff } \begin{cases} \text{for all } a_{-i}, b_{-i} \in X_{-i}, \\ (z_i, a_{-i}) \succsim (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \succsim (y_i, b_{-i}). \end{cases} \tag{6.28}$$

Intuitively, if $(x_i, y_i) \succsim_i^* (z_i, w_i)$, it seems reasonable to conclude that the preference difference between x_i and y_i is not smaller that the preference difference between z_i and w_i. Note that, by construction, $\succsim_i^*$ is reflexive and transitive.

Contrary to our intuition concerning preference differences, the definition of $\succsim_i^*$ does not imply that there is any link between two "opposite" differences (x_i, y_i) and (y_i, x_i). Henceforth we introduce the binary relation $\succsim_i^{**}$ on X_i^2.

Definition 6.14 (Marginal trace on differences $\succsim_i^{**}$)
The marginal trace on differences $\succsim_i^{**}$ is the relation on the pairs of levels X_i^2 defined by:

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \text{ iff } [(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)]. \tag{6.29}$$

It is easy to see that $\succsim_i^{**}$ is transitive and *reversible*, i.e.

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow (w_i, z_i) \succsim_i^{**} (y_i, x_i). \tag{6.30}$$

The relations $\succsim_i^*$ and $\succsim_i^{**}$ both appear to capture the idea of comparison of preference differences between elements of X_i induced by the relation $\succsim$. Hence, they are good candidates to serve as the basis for the definition of the functions p_i. They will not serve this purpose well however, unless they are complete. Before turning to the study of models based on traces on differences, it may be useful to emphasise that, by definition, preferences have some monotonicity properties with respect to their traces. We collect these properties in the following proposition (Bouyssou and Pirlot, 2002b, Lemma 3, p. 689).

Proposition 6.9 (Responsiveness w.r.t. traces on differences)
For all $x, y \in X$ *and all* $z_i, w_i \in X_i$,

1. $[x \succsim y$ *and* $(z_i, w_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \succsim (w_i, y_{-i})$,
2. $[x \succ y$ *and* $(z_i, w_i) \succsim_i^{**} (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \succ (w_i, y_{-i})$,

These statements tell us how a preference relation responds when a difference of preference on a criterion is substituted with a larger one. The preference cannot be reversed with respect to both $\succsim_i^*$ and $\succsim_i^{**}$ when enlarging the difference between the compared alternatives on any criterion. Furthermore, it is not impossible that, using $\succsim_i^*$, a strict preference ($x \succ y$) becomes an indifference ($(z_i, x_{-i}) \sim (w_i, y_{-i})$), which is impossible when using $\succsim_i^{**}$. Note that these are simply consequences of the definition of $\succsim_i^*$, not really a property of the preference; they add credit to our interpretation of relations $\succsim_i^*$ and $\succsim_i^{**}$ as comparing differences of preference on X_i.

These monotonicity properties of the preference with respect to the relations $\succsim_i^*$ or $\succsim_i^{**}$ are similar to those observed with respect to marginal preferences (proposition 6.1) and marginal traces (proposition 6.4).

6.3.3 A general family of models using traces on differences

In the same spirit as the generalisation of the decomposable model to the models based on marginal traces, we envisage a very general model based on preference differences here. It formalises the idea of measuring "preference differences" separately on each dimension and then combining these (positive or negative) differences to find out whether the aggregation of these differences leads to an advantage for x over y. More formally, this suggests a model in which:

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \ldots, p_n(x_n, y_n)) \geq 0 \qquad (D0)$$

where p_i are real-valued functions on X_i^2 and G is a real-valued function on $\prod_{i=1}^n p_i(X_i^2)$.

As already noted by Goldstein (1991), all binary relations satisfy model ($D0$) when X is finite or countably infinite. Necessary and sufficient conditions for the non-denumerable case are well-known (Bouyssou and Pirlot, 2002b).

As for the variants of model ($L0$), it is easy to impose conditions on G that will result in simple properties of $\succsim$; we have for instance:

- $\succsim$ is reflexive iff it has a representation in model ($D0$) with

$$G([p_i(x_i, x_i)]) \geq 0, \text{ for all } x_i;$$

- $\succsim$ is independent iff it has a representation in model ($D0$) with

$$p_i(x_i, x_i) = 0 \text{ for all } x_i;$$

in addition, $\succsim$ is reflexive iff $G(\mathbf{0}) \geq 0$ and $\succsim$ is irreflexive iff $G(\mathbf{0}) < 0$.

- $\succsim$ is complete iff it has a representation in model $(D0)$ with skew-symmetric p_i, i.e.
$$p_i(x_i, y_i) = -p_i(y_i, x_i) \text{ for all } x_i, y_i;$$
and G odd, i.e. $G(-\mathbf{p}) = -G(\mathbf{p})$ for all $\mathbf{p} = (p_1, \dots, p_n)$.

Again, as for the models based on marginal traces, the monotonicity of G is related to the properties of traces on differences (6.28) and (6.29). The axioms needed to guarantee the monotonicity of G are very similar to $AC1$, $AC2$ or $AC3$ because traces are involved.

Definition 6.15
We say that relation $\succsim$ on X satisfies:
$RC1_i$ *if*
$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \succsim (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(x_i, c_{-i}) \succsim (y_i, d_{-i})\\ \text{or}\\ (z_i, a_{-i}) \succsim (w_i, b_{-i}),\end{array}\right.$$
$RC2_i$ *if*
$$\left.\begin{array}{r}(x_i, a_{-i}) \succsim (y_i, b_{-i})\\ \text{and}\\ (y_i, c_{-i}) \succsim (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(z_i, a_{-i}) \succsim (w_i, b_{-i})\\ \text{or}\\ (w_i, c_{-i}) \succsim (z_i, d_{-i}),\end{array}\right.$$
for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

$RC3_i$ *if $\succsim$ satisfies $RC2_i$ and when one of the conclusions of $RC2_i$ is false then the other holds with $\succ$ instead of $\succsim$.*

We say that $\succsim$ satisfies $RC1$ (resp. $RC2$) if it satisfies $RC1_i$ (resp. $RC2_i$) for all $i \in N$. We also use $RC12$ as shorthand for $RC1$ and $RC2$.

Condition $RC1_i$ implies that any two ordered pairs (x_i, y_i) and (z_i, w_i) of elements of X_i are comparable in terms of the relation $\succsim_i^*$. Indeed, it is easy to see that supposing $Not[(x_i, y_i) \succsim_i^* (z_i, w_i)]$ and $Not[(z_i, w_i) \succsim_i^* (x_i, y_i)]$ is the negation of $RC1_i$. Similarly, $RC2_i$ implies that the two opposite differences (x_i, y_i) and (y_i, x_i) are linked. In terms of the relation $\succsim_i^*$, it states that if the preference difference between x_i and y_i is not at least as large as the preference difference between z_i and w_i then the preference difference between y_i and x_i should be at least as large as the preference difference between w_i and z_i (Bouyssou and Pirlot, 2002b, Lemma 1).

Proposition 6.10 (Completeness of the traces on differences)
We have:

1. $[\succsim_i^*$ *is a weak order*$] \Leftrightarrow RC1_i$,
2. $[\succsim_i^{**}$ *is a weak order*$] \Leftrightarrow [RC1_i$ *and* $RC2_i]$.

Here again (as for the models based on marginal traces, see section 6.2.6) the links between $RC1$, $RC2$ and properties of $\succsim_i^*$ and $\succsim_i^{**}$ play a fundamental role in the construction of a representation of a preference relation in model $(D0)$ with a monotone G function. Axiom $RC2$ introduces a *mirror effect* on preference differences: under $RC2_i$, the difference of preference (y_i, x_i) is the mirror image of (x_i, y_i) (Bouyssou and Pirlot, 2002b, Theorem 1).

Proposition 6.11 (Representation in model D)
A preference relation $\succsim$ on X admits a representation in model $(D0)$ with G nondecreasing in all its n arguments iff $\succsim$ satisfies $RC1$. It admits such a representation with, in addition, $p_i(x_i, y_i) = -p_i(y_i, x_i)$, iff $\succsim$ satisfies $RC1$ and $RC2$.

The construction of a representation under the hypotheses of the above proposition helps to make this proposition more intuitive. We outline this construction below.

Suppose that $\succsim$ satisfies $RC1$. We know, by proposition 6.10.1 that $\succsim_i^*$ is a weak order on the set of pairs of levels X_i^2 for all i. Select, for all i, a real-valued function p_i that represents the weak order $\succsim_i^*$, i.e. that satisfies:

$$p_i(x_i, y_i) \geq p_i(z_i, w_i) \text{ iff } (x_i, y_i) \succsim_i^* (z_i, w_i),$$

for all $x_i, y_i, z_i, w_i \in X_i$. Then define G as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} \exp \sum_{i=1}^n p_i(x_i, y_i) & \text{if } x \succsim y \\ -\exp[-\sum_{i=1}^n p_i(x_i, y_i)] & \text{otherwise.} \end{cases} \tag{6.31}$$

It can easily be shown that G is well-defined. The choice of the exponential function and the sum operator is purely arbitrary; any other increasing function defined on the set of real numbers and taking positive values would do as well. The role of such a function is to ensure that, in each of the two sub-domains $x \succsim y$ and "otherwise", function G is increasing in the p_i's; since the relation $\succsim$ is itself non-decreasing with respect to the relations $\succsim_i^*$ for all i (as implied by proposition 6.9), raising the value of a p_i (which represents $\succsim_i^*$) may only result in remaining in the same sub-domain or passing from the domain "otherwise" to the domain "$x \succsim y$"; the value of G is negative in the former sub-domain and positive in the latter and in each sub-domain, G is increasing. This proves that G is increasing in all its arguments p_i.

The second case, in which $\succsim$ satisfies $RC1$ and $RC2$ is dealt with similarly. Since in this case $\succsim_i^{**}$ is a weak order, we use functions p_i that represent $\succsim_i^{**}$ instead of $\succsim_i^*$. We may, moreover, exploit the reversibility property (6.30) of $\succsim_i^{**}$ to ensure that we may choose a skew-symmetric function p_i to represent $\succsim_i^{**}$. Then we define G as in (6.31). In the same case, we may also obtain a representation in which G is increasing (instead of non-decreasing) by defining G as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} \exp \sum_{i=1}^n p_i(x_i, y_i) & \text{if } x \succ y \\ 0 & \text{if } x \sim y \\ -\exp\left[-\sum_{i=1}^n p_i(x_i, y_i)\right] & \text{otherwise.} \end{cases} \tag{6.32}$$

Combining the various additional properties that can be imposed on $\succsim$, we are led to consider a number of variants of the basic $(D0)$ model. These models, labelled $(D1)$ to $(D11)$, can be fully characterised using the axioms $RC1$, $RC2$ and $RC3$. The definition of the models, as well as their characterisation are displayed in table 6.2.

Remark 6.3.1 (Importance of marginal traces on differences)
In models $(L0)$ to $(L8)$ (see table 6.1, p. 268), both the understanding of the models and the basis for eliciting them rely on a fundamental object: the marginal

Table 6.2: Main models using traces on differences and their characterisation.

Models	Definition	Conditions
$(D0)$	$x \succsim y \Leftrightarrow G([p_i(x_i, y_i)]) \geq 0$	$\varnothing$
$(D1)$	$(D0)$ with $p_i(x_i, x_i) = 0$	
$\Updownarrow$		ind.
$(D2)$	$(D0)$ with p_i skew symmetric	
$(D3)$	$(D0)$ with p_i skew symmetric and G odd	cpl., ind.
$(D4)$	$(D0)$ with $G(\nearrow)$	
$\Updownarrow$		$RC1$
$(D8)$	$(D0)$ with $G(\nearrow\!\!\nearrow)$	
$(D5)$	$(D1)$ with $G(\nearrow)$	
$\Updownarrow$		$RC1$, ind.
$(D9)$	$(D1)$ with $G(\nearrow\!\!\nearrow)$	
$(D6)$	$(D2)$ with $G(\nearrow)$	
$\Updownarrow$		$RC12$
$(D10)$	$(D2)$ with $G(\nearrow\!\!\nearrow)$	
$(D7)$	$(D3)$ with $G(\nearrow)$	cpl., $RC12$
$(D11)$	$(D3)$ with $G(\nearrow\!\!\nearrow)$	cpl., $RC3$

$\nearrow$ means nondecreasing, $\nearrow\!\!\nearrow$ means increasing
cpl. means completeness, ind. means independence

traces $\succsim_i^{\pm}$. Here, obviously, the same role is played by the traces on differences $\succsim_i^*$ or $\succsim_i^{**}$. It is indeed tempting to interpret the functions p_i as being numerical representations of $\succsim_i^*$ or $\succsim_i^{**}$; it is always possible to impose that the p_i functions used in models $(D1)$ to $(D11)$ represent either one or the other of those traces. •

An alternative strategy for eliciting a preference model, relies on the elicitation of a relation comparing differences of preferences on each dimension, in contrast to the elicitation of marginal traces for models $(L1)$ to $(L8)$ or, when preferences are assumed to be transitive, in contrast to the elicitation of the marginal preferences. In order to apply such a strategy, the basic property that should reasonably be required of the preference is that its traces on differences be complete, which can be tested using axioms $RC1$ and/or $RC2$. Methods of aggregation based on pairwise comparisons, such as the numerous versions of the majority rule considered in chapter 5, are likely to lead to preferences that fit into the "D" models.

Remark 6.3.2 (Responsiveness with respect to traces on differences) Proposition 6.9 expresses the fact that all preferences are positively (more precisely, non-negatively) responsive with respect to their traces on differences; this is implied by the very definition of the traces. The response however is not *strictly* positive in general. As can be seen in table 6.2, when the preference is complete, $RC3_i$ is linked to the way the preference reacts to a strict increase of a preference difference. Complete preference relations satisfying $RC3_i$ enjoy the following positive responsiveness property (Bouyssou and Pirlot, 2002b, Lemma 3.5, p. 689):

$$[x \succsim y \text{ and } (z_i, w_i) \succ_i^{**} (x_i, y_i)] \Rightarrow (z_i, x_{-i}) \succ (w_i, y_{-i}). \tag{6.33}$$

For such a preference, indifference is "thin", since, in case x and y are indifferent, increasing the difference of preference (x_i, y_i) or reducing the difference (y_i, x_i) converts indifference into strict preference.

Note that positive responsiveness is not the rule for all preferences. It is not the case for instance in example 6.3 (statistical test of comparison of means) in which the marginal traces on differences can be represented by the algebraic differences of the values $a_i - b_i$. Clearly in this example, indifference is not thin, due to the fact that the values of the means cannot be significantly distinguished unless their difference reaches some threshold (see equation (6.14)). On the contrary, the usual additive value model (6.1) is positively responsive; its marginal traces on differences can be represented by the differences of marginal utilities $u_i(x_i) - u_i(y_i)$. •

6.3.4 Eliciting models using traces on differences

We suppose that $\succsim$ is reflexive and satisfies $RC1$, i.e., we are in model $(D5)$ (equivalent to $(D9)$). In this model, $\succsim_i^*$ is a weak order on the "differences of preference" $(x_i, y_i) \in X_i^2$, for all i, and the functions p_i may be chosen to be numerical representations of $\succsim_i^*$. To each pair of alternatives $x, y \in X$, a profile $\overline{p} = (p_1, \ldots, p_n)$ of differences of preferences ($p_i = p_i(x_i, y_i)$, for $i = 1, \ldots, n$) is henceforth associated. The function G may be conceived of as a rule that assigns a value to each profile; in model $(D5)$, G is just assumed to be nonincreasing (not necessarily increasing if we choose to represent $\succsim$ into model $(D5)$ instead of the equivalent model $(D9)$) and therefore we may choose a very simple form of G that codes profiles in the following way:

$$G(\overline{p}) = \begin{cases} +1 & \text{if } \overline{p} \text{ corresponds to } x \succ y; \\ 0 & \text{if } \overline{p} \text{ corresponds to } x \sim y; \\ -1 & \text{if } \overline{p} \text{ corresponds to } Not[x \succsim y]. \end{cases} \tag{6.34}$$

The strategy for eliciting such a model (directly) may thus be as follows:

1. for all i, elicit the weak order $\succsim_i^*$ that ranks the differences of preference; choose a representation p_i of $\succsim_i^*$

2. elicit the rule (function) G that assigns a category (coded $+1$, 0 or -1) to each profile $\overline{p}$.

The second step of the elicitation strategy is comparable to that used for eliciting function U in the decomposable model (page 256), since the monotonicity of G in its arguments can be exploited.

The initial step, however, is more complex than with the decomposable model, because we have to rank-order the set X_i^2 instead of X_i. If it may be assumed that the difference of preference is reversible (see (6.30)), almost half of the work can be saved since only the "positive" (or only the "negative") differences must be rank-ordered [4]. The difficulty that remains even in the reversible case, may motivate the consideration of another family of models that rely both on marginal traces and on traces on differences. In some of these models, $\succsim_i^*$ reacts positively (or non-negatively) to marginal traces and therefore, the elicitation of p_i may benefit from its monotonicity w.r.t. marginal traces. This family of models is presented in section 6.4.

Models $(D4)$, $(D5)$, $(D6)$ and $(D7)$, in which G is a nondecreasing function, can be elicited in a similar fashion. The situation is different when a representation is sought with G increasing, in particular for model $(D11)$. The definition of G by (6.34) is no longer appropriate for such representations, and defining G requires more care and effort. We do not analyse this point.

6.3.4.1 Testing whether preferences fit into model $(D5)$

In view of the characterisation of $(D5)$ (see table 6.2), a preference satisfies $(D5)$ iff the differences of preferences can be rank-ordered (according to $\succsim_i^*$) and the preference $\succsim$ is monotone w.r.t. the orders on the differences of preference $\succsim_i^*$. $(D5)$ might be considered as a default model if a model based on preference differences was previously chosen. The elicitation strategy outlined above could be pursued until consistency problems are encountered in the elicitation process: e.g. contradictions between the client's answers and consequences of the monotonicity of $\succsim$ applied to previous answers. If no such contradiction has been met when the elicitation is completed, the validation of the model may consist in partially testing the consistency of the model by asking redundant questions aimed at detecting non-monotonicity of $\succsim$ w.r.t. the elicited $\succsim_i^*$. Detected contradictions may lead either to reject the model or to revise the elicitation of some $\succsim_i^*$.

Preliminary questions may lead to assuming a more structured model, such as e.g.

- $(D6)$ (equivalent to $(D10)$) if, in addition, the decision maker feels that the difference of preference (x_i, y_i) is exactly the opposite of the difference of preference (y_i, x_i) for all x_i, y_i (this may be partially tested by asking appropriate questions);

- $(D7)$ (equivalent to $(D10)$) if, in addition to the hypotheses of $(D6)$, the decision maker feels that the preference is complete (this can be partly tested).

[4] In the case of a tie, i.e. whenever $(x_i, y_i) \sim_i^* (z_i, w_i)$, one has, however, to explicitly look at the relation between the reverse differences (y_i, x_i) and (w_i, z_i) since all cases ($\succsim_i^*$, $\sim_i^*$ or $\precsim_i^*$) can possibly occur.

Testing (or questioning about) Model $(D11)$ seems to be more difficult. This model departs from model $(D7)$ or, equivalently $(D10)$ because preference $\succsim$ reacts positively to any improvement of the difference of preference on any dimension i. More precisely, if $x \sim y$ and the difference of preference (x_i, y_i) is substituted by a larger one w.r.t. $\succsim_i^{**}$, say (z_i, w_i) with $(z_i, w_i) \succ_i^{**} (x_i, y_i)$, the preference becomes strict between the transformed alternatives, i.e. $(z_i, x_{-i}) \succ (w_i, y_{-i})$ (see Bouyssou and Pirlot, 2002b, Lemma 3.5). Partially testing this condition does not make much sense since this condition may, of course, hold in some cases in model $(D10)$; the fact that it holds in *all* cases is characteristic of model $(D11)$.

6.3.5 Examples of models that distinguish no more than three classes of differences

The family of models using traces on differences provides an appropriate framework for describing the procedures examined in section 5.2, i.e. procedures that aggregate a profile of preference relations into one relation. In this section, we show that the simple majority (or Condorcet method), weighted majority, qualified majority and the lexicographic method can be represented in some of the models $(D1)$ to $(D11)$. We consider, in addition, a variant of the ELECTRE I procedure in which the profile of preferences on each dimension are not weak orders but semiorders. In each of these cases, the relation that orders the differences of preference on each criterion is revealed by the global preference relation.

The above rules can also be described in another, more detailed, framework that will be discussed below in section 6.4, where we will come back to all of these rules. In section 6.5, we will study, a class of relations that encompass all the rules and are called *concordance relations.*

First of all, the definitions of the various majority rules and the lexicographic method discussed in section 5.2 require to be slightly adapted to our conjoint measurement context. We do not start with a profile of preference relations here, but, instead, with a global preference relation $\succsim$ that—we assume—can be obtained through the application of some sort of a majority or lexicographic rule to a profile of a priori preference relations on each dimension. More formally, we say that a relation $\succsim$, defined on a product set $X = \prod_{i=1}^n X_i$ is the result of the application of a majority or a lexicographic rule if there is a relation S_i on each X_i such that $\succsim$ can be obtained by aggregating the n relations S_i using that rule. These S_i's will usually be *weak orders*, but we will also consider more general structures such as semiorders. There can be some sort of relationship between S_i and the revealed marginal preferences $\succsim_i$ induced by $\succsim$ on X_i. This relationship will be examined in section 6.4.3. In the sequel, we refer to S_i as to the *a priori preference relation* on X_i. Such relations may have been obtained as suggested in chapter 3

Take the example of the simple majority rule. We say that $\succsim$ is a simple majority preference relation if there are relations S_i that are weak orders on the corresponding X_i such that:

$$x \succsim y \text{ iff } \begin{cases} \text{the number of criteria on which } x_i \; S_i \; y_i \\ \text{is at least as large as} \\ \text{the number of criteria such that } y_i \; S_i \; x_i. \end{cases} \tag{6.35}$$

There is apparently a difference with the procedures defined in section 5.2. In that section, the weak orders (that are denoted there by $\succsim_i$ and correspond here to the S_i relations) are defined on the set of alternatives X; it is easy to extend our relation S_i, defined on X_i, to a relation ${S_i}^X$ on X just by saying that $x \; {S_i}^X \; y$ iff $x_i \; S_i \; y_i$, where x_i (resp. y_i) is the evaluation of x (resp. y) on the ith dimension. In other words, X_i can be interpreted as representing the aspect of the alternatives that is relevant for ranking them according to dimension i; this ranking is S_i.

We emphasise that the relations denoted by $\succsim_i$ in section 5.2 are not to be confused with the marginal preferences induced by $\succsim$ as defined by equation (4.7) in section 4.3.6 (although there may exist relationships between them as we shall see in section 6.4.1).

In the rest of this section, P_i will denote the asymmetric part of a relation S_i defined on X_i and its symmetric part will be denoted by I_i. In the first five examples, the S_i's are assumed to be weak orders.

We refer the reader to section 5.2 for a comparison of the social choice and conjoint measurement perspectives on the procedures described below.

6.3.5.1 Simple majority or the Condorcet method

A relation $\succsim$ on X is a *simple majority relation* (see section 5.2.1 for a social choice viewpoint on simple majority) if there is a weak order S_i on each X_i such that

$$x \succsim y \text{ iff } |\{i \in N : x_i \; S_i \; y_i\}| \geq |\{i \in N : y_i \; S_i \; x_i\}|. \tag{6.36}$$

In other words, $x \succsim y$ if the "coalition" of criteria on which x is at least as good as y is at least as large as the "opposite coalition", i.e. the set of criteria on which y is at least as good as x. The term "coalition" is used here for "set", in reference to social choice. We apparently do not distinguish between the case in which x_i is better than y_i ($x_i \; P_i \; y_i$) and that in which they are indifferent ($x_i \; I_i \; y_i$). Note that the criteria for which x_i is indifferent to y_i appear in both coalitions and hence cancel each other. We could thus define a simple majority relation equivalently by $x \succsim y$ iff $|\{i \in N : x_i \; P_i \; y_i\}| \geq |\{i \in N : y_i \; P_i \; x_i\}|$.

Such a relation can be represented in model ($D11$) by defining

$$p_i(x_i, y_i) = \begin{cases} 1 & \text{if} \quad x_i \; P_i \; y_i \\ 0 & \text{if} \quad x_i \; I_i \; y_i \\ -1 & \text{if} \quad y_i \; P_i \; x_i \end{cases} \tag{6.37}$$

and

$$G([p_i]) = \sum_{i \in N} p_i. \tag{6.38}$$

Indeed $x \succsim y$ iff $G([p_i(x_i, y_i)]) = |\{i \in N : x_i \; P_i \; y_i\}| - |\{i \in N : y_i \; P_i \; x_i\}| \geq 0$, which is clearly equivalent to definition (6.36).

This representation of a simple majority relation can furthermore be called *regular*, in the sense that the functions p_i are numerical representations of the weak orders $\succsim_i^{**}$; the latter having exactly three equivalence classes, namely, the set of pairs (x_i, y_i) such that $x_i \; P_i \; y_i$, the set of pairs for which $x_i \; I_i \; y_i$ and the set of pairs such that $y_i \; P_i \; x_i$. Note that the relation $\succsim_i^*$ distinguishes the same three classes; hence $\succsim_i^* = \succsim_i^{**}$.

6.3.5.2 Weighted simple majority or the weighted Condorcet method

A relation $\succsim$ on X is a *weighted simple majority relation* (see section 5.2.2 for a social choice viewpoint on weighted majority) if there is a vector of normalised weights $[w_i]$ (with $w_i \geq 0$ and $\sum_{i \in N} w_i = 1$) and a weak order S_i on each X_i such that

$$x \succsim y \text{ iff } \sum_{i \in N : x_i S_i y_i} w_i \geq \sum_{j \in N : y_j S_j x_j} w_j. \tag{6.39}$$

The coalitions of criteria are weighted in this model: they are assigned a value that is the sum of those assigned to the criteria belonging to the coalition. As in the simple majority rule, the preference of x over y results from the comparison of the coalitions: $x \succsim y$ if the coalition of criteria on which x is at least as good as y *does not weigh less* than the opposite coalition. As for simple majority, we could have defined the relation using strict a priori preference, saying that $x \succsim y$ iff $\sum_{i \in N : x_i P_i y_i} w_i \geq \sum_{j \in N : y_j P_j x_j} w_j$.

A representation of a weighted majority relation in model $(D11)$ is readily obtained. Let:

$$p_i(x_i, y_i) = \begin{cases} w_i & \text{if} \quad x_i \; P_i \; y_i \\ 0 & \text{if} \quad x_i \; I_i \; y_i \\ -w_i & \text{if} \quad y_i \; P_i \; x_i \end{cases} \tag{6.40}$$

and

$$G([p_i]) = \sum_{i \in N} p_i. \tag{6.41}$$

We have that $x \succsim y$ iff $G([p_i(x_i, y_i)]) = \sum_{i \in N : x_i P_i y_i} w_i - \sum_{j \in N : y_j P_j x_j} w_j \geq 0$.

This representation is *regular* since p_i is a numerical representation of $\succsim_i^{**}$ and $\succsim_i^{**}$ has only three equivalence classes as in the case of simple majority.

6.3.5.3 Weighted qualified majority

A relation $\succsim$ on X is a *weighted qualified majority relation* if there is a vector of normalised weights $[w_i]$ (i.e. with w_i non-negative and summing up to 1), a weak order S_i on each X_i and a threshold δ between $\frac{1}{2}$ and 1 such that

$$x \succsim y \text{ iff } \sum_{i \in N : x_i S_i y_i} w_i \geq \delta. \tag{6.42}$$

In contrast to the previous models, the preference does not result from a comparison of coalitions, but from stating that the coalition in favour of an alternative is *strong enough*, i.e. that the measure of its strength reaches a certain threshold δ (typically above 0.5). Even when δ is set to 0.5, this method is not equivalent to weighted simple majority, with the same weighting vector $[w_i]$; this is due to the inclusion of the criteria on which x and y are indifferent in both coalitions in favour of x over y and in favour of y over x. Take for example two alternatives x,

y compared on five points of view; suppose that the criteria are all assigned the same weight, i.e. $w_i = 1/5$, for $i = 1, \ldots, 5$. Assume that x is preferred to y on the first criterion ($x_1\ P_1\ y_1$), x is indifferent to y on the second and third criteria ($x_2\ I_2\ y_2$; $x_3\ I_3\ y_3$) and y is preferred to x on the last two criteria ($y_4\ P_4\ x_4$; $y_5\ P_5\ x_5$). Using the weighted majority rule (equation (6.39)), we obtain $y \succ x$, since the coalition in favour of x against y is composed of criteria 1, 2, 3 (weighting 0.6) and the opposite coalition contains criteria 2, 3, 4, 5 (weighting 0.8). Using the weighted qualified majority with threshold δ up to 0.6, we obtain that $x \sim y$, since both coalitions weigh at least 0.6.

Note that when the criteria have equal weights ($w_i = 1/n$), weighted qualified majority could be called simply *qualified majority*; the latter has the same relationship with weighted qualified majority that weighted simple majority has with simple majority.

Remark 6.3.3 (Strict weighted qualified majority)
There is another way of defining a weighted qualified majority, denoting the preference by $\succ$ and using P_i, the asymmetric part of S_i, in the sum in definition (6.42):

$$x \succ y \text{ iff } \sum_{i \in N : x_i P_i y_i} w_i \geq \delta.$$

With this rule, that could be called *strict weighted qualified majority*, only those criteria on which x is strictly preferred to y enter into the coalition; the criteria on which x and y are tied (from the preference point of view) don't count in the comparison of these alternatives. The resulting preference $\succ$ is irreflexive, since, when comparing x with x, the coalition of criteria stating that x is strictly preferred to x is empty. Furthermore, this preference is asymmetric when the threshold δ is strictly larger than 0.5; this results from the following fact: the coalition of criteria stating that x is strictly preferred to y weighs more than 0.5 if and only if the opposite coalition, the one stating that y is strictly preferred to x weighs less than 0.5. The "asymmetric intuition" behind this kind of rule leads to excluding values of δ less than or equal to $\frac{1}{2}$. •

Weighted qualified majority relations are a basic component of the ELECTRE I and ELECTRE II methods (Roy, 1971) as long as there are no vetoes (see also section 5.2.3.5).

Any weighted qualified majority relation admits a representation in model ($D8$). Let:

$$p_i(x_i, y_i) = \begin{cases} w_i - \frac{\delta}{n} & \text{if } x_i\ S_i\ y_i \\ -\frac{\delta}{n} & \text{if } Not[x_i\ S_i\ y_i] \end{cases} \tag{6.43}$$

and

$$G([p_i]) = \sum_{i \in N} p_i. \tag{6.44}$$

We have that

$$x \succsim y \text{ iff } G([p_i(x_i, y_i)]) = \sum_{i \in N: x_i S_i y_i} (w_i - \frac{\delta}{n}) - \sum_{j \in N:\, Not[\, x_j S_j y_j\,]} \frac{\delta}{n}$$
$$= \sum_{i \in N: x_i S_i y_i} w_i - \delta \tag{6.45}$$
$$\geq 0.$$

In this representation, p_i is a numerical representation of $\succsim_i^*$ but not of $\succsim_i^{**}$. The former has two equivalence classes: the pairs (x_i, y_i) that are in S_i form the upper class of the weak order; those that are not in form the lower class. Note that there are no further distinctions between pairs; all pairs in the upper class contribute the same amount $w_i - \frac{\delta}{n}$ to the value of the coalition, while the pairs of the lower class all contribute the same amount $-\frac{\delta}{n}$. The comparison of preference differences in this model is thus rather poor (as is the case, of course, in the previous two models).

The relation $\succsim_i^{**}$ is also a weak order; it has three equivalence classes. It makes a distinction between $x_i \; P_i \; y_i$ and $x_i \; I_i \; y_i$ (a distinction that $\succsim_i^*$ does not): both cases play the same role when comparing (x_i, y_i) to other pairs (since what counts in formula (6.42) is whether or not (x_i, y_i) belongs to S_i); this is no longer the case when comparing (y_i, x_i) to other pairs since then, $x_i \; I_i \; y_i$ counts in the coalition in favour of y against x while $x_i \; P_i \; y_i$ does not.

Are there representations of a weighted majority relation in models which are more constrained than $(D8)$. The answer is positive in view of the fact that $\succsim_i^{**}$ is a weak order for all i, hence that $\succsim$ satisfies axioms RC12 (Proposition 6.10.2). Consequently, there are representations of $\succsim$ in model $(D10)$ and possibly in more constrained ones.

In such models, however, G may no longer—in general—be taken to be the sum of the p_i's. Indeed, in all more constrained models considered in table 6.2, $p_i(x_i, x_i) = 0$ for all x_i and $p_i(x_i, y_i) < 0$ whenever x_i is not at least as good as y_i (i.e. when $y_i \; P_i \; x_i$). Suppose for simplicity that all criteria have equal positive weight ($w_i = 1/n$) and suppose that threshold δ is less than $\frac{n-1}{n}$ so that unanimity is not required for preference. Take a pair of alternatives that are indifferent on all criteria but one, say criterion 1 (thus $x_i = y_i$ for all $i \neq 1$). We may assume without being restrictive that x_1, y_1 are such that $y_1 \; P_1 \; x_1$ and hence that $p_i(x_i, y_i) < 0$. We have $x \succsim y$ since the former alternative is at least as good as (in fact indifferent to) the latter on $n-1$ criteria (and worse only on the first criterion). Using an additive representation $G([p_i(x_i, y_i)]) = \sum_{i \in N} p_i(x_i, y_i)$ with $p_i(x_i, y_i) = 0$ for all $i \neq 1$ and $p_1(x_1, y_1) < 0$ would lead us to conclude that x is not preferred or indifferent to y ($Not[\, x \succsim y\,]$) since we have:

$$p_1(x_1, y_1) + \sum_{i \neq 1} p_i(x_i, y_i) = p_1(x_1, y_1) < 0$$

It is however possible to get a representation of $\succsim$ in model $(D6)$ that is equivalent to model $(D10)$ using a function G that is not the sum of its arguments p_i (see table 6.2). Take for p_i a numerical representation of $\succsim_i^{**}$, for instance:

$$p_i(x_i, y_i) = \begin{cases} w_i & \text{if} \quad x_i \; P_i \; y_i \\ 0 & \text{if} \quad x_i \; I_i \; y_i \\ -w_i & \text{if} \quad y_i \; P_i \; x_i \end{cases} \tag{6.46}$$

and define G for instance by:

$$G([p_i]) = 1 - \sum_{i \in N: p_i < 0} p_i - \delta. \tag{6.47}$$

Since the weights w_i sum up to one, $1 - \sum_{i \in N: p_i < 0} p_i = \sum_{i \in N: x_i S_i y_i} w_i$ and thus $x \succsim y$ iff $G([p_i(x_i, y_i)]) = \sum_{i \in N: x_i S_i y_i} w_i - \delta \geq 0$, which is exactly the same expression as in equation (6.45). The difference between these models is in the different decompositions of the function mapping $(x_1, y_1, \ldots, x_n, y_n)$ onto $G(p_1(x_1, y_1), \ldots, p_n(x_n, y_n))$ into G and the p_i's. The two models for a weighted qualified majority relation described above provide two such decompositions; they vary in the properties of the p_i's, the crucial one being the requirement in the latter that $p_i(x_i, x_i) = 0$.

Let us turn to examining the properties of G as defined by equation (6.47). G is a nondecreasing function of its arguments; it is not odd since when $p_i = 0$ for all i, we should have $G([p_i]) = 0$ (since $G([-p_i]) = -G([p_i])$ implies $G([p_i]) = 0$). G is not strictly increasing since the expression that defines G in terms of the p_i's (equation (6.47)) makes no difference between $p_i = 0$ (in case $x_i \; I_i \; y_i$) and $p_i = w_i$ (in case $x_i \; P_i \; y_i$). With the above definitions of p_i and G we thus have a representation of $\succsim$ in model ($D6$). Since models ($D6$) and ($D10$) are equivalent, one can also build a representation in model ($D10$) by choosing a function G that is increasing instead of nondecreasing. This can be achieved through the general construction scheme outlined in section 6.3.3, using for instance equation (6.32) as a definition of G.

Are there representations in models such as ($D7$) or ($D11$) or do we have to conclude that a weighted qualified majority relation does not, in general, satisfy the axioms for these models? Examining the properties of a weighted qualified majority relation $\succsim$, one readily sees that $\succsim$ is not necessarily complete. Take for instance the case where there are four criteria and two levels on each criterion, i.e. $X_i = \{a_i, b_i\}$, with $a_i \; P_i \; b_i$, for $i = 1, \ldots, 4$. Let the criteria have equal weights ($w_i = 0.25$) and δ be equal to 0.75. Consider the alternatives $x = (a_1, a_2, b_3, b_4)$ and $y = (b_1, b_2, a_3, a_4)$. We have neither $x \succsim y$, nor $y \succsim x$, since the first two criteria are in favour of x and the last two in favour of y; both coalitions weigh 0.5, none reaches the threshold of 0.75.

Property $RC3$ is also not fulfilled, in general, by weighted qualified majority relations. Consider a case with three criteria and two levels on each criterion ($X_i = \{a_i, b_i\}$, with $a_i \; P_i \; b_i$, for $i = 1, \ldots, 3$). Take equal weights for all criteria ($w_i = 1/3$) and set the threshold δ to 2/3. We have $(a_1, a_2, b_3) \succsim (a_1, a_2, a_3)$ since these alternatives have common levels on two criteria. We apply $RC2_1$ to two "copies" of this preference, substituting a_1 with b_1 yielding:

$$\left.\begin{array}{r}(a_1,a_2,b_3) \succsim (a_1,a_2,a_3)\\ \text{and}\\ (a_1,a_2,b_3) \succsim (a_1,a_2,a_3)\end{array}\right\} \Rightarrow \left\{\begin{array}{l}(b_1,a_2,b_3) \succsim (a_1,a_2,a_3)\\ \text{and}\\ (a_1,a_2,b_3) \succsim (b_1,a_2,a_3).\end{array}\right.$$

The first preference on the righthand side is false; the second one is true (in agreement with $RC2_1$, which is a property of weighted qualified majority relations as already observed), but it is not strict (indeed, it is easy to see that $(a_1, a_2, b_3) \sim (b_1, a_2, a_3)$), contrary to $RC3_1$.

Another way of obtaining an intuition about model $(D11)$ and $RC3$ is through noticing that in model $(D11)$, the class of indifferent alternatives is "thin" in the following sense: take two alternatives x and y that are indifferent ($x \sim y$); let (z_i, w_i) be a pair of levels on X_i that represent a difference of preference larger than (x_i, y_i), the pair of levels shown by x and y on X_i; we thus have $(z_i, w_i) \succ_i^* (x_i, y_i)$. Substituting x_i (resp. y_i) by z_i (resp. w_i) in x (resp. y) transforms indifference into strict preference: $(z_i, x_{-i}) \succ (w_i, y_{-i})$. Indifference is broken in the same way as soon as $(z_i, w_i) \succ_i^{**} (x_i, y_i)$, thus even when $(z_i, w_i) \sim_i^* (x_i, y_i)$ but $(y_i, x_i) \succ_i^* (w_i, z_i)$ (see section 6.3.8 for another example).

In the case of a weighted qualified majority preference relation $\succsim$, we may therefore not hope to have a representation of $\succsim$ either in model $(D7)$ (unless $\succsim$ is complete), or in model $(D11)$ (unless $\succsim$ is complete and satisfies $RC3$). The most constrained model that $\succsim$ fits in is $(D6)$ and its equivalent strictly increasing version $(D10)$. Obtaining a representation of $\succsim$ in this model is of no practical interest since such a representation is highly artificial. It can however be obtained in the generic way suggested by formula (6.31): take a skew symmetric representation of $\succsim_i^{**}$ for p_i (as in (6.46)) and define G by:

$$G([p_i(x_i, y_i)]) = \left\{\begin{array}{ll} \exp \sum_{i=1}^n p_i(x_i, y_i) & \text{if } \sum_{i: x_i S_i y_i} w_i \geq \delta \\ -\exp - \sum_{i=1}^n p_i(x_i, y_i) & \text{if} \sum_{i: x_i S_i y_i} w_i < \delta \end{array}\right. \tag{6.48}$$

This definition fulfills all the requirements of model $(D10)$, but it offers no hint for constructing it since it presupposes the knowledge of $\succsim$ (here via a representation in model $(D8)$) to determine the adequate sub-domain.

See section 5.2.3 for a social choice viewpoint on qualified weighted majority.

Remark 6.3.4 (Majority models with semiordered a priori preferences) In the variants of majority rules defined in sections 6.3.5.1 and 6.3.5.2 and in this section, we considered an a priori preference relation S_i on each dimension and we assumed that this relation is a weak order. The reader might have noticed that this assumption (in particular, the transitivity of S_i) was not needed for obtaining a representation of the majority relations in the models described. We can thus relax the hypothesis made on S_i to encompass other types of preference relations. This may prove useful since, in the ELECTRE methods, when there is no veto, it may occur that the a priori preferences are semiorders. This is the situation described in section 3.7.1.2 of chapter 3. Often, the client may feel that a small difference between the evaluations of alternatives on a dimension is not a sufficient reason for saying that an alternative is better than another on that criterion. This lack of discrimination of the dimension may be due to the imprecision of the evaluation process. Let us consider, for instance, one of the criteria retained in the choice of a car example described in Bouyssou et al. (2000, chapter 6). The client wants to buy a second hand car and he evaluates the potential cars on several criteria, including their annual utilisation cost. Due to the uncertainty involved in the estimation of such a cost (annual mileage, price of gasoline, maintenance costs, credit costs, etc), he might consider that a difference of less than 100€ is not significant. If X_1 denotes the set of costs the client could afford, its a priori preference would be modelled as follows: he would say that two cars x, y differing in cost by less than 100€ are a priori indifferent on criterion 1; representing the evaluation of x (resp. y) on criterion 1 by x_1 (resp. y_1), we would write that $x_1\ I_1\ y_1$ as soon as $|x_1 - y_1| \leq 100$. The client would also say that level x_1 is a priori strictly preferred to level y_1 ($x_1\ P_1\ y_1$) whenever x_1 is at least 100€ cheaper than y_1 ($x_1 < y_1 - 100$). Since the a priori preference S_1 on criterion 1 occurs when either P_1 or I_1 occurs, we would thus have the following description of S_1:

$$x_1\ S_1\ y_1 \text{ iff } x_1 \leq y_1 + 100 \tag{6.49}$$

Relation S_1 is a semiorder. In all variants of majority relations studied in section 6.3.5, the a priori preferences S_i were assumed to be weak orders. Relaxing this hypothesis into the assumption that S_i are semiorders, does not raise any problems with the definitions of the variants of majority relations introduced so far. In the case of weighted qualified majority for instance, we could simply apply the same definition, obtaining what could be called a *qualified majority relation with semiordered a priori preferences on the attributes.* With this relaxed definition, nothing changes in the possibility of representing $\succsim$ in models ($D1$) to ($D11$); in particular, the same forms of representation in models ($D6$) and ($D10$) are valid in case $\succsim$ has semiordered a priori preferences (see section 6.3.5.3). And the same is true of course for the other variants of majority relations. •

6.3.5.4 Lexicographic preference relations

A lexicographic procedure supposes that the criteria are linearly ordered and are considered in that order when comparing alternatives (see section 5.2.4): in this order, the first criterion that favours one alternative with respect to another determines the global preference. Denoting a linear order on the set of criteria by $>_\ell$, we rank-order the criteria according to it: $1^\ell >_\ell 2^\ell >_\ell \ldots >_\ell n^\ell$. We thus have the following definition: a relation $\succsim$ on X is a *lexicographic* preference relation if there is a linear order $>_\ell$ on the set of criteria and a weak order (or a semiorder) S_i on each X_i such that:

$$x \succ y \text{ if } \begin{cases} x_{1^\ell}\ P_{1^\ell}\ y_{1^\ell} \text{ or} \\ x_{1^\ell}\ I_{1^\ell}\ y_{1^\ell} \text{ and } x_{2^\ell}\ P_{2^\ell}\ y_{2^\ell} \text{ or} \\ x_{i^\ell}\ I_{1^\ell}\ y_{i^\ell}\ \forall i = 1, \ldots, k-1 \text{ and } x_{k^\ell}\ P_{k^\ell}\ y_{k^\ell}, \\ \text{for some } k \text{ such that } 2 \leq k \leq n. \end{cases} \tag{6.50}$$

and $x \sim y$ if $x_{i^\ell}\ I_{i^\ell}\ y_{i^\ell}$, for all $i \in N$. In other words, $x \sim y$ if x_i is a priori indifferent to y_i, for all i; $x \succ y$ if, for the first index k^ℓ for which x_{i^ℓ} is not a priori indifferent to y_{i^ℓ}, one has x_{k^ℓ} a priori preferred to y_{i^ℓ}.

Such a relation can be viewed (as long as there is only a finite number of criteria) as a special case of a weighted majority relation. Choose a vector of weights w_i as follows: for all $i \in N$, let w_{i^ℓ} be larger than the sum of all remaining weights (in the order $>_\ell$), i.e.:

$$\begin{aligned} w_{1^\ell} &> w_{2^\ell} + w_{3^\ell} + \ldots + w_{n^\ell} \\ w_{2^\ell} &> w_{3^\ell} + \ldots + w_{n^\ell} \\ &\ldots \\ w_{(n-1)^\ell} &> w_{n^\ell} \end{aligned} \tag{6.51}$$

Using these weights in (6.40) and (6.41), which define a representation for weighted majority relations, one obtains a representation for lexicographic relations in model ($D11$). To illustrate the definition of weights appropriate for lexicographically ordering alternatives, we adapt the example presented in section 5.2.4. Consider a case with three criteria and the following linear order $>_\ell$ on the criteria: criterion 2 is more important than criterion 1, which in turn is more important than criterion 3 ($2 >_\ell 1 >_\ell 3$). Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$ be three alternatives with the following a priori weakly ordered preferences on each criterion: $a_1\ I_1\ b_1\ P_1\ c_1$, $c_2\ P_2\ b_2\ I_2\ a_2$, $b_3\ P_3\ a_3\ I_3\ c_3$, as in the example in section 5.2.4. The following weights constitute an appropriate choice for obtaining the lexicographic ordering of the three alternatives, i.e. $c \succ b \succ a$. Let:

$$w_2 = 4\ w_1 = 2\ w_3 = 1.$$

Using formula (6.40) and (6.41), we obtain:

$$\begin{array}{lcccl} G([p_i(a_i,b_i)]) &=& 0+0-1 &=& -G([p_i(b_i,a_i)]) \\ G([p_i(a_i,c_i)]) &=& 2-4+0 &=& -G([p_i(c_i,a_i)]) \\ G([p_i(b_i,c_i)]) &=& 2-4+1 &=& -G([p_i(c_i,b_i)]), \end{array}$$

which represents the lexicographic ordering of these three alternatives correctly.

Note that any set of weights such that w_{i^ℓ} is larger than the sum of all the remaining weights leads to the same relation $\succsim$.

6.3.5.5 Other forms of weighted qualified majority

Instead of imposing a threshold above 0.5 for defining a weighted qualified majority, as in section 6.3.5.3, we may alternatively impose a relative majority threshold, in an additive or a multiplicative form. A preference relation $\succsim$ on X is a *weighted majority relation with additive threshold* if there is a vector of normalised weights $[w_i]$ (with $w_i \geq 0$ and $\sum_{i \in N} w_i = 1$), a weak order or semiorder S_i on each X_i and a non-negative threshold γ such that

$$x \succsim y \text{ iff } \sum_{i \in N: x_i S_i y_i} w_i \geq \sum_{j \in N: y_j S_j x_j} w_j - \gamma. \tag{6.52}$$

A relation $\succsim$ is a *weighted majority relation with multiplicative threshold* $\rho \geq 1$ if

$$x \succsim y \text{ iff } \sum_{i \in N: x_i S_i y_i} w_i \geq \frac{1}{\rho} \sum_{j \in N: y_j S_j x_j} w_j, \tag{6.53}$$

with $[w_i]$ and S_i as in the case of an additive threshold.

It is easy to provide a representation of a weighted majority relation with additive threshold in model ($D10$); define for instance p_i by equation (6.46) and G by:

$$G([p_i]) = \sum_{i \in N} p_i - \gamma. \tag{6.54}$$

In this representation, G is not an odd function due to the presence of the $-\gamma$ term; we thus have a representation in model ($D10$), since the p_i's are skew-symmetric and G is increasing in the p_i's. Despite the fact that the representation above is not in model ($D7$) or ($D11$), relation $\succsim$ is complete: we have $x \succsim y$ or $y \succsim x$, or both, for all x, y, since $\sum_{i \in N: x_i S_i y_i} w_i \geq \sum_{j \in N: y_j S_j x_j} w_j$ or $\sum_{i \in N: x_i S_i y_i} w_i < \sum_{j \in N: y_j S_j x_j} w_j$ and γ is non-negative. There must thus be (according to table 6.2) a representation of $\succsim$ in model ($D7$). A representation that would appear natural is not obvious; we always have the opportunity of defining G according to the general construction scheme provided by (6.31); using the p_i's defined by (6.46), it adapts as follows:

$$G([p_i(x_i, y_i)]) = \begin{cases} \exp \sum_{i=1}^{n} p_i(x_i, y_i) & \text{if } \sum_{i \in N} p_i \geq \gamma \\ -\exp[-\sum_{i=1}^{n} p_i(x_i, y_i)] & \text{if} \sum_{i \in N} p_i < \gamma \end{cases} \tag{6.55}$$

This form guarantees the oddness of G as soon as the relation $\succsim$ is complete.

There is, in general, no representation of such a preference in model ($D11$) since we cannot assume that indifference is "thin" unless $\gamma = 0$ or γ is smaller than w_*, the smallest of the weights w_i, $i \in N$. Indeed, x and y are indifferent iff $-\gamma \leq \sum_{i \in N} p_i \leq \gamma$. Any alternative x is indifferent to itself ($x \sim x$); if γ is at least as large as some weight w_j, we can build an alternative $y = (y_j, x_{-j})$ by changing x only on criterion j on which we substitute level x_j by any y_j that is a priori preferred to x_j ($y_j \; P_j \; x_j$). Comparing y to x, we see that $-\gamma \leq \sum_{i \in N} p_i = w_j \leq \gamma$; this means that y is indifferent to x, which violates the positive responsiveness property.

Finally, note that the representations using the p_i's defined by (6.46) are, in general, regular in the sense that such p_i's are numerical representations of the weak orders $\succsim_i^{**}$ on differences of preference. This would fail to be the case only in very degenerate situations in which a criterion would have no influence whatsoever on preference $\succsim$; such a criterion would never make any difference and could be eliminated (see the notion of *influent* criterion in section 6.5).

Turning to weighted majority relations with multiplicative threshold as defined by (6.53), one observes that $\succsim$ is complete and can be represented in model ($D10$), for instance through defining p_i by equation (6.46) and G by:

$$G([p_i]) = \sum_{i: p_i > 0} p_i + \frac{1}{\rho} \sum_{i: p_i < 0} p_i + (1 - \frac{1}{\rho}) \sum_{i: p_i = 0} w_i. \tag{6.56}$$

Since $\succsim$ is complete, it is possible, as in the additive threshold case, to provide a representation in model ($D7$). The preference $\succsim$ does not fit in model ($D11$) since, in general, indifference is not "thin".

Remark 6.3.5 (Asymmetric preference relations: the TACTIC method)
Constructing preference relations using these rules resembles what is known as the TACTIC method; it was proposed and studied in Vansnick (1986a) with the possible adjunction of vetoes that we shall consider later in remark 6.3.8, p. 297. In the original version of TACTIC the preference is defined as an asymmetric relation $\succ$ (a *strict* preference) either by:

$$x \succ y \text{ iff } \sum_{i \in N: x_i P_i y_i} w_i > \sum_{j \in N: y_j P_j x_j} w_j + \gamma. \tag{6.57}$$

or by:

$$x \succ y \text{ iff } \sum_{i \in N: x_i P_i y_i} w_i > \frac{1}{\rho} \sum_{j \in N: y_j P_j x_j} w_j, \tag{6.58}$$

where P_i denotes the asymmetric part of the weak order or the semiorder S_i. The reflexive relation $\succsim$ defined by (6.52) can be obtained from the irreflexive one defined by (6.57) just by saying that $x \succsim y$ if and only if $Not[y \succ x]$. Indeed, assuming (6.52), we have

$$\begin{aligned} Not[y \succ x] \quad & \text{iff} \quad \sum_{i \in N: x_i P_i y_i} w_i \geq \sum_{j \in N: y_j P_j x_j} w_j - \gamma \\ & \text{iff} \quad \sum_{i \in N: x_i S_i y_i} w_i \geq \sum_{j \in N: y_j S_j x_j} w_j - \gamma, \end{aligned}$$

since the term $\sum_{i \in N: x_i I_i y_i} w_i$ appears in the two following expressions and can be cancelled:

$$\sum_{i \in N: x_i S_i y_i} w_i = \sum_{i \in N: x_i P_i y_i} w_i + \sum_{i \in N: x_i I_i y_i} w_i$$

and

$$\sum_{j \in N: y_j S_j x_j} w_j = \sum_{j \in N: y_j P_j x_j} w_j + \sum_{j \in N: y_j I_j x_j} w_j,$$

where I_i is the symmetric part of S_i. The reflexive and complete relation $\succsim$ derived from definition (6.57) of its asymmetric part is thus a weighted majority relation with additive threshold as defined by (6.52); hence it admits representations in models ($D10$) and ($D7$), but not in ($D11$) [5].

Remark 6.3.6 (Duality)
Usually, if a relation $\succsim$ is defined by $x \succsim y$ if and only if $Not[y \succ x]$, it is called the *dual* of $\succ$; of course, $\succ$ is also the dual of $\succsim$. Duality transforms irreflexive relations into reflexive ones (and conversely); it transforms asymmetric relations into complete ones (and conversely). If we interpret $\succ$ as a "better than" relation, its dual $\succsim$ interprets as an "at least as good" relation. •

[5] Note that it is possible to develop a theory of models based on marginal traces or on marginal traces on differences for irreflexive relations; the characterisation of such models is straightforward, using the results obtained for reflexive relations. See Bouyssou and Pirlot (2002a) for an illustration in a particular context.

The multiplicative versions (6.53) and (6.58) are not quite related in the same way: due to the multiplicative threshold, there is no cancellation of the term $\sum_{i \in N: x_i I_i y_i} w_i$ and hence, assuming (6.58), we have:

$$Not[y \succ x] \text{ iff } \sum_{i \in N: x_i P_i y_i} w_i \geq \frac{1}{\rho} \sum_{j \in N: y_j P_j x_j} w_j. \tag{6.59}$$

Such a relation, the dual of $\succ$, is a variant of the weighted majority relation with multiplicative threshold defined by (6.53). It admits a (simpler) representation in model ($D10$), using (6.46) as a definition of p_i and defining G by:

$$G([p_i]) = \sum_{i:p_i>0} p_i + \frac{1}{\rho} \sum_{i:p_i<0} p_i + (1 - \frac{1}{\rho}) \sum_{i:p_i=0} w_i. \tag{6.60}$$

Since the relation is complete, it also admits a representation in model ($D7$).

For more information about TACTIC, see also sections 5.2.1.4 and 5.2.1.4; note that definitions (6.52) and (6.53) both reduce to that of weighted simple majority (see section 6.3.5.2) when $\gamma = 0$ and $\rho = 1$ respectively. •

Table 6.3 provides a summary of the main models applicable to preferences that distinguish no more than three classes of differences of preference on each dimension.

Aggregation rule	General model	Special models
Weighted simple majority* (see 6.3.5.2)	($D11$)	($D11$) + additive
Weighted qualified majority* (see 6.3.5.3)	($D10$)	($D8$) + additive
Lexicographic (see 6.3.5.4)	($D11$)	($D11$) + additive
Weighted majority with add. threshold (see 6.3.5.5)	($D7$)	($D10$) + additive (with constant: eq. (6.52))
Weighted majority with mult. threshold (see 6.3.5.5)	($D7$)	($D10$) + linear (eq. (6.56))

Table 6.3: Models distinguishing no more than three classes of differences of preferences.

6.3.6 Examples of models using vetoes

Vetoes could be introduced in all the examples dealt with in the previous section. We shall only consider the case of qualified weighted majority relations (see section 6.3.5.3) with vetoes (the relations that are the basic ingredients of the ELECTRE I and II methods) and of weighted majority relations with thresholds (see section 6.3.5.5) and vetoes (these relations are fundamental in TACTIC). This section responds to section 5.4.6 of chapter 5; we use the notations introduced there.

* Also with semiordered a priori preferences, see remark 6.3.4.

The intuition one can have about a *veto* is the following. Consider an alternative x and a criterion i on which the level of the performance of x, x_i, is much worse than the level y_i of another alternative y. A veto of y on x on criterion i consists in rejecting the possibility that x be globally preferred to y, irrespective of the performances of x and y on the criteria other than i. In other words, a veto on criterion i forbids the declaration that $x \succsim y$ if (x_i, y_i) is a "negative" difference that is "large enough in absolute value", with respect to relation $\succsim_i^*$ or $\succsim_i^{**}$ (in the latter case, this is equivalent to saying that (y_i, x_i) is a large enough "positive" difference). Of course, if the difference (x_i, y_i) leads to a veto forbidding the declaration that x preferred to y, it is certainly because we do not have $x_i \; S_i \; y_i$, but, instead, $y_i \; P_i \; x_i$, and "even more". We thus define the veto relation V_i as a subset of relation P_i consisting of all pairs $(y_i; x_i)$ such that the presence of the reverse pair (x_i, y_i) for two alternatives x and y prohibits $x \succsim y$; V_i is an asymmetric relation.

Suppose that, for all i, X_i is a subset of the set of real numbers (X can be seen, in a sense, as a performance table, as in section 5.4) and that S_i is a semiorder determined by the following condition:

$$x_i \; S_i \; y_i \Leftrightarrow x_i \geq y_i - \tau_{i,1} \tag{6.61}$$

where $\tau_{i,1}$ is a non-negative threshold. This is similar to the situation described in section 6.3.5.3 using the example of the cost (except that the cost was to be minimised; here we prefer the larger values): the values x_i and y_i are indifferent ($x_i \; I_i \; y_i$) if they differ by less than the threshold $\tau_{i,1}$; x_i is strictly preferred to y_i ($x_i \; P_i \; y_i$) if it surpasses y_i by at least the value of the threshold. In such a case, a convenient way of defining the veto relation V_i, a subset of P_i, is by means of another threshold $\tau_{i,2}$ that is larger than $\tau_{i,1}$. We say that the pair (y_i, x_i) belongs to the veto relation V_i if the following condition is satisfied:

$$y_i \; V_i \; x_i \Leftrightarrow y_i > x_i + \tau_{i,2}. \tag{6.62}$$

Clearly, the veto relation defined above is included in P_i. Assume indeed that $y_i \; V_i \; x_i$; since $\tau_{i,2}$ is larger than $\tau_{i,1}$, we have $y_i > x_i + \tau_{i,2} > x_i + \tau_{i,1}$, yielding $y_i \; P_i \; x_i$. We call $\tau_{i,2}$, a *veto threshold*; the relation V_i defined by (6.62) is a strict semiorder, i.e. the asymmetric part of a semiorder; it is contained in P_i that is also a strict semiorder, namely, the asymmetric part of the semiorder S_i. In such a situation, when comparing an arbitrary level x_i to a fixed level y_i, we can distinguish four relative positions of x_i with respect to y_i that are of interest. These four zones are shown on figure 6.7; they correspond to relations described above, namely:

If x_i belongs to:	then:
Zone I	$x_i \; P_i \; y_i$
Zone II	$x_i \; I_i \; y_i$
Zone III	$y_i \; P_i \; x_i$ and $Not[y_i \; V_i \; x_i]$
Zone IV	$y_i \; P_i \; x_i$ and $y_i \; V_i \; x_i$

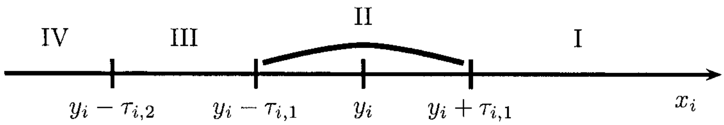


Figure 6.7: Relative positions of an arbitrary level x_i with respect to a fixed level y_i.

6.3.6.1 Weighted qualified majority with veto

Starting with both an a priori preference relation S_i (a semiorder) and an a priori veto relation V_i (a strict semiorder included in P_i) on each set X_i, we can define a global preference relation of the ELECTRE I type as follows:

$$x \succsim y \text{ iff } \begin{cases} \sum_{i \in N: x_i S_i y_i} w_i \geq \delta \\ \text{and} \\ \text{there is no dimension } i \text{ on which } \ y_i \ V_i \ x_i; \end{cases} \tag{6.63}$$

in this expression, $(w_1, \ldots, w_n)$ denotes a vector of normalised weights and δ, a majority threshold that belongs to the $[\frac{1}{2}, 1]$ interval. The global preference of the ELECTRE I type is thus a weighted qualified majority relation (in which the a priori preferences may be semiorders instead of weak orders) that is "broken" as soon as there is a veto on any single criterion, i.e. as soon as the performance of an alternative on some dimension is sufficiently low in comparison to the other.

It is not difficult to provide a representation of such a preference relation $\succsim$ in model ($D8$). Let:

$$p_i(x_i, y_i) = \begin{cases} w_i & \text{if } \ x_i \ S_i \ y_i \\ 0 & \text{if } \ y_i \ P_i \ x_i \text{ but } Not[y_i \ V_i \ x_i] \\ -M & \text{if } \ y_i \ V_i \ x_i, \end{cases} \tag{6.64}$$

where M is a large positive constant and

$$G([p_i]) = \sum_{i \in N} p_i - \delta. \tag{6.65}$$

If no veto occurs in comparing x and y, then $G([p_i(x_i, y_i)]) = \sum_{i: x_i S_i y_i} w_i - \delta$, which is the same representation as for the weighted qualified majority without veto (section 6.3.5.3). Otherwise, if on at least one criterion j, one has $y_j \ V_j \ x_j$, then $G([p_i(x_i, y_i)]) < 0$, regardless of x_{-j} and y_{-j}. The effect of the constant M in the definition of p_i is to make it impossible for G to reach 0 whenever any of the terms p_i is equal to $-M$; it is sufficient that M be larger than 1 to ensure this effect since the sum of all weights w_i is equal to 1 and cannot balance a penalty (represented by M) larger than 1.

The above-mentioned representation of an ELECTRE I type of preference relation in model $(D8)$ is regular since p_i, as defined by (6.64), is a numerical representation of the weak order $\succsim_i^*$ on the differences of preference. This order distinguishes three equivalence classes of differences of preference, namely those corresponding respectively to the cases where $x_i \; S_i \; y_i$, $y_i \; P_i \; x_i$ but $Not[\, y_i \; V_i \; x_i \,]$ and $y_i \; V_i \; x_i$.

The representation given above is probably the most natural and intuitive. Since the set of relations that can be described by (6.63) contains the weighted qualified majority relations, it is clear from section 6.3.5.3 that one cannot expect that weighted qualified majority relations with veto admit a representation in model $(D7)$ or $(D11)$. Nevertheless, they admit a representation in model $(D6)$ and in its strictly increasing yet equivalent version $(D10)$. For a representation in model $(D6)$, we may choose a numerical representation of the weak order $\succsim_i^{**}$ for p_i, which determines five equivalence classes of differences of preference, namely:

$$p_i(x_i, y_i) = \begin{cases} M & \text{if} \quad x_i \; V_i \; y_i \\ w_i & \text{if} \quad x_i \; P_i \; y_i \text{ and } Not[\, x_i \; V_i \; y_i \,] \\ 0 & \text{if} \quad x_i \; I_i \; y_i \\ -w_i & \text{if} \quad y_i \; P_i \; x_i \text{ and } Not[\, y_i \; V_i \; x_i \,] \\ -M & \text{if} \quad y_i \; V_i \; x_i, \end{cases} \tag{6.66}$$

where M is a positive constant larger than w_i. The function G can be defined by

$$G([p_i(x_i, y_i)]) = \begin{cases} \sum_{i: x_i S_i y_i} \min(p_i(x_i, y_i), w_i) - \delta & \text{if, for all } j \in N, \; Not[\, y_j \; V_j \; x_j \,] \\ -1 & \text{if, for some } j \in N, \; y_j \; V_j \; x_j. \end{cases} \tag{6.67}$$

Using a representation of $\succsim_i^{**}$ for p_i forces us to define G in a tricky way since, when $x_i \; P_i \; y_i$, G should not make any distinction between the sub-cases $Not[\, x_i \; V_i \; y_i \,]$ and $x_i \; V_i \; y_i$; the fact that the pair of levels (y_i, x_i) is in the veto relation V_i only intervenes when determining whether x is preferred to y ($x \succsim y$) and not when determining whether y is preferred to x. This leads us to write $\min[p_i(x_i, y_i), w_i]$ instead of simply writing $p_i(x_i, y_i)$ in the definition of G. In this way, the value M of p_i is truncated to w_i by function G. With this definition, G is nondecreasing in all its arguments p_i. Note that the value $-M$ never appears in the sum since the latter only adds up the weights of the criteria on which x_i is at least as good as y_i. Thus the only constraint on M is to be larger than the maximal value w^* of the weights w_i is ; this has to be imposed in order to obtain, with p_i, a numerical representation of the weak order $\succsim_i^{**}$.

A strictly increasing representation (in model $(D10)$) is obtained using the usual construction, for instance equation (6.48).

6.3.6.2 Weighted relative majority with additive threshold and veto

A veto relation can be defined and used as above to discard preferences in each of the models of majority described in section 6.3.5. We consider a weighted majority relation with additive threshold described by equation (6.52) as a further example. We can "add" a veto in the same way as in (6.63), defining $\succsim$ by:

$$x \succsim y \text{ iff } \begin{cases} \sum_{i \in N: x_i S_i y_i} w_i \geq \sum_{j \in N: y_j S_j x_j} w_j - \gamma \text{ and} \\ \text{there is no dimension } i \text{ on which } y_i \; V_i \; x_i. \end{cases} \tag{6.68}$$

We easily obtain a natural representation of such a relation in model ($D9$) by modifying the definition (6.40) of p_i in a weighted simple majority into:

$$p_i(x_i, y_i) = \begin{cases} w_i & \text{if} \quad x_i \; P_i \; y_i \\ 0 & \text{if} \quad x_i \; I_i \; y_i \\ -w_i & \text{if} \quad y_i \; P_i \; x_i \text{ and } Not[\, y_i \; V_i \; x_i \,] \\ -M & \text{if} \quad y_i \; V_i \; x_i. \end{cases} \tag{6.69}$$

and defining G by

$$G([p_i]) = \sum_{i \in N} p_i + \gamma; \tag{6.70}$$

the positive constant M has to be chosen large enough to make G negative as soon as there is a veto on any one criterion (e.g. M larger than $1 + \gamma$). To obtain a representation in model ($D6$), it is sufficient to define p_i by (6.66) and G using the same trick as in (6.67), yielding:

$$G([p_i(x_i, y_i)]) = \sum_{i \in N} \min[p_i(x_i, y_i), w_i] + \gamma. \tag{6.71}$$

Here too, terms equal to $+M$ should not show up in the sum; hence p_i has to be truncated in order not to go above the value of weight w_i; in contrast to (6.67), the value $-M$ plays its role, when there is a veto, by driving G to the negative numbers.

A representation in model ($D10$) can also be obtained using the standard construction. In general, the preference will not be a complete relation and hence will not fit into models ($D7$) or ($D11$).

Remark 6.3.7
"Adding" vetoes to a previously defined preference $\succsim$, as was done in the last two subsections can have two kinds of effects on a pair of alternatives x and y. If we initially had $x \succ y$, a veto can break the strict preference, yielding incomparability between x and y; if this occurs when the initial preference was a complete relation, the latter property will be lost. Another case is when x and y are indifferent with respect to the initial preference ($x \sim y$); in this situation, vetoes may either make x and y incomparable by breaking both the preference $x \succsim y$ and the preference $y \succsim x$, or they may break only one, say $x \succsim y$; in this case, the introduction of vetoes turns indifference into strict preference. Vetoes can only delete preference arcs; they never create new ones. •

Remark 6.3.8 (Adding vetoes to asymmetric preferences)
In the TACTIC method (Vansnick, 1986a), the preference is defined as an asymmetric relation $\succ$; in the absence of veto, it is defined by formula (6.57), in case

of an additive threshold, or by (6.58), in case of a multiplicative threshold. We consider only the additive threshold case here.

Remark 6.3.5 has shown that, starting with an irreflexive relation $\succ$ defined by (6.57), we obtain a reflexive relation $\succsim$, using $x \succsim y$ if and only if $Not[y \succ x]$; this relation, the dual of $\succ$, is a weighted majority relation with additive threshold in the sense of (6.52).

The original definition of the preference relation in the TACTIC method involves a veto; it is defined as

$$x \succ y \text{ iff } \begin{cases} \sum_{i \in N: x_i P_i y_i} w_i > \sum_{j \in N: y_j P_j x_j} w_j + \gamma \\ \text{and} \\ \text{there is no dimension } i \text{ on which } y_i\ V_i\ x_i, \end{cases} \tag{6.72}$$

with V_i, a strict semiorder included in P_i; remember that $y_i\ V_i\ x_i$ is interpreted as "y_i is much better than x_i".

Using remark 6.3.5, the dual $\succsim$ of the relation just defined is such that:

$$x \succsim y \text{ iff } \begin{cases} \sum_{i \in N: x_i S_i y_i} w_i \geq \sum_{j \in N: y_j S_j x_j} w_j - \gamma \\ \text{or} \\ \text{there is a dimension } i \text{ on which } x_i\ V_i\ y_i. \end{cases} \tag{6.73}$$

Although the first condition alone determines a weighted majority relation with additive threshold (as established in remark 6.3.5), the relation $\succsim$ defined by (6.73) is not a weighted majority relation with additive threshold *and veto*; it could be called instead a weighted majority relation with additive threshold *and bonus*, since the veto condition, which removes arcs from $\succ$, adds arcs to its dual as soon as x is "much better" than y on any single dimension i (according to the interpretation of V_i). We emphasise that the dual of a relation with veto is not a relation with veto, but a relation *with bonus*. The intuition behind this type of preference is that x is declared preferred to y as soon as there is a "large" preference difference in favour of x on any dimension. •

6.3.7 Other examples of preferences that distinguish five classes of differences

The relations defined by using vetoes described in the previous section, make up a very particular subclass of relations for which five classes of differences of preference can be distinguished. Using vetoes, the lowest class of the relation $\succsim_i^{**}$ on differences of preference intervenes in a particular way that could be qualified as "conjunctive and negative"; we declare that x is preferred to y if some condition is fulfilled (involving neither the highest nor the lowest class of $\succsim_i^{**}$, but just the fact that the pairs (x_i, y_i) are either above or below the "null" level (x_i, x_i)) *and for each criterion*, a requirement of "*non* veto" is satisfied. It is obviously possible to conceive interventions of the highest and lowest classes of $\succsim_i^{**}$ that are much less radical in the determination of a global preference. To illustrate this, we briefly present an example of a preference $\succsim$ determined by a relation $\succsim_i^{**}$ with five equivalence classes, not using vetoes.

The thresholds $\tau_{i,1}$ and $\tau_{i,2}$ that we introduced in section 6.3.6 (formulas (6.61) and (6.62)) can be used with totally different meanings, for instance, with the semantic of the (P, Q, I) preference structure introduced in chapter 3, section 3.7.1.1 (we consider a very special case here, in which the relations can be defined using two constant thresholds). Let us recall the interpretation of the three binary relations that appear in this structure: P represents clear-cut strict preference (an asymmetric relation); I is indifference (a symmetric relation) and Q (an asymmetric relation) represents weak preference, i.e. a state of hesitation between strict preference (P) and indifference (I); this system of relations is assumed to be complete, i.e. any pair of objects (x, y) either belongs to one of the three relations or the opposite pair (y, x) belongs to P or Q. Suppose that, for all i, X_i is the set of real numbers. A convenient way of determining a (P, Q, I) structure (P_i, Q_i, I_i) on X_i is by means of a pair of thresholds $\tau_{i,1}$, $\tau_{i,2}$ (with $0 \leq \tau_{i,1} < \tau_{i,2}$) that we use to delimit the categories of pairs of levels[6] (x_i, y_i) in the following way:

- P_i: level x_i is strictly preferred to level y_i:

$$x_i \; P_i \; y_i \;\text{ if }\; x_i \geq y_i + \tau_{i,2} \tag{6.74}$$

- Q_i: level x_i is weakly preferred to level y_i:

$$x_i \; Q_i \; y_i \;\text{ if }\; y_i + \tau_{i,1} \leq x_i < y_i + \tau_{i,2} \tag{6.75}$$

- I_i: level x_i is indifferent to level y_i:

$$x_i \; I_i \; y_i \;\text{ if }\; y_i - \tau_{i,1} < x_i < y_i + \tau_{i,1} \tag{6.76}$$

  or, in other words, if the absolute value of the difference of x_i and y_i is smaller than $\tau_{i,1}$ ($|x_i - y_i| < \tau_{i,1}$);

symmetrically, if (x_i, y_i) does not belong to any of the relations P_i, Q_i or I_i, then we have:

$$y_i \; Q_i \; x_i \;\text{ if }\; y_i - \tau_{i,2} < x_i \leq y_i - \tau_{i,1} \tag{6.77}$$

$$y_i \; P_i \; x_i \;\text{ if }\; x_i < y_i - \tau_{i,2}. \tag{6.78}$$

The above-described situation is illustrated in figure 6.8; it corresponds to a particular case of a (P, Q, I) interval order (Tsoukiàs and Vincke, 2003) but also of a *pseudo-order*, a structure mentioned in section 3.7.1.3 (see also Roy and Vincke, 1987); this pseudo-order is particular since it admits a representation with two constant thresholds. Using such a definition, we thus build exactly five classes of differences of preference on each set X_i and we may decide to combine them to obtain a global preference through a model based on traces on differences (($D1$) to ($D11$)). To make it more concrete, let the functions p_i be defined as follows:

[6] We use the term "level" in a rather improper way here since—as we recall—there is no a priori ordering on the sets X_i; in this context, the term "level" designates an element of a set, the set of symbols used to characterise the alternatives on dimension i, i.e. the co-domain of the scale associated with i (see section 2.3.3).

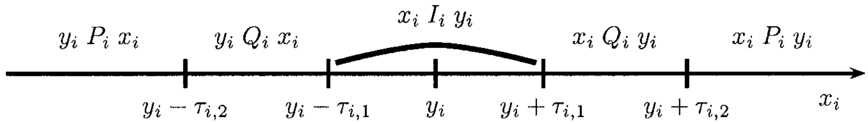


Figure 6.8: Relative positions of an arbitrary level x_i with respect to a fixed level y_i in a (P, Q, I) preference structure with two thresholds.

Case	$p_i(x_i, y_i)$	$p_i(y_i, x_i)$
$x_i \; P_i \; y_i$	2	−2
$x_i \; Q_i \; y_i$	1	−1
$x_i \; I_i \; y_i$	0	0
$y_i \; Q_i \; x_i$	−1	1
$y_i \; P_i \; x_i$	−2	2

and let the p_i's be additively aggregated:

$$x \succsim y \text{ iff } \sum_{i \in N} p_i(x_i, y_i). \tag{6.79}$$

Observing a preference of this type would reveal in particular that, in comparing alternatives x and y belonging to X,

- any difference (x_i, y_i) belonging to category Q_i can be exactly compensated by a difference (y_j, x_j) belonging to category Q_j on another dimension j; for instance, let x and y be two alternatives such that $x_i \; Q_i \; y_i$ and $y_j \; Q_j \; x_j$ while, on the other dimensions $k \neq i, j$, $x_k = y_k$; in such a case x and y cannot be distinguished; they are not only indifferent ($x \sim y$) but they also compare to all third-party alternatives in the same manner: $x \succsim z$ iff $y \succsim z$ and $z \succsim x$ iff $z \succsim y$.

- any difference (x_i, y_i) belonging to relation P_i can be exactly compensated by a difference (y_j, x_j) such that $y_j \; P_j \; x_j$ on another dimension j or by differences $y_j \; Q_j \; x_j$ and $(y_k \; Q_k \; x_k)$ on two dimensions j and k different from i; for instance, let x and y be two alternatives such that (x_i, y_i) belongs to P_i, (y_j, x_j) belongs to Q_j and (y_k, x_k) belongs to Q_k while, on the other dimensions $l \neq i, j, k$, $x_l = y_l$; in such a case x and y should be declared indifferent and compare in the same manner with respect to all third-party alternatives: $x \succsim z$ iff $y \succsim z$ and $z \succsim x$ iff $z \succsim y$.

Clearly, the above-defined model belongs to the class $(D11)$.

Remark 6.3.9
There are of course many other ways of defining models of preference that distinguish five classes of differences. The preference we have just defined belongs to

Fishburn's model (6.27) since it can be represented combining the p_i's in an additive manner. It also belongs to Tversky's additive difference model (6.26) since $p_i(x_i, y_i)$ can be obtained by recoding the arithmetic difference $x_i - y_i$ by means of a function Φ_i:

$$p_i(x_i, y_i) = \Phi_i(u_i(x_i) - u_i(y_i)) = \Phi_i(x_i - y_i). \tag{6.80}$$

This is not the most general case:

- for an observed preference that admits a representation in model ($D11$), it may be impossible to find p_i's such that the preference can be represented by means of a sum of these p_i's as in model (6.27);
- the possibility or impossibility of decomposing the p_i functions using an order on the sets X_i (the latter being possibly represented by partial value functions u_i on X_i) will be examined in section 6.4; a special case is Tversky's model in which p_i is a function of the difference $u_i(x_i) - u_i(y_i)$. •

6.3.8 Examples of preferences that distinguish a large variety of differences

Contrary to the examples discussed so far in which the relations $\succsim_i^*$ or $\succsim_i^{**}$ distinguish a small number of classes of preference differences (typically three or five classes for $\succsim_i^{**}$ in the examples given above), there are very common cases where there is a large number of distinct classes, possibly an infinite number of them.

The most common model, the additive value model, usually belongs to the class of models in which $\succsim_i^{**}$ makes subtle distinctions between differences of preferences. Indeed its definition, equation (6.1), p. 238, can be rewritten as follows:

$$x \succsim y \text{ iff } \sum_{i=1}^{n} (u_i(x_i) - u_i(y_i)) \geq 0. \tag{6.81}$$

The difference $u_i(x_i) - u_i(y_i)$ can often be interpreted as a representation $p_i(x_i, y_i)$ of $\succsim_i^{**}$; the preference then satisfies model ($D11$). Let us take a simple example; assume that $X_i = \mathbb{R}$, that the number of dimensions n is equal to 2 and that $u_i(x_i) = x_i$ for $i = 1, 2$. The preference is defined by:

$$\begin{aligned} x \succsim y &\text{ iff } x_1 + x_2 \geq y_1 + y_2 \\ &\text{ iff } (x_1 - y_1) + (x_2 - y_2) \geq 0. \end{aligned} \tag{6.82}$$

In such a case, $p_1(x_1, y_1) = x_1 - y_1$ is a numerical representation of the relation $\succsim_1^{**}$ on the differences of preference on the first dimension X_1 (and similarly for $x_2 - y_2$ on X_2). The pair (x_1, y_1) corresponds to an at least as large difference of preference as (z_1, w_1) iff $x_1 - y_1 \geq z_1 - w_1$; indeed, if $(z_1, a_2) \succsim (w_1, b_2)$ for some "levels" a_2, b_2 in X_2, then substituting (z_1, w_1) by (x_1, y_1) results in $(x_1, a_2) \succsim (y_1, b_2)$ and, conversely, if $(y_1, c_2) \succsim (x_1, d_2)$ for some c_2, d_2 in X_2, then $(w_1, c_2) \succsim (z_1, d_2)$ (by definition of $\succsim_1^{**}$, see (6.29) and (6.28)). We furthermore know that both preferences obtained after these substitutions are strict as soon as $(x_1, y_1) \succ_1^{**} (z_1, w_1)$, i.e. as soon as $x_1 - y_1 > z_1 - w_1$. This strict responsiveness property of $\succsim$ is characteristic of model ($D11$), in which indifference is "thin" as was already mentioned at the end of section 6.3.5.3. Indeed if $(z_1, a_2) \succsim (w_1, b_2)$, we must have:

$$(z_1 - w_1) + (a_2 - b_2) = 0$$

and substituting (z_1, w_1) by (x_1, y_1) results in $(x_1 - y_1) + (a_2 - b_2) > 0$ as soon as $x_1 - y_1 > z_1 - w_1$.

Thus, any increase or decrease of $p_i(x_i, y_i)$ breaks indifference. This is also the case with the additive difference model (6.26) (with $p_i(x_i, y_i) = \Phi_i(u_i(x_i) - u_i(y_i))$ and the nontransitive additive model (6.27).

Remark 6.3.10 (From ordinal to cardinal)
The framework based on marginal traces on differences that we studied in this section 6.3 is general enough to encompass both "noncompensatory" and "compensatory" preferences, for instance, preferences based on a majority or a lexicographic rule (three classes of differences of preference) and those represented in an additive manner (that can potentially distinguish an unbounded number of differences). A weighted qualified majority rule, for instance, can be said to be *ordinal* or *purely non-compensatory*; from the representation of the procedure (equations (6.43–6.44)), one can see that the full weight w_i associated to a dimension is credited to an alternative x, as compared to an alternative y, as soon as the preference difference $p_i(x_i, y_i)$ is in favour of x on that dimension. In this model, the preference difference $p_i(x_i, y_i)$ is positive as soon as x_i is preferred to y_i, w.r.t. some a priori preference relation S_i on X_i, hence the denomination of "ordinal".

Contrarily, in the additive value model (equation (6.81)), a large difference of preference on one dimension can be compensated by a conjunction of small differences of opposite sign on other dimensions: the procedure is compensatory and it uses the full power of the numbers p_i in arithmetic operations such as sums and differences; we call it "cardinal".

Between these two extremes, the other procedures can be sorted in the increasing order of the number of classes of differences of preference they allow to distinguish. This can be seen as a picture of a transition from "ordinal" to "cardinal" or, alternatively, from noncompensatory to compensatory procedures. Of course, the type of model is determined by the richness of the preferential information available. •

The family of models based on marginal traces on differences encompasses aggregation procedures ranging from those using purely ordinal information (like majority rules) to those relying on cardinal information (like the additive value model). Their description in a common framework enables to break from a vision of pure opposition between ordinal and cardinal procedures; it allows us to view the existing aggregation procedures more as a continuum.

6.4 Models using traces on differences and marginal traces

In many of the examples examined in section 6.3.5, the functions p_i are representations of $\succsim_i^*$ or $\succsim_i^{**}$ and they can be expressed in terms of functions u_i defined on X_i. This is, in particular, the case with the additive value model discussed in the previous subsection: $p_i(x_i, y_i) = u_i(x_i) - u_i(y_i)$. In this section, we examine the possibility of further decomposing the model on differences introduced in section 6.3.

Consider any relation $\succsim$ on X. As we have seen in section 6.3.3, any relation admits a representation in model $(D0)$ (equation (0)):

$$x \succsim y \Leftrightarrow G(p_1(x_1, y_1), p_2(x_2, y_2), \dots, p_n(x_n, y_n)) \geq 0.$$

It is always possible to further decompose each p_i using a real-valued function u_i defined on X_i. The latter, in this trivial model, is just a numeric label assigned to each element of X_i; all elements that are not distinguished by the marginal trace $\succsim_i^{\pm}$ may receive the same label or, in other words, the fact that $u_i(x_i) = u_i(y_i)$ implies that $x_i \sim_i^{\pm} y_i$ is the only requirement imposed on u_i. We may then unambiguously define the function of two variables φ_i on $u_i(X_i) \times u_i(X_i)$ by

$$p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i)). \tag{6.83}$$

We thus have the general model using marginal traces and traces on differences, that we label $(L0D0)$:

$$x \succsim y \Leftrightarrow G([\varphi_i(u_i(x_i), u_i(y_i))]) \geq 0. \tag{L0D0}$$

Of course this definition, which makes sense in all cases, becomes interesting and useful when φ_i enjoys some properties such as non-decreasingness in its first variable and non-increasingness in its second variable; such a property brings it closer to an algebraic difference (and thus closer to Tversky's model (6.26)). Combining the variants of model $(D0)$ (studied in table 6.2) with monotonicity properties of φ_i may indeed lead to interesting models.

A model in which the term $p_i(x_i, y_i)$ is substituted with $\varphi_i(u_i(x_i), u_i(y_i))$ corresponds to each of the 12 models $(D0)$ to $(D11)$ studied in section 6.3. In order to bring the function φ_i "closer" to a subtraction, we envisage two variants of each of these models. In the first one, we impose that φ_i be nondecreasing in its first argument and nonincreasing in its second argument. This defines models $(L1D0)$ to $(L1D11)$. In the other variant, we impose that φ_i be increasing in its first argument and decreasing in its second argument. This defines models $(L2D0)$ to $(L2D11)$.

An interesting feature is that the axioms to be added to those shown in table 6.2 to characterise the newly defined models, are precisely axioms $AC1$, $AC2$, $AC3$ and $AC4$ that were used in the models based on marginal traces. The "RC" and the "AC" axioms do not interact: they are independent (see Bouyssou and Pirlot, 2004a).

Table 6.4: Models $(L1D0)$ to $(L1D11)$: Definition and characterisation.

Models	Definition	Conditions
$(L0D0)$	$x \succsim y \Leftrightarrow G([\varphi_i(u_i(x_i), u_i(y_i))]) \geq 0$	$\varnothing$
$(L1D0)$	$(L0D0)$ with $\varphi_i(\nearrow, \searrow)$	$\varnothing$
$(L1D1)$ $\Updownarrow$ $(L1D2)$	$(L1D0)$ with $\varphi_i(u(x_i), u_i(x_i)) = 0$ $(L1D1)$ with φ_i skew symmetric	ind.
$(L1D3)$	$(L1D2)$ with G odd	cpl., ind.
$(L1D4)$ $\Updownarrow$ $(L1D8)$	$(L1D0)$ with $G(\nearrow)$ $(L1D0)$ with $G(\nearrow\!\!\nearrow)$	$RC1$, $AC123$
$(L1D5)$ $\Updownarrow$ $(L1D9)$	$(L1D1)$ with $G(\nearrow)$ $(L1D1)$ with $G(\nearrow\!\!\nearrow)$	$RC1$, ind., $AC123$
$(L1D6)$ $\Updownarrow$ $(L1D10)$	$(L1D2)$ with $G(\nearrow)$ $(L1D2)$ with $G(\nearrow\!\!\nearrow)$	$RC12$, $AC123$
$(L1D7)$	$(L1D3)$ with $G(\nearrow)$	cpl., $RC12$, $AC123$
$(L1D11)$	$(L1D3)$ with $G(\nearrow\!\!\nearrow)$	cpl., $RC3$, $AC123$

$\nearrow\!\!\nearrow$ means increasing, $\nearrow$ means nondecreasing, $\searrow$ means nonincreasing
cpl. means completeness, ind. means independence

The definition and characterisation of the various models "$(L1 - Dj)$" (for $j = 0$ to 11) that we consider are provided in table 6.4. The table only describes models $(L1D0)$ to $(L1D11)$. The models where φ_i is assumed to be increasing in its first argument and decreasing in its second, i.e. models $(L2D0)$ to $(L2D11)$, are equivalent to the corresponding $(L1Dy)$ model with the exception of the last one: $(L2D11)$ is not equivalent to $(L1D11)$. The characterisation of this model can be found in table 6.5.

6.4.1 Relationship between marginal traces and traces on differences

As suggested by the axioms used to characterise the variants of model $(L0D0)$ (see tables 6.4 and 6.5), these models use both marginal traces $\succsim_i^{\pm}$ (introduced

Table 6.5: Characterisation of model $(L2D11)$.

Model	Definition	Conditions
$(L2D11)$	$x \succsim y \Leftrightarrow G([\varphi_i(u_i(x_i), u_i(y_i))]) \geq 0$, with $\varphi_i(\nearrow\!\nearrow, \searrow\!\searrow)$ and skew symmetric, G odd and increasing	cpl., $RC3$, $AC4$

$\nearrow\!\nearrow$ means increasing, $\searrow\!\searrow$ means decreasing, cpl. means completeness

in section 6.2.5, formulas (6.17)) and traces on differences $\succsim_i^*$ and $\succsim_i^{**}$ (formulas (6.28) and (6.29)). It is quite important and also quite simple to understand how those traces are related: in fact, $\succsim_i^{\pm}$ is not only the marginal trace left by the relation $\succsim$ on X_i but, at the same time, it is the marginal trace left by $\succsim_i^*$ and $\succsim_i^{**}$ on X_i. Indeed, using the original definitions of the involved relations, we can easily verify that we have:

$$x_i \succsim_i^{\pm} y_i \text{ iff } \forall z_i \in X_i, (x_i, z_i) \succsim_i^* (y_i, z_i) \\ \text{and } \forall w_i \in X_i, (w_i, y_i) \succsim_i^* (w_i, x_i). \tag{6.84}$$

The latter expression implies that $\succsim_i^{\pm}$ is the marginal trace of both $\succsim_i^*$ and $\succsim_i^{**}$. This is true without any assumption on $\succsim$. When the traces of $\succsim$ are assumed to be weak orders, the weak orders $\succsim_i^*$ and $\succsim_i^{**}$ react monotonically with respect to their traces $\succsim_i^{\pm}$ (which are also weak orders). Table 6.4 shows us that for models $(L1D4)$ and those more constrained, both $\succsim_i^*$ and $\succsim_i^{\pm}$ are complete relations (since $RC1$ and $AC123$ hold), hence they are weak orders. For model $(L1D6)$ and more constrained ones, we have in addition that $\succsim_i^{**}$ is also a weak order. In model $(L1D4)$, one may thus take numerical representations of the weak orders $\succsim_i^*$ for the functions p_i and these functions factorise as $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$; the functions u_i may furthermore be taken to be numerical representations of the weak orders $\succsim_i^{\pm}$; G can be assumed to be nondecreasing in the p_i's. In model $(L1D6)$, we may choose functions p_i that represent the weak orders $\succsim_i^{**}$, the rest of the properties of model $(L1D4)$ remaining true. These facts have important consequences for the elicitation of such models as we shall see in section 6.4.2.

Another feature shown in tables 6.4 and 6.5 is that the strict monotonicity of G or the φ_i's is not linked to observable characteristics of preference $\succsim$, unless we consider the more constrained of the models, i.e. models $(L1D11)$ and $(L2D11)$. In the former model, G strictly responds to any improvement or depreciation of a difference of preference on any dimension i; in the latter model, not only does G react in that way, but it is also the case for φ_i, for all i: φ_i strictly responds to any improvement or depreciation of any of the compared alternatives on dimension i. The practical consequences of this feature of the models are however relatively limited: in these models, the indifference is "thin", with, as we shall see, slightly different behaviours depending on which of the two models the preference belongs to. The concept of preference relations with thin indifference has been already discussed, on p. 288, as a consequence of $RC3$ (for a preference that is a complete relation). Examples of preferences with thin indifference have been presented on p. 302. Models $(L1D11)$ and $(L2D11)$ both satisfy $RC3$ and are complete; there is nothing specific regarding thinness of indifference, with model $(L1D11)$, contrary to model $(L2D11)$. Indeed, suppose that x and y are tied, i.e. $x \sim y$;

- in model $(L1D11)$, indifference is broken as soon as any difference of preference (x_i, y_i) is substituted with a non-equivalent (w.r.t. $\succsim_i^{**}$) one;
- in model $(L2D11)$ indifference is also broken when any level x_i or y_i is changed into a non-equivalent (w.r.t. $\succsim_i^{\pm}$) one.

In the latter model, variations of levels produce variations in the traces on differences which in turn can break ties.

6.4.2 Eliciting models using both marginal traces and traces on differences

6.4.2.1 Procedure

The strategy suggested by the models using traces on differences for eliciting $\succsim$ can be further refined with these models. In the models using traces on differences, it is natural, as emphasised in remark 6.3.1, to base the elicitation of the preference on the elicitation of the relation on preference differences ($\succsim_i^*$ or $\succsim_i^{**}$). Here, we may further wish to use the possible decomposition of the relation on preference differences on each dimension i as a function of traces on the set X_i.

Due to the existence of the ordering $\succsim_i^{\pm}$ on X_i, we may represent all pairs (x_i, y_i) (where x_i, y_i belong to X_i) in a system of orthogonal axes; on both axes, we rank the elements of X_i in increasing order w.r.t. $\succsim_i^{\pm}$, e.g. by assigning the value $u_i(x_i)$ to x_i (where $u_i(x_i)$ is chosen to be a numerical representation of $\succsim_i^{\pm}$). Each pair (x_i, y_i) can thus be represented in Cartesian coordinates by the point $(u_i(x_i), u_i(y_i))$. We will be interested in the *indifference curves* of $\succsim_i^*$ (and $\succsim_i^{**}$), i.e. the equivalence classes of this relation.

Let us consider two simple examples of relations on preference differences on dimension i. In the first one, the relation $\succsim_i^{**}$ (or $\succsim_i^*$) responds strictly to the marginal trace $\succsim_i^{\pm}$; in the second, the response is not necessarily strict.

Example 6.5 (Strict responsiveness)
Let $X_i = \{1, 2, 3, 4, 5\}$ and suppose that $\succsim_i^{\pm}$ is the usual order on X_i. Let $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ be defined by

$$p_i(x_i, y_i) = x_i - y_i \tag{6.85}$$

and suppose that p_i is a numerical representation of the weak order $\succsim_i^{**}$. In such a case, the equations of the *indifference curves* of $\succsim_i^{**}$ are $x_i - y_i = k$, for all possible constants k. These "curves" are represented in figure 6.9. One observes that they define increasing functions mapping X_i into X_i. Indeed, for each equivalence class of $\succsim_i^{**}$, to each x_i corresponds at most one y_i such that (x_i, y_i) belongs to that class (for instance, in the class $p_i = 2$, to $x_i = 3$ corresponds $y_i = 1$, but no y_i can be associated to $x_i = 1$). Moreover, if (x_i, y_i) belongs to an indifference curve, and you increment x_i, positioning yourself at $z_i \succ_i^{\pm} x_i$, then if there is a point on the indifference curve corresponding to z_i, it must be a level w_i above y_i and in any case, (z_i, y_i) belongs to an indifference curve that is below (w.r.t. $\succsim_i^{**}$) that passing through (x_i, y_i).

The case in which $\succsim_i^{**}$ is a weak order may be simpler compared to the case in which only $\succsim_i^{*}$ is complete. Indeed, $\succsim_i^{**}$ is a reversible relation and thus about[7] "half of the relation" has to be described since, as is the case in the example given above, when we know that (x_i, y_i) and (z_i, w_i) belong to the same indifference curve, then we also know that (y_i, x_i) and (w_i, z_i) belong to the same curve (usually another one). The diagonal of X_i^2, that is the set of all pairs (x_i, x_i), is the only indifference curve that contains both (x_i, y_i) and (y_i, x_i). In such a case,

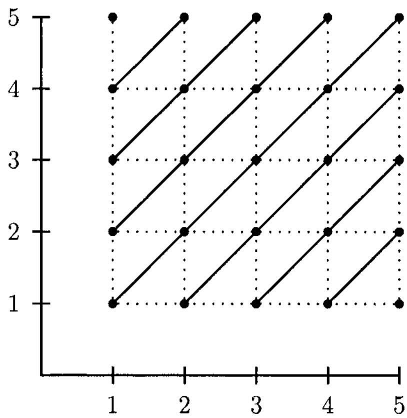


Figure 6.9: The indifference curves of $p_i(x_i, y_i) = x_i - y_i$.

an elicitation procedure of $\succsim_i^{*}$ based on a preliminary elicitation of $\succsim_i^{\pm}$, could be designed as follows: start from any pair (x_i, y_i) and try to list the pairs in its equivalence class by gradually incrementing the value of x_i; start with the value z_i just above x_i, and ask which value w_i (there is at most one such value) is such that (z_i, w_i) is indifferent to (x_i, y_i). Then go ahead incrementing z_i. When the procedure finishes, start again at x_i and decrement it. Here is a numerical illustration on the example given above. We ask for indifference judgements based on the relation $\sim_i^{*}$. Let us start for instance with $(x_i, y_i) = (3, 1)$. Asking which pair of type $(4, w_i)$ is indifferent to $(3, 1)$, we obtain $w_i = 2$; then we find that $(5, 3)$ is indifferent to $(4, 2)$ (and by transitivity, to $(3, 1)$). Decrementing x_i starting again downwards from $(3, 1)$, we obtain no other pair since no pair $(2, w_i)$ is indifferent to $(3, 1)$ if w_i is only allowed to take values in $\{1, 2, 3, 4, 5\}$. To describe the other curves, just remove the pairs that have already been assigned to a curve and start the same procedure from one of the unassigned pairs. ◇

[7] In case of ties in the relation $\succsim_i^{*}$, the remark in the footnote on p. 281 also applies here

Example 6.6 (Non strict responsiveness)
Our second example is built on the set $X_i = \{1,2,3,4\}$ (with $\succsim_i^{\pm}$, the natural order) by defining the representation p_i of $\succsim_i^*$ as follows:

$$p_i(x_i, y_i) = \begin{cases} 0 & \text{if } x_i - y_i = 0 \\ 1 & \text{if } x_i - y_i = 1 \text{ or } 2 \\ 2 & \text{if } x_i - y_i = 3 \end{cases} \tag{6.86}$$

and $p_i(x_i, y_i) = -p_i(y_i, x_i)$. Here the indifference "curves" should be called *indifference strips*; as the indifference curves in example 6.5, they are nondecreasing in some sense. The indifference strips are shown in figure 6.10. In contrast to

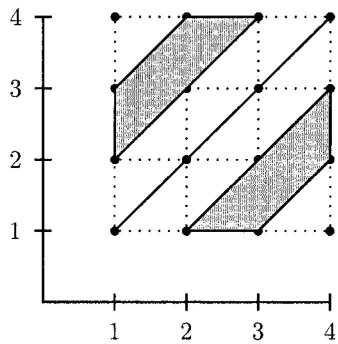


Figure 6.10: The indifference strips in example 6.6.

example 6.5, the equivalence classes of $\succsim_i^{**}$ are not functions: for any x_i, there may be several pairs (x_i, y_i), (x_i, z_i) that are indifferent, while y_i is not equal to z_i; for all x_i; there is an interval (possibly empty) of values such that all pairs (x_i, y_i), with y_i in the interval, are indifferent. The lower and upper boundaries of these intervals form a nondecreasing function of x_i. For example, in figure 6.10, the boundaries of the class containing pair $(3,2)$ are:

- lower boundary: $(2,1)$, $(3,1)$, $(4,2)$;
- upper boundary: $(2,1)$, $(3,2)$, $(4,3)$.

Thus, in this example, the interval for $x_i = 1$ is empty, that for $x_i = 2$ contains a single pair and the intervals for $x_i = 3$ and $x_i = 4$ both include two elements.

Based on the preliminary knowledge of $\succsim_i^{\pm}$, an elicitation procedure of $\succsim_i^{**}$ could run as follows: start with an arbitrary pair, say $(3,2)$. Ask which pairs $(3, w_i)$ are indifferent to $(3,2)$; we then obtain $(3,1)$ and $(3,2)$. Afterwards, increment $x_i = 3$, asking for pairs of type $(4, w_i)$ indifferent to $(3,2)$; we obtain $(4,2)$ and $(4,3)$. We start again from $(3,2)$, decrementing $x_i = 3$ and asking for pairs $(2, w_i)$ indifferent to $(3,2)$; we obtain the single $w_i = 1$. Finally, one asks for pairs $(1, w_i)$

indifferent to $(2, 1)$ and we don't get any value w_i. The indifference class containing $(3, 2)$ is now completely known. We remove the pairs that belong to it from further consideration and start the same procedure from an arbitrary pair that has not yet been assigned to an indifference class. If $\succsim_i^{**}$ is reversible as in this example, it suffices to do half of the job, as in example 6.5. $\diamond$

6.4.2.2 Peculiarities of the elicitation of models $(L - D)$

To build a representation of a preference $\succsim$ in one of the models from $(L1D5)$ and those more constrained, we might, in theory, apply the strategy outlined in section 6.3.4. When both marginal traces and traces on differences are weak orders (i.e. in models from $(L1D5)$ and those more constrained), we may exploit what precedes to ease the elicitation of the traces on differences using marginal traces. In principle, the more the model is constrained the lower the complexity of the elicitation; in particular:

- in models where p_i is skew-symmetric, we may assign the value 0 to $p_i(x_i, x_i)$, for all x_i and elicit either the "positive" or the "negative" part of p_i (exploiting the fact that $p_i(x_i, y_i) = -p_i(y_i, x_i)$);

- when $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ is increasing in its first argument and decreasing in its second one, the indifference curves are functions and their elicitation might possibly be considered to be easier (compared to the description of "strips of indifference" in the non-strictly monotone case).

The latter advantage might however be questioned for two reasons. First, it is not that clear, that eliciting indifference strips is more complex in terms of numbers of mental operations, than eliciting indifference functions: we lack a full proper theory on the complexity of eliciting empirical structures (in contrast to the complexity of logical decision problems, which is well-studied (Garey and Johnson, 1979)). The second objection is more serious. Consider all models $(L - D)$ in which $\succsim_i^{\pm}$ and $\succsim_i^*$ (or possibly $\succsim_i^{**}$) are weak orders. In the models in which $p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i))$ is non-decreasing in its first argument and non-increasing in its second argument, it is always possible (as long as X_i is a finite or denumerable set, which we assume) to choose a numerical representation of the weak order $\succsim_i^*$ (or $\succsim_i^{**}$) on differences of preferences for p_i and a numerical representation of the marginal trace $\succsim_i^{\pm}$ for u_i. This is not always the case if we impose that $\varphi_i(u_i(x_i), u_i(y_i))$ is increasing in its first argument and decreasing in its second argument. We shall not enter into the—rather technical—discussion of this issue that we call the *regularity* of the representation elsewhere; the interested reader is referred to Bouyssou and Pirlot (2004a, section 5.4.2) for more detail. The disadvantage of a non-regular representation is obvious: if p_i is not a numerical representation of $\succsim_i^*$ (or $\succsim_i^{**}$), an elicitation procedure as the one outlined for example 6.5 loses its justification.

6.4.3 Models distinguishing no more than five classes of differences revisited

We start with the weighted majority model then address more general models in which $\succsim_i^*$ or $\succsim_i^{**}$ have at most three classes of equivalence.

6.4.3.1 The weighted majority model revisited

In the weighted (simple) majority model (see section 6.3.5.2), we declare that $x \succsim y$ if and only if the sum of the weights of the dimensions saying that x is at least as good as y is not less than the corresponding sum for dimensions stating that y is at least as good as x. The numerical representation of $\succsim$ in model $(D11)$ described by equations (6.40) and (6.41) can easily be transformed into a representation in model $(L1D11)$; it is sufficient to define φ_i by

$$\varphi_i(u_i(x_i), u_i(y_i)) = \begin{cases} w_i & \text{if } u_i(x_i) > u_i(y_i) \\ 0 & \text{if } u_i(x_i) = u_i(y_i) \\ -w_i & \text{if } u_i(x_i) < u_i(y_i), \end{cases} \tag{6.87}$$

where u_i is any numerical representation of the weak order S_i on X_i (see section 6.3.5.2). With this definition, the value of the function $\varphi_i(u_i(x_i), u_i(y_i))$ is the same as that of $p_i(x_i, y_i)$ defined by (6.40). The aggregation of the φ_i's is carried on additively as in (6.41), i.e. $G([\varphi_i]) = \sum_{i \in N} \varphi_i$.

One can immediately see that in the above representation, φ_i is not strictly monotonic in its arguments. Indeed, as soon as we have, for instance, $u_i(x_i) > u_i(y_i)$, $\varphi_i(u_i(x_i), u_i(y_i))$ is equal to the weight w_i and its value does not respond to any further increase in its first argument. Similarly, the value of φ_i does not increase when decreasing the value of its second argument. This shows that (6.87) does not define a representation of majoritarian preferences in the strictly responsive model $(L2D11)$.

Is there another way of building a representation of such a preference that would yield a representation in model $(L2D11)$ [8]? The answer is, in general, negative for the following reason: suppose that x is indifferent to y while $u_j(x_j) > u_j(y_j)$ for some j; we have $G([\varphi_i(u_i(x_i), u_i(y_i))]) = 0$ to represent the indifference of x and y. Assume that there exists z_j with $u_j(z_j) > u_j(x_j)$. According to the definition of the weighted simple majority rule (6.39), we have $(z_j, x_{-i}) \sim y$. Postulating the existence of a strictly monotonic representation of $\succsim$, would lead to $G(\varphi_j(u_j(z_j), u_j(y_j)), [\varphi_i(u_i(x_i), u_i(y_i))]_{i \neq j}) > 0$, which implies $(z_j, x_{-i}) \succ y$, a contradiction.

Remark 6.4.1 (Regularity of the representation)
Another issue about the representation is related to its *regularity*. In models $(L1Di)$ or $(L2Di)$, regularity is twofold: in equation $(L0D0)$, it can or not be that $\varphi_i(u_i(x_i), u_i(y_i))$ represents the weak order $\succsim_i^*$ or $\succsim_i^{**}$ and it can or not be that $u_i(x_i)$ represents the weak order $\succsim_i^+$. The representation is *regular* if both are true, which presupposes at least model $(L1D4)$ (in order to be sure that $\succsim_i^*$ and

[8] This is the only case of interest since we know that models $(L1Di)$ and $(L2Di)$ are equivalent for $i = 1$ up to $i = 10$. So, if there is a representation in $(L1Di)$, there is one in $(L2Di)$.

$\succsim_i^+$ are both weak orders). It is always possible to obtain regular representations functions φ_i that are monotonic but not strictly monotonic. This is no longer true when we want a representation in models ($L2Di$) for $i = 4$ to 10 since all pairs x_i, y_i for which $x_i \ P_i \ y_i$, i.e. for which $u_i(x_i) > u_i(y_i)$, are equivalent with respect to $\succsim_i^*$. The existence of regular representations is an advantage from the point of view of the elicitation of the preferences, since, as shown in section 6.4.2, one may rely upon the traces $\succsim_i^*$ and $\succsim_i^+$ to build a numerical representation of the preference. •

6.4.3.2 Other models

In all models studied in sections 6.3.5 and 6.3.6, the relations $\succsim_i^+$ are weak orders for all i; they thus admit a numerical representation that we denote by u_i. For all these models, as soon as p_i is a representation of the weak order $\succsim_i^*$ or $\succsim_i^{**}$, it always makes sense to define φ_i by setting $\varphi_i(u_i(x_i), u_i(y_i)) = p_i(x_i, y_i)$. By doing this, starting from a representation of $\succsim$ in model Di for some $i \in N$, one obtains a representation in the corresponding model ($L1Di$). For all $i \neq 11$, this implies that a representation also exists in the corresponding model ($L2Di$), since the latter is equivalent to the former. The picture is not the same for models ($L1D11$) and ($L2D11$) as already observed for weighted majority preferences: there is a representation in model ($L2D11$) iff there is one in model ($L1D11$) and indifference is "thin". Regarding the regularity of the representation, as in the case of weighted majority, it is seldom possible to guarantee both regularity and the strict monotonicity of φ_i whilst regularity and (non-strict) monotonicity of φ_i are perfectly compatible.

6.5 Models with weakly differentiated preference differences

In section 6.3.5 and subsection 6.4.3.1, we investigated a variety of models in which the number of classes of differences of preference is reduced to at most three. Can one provide a unified framework for discussing and understanding all these variants of a majority rule? It is our aim in this section to briefly describe such a framework. All the preferences described in the above-mentioned sections have some right to be called *concordance* relations. The term "concordance" was introduced by Roy (1968, 1971) in the framework of the ELECTRE methods (see also Roy (1996), Roy and Bouyssou (1993, sections 5.2 and 5.3) and Roy (1991); Roy and Vanderpooten (1996).). It specifies an index (the so-called *concordance index*) that measures the strength of the coalition of criteria stating that an alternative x is at least as good as an alternative y. Here, we use this term in the same spirit for qualifying a preference relation that results from the comparison of the strengths of coalitions of criteria: we have all preference relations studied in section 6.3.5 in mind[9] and subsection 6.4.3.1.

[9] The lexicographic preference described in subsection 6.3.5.4 enters into this framework but can be seen as a limit case.

An earlier investigation of preference relations of this type in a conjoint measurement framework is that of Fishburn (1976) through its definition of *noncompensatory* preferences (see also Bouyssou and Vansnick (1986)). More recently, Fargier and Perny (2001) (see also Dubois, Fargier, Perny, and Prade, 2001a; Dubois et al., 2003 and Dubois, Fargier, and Perny, 2002) have proposed a characterisation of concordance relations that relies on an axiom inspired from neutrality and monotonicity conditions used in Social Choice Theory, which strengthens Fishburn's noncompensation condition.

Although it has long been thought that noncompensatory preferences provided the adequate framework for the analysis of preferences resulting from ordinal aggregation methods (i.e. methods in which the only thing that matters in comparing x to y on a dimension is whether x is ranked above or below y if not x and y are tied on that dimension), it was recently shown in Bouyssou and Pirlot (2002a), that this is not totally true and that a slightly broader framework is needed. In this paper (see also Bouyssou and Pirlot, 2005a), a precise definition of concordance relations is proposed and the relations that fulfill it can be described within the family of models that rely on traces on differences (sections 6.3.3 and 6.4). It is the goal of this section to outline these results (we mainly follow Bouyssou and Pirlot, 2005a). Similar ideas have been developed by Greco, Matarazzo, and Słowiński (2001a)

6.5.1 Concordance relations

In a conjoint measurement context, a concordance relation is characterised by the following features.

Definition 6.16 (Concordance relation)
A reflexive relation $\succsim$ on X is a concordance relation if there are:

- *a complete binary relation S_i on each X_i,*
- *a binary relation $\trianglerighteq$ between subsets of N, the union of which is N, which is monotonic with respect to inclusion, i.e. such that for all $A, B, C, D \subseteq N$,*

$$[A \trianglerighteq B, C \supseteq A, B \supseteq D, C \cup D = N] \Rightarrow C \trianglerighteq D, \tag{6.88}$$

such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow S(x,y) \trianglerighteq S(y,x), \tag{6.89}$$

where $S(x,y) = \{i \in N : x_i \; S_i \; y_i\}$.

In this definition, we interpret S_i as the a priori preferences on the scale co-domain X_i of each dimension; in cases of practical interest, S_i will usually be a weak order or a semiorder (but we do not assume this to begin) and the global preference of x over y results from the comparison of the coalitions of criteria $S(x,y)$ and $S(y,x)$. The former can be seen as the list of reasons for saying that x is at least as good as y, while the latter is a list of reasons supporting conversely that y is at least as good as x. A fundamental ingredient amalgamated in a concordance preference is a way of comparing coalitions of criteria: we assume that there is a relation $\trianglerighteq$ on the power set of the set N that allows us to decide whether a subset of criteria constitute a stronger argument than another subset of criteria; the interpretation of such a relation is straightforward when the compared subsets are the lists of dimensions $S(x, y)$ and $S(y, x)$ involved in the comparison of two alternatives x and y. Note that $\trianglerighteq$ enables us only to compare "complete" coalitions of criteria, i.e. those whose union is N.

The weighted majority relation (section 6.3.5.2), typically, fulfills the requirements for a concordance relation as defined above. In this example, the strength of a subset of criteria can be represented by the sum of their weights and, comparing $S(x, y)$ to $S(y, x)$ amounts to comparing two numbers, namely the sums of the weights of the dimensions that belong respectively to $S(x, y)$ and $S(y, x)$. In such a case, $\trianglerighteq$ can be extended to a weak order on the power set of N and this weak order admits a numerical representation that is additive with respect to individual dimensions:

$$S(x, y) \trianglerighteq S(y, x) \text{ iff } \sum_{i \in S(x,y)} w_i \geq \sum_{i \in S(y,x)} w_i. \tag{6.90}$$

In our general definition however, we neither postulate that $\trianglerighteq$ is a weak order nor that it can be additively represented on the basis of "weights" of individual criteria. We only impose a quite natural property (6.88) on the relation $\trianglerighteq$, namely that it is monotonic with respect to the inclusion of subsets of criteria. Suppose that we start with a list of arguments A (e.g. $S(x, y)$) that is at least as strong as a list B (e.g. $S(y, x)$)—we thus start with $A \trianglerighteq B$. This relation should be preserved when enlarging the list A into a list C that contains A or, on the opposite, when contracting the list B into a subset D of B. This is the minimal requirement we can impose on a relation comparing the strengths of coalitions.

The interesting feature of concordance relations, in the sense of definition 6.16 is that they can easily be characterised within the family of models (Dk) that rely on preference differences. The main result, obtained in Bouyssou and Pirlot (2005a, Theorem 1), establishes that concordance relations are exactly those preferences for which the traces on differences $\succsim_i^{**}$ are weak orders and have no more than three equivalence classes. This result will be part 1 of theorem 6.7 stated below on p. 316. Consequently, concordance relations form a subclass of the relations belonging to model ($D6$) (or equivalently to model ($D10$)).

6.5.1.1 The relation $\trianglerighteq$

As a consequence of this result, *all* preferences described in section 6.3.5 (see also table 6.3) admit a representation as a concordance relation, i.e. can be described by means of equation (6.89), i.e.:

$$x \succsim y \Leftrightarrow S(x, y) \trianglerighteq S(y, x),$$

for some $\trianglerighteq$ and some S_i satisfying the requirements of definition 6.16. We emphasise that this is true, not only for simple weighted majorities (section 6.3.5.2), but also for *qualified* majorities (section 6.3.5.3) or lexicographic preferences (section 6.3.5.4) that are not primarily defined through comparing coalitions (qualified majority is defined through comparing the "pros" in favour of x against y to a threshold; lexicographic relations arise from considering the most important criterion and only looking at the others when alternatives are tied on the most important one). Part 1 of theorem 6.7 says that all these relations can *also* be represented according to equation (6.89) using an appropriate definition of $\trianglerighteq$ and S_i. Of course, we cannot ensure that $\trianglerighteq$ can be represented, in general, according with equation (6.90), i.e. in an additive manner.

6.5.1.2 The relations S_i

Are these relations determined by the preference $\succsim$? Indeed they are; S_i can be defined as follows:

$$x_i \; S_i \; y_i \Leftrightarrow (x_i, y_i) \succsim_i^* (x_i, x_i). \tag{6.91}$$

The interpretation of this definition is clear (at least for reflexive and independent preferences $\succsim$ with which all "null differences" (x_i, x_i), for $x_i \in X_i$, are indifferent with respect to relation $\succsim_i^*$): $x_i \; S_i \; y_i$ means that the difference of preference (x_i, y_i) is "non negative", in the sense that it is at least as large as the "null difference" (x_i, x_i) or any other null difference (z_i, z_i).

It can be shown that S_i is complete but not necessarily transitive for a general concordance relation $\succsim$; the marginal traces $\succsim_i^+$ and $\succsim_i^-$ are included in S_i, which in turn is contained in the marginal preference $\succsim_i$. Note that in general concordance relations, the marginal traces are not necessarily complete (hence not necessarily weak orders) and the marginal preferences cannot be guaranteed to be transitive or complete.

For more constrained concordance relations, namely for those that admit a representation in model ($L1D6$), S_i can be proved to be a semiorder (Bouyssou and Pirlot, 2005a, theorem 4 and lemma 10). Remember that in such models, the marginal preferences $\succsim_i$ is also a semiorder (proposition 6.6). It would however be wrong to infer that $S_i = \succsim_i$ for concordance relations representable in model ($L1D6$), as will be shown by the second example below. In the still more constrained model ($L2D11$), S_i and $\succsim_i$ will be weak orders and, at this stage, it is true that S_i equals the marginal preferences $\succsim_i$ as well as the marginal trace $\succsim_i^{\pm}$.

We give two examples that illustrate the relatively subtle relationships between all these relations.

Example 6.7

Consider alternatives that can be described by two dimensions; the co-domain of their associated scale is the integer interval $[0, 10]$. Equal weights $w_i = 0.5$ are associated to both dimensions and the decision rule that determines whether $x = (x_1, x_2)$ is preferred at least as much as $y = (y_1, y_2)$ consists in checking whether for both dimensions x_i is not less than $y_i - 1$. The rationale for this rule is that the client does not perceive a clear difference of performance on a dimension unless x_i and y_i differ by at least two units. The global preference results from unanimous agreement to say that x is at least as good as y on both dimensions.

We thus have:

$$x \succsim y \Leftrightarrow \sum_{i:x_i \geq y_i - 1} w_i = 1. \tag{6.92}$$

So, for example, we have $(1,2) \sim (2,3)$ and $(1,2) \sim (2,1)$ but not $(2,3) \succ (2,1)$, which implies that the symmetric part of the global preference—interpreted as indifference—is not transitive. The marginal traces $\succsim_i^+$ and $\succsim_i^-$ are the natural order on the integers of $[0,10]$ since any advantage $x_i > y_i$ can make a difference, in an appropriate context; for instance: $x_1 = 2 > y_1 = 1$ yields $(2,4) \succsim (0,5)$ while $Not[(1,4) \succsim (0,5)]$; hence $2 \succ_1^+ 1$. The marginal preferences $\succsim_i$ are the semiorders defined by $x_i \succsim_i y_i$ iff $x_i \geq y_i - 1$. We indeed have $(x_i, a_{-i}) \succsim (y_i, a_{-i})$ as soon as this condition is fulfilled. The trace $\succsim_i^{**}$ on differences has three equivalence classes: the class of pairs (x_i, y_i) such that $x_i - y_i$ is strictly larger than 1, that for which $x_i - y_i$ is either $-1, 0$ or 1 and, finally, that for which $x_i - y_i$ is strictly smaller than -1. In this example, S_i is equal to the marginal preference $\succsim_i$, since we have $(x_i, y_i) \succsim_i^* (y_i, y_i)$ iff $x_i \geq y_i - 1$. ◇

Example 6.8

This example is a variant of the previous one. We consider three dimensions instead of two, with scales valued in the integer interval $[0,10]$. The weights attached to the dimensions are equal ($w_i = \frac{1}{3}$) and the preference of alternative $x = (x_1, x_2, x_3)$ over $y = (y_1, y_2, y_3)$ results from the following qualified majority rule:

$$x \succsim y \Leftrightarrow \sum_{i:x_i \geq y_i - 1} w_i \geq 0.6. \tag{6.93}$$

Note that setting the threshold to any value between 0.34 and 0.66 would not make any difference to the preference relation $\succsim$. The only difference with example 6.7, in terms of relations derived from the preference $\succsim$, is the marginal preference $\succsim_i$; in this example, all levels x_i are indifferent with respect to the marginal preference. Indeed we have for all a_{-i}, x_i, y_i, $(x_i, a_{-i}) \sim (y_i, a_{-i})$ since

$$\sum_{j:x_j \geq y_j - 1} p_j \geq \sum_{j \neq i} w_i = \frac{2}{3} > 0.6;$$

in this case the common levels a_{-i} ensure on their own that the required majority threshold is reached, whatever happens on dimension i. In this example, the marginal preferences $\succsim_i$ are different from the S_i that are such that $x_i \, S_i \, y_i$ iff $x_i \geq y_i - 1$ as in example 6.7. This again illustrates (as already shown in section 6.2.5) that *ceteris paribus* reasoning can be insufficient, even with quite reasonable preferences. Note also that the present example is not covered by Fishburn's theory of noncompensatory preferences (Fishburn, 1976), because Fishburn's axioms imply that the marginal preferences and the S_i relations are identical; the concordance relations in the sense of definition 6.16 are thus significantly more general. ◇

We summarise the results given above in the following theorem that is based on Bouyssou and Pirlot (2005a, theorems 2 and 4). Note that this paper provides conditions, expressed in terms of the relation $\succsim$, that are equivalent to requiring that the traces on differences $\succsim_i^{**}$ have at most three equivalence classes.

Theorem 6.7 (Concordance relation)

1. *A relation $\succsim$ on X is a concordance relation iff it is reflexive, satisfies $RC12$ and its traces on differences $\succsim_i^{**}$ have at most three equivalence classes.*
2. *The relations S_i that intervene in the definition of concordance relations are semiorders iff $\succsim$ satisfies, in addition, $AC123$.*
3. *These relations are weak orders as soon as $\succsim$ satisfies $AC4$.*

6.5.2 Relationship with actual outranking methods

The above results and examples echo the practice of building concordance relations in the ELECTRE I and II methods (section 6.3.5.3 and, particularly, remark 6.3.4) or the TACTIC method (section 6.3.5.5). In the process of building an outranking relation *à la* ELECTRE, a priori preferences on each dimension are used to determine whether level x_i is not worse than level y_i and if this the case, dimension i enters the coalition of dimensions $S(x, y)$ that is in favour of saying that x is globally not worse than y. Such a process is likely to lead to relations S_i as defined by (6.91).

This is also to be connected to the respect of the dominance relation by preferences that satisfy $AC123$ (section 6.2.8). If we interpret S_i as the a priori preference of the client on dimension i, the concordance relation of an ELECTRE I method, is compatible with the dominance relation with respect to the S_i's. In addition, of course, the differences of preference on each dimension are weakly differentiated. The further introduction of vetoes may contribute towards refining the discrimination between differences of preference. Models that encompass the latter will not be discussed in detail here. Observe simply that outranking relations resulting from the application of the ELECTRE I or II methods, are representable in the subclass of model $(L1D6)$ in which the traces on differences of preferences $\succsim_i^{**}$ have at most five equivalence classes. Models with vetoes constitute a very peculiar subclass of that class, as emphasised in section 6.3.7. Greco et al. (2001a) have characterised a slightly restrictive version of concordance relations with vetoes (see the discussion section in Bouyssou and Pirlot, 2005a).

Recently, Bouyssou and Pirlot (2005c) have modified definition 6.16 of concordance relation in order to cover concordance relations with vetoes; these are called *concordance-discordance* relations.

Definition 6.17 (Concordance-discordance relation)
A reflexive relation $\succsim$ on X is a concordance-discordance relation if there are:

- *a complete binary relation S_i on each X_i,*
- *an asymmetric relation V_i included in S_i,*
- *a binary relation $\trianglerighteq$ between subsets of N, the union of which is N, which is monotonic with respect to inclusion, i.e. such that for all $A, B, C, D \subseteq N$,*

$$[A \trianglerighteq B, C \supseteq A, B \supseteq D, C \cup D = N] \Rightarrow C \trianglerighteq D,$$

such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow [S(x,y) \trianglerighteq S(y,x) \text{ and } V(y,x) = \varnothing],$$

where $S(x,y) = \{i \in N : x_i \; S_i \; y_i\}$ *and* $V(y,x) = \{i \in N : y_i \; V_i \; x_i\}$.

This definition clearly encompasses the models of preference involving a veto that were described in section 6.3.6. Concordance-discordance relations can be characterised in the same spirit as concordance relations (theorem 6.7). Besides being reflexive and satisfying $RC12$, concordance-discordance relations have traces on differences $\succsim_i^{**}$ that determine at most five classes of differences; if there are indeed five classes, all "positive" differences play the same role, while the largest "negative" differences trigger a veto. The axioms characterising concordance-discordance relations in Bouyssou and Pirlot (2005c) express essentially these characteristics.

The models based on weakly differentiated differences of preference described above, thus come quite close to a realistic description of the practice of building outranking relations *à la* ELECTRE.

6.5.2.1 Elicitation issues

It has just been suggested that the elicitation of the relations S_i could be rather direct; determining whether $x_i \; S_i \; y_i$ amounts to determining whether dimension i joins the coalition $S(x,y)$ that will be compared to $S(y,x)$ to decide whether x is at least as good as y. Eliciting the relation $\trianglerighteq$ on the coalitions of dimensions might be more delicate. In practice, this relation is usually assumed to be additively representable, which means that coalitions can be compared by comparing their weights; the weight of a coalition is computed as the sum of the weights of the dimensions that belong to the coalition. The weights of individual dimensions can be determined for instance by using Simos' "cards method" (see section 4.4.2, page 149) or by using one of the other methods discussed in section 4.4.

The existence of such an additive representation of $\trianglerighteq$ is certainly not guaranteed, in general, for concordance relations, even in the most constrained of the variants considered in theorem 6.7. We are not aware of any characterisation of concordance relations for which the relation $\trianglerighteq$ would be guaranteed to admit an additive representation [10]; it is likely that the axioms used in such a characterisation would be barely interpretable (such as those for additive representation of value functions in the case of a finite set of alternatives; see section 4.3, page 131). In the absence of an indication of the existence of an additive representation of $\trianglerighteq$, the monotonicity of $\trianglerighteq$ is the sole property that could be exploited to simplify the elicitation of the relation on the power set of the set of dimensions.

The models based on weakly differentiated differences of preferences are

[10] Note that the problem of characterising relations $\trianglerighteq$ that admit an additive representation is similar to the characterisation of comparative probabilities that admit a representation by means of an additive probability measure (on this—much studied—issue, see de Finetti, 1931; Fishburn, 1996; Kraft, Pratt, and Seidenberg, 1959 and a recent survey by Regoli, 2000).

an ideal framework both for

- understanding the characteristics of the preferences obtained through a large number of aggregation procedures based on pairwise comparisons of alternatives;

- proposing new procedures of this type and analysing them easily.

6.6 Models for valued preferences

So far in chapter 6, we have adopted a classical conjoint measurement point of view, that led us to describing a preference relation on a product set of alternatives by means of several types of models. For the reader's convenience, the rules that define $\succsim$ in various families of models studied in the previous subsections, are summarised in table 6.6; these families of models include:

- models based on marginal traces (models "(L)") analysed in subsection 6.2;

- models based on traces on differences (models "(D)") analysed in subsection 6.3;

- models based on both traces (models "$(L-D)$") analysed in subsection 6.4.

Models	Representation of $\succsim$
Model "(L)"	$F([u_i(x_i)],[u_i(y_i)]) \geq 0$
Model "(D)"	$G([p_i(x_i,y_i)]) \geq 0$
Model "$(L-D)$"	$G([\varphi_i(u_i(x_i),u_i(y_i))]) \geq 0$

Table 6.6: Representations of three models of preference.

A glance at table 6.6 shows that, in all cases, the preference relation is obtained by "cutting" a function (F or G) at a single level—that is chosen to be 0 only for convenience. It is tempting to ask whether the functions F or G could not be used for representing preference structures richer than just a binary relation $\succsim$ on the set of alternatives. The first thing that comes to mind is that the values of F or G could be used, for instance, for representing preference intensity or the credibility of the preference. This immediately points towards a number of models that have been evoked in earlier sections of this chapter, namely, the additive measurement of differences, ELECTRE III and PROMETHEE II.

In this section, we shall briefly show that these models can be represented as models (L), (D) or $(L-D)$, the main difference with the latter being in the way the values taken by G or F are related to a more complex preference structure. Since the complete characterisation of such models is still under development, we shall only indicate some general ideas of a theory of such structures in a conjoint measurement perspective.

6.6.1 The measurement of preference differences

In section 4.3.9, we introduced the model of measurement of preference differences which involves two relations $\succsim$ and $\succsim^*$; the former denotes the usual preference relation on the set of alternatives and the latter is a relation that compares pairs of alternatives; $(x,y) \succsim^* (z,w)$ states as "the difference of preference between x and y is at least as large as that between z and w". In the *additive measurement of differences of preferences*, there is a function u on the set of alternatives that can be decomposed as a sum of partial value functions $u_i(x_i)$, i.e. $u(x) = \sum_i u_i(x_i)$, and that satisfies the following two conditions:

$$x \succsim y \Leftrightarrow u(x) \geq u(y) \tag{6.94}$$
$$(x,y) \succsim^* (z,w) \Leftrightarrow u(x) - u(y) \geq u(z) - u(w) \tag{6.95}$$

Bearing the "(L)" and the "(D)" models in mind, we can consider two natural generalisations of the additive differences of preference models. Using the ideas of models on levels (models "(L)"), yields the following representation of the pair of relations $\succsim$, $\succsim^*$:

$$x \succsim y \Leftrightarrow F([u_i(x_i)],[u_i(y_i)]) \geq 0 \tag{6.96}$$
$$(x,y) \succsim^* (z,w) \Leftrightarrow F([u_i(x_i)],[u_i(y_i)]) \geq F([u_i(z_i)],[u_i(w_i)]). \tag{6.97}$$

We will refer to this model as the (L^*) model; it is obtained from the additive differences of preference model (6.94) and (6.95) through substituting the differences $u(x) - u(y)$ and $u(z) - u(w)$, respectively by $F([u_i(x_i)],[u_i(y_i)])$ and $F([u_i(z_i)],[u_i(w_i)])$.

Another avenue of generalisation is offered by substituting the differences $u(x) - u(y)$ and $u(z) - u(w)$, respectively by $G([p_i(x_i, y_i)])$ and $G([p_i(z_i, w_i)])$, yielding:

$$x \succsim y \Leftrightarrow G([p_i(x_i, y_i)]) \geq 0 \tag{6.98}$$
$$(x,y) \succsim^* (z,w) \Leftrightarrow G([p_i(x_i, y_i)]) \geq G([p_i(z_i, w_i)]). \tag{6.99}$$

We refer to the latter as the (D^*) model. In both the (L^*) and (D^*) models, in the absence of any additional specification, relation $\succsim$ has no special property, as was the case for model $(L0)$ defined on p. 263 or model $(D0)$ defined on p. 276. The status of relation $\succsim^*$ is different. Any valued relation on a set induces a weak order on the pairs of elements of this set: the pair (x,y) comes before the pair (z,w) in this weak order iff the value attached to (x,y) in the valued relation is larger than that attached to (z,w). Considering F (resp. G) as a valued relation on the set of alternatives X, we see that equations (6.97) (resp. (6.99) exactly define the weak order induced by F (resp. G) on X^2; hence, $\succsim^*$ is a weak order both in the (L^*) and (D^*) models.

It is not difficult to define a third family of models that we will call the $(L^* - D^*)$ model; it is obtained from (6.98) and (6.99) through decomposing $p_i(x_i, y_i)$ into $\varphi_i(u_i(x_i), u_i(y_i))$ for all alternatives x, y.

In the axiomatic analysis of such models, we would of course be interested in conditions that make:

- F nondecreasing (resp. increasing) in its first n arguments and nonincreasing (resp. decreasing) in its last n arguments;
- that $F([u_i(x_i)], [u_i(x_i)]) = 0$ and that

$$F([u_i(x_i)], [u_i(y_i)]) = -F([u_i(y_i)], [u_i(x_i)]);$$

- G nondecreasing (or increasing) in its n arguments;
- that $p_i(x_i, x_i) = 0$, $p_i(x_i, y_i) = -p_i(y_i, x_i)$ and that G is odd;
- $\varphi_i(u_i(x_i), u_i(y_i))$ nondecreasing (resp. increasing) in its first argument $u_i(x_i)$ and nonincreasing (resp. decreasing) in its second argument $u_i(y_i)$.

These questions correspond exactly to those we solved in the simpler case in which the analysis is only concerned with a single preference relation $\succsim$, without taking the measurement of global differences of preference into account. One can be confident that the axiomatic analysis of such models could be achieved using the tools presented in the previous sections, namely, various sorts of traces. Since such a study has not yet been completed, we do not develop these formal aspects further; instead, we simply suggest a framework that encompasses several valued relations showing up at some stage of practical multiple criteria methods.

6.6.2 Conjoint measurement models for ordinally valued or fuzzy preference relations

Assume that a value $S(x, y)$ is attached to each pair of alternatives $(x, y) \in X$. Assume further that the interpretation given to the values is *ordinal.* This means that another set of values S' is equivalent to S provided that, for all alternatives x, y, z, w, we have $S(x, y) \geq S(z, w)$ if and only if $S'(x, y) \geq S'(z, w)$; in other words, two ordinally valued relations S and S' are equivalent iff they induce the same (weak) order on the pairs of alternatives:

$$(x, y) \succsim^* (z, w) \Leftrightarrow S(x, y) \geq S(z, w).$$

Thus, an ordinally valued relation on X is equivalent to some relation $\succsim^*$ on the set of pairs of alternatives X^2. This relation is a weak order by construction.

Among several commonly used practical methods, which pass through the construction of valued relations as an intermediary stage on their way to the elicitation of a preference, are PROMETHEE II and ELECTRE III, not to speak of the additive measurement of differences of preferences briefly discussed in section 4.3.9. Let us interpret these valued relations in an ordinal way; we thus try to describe these relations as representations of a relation $\succsim^*$ comparing the differences of preference between pairs of alternatives.

Consider a weak order $\succsim^*$ on X^2; we can view it as a weak order on the set $Y = \prod_{i=1}^n Y_i$, where $Y_i = X_i \times X_i$. Proposition 6.8 (p. 271) applies to this situation; assuming that $\succsim^*$ satisfies the classical weak separability condition formulated for the Cartesian product Y, it tells us that $\succsim^*$ can be represented by means of a nondecreasing function U of n variables and n functions $p_i(x_i, y_i)$ defined on $X_i \times X_i$ such that:

$$(x, y) \succsim^* (z, w) \Leftrightarrow U([p_i(x_i, y_i)]) \geq U([p_i(z_i, w_i)]). \qquad (6.100)$$

If we view the p_i functions as valued relations on X_i (representing the marginal traces $\succsim_i^*$ of relation $\succsim^*$), we understand that the model given above provides a framework for studying the aggregation of n valued relations into one (ordinally) valued relation (this is to be put in relation with section 5.3, p. 192). Assuming that $U([p_i(x_i, x_i)]) = 0$ for all x, yields a relation $\succsim$ on X, defined as $x \succsim y$ if $(x, y) \succsim^* (x, x)$ with the numerical representation $x \succsim y$ iff $U([p_i(x_i, y_i)]) \geq 0$.

Under appropriate further conditions, the p_i's decompose as:

$$p_i(x_i, y_i) = \varphi_i(u_i(x_i), u_i(y_i)),$$

with φ_i nondecreasing in its first argument and nonincreasing in its second. When the p_i's have been substituted by the φ_i's, model (6.100) offers a framework for dealing with the aggregation of performance tables into a valued relation (see sections 5.4, p. 202 and 5.5, p. 224).

As particular cases of the latter model, we have:

- the additive preference differences model, in which,

$$\varphi_i(u_i(x_i), u_i(y_i)) = u_i(x_i) - u_i(y_i)$$

  and U is a sum;

- the relation $S(x, y)$ of PROMETHEE II (see p. 146), in which,

$$\varphi_i(u_i(x_i), u_i(y_i)) = w_i P_i(g_i(x) - g_i(y))$$

  (equations 4.17 and 4.18) and U is a sum;

- the relation $S(x, y)$ of ELECTRE III that also belongs to the model with complicated φ_i's and additive U.

Note that in all the cases stated above, U is additive. If appropriate conditions are fulfilled, the relation $\succsim^*$ admits a unique additive representation (see section 4.3.4, p. 131). The theory of additive value functions could then apply; in particular, the p_i's could be elicited as marginal value functions on the Cartesian products $X_i \times X_i$ (see section 4.3.7, p. 135). Having modelled things in this way, how do we help the client? The additive preference difference model is, in a sense, trivial. Since in this model, it occurs that the relation $\succsim$ (defined above using $\succsim^*$) is a weak order, deriving a recommendation to the client is relatively straightforward (see chapter 7 for a thorough discussion). When this is not the case, as with PROMETHEE II, an "exploitation procedure" has to be applied to $S(x, y)$ or $\succsim^*$ in order to derive a recommendation. Classically, the *net flow* method is used with PROMETHEE II, but it clearly makes usage of the numerical value of S, which is hardly arguable since we assumed that S is an ordinally valued relation (see the discussion in section 5.3.5, p. 200 and also section 5.3.3, p. 197).

There is an alternative way of looking at this issue. Consider that the intensity of the preference relation, $\succsim^*$, is the main preferential object, the one that is perceived by the client. Under this hypothesis, once this information has been elicited from the client, we have to transform it into "decisive information". For instance, in case a ranking of the alternatives is needed—and under the assumption that this ranking is not the client's preference $\succsim$—it would be advisable to study models of procedures transforming an (ordinally) valued relation into a ranking (i.e. a weak order), imposing rationality assumptions on such procedures. One could envisage, for example, not to use the numerical representation of the valued relation S obtained in the elicitation of $\succsim^*$ but to change it into an equivalent one that is needed in the process of building a final ranking (see also the discussion of exploitation procedures in chapter 7, section 7.4).

6.7 Reconciling Social Choice and Conjoint measurement

In chapters 5 and 6, we have described two quite contrasted approaches that we believe are relevant for understanding the relationships between a preference on a set of dimensions and the evaluations of these alternatives on a complete family of criteria. The approach developed in chapter 5 finds inspiration in Social Choice Theory and aims at characterising aggregation procedures. Conjoint measurement, presented in chapter 6, characterises families of preferences that can be represented in specific models.

Although conceptually different, the two approaches shed some light, from various angles, on the aggregation issue. In this section, we try to emphasise correspondences at various levels between the two approaches. The reader should be aware that "correspondence" does not mean "equivalence": in particular, bear in mind that chapter 5 characterises *procedures* while chapter 6 characterises *preference relations.*

1. The *decomposable model* (section 6.2.1) corresponds to the aggregation of weak orders into a weak order, a special case of the aggregation of binary relations into a binary relation (sections 4.2.2 and 5.2), encompassing, in particular, the Borda (section 4.2.2) and lexicographic (section 5.2.4) methods. A preference $\succsim$ fitting with the decomposable model can be described by $x \succsim y$ iff $u(x) = U([u_i(x_i)]) \geq u(y) = U([u_i(y_i)])$. A general procedure for aggregating weak orders into a weak order would start with a profile of relations $(S_1, \ldots, S_i, \ldots, S_n)$, that are weak orders, and output a weak order $\succsim$. If u_i is any representation of the weak order S_i in the input profile and if u represents $\succsim$, then we can see U as an aggregation procedure; using the decomposable model, this aggregation procedure is quite a general one but the resulting preference is at least weakly separable (definition 6.9).

2. *Models based on marginal traces (L models).* There are examples of aggregation procedures in section 5.2 yielding preferences that fit with the more general models based on marginal traces (section 6.2.7) and are not decomposable. From proposition 6.8, we know that they are not weak orders (otherwise they would fit with the decomposable model). The preferences resulting from a majority rule (sections 5.2.1 to 5.2.3) are examples of such procedures, but their description in the model based on marginal traces is not the most appropriate one; they fit better into the model based both on marginal traces *and* on traces on differences (see below, item 4).

3. *Models based on traces on differences (D models).* They are studied in section 6.3 and correspond to the aggregation of relations or valued relations into a relation (sections 5.2 and 5.3). A preference $\succsim$ that can be described within a D model satisfies $x \succsim y$ iff $G([p_i(x_i, y_i)]) \geq 0$. A general procedure aggregating relations into a relation is like the one described in item 1 except that the relations S_i are not necessarily weak orders. If p_i represents the relation S_i (i.e. $p_i(x_i, y_i) = 1$ if $x_i \ S_i \ y_i$ and 0 otherwise), then G can be viewed as an operator aggregating relations.

4. The *models based on marginal traces and on traces on differences ($L - D$ models)* are studied in section 6.4; the p_i function of the D models is further decomposed into $\varphi_i(u_i(x_i), u_i(y_i))$. The $L - D$ models correspond to the procedures that aggregate performance tables into a relation (section 5.4); the input of such a procedure is a profile of functions $(g_1, \ldots, g_i, \ldots, g_n)$. If we interpret x_i as $g_i(x)$ (the evaluation of alternative x on dimension i) and y_i as $g_i(y)$, the function $\varphi_i(u_i(x_i), u_i(y_i))$ appearing in the $L - D$ model can be seen as a way of coding the difference of preference between x_i and y_i on dimension i. Function G then aggregates these differences, determining whether the balance is positive; if so, then $x \succsim y$. All procedures studied in section 5.4 fit with this interpretation of models $L - D$, but they also fit with the more parsimonious decomposable model (item 1), which thus provides a more appropriate framework for them. The procedures studied in section 5.4 do not however illustrate all the ways of aggregating a performance table into a relation; there are reasonable ones that do not fit with the decomposable model. Consider, for instance, PROMETHEE II, which was described as a procedure for aggregating fuzzy relations into a relation in section 5.3.2.3. The fuzzy relations $S_i(x, y)$ on each dimension are built through recoding the differences $g_i(x) - g_i(y)$ using formula (5.3). One could thus also interpret PROMETHEE II as aggregating the performance table associated with the functions g_i into a relation. This leads to interpreting $S_i(x, y)$ as the $\varphi_i(u_i(x_i), u_i(y_i))$ function of a $L - D$ model; this model will not, in general, be a decomposable one.

   Note that the $L - D$ models also provide an adequate framework corresponding with the aggregation of linguistic performance tables into a relation (section 5.5). Remember that no particular structure is required on the components X_i of a conjoint measurement model. The "levels" of the linguistic evaluation scales can thus be represented by the sets X_i. An ordering of the levels of the linguistic scales, which is not pre-existent, is possibly induced by the preference of the client (it is the case in the $L-D$ models satisfying $RC1$ and $AC123$; the induced weak order is the marginal trace $\succsim_i^{\pm}$).

5. The valued version of the $L-D$ model, briefly described in section 6.6.1, can be put in correspondence with a family of aggregation procedures that has not been discussed, but is the "valued version" of the aggregation of a performance table into a relation (section 5.4). Take for instance the "weighted sum procedure" (section 5.4.4); in the resulting relation, x is ranked before y if the weighted sum $s_x(\mathbf{g})$ of the evaluations $\mathbf{g}(x)$ of x is larger than the weighted sum $s_y(\mathbf{g})$ of the evaluations of y. Instead, one could consider the procedure that would associate the difference $s_x(\mathbf{g})-s_y(\mathbf{g})$ to each pair (x, y), which can also be written as $\sum_{i\in N} w_i(g_i(x) - g_i(y))$ [11]. By doing this, we would have aggregated a performance table into a valued relation that can obviously be put in correspondence with the measurement of differences of preferences (section 4.3.9). It also corresponds to a particular case of the valued $L-D$ model, namely, the case in which the functions φ_i are differences and G is a sum of those differences. Similar things can be said about the "min" procedure (section 5.4.3) as well as about "leximin" and "leximax" (section 5.4.5). In all these cases, the corresponding valued $L-D$ model could be called "decomposable" since the value associated with the pair (x, y) is a difference of scores of the type $U([u_i(x_i)]) - U([u_i(y_i)])$. It is not difficult to find examples in which the $L-D$ model does not decompose into an algebraic difference. An appropriate "part" of the PROMETHEE II or of ELECTRE methods can be viewed as such examples. PROMETHEE II, for instance, can be described as associating a value $S(x, y)$ to each pair (x, y), namely a weighted sum of the $S_i(x, y))$ (formula (5.3)):

$$S(x,y) = \sum_{i\in N} S_i(x,y) = \sum_{i\in N} P_i(g_i(x) - g_i(y)); \qquad (6.101)$$

then a score [12] is computed that is the "net flow" of the valued relation S at each "node" x:

$$\Phi(x) = \sum_{y\in\mathcal{A}} S(x,y) - S(y,x). \qquad (6.102)$$

If we consider the intermediary step of computing the relation S from the performance table as a procedure *per se*, we see that we can analyse it in the framework of the $L-D$ models.

Going through the various models described in this chapter and overviewing their inter-relationships, prompts two further remarks that will respectively relink aggregation models looking back towards chapter 3 and forward towards chapter 6.

[11] Instead of associating the difference of the weighted sums to a pair, we could associate the difference of the Borda scores. This method has been characterised in Marchant (1998, 2000).

[12] The reader can verify that this score is the same as that in formula (5.4). It is not uninteresting to note that a procedure can sometimes be analysed in several different ways.

Looking back towards chapter 3. What is the "input" of the aggregation procedures considered in the present chapter? On which objects are the preferences represented in our conjoint measurement models defined? We did not discuss much how the data needed for describing the alternatives were prepared for use in all these procedures or models.

The preparation of the inputs obviously has a lot to do with chapter 3. In chapter 5, we considered various types of data as inputs of aggregation procedures, especially: a binary or a valued relation on each dimension; a performance table containing the evaluations of all alternatives on all dimensions. In chapter 3, we mainly examined how relational information can be transformed into numerical representations and, vice versa, how the evaluations of the alternatives on a dimension can be transformed into a relation comparing the alternatives from the point of view associated with this dimension. This could be described as *preference modelling* on a single attribute.

It is one of the aims of chapter 3 to provide ways of preparing the information on the alternatives for use in aggregation procedures. The circumstances of the decision aiding process (type of information available, culture of the client, type of recommendation required, etc.) may of course influence the type of aggregation procedure that will be chosen and, accordingly, the type of preparation of the input data. Comparing sections 4.2.2 and 5.3.2 illustrates the fact that using, for instance, the Borda procedure applied to weak orders or the variant of the Borda method applied to performance tables may depend on the type of raw data available but also on the preference modelling phase and the interpretation of the meaning of the data.

In conjoint measurement models, the perspective is apparently different since the global preference is presented as a primitive of the model. Any alternative, say x, is described as a vector $(x_1, \dots, x_n)$ of a Cartesian product. The level x_i is usually interpreted as the evaluation $g_i(x)$ of alternative x on dimension i (the scale of evaluation being possibly numeric, linguistic, ...).

A preliminary recoding of the "original" evaluation scale is not excluded by the conjoint measurement approach. In any case, the vector $(x_1, \dots, x_n)$ should offer a complete, unambiguous description of the alternative x (see p. 128); in particular, two alternatives associated with the same vector should be indistinguishable from the point of view of the preference. Recoding a linguistic descriptor $x_i \in X_i$ into a numerical descriptor or a numerical descriptor into another numerical descriptor can ease the elicitation of the preference. Consider for instance the problem of choosing a place to live. Let x_i denote the distance of house x from the centre of the city, which constitutes point of view i. The client's preference does not decrease or increase monotonically with x_i; the centre of the city is not an ideal place to live (too much traffic, noise, pollution, ...); being far from the centre is not good either (long journey to reach the office, ...); there might be an ideal distance Δ corresponding to residential suburbs and the preference on this viewpoint will decrease as the absolute value $|x_i - \Delta|$ of the difference between Δ and the distance to the centre of the city increases. Suppose that we have determined that the global preference of the client on the houses he could possibly buy and live in can be modelled within a model based on marginal traces, say $L3$, for which the marginal trace $\succsim_i^{\pm}$ is a weak order; we can choose a numerical representation of $\succsim_i^{\pm}$ for u_i. Recoding the "distance from the centre" attribute as $|x_i - \Delta|$ could ease the elicitation process. Indeed, knowing that the preference decreases when $|x_i - \Delta|$ increases provides us with a first approximation of a representation u_i of the marginal trace $\succsim_i^{\pm}$ since we know that $u_i(x_i')$ is a nonincreasing function of $x_i' = |x_i - \Delta|$; we just have to determine the intervals where u_i would possibly be constant as a function of x_i'.

In the same spirit, for D or $L - D$ models, it could be advisable to find a "pre-model" of the functions $p_i(x_i, y_i)$ or $\varphi_i(u_i(x_i), u_i(y_i))$. If a scale X_i is poorly structured (linguistic or ordinal), it may make sense to build a relation $S_i = (P_i, I_i)$ (that is not necessarily a weak order) telling us which level x_i is certainly not worse than level y_i. This is essentially what is done in formula (6.49) when determining that cost x_i is not worse than y_i as long as it does not exceed y_i by more than 100€. This introduces a constraint on the traces of differences $\succsim_i^*$ or $\succsim_i^{**}$ in models D or $L - D$ satisfying $RC1$ (hence for which $\succsim_i^*$ is a weak order) since pairs (x_i, y_i) such that $x_i \; S_i \; y_i$ should be ranked (by $\succsim_i^*$) before pairs (z_i, w_i) such that $Not[z_i \; S_i \; w_i]$. In other words, in such a case, $p_i(x_i, y_i)$ should be larger than $p_i(z_i, w_i)$, provided we choose a numerical representation of the weak order $\succsim_i^*$ for p_i. This again can ease the elicitation process.

Pointing forward to chapter 7. The recommendation to make to the client is not always a straightforward consequence of the output of the aggregation phase (it is even seldom the case as was announced in chapter 2 and is elaborated on in chapter 7). Here, we only present one blatant example for which a post-aggregation phase is needed. If we aggregate relations into a relation, say, by means of a majority method (or, more generally, in the language of conjoint measurement, if the preference fits with a D or $L - D$ model and distinguishes few classes of preference differences) it may occur that the output preference relation has circuits. It is of course arguable that a procedure that may lead to (or a model that admits) circuits is not appropriate for decision analysis. There is another way of dealing with such a drawback. The idea is to accept cycles and other "defects" in the preferences modelled and to exploit this information further in order to derive a recommendation. Examples of *exploitation procedures* for this case are presented in chapter 7, as well as considerations showing that an exploitation phase is needed for all procedures and models.

7

MAKING RECOMMENDATION

7.1 Introduction

7.1.1 Position of the problem

In chapters 4, 5 and 6, we presented various preference models for alternatives evaluated on several attributes/criteria. The presentation in these chapters emphasised the underlying logic of these models, their axiomatic analysis and their possible implementation. Two main types of preference models were envisaged:

- preference models based on *value functions* leading to a weak order on the set of alternatives,
- preference models in which incomparability and/or intransitivity may occur.

These preference models are tools built by the analyst in the course of the decision aiding study, the main phases of which were described in chapter 2. Having built one or several preference models does not mean that the analyst'work is over: this is only a step in the elaboration of a recommendation and its possible implementation in the decision process.

Going from a formal preference model to a recommendation requires many different tasks. Some of them are rather informal, involving, e.g., a good communication strategy with the actors in the decision process, the need for transparency in this process, a sound management of multiple stakeholders, etc. This chapter discusses the formal tasks that are involved in the elaboration of a recommendation. Sections 7.2, 7.3 and 7.4 will be devoted to the elaboration of a recommendation on the basis of the preference models analysed in the previous chapters. Section 7.5 will be devoted to the management of imprecision, uncertainty and inaccurate determination in order to reach *robust* conclusions.

It should be clear at this stage that using the well-structured preference models that are induced by value functions will make the elaboration of a recommendation will be much easier. This does not mean that such models are always adequate. We saw in chapters 5 and 6 that their construction often requires a delicate analysis contrary, e.g., to more ordinal preference models, e.g. majoritarian models as introduced in section 5.2.2 of chapter 5. Here, the analyst faces a difficult tradeoff between the ease of construction of the preference model and the ease of using it to derive a recommendation. The analysis of this tradeoff depends on the nature of the decision process.

7.1.2 What kind of recommendation?

The aim of this chapter is to study how an analyst can use formal preference models to tentatively build a *recommendation*. The nature of this recommendation that is looked for will therefore be of crucial importance in this phase of the decision aiding study. As should be apparent from section 2.3 of chapter 2, the central element here is the *problem statement* Π that has been agreed upon at the problem formulation stage of the decision aiding process.

Among the various problem statements presented in chapter 2, in this chapter our attention will be focused on the ones aiming at partitioning the set of alternatives. Depending on:

- whether or not the categories are ordered,
- whether or not the categories are predefined and,
- the number of categories,

several problem statements arise. For instance, the situation in which categories are neither ordered nor predefined, calls for the use of *clustering* techniques. Because our basic material in this chapter will be one or several relations comparing alternatives in terms of preference, we will mostly restrict our attention to problem statements involving ordered categories. This roughly leaves us with three main problem statements, i.e., the three "purposeful" problem statements introduced in chapter 2:

Choosing : at most two categories that are ordered and not predefined,

Ranking : ordered categories that are not predefined,

Sorting : ordered categories that are predefined.

The first two problem statements lead to a *relative* evaluation. They are concerned with the fact that an alternative is or not preferable to another, without taking a position on the "intrinsic desirability" of the alternatives that are compared, which would require the categories to be predefined. Alternative a may be found preferable to alternative b while a and b may both be rather poor. The third problem statement deals with *absolute* evaluation: it will lead to a judgement on the intrinsic desirability of the alternatives.

Before recalling the essential elements of these three problem statements, it is important to note that the distinction between absolute and relative evaluation is sometimes blurred in practice. The analyst having opted for a "choosing" problem statement might well be lead to start a new phase of the decision aiding study after realising that all alternatives that were considered are not likely to contribute much to the decision process. Similarly, if the analyst has succeeded in isolating a set of "desirable" alternatives, their relative evaluation will be necessary if only one of them can be implemented. In such cases, the problem statement is likely to evolve during the decision aiding process.

7.1.2.1 Choosing

The first problem statement, *choosing*, is quite familiar in Operational Research and in Economics. The analyst'task is formulated in such a way that he either tries

- to isolate, in the set A of potential alternatives, a subset A' that is likely to contain the most desirable alternatives in A given the information available or

- to propose a procedure that will operate such a selection.

Examples in which such a problem statement seems appropriate are not difficult to find:

- a recruiter wants to select a unique applicant,

- an engineer wants to select the best possible technical device,

- a patient wants to choose the best possible treatment among those offered in a hospital,

- a manager wants to optimise the supply policy of a factory,

- a consultant wants to screen a large number of possible sites to set up a new factory. Only the most promising ones will be subjected to detailed on-site studies.

In all these examples, the selection is to be made on the sole basis of the comparison of potential alternatives. In other words, the "best" alternatives are not defined with respect to external norms but with respect to a set of alternatives A; the evaluation is only *relative*. Therefore, it may occur that the subset A', while containing the most desirable alternatives within A, only contains poor ones.

7.1.2.2 Ranking

The second problem statement, *ranking*, is also familiar in Operational Research and Economics. The problem is formulated in such a way that the analyst tries

- to rank order the set of potential alternatives A according to their desirability or,

- to propose a procedure that will operate such a ranking.

It is not difficult to find examples in which this problem statement seems appropriate:

- a sports league (e.g., soccer or basketball) wants to rank order the teams at the end of the season,
- an academic programme has to select a number of applicants given the programme's size. A competitive exam is organised which leads to rank ordering the applicants according to an "average grade". Applicants are then selected in the decreasing order of their average grades until the size of the programme is reached,
- an R&D department has to finance a number of research projects subject to a budget constraint. Research projects are then rank ordered and financed till the budget constraint is binding.

The evaluation is performed, as in the preceding problem statement, on a *relative* basis: the top ranked alternatives are judged better than the others while nothing guarantees that they are "satisfactory". The ranking of the alternatives is not defined with respect to outside norms but with respect to the comparison of the alternatives in A among themselves.

Ideally we would like to be in a position to rank order the set A of alternatives from the best to the worst alternatives. Remember from chapter 3 that this amounts to defining a complete and transitive binary relation on A. As we will show below, this is not always an easy task. Therefore, some techniques do not insist on obtaining a *complete* relation leaving the possibility of incomparable alternatives (see the case study described in Bouyssou et al., 2000, ch. 9).

Remark 7.1.1
Our definition of the ranking problem statement does not prevent the relative position of two alternatives a and b from depending upon their comparison with other alternatives, e.g., c. Methods using ranking techniques allowing for such comparisons violate the famous independence condition introduced by Arrow (1963) (see section 5.2.1.2 in chapter 5). •

Remark 7.1.2
The distinction between this problem statement and the preceding one, choosing, is often subtle. Both are based on the comparison of alternatives amongst themselves. Intuitively, one would expect that the alternatives in set A' in a choosing problem statement should be ranked in the first equivalence class of the ranking within the ranking problem statement. This is misleading however since the ranking problem statement aims at providing much richer information than the choosing problem statement. In a location study, the elements in A' may be promising and worth a further detailed study. This does not mean that they appear as *equally* promising. On the contrary, A' may contain sites that are quite different, e.g., isolated sites and sites close to city or to a recreational facility. •

7.1.2.3 Sorting

The third problem statement, *sorting*, is designed to deal with absolute evaluation. The problem is here formulated in such a way that the analyst tries

- to partition the set of alternatives into several categories, the definition of these categories being intrinsic,
- to propose a procedure that will generate such a partition.

The essential distinctive characteristics of this problem statement therefore lie in the definition of the categories. Two main cases arise.

The definition of the categories may not refer to the desirability of the alternatives. Many problems that arise in pattern recognition, speech recognition or diagnosis are easily formulated in this way. We may, e.g., want to decide whether an image reveals the presence of roads, whether a certain sound is to be interpreted as "yes" or "no", whether a patient has a certain disease, etc. Such situations call for the use of *classification* techniques that are beyond the scope of the present volume. In those situations, a category is often defined with respect to one or several of its prototypical elements. Alternatives are then assigned to the categories according to their "proximity" to the prototypical elements.

In this chapter, we will exclusively be concerned with the case in which the absolute evaluation that sought involves the *desirability* of the alternatives, e.g., a credit manager may want to isolate "good" risks and "bad" risks, an academic programme may wish to enroll only "good" students, etc. A crucial problem here will lie in the definition of the categories, i.e., of the *norms* defining what is a "good" risk, what is a "good" student. Note that the traditional "classification" methods used, e.g., in machine learning or pattern recognition are not always well adapted to deal with the case of categories conveying information on the desirability of the alternatives (on this point, see Greco, Matarazzo, and Słowiński, 1999b, 2001c).

Remark 7.1.3

As already mentioned, the distinction between absolute and relative evaluation is often more subtle than presented above and it is often the case that absolute and relative considerations are mixed. For instance, in a choosing problem statement, the set A' may contain alternatives that are "obviously" very poor, leading the analyst not to recommend any alternatives in A, but to foster a reformulation of the problem that aims at enlarging the set of alternatives (for a formal study of selection procedures that may end up with an empty choice set, see Aizerman, 1985; Aizerman and Aleskerov, 1995). Similarly, in a ranking problem statement, it is often the case that alternatives are rank ordered using "average grades" as is customary in many academic programmes. Although these average grades have, most often, an ordinal meaning (see Bouyssou et al., 2000, ch. 3) some grades (e.g., the middle of the grading scale) may have a special meaning involving an element of absolute evaluation. •

Remark 7.1.4

A frequent misunderstanding is to confuse "absolute evaluation" with the measurement of desirability on a "cardinal" evaluation scale, the frequently mentioned example being the grades assigned to students. We saw in Bouyssou et al. (2000, ch. 3) that the case of grades is probably more complex than it appears at first sight. Let us simply observe that absolute evaluation can be conceived independently of the construction of any "cardinal" evaluation scale. Suppose that some alternatives are judged "satisfactory", which is an absolute judgement. The analyst, having built a preference relation on the set of alternatives, might be led to consider that all alternatives that are preferable to the satisfactory ones are also satisfactory. This is independent from any "cardinal" evaluation scale. •

An appropriate definition of the problem statement is an essential part of the formulation of the problem. It is crucial to know whether the desired evaluation is *absolute* as in the sorting problem statement or *relative* as in the choosing and ranking problem statements. The use of a problem statement involving absolute judgements calls for the modelling of *norms*. This is not necessary when looking for a relative evaluation. In such a case however, the best alternatives may not be desirable. The main difference between the choosing and ranking problem statements lies in the richness of the output.

We refer the reader to Bana e Costa (1996), Roy (1996) and Roy and Bouyssou (1993) for a thorough analysis of these three problem statements. The aim of this chapter is to describe a number of techniques that the analyst can use in order to build a recommendation in one of these three problem statements on the basis of the preference models that were introduced in chapters 5 and 6.

In section 7.2, we tackle the simple case in which the preference model takes the form of a value function. Section 7.3 is devoted to the case of making a recommendation on the basis of *several* value functions. Such a situation frequently arises when using Linear Programming-based assessment techniques of, e.g., an additive value function. In section 7.4 we deal with the more delicate case of deriving a recommendation on the basis of less well-structured preference models, e.g., those obtained by using ELECTRE, TACTIC or PROMETHEE, belonging to the family of the so-called outranking methods. A final section (7.5) will deal with the general problem of deriving robust conclusions.

7.2 Deriving a recommendation with a value function

Many of the preference models envisaged in chapter 6 are based on value functions. This means that the analyst has built a real-valued function V such that alternative a is judged at least as good as alternative b when $V(a) \geq V(b)$. In chapters 5 and 6, several techniques were presented to assess such a function. Many of them imply a particular functional form for V, e.g., an additive value function in the case of the comparison of multi-attributed alternatives.

The value function V induces a binary relation $\succsim$ on the set of alternatives A, interpreted as an "at least as good" relation letting, for all $a, b \in A$:

$$a \succsim b \Leftrightarrow V(a) \geq V(b). \tag{7.1}$$

Such a relation $\succsim$ is complete and transitive. It is therefore simple to use it to build a recommendation involving only a relative evaluation of the alternatives, the hard work involved in the assessment of a value function being rewarded at this stage of the decision aiding process.

In this section, we suppose that the value function V is only constrained by (7.1). This means that any increasing transformation of V would carry the same information as V (in the language of chapter 3, we suppose that V defines an ordinal scale).

Remark 7.2.1
It should be mentioned here that all that follows does not depend on the way value function V was obtained. The analysis in this section applies as soon as a preference model is, explicitly or implicitly, defined by (7.1). This is frequently the case in Operational Research, Economics or Statistics. •

7.2.1 Relative evaluation

If the analyst has opted for a problem statement involving only a relative evaluation of the alternatives, i.e., choosing or ranking, the well-behaved relation $\succsim$ between alternatives induced by V offers a direct way to build recommendations.

7.2.1.1 Choosing

In a choosing problem statement, it is natural to look for alternatives that would be "at least as good" as all other alternatives, i.e., to identify the set $G(A, \succsim)$ of *greatest alternatives* in A given the binary relation $\succsim$ defined by:

$$G(A, \succsim) = \{a \in A : a \succsim b, \forall b \in A\}.$$

Since $\succsim$ is complete and transitive, $G(A, \succsim)$ will, in general [1], be nonempty. Finding the alternatives in $G(A, \succsim)$ is equivalent to finding the solutions to the following optimisation problem:

$$\max_{a \in A} V(a).$$

Note that the set of solutions of this optimisation problem is unchanged if V is replaced by any value function satisfying (7.1), i.e., by any value function obtained from V applying to it an increasing transformation. Again, if a relative evaluation is sought, the only element that really matters is $\succsim$.

The set $G(A, \succsim)$ may contain more than one element. In this case, all alternatives in $G(A, \succsim)$ are indifferent and compare in the same way to all other alternatives. Therefore, the preference model defined by V offers no means of distinguishing between them. All alternatives in $G(A, \succsim)$ are strictly preferred to all alternatives outside $G(A, \succsim)$. The rejection of the latter therefore seems fully justified: all recommended alternatives are judged strictly better than all rejected alternatives.

[1] This is true when A is finite. The general case may be more tricky: while the relation $\geq$ on $\mathbb{R}$ is complete and transitive, $G(\geq, \mathbb{R})$ is clearly empty. The same is true with $\geq$ on the open $]0, 1[$ interval.

With a value function at hand, the analyst's task in a choosing problem statement is therefore clear: it consists in identifying the set $G(A, \succsim)$ of greatest elements in A. This is easy when A is "small" (i.e., finite and of limited cardinality). When A is not "small", the analyst will resort to one of the classical optimisation techniques developed in Operational Research, depending on the structure of A and the properties of V (bearing in mind that we may apply any increasing transformation to V).

Remark 7.2.2
As observed in chapters 5 and 6, the task of assessing a value function is not always easy. Several value functions may appear as a reasonable preference model, leading to several possible relations $\succsim$ an, thus, several sets $G(A, \succsim)$. The elaboration of the recommendation should this take into account. Such "robustness" considerations will be discussed in sections 7.3 and 7.5.

Remark 7.2.3
The set of maximal alternatives $M(A, \succsim)$ in A, given the binary relation $\succsim$, is defined by:

$$M(A, \succsim) = \{a \in A : Not[b \succ a], \forall b \in A\}.$$

where $\succ$ is the asymmetric part of $\succsim$. It is often presented as the central notion in a choosing problem statement. It is important to note that, when $\succsim$ is complete, we always have $G(A, \succsim) = M(A, \succsim)$.

When A is finite, it is easy to show that $M(A, \succsim)$ is nonempty when $\succsim$ has no circuit in its asymmetric part $\succ$. For finite sets, the absence of any circuit in $\succ$ is, in fact, a necessary and sufficient condition for $M(B, \succsim)$ to be nonempty for all nonempty sets $B \subseteq A$.

As soon as $\succ$ has no circuit, building a recommendation in a choosing problem statement does not raise conceptual difficulties, even when $\succsim$ cannot be be represented using a value function in the sense of (7.1). In fact, all what is in fact needed is that $\succ$ has no circuit. •

Remark 7.2.4
The determination of $G(A, \succsim)$ does not become easier if we suppose that V defines a scale that is stronger than an ordinal scale, e.g., because it allows to compare preference differences. This type of richer information may however ease the interpretation of $G(A, \succsim)$, giving an indication of the "distance" between selected and rejected alternatives. •

7.2.1.2 Ranking

Let us now envisage the case of a ranking problem statement. The hard work of building a value function also pays off here since the binary relation $\succsim$ induced on A by the value function V (or by any increasing transformation of V) rank orders the alternatives from the best to the worst, which is precisely what is wanted. Apart from the necessity of conducting a robustness analysis (see sections 7.3 and 7.5), no additional work is required.

Remark 7.2.5
As in the choosing problem statement, having a value function V that is more constrained than an ordinal scale may ease the interpretation of the ranking. If, e.g., the difference between $V(a)$ and $V(b)$ conveys information on the "preference difference" between a and b, this may be used to analyse the proximity of the various indifference classes of $\succsim$. It cannot be overemphasised that this is only legitimate if such information has been modelled in the definition of V (see section 6.6.1 of chapter 6). •

When a value function is defined on the set of alternatives, it is easy to derive a recommendation in the choosing or ranking problem statement. The main difficulty lies in the definition of the value function.

7.2.2 Absolute evaluation: Sorting

In both problem statements involving only a *relative evaluation* of alternatives, we have seen that the value function model provided an almost immediate way of deriving a recommendation. The situation is slightly more complex in a sorting problem statement, which calls for an *absolute evaluation*. It is thus necessary to define the "norms" that will give sense to such an evaluation, whereas the assessment of a value function usually does not require such an analysis.

The general problem of defining what is "good" and "bad" on the basis of a preference model is complex (and often involves the definition of a "neutral" point, see Rescher, 1969; von Wright, 1963, 1972). We will only envisage the, frequent, case in which the absolute evaluation that is sought takes the form of a sorting of the alternatives between several categories. We consider the case of r ordered categories $C^1, C^2, \ldots, C^r$, with C^1 containing the least desirable alternatives. The definition of each category involves the definition of norms. These norms usually take two distinct forms. They may be modelled as *prototypes* of alternatives belonging to a category or as *limiting profiles* indicating the limit of each category. Such norms may result from conventions, interaction with the decision maker or the analysis of past decisions. The definition of such norms is discussed in some detail in section 7.3.4.

Remark 7.2.6
These two ways of defining categories in a sorting problem statement are easily illustrated by considering the case of the evaluation of students in an academic programme. A "good" student may be defined using examples of past students in the programme. This would define the prototypes of the category of "good students". Alternatively, we could define, as is done in the French *baccalauréat*, an average grade above which, students are considered to be "good". E.g., in the French *baccalauréat* an average grade above 16 on a scale going from 0 to 20 implies that the exam is passed magna cum laude. •

7.2.2.1 Limiting profiles

When each category C^k is delimited by a limiting profile π^k, an alternative a should belong at least to the category C^k when it is preferred to π^k. It then becomes easy to use a value function to sort the alternatives: alternative $a \in A$ will belong to C^k if and only if $V(\pi^k) < V(a) < V(\pi^{k+1})$, where the unlikely cases of equality are dealt with conventionally, depending on the definition of the limiting profiles π^k. Note that the definition of a limiting profile implies that there is only one such profile per category. The main problem here lies in the definition of the limiting profiles π^k. We shall come back to this point in section 7.3.4.

7.2.2.2 Prototypes

The situation is more delicate when categories are defined via prototypes. Suppose that category C^k has been defined by a set P^k of prototypes. A first step in the analysis consists in checking whether this information is consistent with the value function V, i.e., if the prototypes defining a category C^k are all preferred to the prototypes defining the category $C^{k'}$ when $k > k'$.

When this consistency test fails, the analyst may wish to reconsider the definition of V or of the various prototypes. When the prototypes are consistent, we may easily associate to each category C^k, its lowest prototype L^k and its highest prototype H^k in terms of the value function V. If $V(a) \in [V(L^k); V(H^k)]$, alternative a should be assigned to the category C^k. If this simple rule allows to assign each alternative to a well-defined category, no further analysis is required. When this is not the case, i.e., when there are alternatives $a \in A$ such that $V(a)$ falls between two intervals, we may either try to refine the information defining the categories, e.g., try to ask for new prototypes, or apply a simple rule e.g., replacing the intervals $[V(L^k); V(H^k)]$ by the interval $[(V(H^{k-1})+V(L^k))/2; (V(H^k)+V(L^{k+1}))/2]$. Ideally we would need a similarity measure on the set of alternatives, that would allow to classify a as a member of C^k if a is close to one or several of the prototype alternatives defining C^k. The simple rule envisaged above amounts to using V as a very rough similarity measure since this amounts to saying that a is more similar to b than it is to c if $|V(a) - V(b)| < |V(a) - V(c)|$. It should however be noted that the assessment procedures of V envisaged above do not guarantee that such a measure is appropriate. In general, this would call for the modelling of "preference differences" between alternatives, e.g., using a model in which:

$$a \succsim b \Leftrightarrow V(a) \geq V(b) \text{ and} \tag{7.2}$$

$$(a,b) \succsim^* (c,d) \Leftrightarrow V(a) - V(b) \geq V(c) - V(d), \tag{7.3}$$

where $\succsim^*$ is a binary relation on A^2 such that $(a,b) \succsim^* (c,d)$ is interpreted as "the preference difference between a and b is at least as large as the preference difference between c and d". Preference models satisfying (7.2) and (7.3) were presented in section 4.3.9 of chapter 4 and section 6.6.1 of chapter 6. They are thoroughly analysed in Krantz et al. (1971, ch. 4). Again, a common mistake here is to use any V satisfying (7.2) as if it would automatically satisfy (7.3). Note that the fact that V in (7.2) defines an interval scale (e.g., if V is a value function obtained using an expected utility model, see Fishburn, 1970, ch. 8), does not guarantee that (7.3) holds.

When a value function is defined on the set of alternatives, the derivation of a recommendation in the sorting problem statement calls for the definition of the various categories. This may be done either by using limiting profiles or prototypical elements. In the former case, the derivation of a recommendation is straightforward. In the latter case, the situation is more complex, unless the value function has been defined to model preference differences between alternatives.

Remark 7.2.7
As already emphasised, the assessment of a value function V is often a difficult task. Therefore, the situation in which *several* functions V appear as reasonable preference models is not exceptional. The following section deals with this situation in the special case of additive value functions. In such a case, it is often possible to reach an *explicit* definition of the set of all acceptable value functions. Formal techniques can then be used to derive robust recommendations taking the fact that the preference model is not perfectly defined into account. Besides this special case, the possibility of an explicit definition of the set of all acceptable value functions is quite unlikely. The need for robust conclusions remains, however. Using a value function as if "small" differences were not significant is often helpful in this respect. The definition of a threshold allowing to separate significant from insignificant differences is not straightforward. The use of such a threshold should be considered as a technical device allowing the analyst to cope with the likely imprecision of the value function assessed. A more general view on robustness is presented in section 7.5. •

7.3 Deriving a recommendation with a set of value functions

In the preceding section, we envisaged the case in which the analyst has to build a recommendation on the basis of a *single* value function V (or, more precisely, of a set of value functions representing the same preference relation $\succsim$ in the sense of (7.1); these value functions can all be deduced from V using increasing transformations, see chapter 3). This amounts to separating the assessment phase of V from the elaboration phase of the recommendation. Motivated by the assessment of an additive value function via Linear Programming, in this section we envisage techniques for which this separation is not so clear in that the assessment procedure is no longer oriented towards the definition of a single value function V. The analyst will then have to build a recommendation on the basis of *several* value functions that cannot be deduced from one another using an increasing transformation. In this section, the analysis is closely related to section 7.5 in which the general problem of defining and deriving robust conclusions is tackled.

7.3.1 Motivation: Linear programming assessment of additive value functions

In order to motivate our study of techniques designed to derive a recommendation on the basis of several value functions, it is instructive to realise that such a situation is common when one tries to assess an additive value function using Linear Programming techniques. This will also allow us to illustrate a "learning by example" strategy for the assessment of a preference model based on regression.

7.3.1.1 Beyond standard sequences

In chapter 6 (see also Bouyssou et al., 2000; Krantz et al., 1971; Wakker, 1989), we have presented the theory underlying the additive value function model. Consider a set of alternatives A evaluated on a set of attributes. Let us denote by X_i the set of possible levels for attribute $i \in N$. This set of levels on attribute $i \in N$ may well contain levels that are not encountered within set A: this will be the case as soon as the analyst wishes to build a preference model that can evaluate alternatives that are outside A either because the recommendation will take the form of a methodology for evaluating alternatives or because set A may evolve during the study. If the set of attributes adequately describes the consequences of the potential alternatives, each alternative is modelled as an element of $X = X_1 \times X_2 \times \ldots \times X_n$. At this stage, the analyst may wish to build a preference model allowing to compare all the elements of X. An additive value function model takes the following form:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i), \tag{7.4}$$

where u_i is a real-valued function on X_i.

The reader is referred to section 6.1 of chapter 6 for a detailed analysis of the properties of this model. An analyst willing to make use of such a model should therefore assess the functions u_i, called marginal value functions. The results in chapter 6 give useful hints on how such a value function may be assessed, which exemplifies the interest of the axiomatic analysis of a preference model.

The main tool envisaged in chapter 6 to assess such a value function is the *standard sequence technique* that directly follows from theorems 6.1 and 6.2. Let us recall here that this technique leads to choose a *reference point* $(x_1^0, x_2^0, \ldots, x_n^0)$ in X and a *reference level* x_1^1 on attribute $i = 1$. A standard sequence on attribute $j = 2$ is a set of levels $x_2^1, x_2^2, \ldots, x_2^k \in X_2$ such that:

$$\begin{gathered}
(x_1^0, x_2^1, x_3^0 \ldots, x_n^0) \sim (x_1^1, x_2^0, x_3^0 \ldots, x_n^0), \\
(x_1^0, x_2^2, x_3^0 \ldots, x_n^0) \sim (x_1^1, x_2^1, x_3^0 \ldots, x_n^0), \\
(x_1^0, x_2^3, x_3^0 \ldots, x_n^0) \sim (x_1^1, x_2^2, x_3^0 \ldots, x_n^0), \\
\ldots \\
(x_1^0, x_2^k, x_3^0 \ldots, x_n^0) \sim (x_1^1, x_2^{k-1}, x_3^0 \ldots, x_n^0).
\end{gathered}$$

Using (7.4), we have:

$$\begin{aligned}
u_1(x_1^1) - u_1(x_1^0) &= u_2(x_2^1) - u_2(x_2^0), \\
&= u_2(x_2^2) - u_2(x_2^1), \\
&\ldots \\
&= u_2(x_2^k) - u_2(x_2^{k-1}).
\end{aligned}$$

We can always take $u_i(x_i^1) = 1$ and $u_i(x_i^0) = 0$, for all $i \in N$. Therefore the standard sequence on attribute 2 built above leads to $u_2(x_2^1) = 1$, $u_2(x_2^2) = 2$, ..., $u_2(x_2^k) = k$.

The logic of the assessment procedure is then to build a standard sequence on each of the attributes which is different from the reference attribute 1.

Once this is done, we can use the information collected on any attribute other than the reference attribute to build a standard sequence on the reference attribute. We may, e.g., use attribute 2 in order to define a standard sequence on the first attribute. This implies finding values $x_1^2, x_1^3, \dots, x_1^k \in X_2$ such that:

$$(x_1^2, x_2^0, x_3^0 \dots, x_n^0) \sim (x_1^1, x_2^1, x_3^0 \dots, x_n^0),$$
$$(x_1^3, x_2^0, x_3^0 \dots, x_n^0) \sim (x_1^2, x_2^1, x_3^0 \dots, x_n^0),$$
$$\dots$$
$$(x_1^k, x_2^0, x_3^0 \dots, x_n^0) \sim (x_1^{k-1}, x_2^1, x_3^0 \dots, x_n^0).$$

This will enable the assessment of a number of points on the graph of the marginal value function on the reference attribute 1.

The logic of the assessment procedure derived from theorems 6.1 and 6.2 is then to assess more and more points considering more finely grained standard sequences. A limiting process then unambiguously defines the functions u_i. The resulting u_i functions are unique up to the choice of the origin and that of a common unit. Indeed, the only arbitrary choices made above were the definition of the reference point $(x_1^0, x_2^0, \dots, x_n^0)$ (defining the origin) and the definition of the reference level x_1^1 (defining the common unit).

This assessment procedure results directly from theoretical considerations. It is worth noting here that this procedure:

- requires that the set X_i to be *rich* in that on each attribute $i \in N$ there must be a level x_i^r such that the "difference" between x_i^r and x_i^{r-1} exactly offsets the "difference" between x_1^1 and x_1^0 (this is often called a "solvability" assumption). Practically, this excludes using such an assessment procedure when some of the sets X_i are discrete,

- relies on *indifference* judgements which, a priori, are less firmly established than preference judgements,

- relies on judgements concerning fictitious alternatives which, a priori, are harder to conceive than judgements concerning real alternatives,

- is such that the various assessments are thoroughly intertwined and an imprecision on the assessment of x_2^1, for instance, will propagate to many assessed values.

It may thus be useful to resort to other kinds of assessment techniques.

The assessment of an additive value function using the standard sequence technique is technically and cognitively demanding. In particular, it is not appropriate when some attributes have an underlying discrete structure or when it appears to be difficult to compare unrealistic fictitious alternatives.

7.3.1.2 Linear Programming assessment techniques

In practice, it is not restrictive to suppose that the sets X_i are bounded so that there is a worst value $\underline{x}_i$ and a most preferable value $\overline{x}_i$. Using the uniqueness properties of the u_i, it may always be assumed that:

$$u_1(\underline{x}_1) = u_2(\underline{x}_2) = \dots u_n(\underline{x}_n) = 0 \text{ and} \tag{7.5}$$

$$\sum_{i=1}^{n} u_i(\overline{x}_i) = 1. \tag{7.6}$$

Two main cases arise (see figures 7.1 and 7.2):

- set X_i is discrete and we have $X_i = \{\underline{x}_i, x_i^1, x_i^2, \dots, x_i^{r_i}, \overline{x}_i\}$. We therefore have to assess $r_i + 1$ values of u_i,
- set X_i has a continuous structure. It is hardly restrictive in practice to assume that $X_i \subset \mathbb{R}$. Instead of assessing u_i we may opt for the assessment of a piecewise linear approximation of u_i partitioning the set X_i into $r_i + 1$ intervals and assuming that u_i is linear on each of these intervals. Note that the approximation of u_i can be made more precise by simply increasing the number of these intervals[2].

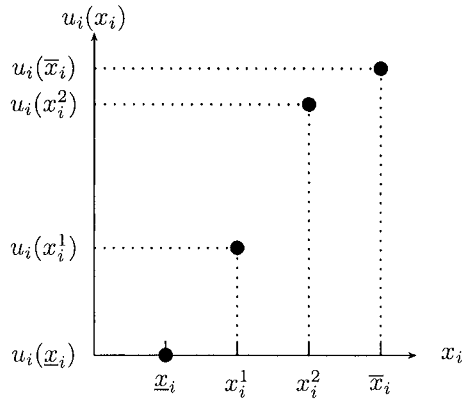


Figure 7.1: Value function when X_i is discrete.

With these conventions, the assessment of model (7.4) amounts to giving a value to $r_i + 1$ points on each function u_i subject to conditions (7.5-7.6). Taking these conditions into account, this gives a total of $\sum_{i=1}^{n}(r_i + 1) - 1$ unknowns. Any judgment of preference linking x and y translates into a *linear inequality* between these unknowns. Similarly any judgment of indifference linking x and y translates into a *linear equation* between these unknowns.

[2] It is, of course, not compulsory to partition the set X_i into intervals of equal length.

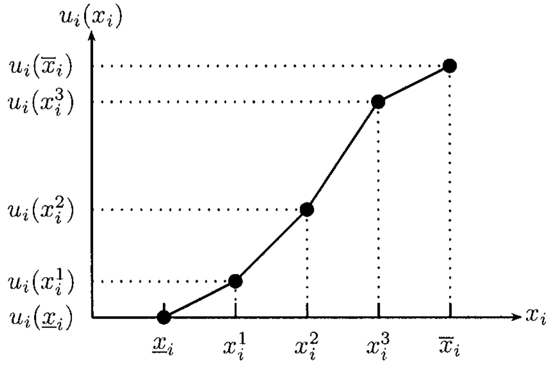


Figure 7.2: Value function when X_i is continuous.

Linear Programming (LP) offers a powerful tool for testing whether a system of linear constraints has solutions. Therefore, alternative assessment procedures may be conceived on the following basis (which is closely related to the theoretical analysis of model (7.4) in the finite case proposed by Scott (1964), see theorem 6.3):

- obtaining judgments in terms of preference or indifference linking several alternatives in X,
- converting these judgments into linear constraints,
- testing, using LP, whether this system of constraints has a solution.

If the system has no solution then one may envisage either to propose a solution that will be "as close as possible" to the information obtained, e.g., violating the minimum number of constraints or to suggest the reconsideration of certain judgements. If the system is compatible, one may explore the set of all solutions to this system, since they are all candidates for the establishment of model (7.4). These various techniques depend on

- the choice of the alternatives in X that are compared: they may be real or fictitious, they may differ on a different number of attributes,
- the way of dealing with the inconsistency of the system of linear constraints and to eventually propose that some judgments be reconsidered,
- the way of exploring the set of solutions of the system and to use this set as the basis for deriving a recommendation.

Linear programming offers a simple and versatile technique to assess an additive value function. All restrictions generating linear constraints on the parameters of the value function can easily be accommodated.

7.3.1.3 Variants of LP based assessment

The idea that the assessment of an additive value function can be obtained via the solution of a system of linear constraints has generated numerous studies (for a thorough overview, see Belton and Stewart, 2001). We will look at the two most useful and well-known techniques that have been developed within this framework here.

7.3.1.3.1 UTA (Jacquet-Lagrèze and Siskos, 1982) UTA (which is the French acronym for additive utility) is one of the oldest technique belonging to this family of assessment techniques. In UTA it is assumed that there is a subset $A_{Ref} \subset A$ of reference alternatives that the decision maker knows well either because he has experienced them or because they have received particular attention. The technique amounts to asking the decision maker to provide a weak order on A_{Ref}. Each preference or indifference relation contained into this weak order is then translated in a linear constraint:

- $x \sim y$ yields an equation $V(x) - V(y) = 0$ and
- $x \succ y$ yields an inequality $V(x) - V(y) > 0$,

where $V(x)$ and $V(y)$ can be expressed as a linear combination of the unknowns as noted earlier. Strict inequalities are then translated into nonstrict ones as is usual in Linear Programming, i.e., $V(x) - V(y) > 0$ becomes $V(x) - V(y) \geq \epsilon$ where $\epsilon > 0$ is a very small positive number that should be chosen according to the precision of the arithmetics used by the LP package.

The testing of the existence of a solution to the system of linear constraints is carried out via standard Goal Programming techniques adding appropriate deviation variables. In UTA, each equation $V(x) - V(y) = 0$ is translated into an equation $V(x) - V(y) + \sigma_x^+ - \sigma_x^- - \sigma_y^+ + \sigma_y^- = 0$, where $\sigma_x^+, \sigma_x^-, \sigma_y^+$ and σ_y^- are nonnegative deviation variables. Similarly each inequality $V(x) - V(y) \geq \epsilon$ is written as $V(x) - V(y) + \sigma_x^+ - \sigma_x^- - \sigma_y^+ + \sigma_y^- \geq \epsilon$. It is clear that there will be a solution to the original system of linear constraints if there is a solution of the LP in which all deviation variables are zero. This can easily be tested using the objective function

$$\text{Minimise } Z = \sum_{x \in A_{Ref}} \sigma_x^+ + \sigma_x^- \tag{7.7}$$

Two cases arise

1. If the optimal value of Z is 0, there is an additive value function that represents the preference information. It should be observed that, except in exceptional cases (e.g., if the preference information collected is identical to the preference information collected with the standard sequence technique) there is an infinite number of such value functions (that cannot be deduced from one another by an increasing transformation since we have normalised the value functions using (7.5) and (7.6)). The value function given as the "optimal" one using LP does not have a special status, since it is highly

dependent upon the arbitrary choice of the objective function and upon the implementation of the LP algorithm. Instead of minimising the sum of the deviation variables, we could have as well minimises the largest of these variables and still preserving linearity using standard tricks in LP. The whole polyhedron of feasible solutions of the original constraints corresponds to adequate additive value functions: we have an entire set $\mathcal{V}$ of additive value functions representing the information collected on the set of reference alternatives A_{Ref}.

Using standard techniques in LP, several functions in $\mathcal{V}$ may be obtained, e.g., the ones maximising or minimising, within $\mathcal{V}$, $u_i(\overline{x}_i)$ for each attribute (see Jacquet-Lagrèze and Siskos, 1982). The size of $\mathcal{V}$ is dependent on the choice of the alternatives in A_{Ref}.

2. If the optimal value of Z is strictly larger than 0, there is no additive value function representing the preference information available. Note that, in general, the value function derived from the optimal solution of the LP, is highly dependent upon the choice of the objective function and there is no guarantee that it leads to the minimum possible number of violations with respect to the information provided (this would require solving an Integer Linear Programme). This absence of a solution to the system of linear constraints might be due to several factors:

   - the piecewise linear approximation of the u_i for the "continuous" attributes could be too rough. It is easy to test whether an increase in the number of linear pieces on some of these attributes may lead to a nonempty set of additive value functions.
   - the information provided by the decision maker could be of poor quality. It might then be interesting to present one additive value function (e.g., one may present an average function after some post-optimality analysis) in the pictorial form of figures 7.1 and 7.2 to the decision maker and to let him react to this information either by modifying his initial judgments or even by letting him react directly to the shape of the value functions. This is the solution implemented in the well-known PREFCALC system (Jacquet-Lagrèze, 1990).
   - the preference information provided by the decision maker might be inconsistent with the conditions implied by an additive value function. The system should then help to locate these inconsistencies and allow the decision maker to reflect on them.

Even when a perfect restitution of the information provided by the decision maker is not possible, the "optimal" additive value that has been obtained may still be considered as an adequate model. Again, since the objective function introduced above is somewhat arbitrary, it is highly recommended to perform a post-optimality analysis, considering additive value functions that are "close" to the optimal solution. This can easily be done using alternative objective functions, e.g., maximising or minimising $u_i(\overline{x}_i)$, and introducing the linear constraint:

$$Z \leq Z^* + \rho,$$

where Z^* is the optimal value of the original linear programme and ρ is a small positive number. As in the above case, the result of the analysis is a set $\mathcal{V}$ of additive value functions defined by a set of linear constraints. A representative sample of additive value functions within $\mathcal{V}$ may be obtained as above.

It should be noted that many variants of UTA can be conceived building on the following comments. They include:

- the addition of monotonicity properties of the u_i with respect to the underlying continuous attributes,
- the addition of constraints on the shape of the marginal value functions u_i, e.g., requiring them to be concave, convex or S-shaped,
- the addition of constraints linked to a possible indication of preference intensity for the elements of A_{Ref} given by the decision maker, e.g., the difference between x and y is larger than the difference between z and w.

With UTA, the assessment of an additive value function rests on a weak order given by the decision maker on a subset A_{Ref} of A. This leads either to a whole set of additive functions $\mathcal{V}$ compatible with the information or to the conclusion that there is no compatible additive value function. In the first case, interaction with the decision maker can help reduce the size of $\mathcal{V}$. In the second case, it should be remembered that the objective function of the optimisation model used to test the compatibility of the constraints is arbitrary. No particular status should be given to the value function derived from the optimal solution of the LP. Interaction and post-optimality analysis should extensively be used to delineate an adequate set $\mathcal{V}$.

7.3.1.3.2 MACBETH (Bana e Costa and Vansnick, 1994) It is easy to see that equation (7.4) may equivalently be written as:

$$x \succsim y \Leftrightarrow \sum_{i=1}^{n} w_i v_i(x_i) \geq \sum_{i=1}^{n} w_i v_i(y_i), \tag{7.8}$$

where

$$v_1(\underline{x}_1) = v_2(\underline{x}_2) = \ldots v_n(\underline{x}_n) = 0, \tag{7.9}$$

$$v_1(\overline{x}_1) = v_2(\overline{x}_2) = \ldots v_n(\overline{x}_n) = 1 \text{ and} \tag{7.10}$$

$$\sum_{i=1}^{n} w_i = 1. \tag{7.11}$$

Categories	Description
$C1$	weak
$C2$	strong
$C3$	extreme

Table 7.1: Definition of categories in MACBETH.

With such an expression of an additive value function, it is tempting to break down the assessment into two distinct parts: a marginal value function v_i is assessed on each attribute and then, *scaling constants* w_i are assessed taking the shape of the value functions v_i as given. This is the path followed in MACBETH.

The assessment procedure of the v_i is conceived so as to avoid comparing alternatives differing on more than one attribute. The trick here is that MACBETH asks for judgments related to the *difference* between the desirability of alternatives. Value functions v_i are approximated on each attribute in a way similar to that used in UTA: each point on the function is assessed for discrete attributes, a piecewise linear approximation is used for continuous ones. MACBETH asks the decision maker to compare pairs of levels on each attribute. If no difference is sensed between these levels, they receive an identical marginal value level. If a difference is felt between x_i^k and x_i^r, MACBETH asks for a judgment qualifying the *strength* of this difference. The method and the associated software propose three different semantical categories (see table 7.1), with the possibility of using intermediate categories, e.g., between weak and strong (giving a total of six distinct categories, taking an hesitation between a weak difference and no difference at all into account).

This information is then converted into linear inequalities using the natural interpretation that if the "difference" between the levels x_i^k and x_i^r has been judged larger than the "difference" between $x_i^{k'}$ and $x_i^{r'}$, then it should follow that $v_i(x_i^k) - v_i(x_i^r) > v_i(x_i^{k'}) - v_i(x_i^{r'})$.

The software associated to MACBETH offers the possibility of comparing all pairs of levels on each attribute for a total of $(r_i + 1)r_i/2$ comparisons. Using standard Goal Programming techniques as in UTA, the test of the compatibility of a marginal value function with this information is performed via solving a Linear Programme.

If there is a marginal value function compatible with the information, a post-optimality analysis is performed and a "central" function is proposed to the decision maker who has the possibility of modifying it. If not, the results of the LP are exploited so as to propose modifications of the information that would make it consistent.

The assessment of the scaling constants w_i is made using similar principles.

The decision maker is asked to compare the following $(n+2)$ alternatives by pairs:

$$\begin{aligned}
&(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-1}, \underline{x}_n),\\
&(\overline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-1}, \underline{x}_n),\\
&(\underline{x}_1, \overline{x}_2, \dots, \underline{x}_{n-1}, \underline{x}_n),\\
&\qquad\dots\\
&(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{n-1}, \overline{x}_n) \text{ and}\\
&(\overline{x}_1, \overline{x}_2, \dots, \overline{x}_{n-1}, \overline{x}_n),
\end{aligned}$$

placing each pair in a category of difference. This information immediately translates into a set of linear constraints on the w_i. They are processed as before.

Remark 7.3.1
It should be noted that once the marginal value functions v_i are assessed, it is not necessary to use the levels $\underline{x}_i$ and $\overline{x}_i$ to assess the w_i since they may well lead to alternatives that are too unrealistic. The authors of MACBETH suggest to replace $\underline{x}_i$ by a "neutral" level which appears neither desirable nor undesirable and $\overline{x}_i$ by a desirable level that is judged satisfactory. Although this has an impact on the quality of the dialogue with the decision maker, this has no consequence on the underlying technique used to process information. •

As in UTA, many variants of the method are easy to conceive through the exploitation of various types linear restrictions on the v_i and/or on the w_i. The result of MACBETH, as in UTA is an entire set $\mathcal{V}$ of additive value functions (again, since these functions are normalised using (7.9), (7.10) and (7.11), they cannot be deduced from one another by an increasing transformation). The originality of MACBETH, breaking down the assessment phase into two different steps, often allows to reduce the size of $\mathcal{V}$ compared with UTA, e.g., it is often the case that interaction with the decision maker allows to specify a unique marginal value function on each attribute.

With UTA or MACBETH the result of the assessment procedure is a set $\mathcal{V}$ of additive value functions. The formulation of a recommendation should take the whole set of additive value functions in $\mathcal{V}$ into account.

This raises the problem of formulating a recommendation on the basis of a set $\mathcal{V}$ of value functions that cannot be deduced from one another by using an increasing transformation. What is sought here is a way of deriving "robust" recommendations in spite of the fact that our assessment techniques have not allowed to isolate a single value function.

Remark 7.3.2
As an alternative, we could also try to aggregate this information using the techniques envisaged in chapter 5. This is rarely appropriate however, since most of the techniques presented in chapter 5 require some form of interaction with the decision maker. Here, the existence of several value functions stems from the assessment procedure of the preference model and, therefore, occurs after the main phase of interaction with the decision maker. •

7.3.2 Choosing with a set of additive value functions

Suppose for example that, because we have assessed an additive value function with UTA or MACBETH, we have an entire set $\mathcal{V}$ of value functions compatible with the available information. Two main ways of exploiting this set $\mathcal{V}$ may be envisaged within a choosing problem statement.

The simplest way of using the set $\mathcal{V}$ is to consider that an alternative $a \in A$ should be included in the set $A' \subseteq A$ of recommended alternatives as soon as there is one additive value function in $\mathcal{V}$ such that using this function, a is at least as good as any other alternative in A. This is illustrated in section 7.3.5.

When the set $\mathcal{V}$ comes from Linear Programming-based assessment techniques, such a test is easily performed using LP, since the elements in $\mathcal{V}$ correspond to the solution of a set of linear constraints. In fact, we only have to test whether the system of inequalities $V(a) \geq V(b)$, for all $b \in A$, is consistent for some $V \in \mathcal{V}$. This requires solving a linear programme for each alternative $a \in A$.

The above mentioned technique is very cautious and is likely to lead to quite large choice sets. A more refined analysis is based on the "proportion" of value functions $V \in \mathcal{V}$ for which an alternative is optimal. The "more functions" V in $\mathcal{V}$ give a as the optimal solution, the more confident we are in the fact that a can be recommended (implementing such an approach, would require making the way of "counting" the functions in $\mathcal{V}$ precise and of making an hypothesis stating that all functions in $\mathcal{V}$ play a similar role). In general, such an analysis would require an enormous amount of computation (see Bana e Costa, 1986, 1988), even when $\mathcal{V}$ is defined by the solutions of a set of linear constraints. A possible solution would be to sample a few value functions within $\mathcal{V}$.

When $\mathcal{V}$ is defined by linear constraints, Jacquet-Lagrèze and Siskos (1982) suggested that a finite subset $\mathcal{V}'$ of $\mathcal{V}$ that is "representative" of the whole set $\mathcal{V}$ can be built considering on top of the "optimal" value function $2 \times n$ functions respectively obtained by maximising and minimising $u_i(\overline{x}_i)$ for each attribute. This set $\mathcal{V}'$ is easily obtained using LP.

When using techniques such as MACBETH, it may also occur that the shape of the single attribute value functions u_i are assessed with sufficient confidence but that the scaling constants w_i are only known through a number of inequalities. This case has been thoroughly studied in Bana e Costa (1986, 1988), Bana e Costa and Vincke (1995), Carrizosa, Conde, Fernandez, and Puerto (1995), Eum, Park, and Kim (2001), Fishburn (1964); Hazen (1986), Henggeler Antunes and Clímaco (1992), Kirkwood and Corner (1993), Kirkwood and Sarin (1985), Kmietowicz and Pearman (1981), Mateos, Jiménez, and Ríos-Insua (2003), Ríos-Insua (1990) and Ríos-Insua and French (1991).

LP offers simple means of deriving a recommendation in the choosing problem statement, on the basis of a entire set $\mathcal{V}$ of additive value functions. The more refined analysis based on the proportion of the value functions within $\mathcal{V}$ that put each alternative in the first place is computationally intensive, except under special circumstances.

7.3.3 Ranking with a set of additive value functions

Here again, the crudest way of using the information contained in $\mathcal{V}$ would be to build a partial preorder (i.e., a reflexive and transitive relation) T such that:

$$a\,T\,b \Leftrightarrow V(a) \geq V(b) \text{ for all } V \in \mathcal{V}, \tag{7.12}$$

i.e., letting a be ranked before b if it is so for every admissible function V in $\mathcal{V}$.

Testing if $a\,T\,b$ can easily be done using LP when $\mathcal{V}$ is defined via linear constraints. The use of such a technique is however limited since it implies solving $n(n-1)$ linear programmes when $|A| = n$. Furthermore, such a unanimity argument is likely to lead to a very poor recommendation: many alternatives will be incomparable when $\mathcal{V}$ is large.

When $|A|$ is too large to allow the use of the technique described above or when a richer result is sought, one may either try to restrict the domain $\mathcal{V}$ through emphasising interaction with the decision maker during the assessment phase, or work with the representative set of value functions $\mathcal{V}'$ introduced above. Quite interesting examples of such techniques can be found in Siskos (1982). This is illustrated in section 7.3.5.

The case in which the value functions in $\mathcal{V}$ only differ in the assessment of the scaling constants w_i has been thoroughly studied in the literature (see Bana e Costa, 1990; Bana e Costa and Vincke, 1995; Carrizosa et al., 1995; Kirkwood and Sarin, 1985, for thorough overviews).

Remark 7.3.3
Suppose that we have obtained a finite representative sample $\mathcal{V}'$ of $\mathcal{V}$. At this stage, it is tempting to consider that alternative a should be ranked higher than alternative b if there are more value functions in $\mathcal{V}'$ leading to $a \succsim b$ than to $b \succsim a$. This amounts to replacing a "unanimity" argument by a "majority" one.

Although quite simple, such a "majority" argument is quite deceptive. Indeed, as detailed in chapter 5, simple examples show that, in general, it does not lead to compare alternatives in a transitive way, which is a basic requirement of the ranking problem statement. •

Remark 7.3.4
It may be interesting to detail the links between the ranking technique evoked here and the choice technique detailed above. Suppose that $a \in A'$, i.e., that for some $V \in \mathcal{V}$, we have $V(a) \geq V(b)$, for all $b \in A$. This obviously implies that a cannot be strictly beaten by any other alternative using the relation T defined by (7.12). Hence, a must belong to set $M(A, T)$ of maximal elements in A for T. •

7.3.4 Sorting with a set of additive value functions

In the techniques envisaged so far we did not consider the definition of the "norms" that are necessary to sort alternatives. A useful technique, in the spirit of UTA, consists in assessing the additive value function using examples of alternatives belonging to each of the ordered categories, that we called prototypes in section 7.1. Such examples may come from past decisions or may be obtained from the decision maker as prototypical examples of each category. We may then try to infer limiting profiles and an additive value function on the basis of such information.

This amounts to assessing an additive value function V and thresholds s_k such that, for all prototypes π_j^k of category C^k we have $V(\pi_j^k) \in [s^k, s^{k+1}[$. This is the basis of the UTADIS technique (see Jacquet-Lagrèze, 1995; Zopounidis and Doumpos, 2000b, 2001, 2002) and its variants (Zopounidis and Doumpos, 2000a).

Basically UTADIS replaces the weak order on a subset of reference alternatives as used in UTA, by a number of prototype alternatives for each ordered category. Such a technique extends the traditional methods of discrimination used in Statistics considering the possibility of nonlinear value functions. As in Statistics, the assessment may use "cost of misclassification" which simply amounts to weighting the deviation variables in the LP used to assess the value function V appropriately. As in UTA, this leads to a whole set of possible additive value functions with associated limiting thresholds.

The way to make use of such information to build a recommendation has not been thoroughly studied in the literature. The most obvious way of doing so seems to be to consider a subset $\mathcal{V}'$ of representative additive value functions as suggested above. For each alternative $a \in A$, it is easy to compute a set of possible assignments using $\mathcal{V}'$. One may then, for example, use the frequency with which each alternative is assigned to a category to devise a recommendation. This is illustrated below.

7.3.5 Example: Thierry's choice

7.3.5.1 Thierry's choice (Bouyssou et al., 2000, ch. 6)

In order to illustrate the techniques described above, let us consider the example of the choice of a car presented and discussed at length in Bouyssou et al. (2000, ch. 6). Let us simply recall here the structure of this case.

Thierry, a Belgian engineering student, aged 21 (back in 1993 when the problem was formulated), is passionate about sports cars and driving (he has taken lessons in sports car driving and participates in races). Being a student, he cannot afford to buy either a new car or a luxury second hand sports car; so he decides to explore the middle range segment, 4 year old cars with powerful engines. Thierry intends to use the car in everyday life and occasionally in competitions. His strategy is first to select the make and type of the car on the basis of its characteristics, estimated costs and performances; then to look for such a car in second hand car sale advertisements.

The initial list of alternatives was selected taking an additional feature into account. Thierry lives in town and does not have a garage to park the car in at night. Consequently he does not want a car that would be too attractive to thieves. This explains why he discards cars like VW Golf GTI or Honda CRX. He thus limits his selection of alternatives to the 14 cars listed in table 7.2. As discussed in Bouyssou et al. (2000, ch. 6), Thierry's concerns are very particular. This leads him to select five viewpoints related to cost (criterion 1), performance of the engine (criteria 2 and 3) and safety (criteria 4 and 5). Evaluations of the cars on these

	Trademark and type	Abbreviation
1	Fiat Tipo 20 ie 16V	Tipo
2	Alfa 33 17 16V	Alfa
3	Nissan Sunny 20 GTI 16	Sunny
4	Mazda 323 GRSI	Mazda
5	Mitsubishi Colt GTI	Colt
6	Toyota Corolla GTI 16	Corolla
7	Honda Civic VTI 16	Civic
8	Opel Astra GSI 16	Astra
9	Ford Escort RS 2000	Escort
10	Renault 19 16S	R19
11	Peugeot 309 GTI 16V	P309-16
12	Peugeot 309 GTI	P309
13	Mitsubishi Galant GTI 16	Galant
14	Renault 21 20 turbo	R21t

Table 7.2: List of the cars selected as alternatives.

criteria were obtained from monthly journals specialised in the benchmarking of cars. The official quotation of second hand vehicles of various ages is also published in such journals following the process described in Bouyssou et al. (2000, ch. 6) and Perlias-Bouncke (1998).

The cost criterion evaluates, in €, the estimated expenses incurred by buying and using a car. Criterion 2 ("Accel" in table 7.3) encodes the time (in seconds) needed to cover a distance of one kilometre starting from standstill. The third criterion that Thierry took into consideration is linked to the pick up or suppleness of the engine in urban traffic; this dimension is considered important since Thierry also intends to use his car in normal traffic. The indicator selected to measure this dimension ("Pick up" in table 7.3) is the time (in seconds) needed to cover one kilometre when starting in fifth gear at 40 km/h. This dimension is not independent of the second criterion, since they are generally positively correlated (powerful engines generally lead to quick response times on both criteria); cars that are specially prepared for competition may however lack suppleness in low operation conditions, which is quite unpleasant in urban traffic. So, from the point of view of the user, i.e., in terms of preferences, criteria 2 and 3 reflect different requirements and are thus both necessary.

Criteria 4 and 5 (resp. "Brakes" and "Road-h" in table 7.3) were evaluated using ordinal evaluations reported in several magazines on a scale with levels "serious deficiency", "below average", "average", "above average", "exceptional". He considers 3 such indicators for criterion 4 and 4 for criterion 5. To obtain an overall indicator of braking quality (and also for road-holding), Thierry re-codes the ordinal levels with integers from 0 to 4 and takes the arithmetic mean of the 3 or 4 numbers; this results in the figures rounded to 2 decimals provided in the last two columns of table 7.3.

Note that the first 3 criteria have to be minimised while the last 2 must be maximised. It seems reasonable to consider that the scale of each of these criteria is continuous.

		Crit1 Cost	Crit2 Accel	Crit3 Pick up	Crit4 Brakes	Crit5 Road-h
1	Tipo	18 342	30.7	37.2	2.33	3.00
2	Alfa	15 335	30.2	41.6	2.00	2.50
3	Sunny	16 973	29.0	34.9	2.66	2.50
4	Mazda	15 460	30.4	35.8	1.66	1.50
5	Colt	15 131	29.7	35.6	1.66	1.75
6	Corolla	13 841	30.8	36.5	1.33	2.00
7	Civic	18 971	28.0	35.6	2.33	2.00
8	Astra	18 319	28.9	35.3	1.66	2.00
9	Escort	19 800	29.4	34.7	2.00	1.75
10	R19	16 966	30.0	37.7	2.33	3.25
11	P309-16	17 537	28.3	34.8	2.33	2.75
12	P309	15 980	29.6	35.3	2.33	2.75
13	Galant	17 219	30.2	36.9	1.66	1.25
14	R21t	21 334	28.9	36.7	2.00	2.25

Table 7.3: Data for the "choosing a car" problem.

7.3.5.2 Using UTA

Suppose that Thierry already has some knowledge about the 14 cars he wishes to evaluate, e.g., because he has driven some of them or because he has read Bouyssou et al. (2000, ch. 6). He feels able to express the following preferences:

$$\text{P309-16} \succ \text{Sunny} \succ \text{Galant} \succ \text{Escort} \succ \text{R21t}.$$

Let us assume that Thierry only wishes to build a preference model that will allow him to evaluate the 14 cars at hand. It is then reasonable to take, for each of the 5 criteria, $\underline{x}_i$ (resp. $\overline{x}_i$) as the worst (resp. best) value encountered in table 7.3 for this criterion. Let us also suppose that, as a first attempt, we wish to fit an additive value function model in which each of the marginal value function has two linear pieces to the information provided. For simplicity, the breakpoint $\hat{x}_i$ is taken as $(\underline{x}_i + \overline{x}_i)/2$, for all criteria. Table 7.4 summarises this. Using this information,

	$\underline{x}_i$	$\hat{x}_i$	$\overline{x}_i$
Cost	21 334	17 587.5	13 841
Accel.	30.8	29.4	28
Pick up	41.6	38.15	34.7
Brakes	1.33	1.995	2.66
Road-h.	1.25	2.25	3.25

Table 7.4: Additional data for the "choosing a car" problem.

we can express the utility of each of the 5 cars that were rank ordered, introducing

two decision variables per criteria, y_{i1} giving the level of utility of breakpoint $\hat{x}_i$ and y_{i2} that of the best value $\overline{x}_i$. Using linear interpolation as explained above, we obtain:
$V(\text{P309-16}) = 0.987y_{11} + 0.013y_{12} + 0.214y_{21} + 0.786y_{22} + 0.029y_{31} + 0.971y_{32} + 0.496y_{41} + 0.504y_{42} + 0.5y_{51} + 0.5y_{52}$,
$V(\text{Sunny}) = 0.836y_{11} + 0.164y_{12} + 0.714y_{21} + 0.286y_{22} + 0.058y_{31} + 0.942y_{32} + y_{42} + 0.75y_{51} + 0.25y_{52}$,
$V(\text{Galant}) = 0.902y_{11} + 0.098y_{12} + 0.429y_{21} + 0.638y_{31} + 0.362y_{32} + 0.496y_{41}$,
$V(\text{Escort}) = 0.409y_{11} + y_{21} + y_{32} + 0.992y_{41} + 0.008y_{42} + 0.5y_{51}$,
$V(\text{R21t}) = 0.643y_{21} + 0.357y_{22} + 0.58y_{31} + 0.42y_{32} + 0.992y_{41} + 0.008y_{42} + y_{51}$.

In order to test whether the information provided by Thierry is compatible with an additive value function, we may then solve the following Linear Programme:

$$\min Z = \sum_{i=1}^{5} \sigma_i^+ + \sigma_i^- \qquad (F)$$

subject to

$$\begin{cases} V(\text{P309-16}) - V(\text{Sunny}) + \sigma_1^+ - \sigma_1^- - \sigma_2^+ + \sigma_2^- \geq \epsilon, \\ V(\text{Sunny}) - V(\text{Galant}) + \sigma_2^+ - \sigma_2^- - \sigma_3^+ + \sigma_3^- \geq \epsilon, \\ V(\text{Galant}) - V(\text{Escort}) + \sigma_3^+ - \sigma_3^- - \sigma_4^+ + \sigma_4^- \geq \epsilon, \\ V(\text{Escort}) - V(\text{R21t}) + \sigma_4^+ - \sigma_4^- - \sigma_5^+ + \sigma_5^- \geq \epsilon, \\ y_{i2} - y_{i1} \geq 0, \text{ for } i = 1, 2, \ldots 5, \\ \sum_{i=1}^{5} y_{i2} = 1, \\ y_{ik} \geq 0, \text{ for } i = 1, 2, \ldots 5 \text{ and } k = 1, 2, \\ \sigma_i^+, \sigma_i^- \geq 0, \text{ for } i = 1, 2, \ldots 5, \end{cases} \qquad (C)$$

where the values $V(\text{Cars})$ are as given above and ϵ is a small positive number, e.g., 0.01.

Using a standard LP package, the reader will easily check that the optimal value of Z is 0, so that there is an additive value function compatible with the available information.

It is worth recalling that the optimal values of the variables y_{ij} have no special status since the objective function (F) is arbitrary: we could have decided to minimise the largest of the deviation variables instead of minimising their sum.

7.3.5.3 Choosing

Since the set of alternatives in this example is small, we can test whether it is possible to obtain any alternative as the most preferred one, given the information obtained. This requires to express the value for each of the 14 alternatives as a linear function of the y_{ij}. For each of these 14 alternatives, we add to the constraints (C) new ones expressing that the alternative under consideration is preferred or indifferent to all others. We then test, using LP, if the resulting system of constraints is compatible. If the answer is positive this means that, given the available information, the alternative under consideration appears as a potential choice. Table 7.5 shows that the information available is compatible with the choice of 6 among the 14 possible cars.

This relatively disappointing result shows the importance of interaction before directly using the results obtained with UTA and/or of a more sophisticated analysis exploiting the "frequency" with which each alternative appears as at least as good as all others. The analysis below will allow us to draw more conclusions on this point.

	Abbrev.	Can be chosen
1	Tipo	No
2	Alfa	Yes
3	Sunny	No
4	Mazda	No
5	Colt	Yes
6	Corolla	Yes
7	Civic	No
8	Astra	No
9	Escort	No
10	R19	Yes
11	P309-16	Yes
12	P309	Yes
13	Galant	No
14	R21t	No

Table 7.5: Potentially optimal alternatives.

7.3.5.4 Ranking

It would be very time consuming to test for each pair of alternatives whether all the value functions in $\mathcal{V}$ rank the elements of this pair in the same way. We instead use a subset $\mathcal{V}'$ of $\mathcal{V}$ consisting of all the functions obtained minimising and maximising the values y_{i2} for all criteria as well as the "optimal" additive value function. This gives $1 + 2 \times 5 = 11$ rankings in total. They are summarised in table 7.6. Although taking the intersection of these 11 rankings would result in a very poor relation, compared to table 7.5, table 7.6 reveals that P309-16 seems to be a very good alternative whatever the value function chosen, with alternatives R19, P309 and Sunny as close contenders. Clearly alternatives Mazda, Escort, Galant and R21t are quite poor. Although the choice of Alfa, Colt and Corolla is compatible with the information available, table 7.6 leads to believe that their choice is rather unfrequent with the set $\mathcal{V}$.

7.3.5.5 Sorting

Let us suppose that, instead of a ranking, Thierry is simply able to divide the reference set into "good" (P309-16 and Sunny), "acceptable" (Galant, Escort) and "bad" cars (R21t), therefore creating three ordered categories. We may then

	Abbrev.	Opt.	min y_{12}	max y_{12}	min y_{22}	max y_{22}	min y_{32}	max y_{32}	min y_{42}	max y_{42}	min y_{52}	max y_{52}
1	Tipo	6	2	11	6	14	7	11	6	6	13	2
2	Alfa	5	6	4	5	8	3	14	5	5	14	6
3	Sunny	2	5	2	3	3	2	3	2	2	3	5
4	Mazda	11	8	8	11	12	11	7	11	9	7	8
5	Colt	9	7	7	9	6	9	6	9	10	6	7
6	Corolla	7	9	6	7	13	6	8	7	14	8	9
7	Civic	10	12	12	10	2	10	2	10	7	2	12
8	Astra	8	11	10	8	4	8	4	8	12	4	11
9	Escort	13	13	13	13	10	13	12	13	11	10	13
10	R19	4	1	3	4	7	3	10	4	3	12	1
11	P309-16	1	3	1	1	1	1	1	1	1	1	3
12	P309	3	4	4	2	5	3	5	3	4	5	4
13	Galant	12	10	9	12	9	12	9	12	8	9	10
14	R21t	14	14	14	14	11	14	13	14	13	11	14

Table 7.6: Ranks of alternatives using $\mathcal{V}'$.

exploit this information in the spirit of the UTADIS method mentioned earlier. This amounts to solving the following linear programme:

$$\min Z = \sum_{i=1}^{5} \sigma_i + \sigma'_3 + \sigma'_4 \qquad (F')$$

subject to

$$\left\{\begin{array}{l} V(\text{P309-16}) + \sigma_1 \geq s_1 + \epsilon, \\ V(\text{Sunny}) + \sigma_2 \geq s_1 + \epsilon, \\ V(\text{Galant}) - \sigma_3 \leq s_1, \\ V(\text{Galant}) + \sigma'_3 \geq s_2 + \epsilon, \\ V(\text{Escort}) - \sigma_4 \leq s_1, \\ V(\text{Escort}) + \sigma'_4 \geq s_2 + \epsilon, \\ V(\text{R21t}) - \sigma_5 \leq s_2, \\ y_{i2} - y_{i1} \geq 0, \text{ for } i = 1, 2, \ldots 5, \\ \sum_{i=1}^{5} y_{i2} = 1, \\ y_{ik} \geq 0, \text{ for } i = 1, 2, \ldots 5 \text{ and } k = 1, 2, \\ \sigma_i \geq 0, \text{ for } i = 1, 2, \ldots 5, \sigma'_3, \sigma'_4 \geq 0, \\ s_1 \geq s_2 \geq 0, \end{array}\right. \qquad (C')$$

where s_1 and s_2 are the thresholds used to separate the three categories and ϵ is a small positive number, e.g., 0.01. In view of the results already obtained, it should not be a surprise that the optimal value of this LP is 0.

It is interesting to test what the constraints imply for the assignment of the $14 - 5 = 9$ cars that are not in the reference set using LP. This is summarised in table 7.7. In this example, the assignment of two cars not in the reference set (Colt and P309) is constrained by (C').

	Abbrev.	Bad	Acceptable	Good
1	Tipo	YES	YES	YES
2	Alfa	YES	YES	YES
4	Mazda	YES	YES	YES
5	Colt	NO	YES	YES
6	Corolla	YES	YES	YES
7	Civic	YES	YES	YES
8	Astra	YES	YES	YES
10	R19	YES	YES	YES
12	P309	NO	YES	YES

Table 7.7: Sorting the alternatives not in the reference set.

7.4 Deriving a recommendation with other preference models

As argued in chapters 5 and 6, using a value function is not always appropriate to adequately model preferences. Several extensions of this central model were proposed in these chapters. In most of them, we have seen that the "more ordinal" aggregation at work could well lead to preference structures that are not transitive and may include incomparability. As stressed in chapter 4, this additional flexibility at the level of preference modelling may ease the analyst's work and the acceptation of the model. It nevertheless raises difficult problems when it comes to establishing a recommendation. The aim of this section is to briefly envisage a number of techniques that can be used for such a purpose.

7.4.1 The extent of the problem

Suppose that you have built a preference relation on a set of alternatives using one of the techniques presented in chapters 5 and 6 that does not guarantee the transitivity or the completeness of the result. This does not necessarily mean that *any* preference structure can be obtained with such a method (e.g., only certain types of intransitive or incomplete relations could occur). Below, we prove that for a number of well known techniques, this is unfortunately true, thereby showing the difficulty of building a recommendation on such a basis.

7.4.1.1 Simple majority

Consider simple majority, i.e., the simplest "ordinal" technique for comparing alternatives as introduced in section 5.2.1 of chapter 5. On each criterion, we suppose that alternatives can be compared using a weak order. Simple majority amounts to declaring that:

$$x \succ y \Leftrightarrow |P(x,y)| > |P(y,x)| \text{ and}$$
$$x \sim y \Leftrightarrow |P(x,y)| = |P(y,x)|,$$

where $P(x, y)$ denotes the set of criteria on which x is preferred to y. Clearly, a relation $\succsim$ obtained in such a way is always complete.

Let T be any complete binary relation on a finite set of alternatives A. Besides completeness, no hypothesis is made on T; it may be the most intransitive relation you can think of, with circuits of any length in its asymmetric part. The surprising and disturbing fact, proved by McGarvey (1953), is that it is always possible to see T as the result of a simple majority aggregation.

The proof of this result is simple and instructive. Take any complete relation T on the finite set A. Consider any two alternatives a, b in A and label the other alternatives in A arbitrarily $x_1, x_2, \ldots, x_{k-2}$. Considering only two criteria for which, using obvious notations:

$$\begin{aligned} a > b > x_1 > x_2 > x_3 > \ldots > x_{k-2}, \\ x_{k-2} > x_{k-3} > \ldots > x_2 > x_1 > a > b, \end{aligned} \tag{7.13}$$

we have $|P(a,b)| = 2$, $|P(b,a)| = 0$, $|P(a,x)| = |P(x,a)| = 1$, $|P(b,x)| = |P(x,b)| = 1$ and $|P(x,y)| = |P(x,y)| = 1$, $\forall x, y \in A \setminus \{a, b\}$.

Similarly considering two criteria such that:

$$\begin{aligned} a > b > x_1 > x_2 > x_3 > \ldots x_{k-2}, \\ x_{k-2} > x_{k-3} > \ldots > x_2 > x_1 > b > a, \end{aligned} \tag{7.14}$$

we obtain $|P(a,b)| = |P(b,a)| = 1$, $|P(a,x)| = |P(x,a)| = 1$, $|P(b,x)| = |P(x,b)| = 1$ and $|P(x,y)| = |P(x,y)| = 1$, $\forall x, y \in A \setminus \{a, b\}$.

Now consider all $k(k-1)/2$ distinct ordered pairs in A. If $a\ T\ b$ and $Not[\,b\ T\ a\,]$, we introduce two criteria satisfying (7.13). If $b\ T\ a$ and $Not[\,a\ T\ b\,]$, we introduce two criteria satisfying (7.13) interchanging the roles of a and b. Otherwise, since T is complete, we have $a\ T\ b$ and $b\ T\ a$. We then introduce two criteria satisfying (7.14). Using simple majority on such $k(k-1)/2$ criteria will then yields the relation T.

Remark 7.4.1

The algorithm described above amounts to considering $k(k-1)/2$ criteria to obtain the complete relation T. In many cases, a much lower number of criteria can be used. The determination of the minimal number of criteria for the result to hold raises difficult combinatorial questions (see Stearns, 1959). •

With simple majority, *any* complete relation on a finite set of alternatives may be obtained. Therefore, when devising a procedure designed to build a recommendation on the basis of a simple majority aggregation, this procedure has to deal with *any* complete relation.

7.4.1.2 ELECTRE I (Roy, 1968)

As we saw in section 5.2.3.5 of chapter 5, ELECTRE I, leads to building a relation S on a finite set of alternatives evaluated on a set N of criteria. For each criterion $i \in N$, ELECTRE I uses the following ingredients:

- a weak order S_i on X_i,

- a positive weight w_i,
- a binary relation V_i on X_i included in the asymmetric part of S_i.

Defining for all $x, y \in A$, $S(x,y) = \{i \in N : x_i \; S_i \; y_i\}$, i.e., the set of criteria for which x is "at least as good as" y, we have in ELECTRE I:

$$a \; S \; b \Leftrightarrow \begin{cases} \sum_{i \in S(a,b)} w_i \geq s \text{ and} \\ Not[\, b_i \; V_i \; a_i \,], \text{ for all } i \in N \end{cases} \tag{7.15}$$

where $s \in [1/2; 1]$ is the concordance threshold.

It is easy to build examples in which S is incomplete and intransitive. More is in fact true since it is possible to show (see Bouyssou, 1996) that *any* reflexive relation on a finite set of alternatives may be obtained with ELECTRE I. This is easily shown by considering a situation in which:

- there is a criterion on which all alternatives are indifferent and to which a large weight is assigned,
- for each ordered pair (a, b) of alternatives such that $Not[\, a \; S \; b \,]$ there is a criterion to which little weight is assigned and on which we have $b \; V_i \; a$.

In fact, the patient reader will easily prove that a similar conclusion still holds, with a more complex construction, when all relations V_i are assumed to be empty (see Bouyssou, 1996). Therefore the situation is even worse with ELECTRE I than with simple majority: any reflexive relation can occur!

7.4.1.3 ELECTRE III (Roy, 1978)

In Bouyssou (1996) it is shown that the situation is not simpler with aggregation methods such as ELECTRE III (Roy, 1978) or PROMETHEE (Brans, Mareschal, and Vincke, 1984; Brans and Vincke, 1985) leading to a valued preference relation. With ELECTRE III, any reflexive (i.e., such that $R(a,a) = 1$, for all $a \in A$) valued relation on a finite set may be obtained. The situation is slightly more complex with PROMETHEE. It is nevertheless true that if P is any irreflexive (i.e., such that $P(a,a) = 0$, for all $a \in A$) valued relation on a finite set A, then, for some $\lambda \in [0; 1]$ it is possible to obtain the valued relation $[\lambda P]$ (defined by letting $[\lambda P](a,b) = \lambda P(a,b)$, for all $a, b \in A$) as the result of PROMETHEE.

For many aggregation methods that does not imply transitivity or completeness, any preference structure can, in fact, be obtained. Techniques designed to build recommendations should therefore be able to deal with *any* such structure. They are therefore quite unlikely to give satisfactory results in all cases.

The difficulty of building adequate procedures dealing with all kinds of incomplete and/or intransitive relations is illustrated in the next section.

7.4.2 How things may go wrong: examples

Many techniques for building recommendations on the basis of a non-necessarily transitive or complete binary relation have been proposed in the literature on MCDM. Most of them were justified on an ad hoc basis. In view of the results in the preceding section, it should be expected that the intuition supporting these techniques might not work appropriately in all cases. We illustrate this crucial point using two examples.

Example 7.1 (Choice procedures and dominated alternatives)
Consider a set of alternatives $A = \{a, b, c, d\}$ evaluated on three criteria. Suppose that, on each criterion, alternatives are weakly ordered by a binary relation S_i. Suppose that the preference on each criterion are such that, using an obvious notation for weak orders:

$$a\ P_1\ b\ P_1\ c\ P_1\ d,$$
$$c\ P_2\ d\ P_2\ a\ P_2\ b,$$
$$d\ P_3\ a\ P_3\ b\ P_3\ c,$$

where P_i denotes the asymmetric part of S_i.

Alternative b is *strongly dominated* by alternative a (a is strictly preferred to b on all criteria). Intuitively, this gives a decisive argument not to include b in the set of recommended alternatives.

Suppose then that the above information is aggregated into a binary relation S using simple majority. It is not difficult to see that S is such that (see figure 7.3):

$$a\ P\ b, a\ P\ c,$$
$$b\ P\ c,$$
$$c\ P\ d,$$
$$d\ P\ a, d\ P\ b,$$

where P denotes the asymmetric part of S. Observe that the same result is obtained with ELECTRE I using equal weights, a concordance threshold $s \in [1/2; 2/3[$ and no veto. It is obvious that S is not well suited to select a subset of alternatives since its asymmetric part P contains a circuit involving all alternatives ($a\ P\ b, b\ P\ c, c\ P\ d, d\ P\ a$). The simplest way to get rid of such a circuit is

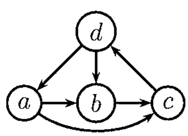


Figure 7.3: Majority relation in example 7.1.

to consider that all alternatives included in a circuit should be considered "equivalent". This can be done by considering the *transitive closure* of the relation, i.e., the smallest transitive relation containing it. But using the transitive closure of S would then lead to consider that all alternatives are equivalent and, hence, to propose the whole set A as the set of recommended alternatives. This does

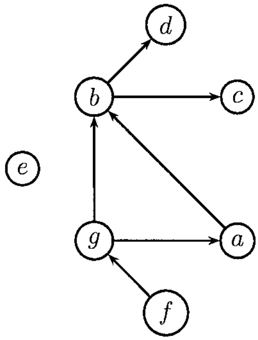


Figure 7.4: Relation P in example 7.2. The relation P is weakly complete, all non depicted arrows point downwards.

not appear to be sound since we have observed that there are quite compelling arguments showing that b should *not* be recommended. It should be noted that, the selection procedure of ELECTRE I (Roy, 1968), introduced below, would also lead to recommending the whole set A for this example.

This example also illustrates that separating the phase of construction of S from that of the construction of a recommendation may be deceptive. In our example, the fact that b is strongly dominated is only apparent considering the original information and not just the relation S. ◇

Example 7.2 (Ranking procedures and monotonicity)
Let $A = \{a, b, c, d, e, f, g\}$. Using the results in the previous section, we know that with simple majority and ELECTRE I, we might end up with a complete binary relation S such that (see figure 7.4):

$$\begin{gathered} a \; P \; b, a \; P \; f \\ b \; P \; c, b \; P \; d, b \; P \; e, b \; P \; f \\ c \; P \; a, c \; P \; e, c \; P \; f, c \; P \; g \\ d \; P \; a, d \; P \; c, d \; P \; e, d \; P \; f, d \; P \; g \\ e \; P \; a, e \; P \; f, e \; P \; g \\ f \; P \; g, \\ g \; P \; a, g \; P \; b, \end{gathered}$$

where P denotes the asymmetric part of S.

In order to obtain a ranking on the basis of such information, one may use a measure of the "desirability" of each alternative. A simple measure of the desirability of an alternative x consists in counting the number of alternatives y such that $x \; S \; y$ minus the number of alternatives z such that $z \; S \; x$. This measure is called the *Copeland score* of an alternative (Laslier, 1997).

A simple way of building a ranking on A goes as follows. Define the first equivalence class of the ranking as the alternatives that have obtained a maximal Copeland score. Remove these alternatives from the set of alternatives. Define the second equivalence class of the ranking as the alternatives with maximal Copeland scores in the reduced set. Repeat this procedure as long as there are unranked alternatives. Such a ranking procedure is intuitively appealing and leads to the following ranking, using obvious notations:

$$d \succ c \succ e \succ [a, g] \succ b \succ f,$$

which does not seem unreasonable.

Consider now a relation identical to the one above except that $a \; P \; d$ is added. Intuition suggests that the position of a has improved and we should reasonably expect that this is reflected in the ranking obtained on the basis of this new relation. But applying the same ranking method as before now leads to:

$$[b, c, d] \succ e \succ [a, f, g].$$

Such a result is quite disappointing since, before a was improved, a was ranked before b while, after the improvement of a, b is ranked before a. ◇

These two examples illustrate the following points.

The definition of sound procedures for deriving a recommendation on the basis of a non necessarily transitive or complete binary relation is a difficult task. Intuitively appealing procedures may sometimes produce very disappointing results.

This raises the question of how to analyse and compare the various procedures that have been proposed in the literature for such a purpose. The literature on MCDM is quite poor in this respect. Most often, the authors of methods have advocated an "intuitively reasonable" procedure. As shown above, "intuition" may hide major difficulties.

A similar problem arises in Social Choice Theory. Although the literature on Social Choice Theory is much richer than the literature on MCDM, it is mainly restricted to the choosing problem on the basis of a *complete* binary relation, with McGarvey's result in mind. Furthermore, the attention of Social Choice theorists has mainly been concentrated on the case of *tournaments*, i.e., complete and antisymmetric relation (an excellent account of this literature can be found in Laslier, 1997).

Two main routes may be followed to study the difficult problem of deriving a recommendation on the basis of a non necessarily complete and transitive binary relation. The first one (see, e.g., Bouyssou and Vincke, 1997; Vincke, 1992a) consists in defining a list of properties that seem "desirable" for such a technique (for example, never select a dominated alternative or respond to the improvement of an alternative in the expected way). Given such a list of properties one may then try:

- to analyse whether or not they are satisfied by a number of techniques,
- to establish "impossibility theorems", i.e., subsets of properties that cannot be simultaneously fulfilled,

- to determine, given the above-mentioned impossibility theorems, the techniques that satisfy most properties.

The second one (see, e.g., Bouyssou, 1991, 1992a,b, 1995, 1997; Bouyssou and Perny, 1992; Bouyssou and Pirlot, 1997; Pirlot, 1995) consists in trying to find a list of properties that would "characterise" a given technique, i.e., a list of properties that this technique would be the only one to satisfy. This allows to emphasise the specific features of an exploitation technique and, thus, to compare it more easily with others.

These two types of analysis are not unrelated: ideally they should merge at the end, the characterising properties exhibited by the second type of analysis being parts of the list of "desirable" properties used in the first type of analysis. Both types of analysis have their own problems. In the first, the main problem consists in defining the list of "desirable" properties. These properties should indeed cover every aspect of what seems to be constitutive of an "appropriate" technique. In the second, the characterising properties will only be useful if they have a clear and simple interpretation, which may not always be the case when analysing a complex technique.

A thorough analysis of the problem would be rather lengthy and technical. Our aim in this section will therefore be twofold. We shall first try to present the procedures that have been proposed in the literature in a critical manner, warning the reader against common pitfalls. Second, we shall try to offer an introduction to the growing but quite technical literature on the subject.

7.4.3 Choice procedures

Let A be a set of alternatives. Suppose that you have built a preference relation S on A using an aggregation technique. Let us call $\mathcal{S}$ the set of all conceivable preference relations that can be obtained using such a technique. As shown above, $\mathcal{S}$ consists of all reflexive binary relations if one is using ELECTRE I, all complete binary relations if one is using simple majority and all reflexive valued relations if one is using ELECTRE III. A choice procedure C is a function associating a nonempty subset $C(S)$ of A with each element S of $\mathcal{S}$. The choice set $C(S)$ should:

- be as small as possible given the available information,
- be such that there are clear arguments to justify the elimination of the alternatives in $A \setminus C(S)$, i.e., the alternatives that are not selected,
- be such that there is no built-in bias in favour of some alternatives, i.e., that the only arguments that can be taken into account in the determination of $C(S)$ are how these alternatives are related in terms of the relation S. Technically, this leads to requiring that C is *neutral*[3], i.e., that $C(S) =$

[3] The "neutrality" condition for choice procedures is different from the neutrality condition introduced in chapter 5. We use a similar term however, because the idea underlying these two conditions is similar: alternatives should not be treated differently because of their label. A similar remark holds for the conditions of monotonicity and faithfulness introduced below.

$\sigma[C(S^\sigma)]$, where σ is any one-to-one function on A and S^σ is the binary relation in $\mathcal{S}$ such that, for all $a, b \in A$, $S(a, b) = S^\sigma(\sigma(a), \sigma(b))$.

- react to the improvement of an alternative in the expected direction. Technically, the procedure should be *monotonic*, i.e., if $a \in C(S)$ and S' is identical to S except that $[a\ S'\ b$ and $Not[a\ S\ b]]$ or $[Not[b\ S'\ a]$ and $b\ S\ a]$, for some $b \in A$, then we should have $a \in C(S')$.

Below, we distinguish the case in which $\mathcal{S}$ is a set of crisp (i.e., non valued) binary relations from the case in which $\mathcal{S}$ is a set of valued relations.

Remark 7.4.2
We have defined a choice procedure as a function from the set of all possible relations $\mathcal{S}$ to the set of nonempty subsets of A. It is important to realise that this very definition implies that the only information that is taken into account by C is the relation S on A. This, in particular, implies that:

- the choice set $C(S)$ may depend on the behaviour of S on the whole set A. Adding or removing alternatives from A may have a dramatic influence on the result of the choice procedure. The fact that an alternative a belongs to the choice set and that an alternative b is rejected may depend on the comparison of a and b with respect to other alternatives. It may even depend on the comparison of two alternatives distinct from a and b. Although such a dependence is almost inevitable as soon as the choice procedure has to deal with relations S having no remarkable transitivity properties, it may lead to undesirable effects. Indeed, the result of the choice procedure will be dependent on the set of alternatives A, whereas, in practice, the definition of this set can always be modified, e.g. adding very poor alternatives.

- the relation S contains all the information used by C. In particular, this excludes the use of some "reference points", i.e., of alternatives playing a particular role, as advocated by Dubois et al. (2003). When such reference points are taken into account, the separation between the phases of building a relation S and exploiting it in order to build a choice set is blurred. Indeed, it is then tempting to compare alternatives only to the reference points and not amongst themselves. Such approaches may offer an interesting alternative to the use of choice procedures. They have not been worked out in much detail to date. In particular, the selection in practice of appropriate reference points does not seem to be an obvious task. •

7.4.3.1 Crisp relations

Let $S \in \mathcal{S}$. We shall always denote by P (resp. I) the asymmetric (resp. symmetric) part of S and J the associated incomparability relation.

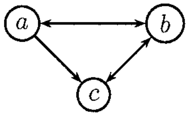


Figure 7.5: Refining the set of greatest alternatives $\{a, b\}$.

7.4.3.1.1 Procedures based on covering relations Suppose that there exists $a \in A$ such that $a\ P\ b$, for all $b \in A \setminus \{a\}$. Such an alternative is usually called a *Condorcet winner*. In this case, letting $C(S) = \{a\}$ seems to be the only reasonable choice. In fact, by construction:

- when there is a Condorcet winner, it is necessarily unique,
- there is *direct evidence* that a is better to all other alternatives.

Unfortunately, the existence of a Condorcet winner is an unlikely situation and we must agree on what to do in the absence of a Condorcet winner.

A simple extension of the notion of a Condorcet winner is that of *greatest* alternatives already introduced. Remember that an alternative $a \in A$ belongs to the set $G(A, S)$ of greatest alternatives in A given S if $a\ S\ b$, for all $b \in A$. If a belongs to $G(A, S)$, we have direct evidence that a is at least as good as any other alternative in A. Contrary to the case of Condorcet winners, there may be more than one greatest alternative. When the set of greatest alternatives is nonempty, it is tempting to put all alternatives on $G(A, S)$ in $C(S)$.

This seems a natural choice. Indeed, all greatest alternatives are indifferent, so there is no direct evidence that would allow to further refine the choice set $C(S)$. Contrary to the case in which S is a weak order, it should however be noted that there might be *indirect* evidence that allows to distinguish between greatest alternatives. As shown in the following example, indirect evidence may be usefully employed to narrow down the set of selected alternatives.

Example 7.3
Suppose that $A = \{a, b, c\}$ and S be such that $a\ I\ b$, $b\ I\ c$ and $a\ P\ c$ (see figure 7.5). Although both a and b belong to $G(A, S)$, we can use the way a and b compare to a third alternative, c, to distinguish between them. In our example, since $a\ P\ c$ while $b\ I\ c$, it is very tempting to use this indirect evidence to conclude that that $C(S)$ could be narrowed down to $\{a\}$. ◇

Unfortunately, there is no clear-cut way of defining what should count as an *indirect evidence* that an alternative is better to another and to balance it with the direct evidence.

Suppose first that $a\ P\ b$ so there is direct evidence that a is superior to b. If, for all $c \in A$, we have $c\ P\ a \Rightarrow c\ P\ b$, $c\ I\ a \Rightarrow c\ S\ b$, $b\ P\ c \Rightarrow a\ P\ c$ and $b\ I\ c \Rightarrow a\ S\ c$, there is no indirect evidence that b could be superior to a. In such a case, we say that a strongly covers b ($a\ SC\ b$) and it seems that the selection of b would be quite unwarranted. A cautious selection would then seem to be to select all alternatives that are not strongly covered by any other, i.e., the set $M(A, SC)$ of maximal alternatives in A for SC. When A is finite, $M(A, SC)$ is always nonempty since the strong covering relation is asymmetric and transitive and, thus, has no circuit. Therefore letting $C(S) = M(A, SC)$ defines a selection procedure. Note that the use of this selection procedure would allow to avoid selecting a strongly dominated alternative as was the case with the procedure envisaged in example 7.1 since, in this example, a strongly covers b. With such a procedure, the rejection of the elements in $A \setminus C(S)$ would seem fully justified since for each $b \in A \setminus C(S)$, there would be an $a \in C(S)$ such that $a \ P \ b$. We leave to the reader the, easy, task of showing that this selection procedure is neutral and monotonic.

The relation SC is likely to be rather poor so, that the above procedure is quite stringent and may result in large choice sets. In order to reject an alternative, it is necessary to have direct evidence against it and no indirect evidence in its favour. In example 7.3, it would not allow to distinguish between the two greatest alternatives a and b since there is no direct evidence for a against b.

A less stringent procedure would consist in saying that the selection of b is unwarranted as soon as there is an alternative a such that there is direct evidence that a is at least as good as b while there is no indirect evidence that b is better to a. This would lead to the definition of a covering relation in which a weakly covers b ($a \ WC \ b$) as soon as $a \ S \ b$ and for all $c \in A$, we have $c \ P \ a \Rightarrow c \ P \ b$, $c \ I \ a \Rightarrow c \ S \ b$, $b \ P \ c \Rightarrow a \ P \ c$ and $b \ I \ c \Rightarrow a \ S \ c$. Therefore, the weak covering relation WC is identical to the strict covering relation SC except that $a \ I \ b$ is compatible with $a \ WC \ b$. Contrary to SC, the relation WC is not asymmetric. It is reflexive and transitive so its asymmetric part has no circuit. When A is finite, letting $C(S) = M(A, WC)$ therefore defines a selection procedure. For each non selected alternative b, there is a selected alternative a such that either $a \ P \ b$ or $a \ I \ b$, while there is no indirect evidence that b might be superior to a. The theoretical properties of this choice procedure are quite distinct from the one relying on the strong covering relation (Dutta and Laslier, 1999; Peris and Subiza, 1999), while remaining neutral and monotonic. It seems to qualify as a natural benchmark for all choice procedures.

A weakness of the procedure given above is that when a and b are incomparable, it is impossible to distinguish between them even when there is strong indirect evidence that one is better to the other. It is possible to modify the definition of the weak covering relation requiring only that there is no direct evidence against a, i.e., that $a \ S \ b$ or $a \ J \ b$ (remember that J is the incomparability relation associated to S; with WC, it is impossible to have $a \ WC \ b$, while $a \ J \ b$), while still requiring that there is no indirect evidence that b is superior to a. This very weak covering relation is still reflexive and transitive. Taking the maximal alternatives in A for the very weak covering relation therefore defines a selection procedure. It refines the above selection procedure based on the weak covering relation. This is however a price to pay. Using such a choice set does not prevent the existence of a non selected alternative b such that there is no alternative in the choice set for which there is direct evidence that it is at least as good as b. Therefore, the narrowing of the choice set, considering the very weak covering relation, may be judged unsatisfactory.

We refer to Dutta and Laslier (1999), Laslier (1997) and Peris and Subiza (1999) for a thorough study of the properties of choice sets that are based on some idea of "covering" i.e., mixing direct and indirect evidence to justify the selection of $C(S)$.

7.4.3.1.2 Procedures based on kernels Quite a different path was taken by Roy (1968) and Roy and Skalka (1984) in the ELECTRE I and ELECTRE IS methods (a similar idea is already detailed in von Neumann and Morgenstern, 1947, in the context of Game Theory). Note that the selection procedure is clear as soon as S is transitive. In fact, in such a case, the set of maximal elements in A, i.e., $M(A, S) = \{a \in A : Not[b\ P\ a] \text{ for all } b \in A\}$ is always nonempty and such that, for all $b \notin M(A, S)$, there is an alternative $a \in M(A, S)$ such that $a\ S\ b$. In fact, when S is transitive, the set $M(A, S)$ coincides with the set of maximal alternatives for the weak covering relation since, in this case, $S = WC$.

For $B \subseteq A$, we say that B is *dominating* if for all $c \notin B$ there is an alternative $b \in B$ such that $b\ S\ c$. Therefore the selection of the alternatives in a dominating subset always justifies the non selection of the other alternatives. By construction, the set A itself is dominating. When A is finite, there are therefore dominating subsets of minimal cardinality. If there is only one such dominating subset, it is a good candidate for the choice set $C(S)$. When S has circuits, there may be more than one dominating subset of minimal cardinality. Taking their union will generally result in quite an undiscriminating procedure. This is illustrated in the following example.

Example 7.4
Let $A = \{a, b, c, d, e\}$. Suppose that S is such that $a\ P\ b$, $b\ P\ c$, $c\ P\ d$, $d\ P\ e$ and $e\ P\ a$ (see figure 7.6). This relation has 5 dominating subsets of minimal cardinality, i.e., $\{a, c, e\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, e\}$ and $\{b, d, e\}$. The union of the minimal dominating subsets is A. ◇

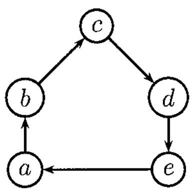


Figure 7.6: Relation P in example 7.4.

B. Roy therefore suggested to consider the relation S' obtained by reducing the circuits in S, i.e., to consider all alternatives that are involved in a circuit as a single alternative. Working with S' instead of S amounts to considering that all alternatives involved in a circuit compare similarly with alternatives outside the circuit. This is frequently a strong hypothesis implying the loss of a lot of information, as shown in example 7.4. The following example illustrates the process of reducing the circuits of S.

Example 7.5
Let $A = \{a, b, c, d, e, f\}$ and consider the binary relation S such that:

$$\begin{gathered} a\ S\ b, a\ S\ c, a\ S\ d, a\ S\ e, a\ S\ f, \\ b\ S\ c, b\ S\ f, \\ c\ S\ a, c\ S\ e, \\ d\ S\ e, \\ e\ S\ f. \end{gathered}$$

represented in figure 7.7 In order to build the relation S' obtained by reducing

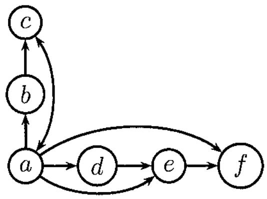


Figure 7.7: Relation S in example 7.5.

the circuits in S we need to find the maximal circuits in S (i.e., circuits that are not included in other circuits). There is only one circuit in S: $a\ S\ b$, $b\ S\ c$ and $c\ S\ a$. Therefore the three alternatives a, b and c are replaced by a single one, say x, and there is an arc from x to another alternative if there is an arc in S going from either a, b or c to this alternative. Similarly there is an arc going from an alternative to x if there was an arc going from this alternative to either a, b or c in S. Therefore the binary relation S' (see figure 7.8) is such that:

$$\begin{gathered} x\ S'\ d, x\ S'\ e, x\ S'\ f, \\ d\ S'\ e, \\ e\ S'\ f. \end{gathered}$$

In the relation built in example 7.2, there is a circuit going through all alternatives ($d\ P\ c$, $c\ P\ a$, $a\ P\ b$, $b\ P\ e$, $e\ P\ f$, $f\ P\ g$, $g\ P\ b$ and $b\ P\ d$). In such cases, the reduction of circuits involves a huge loss of information. ◇

A famous result of Graph Theory (Berge, 1970; Roy, 1969–70) implies that when a graph has no circuit, it has a unique *kernel*, defined as a dominating subset that is internally stable, i.e., such that there is no arc between any of its elements (this

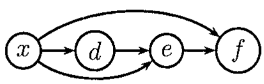


Figure 7.8: Relation S' in example 7.5.

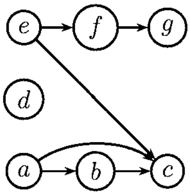


Figure 7.9: Relation S in example 7.6.

implies that the kernel is a minimal dominating subset). Reducing the circuits and taking the kernel of the relation is the selection procedure proposed in ELECTRE I. It is easy to verify that it is neutral and monotonic.

Example 7.6
Let $A = \{a, b, c, d, e, f, g\}$. Consider the relation S defined by figure 7.9. This relation has no circuit. Its unique kernel is $C(S) = \{a, d, e, g\}$ ◇

The selection procedure defined by the kernel is distinct from the one consisting of alternatives that are maximal for the weak covering relation. Indeed in example 7.6, the set of maximal alternatives for the weak covering relation is $\{a, d, e, f\}$ (g is covered by f, b and c are covered by a). This shows that covered alternatives may be selected in the kernel. Most importantly, as we have seen, the reduction of circuit may involve an important loss of information and can even lead to selecting dominated alternatives as in example 7.1.

The procedure in ELECTRE IS (see Roy and Bouyssou, 1993; Roy and Skalka, 1984) amounts to a more sophisticated reduction of the circuits that takes the way the relation S has been defined into account. In particular, it can help to avoid the selection of dominated alternatives.

Remark 7.4.3
A related selection procedure was suggested by Hansen, Anciaux-Mundeleer, and Vincke (1976) and Vincke (1977) in order to avoid the reduction of circuits, which, as we have seen, can lead to a significant loss of information. A quasi-kernel of a graph is a set of vertices that is internally stable (alternatives in a quasi-kernel are incomparable) and quasi-dominating, i.e., such that for any alternative b outside the quasi-kernel, there is one alternative a in the quasi-kernel such that either $a\ S\ b$ or $a\ S\ c$ and $c\ S\ b$, for some alternative c. Thus, a quasi-kernel may not be dominating but all alternatives outside the quasi-kernel can be reached via a path of length at most 2.

It is well-known (Lovász and Chvátal, 1974) that all graphs have at least one quasi-kernel. There may however be several quasi-kernels. Hansen et al. (1976) suggest to consider the selection of a quasi-kernel of minimal weakness, i.e., such that the set of alternatives that are not dominated by one alternative in the quasi-kernel is of minimal cardinality. This raises difficult combinatorial problems however. •

Choice procedures based on covering relations take the indirect evidence that an alternative is at least as good as another into account. There are several ways to define what should count as indirect evidence. Procedures based on kernels imply getting rid of circuits, which may involve a considerable loss of information. They may lead to the selection of covered alternatives.

7.4.3.1.3 Other procedures The use of covering relations and of the notion of kernel are far from being the only possible choices to devise a selection procedure (Laslier, 1997; Peris and Subiza, 1999; Schwartz, 1986). Some of the possibilities that we do not investigate here are:

- selection procedures based on the consideration of relations close to S for which the choice is simple, e.g, orders or weak orders (see Barthélémy, Guénoche, and Hudry, 1989; Laslier, 1997),
- selection procedures based on scores, e.g., Copeland scores (see Henriet, 1985; Rubinstein, 1980; van den Brink and Gilles, 2003),
- selection procedures that directly operate on the evaluations of the alternatives without building a relation S as an intermediate step. This was studied in section 5.6 of chapter 5 (see also Fishburn, 1977).

7.4.3.2 Valued relations

The literature on selection procedures on the basis of valued preference relations is extensive (Banerjee, 1993; Barrett, Pattanaik, and Salles, 1990; Basu, Deb, and Pattanaik, 1992; Bisdorff, 2000; Bouyssou, 1992a, 1997; Bouyssou and Pirlot, 1997; Dasgupta and Deb, 1991; De Donder, Le Breton, and Truchon, 2000; Dutta and Laslier, 1999; Dutta, Panda, and Pattanaik, 1986; Fodor, Orlovski, Perny, and Roubens, 1998; Fodor and Roubens, 1994; Herrera and Herrera-Viedma, 2000; Kitainik, 1993; Lahiri, 2002; Litvakov and Vol'skiy, 1986; Montero and Tejada, 1988; Nurmi and Kacprzyk, 1991; Pattanaik and Sengupta, 2000; Perny, 1995; Perny and Roubens, 1998; Roubens, 1989; Sengupta, 1999) and it would be illusory to attempt summarising it here.

Let us simply mention here that this diversity is due to numerous factors:

- the variety of possible interpretations of the valued relation which goes from interpretations in terms of "credibility" to probabilistic or "intensity of preference" interpretations,
- the different ways in which to interpret the numbers in the valued relation which goes from a purely ordinal interpretation to more cardinal interpretations. Indeed the number $S(a, b)$ may, depending on the context, be interpreted as the (weighted) number of criteria on which a is judged at least as good as b or simply as a "credibility" index of the proposition "a is at least as good as b",
- the various ways of defining classical properties (completeness, transitivity) of binary relations for the valued case; these various definitions are not always equivalent,

- the difficulty to define "strict preference", "indifference" and "incomparability" on the basis of a valued relation.

We briefly envisage three different types of techniques here.

7.4.3.2.1 Use of λ-cuts Any selection procedure designed for crisp relations may be applied to a valued relation considering various λ-cuts of the valued relations, i.e., the crisp relation S_λ defined by:

$$a\ S_\lambda\ b \Leftrightarrow S(a,b) \geq \lambda. \tag{7.16}$$

The definition of the λ-cuts of a valued relation only uses the ordinal properties of the valuations. With a strictly ordinal interpretation of valued relations, the set of all λ-cuts of a valued relation contains the same information as the valued relation itself. A cautious attitude is therefore to study the result of selection procedures for crisp relations when applied to the set of all λ-cuts of the valued relation (in practice, one may want to consider only the λ-cuts corresponding to relatively high values of λ). This raises the problem of aggregating this information. This problem is all the more serious that it is easy to build examples in which two distinct λ-cuts of S may result in vastly different crisp relations, even when the two values of λ are "close".

7.4.3.2.2 Fuzzyfication of crisp procedures Another class of procedures consists in "fuzzyfying" the definition of various selection procedures for crisp relations using a particular interpretation of logical connectives (AND, OR, NOT) in a valued framework. This is a classical procedure in "fuzzy" mathematics.

Suppose, for instance, that S is a valued preference relation interpreted as an "at least as good as" preference relation. The set of maximal elements in A given a crisp relation S has been defined as $M(A, S) = \{a \in A : \forall b \in A,\ Not[\,b\ P\ a\,]\,\}$. This set may be empty. When it is not, we have seen that the alternatives in this set may be seen as reasonable candidates for choice. The "fuzzyfication" of the concept of the set of greatest alternatives amounts to attaching to each alternative in A the credibility that it belongs to the set of greatest elements[4].

Given the relation S, we have $a\ P\ b$ if $[a\ S\ b$ and $Not[\,b\ S\ a\,]\,]$. Interpreting AND as "min" and NOT as "1−", we obtain the degree of credibility of the proposition "a is strictly preferred to b" as $P(a,b) = \min(S(a,b), 1 - S(b,a))$. Now we are looking for alternatives in A for which, for all $b \in A$, it is not true that $b\ P\ a$. Interpreting "for all" as "min", which is consistent with our interpretation of AND, we obtain:

$$\begin{aligned}\mu(a) &= \min_{b\in A}(1 - P(b,a)),\\ &= \min_{b\in A}(1 - \min(S(b,a), 1 - S(a,b))),\\ &= \min_{b\in A}\max(1 - S(b,a), S(a,b)),\end{aligned}$$

[4] An alternative approach in which a credibility degree is attached to *subsets* of alternatives was explored by Kitainik (1993)

that may be interpreted, given our particular choice of valued connectives, as the credibility of the proposition "there is no alternative in A that is strictly preferred to a". One may then select alternatives for which this score is maximal (which would imply that any difference in μ is significative) or the alternatives in A for which the value of μ exceeds a certain threshold. Note that the choice of "1−" as a valued interpretation of NOT is not fully compatible with a purely ordinal interpretation of the valued relation S.

Such fuzzyfication techniques were first proposed by Orlovski (1978), using a different definition of P (see Fodor and Roubens, 1997, for a detailed analysis). The set of maximal alternatives for the strong or weak covering relation and/or the set of greatest elements may be fuzzyfied in a similar manner (see Perny, 1995).

In fact, this technique allows to transfer any crisp definition into a definition adapted to the valued case almost immediately, once fuzzy connectives have been agreed upon. This is not an easy choice however (see, e.g., Alaoui, 1999; Bisdorff, 2000; Fodor et al., 1998; Fodor and Roubens, 1994; Kitainik, 1993; Perny and Roubens, 1998; Perny and Roy, 1992).

7.4.3.2.3 Procedures based on scores Another class of procedures associate to each alternative $a \in A$ a "measure of its desirability" in A, we shall say a score, given a valued relation S. Many such scores can be envisaged, e.g., the Net Flow score which is the analogue of the Copeland scores for crisp relations obtained by letting the score of alternative $a \in A$ be $Score_{NF}(a, S) = \sum_{b \in A \setminus \{a\}} (S(a, b) - S(b, a))$ which was axiomatised by Bouyssou (1992a), or the Minimum in Favour score $Score_{\min}(a, S) = \min_{b \in A \setminus \{a\}} S(a, b)$, characterised in Bouyssou (1995) and Bouyssou and Pirlot (1997). This procedure based on min can be refined in many ways (see Dubois, Fargier, and Prade, 1996; Dubois, Fortemps, Pirlot, and Prade, 2001b, for several lexicographic variants of min). They have been studied in Fortemps and Pirlot (2004).

The choice of an adequate score is dependent upon the interpretation of the valuations $S(a, b)$. For instance, the use of the Net Flow score appears adequate only if it is supposed that the credibility $S(a, b)$ is measured on a scale that is stronger than an ordinal scale, so that adding and subtracting credibility indices is meaningful.

Remark 7.4.4

It is instructive to show how the axioms used to characterise the choice procedures based on the Net Flow score (Bouyssou, 1992a) and the Minimum in Favour score (Bouyssou, 1995; Bouyssou and Pirlot, 1997) make hypotheses on the nature of the valuations of the fuzzy relation.

The selection procedure based on the Net Flow score is characterised by an axiom implying that all circuits of length 2 or 3 in the fuzzy relation can be eliminated without affecting the selection, together with neutrality and monotonicity requirements. Technically this means that if two fuzzy relations S and R are identical except that:

$$R(a, b) = S(a, b) + \epsilon \text{ and } R(b, a) = S(b, a) + \epsilon \text{ or}$$
$$R(a, b) = S(a, b) + \epsilon, R(b, c) = S(b, c) + \epsilon \text{ and } R(c, a) = S(c, a) + \epsilon,$$

then S and R should lead to identical selections. Clearly, such an axiom is only adequate if the valuations are "cardinal" for it to make sense to add a constant ϵ to some of them.

Similarly, the selection procedure based on the Minimum in Favour score is essentially characterised by an "ordinality" axiom stating that if it is possible to go from S to R via an increasing transformation on $[0;1]$ then S and R should lead to identical selections. •

7.4.3.2.4 Examples Some of the selection procedures for valued relations envisaged so far are illustrated below.

Example 7.7
Let $A = \{a, b, c, d, e, f\}$ and consider the valued binary relation S defined by:

S	a	b	c	d	e	f
a	1.0	0.5	0.5	0.3	0.3	0.6
b	0.8	1.0	0.1	0.5	0.4	0.9
c	0.2	0.2	1.0	0.8	0.2	0.3
d	0.1	0.0	0.2	1.0	0.9	0.5
e	0.7	0.6	0.1	0.1	1.0	0.8
f	0.1	0.4	0.2	0.0	0.0	1.0

Taking the Minimum in Favour score $Score(a, S) = \min_{b \in A \setminus \{a\}} S(a, b)$ leads to the unique choice of alternative a which has a maximal score of 0.3. Fuzzyfying the quantifier "for all" using "min", this degree can be interpreted as the credibility of the proposition "a is at least as good as any other alternative in A", i.e., that a belongs to the greatest alternatives for S in A.

The net flow score $Score(a, S) = \sum_{b \in A \setminus \{a\}} (S(a, b) - S(b, a))$ leads to the unique choice of b with a score of $3.7 - 2.7 = 1$.

Taking $P(x, y) = \min(S(x, y), 1 - S(y, x))$, we obtain:

P	a	b	c	d	e	f
a	0.0	0.2	0.5	0.3	0.3	0.6
b	0.5	0.0	0.1	0.5	0.4	0.6
c	0.2	0.2	0.0	0.8	0.2	0.3
d	0.1	0.0	0.2	0.0	0.9	0.5
e	0.7	0.6	0.1	0.1	0.0	0.8
f	0.1	0.1	0.2	0.0	0.0	0.0

The fuzzyfication of the set of maximal elements envisaged above therefore leads to the unique choice of c with the credibility that it belongs to the set of maximal alternatives of $1 - 0.5 = 0.5$.

Note that on the example above, the three procedures give different results. Unless S has some remarkable properties, this usually cannot be avoided. ◇

The definition of a selection procedure for valued relations should take the nature of the valuation (e.g., the nature of the scale on which they are measured) and their interpretation (e.g., large preference or strict preference) into account. It is always possible to apply a selection procedure designed for crisp relations to a valued relation through the use of λ-cuts. Using the family of λ-cuts of a valued relation and applying a choice procedure for crisp relations to each of these relations may be seen as a very cautious attitude that does not seek to exploit the finely grained information provided by the valued relation. It nevertheless raises the problem of aggregating the results obtained at each cut; this aggregation problem is all the more serious that the application of a choice procedure to two λ-cuts obtained for values of λ that are close to each other may lead to quite different results. If more refined procedures are applied, care must be taken to ensure the compatibility of the operations performed on the valuations with the way in which they were obtained. In some techniques, e.g., ELECTRE III or PROMETHEE, the precise nature of the valuations is not easy to determine, as was emphasised in section 5.3.1 of chapter 5.

7.4.4 Ranking procedures

7.4.4.1 Crisp relations

Let A be a set of alternatives. Suppose that you have built a crisp relation S on A using some kind of aggregation technique. Let $\mathcal{S}$ be the set of all conceivable preference relations that can be obtained using such a technique. A ranking procedure $\succsim$ is a function associating a reflexive and transitive binary relation $\succsim(S)$ on A with each element S of $\mathcal{S}$. The task of building a transitive result on the basis of a binary relation, that might not be transitive or complete is not easy: we are in fact looking for a much richer result than that obtained using choice procedures.

Remark 7.4.5
Our definition of a ranking procedure does not imply that $\succsim(S)$ is necessarily complete. This is in accordance with our definition of the ranking problem statement above. When using a ranking procedure $\succsim$ that might lead to an incomplete relation $\succsim(S)$, it will be important to analyse the conditions under which incomparabilities could occur. Although always asking for a complete and transitive relation may be overly demanding, having many incomparabilities in $\succsim(S)$ is unlikely to be much helpful. •

Remark 7.4.6
Remark 7.4.2 about choice procedures fully applies here. Indeed, our definition of ranking procedures implies that $\succsim(S)$ depends on the behaviour of S on the entire set A and S is the only information used by $\succsim$. Again the fact that the positions of a and b in $\succsim(S)$ could depend on how they compare with respect to other alternatives and, more generally on the whole relation S, although inevitable in this approach, may be criticised. Similarly a ranking procedure $\succsim$ does not make use of "reference points" to rank alternatives. •

We expect such a ranking procedure to be:

- *neutral*, i.e., insensitive to the labelling of the alternatives,

- *faithful*, i.e., if S is a reflexive and transitive relation, we should have $\succsim(S) = S$,
- *monotonic*, i.e., the position of a in the ranking $\succsim(S)$ should not decrease if S is substituted by a relation S' in which the position of a has improved (see example 7.2).

Clearly, this list is only partial, e.g., we would also expect the ranking $\succsim(S)$ to be linked to the covering relations defined above (or to have links with the underlying weak order of S when S is a semiorder, see Vincke, 1992a).

Several types of ranking procedures have been suggested in the literature:

1. Ranking procedures based on the transitive closure of S,
2. Ranking procedures based on scores, e.g., the Copeland score,
3. Ranking procedures based on the repeated use of a choice mechanism (as in example 7.2),
4. Ranking procedures based on distances.

We briefly illustrate each type of procedure below.

7.4.4.1.1 Procedures based on the transitive closure Let S be a reflexive binary relation on A. A simple way to obtain a reflexive and transitive relation $\succsim(S)$ on the basis of S is to take its transitive closure $\widehat{S}$, i.e., the smallest transitive relation containing S. This defines a ranking procedure; it is easy to see that it is neutral, faithful and monotonic. In view of our discussion of choice procedures, the main defect of this ranking procedure should be apparent. All alternatives that are involved in a circuit of S will be equally ranked if we let $\succsim(S) = \widehat{S}$. This often results in a very poor information. As suggested in Schwartz (1972) and Schwartz (1986), this phenomenon is somewhat less severe if the transitive closure is taken on the asymmetric part P of S. This is however a price to pay, since indifferent alternatives in S that are not included in a circuit of P will then appear incomparable in $\succsim(S)$. This calls for the use of techniques allowing to deal with such situations (see Perny, 1992).

A closely related ranking procedure is the one used in ELECTRE II (Roy and Bertier, 1973). It was originally designed to produce a reflexive and transitive relation on the basis of two nested reflexive relations. We present it below in the special case in which there is only one relation (the role of the second one being only to possibly refine the equivalence classes that are obtained).

Consider any reflexive relation S on A. The ranking procedure of ELECTRE II first consists, as with ELECTRE I, in reducing the eventual circuits in S, replacing all alternatives involved in a circuit by a single vertex in the associated graph. Once this is done, we obtain, by construction, a relation with no circuit. We use this relation to build two weak orders. In the first one, T_1, the first equivalence class consists of the maximal elements (there is no element that is strictly preferred to them) of the relation with no circuit. These elements are then removed from the

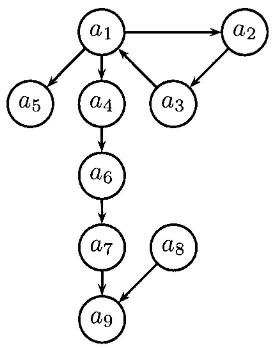


Figure 7.10: Relation S in example 7.8.

set of alternatives. The second equivalence class of T_1 consists of the maximal elements of the relation among those remaining and so forth.

The second weak order T_2 is obtained in a dual way, building the last equivalence class consisting of the minimal elements first (they are preferred to no other element) in the relation with no circuit, removing these elements from the set of alternatives and building the penultimate equivalence class of T_2 as the minimal elements among those remaining and so forth. Let us illustrate this process using a simple example.

Example 7.8
Let $A = \{a_1, a_2, \ldots, a_9\}$ and let S be such that (see figure 7.10):

$$\begin{gathered} a_1 \; S \; a_2, a_1 \; S \; a_4, a_1 \; S \; a_5, \\ a_2 \; S \; a_3, \\ a_3 \; S \; a_1, \\ a_4 \; S \; a_6, \\ a_6 \; S \; a_7, \\ a_7 \; S \; a_9, \\ a_8 \; S \; a_9. \end{gathered}$$

The relation S has a circuit: $a_1 \; S \; a_2$, $a_2 \; S \; a_3$, $a_3 \; S \; a_1$. We therefore replace S on A with the relation S' on A' defined by (see figure 7.11):

$$\begin{gathered} b \; S' \; a_4, b \; S' \; a_5, \\ a_4 \; S' \; a_6, \\ a_6 \; S' \; a_7, \\ a_7 \; S' \; a_9, \\ a_8 \; S' \; a_9, \end{gathered}$$

where a_1, a_2 and a_3 have been replaced by b. The relation S' has no circuit. Its set of maximal elements consists of $\{b, a_8\}$. Once these elements have been removed, the set of maximal elements is $\{a_4, a_5\}$. At the next iteration, we obtain $\{a_6\}$,

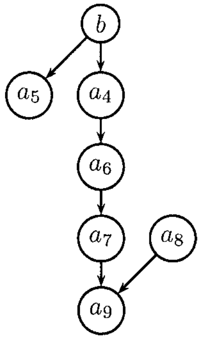


Figure 7.11: Relation S' in example 7.8.

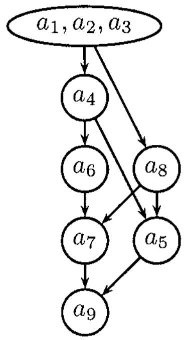


Figure 7.12: Relation $\succsim(S)$ in example 7.8. Transitivity arcs are omitted.

then $\{a_7\}$ and $\{a_9\}$. Therefore the weak order T_1 is, using obvious notation:

$$[a_1, a_2, a_3, a_8]\ T_1\ [a_4, a_5]\ T_1\ a_6\ T_1\ a_7\ T_1\ a_9.$$

In a dual way, we obtain the weak order T_2:

$$[a_1, a_2, a_3]\ T_2\ a_4\ T_2\ a_6\ T_2\ [a_7, a_8]\ T_2\ [a_5, a_9]. \qquad \diamond$$

In general, T_1 and T_2 are not identical. The reflexive and transitive relation $\succsim(S)$ is then taken to be the intersection of these two weak orders. In our example we would obtain, abusing notation (see figure 7.12):

$$\begin{gathered}
[a_1, a_2, a_3] \succ a_4 \succ a_6 \succ a_7 \succ a_9, \\
[a_1, a_2, a_3] \succ a_8, \\
a_4 \succ a_5, \\
a_8 \succ a_7, a_8 \succ a_5, \\
a_5 \succ a_9.
\end{gathered}$$

What can be said of this result? First observe that the rationale for building two weak orders and for defining $\succsim(S)$ as their intersection is to introduce incomparability between alternatives that are difficult to compare using S. This is, for

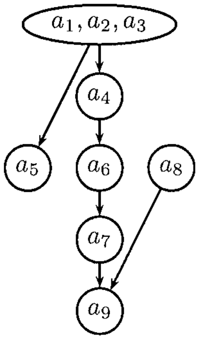


Figure 7.13: Transitive closure $\widehat{S}$ of S in example 7.8 .Transitivity arcs are omitted.

instance, the case between a_5 and all alternatives except a_1 or between a_8 and all alternatives except a_9. In this respect the success of the procedure is only limited since we finally conclude that $[a_1, a_2, a_3] \succ(S)\ a_8$, $a_8 \succ(S)\ a_7$, $a_4 \succ(S)\ a_5$ and $a_5 \succ(S)\ a_9$.

Let us also note that we would have obtained a similar result starting with the transitive closure $\widehat{S}$ of S instead of S. Observe that, simply taking $\succsim(S) = \widehat{S}$, would have probably been a better choice in this example (see figure 7.13).

The final result of the ranking procedure is obtained by taking the intersection of two weak orders. Since it is well-known that there are reflexive and transitive relations that cannot be obtained in such a way (Dushnik and Miller, 1941), this procedure is not faithful. We leave the proof that this procedure is indeed neutral and monotonic to the reader (it is detailed in Vincke, 1992a).

Taking the transitive closure of a relation leads to a ranking procedure that is quite undiscriminating. Applying transitive closure to the asymmetric part of S somewhat alleviates the problem but calls for the application of techniques designed to deal with indifferent alternatives. Ranking procedures building a reflexive and transitive relation on the basis of the intersection of two weak orders are not faithful.

7.4.4.1.2 Copeland scores We have seen that the procedure suggested in ELECTRE II does not satisfy all the requirements we intuitively would like to see satisfied. A simpler ranking procedure consists in rank ordering the elements in A according to their Copeland scores, i.e., the number of alternatives that they beat minus the number of alternatives that beat them. In our earlier example, this would, abusing notation, give the weak order (see figure 7.14):

$$a_1 \succ a_8 \succ [a_2, a_3, a_4, a_6, a_7] \succ a_5 \succ a_9.$$

We cannot expect faithfulness with such a procedure, since the result of the

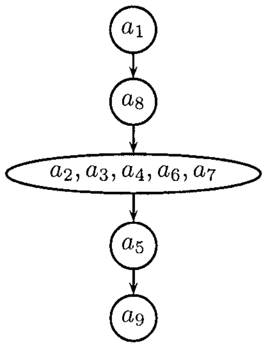


Figure 7.14: Relation $\succsim(S)$ using Copeland scores.

procedure is obviously complete (note that the procedure treats indifference and incomparability similarly). On the other hand, such a procedure is neutral and monotonic.

The ranking procedure based on Copeland scores was characterised by Rubinstein (1980) (for the case of tournaments, i.e., complete and antisymmetric relations) and Henriet (1985) (for the case of complete relations). It is not difficult to extend Henriet's result to cover the case of an arbitrary reflexive relation (see Bouyssou, 1992b). The main distinctive characteristic of this ranking procedure is that it is insensitive to the presence of circuits in S since the contribution of this circuit to the Copeland scores of the alternatives in the circuit is always zero.

Ranking procedures based on a score always lead to a *complete* and transitive relation. They are not faithful.

Remark 7.4.7

Observe that we could have weakened faithfulness requiring only that $\succsim(S) = S$, when S is *complete* and transitive. We leave the easy task of showing that the ranking procedure based on Copeland scores is indeed faithful in this weaker sense to the reader. •

Remark 7.4.8

An alternative way of building a ranking procedure consists in using *several* scores (see Bouyssou and Perny, 1992). Each score is used to build a weak order on A and $\succsim(S)$ is taken as the intersection of these weak orders. The result of this type a procedure is a reflexive and transitive relation that can have at most dimension k, where k is the number of scores involved. Hence, such techniques are not faithful. •

7.4.4.1.3 Ranking by repeated choice A possible way of combining the simplicity of such a ranking procedure with a move towards faithfulness consists in using the Copeland scores iteratively to build two weak orders T_1 and T_2. This would consist here in building the first equivalence class of a weak order T_1 with the

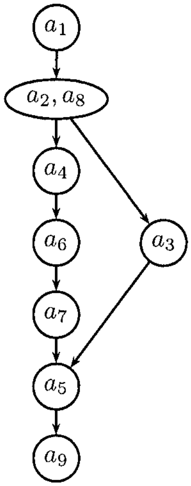


Figure 7.15: Relation $\succsim(S)$ using repeated choice based on Copeland scores.

alternatives having the highest Copeland scores, and iterating the procedures after having removed the already-ranked alternatives. For the relation in example 7.8, we would obtain:

$$a_1 \; T_1 \; [a_2, a_4, a_8] \; T_1 \; a_6 \; T_1 \; a_7 \; T_1 \; [a_3, a_5, a_9].$$

Using a dual principle, we could also build a weak order T_2 the last equivalence class of which consists of alternatives having minimal Copeland scores and reiterate the process on the set of unranked alternatives. This would yield:

$$[a_1, a_2, a_3, a_8] \; T_2 \; a_4 \; T_2 \; a_6 \; T_2 \; [a_5, a_7] \; T_2 \; a_9.$$

Taking the intersection of these two weak orders is a much simplified version of the ranking procedure implemented in ELECTRE III (Roy, 1978). This leads to, abusing notation, (see figure 7.15):

$$\begin{array}{c} a_1 \succ [a_2, a_8] \succ a_4 \succ a_6 \succ a_7 \succ a_5 \succ a_9, \\ [a_2, a_8] \succ a_3 \succ a_5. \end{array}$$

Such a result does not seem to lead us closer to an adequate restitution of the uncertain positions of a_8 and a_5 within S. Furthermore, as observed in example 7.2, such a ranking procedure is not monotonic, which seems to be quite a serious shortcoming.

Ranking procedures based on the iteration of choice mechanisms are quite unlikely to respect monotonicity except in trivial cases (Bouyssou, 2004; Juret, 2003; Perny, 1992). This tends to severely limit their interest.

7.4.4.1.4 Use of distances Suppose that you have defined a distance d on the set of binary relations in S. A natural way of obtaining a ranking procedure would seem to be to find the reflexive and transitive relation T at minimal distance from S. This idea dates back at least to Barbut (1959), Kemeny (1959), Kemeny and Snell (1962) and Slater (1961). Although this may seem the most natural way of defining a ranking procedure, this approach raises new problems:

- the determination of a transitive relation at a minimum distance from crisp or valued binary relations raises deep combinatorial questions and quite difficult algorithmic problems (see Barthélémy et al., 1989; Barthélémy and Monjardet, 1981, 1988; Bermond, 1972; Charon, Hudry, and Woirgard, 1996; Charon-Fournier, Germa, and Hudry, 1992; Hudry, 1989; Monjardet, 1990). From a practical point of view, this tends to limit the use of such techniques to small sets of alternatives.

- it is likely that many quite distinct relations are at minimum distance of S. The definition $\succsim(S)$ on the basis of this family of relations is far from obvious. Systematically taking the intersection of all such relations will often yield a result containing more incomparabilities than we would have liked.

- the choice of the distance function should be analysed with care (see Roy and Słowiński, 1993) as soon as one is no loner faced with the, easy, case of a distance between tournament and linear orders for which the distance based on the symmetric difference is an obvious choice (see Barthélémy, 1979).

- the normative properties of such procedures are not easy to analyse (see, however, Young and Levenglick, 1978).

Remark 7.4.9
The analysis above shows that it is very difficult to devise a ranking procedure that is fully satisfactory. This difficulty is related to Arrow-like theorems introduced in section 5.2.1.3 of chapter 5. Indeed, suppose that you have defined a "very nice" ranking procedure. You could then proceed as follows to rank order alternatives. Use one of the majoritarian aggregation methods introduced in section 5.2.3 chapter 5 to build a relation S. This relation S will mainly depend on "ordinal" considerations but will not have remarkable transitivity properties. Applying your nice ranking procedure to S will lead to a reflexive and transitive binary relation $\succsim(S)$. Clearly, this two-step process (building S and then applying $\succsim$ to S) may be viewed as a one-step process associating a reflexive and transitive relation to a profile of evaluations. But then, Arrow-like theorems apply to this one-step process. The fact that there does not seem to be a ranking process that would be fully satisfactory within this framework is therefore unsurprising.

This does not mean, however, that such ranking procedures are useless. Once their shortcomings are acknowledged, they may indeed be useful tools for the analyst in order to elaborate a recommendation. Discovering, for instance, that a subset of the entire set A is almost always ranked in a similar way using several such procedures, may be used as building block by the analyst in order to come up with recommendations. •

7.4.4.2 Valued relations

We have shown the difficulty of devising a satisfactory ranking procedure for crisp relations. The situation is not easier with valued relations. An overview of ranking procedures for valued preference relation may be found in Fodor and Roubens (1994), Fodor et al. (1998), Perny (1992) and Perny and Roubens (1998). Working with valued relations allows to better discriminate between alternatives. This increased discrimination is often obtained at the cost of performing operations on the valuations that are not always compatible with a strictly ordinal interpretation of these numbers.

Since these procedures do not appear to be significantly more satisfactory than the ones envisaged above, we do not study them in detail here. As in the case of crisp relations, ranking procedures for valued preference relations may be based:

- on *scoring functions.* The main difficulty here will be that the result of the ranking procedure will always be complete and that it is necessary to use a scoring function that is somehow compatible with the nature of the valuations. The ranking procedure based on the Net Flow score has been characterised in Bouyssou (1992b) using axioms that are very similar to the ones used for the associated choice procedure, i.e., interpreting the valuations in a "cardinal way" (this result has been extended in Bouyssou and Perny (1992) to cover the case of the intersection of two procedures based on scores). Note that this ranking method is at work in the PROMETHEE method (Brans et al., 1984; Brans and Vincke, 1985). Similarly the ranking procedure based on the Minimum in Favour score was characterised in Bouyssou (1991), Bouyssou and Pirlot·(1997) and Pirlot (1995) using axioms compatible with an ordinal interpretation of the valuations.

- on the *repeated use of a choice procedure* as in ELECTRE III (Roy, 1978) or in MAPPAC and PRAGMA (Matarazzo, 1986, 1988, 1990). As first shown in Perny (1992), such procedures are quite unlikely to be monotonic, which tends to seriously limit their interest.

- on a *transitive relation close to the valued relation.* Such procedures often raise the same kind of difficulties as the ones evoked in the crisp case (see page 379).

7.4.5 Sorting procedures

We have seen that the lack of transitivity and/or completeness raised quite serious difficulties when it comes to devising choosing and ranking procedures. These difficulties are somewhat less serious here. This is because, with sorting procedures, the assignment of an alternative only depends on its comparison to carefully selected reference actions defining the categories. The use of such reference points implies that, contrary to the case of choice and ranking procedures, the distinction between the phase of building a relation S and then using this relation in order to reach conclusions is blurred with the sorting problem statement. Reference points are used from the beginning and the relation S is mainly used to compare the alternatives in A to these reference points.

To keep things simple, we only deal with the case of a crisp relation S here. As in the case of value functions, we only consider the case of ordered categories.

Early attempts to propose sorting procedures are Massaglia and Ostanello (1991), Moscarola and Roy (1977) and Roy (1981). A more general approach to the problem was suggested in Roy and Bouyssou (1993) and Yu (1992a) with the so-called ELECTRE TRI approach that we present below.

7.4.5.1 An overview of ELECTRE TRI

We consider the case of r ordered categories $C^1, C^2, \ldots, C^r$, with C^r containing the most desirable alternatives. We suppose, for the moment, that each category C^k is delimited by a limiting profile π^k. It is not restrictive to suppose that π^{k+1} strictly dominates[5] π^k, for all k. Furthermore, we can always find an alternative π^{r+1} that strongly dominates[6] all other alternatives in A and, conversely, an alternative π^1 that is strongly dominated by all other alternatives (see figure 7.16). How can

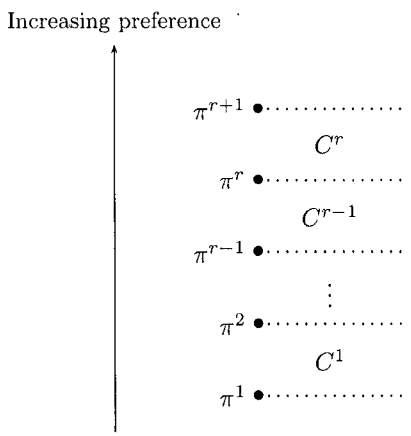


Figure 7.16: Sorting with r ordered categories.

we use a preference relation between the alternatives in A and the set of limiting profiles to define a sorting procedure? Intuitively, since π^k is the lower limit of category C^k, we can apply the following two rules:

- if an alternative a is preferred to π^k, it should at least belong to category C^k,
- if π^k is preferred to a, a should at most belong to category C^{k-1},

[5] I.e., π^{k+1} is at least as good as π^k on all criteria and strictly better on some criterion.

[6] I.e., it is strictly better on all criteria.

the case in which a is indifferent to π^k is dealt with conventionally depending on the definition of the limiting profiles π^k.

When the relation S is complete and transitive, these two rules lead to unambiguously assign each alternative to a single category.

The situation is somewhat more complex when S is intransitive or incomplete. When S is compatible with the dominance relation (which is not a very restrictive hypothesis), as we have supposed that π^k strictly dominates π^{k-1}, it is possible to show (see Roy and Bouyssou, 1993, ch. 5) that when an alternative a is compared to the set of limiting profiles $\pi^1, \pi^2 \ldots, \pi^{r+1}$, three distinct situations can arise:

1. $\pi^{r+1}\ P\ a, \pi^r\ P\ a, \ldots, \pi^{k+1}\ P\ a, a\ P\ \pi^k, a\ P\ \pi^{k-1}, \ldots, a\ P\ \pi^1$. In such a case, there is little doubt on how to assign a to one of $C^1, C^2, \ldots, C^r$. Since $a\ P\ \pi^{k-1}$, a should be assigned at least to category C^k. But since $\pi^k\ P\ a$, a should be assigned at most to C^k. Hence, a should belong to C^k.

2. $\pi^{r+1}\ P\ a, \pi^{r+2}\ P\ a, \ldots, \pi^{\ell+1}\ P\ a, a\ I\ \pi^\ell, a\ I\ \pi^{\ell-1}, \ldots, a\ I\ \pi^{k+1}, a\ P\ \pi^k, \ldots, a\ P\ \pi^1$. The situation is here more complex. Since $\pi^{\ell+1}\ P\ a$, alternative a must be assigned at most to category C^ℓ. Similarly since $a\ P\ \pi^k$, a must be assigned at least to category C^k.

   The fact that a is indifferent to several consecutive limiting profiles is probably a sign that the definition of the categories is too precise with respect to the binary relation that is used by the sorting procedure: the profiles are too close to one another. This would probably call for a redefinition of the categories and/or for a different choice for S. In such a situation, an optimistic attitude consists in assigning a to the highest possible category, i.e., C^ℓ. A pessimistic attitude would assign a to C^k.

3. $\pi^{r+1}\ P\ a, \pi^r\ P\ a, \ldots, \pi^{\ell+1}\ P\ a, a\ J\ \pi^\ell, a\ J\ \pi^{\ell-1}, \ldots, a\ J\ \pi^{k+1}, a\ P\ \pi^k, \ldots, a\ P\ \pi^1$. In this situation, a is incomparable to several consecutive profiles. This is a sign that, although we are sure that a must be assigned at most to category C^ℓ and at least to category C^k, the relation S does not provide enough information to opt for a category within this interval. Again, an optimistic attitude in such a situation consists in assigning a to the highest possible category, i.e., C^ℓ. A pessimistic attitude would be to assign a to C^k.

The assignment procedure described above is the one introduced in ELECTRE TRI (Roy and Bouyssou, 1993; Yu, 1992a) in which a is assigned to one of $C^1, C^2, \ldots, C^r$ using an optimistic procedure and a pessimistic procedure. Alternative a is always assigned to a higher category when using the optimistic procedure than when using the pessimistic procedure. One can verify that this procedure coincides with the one suggested in 7.2.2 when S is defined by a value function.

Another interesting special case of this procedure arises when S is identical to a dominance relation. In this case, the optimistic procedure suggested above coincides with a disjunctive sorting procedure. In fact a will be assigned to C^ℓ as soon as $\pi^{\ell+1}\ P\ a$ and $Not[\pi^\ell\ P\ a]$, which means that ℓ is the highest category such that, on some criterion $i \in N$, a is better than π^ℓ. Conversely, the pessimistic procedure coincides with a conjunctive assignment strategy: a will be assigned to C^k as soon as $Not[a\ P\ \pi^{k+1}]$ and $a\ P\ \pi^k$, which amounts to saying that k is the lowest category such that a dominates π^k.

It is worth noting that although the authors of this method have coupled this procedure with a particular definition of S (a crisp relation based on a concordance discordance principle), it can be applied to any relation that is compatible with a dominance relation.

Remark 7.4.10
We refer Greco, Matarazzo, and Słowiński (2001b) and Słowiński, Greco, and Matarazzo (2002) for an axiomatic analysis of the sorting model in which (see also the pioneering work of Goldstein, 1991):

$$a \in C^k \Leftrightarrow s^k < F(u_1(h_1(a)), u_2(h_2(a)), \ldots, u_n(h_n(a))) < s^{k+1},$$

where F is a real-valued function on $\mathbb{R}^n$ which is nondecreasing in each of its arguments, u_i are real-valued functions on $\mathbb{R}$, s^k are real numbers and $h_i(a)$ is the evaluation of alternative $a \in A$ on the ith criterion.

Taking F as a sum shows that the above model contains the UTADIS technique introduced in section 7.3.4. It is not difficult to show that the same is true with the ELECTRE TRI technique described above. A complete axiomatic analysis of ELECTRE TRI was recently proposed in Bouyssou and Marchant (2005a) and Bouyssou and Marchant (2005b). •

Remark 7.4.11
Each alternative is assigned to a category in ELECTRE TRI. Such an assignment may hinder the fact that some assignments may be more well-established than others. This clearly calls for a robustness analysis before coming to conclusions. An interesting way of having a "built-in" robustness analysis within a sorting procedure is to compute the credibility, between 0 and 1, that each alternative belongs to each category. Alternatives for which this credibility is close to 1 for a given category and close to 0 for all other categories are then seen to be "robustly" assigned. Such assignment procedures allow to explicitly model the fact that the definition of the categories may not allow to unambiguously assign each alternative. These types of techniques are detailed in Perny (1998). •

Remark 7.4.12
When first confronted with ELECTRE TRI, many people have the impression that this method, while preserving an "ordinal" character, provides a way out of the problems caused by incompleteness and/or intransitivity. Indeed, the result of ELECTRE TRI is an assignment of the alternatives in A among ordered categories on the basis of an outranking relation built using the concordance-discordance principles. This seems quite close to obtaining a weak order on A. Such a way of ranking alternatives is in the spirit of the use of "reference points" for choosing or ranking alternatives as advocated in Dubois et al. (2003).

It should however be noted that, unsurprisingly, sorting methods à la ELECTRE TRI do not offer a "miraculous way out" of the problems of ordinal aggregation uncovered by Arrow-like theorems. Indeed, the appearance of transitivity of

	h_1	h_2	h_3
a_1	11	9	10
a_2	10	11	9
b	9	10	11

Table 7.8: Evaluation of two alternatives and the limiting profile.

the result of ELECTRE TRI is due to the fact that alternatives are only compared to the limiting profiles and are not compared between themselves. If this were the case, intransitivities would inevitably reappear. Let us illustrate this point using a simple example.

Consider two alternatives a_1 and a_2 evaluated on a family of three criteria and let us suppose that we want to sort these two alternatives into two categories C^1 and C^2, b being the limiting profile between C^1 and C^2. The evaluations of a_1, a_2 and b for the three criteria are given in table 7.8. Suppose that we compare these alternatives using simple majority (therefore considering that all criteria are of equal importance). We obtain: $b \; P \; a_1$ and $a_2 \; P \; b$, so using both the pessimistic and the optimistic versions of ELECTRE TRI, we should conclude that $a_1 \in C^1$ and $a_2 \in C^2$. This seems to give evidence that a_2 is superior to a_1. However, this evidence is contradicted by the fact that, using the same principles, the direct comparison of a_1 and a_2 would have led to $a_1 \; P \; a_2$, a_1 being better than a_2 on two criteria. •

ELECTRE TRI offers a simple way of using a relation based on a concordance-discordance principle to assign alternatives to ordered categories defined by limiting profiles. Conjunctive and disjunctive sorting procedures are particular cases of ELECTRE TRI. Because alternatives are only compared to carefully selected reference alternatives, the possible incompleteness or intransitivity of the preference relation that is used has less severe consequences than for choosing or ranking procedures. However, this raises the problem of defining these reference alternatives.

7.4.5.2 Implementation of ELECTRE TRI

The ELECTRE TRI procedure described above supposes that the analyst has defined:

- the limiting profile π^k for each category C^k,
- the parameters involved in the definition of S: weights, indifference and preference thresholds, veto thresholds.

This is overly demanding in most applications involving the use of a sorting procedure. In many cases however, it is possible to obtain examples of alternatives that should be assigned to a given category. Like in the UTADIS method described earlier (see 7.3.4), one may use a "learning by examples" strategy to assign a value to these parameters. Several strategies for doing this were investigated in Dias and Clímaco (2000), Dias and Mousseau (2006), Dias, Mousseau, Figueira, and Clímaco (2002), Mousseau et al. (2001), Mousseau and Słowiński (1998), Mousseau et al. (2000) and Ngo The and Mousseau (2002).

Remark 7.4.13
The symmetric part I of the relation S can be interpreted with some precaution as a similarity relation. When this interpretation is accepted, we may extend this type of methods to cover the case of unordered categories defined by prototypical examples, through computing the "similarity" of an alternative with its prototypical elements. This has been investigated in detail in Belacel (2000), Belacel, Hansen, and Mladenović (2002), Belacel, Scheiff, Vincke, and Boulassel (2000), Belacel, Vincke, and Boulassel (1999), Bisdorff (2002), Henriet (2000), Henriet and Perny (1996) and Perny (1998) •

7.5 Robustness of the conclusions

We have seen in section 7.3 that an assessment procedure can lead to several possible value functions. We argued that, in such a case, the derivation of recommendation should take all possible value functions into account. Indeed, we are interested in obtaining recommendations that could be justified using any of the possible value functions, i.e., in what could be called "robust" recommendations.

The interest of this idea of robustness is not limited to the case of an assessment procedure leading to several value functions. As argued in section 2.3.3 of chapter 2 many other sources of uncertainty, imprecision and inaccurate determination interfere with the work of the analyst (see Bouyssou, 1989; Roy, 1989). The way to manage them has generated a research trend in decision aiding under the name of "robustness" problems. The purpose of this section is to introduce the reader to this recent literature.

7.5.1 Introduction

All scientists who have treated real decision problems know that the numerical values used in the models are questionable. On the one hand, this is the case for the information describing the decision situation, traditionally called the data. They often are values built by the analyst according to the model he wants to use, they result from assumptions about the context of the problem, from estimations of badly known or random values, from forecasting of future events. Therefore, it is often the case that several plausible sets of data, possibly very different from each other, can constitute good representations of the situation. On the other hand, this is also the case for the parameters (value functions, weights, thresholds, etc.) which must be (more or less arbitrarily) chosen by the user of the methods described in this book (see, in particular, section 4.4 of chapter 4).

In such a context, working with a unique (the "most plausible") set of values can be very risky. This is particularly true for the decision maker who has to live with the consequences of his/her decision, if the "real" set of values is different from the set used in the determination of the decision.

In Bouyssou et al. (2000, ch. 8), we analysed traditional and less traditional ways of coping with uncertainty in Decision Theory. In Operational Research, stochastic optimisation also takes the presence of multiple data instances which can occur in the future into account. However, these approaches usually require explicit information on the "plausibility" associated to each instance. This explicit information (probabilities, possibilities, fuzzy numbers, etc.) is not known with certainty; it can result from more or less reliable estimations and be based on more or less strong assumptions. Moreover the enormous number of parameters introduced in these approaches can lead to a "black box effect" which is rarely desirable in a decision aiding process.

What the decision maker generally wants is a decision which is relatively good for all (or almost all) of the plausible sets of data and which does not imply too much risk. This is the basis of the concept of robustness that we want to introduce in this section. This concept, which until now was not really integrated into decision aiding methods, is a challenging area of research and is likely to be a very important part of decision aiding techniques.

7.5.2 Robustness versus stability

We would like to avoid any confusion between robustness and stability. A solution (a decision) is said to be *stable* if it resists to some perturbations of the data and parameters which were used to determine it. The stability of a decision generally results from an a posteriori sensitivity analysis which consists in studying how the results vary with (generally small) changes in the data. This means that a solution (a result, a decision) was determined on basis of a particular set of values for the data and the parameters (the most "plausible" ones) and that an a posteriori study of the neighbourhood of that solution is performed. Note also that, generally, for technical reasons, the sensitivity analysis is performed for the perturbations of one parameter at a time.

The idea of robustness leads to consider, a priori, several sets of values of the parameters (possibly rather different from each other) and to look for decisions which are "good" for all or almost all sets of values. No particular set of values is privileged; uncertainty is introduced in the formulation of the problem and it does not necessarily have to be quantified by probabilities or other tools.

7.5.3 Alternative definitions of robustness in the literature

To date there is no specific definition of robustness accepted by the scientific community. Moreover, the idea of robustness is rarely integrated into the decision aiding tools proposed in the literature and, when it is, it is generally assimilated to stability, which is a different property, as explained in section 7.5.2.

However, the word "robustness" is not new: it was introduced in different contexts and with different meanings for the last 30 years.

One of the first papers dealing with a concept of robustness appeared in Management Science in the seventies (Gupta and Rosenhead, 1972) and was devoted to strategic planning. In this context, due to the often appreciable uncertainty about external conditions in the future, it is possible that a best decision based on the state of current knowledge will prove to be less than good during the following years. One way of avoiding this danger is to ensure that the early and irreversible decisions keep as many options of "good" plans open as possible. In Rosenhead, Elton, and Gupta (1972) and Rosenhead (1989), the robustness of a decision is defined as the ratio of two quantities. The first is the number of "good" end-states which remain as open options after the decision. The second is the number of all possible end-states. The exact mathematical formulation and examples of applications can be found in the aforementioned references.

Rosenblatt and Lee (1987) studied a facilities design problem where different versions are possible for the demand of products to be manufactured with these facilities, the objective being to minimise the cost resulting from the manipulations of the material. This paper defines the robustness of a solution as the number of versions where the solution provides a cost that is "not too far" from the optimum, this acceptable distance being expressed as a pre-defined percentage.

Sengupta (1991) introduced a concept of robustness in Data Envelopment Analysis that mixes the idea of stability for small variations of the data (classic sensitivity analysis) and the idea of prudence with regards to possible bad versions.

In Statistics, robustness analysis is used to reduce the influence of outliers on the results provided by regression methods or econometric models.

In Mathematical Programming, Mulvey, Verderbel, and Zenios (1995) introduced a concept of robustness in relation to optimality (the solution must be "close to" the optimum for all possible versions) and another in relation to feasibility (the solution must be feasible for all possible versions). The final solution is calculated by stochastic programming where penalties for less robust solutions are introduced in the objective function .

In the field of Combinatorial Optimisation, the main contributions are those of Kouvelis, Karawarwala, and Gutierrez (1992), Kouvelis and Yu (1997). They propose three different definitions that are all inspired by the idea that a robust solution should avoid any catastrophic result:

1. the first definition (*absolute robustness*) attaches to each solution its worst value among all possible versions. One then tries to find the solution for which this worst value is the best.

2. in the second definition (*robust deviation*) each solution is characterised, for each version, by the difference between its value and the optimal value for this version. The robust solution is then the solution that minimises the largest of these differences.

3. the third definition (*relative robustness*) is similar to the second one except that it uses deviations from the optimal solution expressed in percentage.

Roy (1998) suggests to apply the concept of robustness not only to solutions but, more generally, to "conclusions" (assertions, recommendations). A conclusion is an information deduced from the model and given to the decision maker during the decision process. It can be a solution to the problem, but it can also be a property or a fact that can be useful for the decision maker. A conclusion is called robust if it is true for all (or almost all) the versions of the problem where a version is characterised by a plausible set of values for the parameters of the model used to solve the problem. The reader is referred to Roy (1998) and Vallin (1999) for details.

Vincke (1999a) proposed a theoretical framework for the concept of robustness. It is based on formal definitions of "problem", "instance of a problem", "procedure" and "method". It leads to precise definitions of robust solutions and robust methods, which are illustrated using classical optimisation problems (minimum spanning tree, minimum Hamiltonian path) and preference aggregation problems.

Several recent papers deal with the robustness of the solutions to decision problems (the interested reader will find a list of references available at `http://www.ulb.ac.be/polytech/smg/indexresearch.htm`) but, as already mentioned, the definition of the concept is far from being unique. Robustness may have several meanings, such as flexibility (as in Rosenhead, 1989), prudence (as in Kouvelis and Yu, 1997), stability (as in Roy, 1998), so that several formalisations of the concept should be developed in the future. In the next sections, we would like to illustrate some aspects which constitute stepping stones for the analyst on this subject.

7.5.4 Robustness illustrated: examples

Example 7.9 (Minimum spanning tree)
This first example illustrates the case of a decision situation that has been modelled as an optimisation problem with some uncertainty on the data.

A communication network must be established between 4 cities A, B, C, D at a minimum cost. The costs of the different connections are given in table 7.9 (see also figure 7.17). They are expressed in millions of Euros. However, the total

	AB	AC	AD	BC	BD	CD
Costs	6	2	8	3	5	7

Table 7.9: Costs of the possible connections.

cost could be reduced due to the fact that another project, supported by another budget, could be decided by the government in the near future. The problem is that, for political reasons, it is impossible to know whether this project will concern the connection AB (leading to a reduction of 4 million Euros for the cost of this connection) or the connection CD (leading to a reduction of 3 million Euros). In other words, a decision has to be taken in a context where there are two possibilities for the costs, leading to two versions (one could also speak of two scenarios) of the problem. Table 7.10 summarises the costs of the connections in both versions. It is not difficult to see that the optimal solution in version 1 consists in choosing the connections AB, AC and BD (this is known, in Operational Research, as

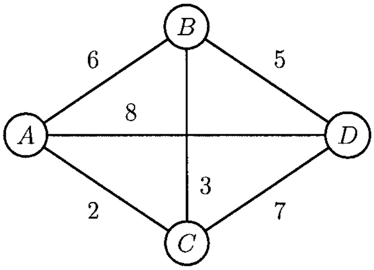


Figure 7.17: Costs of the possible connections.

	AB	AC	AD	BC	BD	CD
Version 1	2	2	8	3	5	7
Version 2	6	2	8	3	5	4

Table 7.10: Costs in the two versions.

the minimum spanning tree problem), giving a communication network costing 9 million Euros. However, this choice is rather risky because, should version 2 occur, the cost of this network would be equal to 13 million Euros, which represents an increase of nearly 50%.

In version 2, the optimal solution consists in choosing the connections AC, BC and CD, also giving a communication network costing 9 million Euros, but with the risk of paying 12 million Euros in version 1. A rapid analysis of this (very simple) example shows that the network consisting in AC, BC and BD costs 10 million Euros in both versions, which is nearly optimal whatever the version is. This last solution could be called "a robust solution" because it is very good in both versions (even if not optimal) and its value does not vary too much (in this case, it does not vary at all) when the version changes.

Remark 7.5.1
In this particular case, the solution $\{AC, BC, BD\}$ is optimal if a probability equal to 0.5 is assigned to each version and the mean cost of each connection is computed. However, the robust solution cannot always be obtained in this way. For example, consider a similar problem where the costs, in both versions, are given in table 7.11. The reader can verify that the only solutions that can be obtained by

	AB	AC	AD	BC	BD	CD
Version 1	7	5	12	11	9	16
Version 2	17	9	10	12	4	3

Table 7.11: New costs in both versions.

assigning probabilities to the two versions of the problem and by minimising the expected cost are, for any set of probabilities, the solution $S_1 = \{AC, BD, CD\}$ or $S_2 = \{AB, AC, BD\}$, while it would not be unreasonable for a decision maker, to consider that solutions $S_3 = \{AC, AD, BD\}$ or $S_4 = \{AC, BC, BD\}$ are more robust, as shown in table 7.12.

	S_1	S_2	S_3	S_4
Version 1	30	21	26	25
Version 2	16	30	23	25

Table 7.12: Comparison of the costs of four solutions in the two versions.

Remark 7.5.2
The solution $S_4 = \{AC, BC, BD\}$ of the initial problem would also be the one given in the approach of Kouvelis and Yu (see section 7.5.2). However, it is not difficult to build an example where the three definitions proposed by these authors do not coincide with a reasonable and intuitive concept of robustness. this is illustrated in the following example. •

◇

Example 7.10 (Choice of projects)
A choice must be made between 6 projects the costs of which depend on some external conditions. To simplify the presentation, let us consider that two versions of the problem are possible and that the estimation of the costs in these two versions are given in table 7.13. We see that the best project (minimising the

Projects	Version 1	Version 2
A	10	60
B	70	20
C	28	29
D	20	30
E	15	31
F	11	32

Table 7.13: Possible costs.

cost) in version 1 is A, which is very bad in version 2. Similarly, the best project in version 2, which is B, is very bad in version 1. Applying Kouvelis and Yu's definition of absolute robustness, we have to associate the worst value to each solution, yielding column 1 of table 7.14. According to this definition, the absolute robust solution (having the best worst value) is C, which indeed can be considered as satisfying for both versions. The robust deviation of each solution is obtained by calculating, in each version, the difference between the value of this solution and the optimal value of this version and by taking the largest difference. This leads to column 2 of table 7.14, where we see that the best solution (minimising the robust deviation) is D. Note that D could reasonably be considered as better than C because it provides a significative improvement in version 1 (see table 7.13) for a slight disadvantage in version 2. For Kouvelis and Yu's third definition of robustness, we have to compute the relative robust deviation of each solution. For this purpose, we compute, in each version, the ratio between the value of each solution and the optimal value of this version and we take the largest of these ratios. This leads to column 3 of table 7.14, where we see that the best solution (minimising the relative robust deviation) is E, which again can reasonably be preferred to C and D. Finally, solution F, that is very close to the previous ones in version 2, is significantly better in version 1 and is in fact very close to the optimum for this version.

In this example, we see that it would not be unreasonable for a decision maker to have a preference for F over all the other solutions, although F will not be proposed by any of Kouvelis and Yu's definitions. This example shows that:

	1	2	3
A	60	40	2
B	70	60	6
C	29	18	1.8
D	30	10	1
E	31	11	0.55
F	32	12	0.6

Table 7.14: Three kinds of robustness.

- in the presence of uncertainty, the concept of robust solutions may be more suitable than that of an optimal solution,
- even in relatively simple optimisation problems, the determination of robust solutions cannot always be reduced in a straightforward manner to an optimisation problem,
- the way to model robustness should integrate aspects of the decision maker's preferences. ◇

Example 7.11 (Weighted absolute majority)
Let us now consider a situation with no uncertainty on "data" but in which numerical values have to be chosen for the various parameters of the decision aiding method.

Four objects a, b, c, d are compared according to three dimensions, yielding the following three rankings:

- first dimension: a better than $\{b, c\}$, better than d.
- second dimension: b better than $\{a, d\}$, better than c.
- third dimension: c better than $\{a, b\}$, better than d,

where $\{x, y\}$ means that x and y are tied. We want to use a weighted absolute majority rule, which requires the following steps (see section 5.2.3 of chapter 5):

- assessing the "weights" of the dimensions;
- building a global preference relation: x is considered at least as good as y iff the sum of the weights of the dimensions supporting this assertion represents at least 50% of the sum of all weights.

According to section 4.4 of chapter 4 (see also 5.2.2), the only way to assess weights in a significant and consistent manner (i.e., a way taking the use of these weights in the next steps of the method into account) is to try to obtain some information about the decision maker's global preference relation. Note that without any such information, we can already conclude from the data that a and b are at least as good as d: whatever the weights, the sum of the weights supporting these assertions represent 100% of the total sum of the weights. Note also that d can never be at least as good as b, but it can be at least as good as a if the weight of the second dimension is larger than 50% of the sum of all weights. Assume that the decision maker has a global strict preference for a over c (i.e. a is at least as good as c but c is not at least as good as a). This information leads to the following constraints on the weights, denoting by w_i the weight of the ith dimension:

$$\begin{cases} w_1 + w_2 \geq 0.5(w_1 + w_2 + w_3), \\ w_3 < 0.5(w_1 + w_2 + w_3). \end{cases}$$

As the weights are clearly defined up to a positive multiplicative constant, we can assume that the total sum is equal to one and these two constraints are equivalent to the unique constraint:

$$w_1 + w_2 > 0.5.$$

With any set of weights satisfying this constraint we obtain a global preference relation respecting the information given by the decision maker. For example, choosing $w_1 = w_2 = w_3 = 1/3$ yields:

$$\begin{array}{c} a\ I\ b, a\ P\ c, a\ P\ d, \\ b\ I\ c, b\ P\ d, c\ P\ d, \end{array}$$

where P and I respectively denote the global strict preference relation and the global indifference relation.

However, choosing another set of weights compatible with the available information will lead to different relations P and I. The central question is therefore to know what a robust conclusion is in such a problem. This clearly depends on the definition of robustness, which, in turn, depends on the definition of "contradictory" results. For instance, an inversion of strict preference can be considered as "less acceptable" than the transformation of an indifference into a strict preference.

As our example is very simple, let us enumerate all the possible results that can be obtained for all the possible sets of weights such that

$$\begin{cases} w_1 + w_2 + w_3 = 1, \\ w_1 + w_2 > 0.5. \end{cases}$$

Let us consider each ordered pair (x, y) of alternatives and compute for what weights we obtain the proposition "x is at least as good as y". This is done in table 7.15. Using this table, it is easy to see that 4 different results can be obtained,

(a, b)	:	$w_2 \leq 0.5$	(b, a)	:	$w_1 \leq 0.5$
(a, d)	:	always	(d, a)	:	$w_2 \geq 0.5$
(b, c)	:	$w_1 + w_2 \geq 0.5$	(c, b)	:	$w_2 \leq 0.5$
(b, d)	:	always	(d, b)	:	never
(c, d)	:	$w_2 \leq 0.5$	(d, c)	:	$w_2 \geq 0.5$

Table 7.15: Conditions on weights.

depending on the choice of weights. They are presented in table 7.16 (remember that we know that $a\ P\ c$). If we are very strict and decide that a result is robust

Weights	Results
$w_1 < 0.5$ and $w_2 > 0.5$	$b\ P\ a, a\ I\ d, b\ P\ c, b\ P\ d, d\ P\ c$
$w_1 \leq 0.5$ and $w_2 = 0.5$	$a\ I\ b, a\ I\ d, b\ I\ c, b\ P\ d, c\ I\ d$
$w_1 \leq 0.5$ and $w_2 < 0.5$	$a\ I\ b, a\ P\ d, b\ I\ c, b\ P\ d, c\ P\ d$
$w_1 > 0.5$ and $w_2 < 0.5$	$a\ P\ b, a\ P\ d, b\ I\ c, b\ P\ d, c\ P\ d$

Table 7.16: Possible results.

only if it remains unchanged for all possible sets of weights, then the only robust conclusion is $b\ P\ d$ (to which we could add two "negative" robust conclusions: $Not[d\ P\ a]$ and $Not[c\ P\ b]$).

But if we accept to relax the definition and refuse only the inversion of strict preference, we can also accept the global preference for a over d and for b over c as robust. In a choice problem, a robust prescription could be the elimination of c and d, as they are both globally not as good as a and b.

This example again shows that robustness is not an objective concept: it depends on what the decision maker considers as "different results" (here, the inversion of strict preference). This is why we consider that the concept of robustness should be taken into account as early as possible in the decision aiding process: ideally, it should be defined in the modelling step of the problem (see section 2.3.2 of chapter 2).

Note that, due to the small number of dimensions in the example, it was possible here to enumerate all the versions compatible with the available information. This is generally not the case and a difficult question is how to build a representative set of versions. ◇

Example 7.12 (Linear Programming)

Note that the previous example also illustrated the fact that the concept of robustness can be applied to prescriptions and not only to solutions of the problem (as was the case in example 7.9). More generally, it can be applied to any kind of information, even in classical optimisation problems, as illustrated below.

Suppose that you have to produce a mix of two products A and B. The total quantity of A and B to be produced is 30 tons; for technical reasons, you cannot produce more than 20 tons of the same product. The profit associated to each product depends on the market conditions and two representative versions are considered.

In the first version, the profit made on product A is 20€ per ton and the profit made on product B is 10€ per ton. In the second version, the profits are respectively 10€ per ton for product A and 30€ per ton for product B.

A traditional tool used for treating such a problem is linear programming. Denoting by x and y the respective quantities of A and B in the production plan, we have to determine the values of x and y that maximise $(ax + by)$ under the constraints

$$\begin{cases} 0 \leq x \leq 20 \\ 0 \leq y \leq 20 \\ x + y = 30 \end{cases}$$

where $a = 20$, $b = 10$ in the first version and $a = 10$, $b = 30$ in the second version. In such a context, assertions such as:

- there exists a solution giving a value at least equal to 50 to the objective function,
- the value of the objective function is less than 700,
- the solution $x_1 = x_2 = 15$ cannot be optimal,

can be qualified as robust because they are true whatever the version.

However, the conclusions that are true for all the possible versions will generally be of minor interest to the decision maker because there are too general (this is the reason why Roy, 1998, proposed several variants of robustness).

Again, the choice of a robustness concept will depend on the context and on the decision maker's preferences. In an optimisation problem (as in examples 7.9 and 7.12), he may want to obtain, for instance:

- a solution that is feasible in all the versions and gives in each version a value of the objective function that is within 10% of the optimal value of the objective function for that version,
- a solution that belongs to the 10% best feasible solutions in each version,
- a solution that is feasible in 95% of the versions and quasi-optimal (within 5% of the optimum) in all the versions in which it is feasible,
- a solution that is feasible in "most" of the versions, "very good" in "many" versions and "not too bad" in the others (the terms between inverted commas having to be progressively formalised during the decision aiding process).

In a more general decision problem (as in example 7.11), there are of course many more possibilities. A dialogue with the decision maker about these aspects seems to be necessary in a decision aiding perspective. ◇

7.5.5 Robust methods

Let us consider again the preference aggregation problem presented in example 7.11. We fixed an aggregation method (the weighted absolute majority rule) and we studied the set of results that could be obtained using this method on the basis of the available information. Then, given a particular definition of robustness, we obtained a set of robust prescriptions. Another approach would consist in trying to build a method which always gives robust results, the definition of robustness having been fixed in advance. Assume for example that the decision maker considers an aggregated preference relation as robust if it is unchanged from one version to another, except the eventual replacement of strict preferences by indifferences or vice versa (remember that, in this example, a version is characterised by a set of weights for the different dimensions). A method providing only robust aggregated preference relations could be qualified as robust for this problem (the reader will find an example in Vincke, 1999a). However, the search for robust methods may lead to ad hoc methods which are not very interesting in practice (pushing to the limit, if the method imposes the values of the weights, there is only one version and the result will certainly be robust). This can be remedied by the introduction of a concept of "neutrality", which was proposed in Vincke (1999a) (note that this term has here a meaning that is different from the one in section 7.4.3 and chapter 5).

Sörensen (2003) and Sevaux and Sörensen (2004), in the field of scheduling problems, propose a robust tabu search technique for combinatorial optimisation problems and suggest a distinction between two kinds of robustness for the solutions: the robustness of their structures and the robustness of their performances.

7.5.6 Back to Thierry's choice

Considering again the example in section 7.3.5 (see also Bouyssou et al., 2000, ch. 6), we can identify several sources of uncertainty that justify some robustness considerations.

First of all, as in all decision problems, the so-called data (see table 7.3) cannot be considered as completely and precisely known. These "data" depend on the origin of the information (here, journals specialised in used cars), on the chosen scales for each dimension and on some preliminary calculations made on the raw data to summarise them (see Bouyssou et al., 2000, ch. 6.1.1; this is especially true for criteria 4 and 5). Explicitly taking these uncertainties into account could lead the analyst to replace the numbers in table 7.3 by intervals and to consider that each element of the Cartesian product of these intervals defines a different version of "Thierry's choice" problem.

Using intervals in the UTA approach (see section 7.3.1.3.1) would lead to a lower bound $\underline{V}(x)$ and an upper bound $\overline{V}(x)$ for each alternative x and the preference:

$$\text{Sunny} \succ \text{Galant}$$

given by the decision maker would lead to an inequality such as:

$$\underline{V}(\text{Sunny}) > \overline{V}(\text{Galant})$$

that could be integrated in the linear programme that is solved in order to build the value function. Note that, as explained in chapter 3 (and, in particular, sections 3.7.4 and 3.7.5), there are several ways to express preferences between intervals and the preference here above could also be represented by the two following inequalities:

$$\begin{cases} \underline{V}(\text{Sunny}) > \underline{V}(\text{Galant}), \\ \overline{V}(\text{Sunny}) > \overline{V}(\text{Galant}). \end{cases}$$

Moreover, as already mentioned in section 7.3.1.3.1, the choice of the objective function of the linear programme is somewhat arbitrary, so that it is important to look for conclusions that, insofar as possible, resist the arbitrariness in the choice of the objective function. In a choice problem, an interesting question is to know whether there exists, for a given alternative x, a specification of the model leading to the choice of that alternative. On the basis of table 7.5, we can say that "Tipo is not the best" is a robust conclusion (because there is no value function compatible with the available information that leads to the choice of Tipo). The conclusion "Alfa is the best" is not robust. Indeed, while table 7.5 shows that the assertion is true for some value functions, table 7.6 reveals that there are value functions compatible with the available information for which it is not true. Similarly, in a ranking problem, an interesting question is to know whether some global preferences between pairs of alternatives are valid for all (or almost all) value functions (see section 7.3.5)

Besides the uncertainties on the data and on the parameters of the decision aiding model, a third level of uncertainty is connected to the choice of the model itself. In Bouyssou et al. (2000, ch. 6), the "choosing a car" problem was treated with different methods (e.g., methods using value functions and methods using outranking relations): the robustness of a conclusion or a prescription can also be studied in relation to this diversity. When this is done, one should note that the choice of the decision aiding method has an influence on the definition of the necessary data (since two different methods may require different data). This shows that the distinction between the so-called "data" and the so-called "parameters of the method" (which we made in examples 7.9 and 7.11) is not always so clear.

7.5.7 Robustness and MCDA

In the case where the decision problem is modelled as an optimisation problem and where a finite number of versions (sets of values for the data and the parameters of the model) has to be taken into account, one could argue that there are some similarities between searching for a robust solution of the optimisation problem (that is a solution which is good in most versions and not too bad in others) and searching for a compromise solution of a multicriteria problem where the versions play the role of criteria. A concept such as efficiency (i.e., the search for non-dominated solutions) could be used to select the candidates that qualifies as robust solutions and multicriteria methodologies could be applied to determine robust solutions. The interested reader will find an illustration of this approach in Hites, De Smet, Risse, Salazar, and Vincke (2003), where the robustness of a solution does not only depend on its worst performance (as in Kouvelis and Yu, 1997) but simultaneously on its good and bad performances (without trivially applying an arithmetic or a weighted mean the drawbacks of which were abundantly illustrated in Bouyssou et al., 2000. See also the concept of generalised Lorenz dominance used by Perny and Spanjaard, 2003 for the same kind of problem).

Despite the similarities between searching for a compromise solution of a multicriteria problem and searching for a robust solution of a multiversion optimisation problem, one should avoid considering that the only difference is the vocabulary (on this subject, see Hites et al., 2003). In the formulation of the problem, the family of criteria is built in such a way that the decision maker's opinion is as well represented as possible (see the concept of consistent family of criteria proposed in Roy and Bouyssou, 1993, ch. 2), while the set of versions is often, at least partially, imposed by external conditions. Moreover, the number of versions can be infinite (e.g., if the values of the parameters are defined using intervals) and the concepts of relative importance or preferential independence are not easy to transpose. Finally, most decision problems are simultaneously multicriteria and multiversion. In conclusion, it seems clear that the concept of robustness justifies the development of a *specific* theoretical framework and of new methodologies. This is an open field of research for the future.

7.5.8 Summary and open questions

1. Ignorance and uncertainty constitute an inevitable feature of all decision or evaluation problems. They find their origin, in particular, in:
   - the attitudes of the actors,
   - the fact that the model is not reality,
   - the incomplete or imprecise knowledge of the environment,
   - the imprecision of the measurement instruments,
   - the fact that the choice of a precise decision aiding model is somewhat arbitrary,
   - the imperfections of the communication between the actors.
2. Traditional tools (and, in particular probabilistic tools) are not completely satisfactory to cope with all these uncertainties. "Much of what is not known cannot be expressed in terms of probabilities" (Rosenhead et al., 1972). The fact that there is always an irreducible part of uncertainty or ignorance that cannot be quantified and reduced to an optimisation problem is included in the idea of robustness.
3. Robustness is, like "preference" or "importance", a property that de-

pends on the actors and the context: it has to be modelled in the course of the decision aiding process.

4. If the situation requires the use of the notion of robustness, it should be integrated in the very beginning of the decision aiding process, in the structuring step (see section 2.3.2 of chapter 2). This implies a careful reflection about acceptable assumptions, reasonable requirements and, finally, a good knowledge of the situation. It is also an interesting communication tool to improve the mutual understanding between the decision maker and the analyst.

5. Robustness ideas and multicriteria concepts present some similarities, but searching for robust solutions is not simply a particular application of multicriteria methodology.

6. Classifying and characterising multiversion situations in function of the various sources and types of uncertainties is an open research question. It is likely that the concept of robustness and its implementation should depend on this classification.

7. Decision aiding may first consist in trying to reduce the uncertainties, in working on the set of versions, instead of immediately searching for robust conclusions or solutions.

8. Decision aiding may also consist in building robustness indicators or providing structured sets of solutions and mechanisms of adaptation to the evolving circumstances.

9. An interesting question is that of the dependence or independence among the various versions of a problem (in particular in the case in which the data are defined using intervals).

Taking the concept of robustness into account in decision aiding tools and techniques calls for the development of specific concepts and tools. This development is likely to be of central importance in the next few years. It will considerably enrich the toolkit of analysts.

8

CONCLUSION AND PERSPECTIVES

8.1 Did we keep our promise?

This book follows a volume published in 2000 by the authors and Patrice Perny. Five years ago we wrote (see Bouyssou et al., 2000, p. 244):

> *At this point it should be apparent that research on formal decision and evaluation methods should not be guided by the hope of discovering models that would be ideal under certain types of circumstances. Can something be done then? In view of the many difficulties encountered with the models envisaged in this book and the many fields in which no formal decision and evaluation tools are used, we do think that this area will be rich and fertile for future research.*
>
> *Freed from the idea that we will discover* THE *method, we can, more modestly and more realistically, expect to move towards:*
>
> - *structuring tools that will facilitate the implementation of formal decision and evaluation models in complex and conflictual decision processes;*
> - *flexible preference models able to cope with data of poor or unknown quality, conflicting or lacking information;*
> - *assessment protocols and technologies able to cope with complex and unstable preferences, uncertain tradeoffs, hesitation and learning;*
> - *tools for comparing aggregation models in order to know what they have in common and whether one is likely to be more appropriate in view of the quality of the data?*
> - *tools for defining and deriving "robust" conclusions.*
>
> *To summarise, the future as we see it: structuring methodologies allowing for an explicit involvement and participation of all stakeholders, flexible preference models tolerating hesitations and contradictions, flexible tools for modelling imprecision and uncertainty, evaluation models fully taking incommensurable dimensions into account in a meaningful way, assessments technologies incorporating framing effects and*

learning processes, exploration techniques allowing to build robust recommendations (see Bouyssou et al., 1993). Thus, "thanks to rigourous concepts, well-formulated models, precise calculations and axiomatic considerations, we should be able to clarify decisions by separating what is objective from what is less objective, by separating strong conclusions from weaker ones, by dissipating certain forms of misunderstanding in communication, by avoiding the trap of illusory reasoning, by bringing out certain counter-intuitive results" (Roy and Bouyssou, 1991, see).

This "utopia" calls for a vast research programme requiring many different types of research (axiomatic analyses of models, experimental studies of models, clinical analyses of decision/evaluation processes, conceptual reflections on the notions of rationality and performance, production of new pieces of software, etc.).

The authors are preparing another book that will hopefully contribute to this research programme. It will cover the main topics that we believe to be useful in order to successfully implement formal decision/evaluation models in real-world processes :

- *structuring methods and concepts,*
- *preference modelling tools,*
- *uncertainty and imprecision modelling tools,*
- *aggregation models,*
- *tools for deriving robust recommendations.*

If we managed to convince you that formal decision and evaluation models are an important topic and that the hope of discovering "ideal" methods is somewhat chimerical, it is not unlikely that you will find the next book valuable.

Well, the "next" book is now in your hands. Did we manage to keep the promise that we made five years ago? Although you remain the ultimate judge, we think that it has been kept, at least partially.

While writing this book, we quickly realised that we could not give an exhaustive view of all the current trends of research in decision aiding. Significant parts of the field, such as decision under uncertainty and combinatorial optimisation, had to be neglected. We finally decided to concentrate on "multiple criteria", although we are well aware that in many important situations this may not be the central issue. Yet, our feeling is that, we have contributed to the "utopia" announced earlier. Indeed:

1. Our presentation, although it is not exhaustive, is carried out within a unique frame that can be extended to most (all?) decision and evaluation models: the establishment of a "decision aiding methodology". This is a step towards *a coherent structure of reasoning about theories and practices concerning deciding and aiding to decide.* We tried to show that different perspectives on practice as well as different decision theories can be unified within a "unique methodology", the layout of which is introduced in this book.

2. This book summarises concepts, findings and results obtained by the authors and the larger OR/MS community over the last 15 years in a research project which was summarised in the "Manifesto of a new MCDA era", in which we claimed (Bouyssou et al., 1993):

   > *So what? We feel that at the beginning of the new age of MCDA some priorities have to be settled. We do not need new methods that just extend old ones or complicate already existing procedures. We do not need conventional examples and applications that do not allow us to learn more about MCDA. We believe that two main subjects should be explored:*
   >
   > - *theoretical and axiomatic foundations of MCDA at all levels (approach, methodology, methods);*
   > - *conceptual and operational validation of the use of MCDA in real world problems.*

   The results obtained since then are sufficiently encouraging not only to keep going on, but also to try to summarise them in this volume.

8.2 Decision Aiding Methodology using stepping stones

Hopefully reading this book has helped you realise that a decision aiding methodology is not just a collection of methods with some underlying theory. Indeed, we cannot reduce decision aiding to the mere application of some formal methods that "faithfully" report the decision maker's problem, preferences and values. As discussed extensively in chapter 2 decision aiding is a process, during which a number of "shared cognitive artefacts" are constructed through the interaction of the participating actors, that is, at least, the client and the analyst. The main cognitive artefacts are:

- a representation of the problem situation;
- a problem formulation;
- an evaluation model;
- a final recommendation.

Each of such artefacts contains precise elements of information, the presence of which must have a justification. Such a justification comes from:

- the fact that the client and the analyst agree that these artefacts are relevant for the decision process for which the decision aiding was requested;
- the fact that such elements constitute a consistent body of information, where consistency is provided by axioms and theorems established in Decision Theory and Operational Research.

The content of these cognitive artefacts is not the result of a straightforward process, but the reasoned result of the interactions between the client and the analyst. There are no "objective" elements within such an artefact, but elements "subjectively" chosen as useful by the client and the analyst. This is again illustrated by the following example.

Example 8.1
A regional authority wants to establish a health care policy. In doing so, it may consider the opinion of its "health officers" relevant. This is a choice. Not all regional authorities will have the same attitude. We have to take this option into account when providing decision aiding in such a situation. If, in order to implement the above policy, the regional authority decides to listen to the trade unions or the individuals, this is again a choice which will affect the information collected and possibly the outcome of the process. Furthermore, the way in which such an opinion will be considered is a choice: it can be considered to be a constraint (hard or soft), it can be considered to be a criterion among others or it can finally simply be ignored. These are examples of critical options to be considered when involved in a decision aiding process. Last, but not least, in the case the client (for the same problem situation: the new health care policy in a certain region) is not the regional authority, but another actor involved in the process, all the above choices could be totally different. Aiding somebody to decide means being able to assist him in all such choices. Aiding a client in a decision process is not only the construction of a model comparing policies, but also the process with which these policies are conceived, shaped and analysed. $\diamond$

Decision aiding is always viewed as a decision process in which a "client" asks for the advice of an "analyst". In this book, we have chosen a simplified presentation of this process in which the client and the analyst are seen as two interacting individuals. However, a client is not necessarily a decision maker (he could for instance be an adviser to the decision maker). Furthermore a client is not necessarily an individual, but could be a collective body (a board of directors, a committee, a group of experts, a social group etc.). The motivation for asking advice is not necessarily "to make a decision", but to construct an argumentation or a justification. Finally, an analyst is not necessarily an individual, but may be a group of analysts. There might be a "chain" of analysts, each being the client of another. A real decision aiding process is always a complex reality of interactions occurring within real decision processes. Our simplified representation of such a process has been conceived for two reasons:

1. our aim is not to make a "sociological" analysis of the decision aiding process (while this is also an important field of research), but to identify which cognitive artefacts characterise the process in order to be able to conduct it; in other words we try to provide a guide, a handbook, some stepping stones for those who, for some reason, are in the position of analyst;

2. even in the most complex decision aiding situations there will always be two distinct actors (almost always two individuals) who will have to argue about what the problem is, how to formulate it and how to solve it; they represent the "not further decomposable" units of the decision aiding process.

8.3 Decision aiding approaches and tools

In this book we have discussed a number of different decision aiding approaches, i.e. different perspectives about conducting of a decision aiding process. They are essentially different insofar as their assumptions about the origin and the nature of the rationality model to be introduced in the decision aiding process are concerned. We claim that such approaches are not collections of methods, although this is a common way to classify decision methods. Indeed, what we distinguish are not methods, but how these methods are or can be used. For instance, optimisation methods can be used in a constructive way, while outranking methods can be used in a normative way. Normative, descriptive, prescriptive and constructive approaches represent general directions on how a decision aiding process is conducted and therefore represent a key part of a decision aiding methodology.

This having been said, we consider that a decision aiding methodology also contains a toolbox of methods, protocols of interaction, procedures, algorithms and concepts. When facing a problem situation, the analyst has to use such a toolbox. The issue is then how to use it consistently in order to provide the client with a useful, meaningful and legitimated recommendation. Indeed, this book mainly aims at providing elements allowing to construct and use formal models of different natures. We wanted the reader of this book to be able to use formal models and tools in a reasoned and informed way. We tried to provide some stepping stones in this direction. More precisely, we decided to cover the main tools used for the construction of decision support models and methods in the presence of multiple criteria. We distinguished three classes of such tools:

- preference modelling tools;
- preference aggregation tools;
- final recommendation tools.

The reader may have already noticed that several of the tools we discuss can also be used when where multiple criteria are not present. Preferences are modelled in any type of decision support model and under any approach. Aggregation procedures are extremely common in many situations in which no criterion is modelled (such as when we aggregate uncertainties or measures). Some algorithms presented in chapter 7.1 are derived from graph theory and, as such, have wider applications than the ones discussed here. We have not discussed such extensions in this book, although they may prove important features of a decision aiding methodology.

8.4 Stepping stones for preference modelling

Modelling preferences is the essential and elementary activity of any decision aiding process. Preferences always refer to somebody and to a given problem situation and formulation. As such they always represent the "subjective" dimension of any decision support model. *There are no "objective preferences", as is no "objective decision support".*

There are two different problems in preference modelling. In the first one objects, for which preferences have to be expressed, are described using one or several attributes to which a "measurement scale" is associated. From this information we try to derive a preference model. Consider the case in which two objects have respective lengths of 10 cm and 12 cm: we want to know which one is preferred to the other and under which preference model.

In the second one, we already have a set of preference statements (possibly expressed directly by the client) and we want to know whether there is a preference model that captures such statements. Furthermore, we want to know whether a numerical representation equivalent to such a model exists, i.e., one or more real valued functions on the set of objects for which the preference statements have been expressed, such that the relations between the numerical values are equivalent to the preference statements. Consider the case in which the client claims that a is indifferent to b which is indifferent to c, but a is preferred to c. We are looking for one or more functions $u : \{a, b, c\} \rightarrow \mathbb{R}$ that will associate a real number to each of the objects in such a way that we can represent the above-mentioned statements comparing these numbers in some way.

In both cases the analyst has to pay attention to:

- the properties the numerical scales have or could have; this is important when such information (the scales) have to be further used in the decision aiding process since it affects the meaningfulness of the manipulations we carry out (as for instance when we aggregate measures or preferences);

- the properties that preference models fulfill (such as completeness or transitivity), since again these can allow the use of certain numerical representations and/or of certain methods;

- the fact that although there is a limited number of preference models available in the literature, they are sufficiently flexible to cover most of the preference statements a client can address within a problem situation, including situations of conditional preferences, ambiguity, uncertainty and/or inconsistency; it is important therefore, to look carefully for the most appropriate model;

- the fact that numerical representations of preferences are a very elegant and easy to handle tool, but by no means the only way to elaborate recommendations; it is possible to work with the preference statements modelled in a different way without necessarily looking for a numerical representation, which might not even exist.

8.5 Stepping stones for preference aggregation

Aggregating preferences is one of the main technical problems in Multiple Criteria Decision Analysis methods. Indeed a large part of this book are dedicated to this problem (chapters 4–6).

The point of view adopted is to help the analyst to make better use of a toolbox of techniques, rather than describing a number of methods exhaustively (although several of them are briefly described in the text). As with preference modelling, we consider two different perspectives of the preference aggregation problem.

The first perspective focuses on *aggregation procedures* and is inspired by related results in social choice theory. Given preferences expressed on several criteria, we try to construct or identify the most appropriate procedure to perform a synthesis of the preferences and the most appropriate protocol to obtain the necessary preferential information that such a procedure may require. Despite their differences, most of the procedures that were analysed share several common features that we tried to uncover in the text, while emphasising the specific characteristics of a number of well-known techniques.

The second perspective focuses on the client's *preferences* and the models that can represent them. The idea here is, how to interpret the client's global preference statements when there is an underlying multi-attribute structure. We obtain sets of conditions that preferences have to satisfy in order to be represented using a number of models. Such conditions allow the comparison of models and, most importantly, give hints on how to assess them. Such a perspective is clearly inspired by conjoint measurement theory and extensively discussed in chapter 6.

We can summarise some stepping stones for the analyst as follows.

- There is no unique and/or universal procedure or model to aggregate preferences. A preference aggregation procedure has to be discussed, chosen, validated and justified as appropriate within the decision aiding process (in the evaluation model) given the information available and the problem formulation adopted. Chapters 4–6 show that this is not an impossible task.

- A preference aggregation procedure tends to impoverish the information available before aggregation. This means that from "poor" information we cannot construct a "rich" result without adding information. The client has to be aware that in order to obtain a "rich" result he has to provide more information and this can be costly (not only in monetary terms) and painful.

- Almost all preference aggregation procedures make use of specific parameters (e.g., tradeoffs, importance coefficients, thresholds, beliefs) that have to be assessed. Quite often, there are specific protocols to assess such information, which take the client's cognitive effort and the biases possibly arising from the client/analyst interaction into account. The analyst has to take care in using them appropriately.

- It is not uncommon that applying a preference aggregation procedure implies making several hypotheses that are difficult to verify. These hypotheses have to be explained to the client who has to understand their consequences. The client should understand the logic of the models that were used. He should feel the owner of the models.

- The axioms characterising preference aggregation procedures and models are not just their mathematical description. They have to be seen as properties that may or may not be desirable or required. Knowing the axioms characterising a certain procedure is much like knowing the "properties" of a cruciform screwdriver: they give hints on how, where and when to use it.

- Despite their variety, preference aggregation procedures and models share several common features and can be sorted into in a limited number of classes. The analyst tempted by the creation of a new ad-hoc procedure should take care to verify whether what he is trying to do makes sense and has not already been analysed in the literature (and the book you have in your hands gives a reasonable sample).

8.6 Final stepping stones

Decision aiding is a process which starts in the real world (recognising the problem situation) and step by step moves towards formal modelling (indeed, it is based on the use of a formal language). However, at a certain point, it has to come back to the reality and formulate a final recommendation for the client.

From such a perspective, the results obtained through the use of the evaluation model remain in the abstract and formal world and do not necessarily represent something which can be directly used by the client for his concerns and purposes. After all, a client asks for your advice in order to buy a car, not to learn his value function for cars.

The return to reality is a mix of formal and informal activities. Informal for the validation and legitimation of the results regarding the decision process for which the aid was requested. Formal for the elaboration of final recommendation from the rough result of the evaluation model. Chapter 7.1 is dedicated to this last part of formal decision aiding activities. The problem here is that the evaluation model can elaborate a synthetic representation of the client's preferences, beliefs, judgements and assessments, but may not provide a direct answer to the problem statement agreed upon in the problem formulation. If this is the case (and it often is), then we still need one further step to obtain such a specific answer. This means, e.g., going from a global value function to a best choice; establishing a subset of "good" candidates (not identical to the subset of the better ones) from some pairwise comparisons between candidates and profiles, etc. In performing such a final step, the analyst should take into account the fact that:

- most of the procedures elaborating the final recommendation are algorithms, which fulfil some specific properties (and not others) that should be analysed with care. There is no straightforward procedure in performing this step (as was the case for the previous ones). It has to be chosen and justified.

- It is not uncommon that several such procedures will (again) introduce some arbitrary hypotheses. This has to be discussed with the client, who has to understand them and agree on their use.

- Sensitivity analysis (in the sense of analysing the behaviour of the recommendation with respect to perturbations of the evaluation model parameters) is an essential activity to be pursued at this step. It enable the analyst to give a more convincing result to the client and to construct a reasoned argumentation.

- Robustness analysis (in the sense of establishing whether the recommendation will still hold under different scenarios of information and combinations of parameters) is a critical activity to be performed at least at this stage and for which sensitivity analysis is not a substitute. Being able to provide a robust recommendation can be of invaluable help to the client and should be the ultimate aim of a decision aiding process. We may even claim that robustness should be an issue to consider when formulating the problem.

8.7 And after all this?

Our hope "after all this" is that the stepping stones we presented and justified in this text will be really helpful to the analysts who try decision aiding in the real world. We will be happy to receive feedback on this point.

On the other hand this book concludes (we hope positively) an experience started over 10 years ago, trying to condensate theoretical and practical knowledge about decision aiding into a methodology. We are aware that despite our efforts (and the efforts of a whole community carrying out research and practice in decision aiding), questions of capital importance remain unanswered. They deal with both the theoretical foundations of our discipline and the practical carrying out of decision aiding processes. Indeed, what is presented here remains far from a "ready to use methodological compendium".

This means more research. We need to further investigate theoretical questions in preference modelling, in decision making under uncertainty, in axiomatising protocols, algorithms and models. We need further research in order to understand the dynamics of decision aiding processes and the relations between their cognitive artefacts. We also need more insight into our practical experiences and professional activities to enhance our knowledge about successes and failures.

In other words: there is still a lot of work to be done. But this is another story. Our hope is that we have motivated you enough to contribute. Who knows; it may be that one day, some of us continue it. Until then, so long...

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