Modern Microeconomics

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1st edition

ISBN 978-87-403-0419-0

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Preface

This book aims at fulfilling the curriculum requirement of the master's degree students of Microeconomics, a most practical subject with its many applications. This book will be very useful for microeconomic policy making from the local to the global level. It is mainly an attempt to explain an individual's behavior and a firm's production function. It elucidates issues like individual utility, saving, income, consumption, information economics, and game theory. General equilibrium, Pareto efficiency, and social welfare are some of the significant topics covered. The various dimensions of the production function, the economics of games, information and logic, and market failure comprise the core of this book.

This book will help students to think, analyze and apply microeconomic issues practically. Various industry-related examples such as prices, output, employment and games are used to make understanding the microeconomic issues in detail easier. By providing insights to students, teachers, policy makers, planners and academicians to think about various current microeconomic issues at different levels, this book will ultimately help to solve some of these issues. The consumer choices, preferences, and risks operate at much more complicated levels in microeconomic analysis. Each individual has a unique utility function, making it difficult to study each individual's behavior. Similarly, each firm always tries to produce more commodities to sell in markets in order to obtain the highest possible profits. Therefore, general conclusions are derived from different case studies in this book.

This advanced microeconomics book provides fundamentals of the basic microeconomic identities. It will also assist students from other educational streams to understand microeconomic issues. This subject of microeconomics will be easy to understand by the book's use of examples dealing with issues relevant today.

This book is divided into three parts. The first part explains the topics related to consumer behavior. The second part deals with the industry where production function and game theory are explained. Both these parts are equally important because the first part provides the basis for understanding the second part. The second part needs a slightly more comprehensive knowledge of microeconomics. The third part of the book gives an explanation of Walrasian equilibrium, Pareto efficiency, social welfare and market failure. Besides these, some current issues such as game theory, the Cournot model, the Lemon theory and signalling are also explained.

The basic concepts of microeconomics are covered in the first chapter. The concepts of consumer preference with revealed preference, rational choice and utility maximization are introduced here. The demand function and indirect utility function are also explained in detail. The expenditure function and the Hicksian demand function are also covered in this chapter. The Von Neumann-Morganstern utility function explains the lottery framework of the consumers. The best and worst lotteries with different probabilities are compared. Lastly, the risk aversion of consumers as a part of their utility function is discussed.

The second chapter covers the production function. The specifications of technology, input requirement set, convex technology and technical rate of substitution are discussed in this chapter. Homogeneous and homothetic production functions, and the duality of cost are also covered. The last part of this chapter explains what factor share is and the contribution of Kalecki and Kaldor in the study of the production function.

The third chapter elucidates what information economics is, and explains the economics of game theory. Every game has few players but each game is played with certain rules. A Nash equilibrium is an outcome of the various types of games. This chapter explains the various types of games such as co-operative and non-cooperative game, welfare game and the principal agent model. Firms' production- related games are explained along with the Bertrand and the Cournot models in the second part of this chapter. The game of entry deterrence forms the last part.

The fourth chapter describes game theory and its applications. The moral hazard and principal agent problem is explained in detail. Lastly, the Lemon theory, pooling and separating equilibrium, signalling and screening are also explained. These topics are part of advanced microeconomics and are important from a policy point of view.

The fifth chapter covers general equilibrium and social welfare. The Walrasian equilibrium with exchange and production is explained. The second section deals with the existence, uniqueness and stability of equilibrium, and explains the welfare properties of a general equilibrium. The last section deals with the measurement of welfare, where market failure and externality are explained with different examples.

Acknowledgement

Many researchers and academicians have made their unique contributions to the field of Advanced Microeconomics. This book of mine is but a tiny contribution to the vast knowledge available out there. But sincere efforts have been made to study the consumer and the firm and the factors affecting them. My words fall short to express my deep sense of appreciation to my research guide, Professor Dr. Neeraj Hatekar of the Department of Economics at the University of Mumbai. During my postgraduate study, he gave me insight into the economics of game theory – the theory as well as its applications. His research in game theory and its application in different fields of economics forced me to think and write about microeconomics. His continuing support in my research endeavors is a genuine source of inspiration.

I have been inspired to write this book because of Professor Dr. Indira Hirway, and the Director of the Center for Development Alternatives (CFDA) in Ahmadabad, India. Her work on labor and gender economics and time use study has helped me understand the various issues at the micro level in detail. She made many efforts to teach me the theory and application of microeconomics. During field work in Gujarat, all microeconomic issues were discussed by our team. This discussion helped me to understand microeconomics issues in the most practical way.

I wish to express my heartfelt gratitude to Dr. Sangita Kohli, the Principal of the S.K. Somaiya College of Arts, Science and Commerce, for her continued support and encouragement, right from the planning and writing of this book. I am also thankful to Mrs. Charlotte Braganza, the Vice-Principal at the S.K. Somaiya College of the University of Mumbai, for her consistent support in my research work. By making me a coordinator of the M.A. (Economics) course, she inspired me to work harder, enabling me to find solutions to microeconomic problems and to deepen my knowledge in the subject as a whole.

I would like to thank Dr. Raji Ramesh at the Department of English for his valuable suggestions and help during the research work. I owe a very special gratitude to Mrs. Smitha Angane at the Department of Statistics and Mathematics who has always encouraged me to concentrate on my study rather than on various administrative issues at different levels.

I would like to extend my deep appreciation to the administrative staff of the S.K. Somaiya College at the University of Mumbai, particularly to the librarian, Mr. Sanam Pawar, and to Mr. Mane for their immense help which have allowed me to fulfill the requirements smoothly. I am thankful to my friend Mr. Srinivasan Iyar for some very fruitful discussions on various aspects of this book. Mr. Amit Naik and Mr. Anant Phirke have been a continuous source of inspiration and lent a hand when needed. Their affection and backing served to encourage me during my research work. I must also acknowledge the support of my numerous friends and associates, particularly, Mr. Rajendra Patil and Mr. Rajendra Ichale.

Finally, I would like to express my affectionate appreciation of my mother and father. They have always defended my study and encouraged me to study different issues in microeconomics and to write about these issues. I am especially thankful to my aunt and my uncle, who is keen to understand the various micro economic behaviors of consumers. My brother, Mr. Shantaram Rode, constantly provided moral support during difficult times. I am thankful to many of my friends from different organizations and colleagues. Without their help and support, this work would not have seen the light of day. Last but not the least, I would like to thank my postgraduate and undergraduate students

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1 Consumer preference and utility

1.1 Introduction

Generally, consumers prefer a certain number of commodities. Some of these commodities are normal goods and can be substituted by others. The price of such commodities has an impact on demand. At the same time, the substitution of some normal goods is not possible. Such commodities are bought at fixed intervals and they are part of the basic consumer basket such as food, water, power and gas. Consumers buy normal goods at a regular price and as a proportion of their income. But the prices of goods can change and so can consumers' income. The supply of commodities can also be affected by a number of factors. The consumer decides what to purchase and what not to purchase. A consumer's preference changes when the consumer decides to maximize utility at a lower price level. For this purpose, any consumer prefers to have a number of options. Such preferences and choices are the subject matter of this chapter.

1.2 Preference relations

Any consumer preference has two important relations. The strict and weak preference relation decides the overall preference. The strict preference relation is defined as

$$x > y \rightarrow x \geq y \text{ but } y \geq x \tag{1}$$

This means x is preferred to y. Here, y is also considered to be equal to x. The consumer can either prefer x or y. For example, as a commodity, an apple is as good as an orange. It is up to the consumer which fruit s/he chooses. But it is not always true that when choosing an apple over an orange, the consumer always thinks that it is better to buy an apple than to buy an orange. Sometimes an orange is as good as an apple. It depends on the consumer's perception of such a good. The indifference relation ~ is defined as the tilde and it means approximately

$$x \sim y \rightarrow x \geq y \text{ and } y \geq x \tag{2}$$

In other words, x is indifferent to y; alternatively, x and y are of the same order of magnitude. The chosen quantity of x will give equal satisfaction of y to the consumer, as commodity x is equal to y in terms of consumption. At this point, we can ignore the taste, color and size of each commodity. Sometimes, it is difficult to differentiate commodities from each other. Each consumer is assumed to be rational in their thinking. A consumer will always try to maximize their own utility. Quite often the consumer will choose the commodity which has a lower price.

The rationality hypothesis has two basic assumptions:

Completeness

Completeness means $x, y \in x$, we have $x \geq y$ or $y \geq x$ or both. Any commodity is always preferred to its close substitute. They are very different from each other in terms of characteristics. For example, Colgate toothpaste may be preferred to such substitutes as Close Up, Pepsodent or Sensodyne. The qualities of each toothpaste such as size, color, shape, and price are important for a consumer to prefer it but the choice also depends on the income and the taste of the same consumer.

Transitivity

Sometimes, more commodities are available as substitutes. It means $x, y \in x$; we can say that if $x \geq y$ and $y \geq z$ then $x \geq z$. Completeness means that a consumer is able to express a preference or s/he is indifferent to any pair of consumption bundles. In other words, the consumer remains indifferent after consuming any commodity which s/he has demanded. There are a number of reasons for such preferences. These reasons ensure that there are no problems in the preference ordering. Transitivity is slightly different from completeness. It implies that it is impossible to face the decision maker with a sequence of pair-wise choices in which preferences appear to be cyclical. To illustrate: an apple is as good as an orange and an orange is as good as a mango. But then an apple is preferred to a mango. Transitivity explains that consuming any goods will give equal satisfaction to the consumer. The concept will be clearer in the following assumption.

Reflexivity

Reflexivity means any commodity is as good as any other commodity. There is no difference in consumer satisfaction when any commodity is consumed. For example, if $x > y \geq z$ then, $x > z$. The preferences do not change. The consumer prefers a bundle and any bundle is preferred or is indifferent to itself. The two bundles are indifferent to themselves which seems to be trivially true. The consumer remains indifferent after preferring either x, y or z. However, the implication is a little important; it ensures that every bundle belongs to at least one indifference set, namely, the set that contains itself, if nothing else. In the everyday consumption basket, a consumer often purchases different close substitutes of commodities based on taste, color, size, etc. and on their income.

Nonsatiation

Nonsatiation adds more characteristics of a good because it is preferred by the consumer. Each consumer expects something different from the earlier purchased commodity. Consumers regularly purchase commodities from the market and they have perfect knowledge of available commodities. Some consumers expect higher discounts on some purchased goods. In the modern world, such a discount is offered by all sellers. Alternatively, if $x > y$, x contains more of at least one good and no less of any other. Sometimes, a small gift is given as an additional commodity. It is difficult to specify a discount for different types of commodities. It is the policy of any firm to try to increase sales of commodities. The nonsatiation characteristic explains that a consumer is assumed to be never satisfied with goods. Consumers always search for something extra above what they usually buy from the market. Each manufacturer also changes the size, color, taste, and the type of packaging of a commodity. This implies that none of the goods is in fact a bad commodity. For example, a consumer will purchase Colgate toothpaste but at the same time will look at what kind of discount is given on the toothpaste. The consumer will see that a free toothbrush can be obtained with each tube of toothpaste bought, or that there is an additional quantity of toothpaste available at the same price. The consumer buys the brand if the discount or gift is desired enough, otherwise they will purchase the toothpaste brand with the lowest price. Sometimes, shops in malls offer discounts on different occasions. Most of the people who visit such places go there to get commodities at bargain prices. But most economists have different opinions on the quality and quantity of goods at such places, and at this point, we will not be discussing this issue. It is the consumer who decides what to purchase and what not to purchase.

Continuity

Consumers always try to switch to different commodities or to close substitutes. They get more satisfaction from consuming different commodities at lower prices. But other factors may make them worse off, such as changes in price, size, etc. Suppose there are two goods in a consumer's consumption bundle; if we reduce the amount of one good and increase the amount of the other good to compensate, the consumer is left with a consumption bundle very different to the first. This is the compensation variation which we will study in the next section.

Strict convexity

Strict convexity assumes that consumer preferences are related to two commodities. Given the feasible set is convex; the consumer's optimal point will be a unique local point. This is shown in the following figure, Figure 1.1. The downward sloping line shows the indifference curve. The smaller the x amount, the larger the y amount preferred by the consumer. In other words, the change in marginal utility of x relative to change in marginal utility of y is almost the same. This is the consumer preference curve and a significant property of the indifference curve.

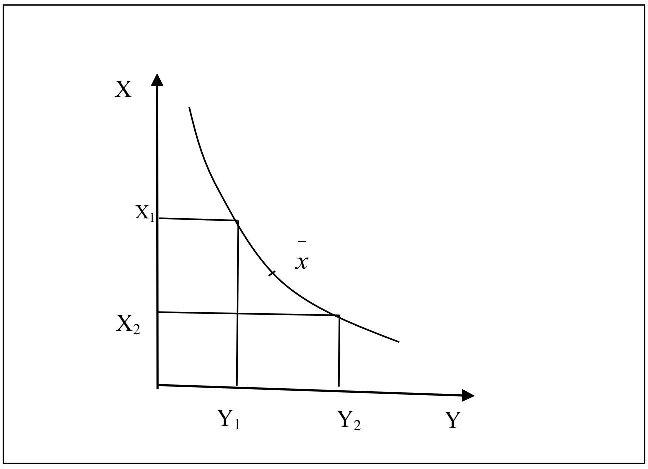


Figure 1.1. Indifference curve with two commodities

1.3 Utility function

In microeconomics, the consumer's utility from consuming different commodities can be measured. Suppose for the utility function u(x) we assign a numerical value to each element of x then the ranking of the elements of x is marked in accordance with the individual's preference. Then, a function u: $X \to R$ is a utility function. This represents the preference relation $\gg$ if for all x, y ϵ x , $x \geq y \leftrightarrow u(x) \geq u(y)$ for any strict increasing function f. $R \to R$ and U(x) = f(u(x)) is a new utility function representing the same preferences as u (.). It is only the ranking that matters in making alternative choices. After all, it is the consumer's preference and they are assigned some numbers. We can measure this with different examples. The ordinal utility function is known as the property of the utility function. They are constant for any strictly increasing transformation preference relation and are associated with a utility function. This is an ordinal property of utility function.

Property 1: The consumer preference relation ≥ can be represented by a utility function only if it is rational.

Solution: Let's suppose a utility function that represents consumer preferences ≥ then ≥ must be complete and transitive. This can be explained with the help of the properties.

Completeness

Suppose utility {u (.)} is a real valued function defined as x then it must be that for any $x, y \in x$ either $u(x) \geq u(y)$ or $u(y) \geq u(x)$. But u(.) is a utility function and it is representing $\geq$; this implies that $x \geq y$ or that $y \geq x$. Here, $\geq$ must be complete.

Transitivity

Let's suppose that $x \geq y$ and $y \geq z$ because u(.) represents $\geq$. We must have $u(x) \geq u(y)$ and $u(y) \geq u(z)$. Therefore, $u(x) \geq u(z)$. This is because u(.) represents $\geq$ and this implies $x \geq z$. It is shown that $x \geq y$ and $y \geq z$ imply $x \geq z$; in this way transitivity is established. These two properties can also be explained with the consumer choice rule.

Consumer choice rule

The preferences for commodities are nothing but the choices made by the consumer. A consumer has a routine number of choices. Choices for particular commodities are perfect and they are done after some mental exercise or psychology. When choosing preferences, a consumer is ready to choose different products depending on their habits and likes. In other words, the preference is a choice or it is a decision made by the consumer. A consumer often faces a tradeoff between purchasing small quantities of a high quality and expensive product and purchasing large quantities of a lower quality and hence cheaper product (Epple & Sieg, 2010).

In the simplest model, a sequence of people, each in turn chooses one of two options, A or B, with each person observing their predecessor's choices. They have common preferences over the two choices but do not know which is better. Rather, they receive independent and equally strong private binary signals about the right choice. In this setting, rational agents herd. Once the pattern of signals leads to two more choices of one action than the other, all subsequent people ignore their signals and take that same action (Eyster & Rabin, 2010).

The choice behavior is a choice structure sets and each set has two types. Firstly, S is a family of nonempty subsets of x. Every element of S is a set $єx. The element $єs is made up of budget sets. The S set is a list of all choices available to a consumer. It includes all subsets of x. Secondly, the c is a choice rule that assigns a non-empty set of chosen elements c(h,i). Each budget set is written as $ єs. When c ($) contains a single element of c ($) then it is the alternative in $. The consumer might choose such alternative. A consumer repeatedly faces the problem of choosing an alternative from set $.

1.4 Lexicographic ordering

Lexicographic ordering is also known as lexical order, a mathematical concept that supports the assumption of continuity. It works with a numerical representation of consumers' preference ordering. Sometimes, the numbers are not true because they are individual specific ordering and they get changed from time to time. Lexicographic ordering assumes the simple utility maximization and function. It satisfies the four assumptions of lexicographic ordering - continuity, transitivity, reflexivity and non-satiation. If any consumer orders two commodities and both commodities are necessary to him then the commodities will be consumed in a particular proportion, and are ordered in a different form. Consumption of both commodities will add to the consumer's utility function, which can be shown to satisfy the first four assumptions of completeness, transitivity, reflexivity and non-satiation. It can also be shown to give rise to a well-defined demand function, which implies that the continuity assumption is not necessary for their existence.

Theorem

As pointed out in the above paragraph, we assume that there are two goods ordered by the consumer because the consumer likes these commodities. The ordering can take place in the following form.

$$x' = (x_1', x_2') \,\&\, x'' = (x_1'', x_2'') \tag{3}$$

a) $x_1' > x_1''$ implies $x' > x''$ (4)

b) $x_1' > x_1''$ and $x_2^1 > x_2''$ implies $x' > x''$ (5)

In the model, a consumer prefers a bundle with more of the first good in their consumption basket, regardless of how many there are of the second good. Sometimes, the consumer's bundle contains the same quantity of first and second goods. We can take an example of a drinking man. He requires wine and bread in his consumption basket. As a drinking man, he prefers more wine in his bundle but at the same time, bread is also important for him. He will choose to have different combinations of wine and bread. Sometimes, he will try to ensure to have the same amount of bread and wine in his consumption bundle, sometimes, more of one or the other. This is called lexicographic ordering.

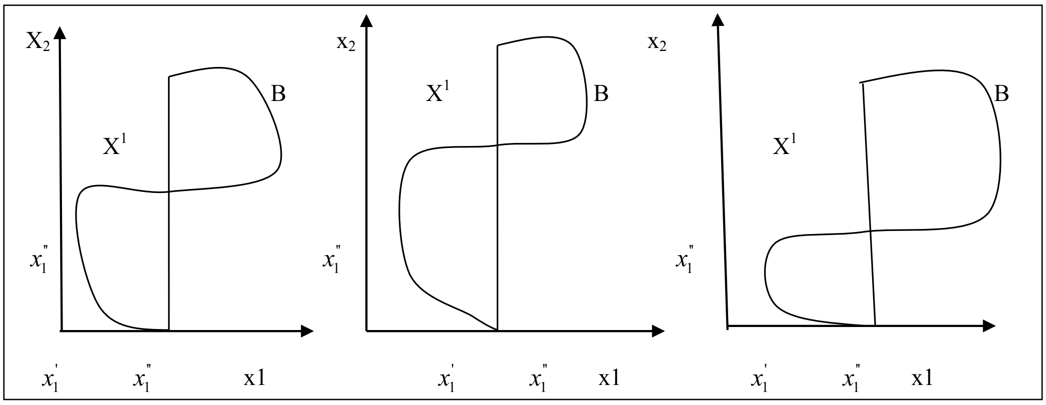


Figure 1.2. Lexicographic ordering for commodities

If we consider a consumption bundle $x'(x_1^{'}, x_2^{''})$ with wine and bread, then all the points in B are preferred to wine. The other quantity preferred is of bread. There are three figures presented with different combinations of bread and wine. The first figure shows an equal proportion of wine and bread in the consumption basket. The second diagram shows less wine and more bread is preferred. The third diagram shows that more wine and less bread is preferred. Each diagram shows an individual's preference and it is difficult to measure such choices. Such ordering is difficult to study all the time. Therefore, it has been criticized in the following way.

Criticism

Suppose the drinker chooses wine arbitrarily, then the other points in the shape are for bread only. The lexicographic ordering is criticized. Firstly, it does not satisfy the continuity assumption. To show the continuity assumption, the points should make up a continuous curve. But Figure 1.2 shows the indifference set which are the assumed points and not a continuous curve. The indifference curve assumes that the two commodities are indifferent. Suppose we reduce a small amount of the wine in the bundle to replace the bread then we find that no amount of bread can be replaced for wine. Therefore there is no continuity in the lexicographic ordering. It is common knowledge that drinking men or women prefer more wine to bread. Secondly, it is not possible to represent the use of the utility function through lexicographic ordering. If we divide the real line into non-empty disjoint bounded intervals, the set of these intervals is not countable. The positive half of the real line is countable and must be false. It is difficult to collect data of lexicographic ordering. Similarly, inter and intra individual choices differ all the time. The theory is thus only known for its contribution in advanced microeconomics.

1.5 Demand function

From the lexicographic ordering, we can assume that the drinking man or woman has M income. Suppose s/he faces a price p_1 for one bottle for wine and price p_2 per loaf of bread then s/he is free to spend her/his entire income on wine. The demand function can be written as

$$X_1 = M/P_1 \ , \ X_2 = 0 \tag{6}$$

Here, $X_2 = 0$ because the drinking man or woman does not spend their income on bread. The demand is just a rectangular hyperbola (x_1, p_1). The demand function for bread (x_2, p_2) is a space in the vertical axis.

The existence of the utility function

We have already observed that the lexicographic ordering satisfies the completeness, reflexivity, transitivity, and nonsatiation assumptions. But lexicographic ordering does not satisfy the demand of goods. It only gives the preference of two commodities. The continuity assumption guarantees that a continuous increasing utility function can be found to represent the preference ordering.

Figure 1.3 shows that the indifference curve is continuous. There are two commodities x_2 and x_1 which are shown on the x and y axis. The equilibrium point E is intersecting at a 45^0 line. Any point on the indifference curve (x^0) is associated with real number $u(x^0)$. In the diagram, E is a point at which the indifference curve x^0 cuts the 45^0 line. The utility (u) values on the axis are a utility function for the consumer. The indifference bundles have the same utility values but the higher preferred bundles have higher utility values.

In the figure, the utility function is u (x'). In a given bundle $x^0 = (x^0_{1 \dots\dots\dots} . x^0_n)$, the consumer chooses the $\underline{X}^0$ and $\bar{X}^0$. We have assumed that the two commodities are indifferent. This means that $\bar{X}^0 = \underline{X}^0$. This assumption satisfies the continuity, transitivity, and other assumptions. We also assumed that $u(x^0) = \bar{X}^0$ is equal to $u(X^0) = \tilde{X}^0$. These points are shown below.

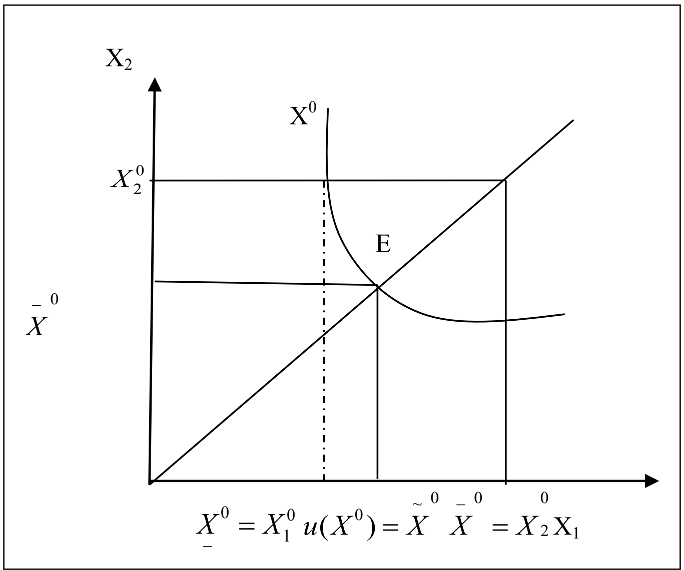


Figure 1.3. Consumer utility function

The transitivity and nonsatiation assumptions show that x values for $1.x > x^0$ are strictly greater than the complementary sub-interval for which $x^0 > 1.x$. The formal rule interval has a lower bound at lower levels. The latter rule interval has an upper bound. These bounds must be the same.

The Utility theorem

The u(x) constructed for two bundles x^0, x^1 satisfies the definition of a utility function.

$$u(x^0) \geq u(x') \Leftrightarrow X^0 \geq X^1 \tag{7}$$

The above function can be proved in two ways:

a) $u(x0) \geq u(x^1)\ x \geq x^1$ (8)

We assume that $u(x^0) \geq x^1$ but $x^1 > x^0$, then $1.u(x^0) \geq 1.u(x^1)$. By the transitivity assumption, $1.u(x^1) \sim x^1 > x^0 \sim 1.u(x^0)$. The non-satiation assumption gives the contradiction that is $U(x^1) > u(x^0)$.

b) If $x^0 \geq x^1 \Leftrightarrow u(x^1)$ (9)

Suppose $x^0 \geq x^1$ but $u(x^1) > u\ (x^0)$ then $1.u(x^1) > 1.u(x^0)$. By applying the chain rule, $x^1 \sim 1.u\ (x^1) > 1.u(x^0)$ gives the contradiction. The u(x) is a continuous function. In order to prove the u(x), let us assume a continuous function. A function u(x), $X \in R^n_+$ is continuous with R^n_+. This is only true for each pair of subsets of function values of u_1 and u_2. If $u_1$and u_2 are separated then $u^{-1}(u_1)$ and $u^{-1}(u_2)$ are also separated. Suppose the two sets are separated then no point in one set is a boundary point of the other. The u_1 and u_2 are separated in this case. These subsets lie on either side of (x^0). They are separated and the subsets do not belong to them. Since $x^0, \underline{u}$ and $\bar{u}$ are arbitrary, the function u(x) is continuous.

1.6 Revealed Preference Theory

The theory of revealed preference allows us to use information about consumer choices to interpret how the consumer must rank bundles if they are maximizing utility with budget constraints (Besanko & Braeutigam, 2002). Alternatively, the revealed preference model explains how the consumer spends their income with given prices of different commodities. The change in price allows the consumer to buy more or fewer commodities. But an increase in income also helps the consumer to buy more of one commodity or buy more of another commodity. The model is based on certain assumptions.

Assumptions

1. The consumer spends their entire income on only two commodities.
2. The consumer chooses one commodity for each price vector p and income situation.
3. There is one and only one price p and income combination at which bundle x is chosen by the consumer.
4. The consumer's choices are consistent. When X^o with price p^0 in a bundle, x^1 is chosen then x^0 will be no longer be a feasible alternative.

The Model

Let us assume that p^0 will be the price vector at which x^0 can be purchased. The consumer chooses x^1 when X^0 was chosen. The cost for the consumer of this consumption bundle is $p^0x^1 > p^0x^0$. The consumer chooses x^0 when $p^0 x^0 = M_0$. Suppose the price changes and p^0 increases to p^1. This a rise in prices can be seen in some commodities. At the new price level p^1x^0, the consumer is not happy because s/he has to pay more in terms of money. But the consumer's choices are changing and when $p^1x^0 > p^1x^1$, the consumer is still better off. Similarly $p^1x^0 \geq p^0x^1$ implies that $p^1x^1 < p^1x^0$. At this new adjustment, the previous bundle of consumption gives more satisfaction to the consumer. It also means that $p^0x^0 \geq p^0x^1 \rightarrow p^1x^1 \leq p^1x^0$.

1.7 The Weak Axiom of Revealed Preference (WARP)

In the Weak Axiom of Revealed Preference (WARP), we have assumed two commodities. Now x^0 is chosen at p^0, M^0 where B_0 is at equilibrium. Suppose x^1 is chosen at p^1m^1. The price effect shows a choice structure. This satisfies the weak axiom of revealed preference. There may be a rational preference consistent with these choices.

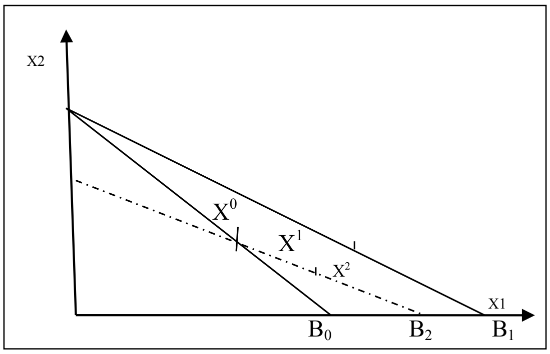


Figure 1.4. Income and substitution effect

In Figure 1.4,

B_0 = a consumers' budget line defined by P_0 and $M_{0,}$
X_0 = the initial bundle chosen by consumer on $B_{0,}$

B_1= the budget line after the fall in p_1 with $\bar{M}$,
X^1= the new bundle chosen on B_1

In the figure, line B_1 shifts to B_2. The bundle x^2 is just right of x^0.Therefore x^0 and x^2 are the substitution effect. Both goods are alternatively purchased. The x^0 and x^1 is the income effect which is shown in the diagram, which is due to a fall in price p_1. The p^0x^0 is the price vector as well as the consumption vector. P^1x^1 is the new price vector and consumption vector. The consumer's income is adjusted up to m_2x^0. Goods can be purchased at the new prices p^1, so that $p^1x^0 = M_2$. The price vector p^1 and the compensated money income is M_2. The consumer chooses x^2, because all income has been spent. We have

$$p^1x^2 = M_2 \tag{10}$$

The compensating change in M ensures that

$$P^1x^0 = M_2 = P^1x^2 \tag{11}$$

Now x^2 is chosen even though x^0 is still available to the consumer. Normally, a consumer expects to buy new commodities even when old commodities still exist. This means tastes and preferences change. From equation (10), we have

$$p^0x^0 < p^0x^2 \tag{12}$$

Here, x^2 is not purchased by the consumer who purchased x^0. This could be because the new choice of commodity is not suitable to the consumer. Considering equation (11)

$$p^1x^0\text{-}p^1x^2 = p^1(x^0\text{-}x^2) = 0 \tag{13}$$

Equation (12) can be written as

$$p^0x\text{-}p^0x^2 = p^0(x^0\text{-}x^2)\ 0 < 0 \tag{14}$$

Subtracting equation (14) from (13) gives the following equation

$$p^1\ (x^0\text{-}x^2)\ \text{-}\ p^0(x^0\text{-}x^2)$$

$$= (p^1\text{-}p^0)\ (x^0\text{-}x^2) > 0$$

And multiplying the above equation by (-1) results in the following equation:

$$(p^1\text{-}p^0)\ (x^2\text{-}x^0) < 0 \tag{15}$$

This prediction applies irrespective of the number and direction of price changes. In case of a change in the j^{th} price, only p^1 and p^0 differ in p_j. Therefore, equation (15) can be written as

$$\begin{aligned}&\sum_1 (p_i^1 - p_i^0)(x_i^2 - x_i^0)\\ &= (p_i^1 - p_i^0)(x_i^2 - x_i^0) < 0\end{aligned} \tag{16}$$

We can also derive Slutsky's equation from the behavioral assumption. Therefore, $M_2 = p^1x^0$ and $M_0 = p^0x^0$, the compensating reduction in M is

$$\begin{aligned}\vartriangle M &= M_0 - M_2\\ &= p^0x^0 - p^1x^0\\ &= o^0(p^0 - p^1)x^0\\ &= -(p^1 - p^0)x^0\end{aligned} \tag{17}$$

The case of Δ in P_i gives us

$$\Delta M = -\Delta p_i x_i \tag{18}$$

The above equation explains that the change in price is equivalent to the change in money income. The government offsets the effect of a price rise by giving a dearness allowance to public sector workers. Such dearness allowance increases the money income up to the level of increase in price. The price effect of p_i on X_j is $(x_j^i - x_j)$ and this can be partitioned into the substitution effect $(x_j^2 - x_j)$ and the income effect $(x_j^1 - x_j)$. Dividing by Δ_{Pi}, the equation becomes

$$\frac{x_j^1 - x_j^0}{\Delta p_i} = \frac{x_j^2 - x_j^0}{\Delta p_i} + \frac{x_j^1 - x_j^2}{\Delta p_i} \qquad (19)$$

But from equation (18) we have $\Delta M = -\Delta p_i x_j^0$

Therefore, $\Delta p_i = -\Delta M / X_i^0$, substituting this into the second term of equation (19), we have on the right hand side

$$\frac{x_j^1 - x_j^0}{\Delta p_i} = \frac{x_j^2 - x_j^0}{\Delta p_i} - x_i^0 . \frac{x_j^1 - x_j^2}{\Delta M} \qquad (20)$$

The above equation helps to show the utility maximizing theory of the consumer. The revealed preference theory and utility maximization are equal in their nature.

1.8 Indirect utility function

There is a direct and an indirect utility function for the consumer. The direct utility function (DUF) is a function whose arguments are the quantities consumed of different goods and two of its basic properties are (increasing) monotonicity and quasiconcavity. An indirect utility function (IUF) is a function whose arguments are the normalized prices of the goods. The corresponding properties are (decreasing) monotonicity and quasiconvexity. For both types of functions, the indifference curves (i.e. the contours) are convex to the origin. These well-known observations suggest a simple method for obtaining one utility function from another: reversing the sign. Reversing the sign of a DUF that satisfies the basic axioms of consumer theory gives rise to an IUF that also satisfies basic axioms, and vice versa. We shall refer to such a pair of functions as a 'mirror pair' (Moffatt & Moffatt, 2011).

Microeconomic theory explains that each consumer maximizes their utility subject to price and income. These are defined by an indirect utility function which summarizes the consumers' preferences and the technologies. Weak concavity assumptions of the indirect utility function allow one to prove differentiability of optimal solutions and stability of the steady state. This study shows that if the consumption good production function is concave-γ and the instantaneous utility function is concave-ρ, then the indirect utility function is weakly concave, and its curvature coefficients are bounded from above by a function of γ and ρ. (Venditti, 2012). We will consider such notations in the following paragraph.

The utility maximization bundle can be written as x (p, y). The level of utility maximization is chosen at the highest level by the consumer's budget constraints. It is facing price p and income y. For any group of individuals, price and income give different budget constraints. Such combinations give the different combination of indifference curves to consumers. The real value function shows the relationship between price, income and maximum value of utility. It can be summarized by a real valued function as $V : R^n_+ * R_+ \to R$

$$U(p, y) = \max_{x \in R_+} u(x) \quad \text{subject to} \quad p.x \leq y \tag{21}$$

This is called the indirect utility function. When u(x) is continuous, u(x, y) is well defined for all $p \gg 0$ and $y \geq 0$. This is because a solution to the maximization problem exists. The consumer achieves the maximum level of satisfaction subject to price (p) and income (y). This is true for all consumers, and can be further written as

$$U(p, y) = u(x(p, y)) \tag{22}$$

The u (p, y) gives the utility level of the highest indifference curve. Such utility function is shown in the diagram. The consumer can reach the highest indifference curve with given prices p and income y. The price p and y are sufficient to guarantee that u (p, y) will be continuous in p and y on $R^{"}_{++} * R^{+}_{+}$. The continuity of u (p, y) follows positive prices. A small change in any of the parameters (p,y) fixes the location of the budget constraint. It will only lead to small changes in the maximum level of utility which the consumer can achieve.

Properties

The utility of x can be written as u(x). It is continuous and strictly increasing on R^{n}_{+} . There are six properties of the indirect utility function. Such properties are explained along with proof as follows. **Property 1. Continuous on $R^{n}_{++} * R_{+}$**

Proof

The utility of x, u(x) is a continuous function. There might be a change in income and in price. The change in price is always observed with a change in income. But a first change in price is also possible. It means $\Delta P = \Delta Y$, now $\Delta Y = \Delta M$. Therefore, the consumer achieves the same utility which they were getting before the change in income and price. This is also called the compensation variation. A change in income is equivalent to the change in the consumer's budget constraint. The utility level remains unaffected and it remains continuous over a period of time. This is elaborated in more detail in the following property.

Property 2. Homogenous of degree zero in (p,y)

Proof

The proof of property 1 and 2 of the indirect utility function is given in a simple form.

$$U (p, y) = U (tp, ty) \text{ for all } t > 0 \tag{23}$$

Suppose v (tp, ty) = [max u(x) subject to t.p.x ≤ ty) which is clearly equivalent to max u(x) subject to p.x ≤ y. This is because we can divide both sides of the constraints by t, where t > o without affecting the set of bundles satisfying it. Consequently, v (tp, ty) = [max u(x) subject to p.x≤ y] = U (p, y). This is the simple proof of the above property. The utility of x is subject to price and income.

Property 3. It is strictly increasing in y

Such property is easy to prove. It is therefore not explained here but the proof can be explained in the following property.

Property4. The Indirect Utility Function is decreasing in p

Proof

Properties 3 and 4 are explained simultaneously as follows. A consumer will always want to increase their utility. The consumer's budget constraint can never cause the maximum level of achievable utility to decrease. At a given level of income and price, the consumer always achieves the highest utility on their indifference curve. We assume that the utility is strictly positive and differentiable, where (p, y) ≫ 0 and that u (0) is differentiable with $(\partial u / x) / \partial xi > 0$ for all x≫0.

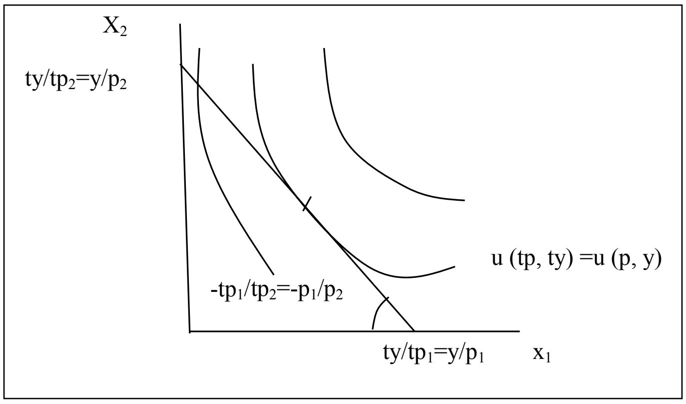


Figure 1.5. Indirect utility functions in prices and income

The homogeneity of the indirect utility function can be defined in terms of prices and income. Here, u(.) is strictly increasing in this utility function. The utility constraint must stick to the optimum level. Therefore, the utility function is equivalent to

$$U\ (p, y) = \max u(x) \quad \text{subject to} \quad p.x = y \tag{24}$$

$$x \in R^n_+$$

Adding Lagrangian to equation (24), it becomes

$$l\ (x, \lambda^* \varepsilon R) = u(x) + \lambda\ (y - p.x) \tag{25}$$

For (p, y≫0), let us assume that x* = x (p, y). We have to further assume that x*≫ 0 to be able to solve equation (24). Applying Lagrange's theorem to conclude that there is a $\lambda^* \varepsilon R$ such that

$$\frac{\partial l(x^*, \lambda^*)}{\partial x_i} = \frac{\partial u(x^*)}{\partial x_i} - \lambda^* p_i = 0 \qquad I = 1,\ldots\ldots\ldots\ldots n \tag{26}$$

Both p_i and $\delta u(x^*)/\delta x_i$ are positive in the above equation. Applying the Envelope theorem to establish that v (p, y) is strictly increasing in y, the partial derivative of the minimum value function v (p, y) with respect to y is equal to the partial derivative of the Lagrangian. This is with respect to y evaluated at (x*, λ^*),

$$\frac{\partial v(p, y)}{\partial y} = \frac{\partial l(x^*, \lambda^*)}{\partial y} = \lambda^* > 0 \tag{27}$$

Thus v (p, y) is strictly increasing with income. This is because v is continuous and increasing.

The elementary proof of the equation does not rely on any additional hypothesis. If $p^0 \geq p^1$ and the equation can be solved when $p = p^0$. Then this is $x^0 \geq 0$, $(p^0\text{-}p^1)$, $x^0 \geq 0$, hence $p^1.x^0 \leq p^0 .x^0 \leq y$. The x^0 is feasibly set when $p = p^1$. The conclusion of the above property is that $v\ (p^1, y) \geq u(x^0) = v\ (p^0\ y)$. This is the desirable conclusion from the previous property.

Property 5. Quasi convex in (p, y)

In order to prove this property, we need to assume that a consumer would prefer one of any two extreme budget sets. The point is to show that v (p, y) is quasiconvex in the vector of prices and income (p, y). This proof is concentrated on the budget sets.

Suppose β^1 is the budget set available to a consumer available when prices and income are (p^1y^1) $(p^2\,y^2)$ and (p^ty^t) respectively. The available prices and incomes are denoted as $p^t \equiv tp^1 + (1\text{-}t)p^2$ and $y^t \equiv y + (1\text{-}t)\ y^2$. Here, we have taken three probabilities.

$\beta^1=\{x/p^1\ x \leq y^1\}$

$\beta^2\{x/p^2\ x \leq y^2\}$

$\beta^t=\{x/p^t\ x \leq y^t\}$

The choice is made by a consumer when they face the budget constraint β^t. It is a choice made when the consumer faces either β^1 or budget set β^2. At each level of utility, the consumer can achieve their utility β^t budget set. The maximum level of utility is achieved over the β^t budget set .

The maximum level of utility a consumer can achieve is at β^1 or at β^2 budget set. Let's assume that the maximum level of utility is achieved at β^t. We know that $u(p^t,y^t) \leq$ max [max $[v[p^1,y^1), v(p^2,y^2)]$ $\forall t\in[0,1]$. This is equivalent to the statement that u (p, y) which is quasi-convex in (p, y). We want to show that if $x\in\beta^t$ then $x\in \beta^1$ or $x\in \beta^2$ for all $t \in [0, 1]$. Suppose we choose either extreme value for t, then the β^t budget set coincides with either the β^1 or β^2 budget set. It remains to be seen that they hold for all $t\in (0, 1)$.

Suppose $X\in \beta^1$ and $x\in \beta^2$ then

$P^1.x>y^1$ a and $p^2.x>y^2$

This is because $t\in(0, 1)$. We multiply the first equation by t and the second equation by (1-t). We preserve the inequality to obtain following equation.

$tp^1.x>ty^1$

and $(1\text{-}t)\ p^2.x > (1\text{-}t)\ y^2$

Adding the above two equations, we obtain

$(tp^1+ (1\text{-}t)\ p^2).x > ty^1+ (1\text{-}t)\ y^2$

Or $P^t.x > y^t$ (28)

From (28) we learn that x is not equal to β^t, contradicting our original assumption. We can conclude that if $x\in\beta^t$ then $x\in\beta^1$ or $x\in\beta^2$ for all $t\in (0, 1)$. It can be derived that v (p, y) is quasiconvex in (p, y).

Property 6. Roy's identity

Roy's identity explains that a consumer's Marshallian demand for goods is simply the ratio of the partial derivatives of indirect utility I, with respect to pi and y after a change of sign. We have assumed that x*= x (p, y) is a strictly positive solution. If we apply the envelope theorem to evaluate ∂U (p, y)/∂pi, then it gives us the following equation

$$\frac{\partial u(p,y)}{\partial p_i} = \frac{\partial l(x^*,\lambda^*)}{\partial p_i} = -\lambda^* X_i^* \tag{29}$$

According to the above equation $\lambda^* = \partial u(p,y)/\partial y > 0$, hence equation (29) can be interpreted as

$$-\frac{\partial u(p,y)/\partial p_i}{\partial u(p,y)/\partial y} = x_i^* = x_i(p,y) \tag{30}$$

This is proof of the property and is a desired function.

1.9 Expenditure function

The expenditure function assumes that the prices of commodities are fixed. In order to achieve a utility level, the consumer has to make certain expenditures at a given set of prices. At any given moment, consumers come across a variety of commodities available at a variety of prices. Most consumers have a fixed level of income. Therefore, each consumer decides how much to spend on different commodities to achieve a particular level of utility. Figure 1.6 shows that all bundles of x require the same level of expenditures. The consumer faces the prices p = (p_1, p_2). The isoexpenditure curves are defined implicitly by e = $p_1x_1 + p_2x_2$ for the different levels of total expenditures where e > 0. Therefore, there will be the same slope $-p_1/p_2$, but different horizontal and vertical intercepts e/p_1 and e/p_2 respectively. An iso-expenditure curve contains bundles which cost more. A consumer shifts upward on isoexpenditure lines. If we fix the level of utility at u, then the indifference curve u(x) = u. This curve gives all bundles which yield the same level of utility to the consumer. The indifference curve u indicates e3 point. Money is insufficient at these prices to achieve maximum utility u. In the diagram, each of the curves e1 and e* has at least one point in common with u. This point shows that the level of total expenditures is sufficient for the consumer to achieve utility u. Sometimes, the consumer makes regular minimum expenditures on various commodities to achieve a fixed utility level. The consumer also knows that s/he cannot make any more regular purchases of such commodities.

The expenditure function only explains that the consumer requires the minimum expenditure to achieve utility u. It is in the form of the purchase of various goods and services. This is the lowest possible expenditure curve that has at least one point in common with the indifference curve u. The level of e* is the bundle that costs the least. The consumer achieves utility u at prices p, which we can call the equilibrium point. It will be the bundle $x^h = (x_1^h(p,u).x_2^h(p,u))$. The minimum expenditure u is required to achieve utility u at prices p by e (p, u). It means that the expected minimum utility is a function of the price level. The level of expenditure is equal to the cost of bundles x^h. This can be represented in an equation as e (p, u) = $p_1 x^h{}_1$ (p.u) $p^2 x_h{}^2$ (p, u) = e*

The expenditure function is the minimum value function, and can be expressed as follows:

$$e(p,u) = \min_{x\varepsilon^n} p.x \text{ Subject to } u(x) \geq u \quad (31)$$

the lowest level of expenditures required to achieve utility level u.

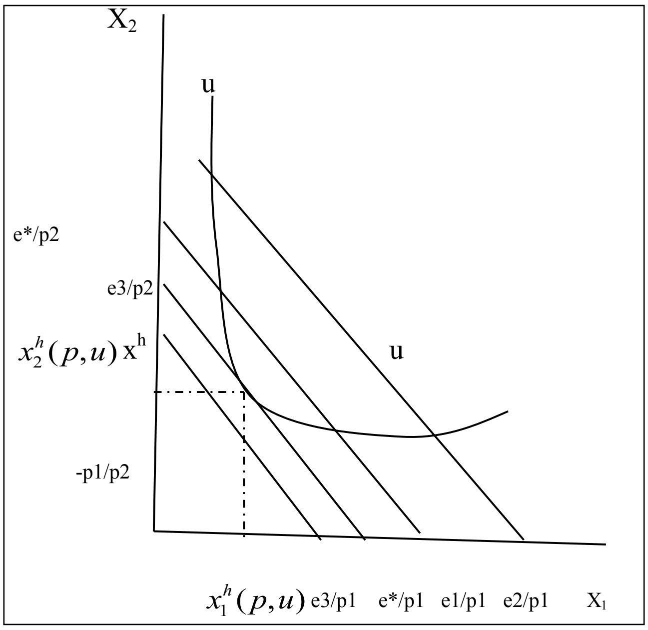


Figure 1.6. Expenditure function with minimum price

Properties of the expenditure function

If u (.) is continuous and strictly increasing then e (p, u) can be defined into seven properties. They are as follows:

Property 1: Zero when u takes on the lowest level of utility in u

The lowest value in utility is u (0). It is because u (.) is strictly increasing on R_+^h.Consequently (p, u (o)) =0. This is because x = o attains utility u (o). It requires the expenditure of p_{0bv} = 0.

Property 2: Continuous on its domain $R_{++}^n * u$

Such property follows from the theorem of the maximum.

Property3: For all p≫0, strictly increasing and unbounded above in u

The third property shown via the additional hypothesis $X^h(p, u) \gg 0$ which is differentiable ∀ p≫0, u > u (0) and that u (.) is differentiable with ∂u(x)/∂x>0 ∀i on R_{++}^n. We have assumed that the u (.) is continuous and strictly increasing. The p≫ 0 is the constraint and is binding. For u (x') > u, there is at ∈ (0, 1) which is close enough to 1. It is u (tx^1) > u. Moreover u ≥ u (0) implies that u(x^1) > u (0) so that $x^1 \neq 0$. Therefore, p.(tx^1) < p.x^1, because p.x^1> 0 when the constraint is not binding. There is a strictly cheaper bundle that also satisfies the constraint. If we write it in a different way, then

$$E\ (p, u) \equiv \min p.x \qquad \text{subject to } u\ (v) = u \tag{32}$$

$x \in R^n_+$

The Lagrangian function is used for the above and can be written as

$$l(x, \lambda) = p.x + \lambda\,[u - u(x)] \tag{33}$$

Now for p≫ and u > u (0), we have $x^* = x^h(p, u) \gg 0$

We need to solve equation 33. There is a $\lambda *$ shown as

$$\frac{\partial l(x^*, \lambda^*)}{\partial x_i} = p_i - \lambda * \frac{\partial u(x^*)}{\partial x_i} = 0 \qquad i=1\ldots\ldots\ldots\ldots\ldots n \tag{34}$$

P_i and $\frac{\partial u(x^*)}{\partial x_i}$ are positive. Due to this hypothesis, the envelope theorem can be used to show that e (p, u) is strictly increasing in u. According to the envelope theorem, the partial derivative of the minimum value function e (p, u) with respect to u is equal to the partial derivative. It is Lagrangian with respect to u and evaluated at (x^*, λ^*), hence

$$\frac{\partial e(p,u)}{\partial u} = \frac{\partial l(x^*, \lambda^*)}{\partial u} = \lambda^* > 0 \tag{35}$$

Suppose, we hold for all u > u (0) , e (.) is continuous, we may conclude that for all p≫0, e (p, u) is strictly increasing in u on u (which includes u (0), that e is unbounded in u. This can be shown to follow from the fact that u (x). It is continuous and strictly increasing.

Property 4: The expenditure function is increasing in p.

Proof of the above property is shown in property 7.

Property 5: Homogenous to degree 1 in p.

Proving this property true is simple. The rise in price level (p) and income (y) are almost the same. Therefore, it is called homogenous to degree one.

Property 6: Concave in p

Suppose p^1 and p^2 are assumed to be two positive price vectors. Let $t \in (0, 1)$ and $pt = tp + (1\text{-}t)\, p^2$ be any convex combination of p^1 and p^2.

The expenditure function will be concave in prices if

$$te(p^1,u) + 1 - t)\, e\,(p^2,u) \leq e\,(p^t,u) \tag{36}$$

Now we will focus on what it means to reduce the expenditure at given prices. Suppose particular x^1 minimizes expenditure to achieve u when prices are p^1. Similarly x^2 minimizes expenditure to achieve u when prices are p^2. Therefore, x* is the minimum expenditure to achieve u when prices are p^2. The cost of x^1 at process p^1 must not be more than the cost at price p^1 of any other bundle x that achieves utility u. Similarly, the cost of x^2 at prices p^2 must not be more than the cost at price p^2 of any other bundle x, which achieves utility u if,

$$P^1x^1 \leq p^1x \qquad \text{and} \qquad P^2x^2 \leq p^2.x$$

For all x that achieve u, the relation must also hold for x*. This is because x* achieves u as well.

To maximize expenditure to achieve u at given prices, we know that

$$P^1x^1 \leq p^1x* \qquad \text{and} \qquad P^2x^2 \leq p^2.x*$$

If $t \geq 0$ and $(1\text{-}t) \geq 0$, we can multiply the first of these by t, the second by (1-t) and add them. If we then substitute from the definition of pt, we obtain,

$$tp^1.x^1 + (1\text{-}t)\, p^2.x^2 \leq p^t.x* \tag{37}$$

In the above equation, the left hand side is a convex combination of the minimum levels of expenditure. It is necessary at prices p^1 and p^2 to achieve utility u. The utility u must be constant at each change in price. The right hand side is the minimum expenditure needed to achieve utility u at the convex combination of those prices.

Equation (37) explains that

$$te(p^1,u) + (1- t)\, e\,(p^2, u) \leq e\,(p^t, u)\ \forall t \in [0,1] \tag{38}$$

What we intended to show from the previous equations is that such function is concave in p.

Property 7: Shepherd's Lemma: e (p, u) is differentiable in p at (p^0,u^0) with $p^0 \gg 0$and

$$\frac{\partial e(p^0,u^o)}{\partial p_i} = x_i^h(p^0,u^o) \qquad I = 1\ldots\ldots n \tag{39}$$

In order to use the above property, we can use the Envelope theorem. But now we differentiate it with respect to pi. It gives us the following equation.

$$\frac{\partial e(p\ ,u\ )}{\partial p_i} = \frac{\partial l(x^*,\lambda^*)}{\partial p_i} = x_i^* \equiv x_i^h(p,u\ ) \tag{40}$$

This is required because $x^h(p,u) \geq 0$. It is also possible to prove property 4 (Jehle & Reny, 2001).

All the properties of the expenditure function are equally important and they help us to understand the expenditure function in detail.

1.10 The expenditure minimization problem

Every consumer tries to minimize their total expenditure to increase utility. Sometimes, a consumer prefers substitutes to reduce their expenditure. Most substitutes are available at a lower price. The Expenditure Minimization Problem (EMP) explains the price and utility function. It is explained as $p \gg 0$,and u > u (0).

Min p.x $\qquad$ x ≥ 0 $\qquad$ subject to u (z) ≥ u

A consumer has unlimited wants. We know that a minimum level of wealth is required to achieve utility u. Sometimes, this is used as the cutoff point to achieve minimum utility out of wealth. We can observe such cut-off point for every family as an efficient use of the family's purchasing power while reversing he roles of objective function and constraints.

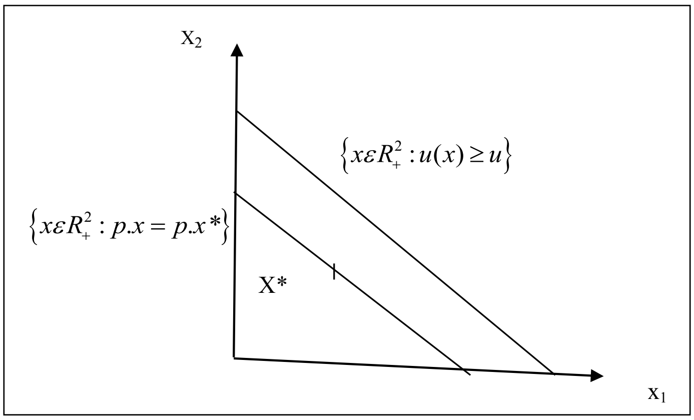


Figure 1.7. The expenditure minimization problem (EMP)

We assume that u (.) is a continuous utility function. This represents a locally nonsatiated preference relation ≥ defined on the consumption set R_+^L . Figure 1.7 shows that the optimal consumption bundle x* is the least costly bundle. It still allows the consumer to achieve the utility level u. The consumer gets maximum satisfaction from the desired bundle of goods.

From the geometric point of view, it is the point in the set $\{x\varepsilon R_+^L : u(x) \ge u\}$. It lies on the lowest possible budget line associated with the price vector p. This is shown in the figure as x* point. If we assume that u (.) is a continuous utility function representing a locally nonsatiated preference relation, ≥ is defined on the consumption set as x = R_+^L. The price vector is P.

1. If x* is optimal in the utility maximization problem when wealth is w > 0, then x* is optimal in the EMP when the required utility level is u (x*). The minimized expenditure level in the EMP is exactly w. Sometimes, a consumer cannot afford an expenditure which is above what they can spend.
2. Suppose x* is optimal in the EMP when the required utility level is u > u (0) then x* is optimal in the UMP when wealth is p.x*.

The minimized utility level in this UMP is exactly u. To prove the above proposition,

i) Suppose x* is not optimal in the EMP with required utility level u(x*). Then there exists an x' such that

$$u(x^{'}) \ge u(x^*) \,\&\, p.x^* \le w \tag{41}$$

By local nonsatiation, we can write that x" is very close to x'. It also means u(x") > u (x') and p.x"< w.

The above notation implies that $x'' \in B_{pw}$ and u(x'') > u(x*). It is contracting the optimality of x* in the UMP. The x* must be optimal in the EMP when the required utility level is u(x*).The minimized expenditure level is p.x*. The x* solves the UMP when wealth is w. In Walras's law, we have p.x* = w.

ii) If u > u (0) and x* ≠ 0; hence p.x* > 0.

If x* is not optimal in the UMP when wealth is p.x*, there exists an x' such that u (x') > u(x*) and p.x' ≤ p.x*. Let's consider bundle x"= ∝ x' where ∝∈ (0, 1). Here x" is a scaled down version of x' through the continuity of u (.), if ∝ is close enough to 1, then the u(x") > u(x*) and p.x"< p.x*. This contradicts the optimality of x* in the EMP. Therefore, x* must be optimal in the UMP when wealth is p.x*. The maximized utility level is therefore u (x*). The required utility level is u, then u(x*) = u. According to the utility maximization problem when p≫0, the solution to the EMP exists under general conditions. The constrained set needs to be non-empty. It means u (.) must attain values at least as large as u for some x. The condition will be satisfied for any u > u (0) if u (.) is unbounded.

1.11 The Hicksian demand function

The Hicksian welfare measures can be used for the evaluation of any change of state as long as the agent's indirect utility for income is well defined before and after the change (Weber, 2010). The set of optimal commodity vectors in the EMP is denoted as $h(p,u) \subset R_+^L$. This is known as the Hicksian or compensated demand correspondence function. Figure 1.8 shows the solution set h (p, u) for two different price vectors p and p'. The basic properties of the Hicksian demand function is explained as follows:

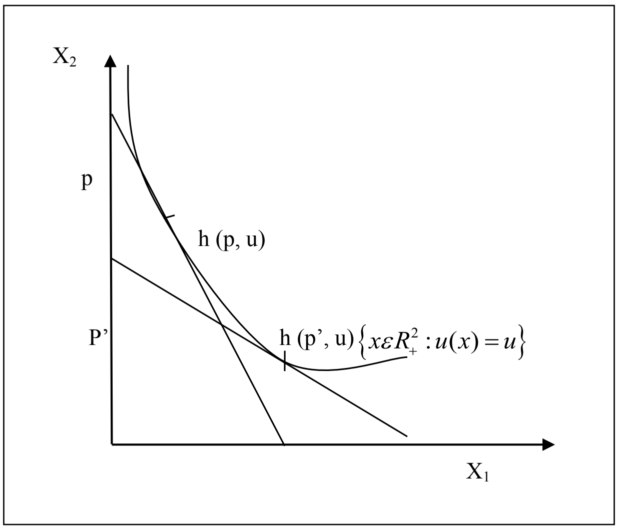


Figure 1.8. The Hicksian demand function

Suppose u(.) is a continuous utility function representing a locally nonsatiated preference relation ≥ defined on the consumption set $X = R_+^L$. Then, for any p≫ 0, the Hicksian demand correspondence h (p, u) possesses the following two properties. Homogeneity of degree zero in P follows because of the optimal vector. The minimizing p.x is subject to u (x) ≥ u. It is the same as that for minimizing ∝p.x and subject to this constraint for any scalar ∝ > 0.We will explain the properties and proof in the following paragraph.

Property 1: No excess utility for any x ∈ h (p, u), u(x) = u

Proof

This property follows from the continuity of u (.). Suppose there exists an x∈ h (p, u), such that u(x) > u. If we consider a bundle x'= ∝ x, where ∝∈ (0, 1), continuity for ∝ is close enough to 1.u (x') ≥ u and p.x'<p.x1, contradicting x being optimal in the EMP with required utility level u.

Property 2: Convexity /Uniqueness

If ≥ is strictly convex, then u (.) is strictly quasiconcave. There is a unique element in h (p, u) if there is a utility function $u(x_1, x_2) = \propto \ln x_1 + (1-\propto) \ln x_2$.. Substituting x_1 (p, w) and x_2 (p, w) into u(x) we have,

$$V(p, w) = u(x(p, w))$$

$$= \left[\alpha \ln \alpha + (1-\alpha)\ln(1-\alpha)\ln p_2\right] \quad (42)$$

As the UMP, when u (.) is differentiable, the optimal consumption bundle in the EMP can be characterized using the first order condition. The first order condition is similar to that of the UMP.

Proposition 1: If we assume that u (.) is differentiable and it shows that the first order condition for the EMP is

$$p \geq \lambda \nabla u(x^*) \qquad \text{and} \qquad x^*\left[p - \lambda \nabla u(x^*)\right] = 0$$

For some $\lambda \geq 0$, compare this with the first order conditions for the UMP.

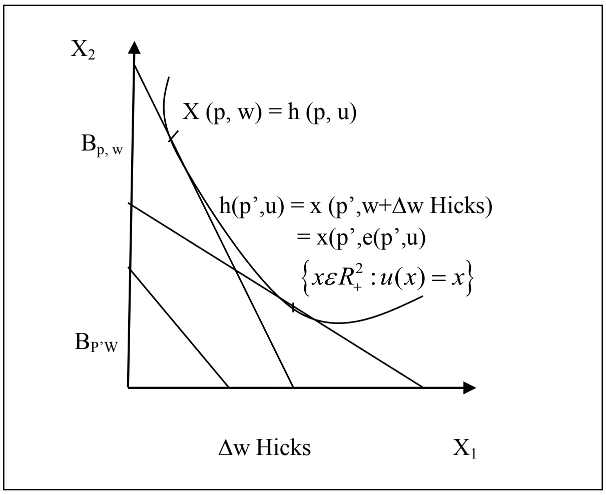


Figure 1.9. The Hicksian demand function and wealth effect

Using the above proposition, we can relate the Hicksian and Walrasian demand correspondence as follows.

H(p,u) = x(p,e(p,u)) and x(p,w) = h(p,v(p,w)) (43)

The first of these relations explains the use of the term compensated demand correspondence to describe h (p, u). Suppose the price change h(p, u) gives the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep their utility level at u. The government helps consumers by subsidizing goods. In India, this is done through the public distribution system. This type of wealth compensation to consumers is depicted in figure 1.9, and is known as the Hicksian wealth compensation.

Figure 1.9 shows that the consumer's initial situation at the price wealth pair (p, w); prices then change to p', where $p_1' = p_1$ and $p_2' > p_2$. The Hicksian wealth compensation is defined as the amount

$$\Delta w_{Hicks} = e(p',u) - w \tag{44}$$

Therefore, the demand function h (p, u) keeps the consumer's utility level fixed as prices change. In contrast to the Walrasian demand function, money wealth is fixed but utility is allowed to vary.

As with the value functions of the EMP and UMP, the relations allow us to develop a tight linkage between the properties of the Hicksian demand correspondence h (p, u) and the Walrasian demand correspondence x (p, w).

Hicksian demand and the compensated law of demand

A property of the Hicksian demand is that it satisfies the compensated law of demand. The price and demand of commodities move in opposite directions. The price change is accompanied by the Hicksian wealth compensation.

Proposition 1: If u (.) is a continuous utility function representing a locally nonsatiated preference relation $\geq$ and that h (p, u) consists of a single element for all p$\gg$ 0, then the Hicksian demand function h (p, u) satisfies the compensated law of demand: For all p' and p'',

$$(p''-p')[h(p'', u)-h(p', u)] \leq 0 \tag{45}$$

Proof

For any p$\gg$0, the consumption bundle h (p, u) is optimal in the EMP and as such, achieves a lower expenditure at prices p than any other bundle that offers a utility level of at least u. Therefore,

$$P''.h(p'', u) \leq p''.h(p', u)$$

and

$$P'.h\ (p'', u) \geq p'.h\ (p', u) \tag{46}$$

Subtracting these two inequalities yields the results.

Hicksian demand and expenditure functions for the Cobb-Douglas utility function

We assume that the consumer has a Cobb-Douglas utility function over two goods, that is, $u(x_1, x_2) = x_1^{\alpha} x_1^{1-\alpha}$. By deriving the first order conditions for the EMP and substituting from the constraints u (h_1 (p, u), h_2 (p, u) = u, we obtain the Hicksian demand function.

$$h_1(p,u) = \left[\tfrac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} u$$

&

$$h_2(p,u) = \left[\tfrac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} u$$

If we calculate it as follows

E (p, u) = p.h (p, u) yields the following equation

$$e(p,u) = [\alpha^{-\alpha}(1-\alpha)^{1-\alpha}]p_1^{\alpha} p^{\frac{1}{2}-\alpha} u \tag{47}$$

The above function is the Hicksian demand and expenditure functions for the Cobb-Douglas utility function (Mas-Colell, Whinston & Green, 2004).

1.12 The Von Neumann-Morganstern utility function

Introduction

A consumer has different choices for clothes, toothpaste, soap, biscuits, air tickets, bonds and debentures in their day-to-day life. Sometimes, such choices are made under uncertainty. Microeconomics is interested in this kind of choice behaviors of individuals. To understand the theorem, we use the example of lotteries. This is because the choice of lotteries is made under uncertainty. Consumers do not know whether such choices will yield them a prize or not.

Lotteries

Economists view lotteries as either a consumer commodity or as a source of public revenue. As a commodity, lotteries are notable for their broad market penetration and rapid growth (Clotfelter & Cook, 1990). Alternatively, a consumer's buying a certain lottery is telling because it tells us how the consumer makes their choice. An individual selects a certatin lottery to win that lottery's prize. The choice of lottery often leads to winning or not winning the prize.

Let's denote a lottery as

$$P0\ x \oplus (1-\ p)\ 0\ y \tag{48}$$

The above notation shows that a consumer receives a prize x with probability p and prize y with probability 1- p, and shows a probability function with a chance of winning or not winning the lottery. Winning the lottery could mean a further chance to play in the lottery, or winning goods or money. Presently, money is usually the prize for winning a lottery. Modern microeconomics studies the risky behaviors of consumers under different circumstances. There are a number of risky behaviors such as drinking alcohol, smoking, drinking and driving. The cost of accidents or injury is much higher for the risk-adverse consumer. Now, let's consider the consumer's behavior with lotteries. When a consumer buys a lottery ticket, they take certain risks. This consumer behavior is put into a number of frameworks. A number of assumptions are made to understand the consumer's perception of the lottery.

1. $P0\ x \oplus (1\text{-}p)\ 0\ y \sim x.$ (49)
   The above notation shows that getting a prize with probability one is the same as getting the prize for certainty.

2. $P0\ x \oplus (1\text{-}p)\ 0\ y \sim (1\text{-}p)\ 0y \oplus p0x$ (50)
   This means that the consumer does not care about the order in which the lottery is described.

3. $q0(P0\ x \oplus (1\text{-}p)0\ y) \oplus (1\text{-}q)0y \sim (qp)0x \oplus (1\text{-}qp)0y.$ (51)

This notation describes the perception of a lottery dependent on the net probabilities of receiving the various prizes.

The above notations are different from each other. The first and second assumptions are harmless and yield satisfactory results. Assumption 3 is called the reduction of compound lotteries. The consumer treats compound lotteries differently from one lottery. The third assumption is not important to understand the lottery framework.

In the lottery framework, we can define that ℓ is a space of lotteries available to a consumer. There are a number of lotteries available to the consumer at different points and places. Suppose the consumer is offered two lotteries at a particular place then he will choose the best one in that place. From the previous discussion, we assume the revealed preference and choice rule. It is assumed that they are complete, reflexive and transitive. The consumer can choose one lottery but the prizes are not restrictive. The consumer may get another lottery ticket, or a car or money as a prize. Let's assume that there are three prizes: x, y and z. The probability of winning each prize is one third. Considering the reduction of compound lotteries, we get the lottery as follows,

$$\frac{2}{3}0\left[\frac{1}{2}0x\oplus\frac{1}{2}0y\right]\oplus\frac{1}{3}0z \qquad (52)$$

According to the third assumption, the consumer only cares about the net probabilities involved. This is equivalent to the original lottery notation.

Utility expected

There is a continuous utility function., which we denote as u, and which describes the consumer's preferences. This can be explained as

$$P0\ x \oplus (1\text{-}p)\ 0\ y) > q0w \oplus(1\text{-}q)\ 0z \text{ if and only if}$$

$$u\ (p0\ x \oplus (1\text{-}p)\ 0\ y) > u\ (q0w \oplus(1\text{-}q)\ 0z) \qquad (53)$$

The above utility function is not unique. Any monotonic transformation would do as well. Under certain assumptions, we can find a monotonic transformation of the utility function. This is a very convenient property. The expected utility is

$$u\ (P0\ x \oplus (1\text{-}p)\ 0\ y) = pu(x) + (1\text{-}p)\ u(y) \tag{54}$$

The expected utility means that the utility of a lottery is the consumer's expectation of the utility from its prizes. The utility of any lottery can be computed as each outcome and summing up over the outcome. We can say that the probability of winning a lottery from x or y adds to the utility of the consumer. The utility is a linear transformation over the probability and is separable from the outcome. The utility function and its existence are not an issue. A well-behaved preference ordering can be represented by a utility function. We have already discussed this in the first part of this book. But we can still prove it. Additional axioms are required and they are as follows

$$\{P \text{ in } [0, 1]: p0x \oplus (1\text{-}p)\ 0y \geq z\} \text{ and } \{p \text{ in } [0, 1]: z \geq p\ 0x \oplus (1\text{-}p)\ 0\ y\}$$

The above equation is a closed set for all x, y and z in ℓ

$$\text{If } x \sim y \text{ then } p0\ x \oplus (1\text{-}p)\ 0\ z \sim p\ 0\ y \oplus (1\text{-}p)\ 0\ Z \tag{55}$$

Assumption 1 explains the continuity which is innocuous. Assumption 2 explains that lotteries with indifferent prices are indifferent. From equation (55), suppose the lottery is given as $p0x\oplus (1\text{-}p)0\ z$ and we know that $x \sim y$, then substituting y for x to construct a lottery, the result is $p0y\oplus(1\text{-}p)0z$. The consumer regards this lottery as being equivalent to the original lottery. This is the only possible assumption. For technical things, we can make further assumptions in the following paragraphs.

To better understand the theory, let's assume that there is a worst lottery w and a best lottery b. In the worst lottery, the consumer never wins a prize. For any x in the lottery space (ℓ), $b \geq x \geq w$.

A lottery $p0b \oplus(1\text{-}p)\ 0w$ is preferred to $q0b \oplus(1\text{-}q)\ 0\ w$ if and only if $p > q$. This means that the probability of winning a prize in the best lottery is higher. A consumer always thinks positively and will prefer the best lottery. The consumer will have heard about it or has asked the seller, friend or relative. The consumer's choice is bounded for the consumer to get the best price for the best lottery.

The above three assumptions are purely for convenience. Assumption 4 can be derived. A consumer prefers the best lottery to the worst lottery. Sometimes they use their experience and knowledge. There is a higher probability for the consumer to win a prize in the best lottery, and the expected prize could be bigger and made known to the public before the consumer has chosen to participate in this lottery.

Expected utility theorem

If (ℓ, ≥) satisfies the above axioms then there is a utility function u defined on ℓ. This satisfies the expected utility property

$$U\ (p0\ x \oplus (1\text{-}p)\ 0\ y) = Pu\ (x) + (1\text{-}p)\ u(y) \tag{56}$$

The proof of the above theorem is given as follows.

Define u (b) = 1 and u (w) = 0.

This means that the best lottery gives a utility equal to 1, and the worst lottery gives a utility equal to 0. Now we need to find the utility of an arbitrary lottery z. The utility of set u (z) = Pz where Pz is defined by

$$P_z 0\ b \oplus (1\text{-}Pz)\ 0\ w \sim z \tag{57}$$

The consumer is indifferent between z and a gamble between the best and the worst outcome of the lottery. The best outcome has a probability Pz.

There are two things which are important for the prize of the z lottery:

1. Pz is existing in each form of the selected lottery. There are two sets {p in [0,1] : p 0 b ⊕(1-p) 0 w ≥ z} And {p in [0, 1]: z ≥ p 0 b ⊕ (1-p) 0 w} are closed and non-empty. Every point in [0, 1] is in one or the other of the two sets. This is the probability of winning a prize in the best. Since the unit interval is connected, there must be some p in both, because each probability will result in a prize or no prize. But every lottery has the desired outcome therefore, it is written as Pz.
2. Pz is unique in the lottery framework. If we assume that Pz and P'z are two distinct numbers then each number satisfies the lotteries in the framework. The fourth assumption is that the lottery gives a bigger probability of getting the best price. But at the same time, other lotteries cannot be indifferent to the one for which the probability of winning it is smaller. Therefore, Pz is unique and the utility is well defined. A consumer cannot define the differences of probability between the two lotteries.

There is a need to check the utility (u) from the lottery to each consumer. Each lottery has the expected utility property. It follows from some simple substitutions that

$$P0\ x \oplus (1\text{-}p)\ 0\ y$$

$$\sim_1\ p\ 0\ [p_x 0b \oplus (1\text{-}P_x)\ 0\ w] \oplus (1\text{-}p)\ 0[P_y 0b \oplus (1\text{-}p_y)\ 0\ w]$$

$$\sim_2\ [pp_x + (1\text{-}p)p_y\ ]\ 0\ b \oplus (1\text{-}Pp_x) - (1\text{-}p)\ p_y]0\ w$$

$$\sim_3\ [pu(x) + (1\text{-}p)\ u(y)]0b \oplus (1\text{-}Pu(_x) - (1\text{-}p)\ u(y)]\ 0\ w \qquad (58)$$

From the above equations, one uses the utility framework and the definition of Px and Py. Substitution 2 uses lottery framework 3, and explains only the net probability of obtaining the best lottery or the worst lottery. It matters to us because substitution 3 uses the construction of the utility function. It follows from the construction of the utility function that

$$U\ (p0x \oplus (1\text{-}p)\ o\ y) = pu\ (x) + (1\text{-}p)\ u(y) \qquad (59)$$

Again we are using the simple probability between the x and y lottery. Now we need to verify that u is utility function for the consumer. Suppose that the probability of winning the lottery x > y then the following equations are possible.

$$U(x) = P_x \text{ such that } x \sim P_x 0\ b \oplus (1\text{-}p_x)\ 0\ w$$

$$U(y) = P_y \text{ such that } y \sim P_y 0\ b \oplus (1\text{-}p_x)\ 0\ w$$

This means that the utility from x is greater than the utility from y.

Uniqueness of the expected utility function

The expected utility function is u: $\ell \rightarrow R$. Any monotonic transformation of u will also be a utility function. This describes the consumer's choice behavior.

Suppose u (.) is an expected utility function then V (.) = au (.) + c where a > 0. This means that any final transformation of an expected utility function is also an expected utility function. This is the linear function of the expected utility function.

It is clearer when we transform the entire utility function in terms of the linear expected utility function.

$$\begin{aligned} V\ (p\ 0\ x \oplus (1\text{-}p)\ 0y) &= au\ (p0\ x \oplus (1\text{-}p)\ 0\ y + c \\ &= a\ [Pu(x) + (1\text{-}p)\ u(y) + c \\ &= p\ [au(x) + c] + (1\text{-}p)\ [au(y) + c] \\ &= Pv(x) + (1\text{-}p)\ v(y) \end{aligned} \tag{60}$$

Equation (60) is very similar to equation (59). We have added the vector in the above function. Looking at the above equation, it is not hard to see that any monotonic transformation of u has the expected utility property. It must be the final transformation and can be stated in another way.

Uniqueness of expected utility function

An expected utility function is unique up to the line of transformation. The proof of such transformation is given as follows.

In this explanation, the monotonic transformation preserves the expected utility property. But the condition is that it must be an affine transformation. Let's assume that f: $R \rightarrow R$ which is a monotonic transform of u. Such monotonic transformation has the expected utility property. Then

$$F\ (u\ (p\ 0\ x \oplus (1\text{-}\ p)\ 0\ y)) = pf\ (u(x)) + (1\text{-}p)\ f\ (u(y)),$$

Or

$$F\ (pu(x) + (1\text{-}p)\ u(y)) = pf\ (u(x)) + (1\text{-}p)\ f\ (u(y)). \tag{61}$$

We understand that the above two equations are the same and they are equivalent to the definition of an affine transformation.

1.13 Measures of Risk Aversion

We are given various axioms which satisfy the consumer's choice behavior. We can find a representation of utility that has the expected utility property. The consumer always gambles and their behavior is geared towards more utility through winning a prize. Therefore ,we need to have a particular representation of this utility function for money. For example, to compute the consumer's expected utility of a gamble, we have already taken the simple probability as, p 0 x ⊕(1-p) 0 y. Now we represent it in terms of utility function as Pu(x) + (1-p) u(y).

The consumer prefers to get the expected value of the lottery. The utility of the lottery u is (p 0 x) 0 y, viewed another way, as pu(x) + (1-p) u(y).

The consumer prefers to get the expected value of the lottery. The utility of the lottery u (p 0 x ⊕(1-p) 0 y) is less than the utility of the expected value of the lottery px+ (1-p) y. Such behavior is called risk aversion by the consumer. There are two types of consumers: risk-averse and risk lovers. A consumer who is risk loving prefers to get the expected value of a lottery. The preferences of such consumers are different. from the preferences of risk-averse consumers. Their value judgments for winning in different lotteries are different. In the following figure, the concavity of the expected utility function is equivalent to risk aversion.

A risk-averse agent decides how to allocate his total wealth between investments in an asset with stochastic return (the risky asset) and an asset with deterministic return (the safe asset), so as to maximize the expected utility of return. If the return on the risky asset is less than that on the safe asset, the agent concentrates all his investments in the safe asset. On the other hand, if the mean return on the risky asset is greater than the safe return, the agent invests a positive fraction of his wealth in the risky asset (Roy & Wagenvoort, 1996).

The more concave is the expected utility function, the more risk averse is the consumer. The graph of this utility function in this region must lie below the function. In order to normalize the second derivative we need to divide it by the first, and we get a reasonable measure. It is known as the Arrow-Pratt measure of (absolute) risk aversion.

$$x_2' = -\frac{p}{1-p} \tag{62}$$

Let's now assume a gamble by a pair of numbers (x_1, x_2) where the consumer gets x_1 if some event (E) occurs and x_2 if the event (E) does not occur. We define the consumer's acceptance set as consisting of both expectations, which is the simple probability function. The consumer plays this gamble or a set of gambles. The consumer would accept an initial wealth level w. We assume that the consumer uses his wealth to buy a lottery ticket. If the consumer is risk averse, the acceptable set will be a convex set. The boundary of this set and the set of indifferent gambles can be given by an implicit function $x_2(x_1)$ as shown in figure 1.10.

If the consumer behavior can be described by the maximization of expected utility, then $x_2(x_1)$ must satisfy the identity, and is represented as follows

$$Pu\ (w+x_1) + (1\text{-}p)\ u\ (w+x_2(x_1)) \equiv u\ (w) \tag{63}$$

The slope of the acceptance set boundary at (0, 0) can be found by differentiating this identity with respect to x_1 and evaluating this derivative at $x_1 = 0$.

$$Pu'\ (w) + (1\text{-}p)\ u'\ (w)\ x'_2(0) = 0 \tag{64}$$

Suppose we solve the above equation for the slope of the acceptance set, then we find it as

$$x_2^{'} = -\frac{p}{1-p}$$

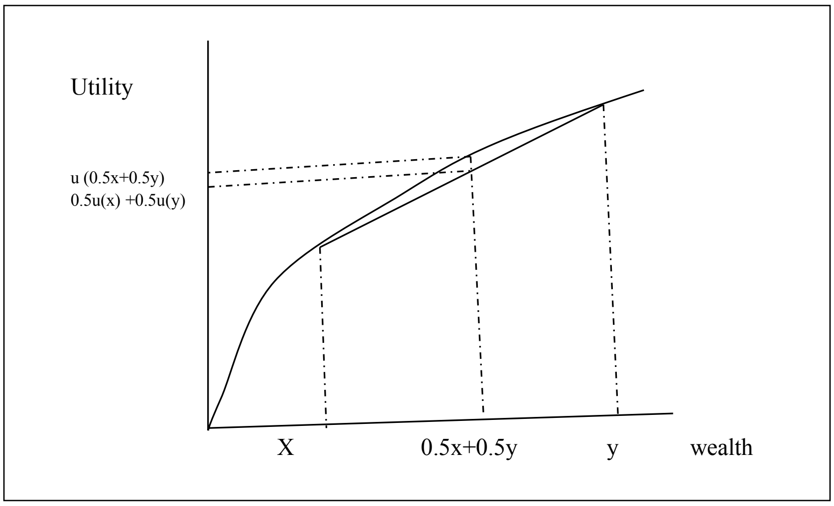


Figure 1.10. Relationship of wealth and expected utility

Figure 1.10 shows that the expected utility from both the lotteries 0.5u(x) + 0.5u(y) is equal. If we compare this with u (0.5x+0.5y) then such utility is slightly more than the equal probability of x and y.

The utility of x and y are different. In terms of wealth, the utility of the y lottery is much more than the utility of the x lottery. It is up to the consumer to decide how to choose between the two lotteries.

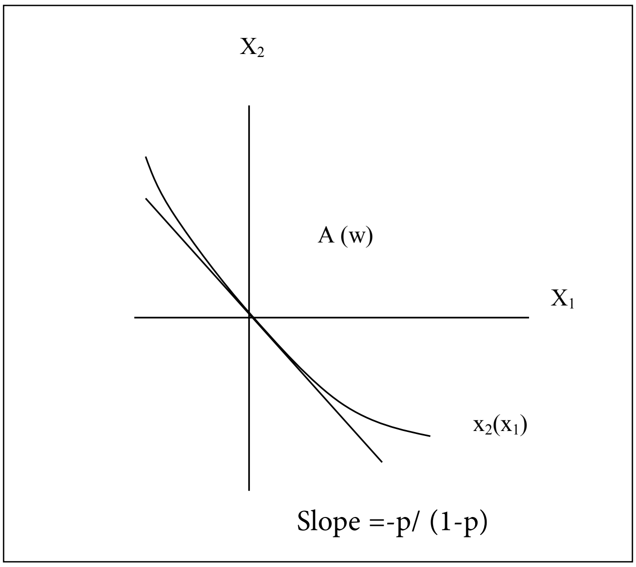


Figure 1.11. Trade off among lotteries to consumer

Figure 1.11 shows that the slope of the acceptance set at (0, 0) gives odds. At the same time, it gives us a nice way of drawing out probabilities. We need to find the odds at which a consumer is just willing to accept a small count on the event in question. We can prove this with the help of two consumers and probabilities. Suppose there are two consumers with identical probabilities on the event E. Let's assume further that consumer i is more risk averse than consumer j. If consumer i's acceptance set is contained in consumer j's acceptance set then this is a global statement of risk aversion, and means that consumer j will accept any gamble that consumer i will accept. Consumer i is locally more risk averse than consumer j then i's acceptance set is contained in j's acceptance set in a neighborhood of the point (0, 0). By differentiating the identity with respect to x_1 and evaluating the resulting derivative at zero, we find the following equation

$$Pu''(w) + (1-p)\, u''(w)\, x'_2(0) + (1-p)\, u'(w)\, x''_2(0) = 0$$

We can use the fact that $x'_2(0) = -p/(1-p)$, we have

$$x_2''(0) = \frac{p}{(1-p)^2}\left[-\frac{u''(w)}{u'(w)}\right] \quad (66)$$

The above equation is a proportion of the Arrow–Pratt measure. Consumer j will take more risk with small gambles than consumer i. But this is only possible if consumer i has a larger Arrow-Pratt measure of local risk aversion (Varian, 2009). Risk aversion has many applications. Consumers always try to reduce any risks to improve their economic gains.

Questions

Question1. Explain the consumer preference relation in detail.

Question 2. Define the following terms in detail.

a) Completeness b) transitivity c) reflexive d) Nonsatiation e) Strict convexity

Question 3. Define the utility function of consumer preference.

Question 4. How is consumer preference different from the choice rule?

Question 5. What is lexicographic ordering? Why is it criticized by economists?

Question 6. Explain briefly the revealed preference theory.

Question 7. Explain the weak axiom of revealed preference with reference to the substitution and income effect.

Question 8. Critically examine the indirect utility function along with its various properties.

Question 9. Examine the expenditure function along with its properties.

Question 10. Write a short note on the following:

a) Hicksian demand function b) Expenditure minimization problem

Question 11. Critically examine the Von Neumann-Morganstern utility function.

2 The Production Function

2.1 Inputs to output function

Every firm uses inputs and produces output. Inputs used in the production function include labor, machinery, land, etc. From the available resources, a firm produces a maximum quantity of goods, called the Production Possibility Frontier (PPF). The PPF shows the different combinations of the inputs to output. The combinations of the production function change with different inputs. The time factor-the time dimension of specific inputs used in output.-is also included in the production function. The production possibility frontier differs in terms of the nature of output. Some commodities require different proportions of capital and labor, while some production functions have the same proportion of labor and capital. Some production functions require the maximum labor whereas other production functions require more capital. It is the firm which decides whether to replace labor with capital. Most of the time such replacements of the factors of production depend on factor rewards and endowments. In a labor abundant region, more labor intensive goods are produced. At the same time, capital intensive goods are produced in developed countries. But in any production function, capital and labor intensive techniques are used.

2.2 Technology specification

In technology specification, we assume that there is only one firm. Such a firm produces n goods from available inputs. The production of goods usually requires some amount of labor and capital. The production plan is prepared by each firm, and consists of the use of inputs in the production of output. A firm uses Y^i_j units of input and produces Y_j output during a particular time period. The net output of good j is given as

$$Y_j=Y^0_j-Y_j^i \tag{1}$$

If j is positive then the firm produces more of good j that it uses as input. The production plan of any firm is a vector Y in R^n. Y_j is negative if the j^{th} good serves as a net input and positive if the j^{th} good serves as a net output. The production possibility set is the technologically feasible production plan, denoted by Y and which is a subset of R^n. The Y set is the production plan where all input and output patterns are described.

Every firm has two types of production functions. The first is the conventional production function and the second, the new or innovative production function. The innovative production function uses the new technologies and professional skilled manpower. Such technologically innovative production function changes in the short term. There is no fixed market for such goods which can be trendy or fashionable products. In the globalization era, due to competition, a firm uses the innovative production function to sustain a market share. In the short run all the factors of production are fixed. The technology set is fixed therefore only the conventional production set is possible. A firm cannot make the decision to use more capital and hire more skilled manpower. But in the long run, such factors are variable in their nature. Therefore, a firm's technological possibility set may change. A firm makes decisions about the different factors of production and becomes more competitive in the market. A firm plans a dominant strategy in the long term to increase the firm's market share.

In the short run, the production function is denoted as Y(Z), which consists of all feasible net output bundles consistent with the constraints level Z. In the short run, supposing that the factors are fixed then an equation can be written as

$$Y\ (Y_n) = \{Y \text{ in } Y\text{: } Y_n = Y_n\} \tag{2}$$

where Y (z) is a subset of Y and consists of all production plans which are more feasible.

2.3 Input requirement set

Sometimes a firm produces only one good from the available inputs. The net output is (Y-X). Here X is a vector of inputs and produces Y units of output. In the garment units, a firm produces a shirt from the available fabric. For such production, the firm's input requirement set is given as

$$V(Y) = \{X \text{ in } R^n_+\text{: } (Y\text{-}X) \text{ is in } Y\} \tag{3}$$

The input requirement set is the set of all input bundles. These bundles produce at least Y units of output. But a firm also uses land and labor in its production function, something which will not be taken into consideration at this point.

Isoquants

Isoquants are the production plan and possibilities of a firm, and can be defined as

$$Q(Y) = \{X \text{ in } R^n_+\text{:} X \text{ is in } V(Y) \text{ and } X \text{ is not in } V\ (Y') \text{ for } Y'>Y\}$$

Isoquants are the given bundles that produce exactly Y units of output. Input combinations are sometimes changed in the production function while output is kept constant.

Short-run production possibility set

In the short run, the production function is fixed. If a firm uses some proportion of labor and capital then this is written as

$$Y = f\,(K, L) \tag{4}$$

where

K= the amount of capital used in the production function

L = the amount of labor used as input in the production function

$Y\,(k) = \{(L, K)\}$ in Y: $K = k$

This is an example of a short-run production possibility set. In the above production function, more capital than labor is used.

2.4 The transformation function

A production plan Y in y is technologically efficient. Suppose there is no y' in Y then Y'> y and Y'= y. A production plan is efficient if there is a way to produce more output with the same inputs or a way to produce the same output with less input. Every firm prefers only an efficient production function. The transformation function can be written as

$$T: R^n \rightarrow R \tag{5}$$

where T (y) = 0, it is possible when y is efficient. The production function picks out the maximum scalar output as a function of the inputs. The transformation function picks out the maximal vectors of the net output.

Activity analysis in the production function

In activity analysis, we need to assume that there are two inputs required in the production function. The output is produced using the factor inputs: labor and capital. The two different factors of production perform two different activities. Labor and capital are used in different proportions in the production function. From both factors of production, output is produced. Capital cannot be used for the labor-intensive tasks which we assume are specific in nature. The capital intensive techniques use more machinery while labor is used for supervision or for quality control. Sometimes more machines can produce more output. But sometimes the quality of produced goods matters.

For example, in the labor-intensive technique, two units of labor and one unit of capital are used to produce one unit of output. Let's say that in a garment unit, a machine is required for two workers. They perform activities such as sewing and cutting of fabric. At the final stage, one worker can produce a shirt by joining the different parts.

In the capital-intensive technique, two units of capital and one unit of labor are used. In order to make prints of a draft, a computer and printer are required. Such units are capital intensive units. One unit of labor is enough to make a print of a draft. But without the printer, printing of the draft is not possible. In the modern world, firms produce a number of goods with different combinations of labor and capital. There are thus, a number of possibilities of a firm's production set. The production possibility set implies two different sets of production. We can define Y as a set of combinations, and define it as follows

$$Y= \{(10, 10, 20)\ (10, 20, 10)\} \tag{6}$$

Now if we assume that the firm is using the input requirement set, such set can be written as

$$V\ (Y) = \{(10, 20), (20, 10)\}.$$

The input required for production of goods is represented by two subsets. Assuming that the production of Y requires different combinations of inputs, we can write the possible combinations as

$$V(y) = \{(y, 4y), (4y, y)\}. \tag{7}$$

The equation shows that there are two subsets and two possible production functions. Suppose we write it further as

$$V(y) = \{(A, B)\} \tag{8}$$

Now, the use of technique A will require (y, 4y) input of two factors of production. Secondly, using the B type of technique requires (4y, y) input of the factors of production. Such techniques and combinations are shown in figure 2.1.

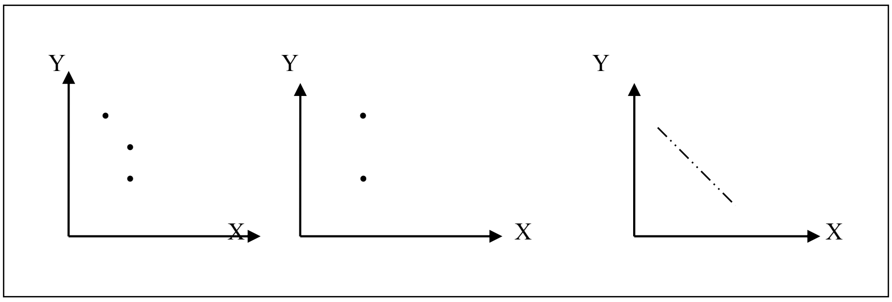


Figure 2.1. Combinations of techniques and output

It is assumed that the firm will either use technique A or technique B to produce output. Output is indicated as Y. The Y set is now shown to be different, and given in the following form

$$V(y) = \{(2, 4), (4, 2), (8, 8)\} \tag{9}$$

In the above set, there are different combinations. The last combination explains that (8, 8) can produce one unit of output.

2.5 Monotonic technologies

We have different input vectors to produce the output. Their combinations can be changed. Let's assume that the input factors are (2, 4) to produce output at any given time. It is also possible to produce one unit of output from (6, 4) input at any particular time. We can say that x is a feasible way to produce y units of output and x' is an input vector. We can further say that x' should be a feasible way to produce y. Therefore, the inputs are required to be monotonic to produce the output.

If x is in V(Y) and x' > x then x' is in V(Y). This is depicted in figure 2.2.

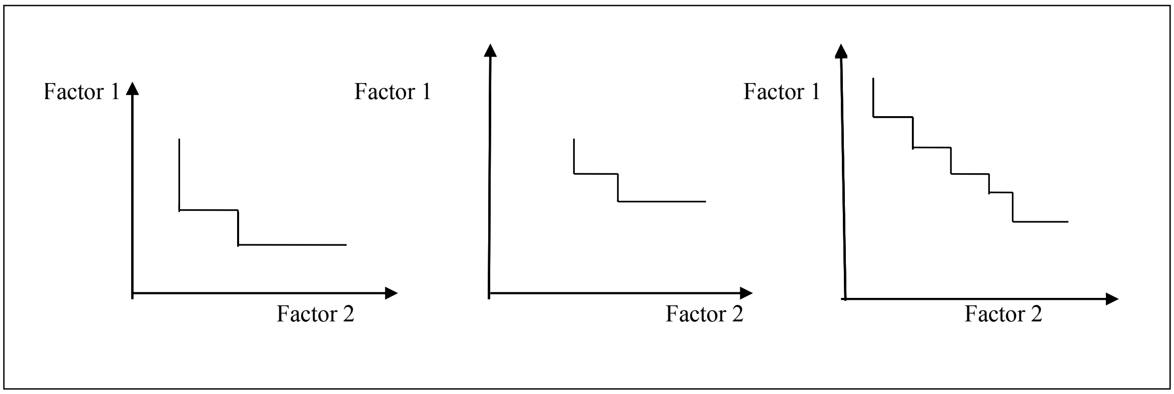


Figure 2.2. Possibilities of production through different sets of inputs

In figure 2.2, we assume that y is in Y and Y < y. Now y' must also be in y. Similarly, if y' < y, then every component of vector y' is less than or equal to the corresponding component of y. The production plan represented by y' produces an equal or smaller amount of all output by using at least as much of all inputs as compared to y. Hence it is natural to assume that if y is feasible, y' is also feasible.

2.6 Convex technology

In the convexity of technology, we need to assume an actual output of the commodity. Suppose we produce 100 units of output of any commodity. If we multiply the vectors (1, 2) and (2, 1) by 100 then we will get the following identity. At this point, we should be able to replicate what we were doing before and produce 100 times as much. In any possible situation, production processes will necessarily allow for the replication. In the process of replication, we can explain that (100, 200) and (200, 100) are the V (100). Suppose we operate 50 processes of activity A and 50 processes of activity B. Then we can use 10 units of good 1 and 150 units of good 2 to produce 100 units of output. Hence (150, 100) should be in the input requirement set.

We could also operate 25 processes of activity A and 75 processes of type B. It implies that 0.25 (100, 200) + 0.75 (200, 100) = (175, 125) It should be V (100) .The probability function is as follows

$$t\,(100, 200) + (1-t)\,(200, 100) = (100t+200\,(1-t)\,,\ 200t + (1-t)\,100)$$

V(100) for t = 0.01, 0.02………, supposing that t is a fraction between 0 and 1.

Convexity

In the convexity of technology, if x and x' are in V(y) then tx + (1-t) x' is in v(y) for all 0 < t < 1, that is, v(y) is a convex set. We can show this in figure 2.3.

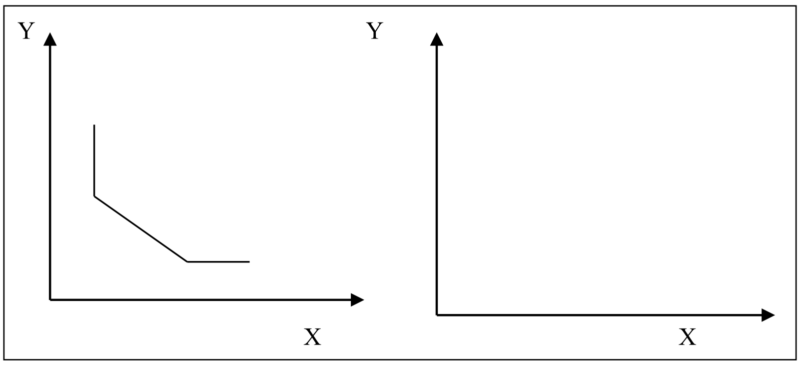


Figure 2.3. Convex set of technology

If a large amount of goods is produced by a single firm then we can replicate small production processes. Technology should be modeled as being convex which increases with each increase in production. If the scale of the underlying activities is large relative to the desired amount of output, convexity may not be a reasonable hypothesis. If one vector of input x produces y units of output then we might use other vectors that is x which also produces y units of output. Here we might use x for half the amount and x'. It is common to assume that if y and y' are both in Y, then ty + (1-t) y' is also in Y for 0 < t < 1. In other words, Y is a convex set. It should be noted that the convexity hypothesis is problematic than the convexity of the input required set. The convexity production set implies a convex input requirement set. Both are expected to be parallel with each other. Suppose the production set y is a convex set, then the associated input requirement set V(y) is also a convex set.

Proof

We can present proof in terms of a simple probability function. We have (ty+ (1-t) y, tx – (1-t) x') in y. This simply requires that (Y-(tx+ (1-t) x)) is in y.

In the above notation it follows that x and x' are in V (Y). It explains that tx + (1-t) x' which is in V (Y), showing that V(Y) is convex. The convex input requirement set is equivalent to a quasiconcave production function. Now, V(Y) is a convex set. This is possible only if the production function F(x) is a quasiconcave function. The proof is given as V(Y) = x: f(x) > y, which is just the upper contour set of (x).

2.7 Regular technology

Few production functions require regular technology to produce output. Sometimes, it may be difficult to adjust output with new technology. In the regular technology set, the weak regularity condition concerning v(y) is explained as regular technology. The v(y) is a closed non-empty set for all y ≥ 0.

It is assumed that v(y) is a non-empty set which requires that there be some conceivable way to produce any given level of output. Suppose we have a sequence (x^1) of input bundles that can each produce y. This sequence converges to an input bundle x^0. We can say that the input bundle in the sequence gets arbitrarily close to x^0. Suppose v(y) is a closed set then this limit bundle x^0 must be capable of producing y. This means that the input requirement set must include its own boundary. This is sometimes true in a production set.

Parametric representations of technology

Firms use different ways to produce any given level of output. The choice of a firm about which inputs to use affects the level of output. These parametric technological representations should not necessarily be thought of as a literal depiction of production possibilities. Production possibilities are sometimes viewed as engineering data, which describe the physically possible production plans. Firms regularly change the input set to produce the same quantity of output.

Transformation function

In the transformation function, there is an n dimensional analog of a production function. Such a function is useful in our study for general equilibrium theory. A production plan of any firm is y in Y. It is a technological plan. Capital is used to increase the share of technology. Suppose there is no y' in Y then y'≥ y can be possible option. Similarly y'≠ y_i means a production plan is efficient if there is no way to produce more output with the same inputs. Alternatively, producing the same output with less input can also mean that a production plan is efficient.

We often assume that we can describe the set of the technologically efficient production plan by a transformation function. We can write this as T: $R^n \to R$ where T(y) = 0.

Suppose y is efficient, then the production function picks the maximum scalar output which is a function of the inputs. The transformation function picks out the maximal vector of net output. In terms of Leontief technology, the transformation function can be written as

$$T(y, x1, x2) = y - \min(ax_1, bx2)$$

2.8 Cobb-Douglas technology

In Cobb-Douglas technology, the production function is defined in terms of the technology set. Now all the competitive firms use maximum technology in their production functions. If we assume that the parameter is 1 < a < 1 then the Cobb Douglas technology is defined as follows

$$Y= \{(y\text{-}x_1\text{-}x_2) \text{ in } R^3\text{: } y \leq x_1^0 x_2^{1-\alpha}\}$$

$$V(y) = \{(x1, x2) \text{ in } R_+^2 : y \leq x_1^a x_2^{1-a}\}$$

$$Q(y) = \{(x_1, x_2) \text{ in } R_+^2 : y \leq x_1^a x_2^{1-a}\}$$

$$Y\ (z) = (y - x_1 - x_2) \text{ in } R^3 :\ y \leq x_1^a x_2^{1-a}, x_2 = 2\}$$

$$T(y\ x_1, x_2) = Y\text{-}\ x_1^a x_2^{1-a}$$

$$F(x_1, x_2) = x_1^a x_2^{1-a} \tag{10}$$

The above equation shows the possible production, quantity of inputs, technology set of any firm.

Leontief technology

If we let a > 0 and b > 0 be the parameter, then the Leontif technology is defined as

$$Y = \{(y - x_1 - x_2) \text{ in } R^3 :\ y \leq \min(ax_1, bx_2)\}$$

$$V(y) = \{(x_1, x_2) \text{ in } R_+^2 :\ y \leq \min(ax_1, bx_2)\}$$

$$Q(y) = \{(x_1, x_2) \text{ in } R_+^2 :\ y = \min(ax_1, bx_2)\}$$

$$T(y, x1, x2) = y\text{-}\min(ax_1, bx_2)$$

$$F(x1, x2) = \min\ ((ax_1, bx_2) \tag{11}$$

Both the Leontief technology and Cobb-Douglas production function are explained in figure 2.4.

Figure 2.4(A) shows that the Cobb-Douglas production function is convex in nature. In this figure, the labor and capital are indifferent. If labor's proportion is reduced then automatically the proportion of capital rises. But the production will be indifferent at different points. A producer has more flexibility to adjust both inputs and produce output. Such a production function has a general shape.

In the Leontief technology, both factors of production are used in the same proportion. A change in one factor may cause an equal change in the other factor. Sometimes, they change in proportion with each other.

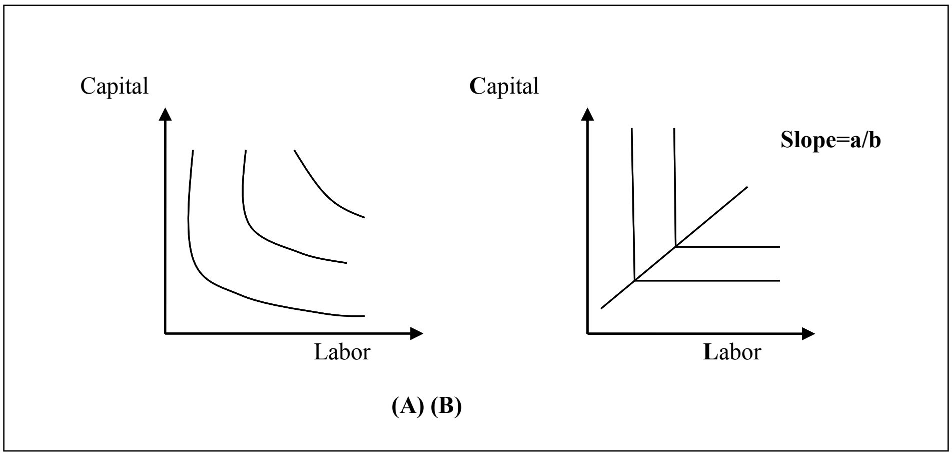


Figure 2.4. Cobb-Douglas and Leontief technology

Figure 2.4 A depicts the production function of the Cobb-Douglas technology which has a general shape. They are isoquants of the general production function. Figure 2.4 B shows the general shape of Leontief technology. Both factors of production are used in the same proportion.

2.9 Leontief technology

In some industries, the production is diverse in nature. Therefore, the production process may require the same factor proportion. Sometimes the factors remain constant. For the delving of farm land it is not always enough to add more workers. We must also use machines such as a tractor to cultivate the land. But adding more tractors will not help to cultivate more land, and will not increase the farm production. We need to increase the number of workers in proportion to the increase in tractors. This means that labor and capital must work together in a constant proportion. The proportion is one tractor for one worker. We have assumed that the worker knows how to drive a tractor.

In a particular production function Y = f (K, L). Such a production function exhibits the fixed proportion function.

Property

In order to produce a single unit of output, we need V units of capital and U units of labor. There is no flexibility in technique here. The coefficient V and U are the fixed input requirement in order to produce a single unit of output. Suppose, to produce Y units of output we need VY units of capital and UY units of labor.

In other words,

$$K = VY \tag{12}$$

This is the capital requirement and

$$L = UY \tag{13}$$

The above equation also shows the labor requirement. If we combine the two equations then the result is presented as follows and it is the only technique available as L/K = U/V. The ratio of labor to capital is nothing but the unit of capital to the unit of labor. The implied L shape of isoquants of such a production function is shown in figure 2.5.

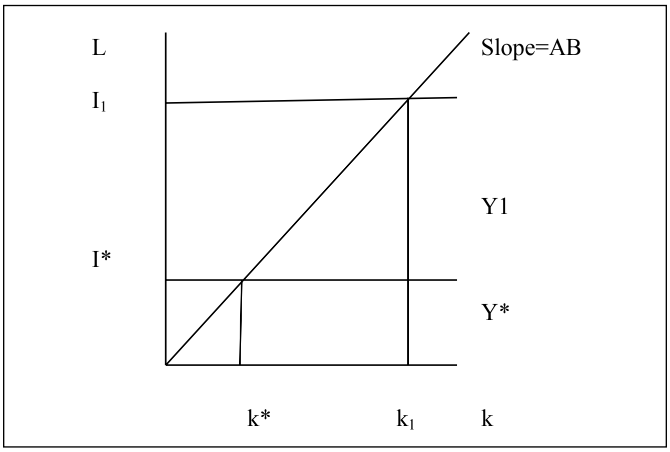


Figure 2.5. Possibilities of production

Leontief isoquants

Leontief isoquants are explained with no substitution effect. Such a function for a no substitution case can be written as follows

$$Y = \min(VK, UL) \tag{14}$$

The above production function is referred to as a Leontief production function. This production function was introduced by Wassily Leontief in 1941. Since that time, the Leontief production function has been used widely in the technology set.

A production possibility set in terms of Leontief technology is explained as follows. Let a > 0 and b > 0; both are the parameters in this function. The Leontief technology is defined in the following manner.

Input requirement set

A firm produces the output but uses the input set. We can write this as the output as a function of input. The net output bundle is (y - x). In this function, x is a vector of inputs that can produce a y unit of output. Now, we can define a special case of a restricted production set. The input required set can be written as

$$V(y) = \left\{ x in R_+^n : (y, -x) in Y \right\} \tag{15}$$

Equation (15) explains that the input requirement set is the set of all input bundles. Such bundles produce at least y units of output. We can take note that the requirement is defined differently. In a production possibility set, the measured inputs are expressed as positive rather than as negative numbers. Such numbers and functions are explained in terms of the Leontief technology set as

$$V(y) = \left\{ (x_1, x_2) in R_+^2 : y \le \min(ax_1, bx_2) \right\} \tag{16}$$

In terms of the production of commodities, one can define the isoquants as

$$Q(y) = \{x \text{ in } R^n_+: x \text{ in } V(y) \text{ and } x \text{ is not in } V(y') \text{ for } y' > y\} \tag{17}$$

The isoquant shows all input bundles that produce exactly y units of output. The isoquant in terms of Leontief technology is defined as

$$Q(y) = \left\{(x_1, x_2) in R^2_+ : y = \min(ax_1, bx_2)\right\} \tag{18}$$

The production function of a firm is defined differently. Suppose the firm has only one output, we can define that firm's production function as

$$F(x) = \{y \text{ in } R: y\} \tag{19}$$

This is the maximum output associated with x in y.

The production function in terms of Leontief technology is defined in the following equation.

$$f(x_1,x_2) = \min(ax_1, bx_2) \tag{20}$$

This is the most widely used production function.

2.10 The technical rate of substitution

The technical rate of substitution in two-dimensional cases is just the slope of the isoquant. The firm has to adjust x_2 to keep producing a constant level of output. If x_1 changes by a small amount then x_2 needs to be kept constant. In n dimensional case, the technical rate of substitution is the slope of an isoquant surface, measured in a particular direction. Let assume that $x_2(x_1)$ is the implicit function. This tells us how much of x_2 it takes to produce y. If we use x_1 units then the effect will be different. By definition the function $x_2(x_1)$ has to satisfy the identity.

$$F(x_1,x_2(x_1)) \equiv y \tag{21}$$

If we take a derivative of the above function then we can express it as $\partial x2(x^*_1)/\partial x_1$. If we are differentiating the above equation, then

$$\frac{\partial f(x^*)}{\partial x_1} + \frac{\partial f(x^*)}{\partial x_2}\frac{\partial x_2(x_1^*)}{\partial x_1} = 0$$

Or

$$\frac{\partial x_2(x^*)}{\partial x_1} = -\frac{\partial f(x^*)/\partial x_1}{\partial f(x^*)/\partial x_2} \tag{22}$$

The above equation gives us the technical rate of substitution. There is another way to derive the technical rate of substitution.We show this in figure 2.6. The technical rate of substitution measures the change in one input. This change in input gets adjusted to keep the output constant. There are a number of firms practising this. They also adjust another input in production. Sometimes, firms only hire labor for production. But strikes, labor unions and labor disputes force firms to use technology in their production function. Therefore, firms employ more capital and machinery as a factor of production. It is interesting to see how firms substitute labor for capital. Technology change among capital equipment suppliers lowers the costs over time of the firm's increasing delivery speed, using more flexible manufacturing methods, reducing the probability of defects, reducing costs of redesign and controlling production costs (Milgrom & Roberts, 1990). When changing such composition, some firms always keep the output constant. This is shown as follows.

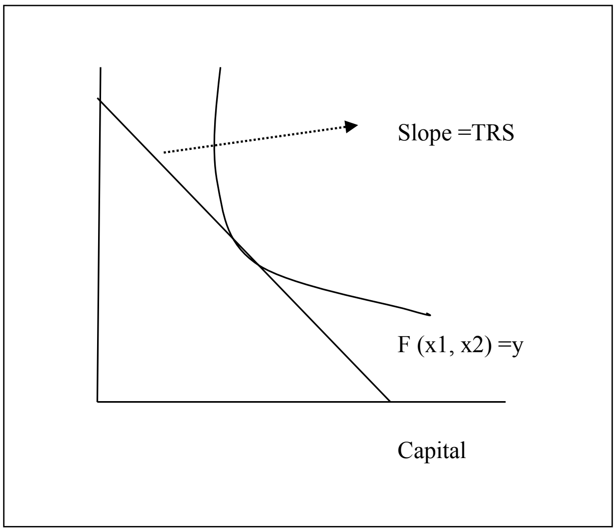


Figure 2.6. Production function with labor and capital

This can be presented in terms of the derivative of the two factors of production.

$$0 = \frac{\partial f}{\partial x_1}\partial x_1 + \frac{\partial f}{\partial x_2}\partial x_2 \tag{23}$$

After solving the above equation, we get the following

$$\frac{\partial x_2}{\partial x_1} = -\frac{\partial f / \partial x_1}{\partial f / \partial x_2} \tag{24}$$

Equation (24) shows the implicit function. The total differential method may be used to calculate the technical rate of substitution. The first method of calculation is wide and rigorous. But the second method is self-generated. But both methods are complete in their nature and both are useful.

Technical Rate of Substitution for Cobb-Douglas technology

In the technical rate of substitution for Cobb-Douglas technology, we need to derive the technical rate of substitution. Suppose the given function is defined as $f(x_1, x_2) = x_1^a x_2^{1-a}$,

$$\frac{\partial f(x)}{\partial x_1} = a x_1^{a-1} x_2^{1-a} = a[\frac{x_2}{x_1}]^{1-a}$$

$$\frac{\partial f(x)}{\partial x_2} = (1-a) x_1^a x_2^{-a} = (1-a)[\frac{x_1}{x_2}]^a$$

It can be further explained as follows

$$\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial f / \partial x_1}{\partial f / \partial x_2} = -\frac{a}{1-a}\frac{x_2}{x_1} \tag{25}$$

2.11 Elasticity of substitution

The elasticity of substitution measures the curvature of an isoquant. Such substitution is given as follows

$$\sigma = \frac{\%\Delta in \frac{x_2}{x_1}}{\%\Delta in TRS}$$

$$= \frac{TRS}{\frac{x_2}{x_1}} \cdot \frac{\partial(\frac{x_1}{x_2})}{\partial TRS} \tag{26}$$

It is often convenient to calculate σ through using the logarithmic derivative. In general, suppose y = g(x), then the percentage change in y induced by a small percentage change in x is defined as

$$\varepsilon = \frac{\frac{\partial y}{y}}{\frac{\partial x}{x}}$$

$$= \frac{\partial y}{\partial x} \cdot \frac{x}{y} \tag{27}$$

The equation shows that x and y are positive. Equation (27) can be written differently as

$$\varepsilon = \frac{\partial \ln y}{\partial \ln x} \tag{28}$$

We can use a total differential function to rewrite the above equation as

$$\partial \ln y = \frac{1}{y} \partial y \tag{29}$$

$$\partial \ln y = \frac{1}{x} \partial x$$

From the above two equations, we can derive the following equation

$$\varepsilon = \frac{\partial \ln y}{\partial \ln x} = \frac{\partial y}{\partial x} \cdot \frac{x}{y} \tag{30}$$

The equation can be further modified. By applying the elasticity of substitution, it can be written as

$$= \frac{\partial \ln(x_2 / x_1)}{\partial \ln |TRS|} \tag{31}$$

The above function is an elasticity of substitution for the Cobb-Douglas production function. We have already seen that

$$TRS = -\frac{a}{(1-a)}\frac{x_2}{x_1} \tag{32}$$

which can be written alternatively as

$$\frac{x_1}{x_2} = -\frac{1-a}{a} TRS \tag{33}$$

Equation (33) can be further written after taking log as follows

$$\ln\frac{x_2}{x_1} = \ln\frac{1-a}{a} + \ln|TRS| \tag{34}$$

The equation implies that

$$\sigma = \frac{\partial \ln(x_2 / x_1)}{\partial \ln|TRS|} = 1 \tag{35}$$

2.12 Variation in scale

In the variation of scale, we need to understand the response of output to changes in inputs. Here, the change in any input can arise from two basic points. Firstly, the change in the scale of production can take place by varying all inputs in the same proportion. Secondly, a change in relative input proportion can take place.

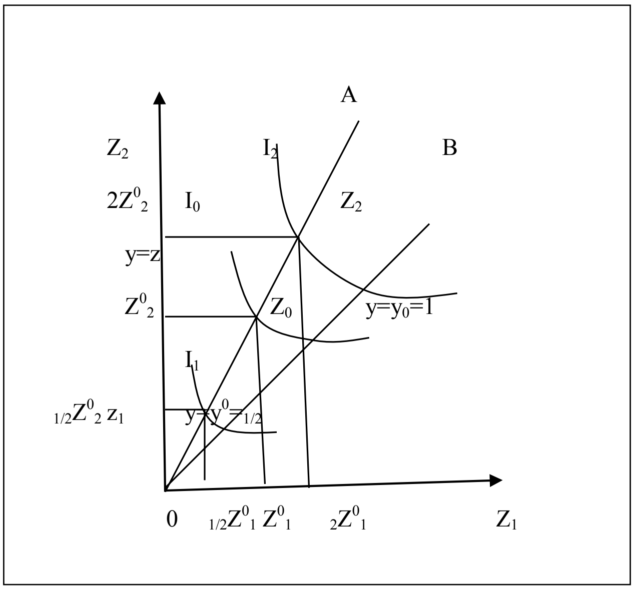


Figure 2.7. Input to output proportion

Figure 2.7 shows movements along the OA and OB points. The movement from one point to another point is explained in the following way. If we consider y point then the variation in y with scale parameters is in proportion, and kept constant. The value implied by the initial z by the elasticity of scale is written as follows

$$E = \frac{\text{Percentage change output (y)}}{\text{Percentage change in scale parameter (s)}}$$

Alternatively, this can be written as

$$E = \frac{\partial y}{y} \cdot \frac{s}{\partial s}$$

$$= \frac{\partial y}{\partial s} \cdot \frac{s}{y} \qquad (36)$$

Suppose that E > 1 then it shows increasing returns to scale. At the same time, if E = 0 then it shows constant returns to scale. When E < 1, there is a decreasing return to scale.

Since $\frac{\partial y}{\partial s}$ depends on the input mix, the returns to scale for the production function may depend on the input mix and s. In figure 2.7, I_0 and I_2 isoquants show constant returns to scale. The rays along OA and OB also show increasing returns to scale.

2.13 Revised technical rate of substitution

If we assume that technology is constant then a firm produces output with the help of inputs. The production function can be written as

$$Y^* = f(x^*_1, x^*_2).$$

It is constant and for a particular time. Suppose we want to increase the amount of capital input and decrease the amount of labor input. The output is maintained at a constant level. It is determined by the Technical Rate of Substitution or TRS. In two-dimensional cases, TRS is nothing but the slope of isoquants. It is interesting to see how one can adjust x_2 to keep output constant while decreasing x_1. We can see this in figure 2.8.

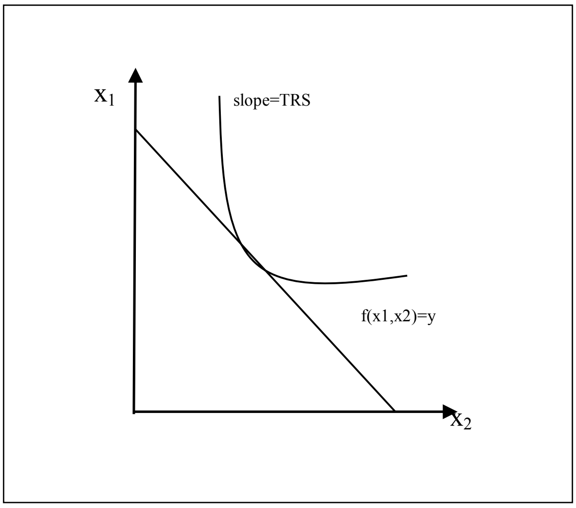


Figure 2.8. Derivation of the technical rate of substitution (TRS)

In figure 2.8, we have the derived technical rate of substitution. It shows a small change in the vector of inputs. We can write this as

$$dx = (dx_1, dx_2).$$

The associated change in the output is approximated by

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{df}{dx_2} \partial x_2 \tag{37}$$

The above equation is known as the total differentiation of function f(x). Now we consider $dx_1$and dx_2 adjusting along the isoquant while the output remains constant. The function can be derived as

$$0 = \frac{\partial f}{\partial x_1} dx_1 + \frac{df}{dx_2} \partial x_2$$

$$0 = \frac{\partial x_2}{\partial x_1} = -\frac{df / \partial x_1}{df / dx_2} \tag{38}$$

The above equation is a slope of the isoquant. The TRS for Cobb-Douglas technology can be derived as follows.

Given that $f(x_{1,} x_2) = x_1^a x_2^{1-a}$ we can take the derivative of the above function

$$\frac{\partial f(x)}{\partial x_1} = a x_1^{a-1} x_2^{1-a}$$

$$= a(\frac{x_2}{x_1})^{1-a}$$

$$\frac{\partial f(x)}{\partial x_2} = (1-a) x_{,}^{a} x_2^{-a}$$

$$= 1 - a(\frac{x_1}{x_2})^{a}$$

In the above equation it follows that

$$\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial f / dx_1}{\partial f / dx_2}$$
$$= -\frac{a}{1-a}.\frac{x_2}{x_1} \tag{39}$$

The above equation is the technical rate of substitution.

2.14 Homogenous and heterogeneous production function

We have seen that the production function is homogenous to degree one. Suppose we multiply all input by s, this leads to an increase in output by factor s^t.

Suppose f (SZ) = S^tfi (Z)

The production function is homogenous to degree t. In the production function T = 1. The production function is linearly homogenous. The homogenous production function of degree 1 can be defined as

$$E = \frac{\partial y}{\partial s}.\frac{s}{y}$$
$$= \frac{\partial f(sz)}{\partial s}.\frac{s}{f(sz)}$$
$$= \frac{\partial s^t fi(z)}{\partial s}.\frac{s}{S^t fiz}$$
$$= ts^{t-1} fi(z).\frac{s}{s^t fiZ} = t \tag{40}$$

The above linear homogenous function is equal to 1, and can also be written as t = 1. The linear homogenous function has constant returns to scale, which all have input combinations.

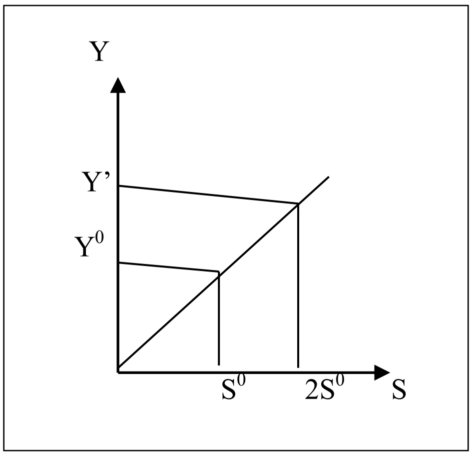


Figure 2.9. Linear homogenous production function

The two sides of the equation must be equal. If we take a partial derivative of the left hand side then

$$\frac{\partial f(Z_1, Z_2)}{Z_i}$$

$$= \frac{\partial f(z_1, z_2)}{z_i} \cdot \frac{\partial(sz_i)}{\partial z_i}$$

= fi(sz)s (41)

The partial derivative of the right hand side is S^tf(z), we have fi(sz) s = s^tfi(z). This can be written alternatively as

fi(sz) = s^{t-1}fi(z)

The above function is homogenous to degree t, which have partial derivatives which are homogenous to degree t -1. We have assumed that t = 1. This is a linear homogenous function, and is independent of scale. The slope of the isoquant is

$$S_Z = -\frac{fi(sz)}{f_2(sz)}$$

$$= -\frac{fi(z)}{f_2(z)} \qquad (42)$$

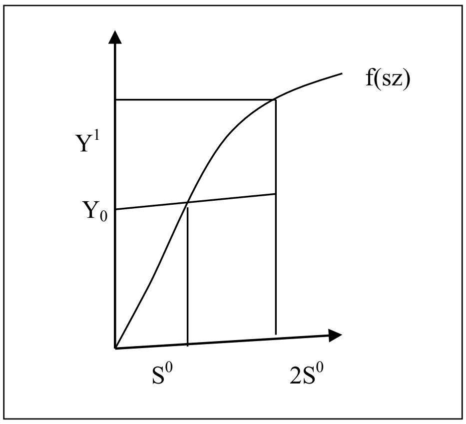


Figure 2.10. Slope of the isoquant and the production function

Equation (42) holds for all s which are homogenous. The derivative of the left hand and the right hand sides of the equation with respect to two must be s. The left hand side of equation (42) is

$$\frac{\partial f(sz_1, sz_2)}{\partial s_i}$$

$$= \sum_i \frac{\partial f(sz_1, sz_2)}{\partial (sz_i)} \cdot \frac{\partial (sz_i)}{\partial s}$$

=fi (s.z) zi

$$= s^{t-1} \sum_i fi(z)zi \tag{43}$$

The right hand side of equation (42) can be written as $Ts^{t-1}f(z)$, and is homogenous to degree t.

$$\sum_i fi(z)zi = tf(s) \tag{44}$$

2.15 The Envelope theorem for constrained optimization

The Envelope theorem is explained in terms of Shepherd's Lemma. We can apply a version of the envelope theorem, which is appropriate for the following case.

The Envelope theorem is a general parameterized constrained maximization problem of the form

$$M(a) = \max_{x_1, x_2} g(x_{1,} x_{2,} a) \tag{45}$$

and can be explained as $h(x_{1,}x_{2,}\ a) = 0$. In the case of the cost function, the function is written as

$$g(x_{1,} x_{2,} a) = w_1 x_1 + w_2 x_2, h(x_{1,} x_2, a) = f(x_{1,} x_2) - y \tag{46}$$

In the above function price is denoted by a. The Lagrangian for this problem can be written as

$$\ell = g(x_{1,} x_{2,} a) - \lambda h(x_{1,} x_2, a) \tag{47}$$

For the above function, we need to take the first order condition, as follows

$$\frac{\partial g}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$$

$$\frac{\partial g}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$$

Similarly,

$$H(x_{1,}\ x_{2,}\ a) = 0 \tag{48}$$

The above condition determines the optimal choice function ($x_1(a)$, $x_2(a)$), and determines the maximum value function

$$M(a) \equiv G(x_1(a), x_2(a), a) \tag{49}$$

The Envelope theorem gives us a formula for the derivative of the value function with respect to the parameters in the maximization problem. The formula is given as

$$\frac{dM(a)}{da} = \frac{\partial l(X,a)}{\partial a}\Big|_{X=X(a)}$$

$$= \frac{\partial g(x_1,x_2,a)}{\partial a}\Big|_{xi=xi(a)} - \lambda\frac{\partial h(x_1,x_2,a)}{\partial a}\Big|_{x_i=x_i(a)} \tag{50}$$

The interpretation of partial derivatives needs special care. They are the derivatives of g and h with respect to a holding x_1 and x_2. They are fixed at their optimal values. The proof of the envelope theorem is straightforward and is calculated in the following equation.

Differentiating the identity above to get

$$\frac{dM}{da} = \frac{dg}{dx_1}\frac{dx_1}{da} + \frac{dg}{dx_2}\frac{dx_2}{da} + \frac{dg}{da}, \tag{51}$$

If we substitute from the first order condition of equation (51) then we get the following equation

$$\frac{dM}{da} = \lambda\left[\frac{dh}{dx_1}\frac{dx_1}{da} + \frac{dh}{dx_2}\frac{dx_2}{da}\right] + \frac{dg}{da} \tag{52}$$

From equation (52), we can observe that the optimal choice function must identically satisfy the constraints, as follows

$$H(x_1(a), x_2(a), a) \equiv 0. \tag{53}$$

Suppose we differentiate this identity with respect to a, then we have

$$\frac{dh}{dx_1}.\frac{dx_1}{da} + \frac{dh}{dx_2}\frac{dx_2}{da} + \frac{dh}{da} \equiv 0 \tag{54}$$

Now we substitute (54) into (53) to get the following equation

$$\frac{dM}{da} = -\lambda\frac{dh}{da} + \frac{dg}{da} \tag{55}$$

Equation (55) is required for further interpretation of the results. which can be used for the cost minimization problem. In the cost minimization problem, the parameter can be chosen to be the factor price w_i. The optimal value function M (a) is a cost function and is presented as c (w, y).

The Envelope theorem explains that

$$\frac{dc(w,y)}{dw_i} = \frac{dl1}{dw_i} = x_i \Big|_{x_i = x_i(w,y)} = x_i(w,y), \tag{56}$$

The above function is simply a Shephard's Lemma. The proof is given as follows. Let us assume that x_i(w,y) , the firm's conditional factor demand for input i. Suppose the cost function is differentiable at (w,y) and wi > 0 for I = 1….n., we can derive the function as

$$x_i(w,y) = \frac{\partial c(w,y)}{\partial w_i} \quad \text{I = 1…..n} \tag{57}$$

Proof

Let's assume that x∗ is a cost minimizing bundle. It produces y at price w∗. We define the function as

$$\text{g (w) = c(w,y) - wx*} \tag{58}$$

In the above equation, c (w, y) shows that it is easy to produce y. This function is always negative. at w = w∗, g (w∗) = 0. Since this is a maximum value of g (w) and is already given, its derivative must vanish.

$$\frac{\partial g(w^*)}{\partial w_i} = \frac{\partial c(w^*, y)}{\partial w_i} - x_i^* = 0. \quad \text{I = 1…….n} \tag{59}$$

Hence, the cost minimizing input vector is given by the vector of derivatives. This is a cost function with respect to the prices. We can explain Shepherd's Lemma after adding the above proof. Shepherd's Lemma is an important result in microeconomics, and can be applied in the consumer choice preference and theory of firms. It helps one to understand how a consumer chooses commodities in their consumption bundle. The lemma states that if the indifference curve of the expenditure or cost function is convex, then the cost minimizing point of a given good (i) with the price p_i is unique.

The ideal case is that the consumer will buy a unique amount of each item, which minimizes the price for obtaining a certain level of utility given the price of goods in the market. We have already seen this in the first chapter. This theory and proof was named after Ronald Shepherd. He presented his research paper in 1953, in which he gave a proof using the distance formula. But the formula was already used by John Hicks (1939) and Paul Samuelson (1947) in their model. The lemma gives a precise formulation for the demand of each good in the market with respect to a certain level of utility and prices. The derivative of the expenditure function (e (p, u)) with respect to price is given as follows

$$hi(u,p) = \frac{\partial e(p_1 u)}{\partial p_i} \tag{60}$$

where

hi (u, p) is the Hicksian demand for good i.

E (p, w) shows an expenditure function and both functions are in terms of a price (a vector p) and utility u.

Although Shepherd's original proof used the distance formula, Shepherd's Lemma is proved in the Envelope theorem.

2.16 Duality of cost and the production function

In any given cost function, we can solve for a technology set. This means that the cost function contains essentially the same information that the production function contains. The concept is defined in terms of the properties of the production function which has a duel definition, in terms of the properties of the cost function and vice versa. This general observation is known as the principle of duality, and is broadly defined as a set VO(y) which is outer bound to the true input requirement set V(y). In a given data (W^t,X^t,Y^t), VO(y) is defined to be as follows

$$VO(y) = \{x : W^t x \geq W^t X^t \text{ for all } t \text{ such that } y^t \leq y\} \quad (61)$$

The above equation is straightforward to verify that VO(y) is a closed monotonic and convex technology. The equation includes any technology that could have generated the data (W^t,X^t,Y^t) for t = 1……..T In some sense, it should "approach" the true input requirements set. In order to make this precise, the factor prices vary over all possible price vectors w ≥ 0.The natural generalization of VO becomes

$$V^*_{(y)} = \{x : wx \geq wx(w, y) = c\ (w, y) \text{ for all } w \geq 0\} \tag{62}$$

The relationship between v*(y) and the true input requirement set V(y) in general v*(y) will strictly contain v(y). For example, in part (A) of figure 2.11, the left hand side area cannot be ruled out of V*(y) since the points in this area satisfy the condition that wx ≥ c (wy).

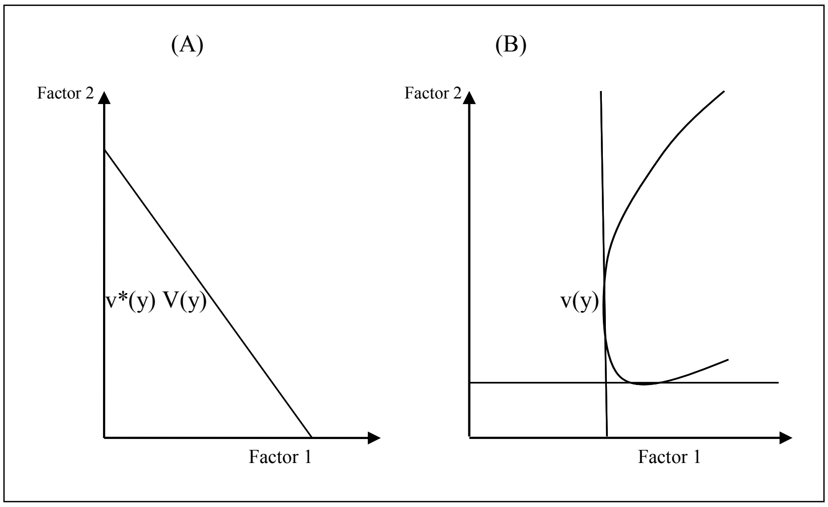


Figure 2.11. Input relation set

The relationship between v(y) and v*(y) is important. In the figure, v*(y) will strictly contain V(y).

The same is true for part B. The cost function can only contain information about the economically relevant sections of v(y). Factors that could actually be the solution to a cost minimization problem could be conditional factor demands. If we assume that the technology is convex and monotonic, it is proved as follows.

In this case $v^*(y)$ will be equal to $V(y)$. This is because in the convex, monotonic case, each point on the boundary of $V(y)$ is a cost minimizing factor. The demand for some price vectors is $w \geq 0$. Thus, the set of points is where $wx \geq c(w, y)$. For all $w \geq 0$, the inputs requirement set will be more formally describe.. If we assume that $V(y)$ is a regular, convex, monotonic technology, then $V^*(y) = V(y)$. The proof is given as follows. From the above equation, we already know that $V^*(y)$ contains $V(y)$, so we have to show that if x is in $V^*(y)$ then x must be in $V(y)$. We can say that x is not an element of $V(y)$ since $V(y)$ is a closed convex set satisfying the monotonic hypothesis. We can apply a version of the separating hyperplane theorem to find a vector. $W^* \geq 0$ is because $w^*x < w^*z$ for all z in $V(y)$. Let us assume that z^* is a point in $V(y)$. Costs are minimized at the price w^*. Now we have $w^*x < w^*z^* = C(w^*, y)$. But we cannot define x is in $V^*(y)$, according to the definition of $v^*(y)$.This proposition shows that if the original technology is convex and monotonic, then the cost function associated with the technology can be used to completely reconstruct the original technology.

Let's start with some technology $V(y)$, which may possibly be not convex. We find that it is the cost function $c(w, y)$, which thus generates $V^*(y)$. We also find the cost function $c(w, y)$ and then generate $v^*(y)$. We know from the above results that $v^*(y)$ will not necessarily be equal to $v(y)$, due to the convexity and monotonic properties. However, we need to define this differently as

$$C^*(w, y) = \min Wx \tag{63}$$

This means that X is in $V^*(y)$, when $C(w, y)$ equals $c^*(w, y)$. It follows from the definition of the function that $C^*(w, y) = C(W, Y)$. We can prove the theorem as follows.

It is easy to see that $c^*(w, y) \leq c(w, y)$ since $V^*(y)$ always contains $V(y)$. The minimal cost bundle in $V^*(y)$ must be at least as small as the minimal cost bundle in $V(y)$.Suppose that for the same price w', the cost minimizing bundle x' in $V^*(y)$ has the property that

$$W'x' = c^*(w', y) < C(w'y).$$

But this cannot happen since by definition of $V^*(y)$,

$$W'X' \geq C(W', y) \tag{64}$$

This proposition shows that the cost function for the technology $V(y)$ is the same as the cost function for its convex $V^*(y)$.

From the above discussion, we can explain that at any given cost function, the input requirement set is v(y*). If the original technology is convex and monotonic then the constructed technology will be identical to the original technology. Suppose the original technology is non-convex and non-monotonic, the constructed input requirement will be a convex monetized version of the original set. The constructed technology will have the same cost function as the original technology.

2.17 Michael Kalechi's theory

In advanced microeconomics, Michael Kalechi's theory is well known. The theory states that the profit share is a function of the degree of the monopoly power of the firm. That is,

$$\Pi b0 = f(\delta m) \quad (65)$$

where

Π_b = Profit share of the firm

δ_m = Degree of the firm's monopoly

In a capitalist economy, the production capacity of an individual firm is important. The actual production may increase or decrease but the capacity of the firm does not change in the long term. Therefore, excess capacity is a normal feature of the capitalist economy. As a consequence of this, production decreases or remains constant. The price of a commodity is determined by the full cost principle. The prime cost is defined as the cost consisting of wages of workers plus the cost of raw materials. The markup pricing is defined as

$$Mp = P + OC \qquad (66)$$

Where

Mp = Markup price

P = Profit in the form of dividend

OC = Overhead costs and may consist of interest, depreciation and salaries.

The above formula of markup pricing can be modified for a single monopoly firm as follows

$$P = K.AVC \qquad (67)$$

where

P = Price

K = Markup price

AVC = Average variable cost

The prime cost is defined as follows

$$K = \frac{Ap}{Apc} \qquad (68)$$

where

K = Mark up price/prime cost

Ap = Aggregate proceeds

Apc = Aggregate prime cost

$$K\ (W+R) = (W+O+R+E) \qquad (69)$$

Where

K = Markup price

W= Wage bill

R = Raw material cost

O = Overhead cost

E = Entrepreneur's income

$$KW + KR = (W + O + R + E) \tag{70}$$

If we rearrange the above equation then

$$O + E = KW + KR - (W+R)$$

$$O + E = (K-1)(W+R)$$

$$K = \frac{W + O + R + E}{W + R} \tag{71}$$

From the above equation we get

$$O + E + W + R = (W+R) K$$

$$O + E = KW + KR - W - R$$

$$O + E = KW - W + KR - R$$

$$O + E = W (K-1) + R (K-1)$$

$$O + E = (W + R)(K-1) \tag{72}$$

The left hand side of the equation represents the sum of the overhead costs and profits. We can assume that A is a value added by the production process.

$$A = W + O + E \tag{73}$$

Substituting (O + E) in the above equation, we get

$$A = W + (W + R)\,(K - 1) \tag{74}$$

We can divide the above equation by W on both sides

$$\frac{A}{W} = \frac{W + (K-1)(W+R)}{W} \tag{75}$$

Taking the reciprocal effect from the above equation

$$\frac{W}{A} = \frac{W}{W + (W+R)(K-1)} \tag{76}$$

Dividing further by W

$$= \frac{1}{1+(K-1)(1+\frac{R}{W})}$$

$$= \frac{1}{1+(K-1)(1+j)}$$

$$\text{where } j = \frac{R}{W} \tag{77}$$

This is the ratio of the cost of raw materials to the wage bill. Equation (77) indicates that the share of wages in the national income is the function of two variables k and j. Under pure competition,

P = AC = MC

∴ MC = AVC

$$\therefore K = \frac{P}{AVC} \tag{78}$$

Suppose P = AVC, then K = 1.

If the market deviates, then pure competition is k > 1. In perfect competition, there is a gap between the price and the marginal cost. This will further lead to an increase in the k. Now k represents the degree of monopoly in the market. The share of the wages in the national income is higher when the monopoly power is greater. At the same time $\frac{R}{W} = j$ is low.

Criticism

Michael Kalechi's model is criticized on various points. According to Kalechi, there is growth of industry in the long term. This affects capital in the long term, and will reduce the share of wages to national income. This effect is observed more often with agricultural products than with manufactured items. The wage bill and its share remain stable in the long term. Kalechi's model was also criticized by Bauer, who said that Kalechi lumped together the overheads and profits in his theory.

2.18 Neo-Keynesian model of distribution (Kaldor Model)

The Neo-Keynesian model of distribution is the full-fledged macroeconomic model. The model uses the concept of aggregate income, saving and distribution of income, and assumes a condition of full employment. It is endogenously constant-given by the state of confidence in the economy and it is not determined by the distribution of income. There are two income categories: wages and profits. Wages include payments to manual labor and salaries. Profits of firm include the returns to the entrepreneur and income of the owner. Constant saving propensities are assumed in the model, which states that the average propensity to consume of each group is the same. Moreover, the propensity to save out of profit is also explained in the model. With these assumptions, Kaldor proceeds to establish the consistency of the relative factors share and capital output ratio. The model can be expressed using the following symbols

$$Y = wp \tag{79}$$

where

Y = Wage income

w = Wage bill

p = Profit

Y = S and S = I (80)

S = Sw + Sp (81)

where

S = Saving

Sp = Average and marginal propensity to save out of profits

Sw = Aggregate saving out of wage income $s_w w$

Now y = wp , conversely w = y - p (82)

Substituting equation (81) into equation (80)

I = Sw+ Sp

= SwW+ SpP

= Sw(y-p) + SpP

= (SwY-SwP) + SpP

= (Sp-Sw) p +SwY (83)

Dividing both sides by Y

$$\frac{I}{Y} = (Sp - Sw)\frac{P}{Y} + Sw$$

$$\therefore \frac{P}{Y} = \frac{1}{(Sp - Sw)} . \frac{I}{Y} - \frac{Sw}{(Sp - Sw)} \quad (84)$$

The assumption is that Sp and Sw are a constant share of the profits in NI, and a direct function of the share of I^t in NI.

An important assumption is made where Sp > Sw. We have stated the equilibrium as I = S. Suppose the ratio of I^t to national income increases then the equality is affected. But suppose investment (I) is not equal to saving(S). The price will rise. The steps are given as follows

$$\frac{I}{Y}\uparrow \rightarrow P\uparrow \rightarrow \frac{\Pi}{Y}\uparrow \rightarrow Sp > Sw \rightarrow S\uparrow \rightarrow S = I$$

When I/Y increases, p also increases. This leads to an increase in π/y but at the same time Sp > Sw where saving (S) increases and continues to increase until saving equals investment (S = I). The only way saving can be brought to equality with I^t is through a change in the distribution of income in favor of profits and away from wages. The mechanism explained above can be shown graphically in figure 2.12.

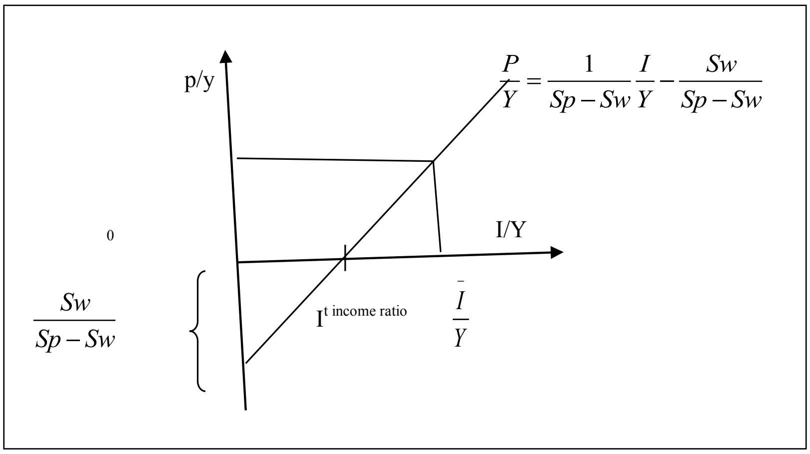


Figure 2.12. Linear relations in profit and wages

Figure 2.12 depicts a linear relationship between the rate of profit and the rate of I^t .Any change in the rate of I^t leads to a corresponding change in the rate of profit. The rate of savings increases until I^t decreases. Then there is a decrease in aggregate demand, which further leads to a decrease in prices and an increase in π.

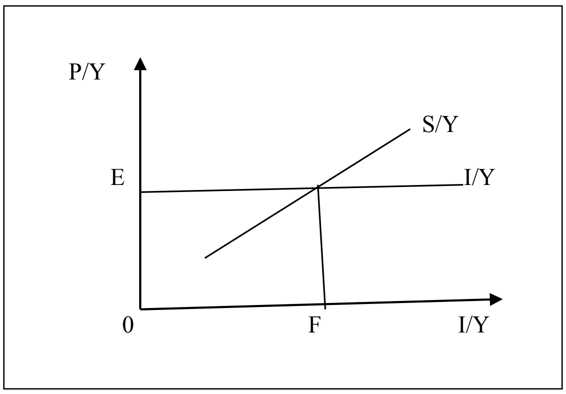


Figure 2.13. Linear relations in profit and income

Since I_t is an exogenous given and independent of P/Y, the ratio of I/Y is constant and it is shown as a horizontal line in the figure. On the other hand Sp > Sy, therefore S/Y increases with P/Y, so that the S/Y function has a positive slope. In the above figure, the ratio of I/Y = OF means that the ratio of P/Y = OE. Any increase in I/Y will increase the share of profits and lower the share of income. On the other hand, the net increase in aggregate propensity to save will increase with an increase in the wage share. The stability of the system depends on the relative saving propensity of workers and capitalists.

Equation (84) explains that

$$\frac{P}{Y}=\frac{1}{Sp-Sw}\frac{I}{Y}-\frac{Sw}{Sp-Sw} \qquad (85)$$

Suppose (Sp - Sw) is positive, then a very small change in I/Y will tend to produce a large change in P/Y. The system is stable if Sp = Sw. It is necessary that Sp ≠ Sw and secondly, Sp > Sw. If Sw = 0 the above equation becomes

$$\frac{P}{Y}=\frac{1}{Sp}\frac{I}{Y}$$

$$\text{That is } P=\frac{1}{Sp}.I \qquad (86)$$

There are two interesting conclusions that emerge from the above discussion. Since Sw = 0 it means that most people spend all their earnings. Therefore, the average saving out of income may be positive. Secondly, since $P = \frac{1}{Sp}.I$ the capitalists earn from what they spend from their expenditures. This is because the income of the capitalists comes directly from their expenditures. An increase in expenditures of the capitalists raises the level of real profit by the extent of expenditures adopted for their propensity to save. Thus two basic features explain this basic idea. The rate of I^t is the propensity to save out of profit.

Questions

Question 1. Explain the different types of production function.

Question 2. How does activity analysis change the production function? Give a short explanation.

Question 3. Write a note on each of the following:

a) Monotonic technology
b) Convex technology
c) Regular technology
d) Leontief technology
e) Cobb-Douglas production function
f) Variation in scale
g) Duality of cost and the production function

Question 4. Give a brief explanation of the technical rate of substitution.

Question 5. An elasticity of substitution is always positive for two factors of production. Explain.

Question 6.What is the difference between homogenous and heterogeneous production functions?

Question 7. Critically discuss the Envelope theorem of constrained optimization.

Question 8. In Michael Kalechi's theory, the monopoly power of a firm affects the firm's profits. Explain in detail.

Question 9. Discuss critically the Neo-Keynesian model of distribution.

3 Game Theory

3.1 Introduction

Game theory is widely used in different subjects. Because of the complexities in economic gains, game theory is being used more. In mathematics, game theory is used to explain the relationship between variables. A decision tree is drawn to understand the gains from any game. The games may be played by individuals, groups or firms. Every individual is an economic agent in nature. At each point of a game, an individual expects some economic gains out of different actions. The agents or firm get benefits to move ahead, collude or disagree with the opposite party in the game. Game theory is a theory of decision making under conditions of uncertainty and interdependence. In game theory, human behavior within a strategic situation is studied and a mathematic model is created. A strategic game consists of a set of players, which may be a group of nodes or an individual node. A set of actions is available for each player to make a decision and to choose preferences over the set of action profiles for each player. In any game, utility represents the motivation of players. Applications of game theory always attempt to find equilibriums. If there is a set of strategies with which no player can make a profit by changing their strategy while the other players keep their strategies unchanged, then that set of strategies and the corresponding payoffs constitute the Nash equilibrium (Valli & Dananjayan, 2010). We will study all strategies with a number of examples.

3.2 The rules of the Game

A game should be systematic and have certain rules. The actions of the individual agents are the subject matter of game theory. Decision theory helps one person to arrive at a particular decision, also, under uncertainty. The general model of action seems to be the heart of game theory. For instance, it could apply to any type of player and not just to an individual. So long as the state or the working class has a consistent set of objectives/preferences, then we could assume that it (or they) also acts instrumentally so as to achieve these ends. Likewise it does not matter what ends a person may work towards: they can be selfish, weird, altruistic or whatever, as long as they consistently motivate then people can still act in such a manner as to satisfy them best (Hargreaves Heap & Varonfakis, 2004).

Game theory makes use of decision theory. While in decision theory there are no players and actions, in game theory there are different players and actions, payoffs and information. Players try to maximize their payoffs, while formulating plans which are also known as strategies. These alternative strategies or actions are dependent on the information received by the individual player. Each player's actions are different from those of other players, and are influenced by information received at different levels and times. The strategies and actions taken by each player achieve different equilibriums at different points. Game theory experts explain the strategies of and games played by and payoffs to different players. Such common strategies are useful in formulating policies at the micro and the macro level. In game theory, the players of any game are the individuals and firms. The goal of the individual player is to maximize their utility. To maximize their utility, the player chooses an action from alternative actions. A firm chooses a particular strategy among various alternative strategies to get more profit.

Games are a way of modeling strategic interactions, that is, situations in which the consequences of an individual's actions depend on the actions taken by others. This mutual interdependence is recognized by those involved (Bowel., 2004). Game theory is the study of games, also called strategic situations. These are decision problems with multiple decision makers, whose decisions impact one another. Game theory is divided into two branches: non-cooperative game theory and cooperative game theory. The actors in non-cooperative game theory are individual players, who may reach agreements only if they are self-enforcing. The non-cooperative approach provides a rich language and develops useful tools for analyzing games. One clear advantage of the approach is that it is able to model how specific details of the interactions among individual players may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to these details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt. The actors in cooperative game theory are coalitions of a group of players. There are two things that stand out: a coalition has been formed and that there is a feasible set of payoffs available to the coalition members. Given the coalitions and their sets of feasible payoffs as primitives, the question tackled is the identification of final payoffs awarded to each player (Serrano, 2007)

In game theory, an action by any individual player is simply denoted as $a_{i\,.}$ The numbers of actions are taken in the action set Ai = {a}. The set consists of actions taken by individual players at i. The actions of an individual player is the combination of ordered set a ={a_i} where (i = 1…n). All players in the game play to maximize their utility. The player i receives the return $\pi 1(S_i \ldots\ldots S_n)$ after playing a game. Two important points can be raised here, namely:

1. The utility of player i received after all players have picked their strategies and the game has been played out by two or more players.
2. A player expects some gain after playing any game. This means that the expected utility is a function of the strategies chosen by the individual and the other players $\Pi_i = (S_s, S_o)$. Here, the self is denoted as s while the other player is denoted as o. The profit is for player i^{th}.

Each game has an interesting outcome. Values can be given for actions, payoffs and other variables after the game is played. Such actions and payoffs are useful for the development of microeconomic models. Microeconomists are keen to understand who is playing what game and for what benefit.

Most of the actions and payoffs are used for decision theory. These decisions, actions and payoffs are presented in a graphical way to depict the order of play of players. They are very useful in discouraging certain risky behaviors or illegal transactions, market failures, etc. Particular laws can be made to prohibit such risky actions. The aim behind any law is to increase the welfare of individuals and allow the system to work efficiently. Game theory also studies the actions of the other players.

At the same time, the observer plays the game by assuming themselves in everybody's position. An individual player i has particular strategies and there are certain rules which explain how to choose a particular action based on a given information set. The particular strategies at any particular time are the set of strategies available to the player. Such strategies are $S = (S_i)$, and strategy combinations, $S = (S_i \ldots S_n)$. It consists of one strategy for each of the n players in the game. An equilibrium $S^*= (S^*1, \ldots S^*n)$ strategy combination consists of the best strategy for each of the n individual players in the game. Most of the equilibrium strategies are visible because players pick the strategies to maximize their own payoffs.

3.3 The prisoner's dilemma: A dominant strategy

The prisoner's dilemma is usually defined between two players. Game theory assumes that both players act rationally. Realistic investigations of collective behavior, however, require a multi-person model of the game that serves as a mathematical formulation of what is wrong with human society. The study may lead to a better understanding of the factors stimulating or inhibiting cooperative behavior within social systems. It recapitulates characteristics fundamental to almost every social intercourse.

Various aspects of the multi-person prisoner's dilemma have been investigated in the literature (Szilagyi, 2003). The prisoner's dilemma gives the dominant strategy. For our purposes, we use the notation: S-Smita is the combination of strategies of every player except player Smita. The player i's best response or best reply to the strategies S-i, is chosen by the other players. At the same time s*i is the best strategy chosen by player i^{th} that yields them the greatest payoff, and can be written as follows

$$\Pi_i(S^*i,\ s\text{-}i) \geq \pi_i(S'_i,\ S\text{-}_i), \forall s\text{-}_i, \forall si \neq s^*_i \tag{1}$$

The above notation explains that the best strategy (s*i) gives the highest gain. The individual chooses the best strategies which become the dominant strategies. Therefore, the dominant strategies are also the equilibrium strategies. These equilibrium strategies are the best response to the strategies of the other players. However, in the welfare game, the dominant strategy does not exist. The players work out others' actions as the best strategies. The government's welfare programs include NREGA in India. The work sanctioned under this program was meant to provide benefits to poor rural people. We will discuss this later when we discuss the welfare game.

The prisoner's dilemma is based on certain assumptions, which are:

1. The prisoner's dilemma is a two-person game. There are many persons interacting with each other.
2. There is no communication between the two prisoners. If they communicate and commit to co-operate then the outcome will be different.

In the prisoner's dilemma, two prisoners interact only once. Suppose Smita and Deepa are the prisoners. If both of them have committed the crime and were caught by the police then they have a number of strategies available. If both Smita and Deepa confess, each is sentenced to ten years in prison. But they are given a choice to confess or to deny. If both Smita and Deepa deny their involvement in the crime then both are sentenced to two years in prison. Here, there is a chance to apply the prisoner's dilemma. If Smita confesses then she would be released. At the same time, Deepa would be sentenced to 15 years in prison. The prisoner's dilemma is a two-by-two game. This is because two players have two different actions and both will try to get more benefit out of their decision. Their strategies could be to confess, or to deny, or something else.

			Smita	
		Deny		Confess
	Deny	2, 2		15, 0
Deepa				
	Confess	0, 15		10, 10
Payoff: Deepa, Smita				

Figure 3.1. The prisoner's dilemma

Smita and Deepa each have their own dominant strategy. Deepa does not know what action is chosen by Smita. If Deepa chooses the action 'deny', Smita faces the payoff for 'deny' and if she chooses 'confess' then she will face payoff = 0. Suppose Deepa chooses 'confess' then she will face Smita's 'deny' payoff of 15. Suppose she choose 'confess' then the payoff is 10. In the above example, Deepa is not better off with 'confess'. She gets a 0 payoff. The dominant strategy in the above example is (confess, confess). The equilibrium payoff for both the players is (10, 10). But if both Deepa and Smita choose 'deny' then they get (2, 2), the same payoffs. If equilibrium is the dominant strategy then the information set does not matter.

The prisoner's dilemma is useful in various economic theories. It is used in oligopoly pricing, auction bidding, salesmen's efforts, political bargaining and arms races. Game theory and their examples are unrealistic to many scholars. They do not encounter such kinds of incidences. However, thinking in terms of game theory can also be useful. We need to understand the strategies of players at different points. A common observation in experiments involving finite repetition of the prisoner's dilemma is that players do not always play the single-period dominant strategies ("finking") but instead, achieve some measure of cooperation. Yet finking at each stage is the only Nash equilibrium in the finitely repeated game. We show here how incomplete information about one or both players' options, motivation or behavior can explain the observed cooperation. Specifically, we provide a boundary on the number of rounds at which finking may be played, when one player may possibly be committed to a "Tit-for-Tat" strategy (Kreps et.al, 1982).

Types of games

There are many different types of games in game theory. The cooperative and the non-cooperative, the welfare, the dominant strategy and repeated strategies games are the more useful games. We need to understand the behavior of agents in all these game types.

Cooperative and non-cooperative game

In the cooperative game, the players can make a commitment to each other. Sometimes they go beyond that commitment. This is opposite to the non-cooperative game where the players are not bound by their commitments. All games in the cooperative form are based on the principle of Pareto optimality, fairness, and equity. Most non-cooperative games are economic in nature. The players try to maximize their utility subject to stated constraints. The cooperative game theory focuses on the properties of the outcome rather than on the strategies. These strategies often achieve the outcome. The cooperative game is used to model bargaining in applied microeconomics. In such cases, the game is more appropriate than the non-cooperative game. The prisoner's dilemma is a non-cooperative game. Such a model can be allowed for players not only for commitment but to make binding constraints. Cooperative games often allow players to split the gains, by allowing players to make side payments.

The different types of cooperative and non-cooperative games are:

I. Cooperative game without conflict: The members of a workforce choose equally heavy tasks to undertake, to best coordinate with each other. There is unity in the workforce even though they have diverse strategies.

II. Cooperative game with conflict: a game where the workforce has different strategies but common strategies are found such that there will be cooperation among the players

III. Non-cooperative game with conflict: The prisoner's dilemma is the best example. Here, two players have different strategies which remain independent of each other.

IV. Non-cooperative game without conflict: Two companies set a product standard without communicating with each other.

We will give a few examples for each of the different types of games, except for the non-cooperative game with conflict an example of which has already been given.

Repeated Dominance: The Battle of the Bismarck Sea

This refers to a strategy combination found by deleting the weak dominant strategy from the alternative strategies. One of the players makes recalculations to find out which remaining strategy is weakly dominant, deleting one strategy and continuing the process until only one strategy remains for each player.

When people interact over time, threats and promises concerning future behavior may influence current behavior. Repeated games capture this fact of life, and hence have been applied more broadly than any game theoretic model not only in virtually every field of economics but also in finance, law, marketing, political science and sociology (Gibbons., 1997). We have given the example of the Battle of Bismarck Sea. The Battle of the Bismarck Sea is an example given in the repeated dominance strategies. It is set in the South Pacific in 1943. General Imamura has been ordered to transport Japanese troops across the Bismarck Sea to New Guinea. But General Kenney wants to bomb the troops' transport. It is his strategy and it was chosen at this point. In order to do this, Imamura must choose between a shorter northern route and a longer southern route to New Guinea and Kenney must decide where to send his planes to look for the Japanese. Now there are alternative strategies planned. Suppose Kenny sends his planes to the wrong route. He can recall them but the number of days for bombing will be reduced.

In the example, the players are Kenney and Imamura and they each have the same action set {North, South}. Their payoffs are never the same. Imamura loses exactly what Kenney gains. Such game can be shown in the following.

		Imamura	
		North	South
Kenney	North	2,-2	2, -2
	South	1,-1	3,-3

Payoff: Kenney, Imamura

Figure 3.2. The Battle of the Bismarck Sea

In figure 3.2, neither Kenney nor Imamura has a dominant strategy. Kenney would choose 'North' if he thought Imamura would choose 'South'. But he would choose 'South' if he thought Imamura would choose 'North'. This is a policy dilemma faced by both players. Suppose Imamura thought that Kenney would choose 'South', he would be indifferent between actions if he thought Kenney would choose 'North'. But we can find a plausible reasonable, probable equilibrium, using the concept of weak dominance.

Suppose strategy S_i is weakly dominated then there exists some other strategy S^*_i for players i. There could be a possible better yielding higher payoff in some strategy combination. The strategy never yields a lower payoff. Mathematically, S'_i is weakly dominated if there exists S''_i, such that

$$\Pi_i\ (s''_i{}^*, s_{-i}) \geq \pi i(s'_i\ , s_{-i})\ \forall S_{-i} \text{ and}$$

$$\Pi_i\ (s''_i{}^*, s_{-i}) > \pi i(s'_i\ , s_{-i}) \text{ for } S_{-i} \qquad 2)$$

One might define a weak dominant strategy equilibrium as the strategy profile found by deleting all the weakly dominant strategies of each player. Weakly dominant strategies do not help much in the Battle of the Bismarck Sea game. However, Imamura's strategy of 'South' is weakly dominated by the strategy 'North' because his payoff from 'North' is never smaller than his payoff from 'South'. The payoff is greater if Kenney picks 'South'. For Kenney, however, neither strategy is weakly dominant.

Kenney decides that Imamura will pick 'North' because it is weakly dominant. Therefore, Kenney eliminates it. Imamura chooses 'South' after considering his options. Having deleted the column n the table, Kenney has a strong dominant strategy: he chooses 'North' which achieves payoffs greater than if he chose 'South'. The strategy combination (North, North) is an iterated dominant equilibrium and indeed (North, North) was the outcome in 1943.

If Imamura moved first (North, North) then it would be the only equilibrium which is important for each player. This is because he is moving first and by doing so, gives the other player more information before he acts. Suppose Kenney has cracked the Japanese code and found Imamura's plan then there is an information leak and both players can move simultaneously. The Battle of the Bismarck Sea is special because the payoffs of the players always sum up to zero. This feature is important enough to deserve a name.

A zero-sum game

A zero-sum game is a game in which the sum of the payoffs of all the players is zero, whatever strategies they choose. If the game is not zero-sum then it is a variable-sum game. In a zero-sum game, what one player gains, another player must lose. From the above example, the repeated dominance strategy: the Battle of Bismarck Sea is an example of a zero-sum game. But the prisoner's dilemma game is not an example. It is not possible to change the payoffs and make the prisoner's dilemma game a zero-sum game. There is a need to change the essential character of the games.

Nash Equilibrium

The Nash equilibrium is the standard equilibrium concept in modern microeconomics. It is a widely used concept in various microeconomic models. Suppose the model does not assume a specific equilibrium concept but it is used then it is a Nash equilibrium or some refinement of the Nash equilibrium. The Nash equilibrium is a certain kind of rational expectations equilibrium. The Nash equilibrium is a situation in which each player chooses an optimal strategy given the strategy chosen the other player (Salvatore, 2003). In a Nash equilibrium, each player's strategy choice is a best response to the strategy choice of others.

The strategy combination S* is a Nash equilibrium. Suppose a player has no incentive to deviate from his strategy then he has incentive to deviate from his strategy as long as the other players do not deviate.

$$\forall_i \, , \Pi_i \, (s_i^*, s^*_{-i}) \geq \pi_i(s'_i \, , s^*_{-i}) \; \forall S_{-I} \qquad (3)$$

Suppose one child and adult are asked to stand on a box with a special panel at one end and a food dispenser at the other. The adult person has a strong dominant strategy to get the food whereas the child has a weak dominant strategy. Suppose the adult gets to the food dispenser first then the child will get only a reward worth 1 unit. Suppose the child arrives first. He then eats more units of food. But if both arrive at the same time, then the child gets fewer units of food.

		Child	
		Press	Wait
Adult	Press	10, 2	[8], [8]
	Wait	18,-1	0, [0]
Payoff: Adult, Child			

Figure 3.3. Nash equilibrium strategies

The figures in brackets show no dominant strategy equilibrium. This is because what the adult will choose depends on what they thinks that the child will choose. Suppose the adult believes that the child will press the panel. The adult will wait by the dispenser. But the adult believes that the child would wait and then he would press. There does exist an iterated dominance equilibrium (press, wait). We will use a different line of reasoning to justify such outcomes. The example shows that the strategy combination (press, wait) is a Nash equilibrium. The way to approach Nash equilibrium is to propose a strategy combination. It is required to test whether each player has a best response to the other's strategy.

Suppose the adult picks 'press' and the child faces a choice between payoff 2 from choosing 'press' and 8 from choosing 'wait'. At the same time, if the adult is willing to choose 'wait' and the child picks 'wait' then the adult has a choice between a payoff of 8 from 'press' and 0 from 'wait'. This confirms that {press, wait} is the Nash equilibrium and in fact, it is the unique Nash equilibrium.

The Nash equilibrium is either weak or strong. It is required that no player be indifferent between their equilibrium strategy and some other strategy. Every dominant strategy is a Nash equilibrium but not every Nash equilibrium dominant strategy is equilibrium. Suppose the strategy is a dominant strategy then it is a best response to any strategies that the other players pick. It includes their equilibrium strategy. If a strategy is part of a Nash equilibrium it needs only be a best response to the other player's strategy.

3.4 Equilibrium strategies

The Nash equilibrium lacks the "$\forall$ s-i" of the dominant strategy equilibrium. The Nash equilibrium strategy needs only to be a best response to the other Nash strategies, and not to all possible strategies. Although we deal with best responses, the moves are actually simultaneous. The players therefore, are predicting each other's moves.

The Battle of Politics

In the Battle of Politics, we assume that there are two political parties maximizing their payoffs. These payoffs are useful to get a majority during an election. In India, no party gets the majority. Therefore, a coalition is the only way to form a government. Coalition politics are different from the one-party-dominates political strategy. In a coalition government, different parties have different manifestos and they do not compromise on them. But policy formation at the national level can affect the manifestos of local parties. Therefore, the local party may oppose the policies of the national party. In India, we found that the Trinamool Congress Party always opposes the National Congress Party on certain issues. The Trinamool Congress Party is a local political party in West Bengal. Take the example of petrol prices. Any proposed hike in petrol prices is opposed by the Trinamool Congress Party.

In India, subsidy is given on petrol. Now the Congress government is formulating policies to reduce this petrol subsidy in order to reduce the fiscal deficit at the central level. Most government oil companies determine the petrol prices which is based on the current international petrol prices. If the price of petrol rises in the international market, then these companies have no choice but to increase the price of petrol. If this is not done, the total subsidy amount rises, affecting the government's fiscal deficit. Therefore, the government is involved in the price fixation of petrol. While doing this, the government does not discuss with or take the opinion of the other political parties. This sometimes creates instability in coalition politics.

		Congress	
		Petrol subsidy	market prices
T Congress	Petrol subsidy	20, 10	-10,-10
	Market price	-50,-50	10, 20

Payoff: Congress, T Congress

Figure 3.4. Battle of politics

It is obvious that a political party like the Trinamool Congress Party will oppose such a decision to raise the price of petrol. If both parties will not agree to an increase in the price of petrol then there will be a conflict between the two parties. The Nash equilibrium is sometimes justified by the repetition of the game. Suppose the Congress government does not discuss with its coalition parties and does this period after period then there will be a problem of instability. Eventually, they will settle on a decision and then the Nash equilibrium is reached.

Each Nash equilibrium in the Battle of the Politics is Pareto efficient. There is no other strategy combination to increase the payoff of one party without decreasing that of the other. The Congress party moves first in deciding to raise petrol prices. Because of its commitment, the Trinamool Congress Party would have to agree to raise the price of petrol. The petrol price rise will benefit the Trinamool Congress Party (20,10) more than it will the Congress government (20, 10). But if the market price is accepted by the Congress government then the benefits are (10, 20). Such efforts at discussion will help to smooth out the work of government and contribute to its financial stability. If the Trinamool Congress Party supports the market price and the Congress government goes for a petrol price hike then the benefits are (-50,- 50), the same for both parties.

In political games, the party which moves first has a first mover advantage. This is equivalent to a commitment. The Battle of Politics has many economic applications, and is often used in industry. Suppose two firms have different preferences but both want a common standard to encourage consumers to buy the products. But the two firms prefer different terms. In a coalition government, political parties may work together even though they have different preferences. On the basis of the political party manifesto, competition takes place among the parties. Any political party which wins the majority in an election emerges as the strongest party and forms a government. It is easy to understand that the political party which won the maximum number of seats during an election makes the decision at different levels of government.

The normal form and outcome matrix

The game with several moves in sequence requires more care in presentation than single-move games. The strategies are the same as actions in pure coordination and the outcomes are in the two-by-two form. This is related to strategy combinations of payoffs and actions and outcomes. These two mappings are called the normal form and outcome matrix and in more complicated games, they are distinct from each other.

The normal form shows what payoff results from each possible strategy combination. The outcome matrix shows what outcome results from each possible action combination. The definitions below are used to denote the number of players. Let k be the number of variables in the outcome vector and p be the number of strategy combinations and q the number of action combinations.

The normal form of the game consists of:

1. All possible strategy combinations of S where S_1……Sn.
2. Payoff functions mapping S^i on to the payoff n where π^i (i = 1,….n)

The outcome matrix consists of the following two outcomes.

1. All possible action combinations are a^1…..a^n
2. Outcome functions mapping a^i into the outcome k where z^i, = (1,…….n)

Pure coordination

We have used different examples for different kinds of games. The players in these games are Deepa and Smita. They are competitive with each other and they want higher payoffs out of their actions. Both would prefer a large payoff out of their individual strategy but their individual strategy is dependent on the other's strategy.

		Deepa	
		Large	Small
Smita	Large	20, 20	-10,-10
	Small	-10,-10	10, 10

Payoff: Deepa, Smita

Figure 3.5. Pure coordination strategies

In the pure coordination game, the follow-the-leader-one principal strategy is applied. In this game, Smita moves first. She is committing herself to a certain disk size, no matter what Deepa chooses. The new game has an outcome matrix identical to pure coordination. This is because Deepa's strategies are no longer single actions. Deepa's strategy set has four elements, explained as follows.

$$\left\{ \begin{array}{l} \text{(Large, Large)} \\ \text{(Large, Small)} \\ \text{(Small, Large)} \\ \text{(Small, Small)} \end{array} \right\}$$

Follow-the-leader-one illustrates how adding a little complexity can make the normal form too unclear. But it can be very useful.

		Deepa			
		L/L,L/S	L/L, S/S	S/L, L/S	S/L ,S/S
Smita	Large	20,20(X)	20,20 (Y)	-10,-10,	-10,-10
	Small	-10,-10	10,10	-10,-10 ,	10,10(Z)

Payoff: Smita, Deepa

Figure 3.6. Follow the leader strategy I

From the above, we have given equilibrium strategy outcomes. They are as follows:,

X {Large (large, large)}both pick large

Y {Large (large, small)} both pick large

Z {Small (small, small)} both pick small

In figure 3.6, X, Y and Z are Nash equilibrium strategies. In equilibrium X, Deepa will choose 'large' regardless of what Smita chooses so Smita is quite happy to choose 'large'. Deepa would be stupid to choose 'large' if Smita chooses 'small' first. But that event will never happen in pure cooperative equilibrium. At equilibrium Y, Deepa will choose whatever Smita chooses. So Smita chooses 'large' to maximize her payoff. At this point, payoff 2 is better than payoff of 1. In equilibrium Z, Smita deliberately chooses 'small'. This is because she knows that Deepa will respond with 'small' whatever she does. Deepa is willing to respond with 'small' because Smita chooses 'small' in equilibrium. Equilibrium X and Z are not completely sensible at this point.

The extensive form and decision tree game

We use the extensive form and decision tree to solve the problem. The game tree is the same as the extensive form. There is an outcome at each node. Suppose the outcome out of equilibrium is defined as a payoff then the extensive form is equilibrium to the decision tree. We are interested to see why equilibrium X and Z are unsatisfied even though they are Nash equilibriums. We need to understand this through a decision tree, as follows

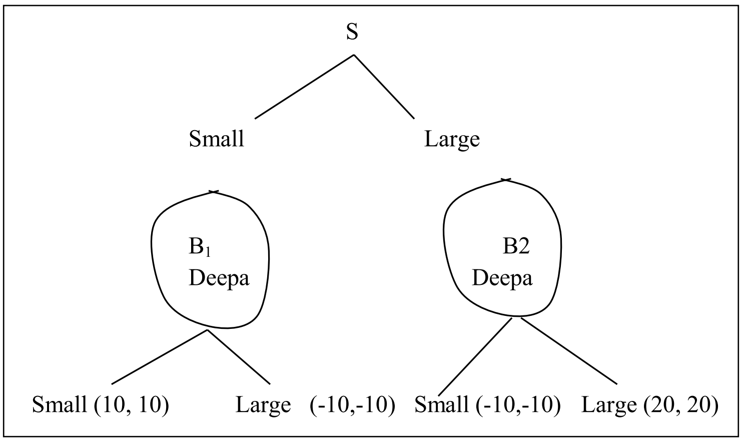


Figure 3.7. Follow the leader I

In figure 3.7, the game actually reaches node of B1 or either B2. Deepa would make her dominant action at B2. This is because the payoff is higher. As far as X and Z equilibrium are concerned, the Nash equilibrium is observed at Y.

The extensive form of pure coordination is presented in figure 3.8. It shows that the dotted lines are the extensive form. This is a Follow-the-Leader I strategy. In this game, each player makes a single decision between two actions. The movements of the players are simultaneous. Suppose Smita does not inform Deepa and she moves first. Deepa understands her movement and the game reaches some node. In the figure, such information set is marked by the dotted lines. But Deepa does not know which exact node is reached in this game.

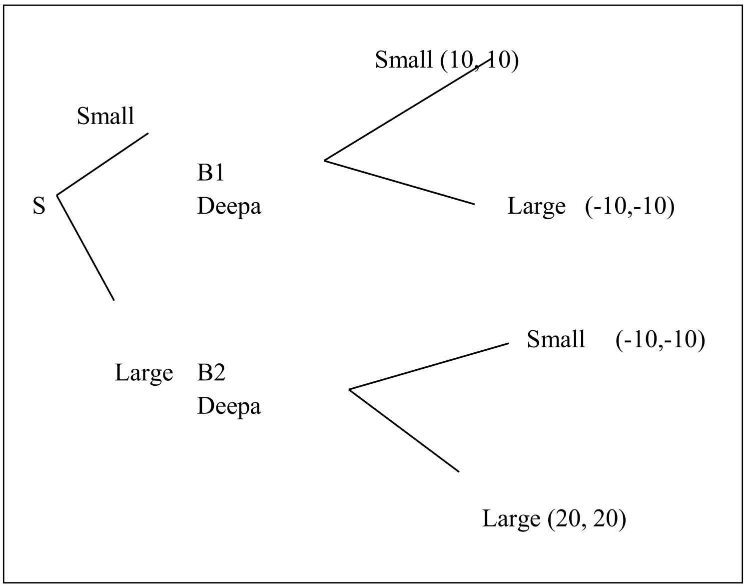


Figure 3.8. The extensive form of pure coordination

Mixed strategy: The welfare game

In game theory, the Nash equilibrium is the most desired outcome. The Nash equilibrium is useful to provide predictions of outcome. It does not require dominant strategies. Some games do not have the Nash equilibrium. It is realistic and useful to expand the strategy space. It includes a random strategy where Nash equilibrium almost always exists. These random strategies are called mixed strategies. A pure strategy map of a player includes the possible information sets to one action, as

$$Si: Wi \rightarrow ai \tag{4}$$

A mixed strategy map for each player's possible information sets to probability distribution actions

$$Si: wi \rightarrow m\ (ai) \text{ where } m \geq o \text{ and}$$

$$\int_{Ai} m\ (ai)\ dai = 1 \tag{5}$$

An expanded game theory version allows mixed strategies. It is called the mixed extension of the game. A pure strategy is a rule that tells the other player what action to choose. A mixed strategy constitutes a rule that tells him what dice to throw to choose an action. If a player pursues a mixed strategy, he might choose any of several different actions in a given situation. The unpredictability can be helpful to him. Such a mixed strategy occurs frequently in the real world.

In American football games, the offensive team has to decide whether to pass or to run. The passing generally gains the team more yards but what is most important is to choose an action which is not expected by their team. Teams decide to run part of the time and pass part of the time. This seems random to observers but rational to game theorists.

The welfare game

In a developing country like India, the government sanctions a number of welfare programs for poor people. We can call it the welfare state criteria or political economy. Some welfare programs become popular because many people participate in such programs. Most of the time, people and local representatives demand such welfare programs. Welfare programs in India really help in creating different kinds of rural infrastructure and in sustaining the ecological environment. But some social welfare programs fail because of lower participation by rural people. In this welfare game, we use the example of the government and a destitute person. In the welfare game model, the government wishes to aid this destitute person who is poor, from a rural area and is always searching for work. There is a dilemma as to whether a destitute person searches for work only if he cannot depend on government aid. The destitute person may not succeed in finding a job even if he tried. In this game neither the government nor the destitute person has a dominant strategy. Both depend on each other for benefits. We can observe that no Nash equilibrium exists in pure strategies, and explain this with the example.

		Destitute Person	
		Work (p_w)	Idle ($1-p_w$)
Government	Aid (Pa)	3, 2	-1, 3
	No Aid (1-pa)	-1, 1	0, 0

Payoff: Government, Destitute Person

Figure 3.9. Welfare game with two players

In figure 3.9, each strategy combination must be examined in turn to check for Nash equilibrium.

The strategy combination (Aid, Try to work) (Aid, bB Idle) (No Aid, Be Idle) (No Aid, Try to work) is not Nash equilibrium.

The government plays Aid with probability Pa and the destitute person tries to find work with probability Pw. The government's expected payoff is

$$
\begin{aligned}
E\pi \text{ Govt} &= Pa[\ 3Pw+ (-1)\ (1-Pw)] +[1-Pa]\ [-1Pw+ [1-Pw]] \\
&= Pa[3Pw-1+Pw] - Pw+PaPw \\
&= Pa[5Pw-1] - Pw \qquad (6)
\end{aligned}
$$

If only pure strategies are allowed, Pa equals zero or one. But in the mixed extension of the game, the government's action of Pa lies on the continuum from zero to one. The pure strategies being the extreme values differentiating the payoff function with respect to the choice variable. This is done to obtain the first order condition.

$$0= \frac{dE\pi \text{ govt}}{d-Pa} = 5\ Pw - 1$$

$$Pw = 0.2 \qquad (7)$$

The above example shows that in the mixed strategy equilibrium, the destitute person selects the option to try to work 20 percent of the time to obtain the strategy. The number of strategies and payoffs for the government is explained as follows.

Firstly, the optimal mixed strategy exists for the government. Secondly, suppose the destitute person works more than 20 percent of the time then the government always selects Aid. But alternatively, if the destitute selects 'Try to work' less than 20 percent of the time, the government never selects 'Aid'. It is a clear strategy for the government to select 'Aid' for the destitute person. Thirdly, for the government, the mixed strategy is that the destitute person must select 'Try to work' with probability of exactly 20 percent. In a developing country like India, there is often a demand for people to work and it is expected that they would be willing to spend some time on a government welfare program. If this is not so then there will be a wastage of public resources. Now it is the government which decides whether to start the aid program or not. In order to obtain the probability of the government choosing 'Aid', we must calculate the payoff for the destitute person, as follows

$E\pi$ destitute = Pa (2Pw+3[1-Pw]) + (1-Pa)

(1Pw+0[1-Pw]

= 2PApW+3Pa-PaPw+Pw-PaPw

= -Pw (2Pa-1) +3Pa

If we take the first order condition then

dE π Destitute

---------------- - = -(2da-a) = 0

dDw

Da = ½ = 0.5 (8)

Suppose the destitute person selects 'Try to work' with 0.2 probability then the government is indifferent to select 'Aid'. There is a probability of (100, 0) or anything in between. In the mixed strategy of the Nash equilibrium, the government selects 'Aid' with 0.5 probability. The destitute selects 'Try to work' with 0.2 probability. The equilibrium outcome could be any of the four entries in the outcome matrix. The entries having the highest probability of occurrence are (No Aid, Be Idle) and (Aid, Be Idle) each with 0.4 = 0.5 (1-0.2) probability.

From the above probabilities it is not clear whether the government will select the Aid program for the destitute. Sometime there is work but the destitute person is not ready to work and prefers to remain idle. Therefore, most of the time, the aid program is evaluated. If the aid is helping the destitute then it is continued; otherwise, it is stopped. Sometimes, the effectiveness of the aid program is more important but this depends on a number of things. For example, the possibility of migration in urban and rural area, alternative job possibilities, learning new skills, etc. The government evaluates such aid programs at different levels and time. The evaluation is done by independent research agencies. The aid is sanctioned after a careful evaluation of the various reports.

Correlated strategies

Cooperative games encourage participation and collaboration. The correlated equilibrium strategies are the solution rather than the Nash equilibrium. This strategy was first presented by mathematician Robert Aumann in 1974. He pointed out that each player chooses an action according to their observation. Such observations have the value of the same public signal. A strategy becomes an action to every possible observation a player can make.

Suppose not a single player would want to deviate from the standard strategy then the distribution is called the correlated strategies. A correlated equilibrium is given for two players tossing a coin. The strategies are simple for Smita and Deepa. Smita can choose 'stay' if it comes up heads. For Deepa, she can choose 'stay' if it comes up tails. Each player's strategy is a best response to the other's. The probability of each choosing 'stay' is half, that is, 0.5. The expected payoff for each is 1.00. A simple probability function exists in this game. Usually the randomizing device is not modeled explicitly when a model refers to the correlated equilibrium. An extrinsic uncertainty refers to uncertainty over variables that do not affect preferences such as endowment or production. In order to model a correlated strategy outcome, we need to specify a move by Nature.This gives each player equal probability, the ability to commit first to an action such as 'stay'. This is often realistic because it amounts to a zero probability of both players entering the industry at exactly the same time. But no one knows in advance who will be the lucky starter. A firm has no prior advantage but the outcome is efficient. The mixed strategies cannot be used for correlation strategies. In an ordinary mixed strategy, the mixed probabilities are statistically independent. In correlated strategies, they are not.

The cheap talk refers to costless communication before the game begins. In pure coordination, cheap talk instantly allows the players to make the desirable outcome. Without communication, the only symmetric equilibrium is mixed strategies. Suppose both players know that making an inconsistent announcement will lead to the wasteful mixed strategy outcome then they are willing to announce whether they will go to the ballet or to the prize fight. Their chances of coming to an agreement are high. Therefore, communication can help reduce inefficiency even if the two players are in conflict. But again, such strategies depend purely on the communication between two players. The goal is not to win as a player but as a team of players. In discovering effective cooperative game patterns is an elusive and important problem that is personal communication (El-Nasr et.al, 2010). Most of the time communication is the last strategy which is used as a solution to the problem.

3.5 The Cournot model

A French economist, A. Augustin Cournot, gave the duopoly model in his book. According to him, the model has a unique equilibrium when the demand curves are linear. The model explains that two firms choose the output levels in competition with each other. The Cournot model has a continuous strategy. The format is game-assign the game a title, list of players, information classification. The order of play and payoff function is explained in the following paragraph.

The model is based on the following assumptions:

- There are two sellers to produce and sell in a homogenous product.
- Each firm produces maximum quantity and is unaware of the rival's plan of production.
- The cost of production of each firm is zero.
- The price is decided by market forces only. There is no arbitrary price decided by any of the firms.
- There are large numbers of buyers for each firm's product.
- The entry of firms is blocked.
- While producing or supplying the output, one firm thinks of the constant output of the other firm.

On the basis of these assumptions, a firm tries to maximize profits subject to the price and the quantity produced by the rival firm. The first duopolistic firm maximizes its profit π_1 with respect to quantity q_1.The second duopolistic firm maximizes its profit π_2 with respect to quantity q_2.This can be shown with the help of a derivative as follows

$$\pi_1 = TR_1 - TC_1$$

$$\pi_2 = TR_2 - TC_2 \tag{9}$$

The first duopolistic firm maximizes its profit with respect to q_1

$$\frac{\partial \Pi_1}{\partial q_1} = \frac{\partial TR_1}{\partial q_1} - \frac{\partial TC_1}{\partial q_1} \tag{10}$$

The second duopolistic firm maximizes its profit with respect to q_2

$$\frac{\partial \Pi_2}{\partial q_2} = \frac{\partial TR_2}{\partial q_2} - \frac{\partial TC_2}{\partial q_2} \tag{11}$$

Setting the approximate partial derivative for Equation (10) and (11) which is equal to zero, for the first firm it is

$$\frac{\partial TR_1}{\partial q_1} - \frac{\partial TC_1}{\partial q_1} = 0 \tag{12}$$

For the second firm, it is

$$\frac{\partial TR_2}{\partial q_2} - \frac{\partial TC_2}{\partial q_2} = 0 \tag{13}$$

The first order condition for the first and second firms is as follows

$$\frac{\partial TR_1}{\partial q_1} = \frac{\partial TC_1}{\partial q_1}$$

$$\frac{\partial TR_2}{\partial q_2} = \frac{\partial TC_2}{\partial q_2} \tag{14}$$

The second order condition, MR < MC by the second order and partial derivatives

$$\frac{\partial^2 TR_1}{\partial\ q^2{}_1} < \frac{\partial^2 TC_1}{\partial\ q^2{}_1}$$

$$\frac{\partial^2 TR_2}{\partial\ q^2{}_2} < \frac{\partial^2 TC_1}{\partial\ q^2{}_2} \tag{15}$$

The Cournot game is a non-cooperative game. It is not necessary that q1+ q2 = q. In order to find the Nash equilibrium in a Cournot game, we need the reaction curve. Based on the assumption of the Cournot model, economists have given a better solution in terms of the reaction curve. The reaction function expresses the output of each duopolistic firm which is a function of his rival's output.

$$q_1 = f(q_2) \tag{16}$$

$$q_2 = f(q_1) \tag{17}$$

The first reaction function gives the value of q1. This maximizes the π_1 for any specified value of q_2. The second reaction function shows the value of q2 which maximizes the π_2 for any specified value of q_1. The demand function and cost function are given as follows

Demand function $\quad p = A - B\ (q_1+q_2) \quad (18)$

$$c1 = a1q1 + b1q^2_1$$

Cost function $\quad c2 = q_2q_2 + b2q^2_2 \quad (19)$

All parameters are positive. The profit of a duopolistic firm is calculated as follows

$$TR_1 = p.q_1$$

$$= A - B(q_1+q_2).q_1$$
$$= Aq_1 - B(q_1+q_2)q_1 \quad (20)$$

$$TR_2 = p.q_2$$

$$= A - B\ (q_1+q_2).q_2$$
$$= Aq_2 - B\ (q_1+q_2)q_2 \quad (21)$$

Π_1=TR_1-TC_1

$$= Aq_1-B(q_1+q_2)q_1-a_1q_1-b_1q^2{}_1 \quad (22)$$

We have substituted cost from equation 19.

Π_2=TR_2-TC_2

$$= Aq_2-B(q_1+q_2)q_2-q_2q_2-b_2q^2{}_2 \quad (23)$$

Duopolistic I maximizes $\pi_1$with respect to q_1

$$\Pi_1 = Aq_1-B\ (q_1+q_2)q_1-q_1-a_1q_1-b_1q^2{}_1$$

$$= Aq_1-B\ q^2{}_1+Bq_1q_2-a_1q_1-b_1q_1{}^2 \quad (24)$$

$$\frac{\partial \Pi_1}{\partial q_1} = A - 2Bq_1 - Bq_2 - a_1 - 2b_1q_1$$

$$= A - Bs\ (2q_1+q_2)-a_1-2b_1q_1 \quad (25)$$

Duopolistic II maximizes π_2 with respect to q_2

$$\Pi_2 = Aq_2-B\ (q_1+q_2)q_2-a_2q_2-b2q_2{}^2$$

$$\frac{\partial \Pi_2}{\partial q_2} = Aq_2 - Bq_1q_2 + Bq_2^2 - a_2q_2 - b_2q_2^2$$

$$=A-B\ q_1+2B\ q_2-a_2-2b_2q_2$$

$$=A-B\ (q_1+2q_2)-a_2-2b_2q_2 \quad (26)$$

Setting appreciate P.D = 0

$$\Pi_1 = A - B\ (2q_1+q_2) - a_{1-}2b_1q_1 = 0$$

$$\Pi_2 = A - B\ (q_1+2q_2) - a_{1-}2b_2q_2 = 0 \quad (27)$$

The corresponding reaction functions are for q_1

$$A - 2Bq_1 - Bq_2 - a_1 - 2b_1q_1 = 0$$

$$A - Bq_2 - a_1 = 2Bq_1 + 2b_1q_1$$

$$A - Bq_2 - a_1 = 2(B+b_1)\ q1$$

$$\frac{A - Bq_2 - a_1}{2(B+b_1)} = q_1$$

$$q_1 = \frac{A - a_1}{2(B+b_1)} - \frac{B}{2(B+b_1)}.q_2 \tag{28}$$

For the second firm, q_2

$$A - B\ (q_1+2q_2) - a_2 - 2b_2q_2 = 0$$

$$A - 2Bq_2 - Bq_1 - a_2 - 2b_2q_2 = 0$$

$$A - Bq_1 - a_2 = 2Bq_2 + 2b_2q_2$$

$$A - Bq_1 - a_2 = 2(B + b_2)\ q2$$

$$\frac{A - Bq_2 - a_2}{2(B+b_2)} = q_2$$

$$q_2 = \frac{A - a_2}{2(B+b_2)} - \frac{B}{2(B+b_2)}.q_2 \tag{29}$$

Since b, $b_1$and b_2 are all positive, a rise of either duopolistic firm's output will cause a reduction of the other's optimistic output. The reaction functions are linear and this is shown in the following diagram. Equilibrium is shown by the interaction of the points for the reaction curve at point 'e'.

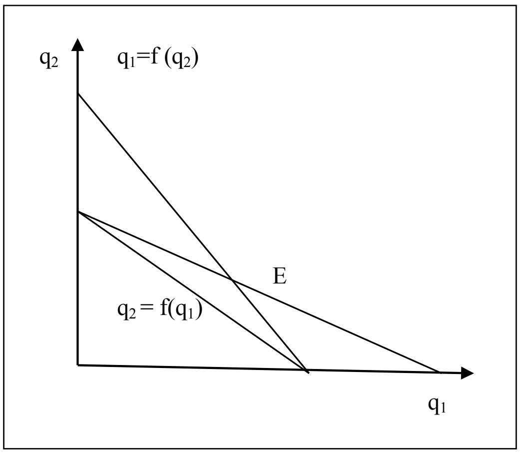


Figure 3.10. Interaction and reaction curve

The Cournot-Nash equilibrium is at E. In the Cournot model, the Nash equilibrium has the property of stability.

Criticism

The Cournot model is criticized on various points. Firm A believes that if it changes q1, another firm will not respond by changing q. The strategies are decided in terms of prices rather than quantities. The Nash equilibrium is very different. The Cournot model assumes that firms pick quantities rather than prices. That means an auctioneer chooses the price to equate supply and demand.

Cournot duopoly model: continuous strategies

The earliest duopoly model was developed in 1938 by the French economist Augustin A. Cournot. He noted in Chapter Seven of his book that this game has a unique equilibrium when demand curves are linear. The Cournot model has a continuous strategy space even without mixing. If a game has a continuous strategy set then it is not always easy to depict the strategic form and the outcome matrix has an extensive form as a tree. In order to present the Cournot duopoly model, a new notation will be useful.

I. The Cournot game model is a duopoly model in which two firms chooses output levels in competition with each other.
II. There are two players: the firms Apex and Brydox.
III. Apex and Brydox simultaneously choose quantities qa and qb from the set $(0, \infty)$
IV. Production cost is zero.

Demand is a function of the total quantity sold in the market.

$$Q = qa + qb \tag{30}$$

$$P(Q) = 120 - qa - qb \tag{31}$$

This can be shown with the help of the following.

Suppose the game is cooperative then the firms would end up producing somewhere on the 45^0 line in the figure. Total output is the monopoly output and it maximizes the sum of payoffs. More specifically, the monopoly output maximizes $PQ = (120 - Q)\,Q$ with respect to the total output of Q, resulting in the first order condition as $120 - 2q = 0$. This implies a total output of 60 and a price of 60. In order to decide how much of that output of 60 should be produced by each firm, the firm's output should be located on the 45^0 line. This output would be a zero sum cooperative game. This is an example of bargaining between firms.

But since the Cournot game is a non-cooperative game, the strategy combinations are such that qa + qb = 60. This is not necessarily in equilibrium despite their Pareto optimality. Each firm produces about the quantity it wants to produce and is unaware of its rival's plan of production. In order to find the Nash equilibrium in a Cournot game, we need a reaction function. If Brydox produces output then Apex would produce the monopoly output of 60.

If Brydox produced qb = 120 or greater, the market price would fall to zero and Apex would choose to produce zero. The best response function is found by maximizing Apex's payoff. It is given in the following equation with respect to his strategy qa.

$$\pi_{Apex} = 120qa - q2a - qaqb \tag{32}$$

$$\pi_{Brydox} = 120qb - qaqb - q2b \tag{33}$$

This generates the first order condition as follows

$$120-2qa-qb=0 \tag{34}$$

or

$$qa = 60 - qb/2$$

The reaction function of the two firms is labeled Ra and Rb in the figure. They cross point c which is the Cournot-Nash Equilibrium. It is also Nash equilibrium when the strategies consist of quantities. Algebraically, if it is solved then the two reaction functions for qa and qb generate the unique equation (qa = 40, qb = 40). The equilibrium price is also 40, coincidentally. In the Cournot game, the Nash equilibrium has the particularly property of stability.

If we assume that the initial strategy combination is point x in the figure, then it moves the profile closer to equilibrium. But this is special to the Cournot game and the Nash equilibrium is not always stable in this way.

Criticisms of the model

The above model is criticized on the following points:

1. In the Nash equilibrium, Apex believes that if he changes qa, Brydox will not respond by changing qb.
2. Another objection is that the strategy set are specified to be quantities.
3. If strategies are prices rather than quantities, the Nash equilibrium is much different.
4. What happens when one firm's costs are positive and information is incomplete?

3.6 Solution to the Cournot model by the Stackelberg equilibrium

The Stackelberg equilibrium differs from the Cournot equilibrium. In the Stackelberg equilibrium, the firm that gets to choose its quantity first is the Stackelberg leader and the other player is the Stackelberg follower. The distinguishing characteristic of the Stackelberg equilibrium is that one player gets to commit first. In figure 3.11, Apex moves first intertemporally. Suppose the moves were simultaneous but Apex could not ommit itself to a certain strategy. The same equilibrium would be reached as long as Brydox is not able to commit itself.

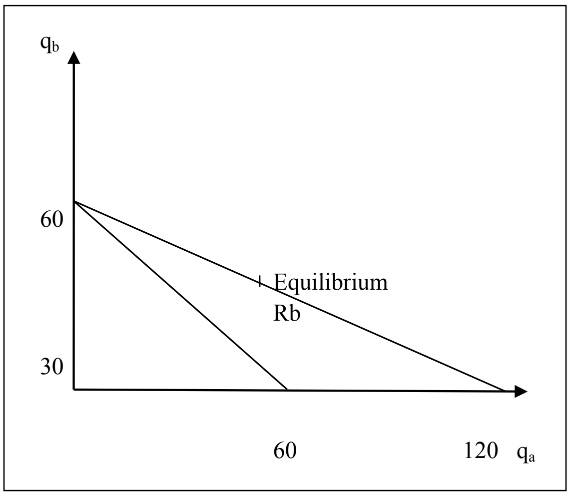


Figure 3.11. Solution to the Cournot model

Algebraically, since Apex forecasts Brydox's output to be Qb = 60 - qa/2 from the analog of equation (34), Apex can substitute this into his payoff function in (34) and obtain

$$\pi_a = 120qa - q^2_a - q_a(60 - q_{a/}2) \qquad (35)$$

If the above function maximizes with respect to $q_{a\,,}$ then it yields the first order condition

$$120 - 2qa - 60 + qa = 0 \qquad (36)$$

which generates qa = 60. Once Apex chooses this output, Brydox chooses his output which is to be qb = 30. The market price is 30 for both firms. Apex has benefited from his status as the Stackelberg leader.

3.7 The Bertrand paradox

The Bertrand paradox was developed in 1883. It is an extension of the Cournot model. The Bertrand paradox model is based on assumptions which are as follows.

- Two firms produce identical goods which are 'non-differentiated'. Such goods are perfect substitutes in the consumer's utility function.
- Consumers buy from the producer who charges the lowest price.
- Each firm faces a demand schedule which is equal to half of the market demand at the common prices.
- Firms always supply commodities according to the demand they face.

The market demand function is q = D (p). Each firm incurs a cost c per unit of production. The profits of firm i arer

$$\pi^i(P_i, P_j) = (P_i - C)\, D_i(P_i, P_j) \qquad (37)$$

where the demand for the output of firm i is denoted by and is given by

$$D_i(P_i, P_j) = \begin{cases} D(P_i) & \text{if } P_i < P_j \\ \frac{1}{2} D(P_i) & \text{if } P_i = P_j \\ 0 & \text{if } P_i > P_j \end{cases}$$

The aggregate profit is defined as

$$\underset{Pi}{\text{Min}}\ (P_j - C)\ D\ (P_i)$$

Such aggregate profit cannot exceed the monopoly profit

$$\pi^m = \underset{p}{\mathrm{Max}}\ (P-C)D(P) \tag{38}$$

Each firm can guarantee itself a non-negative profit. This is possible by charging a price above the marginal cost. Therefore, the profits of the firm are explained as

$$0 \leq \Pi^1 + \Pi^2 \leq \Pi^m \tag{39}$$

The firm chooses their prices both simultaneously and non-cooperatively. A Nash equilibrium in prices is sometimes referred to as a Bertrand equilibrium. This is a pair of prices (P^*_1, P^*_2). Each firm's price maximizes that firm's profit given the other firm's price.

Formally, for all i = 1, 2 and for all p_i

$$\Pi^i (P^*_i, P^*_j) \geq \Pi^i (P_i < P^*_j) \tag{40}$$

The Bertrand paradox states that the unique equilibrium when two firms charge the competitive price.

$$P_1^* = P_2^* = C \tag{41}$$

The proof is as follows. Consider for example

$$P_1^* \rangle P_2^* \rangle C \tag{42}$$

The first firm has no demand and its profit is zero. On the other hand, if the first firm charges

$$P_1 = P_2^* - \varepsilon \tag{43}$$

Now ε is positive and small. It obtains the entire market demand, $D(P_2^* - \varepsilon)$ and has a positive profit margin of

$$P_2^* - \varepsilon - C \tag{44}$$

Therefore, the first firm cannot be acting in its own best interest if it charges P_1^*.

Now suppose that

$$P_1^* = P_2^* > C \tag{45}$$

then the profit of the first firm is

$$D(P_1^*)(P_1^* - C)/2 \tag{46}$$

If the first firm reduces its price slightly to $P_1^* - \varepsilon$ its profit becomes

$$D(P_1^* - \varepsilon)(P_1^* - \varepsilon - C) \tag{47}$$

It is greater for a small ε. In the above situation, the market share of the firm increases in a discontinuous manner. This is because no firm will charge less than the unit cost C. The lowest price firm would make a negative profit. We are left with one or two firms charging exactly C. In order to show that both firms do charge C, suppose

$$P_1^* > P_2^* = C \tag{48}$$

The second firm which makes no profit could raise its price slightly. But still it can supply all the demand and make a positive profit – a contradiction.

The conclusion of this simple model is as follows. We have written it in points.

I) That a firm's price is at marginal cost.
II) That firms do not make profits.

We call these two situations the Bertrand Paradox because it is hard to believe that firms in industries will never succeed in manipulating the market price to make profits.

In a symmetric case conclusions I and II do not hold. Indeed, the following can be shown

III) That both firms charge price p = c2, and
IV) That firm 1 makes a profit of (c_2-c_1) D (c_2) and firm 2 makes no profit.

Thus firm 1 charges above the marginal cost and makes a positive profit. The Bertrand equilibrium is no longer welfare–optimal. But again, the conclusion is a bit strained. Firm 1 makes very little profit if C2 is close to C1 and Firm 2 makes no profit at all.

Solution to the Bertrand paradox

In the above model, we have made three crucial assumptions. Now in order to prove the Bertrand paradox, we need to relax one from the above three assumptions.

The Edgeworth solution

Francis Edgeworth solved the Bertrand paradox in 1897 by introducing capacity constraints by which firms cannot sell more than they are capable of producing. To understand this idea, suppose that Firm 1 has a production capacity smaller than D(c.) The equilibrium is $(P^*_i, P^*_2) = (c, c)$. It is still an equilibrium price system. At this price both firms make zero profit.

Suppose that Firm 2 increases its price slightly then Firm 1 faces demand D(c) but it cannot satisfy this demand. The rational way is that some consumers must move to Firm 2. Firm 2 has a non zero demand. It sells at a price greater than its marginal cost therefore it makes a positive profit. Consequently, the Bertrand solution is no long term equilibrium. The general rule is that in models with capacity constraints, firms make positive profits. The market price is greater than the marginal cost.

3.8. Intertemporal dimensions

The second crucial assumption in the above model is that it is a one-shot game. It does not always seem to reflect economic reality as it stands. In the Bertrand solution, the equilibrium is P1= P2 > C. But there is not an actual equilibrium in the model.

The answer is that Firm 1, for example, could benefit from a slight decrease in its price (i.e. to P-ϵ) and from its resulting takeover of the entire market. Nothing happens after that because of Bertrand's crucial assumption of the one-shot game. In reality, Firm 2 would probably decrease its price in order to regain its share of the market. If we introduce this temporal dimension and the possibility of reaction, it is no longer clear that Firm 1 would benefit from decreasing its price below P2.

Firm 1 would have to compare the short-run gain (the rise of its market share) to its long-run loss in the price war.

Subgame perfectness

The perfect equilibrium of Follow the Leader I

Subgame perfectness is an equilibrium concept based on the ordering of moves and the distinction between an equilibrium path and equilibrium. The equilibrium path is the path through the game tree that is followed in equilibrium, but the equilibrium itself is a strategy combination. It includes the player's responses to other player's deviations from the equilibrium. The path perfectness is a way to eliminate some of the weaknesses of the Nash equilibrium. A flow of the Nash equilibrium was revealed in the game follow the leader I, which has three pure strategies. The Nash equilibrium is one of the reasonable strategies. The players are Smita and Deepa who must choose disk sizes. Both their payoffs are greater if they choose the same size and greatest if they coordinate on the size 'Large'. Smita moves first so her strategy becomes more complicated. Although she must specify an action for each information set, Deepa's information set depends on what Smita chooses. Deepa's strategy set is (large, small). It specifies that she chooses 'large' if Smita chooses 'large'. She chooses 'small' if Smita chooses 'small'. We found the following three Nash equilibriums.

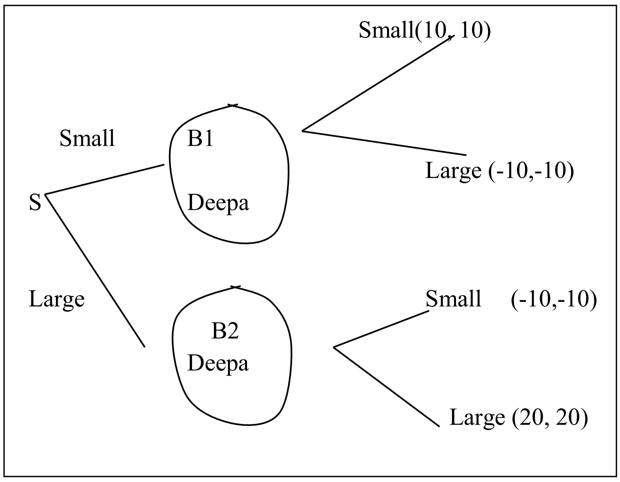
Equilibrium Strategy outcome

X {large (large, large)} both pick large

Y {large (large, small)} both pick large

Z {small (small, small)} both pick small

From the above outcomes, equilibrium Y is a reasonable strategy for both players. This is because the order of the moves should matter to the decision players. The problem with the normal form and thus with the simple Nash equilibrium is that it ignores who moves first. Smita moves first and it seems reasonable that Deepa should be allowed. In fact, she should be required to rethink her strategy after Smita moves.



3.12. Follow the leader I strategy

Now consider Deepa's equilibrium Z strategy of (Small, Small). If Smita deviated from the equilibrium and she chooses 'large' then it would be unreasonable for Deepa to stick to the response of 'large'. She would indeed choose 'large' and Z would not be the equilibrium. A similar case shows that (Large, Large) is an irrational strategy for Deepa and we are left with Y as the unique equilibrium. We say that equilibrium X and Z are Nash equilibrium. But they are not "Perfect" Nash equilibrium. A strategy combination is a perfect equilibrium if it remains equilibrium on all possible paths. Both the equilibrium path and their paths branch into different subgames. A subgame is a game consisting of a node which is a singleton in every player information partition. A strategy combination is a subgame perfect Nash equilibrium if (a) it is Nash equilibrium for the entire game and (b) its relevant action rules are Nash equilibrium for every subgame. The extensive form of Follow-the-Leader I has three subgames:

- The entire game
- The subgame starting at node B1
- The subgame which is starting at node B2

Strategy combination x is not a subgame perfect equilibrium because it is only Nash in subgames (1) and (3). But it is not Nash equilibrium in subgame 2. The strategy combination Z is not a subgame perfect equilibrium. It is because only Nash in subgames (1) and (2) is not in subgame (3). But strategy combination Y is Nash in all three subgames.

One reason why perfectness is a good concept is because equilibrium behavior is irrational in a non-perfect equilibrium. A second justification is that a weak Nash equilibrium is not robust to small changes in the game.

An example of Perfectness Entry Deterrence I

In this game, we assume that there are two players. The first firm is the new entrant firm and the other is the senior firm. The information is perfect and uncertain. These are the following two actions and events observed in this game.

- The new entrant firm decides whether to enter or stay out.
- If the new entrant firm decides to enter, the senior firm can collude with it or fight back by cutting the price drastically.

The following are the payoffs to both in this game.

Market profits are 100 at the monopoly price and 0 at the fighting price. Entry costs 10. Collusion shares the profits evenly.

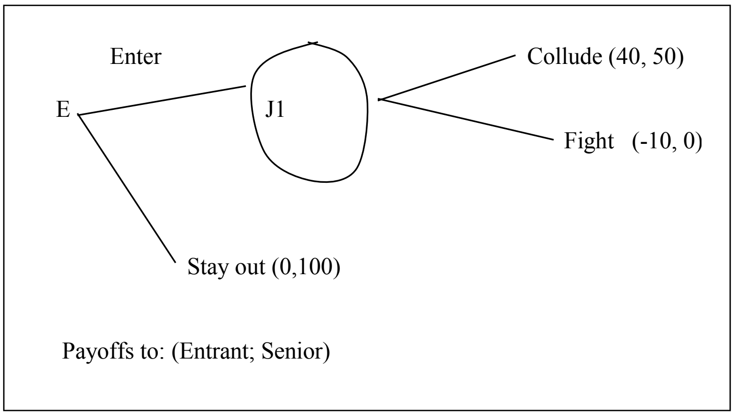
The strategy sets can be discovered from the order of actions and events. They are {enter, stay out} for the entrant and {collude if entry occurs, fight if entry occurs} for the senior firm. The game has the two Nash equilibriums indicated (enter, collude) and (stay out, fight). The equilibrium (stay out, fight) is weak because the senior firm would just as soon collude given that the new entrant is staying out.

		Collude	Fight
New Entrants	Enter	40, 50	-10, 0
	Stay out	0, 100	0, 100

Payoffs to {new entrant, senior}

3.13. Entry deterrents strategy I

Once he has chosen 'enter', the senior firm's best response is 'collude'. The threat to 'fight' is not credible and would be employed only if the senior firm could bind itself to 'fight' in which case, he never does fight. It is because the new entrant chooses 'stay out'. The equilibrium (stay out, fight) is Nash but not subgame perfect because if the game is started after the new entrant has already entered, the senior firm's best response is 'collude'. This does not prove that collusion is inevitable in duopoly but collusion is the equilibrium for Entry Deterrents Strategy I



3.14. Equilibrium for entry deterrents strategy I

Suppose there is effective communication between the two players then the game will change. The senior firm might tell the new entrant that 'entry' would be followed by 'fight'. But the new entrant could ignore this non-credible threat. Suppose the senior firm could commit itself to 'fight' entry, the threat would become credible. A game in which a player can commit himself to a strategy can be modeled in two ways.

Firstly as a game in which non-perfect equilibrium is acceptable. Secondly, we assume that by changing the game the action of both the players changes, too. The second approach is better than the first. This is because the modeler usually wants to let players commit to some actions and not to others. A player can do this by carefully specifying the order of play. Allowing equilibrium to be non-perfect forbids such discrimination and usually multiplies the number of equilibriums. Subgame perfectness is too restrictive and it still allows too many strategy combinations to be equilibrium in games of asymmetric information. A subgame must start at a single node and it should not cut across any player's information set. It is often the only subgame which will be the whole game and imposing subgame perfectness. It does not restrict equilibrium at all.

3.9 The folk theorem

This is part of the conventional wisdom of game theory that threatens of mini-max punishments. It can sustain any individually rational collusive allocation as the Nash equilibrium of an infinitely repeated game. Games in which players meet in strategic interactions repeatedly are referred to as repeated games (Bierman & Fernandez, 2005). It is not possible to assign authorship of the result. The firm being punished is making its best response to the action of the punisher.

In an infinitely repeated n person game with finite action sets at each repetition, any combination of actions is observed in any finite number of repetitions. It is the unique outcome of some subgame perfect equilibrium. The three conditions are given and they are explained in detail as follows.

Condition 1: The rate of time perfectness is zero or positive and sufficiently small.

Condition 2: The probability that the game ends at any repetition is zero or positive and sufficiently small.

Condition 3: The set of payoff combinations that strictly Pareto dominates. The mini-max payoff combinations in the mixed extension of the one-shot game has n dimensions.

The folk theorem explains that a particular behavior arises in a perfect equilibrium. It is meaningless in an infinitely repeated game. This applies to any game that meets condition one to three. If an infinite amount of time always remained in any game, a way can be found to make one player willing to punish some other player for the sake of a better future, even though the punishment hurts the punisher as well as the one punished. Any infinite interval of time is insignificant; it is compared to infinity. The threat of future reprisal makes the players willing to carry out the punishments needed. This is a most practical example that can be observed happening in industry.

The folk theorem has emerged as the best theorem in modern microeconomics. It implies that in repeated games, any outcomes are a feasible solution concept. Under that outcome, the player's mini-max conditions are satisfied. The mini-max condition states that a player will minimize the maximum possible loss. An outcome is said to be feasible if it satisfies this condition for each player of the game. A repeated game is one in which there is not a final move but there is a sequence of rounds. Each player may gather information and choose moves. In mathematics, the theorem is believed and discussed but it has not been published.

A grim trigger strategy is a strategy which punishes an opponent for any deviation from some certain behavior. Therefore, all players of the game must have a certain feasible outcome. The players need only adhere to an almost grim trigger strategy. Any deviation from the strategy will bring about the intended outcome. The punishment is meted to such a degree that any gains made by the deviator on account of the deviation are exactly cancelled out. Thus, there is no advantage to any player for deviating from the course which will bring out the intended and arbitrary outcome. The game will proceed in exactly the manner to bring about that outcome. The following conditions are important for our understanding.

Condition 1: Discounting

We know that discounting the present gain from confessing is weighted more heavily. The future gains from cooperation are taken more lightly. If the discounting rate is very high, the game almost returns to being one-shot. Suppose the real interest rate is high then a payment next year is a little better than a payment a hundred years later. Therefore, next year is practically irrelevant. Any model that relies on a large number of repetitions also relies on the discount rate not being too high. The alarming strategy imposes the heaviest possible punishment for the prisoner's dilemma. For consumer goods, a discount is given because short-terms gains are higher compared to long-term gains. There are more innovations possible in the long term.

Condition 2: A probability of the game ending

Time preference is fairly straightforward. The assumption is that the game ends each period with probability Q. There are different examples where the game ends with time. It does not make a drastic difference if $Q > 0$. The probability of the game ending is taken as one or it is put less than the expected number of repetition. It is still behaving like a discounted game. This is because the expected number of future repetitions is always large. It does not matter how many have already occurred. We always believe that there is an end to each game.

The following two situations are different from each other.

1. The game will end at some uncertain data before T. In statistics, this is assumed as T.
2. There is a constant probability of the game ending.
3. Under the game theory, each game is like a finite game. This is because as time passes, the maximum amount of time still to run shrinks to zero, even though the game will probably end at T. If it lasts until T, the game looks exactly the same as at time zero.

Condition 3: The dimensionality condition

The mini-max payoff is the payoff of the result. Suppose all the other players pick strategies solely to punish player i then he protects himself as best he can. There are different methods of risk diversions. The dimensionality condition is needed for games with three or more dimensions. It is satisfied for each game which has the same payoff. The desired behavior requires some way for the other players to punish a deviator, without punishing themselves. This is observed in terms of different dimensions in the game.

3.10 Conclusion

We saw that game theory is different from decision theory. Decision theory makes use of the decision tree to arrive at a particular decision. But game theory is different; it has different players, strategies and benefits. There are players who decide which strategy they have to choose to get the maximum benefits. There are different types of games such as cooperative and non-cooperative, Follow the Leader, and welfare game. The Nash equilibrium is expected as an outcome of the cooperative game. But there are certain games where the Nash equilibrium is not possible. The welfare game and strategies are completely dependent on cooperative strategies. The Cournot and Stackelberg models of duopoly explain the strategies of two firms and their market share. The players Smita and Deepa are independent but their strategies are chosen depending on each other's move. Such strategies are selected to get the maximum benefit from the rival.

Questions

Question 1. How is decision theory different from game theory?

Question 2. Explain the prisoner's dilemma with the help of an example.

Question 3.Discuss the following types of games with examples.

a) Cooperative and non-cooperative game
b) Repeated dominance
c) Nash equilibrium
d) Battle of Politics
e) pure coordination strategies

Question 4. Discuus critically the welfare game with the example of the destitute person and the government.

Question 5. Explain in detail the duopoly model

Question 6. What solution is provided by Stackelberg to Cournot's duopoly model?

Question 7. What is the Bertrand paradox? Explain in detail along with solutions.

Question 8. Write a note on folk theorem.

4 Information Economics

4.1 Introduction

In the production game, there are two players: the principal and the agent. Both have different strategies to earn the maximum benefits. The principal chooses the strategies which are to his best interest and knowledge. At the same time, the agent wants to get more benefits from his efforts. Information is asymmetric, incomplete and uncertain. There are different contracts provided by the principal to the agent. It depends on both which contract both players will choose or reject. There are economic interests that are involved in each other's efforts.

A firm appoints a salesman but the efforts of the salesman are important to get the maximum benefit. Sometimes, with pushover effort, commodities will be sold. But if the efforts do not give the maximum benefit then the company must believe that the agent is telling the truth. The cooperative strategies will work here to get the benefits for both the company and the salesman. Similarly, signalling is a costly action for workers. If the worker has given a signal and is screened then there is a difference between workers with more or less education.

4.2 The asymmetric information model

In this chapter, we use the principal agent model to analyze asymmetric information. There are two players in this game, for our purposes, the principal and the agent. The principal hires an agent to perform certain tasks. The agent acquires an informational advantage about actions at some point in the game. This information may be available from friends, agents, newspapers and employment exchange, etc. The principal or uninformed player is the player who has the low caliber information partition. The agent or informed player is the player who has the finer information partition.

In each model, the principal (P) offers the agent (A) a contract which the latter accepts or rejects. Nature (N) makes a move or the agent chooses an effort level. The moral hazard models are games of complete information with uncertainty. In the game of moral hazard with hidden actions, the agent moves before nature does and in the game of moral hazard with hidden information, the agent moves after Nature and conveys a message to the principal about Nature's move.

Adverse selection models have incomplete information. Nature moves first and picks the types of the agent, generally, according to his ability to perform the task. In the simplest model, the agent simply accepts or rejects the contract.

In the signalling model, suppose the agent sends the signal before the principal offers a contract and conducts screening. The simple difference between the signal and the message is that a signal is a costless statement but a costly action.

Example:

We consider an employer (the principal) hiring a worker (the agent). The employer knows the worker's ability but not his effort level. At the initial stage, the principal does not know the worker's ability. But after the worker accepts a contract, the principal discovers his ability. The problem arises of the moral hazard with hidden information.

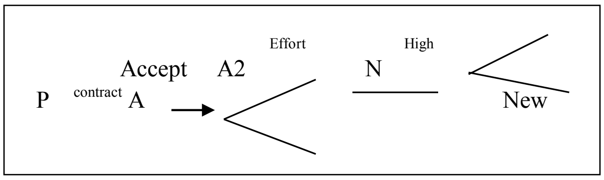
The worker knows his ability from the start but his employer does not. The problem arises of adverse selection, which appears to be widespread in the labor market. A principal action consists of an incentive scheme that specifies a reward to the agent as a function of some (verifiable) performance measure that is correlated with the agent's effort. Depending on the application of interest, the reward can be a monetary payment, the transfer of an asset, the choice of a policy or a combination of any of these (Pavan & Calzolari, 2010).

In addition to this, the worker knows his ability from the star. But he can also be trained. The agent's education is observable by the employer before they sign a contract. The problem is then signalling. The agent's utility functions also include intrinsic utility from the action, utility from consuming the public goods, monetary rewards or costs and utility from the esteem of others (Daugherty & Reinganum, 2010).

If the worker acquires his training as a result of the contract offered by the employer, the problem is screening.

1. Moral hazard with hidden actions

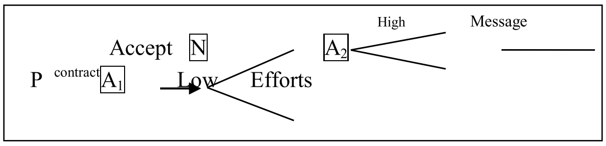
Moral hazard is defined as the tangible loss producing propensities of the individual assured or as that which "comprehends all of the non-physical hazard risks" (Pauly, 1968).



In the moral hazard with hidden information game, the principal signs a contract with the agent, a contract which is acceptable to both. The contract will require the agent to exert particular efforts, which will be intangible.

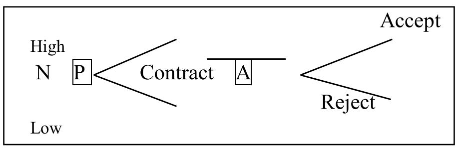
2. Moral hazard with hidden information

In this game, the principal signs a contract with the agent. The agent is expected to work hard. This is because nature shows high efforts. The agent gives the message through his work. The efforts are higher in this game. The information is complete because the principal can observe the efforts of the agent.



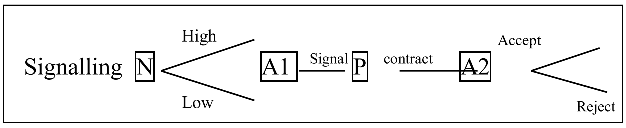
3. Adverse selection

Under the adverse selection model, the principal and the agent need to sign the contract, which would require the agent to exert high efforts. The principal allows the agent to sign the contract without observing the agent's efforts. The information is incomplete; this is adverse selection of the agent by the principal.



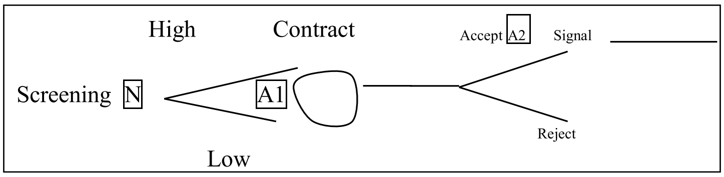
4. Signalling

In this game, Nature begins the game by choosing different options. The principal is ready to sign the contract with the agent which requires high efforts of the agent. The agent has to give the signal to sign the contract. When the signal is received, the principal signs the contract with the agent. This is called signalling. Therefore, the information is incomplete. The game is presented in signalling form.



5. Screening

In this game, the principal needs to sign a contract with the agent. Before signing the contract, he will first screens the agent. He acquires the needed information and screens the agent for high efforts. If the agent is willing to exert high efforts then the contract is signed.



4.3 The principal-agent model: The production game

After hiring an agent, the principal discovers the productive capabilities of the agent. On the basis of previous experience in the market, principals have conditional probability assessments over the productive capacity of employees given various combinations of signals and indices at any point of time when confronted with an agent applicant with certain observable attributes (Spence., 1973). In the principal-agent model, we assume that the principal is a manager and the agent is a worker. The terms are used alternatively in different examples. In the moral hazard game, it is easier for the manager to observe the worker's output than his efforts. Therefore, the manager offers a contract to pay the worker based on output. Output is dependent on the worker's efforts.

The principal is the one whose goal it is to reduce some of the risk with heterogeneous agents. The agents have mean variance preferences. An agent's degree of risk aversion is private information and hidden to the principal. The principal only knows the distribution of risk aversion coefficients which puts him at an informational disadvantage. If all agents were homogeneous, the principal, when offering a structured product to a single agent, could (perhaps) extract the indifference (maximum) price from each trading partner.

In the presence of agent heterogeneity this is no longer possible, either because the agents would hide their characteristics from the principal or prefer another asset offered by the principal but designed and priced for another customer (Horst & Moreno-Bromberg, 2008).

In a principal-agent situation, the agent chooses an action "on behalf of" the principal. The resulting consequence depends on a random state of the environment as well as on the agent's action. After observing the consequence, the principal makes a payment to the agent according to a pre-announced reward function, which depends directly only on the observed consequence. This last restriction expresses the fact that the principal cannot directly observe the agent's action, nor can the principal observe the information on which the agent bases his action. This situation is one of the simplest examples of decentralized decision-making in which the interests of the decision-makers do not coincide (Roy, 1985).

The monetary value of output by q (e) which is increasing in effort e. At this point, q is the quantity produced by the worker. The agent's utility function U(e,w) is decreasing in effort and increasing in the wage while the principal's utility function V(q-w) is decreasing in effort and increasing in the wage. The principal's utility function V(q-w) is increasing in the difference between output and the wage. It is net profit for the principal. In many circumstances, a principal may have relevant private information when he proposes a contract to an agent. We analyze such a principal-agent relationship as a non-co-operative game. The principal proposes a contract which is accepted or rejected by the agent. The contract is executed if accepted; otherwise, the reservation allocation takes effect. This allocation may be determined by a pre-existing contract or it may simply be the non-trade point. The study assumes that the principal's information directly affects the agent's payoff. Before solving the game, we discuss Pareto efficiency with asymmetric information. We define an incentive-compatible allocation to be weakly inter- inefficient if there exists no alternative incentive compatible allocation that both parties prefer for all possible beliefs that the agent might have about the principal's private information (Maskin & Tirole, 1992).

I: A flat wage under certainty

The principal and the agent are the players in this game. There is asymmetric, complete and certain information. Actions and events are divided into three types. Firstly, the Principal offers the worker a wage. Secondly, the agent decides whether to accept or reject the contract. Thirdly, if the agent accepts the contract then he exerts effort e. These efforts cannot be observed by the principal. Therefore, output equals q(e), observed by both players where q'> 0. The payoffs in this game to both players are explained as follows. Firstly, if the agent rejects the contract, then profit for the (π) agent is $\bar{U}$ and profit (π) for the principal is zero. Secondly, if the agent accepts the contract, then profit (π) for the agent is U (e, w) and profit (π) for the principal is V (q-w). Each depends on the other to achieve profits and wages.

The common assumption in most principal-agent models is that either the principal or the agent is a perfect competitor. When the principal decides to employ the agent then the principal's equilibrium profit equals zero. Many agents compete to work for the principal so the agent's equilibrium utility equals the reservation utility $\bar{U}$.The outcome of the above game is simple and inefficient. If the wage is non-negative then the agent accepts the job and exerts zero effort. The principal's best response is to offer a wage of zero.

II: An output-based wage under certainty

The principal and the agent are the players in production game two. The information is again asymmetric, complete and certain. Three actions and events occur. Firstly, the principal offers the worker a wage function w(q). Secondly, the agent decides whether to work or not to work. If he accepts the work then he exerts effort e, which is unobserved by the principal. Thirdly, output equals q(e) and is observed by both players.

The payoff is explained as follows. Suppose the agent rejects the contract, then profit (π) for the agent is equal to $\bar{U}$ and profit (π) for the principal is zero. If the agent accepts the contract , then π for the agent is u(e,w) and π for the principal is V (q-w). The principal must offer the wage based on quantity produced. The principal collects all the gains from trading. He wishes to pick the effort level e* of the worker that generates the efficient output level q*. The contract must provide the agent with utility $\bar{U}$ in equilibrium, but any $U(e,w(q)) < \bar{U}$ for $e \neq e^*$. This will make the agent pick e = e*. Such a contract is called as a forced contract, so-called because it forces the agent to pick a particular effort level.

III : An output-based wage under uncertainty

In this game, the players are the principal and the agent. The information is asymmetric, complete and uncertain. There are three possible actions. Firstly, the principal offers the worker a wage function w(q). Secondly, the agent decides whether to accept or reject the contract. Suppose he accepts the contract then he exerts effort e, which the principal cannot observe. Thirdly, ,the agent sees his efforts are but his efforts are unknown by the principal. Nature chooses the state of the world $\theta \in R$. According to the probability density of (θ), output equals q(e,θ). The payoffs are as follows. Suppose the agent rejects the contract, then π agent = $\bar{U}$ and π forprincipal is zero. Similarly, if the agent accepts the contract, then π for agent is EU (e,w) and π for principal is EV (q-w). The principal cannot just choose an output level and tell the agent to produce it. This is because unlike in production game II, the principal cannot deduce that e = e* just by looking at the output, which is q(e,θ) not just q(e). The optimal wage contract might specify the highest wage for q*, but not necessarily the outcome because it will usually not specify a zero wage for a slightly lower output. Nature might be to blame if the agent is at risk to be blamed if the agent is risk averse.The agent's expected utility equals $\bar{U}$. It is more expensive for the agent to bear the risk and the principal wants to insure the agent by keeping low the risk imposed on him. The tradeoff between incentives and insurance is given as follows.

In terms of a linear equation

$$w = \alpha + \beta q \text{ (linear contract)}$$

This means that wage is a function of the quantity produced by the worker. The wage is directly proportional to the worker's output.

$$w\ \alpha, \text{ if } q < \bar{q} \text{ (threshold contract) and}$$

$$\beta\ , \text{ if } q \geq \bar{q} \quad (1)$$

The source of the moral hazard is observable. But the fact is that the contract cannot be conditioned on effort. Effort is non-contractible. Production game III applies even if the principal can see very well that the agent is slacking. But he cannot prove it in court. In the principal-agent problem, the wage is set without the principal knowing the agent's efforts. This is the problem with this model.

4.4 Optimal contracts: The Broadway game

The relation between output and compensation

An investor advances funds to a producer to produce a Broadway show which may succeed or fail. The producer has the choice to appoint a corrupt or an honest manager. The funds advanced means a direct gain to him if the manager he hires is corrupt. If the company is correct in decision making, then the revenue is Rs. 500 crore were the manager not corrupt and Rs. 100 crore if he were corrupt. If the company fails to recruit an honest manager then the revenue is Rs.-100 crore in either case. Extra expenditure on a fundamentally flawed show is useless.

In the Broadway game, the players include the company and the investors. The information is asymmetric, complete and uncertain. There are three possible actions and events. Firstly, the investors offer a wage contract w (q) as a function of revenue q. Secondly, the producer accepts or rejects the contract. Thirdly, the producer chooses to be corrupt or to be honest. Nature picks the state of the world to be a success or a failure with equal probability. The resulting revenue q is shown in the table.

Payoffs to both are calculated as follows. The company's payoff is U, if he rejects the contract, where U > 0 and U"< 0, and the investors' payoff is 0. Otherwise, the profit of the manager is U (w (q) +50) if he is corrupt. Suppose he is not corrupt, then profit is U (w (q)). These probabilities have an effect on the profitability of the company. In other words, if the manager is corrupt, then he will get some wages and Rs. 50 lakhs.

The profits for the investor will be affected.

$$\pi \text{ investors} = q - w(q)$$

		Failure	Success
Effort	Honest	-100	+500
	Corrupt	-100	+100

Figure 4.1. The Broadway game: profits from the show state of the world (Rs. Crore)

The Broadway game illustrates that options contracts do not always give higher wages for better performance. An optimal contract here is the boiling–in-oil contract. Investors are risk neutral and the company is risk averse. The company should bear as little risk as possible while providing incentives. The boiling-in-oil contract is an application of the sufficient statistics condition. It states that if the manager's utility function is separable in effort and money, the wages should be based on whatever evidence indicates effort and only incidentally on output. In the spirit of the three-step procedure, what the principal wants is to induce the agents to cause the appropriate effort. Suppose both the company and the investor are risk-adverse then risk sharing would change the part of the contract that applies in equilibrium. The optimal contract would then provide w (-100) < w(+500) to share the risk. The company would have a lower marginal utility of wealth when output was +500. Therefore, the company would be better able to pay an extra rupee of wage in that state than when output was -100. Suppose the producer is paid enough when output is -100 or +500, the manager's expected utility equals his reservation. The expected utility is $\bar{U}$ and his boiled-in-oil point is forced down to an arbitrarily low level of utility. Suppose the output is +100 then he refrains from being corrupt. The set of possible outcomes under the optimal effort is different from under any other effort. Certain outputs show without any doubt that the manager refrains from being corrupt The heavy punishments imposed only for those outputs achieve the first best because a non-corrupt manager has nothing to fear. If the manager shirks instead of working then certain low outputs become possible and certain high outputs become impossible. In this case, when output shifts when behavior changes, boiling–in-oil contracts are useful. There are four conditions favoring boiling-in-oil contracts. Firstly, the agent is not very risk averse. Secondly, there are outcomes with high probability under shirking that have a low probability under optimal effort. Thirdly, the manager can be severely punished. Fourthly, it is credible that the company will carry out the severe punishment.

Selling the store

Another first-best contract that can be used is selling the store. Under this arrangement, the agent buys the entire output for a flat fee to the principal, becoming the residual claimant since the manager keeps every additional rupee of output that his extra effort produces. This is equivalent to fully insuring the company since his payoff becomes independent of the actions of the agent and Nature. Selling the store takes the form of the company paying the investors 200 = 0.5 (-100) + 0.5(500) and keeping all the profits for itself. The drawbacks are firstly, the company will not be able to afford to pay the investors the flat price of Rs. 200 crore. Secondly, the company might not be risk averse and incurs a heavy utility cost in bearing the entire risk. These two drawback are why producers go to investors in the first place.

		Failure	minor success	big success
Effort	No corruption	-100	+50	+500
	Speculator	-100	-100	+100

Figure 4.2. The Broadway game: profits from the show with three phenomena

Under the optimal contract

$$W\ (-100, -w\ (+50) = w\ (+500) > w\ (+100) \tag{2}$$

Because only the datum q = +100 is proof that the manager is corrupt when the information set is refined. Therefore, before the agent takes his action both he and the principal can tell whether the show will be a big success or not.

If the company deems the investment profitable even if it knew in advance that the show would not be a big success, this refinement does not help it. The behavior of the manager changes, however, because on observing {failure, minor success} the manager is frees to become corrupt without the boiling-in-oil output of +100. He would still refrain from corrupting if he observed {big success} but the contract can make the manager not corrupt if he observes {failure, minor success} and if the parameters were such that the company needs the returns from even minor successes to reach the breakeven point. Therefore, the company would not make the investment in the first place were the information sets refined and the gains from trade would be lost.

4.5 Moral hazard: Hidden information

In the moral hazard game, information is complete, but under hidden information the agent sees some move of Nature that the principal does not. From the principal's point of view, agents come in several types depending on what they have seen. His chief concern is to discover the agent's type. The agent may exert effort contractibility which is unimportant. When the principal is ignorant. he does not know which effort is appropriate.

IV: Hidden information

There are two players in this game: the principal and the agent. The information is asymmetric, complete and uncertain. The actions and events are as follows. Firstly, the principal offers the worker a wage contract in the form w (q, m). Secondly, the agent accepts or rejects the principal's offer. Thirdly, Nature chooses the state of the world θ according to probability distribution F (θ). The agent observes θ, but the principal does not. Fourthly, if the agent accepts, he exerts effort e and sends a message m, both observed by the principal. The output is q (e, θ). The payoffs are calculated as follows**.** If the agent rejects the contract, then π for the agent is $\bar{U}$and π for the principal is zero. Secondly, if the agent accepts the contract, then π for the agent is U (e, w, θ) and π for the principal is V (q-w). The principal would like to know θ. He would be delighted to employ an honest agent who always chooses m = θ, but in a non-cooperative game, the agent's words are worthless, the principal must try to design a contract that either provides incentives for truthfulness or takes lying into account.

Pooling and separating equilibrium

In the hidden actions model, the principal tries to construct a contract which will induce the agent to take a single appropriate action. In the hidden information model, the principal tries to make different actions attractive under different states. Therefore, the agent's choice depends on the hidden state. If all types of agent picked the same action then the same strategy is chosen at all points. The equilibrium is pooling; otherwise, it is separating.

A single equilibrium – even a pooling option – can include several contracts. In a pooling equilibrium, the agent always uses the same strategy, regardless of agent type. If the agent's equilibrium strategy is mixed, then the equilibrium is pooling. The agent always picks the same mixed strategy, even though the messages and efforts would differ across the realization of the game.

A separating contract need not be fully separating. If agents who observe $\theta < 4$ accept contract c, but other agents accept contract c2, the equilibrium is separating but does not separate out every type. We say that the equilibrium is fully revealing if the agent's choice of contract always conveys his private information to the principal. The pooling and fully revealing equilibrium is synonymously called semi-separating, partially separating, partially revealing or partially pooling equilibrium.

The principal's problem is to maximize his profit subject to the following constraints.

1. Incentive compatibility

In this game the agent picks the desired contract and actions. Under hidden information, the incentive compatibility constraint is sometimes called the self selection constraint because it induces the different types of agents to pick different contracts.

Equilibrium is defined as follows. The principal offers

$W1 = w1\ (q=0) = 3, \qquad w1\ (q=10) = 3$

$W2 = w2\ (q=0) = 0, \qquad w2\ (q=10) = 4$

The agent chooses contract 1 which pays a low wage (w1).

The second agent chooses contract two which pays a high wage (w2).

2. Participation

In this game, the agent prefers the contract to his reservation utility.

The equilibrium must also satisfy a part of the competition constraints, which are not found in the hidden action models: either a non-pooling constraint or a non-separating constraint. If one of the several competing principals wishes to construct a pair of separating contracts c1 and c2 he must construct them so that not only do agents choose c1 and c2**,** depending on the incentive compatibility**.** If the agents prefer (c1,c2) a pooling contract, then c3 is a non-pooling contract.

4.6 Pooling and separating equilibrium: the salesman game

Let's consider the example of a manager and a salesman. The manager of a company tells the salesman to investigate a potential customer, who is either a pushover or a windfall. If a customer is a pushover, the efficient sales effort is low and sales should be moderate. If a customer is a windfall, the effort and sales should be higher.

In the salesman game, the players are the manager and the salesman. The information is asymmetric, complete and uncertain. The manager is uninformed in this game. The actions and the events are as follows. Firstly, the manager offers the salesman a contract of the form w (q,m), where q is sales and m is a message. Secondly, the salesman decides whether or not to accept the contract. Thirdly, Nature chooses whether the customer is a windfall or a pushover with a probability 0.2 and 0.8 respectively. We denote the state variable "customer status" by θ. The salesman observes the state, but the manager does not. Fourthly, when the salesman has accepted the contract he chooses his sales level q, which implies a measure of his effort. If the salesman rejects the contract, his payoff is $\bar{U}$=8 and the manager's is zero. If he accepts the contract, the π of the manager = q - w. The profit for the salesman is

π salesman = U(q, w, θ) where ,

$$\frac{\partial U}{\partial q} < 0, \frac{\partial^2 U}{\partial q^2} < 0, \frac{\partial U}{\partial w} > 0, \frac{\partial^2 U}{\partial w^2} < 0 \tag{3}$$

In Figure 4.3, the manager's indifference curves are straight lines with slope -1. This is because the manager is acting on behalf of a risk-neutral company. Suppose the wage and the quantities both rise by a rupee, profits are unchanged. The profits do not depend directly on whether θ takes the pushover or windfall value. The salesman's indifference curves also slope upward. This is because he must receive a higher wage to compensate for the extra effort. They are convex because the marginal utility of a rupee is decreasing and the marginal disutility of effort is increasing.

The salesman has two sets of indifference curves. The solid lines are pushovers and the dotted lines are windfalls. Because of the participation constraint, the manager must provide the salesman with a contract giving him at least his reservation utility of 8, which is the same in both states.

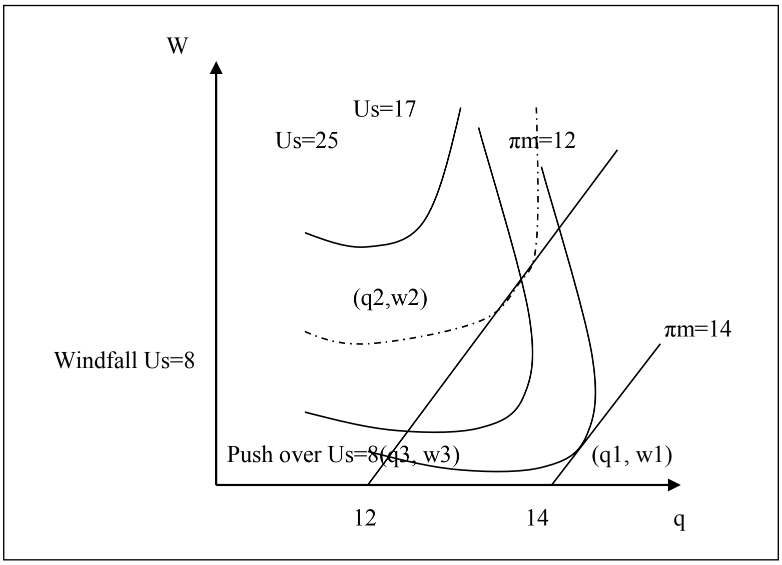


Figure 4.3. Indifference curves and efforts of salesman

In the case of a windfall customer, the manager would like to offer a contract that puts the salesman on the dotted indifference curve U = 8 and the efficient outcome is (q2, w2), the point where the salesman's indifference curve is tangent to one of the manager's indifference curves. At that point, if the salesman sells an extra rupee he requires an extra rupee of compensation. If it were common knowledge that the customer was a windfall, the principal could choose w2 so that u (q2, w2, windfall) = 8 and offer the forcing contract.

$$W = \begin{cases} 0 & \text{if } q<q2 \\ w2 & \text{if } q \geq q2 \end{cases}$$

Let's assume that the salesman accepts the contract and he chooses q - q2. But if the customer were actually a pushover, the salesman would still choose q = q2, an inefficient outcome that does not maximize profits. At this point, profits would be not maximized because the salesman achieves utility and he would be willing to work for less.

The revelation principle says that in searching for the optimal contract that induces the agent to truthfully reveal what kind of customer the salesman faces. If he required more effort to sell any quantity to the windfall customer, then the salesman would always want the manager to believe that he faced a windfall customer to extract the extra pay necessary to achieve a utility of 8. The only optimal truth- telling contract is the pooling contract. It pays the intermediate wage of w3 for the intermediate quantity of q3 and zero for any other quantity. The message is not important with this type of contract.

The pooling contract is a second-best contract. It is a compromise between the optimum for pushovers and the optimum for windfall customers. The point (q3,w3) is closer to (q1,w1) than to (q2,w2). This is because the probability of a pushover is higher and the contract must satisfy the participation constraints

$$0.8\ U\ (q3, w3, \text{pushover}) + 0.2\ U\ (q3, w3, \text{Bonanza}) \geq 8$$

The nature of the equilibrium depends on the shapes of the indifference curves. This is shown in figure 4.4. The equilibrium in this diagram is separating, not pooling.

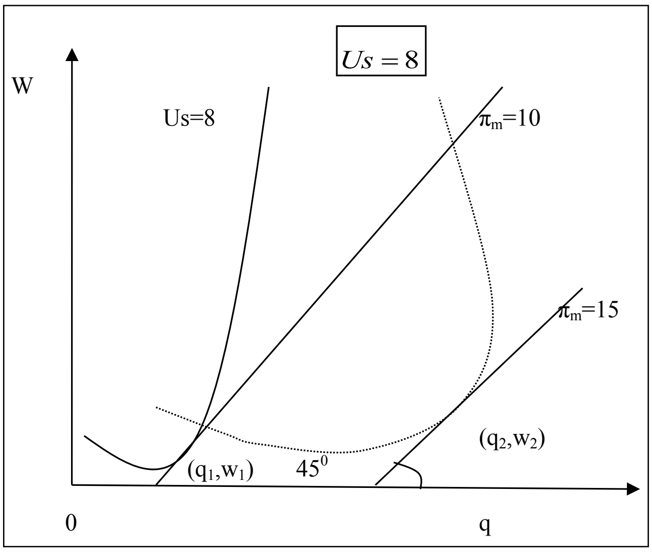


Figure 4.4. Indifference curves and separating equilibrium

Indifference curves for a separating equilibrium

The revelation principle narrows attention to contracts that induce the salesman to tell the truth. In Figure 4.4, an indifference curve induces the salesman to be truthful. The incentive compatibility constraint is satisfied. Suppose the customer is of the windfall type, but the salesman claims to observe a pushover and chooses q1, the salesman's utility is less than 8. It is because the point (q1, w1) lies below the $U = 8$ indifference curve. If the customer is a pushover and the salesman claims to observe a windfall, then although (q2, w2) does yield the salesman a higher wage than (q1, w1) the extra income is not worth the extra effort, because (q2, w2) is far below the indifference curve U = 8.The equilibrium in the salesman game is either pooling or separating and depends on the utility function of the salesman. The revelation principle can be applied to avoid having to consider contracts. In such a contract, the manager must determine if the salesman is lying.

4.7 Efficiency wage hypothesis

The well-known microeconomics model of the efficiency wage was developed by Shapiro and Stiglitz in 1984. In the model, they showed how involuntary unemployment can be explained by a principal agent model. When all workers are employed at the market wage, a worker who is caught shirking and fired can immediately find another job. Therefore, the threat of firing is ineffective in the workplace. The economists Becker and Stigler (1974) have suggested that workers post performance bonds. Suppose the workers are poor, then it is impractical to require them to post bonds. These workers would choose low effort contracts and receive a low wage. This is presented in figure 4.5.

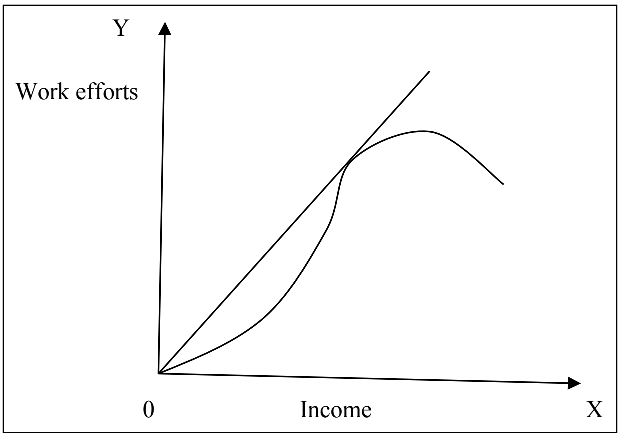


Figure 4.5. Efficiency wage hypothesis

In the figure , the income and work efforts are positively correlated, but to a certain points. Work efforts decline after some time and shirking will take place. To induce a worker not to shirk, the firm can offer to pay him a premium over the market clearing wage, which he loses if he is caught shirking and then fired. If one firm finds it profitable to raise its wages however, so will all firms and one might think that after the wages are equalized, the incentive not to shirk would disappear. But when a firm raises its wages, its demand for labor falls. When all firms raise their wages, the market demand for labor falls. The effect would be unemployment. If all firms pay the same wages, a worker has an incentive not to shirk. This is because if the worker is fired, he would remain unemployed. If there is a random chance of leaving the unemployment pool, the unemployment rate rises sufficiently highly. Therefore, workers choose not to risk being caught shirking.

A firm can choose an option to pay high wages to increase the threat of dismissal. In Shapiro's and Stiglitz's theory, unemployment is generated by these "efficiency wages". The firms behave paradoxically, they pay workers more than necessary to attract them and outsiders who offer to work for less are turned away. It means "overqualified" job seekers are unsuccessful and stupid managers are retained by firms. Trustworthiness matters more than talent in some jobs. Firms are unwilling to hire someone who is talented and intellectual because he could find another job easily. But at this point, too much volatility is observed in the market. In the long run, technology and knowledge are flexible.

4.8 Adverse selection

In a moral hazard with asymmetric information and adverse selectio game, the principal tries to sort out agents with different characteristics. The moral hazard with hidden information is structurally similar to adverse selection. The emphasis is given to the agent's actions rather than to his choice of contract. The agent accepts the contract before acquiring information.

V: Adverse Selection

In this game, the players are the principal and the agent. The information is asymmetric, incomplete and uncertain. The following actions and events occur. Firstly, Nature chooses the agent's ability a, which is unobserved by the principal, according to distribution F(a). Secondly, the principal offers the agent one or more wage contracts w1 (q) or w2 (q). Thirdly, the agent accepts one contract or rejects them all. Fourthly, Nature chooses a value for the state of the world θ, according to distribution G (θ). Output is then q = q (a, θ). If the agent rejects all contracts, then the profit of the agent is written as agent = π and the profit (π) of the principal is zero. Suppose the agent accepts the contract then the profit π for agent is equal to U(w) and π profit for the principal is equal to v(q-w).

Under certainty, the principal would provide a single contract. He simply specifies high wages for high output and low wages for low output. But unlike under moral hazard, either high or low output might be observed in equilibrium. Under adverse selection with uncertainty, multiple contracts may be better than a single contract. The principal might provide a contract with a flat wage to attract the low-ability agents and an incentive contract to attract the high-ability agents.

4.9 Lemon models

George Arthur Akerlof, an American economist who was a professor of economics at the University of California Berkeley won the 2001 Nobel Prize in Economics. He shared the prize with Michael Spence and Joseph E. Stiglitz. George Arthur Akerlof is known for his article "The Market for Lemons: Quality Uncertainty and the Market Mechanism" which was published in the Quarterly Journal of Economics in 1970. In the model of shoddy used cars (lemons), adverse selection arises because the car's quality is better known to the seller than to the buyer. The principal contracts to buy from the agent a car whose quality might be high or low. In the used car market, the seller has private information about his own type before making any kind of agreement. If instead, the seller agrees to resell his car to where he first bought it, the model would be moral hazard with hidden information because there would be no asymmetric information at the time of contracting. The game will have one buyer and one seller. This will simulate competition between buyers because the seller moves first. If the model had symmetric information then there would be no customer surplus. Suppose there are many sellers, then nature randomly assigns a type as a population of sellers of different types. They are drawn by nature to participate in the game.

An example of used cars captures the essence of the problem. From time to time one hears either mention of or surprise at the large price difference between new cars and those which have just left the showroom. The usual lunch table justification for this phenomenon is the pure joy of owning a "new" car. We offer a different explanation. Suppose (for the sake of clarity rather than reality) that there are just four kinds of cars. There are new cars and used cars. There are good cars and bad cars (which in America are known as "lemons"). A new car may be a good car or a lemon, and of course the same is true of used cars (Akerlof, 1970).

Basic Lemon Model

In the lemon model it is assumed that there are two players: a buyer and a seller. The information is asymmetric, incomplete and certain. The buyer is uninformed. The following actions and events occur. Firstly, Nature chooses quality type θ for the seller, according to the distribution F(θ). The seller knows θ, but while the buyer knows F, he does not know the θ of the particular seller he faces. Secondly, the seller offers a price p. Thirdly, the buyer accepts or rejects the offer.

If the buyer rejects the offer, both players receive payoffs of zero. Otherwise,

$$\pi \text{ buyer} = V(\theta) - P \text{ and}$$
$$\pi \text{ seller} = P - U(\theta)$$

Where value (V) and utility (U) will be defined later.

The payoffs of both players are normalized to equal zero if no transaction takes place. The seller gains price (p) if the sale takes place but he loses utility U (θ)) for giving up the car.

I: Identical tastes, two types of sellers

The model assumes that good cars have quality Rs.6 lakh and bad cars (lemons) quality is Rs 2 lakh, so θ∈ {Rs 2 lakh, Rs 6 lakh). Let's say half the cars in the world would be of each type. Assume that both players are risk neutral and they value quality at Rs. hundred per unit, so after a trade, the payoffs are π buyer = θ - p and π seller = p- θ.

If the buyer could observe quality at the time of his purchase, the buyer would be willing to accept a contract to pay Rs 6 lakh for a good car and Rs 2 lakh for a lemon.

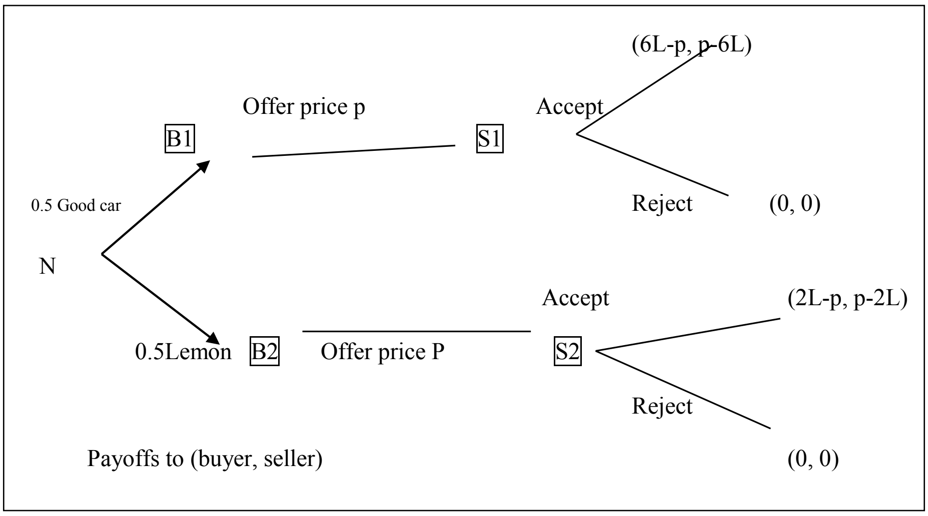


Figure 4.6. Two types of sellers and payoff

If a buyer cannot observe quality, we assume that he cannot enforce a contract based on his discoveries once the purchase is made. Given these restrictions, if the seller offers to sell for Rs 4 lakh, a price equal to the average quality, then the buyer will deduce that the seller does not have a good car. The very fact that the car is for sale demonstrates its low quality. Knowing that for Rs 4 lakh he would be sold only lemons, the buyer would refuse to pay more than Rs 2 lakh.

It is possible to suggest to the owner of a good car that he can wait until all the lemons have been sold, then sell his own car since everyone knows that only good cars have remained unsold. If this were anticipated, the owners of lemons would also hold back and wait for the price to rise. Such a game could be formally analyzed as a war of attrition. The outcome that half the cars are held off the market is interesting, though not startling.

II: Identical tastes: a continuum of types of sellers

In the Lemons II model, the game is generalized by allowing the seller to be any of a continuum of types. We will assume that the quality types are uniformly distributed between Rs 2 lakh and Rs 6 lakh. The value of a car is considered in Indian money only. The average quality is θ = 4 lakh, the price a buyer would be willing to pay for a car of unknown quality, if all cars were on the market. The probability density is zero except on the support [Rs 2 lakh, Rs 6 lakh], where it is $f(\theta)$ = 1/(Rs 6 lakh-Rs 2 lakh).

After substituting the uniform density for $f(\theta)$

$$F(\theta) = \frac{\theta}{Rs.4lakh} - 0.5 \qquad (4)$$

The payoff functions are the same as in the Lemons I model. The equilibrium price must be less than Rs 4 lakh in Lemons II. This is because as in Lemons I, all cars are not put on the market at that price. The owners are willing to sell only if the quality of their cars is less than Rs 4 lakh. Therefore, the average quality of all used cars is Rs 4 lakh. The cars for sale at Rs 3 lakh are of average quality. The price cannot be Rs 4 lakh when the price for average quality cars is Rs 3 lakh. Therefore, the price must drop to at least Rs 3 lakh. If this happens, then owners of cars with values from Rs 3 lakh to Rs 4 lakh pull their cars off the market. The average price of those cars remaining is Rs 2.5 lakh. The acceptable price falls to Rs 2.5 lakh and the effect continues until the price reaches its equilibrium level of Rs 2 lakh. But at a price equal to Rs 2 lakh, the number of cars on the market is small. The market has completely reached the stage of collapse.

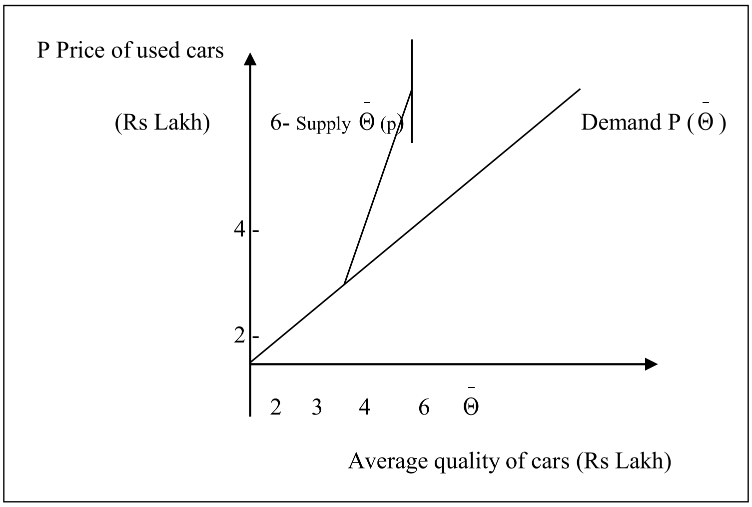


Figure 4.7. Price and quality of used cars

Figure 4.7 shows that each price leads to a different average quality $\bar{\Theta}$ (P) and the slope of $\bar{\Theta}$ (p) is greater than one because the average quality does not rise proportionately with the price. If the price rises, the quality of marginal cars offered for sale equals the new price. The quality of the average car offered for sale is much lower. In equilibrium, the average quality must equal the price so the equilibrium lies on the 45 degree line through the origin as shown in the figure. The line is a demand schedule of sort just as $\bar{\Theta}$ (p) and it is a supply schedule. The only intersection is the point (Rs 2 lakh, Rs 2 lakh).

Heterogeneous Tastes: III and IV

We are separately discussing each point in the following paragraphs.

III: Buyers value cars more than sellers

In this heterogeneous taste model, we assume that sellers value their cars at exactly their quantities. But buyers have valuations 20 percent greater and they outnumber the sellers. The payoffs if a trade occurs is π buyers = 1.20 – p, and π sellers = P - Q.

In equilibrium, the sellers will try to capture the gains from trading.

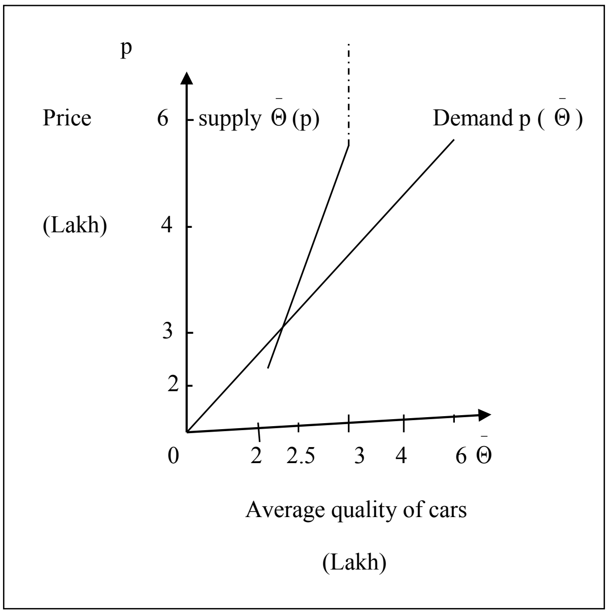


Figure 4.8. Equilibrium of buyers and sellers

The equilibrium condition is no longer that price and average quality lie on the 45^0 line. But it lies on the demand schedule P ($\bar{\Theta}$) which has a slope of 1.2 instead of 1.0. The demand and supply schedules intersect only at [P=Rs.3lakh, $\bar{\Theta}$ (p) =2.5lakh]. This is because buyers are willing to pay a premium; we only see partial adverse selection. The equilibrium is partially pooling. The outcome is inefficient because in a world of perfect information all the cars would be owned by the buyers who value them more. But under adverse selection they only end up owning the low quality cars.

IV: Sellers valuation differ

If we assume that a particular seller decides to trade, then the valuation of one unit of quality is $1+\epsilon$, where the random disturbance ϵ, can be either positive or negative and has an expected value of zero. A disturbance could arise because of the seller realizes he has made a mistake–he did not realize how much he would enjoy driving when he sold the car. He may have travelled using the car regularly but because conditions have changed–he has switched to a job closer to home, and did not need the car anymore. Or for his current job, he may have been transferred to another state, and will be unable to take the car with him. The payoffs if a trade occurs are π buyer = Θ - P and π seller = p - $(1+\epsilon)$ Θ.

If ϵ = -0.15 and Θ = Rs 2 lakh then Rs 1.7 lakh is the lowest price at which player I would sell his car. The average quality of cars offered for sale at price P is the expected quality of cars valued by their owners at less than P, that is,

$$\bar{\Theta}(p) = E(\Theta \mid (1+\varepsilon)\Theta \le P) \qquad (5)$$

Suppose that a large number of new buyers exist in the market. There are more new buyers than there are sellers. Their valuation of one unit of quality is Rs 100 approximately. The demand schedule is a 45^0 line through the origin. One possible shape for the supply schedule is $\bar{\Theta}$ (p). We need to specify the distribution of the disturbances.

If P > Rs 6 lakh then some cars owners would be reluctant to sell their cars. This is because they receive a positive disturbance to their valuations. The average quality of cars on the market is less than Rs 4 lakh even at p = Rs 6 lakh. On the other hand even if P = Rs 2 lakh, some sellers with low quality cars and negative realizations of the disturbance still sell their cars in the market.

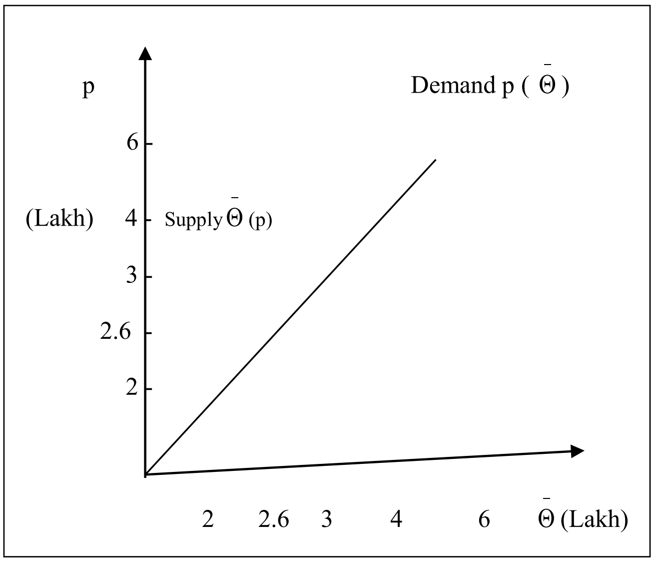


Figure 4.9. Sellers valuation for the used cars

The average quality remains above Rs 2 lakh. The equilibrium is drawn at (pRs 2.6 lakh, $\bar{\Theta}$ = Rs 2.6 lakh). Some used cars are sold but the number is insignificant. Some of the sellers have high quality cars but negative disturbances are responsible for the lower price they get for it. They would like to sell their cars to someone who values them more. Sometimes they expect that the future car owner will keep the car clean and drive carefully. But still they will not sell cars at a price of Rs 2.6 lakh. All four Lemons models could be used when the quality of the cars is not known to the buyer. As a result, there would be fewer cars traded.

4.10 Adverse selection under uncertainty: Insurance game III

The term "Adverse Selection" is similar to the term "Moral Hazard". These terms are often used in the insurance sector. An insurance company pays more if there is an accident so insurance benefits accident-prone customers more than others. A firm's customers are adversely selected to be accident prone. Under moral hazard, assume that someone named Harish chooses to be careful or to be careless. Under adverse selection, Harish cannot affect the probability of a theft, which is chosen by Nature. Harish is either safe or unsafe and while he cannot affect the probability that his car will be stolen, he does know what the probability is of a car getting stolen.

Insurance game III

The players in this game are Harish and two insurance companies. The information is asymmetric, incomplete and uncertain. The insurance companies are uninformed.

The likely actions and events that will take place are as follows. Firstly, Nature chooses Harish to be either safe, with probability 0.6 or unsafe with probability 0.4. Harish knows his type but the insurance companies do not. Secondly, each insurance company offers its own contract (x, y) under which Harish pays premium X unconditionally. He receives compensation y if there is any theft. Thirdly, Harish picks a contract and lastly, Nature chooses whether there is a theft, using probability 0.5 if Harish is safe and 0.75 if he is unsafe.

The likely payoffs in this game are as follows. Harish's payoff depends on his type and the contract (x, y) that he accepts. Let U'> 0 and U"< 0.

$$\pi_H \text{ (safe)} = 0.5 \text{ U } (12 - x) + 0.5 \text{ U } (0 + y - x) \tag{6}$$
$$\pi_H \text{ (unsafe)} = 0.25 \text{ U } (12 - x) + 0.75(0 + y - x) \tag{7}$$

The companies' payoffs depend on what types of customers accept their contracts.

Company payoff types of customers

0	No customers
0.5x + 0.5(x - y)	Just safe
0.25x + 0.75(x - y)	Just unsafe
0.6[0.5x + 0.5(x - y)] +0.4[0.25x + 0.75(x - y)]	Unsafe and safe

Harish is safe with probability of 0.6 and unsafe with probability of 0.4. We have assumed these numbers and probabilities. Without insurance, Harish's rupee wealth is Rs 12 lakh if there is no theft and 0 if there is theft. His endowment in state space is w = (12, 0). If Harish is safe, a theft occurs with probability 0.5, but if he is unsafe, the probability is 0.75. If an insurance company knew that Harish was safe it could offer him insurance at a premium of Rs .6 thousand with a payout of Rs 12 lakh after a theft. leaving Harish with an allocation of (6, 6). This is the most attractive contract that is profitable because it fully insures Harish. Whatever the state, his allocation is Rs 6 lakh.

Insurance Game III: Non-existence of pooling equilibrium

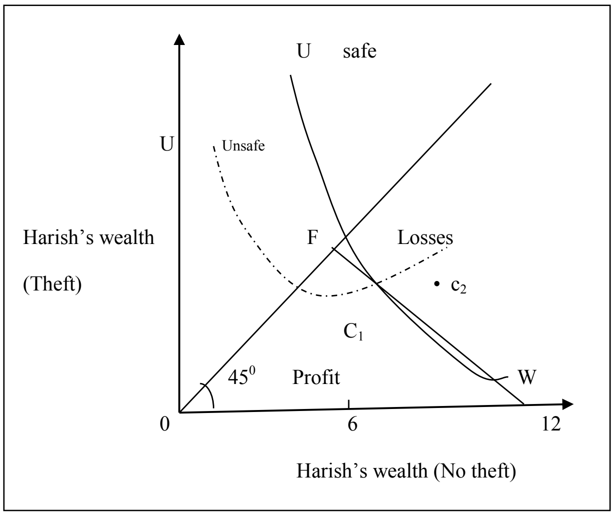


Figure 4.10. Indifference curve for player and insurance company

Figure 4.10 shows the indifference curves of Harish and an insurance company. The insurance company is risk neutral. Its indifference curve is the straight line Wf. Suppose Harish is a customer regardless of his type. The insurance company is indifferent between W and C1, where its expected profits are zero. Harish is risk averse and his indifference curves are closest to the origin along the 45 degree line where his wealth in the two states is equal. He has two sets of indifference curves: solid, if he is safe, and dotted, if he is unsafe.

To make zero profits, the equilibrium must lie on the line WF. It is easy to think about these problems by imagining an entire population like Harish, whom we will call customers. They pick a contract C1 anywhere on WF. We can draw indifference curves for the unsafe and safe customers that pass through C1. Safe customers are always willing to trade theft wealth for no theft wealth at a higher rate than unsafe customers. At any point, the slope of the solid (safe) indifference curve is steeper than that on the dotted (unsafe) curve. We can insert another contract C2 between them and just barely to the right of wF. The safe customers prefer contract C2 to C1. But the unsafe customers stay with C1 so C2 is profitable, as C2 attract the more safe or cautious customers. Our argument holds for any pooling contract. No pooling equilibrium exists.

We next consider whether a separating equilibrium exists, in figure 4.11. The zero profit condition requires that the safe customers take contracts on WC4 and the unsafe on WC3.

Insurance Game III: A separating equilibrium

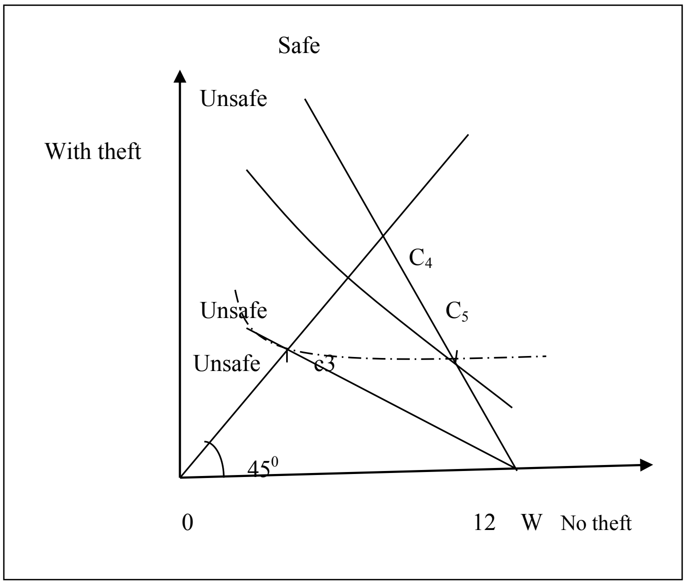


Figure 4.11. Indifference curve with separating equilibrium

The unsafe will be completely insured in any equilibrium, although at a high price. On the zero profit line WC3, the contract they like best is C3. The safe customers would prefer contract C4 but C4 uniformly dominates C3. It would also attract the unsafe customers and generate losses. The assumption on which the equilibrium is based is that the proportion of safes to unsafes is 0.6. The zero profit line is for pooling contracts WF and C6 would be unprofitable.

It is assumed that the proportion of safe customers to unsafe ones is higher. The zero-profit line for pooling contracts would be WF' and C6 lying to its left is profitable. Since neither separating pair likes (C3, C5) nor a pooling contract like C6 is equilibrium, no equilibrium whatsoever exists.

Insurance game III: No equilibrium exists

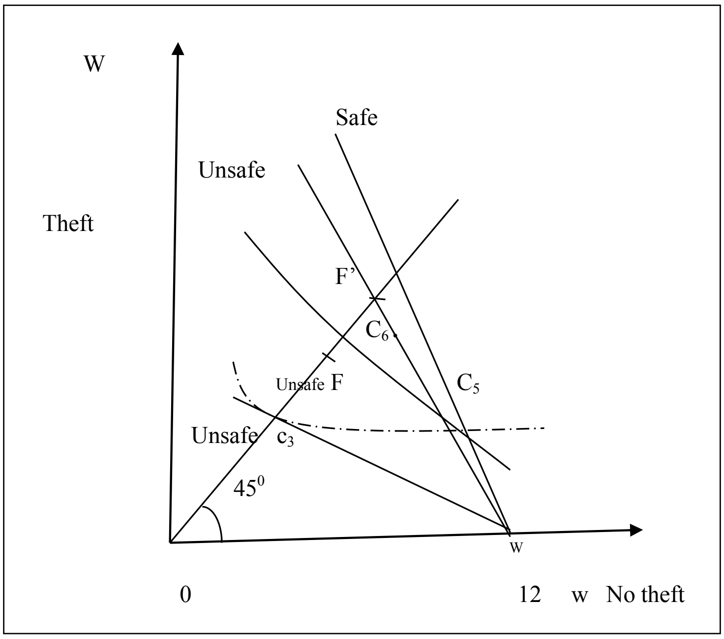


Figure 4.12. Indifference curve without equilibrium

If separating contracts are offered, the company is willing to offer a superior pooling contract. But if a pooling contract is offered, the company is willing to offer separating contracts. A monopoly would have a pure strategy equilibrium, but in a competitive market only a mixed strategy Nash equilibrium exists.

4.11 Signalling

In advanced microeconomics, game theory helps players to maximize their utility. Signalling is a way for an agent to communicate his type under adverse selection. The signalling contract specifies a wage that depends on an observable characteristic. The agent chooses the signal for himself after Nature chooses his type. If the agent chooses his signal before the contract is offered, then he is signalling to the principal. If he chooses the signal afterwards, the principal is screening him. Inducing truthful communication then requires a form of team incentives (Friebel & Raith, 2010).

A signalling game is an extensive form of game between two persons: the sender and the receiver (Jager, 2008). Signalling games refer narrowly to a class of two-player games of incomplete information in which one player is informed and the other is not. The informed player's strategy set consists of signals contingent on information and the uninformed player's strategy set consists of actions contingent on signals. More generally, a signalling game includes any strategic setting in which players can use the actions of their opponents to make inferences about hidden information (Sobel, 2007).

A signalling game is a two-stage game with incomplete information on one side where the informed party (Player 1, or Sender) chooses a "message" m from some set M and the uninformed party (Player 2, or Receiver) responds with an action a from some set A. Here it is assumed that without a substantial loss of generality, that the set of feasible messages of the Sender does not depend on his private information and that the set of feasible responses for the Receiver does not depend on the message sent by the Sender (Battigalli, 2004).

The informal players move first: Signalling

This concept was introduced by Spence in 1973. He introduced the idea of signalling in the context of education. The series of models are constructed which formalize the notion that education is useless to increase a worker's ability. But it is useful to demonstrate that ability to employers. Let half of the workers be the type "high ability" and half "low ability"where the ability of a worker is a number denoting the rupee value of a worker's output. The output is assumed to be a non-contractible variable.

Employers cannot observe their workers' ability. They do know the distribution of abilities and they know the workers' education levels. Workers choose their education levels before employers choose compensation schemes to attract them. The employer's strategies are the sets of contracts they offer with wages as functions of the worker's education level.

Education I

In this game, the players are a worker and two employers. The information is asymmetric, incomplete and certain. The likely actions and events are as follows. Firstly, Nature chooses the worker's ability $a\epsilon\{2,5.5\}$. These are the low and the high ability workers categories, each having probability of 0.5. The variable a is observed by the worker, but not by the employers. Secondly, the worker chooses education level $y\epsilon\ \{0,1\}$. Thirdly, the employers each offer a wage contract w(y). Fourthly, the worker accepts a contract or rejects.

The worker‘s payoff is his wage minus his cost of education and the employer's pay off is his profit.

$$\pi \text{ for worker} = \begin{cases} w - 8\,y/a & \text{if the worker accepts contract } w \\ 0 & \text{if the worker rejects both contracts} \end{cases}$$

$$\pi \text{ for worker} = \begin{cases} a - w & \text{For the employer whose contract is acceptable} \\ 0 & \text{For the other employer} \end{cases}$$

Education is costly if the worker's ability is low. As in any hidden information game, we must think about both pooling and separating equilibriums. Education I has both pooling and separating equilibriums. We will call the pooling equilibrium PE 1.1, at which both types of workers pick zero education and the employers pay the zero-profit wage that is 3.75 = (2 + 5.5)/2, regardless of the education level of workers. The equilibrium needs to specify the employers' belief when he observes y = 1. In PE 1.1, the beliefs are "passive conjectures". The employer believes that a worker who chooses y – 1 is a low-ability worker with a probability of 0.5. Given this belief, both types of workers realize that education is useless. The model reaches the unsurprising outcome that workers do not bother to acquire more unproductive education.

$$\begin{array}{l} \text{Polling equilibrium} \\ \text{(PE.1.1)} \end{array} \left\{ \begin{array}{l} Y\ (\text{low}) = y\ (\text{High}) = 0 \\ w\ (0) = w\ (1) = 3.75 \\ \text{prob}\ (a = \text{low}/y = 1) = 0.5 \end{array} \right\}$$

Under the belief probability of (a = Low/y = 1) = 0 for example, the employer believes that any worker who has acquired an education has a high ability, so the pooling is not a Nash equilibrium. The high-ability workers are tempted to deviate and acquire more education. This leads to the separating equilibrium of which signalling is best known. The high-ability worker acquires more education to prove to the employers that he really is a high-ability worker.

$$\begin{array}{ll} \text{Separating Equilibrium} & \left\{ \begin{array}{l} \text{Y(Low) = 0, y(High) = 1} \\ \text{w(0) = 2,W(1) = 5.5} \end{array} \right\} \\ \text{(SE. 1.2)} & \end{array}$$

A pair of separating contracts must maximize the utility of the highs and lows subject to the participation constraints that firms can offer the contracts without incurring losses. The incentive compatibility constraints show that the lows are not attracted to the high contract. The non-pooling constraints show that the highs would prefer not to pool. The participation constraints are

$$W\ (0) \leq a_1 = 2 \text{ and } w\ (1) \leq aH = 5.5 \tag{8}$$

The incentive compatibility constraints shows that

$$U1(y = 0) \geq U1\ (y\ 1) \tag{9}$$

Since in SE 1.2, the separating wage of the lows is 8, the separating wage of the highs is 5.5 from (9). The incentive compatibility constraint is satisfied at this point.

For education I, the wage must equal 3.75 in a zero profit pooling contract,

$$W(1) - 8/5.5 \geq 3.75 - 0 = 3.75 \tag{10}$$

Constraint (10) is satisfied by SE 1.2.

Given the worker's strategy and the other employer's strategy, each employer must pay the worker his full output or lose him to the other employers. Given the employer's contracts, the low-ability worker has a choice between the payoff (2 = 2 - 0) for ignorance and 1. 5 = (5.5 - 8/2) for education, so he picks ignorance. The high-ability worker has a choice between the payoff 2 = (2- 0) for ignorance and it is 4.05 = (5.5-8/5.5) for education. Therefore, the high-ability worker picks education.

Now we can use Bayes' rule to interpret how employers see the problem. Education means the agent is a high-ability worker and lack of education means he is a low-ability worker. This does not mean that a worker cannot deviate but deviation will not change the employer's belief. If a high-ability worker deviates by choosing y = 0 and tells the employer he has a high ability who would rather pool than separate. The employer does not believe him and offers him the low wage of 2. He does not get offered the pooling wage of 3.75 or the high wage of 5.5.

The low-ability worker could never benefit from deviating from PE 1.1. Under the passive conjectures specified, the low-ability worker gets a payoff of 3.75 in equilibrium versus - 0.25 = (3.75-8/2) if he deviates and becomes educated. Under the most favorable belief possible, a worker who deviates is high with probability of 1, the low-ability worker would get a wage of 5.5. Suppose the worker deviated, then his payoff from deviating would be 1.5 = (5.5 - 8/2). A worker who acquires more education has high ability and does not support the pooling equilibrium.

Education II: Modeling trembles so nothing is out of equilibrium

Suppose nature's move in Education I is replaced then the following actions and events can occur. Nature chooses a worker's ability to be aϵ{2, 5.5}. Each worker has ability with probability 0.5. Nature then observes a worker with free education. The payoff is as follows

$$\pi \text{ worker} = \begin{cases} W,\ 8y/a & \text{if the worker accepts contract w (ordinarily)} \\ w & \text{if the worker accepts contract w (with free education)} \\ 0 & \text{if the worker does not accept a contract} \end{cases}$$

With probability 0.001 the worker receives free education, regardless of the worker's ability. If the employer sees a worker with education, he knows that the worker might be one of those rare types, in which case the probability that the worker with low ability is 0.5. Both y = 0 and y = 1 can be observed in any equilibrium.

The separating equilibrium does not depend on beliefs and remains at equilibrium. The pooling equilibrium explains that all workers behave in the same way. But the small number with free education may behave differently. The two types that are of greatest interest are those high- and low-ability workers separated from the workers whose education is free. The small amount of separation allows the employers to use Bayes' rule and eliminates the need for exogenous beliefs.

Education III: No separating equilibrium: Two pooling equilibrium

Now we modify the education I game by changing the possible worker abilities from {2, 5.5} to {2, 12}. The separating equilibrium vanishes but a new pooling equilibrium emerges. In PE 3.1 and 3.2 both pooling contracts pay the same zero-profit wage of 7 = (2 + 12/2). Both types of agents acquire the same amount of education, but the amount depends on the equilibrium.

Pooling equilibrium (PE 3.1)
$$\left\{ \begin{array}{l} y\text{ (Low)} = y\text{ (High)} = 0 \\ W\text{ (O)} = W\text{ (1)} = 7 \end{array} \right\}$$

Prob (a = Low/y = 1)0.5(passive conjectures)

Pooling equilibrium (PE 3.2)
$$\left\{ \begin{array}{l} Y\text{ (Low)} = y\text{ (High)} = 1 \\ W\text{ (O)} = 2\ W\text{ (1)} = 7 \\ \text{Prob (a = Low/y = 0)1} \end{array} \right\}$$

The figure shows that both types of workers receive the same wage but they incur the education costs anyway. Each type is afraid to do without education. This is because the employers would pay him not the average pooling wage but the wage appropriate to the low-ability worker, even though the two types of workers adopt the same strategies. The equilibrium contract offers different wages for different education. The implied threat to pay a low wage to an uneducated worker never needs to be carried out, so the equilibrium is still pooling. The perfectness does not rule out threats based on beliefs. The model imposes these beliefs on the employer and he would carry out his threats because he believes they are the best responses. The employer receives a higher payoff under the same beliefs than under others. But he is not free to choose his beliefs. We end up with PE 3.1 because the only rational belief that if y = 0 is observed, the worker has an equal probability of being a high-ability or a low-ability worker. To eliminate PE 3.1 requires less reasonable beliefs and a probability 0.001 that a low-ability gets free education together with the probability zero that a high-ability worker does.

4.12 Screening

The theory of screening is unique in microeconomics. It is also known as the theory of adverse selection or discrimination, and represents a major accomplishment of the economics of information in the last two decades. This theory is often cast in a framework with two parties, a principal and an agent. The principal offers a contract, which the agent decides to accept or reject. The agent has private information about some parameter of his utility function. This parameter determines his "type." The parameter affects the principal's payoff, at least indirectly, since the agent's type establishes the class of contracts that he will accept. The literature has developed this model both in the abstract and as applied to a variety of interesting economic problems, e.g., labor contracts, optimal taxation, price and quality discrimination, insurance contracts, educational screening, auctions, public goods, and regulation of monopoly. An important hypothesis of the model is that the principal is not informed, which means he does not possess private information when contracting. Thus, the asymmetry of information is one-sided. One can think of many circumstances, however, where such an assumption is too restrictive. For example, in the literature on public good mechanisms, the informational deficiency usually emphasized is the government's (principal's) lack of knowledge of consumers' (agents') preferences. But at the time the government institutes a mechanism for eliciting those preferences, it may well know more than the consumers themselves know about the cost of supplying certain goods (Maskin & Tirole, 1990). In the previous section, we have discussed that if the agent chooses his signal before the contract is offered, he is signalling to the principal. If he chooses the signal afterwards, the principal is screening him.

Education IV: Screening with discrete signal players

In this game, a worker and the two employers are the players. The information is asymmetric, incomplete and uncertain. The actions and events that are likely are as follows. Firstly, Nature chooses for each worker, ability a ϵ {2, 5.5}, with probability 0.5. Employers do not observe this ability, but the worker does. Secondly, each employer offers a wage contract w(y). Thirdly, the worker chooses the education level y ϵ {0,1}. Fourthly, the worker accepts the contract. Lastly, the output equals a.

The payoffs are explained as follows.

$$\pi \text{ worker} = \begin{cases} w - 8\,y/a & \text{if the worker accepts contract } w \\ 0 & \text{if the worker rejects both contracts} \end{cases}$$

$$\pi \text{ employer} = \begin{cases} a - w & \text{for the employer whose contract is accepted} \\ 0 & \text{for the other employer} \end{cases}$$

Education IV has no Nash pooling equilibrium. This is because if one employer tried to offer the zero profit pooling contract w (0) = 3.75, the other employer would offer w (1) = 5.5 and draw away all the high-ability workers. The unique equilibrium is

$$\text{Separating equilibrium} \quad \left\{ \begin{array}{l} y\,(\text{low}) = 0,\ y\,(\text{high}) = 1 \\ \\ \end{array} \right\}$$

(SE 4.1) w (0) = 2, w (1) =5.5

The uninformed player moves first in the above game. His beliefs after seeing the move of the informed player are irrelevant. The informed player is fully informed. His beliefs are not affected by what he observes. This is very much like the simple adverse selection in which the uninformed player moves first, offering a set of contracts, after which the informed player chooses one of them.

Education V: Screening with continuous signal players 1

The players in this game are a worker and two employers. Information is asymmetric, incomplete and uncertain. The actions and events are as follows.

Firstly, Nature chooses to give each worker, ability aϵ {2, 5.5}, with probability 0.5. The employers do not observe this ability but the worker does. Secondly, each employer offers a wage contract w(y). Thirdly, the worker chooses the education level y ϵ (0, 1). Fourthly, the worker chooses a contract or rejects both of them. Lastly, the output equals a. The payoff is as follows

$$\pi \text{ worker} = \begin{cases} w - 8y/a & \text{if the worker accept contract} \\ 0 & \text{if the worker rejects both contracts} \end{cases}$$

$$\pi \text{ employer} = \begin{cases} a - w & \text{for the employer whose contract is accepted} \\ 0 & \text{for the other employer} \end{cases}$$

A pooling equilibrium generally does not exist in screening games with continuous signals and separating equilibriums are also sometimes lacking.

Education V, however, has a separating Nash equilibrium with a unique equilibrium path.

$$\begin{array}{ll} \text{Separating equilibrium} & \\ \text{(SE. 5.1)} & \left\{ \begin{array}{c} y \text{ (low)} = 0 \;\; y \text{ (high)} = 0.875 \\ w = 2 \text{ if } y \leq 0.875 \\ 5.5 \text{ if } y \geq 0.875 \end{array} \right\} \end{array}$$

In any separating contract, the low-ability workers must be paid a wage of 2 for an education at 0. The separating contract for the high-ability workers must maximize their utility subject to the constraints discussed in Education I.

No pooling equilibrium in Education V:

Education V's pooling equilibrium would require the outcome {y = 0, w (0) = 3.75} which is shown as C1 in figure 4.13. If one employer offered a pooling contract requiring more than zero education (such as in PE 3.2), the other employer could make the more attractive offer of the same wage for zero education. The wage is 3.75 to ensure zero profits.

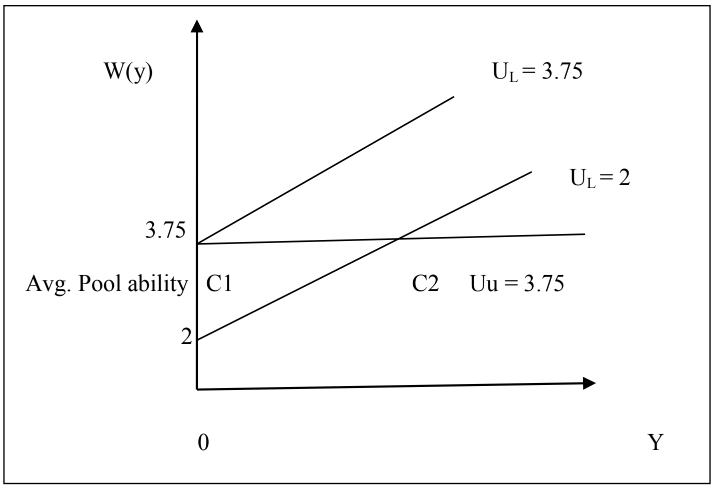


Figure 4.13. Pooling equilibrium with education

The wage function is equal to the wages for positive education levels. It can take a variety of shapes so long as the wage does not rise as fast as with education. The high-ability workers are tempted to become more educated.

In a Nash equilibrium, no exchange can offer a pooling contract. This is because the other employer could always profit by offering a separating contract paying more to educate one. Such separating contract is C2 in the figure which pays 4.9 to workers with an education of y = 0.5 and yields a payoff of 4.17 = [4.9-(8*0.3*5.5] to the high-ability workers and 2.9 = 4.9-8*0.5/2 to the high-ability workers and 2.9 = 4.9-8-0.5/2 to the low-ability workers. Only the high-ability workers will prefer C2 to the pooling contract C1. This yields payoffs of 3.75 to both highs and lows. Only the highs accept C2, which yields positive profits to the employer (Rasmusen, 2007). This model is used in different sectors of the economy. It is the employers who decide to use the information of workers and pool or separate them based on the available information.

Questions

1. Explain the asymmetric information model and give an example.
2. Critically examine the principal-agent model along with the different production games.
3. Explain the pooling and separating equilibrium along with the salesman game in detail.
4. Write a note on the efficiency wage hypothesis.
5. Discuss the Lemon theory along with its different types of sellers and buyers.
6. What is adverse selection under uncertainty with separating and pooling equilibrium?
7. Explain the signalling along with education.
8. Write a brief note on screening.

5 General equilibrium and welfare economics

5.1 Introduction

The Walrasian model is an important model of classical economics. The model assumes that there are households and firms which are exchanging goods. This exchange of goods depends on the bargaining of the two individuals. Thus, households supply services to firms and the firms supply goods to the households resulting in a circular flow in the economy. The prices of commodities are determined on the basis of the market mechanism. There is no scope for individuals to bargain for price. There is a general equilibrium in the competitive economy. General equilibrium analysis seeks to determine equilibrium prices simultaneously in all the markets (Sen, 1998). But markets do not work efficiently due to monopoly. In some cases, a monopoly in a market leads to inefficiency in resource allocation and mobilization. Therefore, the government plays an important role in ensuring market efficiency. Externality exists in the production function of a firm. Negative externalities reduce the production of the firm. Again the role of government is important to correct the negative externality and to improve the allocation of resources.

5.2 The Walrasian equilibrium of a competitive economy

The basic assumption of the Walrasian equilibrium is that the economy is an exchange economy. where goods are exchanged for other goods. It is further assumed that there are a number of goods produced by firms and consumed by households. The economy consists of households and firms. Firms produce n goods and households provide services to firms. Let's assume that $\hat{X}_{hj}$ is the net demand for the j^{th} good of the h^{th} household, where H = 1,2,….H. Similarly y_{ij} is the net supply of the good by the i^{th} firm where I = 1,2,…M. The supply of goods by firms and demand of goods by households is measured and they are negatively correlated. A household's strictly quasiconcave utility function is presented as $u_h(\hat{X}^h)$ where $\hat{X}^h = (\hat{X}_{h1}, \hat{X}_{h2}, \ldots\ldots \hat{X}_{hn})$. Each household maximizes its utility subject to its budget constraints. Each household has scarce resources and this affects the household budget. A household's budget constraint is presented as

$$\sum_{j=1}^{n} P_J \hat{X}_{hj} \le W_h \quad \text{h = 1, 2….H} \tag{1}$$

where W = household wealth which is measured in terms of rupees.

Each household owns wealth in the form of physical and financial assets. Financial assets consist of shares, bonds, debentures, etc. Firms provide financial investment opportunities to households. Households invest in a given vector of shares in firms. This is denoted by $\beta_h = (\beta_{h1}, \ldots\ldots \beta_{hm})$ with $1 \geq \beta_{hi} \geq 0$ all h, i. We assume that $\sum_h \beta_{hi} = 1$. A firm's profits are positively correlated to the households' wealth. Firms provide dividends on the shares owned by the households.

$$W_h = \sum_{i=1}^{m} \beta_{hi} \pi_i$$

where π_i is the profit of the i[th] firm which has earned it out of production. A household's financial wealth consists of the sum of its shares in the firm and dividends paid by the firm.

Firms wish to maximize profits from output. This output is multiplied by the price level in the market. The sale of the output in the market provides profits to the firm, and can be defined as

$$\pi_i = \sum_{j=1}^{n} P_i y_{ij} \qquad i = 1, 2 \ldots M \qquad (2)$$

But the profits of a firm are subjected to the production possibilities frontier. A firm produces output after using certain inputs. Such output satisfies the following condition

$$F_i(y_i) \geq 0 \qquad i = 1,2 \ldots\ldots\ldots\ldots\ldots\ldots M \qquad (3)$$

where $y_i = (y_{i1}, y_{i2}, y_{in})$. The output of a firm is always greater than zero. The firm's production function (f_i) exhibits strictly diminishing returns. The diminishing returns of a firm is a long term phenomena. Each firm's production set is strictly convex.

A firm produces output which is demanded by households. The household's net demand function is presented as

$$\hat{X}_{hj} = D_{hj}(p_{1,} p_2, \ldots\ldots p_n) \qquad h = 1, 2 \ldots\ldots H \qquad j = 1, 2 \ldots\ldots n \qquad (4)$$

And each firm's net supply function can be represented as

$$Y_{ij} = s_{ij}\ (p_1, p2, \ldots\ldots\ldots p_n)\ i = 1, 2 \ldots M \qquad j = 1, 2, \ldots\ldots n \qquad (5)$$

The income to firms is a function of the supply of a number of commodities to the household. The goods consist of p_1 to p_n.

Properties

a) The price vector is given for a market and the market mechanism decides the unique demand and supply for a particular good.
b) Prices do not change in the short term. The net demand and supply varies continuously with prices.
c) Suppose all prices change along with demand and supply of commodities then the net demand or supply is unchanged.

Now we will consider commodity j. The demand and availability of supply of this commodity shows the deficit or excess stock of the commodity. This can be presented in equation form as

$$Z_j = \sum_{j=1}^{N} \hat{X}_{hj} - \sum_{i=1}^{M} y_{ij} \qquad j = 1, 2,\ldots\ldots n \qquad (6)$$

We refer to Z_j as the excess demand or deficit for good j, which simply represents the difference between net demand and supply of commodity, and is inversely related. When there is less supply X_{hi}, Z_j will be higher. The excess demand or supply of any commodity is written in the equation.

$$Z_j = Z_j(P_1,P_2, \ldots\ldots\ldots\ldots\ldots P_n) \qquad j = 1,2,\ldots\ldots\ldots n \qquad (7)$$

The demand and supply of any commodity may be positive or negative. These demand and supply curves can be presented in figure 5.1. The demand and supply of a commodity shows the equilibrium price level.

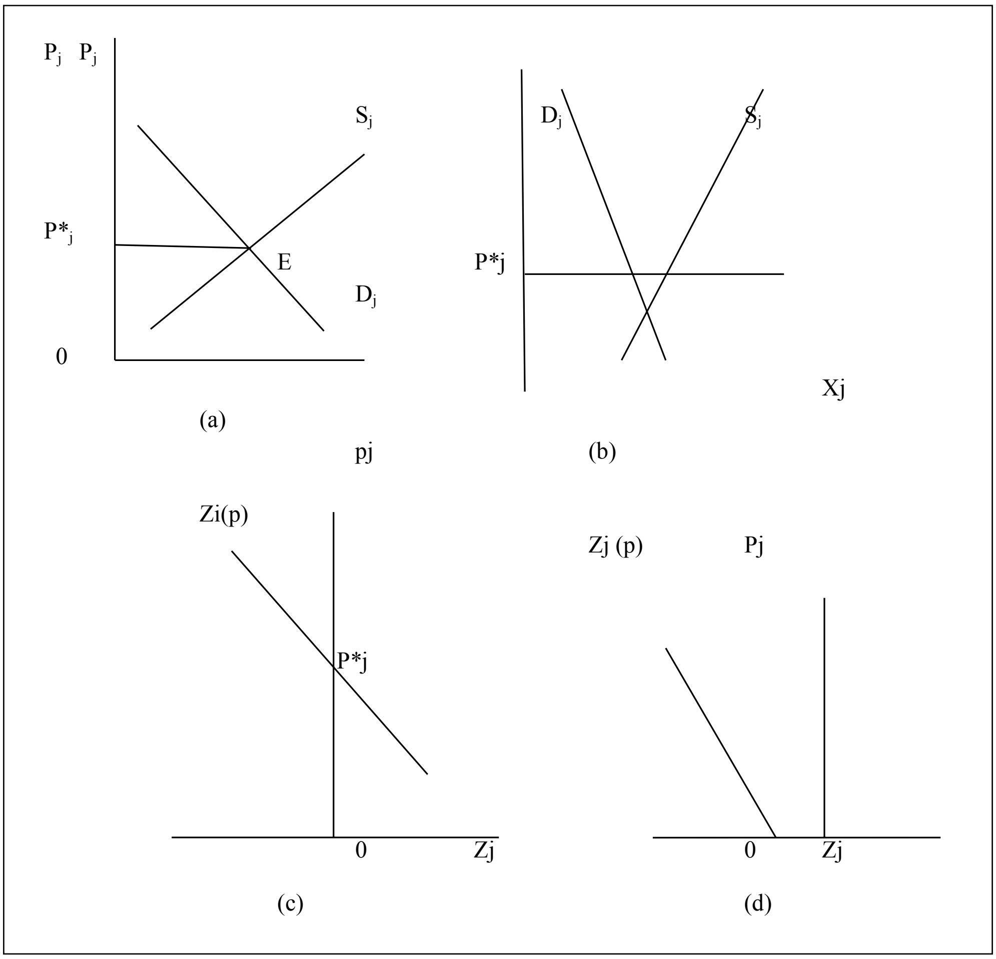


Figure 5.1. Market demand and supply curves

Figure 5.1(a) shows the upward sloping supply curve Si and downward sloping demand curve Di. They intersect at point E, which is the equilibrium of demand and supply. This is the normal market clearing equilibrium of demand and supply. The equilibrium price of commodity j^{th} is shown on the y axis. On the x axis, the quantity demanded is shown. At p, there is no change in price and the equilibrium of demand and supply is achieved. This equilibrium is observed for normal goods. For scarce commodities, prices rise continuously.

Figure 5.1(b) shows the negative demand and supply. The price of the commodity is shown as Pj which is below zero. At this point, both demand and supply should increase, and help prices to rise.

Figure 5.1(c) shows that demand exceeds supply. The price is shown at $P^*_{j,}$.

Figure 5.1 (d) shows the negative demand. No price exists for the commodity. Suppose the commodity is supplied then it will have an automatic impact on the price. An automatic adjustment results in both demand and price.

The purpose of supplying more goods is to keep the price at a certain level. But at the same time, it does not make sense to supply the commodities at a lower price. The relative prices may not remain the same in the long term. When prices start to rise people will buy more substitutes, helping to reduce the prices of commodities in the long term. Therefore, price is the best parameter to bring any market into equilibrium.

For the economy as a whole, the general equilibrium price vector is defined as (p*1, p*2 ….p*n). The vector of excess demands is represented as (z*1, z*2……z*n), with the following properties:

Properties:

1. Any household's net demand for any commodity is $\hat{X}^{*}_{hj}$ .Such demand corresponds to Zj*. This is because

   $$u_h(\hat{X}^{h*}) \geq u_h(\hat{X}^{h}) \text{ all h} \tag{8}$$

   For all $\hat{X}^{h}$ satisfying

   $$\sum_j p_j^* X_{hj} \hat{\leq} W_h$$

   The price of commodities and net demand of commodities is equivalent to the wealth of the household. A household cannot demand goods if that household's wealth is low.

2. Each firm supplies goods in the market. The individual firm's demand and supply corresponds to $Z^{\propto}_{j.}$ This supply is denoted as y_{ij}. The supply of goods must satisfy the criteria of a minimum price; otherwise, the firm will not supply goods in the market. This is shown as

   $$\sum_j p_j^* y_{ij}^* \geq \sum_j p_j^* y_{ij} \text{ all i} \tag{9}$$

Y_{ij} is the standard output of a firm. But firms always produce close to the standard output. Therefore, the actual supply of the output by a firm is y_{ij}. An actual output maximizes a firm's profits subject to the price level p*j in the market. Here we are not considering the negative prices of commodities. In a market, the price is an equilibrium price and it does not change in the short term. We can write the equilibrium price as

$$p_j^* \geq 0 \text{ for all j} \tag{10}$$

In order to determine the general equilibrium, the price level must be in equilibrium in every market. This equilibrium price determines the equilibrium demand and supply. We can write the demand vector as follows

$$z_j^* = 0 \text{ if } p_j^* > 0$$

$$z_j^* \leq 0 \text{ if } p_j^* = 0 \qquad j = 1, 2 \ldots n \qquad (11)$$

We can modify the above two equations as

$$z_j^* \leq 0, p_j^* \geq 0, p_j^* z_j^* = 0 \qquad j = 1, 2 \ldots n \qquad (12)$$

Equations 8, 9, 10, and 12 show a general equilibrium. The prices of different commodities are non-negative and they consist of various sets. These sets constitute the demand of households and the supply of commodities by firms. The demand and supply set are optimal bundles and they correspond to the prices at which the firm supplying goods are willing to sell them for and at which prices the households demanding such goods are willing to pay for. Therefore, no positive nor negative demand for commodities exists.

Walrasian equilibrium

In any economy, we assume that there is excess demand for each commodity. This excess demand function of the Walrasian equilibrium is expressed as

$$Z_j(z_j(p_1,p_2,\ldots\ldots p_n) \qquad j = 1,2,\ldots\ldots n \tag{13}$$

The demand for a particular commodity is continuous, and changes in equal proportion to changes in the price level. Equation (13) shows mapping from a set of price vectors into the set of excess demand vectors. In equation (13) mapping for strictly positive prices is continuous and at the same time it is homogenous of degree zero. If we consider the set of prices corresponding with the demand and supply of commodities then they are the set of vector prices (p_1, p_2, …..p_n). These set of vector prices are bounded below the condition that $p_j \geq 0$. But this same set of vector prices are not bounded above the condition. This can only be possible if we apply the fixed point theorem. At this point, we can only apply the normalization rule at any price vector. It is price vector (p_1, p_2, p_3) in p. For this rule, there is a need for a new price vector and it is further written as $(p_1^1, p_2^1 \ldots\ldots\ldots\ldots p_n^1)$. By the normalization rule the price vector can be written as

$$p_j^1 = p_j \frac{1}{\sum p_j} = p_j \frac{1}{p_e} \qquad (j = 1, 2\ldots\ldots n) \tag{14}$$

In the equation, e = (I, 1……1) and $\sum p_j = p_e$. This means that the cost of a bundle of goods to a household is such that such bundle of goods consists of one unit of each commodity. The bundle can be written as 1/pe, and can be bought for Rs 10. This is the assumed price for a household because it buys this number of bundles of commodities. The normalization rule is useful in determining the set of normalization price vector p'. In equation (14) , the rule is bounded, closed and convex. The price vector P' is bounded and it is $p_j \geq 0$ for all j. All the price vectors are p' positive and they are multiplied by ($1/p_e$) of some p in p and there exist p'j'≥ 0 for all j. We can now show that p' is bounded above the equation, and can be represented as

$$p'_e = \sum_j p'_j = \sum_j p_j \frac{1}{p_e} = 1 \tag{15}$$

We know that the normalization rule for normalized prices is not negative. This means that $p_j^1 \leq 1$ for all j. We can establish the normal price level as

$$o \leq p_j^1 \leq 1 \quad (j = 1, 2) \tag{16}$$

In figure 5.2, the normalization procedure is illustrated for n = 2. It shows that the positive quadrant is the set p corresponding to all pairs of non-negative price vectors (p_1, p_2). The line ab joins the price vectors (0, 1) and (1, 0). This price vector is the locus of price vectors satisfying the different conditions for n = 2.

We can express this as an equation

$P_2 = 1 - p_1$

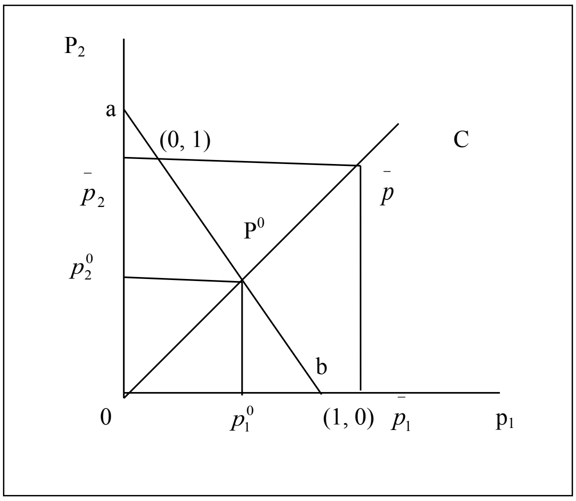


Figure 5.2 .Price and demand vector

Walras's law

The desired net demand vector of a household is represented by $\hat{X}^h$. Yi designates the planned net supply of a firm. The household's net demand $\hat{X}^h$ must satisfy the budget constraints.

$$\sum_h \sum_j p_j \hat{X}_{hj} = \sum_j pj \sum_h \hat{X}_{hj} = \sum_h w_h \tag{17}$$

Equation (17) explains that a household's demand for goods is the total demand for j[th] commodities. This is equivalent to a household's wealth and equivalent to the profits earned by households from the dividends paid out by firms. Therefore,

$$\sum_h w_h = \sum_h \sum_i \beta_{hi} \Pi_i = \sum_h \Pi_i \tag{18}$$

This means that the total wealth of households is equivalent to the profits earned by firms. We assume that $\sum_h \beta_{hi} = 1$. Substituting equation (17) into equation (18)

$$\sum_j p_j \sum_h \hat{X}_{hj} - \sum_j p_j \sum_i yij = \sum_j pj \left[\sum_h \hat{X}_{hj} - \sum_i y_{ij} \right] = \sum_j p_j z_j = 0 \qquad 19)$$

Equation (19) shows that the prices of goods and net demand of households are subtracted from the prices and the output supplied by firms, and is equivalent to the differences between net demand and supply and the prices of the commodities. This is further equivalent to the price vector and excess demand for commodity j. Alternatively, at any price vector, the value of the excess demand is exactly zero. Therefore, it is rendered zero in the above equation. This is known as Walras's law.

General equilibrium proof

We can explain the excess demand function when the continuity and zero degree homogeneity are given. This excess demand function is Zj = Zj(p) and p ϵ p'. The price vector p*ϵ p' exists. We can express it as.

$$z_j^* = z_j(p^*) \le 0 \quad p_j^* \ge 0 \quad p_j^* z_j^* = 0 \quad \text{for all j} \tag{20}$$

There is excess demand which corresponds to utility maximizing choices by consumers, and also corresponds to the profit maximizing choices of firms. We can prove the proposition for n goods which are demanded and supplied in the economy as follows.

1. In an economy, the excess demand is defined as a continuous function. The mapping from the set of normalized price vector p' is the set of excess demand vector.

$$z = \left\{(z_1 z_2 \ldots\ldots, z_n) \middle| z_j = z_j(p), p \varepsilon p' \text{ j} = 1, 2\ldots\text{n}\right\} \tag{21}$$

Equation (21) explains that excess demand is a function of the price level. We can do the mapping of the prices and the net demand.

$$Z: p' \longrightarrow Z$$

We need to define the second continuous mapping. This is the set where excess demand backs into the set p'. If we take the two compositions of mapping then a continuous mapping of the closed bound convex set is at p'. We can do the second mapping which ensures that a fixed point is an equilibrium price vector. The mapping can be defined by

$$P_j = \frac{\max[o, p_j^{'} + k_j z_j(p^{'})]}{\sum_j \max[o, p_j^{'} + k_j z_j(p^{'})]} \text{ for all } p' \epsilon\ p' \ k_j > 0 \text{ fr } j = 1\ldots.n \tag{22}$$

Suppose we consider the initial price vector p ∈ p' and it explains that each corresponding excess demand is $Z_j(p)$. The associated price is pj. A new pj is defined in three ways.

Firstly, we assume that excess demand $Z_j(p')$ is positive, and adds to the initial price p_j. Secondly, suppose the excess demand for commodity k_j (p') is zero then the new price observed for the commodity is exactly equal to the old price. Sometimes the commodity price does not change. It retains its price, period after period or year after year. Commodities such as biscuits, salt, and razor blades have constant prices over a period of time.

Thirdly, suppose the old price persists and there is excess demand then the old price p'_j is some multiple K_j of the excess demand. We can reapply the normalization rule by summing the prices obtained by applying the above rules. We can define the mapping differently as follows

$$K{:}z \rightarrow p' \tag{23}$$

In the above, we have presented a continuous function. We need to reconstitute it and therefore composing mapping is defined as

$$Koz{:}p' \rightarrow p' \tag{24}$$

This is also a continuous function and maps the closed bounded convex set p'. Suppose the p' is given, then we find that z(p') ϵ z. Now we have p = k[z(p')]ϵ p'.

We can then apply Brouwer's fixed point theorem to prove that a price vector p*ϵ p' exists, such that:

$$P^* = k\ [z\ (p^*)] \tag{25}$$

Considering equation (22), we can represent the equation (25) as

$$P_j^* = \frac{\max[o, p_j^* + k_j z_j(p^*)]}{\sum_j \max[o, p_j^* + k_j z_j(p^*)]} \qquad j = 1, n \tag{26}$$

We can prove that p* is an equilibrium price vector. If we assume that the right hand side of the above equation is zero then $P_j^* = 0$ for that j. Since $p_j^* + k_j z_j(p^*) \le 0$ and $k_j > 0$, then

$$Z_j(p^*) \le 0 \text{ if } p_j = 0 \tag{27}$$

If we assume that there are free goods which are either a gift of nature or provided by the government, the right hand side of the equation is positive and $P_j^* > 0$ for jϵN.

If we multiply equation (25) by zj (p*) for each j ϵ N, then we obtain the following equation

$$P_j^* Z_j(p^*) = \frac{P_j^* Z_j(p^*) + k_j[z_j(p^*)]^2}{\sum_j \max[o, p_j^* + k_j z_j(p^*)]} \text{ jϵN} \tag{28}$$

Now we can apply Walras's law and rewrite the equation as

$$\sum_{j\varepsilon N} p_j^* Z_j(p^*) = \sum_j p_j^* z_j(p^*) = 0$$

Since $P_j^* = 0$, jϵ N, therefore we can sum through equation (28) over j∈N, giving us the following equation

$$\sum_{j\in N} \frac{(P_j^* Z_j(p^*) + k_j[z_j(p^*)]^2)}{\sum_j \max[o, p_j^* + k_j z_j(p^*)]} = 0 \tag{29}$$

The denominator can be cancelled out. By applying Walras's law to the above equation, we can get the following

$$\sum_{j \quad N} K_j[z_j(p^*)] \qquad 0 \tag{30}$$

But, since $k_j < 0$, this can clearly hold true only if $z_j(p^*) = 0$ so

$$Z_j(p^*) = 0 \text{ if } p^*_j > 0 \tag{31}$$

P* is an equilibrium price vector. Since z (p*) is an excess demand vector satisfying the requirements (8), (9), p* is non-negative. Therefore, equation (10) is satisfied and (11) and (15) together to hold true equation (1).

Stability of the Walrasian Equilibrium

In a Walrasian equilibrium, stability is an important aspect. The Walrasian equilibrium involves analysis of the movement of prices through successive disequilibrium positions. They are persistent over time. In a Walrasian equilibrium, at least one equilibrium price vector exists which is $p^* = (p_1^*, p_2^*, \ldots.. p_n^*)$. At an initial movement of time t = 0, a price vector p (0) ≠ p∗ exists. Time is varying and continuously changing. Therefore, the price vector is itself a function of time.

We can represent this as

$$p\,(t) = (p^1(t), p^2(t), \ldots. p^n(t)) \tag{32}$$

Equation (32) explains that the price at a particular time is itself a function of price at different times. For stability, we need a function of price at different times and we need a global equilibrium system which is stable as

$$\lim_{t \to \infty} p(t) = p^* \tag{33}$$

We can say that there is an equilibrium price vector p (0). At this price level, we can allow different adjustment processes which are dynamic in nature. We need to discuss stability relative to a particular adjustment process.

Tatonnement process

In an economy, there is an umpire whose job it is to announce a price vector at each different time. Before announcing the prices of different commodities, the umpire collects information. Sometimes, prices are too low or too high, therefore, the umpire decides whether to permit the trading of commodities or not. In a modern economy, the government often acts like this umpire whose job it is to collect the data of prices of all commodities and allow the trading of commodities. When prices rise very fast the government takes steps to reduce the prices. But there are certain rules and the umpire acts according to these rules.

1. Firstly, the umpire announces a new price vector, if the old prices are not at the equilibrium level. There is disequilibrium and the demand and supply of commodities do not match each other.
2. Suppose the demand and supply is not at equilibrium at the equilibrium price then the umpire will not permit the actual trading of commodities.
3. Suppose there is excess demand of commodities and the umpire changes a given price say j^{th} where j = 1, 2 …n. This price is proportionate to the excess demand for the corresponding j^{th} commodity.

At time t the prices p_j (t) are announced by the umpire, in equation form as

$$\frac{dp_j(t)}{dt} = \dot{p}_j = \lambda_j z_j \quad j = 1, 2, n \ \lambda_j > 0 \tag{34}$$

For an equilibrium price vector, Walras's law persists. At the announced price vector, consumers buy different commodities and get maximum satisfaction. Budget constraints satisfy the consumers' net demand. This adjustment is continuous and time cannot restrict such adjustments. Walras's law aggregates households and firms. The households are buying many commodities at the prevailing price and firms are selling the commodities at the existing price level. Suppose there is disequilibrium in either supply or demand then this is reflected on the price level. At equilibrium, trade will take place. Therefore, Walras's law will not be observed here. But for global stability, we require Walras's law to satisfy different conditions. With the help of the tatonnement process, we can formulate the global stability system. But due to a large number of commodities in the market, the tatonnement process would be exhaustive. In order to determine the global stability system, we have to assume that there are only two goods traded in the market. This assumption will help us to provide the tatonnement process in its complete form. We can also assume that there is a demand that exists at an equilibrium price vector (p *> 0) which is greater than zero.

$$Z_j = (p_1^*, p_2^*) = 0 \quad j = 1,2 \tag{35}$$

The excess demand function has the zero degree homogeneity. It helps us to decide the p* which is an equilibrium price vector for any u > 0.

5.3 Stability proposition

In any economy, there are multiple goods produced by firms. But a consumer decides to buy certain commodities. Sometimes, the consumer buys substitutes. The tatonnement process determines the prices of the commodities, as expressed in the equation

$$\lim_{t \to \infty} p(t) = p^* \qquad (36)$$

At a particular price vector p(0) , the goods market is at equilibrium. This equilibrium is globally stable for good 1 and good 2. There are gross substitutes for good 1 and good 2 and this can be written as

$$\frac{\partial z_1}{\partial p_2} > 0, \frac{\partial z_2}{\partial p_1} > 0 \qquad (37)$$

We have defined two goods and assumed that they are close substitutes. But we do not know which goods are substitutes. In order to find the substitutes, we have to determine the global equilibrium price level with a particular demand and supply. This shows that the supply of commodities is more important than the demand by consumers. To decide the stability proposition, we require some propositions, such as

1. We assume that there are two goods and they can be substituted. The equilibrium price vector is unique. If it is assumed that there are two price vectors for two commodities then we can write this as $p^* = (p_1^*, p_2^*)$. Let's assume that there are two price vectors and that they are at equilibrium. For example, p* and p** are at equilibrium. Therefore, Zj(p*) = Zj(p**) = 0 for all j. The gross substitutability property clarifies that p** = up*.

Figure 5.3 shows that p* line is a 45 ^{0} line and is upward slopping. This line is the up* line and shows the set of all price vectors available for consumers. These price vectors are scalar multiples of p*. due to the zero degree homogeneity of the excess demand functions. Supposing that p* is an equilibrium price vector, up* must be for any u > 0.

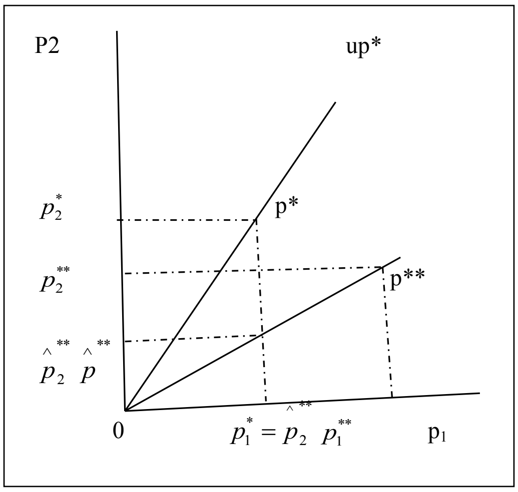


Figure 5.3. Price vector and demand function

In figure 5.3, if we assume that p**≠up*, then the line 0p** must be distinct from 0p*. This effect is shown on two separate lines. Suppose p* is an equilibrium price vector. In the figure it must be any point on the op** line.

Since a point in the above figure can be written as

$\hat{p}^{**} = up^{**}$ for 1 > u > 0

Point $\hat{p}^{**}$ in the figure can be compared with the vector p*,

$$\hat{p}_1^{**} = p_1^* \tag{38}$$

$$\hat{p}_2^{**} = p_2^* \tag{39}$$

Now p* is an equilibrium point as we have pointed out in the diagram. Therefore, the following function is slightly different.

$$Z_1(p_1^*, p_2^*) = 0 = z_2(p_1^*, p_2^*) \tag{40}$$

Here we have applied the gross substitutability. We can then write equations (38) and (39) as

$$\begin{aligned} Z_1(\hat{p}_1^{**}, \hat{p}_2^{**}) < 0 \\ Z_2(\hat{p}_1^{**}, \hat{p}_2^{**}) > 0 \end{aligned} \tag{41}$$

$\hat{p}^{**}$ comprises the lower price than p* of good 2. Equation (41) explains that $\hat{p}^{**}$ is not a true equilibrium. It is because p**= (1/u) p**. Hence an equilibrium price vector can only lie along the line up*.

2. Gross substitutability with Walras's law explains that $p_1^* z_1(p_{1,} p_2) + p_2^* z_2(p_{1,} p_2) > 0$. This equation is written to show the disequilibrium in the price vector. The equilibrium price vector is written as $p = (p_1, p_2)$.

The disequilibrium excess demand may have an effect on prices. But the equilibrium prices must be strictly positive. The property of the equilibrium is explained as follows.

$$Z_1(p_1^*, p_2^*) = 0 = z_2(p_1^*, p_2^*) \tag{42}$$

The new price vector is defined as $\hat{p}$ which is further defined as

$$\hat{p}_1 < p_1^* \ \hat{p}_2 < p_2^* \tag{43}$$

Now we can substitute the new price vector. Therefore, gross substitutability is defined as

$$Z_1(\hat{p}_1, \hat{p}_2) > 0, Z_2(\hat{p}_1, \hat{p}_2) < 0 \tag{44}$$

The equations (43) and (44) are unique. If we combine them, then

$$(\overset{*}{p}_1, \hat{p}_1).Z_1(\hat{p}_1, \hat{p}_2) + \overset{*}{p}_2 - \hat{p}_2).Z_2(\hat{p}_1, \hat{p}_2 > 0 \tag{45}$$

In equation (45) each term must be positive. If we further rearrange the equation we get.

$$\overset{*}{p}_1 z_1 + \overset{*}{p}_2 z_2 > \hat{p}_1 z_1 + \hat{p}_2 z_2 \quad (46)$$

If we apply Walras's law to the above equation then the right hand side of the equation is zero. The proposition is proved because of a change of direction in equation (43).

3. We have already assumed that price vector p* is an equilibrium price vector. From this assumption we can define a distance function: D (P (t) p*). This distance function can be proved in a slightly different way in terms of derivatives as follows.

$dD / dt < 0$, it means p (t) ≠ up*.

The equation also implies that dD/dt = 0.

In the distance function, D is a real number to each price vector p (t). This distance function measures the distance from equilibrium vector p*. This was already known and presented in the tatonnement process discussion. Walras's law and the gross substitute assumption are the two functions which have monotonic values through time. But if a fall in the price level is observed then there is a fall in monotonic values or some scalar multiple. This occurs because there is an equilibrium vector. The time path of the price vector is getting steadily closer to an equilibrium price vector.

5.4 Edgeworth's exchange theory

Introduction

The exchange theory is a most vital and practical theory in advanced microeconomics. The theory was first formulated by F.Y. Edgeworth. He assumed that there are two individuals and they exchange two commodities with each other. In the market, there are many commodities supplied and also, many commodities are exchanged. Two individuals bargain with each other to exchange commodities. Every individual always looks for the best contract at which both would be better off.

Edgeworth's diagram

Edgeworth had a precise formulation of barter processes for the simple two-good economy. The process of barter dealt with by Marshall and Edgeworth consisted of successive bartering between individuals until a position was reached at which no barter was possible for each individual to become better off. Edgeworth graphically showed that the equilibrium reached by the process depended upon the path of bartering as well as the amount of goods initially held by each individual. The process of barter, therefore, constitutes a strong contrast to Walras's tatonnement process. Walras's process is a provisional market process by which competitive equilibrium is attained, and the equilibrium reached by it is determined solely by the initial holdings, independently of the path of the process (Uzawa, 1962).

In an Edgeworth box diagram, there are two individuals 1 and 2. The $\bar{x}_i$ and $\hat{x}_i$ denote individual i's consumption endowment and net demand respectively of commodity x where i = 1,2. The total endowment of commodity x is $\bar{x} = \bar{x}_1 + \bar{x}_2$. In addition to this, individual 2 possesses some amount of commodity y, where $\bar{y}_i$ and $\hat{y}_i$ are i's consumption endowment and net demand. This theory gives the

indifference curves to explain both consumers' preference for two goods. The figure shows the length of the horizontal side of the box. It represents the total endowment of commodity x. The length of the vertical side of the box represents the total endowment of commodity y.

A point in the diagram shows the four consumption values for two individuals. They are (x1, y1, x2, y2). These values satisfy the feasibility conditions as follows.

$$x_1 + x_2 = \bar{x} \tag{47}$$

$$y_1 + y_2 = \bar{y} \tag{48}$$

In the box diagram, we can measure individual 1's consumption of x which is to the right of O_1. This is at the bottom left hand corner of the box diagram. Consumer 2's consumption of y is measured to the left from the origin O_2 at the top right hand of the box. Consumer's consumption of y2 is measured vertically downward from o2 in diagram. There are four consumption coordinates at each point in the box diagram. They are presented in the following figure.

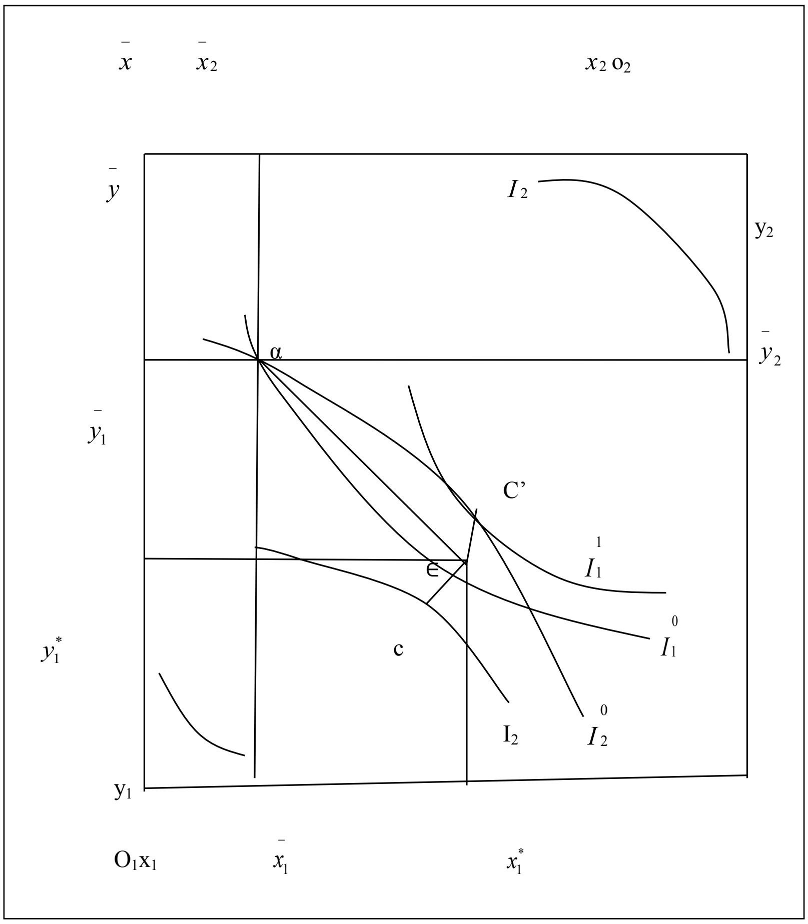


Figure 5.4. Consumption and endowment of commodity

Figure 5.4, *α shows a point which has four consumption values: x1, y1, x2, and y2. Now $\bar{x}$ can be written slightly differently, as $\bar{x} = \bar{x}_1 + \bar{x}_2$. This represents a medium consumption of the total endowment of both commodities. Similarly, for y commodities, the total endowment can be written as $\bar{y} = \bar{y}_1 + \bar{y}_2$. The equation is the median consumption of y commodity by both consumers. In the diagram, we can draw 1's indifference curves with reference to origin $0_1$and 2's indifference curves with reference to the origin O_2.. These indifference curves are given as $I_1$and I_2 in the diagram. Point $\propto$ indicates that the initial endowment point has coordinates $(\bar{x}_1, \bar{y}_1, \bar{x}_1, \bar{y}_2)$. In the diagram, the indifference curves I_1^0 and I_2^0 passes through total endowments of two individuals. Before doing any trade on the I_1^0 and I_2^0 indifference curves, the consumers are located at point $\propto$.

Edgeworth's hypothesis (EH)

Edgeworth's hypothesis is based on two assumptions. Firstly, there are only two individuals in this hypothesis. They will always agree to an exchange of the commodities, so long as no individual which will be worse off. Secondly, both individuals in the hypothesis will not agree to an exchange if it results in any one worse off than what they are now. Edgeworth hypothesis is very clear and there is no possibility of failure in it. Both individuals will agree to exchange commodities and bargain for better gains out of the exchange. Edgeworth's hypothesis assumes that each individual is thinking rationally. At the end of the bargaining, both will agree to exchange commodities. This exchange between the two individuals will make at least one better off and no one worse off.

In figure 5.4, we move along I_2^0 with individual 1 giving individual 2 y in exchange for x. Suppose an individual 1 buys x from individual 2. He is then better off while individual 2 is no worse off. Both individuals can become better off only if individual 1 buys from individual 2 with payment falling between those implied by moving along I_2^0 and those implied by moving along I_1^0. Such movements along two indifference curves allow trading between two individuals, and help both.

The exchange process

In order to decide the exchange process for two individuals, we require the cc' line in the diagram. This line is a locus of points and is sometimes called a contract curve. The line is tangent to the indifference curves of the consumers in the area bounded above by I_2^0 and below by I_1^0. The assumption's strict convexity shows that any given pair of indifference curves for the two parties or two consumers will have no more tangency point. In the diagram, there is no point cc' in the area which is bounded by I_1^0 and I_2^0 which can be a tangency point. All points of cc' must be the points of intersection. Any point of interaction on the cc' line makes one individual better off.

Both individuals should agree by sliding along the two indifference curves. Before an exchange, the $\propto$ point is not an equilibrium point on the cc' line, as there is no equilibrium outcome out of the exchange process between two individuals. On the other hand, a point cc' cannot be improved upon if both individuals reach a particular point. Then a move on the cc' line is not possible. At such point, they are ready to exchange the goods. Here, at least one consumer is better off. In the diagram, any direction of curves leads to a lower indifference curve for at least one individual. It is not possible to move from one point to another point along the cc' line. Such possibility makes one individual worse off. The line cc' in the diagram satisfies Edgeworth's hypothesis. The cc' line is also called a contract curve, a set of possible contracts for exchange of commodities which any individual finally makes.

Criticism

1. **Arbitrary explanation**: The Edgeworth exchange theory does not satisfy how the consumers get from point $\propto$ to a particular point on the cc' line. It is an artificial explanation to make the two consumers better off. Edgeworth suggested that this is the only possible solution or there is an accident which forces the consumers to exchange the commodities and make each other better off.
2. **Artificial explanation**: It is assumed that both consumers bargain with each other to sign a contract which is available on the contract curve. There is no other point other than on the contract curve. It is also not possible to change contracts.
3. **Static theory**: The theory does not give any particular point as equilibrium which is a set of points rather than a particular point. Both consumers are ready to agree to exchange both commodities but at different points. Therefore, there is no particular point of satisfaction or solution. Such a model is called an 'intermediate' model because it results in a set of points.
4. **Narrow view**: The theory explains only two commodities and two consumers. This explanation is narrow and does not represent the entire market behavior. Sometimes both cannot agree to exchange commodities. In the modern world, there would be a number of substitutes which are available. This does not mean that the market has stopped functioning. The theory does not explain if both consumers are ready to exchange commodities with other consumers. Both consumers wants the maximum gain therefore, they continue bargaining until they get satisfaction.

Before bargaining both consumers are at equilibrium at **α**. Suppose both are bargaining and the process ends at point ϵ. Then the line ∝∈ defines a price ratio. Consumer 1 pays consumer 2 the amount $\bar{y}_1 - y_1^*$ in exchange for $X_1^* - \bar{X}_1$ of x, and so the implied equilibrium price of x is

$$p^* = (\bar{y}_1 - y_1^*) / (x_1^* - \bar{x}_1) \tag{49}$$

This equilibrium can be defined by the slopes of the lines from ∝ to the curve cc'. It is below the slope of ∝c'. This shows that consumer 1 is a buyer of x. The lower the price then the better off he is.

5. **Continuity and strict convexity**: The consumer preferences must satisfy continuity and strict convexity. In the diagram, any point on cc' can be generated as follows.
   Max u_1 (X_1, Y_1) subject to u_2 (X_2, Y_2) ≥ u_2^0 (50)

Through endowment constraints and non-negative constraints u_2^0 has a utility value of 2. This varies between the values of utility, and corresponds to indifference curves I_2^0 and I_2. It is hard to construct the cases on the cc' line which coincides with an edge of the box.

5.5 Welfare economics

Welfare economics deals with the allocation of resources between individuals which always tries to make at least one individual better off and no one worse off. In any economy, resources are used efficiently. Therefore, there is competition among individuals to exchange goods and increase welfare. Individual welfare is not different from the social welfare.

Pareto efficient resource allocation

In an economy, there is an allocation A described as the use of resources in an economy. These resources are the consumption bundles of the consumers. The labor in the economy supplies inputs. Firms use labor inputs to produce output.

In Pareto efficient resource allocation, allocation A^1 is Pareto superior to allocation A^2. Suppose A^1 generates at least as much utility for all individuals then A^1will be preferred. The figure shows that all allocations which generate utility combinations $\propto^1$, $\propto^2$ and $\propto^*$. This is shown in the shaded area. They are superior to the allocation and generate $\propto^0$. A Pareto efficient allocation explains that no other feasible allocation exists. This allocation makes individuals much better off and maximizes u'. The utility of individual 1 is subject to the constraints $u^h \geq \bar{u}^h$ where, h = 1,2……H. Such constraints are imposed by technology and the endowments of the economy.

An allocation is Pareto efficient if there is no feasible allocation. If we assume that there is a variance in the minimum required utilities u^h then the Pareto efficient allocation is altered. It is derived from maximizing u'. Figure 5.5 shows the utility frontier FF' which was derived by maximizing u'. Such utility frontier is subject to $u^2 \geq u^2$ for different values of u^2 .

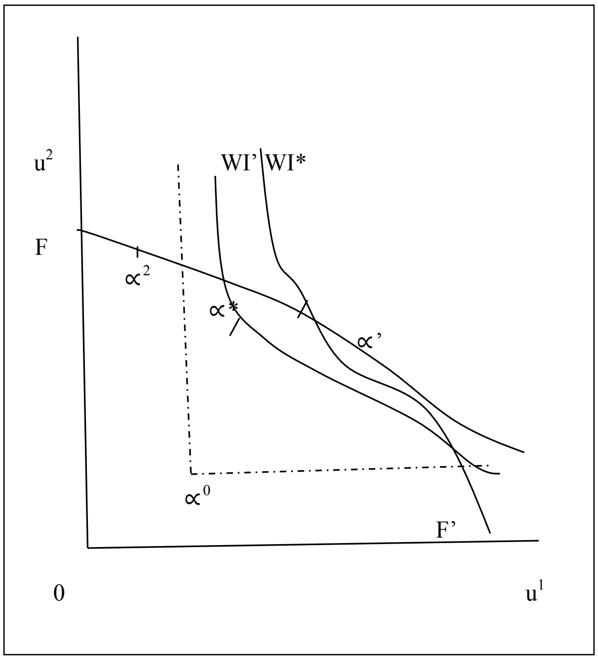


Figure 5.5. Utility maximization frontier

The allocations generate utility combinations on the utility frontier. Such allocations are as $\propto^2$ or $\propto^*$ which are Pareto efficient. The allocations also generate points inside the frontier like $\propto^0$which are Pareto inefficient.

Paretian value judgments

In microeconomics, the concept of Pareto efficiency and Pareto superiority are widely used. It is important to realize that they are a set of value judgments.

Process independence

The strong value judgment assumption explains the process of a particular allocation. It is achieved and it does not matter to show particular allocation. Suppose an individual's allocation mechanism produces the allocation which leaves him/her better off or worse off. Individuals supply their own labor input and become better off. They get remuneration for supplying labor inputs. A centrally planned economy should allocate resources properly.

Individualism

Under the Paretian criteria the only aspect of an allocation which is relevant is its effect on the individuals in society. Thus the output and input mixes of individual firms are of no consequence for welfare purposes. It does not matter per se for example, whether a given output is produced in one large or many small firms or whether the firms are privately or publicly owned or owned by foreigners.

The organization of production is relevant for welfare proposes only in so far as it affects the consumption or labor supplied by individuals.

Non-Paternalism

In non-paternalism, it is assumed that an individual is the best judge of their own welfare. But some individuals do not know their own preferences or do not respect the preferencse of other individuals. Each individual has their own preferences to get maximum satisfaction. We have already seen choice and preference ordering in Chapter One. Such maximization preferences are the basic subject matter of microeconomics. But some preferences are bad and some are good. The bad preferences are consumption of alcohol, tobacco chewing, smoking, drinking and driving, etc. The good preferences are seeking health care, listening to music, getting an education and saving money, etc.

Benevolence

The Pareto efficient criterion is benevolent towards individuals, and increases the utility of one individual and also leads to an improvement in the utility of others. Benevolence seems very weak and therefore it is an uncontroversial value judgment. Similarly, it is not a universally accepted concept. A very rich individual may become even richer, his well-being improved and his utility increased. But at the same time, some individuals in the community can be malnourished or starving. For example, the Indian economy is growing, but still, half of Indian children are malnourished.

5.6 Pareto efficiency conditions

In Pareto efficient conditions, we need to assume a simple economy. Let's assume two consumers, two goods, two inputs, and two firms. The individual h has a certain utility function which is achieved after the consumption of commodity 1 and commodity 2, and can be represented as

$u^h(X_{h1}, X_{h2}, Z_h)$ where X_{h1} is the consumption of commodity 1 and commodity 2. We have assumed h is the supply of an input.

The individual's initial endowments h of the efficient inputs are denoted as $\bar{Z}_h$. We assume that the consumer is not satiated after consuming $\bar{Z}_h$, the consumption bundles of commodities. The marginal utility of consumption of the $\bar{Z}_h$ bundle is positive and can be denoted as

$$u_2^h > 0 \text{ (i = 1,2; h = 1,2).}$$

The marginal utility of h of supplying more of input h is always negative, and is presented as

$$u_2^h < 0.$$

In the Pareto efficient condition, firms produce the total output Xi. There are i[th] commodities produced by firms. The production function of a firm is represented as

$$x_i = f'(Z_{i1}, Z_{i2}) \text{ i = 1, 2} \tag{51}$$

Where Z_{ih} is the quantity of input h, which is used in producing good i. We further state that the marginal product of input u in producing commodity i is positive. We can write this in equation form as

$$f_h^1 = \partial f^i / \partial z_{ih} > 0, \text{ (i = 1, 2; h = 1, 2)}$$

We have taken a simple derivative of the production function. It is also possible that the total consumption of good i by individuals will exceed the total output of firm i. The total use of input h by the firm may exceed the supply.

This can be written as

$$Xi \geq \sum_{h=1}^{2} x_{hi} \qquad i = 1,2 \tag{52}$$

$$Z_h \geq \sum_{i=1}^{2} z_{ih} \qquad h = 1,2 \tag{53}$$

This shows that each individual is maximizing their utility but is no satiated. The marginal products are positive, which means that the materials balance requirements will bind as equalities in a Pareto efficient allocation.

In strict inequalities, increasing the output of commodity 1 could increase the input z_{1h} .An increase in the consumption by individual 1 is also permitted. Suppose equation 2 in the above explanation is a strict inequality equation then the consumption of individual 1 will increase, increasing without any increase in inputs, up to u'. In this condition, the feasible allocations must also satisfy the constraints. The supply of input h cannot exceed individual h's initial endowment where $\bar{Z}_h \geq Z_h$. The preferences are such that the inequality is always strict and the constraint never binds. Suppose Z_h is interpreted as labor then this is not an implausible assumption. Firms use the inputs of labor. We can put this problem and solution in terms of Pareto efficient allocation as

$$\text{Max } u'(x_{11,}x_{12}, z_1) s.t. u^2(x_{21,}x_{22,}z_2) \geq \bar{u}^2 \tag{54}$$

Xh$_i$, z$_i$, h, x$_i$, zh

We need to use equations (51),(52), and (53) and develop a Langrangean function for the Pareto efficiency problem, as follows

$$L = u'(x_{11,}x_{12}, z_1) + \lambda[u^2(x_{21,}x_{22,}z_2) - \bar{u}^2 + \sum_i pi[x_i - \sum_h x_{hi}] + \sum_h w_h[z_h - \sum_i z_{ih}] + \sum_i u_i[f^i(z_{i1}, z_{i2}) - x_i]$$

We can take the first order condition of the above function, and present it as

$$\begin{aligned} \partial L / \partial x_{1i} &= u_i^1 - \rho_i = 0 \\ \partial L / \partial x_{2i} &= \lambda u_i^2 - \rho_i = 0 \end{aligned} \quad \text{where i = 1, 2} \tag{55}$$

where i = 1, 2 (56)

$$\begin{aligned} \partial L / z_1 &= u_i^1 + w_1 = 0 \\ \partial L / z_2 &= \lambda u_2^2 + w_2 = 0 \end{aligned} \quad \text{efficient input supply} \tag{57}$$

(58)

$$\partial L / z_{1h} = u_i f_h^i - w_h = 0 \text{ i}, \qquad h = 1,2 \text{ efficient input use} \tag{59}$$

$$\partial L / x_i = \rho_i u_i = 0 \qquad i = 1, 2 \text{ efficient output mix} \tag{60}$$

The above equations depict the Pareto efficient resource allocation condition, which is sufficient and necessary.

Efficiency consumption

For efficiency in consumption, we need to re-arrange equations (6) and (7), and discuss them further

$$MRS_{21}^{'} = \frac{u_1^1}{u_2^1} = \frac{p_1}{p_2} = \frac{u_1^2}{u_2^2} = MRS_{21}^2 \qquad (61)$$

From equation (55), it is clear that $u_1^1 = p_1$ and $u_1^2 = P_2$.

Presently, equation (56) shows a further derivation

$$\lambda u_1^2 = p_1 \ , \ \lambda u_2^2 = p_2 .$$

We need to divide each of the above equations for good 1 and good 2 to yield equation (61).

The total output is efficiently allocated between two consumers. We must ensure that this allocation equalizes the marginal rate of substitution (MRS_{21}^{h}) between the two goods. Now the MRS_{21}^{h} measures the substitution rate, the rate at which h is willing to substitute commodity 1 for commodity 2. The MRS_{21}^{h} measuring h is a marginal valuation of commodity 1 in terms of commodity 2. The consumption cannot be efficient if the individuals have different marginal valuations of the goods. Initially, there is no Pareto efficiency and equality of MRS. This is a necessary condition for a Pareto efficient allocation of goods among consumers.

5.7 The Edgeworth box diagram

The Edgeworth box diagram is divided into two types. The horizontal side of the box measures a fixed total output of good 1 and the vertical side measures a fixed total output of good 2. Individual 1's consumption of good 1 is measured horizontally from the origin at o^1. Individual 1's consumption of good 2 is vertical from o^1. In the diagram, the consumption of individual 2 is measured from the origin at o^2. The Edgeworth box diagram assumed that there is nonsatiation from consuming commodities. It means that it cannot be efficient to have total consumption of any good which is less than the output of the good. Therefore, we need to restrict attention to consumption bundles for the individuals.

In the Edgeworth box diagram, a single point is defined as the consumption bundle of both individuals. The allocation A^0 has individual 1's allocation and he is getting the consumption bundle $(x_{11,}^{0} x_{12}^{0})$. Individual 2 is getting the consumption bundle $(x_{21,}^{0} x_{22}^{0})$. If we assume that the individual labor supply is constant then we need to draw different indifference curves. We can draw another indifference curve for the two commodities. The individuals have strictly quasi-concave utility functions. Therefore, the indifference curves for individuals are convex to their origin.

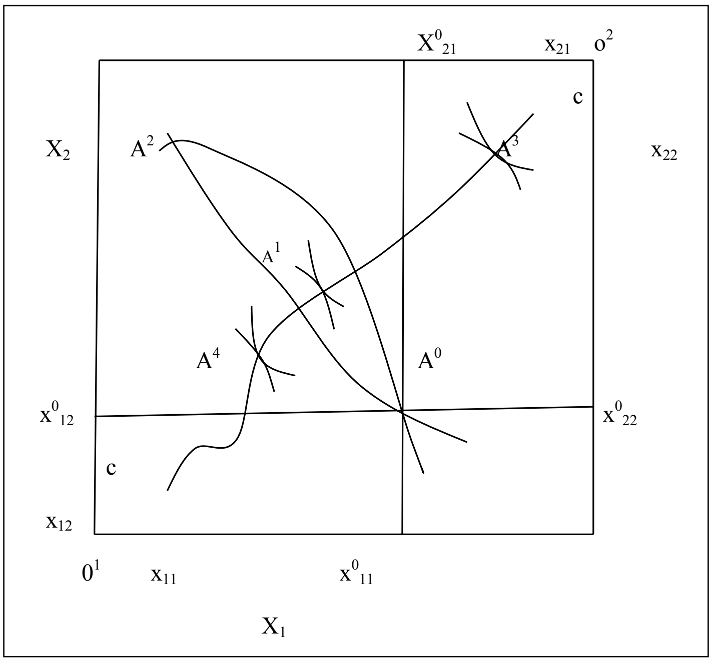


Figure 5.6. Pareto efficient allocations and utility combinations

In the Edgeworth box diagram, the allocation A^0 is not Pareto efficient. It is possible to exchange commodities between two individuals so as to make them both better off. The allocation A' is Pareto superior to A^0. This new allocation puts both individuals on indifference curves which are far from their respective origins. The A^2 allocation in the diagram is inverse of the A^0 allocation. It is a lens-shaped area which is defined by the indifference curves through A^0. Allocation A^2 is superior to A^0. In the diagram, the allocation cannot cross the indifference curves. This is because all allocations are Pareto efficient. There are different allocations in the box diagram. The indifference curves are tangent at A^1, A^3 or A^4. All tangent points in the box diagram are efficient. The indifference curves and their slopes are negative because of the marginal rate of substitution.

In the diagram, point cc is the locus of the points of tangency between the indifference curves. Such points constitute the set of all Pareto efficient allocations of the given total output. It is measured at both sides of the diagram. In an exchange economy, that is, the barter system where goods are exchanged for goods, consumers have fixed endowments of consumption goods. Efficient consumption conditions are required for the Pareto efficiency. In the box diagram, each allocation and point generates the utility combinations which are written as (u1, u2). The Pareto efficient allocations on the curve cc would generate utility combinations of individuals which are considered as utility frontiers. The inefficient allocations would generate combinations inside the utility frontiers.

Efficient input supply

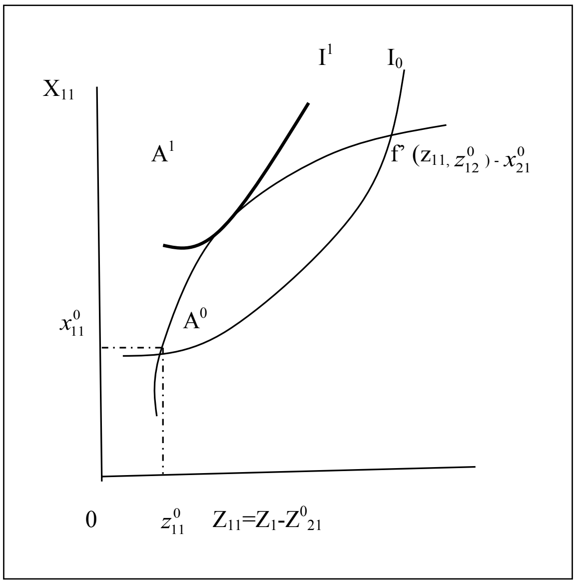


Figure 5.7. Condition for efficient input supply by individual

For an efficient input supply, we need to combine equations (55), (56), (57), and (58), and get the equation

$$-\frac{u_2^h}{u_i^h} = \frac{w_h}{p_i} \tag{62}$$

From equation (59) and equation (60), we can derive another function which is,

$$f_h^1 = w_h / u_i = w_h / p_i$$

We need to consider the efficiency and its further implication, and present this as

$$MRS_{i2}^h = -\frac{u_2^h}{u_i^h} = \frac{w_h}{p_i} = f_h^i \text{ h, i = 1, 2} \tag{63}$$

The MRS in the above equation means h's marginal rate of substitution is between the input supply and the consumption of commodity i. It is further defined as the rate at which h must be compensated by being given more of commodity i. This is possible when the consumer increases supply of Z_h by one unit. The right hand side of equation 63 is the marginal product of Z_h in the production of commodity i. The Pareto efficiency requires that the additional output produced by an extra unit of Z_h is just equal to the marginal cost. This is in terms of good i of Z_h to h. Suppose we assume that h can be compensated by two units of good i for supplying one unit of Z_h, then it can be used to increase output of good i by 3 units. Such allocation cannot be Pareto efficient and is true for the above case.

We have also derived the figure which shows the condition for efficient input supply. This efficient input supply is provided by individual 1. All the consumption levels except x_{11} and all input uses except z_{11} are held constant. In the diagram, the vertical axis plots the consumption of good 1 by individual 1, and is denoted by $x_{11} = x_1 - x_{21}^0$. The horizontal axis plots the use of this input by firm 1, denoted by $Z_{11} = Z_1 - z_{21}^0$. With all other consumption and input use, fixed increases in z_{11} imply equal increases in z_1. Thus we can show the indifference curves of individual 1 in (z_{11}, x_{11}) space as I^o, I^1.

In the diagram, these curves are just the counters of u' $(x_{11,}\ x_{12}^0\ z_1)$ = u'$(x_{1-}\ x_{21\ ,}^0\ x_{21}^0,\ z_{11+}\ z_{21}^0)$. The curve f'- x_{21}^0 plots f' $(z_{11},\ z_{12}^0)$- x_{21}^0 against z_{11}. It shows the effect of variations in z_{11} on the consumption of good 1 by individual 1.

Suppose the initial allocation is A^0 then the consumption of good 1 by individual 1 is x_{11}^0. The use of inputs by firm 1 is z_{11}^0. At allocation A^0, the indifference curve I^0 cuts the curve f'- x_{21}^0. This is because by shifting allocation A', individual 1 achieves a higher utility level of I'. When the indifference curves of individual 1 are tangent to the curve f'- x_{21}^0, this is considered the efficient allocation. The slope of the curve is given by the equation as f'- x_{21}^0 is $\partial[f\,'(z_{11,}z_{12}^0) - x_{21}^0] / \partial z_{11} = f_1^1$. The marginal product of input 1 is the production of good 1 and the slope of the indifference curve is $(x_{11}, z_1\text{-}\ z_{21}^0)$. The space is just individual 1's marginal rate of substitution between commodity 1 and his input supply. Therefore, we have again established the efficient input supply condition. Equation (63) is necessary for the efficiency condition.

Efficient input use

From equation(59), we can use a given total supply of inputs by the firms, giving us the following equation

$$MRTS_{21}^1 = \frac{f_1^1}{f_2^1} = \frac{w_1}{w_2} = \frac{f_1^2}{f_2^2} = MRTS_{21}^2 \tag{64}$$

The ratio of the marginal products is $\frac{f_1^1}{f_2^1}$, the marginal rate of technical substitution. The $MRTS_{21}^1$ of input 1 for input 2 is the production of good i. The $MRTS_{21}^1$ is the rate at which input 1 can be substituted for input 2 without a change in the output of good i.

In figure 5.8, a feasible allocation increase in z_{11} by one unit reduces z21 by one unit. Further, it reduces z_{12} by four units and it increase z_{22} by four units. The net effect will be to increase the output of both goods. The initial allocation cannot be Pareto efficient. The equality of the marginal rate of technical substitution for a firm is necessary for its efficiency. The following diagram shows the Edgeworth box diagram and shows the fixed input supplies of the two individuals which is measured by the lengths of the sides of the box.

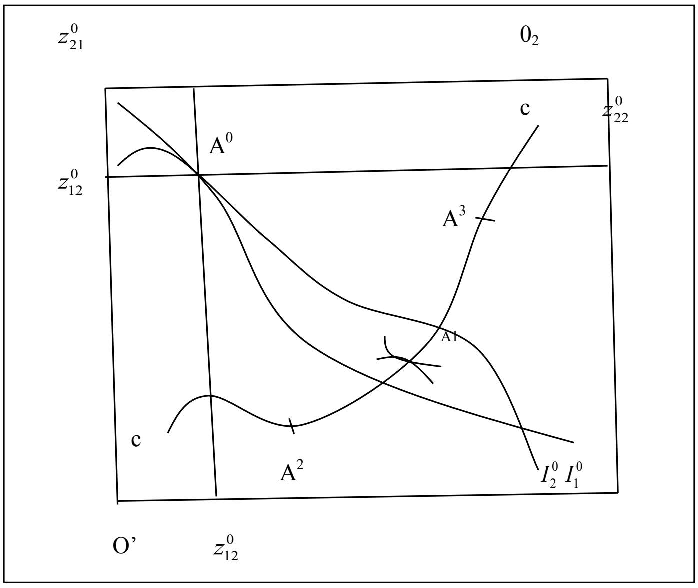


Figure 5.8. The Edgeworth box diagram

Firm 1 uses the input which is measured from the origin 0^2, but it cannot be efficient for the total use of an input to be less than the supply. Now we can restrict our attention to allocations. But the allocation is $\sum_i z_{ih} = z_h$.

In the Edgeworth box diagram, the allocations are defined by a point A^0. Here, we assume that the production function is strictly quasiconcave. The isoquant for firm 1 is the curve I_1^0 and for firm 2 the curve is I_2^0. We have assumed that marginal products are positive. They are intersecting each other at the larger output. The allocation A^0, is the isoquant where two firms have crossing curves where the points are not efficient. Other feasible allocations like A' always exist. At this point, the firm produces more output. At A' point, the isoquants are tangent and these are efficient points. The slope of isoquants is the firm's marginal rate of technical substitution and yields equation (64).

5.8 Welfare functions and the Pareto criterion

In a welfare function, we need a set of judgments to provide complete, transitive and reflexive comparisons of allocations. There is a welfare preference ordering which is similar to individual preference ordering. This welfare preference ordering is further useful to rank all alternative allocations. The welfare preference ordering provides the optimal allocation which is also feasible based on the above properties. Suppose the feasible allocation does not satisfy the above properties then the allocation does not exist. Any existing allocation therefore is ranked higher by welfare preference ordering. A welfare preference ordering function is always used to rank all possible alternative allocations. The optimal allocation is a feasible allocation with the property. This was pointed out in the earlier paragraph. There is no other feasible allocation which is ranked higher by the welfare preference ordering. In the case of welfare preference ordering, it is continuous and such function can be conveniently represented by a Bergsonian social welfare function (swf). In this allocation, w (A) is defined as the allocation which maximizes w over the set of feasible allocations.

Now we can represent the social welfare function for a two-person economy as

$$w = w(x_{11}, x_{12}, z_1, x_{21}, x_{22}, z_2) \tag{65}$$

From the above assumptions, we are free to make the assumption of a non-paternalistic value judgment. This means that an individual is the best judge of their own welfare and will choose the best preference ordering. But now we need to change the above function slightly, and so equation (65) can be rewritten as

$$w = w(u^1(x_{11}, x_{12}, z_1)u^2(x_{21}, x_{22}, z_2)) = w(u^1u^2) \tag{66}$$

where u^h and h^{th} are the individual's utility function and represents their own preference ordering over their consumption bundles and labor supply. If it was felt that individuals were not the best judges of their own welfare then we could specify a social welfare function or swf, the individual's own utility function which was replaced by the welfare function. This is shown to be $g^h(x_{h1,} x_{h2,} z_h)$. Such function represents the paternalistic views on the effect of allocations on individuals.

The benevolence value judgment implies that the partial derivatives of w with respect to u^h or g^h. have positive values. Therefore, in conjunction with non-paternalism, benevolence implies that

$$\frac{\partial w(u^1, u^2)}{\partial u^h} = w_h > 0 \quad (h = 1, 2) \tag{67}$$

Pareto optimality

We have already explained that Pareto swf is a Bergsonian social welfare function. It embodies the value judgment of individualism, non-paternalism and benevolence. A Pareto optimal allocation maximizes a Paretian swf and is subject to production and material balance constraints. A competitive equilibrium is optimal in the Paretian sense that no alternative feasible allocation of commodities can improve the lot of one agent without worsening the conditions of some other individual. Equally important is the converse proposition that any given Pareto optimal allocation can be sustained by a competitive equilibrium. A prime achievement of welfare economics has been to establish conditions that are roughly speaking, necessary and sufficient for the validity of these conclusions in finite economies (i.e. economies in which the numbers of commodities and economic agents are finite). On the other hand, it is known that in non-finite economies, these propositions may fail even when the sufficient conditions of the finite case are met. (McFadden et.al, 1980)

In figure 5.9, the feasible utility combinations are those on or inside the utility frontier FF'. The Paretian swf w (u^1,u^2) gives rise to welfare indifference curves, such as wI^1 wI^*, which have slope

$$\frac{\partial u^2}{\partial u^1}\Big|_{dw=0} = \frac{w_1}{w_2} < 0 \tag{68}$$

The assumption of benevolence implies that the welfare indifference curves are negatively sloped. The higher indifference curves correspond to greater welfare. The utility combination $\propto^*$ on the utility frontier maximizes w over the set of feasible utility combinations. An allocation A* which generates the utility combinations $\propto^*$ is a Pareto optimal allocation. Different value judgments about the relative merits of the two individuals would be represented by a different welfare function. It would give rise to a different welfare function as well as different Pareto optimal resource allocations.

Compensated principal

We have already seen that the Hicks and Kaldor compensation test is ingenious. Successful attempts have been made to extend the set of situations, however, they were not practical. Therefore, it can be compared without the need to specify value judgments, which concern the relative merits of individuals. There is no need to construct a Bergsonian swf. If we assume that $v^h(y_h\ell,a_h\ell)$ then such utility is the individual h's utility he gets in situation ℓ, where $y_h\ell$ is h's income and $ah\ell$ is a vector of attributes of the situation. Now this depends on the context. An individual may set property rights regulating behavior defined implicitly as

$$v^h(y_{h2} - cv_{12}^h, a_2) = v^h(y_{h1}, a_1) \tag{69}$$

There are a number of alternatives assumed in situation 1 and situation 2. We have assumed that situation 2 is better for individual 1 than situation 1. In equation (69) cv_{12}^{h} is assumed to be positive. Suppose the individual is worse off then cv_{12}^{h} is negative. The individual must pay to move from situation 1 to situation 2. We have already seen that the Hicks-Kaldor compensation test recommends a move from situation 1 to situation 2. Compensation can be possible from situation 1 and situation 2. There are gainers in situation 1 and losers in situation 2. Suppose the gainers move from situation 1 to situation 2, then the gainers' move can compensate the losers who would still be better off in the same situation.

We can still recommend the move of the individual under a different situation

$$\sum_{h} cv_{12}^{h} > 0 \tag{70}$$

We do not consider that individual 1 gains in his utility. Our major concern is to examine the individual 1 who has utility gains that are worth more than the utility loss of another. Every individual is the best judge of their own utility, and so, an individual always tries to maximize his utility. Each individual is willing to pay more money to get the maximum utility. Sometimes a change in the price level may force him to pay more. He will pay that much more which can make him better off after the change. Most of the studies of cost-benefit analysis of the public sector investment projects are based on the aggregation of compensating variation. Compensation to farmers is an important subject of discussion in special economic zone projects which require the appropriation of farmers' lands. Industrialists also want to get more benefits from their investments. They expect that the government to provide various benefits for their investment. But the farmers should not have to sacrifice their livelihoods. There should be a good compensation package to farmers. Similarly, the land acquired for the projects such as nuclear plants, transport hubs, and ports can have provided higher environmental and economic value for the farmers. In determining how much compensation farmers should receive, planners should also think about the farmer's loss of income and occupation.

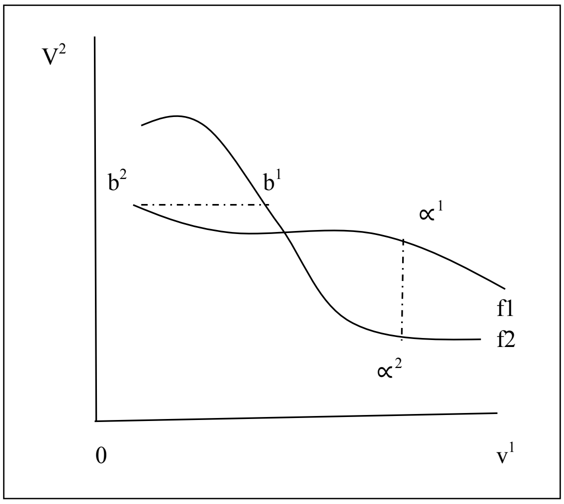


Figure 5.9. Aggregation of compensating variations for projects

Criticism

The theory is criticized on a number of points. Some criticisms are valid and discussed in detail as follows.

1. Interpersonal value judgment

If we assume that a change makes the rich better off and the poor worse off then the change passes the compensation on to rich people. This is because the rich could not compensate the poor and would still be better off. Since some individuals' utilities have increased while others' utilities have decreased, it is impossible to evaluate the change without making a judgment about the relative merits of the individuals and the existing distribution of income.

2. Change in utility

We have assumed that at the initial situation the utility distribution $\propto^1$ is generated. The new situation yields $\propto^2$. In the new situation $\propto^2$ it is possible to redistribute income from individual 1 to individual 2. This can be done by increasing v^2 and reducing v^1. The curve f_2 through $\propto^2$ is a utility feasibility curve which shows the utility distribution. This distribution can be achieved in situation 2 by transferring income between the individuals. In the figure f2 passes through a point b2. At this point, individual 2 has the same utility as in the initial situation and individual 1 has more utility. Therefore, it is impossible for individual 1 to compensate individual 2. If there is a move from situation 1 to situation 2 then individual 2 is better off. There is a potential Pareto improvement and the compensation test is passed.

We can show the reverse argument. Suppose it is assumed that at the initial period, the individual is at situation 2 where $\propto^2$ is observed. There is a move to situation 1 at $\propto^1$ where individual 1 is worse off and individual 2 is better off. These are two different competing situations.

Suppose we apply the compensation principal then we need to examine the utility feasibility curve f1 through $\propto^1$. It can be shown when the utility combination reaches $\propto^1$ by means of transfers of income between the parties. In this case, it is possible to reach b1 where individual 1 has the same utility as can be observed in situation 2. At this point, individual 2 has greater utility. Therefore, a move from situation 2 to situation 1 also passes the compensation test. This compensation test is known as the Scitovsky paradox. The compensation test may lead to cycles mainly because compensation is not actually paid to individuals. We can show this with the help of an equation and a diagram. We can define the compensation variation and its move from situation 2 to situation 1 as

$$v^h(y_{h2}, a_2) = v^h(y_{h1}, cv_{21}^h, a_1) \tag{71}$$

Equation (71) is only possible when $cv_{12}^h = -cv_{21}^h$. and is possible for all individuals. We must be sure that there will be no paradox. Now suppose cv_{21}^h is the compensating variation for the move from situation 2 to situation 1 then it is also the equivalent variation for the move from situation 1 to situation 2. The amount of money must be paid to h in situation 1 to achieve the same utility as the individual would get in situation 2.

Pareto efficiency and competitive markets

In the Pareto efficiency and competitive markets, the condition can be used to investigate for a particular situation. It means that a particular institutional framework leads to an efficient allocation of resources. In this section, we will examine the circumstances in which the equilibrium resource allocation in a market economy is efficient.

5.9 First theorem of welfare economics

The first theorem of welfare economics is based on two assumptions.

1. In an economy, all commodities are competitive. The equilibrium in the economy is Pareto efficient.
2. There is a market for all commodities. Each commodity is produced in the economy and the consumption of the commodity adds to the utility function.

As we have assumed earlier, in an economy, all markets are competitive. Consumers and producers believe that their decisions have no effect on prices. In order to reduce the complexities, we assume a simple economy with two markets and two input markets. Each market has two demanders and one supplier. In both markets the prices of the commodities are regarded as the parameter. Competition is possible because there are large numbers of traders of the commodity from both sides of the market. Individual h earns an income from selling input z_h. An individual's own share in the profits of the two firms in the economy maximizes the utility function u_h (xh_1,xh_2, zh). This is subject to the budget constraints and is defined as

$$\sum_i p_i x_{hi} - w_h z_h - R^h = 0 \qquad \text{h = 1, 2} \tag{72}$$

where $p_{i=}$ the price of the commodity

I,$w_{h\,=}$ the price of input h and

R^h: = the non-wage income of h derived from share ownership.

The proportion of firm i owned by individual h is β_{hi}. The profit of firm i is $\pi_{i.}$ We can define the non-wage income as

$$R^h = \sum_i \beta_{hi} \Pi_i \quad \text{h = 1, 2} \tag{73}$$

Where $0 \le \beta_{hi} \le 1$, similarly $\sum_h \beta_{hi} = 1(i = 1, 2)$. Firm i seeks to maximize its profits which is subject to its price and output. The profit function is defined as

$$\Pi_i = p_i x_i - \sum_h z_{ih} \quad \text{i = 1, 2} \tag{74}$$

We have already pointed out that the profit function is subject to the production function, and is defined as $x_i = f^i(z_{i1},z_{i2})$ .We can show the equilibrium of the economy with the help of the Pareto efficiency.

Consumer's choices

In an economy, consumers choose certain consumption bundles to maximize their utility. The individual indifference curves for two goods are tangent to the budget line. Therefore, the utility out of two goods is given as

$$MRS'_{21} = \frac{u_1^1}{u_2^1} = \frac{p_1}{p_2} = \frac{u_1^2}{u_2^2} = MRS_{21}^2 \tag{75}$$

Equation (75) clearly satisfies the condition of efficient consumption. Each consumer always compares the marginal rate of substitution between two goods. In a competitive economy, the goods have the same price. Therefore, the marginal rate of substitution is always the same.

Supply of inputs

Individual h always compares the marginal rate of substitution between the supply of input h and the consumption of good i. This is considered to be the ratio of the market prices of the input and good i. Therefore

$$MRS_{iz}^{h} = -\frac{u_z^h}{u_i^h} = \frac{w_h}{p_i} \tag{76}$$

Now, firm I which is supplying goods in the market tries to maximize profit (π_i) and chooses z_{ih} to satisfy the following equation.

$$p_i f_h^i = w_h$$

$$h, i = 1, 2 \tag{77}$$

The value of the extra output of good i is produced by an additional unit of input. This input is equal to the cost of a unit of input h. Equation (77) can be modified as.

$$f_h^i = w_h / p_i$$

From equation (76), we can further modify (77) to

$$MRS_{iz}^{h} = -\frac{u_z^h}{u_i^h} = \frac{w_h}{p_i} = f_h^i \tag{78}$$

The efficient input supply satisfies the condition in the equation (78). This is mainly because consumers face the same relative prices for goods. Firms also face the same relative prices of inputs.

Input use

The first principle of any firm is to maximize profits. Profit maximization is possible by reducing the cost, and requires the firm to choose a suitable input mix. Sometimes, a firm has to choose between using more labor or more capital. While doing these adjustments, a firm must see that its isoquant line is tangent to its isocost line. Alternatively, if we divide the condition z1 by the condition z2we get the following equation

$$MRTS_{21}^{1} = \frac{f_1^1}{f_2^1} = \frac{w_1}{w_2} = \frac{f_1^2}{f_2^2} = MRTS_{21}^{2} \tag{79}$$

The efficient input use condition is satisfied. This is because the firm faces the same relative prices for inputs.

Output mix

If we combine equation (77) and equation (75), we get the following equation

$$MRT^{i}{}_{21} = \frac{f_1^2}{f_1^1} = \frac{f_2^2}{f_2^1} = \frac{p_1}{p_2} = \frac{u_1^1}{u_2^1} = \frac{u_1^2}{u_2^2} = MRS^{h}{}_{21} \tag{80}$$

In equation (77) $f_h^i = w_h / p_i$. The marginal rate of transformation between outputs is equal to the consumer's marginal rates of substitution. This substitution is observed between the two goods, and thus, the condition for an efficient output mix is satisfied. We have shown that the equilibrium of this simple competitive economy satisfies the necessary conditions for Pareto efficiency. Suppose the consumer's utility functions are strictly quasi-concave and the production function is convex, then the necessary condition is also sufficient. But for equilibrium, it will also be efficient. Such a condition brings out very clearly the role of prices in achieving an efficient equilibrium. The choices of individuals are guided by the prices. Consumers face the fact that all relative prices are the same. It means that in equilibrium they all place the same relative valuation on goods and inputs. Therefore, no reallocation of goods or inputs can achieve a Pareto improvement. To put it differently, all gains from mutually advantageous trade is the equilibrium and the equilibrium prices have been exhausted.

Limitations

We have to assume that preferences and production possibilities are convex. The efficiency conditions are sufficient as well as necessary. This assumption is substantive. There are many points where such a model is not valid. The equilibrium of a complete set of competitive markets is Pareto efficient. This does not imply that any particular market economy achieves a Pareto optimal allocation. First, the conditions of the theorem cannot not be satisfied. The market economy may not be efficient and therefore, cannot be optimal.

a) **Firms and consumers may not be price takers**
In a market economy, there are monopoly sellers for different products. Consumers will not take their prices as parameters. Therefore, prices will not measure the marginal value of activities to all consumers. The efficiency conditions will be violated. This is because different consumers have different marginal values.

b) **Incomplete Markets**
In a simple economy, there are markets for all commodities. Various commodities are regularly traded in the market. But for certain kind of commodities a market does not exist. For example, the market for clean air does not exist. There are certain commodities which are demanded for future purposes but markets for these are developing very slowly. In such a market no prevailing prices exists and no clear guidance is available to consumers. The marginal valuations of activities are likely to be different and will lead to inefficiency for commodities as well as for the market.

c) **Disequilibrium in markets**
In a market economy, if markets are not helping to set up the single relative price then they will not help the individual in marginal valuations. This will affect the allocations which will then be inefficient.

d) **No other feasible method**
In market economies, if the prices are not at equilibrium then an efficient mechanism is not possible. But at the same time, no other feasible method exists in the market.

e) **Non Pareto optimal**
We saw that if the first theorem of welfare economics condition is satisfied then a market allocation is ensured. This market allocation is Pareto efficient, but the method is not Pareto optimal. The efficient allocation achieved by a market economy may be highly inequitable. It may not maximize welfare functions, which are based on value judgments that favor equity.

5.10 The second theorem of welfare economics: (STWE)

We have pointed out that the first theorem of welfare economics has a number of limitations. The second theorem of welfare economics has certain advantages over the first theorem of welfare economics. To explain, if all consumers have convex preferences and all firms have convex production possibility sets then Pareto efficient allocation can be achieved. The equilibrium of a complete set of competitive markets are suitable for redistribution of initial endowments.

In the second welfare theorem, Pareto efficient allocation is at A* . In such an A* allocation, an individual h has consumption x^{h*}. Firm j produces output y^{j*}. We know that A* is a point where all consumers will have the same marginal rates of substitution between all pairs of commodities. Let's assume that $\dot{p}_i$ denotes the consumers' marginal rate of substitution between commodity i and commodity 1. We can represent this in an equation as

$$\dot{p}_i \equiv MRS^h_{i1}(x^{h*}) \quad (i = 2,\ldots..,n;\ h = 1,\ldots..,H) \tag{81}$$

Assuming that the right hand side of the equation is 1.

$$\dot{p}_i \equiv 1 \tag{82}$$

We can interpret $\dot{p} = 1(1, \dot{p}_2,\ldots, \dot{p}_n)$ as the set of relative prices, with commodity 1. If we redistribute the individual's initial endowments into commodities and shareholdings then in terms of the equation

$$\dot{p}x^{h*} = \dot{R}^h \equiv \dot{p}\,\dot{x}^h + \sum_j \dot{\beta}_{hj}\,\dot{p}\,y^{j*} \quad (h = 1,\ldots,H) \tag{83}$$

where

$\dot{x}^h$ = h's initial endowment after the redistribution,

$\dot{\beta}_{hj}$ = the post redistribution shareholding in firm j

$\dot{p}\,y^{j*}$ = the profit earned by firm j.

There are two apparent problems with this redistribution. In the original explanation, an individual's initial endowments $\bar{x}_h$ consists only of labor time. It is not possible to transfer such endowments from one individual to another. They are inseparable and indivisible from one individual to another individual. An individual utilizes their time for productive purposes.

Firms also produce the goods with constant returns to scale. A firm's breakeven point is at y^{j*}. A firm also faces price and input costs. If we redistribute the shareholding of a firm then there will be no effect on budget constraints. Even if the firm earns a zero profit budget constraints will also not be affected. To solve the problem of an individual and a firm, we need to use transfer T^h where $\sum_h \dot{T} = 0$ which is measured in terms of a number. Suppose $\dot{T}^h > 0$ then individual h must pay tax of $\dot{T}^h$ of good 1. There is a difference between holding and initial holding of goods. If the stock is available, then the individual's holding of good 1 will get automatically reduced. Let's suppose that

$$\dot{x}_{h1} \equiv \bar{x}_{h1} - \dot{T}^h < 0$$

An individual sometimes pays tax after selling some of his holding of other goods. When $\dot{T}^h < 0$, the individual receives a lump sum subsidy by holding on to his good 1. Such holding of goods is allowed to increase. If we assume that a subsidy exists then the lump sum transfers are written as

$$\dot{p}x^{h*} = \dot{R}^h \equiv \dot{p}\,\bar{x}^h + \sum_j \beta_{hj}\,\dot{p}\,y^{j*} - \dot{T}^h = R^h - \dot{T}^h \quad (h = 1, 2,\ldots,H) \tag{84}$$

Where R^h = the individual's full income at the original initial holdings.

We can modify the above as

$$\dot{T}^{h} = p(\bar{x} - \dot{x}^{h}) + \sum_j (\beta_{hj} - \dot{\beta}_{hj})\, p\dot{y}^{j*}$$

We already know that the lump sum transfer approach is equivalent to a redistribution of the initial holding of all goods and shares, if we assume that equation (83) or equation (84) have the same value. Therefore the value of h's full income is in terms of a number equivalent to the cost of the Pareto efficient commodity bundle x^{h*} at the prices $\dot{P}$. Suppose h is maximized at $u^h(x^h)$ then it is subject to budget constraints $\dot{P} x^h \leq \dot{R}^h$. An individual would set their marginal rate of substitution between commodity i and 1 which is equal to the relative price $\dot{p}_i$. Their demand for good i at relative prices $\dot{P}$ would be equal to the amount of good i .Therefore the individual receives A* as the Pareto efficient allocation.

Suppose firm j face the relative prices $\dot{P}$ and it would choose to produce y^{j*}output. The firm will want to produce the optimal quantity to maximize profit. Therefore, Pareto efficiency requires the firm to have a marginal rate of transformation. The marginal rate of technical substitution and marginal products are equal to the relative commodity prices. The relative prices of p_i are equal to the consumer's marginal rates of substitution at the Pareto efficient commodity bundles. The profit of the firm py^j is maximized at the Pareto efficient output y^{j*}. We have already pointed out that relative prices are the basis for the demand and supply of goods. These relative prices are identical to the required Pareto efficient allocation which we have defined at A*. The supply and demand decisions are based on relative prices p and they are compatible. In a market economy, the equilibrium is based on relative price p. This is a suitable choice of endowment which achieves the desired Pareto efficient allocation and is the equilibrium price in a competitive economy.

Criticism

1. **Incomplete and non-competitive markets**
   We know that in the real world, markets are neither competitive nor complete. We already know that the redistribution of the initial endowments allows market to allocate resources efficiently. Suppose a firm produces output based on the prices in the market. It uses prices as its parameter for production. If a firm has monopoly power, then the market price does not matter. Most of the time firms realize that prices are affected by the production decisions they make. Supply plays an important role, too, not just prices.

2. **Convex technology**
   The preferences and technology for a firm are no convex. The relative prices p* does not support the desired efficient allocation. This can be expected in a competitive economy.

3. **Redistribution**

   It may not be possible to make the kind of initial redistributions required for this theorem. It is essential that the redistributions are lump sum. The individual may not be able to alter the amounts paid or received. The taxes which should not affect their behavior at the margin; otherwise, the reverse will happen. It is possible for an individual to alter the amount paid or received under the distribution by changing their demands or supplies. The effective prices individuals face are not the market prices and will differ across individuals. If prices do not adjust to the same set of relative prices $\dot{P}$ then efficiency conditions will be violated. In a simple one-period economy, shareholdings or initial endowments are exogenous and they are not affected by decisions made by individuals. In a more complete model, such decisions would be endogenous. Individuals could accumulate shares by saving and alter their endowments by investing in human capital to raise their skill levels. Redistribution from individuals with large shareholdings or valuable initial endowments is effectively a tax on savings or on human investment. It is no longer a lump-sum tax. The relative prices the individuals face would then vary depending on their shareholdings or skill levels and also across individuals.

4. **Individual abilities**
   We have seen that individuals save money and buy shares of firms. The profit earned from shares is invested to enhance human skills and capabilities. But redistribution from individuals with large shareholdings or initial endowments leads to problems and affects human development and skills. There is no longer a lump-sum tax. This means that at relative prices the individuals face the prices which are dependent on skills or share holdings. It is different for different individuals.

5.10 Market failure and second best

We have seen that in a competitive market economy, the Pareto efficient allocations are in equilibrium. We know that a market may fail to allocate resources efficiently. When a market fails, the role of government becomes very important. We need to understand how a government solves such a problem. There are different markets such as commodity, debt, equity, derivative and real estate markets. The role of government in these markets is important. It is assumed that all markets perform in a certain way. But this is not true in the long term. In each market, over a period of time, a monopoly may develop. Such markets may work inefficiently and this leads to misallocation of resources. There is an interdependence of market and economic agents in the market.

Most of the time, the public and its access to resources gets affected. Such change is damaging to the fundamental rights of the individual. Such changes and monopoly power also affect property rights, information and transaction costs.

The causes of market failure

In any competitive market, individuals not only exchange commodities but they have the right to use the commodities in a particular way for a particular length of time. For example, when any individual buys a car, he is not just buying a physical asset but he is buying the rights to use that asset in a certain specified way. Such rights include driving on public highways, with a certain speed or carrying specific passengers if the car is used for commercial purposes with a valid permit number and for a certain period. He has also the right to park his vehicle at specified parking sites. The owner of a car can prevent other individuals from using the car without his consent.

Similarly when any employer hires a worker then the employer has certain rights to direct the worker to perform certain activities within a specified period of time. Similarly, the employer must provide decent services at the workplace such as clean toilets, electricity, clean air, perhaps a child care center for women employees with children.

Such control is defined as the property right attached to each commodity. When a consumer buys a commodity, the rights get transferred to the individual. In the barter exchange, suppose the two goods are not equal for two consumers then there would be an inefficient allocation. But such inefficient allocation can be corrected by rearranging the consumption vector.

Rearranging the consumption vector will result in at least one individual better off and no one worse off. There is a possibility for mutually advantageous trade between the two individuals. Suppose that inefficiency exists then it is possible to exchange and make one individual better off. But such inefficiency can explain the existence of potentially mutually advantageous trades or profitable production decisions. Hence a number of questions arises. Firstly, why is a particular resource allocation mechanism inefficient? We could rephrase this by asking why such advantageous or profitable exchanges or production decisions do not occur. Given that individuals would wish to make themselves better off by trade or production, inefficiency can only persist. Firstly, individuals do not have sufficient control over commodities to affect profitable or advantageous exchanges and production. Secondly, transaction and information costs exceed the gains from trade. Thirdly, the individuals cannot agree on how to share the gains from their mutually advantageous exchange.

Inefficient control: imperfect excludability and non-transferability

The property right is defined as an individual's control over commodities. The property right can be incomplete because of imperfect excludability or non-transferability. Imperfect excludability emerges when effective control of a commodity is not conferred on a single individual but on a group of individuals. Control over assets means the ability to determine who shall use the assets, in which circumstances and for how long. When control of assets is vested on a group then an individual who wishes to acquire that control enters into a contract with all the individuals in the group. But this is difficult and costly. As such, no individual can acquire exclusive control. These assets are known as common property resources. Examples are common grazing land, ocean fishing, public parks and beaches, rivers and ground water resources, and public transport. The control over assets is defined in terms of the ability to exclude any individual, to determine who shall not use the commodity or asset. Excludability's first requirement is legality. The legal right to exclude must be supported by the ability to enforce that right. In some cases enforcement of the right to exclude is simple and inexpensive.

The owner of a house, car or land has the legal right to exclude others from using or occupying it. In the case of land, a farmer can put up a stone wall or compound to enforce his legal right to the land. We often see boards on land put up by the owners. In the case of cinemas, theaters, football stadiums, the owner must install box offices and gates, print tickets and employ staff to ensure that all who enjoy the entertainment must have tickets. At the same time, those who have paid for lower priced seats must occupy the low-priced seats.

Those individuals who want to prevent unauthorized use of their property must devote some resources for the purpose of detection and punishment of unauthorized uses. Such cost is known as the exclusion cost, and depends on the legal and social framework of the economy and the technology. Sometimes, a person can use alternative strategies such as a high-powered electric wire fence or theft alarms. Such strategies could reduce the exclusion cost. But perfect excludability does not exist. The potential advantage of trade or exchange will not take place. A potentially beneficial production may not occur if individuals making the production decision cannot exclude other individuals. A farmer has little incentive to plant crops if the law permits anybody to harvest the crops without the consent of farmers. Lack of exclusion may affect the benefits from higher output, because the potential lower gain may discourage an individual from incurring more costs which may be necessary to produce the extra output.

Non-transferability arises when the legal right to exclude is vested in a single individual and exclusion costs are low. The owner of the asset does not have the unrestricted legal right to transfer use or ownership to just any individual by any condition or term. Lack of transferability may take the extreme form of a complete absence of the right to transfer any of the property right. Take for example the squatters who occupy government land without permission. The squatters do not have the right to rent or lease the land, but by occupying it, can exclude others from using the land.

In terms of the labor market, individuals own their labor and can hire themselves out for limited periods. But the law does not permit the transfer of permanent control over labor. By law, slavery is illegal. An exchange of manpower will be there when the maximum or minimum price of labor or minimum wage is fixed by law. The trade of commodities will take place in a prescribed format. There are certain restrictions on which individuals may trade their labor, for instance, young children are not allowed to work. For example, youth aged less than 18 years cannot buy alcohol from licensed premises but youth can consume alcohol. A taxi or jeep cannot carry passengers unless they have a licence from the local transport authority. If they carry more passengers than they are allowed to, then the police can fine them or put them in jail.

Information and transaction costs

The exchange of commodities requires an exchange of information. The potential buyers and sellers must be known. The quality of goods or services to be exchanged and the property right must be checked. But getting all information required is costly for the individual. Checking the quality of a commodity and discovery of innovative products are costly affairs. There is also the cost of negotiation and specifying the terms of exchange and of enforcing them. Such information and transaction costs may be high therefore, the potentially advantageous contracts are not made or a contract may be incomplete and leave some potential gains unexploited.

Bargaining problem

Trade will occur when both parties will bargain and agree to sign a contract which is mutually advantageous. But if the gains are not positive then trading parties will fail to bargain. Sometimes, bargains can be very costly. Both parties compete to gain the most and they cannot come together at what benefits them both. If the contract is not signed there are other alternatives available. When both parties can find multiple contracts, the bargaining will fail. The exchange will take place when both parties will be better off. Suppose the trade is taking place at a fixed level in the market then there is no scope for bargaining. It is natural that if both gain after trading that a trade will take place. If someone is worse off after a trade then they will not sign a contract. In a competitive economy, if one price prevails in the market then no buyer or seller will trade. There is also no scope for bargaining. No buyer will pay a price above the market price, nor will any seller sell below the market price.

5.12 Instances of market failure

We have already seen that there are different examples of market failure. Suppose the market is imperfect and it does not allow the parties to maximize their gains, then government efforts are needed. Such efforts increase the efficiency in the market. Before going into detail, we need to understand the example of monopoly and market inefficiency.

Monopoly

The monopoly form of market is based on the profit maximization principle. In a monopoly type market, the marginal revenue is equal to the marginal cost, as shown in figure 5.10. In the figure, q_m and p_m are the monopoly output and price points. Suppose a consumer pays p_m , the price for the commodity purchased then such a situation is Pareto inefficient. The monopoly firm has a marginal cost MC, where MR = MC and AC= MC. But the consumer pays price p_m which is greater than MC. Suppose a consumer pays less than p_m but more than MC for the additional unit then he continues to pay $p_m q_m$. At this point, both the producer and the consumer are better off. At this point, the monopoly firm increases its profits from output. The consumer also gets extra at a price less than the actual value to the consumer.

Pareto efficiency is achieved. This is because the consumer is willing to pay for the unit. In the figure, this is measured by the height of the demand curve, which is equal to the cost of an extra unit.

Also in the figure, q*is an output which shows the demand curve cutting the marginal cost curve. The monopoly price and output are inefficient. There are potential gains to consumers and the monopolist of output which is increased from q_m to the level of q*.

Now the consumer and the monopoly firm can sign a contract, which can lead to an efficient output. Suppose the monopolist agrees to sell output q* at a price p*, a price equal to the marginal cost. The profits of the monopoly firm would fall to $(p_m\text{-}p^*).q_m$. The consumer pays a lump sum to the monopoly firm. Both can be better off. This effect can be measured by the area $p_m$bap*. This area exceeds the $(p_m\text{-}p^*).q_m$ by G. The gains are available for division between the monopolist and consumers.

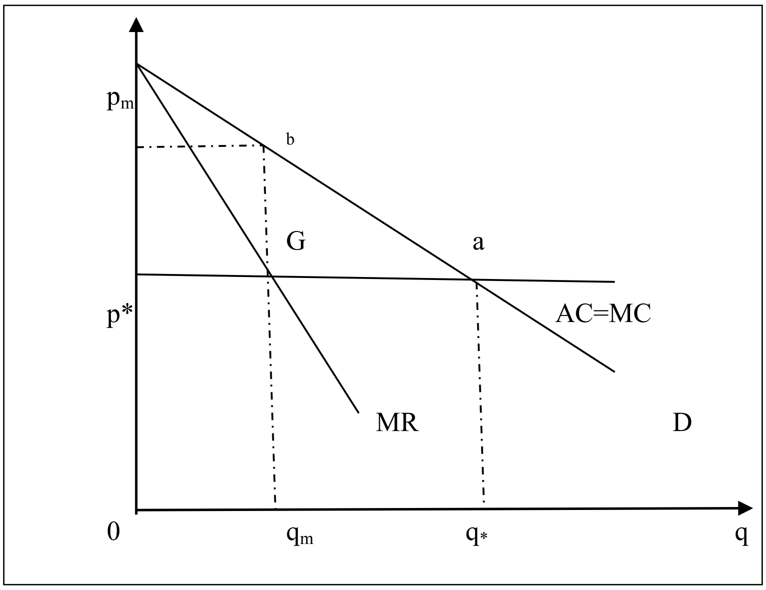


Figure 5.10. Monopoly output and price

In figure 5.10, the marginal cost is higher for the monopolist. It is because a consumer and a producer fail to trade and get mutual satisfaction. They may not agree to the division of the gains from the increase in output. There is a high cost associated with locating and organizing consumers. They may not be able to agree on how the burden of the lump sum payment should get shared. Sometimes it is difficult to prevent consumers who do not contribute to the lump sum payment from enjoying the benefits from the contract with the monopolists.

As a result, monopolists may be forced to make individual contracts with consumers. The monopolist cannot prevent resale at this point; he will set the same price in each contract. Such situation is called the normal inefficient monopoly situation. It is the point where the demand curve is downward sloping, where MR = MC. The consumers will treat the prices as a parameter. An individual consumer cannot follow the monopolist at a lower price. This is because the monopolist is not able to prevent resale.

Externality

An externality means some of the variables affecting a person's decision is utility or profit. Such utility or profit is under the control of another decision taker. For example, a brewery is located downstream of a chemical factory. The chemical factory pumps effluent into the stream, polluting the water. This water is used by the brewery in its production of beer. Effluents in the stream affects the quality of the water, affecting the quality of the beer produced by the brewery. This externality is detrimental but in other causes there may be beneficial externalities. Let's consider a bee keeper whose bees are situated next to an apple grower. The bees will feed off the apple blossoms, and pollinate the apple trees, benefiting the orchard owner. At the same time, the beekeeper benefits. This is an example of externality. We can see that the beekeepers' output of honey depends on the number of apple trees. The output of apples also depends on the number of bees that are able to pollinate the apple trees. We can see that there are producer-producer, consumer-consumer externalities. But producer-consumer and consumer-producer externalities also exist.

Negative externalities can lead to inefficiencies in production. This is because the chemical factory will set the level of output to maximize profit. Such factory will not take into account the effects of the resulting pollution on the profits of the brewery. Let's suppose that the brewery may be willing to pay the chemical factory to reduce the amount of effluent. This payment to reduce effluents reduces the profits of the brewery. Therefore, the firm's output of chemicals and the effluents are produced in a fixed proportion. The reduction in effluents may require a reduction in the output of chemicals. Suppose the reduction in the brewery's cost is above the reduction in the chemical factory's profit. There are potential gains from trading. The initial level of effluent cannot have been efficient. But again it depends on who takes the initiatives.

5.13 The Coase theorem

Suppose we denote x as the firm's effluents. The chemical firm produces output in a fixed proportion. The profit function of the firm is expressed as B(x). The damage inflicted downstream because of the pollution is D(x). In the diagram, the marginal benefit B'(x) and marginal damage D'(x) from pollution are shown. If we assume that the profits of the two firms is measured as the social value of their outputs then the effluent does not impose costs on any individual firm.

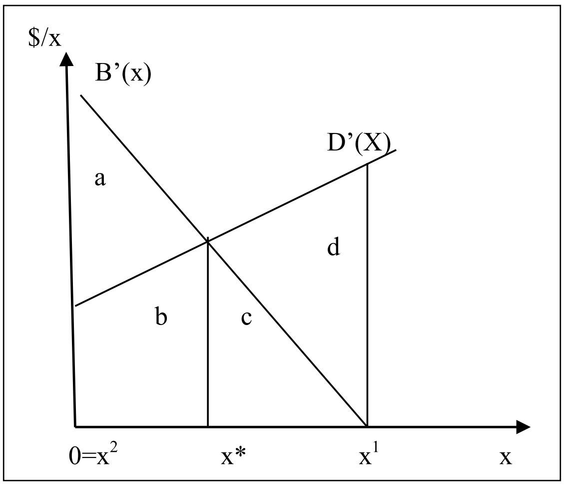


Figure 5.11. Marginal cost and damage by pollutant firm

The profits of the two firms measure the social value of their output. The effluent level of pollution maximizes the total profits of the two firms. The efficient level of pollution maximizes the total profits of two firms. The efficient level of pollution is x* which satisfies the following condition

$$B'(x^*) - D'(x^*) = 0 \tag{87}$$

There are two alternative legal situations that determine how each firm behaves with regards to the stream.

a) Permissive: The chemical firm has the legal right to discharge effluent into the stream. It controls x and would choose a level of x^1 where B'= 0. The level of pollution is largely inefficient because its effects on the brewery are ignored by the chemical firm.
b) Restrictive: Suppose the chemical factory has no legal right to discharge effluent and the brewery can prevent it by a court order. In the diagram, it is shown that the brewery can control the level of pollution and chooses a level x2 = o. The cost from the effluent is minimized. A zero level of pollution is also inefficient. The brewery ignores the effect of its choice on the profits of the chemical factory.

Now x1 and x2 are inefficient points in the diagram. If we assume the pessimistic view and the chemical firm reduces effluents from x1 to x* then the chemical firm's profit is c. The reduction in the brewery's cost is c + d. The contract which shows the brewery paid to the chemical firm is c + θd (0 < θ < 1). This is in exchange for a reduction in pollution from x^1 to x*. An efficient allocation of resources is achieved and both firms are better off. The brewery's profits would increase by (c + d)-(c + θd) = (1 - θ)d. The chemical firm's profit would be –c+(c+ c + θd) = θd. The contract would generate a combined gain from trading of d.

From the legal point of view, a contract by the chemical factory to pay the brewery θ a + b in exchange for an increase in effluents from zero to x* would lead to an efficient level of pollution. The contract makes both parties better off. The payment to the brewery would more than compensate for the increase in cost b. The chemical factory sees an increase in profit a + b, which would more than cover the payment to the brewery. The contract would split the gains from trade between the two firms.

In the Coase theorem, bargaining can achieve an efficient allocation of resources whatever the initial assignment of property rights. Suppose the affected parties can contract with each other, then the externality will be internalized. The party who has the legal right to control the level of pollution will take into account its effects on the other. The initial assignment of rights does not affect the distribution of income. A permissive law can lead to efficient bargaining and increase the polluter's profits by θd and under a restrictive regime, an efficient bargain increases it by θa. The externalities exist because of a number of reasons.

In a small number externality situation, there may be a failure to agree on the division of the gains from a move to a more efficient allocation. In a large number externality situation, the absence of contracting between polluters and victims may arise from any number of reasons. The free rider problem is likely to be important.

The general reductions in pollution always benefits victims in that area. Therefore, individual victims will have a reduced incentive to contract individually with the polluter. A contract between the polluters and voluntary association of victims will thus have benefits. In such a contract it would be difficult to exclude those who do not pay. By law, it is not clear that a polluter has a legal right to pollute or the victims have the legal right to be protected from any pollution.

Going to court is costly for both parties and individuals. The market for pollution is not competitive. A single polluter may have many victims and he may act like a monopolist. Therefore, there is interest in public intervention as a solution to the externality problem. Ideally, the government should be strict with firms that are polluting land, air and water. The future consequences of pollution affect natural resources and people. Firms need to pay taxes and invest funds for environmental sustainability. But if there is corruption at government offices then controlling pollution is a difficult task (Gravelle & Rees, 2008). The model is more practical and widely used to arrive at the right conclusion. It is difficult to explain different examples and prove this theorem. It is important to understand what public intervention can achieve when negative externality exists.

In a developing country like India, environmental issues are widely discussed but the government does not take any action against polluting firms. Sometimes, the policy of moral suasion is most important to control pollution. Production, nature of technology, consumer preferences, land use are also important factors. We are not able to discuss all these issues at this time. Common property resources and the government-owned resources are exploited to great extent but there is no accountability for this exploitation.

Questions

1. Explain Walrasian equilibrium along with properties in a competitive economy.
2. General equilibrium exists for an economy. Explain and give proof.
3. What is the tatonnement process? How does it help achieve stability?
4. Discuss the Edgeworth exchange theory and provide some criticisms.
5. Discuss the Edgeworth box diagram in detail.
6. Write a note on the following:
   a) Welfare function and Pareto criteria
   b) Compensation principle
7. What is the first theorem of welfare economics? Name some of its limitations.
8. What is the second theorem of welfare economics? Why is it criticized?
9. Write a note on market failure and second best.
10. What are the causes of market failure? Give some instances.
11. Critically expound the Coase theory.

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