Fluid Mechanics and its Applications

Arkady Tsinober

An Informal Conceptual Introduction to Turbulence

Second Edition of An Informal Introduction to Turbulence

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FLUID MECHANICS AND ITS APPLICATIONS
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With 115 Figures

Springer

Prof. Dr. Arkady Tsinober
Tel Aviv University
Fac. Engineering
69 978 Tel Aviv
Israel
tsinober@eng.tau.ac.il

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TO MY WIFE

PREFACE TO THE SECOND EDITION

The subject of turbulence remains and probably will remain as the most exciting one for the mind of researchers in a variety of fields. Since publication of the first edition of this book in November 2001 a number of other books on turbulence have appeared, for example Bernard and Wallace (2002), Oberlack and Busse (2002), Foias et al. (2001), Biskamp (2003), Davidson (2004), Jovanovich (2004), Sagaut and Cambon (2008) to mention a few. So one has to ask again the question why a second edition of one book from a field of so many on the same subject? Does it make any difference?

There are additional reasons apart of those given in the first edition. One of the basic premises of this book is that *We absolutely must leave room for doubt or there is no progress and no learning. There is no learning without posing a question. And a question requires doubt... Now the freedom of doubt, which is absolutely essential for the development of science, was born from a struggle with constituted authorities...*R. Feynmann (1964). This is closely related to the term 'conceptual ': the book has now a different title *An informal conceptual introduction to turbulence*. One of the main features of the first edition was indeed its conceptual orientation. The second edition is an attempt to make this feature dominant. Consequently items which are secondary from this point of view were reduced and even removed in favour of those added which are important conceptually. This required addressing in more detail most common misconceptions, which are consequences of the profound difficulties of the subject and which travel from one publication to another. Consequently a one page Appendix D listing some of these misconceptions in the first edition became chapter 9 titled *Analogies, misconceptions and ill defined concepts*. Other main additions include sections on ergodicity, Eulerian versus Lagrangian descriptions, validation of theories in chapter 3; a section on anomalous scaling and ill-posedness of the concept of inertial range in chapter 5; a section on the Tennekes and Lumley balance in chapter 6; and a section on mathematics versus turbulence in chapter 10. Along with a number of minor changes and (hopefully) improvements throughout the whole text, new material was added to the section dealing with issues on the role of strain and its production (which unlike enstrophy production is a local process), nonlocality and fluid particle accelerations all in chapter 6. The Bibliography is changed not much from the first edition, with a marginal number of items dropped and only a few added. Those few are generally recent publications that can help guide a reader through the recently published. A characteristic feature of the text is a considerable number of intext citations which somewhat has been increased in this edition. The main aim is an extensive treatment of the dialogue in the turbulence community with an emphasis on problems of a conceptual nature. All this resulted in an increase by one third as compared to the first edition. This is in spite of the original intention (and the advice by H. Tennekes) not to increase the number of pages and even to make the second edition 'thinner'. It is my opinion that today thick books aiming to describe 'everything' are of little value for several reasons. First, the amount of information is not digestible for an individual. Second, in spite of frequent claims of considerable progress, the Saffman ratio (see the Preface to the first edition below) still remains the only genuinely small parameter in turbulence. Unfortunately, there continues to be a major over-production of publications, without any real breakthrough in understanding. In opinion of this author, the overabundance of literature can only confuse, rather than clarify the real issues, and so it seems that the set of potential readers of thick books (and, unfortunately, not only thick ones) is a pretty small subset of the whole, which is hopefully not a set of measure zero. Thus the next edition with the title *A conceptual introduction to turbulence* should be about half as large as to the first edition, which is a really hard, but hopefully not impossible task.

The present edition was influenced considerably by the Lectures on *Conceptual Aspects of Turbulence and Approaches to Turbulence Research* given by the author at the Imperial College in 2007 and 2008 (http://www3.imperial.ac.uk/_ mathsinstitute/programmes/research/turbulence/marie_curie_chair and/or http://www.eng.tau.ac.il/ tsinober/). As previously the book is intended for as broad a readership as possible with the aim of making it interesting and useful both to graduate students and scientists in all the above mentioned fields. It is hoped that this aim is relatively realistic due to the informal nature of this book, with its emphasis on turbulence as a physical phenomenon, observations, misconceptions and unresolved issues rather than on conventional formalistic elements and models. However, like anything/everything related to turbulence it is not easy.

The list of acknowledgements is again too long to be reproduced here. I am grateful to all those who responded to my queries and requests.

Corrections will be placed at http://tau.eng.ac.il/ tsinober/book. Suggestions and criticisms are very welcomed at tsinober@eng.tau.ac.il.

Tel Aviv, Israel and London, UK A. Tsinober

April, 2009.

PREFACE TO THE FIRST EDITION

Lightly amended

Over the last decade a number of books on turbulence have appeared. To mention a few: Biswas and Eswaran (2000), Bohr *et al.* (1998), Chorin (1994), Durbin and Pettersson (2001), Frisch (1995), Holmes *et al.* (1996), Lesieur (1997), Libby (1996), Mankbadi (1994), Mathieu and Scott (2000), McComb (1990), Piquet (1999), Pope (2000), Scott and Mathieu (2000). So why one more book on the subject? Does it make any difference?

The key words are *informal introduction*. The book, which is *not a textbook*, is essentially an *introduction*, and it is an *informal* introduction that, as far as possible, presents its material in a qualitative form. There are several reasons why an introduction to turbulence should be as informal (in several meanings) as possible.

First, there seems to be little chance in the foreseeable future of creating a pure, formal theory of turbulence – even for its simplest cases. To quote A. N. Kolmogorov (1985): *I soon understood that there was little hope of developing a pure, closed theory, and because of absence of such a theory the investigation must be based on hypotheses obtained on processing experimental data*, see Tikhomirov (1991, p. 487). This refers – in the words of E. Spiegel – to the big T-problem, i.e. the true dynamical problem (most probably) described by the Navier–Stokes equations, which is the main focus of this book; surrounding the big T-problem there are several 'little t-problems' such as turbulent diffusion or more generally behaviour of passive objects in real turbulent or some artificial random velocity field ('passive turbulence'), Burgers and wave "turbulence".

Indeed the heaviest and the most ambitious armoury from theoretical physics and mathematics was tried for more than fifty years, but without much success: genuine turbulence, the big T-problem, as a physical and mathematical problem remains unsolved. There is even no consensus on what is (are) the problem(s) of turbulence[1], neither is there an agreement on what are/should be the aims/goals of turbulence research/theories and what would constitute its (their) solution. Therefore lots of formalisms are avoided, since the methods mostly brought in from linear analysis (such as

[1] *One of the things that I always found troubling in the study of the problem of turbulence is that I am not quite sure what the theoretical turbulence problem actually is... One reason I think we have so much difficulty in solving it, is that we are not really sure what it is* (Saffman, 1991).

various decompositions, perturbation methods, etc.) failed, and genuinely nonlinear analytic methods applicable to turbulence mostly do not exist.

Second, the existing theoretical (mathematical, physico-theoretical and traditional fluid mechanical) material is rather complicated and extremely large in scope. The same is true of the experimental information (laboratory, field and numerical). Many existing books are overloaded with technical details, a notable exception being the most successful course by Tennekes and Lumley (1972) and also the book by Pope (2000). The unwary reader is totally lost in the enormous ocean of existing references. Therefore highly technical information has, for the most part, not been included in this volume. Instead references are given in the text, where appropriate, to the above mentioned books and other sources.

Third, the subject as hardly can be claimed of no other similar subject is of intimate vital interest and importance for scientists in a really enormous variety of fields[2].

Finally, many of the existing texts tend to avoid most of the controversies, contradicting views, unresolved questions (in turbulence there are more of them than 'solutions'); they attempt to 'smooth the angles' as much as possible, in ways that are, to my view, both inappropriate and misleading. Consequently some interpretations/views expressed in this book may appear to some as flagrant/egregious.

For these reasons all the subject(s) will be discussed in an essentially informal form/style, maximally avoiding complicated formalisms, which are useful to a very limited extent only – if at all. Following the advice of Leonardo da Vinci *Remember, when discoursing about water, to induce first experience, then reason* the emphasis is on the physical aspects based in the first place on observations and empirical facts as distinct from intuitive conjectures/hypotheses. This does not mean that the latter are ignored and that the presentation is oversimplified or even easy.

The informal nature of the book allows also to make it 'thin'. The very nature of the problem of turbulence and turbulent flows (absence of a systematic theory, extreme difficulty and enormous scope of the subject) left its mark/imprint on this small book – there are no simple analytical solutions (with one exception – the 4/5 law of Kolmogorov – they do not exist in the field of turbulence), etc. On the contrary, it contains as many, questions and similar things (or even more) as does answers, along with

[2]Mathematics including Applied and Computational, Physics, Engineering (Aeronautical, Naval/Marine, Hydraulic, Civil/Environmental, Chemical/Petroleum, Material Processing, ...), Geo-Astrophysical Sciences (Atmospheric, Meteorology, Oceanography, Fluid Dynamics of Earth Interior, Astrophysics, Cosmology), Bio-Medical Fluid Dynamics. ... *turbulence undoubtedly represents a central principle for many parts of physics, and a thorough understanding of its properties must be expected to lead to important advances in many fields* (Neumann, 1949).

(hopefully) unbiased discussion of the unresolved issues, controversies and major problems. In particular there are no lapses into brevity at difficult places[3].

Due to the above mentioned nature of the problem/subject, the book has to some extent a character of an updated guide to major sources dealing in more detail with its various aspects of the problem.

For the same reasons, visual material is used and/or referenced wherever possible and useful. It is supplied by extended figure captions.

The book is based in part on the graduate course delivered by the author in the Department of Fluid Mechanics, Tel Aviv University, and in part on lectures delivered by the author in Delft Technical University, Swiss Federal Institute of Technology – Lausanne and Zurich, Ecole Normale Supérieure, Université Paris VII and in Laboratoire Modélisation en Mécanique, Université Paris VI, and in part on its revision and updating on the basis of the latest work during the period of the Programme on Turbulence held in the Isaac Newton Institute, Cambridge, January 6 – July 2, 1999 and the research Program on Physics of Hydrodynamics of Turbulence held in the Institute of Theoretical Physics, Santa Barbara, January 31 – June 30, 2000.

It should be emphasized that this is an *informal introduction* only, it is *not a textbook*, but an introduction to turbulence as a physical phenomenon. Those who want to go deep(er) into the field are warned not to underestimate the numerous difficulties, disappointments and even frustration awaiting them. This book standing alone gives only an impression of what turbulence is. It can serve as an introduction to the research and the literature in the field in conjunction with a self-contained text, such as the book written by Pope (2000).

The scope of the book is mostly limited to purely basic aspects of turbulent flows of incompressible fluids. The prerequisites include a basic course in fluid dynamics (including turbulence) and standard knowledge of physics and mathematics at mid-graduate level. Therefore no systematic introduction to general fluid dynamics is included, neither is any material on probabilistic tools[4]. Both are usually included in books on turbulence, and the latter sometimes at a quite elaborate level. However, its use in these books is very limited with most of the highest probability and stochastic tools

[3]*It is much easier to present nice rational linear analysis than it is to wade into the morass that is our understanding of turbulence dynamics. With the analysis, professor and students feel more comfortable; even the reputation of turbulence may be improved, since the students will find it not as bad as they had expected. A discussion of turbulence dynamics would create only anxiety and a perception that the field is put together out of folklore and arm waving* (Lumley, 1987).

[4]The recently published book by Pope (2000) contains well balanced information on all these and other useful tools for treating turbulence.

remaining unused. In order to aid the reader, glossaries of some terms and brief discussions of basic fluid mechanics as well as some other useful information are given in the appendices. More details about the scope of the book are given at the end of the first chapter. This is an introduction in which the most important points are mentioned, and discussed in more detail in the subsequent chapters along with some additional material. The list of references is limited to major sources: books/monographs, collections of essays, review papers and selected specific papers with some emphasis on the most recent ones, overlooking essentially all the rest, which are many indeed. This book is biased unavoidably by my views of what matters, and in this sense the book is to a large extent a personal view of the author on the subject. In selecting the references, I used the (genuinely small) parameter introduced by Saffman (1978), which he called information density, ϵ_I, and defined as the ratio, S/N, in the literature, with $S =$ signal (*understanding*), and $N =$ noise (*mountains of publications*). In order to increase the value of ϵ_I I did all my best to concentrate on the numerator, S, and to reduce the denominator, N, to the best of my knowledge, ability and judgement/understanding. However, absence of references does not necessarily mean that – in my view – they belong to N, but is due to my ignorance and/or the lack of space needed to discuss them here.

The book is intended for as broad a readership as possible with the aim of making it interesting and useful *both* to graduate students and scientists in all the above mentioned fields. It is hoped that this aim is relatively realistic due to the *informal* nature of this book, with its emphasis on turbulence as a physical phenomenon, observations, misconceptions and unresolved issues rather than on conventional formalistic elements and models.

The list of acknowledgements is too long to be reproduced here. I am grateful to all those who responded to my queries and requests and to my hosts in the places mentioned above for their hospitality.

Tel Aviv, Israel A. Tsinober

March 2001.

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CHAPTER 1

INTRODUCTION

About the main features of turbulent flows and the main problems

This introductory chapter comprises the basis and to a large extent is a guide for the rest of the book. It starts with a brief discussion of the history of the subject, which is followed by a number of various representative examples. The principal aim of bringing these examples is to demonstrate the variety of situations in which turbulent flows occur and the diversity of their manifestations. Where appropriate the examples are supplied by comments emphasizing the differences between turbulent and laminar flows and some specific properties of the former. Instead of a definition of turbulence a subsection is devoted to major qualitative universal properties of turbulent flows with cross references to the previous subsection on the representative examples. This is followed by a subsection attempting to give an idea as to why turbulence is such an extremely difficult problem. The last subsection contains an overview of the contents of the following chapters and the rest of the book.

1.1. Brief history

The Rise and Fall of Ideas in Turbulence
Liepmann (1979).

Only a brief outline of some major milestones in turbulence research is given below. Full appreciation of these comes only after reading this book. More details are given in Monin and Yaglom (1971, 1992) and Loitsyanskii (1966); see also Frisch (1995) for interesting historical digressions/excursus and Lumley and Yaglom (2001) for additional comments on the developments during the last century.

1st century AD	Use of the term 'turbulent' in the fable *Lupus et Agnus* by Phaedrus.
1500	Recognition of two states of fluid motion by Leonardo da Vinci and use of the term *la turbolenza*.
1839	'Rediscovery' of two states of fluid motion by G. Hagen.
1883	Osborne Reynolds' experiments on pipe flow. Concept of critical Reynolds number – transition from laminar to turbulent flow regime.
1887	Introduction of the term 'turbulence' by Lord Kelvin.
1895	Reynolds decomposition. Beginning of statistical approach.
1909	D. Riabuchinsky invents the constant-current hot-wire anemometer.
1912	J.T. Morris invents the constant-temperature hot-wire anemometer.
1921, 1935	Statistical approach by G.I. Taylor.
1922	L.F. Richardson's hierarchy of eddies.
1924	L.V. Keller and A.A. Friedmann formulate the hierarchy of moments.
1938	G.I. Taylor discovers the prevalence of vortex stretching.
1941	A.N. Kolmogorov local isotropy, 2/3 and 4/5 laws.
1943	S. Corrsin establishes the existence of the sharp laminar/turbulent interface in shear flows.
1949	Discovery of intrinsic intermittence by G. Batchelor and A. Townsend.
1951	Turbulent spot of H.W. Emmons.
1952	E. Hopf functional equation.
1962	Beginning of quantitative experiments at large Reynolds numbers by H.L. Grant, R.W. Stewart and A. Moilliet.
1967	Bursting phenomenon by S.J. Kline et al.
1972	Beginning of large-scale computing of turbulent flows by S.A. Orszag and G.S. Patterson.
1976	Recapitulation of large-scale coherent structures by A. Roshko.

Most of the developments in turbulence research have occurred since Osborne Reynolds undertook his experiments. During this period, research in turbulence was conducted almost exclusively by the engineering community and in some other practical fields, such as in atmospheric and ocean sciences, and astrophysics. The last three decades have been marked by an increasing involvement of physicists and applied mathematicians though still with pretty limited foci.

1.2. Nature and major qualitative universal features of turbulent flows

1.2.1. REPRESENTATIVE EXAMPLES OF TURBULENT FLOWS

Unlike other complicated phenomena, turbulence is easily observed, but is extremely difficult to interpret, understand and explain.

There exist a number of beautiful collections of images of turbulent flows. To mention some: Corrsin (1961), Fantasy of Flow (1993), Atlas of Visualization (1997), van Dyke (1982) and Werlé (1987). Precise and penetrating in capturing the essential aspects of turbulent flows, the drawings by Leonardo da Vinci deserve a special mention; see Pedretti (1982), Popham (1994) and Richter (1970).

What follows is a selection of such pictures attempting to illustrate the diversity of circumstances in which turbulent flows occur and the variety of their manifestations. A note of warning is that most visualizations employ passive tracers (i.e., objects of Lagrangian nature) which may not reflect the underlying dynamical structure/features of genuine turbulence, see chapters 3, 4 and 9.

Flows in pipes

A qualitative repetition of the Reynolds experiment on his original facility in Manchester is shown in figure 1.1.

In a particular example shown in figure 1.2 the flow becomes turbulent at the value of Reynolds number, Re $\sim$ 2700, though with appropriate measures it can be kept laminar for Re up to 10^5 and, in principle, at much higher values of Reynolds number, since this flow is stable to small enough (infinitesimal) disturbances. The only other flow known to possess a similar property is the Couette flow in a plane channel. Both become turbulent at relatively low Reynolds numbers with disturbances of finite amplitude.

In the laminar regime the pressure difference at distance, l, along the pipe is proportional to the mean velocity, U, whereas in the turbulent regime it is much larger and is approximately proportional to $U^{7/4}$ in pipes with smooth walls and to U^2 in pipes with rough walls, see, for instance, figure 20.18 in Schlichting (1979). The latter means that the rate of energy losses, $\Delta p d^2 U$, i.e., rate of energy dissipation, in the turbulent regime in pipes with rough walls, is proportional to U^3/d and is independent of viscosity. In other words the nondimensional rate of energy dissipation per unit mass $\epsilon \equiv \frac{\Delta p d^2 U}{\rho l d^2 (U^3/d)}$ or simply dissipation, is Reynolds independent. This appears to be true of most turbulent flows at large enough Reynolds numbers, Idelchik (1996).

As mentioned, the visual observations shown in the above and subsequent figures were made by using some dye, i.e., via observing *passive objects*[1], which is not a dynamical quantity such as velocity or vorticity. In order to be able to 'see' some dynamical variable one has to use more

[1]Most visualizations are made in such a way. It has to be noted that what one sees looking at a pattern of a passive scalar in a turbulent flow may have nothing to do with the behaviour of dynamical variables such as velocity and vorticity. We return to this issue in chapters 3, 4, 7 and 9.

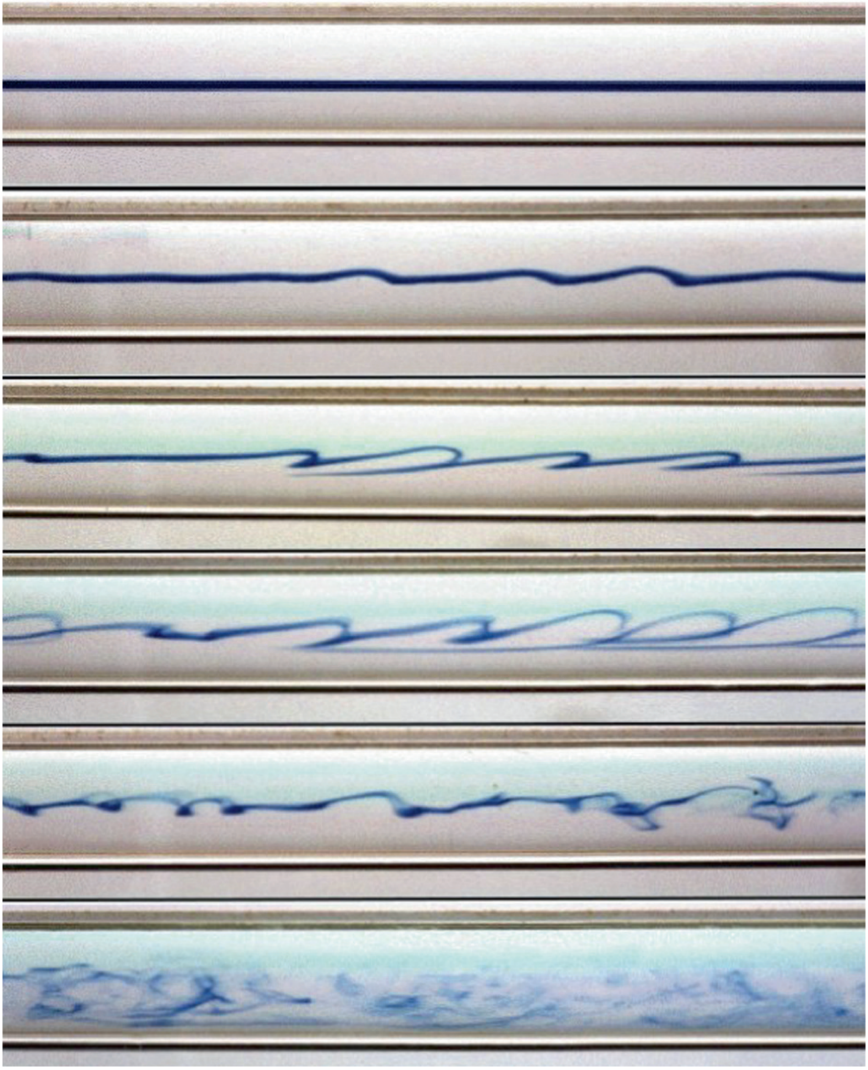

Figure 1.1. Reynolds experiment in a circular pipe. The smallest mean velocity corresponds to the upper frame where the flow regime is laminar, the largest mean velocity corresponds to the lower frame where the flow regime is turbulent. Courtesy of Professor J.D. Jackson, School of Engineering, University of Manchester

elaborate methods such as tracking small enough neutral particles, which is still not a trivial matter, or use the data from direct numerical simulations (DNS) of the Navier–Stokes equations (NSE), see figure 1.3.

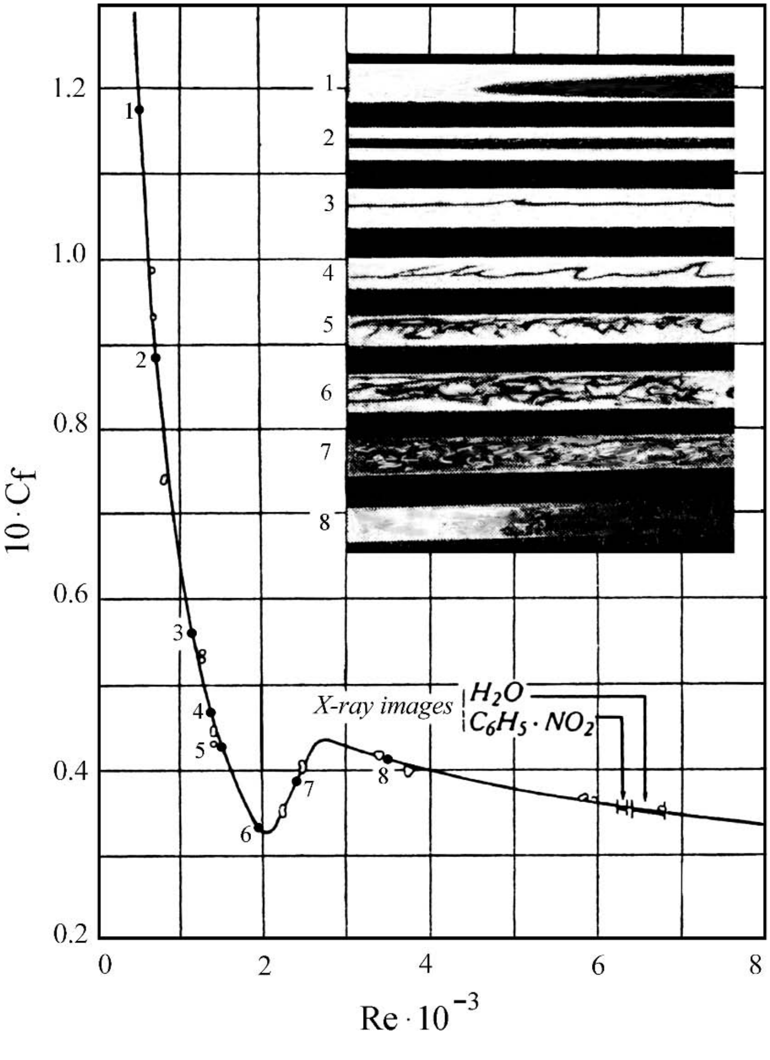

Figure 1.2. Reynolds number dependence of the friction factor, C_f, of flow with mean velocity, U, in a circular pipe of diameter, d, with corresponding flow visualization at particular values of Reynolds number $Re = Ud/\nu$, where ν is the kinematic viscosity of the fluid. The friction factor is defined as $C_f = (\Delta pd)/(l\frac{1}{2}\rho U^2)$, where Δp is the pressure drop on the distance, l, along the pipe, and ρ is the fluid density. Note that there is a range of values of Reynolds number in which the friction factor, C_f, follows the laminar law, $C_f = 64/Re$, but the flow pattern (pictures 4–6) is far from looking as purely laminar. Adapted from Dubs (1939)

One can see the difference between the quiet laminar and the restless turbulent flow regimes by looking at the water jet from a tap, see figure 2 in Corrsin (1961) or figure 4.1 in Mullin (1993). Note that this is more 'pipe turbulence' rather than 'jet turbulence'.

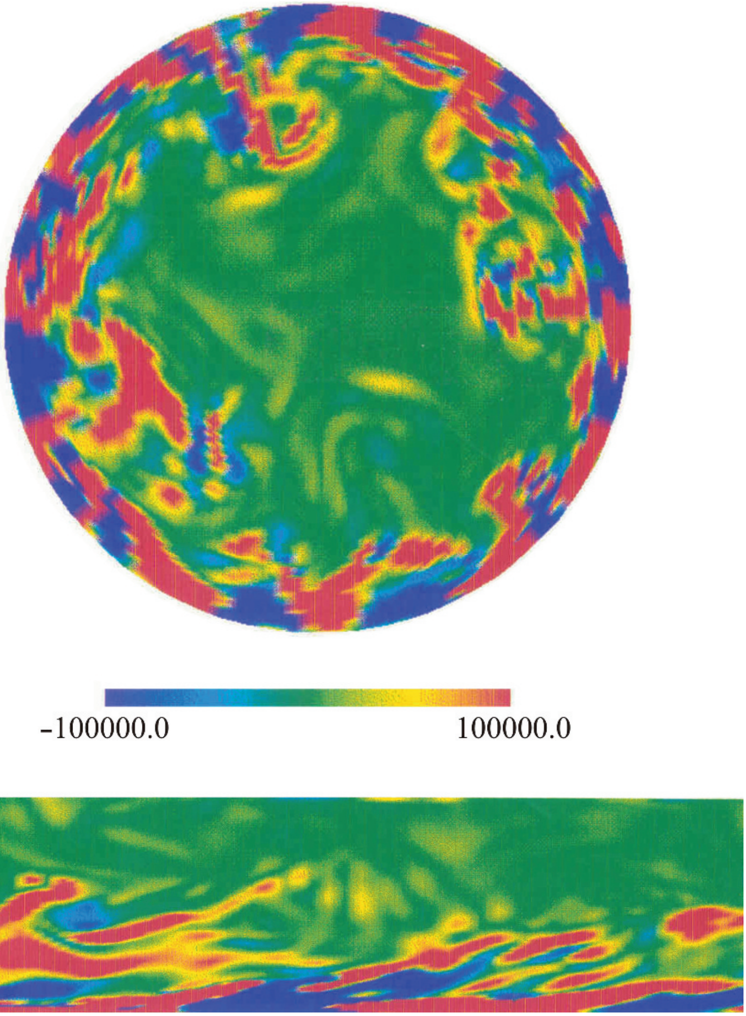

Figure 1.3. Visualization of the field of the enstrophy production $\omega_i\omega_k s_{ik}$ from the data in DNS of NSE in a circular pipe flow at $\mathrm{Re} \approx 7000$, performed by Eggels et al. (1994). One of the prominent features of *all* turbulent flows is that in the *mean* the enstrophy production is always positive, $\langle\omega_i\omega_k s_{ik}\rangle > 0$. Note large regions with *instantaneous negative* enstrophy production, $\omega_i\omega_k s_{ik} < 0$. Courtesy of Professor F.T.M. Niewstadt and Dr. J.M.J. den Toonder

Boundary layers

Boundary-layer flows belong to the same category as the flows in pipe, channels and other *wall-bounded* flows.

The flows shown in figures 1.1–1.5 have many common features and properties, such as their near-wall behaviour. However, turbulent boundary layers possess specific essential features.

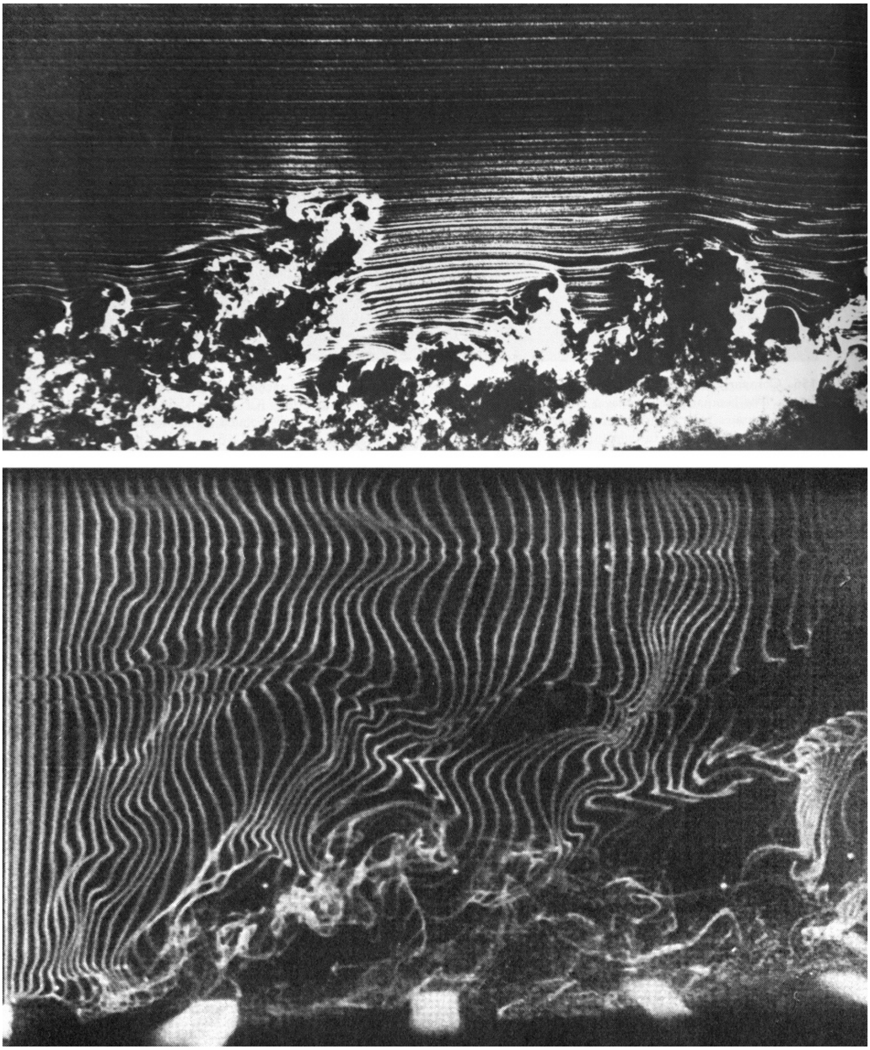

Figure 1.4. Side view of a turbulent boundary layer visualized by (top) smoke traces, courtesy of Professor H. Nagib, and (bottom) by hydrogen bubbles. Courtesy of Professor A.J. Grass

First, these flows are *partly* turbulent[2] in the sense that the turbulent regime coexists with the laminar one (which is usually close to irrotational), as well as with the transitional states between laminar and turbulent regimes (see figure 1.5). Second, the 'boundary' between the laminar

[2]This term comes from Scorer (1978).

Figure 1.5. Coexistence of different flow regimes – laminar, transitional and turbulent – in a boundary layer on an axisymmetric body visualized in a smoke photograph. The upper picture corresponds to a nonspinning body, the lower picture corresponds to a spinning body. In the former regular waves (Tollmien-Schlichting waves) are seen at the early stage of transition. In the latter the T-S waves coexist with the so-called cross-flow vortices. Courtesy of Professor T.J. Mueller (see Mueller et al., 1981)

and turbulent flow regions is strongly corrugated with many space-time scales involved. It is 'fractal'-like, but nonetheless distinct. The largest scales are such that the 'laminar' fluid is found quite close to the wall. Third, the fluid from the laminar region is continuously entrained into the turbulent region through the boundary between the two. That is the entrainment process is such that laminar fluid becomes turbulent in the proximity of the laminar/turbulent boundary. This is one of the basic processes of transition of the flow state from laminar to turbulent. An important overall characteristic of this process is the entrainment rate, but there much more.

The three features mentioned above are also observed in the so-called *free* (of rigid or other boundaries) turbulent shear flows: jets and plumes, wakes, mixing layers and flows in separation regions as well in more complicated situations in geophysical and astrophysical contexts. The simplest example is the entrainment process at the bottom of the turbulent surface (warmer) layer of the ocean. These flows all are also partially turbulent with corrugated boundary between the laminar and turbulent regions, and entrainment of fluid from the former into the latter. It seems that in free shear flows the approximately irrotational fluid can be found deep in the region occupied by the turbulent fluid[3] just as in the turbulent boundary layer, though both observations are mainly based on images of passive scalars. However, frequently (but not always, see chapter 4) they are in agreement, at least qualitatively, with observations made by other methods such as optical (shadowgraph, schlieren and interferometric). A somewhat dissimilar example of the coexistence of different flow regimes is shown in figure 1.6.

Turbulent jets, plumes

Turbulent jet-like flows are ubiquitous in engineering and nature, e.g., jets from aircraft and rocket engines and discharges from smoke stacks, volcanoes and other geologic nozzle eruptions, see figures 1.7–1.8.

Turbulent wakes past bodies and mixing layers

The flow in the wake past a sphere undergoes a number of changes from laminar to fully turbulent[4] as the Reynolds number grows. At $\mathrm{Re} < 1$ the flow is time independent without separation; at $\mathrm{Re} \sim 1 \div 10$ the flow remains steady and symmetric but separates at the back of the sphere in the form of a vortex ring, which becomes larger with increasing Reynolds number; at $\mathrm{Re} \sim 10 \div 10^2$ the flow loses its symmetries: it becomes time dependent but (approximately) periodic, and the vortex ring is deformed into a helical vortex, which rotates around the (former) axis of the flow. The separation line is no more a circle and assumes a complicated form, which is changing in time. At $\mathrm{Re} \sim 10^2 \div 10^5$ the wake becomes (apparently) random and aperiodic but with large-scale structures, which are more complicated than the destroyed helical vortex. The boundary layer (before the line of separation) on the sphere remains laminar. At $\mathrm{Re} >\sim 10^5$ the boundary layer also becomes turbulent. It is noteworthy that the separation zone past the sphere (figure 1.9) for $\mathrm{Re} = 3 \cdot 10^5$ is much narrower due to the onset of turbulence in the boundary layer of the sphere.

This causes a strong *decrease*, by a factor of 6, of its drag, e.g., see figure 1.5 in Schlichting (1979). This is an example when transition to turbulence is reducing the losses instead of increasing them, as in most cases. The reason is that the delayed separation leads to a much narrower wake and hence smaller losses. This feature is also exhibited in changes of

[3] See figures below and figures 109, 117, 134, 151, 158, 166, 167, 174, 176, 177 in van Dyke (1982). Note the figure 151, showing a turbulent wake behind a projectile at supersonic speed with a remarkably sharp boundary between the turbulent flow in the wake and the ambient irrotational flow.

[4] The term *fully developed turbulence* (*voll ausgebildet turbulenz*) is most probably derived from L. Prandtl (1926).

Figure 1.6. Coexistence of different flow regimes – laminar, transitional and turbulent – in a fountain in the center of Washington, photo by the author (1985). Here the process is reverse due to the decrease of Reynolds number in each subsequent step: the turbulent flow regime in the upper part is replaced at the next step by the transitional one with sporadic outbursts of turbulent activity. This in turn is replaced by regular wavelike motions similar to those observed in the initial transitional stage in a boundary layer flow shown in figure 1.5. Note the difference in the meaning of the term 'coexistence' in this example and the one shown in figure 1.5. However, in both cases the changes in the flow regime are related to the changes of the value of the 'local' Reynolds number. Hence the similarity between the two examples

pressure distribution over the surface of the sphere. No such phenomena are observed in flows past bodies with sharp edges, e.g., past a circular disc with its plane normal to the direction of the undisturbed flow. In such a case the normalized drag $C_D = \frac{F_{\text{drag}}}{\rho U^2 d^2}$ (and consequently the energy dissipation)

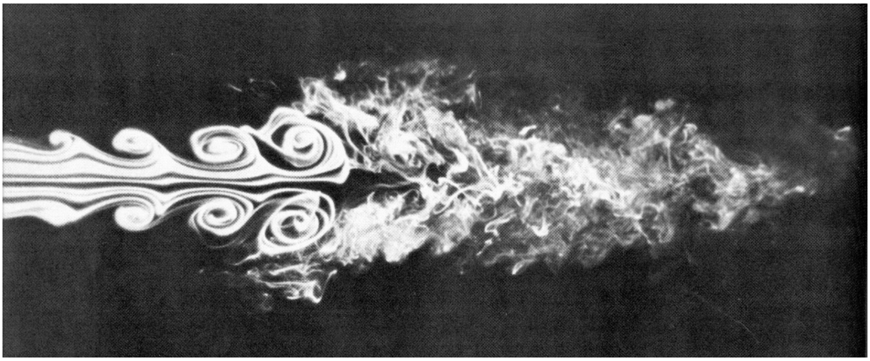

Figure 1.7. Coexistence of different flow regimes – laminar, transitional and turbulent – in a circular jet. Courtesy of Professor H. Nagib

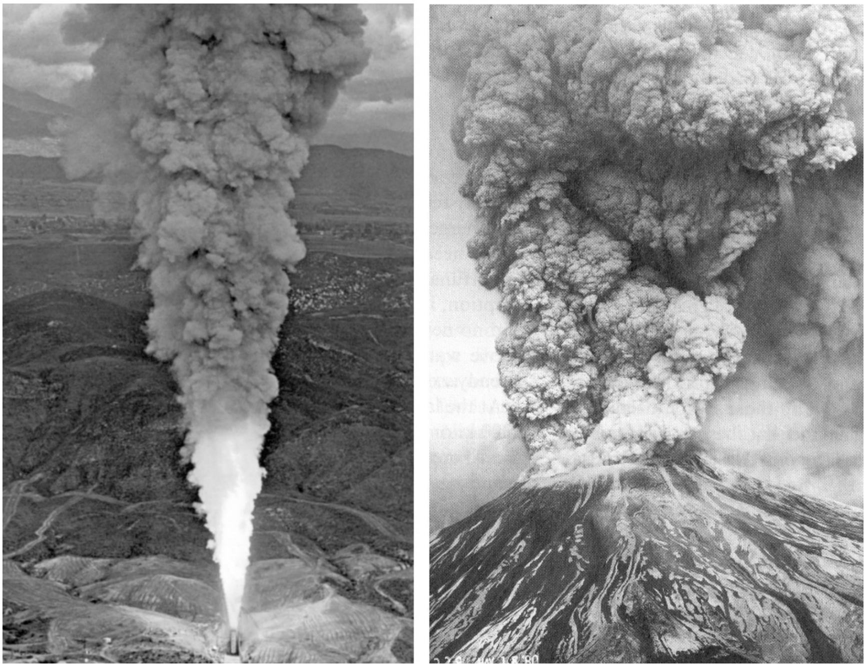

Figure 1.8. Left: a turbulent jet from testing a Lockheed rocket engine in the Los Angeles hills, courtesy of Professor P.E. Dimotakis. Right: eruption of Mount St. Helen volcano on 18 May 1980, US Geological Survey

remains independent of the Reynolds number up to the highest achievable Reynolds numbers of the order 10^6, figure 1.10. Other numerous examples of this kind can be found in Idelchik (1996).

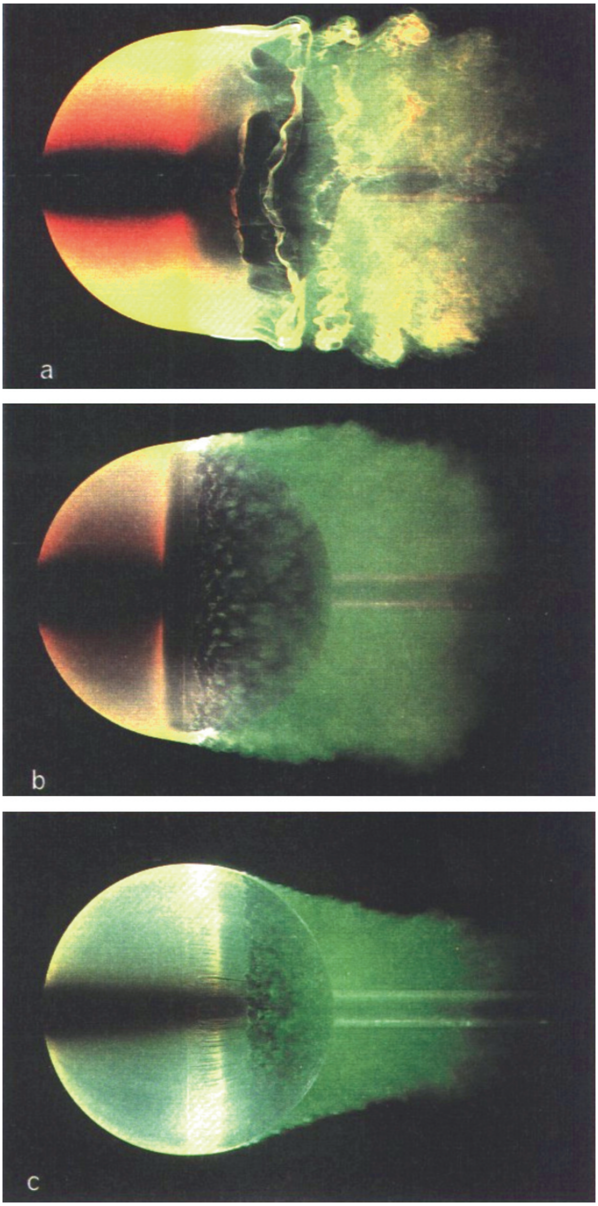

Figure 1.9. Flow in the near wake region past a sphere at three values of the Reynolds number, based on the free stream velocity and the sphere diameter: (a) – $2 \cdot 10^4$, (b) – $2 \cdot 10^5$, (c) – $3 \cdot 10^5$ (Werlé, 1987; by permission from ONERA)

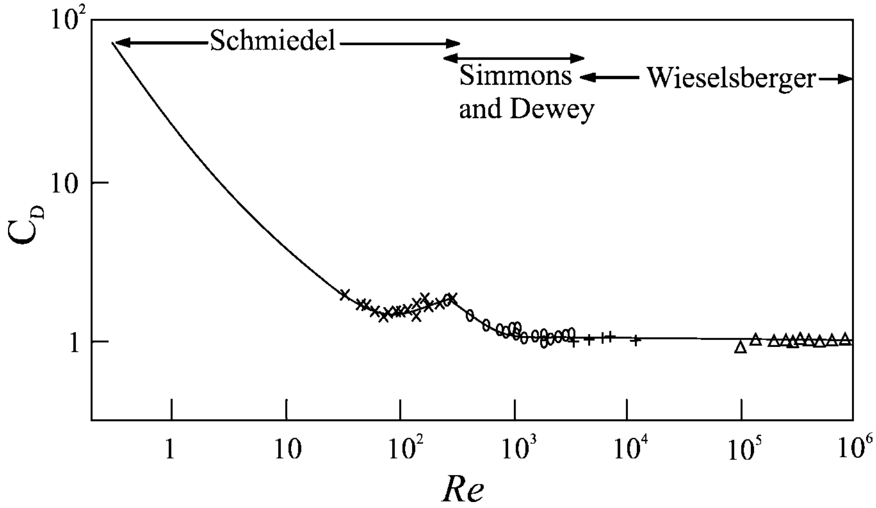

Figure 1.10. Reynolds number dependence of the resistance coefficient, C_D, of a circular disc with its plane normal to the direction of the undisturbed flow. Adapted from Schiller (1932); see also Muttray (1932)

Wakes past bodies exhibit very-large-scale fluctuations (undulations) in the far field in which the wake undulates essentially as a whole (figure 1.11). This is an indication that these large-scale undulations are the result of a large-scale instability not directly related to the turbulent nature of the flow within the wake[5]. This is probably the main reason for the similarity of the large-scale features between the two flows shown in figure 1.11, in spite of a large difference in the value of their Reynolds numbers. Similar large-scale features are observed in other free shear flows such as plumes (figure 1.12) and mixing layers (figure 1.13).

Quasi-isotropic and homogeneous flows

Quasi-isotropic and homogeneous flows are realized in the laboratory in a flow past a grating or a grid such as those shown in figures 1.14 and 1.15.

Another way to produce such flows is via direct numerical simulations of the Navier–Stokes equations with appropriate forcing at the right hand in cubic box with periodic boundary conditions. An example of some results in such a computation is shown in figure 1.16.

Flows as shown in figures 1.11–1.16 are considered by many as of purely academic interest, since they represent an approximation to an idealized situation called homogeneous isotropic turbulent flow. Nevertheless these kinds of flows are of special interest for several reasons. Initially this interest

[5]Though these and similar structures are frequently termed as “coherent structures” of turbulent flows.

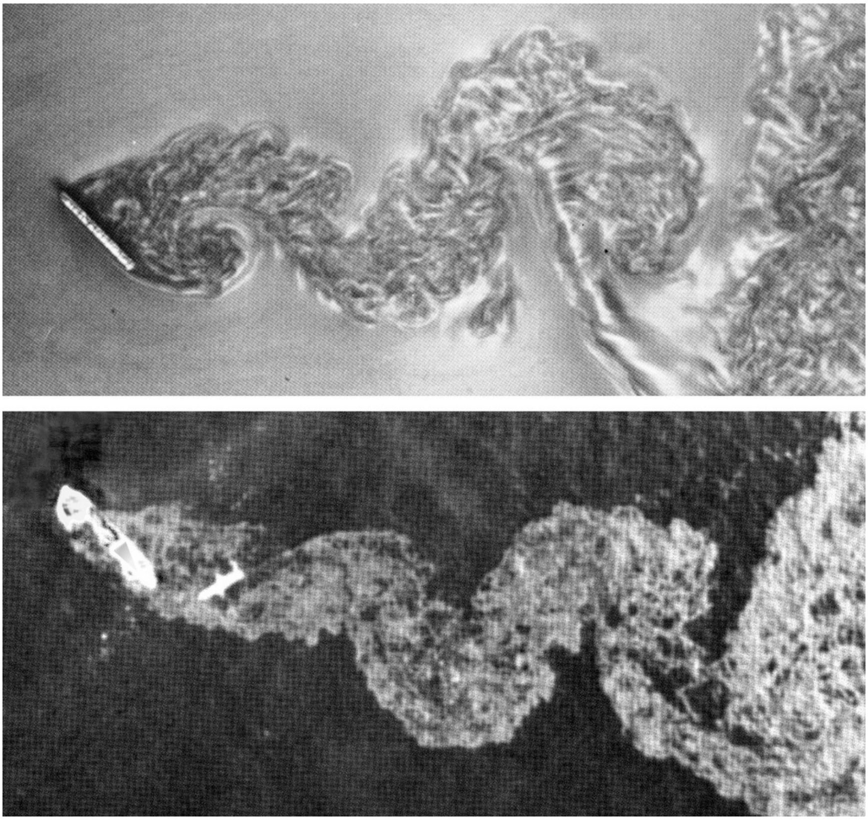

Figure 1.11. Top: a wake past an inclined plate at a Reynolds number 4300. Bottom: a wake of leaking oil past a grounded tanker at a Reynolds number $\sim 10^7$. Courtesy of Professor B. Cantwell

was due to the relative 'simplicity' and analytical convenience of such flows. However, there is one more aspect which makes the flows shown in the last three examples of special interest and importance. Namely, it appears that there exist many universal features at the level of basic physical processes of turbulent flows, which are manifested in this idealized situation, regardless of the origin of a specific turbulent flow, at least qualitatively. The particular significance of these kinds of flows follows from the fact that (quasi-) homogeneous and isotropic flows are free from external influences, like mean shear, centrifugal forces (rotation), buoyancy, magnetic field, etc., which usually act as an organizing factor, favouring the formation of the so-called (large-scale) coherent structures of different kinds (quasi-two-dimensional, helical, etc.). These external influences in many cases have usually a strong

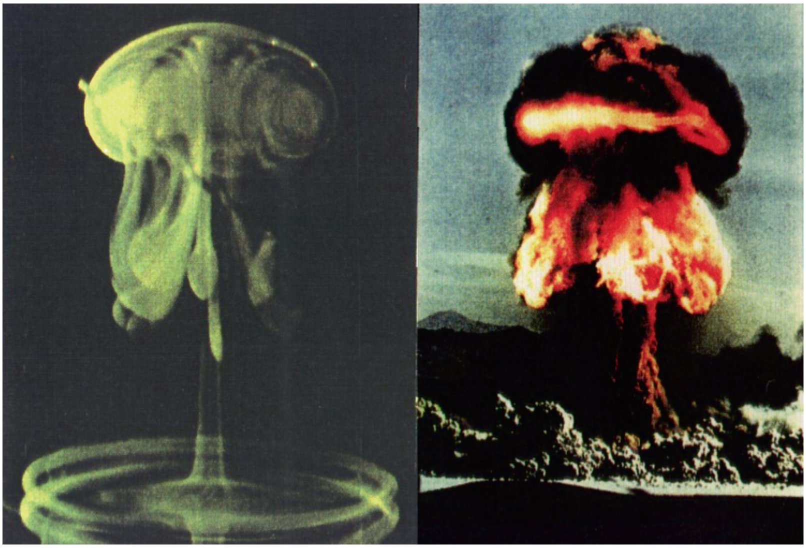

Figure 1.12. Similarity in large-scale flow patterns of a water drop dyed with fluorescein into clear water (inverted) at Reynolds number $\mathrm{Re} \sim 10^2$ and of a nuclear test in Nevada at Reynolds number $\mathrm{Re} \sim 10^9$ (Sigurdson, 1997; by permission CRC Press LLC)

(linear) masking effect on the intrinsic nonlinear turbulent processes in such situations as those described by rapid distortion theory, in which nonlinear interactions between turbulent fluctuations are neglected for the duration of the distortion (Savill, 1987; Hunt and Carruthers, 1990). In other words the nonlinear nature of turbulent flows is manifested more distinctly in (quasi-) homogeneous and isotropic turbulence.

It is argued here that most of the intrinsic properties of turbulent flows, that is their physics, are seen in the cleanest and relatively 'simple' way in quasi-homogeneous isotropic flows. This is the reason for some emphasis given to such flows.

As mentioned there are many factors and influences which cause a real turbulent flow in nature and technology to deviate from this idealized state, sometimes strongly. In the latter case turbulent flows may lose most of their resemblance to the three-dimensional homogeneous isotropic flow, but can be quite similar to the (quasi-) two-dimensional one. This aspect of various influences on turbulence is addressed in chapter. 8.

Before finishing this section it is important to note that all the above examples show the *spatial* intricacy of turbulent flows. The *temporal* behaviour obviously cannot be seen from a single snapshot. An example of time recordings of several quantities is shown in figure 1.17.

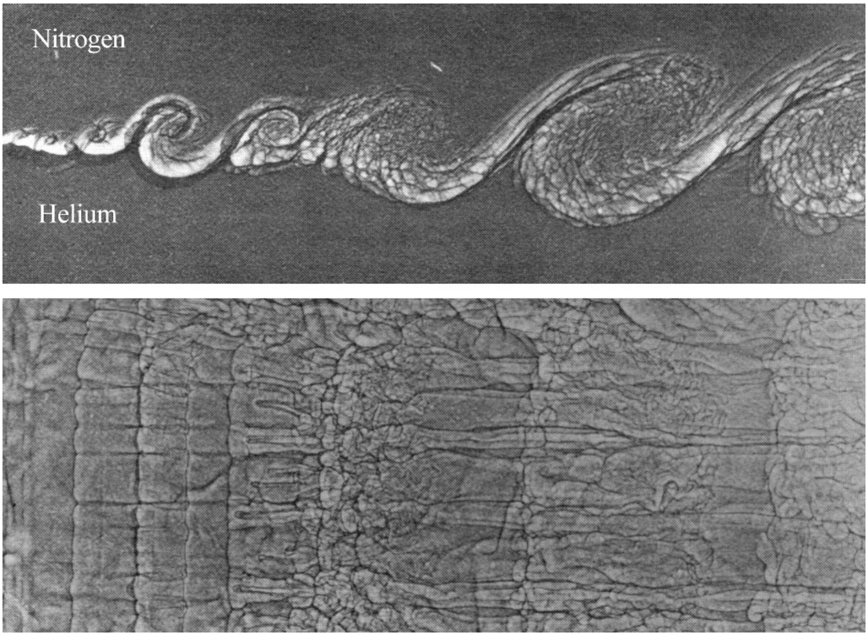

Figure 1.13. An example of a turbulent mixing layer. Top – side view, bottom – (half) plan view, both shadowgraph. The undisturbed velocity of the upper part of the flow (nitrogen) is 10 m/s and the undisturbed velocity of the lower part (mixture of helium and air) is 3.8 m/s. From Konrad (1976)

All the quantities exhibit apparently random temporal patterns, though quite different. Similar differences occur in the spatial variations of these quantities. We will return to these differences and many other related issues in the subsequent chapters.

The above examples of flows show that one can easily observe at least some manifestations of turbulence and even can *describe* some of them qualitatively, i.e., without mathematics. It is natural to use the *qualitative* manifestations of turbulent flows as a first step to 'define' what is turbulence.

1.2.2. IN LIEU OF DEFINITION: MAJOR QUALITATIVE UNIVERSAL FEATURES OF TURBULENT FLOWS

> *Turbulence is a phenomenon which sets in a viscous fluid for small values of the viscosity coefficient ν, ... hence its purest, limiting form may be interpreted as the asymptotic, limiting behavior of a viscous fluid for $\nu \to 0$* (Neumann, 1949).

Figure 1.14. Flow past a grating of five circular cylinders visualized by two colours, by Tetsuo SAGA, The University of Tokyo, Fantasy of Flow (1993), by permission Ohmsha, Ltd

Already from the observation of the examples given in the previous section it is easy to arrive at a conclusion that turbulence is an extremely intricate and complicated phenomenon. Therefore, it would be naïve and hopeless to attempt its definition in a few sentences[6], see the collection of citations in appendix A. *What is turbulence?* showing that this is really not easy if not impossible. Indeed, let us try to give such a 'definition':

> *Turbulence is the manifestation of the spatio-temporal chaotic behaviour of fluid flows at large Reynolds numbers, i.e., of a strongly nonlinear*

[6]Though there are many attempts to do so. In contrast to mathematical theories in which the definition of the main object of the theory precedes the results, in turbulence (as in any field of physics) even if such a definition would be possible it is likely to come *after* the basic mechanisms of turbulence as a physical phenomenon are well understood. In any case there is considerable 'turbulence' in the attempts to define what is turbulence indicating that such attempts at the present stage are conceptually incorrect and futile.

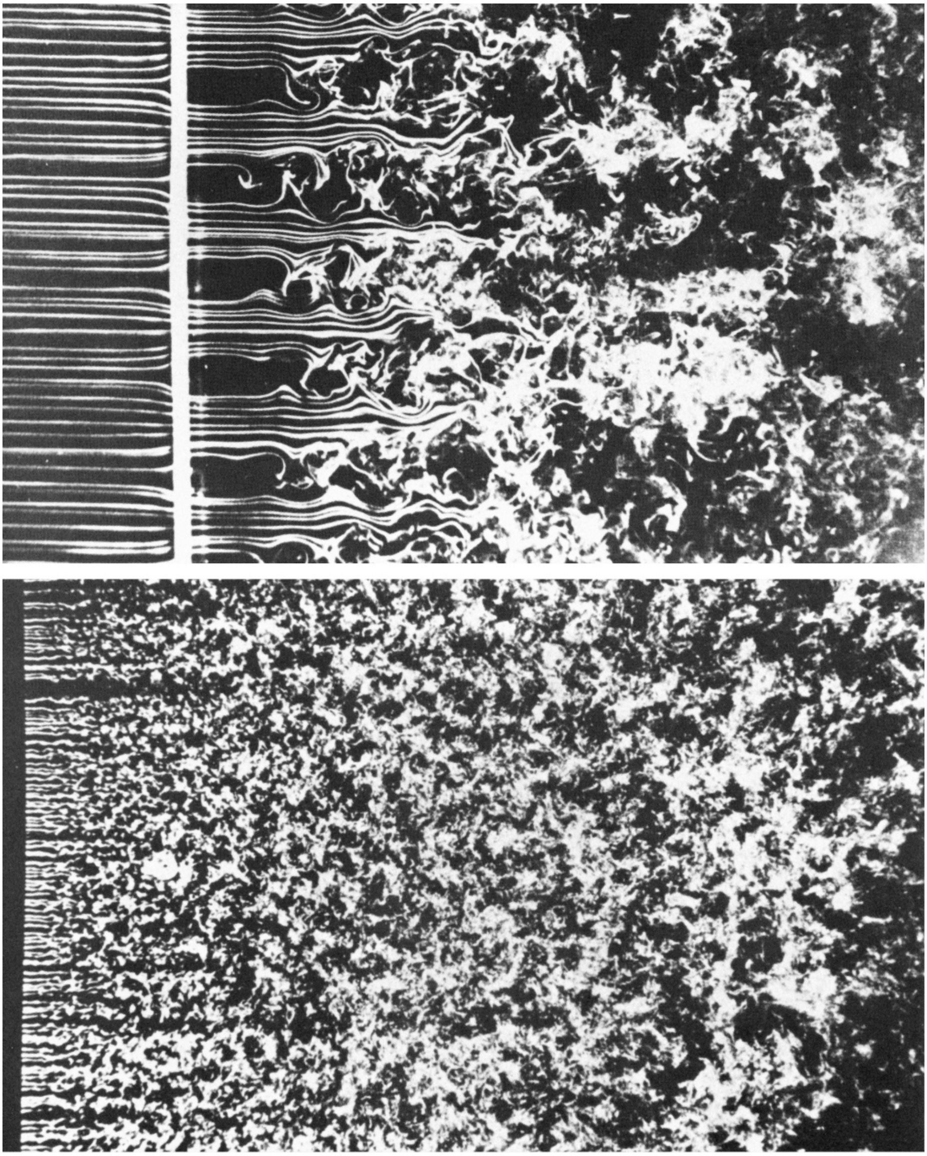

Figure 1.15. Examples of a turbulent flow past a grid. Top – in the proximity of the grid. Bottom – in the far field. Courtesy Professor H. Nagib

dissipative system with an extremely large number of degrees of freedom (most probably) described by the Navier–Stokes equations.

Obviously, it is practically useless and absolutely not sufficient to say this or anything similar. So instead of futile attempts to define what is

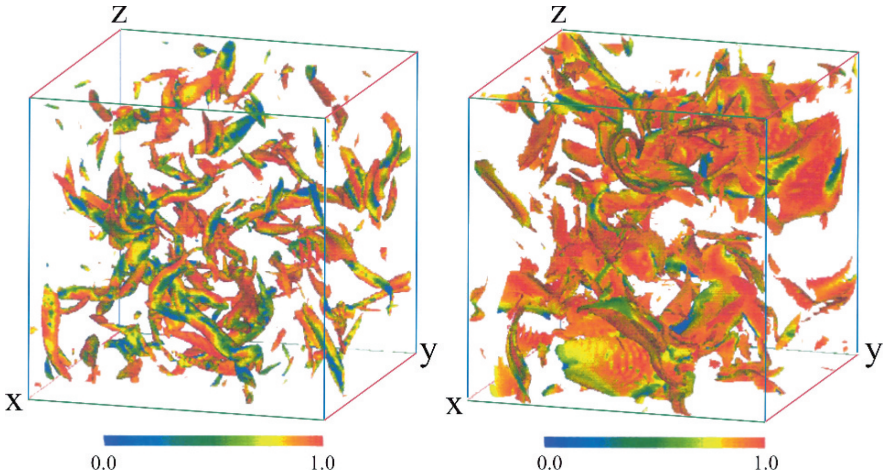

Figure 1.16. Visualizations of the magnitude of (left) the vorticity, ω, and (right) the scalar gradient, G ($G_i \equiv \partial\theta/\partial x_i$), in a direct numerical simulation of the Navier–Stokes equations in a cubic box with periodic boundary conditions at $\mathrm{Re}_\lambda = 58$. The statistically stationary state is achieved by adding an appropriate forcing in the RHS of NSE. The threshold values for both ω and G are larger than 2.5 times of their RMS value. Contour colouring corresponds to the $cos(\boldsymbol{\omega}, \boldsymbol{\lambda}_2)$ and $cos(\mathbf{G}, \boldsymbol{\lambda}_3)$ alignments. The vectors $\boldsymbol{\lambda}_i$ form the eigenframe of the rate of strain tensor, s_{ij}, corresponding to its eigenvalues Λ_i ($\Lambda_1 > \Lambda_2 > \Lambda_3$; $\Lambda_1 > 0, \Lambda_3 < 0$). Red indicates strong alignment. From Ph.D. Thesis by Flohr (1999)

turbulence, one starts by description of its main qualitative features, as did Tennekes and Lumley (1972). The following is an updated list of such features with cross references to the examples given in the previous section.

• – Intrinsic spatio-temporal randomness, irregularity. Turbulence is definitely chaos. However, vice versa, generally, is not true: many chaotic flow regimes are not turbulent, e.g. Lagrangian/kinematic chaos or ‘Lagrangian turbulence’, laminar ‘turbulent’ flows.

One of the most important aspects is that the stochastic/random nature of turbulent flows is its *intrinsic* property (self-stochastization or self-randomization). There is no necessity for external *random* forcing either in the interior of the fluid flow or at its boundaries, nor does one need to start the turbulent flow with some random initial conditions *provided* that the Reynolds number is large enough[7]. The fascinating question is how such

[7]In this sense the random behaviour of nonlinear systems as a response to random forcing is not necessarily turbulence. One of the most popular examples is the Burgers equation: without external random forcing it does not exhibit any chaotic behaviour. Another example is a randomly forced flow at very small Reynolds numbers which though random, is in many respects trivial (as any randomly forced linear system), e.g., there is no interaction between its degrees of freedom/modes.

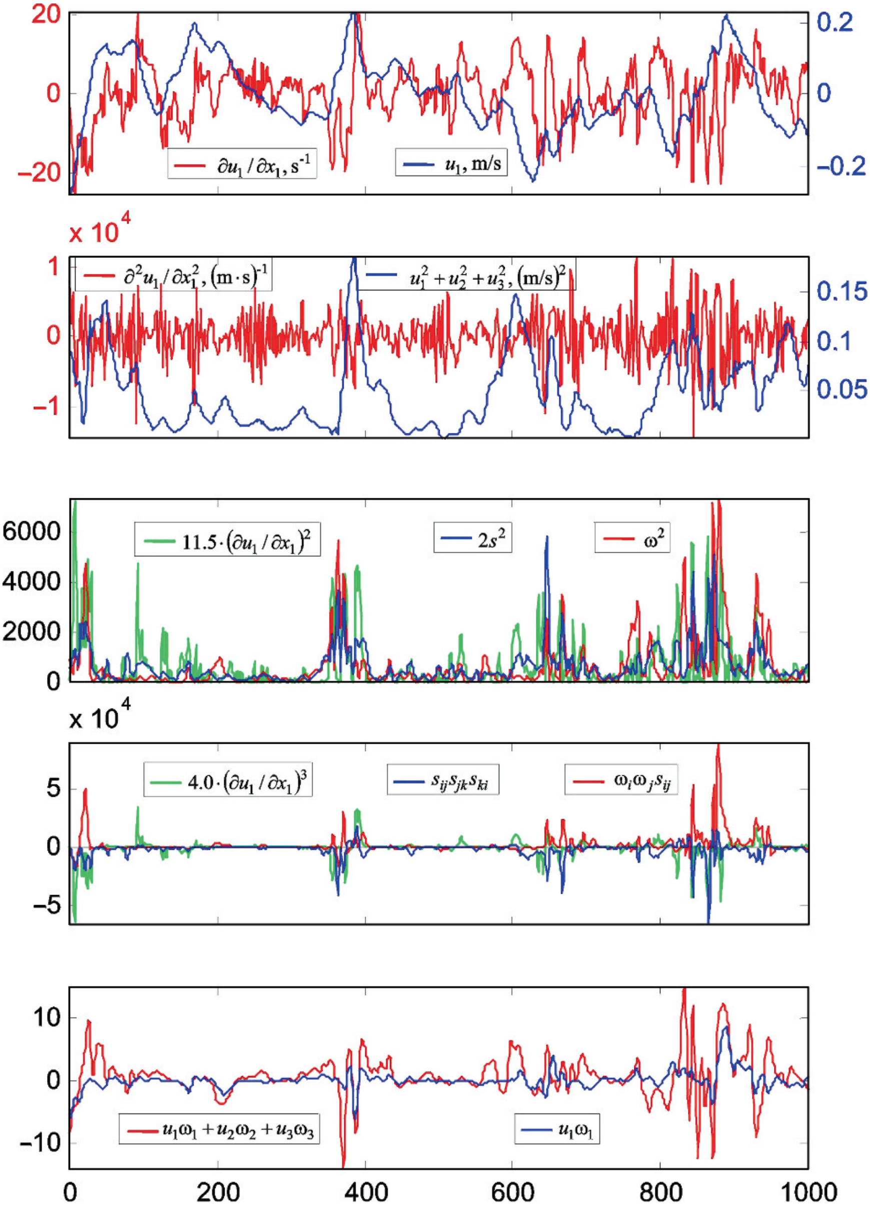

Figure 1.17. An example of a time recording of various quantities in a field experiment at $\mathrm{Re}_\lambda = 10^4$ in the atmospheric surface layer in a point at height 10 m from the surface. The measurements in this experiment included all the three velocity components at five neighboring points. This allowed evaluation all nine velocity derivatives $\frac{\partial u_i}{\partial x_k}$, and to obtain such quantities as instantaneous enstrophy, ω^2, total strain, $s^2 \equiv s_{ik}s_{ik}$, enstrophy production, $\omega_i\omega_k s_{ik}$ and many others; (Kholmyansky et. al., 2001b). Note the intermittent nature of the signals associated with velocity derivatives: peaks which are many hundreds times larger than the mean as well as very low values, are not rare

a behaviour can arise from purely deterministic equations as the Navier–Stokes equations are, deterministic forcing along with smooth initial and boundary conditions. The answer is its extreme sensitivity to disturbances whatever small (initial conditions, boundary conditions, external noise). Turbulence is both a strongly nonlinear (stochastic) oscillator and an amplifier with an (almost) 'infinite' gain.

The feature described here is seen in all the examples given in the previous section.

• – Extremely wide range of strongly interacting degrees of freedom. Turbulent flows are large systems. In atmospheric flows, relevant scales range from hundreds of *kms* to parts of a *mm*, i.e., there exist $\sim 10^{29}$ excited degrees of freedom[8], many of which are strongly interacting. Hence extreme complexity of turbulence (along with its intrinsic randomness) means that description of turbulent flows should be of statistical nature. We emphasize that statistical *description* is not synonymous to statistical *theorization.*

The interaction between the many degrees of freedom, resulting from the *nonlinearity* of turbulent flows is essential (linear systems can have arbitrarily many degrees of freedom as well, but they do not interact, and each degree of freedom lives its own life without knowing anything about other degrees of freedom), but not sufficient: the nonlinear interaction may lead to strongly organized regular behaviour, e.g., solitons in the systems described by the Korteveg–de Vries and Shrödinger equations, and shocks in the Burgers equation.

Again most of the examples given in the previous section clearly show the multi-scale nature of turbulent flows in space. Turbulent flows exhibit also quite complex behaviour in time as well, see figure 1.17.

• – Loss of predictability. Two initially nearly (but not precisely) identical turbulent flows become unrecognizably different on the time scale of dynamical interest. The details of any realization are strongly different from any other realization. This is because any individual realization is extremely sensitive to small perturbations/disturbances. However, *different realizations of the same turbulent flow have the same statistical properties*, such as drag of a sphere or rate of mixing of some contaminant. That is the statistical properties (not only some means, but almost *all* statistical properties) of turbulent flows are insensitive to disturbances – *turbulent flows are statistically stable*, they possess statistically stable properties. This *insensitivity* to disturbances is only *statistical.* In other words turbulent flows

[8] This includes all scales of the atmosphere. Not all of them are considered as turbulent in the atmospheric community. For example, Orlanski (1975) considers only the first several hundred meters as turbulent. With this approach the number of excited degrees of freedom is still enormous $\sim 10^{18}$.

possess both predictable and unpredictable features. The well-known problem of predictability from meteorology is essentially about the dynamics and statistics of an initial error, which is the measure of the differences between some two realizations of a turbulent flow under *almost* (hence the error) the same conditions. An important aspect is that the error also possesses stable statistical properties in the sense that errors corresponding to different pairs of realizations have the same statistical properties.

• – Turbulent flows are highly dissipative, i.e., carry lots of strain. A source of energy is required to maintain turbulence (gradients of mean velocity, buoyancy, or other external forces). The energy supply is usually at large scales, its dissipation is at small ones[9]. Statistical irreversibility is involved, i.e., the processes in turbulent flows are 'one way' in time.

The immediate examples that come to mind are the resistance in pipes and the drag of bluff bodies, which in the turbulent regime are orders of magnitude larger than their laminar counterparts at the same Reynolds number.

• – Turbulent flows are three-dimensional and rotational, i.e., carries lots of vorticity. They are 'random' fields of vorticity, $\omega \equiv curl\ \mathbf{u}$, with predominant vortex stretching, i.e., continuous positive net production of enstrophy, ω^2, by inertial nonlinear processes, which is 'dissipated' by viscosity. An important concomitant process is a positive net production of strain, $s_{ij} = 1/2(\partial u_i/\partial x_k + \partial u_k/\partial x_i)$. Both are the result of *self*-amplification of the field of velocity derivatives in/by turbulent flows, and comprise one of the most basic specific dynamical properties of turbulence. The latter, i.e., production of strain, is directly related to the dissipative nature of turbulent flows, whereas the former, i.e., amplification of vorticity, is related to dissipation only in an indirect manner. Random potential flows are not turbulence.

It is not difficult to observe the three-dimensional nature of turbulent flows, but their rotational nature can be 'seen' from the direct numerical simulations of the Navier–Stokes equations, as shown in figure 1.3.

There is no consensus whether two-dimensional chaotic flows even with many degrees of freedom should be qualified as turbulence. The main reason is that such flows lack the mechanism of vorticity and strain amplification. This issue is discussed in chapter 8.

[9]The common view of turbulence dynamics involves the Richardson–Kolmogorov cascade of energy (the famous poem by Richardson). However, there are numerous examples in which turbulence develops from small scales into larger ones, e.g., in all spatially developing turbulent flows. All the examples given in the previous section on *partly*-turbulent flows are such, both free and wall-bounded. The important point is that the so-called 'cascade' takes place *not* in the physical space. We will return to this point and to the notion of *scale* later in chapter 5 (see also appendix C).

• – Strongly diffusive (random waves are not). Turbulent flows exhibit strongly enhanced transport processes of momentum, energy, passive objects (scalars, e.g., heat, salt, moisture, particles; vectors, e.g., material lines, gradients of passive scalars, magnetic field). It should be emphasized that in respect with passive objects *only* this property is true of a much broader class of systems. Namely, *any* random velocity field and even laminar flows, which are Lagrangian chaotic, exhibit enhanced transport of passive objects.

This aspect is manifested in the large resistance occurring in pipe flows, the enlarged drag of various bodies, and the enhanced heat and mass transfer and rate of mixing in turbulent flows as compared to their laminar counterparts. The latter process is illustrated in figure 1.14. It shows also how two separate passive scalars are mixed – an aspect important for chemical reactions and combustion. Using the terms 'turbulent viscosity' and 'turbulent diffusivity', one can say that they are orders of magnitude larger in turbulent flows than their molecular counterpart. It is the right place to note that turbulence is a property of fluid flows not fluids. For instance, molecular viscosity and diffusivity are properties of fluids (liquids, gases) and are independent of the flow. On the contrary the so-called 'turbulent viscosity' and 'turbulent diffusivity' are properties of fluid flows and depend on the fluid flow in question. The difference is more than essential.

These mostly widely known qualitative features of all turbulent flows are essentially the same, i.e., it is meaningful to speak about *qualitative universality* of turbulent flows. The concept of *qualitative universality* is not just a fuzzy idea: in chapter 10 it will be given a number of quantitative attributes. Indeed these *qualitative* features of turbulent flows are *universal* for all turbulent flows arising in qualitatively different ways and circumstances and generally characterize turbulent flows as a whole. In addition to these general *qualitative* features, there are universal *quantitative* features which are more specific for turbulent flows. They are described later in this book, since they require more specific information and terminology. Many (but not all) *quantitative* properties may vary largely with the range of scales of interest. The *large-scale* properties of turbulent flows depend on *particular* mechanisms generating turbulence and, generally, are *quantitatively* not universal, though as mentioned they are *qualitatively* universal. It is the *small-scale* turbulence which, since Kolmogorov, is believed to possess a number of universal properties independent of the large-scale flow structure, though there is quite a bit of evidence that the latter is not correct. This point of view is not accepted universally: . . . *perhaps there is no 'real turbulence problem', but a large number of turbulent flows and our problem is the self imposed and possibly impossible task of fitting many phenomena into the Procrustean bed of a universal turbulence theory* (Saffman, 1978;

see also Hunt et al., 1994). The issue is one of several continuously debated controversies in the problem of turbulence. This includes the *meaning* of the term 'universality'. For example, one issue involves the invariance of Reynolds number of (some) properties of a particular turbulent flow at large enough Reynolds numbers. Another issue is concerned with the universality of scaling properties of small-scale turbulence, which has remained for more than fifty years one of the most active fields of inquiry. Derivation of scaling properties of fully developed turbulent flows directly from the Navier–Stokes equations analytically is one of the most popular illusive goals of theoretical research. This (scaling) and other *phenomenological* aspects are extensively reviewed in Monin and Yaglom (1971, 1975, 1992, 1996), Frisch (1995), and Sreenivansan and Antonia (1997).

Two notes are in order.

First, the most accepted division of turbulent flows on large and small scales is to a large extent artificial and in some sense even unphysical due to strong coupling between the two and due to an ambiguity of the very term 'scale' and the problematic nature of the decomposition approach to the phenomenon of turbulence and the necessity to handle turbulence as a whole (see chapters 3, 5, 9 and appendix C).

Second, it should be stressed that some of the (possibly universal) properties of turbulent flows (such as scaling) are characteristic of a much broader class of nonlinear systems, others are *specific* to turbulent flows. Both are addressed later in the book with the emphasis on the properties of fluid dynamical turbulence.

1.3. Why turbulence is so impossibly difficult[10]? The three N's

> *Turbulent flow constitutes an unusual and difficult problem of statistical mechanics, characterized by extreme statistical disequilibrium, by anomalous transport processes, by strong dynamical nonlinearity, and by perplexing interplay of chaos and order* (Kraichnan, 1972).
>
> *The experience of 100 years should suggest, if nothing else, that turbulence is a difficult problem, that is unlikely to suddenly succumb to our efforts. We should not await sudden breakthroughs and miraculous solutions* (Lumley, 1999).

1.3.1. ON THE NAVIER–STOKES EQUATIONS

The basic equations and some other essential information are given in appendix C. Here we provide some general notes regarding the Navier–Stokes equations (NSE) and related matters.

[10]See also the collection of citations in appendix B.

Though there exists a set of deterministic differential equations (NSE) probably containing (almost) all of turbulence, most of our knowledge about turbulence comes from observations and experiments (laboratory, field and later numerical)[11]. This was understood long ago by A.N. Kolmogorov: *I soon understood that there was little hope of developing a pure, closed theory, and because of absence of such a theory the investigation must be based on hypotheses obtained on processing experimental data* (see Tikhomirov, 1991)[12]. Much later he wrote that *the observational material is so large, that it allows us to foresee rather subtle mathematical results, which would be very interesting to prove* (Kolmogorov, 1978).

However, the conclusion that NSE are useful as only an experimental tool would be incorrect. It is true that there is little substantial theoretical use of NSE in turbulence, since there is almost no way to use them explicitly in theoretical approaches, e.g., by solving them by 'hand'. However, there are several ways to do this implicitly, i.e., by indirect use of NSE and their consequences. For example, looking at the NSE and their consequences themselves enables us to recognize the dynamically important quantities and physical processes involved. In other words, NSE and their consequences tell us what quantities and relations should be studied. So far this can be done mostly experimentally, but this kind of 'guiding' should also be useful theoretically. The most elegant exception is a set of theoretical results on the *a priori* upper bounds[13] of long time averages of dissipation and global transport of mass, momentum and heat (see Doering, 2009 and references therein). In this sense the NSE tell us more than any estimates of the dimensions of attractors and similar things.

Though the NSE have (at best) a limited kinetic foundation (for gases only), they are commonly believed to be adequate in the sense that their solutions correspond to real fluid flows. This is not obvious, since the NSE are a gradient expansion. So in principle, higher order terms *may* become dominant in regions with large velocity gradients. Also, NSE are the result of coarse graining over the stochastic (molecular) effects. However, one can take the standpoint of continuum mechanics at the very outset. In the latter case one encounters the problem of the relation between the stress tensor and the rate of strain tensor in the fluid flow. The Newtonian fluid is the one in which this relation is linear (see, e.g., Serrin, 1959). There exists

[11] This is the main reason that this book is biased experimentally.

[12] Therefore, the importance of experimental research in turbulence goes far beyond the view of those who think of an experimentalist as a superior kind of professional fixer, knowing how to turn nuts and bolts into a confirmation of their theories. The issue of confirmation/validation of 'theories' in turbulence is far more serious, see chapter 3.

[13] Unfortunately, there are no results on the lower bounds, except trivial values corresponding to the laminar flows.

large empirical evidence that NSE are adequate, at least, at all accessible Reynolds numbers, so we will take the standpoint of continuum mechanics. This does not exclude the possibility that in very 'hot spots', where the strain rate is extremely large, the Newtonian fluids become non-Newtonian (see chapter 10). So far there is no direct evidence of this[14].

1.3.2. ON THE NATURE OF THE PROBLEM

Formally, the problem is to solve the Navier–Stokes equations subject to initial and boundary conditions. At present, it is possible to obtain fully resolved solutions at moderate Reynolds numbers via direct numerical simulations of the Navier–Stokes equations. However, the important point is that looking at the behaviour of a *particular* solution does not solve the problem, since any particular solution (which is not in an analytical form) may not contribute much to the *understanding* of the basic physics of turbulent flows[15]. Ideally, one needs for this *a method of understanding the qualitative content of equations* (Feynmann, 1963). In other words, *nothing less than a thorough understanding of the* [global behaviour of the] *system of all their* [NSE] *solutions would seem to be adequate to elucidate the phenomenon of turbulence* (Neumann, 1949). That is in order to understand the dynamics, or the main characteristics of the dynamics, it is necessary to understand a significant portion of the phase flow (see glossary of terms), especially the unstable solutions[16]. However, at present (if ever) it is impossible due to very high dimension and complicated structure of the underlying attractors (assumed to exist): *one may never be able to realistically determine the fine-scale structure and dynamical details of attractors of even moderate dimension.... The theoretical tools that characterize attractors of moderate or large dimensions in terms of the modest amounts of information gleaned from trajectories* [i.e., particular solutions] ... *do not exist ... they are more likely to be probabilistic than geometric in nature* (Guckenheimer, 1986).

[14]The success of NSE equations for the laminar flows of viscous fluids seems to be well established, but even in this case, *it is, in fact, surprising that the assumption of linearity in the relation between τ_{ij} and s_{ij} as usually employed in continuum theory ..., works as well, and over as large a range, as it does. Unless we are prepared simply to accept this gratefully, without further curiosity, it seems clear that a deeper explanation must be sought* (Goldstein 1972).

[15]There is no consensus on what is (are) the problem(s) of turbulence and what would constitute its (their) solution. Neither is there agreement on what constitutes understanding. It is definitely not the proof of the existence of the smooth solution of the NSE on a three-dimensional domain for all time or anything similar. The discussion of this and related matters is postponed to the chapter 10.

[16]In dynamical systems the unstable solutions give the key to understanding of the global behaviour in some significant parts of the phase space.

However, the situation is not that bad, since in many cases an individual realization of a turbulent flow in a large enough space/time domain allows us to obtain important information of rather general nature. The reason is the property of ergodicity, which is believed to be true in turbulence (and NSE) and has a considerable empirical support[17].

1.3.3. NONLINEARITY

The nonlinearity of turbulence is the most frequently fingered as the main 'guilty party'. There are several 'howevers'.

First, there are nonlinear problems that are completely integrable. The well-known examples, are systems displaying solitons or solitary waves. In these systems the many degrees of freedom are so strongly coupled that they do not display any chaotic/irregular behaviour. Instead they are entirely organized and regular (see Zakharov, 1990 and references therein). By a quite questionable analogy it is thought that the so-called coherent structures in turbulent flows may be treated/ viewed in a similar way.

Second, nonlinearity is frequently blamed for the difficulties in the so-called closure problem which is associated with some form of decomposition, such as the Reynolds decomposition of the flow field into the mean and the fluctuations, or similar decompositions into resolved and unresolved scales associated with large eddy simulations. The essence of the problem is that the equations for the mean field (resolved scales) contain moments of the fluctuations (unresolved scales) due to the nonlinearity of the NSE. However, a similar problem exists for the so-called advection-diffusion equation describing the behaviour of a passive scalar in some flow field. But this equation is linear. The problem arises due to the multiplicative nature of the velocity field, since velocity enters this equation as its coefficients. Finally, a noteworthy caveat is that the inertial interactions as expressed by nonlinearity in the Euler setting, i.e., $(\mathbf{u}\cdot\nabla)\mathbf{u}$ *have a relative nature; they are eliminated in the transformation to the particle attached reference system* (Monin and Yaglom, 1975, p. 532). In other words, in a *pure* Lagrangian setting (see appendix 3) the acceleration term is linear, see further discussion in chapter 3.

1.3.4. NON-INTEGRABILITY

In 1788 Lagrange wrote: *One owes to Euler the first general formulas for fluid motion … presented in the simple and luminous notation of partial differences … By this discovery, all fluid mechanics was reduced to a single*

[17] But no theoretical/mathematical support: *The problem with this ergodicity assumption is that nobody has ever even come close to proving it for the Navier–Stokes equations* (Foias, 1997), though some mathematical results, which are claimed to be relevant to turbulence are given in Foias et al. (2001).

point analysis, and if the equations involved were integrable, one could determine completely, in all cases the motion of a fluid moved by any forces. (Mécanique analitique, Paris, 1788, section X. p. 271).

The *if* in the above citation is crucial: the Navier–Stokes equations are not integrable.

Integrable systems, such as those having a solution 'in closed form' exhibit regular organized behaviour, even those having (formally) an infinite number of strongly coupled degrees of freedom. A prominent example is provided by the solitons in the systems described by the Korteveg–de Vries and Shrödinger equations. Two other examples are the Burgers and the so-called restricted Euler equation, which are integrable equations, and exhibit random behaviour only under random forcing and or initial conditions, otherwise their solutions are not random[18]. That is, these examples represent the response of nonlinear systems to random forcing and which otherwise are not random, and should be distinguished from problems involving genuine turbulence. Navier–Stokes equations at sufficiently large Reynolds number have the property of intrinsic stochasticity in the sense that they possess mechanisms of self-randomization (most probably at all scales) which are not fully understood.

1.3.5. NONLOCALITY

This is probably one of the main reasons the problem of turbulence is so difficult.

Formally, nonlocality is due to the fact that the Navier–Stokes equations are integro-differential for the velocity field, and hence the velocity field is nonlocal in physical or any other space. Physically, it is because of the presence of long-range forces due to pressure. The property of nonlocality of NSE is two-fold. On the one hand, it is due to pressure ('dynamic' nonlocality), since $\nabla^2 p = \rho \frac{\partial^2 u_i u_k}{\partial x_i \partial x_j}$, and therefore pressure is nonlocal due to nonlocality of the operator ∇^{-2}: the pressure is defined in each space point by the velocity in the whole flow field. This aspect of nonlocality is strongly associated with the essentially non-Lagrangian nature of pressure, which is related to the 'memory' of turbulence, i.e., nonlocality in time.

[18]There is no consensus on the meaning of the term integrability, but it is agreed mostly that integrable systems behave nicely and are globally 'regular', whereas the non-integrable systems are not 'solvable exactly' and exhibit chaotic behaviour. See Zakharov (1990) and Kosmann-Schwarzbach et al. (2004) for more examples and discussion on what is integrability. The latter write *It would fit for a course entitled "Integrability" to start with a definition of this notion. Alas, this is not possible. There exists a profusion of definitions and where you have two scientists you have (at least) three different definitions of integrability* but mention the definition by Poincaré: *to integrate a differential equation is to find for the general solution a finite expression, possibly multivalued, in a finite number of functions.*

One cannot get rid of nonlocality by taking the *rot* of the NSE, thereby getting rid of the pressure gradient ∇p, and looking at the resulting equation for vorticity ω. The reason is that the equation for vorticity is nonlocal in vorticity, since it contains the rate of strain tensor $s_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$, $\mathbf{u} = rot^{-1}\omega$ due to the nonlocality of the operator rot^{-1}(‘kinematic’ nonlocality). In other words, since (for incompressible fluids) $\nabla^2 \mathbf{u} = -rot\omega$, the whole flow field is defined in each space point by the vorticity in the whole flow field and boundary conditions on velocity[19]. This in turn means that the large scales as represented by the velocity field and the small scales as represented by vorticity (and strain) should be strongly coupled, as indeed is the case. Note that this coupling is *bidirectional*, i.e., the small scales cannot be seen as passive or as ‘slaved’ to the large scales – the small scales react back in a nonlocal manner. There is no such relation in the case of a passive contaminant (or any other passive object) in a turbulent flow. The relation in this latter case, though also nonlocal, is one-directional: the fluid flow does not ‘know’ anything about the presence of the passive object.

The nonlocality due to the coupling between large and small scales is also manifested in problems related to various decompositions of turbulent flows and in the so-called closure problem. For example, in the Reynolds decomposition of the flow field into the mean and the fluctuations and in similar decompositions associated with large eddy simulations (LES), the relation between the fluctuations and the mean flow (or resolved and unresolved scales in LES, etc.) is not pointwise in space/time, it is a functional. That is the field of fluctuations at each time/space point depends on the mean (resolved) field in the whole time/space domain. Vice versa, the mean (resolved) flow at each time/space point depends on the field of fluctuations (unresolved scales) in the whole time/space domain. This is because the equations for the fluctuations (unresolved scales) contain as coefficients the mean (resolved) field. This means that in turbulent flows, point-wise flow independent ‘constitutive’ relations analogous to real material constitutive relations for fluids (such as stress/strain relations) can not exist, though the ‘eddy viscosity’ and ‘eddy diffusivity’ are frequently used[20] as a crude approximation for taking into account the reaction of fluctuations (unresolved scales) on the mean flow (resolved scales). The fact that the ‘eddy

[19]Note that, just like in the case of vorticity, the whole flow field is determined entirely by the field of strain. This is seen from the equation $\nabla^2 u_i = 2\partial s_{ik}/\partial x_k$, which together with the boundary conditions uniquely defines the velocity field.

Note that the evolution of the rate of strain tensor is governed by the equation (C.17) containing the pressure Hessian $\frac{\partial^2 p}{\partial x_i \partial x_j}$, so, in fact the pressure is present in the equation for vorticity (C.9) either.

[20]But Scorer (1978) is quite critical about *the doubtful meaning of an eddy transfer coefficient.*

viscosity' and 'eddy diffusivity' are flow (and space/time) dependent is just another expression of the strong coupling between the large and the small scales. More details on the issue of nonlocality are given in chapter 6[21].

1.3.6. ON PHYSICS OF TURBULENCE

The difficulties described above are mostly of a formal/technical nature. There is another difficulty of a more general nature. It is the lack of knowledge about the basic physical processes of turbulence and its generation and origin, and poor understanding of the processes which are already known. For example, the underlying mechanisms of predominant vortex stretching, which is why in turbulent flows vorticity is stretched more than compressed, are (at best) poorly understood and essentially not known. Until recently a not less important concomitant process of strain production was mostly neglected by the community. It is this process (rather than vortex stretching) that is directly responsible for the enhanced dissipation of turbulent flows. There are qualitative differences between the two. The enstrophy production is a nonlocal process with predominant stretching, whereas the strain production is a local process with predominant compressing, see chapters 6 and 9. Another example concerns the question on how well defined is the concept of inertial range and the consequences of the so-called 'anomalous scaling'.

1.3.7. ON STATISTICAL THEORIES

> *It has been realized since the beginning that the problem of turbulence is a statistical problem; that is a problem in which we study instead of the motion of a given system, the distribution of motions in a family of systems ... It has not, however, been adequately realized just what has to be assumed in a statistical theory of turbulence* (Wiener, 1939).
> *Thus the analogy with the kinetic theory of gases is of relatively little help in the formulation of the theory of turbulence, and is useful only for a preliminary understanding of the concept of a statistical approach to physical theory* (Monin and Yaglom, 1971).
> *The statistical-mechanical treatment of turbulence is made questionable by strong nonlinearity and strong disequilibrium that result in the creation of ordered structures imbedded in disorder* (Kraichnan and Chen, 1989).

[21] There are also many 'small' **n**'s such as non-gaussianity, non-markovianity, no low-dimensional description, no small parameters and no theory based on first principles as NSE equations, which is a real frustration for a theoretician.

Just as in statistical physics, the statistical approach should be adopted in turbulence not only in 'theories' but also in handling the data from physical and numerical experiments from the outset/start due to the extreme complexity of turbulence phenomenon(a). In both cases certain statistical hypotheses are made. But the former was quite successful in making a number of important predictions, whereas the latter, with few exceptions, such as the Kolmogorov four-fifths law (Kolmogorov, 1941b), was unable to produce genuine predictions based on the first principles. All the rest – in the words of P.G. Saffman – are postdictions. Apart from the above-mentioned reasons for such a failure it should be mentioned that, unlike statistical physics, in turbulence neither 'simple objects' (such that a collection of these objects would adequately represent turbulent flows) 'to do statistical mechanics' with them, nor 'right' statistical hypotheses have so far been found. The question about the very existence of both remains open, for more see, e.g., Monin and Yaglom, 1971, pp. 4–5. The problem seems to be even a bit more complicated as turbulence (being studied by all kinds of statistical methods of description) cannot be considered as just a problem of statistical physics/mechanics only. There is no effective/satisfactory theoretical framework to handle turbulence (nothing new: this was stated by von Neumann in 1949), though it is true that turbulence can be seen also (but only in part!) as a problem of nonequilibrium statistical physics or whatever.

1.4. Outline of the following material

Our main emphasis is on the basic properties of turbulent flows. Therefore origins of turbulence and ways of its creation are discussed only briefly in chapter 2, together with some points concerning the basic differences with transition to chaos. Instability and transition to turbulence and chaos comprise several separate disciplines, in which many thousands of publications already exist.

Chapter 3 is devoted to the problem of describing and studying turbulent flows, with the stress on the principal points. The main additions concern Eulerian versus Lagrangian descriptions (section 3.6) and ergodicity (section 3.7). In addition main technical issues (appendix E in the first edition) are also given an overview in this chapter with a discussion of the issue of experimental 'validation of theories' and measurements at high Reynolds numbers.

Selected kinematic issues are discussed in chapter 4. Namely, we concentrate on those issues of the behaviour of passive objects in random flows and kinematic (Lagrangian) chaos which are relevant for comparison with the dynamical aspects of turbulent flows.

The primary aspects of phenomenology are discussed in chapter 5, with the stress placed on the concepts of inertial range, cascade, decompositions and related matters. The main addition is a section on anomalous scaling and ill-posedness of the concept of inertial range.

Chapters 6 and 7 are devoted to the dynamics of turbulence and its structure, with the emphasis placed on the dynamics of the field of velocity derivatives: vorticity and strain, and their interaction. Special attention is given to nonlocality, non-Gaussianity and geometrical statistics, intermittency and its relation to structure(s). Additional stress and evidence is given to the role of strain in turbulent flows, the so-called Tennekes and Lumley balance, nonlocality and fluid particle accelerations.

An overview of turbulent flows under various influences and physical circumstances, some of which serve also as sources of energy for sustaining the turbulence, is given in chapter 8. These include shear, buoyancy, rotation, (electro-) magnetic field, compressibility and additives. The main additions concern new evidence on the phenomenon of entrainment, qualitative differences between quasi-two-dimensional and pure two-dimensional turbulent flows, and turbulent flows of dilute polymer solutions. For obvious reasons the material of this chapter is limited by only the most important essential features and changes in turbulent flows under various influences.

Chapter 9 is a new one which grew out of a one page appendix D in the previous edition. Its title tells clearly what it is: *Analogies, misconceptions and ill-defined concepts*. The main emphasis is given to differences rather than similarities. The primary reason for this is that understanding of differences is expected to aid better understanding of both systems and avoid misconceptions associated with extending the analogies too far. Second, dealing with conceptual aspects of turbulence research leads necessarily to addressing misconceptions which have arisen during more than a century of turbulence research attempting to achieve some physical understanding/picture of this enigmatic phenomenon.

Chapters 2 to 7 and chapter 9 are concluded by a brief summary.

The former chapter 9 became chapter 10. It is devoted to the recapitulation of some main points with somewhat different emphasis, and to the discussion of issues of general nature not addressed in previous chapters. The main addition there is a section called *Turbulence versus mathematics and vice versa* dealing mainly with the issue of how relevant are the existing mathematical developments to (understanding of) turbulence.

This last chapter is followed by a list of references and appendices. The latter contain glossaries of some terms, and a glossary of essential fluid mechanics containing some not broadly known facts. These are followed by subject and author indices.

1.5. In lieu of a summary

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PROPERTIES OF TURBULENCE

- Intrinsic spatio-temporal randomness, irregularity. Turbulence is chaos (but not necessarily vice versa); its intrinsic property is self-stochastization or self-randomization.
- Loss of predictability, but stable statistical properties.
- Extremely wide range of strongly and nonlocally interacting degrees of freedom ('scales' in time and space).
- Highly dissipative, statistically irreversible.
- Turbulent flows are three-dimensional and rotational with continuous self-production of vorticity and strain.
- Strongly diffusive with enhanced transport of momentum, energy, and passive objects.
- Strongly nonlinear, non-integrable, nonlocal, non-Gaussian.

ORIGINS OF TURBULENCE

Overview of instability, transition and chaos

2.1. Instability

> *Yet not every solution of the equation of motion, even if it is exact, can actually occur in Nature. The flow that occurs in Nature must not only obey the equations of fluid dynamics but also be stable* (Landau and Lifshits, 1959).
> *Kolmogorov's scenario was based on the complexity of the dynamics along the attractor rather than on its stability* (Arnold, 1991).
> *To the flows observed in the long run after the influence of the initial conditions has died down there correspond certain solutions of the Navier–Stokes equations. These solutions constitute a certain manifold* $\mathcal{M} = \mathcal{M}(\mu)$ *(or* $\mathcal{M} = \mathcal{M}(\mathrm{Re})$*) in phase space invariant under phase flow ... The notion of stability here refers to the whole manifold and not to the single motions contained in it* (Hopf, 1948).

It is a common view that the origin of turbulence is in the instability of some basic laminar flow(s). This is understood in the sense that any flow is started at some moment in time from rest, and as long as the Reynolds number (or a similar parameter) is small, the flow remains laminar. As the Reynolds number increases, some instability sets in, which is followed by further (secondary, tertiary ...) instabilities (bifurcations), transition and a fully developed turbulent state[1]. Such sequences of events occur not only throughout the whole flow field, but also at successive downstream locations of a single flow, such as the spatially developing flows as shown in figures 1.4, 1.5, 1.7, 1.8, 1.11 and 1.13 of chapter 1. However, it is important to stress that transition to a turbulent regime may be quite sudden. For example, this

[1] The literature on fluid flow instabilities and transition is vast. Comprehensive reviews and lists of references can be found in Drazin and Reid (2004) and references to the foreword by John Miles, Huerre and Rossi (1998), and Monin and Yaglom (1971, 1992), and also Monin (1986) .

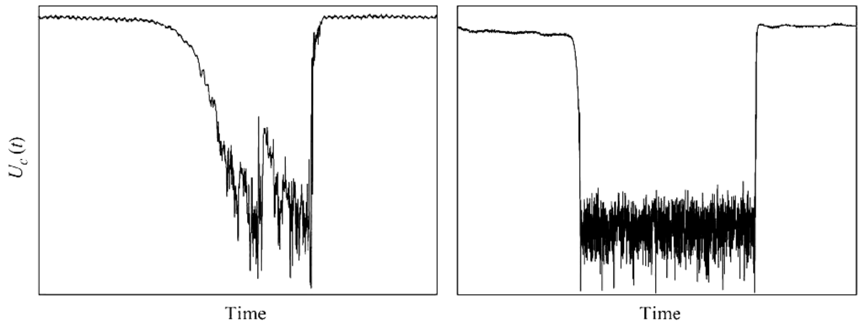

Figure 2.1. Centreline velocity time records as examples of puff (left) and slug (right). Note the abrupt transition between the laminar and turbulent states and vice versa. (Durst and Ünsal, 2006)

may happen 'in no time' in pipe flows under certain conditions[2] (see figure 2.1) or in the process involving the impingement of a laminar vortex ring upon a rigid wall (see figure 2.2) and at the laminar/turbulent 'interfaces' in turbulent spots and in all partly-turbulent flows (see chapter 8). In all these there is a distinct Lagrangian aspect: the abrupt transition of fluid particles (i.e., Lagrangian objects) from the laminar to turbulent state when passing across the laminar/turbulent 'interface'.

From the mathematical point of view the transitions from one flow regime to another with increasing Reynolds number – as we observe them in *physical* space – are believed to be a manifestation of generic structural changes of the mathematical objects called phase flow and attractors in the *phase* space through bifurcations in a given flow geometry (Hopf, 1948). However, partly-turbulent flows (a special feature of these flows is the co-existence of regions with laminar and turbulent states of flow) do not easily 'fit' in this picture. Note that in partly-turbulent flows there is a continuous transition of laminar flow into turbulent as a result of the entrainment process occurring across the boundary between the two (see section 8.2). We emphasize that the above transition is of distinctly Lagrangian nature as it happens with fluid particles which are purely Lagrangian objects!

Whereas the processes by which flows become turbulent are quite diverse, all known qualitative and a number of quantitative properties of

[2]For example in a pipe flow which is held laminar at rather large Reynolds number by special precautions at Re up to 10^5, and then subject to disturbance of finite amplitude. This is possible because *It is thought that pipe flow is stable for all infinitesimal disturbances and it may be that appropriate kinds of disturbance will grow when the amplitude exceeds a value which depends on Reynolds number* (Taylor, 1962). This property is proven rigorously for the plane Couette flow.

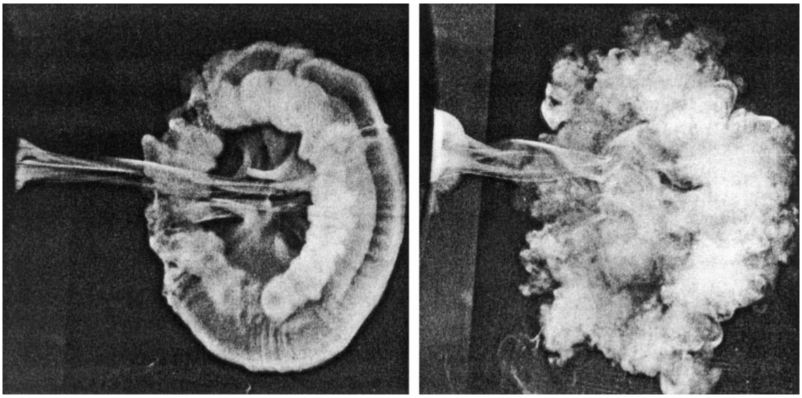

Figure 2.2. Sudden transition of a laminar vortex ring to a turbulent state (Schultz–Grunov, 1980)

many (but not all) turbulent flows do not depend either on the initial conditions or on the history and particular way of their creation, e.g., whether the flows were started from rest or from some other flow and/or how fast the Reynolds number was changed. The qualitative properties of all turbulent flows are the same.

The diversity of the processes by which flows become turbulent is in part due to the sensitivity of the instability and transition phenomena to various details characterizing the basic flow and its environment. For example, the Orr–Sommerfeld equation governing the linear(ized) (in)stability contains the second derivative of the basic velocity profile. Many flows (some of the so-called open flows, such as flows in pipes, boundary layers, jets, wakes, mixing layers) are very sensitive to external noise and excitation. There are essential differences in the instability features of turbulent shear flows of different kinds (wall-bounded – pipes/channels, boundary layers, and free – jets, wakes and mixing layers), thermal, multidiffusive and compositional convection, vortex breakdown, breaking of surface and internal waves and many others[3]. It is important that such differences occur also for the *same* flow geometry, which display in words of M.V. Morkovin *bewildering variety of transitional behaviour*. The specific route may depend on initial conditions, level of external disturbances (receptivity), forcing, time history and other details in most of the flows mentioned above (see figure 2.3 for an example of such sensitivity).

This diversity is especially distinct for the very initial stage, which is the (quasi-) linear(ized) instability. Later nonlinear stages are less sensitive to such details. Hence there is a tendency to universality in strongly nonlinear

[3]For a description of particular examples and references see Tritton (1988, chapters 17, 18, 22–24), Sherman (1990, chapter 13) and Huerre and Rossi (1998).

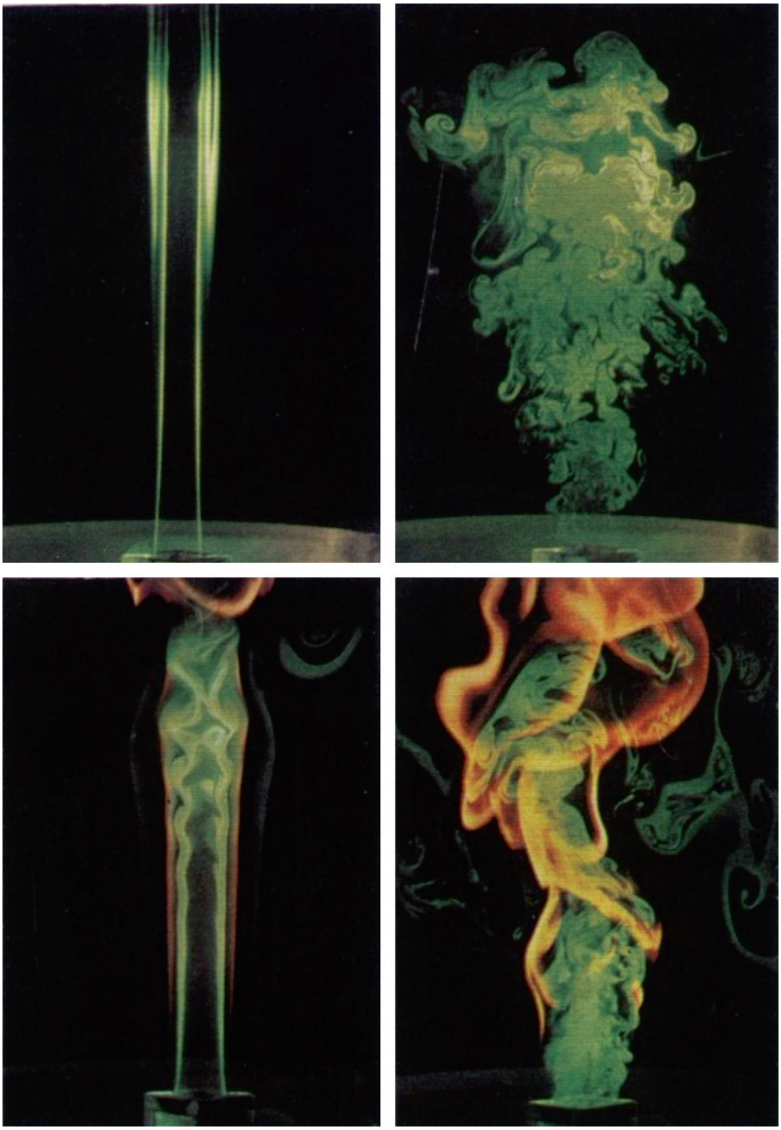

Figure 2.3. Flow patterns of a square jet of cold flow (nitrogen, top) and combusting gas (propane, bottom) exhibiting strong dependence on forcing (4 Hz, left column) at the jet exit by piezoelectric actuators. Courtesy of Professor A. Glezer

regimes, such as developed turbulence[4]. We shall discuss this tendency in more detail in later chapters.

[4]By tendency, it is meant that universality occurs on the qualitative, but not necessarily on the quantitative level.

One of the important common features of processes resulting in turbulence is that all of them tend to enhance the rotational and dissipative properties of the flow in the process of transition to turbulence. The first property is associated with the production of vorticity, whereas the second property is due to the production of strain (see chapters 5, 6 and 8).

2.2. Transition to turbulence versus routes to chaos

One of the main achievements of modern developments in deterministic chaos is the recognition that chaotic behaviour is an intrinsic fundamental property of a wide class of nonlinear physical systems (including turbulence) and not a result of external random forcing or errors in the input of the numerical simulation on the computer or the physical realization in the laboratory. The nonlinear systems and the equations describing them produce an apparently random output 'on their own', 'out of nothing', which is their very nature. However, there is a variety of qualitatively different systems exhibiting such a behaviour just as there is a large diversity of such behaviours.

The qualification of turbulence as a phenomenon characterized by a large number of strongly interacting degrees of freedom enables us to make a clear distinction between transition to turbulence and transition to chaotic behaviour.

The main point of distinction is as follows.

Low-dimensional chaotic systems like the famous Lorenz (1963) system or the spherical pendulum studied by Miles (1984) change their behaviour from simple regular (as periodic) to distinctly chaotic as some parameter of the system changes. However, obviously the number of degrees of freedom of all such systems remains the same, only the character of the interaction of these degrees of freedom changes.

At early stages of transition to turbulence some fluid dynamical systems (but not all)[5] exhibit the same behaviour as found in the so-called 'routes to chaos' in low-dimensional dynamical systems. Namely, they have only a few excited degrees of freedom and a fixed number of them, which are strongly correlated over the whole flow domain. Hence their dynamics is essentially temporal: it is chaotic but rather 'simple', i.e., the chaos is temporal only, the spatial structure of the flow is not changing. However, at more advanced stages of transition, the number of excited degrees of freedom in fluid flows increases rapidly with the Reynolds number (or similar parameters such as

[5] The so-called closed systems, like small aspect ratio Rayleigh-Bènard convection and Taylor–Couette flow, see e.g., Aref and Gollub (1996), Mullin (1993) and Tritton (1988, section 24.7) and references therein.

the Rayleigh number in thermal convection) and in the developed stage, it is $\sim Re^{9/4}$ (see appendix 3 and chapter 5). This steep increase in the number of excited degrees of freedom results in a qualitative change in the behaviour of the flow. It is chaotic as well, but qualitatively different, much more complicated kind of chaos: it is both temporal and spatial and high-dimensional: 'more is different' (Anderson, 1972, 1991, 1995). The idea that the essential feature of transition to turbulence is an increase of the number of excited degrees of freedom dates back to Landau (1944) and Hopf (1948) and is correct, though the details of their scenario appeared to be not precise (see Monin, 1986). However, *Kolmogorov's ideas on the experimentalist's difficulties in distinguishing between quasi-periodic systems with many basic frequencies and genuinely chaotic systems have not yet been formalized* (Arnold, 1991). In other words it is very difficult if not impossible to make such a distinction in practice.

Here is the right place to note that there is an important difference between the number of degrees of freedom roughly proportional to the number of ordinary differential equations necessary to adequately represent a system described by partial differential equations (NSE) and the dimension of the attractor of the system (if such exists). In a particular dynamical system, the former is obviously fixed and is independent of the parameters of the system, whereas the latter is changing with the parameters but is bounded. In turbulence both are essentially increasing with the Reynolds number and become very large at large Reynolds number.

2.3. Many ways of creating turbulent flows

Any turbulent flow is maintained by an external source of energy produced by one or more mechanisms. The mechanisms maintaining/sustaining turbulence, at least some of them, are believed to be closely related (but are not the same) as those by which laminar and transitional flows become turbulent. We address this issue in chapter 8. Here we note that, apart from a great variety of turbulent flows in nature/technology and 'natural' ways resulting from instabilities, turbulent flows can be produced by 'brute force', i.e., by applying external forcing of various kinds both in real physical systems and in computations by adding some forcing in the right hand side of the Navier–Stokes equations[6]. For example, one of the simplest kinds of turbulent flow such as quasi-homogeneous and isotropic, can be established by moving a grid through a quiescent fluid or placing such a grid in a wind tunnel, or oscillating such grids in a water tank. Turbulence is produced by forcing in the interior of the fluid flow (by electromagnetic forces, e.g., in electrolytes, liquid metals or plasma; or other body forces) or at flow

[6]The random force method in turbulence theory is due to Novikov (1963a, 1964).

boundaries, which can be still or moving/flexible, smooth or rough, simple or complex. Similarly turbulent flow can be produced numerically with an infinite versatility by adding a force (random or deterministic) to the right-hand side of the Navier–Stokes equations and/or forcing the flow at its boundaries.

An important point is that the nature of forcing (deterministic, random, temporally modulated or whatever) is secondary in establishing and sustaining a turbulent flow, provided that the Reynolds number is large enough and the forcing is mostly in the large scales. Another important point is that the forcing does not have to be random. Even if a turbulent flow is produced by random forcing, the primary role of such forcing (as of any forcing) is to supply energy to the flow and to trigger the intrinsic mechanisms of self-stochastization or self-randomization of turbulent flow, i.e., creation of randomness out of 'nothing'. This is the reason why both kinds of turbulent flows – those arising 'naturally', e.g., by a simple (time independent and smooth in space) deterministic forcing, or produced by some external random source, are the same qualitatively and in some (but not all) essential respects quantitatively. However, the nature of forcing if, e.g., it is not large-scale, may result in qualitative differences such as in the case of broadband forcing[7]. Similarly, boundary, initial and inflow conditions may cause qualitative difference as well.

At small enough Reynolds numbers, the flow produced by deterministic forcing is not random, it is laminar, but the flow produced by random forcing, though random, is in many respects trivial (as any randomly-forced linear system), e.g., there is no interaction between its degrees of freedom/modes. Strictly speaking this latter is true of pure dynamical flow properties described in an Eulerian setting because some of its 'kinematic' properties as described in a pure Lagrangian setting can be (and usually are) pretty complex and not trivial, see chapters 3, 4 and 9.

Thus a turbulent flow originates not necessarily out of a laminar flow with the same geometry. It can arise from any initial state including a 'turbulent' one, such as random initial conditions in direct numerical simulations of the Navier–Stokes equations. That is, the transition from laminar to turbulent regime is not the only causal relation. This problem is related to a somewhat 'philosophical' question on whether flows become or whether they just are turbulent, and to the unknown properties of the phase flow, attractors and related matters, which are far beyond the scope of this book.

[7]In this case if the forcing is strong enough not only in the large scales it can balance the viscous effects directly, thereby bypassing the nonlinearity, see chaper 6.

2.4. Summary

There is a great variety of ways/routes in which a laminar flow becomes turbulent, just as there are many ways to establish approximately the same turbulent flow. In other words, the view that turbulent flows always develop from laminar ones is too narrow.

Once a flow becomes turbulent, it seems impossible to find out its origin. The reason is due to the chaotic nature and the irreversibility of turbulent flows.

The main difference between the transition to chaos and to turbulence is that in the former the number of degrees of freedom remains fixed, whereas in the latter the number of degrees of freedom increases strongly with increases in the Reynolds number and/or other similar parameters.

CHAPTER 3

METHODS OF DESCRIBING AND STUDYING TURBULENT FLOWS

Deterministic, structural, statistical or something else?

> *While the experimental techniques that have been invaluable in understanding phase transitions promise to be very useful in the study of hydrodynamic phenomena, I suspect that the recent addition to our theoretical arsenal may be less effective than many had hoped* (Martin, 1976).
>
> *I think that the k-space decomposition does actually obscure the physics* (Moffatt, 1990a).
>
> *In contrast to this experimental cornucopia, theory can offer only a few crumbs* (Siggia, 1994).
>
> *... the observational material is so large, that it allows to foresee rather subtle mathematical results, which would be very interesting to prove* (Kolmogorov, 1978).
>
> *Sometimes experiments provide us with so beautiful and clear results that it is a shame on theorists that they cannot interpret them* (Yudovich, 2003).

One of the major problems in describing turbulence stems from its extremely intricate effectively/apparently random behaviour along with a huge number of strongly and nonlocally interacting degrees of freedom. Other reasons why the turbulence problem is so impossibly difficult have been mentioned in chapter 1. One more consideration involves the fact that adequate tools to handle both the problem and the phenomenon of turbulence are not developed enough. In this respect the state of matters is not very much different from the one depicted by von Neumann (1949):

The entire experience with the subject indicates that the purely analytical approach is beset with difficulties, which at this moment are still prohibitive. The reason for this is probably as was indicated above: That our intuitive relationship to the subject is still too loose – not having succeeded at anything like deep mathematical penetration in any part of the subject, we are still quite disoriented as to the relevant factors, and as to the proper analytical machinery to be used.

That is there is no way and no tools so far, if ever, to treat turbulence analytically – turbulence is beyond analytics (TBA). Unfortunately, this is true of other theoretical approaches such as attempts to construct statistical and/or other theories.

Another matter concerns the technical tools, which are of purely experimental and observational nature. Unlike the theoretical issues/problems much essential progress has occurred in developing numerical, laboratory and field experimental approaches to turbulent flows. It is noteworthy that in view of the state of the theoretical field, the experimental research in turbulence was and remains the main source of knowledge of turbulent flows. Therefore, the role of experiments in turbulence goes far beyond the view of those who think of experimentalists as a superior kind of professional fixers knowing how to turn nuts and bolts into a confirmation of other people's 'theories'. From the basic point of view there is almost nothing to be confirmed so far. On the contrary *the essential mathematical complications of the subject were only disclosed by actual experience with the physical counterparts of these equations* (von Neumann, 1949), and ... *the observational material is so large, that it allows to foresee rather subtle mathematical results, which would be very interesting to prove* (Kolmogorov, 1978). These statements remain as valid today as they were earlier. The experiment remains a major exploratory tool in elucidating the properties of turbulence as a physical phenomenon. We recall here that this does not mean that the Navier–Stokes equations are useful as an experimental tool only (see chapter 1, subsection 1.3.1.)

This chapter is devoted to some matters of principle regarding the methods of describing turbulent flows and related issues.

3.1. Deterministic versus random/stochastic or how 'statistical' is turbulence?

> *There is probably no such thing as a most favoured or most relevant, turbulent solution. Instead, the turbulent solutions represent an ensemble of statistical properties, which they share, and which alone constitute the essential and physically reproducible traits of turbulence* (Von Neumann, 1949).
>
> The "statistical" *community ... strongly disputes the possibility of any coherence or order associated to turbulence* (Lesieur, 1997).
>
> *The transition from laminar to turbulent flow is a nonequilibrium phase transition to a more organized motion* (Klimontovich, 1996).

It is quite common to contrapose the 'traditional' statistical and the deterministic/structural approaches in turbulence research. However, contrasting the terms 'deterministic' and 'random' has lost most (but not all) of its meaning with the developments in 'deterministic chaos': it is well established that even simple systems governed by purely deterministic nonlinear sets of equations, such as those described by only three nonlinear ordinary differential equations, as a rule exhibit irregular apparently random/stochastic behaviour. In fact, in respect to turbulence this was known long before the 'discovery of chaos'. However, since Leray (1934) one was not sure about the (theoretical but not observational) possibility that turbulence is a manifestation of breakdown of the Navier–Stokes equations.

Early justifications for the necessity of statistical approaches to turbulence were usually based on the extreme complexity of the individual realizations of turbulent flows:

In … *turbulent motion, an enormous number of degrees of freedom are always excited, and hence the variation with time* (and space) *of any physical value will be described* … *by functions of extremely complicated nature.*

The theory of turbulence by its very nature cannot be other than statistical, i.e., an individual description of the fields of velocity, pressure, temperature, and other characteristics of turbulent flow is in principle impossible. Moreover, such description would not be useful even if possible, since the extremely complicated and irregular nature of all the fields eliminates the possibility of using exact values of them in any practical problems …

In the present-day statistical fluid mechanics, it is always implied that the fluid mechanical fields of a turbulent flow are random fields in the sense used in probability theory. (Monin and Yaglom, 1971, pp. 3–4, 7).

From the very beginning it was clear that the theory of random functions of many variables (random fields), whose development only started at that time, must be the underlying mathematical technique. (A.N. Kolmogorov, 1985 in notes preceding the papers on turbulence in the first volume of his selected papers, English translation, Tikhomirov, 1991, p. 487).

Later the very large dimension (see table 3.1) and complicated (stochastic?) structure of the underlying attractors, assumed to be in existence, was invoked in the justification of the unavoidable necessity of statistical methods of *description* (and 'theories') of turbulent flows: *one may never be able to realistically determine the fine-scale structure and dynamical details of attractors of even moderate dimension* … *The theoretical tools that characterize attractors of moderate or large dimensions in terms of the modest amounts of information gleaned from trajectories* [i.e. particular solutions] … *do not exist* … *they are more likely to be probabilistic than geometric in nature* (Guckenheimer, 1986).

Apart from the extreme intricacy of turbulent flows and their stochastic nature, there is another advantage in favour of the statistical methods of description (not necessarily 'theories'). Namely, all the experience accumulated during the period of a century of studying turbulent flows shows that

TABLE 3.1. Dimensions of "attractors". Here D_L – is the so-called Lyapunov dimension, D_{KL} – is roughly the number of modes to account for 90% of the energy, and D_l – is the well-known Landau estimate, which is roughly proportional to the number of ODE's needed to adequately represent the flow (it was removed from the last Russian and subsequent English editions). This is an updated table based on the one by Sirovich (1997)

	D_L	D_{KL}	$D_l \cdot 10^{-6}$
Isotropic			
Landau			$\mathrm{Re}^{9/4}$
Pipe			
Huang and Huang (1989)	>11		
Sieber (1987)	>10		
Plane Channel			
Keefe et al. (1992)	800	400	26
Sirovich et al. (1991)	1500	4200	100
Webber et al. (1997)	300	650	15

turbulent flows possess stable statistical properties (SSP).[1] It is important to stress that stable statistical properties means not only means (averages) and other simple things, but much more, presumably all statistical properties together with those which are related to what can be called the structure of turbulence, which is not the same as what is called structures of turbulence, e.g., its instantaneous structure-like features (see chapter 7). The existence of SSP seems to be an indication of the existence of what mathematicians call attractors. But again, matters are more complicated than that. Many statistical properties of nonstationary (time-dependent in the statistical sense) turbulent flows are quite similar to those of statistically stationary ones as long as the Reynolds number of the former is not too small at the particular time moment of interest. For example, in a decaying turbulent flow past a grid, almost everything is very similar qualitatively and in many respects quantitatively to what is happening in decaying or statistically stationary-forced turbulence in a cubic box with periodic boundary conditions (see De Bruyn Kops and Riley, 1998; Galanti and Tsinober, 2000; Tsinober et al., 1997 and references therein).

It should be emphasized that our concern here is not with statistical *theories* all of which are using various *ad hoc* assumptions on the nature

[1] *The existence of an asymptotic statistical state is strongly suggested experimentally, in the sense that reproducible statistical results are obtained. However, physical plausibility aside, it is embarrassing that such an important feature of turbulence as its statistical stability should remain mathematically unresolved, but such is the nature of the subject* (Orszag, 1977).

and properties mainly of the small-scale structure and its relation with the rest of the flow, and/or attempting to represent turbulence as a collection of more or less *simple* objects. Our concern is much less ambitious – the focus is on statistical methods of *description and interpretation* of the data from laboratory, field and numerical experiments on turbulent flows via appropriate *processing* of the data. The latter is likely to be a prerequisite for any worthy 'theory' of turbulence. Quoting A.N. Kolmogorov, 1985: *... I soon understood that there was little hope of developing a pure, closed theory, and because of absence of such a theory the investigation must be based on hypotheses obtained on processing experimental data* (Tikhomirov, 1991, p. 487). This view goes back to Leonardo da Vinci: *Remember, when discoursing about water, to induce first experience, then reason.*

It should be stressed that even the simplest nonlinear systems exhibiting chaotic behaviour are analyzed via various statistical means. Also, the so called 'coherent structures' in turbulent flows are looked for using essentially statistical methods, such as conditional statistics though with limited success (Bonnet, 1996). Finally, methods of dimensional analysis, similarity and symmetries (group theoretical methods) and phenomenological arguments are applied exclusively to quantities expressing the statistical properties of turbulent flows.

3.2. On statistical theories, reduced (low-dimensional) representations and related matters

As mentioned, statistical *methods of describing* turbulent flows should not be confused with statistical *theories* of turbulent flows. The latter are outside the scope of this book for the reasons explained above, so only a few brief remarks on this subject are given below.

The natural tendency to simplify the problem is manifested in numerous searches for a reduced description of turbulent flows[2]:

One of the most basic questions in turbulence "theory" (which usually is not asked) is whether there exists a closed representation that is simple enough to be tractable and insightful, but powerful enough to be faithful to the essential dynamics (Kraichnan and Chen, 1989).

An early goal of the statistical theory of turbulence was to obtain a finite, closed set of equations for average quantities, including the mean velocity and energy spectrum. That goal now is viewed to be unrealistic. The goal is now to reduce to a manageable number the many degrees of freedom necessary to describe the flow, to determine the equations governing

[2]The whole issue is closely related to the problem of decomposition/representation of turbulent flows (next section) and their structure (chapter 7).

the dynamics of the reduced degrees of freedom, and to solve those equations analytically or numerically to calculate fundamental quantities that characterize the flow (Frisch and Orszag, 1990).

If we assume as a basic starting point in every theory of turbulence its representation in terms of spectral coefficients, statistical or physical averages, or more generally simple objects conditionally extracted by a weak background, turbulence modelling could be defined reductively, as the art of writing the equations that produce directly such quantities (Germano, 1999).

The emergence of collective modes in the form of coherent structures in turbulence amidst the randomness is an intriguing feature, somewhat reminiscent of the mix between the regular "islands" and the "chaotic sea" observed in chaotic, low-dimensional dynamical systems. The coherent structures themselves approximately form a deterministic, low-dimensional dynamical system. However, it seems impossible to eliminate all but a finite number of degrees of freedom in a turbulent flow – the modes not included form an essential, dissipative background, often referred as an eddy viscosity, that must be included in the description (Newton and Aref, 2003).

These citations represent the most popular view/hope/belief/ assumption and an implicit claim that such a reduction is possible and results in an adequate description of the remaining degrees of freedom, which presumably include some of the so-called 'coherent structures'[3]. Such a belief goes back to the early forties: ... *it is necessary to separate random processes from the nonrandom processes* (Dryden, 1948), and, in fact, is the essence of the concept of 'eddy viscosity' (Boussinesq, 1877; Kraichnan, 1976, 1988). The implication is that such a separation is possible. But it is not obvious at all that such a reduction is possible, as it is seen from the futility of enormous efforts to do so throughout the whole history of turbulence research. The difficulty is a nontrivial one. For example, one even does not know how to separate random gravity-wave motion and genuine turbulence in a stably stratified fluid (Stewart, 1959).

There is, however, a less popular view too:

Most problems in classical stochastic processes are reduced to solubility by statistical independence, or the assumption of a normal distribution (which is equivalent) or some other stochastic model; because of the governing differential equations, the turbulent velocity at two space-time points is, in principle, never independent – in fact, the entire dynamical behaviour is involved in the departure from statistical independence. The equations, in fact, preclude the assumption of any ad hoc model, although this is often done in the absence of a better idea (Lumley, 1970).

[3]This is related to a more general assumption that perhaps(!) large systems actually boil down to a much smaller number of degrees of freedom than actually excited, because many of them are strongly correlated within a group representing a 'coherent structure'.

Perhaps the biggest fallacy about turbulence is that it can be reliably described (statistically) by a system of equations which is far easier to solve than the full time-dependent three-dimensional Navier–Stokes equations (Bradshaw, 1994).

Theoretical estimates of the dimension of the attractor for channel turbulence appear to preclude truly low-dimensional description (Omurtag and Sirovich, 1999).

In spite of these warnings there is a general belief that an adequate reduced (low-dimensional) description is possible[4], e.g., via reduction of the huge number of degrees of freedom by retaining the so-called *relevant/ important* ones, though the meaning of what are the relevant/important modes/degrees of freedom is quite problematic (Holmes et al., 1997; Kraichnan, 1988). Most frequently it is argued that these are 'modes' containing most of the energy, but – at least from the physical point of view – the 'modes', e.g., carrying most of the energy dissipation and vorticity are not less relevant/important in some sense. 'Mixed modes' related to both small and large scales such as eigenfunctions of $\langle u_i(\mathbf{x})\omega_j(\mathbf{x}+\mathbf{r})\rangle$ may appear even more relevant/important[5]. Even in such a case it is not clear whether it is possible to obtain a low-dimensional approximation representing adequately the flow field[6]. For instance, such a 'simple' turbulent flow as the flow in a plane channel at rather low Reynolds number Re $= 3300$, which attractor dimension is estimated to be of the order 10^3 (see table 3.1). Possible exceptions are when the flow, though turbulent, at the outset is strongly dominated by some 'low-dimensional subsystem'/coherent structures (e.g., Holmes et al., 1996, 1997; Lesieur and Metais, 1996; O'Neil and Meneveau, 1997; Jimenez and Simens, 2001). As mentioned in chapter 1, it is quite possible that such large-scale structures are the result of a large-scale instability of the flow as a whole not related directly to the turbulent nature at least of free shear flows. The situation is more complicated in wall-bounded turbulent flows. These matters are discussed in chapter 8.

[4]There are even claims that *the dynamical system which may describe fully developed turbulence can be approximated using just three degrees of freedom,* Robinson (1998). See also Patil, D.J., Hunt, B.R., Klanay, E., Yorke, J.A., and Ott, E. (2001). Local low dimensionality of atmospheric dynamics, *Phys. Rev. Lett.,* **86,** 5878–5881. That is, it may be possible to gain *low- dimensional insight to, and interpretation of, high-dimensional dynamics locally in space and time* (C. Doering, private communication).

[5]Note that two most important quantities: Lamb vector $\omega \times \mathbf{u} \equiv \epsilon_{ijk}\omega_j u_k$ and the vortex stretching vector $W_i = \partial(u_i\omega_j)/\partial x_j$ are precisely of this kind and are closely related to the tensor $u_i(\mathbf{x})\omega_j(\mathbf{x}+\mathbf{r})$.

[6]Inadequate (too) low-dimensional approximations may lead to spurious chaotic behaviour, which disappears when the number of the basic functions becomes large enough and adequate resolution is used (Curry et al., 1984; see also Rempfer, 2000).

We add that in a recent attempt Farge (2007) used orthogonal wavelets to represent what they call the coherent part of velocity and vorticity in a three-dimensional flow in a periodic box. The claim is that out of total $1.4 \cdot 10^7$ degrees of freedom $4 \cdot 10^5$ (2.8% of the total) are sufficient in order to adequately represent the 'coherent' part of the flow. It is still quite a lot. However, the advantage of their approach is that their 2.8% include both large and small scales, so that it seems that these 2.8% represent reasonably the whole flow field rather than its 'coherent' part only.

It is natural to seek a closed representation that is mathematically *simple enough to be tractable and insightful, but powerful enough to be faithful to the essential dynamics* (Kraichnan and Chen, 1989). But this does not justify oversimplified treatment of small scales via methods like eddy viscosity, because the small scales contain a great deal of essential physics of turbulent flows, much of which is not known or poorly understood, and which are intimately and bidirectionally related to the large scales (see chapter 6, section 6.6).

Various models of turbulent flows – a really huge number of them – are all statistical theories in the sense mentioned above. They differ from, e.g., 'physical' theories only by different methods of 'closure' of the resulting equations for the chosen statistical variables. All of them have in common some *ad hoc* assumptions of unknown validity and obscured physical and mathematical justification. In this sense none of the statistical theories are rigorous.

Hans Liepmann wrote in 1979: *Turbulent modelling is still on the rise owing to rapid development of computers coupled with the industrial need for management of turbulent flows. I am convinced that much of this huge effort will be of passing interest only. Except for rare critical appraisals ... much of this work is never subjected to any kind of critical or comparative judgement. The only encouraging prospect is that current progress in understanding turbulence will restrict the freedom of such modelling and guide these efforts toward a more reliable discipline* (Liepmann, 1979).

Liepmann's criticism was directed at (already at that time) the great number of publications which used a variety of assumptions, most of them very remote from any physical basis, to say nothing of any rigorous mathematical foundation. This state of affairs seems to be changed. It was rigorously proved (Fursikov and Emanuilov, 1995 and references therein) that the Keller–Fridman (1925) chain of equations for the moments (and consequently the Hopf equation) has a unique solution for initial conditions in an appropriately chosen functional space[7]. In other words a positive answer was given to the question whether the closure problem has a solution, and

[7]Provided that the corresponding three-dimensional problem for the Navier–Stokes equations does have a unique solution.

an estimate of convergence of approximations for the closure of the infinite chain of equations for moments was given. Thus, it became clear – at least in principle – that turbulence modelling can be put on a rigorous foundation. However, just like direct numerical simulation by itself does not bring understanding, neither does modelling of whatever sophistication.

3.3. Turbulence versus deterministic chaos

> *... the implications for fully turbulent flows are slight ... attempts to understand particular turbulent flows ... have not been significantly modified or aided by the new developments. It is necessary to stress this rather negative fact mainly because of excessive claims that have been made for the new ideas. It has been said that 'the turbulence problem has been solved' ... this can give a false impression.* (Tritton, 1988).

Another rather recent and popular view emerged from the developments in the 'deterministic chaos' already mentioned in chapter 2, i.e., comparatively simple nonlinear systems exhibiting chaotic behaviour. This, however, did not 'solve the problem of turbulence' neither 'the right path was found' as was claimed quite frequently, e.g., *There have been important changes in our understanding of the mechanism whereby turbulence occurs. Although a consistent theory of turbulence is still a thing in the future, there is reason to suppose that the right path has finally been found.* (M.I. Rabinovich, in: L.D. Landau and E.M. Lifshitz, Fluid Mechanics, 2nd edition, Pergamon Press, 1987).

Today it seems that the application of dynamical systems methods and results to turbulence in fluids is hardly appropriate[8]. Methods of dynamical systems theory, after an initial period of euphoria and even claims that the problem of turbulence was solved, have proved to be ineffective/irrelevant for the theory of fully-developed turbulence. Quoting G.K. Batchelor (1989): '... *considerations of the properties of fully-developed turbulence require rather different ideas* ...' (*J. Fluid Mech.,* **205,** 593).

It is now recognized that *despite the considerable successes of the present studies of the application of modern ideas on chaos to well-controlled fluid flows, they appear to have little relevance when applied to the more general problem of fluid turbulence* (Mullin, 1993, p. 93; see also Tritton, 1988, p. 410).

So it is quite plausible that any fluid flow which is adequately represented by a low-dimensional system is not turbulent – a kind of definition of 'non-turbulence'. The immediate examples are low-dimensional chaotic fluid flows.

[8]Though some authors hold an opposite opinion, Bohr et al. (1998), Ruelle (1990).

3.4. Statistical methods of looking at the data only? Or what kind of statistics one needs?

Whatever the origins of real turbulent flows[9], turbulent flow states are so complicated that the use of statistical tools is unavoidable. The question is about what kind of statistics one has to use. It is directly related to the most difficult question on *what are the problems*, i.e., to the skill/art to ask the right and correctly posed questions, which is quite a problem in turbulence research.

At early stages, the interest was in relatively simple quantities like means[10], correlations, spectra, and probability density functions (PDFs) of various quantities. With the digital methods of data acquisition and processing, conditional statistics became a powerful tool of data analysis (Van Atta, 1974; Antonia, 1981).

Standard statistical tools like means and correlations smooth out some important qualitative features of (typical) individual realizations. The 'mean fields', e.g., large-scale averages of velocity or concentration of some species or particles, are smooth whereas the individual realizations are not. They are corrugated, highly intermittent and contain clusters/regions of high level of some quantity/ies (enstrophy, dissipation, passive tracer, reacting species, particles, etc.) surrounded by low level 'voids' of this quantity. In other words, 'standard' 'traditional' statistical methods to a large extent ignore the structure(s) of turbulent flows, which was the main reason for numerous objections against statistical methods often understood as averaging only. More subtle statistical properties of turbulent flows associated with their structure(s) both in small and large scales are important in many applications. For instance, special information on small-scale structure(s) is needed in problems concerning, e.g., combustion, disperse multiphase flow, mixing, cavitation, turbulent flows with chemical reactions, some environmental problems, generation and propagation of sound and light in turbulent environments, and some special problems in blood flow related to such phenomena as hemolysis and thrombosis. In such problems, not only special statistical properties are of importance like those describing the behaviour of smallest scales of turbulence, but also actual 'nonstatistical' features like maximal concentrations in such systems as an explosive gas which should be held below the ignition threshold, some species in chemical reactions, concentrations of a gas with strong dependence of its molecular weight on concentration (such as hydrogen fluoride used in various industries, e.g., in

[9] Whether turbulence *is a priori* random/stochastic because Nature is such or the intricacy of turbulent flows arises out of deterministic equations like NSE or any other unknown reason.

[10] *The true aim of turbulence theory is to predict the mean properties and their dependence on boundary conditions* (Saffmann, 1960).

production of unleaded petrol) and toxic gases. Similarly, problems such as the manipulation (and possibly control) of turbulence and turbulence induced noise require information on large-scale structure(s) of turbulent flows far beyond such simple statistical characteristics as averages, correlations, spectra and PDFs.

In other words statistical methods have their limitations, so that in many cases one has to look not only at the properties of turbulent flows *en masse*, but also at some specific properties of individual realizations like those involved in weather forecasting. After all one does not need ensemble averaging to be sure that the coffee will be well mixed via only one, and pretty short, realization. Likewise, not much can be done statistical-wise to cope with a destructive hurricane or a tropical cyclone. *It may also be that such* (i.e., very rare and exceptionally strong) *events are rather sensitive to details of the physics that do not appreciably affect the character of the majority of events. This does not mean that one should not keep trying, by insight and discernment, to discover useful statistical measures, but rather that statistics will have to be used with that humility and appreciation of the combination of admission of ignorance and decision to ignore detail so successfully used by workers in the past* (Mollo-Christensen, 1973).

Each particular statistical tool has its own limitations; being useful in one context/respect, it may say nothing in many others. A typical example is correlation, widely used in many aspects of turbulence research. Usually if a correlation between two quantities is not small, it reflects some important relation. However, if the correlation is small, it is not necessarily insignificant. For instance, let us have a look at the famous Reynolds stress $\langle u_1 u_2 \rangle$ – the correlation between the velocity fluctuations in the direction of the mean flow (x_1) and those normal to the wall (x_2) in a wall-bounded turbulent flow. The typical value of the corresponding correlation coefficient is $\langle u_1 u_2 \rangle / u_1' u_2' \sim 0.4$. However, the real quantity entering the equation for the mean flow (RANS, see appendix C) is the derivative $d\langle u_1 u_2 \rangle / dx_2$. In a developed turbulent flow with its mean properties independent of the streamwise coordinate, x_1, (flat channel, pipe), $d\langle u_1 u_2 \rangle / dx_2 = \langle (\omega \times \mathbf{u})_1 \rangle \equiv \langle \omega_2 u_3 \rangle - \langle \omega_3 u_2 \rangle$. That is the 'turbulent force' is due to the coupling between large and small scales (again nonlocality, see chapter 6). The corresponding correlations between velocity and vorticity are small: both $\langle \omega_2 u_3 \rangle$ and $\langle \omega_3 u_2 \rangle$ are of order 10^{-2}. However, this does not mean that the coupling between ω and $\mathbf{u}$ is insignificant. Indeed, without such a coupling $d\langle uv \rangle / dx_2 = 0$, so that the mean flow would not 'know' anything about turbulent fluctuations at all and therefore would remain as the laminar one.

Moreover, even if a correlation between two quantities is very small or even precisely vanishing, this still does not necessarily mean that the interrelation/coupling between these two quantities is not existing or is unimportant. For example, in homogeneous turbulent flows, velocity and vorticity, and vorticity and the rate of strain tensor are precisely uncorrelated, $\langle \omega \times \mathbf{u} \rangle \equiv 0$, $\langle \omega_i s_{ij} \rangle \equiv 0$, but their interaction is in the heart of the physics of any turbulent flow. Similarly, the correlation coefficient between $\mathbf{u}$ and $\nabla^2 \mathbf{u}$ is very small ($\sim Re^{-1/4}$) in high Reynolds number flows, but is very significant as directly related to the rate of dissipation of energy in turbulent flows. We will return to this issue in chapter 6.

One more example of the limited value of quantities like correlations and correlation coefficients is provided by a helically-forced turbulent flow (Galanti and Tsinober, 2006). In such a flow, correlations between $\mathbf{u}$ and ω, (and also ω and $curl\omega$) are not vanishing due to lack of reflectional symmetry. Nevertheless these correlations are an order of magnitude smaller that those between $\mathbf{u}$ and $curl\omega$ (and ω and $curlcurl\omega$). This is in spite of the fact that the scales of $\mathbf{u}$ and ω are 'closer' than those of $\mathbf{u}$ and $curl\omega$ in the sense that the characteristic scales of $\mathbf{u}$ and ω differ less than those of $\mathbf{u}$ and $curl\omega$. Moreover, in flows with reflectional symmetry the correlation coefficients between $\mathbf{u}$ and ω (and ω and $curl\omega$) vanish, whereas correlations between $\mathbf{u}$ and $curl\omega$ (and ω and $curlcurl\omega$) remain practically unchanged. The latter is directly related to the rate of dissipation of energy in turbulent flows as, e.g., in homogeneous flows $\langle \mathbf{u} \cdot curl\omega \rangle = -2\langle s_{ij}s_{ij} \rangle$.

Single-point statistics in many cases may be (and usually is) insufficient and even misleading. For example, single-point PDFs of velocity fluctuations are known to be quite close to the Gaussian distribution. In particular, the third moment of velocity fluctuations is close to zero (more precisely its skewness, $\langle u_1^3 \rangle / \langle u_1^2 \rangle^{3/2} \approx 0$), and the flatness, $\langle u_1^4 \rangle / \langle u_1^2 \rangle^2 \approx 3$, as in a Gaussian field. Similarly other higher order odd moments are small, and even moments assume values close to those of a Gaussian field, e.g., $\langle u_1^6 \rangle / \langle u_1^2 \rangle^3 \approx 15$. However, the conclusion that velocity fluctuations are really almost Gaussian would be a misconception, not to mention the field of velocity derivatives (chapters 6 and 7). This is already seen when one looks at *two*-point statistics. For instance, in such a case the odd moments are significantly different from zero (e.g., Frenkiel et al., 1979). This is one of the simplest among numerous examples when multi-point (in space and time) statistics is useful. The widely known two-point correlations for some separation r and/or time t are related to the flow structure(s) larger than $\sim r/t$. An example of application of three-point statistics to structures of the passive scalar in turbulent flow is given by Mydlarski and Warhaft (1998).

3.5. Decompositions/representations

> *Thus, because it is not possible to separate eddies into clearly defined classes according to the source of their energy ... therefore a single coefficient is used to represent the effect produced by eddies of all sizes and descriptions.* (Richardson, 1922).
>
> *... for the very smallest eddies the motion is entirely laminar.* (Batchelor, 1947, p. 535).
>
> *... at the small scales it becomes more difficult to argue for fundamental differences between these two types of flows.* (Southerland et al., 1994).

One of the common approaches both in theory and data analysis is a reductionist one, i.e., some decomposition of the flow field.

The first known decomposition was given by Reynolds (1895), in which the flow field is represented as a sum of a mean and a fluctuative, the latter being just the difference between the mean (assumed to exist) and the instantaneous fields. There have been attempts to extend this approach to a triple decomposition consisting of a mean, 'coherent' and 'random' contributions. Most of these attempts are of heuristic nature since the 'coherent' contribution is well defined only in rather special cases. It is noteworthy that when means (in some sense) exist the Reynolds decomposition is more physically and mathematically natural than its analogues, such as triple decompositions or those associated with large eddy simulations (LES).

The *formal* decompositions employ some suitable basis of expansion of the flow field. These are represented by the Fourier decomposition or its 'relatives' such as Fourier–Weierstrass, Gabor, helical, Littlewood–Paley, the so-called SO(3) decomposition (Cambon and Teissedre, 1985; Kurien and Sreenivasan, (2001b); Biferale and Procacia, 2005) or any other complete basis depending on the geometry of the flow, wavelets, wavepackets, solitons (Arneodo et al., 1999; Farge, 2007; Meneveau, 1991; Frick and Zimin, 1993), and filters and multiscale (Germano, 1999; Leonard, 1974; Meneveau and Katz, 2000; Eyink, 2006), Karhunen–Loève or proper orthogonal decomposition (Holmes et al., 1996, 1997; Sirovich, 1997).

In *heuristic* decompositions the flow field is represented as a 'two-fluid' one, e.g., organized and incoherent or deterministic and random (Cantwell, 1990; Farge and Guyon, 1999; McComb and Watt, 1992; She, 1991), or as a collection of some 'simple' objects – vortex filaments, vortons, 'eigensolutions', etc. (Pullin and Saffman, 1997; Davidson and Pearson, 2005). Another kind of heuristic representations is prompted by the intermittent structure of turbulent flows: breakdown coefficients/multipliers (Novikov, 1971, 1990a) or equivalently (multi)fractals (Frisch, 1995).

There are several difficulties with all/any decompositions (as an essentially linear procedure) mainly due to the nonlinear and nonlocal nature of turbulence[11]. These difficulties are not trivial and seem to be 'generic'. As mentioned above, it is not even known how to separate random gravity-wave motion (which does not produce vertical transport) and genuine turbulence (which does) in a stably stratified fluid (Stewart, 1959). Under turbulent motion/dynamics the interaction of 'modes', whatever they are, is strong. The resulting structure(s) is(are) not represented by the modes of any known decomposition, for example by a Fourier-decomposition of a flow in a box with periodic boundary conditions. The emergence of structures in such a flow, such as the slender vortex filaments, in a random fashion (at random times with random orientation, and to a large extent random shapes) points to the limitation of utilization of Fourier decomposition (or similar), which does not 'see' these or any other structure(s). Another example is the chaotic regime of a system with few degrees of freedom only, e.g., three as in the Lorenz system (Lorenz, 1963), or four in the forced spherical pendulum (Miles, 1984), but with a *continuous* spectrum. Hence the ambiguity of Fourier decomposition (see Liepmann, 1962; Tennekes, 1976 and Lohse and Müller-Groeling, 1996 for other aspects of Fourier-transform ambiguity). All the attempts to find a 'good' decomposition are related to what Betchov (1993) called the 'dream of linearized physicists', i.e., a *superposition* of some, desirably simple elements (e.g., Germano, 1999). The dream is, of course, to find sets consisting of small numbers of weakly interacting elements/objects adequately representing the turbulent field. Those known so far are interacting strongly[12] and most of them nonlocally[13]. This is a reflection of one of the central difficulties in 'solving the turbulence problem' as a whole, in general, and the 'closure problem' such as LES and other reduced descriptions of turbulence (Kraichnan, 1988), in particular, as well as in construction of a kind of statistical mechanics of turbulence generally (Kraichnan and Chen, 1989).

[11]With the exception of the Reynolds decomposition.

[12]Landau wrote (1960, p. 245): *It is well known that theoretical physics is at present almost helpless in dealing with the problem of strong interactions.* The situation in turbulence seems to be not better if not worse.

[13]From time to time there appear claims to decompositions with weakly interacting elements/objects. This appears to be true only if some parameter is small as in RDT-like theories, but not for genuinely nonlinear/strong turbulent flows, for which no hope that small parameter does exist. A recent example, concerns the SO(3) which is the tensorial generalization of the well-known procedure of decomposing a scalar function into components of different irreducible representations using the spherical harmonics. Here too the different components of the decomposition (called anisotropic sectors) interact weakly among themselves and with the isotropic one only in case of weak anisotropy.

3.6. Eulerian versus Lagrangian descriptions

> *One owes to Euler the first general formulas for fluid motion ... presented in the simple and luminous notation of partial differences ... By this discovery, all fluid mechanics was reduced to a single-point analysis, and if the equations involved were integrable, one could determine completely, in all cases the motion of a fluid moved by any forces.* (Lagrange, 1788).
>
> *Of course, fluid mechanics can, in principle, be worked entirely in the Lagrangian frame ... Even neglecting viscous forces ... yield awkward moment equations.* (Corrsin, 1962b).
>
> *... the inertial interactions have a relative nature; they are eliminated in the transformation to the particle-attached reference system ... the use of the viscous Lagrangian equations in turbulence theory is still a matter for the future.* (Monin and Yaglom, 1971).
>
> *Though the Lagrangian description of the flow ... has many attractions ... it is generally unwieldy to work with. Even the kinematic task of determining closed-form solutions for the particle paths ... from an initial position ... is generally intractable.* (Soward and Roberts, 2008).

Practically all the issues just discussed are considered in the Eulerian setting. There are essential differences when one looks at the Lagrangian description.

The Lagrangian[14] description[15] of fluid flows is physically more natural than the Eulerian one[16], since it is related most directly to the motion of fluid elements. Nevertheless, mostly technical difficulties (both in physical and numerical experiments) strongly hindered the use of the Lagrangian approach in most fluid dynamical problems. The traditional problems for which the Lagrangian description is considered especially appropriate are transport and mixing in diverse applications, e.g., geophysical and environmental, cloud formation, chemical technology, combustion and material processing, sedimentation, bio-medical and recently microfluidics, and many others. In most of the above issues the concern is with the kinematic aspects, i.e., with what is called today "passive turbulence", i.e., evolution of passive objects in prescribed velocity fields (see next chapter and chapter 9). Another aspect is associated with the dynamics of inviscid fluids, such as theoretical problems of Euler equations, inviscid vortex dynamics and vortex methods, stability, dynamics of interfaces and surface waves, compressible flows. Though these issues seem to have little to do with genuine

[14]In fact it is also due to Euler, see Lamb, 1932. A detailed account on the 'misnomer' by which the 'Lagrangian' equations are ascribed to Lagrange is found in Truesdell, 1954.

[15]In which the observation is made following the fluid particles wherever they move.

[16]In which the observation of the system is made in a fixed frame as the fluid goes by.

turbulence, there are views/beliefs that such things like possible singularity formation and collapse in Euler flows and that the infinite Reynolds number limit of NSE (or similar equations) is described by singular solutions of Euler equations. We return to these in chapter 10.

There is little (if any) treatment of dynamical aspects of turbulent flows (e.g., those corresponding to those described by NSE in the Eulerian setting) in a pure Lagrangian setting which is one of our main concerns here[17]. Among the reasons is the view that *a principal objective of any theory of fluid motion is the prediction of the spread of matter or "tracer" within the fluid* (Bennet, 2006), though not many in the turbulence community will accept this view. In fact, the main reasons take their origin in the difficulties in handling the pure Lagrangian equations (C.62, C.63, C.66) and related issues. In view of these difficulties it is natural to ask the following questions. Is it true that dynamical issues in turbulence *per se* can be treated satisfactorily in the Eulerian setting only? Is there any need to use for this purpose the Lagrangian setting too? Are there dynamical problems which require such an approach? A plausible answer is that there are important problems/questions of dynamical nature for which Lagrangian information is of utmost importance (as well), i.e., one has to employ both settings. The first example is given by the class of flows where turbulence memory and/or sensitivity to the inflow conditions plays an essential role (e.g., jets, mixing layers, wakes and flows past grids too – the recent example of flows past fractal grids provides especially strong evidence for this, Seoud and Vassilicos, 2007). It has to be mentioned that the issues concerning the conventional Taylor hypothesis and the Random Taylor hypothesis and a number of questions on accelerations belong to this sort of problems too (see chapter 6). Most flows mentioned above belong to the kind of the so called partly-turbulent flows. The main special features of these flows are the coexistence of regions with laminar and turbulent states of flow and continuous transition of fluid particles (purely Lagrangian objects!) from a laminar state into a turbulent one via the entrainment process through the 'boundary' between the two. Hence the necessity of a Lagrangian approach in studying this transition process in proximity to the laminar-turbulent 'interface'. This issue is addressed in chapter 8. Flows with polymer solutions provide another important example where the Lagrangian approach is unavoidable at least for two additional reasons: 1) since the material elements (again purely Lagrangian objects!) in such flows are not passive and 2) there are no equations reliably describing flows of polymer solutions such as NSE for Newtonian fluids. So one needs Lagrangian experimentation with such turbulent flows in the first place (chapter 8). A similar statement is true of flows with any other active additives.

[17]For other issues of Lagrangian aspects of mostly kinematical nature see Falkovich et al. (2001); Toschi and Bodenschatz (2009) and references therein.

On the technical side, since in a pure Lagrangian setting the equations (C.62–C.63) are intractable[18] (so far) in order to obtain true (not modelling!) Lagrangian information, one typically solves the problem in Eulerian setting (i.e., using NSE) and using this information together with the equation (C.64) one can obtain the Lagrangian evolution of any fluid particle[19]. As the Euler information is defined on the computational grid it is necessary to use an appropriate/adequate interpolation scheme (Yeung, 2002)[20]. The issue is, however, not just in technical differences in the Lagrangian and Eulerian settings. They are different conceptually in several aspects. One of the main points is that many flows that are laminar in Eulerian setting (E-laminar) exhibit chaotic behaviour in the Lagrangian setting (L-turbulent) see, chapter 4. It is important that this chaotic behavior is of pure kinematic nature[21]. We stress that the E-turbulence is a dynamical phenomenon, whereas this is not necessarily the case with the L-turbulence which may be a purely kinematic one. In other words, the flow can be purely L-turbulent (i.e., E-laminar) as in the above examples with artificial velocity fields or real flows at very low/zero Reynolds numbers. However, if the flow is E-turbulent (i.e., $\mathrm{Re} \gg 1$) it is L-turbulent as well. An important consequence is that the structure and evolution of passive objects in genuine turbulent flows arises from two (essentially and unfortunately inseparable) contributions: one due to the Lagrangian chaos and the other due to the random nature of the (Eulerian) velocity field itself. Hence, one can expect adequate kinematic simulation or simulation in random and/or multi-scale real E-laminar flows of those properties (Lagrangian) which are insensitive (or weakly sensitive) to the differences between the genuine turbulent velocity fields and those used for the purposes of modelling (quite a non-trivial issue). An important counterexample is the difference between backwards and forwards relative dispersion (with the mean square separation following particle pairs backwards in time[22] being at least twice as large as forwards)

[18]but allow the posing of nontrivial and important questions.

[19]On the theoretical side hybrid formulations have been known since the end of the 19th century, see Lamb (1932); Cartes et al. (2007); Kuznetsov (2008); Soward and Roberts (2008); Ohkitani and Constantin (2008) and references therein. All of them are mostly not ripe yet for handling the issues of turbulence.

[20]There exist several versions of the so-called hybrid (i.e., Lagrangian–Eulerian) numerical approaches which still rely heavily on using the NSE equations.

[21]This qualification includes all artificial velocity fields both random and/or multi-scale or not. The field of particle trajectories is (can be seen) as a passive object: it is a Lagrangian signature of the underlying (and prescribed) velocity field of any nature be it genuinely turbulent, or Lagrangian chaotic such as E-Laminar, synthetic random or not, restricted Euler, kinematic simulations of Lagrangian chaotic evolution, turbulent-like multi-scale fields, including real E-laminar flows at $\mathrm{Re} \sim 0$ from linear Stokes equations with random forcing, flows in porous media, microdevices, to name some.

[22]Following particles backwards in time was introduced by Corrsin (1952, 1972a).

in genuine turbulence (Berg et al., 2006; Sawford et al., 2005; Thomson, 2003 and references therein). Another example is the qualitative difference in alignment properties of a passive vector in genuine (NSE) and Gaussian velocity fields with the same energy spectrum (see chapter 9).

As mentioned, Eulerian and Lagrangian settings are different conceptually, not just/only technically. The Eulerian setting reveals the pure dynamical chaotic aspects of genuine turbulence as contrasted to "mixing" of the kinematical with the dynamical ones in the Lagrangian setting, i.e., in genuine turbulence the latter contains both which seem to be essentially inseparable. This seems to comprise an inherent difficulty in using the Lagrangian setting in handling the dynamical issues of genuine turbulence because of the impossibility of separating the Lagrangian (kinematic) chaos from the genuinely dynamical (Eulerian) stochasticity. Thus with the exception of problems as in examples given above, the Euler setting seems to be preferable for studying dynamical aspects of (e.g., NSE) genuine turbulence as more revealing the dynamical chaotic aspects of genuine turbulence as contrasted to "mixing" of the kinematical with the dynamical ones in the Lagrangian setting. In genuine turbulence the latter contains both.

On the mathematical side there is an important aspect associated with the 'more chaotic' nature of the Lagrangian setting, which is traced back to early Lagrangian simulations (Amsden and Harlow, 1964; see also Harlow, 2004). Namely, one is tempted to conjecture that the pure Lagrangian dynamical equations (C.62–C.63) (so far intractable for viscous flows) are more rich than their Navier–Stokes counterpart (C.4, C.6). The former being equivalent to the latter plus the equation (C.64) relating the Eulerian and Lagrangian descriptions. Though such a conjecture looks plausible, there remain nontrivial issues on the relation between Lagrangian versus Eulerian settings in purely dynamical contexts. One such issue deserves special mention. In the Lagrangian setting the fluid particle acceleration is linear in the fluid particle displacement (see equation C.62) and the 'inertial' effects are manifested only by the term containing pressure. That is, one can hardly speak about things like Reynolds decomposition and Reynolds stresses, turbulent kinetic energy production in shear flows in a pure Lagrangian setting. It seems that nonlinearity in the Lagrangian representation cannot be interpreted in terms of some cascade (as it cannot be maintained by pressure gradient alone) and it is far less clear (if at all) how one can employ decompositions even at the problematic level as done in the pure Eulerian setting (we address the issue of 'cascades' in chapters 5 and 9). Also there is no sweeping of any kind at the outset as there are no terms like the advective terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in the pure Eulerian setting, so one cannot speak about the interaction between advective and diffusive processes in the pure Lagrangian setting. However, in contrast to (C.62), there is interaction between inertial[23] and diffusive effects at the level of vorticity, gradient of passive scalar and passive solenoidal vectors (magnetic field) as is seen, e.g., from the vorticity equation (C.65) and passive scalar (C.67) in the pure Lagrangian setting. A particular manifestation of such interaction is the so called Tennekes-Lumley balance[24] between the enstrophy production and its viscous destruction, production of the energy of the gradient of a passive scalar and its destruction due to diffusivity, and similarly for the energy of the magnetic field.

Further issues related to Lagrangian versus Eulerian settings concern accelerations, the random Taylor hypothesis and some other, see chapter 6. The issues of the relation between the Lagrangian and Eulerian descriptions is discussed in chapters 4 and 9.

3.7. Ergodicity

> *The time average which is produced in a physical or computer experiment corresponds to probability measure invariant under time evolution (in statistical mechanics this would be called an ensemble). For a given differentiable dynamical system, like Hénon map, there are however many invariant probability measures, and one has to decide which one is selected in experiments. My belief is that the choice is produced by the smoothing influence of the small level noise present in physical experiments* (*roundoff errors in computer studies*) (Ruelle, 1983a).
>
> *There is no way to confirm that those turbulence data used in analysis represent typical properties of turbulence* (van Veen et al., 2006).
>
> *The ergodicity of turbulence sounds to me as an assumption which is hard to avoid or test* (Mann, 2006).

For statistically stationary flows ergodicity is (roughly) equivalence of ‘true’ statistical properties (not only means/averages, but ‘almost’ all statistical properties) of an ensemble to those obtained using time series in one very long realization. A similar property is defined in space by replacing time by space coordinate(s) in which the flow domain has an infinite extension, at least in one direction.

Though it is not known whether three-dimensional turbulent flows are ergodic[25], it is common to use the ergodicity hypothesis in turbulence

[23]These inertial effects are due to the interaction of vorticity and the velocity gradients (strain) and are not relative, as is the advection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

[24]The issue of the T-L balance and related are discussed in chapter 6.

[25]Foias et al. (2001) have shown that there are measures on a function space that are time-invariant. However, invariance under time evolution is not enough to specify

research, e.g., in physical and numerical experiments: turbulent flows are just believed to be ergodic. In other words, in statistically stationary situations the time statistics obtained in experiments is believed to correspond to a (unique) probability measure invariant under time evolution. This comprises the essence of the ergodic hypothesis, which is usually expressed in terms of ensemble and is widely used in experiments. A similar statement is made for situations with at least one homogeneous spatial coordinate. In dynamical systems the equivalence of two is used as a definition of ergodicity: *Definition 7.1: An abstract dynamical system is ergodic if for every complex-valued μ-summable function the time mean is equal to the space mean*, Arnold and Avez (1968).

Most mathematical treatments of ergodicity in turbulent flows deal with the so-called stochastic Navier–Stokes equations (SNSE), i.e., with stochastic forcing[26] both in 3D and 2D[27]. In such a case it is natural to expect the property of ergodicity. However, what about a great variety of turbulent flows in which the 'forcing' is not random and in many cases is even not time dependent – just constant in time, such as constant in time overall pressure gradient? Such flows at large enough Reynolds numbers become turbulent due to what can be called intrinsic stochasticity (nobody seems to know what it is precisely). All statistically stationary turbulent flows are massively studied using temporal statistics instead of the 'true' one based on ensembles or probability measures (which are anyhow not accessible). All observed so far statistical (not only average but 'almost' all) properties of many such turbulent flows (but not all) are remarkably reproducible (statistical stability) and – as mentioned – are believed to be ergodic in spite of the fact that a deterministic, say, a constant in time large-scale forcing breaks the ergodicity 'on large scales'.

There seems to exist no direct evidence regarding the validity of the ergodicity hypothesis in turbulent flows. An attempt to obtain such evidence via direct numerical simulations of the Navier–Stokes equations by performing a large number of simulations (or similarly many physical experiments) at different initial conditions representing the members of an ensemble is too laborious and costly. Instead, one can obtain such evidence exploiting the property of a turbulent flow which is both statistically stationary in

a unique measure which would describe turbulence. Another problem is that it is not clear how the objects that the authors have constructed and used in their proofs are relevant/related or even have anything to do with turbulence.

[26] *One of the oldest open problems in theoretical physics is that of describing fully-developed turbulence on the basis of a macroscopic model. The latter is usually taken to be the stochastic Navier–Stokes (NS) equation subject to an external random force that models the energy injection by large-scale modes,* Adzhemyan et al. (2003).

[27] In the latter case it is unlikely that with a deterministic forcing one can expect anything like ergodicity, though there are numerous examples with random forcing which are ergodic.

time and homogeneous in space. In such a flow its temporal and spatial statistical properties should be the same if the ergodic hypothesis is correct. An important consequence is that it is not necessary to perform a large number of time/labor consuming "brute force" experiments with different initial conditions in order to compare the time-statistics of a given observable against the "ensemble" one at a given time[28]. Such an attempt was made by Galanti and Tsinober (2004) in a periodic box with resolution 128^3 uniformly distributed grid points[29]. A deterministic forcing in large scales was used in the form $\mathbf{f} = A\cos z \cos y,\ B\cos x \cos z,\ C\cos y \cos x;\ A = B = C$. This forcing has the property to be locally non-helical, $\mathbf{f} \cdot curl\mathbf{f} = 0$. The Taylor microscale Reynolds number, $\mathrm{Re}_\lambda \approx 145$. In order to have comparable information for the time statistics the equations were run for 2,200,000 time steps (cf. with $128^3 = 2{,}097{,}152$).

Since the large-scale deterministic forcing breaks the ergodicity on large scales the mean (weak) velocity was removed before comparing the temporal and spatial statistics of the velocity field. After such a removal both statistics become very similar for one-point statistics, though some differences remained for two-point statistics, especially at large values of separation between the points. This is most probably due to not large enough scale separation between the spatial integral scale and that of the computational box. Another reason is that the flow is only approximately statistically homogeneous. The temporal and spatial statistics associated with the field of velocity derivatives exhibit much more similarity than those for the velocity field itself. An example of comparison of spatial and temporal statistics for velocity derivatives is shown in figure 3.1, see Galanti and Tsinober (2004) for other numerous examples.

As in other examples of this kind the figures corresponding to time statistics show traces of time evolution, whereas corresponding examples associated with spatial statistics have nothing to do with the time evolution. The similarity between the two can be seen as an indication of equivalence of two formulations of the ergodic hypothesis. The first one corresponds to the 'evolutionary' view on ergodicity, i.e., that the long enough trajectory will sample almost all of the attractor in the phase space. Therefore the statistical properties of statistically stationary flows of an ensemble are equivalent to those obtained using a time series in one very long realization.

[28]Thus, one deals with two different results: one is a statistical analysis over the entire flow field at a certain moment in time, and another one for one position in space over a very long period of time. The first one may not be representative for a longer period of time, while the second one may not be representative for all the points in space. The point is that if the flow is ergodic the two types of statistics should give the same result.

Many ensembles (like the human populations), are not ergodic.

[29]Some similar results concerning scaling of velocity time increments were obtained by Chevillard et al. (2005) and Lévêque et al. (2007).

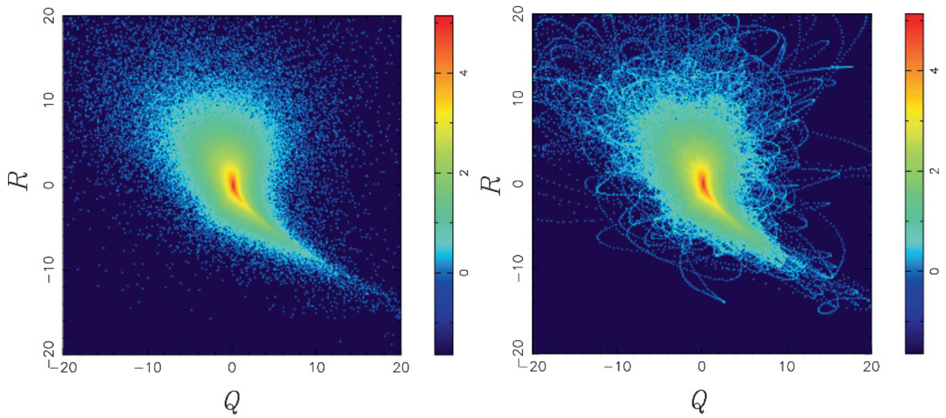

Figure 3.1. The 'tearing drop' pattern, which is the joint PDF of the invariants R, Q of the velocity gradient tensor $\partial u_i/\partial x_j$, $R = -1/3\{s_{ij}s_{jk}s_{ki} + (3/4)\omega_i\omega_j s_{ij}\}$, and $Q = (1/4)\{\omega^2 - 2s^2\}$. On the left is shown temporal statistics corresponding to a time series at single point in space. On the right is shown spatial statistics based on a single time snapshot over the flow domain. Note that the time statistics (left) shows traces of time evolution, whereas nothing of the kind is observed with the spatial statistics (left) as it has nothing to do with the time evolution. Galanti and Tsinober (2004)

Another formulation does not involve the evolutionary aspects and merely states the equivalence of statistical properties of the two.

The results from a long enough in time numerical simulation provides clear evidence that if a turbulent flow is both statistically stationary in time and homogeneous in space, then its temporal and spatial statistical properties are the same. This can be seen as evidence in favor of validity of the ergodic hypothesis in turbulence. One of the 'side' outcomes is a positive addition to the answer to the question (when) do simulations reproduce statistics? At least in some cases one time snapshot is pretty representative.

Is this really the case for all statistically stationary turbulent flows? Can one claim more than that? Whereas it is natural to expect that nonlinear systems driven by a random force should be ergodic, it has to be stressed that the above simulation was made with purely deterministic and constant in time nonhelical forcing. Nevertheless, the flow clearly exhibited strong similarity between its temporal and spatial statistical properties with the exception of the largest scales. A possible 'explanation' is that this happens due to the property of self-randomization of fluid-dynamical turbulence (intrinsic stochasticity).

3.7.1. CHAOTIC BEHAVIOUR VERSUS ERGODICITY

A possible criticism of the point that the forcing is deterministic is that it is very well-known that deterministic forcing can yield a random dynamics even for a few degrees of freedom, let alone for a turbulent flow. The latter is correct, but it is also known that most low-dimensional chaotic systems are not ergodic. Moreover, the issue is broader and is a part of that on differences between ergodicity and randomness. The story goes back to the general belief that any kind of nonlinearity in a system with a large number of degrees of freedom would give rise to ergodicity (e.g., Fermi, 1923 and Orszag and McLauchlin, 1980)[30], and the latter was assumed to serve as the mechanism for the onset of statistical behavior in dynamical systems.

3.7.2. ON DETERMINISTIC LARGE-SCALE FORCING

There is another important and very difficult issue. As mentioned, since large-scale deterministic forcing breaks the ergodicity on large scales, the mean velocity was removed before comparing the temporal and spatial statistics of the velocity field. So one may put forward an objection that ergodicity is a global property of the dynamical system represented by the Navier–Stokes equations and there cannot be a large-scale or a small-scale ergodicity. Another question is about the impact of nonlocality, i.e., direct and bidirectional coupling of large and small scales, especially in case of purely deterministic forcing. It also seems to spoil the cleanness of the ergodicity of turbulent flows. Is it possible to speak about 'approximate', 'small-scale' ergodicity or 'modified' ergodicity?

3.7.3. ARE THERE NON-ERGODIC STATISTICALLY-STATIONARY TURBULENT FLOWS?

There are many flows that cannot be easily qualified as 'cleanly' ergodic: flows in diffusers with separation on one side; flows in 'French washing machine'; confined turbulent convection; all partly-turbulent flows (mixing layers, jets and wakes past bodies – especially axisymmetric ones with spontaneous swirl, boundary layers); properties of these flows depend strongly on the inflow conditions (small oscillations of the body, acoustic excitation, etc.) and on the level of disturbances in the quasi-potential flows outside. Minute changes in both often result in dramatic changes in flows as mentioned above. Sometimes this is considered as 'long memory' of such flows, but there seems to be much more than that as minute changes produce

[30]In this paper the authors *have exhibited a set of dynamic systems having a coupling similar to the Euler equations that seem ergodic over the energy surface for typical parameter choices.*

dramatic changes in the statistical properties of these flows. Some kind of remedy can be imagined in attempting to look at 'ensembles' of such flows, e.g., flows in diffusers with separation with a large enough variety of initial/inflow conditions. In such a way the statistical characteristics would become symmetric and would not feel the one-side separation. There are two problems here. The first problem is of purely theoretical nature in reconciling the statistics based on an ensemble and long time statistics. The latter for a flow for some individual inflow condition would (and is observed to) exhibit a well reproducible (!) strong asymmetry. And second, there seems to be little use for such an approach from a practical point.

A natural question concerns the non-homogeneous flows. One can expect similar results as obtained above for flows with homogeneous coordinates, such as the flow in a plane channel. An obvious conjecture is that the temporal and spatial statistical properties of such a flow will be the same for fixed values of the distance from the wall.

All the above refers to the Eulerian setting. The problem with the Lagrangian setting is that, generally, Lagrangian statistical properties are not (and cannot be) stationary for stationary fields in the Euler setting with the exceptional cases such as homogeneous and stationary Eulerian field implying stationarity of Lagrangian statistical functions dependent on a *single space* point, Lumley (1962a), see also section 9 in Monin and Yaglom (1971). This along with the 'more chaotic nature' of the Lagrangian setting is the likely reason that nonergodicity is encountered in the Lagrangian setting for flows which are ergodic in the Euler setting, see e.g., Girimaji and Pope (1990); Chevillard et al. (2005); Lévêque et al. (2007); Cruzeiro and Malliavin (2008).

3.8. On methods of studying turbulent flows

Methods of studying turbulent flows are usually divided into theoretical and experimental. As discussed in chapter 1 and above in this chapter, in fact, there exist no adequate theoretical methods[31]. However, most valuable is the language and terminology which comes from numerous theoretical approaches. The experimental methods are subdivided into physical (laboratory and field), and numerical. Both are extensively described in the literature with thousands of references, though most books on turbulence contain very little (if any at all) material on the physical methods, whereas the numerical methods, including modelling, are covered extensively.

[31] Since our main concern is with basic aspects of turbulent flows, various methods of modelling are mentioned here briefly in the specific context/question. Namely, whether one can address basic conceptual issues of turbulent flows using such methods. The only exception seems to be the Navier–Stokes equations.

3.8.1. DIRECT NUMERICAL SIMULATIONS OF THE NAVIER–STOKES EQUATIONS

> *Progress in numerical calculation brings not only great good but also awkward questions about the role of the human mind. The human partner in the interaction of a man and a computer often turns out to be the weak spot in the relationship. The problem of formulating rules and extracting ideas from vast masses of computational or experimental results remains a matter for our brains, our minds* (Zeldovich, 1978).
>
> *The ability of a computer code to simulate flow that is difficult to realize in the laboratory has its unfortunate extension the ability of a computational solution to be altogether unphysical* (Aref, 1986).
>
> *It is not sufficient to set up the code and let the computer zip along. It zips all right, but to where?* (Kadanoff, 1997).
>
> *Fearless engineers write gigantic codes that are supposed to produce solutions to the equations: they do not care at least (when they are conscious of the problem, which unfortunately seldom seems to be the case) that what they study are* not *the Navier–Stokes equations, but just the informatic code they produced* (Gallavotti, 2002).
>
> *Once you certify the code, it can go to work, and you really know that the answer is going to be true to a given accuracy* (Jimenez, 2002).

Some of the above citations serve as a warning about taking special care in using numerical simulations. The problem arises due to the extreme sensitivity to very small variations in the initial and boundary conditions and possible different behaviour of the chaotic system approximating the original one[32]. Therefore it is not clear how to interpret numerical experiments designed to test the accuracy of DNS (hence the term numerical *experiments*)[33]. Nevertheless it is impossible to overestimate the importance of direct numerical simulations of the Navier–Stokes equations (Moin and Manesh, 1998; Mathieu and Scott, 2000; and Pope, 2000). This is especially true regarding most of the issues in basic research, since in many contexts

[32] This problem is less serious when handling (some) statistics, but can be acute when attempting to follow a particular realization.

[33] *Whenever they fail in their predictions, scientists tend to blame the poor accuracy of the observations, the lack of computer power and the inadequate parametrization in their numerical models, rather than their own lack of skill in computing the accuracy that can be obtained with present resources. Sloppy reasoning of this kind is responsible for much of the thoughtless expansion and escalation numerical modelers in all branches of science indulge in . . . A calculation that does not include a calculation of its predictive skill is not a legitimate scientific product* (Tennekes, 1993).

one needs neither high Reynolds numbers nor very complex geometry. By its very nature numerical simulation allows one not only to simulate flow that is difficult to realize in the laboratory and to get access to quantities which are not accessible in the laboratory, but also to realize situations which are not reproducible in the laboratory. The immediate examples are the 'unphysical' situations, such as pure two-dimensional turbulence, and three-dimensional turbulence with any desirable forcing.

Experimentalists do not query the importance of computations in turbulence, whereas numerists tend pretty frequently to consider physical (laboratory, field) experiments as superfluous. This is a hubris which has been exhibited many times in the history of science – achieving some goal one is inclined to think that any/every problem can be coped with by the same methods. It seems, however that just as equations cannot replace Nature, experiments cannot be ever fully replaced by computations. Another point is that no sophisticated experiment (laboratory or DNS) by itself brings understanding. This can be brought only by a genuine theory, which seems to be not in existence so far.

3.8.2. PHYSICAL EXPERIMENTS

Not only in basic/fundamental but also in a great many practical problems the most (and perhaps the only) reasonable approach is still to carry out specific *ad hoc* experiments. There seems to be no way to replace experiments in turbulence in the forseeable future and even beyond: nobody will believe in 'theoretical' predictions only. As mentioned, there is a big issue about the very existence of such.

In view of the above, it is natural to ask: why bother *measuring* especially difficult quantities such as velocity derivatives in the age of supercomputers? Indeed, we repeat that many numerical fluid dynamists seem to be inclined to consider physical experiments as superfluous. Yet not every problem in turbulence can be studied by a numerical approach. First, it is useful to remember that Nature is far richer than the Navier–Stokes equations, so that one can encounter surprises at large Reynolds numbers[34]. Turbulent processes in very large systems such as the atmosphere and the ocean are not and never will be accessible to direct numerical computation. Today, it is possible to perform direct numerical simulations of the Navier–Stokes equations at rather moderate Reynolds numbers in simple geometries and the prospects for higher Reynolds numbers are rather modest and not only for the immediate future. Though the scale resolution problem is serious, both in laboratory and numerical experiments, there is

[34] Apart from things like aliasing, numerical viscosity, inadequate BC and other 'small' problems.

an essential difference between the two: inadequate resolution in numerical experiments leads usually to erroneous results of the whole output, whereas in laboratory/field experiments one has the *true* flow and correct results for the scales resolved even when some range of scales is not resolved. Second, even at small Reynolds numbers, it is not always possible to use the numerical approach either. An important example is represented by flows in complex geometries such as flows with rough walls and flows in plant canopies. Similar problems arise in handling flows involving sediment transport and other additives (various particles, bubbles), and especially polymers and surfactants for which even no adequate equations seem to be known. This is a partial list only of the limitations of numerical approaches and the reasons for the importance of physical experiments. One more reason is that it is the physical experiment (dealing with real turbulent flows) that provides the final verdict to the results of both numerical simulations and theories.

The experimental methods of studying turbulent flows are quite elaborate. There is a host of methods of flow visualization providing mostly qualitative information; see references mentioned in section 1.1. Of special interest are methods providing local-in-space and instantaneous-in-time values of various quantities. The main methods include hot-wire anemometry, laser Doppler and ultrasonic anemometry, electromagnetic methods, particle image velocimetry and stereoscopic particle image velocimetry, particle tracking velocimetry, laser induced velocimetry and other methods of local tagging of fluid elements, holographic velocimetry and nuclear magnetic resonance. The advanced ones access all the spatial and temporal velocity and passive scalar derivatives. Selected references can be found in Bonnet et al. (1998); Bruun (1995); Dracos (1996); Gulitski et al. (2007a,b,c); Sanford et al. (1999); Tao et al. (2002); Toschi and Bodenschatz (2009). The review by Corrsin (1963) deserves special mention for anyone intending to perform any experiments in turbulence. There exist also numerous methods to measure temperature, salinity, moisture and other species.

3.8.3. ON VALIDATION OF THEORIES

This is the right place to add a comment on experimental *validation* of 'theories' understood as any theoretical treatment including modelling. This is directly related to the question on how meaningful, and in what sense, is the experimental 'confirmation' of a 'theory'.

The highly-dimensional nature of turbulence is one of the main reasons and obstacles for assessment of conceptual (!!) validity/reliability of any theory let alone low-dimensional (LD) modelling. From a conceptual point of view the main question remains whether it is at all possible and why does it 'work'[35]. Any LD model or any kind of a "theory" that represents a corresponding LD part/aspect of some particular kind/class of turbulent flows – but not necessarily for the right reason – will be (and usually is) inadequate in other flows. Just like simple interpolation/fits polynomials, etc. describe faithfully the behavior of data without any physical reason (pure technical), so many models do precisely the same. Mostly they are postdictions (rather than predictions) and, quite often, successful and useful semi-empirical interpolation schemes. There are many theories – many with contradictory premises – but all agreeing well with *some* experimental data. The issue is more serious as there are many situations in which agreement with experiment may not help too much even if the agreement between "theories" and experiment is excellent as the correspondence with the experimental results may occur for the wrong reasons as happens from time to time in the field of turbulence. For example, there are quantities/properties that are insensitive/invariant to some specific properties of the flow field whether it is real or in some sense synthetic, gaussian/quasi-normal, markovian, etc. For example, addressing issues associated with gaussian/quasi-normal manifestations of turbulent flows (see subsection 6.8.2) and having a perfect agreement with some theory based on quasi-gaussianity and/or quasi-normality an experimentalist may encounter, in fact, a dilemma whether his measurements are perfect or just a nice gaussian noise.

Note that the most of evidence was obtained at moderate Reynolds numbers for finite nontrivial systems, e.g. jets which consist of coexisting turbulent-nonturbulent regions. Theoreticians claim 'explanations' based on infinite objects/boxes. Today they say that the effects of finite box are of special interest.

An outstanding example of different nature is the Kolmogorov 4/5 law which is independent of and insensitive to the nature of the dissipation mechanism as it depends on the mean energy injection rate only. Two more examples are the Yaglom 4/3 law for the passive scalar and the Richardson pair diffusion law which are true for any random isotropic velocity field including the Gaussian one. In the basic context there are numerous misinterpretations of the experimental observations strongly biased by wishful thinking and replacing the sought explanations by mere descriptions of some kind. The so called 'multifractal formalism' is an example of this kind. It is claimed to be an explanation of the 'anomalous scaling' and that the multifractal model is well supported by experimental evidence (Frisch, 1995; Yakhot and Sreenivasan 2005; Eyink, 2008) whereas in fact it is another description of anomalous scaling[36], i.e., of the experimental

[35] And, of course, there is a serious concern about the meaning of the term 'work'.

[36] There are alternative – sometimes standing in contradiction with each other – descriptions which are also *well supported by the experimental evidence.* More details are given in chapters 5 and 10.

evidence. This issue (and some related) is more serious as there are problems concerning the experimental evidence itself. We discuss these matters in chapter 5 especially in the context of the so-called anomalous scaling and ill-posedness of the concepts of inertial range and cascade. Other examples are given in chapters 6, 9 and 10.

3.8.4. ON HIGH-REYNOLDS-NUMBER MEASUREMENTS IN TURBULENT FLOWS

There is little doubt that in order to make any further progress it is vital to have the full vector of velocity, vorticity and the rate of strain tensor (pressure is not so easily accessible). From the general point of view this allows to deal with quantities invariant of the system of reference (separate components are not such) as the most appropriate to characterize the physical processes, which do not know of any frames of reference. In spite of considerable progress in a variety of modern techniques, such as those mentioned above the hot-wire anemometry (more precisely hot-sensor anemometry) remains the only technique capable of measuring the smallest and the fastest physically relevant fluctuations of velocity fluctuations in turbulent flows especially at large Reynolds numbers, see Gulitski et al. (2007 a,b,c) which describe and use a system allowing one to access velocity derivatives *without invoking the Taylor hypothesis* and thereby access accelerations. The HWA technique for 3-D measurements, especially of velocity derivatives, still has more of the *caprice of an art than the complete reliability of a routine laboratory procedure* (Kovasznay, 1959) and *it is this 'caprice of the art' which leads many people not only to worry about the calibration of the instrument but also the person carrying out the measurements* (Perry, 1982). It is still very far from being a routine laboratory procedure and/or just technical issue and requires long term, very nontrivial efforts not only of a technical nature. Apart from the reasons given above and the general fundamental importance of high quality of such measurements, there is an urgent need due to massive use of large eddy simulations for computations of large Reynolds number turbulent flows, the very basis of which requires reliable information on the properties of the small scales in such flows. There is no way other than experiment to obtain such information which so far does not exist. Therefore, no investment seems to be exaggerated for such an endeavor. One of the main challenges for future efforts in technical aspects is reducing the relative errors such as for the acceleration components in a system moving with mean velocity and accessing sub-Kolmogorov scales. It is noteworthy that the latter is of particular importance for the new approach/concept of studying turbulence as an "undecomposable" whole since such concepts as inertial range and cascade are not well-defined, see chapters 5 and 6. Accessing sub-Kolmogorov scales requires substantial improvement of the system such as i) the probe construction requiring miniaturization of its individual arrays as well as of the whole probe in order to minimize the influence of velocity gradients both across the individual arrays as well as the whole probe, and ii) improvement of a number of issues related to the calibration (both hardware and software).

3.9. Summary

There is no basis for contraposing 'statistical' and 'deterministic', just as it seems impossible to separate the structure(s) and the 'random structureless' background from the nonrandom processes. With few exceptions there seem to exist no other way of handling turbulence than by statistical methods. It should be stressed that these include not only the 'traditional' things like means/averages and other simple characteristics, but all kinds of statistics including rather intricate/exquisite ones such as conditional statistics can be, depending on the nature of problems in question and the ability/skill of the researcher to formulate such questions.

The difficulties in using various decompositions are not trivial and seem to be 'generic' as all of them are essentially the tools borrowed from methods used in *linear* problems and therefore not suitable for treatment of turbulence. There is a conceptual necessity to handle turbulence as a whole and undecomposable as it is meaningful as a whole, e.g., due to the N's mentioned above. Separate "components" have at best very limited meaning if at all. For example, it is impossible and conceptually incorrect to say that the turbulent flow is completely laminar or not at smallest scales (Batchelor, 1947; Southerland et al., 1994). It is a matter of principle and a conceptual question whether studying turbulence via (some) decompositions is aiding understanding of its fundamental physics. More than one hundred years of experience seems to be a clear indication that it is not. Any decomposition results in a nontrivial bidirectional relation between the small and the large scales (whatever this means) which is non-local (functional) both in space and time (i.e., history-dependent). Hence there is little chance that this dependence can be local (in several meanings) as has been insisted upon for quite a period of time. Today – also in view of accumulating evidence – it is becoming clear that locality is at best an extremely crude approximation which in many cases is good for empirical purposes, but not as a basis for studying the physics of turbulence.

The question whether adequate low-dimensional description of turbulent flows is possible depends on the meaning of the term 'adequate'. In the strict sense, i.e., from the basic point of view, it seems that there does not exist such a description, though as a (semi) empirical tool it may be definitely more than satisfactory.

The Lagrangian description of fluid flows is physically more natural than the Eulerian one, since it is related most directly to the motion of fluid elements. Though a great variety of dynamical issues in turbulence *per se* can be treated satisfactorily in the Eulerian setting, further insight into the basic physics of turbulent flows requires information on time evolution and associated Lagrangian statistics of such quantities as vorticity, strain, accelerations, etc., as relating the spatial structure (the most popular time snapshots) and the time dimension. There are important problems/questions of dynamical nature for which Lagrangian information is of utmost importance (as well), i.e., one has to employ both settings. Examples include the class of flows where turbulence memory and/or sensitivity to the inflow conditions plays an essential role, partly-turbulent flows involving the entrainment phenomenon and flows with polymer solutions and other additives.

From the conceptual point there is still only partial justification (which is almost all empirical and belief-based) for the assumption that turbulent flows are ergodic.

Not only in basic/fundamental, but also in a great many practical problems the most (and perhaps the only) reasonable approach is still to carry out specific *ad hoc* experiments. There seems to be no way to replace experiments in turbulence in the forseeable future and even beyond: nobody will believe in theoretical predictions only. As mentioned there is a big issue about the very existence of such.

CHAPTER 4

KINEMATICS

Mostly on the behaviour of passive objects

The term *kinematic(s)* is associated with several issues, all of which have in common things which are not directly related to the (Navier–Stokes) dynamics of turbulence. In other words, the dynamics of fluid motion, except incompressibility, does not enter into the problems in question. These issues include the following:

* – Kinematic (statistical) properties/structure of real turbulent and artificial (in some sense, e.g., Gaussian) random flows such as (an)isotropy, (in)homogeneity, etc.
* – Passive objects in random flow fields including artificial ones.
* – Kinematic (Lagrangian) chaos.

In other words, by dynamics we mean dynamics of fluid motions *per se*, e.g., the flow properties associated with the dynamics obeying the NSE.

The first issue is of purely technical nature and is covered extensively in several monographs (e.g., Hinze, 1975; Monin and Yaglom, 1971, 1975; Mathieu and Scott, 2000; Pope, 2000). This aspect is also closely related to the so-called eddy structure identification in turbulent flows (Bonnet, 1996). We do not give here any systematic information on these matters. Instead appropriate references and reminding are made in the text in an *ad hoc* manner throughout the book.

The second and the third issues are described mostly to the extent necessary as a background for comparison with the genuine dynamical features of turbulent flows. The material is included in this chapter, since it is essentially of kinematic nature in the sense mentioned above. Its place here underscores the qualitative difference between the behaviour of passive objects in random flow fields (including artificial ones) and kinematic/Lagrangian chaos, and real fluid dynamical turbulence[1]. The relation to and comparison with a variety of aspects of the dynamics of turbulence is discussed in chapters 6–10.

[1]Especially in view of the claims made in the chaos community that 'the problem of turbulence was solved' with the developments in chaos theory.

4.1. Passive objects in random fluid flows

By definition a passive object in a fluid flow has no dynamical effect on the fluid motion itself, and one is interested in the effects of fluid turbulence (or some artificial random velocity field) on the field of a passive object. Passive objects include passive scalars such as dispersing contaminants, chemical species, temperature, moisture; passive vectors such as material lines, (weak) magnetic field in an electrically conducting fluid; passive surfaces such as material surfaces, and in some cases reacting surfaces and turbulent flames; material volumes.

An essential point is that the evolution of passive objects obeys *linear* equations in which the velocity field does not 'know' anything about the presence of these objects and therefore the velocity field is considered as given *a priori* be it a real fluid flow field or some artificial one. There is no involving phenomenon such as pressure[2]. This does not mean that the problems of the evolution of passive objects are simple. The main complication and simultaneously rich variety of phenomena comes from the fact that the velocity field enters as a coefficient in front of the spatial derivatives, i.e., due its multiplicative character, so that statistical problems become in a sense nonlinear.

The strongly enhanced mixing properties in a turbulent flow are associated with the fact that the fluid particles wander away from their initial positions even in the absence of any mean flow. Equally, or even more important is the fact that *particles which were originally neighbors move apart as the motion proceeds, so that in a diffusive motion the average value of* l^2/l_0^2 *continuously increases* (Taylor, 1938a). Here, $l_0 = l(0)$ is the initial distance between the two fluid particles and $l = l(t)$ is this distance at some subsequent time moment, t. This intuitive idea of the relative diffusion received support in the paper by Cocke (1969). Namely, Cocke proved the following important results[3].

The first result is that the length of an infinitesimal material line element, $l \equiv |\mathbf{l}|$, increases on average in *any* isotropic random velocity field. Similarly, Cocke showed that an infinitesimal material surface element, N, identified by its vector normal, $\mathbf{N}$, increases on average in *any* isotropic random velocity field as well[4]. We stress again that these are purely kinematic results: the flow does not have to be a real one, i.e., to satisfy

[2]Hence 'shocks' in the form of ramp–cliff structures just as in the Burgers equation.

[3]For more details and a review of other related references see Monin and Yaglom (1975), § 24.5; for later references see Bohr et al. (1998); Chaté et al. (1999); Drummond (1993); Girimaji and Pope (1990); Tabor and Klapper (1994); Yeung (1994).

[4]More precisely Cocke showed that $\ln[\langle l(t)\rangle/l(0)] \geq 0$, and $\ln[\langle N(t)\rangle/N(0)] \geq N^2(0)$ for all $t > 0$ with equality holding only if there is no fluid motion at all. Arguments similar to those by Cocke (1969) show that $\langle l^p(t)\rangle \geq l^p(0)$ and $\langle N^p(t)\rangle \geq N^p(0)$ for any $p > 0$ (Monin and Yaglom, 1975, pp. 579–580).

the Navier–Stokes equations and/or to be observable in the laboratory or elsewhere – the only requirement is that the flow should be random and isotropic. For example, this result is true for a Gaussian velocity field as well, which is important for the purpose of comparison of material line elements, which are passive, and vorticity, which is not (see chapters 6–7). On the qualitative level the results by Cocke were confirmed in a number of DNS experiments both for real and artificial flow fields (Drummond, 1993; Girimaji and Pope, 1990; Huang, 1996; Yeung, 1994) and laboratory experiments (Lüthi et al., 2001, 2005; Guala et al., 2005; Liberzon et al., 2005).

In view of the fact that $\mathbf{l}$ and $\mathbf{N}$ satisfy equations (C.1) and (C.2) respectively, the results by Cocke mean that $\frac{D\langle l^2(t)\rangle}{Dt} = \langle l_i l_j s_{ij}\rangle > 0$ and $\frac{D\langle N^2(t)\rangle}{Dt} = -\langle N_i N_j s_{ij}\rangle > 0$, and the mean growth rates $\left\langle \frac{1}{l(t)}\frac{Dl(t)}{Dt}\right\rangle = \langle l_i l_j s_{ij}/l^2\rangle > 0$ and $\left\langle \frac{1}{N(t)}\frac{DN(t)}{Dt}\right\rangle = -\langle N_i N_j s_{ij}/N^2\rangle > 0$. That is, the mean rate of material line (surface) stretching is positive. In other words, there is prevalence of stretching over compressing. It is important to emphasize that $\langle l_i l_k s_{ik}\rangle = 0$ for random s_{ij} and random (and independent of s_{ij}) orientation of l_i. The mean $\langle l_i l_k s_{ik}\rangle > 0$ because though the vector, l_i, and the stretching vector, $W_i^l \equiv l_j s_{ij}$, are completely random in space, they tend to be strictly aligned due to the constraint imposed by equation (C.1), so that their scalar product $\mathbf{l}\cdot\mathbf{W}^l = l_i l_k s_{ik}$ tends to be positive, and the PDF of the cosine of the angle between these two vectors, $\cos(\mathbf{l},\mathbf{W}^l)$, is positively skewed (see below). Similarly, the scalar product $-\mathbf{N}\cdot\mathbf{W}^N = -N_i N_j s_{ij}$ tends to be positive, and the PDF of $\cos(\mathbf{N},\mathbf{W}^N)$, $W_i^N = -N_j s_{ij}$, is positively skewed as well due to the constraint imposed by equation (C.2).

Let us now look at passive vectors in the presence of molecular diffusivity. There are two kinds of such vectors corresponding in some sense to $\mathbf{l}$ and $\mathbf{N}$. The first one, $\mathbf{B}$, is the vector obeying equation (C.36), and is frozen into the fluid in the absence of molecular diffusive effects. A well-known example is the (weak) magnetic field in an electrically conducting fluid. The second kind of passive vectors, $\mathbf{G}$, is the gradient of some passive scalar, θ, i.e., it is governed by the equation (C.33); in the absence of molecular diffusive effects, the surfaces $\theta = const$ become material surfaces.

Cocke's results allow us to expect that the quantities $\langle B_i B_k s_{ik}\rangle$, $-\langle G_i G_k s_{ik}\rangle$ and corresponding rates all should be positive in the presence of molecular diffusive effects as well. It is noteworthy that the similarity in the (statistical) behaviour between the material elements, $\mathbf{l}$ (surfaces, $\mathbf{N}$) and some 'normal' passive vectors, $\mathbf{B}$ (or $\mathbf{G}$) is not at all trivial for two reasons. First, there are many fewer field lines of 'normal' vectors, such as magnetic field lines, than the material ones – at each point there is typically only one such line of $\mathbf{B}$ (or $\mathbf{G}$), but there are infinitely many material lines (surfaces) passing through a point. This may (and does) lead to differences in the statistical properties of the two fields. Second, there is a subtle issue of the singular limiting behaviour of a system with its diffusivity tending to 0. It can be very different from the purely diffusionless case when the diffusivity is put to 0 at the outset, as in the case of material elements, **l**, and surfaces, **N**, though some parameters may be similar (see Childress and Gilbert, 1995; Ott, 1999 and references therein). An additional difference between **l** and **B** is that the latter is solenoidal ($div\mathbf{B} = 0$), whereas the former, generally, is not. Similarly $\mathbf{G} = \nabla\theta$ is a potential vector, whereas the material surface element **N** is not. Nevertheless, there is reasonable evidence that the quantities, such as $\langle B_i B_k s_{ik}\rangle$, $\langle B_i B_k s_{ik}/B^2\rangle$, $-\langle G_i G_k s_{ik}\rangle$ and $-\langle G_i G_k s_{ik}/G^2\rangle$ are positive – a property which can be seen as universal for any random fluid flow, be it real or artificial, such as the Gaussian velocity field. This is indeed the case as observed in numerical simulations (Huang, 1996; Brethouwer et al., 2003; Ohkitani, 1998; Ruetsch and Maxey, 1991, 1992; Tsinober and Galanti, 2001, 2003; Gulitski et al., 2007c).

The net positive stretching in all the mentioned cases is associated with two concomitant processes, tilting and folding (and production of curvature), so that any random fluid flow acts in such a way as to create a fine structure in the field of a passive object. For example, in case of a passive scalar, the positiveness of the term $-\langle G_i G_k s_{ik}\rangle$ is associated with two aspects. First, it represents the rate of production of the 'dissipation' $\chi\left\langle \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_i}\right\rangle \equiv \chi\langle G^2(t)\rangle$ of a passive scalar (see equation [C.35]), so that the latter is continuously amplified by the stretching process reflected in the term $-\langle G_i G_k s_{ik}\rangle$. Production of the gradients $G^2(t)$ of a scalar field is associated with the fine structure of the passive scalar, θ, itself. Second, the term $-\langle G_i G_k s_{ik}\rangle$ is balanced, at least in part, by the 'dissipation', $-\chi\left\langle \frac{\partial G_i}{\partial x_k}\frac{\partial G_i}{\partial x_k}\right\rangle$, of the vector **G** itself (see again the corresponding balance equation [C.35]). The consequence is that the gradients $\frac{\partial G_i}{\partial x_k}$, associated with the fine structure of **G**, are amplified too. An example of such structure both for the passive scalar itself (concentration of a fluorescent dye) and its dissipation is shown in figure 4.1.

It is noteworthy that the continuous stretching even of infinitesimal material elements and other passive vectors by random flows is a statistical tendency; it occurs in the mean only, not every individual element is stretched. This is seen, for example, from the PDF of the rates $l_i l_k s_{ik}/l^2$ and $N_i N_k s_{ik}/N^2$ shown in figure 4.2. Namely, these quantities are *negative* in about 1/3 of the volume occupied by the fluid flow. This latter is associated with the compressing and folding of material elements and the production of their curvature. Similar behaviour is observed in the case of nonzero diffusivity, i.e., for the quantities $\langle B_i B_k s_{ik}\rangle$, $\langle B_i B_k s_{ik}/B^2\rangle$, $-\langle G_i G_k s_{ik}\rangle$ and $-\langle G_i G_k s_{ik}/G^2\rangle$. Examples of positively skewed PDF of corresponding

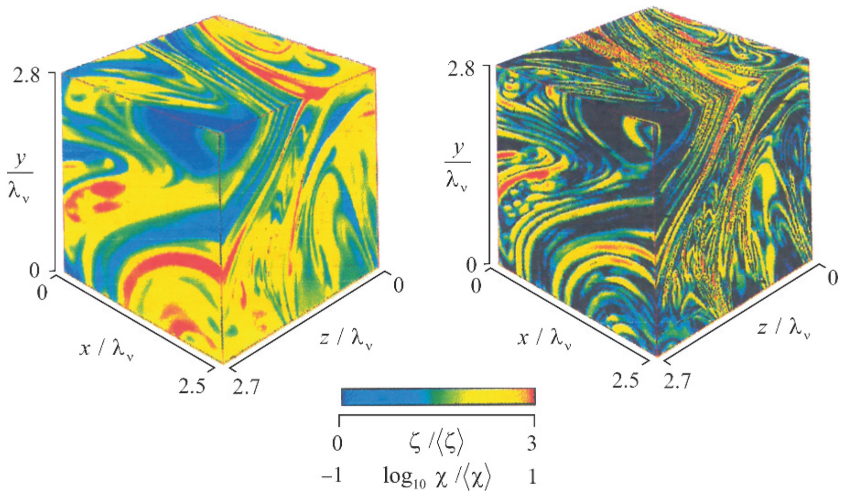

Figure 4.1. Fully resolved three-dimensional data volume of a) the scalar field, θ, and b) its 'dissipation', $(\nabla\theta)^2$ (Frederiksen et al., 1997)

quantities are shown in figure 4.3 along with the PDFs of enstrophy production for the case of a statistically stationary velocity field maintained in a cubic domain with periodic boundary conditions by a deterministic forcing in RHS of NSE, Tsinober and Galanti, 2003. The production $-\langle G_i G_k s_{ik}\rangle$ is balanced by the diffusive term in contrast to the $\langle B_i B_k s_{ik}\rangle$, which is slightly exceeding the dissipative term. The consequence is that, the mean energy of the field $\mathbf{B}$ is continuously growing with time during all the time of simulation. This is akin to the so-called dynamo effect of spontaneous magnetic field amplification by the motion of electrically conducting fluid under certain conditions (see Childress and Gilbert, 1995; Ott, 1999 and references therein)[5]. The growth of $\mathbf{B}$ is due to the linearity of the equation (C.36) and the absence of the Lorenz force $\mathbf{j}\times\mathbf{B}$ in the NSE, which when present 'reacts back' and causes nonlinear saturation of the magnetic field growth (Galanti et al., 1992; Brandenburg, 1995; Brandenburg et. al., 1996). The results of DNS (Tsinober and Galanti, 2001, 2003; Vedula et al., 2001) show that the balance in equations (C.34, C.37) is dominated by the production terms $-\langle G_i G_k s_{ik}\rangle$ and $\langle B_i B_k s_{ik}\rangle$ respectively, with the forcing term at least an order of magnitude smaller. That is, the main contribution to the formation of small-scale structure of passive objects comes mostly from the strain of the velocity field, with a much smaller contribution from

[5]The ABC forcing is strongly helical, $curl\mathbf{F} \parallel \mathbf{F}$, and therefore, along with kinetic energy such a forcing makes an input of helicity into the flow. Presence of nonzero helicity, $\langle\omega\cdot\mathbf{u}\rangle$, is known to aid the process of amplification of the magnetic field. However, this is not the only possibility. The amplification of $\mathbf{B}$ was observed with the nonhelical (NH) forcing, $\mathbf{f}\cdot curl\mathbf{f}=0$, which does not make an input of helicity into the flow.

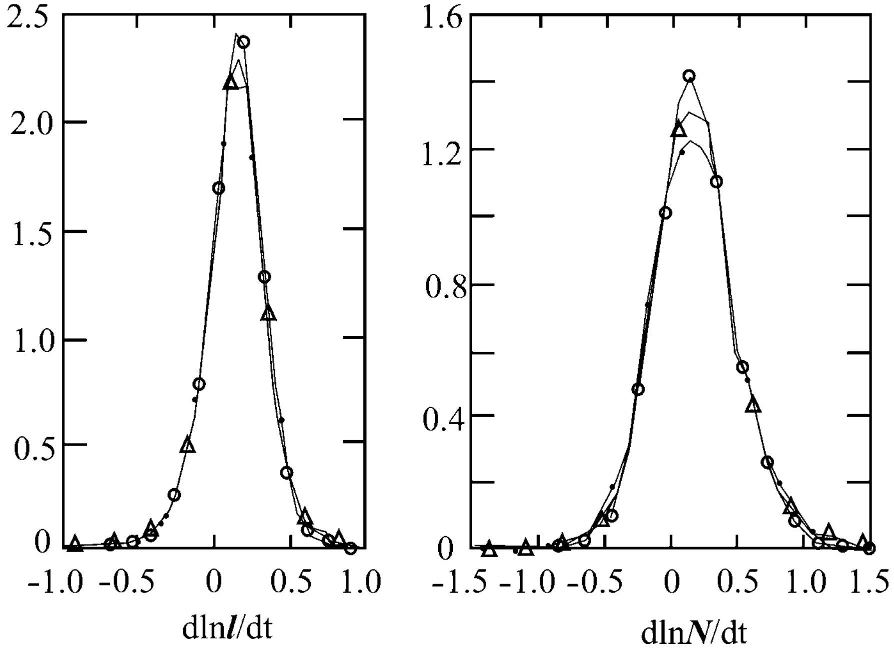

Figure 4.2. PDFs of the mean growth rates of material line and surface elements, adapted from Girimaji and Pope (1990). Note that both PDFs are clearly positively skewed, but contain a considerable negative contribution, corresponding roughly to 1/3 of the volume occupied by the fluid flow. Both distributions correspond to the statistically stationary state of the fluid flow maintained by forcing the flow at large scales in a cubic domain with periodic boundary conditions. Note that even if the rates $\frac{D\ln l(t)}{Dt} = l_i l_j s_{ij}/l^2$ and $\frac{D\ln N(t)}{Dt} = -N_i N_j s_{ij}/N^2$ are statistically stationary (which is approximately the case shown here), e.g., $\langle l_i l_j s_{ij}/l^2\rangle$ and $\langle -N_i N_j s_{ij}/N^2\rangle$ are time independent, the 'energy' $\langle l^2\rangle$ and $\langle N^2\rangle$ is continuously growing with time as follows from Girimaji and Pope (1990). Their results are valid for moderate Reynolds numbers ($\circ, \bullet$, and $\triangle$ correspond to $\mathrm{Re}_\lambda = 38,\ 63$ and 90) and as Cocke's results (1969) for small material line and surface elements; see also Lüthi et al. (2001, 2005); Guala et al. (2005)

the external forcing. This is akin to the so-called Tennekes and Lumley balance for enstrophy production. We discuss this issue in chapter 6 and the differences between the fields G_i, B_i, and the vorticity, ω_i in more detail in chapter 9.

The production processes described by the following terms: $\langle B_i B_k s_{ik}\rangle$, $\langle B_i B_k s_{ik}/B^2\rangle$, $-\langle G_i G_k s_{ik}\rangle$ and $-\langle G_i G_k s_{ik}/G^2\rangle$ are directly associated with the rate of strain, s_{ij}, only. This is because when looking at the energy balance, i.e., of G^2 and B^2, one deals only with the magnitudes $\mathbf{G}$ and $\mathbf{B}$. Among the consequences is a qualitative difference in behavior of the production conditioned on s^2 and ω^2, figure 4.4. However, the *direction* of $\mathbf{G}$ and $\mathbf{B}$, especially their orientation in respect to the eigenframe, λ_i, of the rate of strain tensor, s_{ik}, does depend on vorticity. This is seen immediately

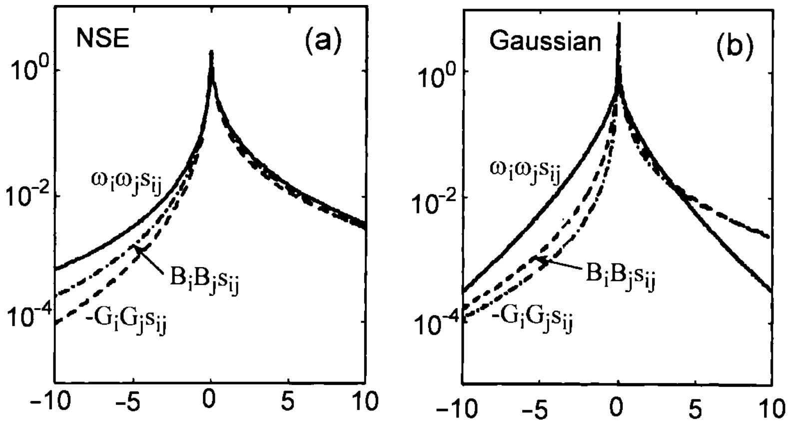

Figure 4.3. PDFs of $-G_iG_ks_{ik}, B_iB_ks_{ik}$ and $\omega_i\omega_ks_{ik}$, Tsinober and Galanti (2003). Here too, as in figure 4.2, the PDFs are clearly positively skewed, and contain a considerable negative contribution. This is true in respect of all the three for the NSE flow field and also for the $-G_iG_ks_{ik}, B_iB_ks_{ik}$ in a Gaussian velocity field, but not for $\omega_i\omega_ks_{ik}$ which is symmetric. The latter reflects a qualitative difference between the NSE and Gaussian velocity fields – in the Gaussian velocity field the mean enstrophy production vanished identically. For other examples and references see Tsinober (2001a)

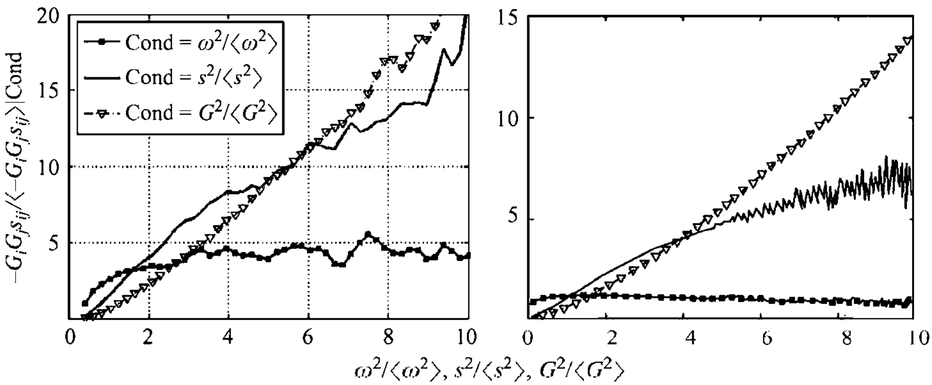

Figure 4.4. Conditional averages of $-G_iG_ks_{ik}$ on ω^2, s^2 and G^2 in A field experiment at large Reynolds number, $\mathrm{Re}_\lambda \sim 10^4$, Gulitski et al. (2007c)

from the vector identity $B_k\frac{\partial u_i}{\partial x_k} = B_ks_{ik} + \frac{1}{2}\epsilon_{ijk}\omega_jB_k$. In other words, many aspects of the behaviour of the vector fields G_i, B_i, such as *alignments* and other geometrical relations, are influenced by vorticity as well[6]. For instance, the alignments reflect important details about how the quantities $-\langle G_iG_ks_{ik}\rangle$ and $\langle B_iB_ks_{ik}\rangle$ and the corresponding rates become positive:

[6]Their and similar statistical properties are denoted by the term *geometrical statistics.*

the orientation of vectors **G** and **B** with respect to the eigenframe, λ_i, of the rate of strain tensor, s_{ik} is of utmost importance. For example, vorticity and strain contribute equally to the tilting, i.e., the rate of change of **G**, **B** and ω, Gulitski et al. (2007a,c).

4.1.1. GEOMETRICAL STATISTICS

Gradient of a passive scalar
Following Betchov (1956), it is convenient to represent the production term $-G_iG_js_{ij}$ in the eigenframe, λ_i, of the rate of strain tensor, s_{ij}, as

$$-G_iG_js_{ij} = -G^2\{\Lambda_i \cos^2(\mathbf{G},\lambda_i). \tag{4.1}$$

Here Λ_i ($\Lambda_1 > \Lambda_2 > \Lambda_3$) are the eigenvalues of the rate of strain tensor and $\cos(\mathbf{G},\lambda_i)$ is the cosine of the angle between **G** and $\lambda_{\mathbf{i}}$. In addition, it is useful to represent the relation (4.1) as $\langle -G_iG_js_{ij}\rangle = \mathbf{G}\cdot\mathbf{W}^G = GW\cos(\mathbf{G},\mathbf{W})$, $W_i^G = G_js_{ij}$, so that the positiveness of $\langle -G_iG_js_{ij}\rangle$ should be associated with the strict alignment between **G** and the corresponding stretching vector $\mathbf{W}^G$ and positively skewed PDF of $\cos(\mathbf{G},\mathbf{W}) = -\{\Lambda_i\cos^2(\mathbf{G},\lambda_i)\{\Lambda_i^2\cos^2(\mathbf{G},\lambda_i)^{1/2}$. The details depend on the mutual orientation of **G** and the eigenframe, λ_i, of s_{ij}, and the behaviour of its eigenvalues, Λ_i.

Since $\Lambda_1+\Lambda_2+\Lambda_3=0$, $\Lambda_1>0$, $\Lambda_3>0$, it is straightforward to see from (4.1) that the positiveness of $\langle -G_iG_js_{ij}\rangle$ is associated with the predominant tendency of alignment between the scalar gradient, **G**, and the eigenvector, $\lambda_{\mathbf{3}}$, corresponding to the compressive (negative) eigenvalue, Λ_3, of the rate of strain tensor, s_{ij}. This tendency is observed in DNS and in laboratory and field experiments, see figure 4.5, figure 1.16 (right).

In other words the amplification of the gradients, G_i, of the passive-scalar field (i.e., increase of the equi-surfaces of the passive scalar) is associated with predominant *compression.* The importance of compression in the process of production of the gradients of the passive-scalar field is reflected in the structure(s) of this field: it is sheet-like (Chen and Cao, 1997; Flohr, 1999; Frederiksen et al., 1997; Su and Dahm, 1996); see figures 4.1 and 1.16 (right). The so-called ramp/cliff structures are also due to the predominant compression (see figure 5 in Warhaft, 2000 or figure 3 in Shraiman and Siggia, 2000; also Celani et al., 2001). This should be contrasted to the predominant *stretching* in case of passive vectors of 'frozen' type, B_i, and also in case of vorticity with structure(s) of a different kind, e.g., tube-like. Of course, due to incompressibility, stretching in some direction is necessarily accompanied by compressing at least in one another direction. However, the above comparison shows that there is hardly any analogy between the behaviour of passive vectors G_i and B_i. On the differences

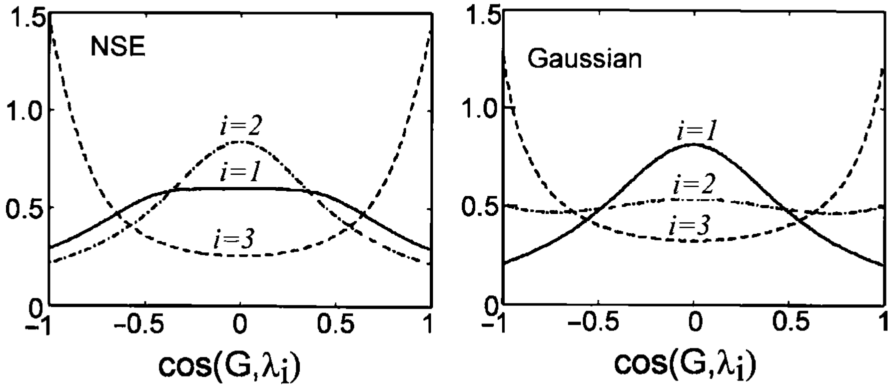

Figure 4.5. PDFs of the cosines of the angle between the scalar gradient, **G**, and the eigenframe of the strain tensor in a DNS simulation and a Gaussian velocity field, Tsinober and Galanti (2003). For other examples and references see Tsinober (2001) and Gulitski et al. (2007c)

between passive vectors of 'frozen' type, B_i, and/or material lines and vorticity, see chapter 9.

'Frozen' passive vectors
In a similar way it is seen from

$$B_iB_js_{ij} = B^2\Lambda_i\cos^2(\mathbf{B},\lambda_i) = \mathbf{B}\cdot\mathbf{W}^B = BW^B\cos(\mathbf{B},\mathbf{W}^B), \qquad (4.2)$$

that the positiveness of $\langle B_iB_js_{ij}\rangle$ is associated with the predominant tendency of alignment between the vector **B** and the eigenvector $\boldsymbol{\lambda_1}$ corresponding to the largest stretching (positive) eigenvalue, Λ_1, of the rate of strain tensor, s_{ij}. Such an alignment was observed in numerical simulations by Drummond and Münch (1990); Girimaji and Pope (1990); Huang (1996) and in PTV experiments, Luthi et al. (2005) for material line elements, l_i.

However, since in real turbulent flows Λ_2 is positively skewed, another possibility would do too: alignment between the passive vector, **B**, and the eigenvector of the rate of strain, $\boldsymbol{\lambda_2}$ corresponding to the intermediate eigenvalue, Λ_2, of the rate of strain tensor, s_{ij} (as does vorticity, see chapters 6 and 7). The results by Ohkitani (1998) and Tsinober and Galanti (2001, 2003) (both with nonzero diffusivity) show such a tendency for alignment between **B** and both $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ with stronger alignment between **B** and $\boldsymbol{\lambda_2}$. Tsinober and Galanti (2001, 2003) observed this tendency in three cases (ABC, NH and compressible) though with some differences, figure 4.6. However, in all three cases the alignment between **B** and the corresponding stretching vector $\mathbf{W}^B$, $W_i^B = B_js_{ij}$ is essentially the same.

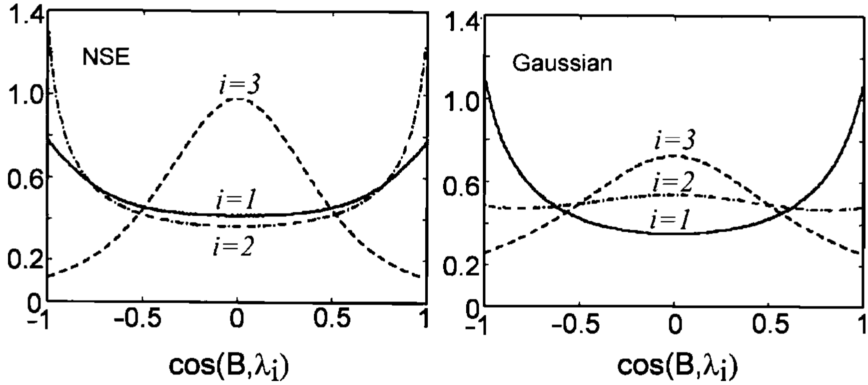

Figure 4.6. PDFs of the cosine of the angles between magnetic field, **B**, and the eigenframe of the strain tensor in a DNS simulation, Tsinober and Galanti (2003). For other examples and references see Tsinober (2001a)

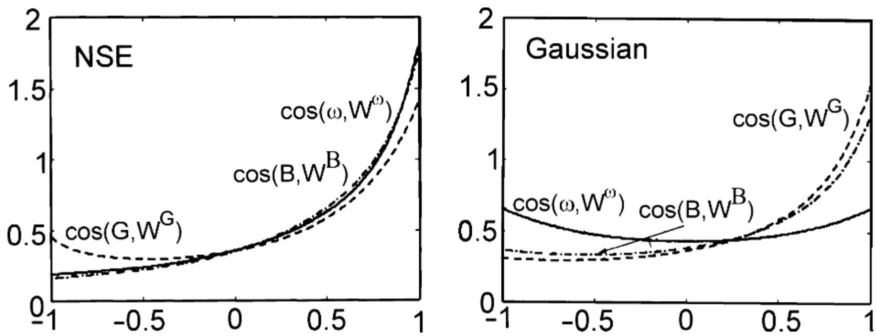

Figure 4.7. PDFs of the cosine of the angle between $\mathbf{B}, \mathbf{G}$ and ω with their stretching vectors for the data as in figures 4.5 and 4.6

This can be seen from figure 4.7, which shows the tendency of strict alignment between **B** and $\mathbf{W}^B$ corresponding to the positiveness of $\langle B_iB_js_{ij}\rangle = \langle \mathbf{B}\cdot\mathbf{W}^B\rangle$ and the positively skewed PDFs of $B_iB_js_{ij} = \mathbf{B}\cdot\mathbf{W}^B$, see (4.2) and figure 4.3. Note that, if Λ_2 is positively skewed, the PDF of $\cos(\mathbf{B},\mathbf{W}^B) = \{\Lambda_i\cos^2(\mathbf{B},\lambda_i)\}/\{\Lambda_i^2\cos^2(\mathbf{B},\lambda_i)\}^{1/2} \sim 1$ in both cases, when either $\cos^2(\mathbf{B},\lambda_1) \sim 1$, or $\cos^2(\mathbf{B},\lambda_2) \sim 1$.

Note, the difference in the alignment properties of **B** and **G** with the eigenframe λ_i. First, there is no tendency of double alignment (i.e., with λ_1 and λ_2) for **G** and this alignment is the same for the NSE and for a Gaussian field. This tendency is also absent for **B** in a Gaussian velocity field.

An important point is, that though the *rate* of production of B^2 (and similarly of G^2), i.e., $B_iB_js_{ij}/B^2 = \Lambda_i \cos^2(\mathbf{B},\lambda_i)$, does not *directly* depend on the magnitude of B, it depends on B via the dependence of alignments, i.e., $\cos^2(\mathbf{B},\lambda_i)$, on the magnitude of $\mathbf{B}$. This dependence is such that the preferential alignments become stronger at larger magnitudes of $\mathbf{B}$ and correspondingly of $\mathbf{G}$.

Two-dimensional flows

It is well known that the behaviour of dynamical variables (velocity, vorticity, etc.) is qualitatively different in three and two dimensions (see chapter 8 and appendix C). When dealing with, e.g., passive scalars, it is believed that in many respects their behaviour is similar in three and two dimensions, *since the conservation properties of the scalar do not depend on dimensionality* (see Holzer and Siggia, 1994; Warhaft, 2000 and references therein). However, there are also differences some of which can be seen via geometrical statistics, e.g., alignments. In two dimensions the rate of strain tensor has only two eigenvectors: one positive, and one negative – there is no intermediate one. Therefore, in the two-dimensional case the positiveness of quantities like $\langle l_il_js_{ij}\rangle, (\langle B_iB_js_{ij}\rangle)$, $\langle N_iN_js_{ij}\rangle$ $(\langle G_iG_js_{ij}\rangle)$ and the corresponding rates is always due to the tendency of alignment of the corresponding vector with the eigenvector corresponding to the positive (negative) eigenvalue.

More details on the behaviour of the alignments can be seen from the equations for the evolution of vectors $a_i^G = \cos(\mathbf{G},\lambda_i)$ and $a_i^B = \cos(\mathbf{B},\lambda_i)$. This approach allows us to distinguish between pure strain effects and rotation. The latter consists of *two* contributions: vorticity and the rotation of the eigenframe, λ_i, of the rate of strain tensor, s_{ij} (see Lapeyre et al., 2001; Brethouwer et al., 2003; Nomura and Post, 1998; Tabor and Klapper, 1994 and references therein).

A final remark is that the viscous effects also influence the alignments. However, practically nothing is known about this influence so far.

4.2. Kinematic/Lagrangian chaos/advection

In the previous section the velocity field was assumed random, so that the chaotic behaviour of passive objects was mainly due to the random nature and the multiplicative character of the velocity field and, as in some problems, due to random forcing.

Since the equations describing the evolution of passive objects are *linear,* it may seem that there is no place for chaotic behaviour of passive objects if the velocity field is not random and is regular and fully laminar, because the chaotic behaviour appears/shows up in *nonlinear* systems. There is, however, no real contradiction or paradox. This apparent contradiction is resolved via the following observations.

Until now we used the Eulerian description, in which the observation of the system is made in a *fixed* frame as the fluid goes by. In this case the motion is characterized by the velocity field $\mathbf{u}(\mathbf{x},t)$ as a function of position vector, $\mathbf{x}$, and time, t. Another way to characterize the fluid flow is the Lagrangian description in which the observation is made *following* the fluid particles wherever they move. Here the dependent variable is the position of a fluid particle, $\mathbf{X}(\mathbf{a},t)$, as a function of the particle label, $\mathbf{a}$ (usually it's initial position, i.e., $\mathbf{a} \equiv \mathbf{X}(0)$) and time, t. The relation between the two ways of description is given by the equation[7]

$$\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t} = \mathbf{u}[\mathbf{X}(\mathbf{a},t); t], \tag{4.3}$$

i.e., the Lagrangian velocity field, $\mathbf{v}(\mathbf{a},t) = \frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t}$, is related to the Eulerian velocity field, $\mathbf{u}(\mathbf{x},t)$, as $\mathbf{V}(\mathbf{a},t) \equiv \mathbf{u}[\mathbf{X}(\mathbf{a},t); t]$.

If the Eulerian velocity field is known/given – as in all problems of kinematic nature – then the equation (4.3) serves for determination of the trajectory of a fluid particle with the initial position $\mathbf{X}(0) \equiv \mathbf{a}$. This equation is *nonlinear* (for almost all) even for very simple fluid flows and is generically non-integrable for all such flows. This seemingly simple equation is generally intractable even in the kinematic setting of determining closed form solutions for the particle paths $\mathbf{X}(\mathbf{a}, t)$ from an initial position $\mathbf{x} = \mathbf{a}$ with given $\mathbf{u}(\mathbf{x}, t)$.

One of the important developments of the so-called deterministic chaos is that (even) simple systems governed by a purely deterministic nonlinear set of equations as a rule exhibit irregular, apparently random/stochastic, behaviour (see, e.g., Mullin, 1993 and references therein). In particular it is well established that the trajectories of fluid particles – their motion is governed by equation (4.3) – exhibit chaotic behaviour even when the Eulerian velocity field is not random but is regular and fully laminar. In other words, though the Eulerian velocity field, $\mathbf{u}(\mathbf{x}; t)$ is not chaotic and is regular and laminar, the Lagrangian velocity field $\mathbf{v}(\mathbf{a},t) \equiv \mathbf{u}[\mathbf{X}(\mathbf{a},t); t]$ is chaotic because $\mathbf{X}(\mathbf{a},t)$ is chaotic. This is generally true of two-dimensional

[7]The relation between the two ways of description can be seen also by looking at any conservative property of fluid particles (i.e., a nondiffusive passive scalar) such as nondiffusive 'dye' or any other (e.g., radioactive) label. Due to its conservative character it is time independent in the Lagrangian description, i.e., has the form $\vartheta(\mathbf{a})$, but is time dependent in some fixed point of space, $\mathbf{x}$, i.e., in the Eulerian description, and has the form $\theta(\mathbf{x},t)$. Hence, both are related via $\vartheta(\mathbf{a}) = \theta[\mathbf{X}(\mathbf{x},t), t]$. Since $\frac{\partial \vartheta(\mathbf{a})}{\partial t} = \frac{D\theta}{Dt} = 0$ it follows that $\frac{\partial \theta}{\partial t} + u_k \frac{\partial \theta}{\partial x_k} = 0$, which is just an expression of the fact that the material derivative of any Lagrangian conservative property should vanish (see Monin and Yaglom, 1971).

time dependent flows, three-dimensional time-independent flows, and, of course, time-dependent ones. In other words, many fluid flows which are laminar in the Eulerian sense (E-laminar) exhibit the so-called Lagrangian (kinematic) chaos or Lagrangian turbulence (chaotic advection), i.e., they are L-turbulent. This chaotic Lagrangian property of simple Eulerian fluid flows can lead to enhanced mixing properties of such flows at any, even very small, Reynolds numbers, i.e., enhanced transport of passive objects occurs not only in genuinely turbulent (E-turbulent) flows, but also in simple E-laminar flows possessing the property of Lagrangian chaos (L-turbulent)[8]. An example out of a great many of such enhanced mixing at $\mathrm{Re} \sim 1$ only is shown in figure 4.8 (see also Aref, 2002; Kim and Stinger, 1992; Sturman et al., 2006 and references therein for further examples).

It has to be stressed that this chaotic behavior is of purely kinematic nature resulting solely from equation (4.3) (and various equations for passive objects – reminding again – linear in the Euler setting) and has nothing to do with dynamics, i.e., genuine (as NSE) turbulence.

The enhanced transport of passive objects by E-laminar but L-turbulent flows is closely related to the property that (almost all) nearby fluid elements separate exponentially in time. That is, the material elements are stretched on average in such flows too, $D\left\langle l^2(t)\right\rangle/Dt = \langle l_i l_k s_{ik}\rangle > 0$ and $\left\langle \frac{1}{l(t)}\frac{Dl(t)}{Dt}\right\rangle = \langle l_i l_k s_{ik}/l^2\rangle > 0$. This stretching property appears to be necessary for the same behaviour of passive vectors with nonzero diffusivity, in spite of the qualitative difference between the two cases (Childress and Gilbert, 1995; Ott, 1999; Zeldovich et al., 1983, 1990 and references therein). As mentioned, this is a subtle issue, as other issues associated with the behaviour of systems described by differential equations in which the diffusivity (i.e., the coefficient in front of the highest derivative) is small and tends to zero: the singular limiting behaviour of the system is usually qualitatively different from the case when the diffusivity (viscosity) is put to zero at the outset. First, one of the reasons is easily seen in the case of incompressible flows. Stretching in one (or two) direction(s) results in compressing in, at least, one other direction bringing fluid elements very close one to another, so that the diffusive (viscous) effects become important for whatever small diffusivity (viscosity). This does not happen if the diffusivity (viscosity) is precisely vanishing at the outset. Second, it is typical for chaotic flows (i.e., L-turbulent, but both E-laminar and/or, of course, E-turbulent) that as the diffusivity becomes smaller, passive objects develop more fine-scale structure, so that the advective and diffusive terms in the corresponding equations remain of the same order in the spot-like

[8]This chaotic property of the trajectories of the fluid particles makes it more difficult to follow them, i.e., much more difficult to utilize the Lagrangian description of even the simplest fluid flows which exhibit Lagrangian chaos.

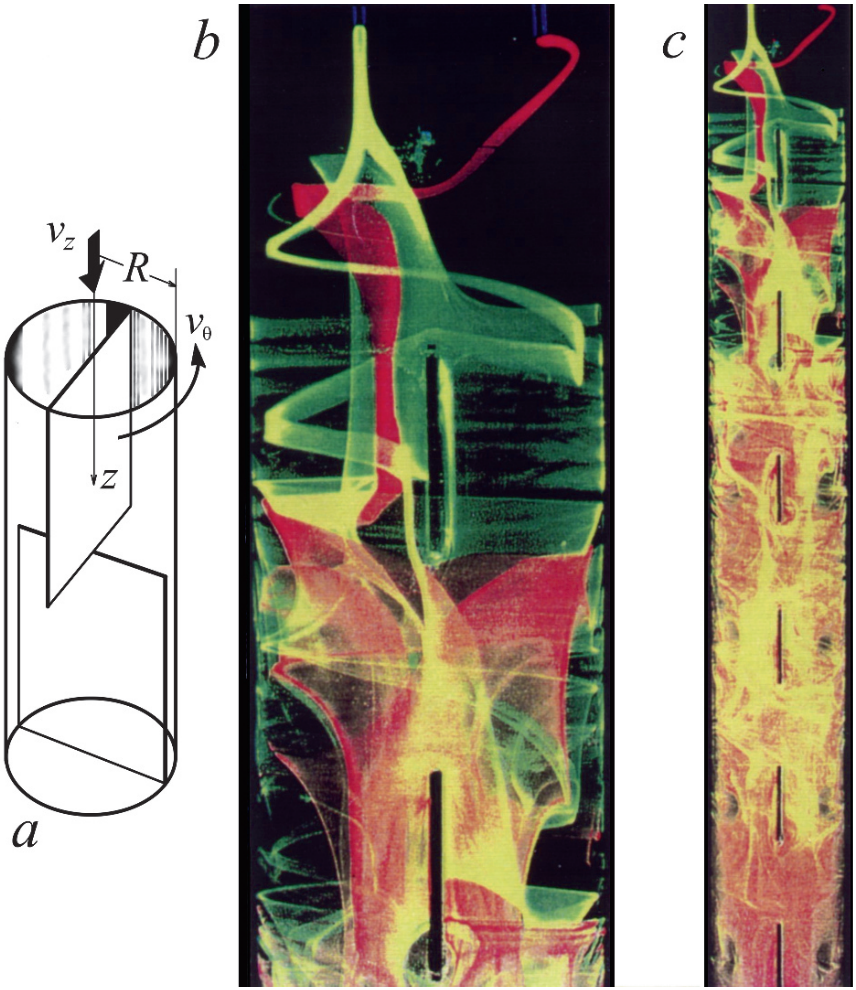

Figure 4.8. Mixing in PPM – partitioned-pipe mixer at very low Reynolds number. $\mathrm{Re}_{PPM:axial} = \langle v_z \rangle R/\nu = 0.3$ and $\mathrm{Re}_{PPM:cs} = v_R R/\nu = 1.8$; here $\langle v_z \rangle$ – average axial velocity and $v_R = \frac{1}{2}(|v_1|_{\max} + |v_1|_{\min})$ – characteristic cross-sectional velocity. $0 < \mathrm{Re}_{PPM:axial} < 0.8$ and $0 < \mathrm{Re}_{PPM:cs} < 8$. a) schematic of the PPM, b) is a close-up of the upper part of c). From Kusch and Ottino (1992). For other examples see Acrivos (1991); Kim and Stringer (1992); Aref and El Nashie (1994) and references therein

(intermittent) regions throughout the whole flow domain. The formation of the small-scale structure is both due to the predominant production of the type $\langle B_i B_k s_{ik} \rangle$, i.e., stretching, and diffusive effects as well as their interaction. The latter is an important issue as the outcome is not just a result of both as if they were additive and independent, e.g., the presence of diffusion changes qualitatively the nature of the production. Hence the small-scale structure developing in diffusionless systems, e.g., as a consequence of Lagrangian chaos with zero diffusivity, is, generally, qualitatively different from that for whatever small but nonzero diffusivity. It seems well established that the positiveness of quantities such as $\langle l_i l_k s_{ik}\rangle$ and $\langle l_i l_k s_{ik}/l^2\rangle$, and $\langle B_i B_k s_{ik}\rangle$ and $\langle B_i B_k s_{ik}/B^2\rangle$ can be seen as a universal qualitative property not only of any Eulerian random fluid flow (be it real fluid turbulence or artificial), but of any fluid flows that are Lagrangian chaotic, many of which are simple laminar in the Eulerian sense. These and similar quantities are closely related to the positiveness of the so-called Liapunov exponents associated with exponential stretching, at least in some part of the flow region[9]. However, the structure or even simpler properties of flow regions where the Liapunov exponents are or should be positive is not known even for simple flows, e.g. three-dimensional time-independent flows. Note that the linearization of the equation (4.3) for a small disturbance $\mathbf{l}$ (i.e., difference of equation (4.3) for $\mathbf{X}+\mathbf{l}$ and for $\mathbf{X}$) is precisely equation (C.1) $\frac{Dl_i}{Dt} = l_k \frac{\partial u_i}{\partial x_k}$, and the RHS of the equation for l^2 is precisely $l_i l_k \frac{\partial U_i}{\partial x_k} \equiv l_i l_k s_{ik}$.

4.3. On the relation between Eulerian and Lagrangian fields

> *Given the marker dispersion the problem is to determine the source(s) of agitation. In general, owing to chaotic advection, this inverse problem is impossible to solve* (Aref, 1984).
> *... the possession of such relationship would imply that one had (in some sense) solved the general turbulence problem. Thus it seems arguable that such an aim, although natural, may be somewhat illusory* (McComb, 1990).
> *What one sees is real. The problem is interpretation.*

The relation between Eulerian and Lagrangian fields is a long-standing and most difficult problem. The general reason is because the Lagrangian field $\mathbf{X}(\mathbf{a},t)$ (and velocities $\mathbf{v}[\mathbf{X}(\mathbf{a},t);t]$) is a complicated functional of the Euler velocity field $\mathbf{u}(\mathbf{x},t)$ resulting from the equation (C.64). Roughly, there is a general relationship in terms of path (Feynman, functional) integrals, but this does not help much (if at all)[10]. Apart from this 'formalistic' issue there is one more important aspect of conceptual nature associated with the 'more chaotic' nature of the Lagrangian setting. This can be seen as an indication that the pure Lagrangian dynamical equations (so far intractable for viscous flows) are more rich than their Navier–Stokes counterpart. The complexity

[9]See Acrivos (1991); Bohr et al. (1998); Childress and Gilbert (1995); Ott (1999); Sturman et al. (2006); Zeldovich et al. (1990) and references therein.

[10]For more on these issues see Monin and Yaglom (1971, **1**, Ch. 9, pp. 568–578), also Bennet (2006, pp. 21–24). The start was made by Corrsin (1959a,b) and Lumley (1962a,b).

of the relation between the Lagrangian and Eulerian fields is seen in the example mentioned in the previous section of Lagrangian (kinematic) chaos or Lagrangian turbulence (chaotic advection) with *a priori* prescribed and *not* random Eulerian velocity field (E-laminar)[11]. In such E-laminar but L-turbulent flows the Lagrangian statistics has no Eulerian counterpart, as in the flow shown in the figure 4.8. Indeed, though the Eulerian velocity field, $\mathbf{u}(\mathbf{x};t)$ is not chaotic and is regular and laminar, the Lagrangian velocity field $\mathbf{v}(\mathbf{a},t) \equiv \mathbf{u}[\mathbf{X}(\mathbf{a},t);t]$ is chaotic because $\mathbf{X}(\mathbf{a},t)$ is chaotic. This shows that, in general, there does not exist a unique relation between Lagrangian and Eulerian statistical properties in genuine turbulent flows as was foreseen by Corrsin in 1959b: *in general, there is no reason to expect that L_{ik}* (the Lagrangian two-point velocity correlation tensor) *and E_{ik}* (the Eulerian two-point velocity correlation tensor) *will be uniquely related.* In other words it may be meaningless to look for such a relation, though there is a considerable number of papers attempting to give such relations mostly for practical purposes[12].

The intricacy of the relation between the Eulerian and Lagrangian fields of the *same* fluid flow has a number of important consequences. In fact, this is a part of a broader issue addressed in chapter 9.

4.4. Summary

The predominance of stretching over compressing of passive objects and formation of intricate structure can be seen as a universal qualitative property of any Eulerian random fluid flow, be it real fluid turbulence or some artificial random field. In the latter case, the non-Gaussianity of a passive field possessing structure arises from a simple Gaussian 'structureless' velocity field. This is a kind of irreversible effect of the randomness of velocity field on passive objects independent of the nature of this randomness.

The predominance of stretching over compressing of passive objects also occurs in all fluid flows that are Lagrangian chaotic (L-turbulent) a set that includes most of the simple laminar flows in the Eulerian sense (E-laminar). Hence, generally, there is no one-to-one relation between the Lagrangian and Eulerian statistical properties in turbulent flows, just as there may be no correspondence between the structure(s) of a passive object (dye) and

[11]This is why Lagrangian description – being physically more transparent – is much more difficult than the Eulerian description, see the citation from Lagrange on page 27.

[12]This was also started by Corrsin (1959a) who proposed the so-called independence approximation to relate the Lagrangian and Eulerian velocity correlations assuming that at large times the probability distributions of particle displacements and of the Eulerian velocity field become statistically independent. Generally this hypothesis (as a host of others) is not correct as is shown in recent experiments by Ott and Mann (2005), see also Weinstock (1976).

the field of a dynamically active variable (velocity, vorticity) in the same fluid flow.

Thus the essential differences in the behaviour of passive and active fields and the intricacy of the relation between them require caution in promoting analogies between the two. We return to this issue in more detail in chapter 9.

CHAPTER 5

PHENOMENOLOGY

And the Kolmogorov 4/5 law

> *Correlations after experiments done is bloody bad. Only prediction is science.* (Hoyle, 1957)
> *Phenomenology – The branch of a science that classifies and describes its phenomena without any attempt at explanation* (Webster's New World Dictionary, College edition, 1962).
> ... *even wrong theories may help in designing machines* (Feynman, 1996).
> *In our present state of understanding, these simple models will be based, in part on good physics, in part on bad physics, and in part on shameless phenomenology* (Lumley, 1992).
> *Our present understanding of anything turbulent is at best phenomenological* ... (Siggia, 1994).

5.1. Introductory notes

There is no definition of what is phenomenology of turbulent flows. In a broad sense, it can be defined by a statement of impotence: it is almost everything except the direct experimental results (numerical, laboratory and field) and/or results (a very small set indeed), which can be obtained from the first principles, e.g., NSE. Phenomenology of turbulence involves use of dimensional analysis, a variety of scaling arguments, symmetries, invariant properties and various assumptions, some of which are of unknown validity and obscured physical and mathematical justification (if any). Thus in the broad sense phenomenology of turbulence includes also most of the semi-empirical approaches and turbulence modelling[1]. Doing all this requires insight into the basic physics of turbulence, hard experimentation and painful efforts of interpretation. The latter may be quite problematic, especially in models having enough free parameters[2] to guarantee the right results not necessarily for the right reasons.

[1]Most of this enormous material is beyond the scope of this small book. Relevant references can be found in Davidson (2004); Frisch (1995); Lesieur (1997); McComb (1990); Meneveau and Katz (2000); Mathieu and Scott (2000); Monin and Yaglom (1971, 1975); Pope (2000); Sagaut and Cambon (2008); Tennekes and Lumley (1972).

[2]Sometimes these parameters are changed according to circumstances – in words of von Karman a kind of "science of variable constants".

It is often claimed that in turbulence research, phenomenology helps to explain some features of turbulent flows, so that there is such a thing as phenomenological understanding of turbulence. This seems too ambitious; phenomenology is mostly a kind of description of some statistical aspects of turbulent flows, which is based on or motivated by some experimental data. The best that can be achieved by phenomenology is formulation of some plausible *a priori* hypotheses, i.e., those *before* experiments are done. The famous Kolmogorov hypotheses belong to this category. We start with an overview of what is called Kolmogorov phenomenology, though it has been discussed in many books, reviews and numerous papers. There are several reasons to do so. First, we will need this material as background to several discussions below and the following chapters. Second, in many cases Kolmogorov has been ascribed things he did not write, so our exposition is based exclusively on his original papers in Russian (Kolmogorov, 1941a,b). This is followed by a discussion of inertial range, cascade and other central phenomenological ideas and related matters.

5.2. Kolmogorov phenomenology and related subjects

The two Kolmogorov's well-known similarity hypotheses are frequently called universality hypotheses (e.g., Frisch, 1995). However, it is important to note that before putting forward these two similarity hypotheses, Kolmogorov formulated the *hypothesis of local isotropy* based on definitions of local homogeneity and isotropy. Together with his definition of local isotropy[3] this hypothesis postulates that at large Reynolds numbers all the symmetries of the Navier–Stokes equations are restored in the statistical sense[4], except for one involving scaling: … *we think it rather likely that in an arbitrary turbulent flow with sufficiently large Reynolds number* $\mathrm{Re} = \frac{LU}{\nu}$ *the hypothesis of local isotropy is realized with good approximation in sufficiently small regions G of the four-dimensional space* (x_1, x_2, x_3, t) *not lying close to the boundaries of the flow or its other special regions.* The text in the English translation of the Kolmogorov papers published in Friedlander

[3]The Kolmogorov (1941a) definition of local isotropy states:

The turbulence is called locally-isotropic in the domain G, *if it is homogeneous and if, besides, the distribution laws mentioned in Definition 1* (see below) *are invariant with respect to rotations and reflections of the original system of coordinate axes* $(x_1, x_2, x_3,)$.

Local homogeneity as defined in Kolmogorov's Definition 1 states the invariance of the distributions F_n on space and time translations and Galilean transformations. Here F_n is a $3n$*-dimensional distribution law of probabilities* for the n velocity increments $\mathbf{w}(P^{(n)}) = \mathbf{u}(P^{(n)}) - \mathbf{u}(P^{(0)})$ between n points $P^{(n)}(x_1^{(n)}, x_2^{(n)}, x_3^{(n)})$ and a certain fixed point $P^{(0)}(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$.

[4]Frisch (1995) presents this in the form of his hypothesis **H1** (p. 74), but omits to mention that it is due to Kolmogorov: there is no presentation of the hypothesis of local isotropy in his book.

and Topper (1961) is reprinted in a slightly less satisfying version in a special issue of the Proc. Roy. Soc. London (1991), **A434**: *in turbulent flow with sufficiently large Reynolds number in sufficiently small regions G of the four-dimensional space* (x_1, x_2, x_3, t) *not lying close to the boundaries of the flow or other singularities of it.* This translation is not very accurate. Specifically, the term *singularities* used for the Russian *osobennosti* is inadequate in this particular case. Therefore the last four words in the above citation were replaced by *its other special regions.*

Naturally, this hypothesis, which we have put forward in such a general and somewhat indefinite form, cannot be proved rigorously. This is followed by an exposition of the turbulence cascade process on a 2/3 page footnote as a *qualitative* justification of the suggested hypothesis. In order *to make its experimental justification possible for individual special cases,* Kolmogorov presents *a number of consequences of the hypothesis of local isotropy* and only then turns to his similarity hypotheses. The first similarity hypothesis states that for *the locally-isotropic turbulence the distributions* F_n *are uniquely determined by the quantities* ν*and* ϵ. The second similarity hypothesis states that, if the separations between the points *are large in comparison with* $\eta = \nu^{3/4}/\langle\epsilon\rangle^{1/4}$, *then the distributions* F_n *are uniquely determined by the quantity* $\langle\epsilon\rangle$ *and do not depend on* ν. It is Kolmogorov's second similarity hypothesis which is introduced in order to cope with the *possible* scale invariance symmetry of NSE[5] at $Re \gg 1$ and which allowed him to find the 2/3 famous exponent; Kolmogorov used the term hypotheses of *similarity*, i.e., scale invariance, intentionally, and avoided using the term *universality*. Namely, straightforward dimensional analysis applied to the second-order structure function in the so-called *inertial range* of scales r, $L \gg r \gg \eta$, resulted in

$$S_2^{\|}(r) \propto C_2 \langle\epsilon\rangle^{2/3} r^{2/3}. \tag{5.1}$$

Here $S_p^{\|}(r) = \langle(\Delta u_{\|})^p\rangle,\ p = 2,$ – is the second-order structure function of the longitudinal velocity increment $\Delta u_{\|} \equiv [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{r}/r$, and C_2 is an 'absolute' constant; $\langle\epsilon\rangle$ is the mean rate of energy dissipation, $\nu -$ kinematic viscosity, and $\eta = \nu^{3/4}\langle\epsilon\rangle^{1/4}$ is the Kolmogorov dissipation scale.

It is noteworthy that Kolmogorov never worked in Fourier space[6]. This was done by his Ph.D. student A. Obukhov (1941), who formulated the

[5]Kolmogorov was aware of the self-similarity before the completion of the first K41 paper, but '*did not know how to determine the exponent* m' (Yaglom A.M., 1994, *Annu. Rev. Fluid Mech.*, **24**, p. 8).

[6]Though it is very common to ascribe to him that he did so. For example, *The idea of similarity in physical space is old – Richardson (1922), for example, expressed it explicitly – but the force of Kolmogorov's theory seems to have oriented attention in later years to wavenumber space.* Sreenivasan (1991, p. 541).

There are other numerous similar ascribings to Kolmogorov in the literature. A recent

$-5/3$ law for the energy spectrum

$$E(k) = C_K \langle \epsilon \rangle^{2/3} k^{-5/3}, \tag{5.2}$$

which in some sense (only) is equivalent to (5.1).

The $2/3$ and $-5/3$ laws are claimed to have received considerable experimental support[7], though looking at the compensated data (i.e., at $S_2^{\|}(r)r^{-2/3}$ and/or $E(k)k^{5/3}$) at large Reynolds numbers (including the famous experiments by Grant et al., 1962) shows that in most of the experiments the exponents in (5.1) and (5.2) are correspondingly larger than $2/3$ and smaller than $-5/3$, see below. These and similar results have various serious consequences concerning the asymptotic behaviour of turbulent flows as $\nu \to 0$ ($Re \to \infty$) such as applicability of some singular solutions of the Euler equation for description of this asymptotic state, see chapter 9. It has to be stressed that in practice one deals *always* with a 'pseudo-limit' $Re \gg 1$ ($\nu \ll 1$) which can make a qualitative difference for several reasons. First, the true limit $Re \to \infty$ ($\nu \to 0$) is singular, so one cannot exclude the possibility that what is observed at whatever large but finite Re can be very much different from what happens in the limit $Re \to \infty$ ($\nu \to 0$) if there is a way to define this limit properly. So strictly speaking, based on whatever experimental evidence, one cannot claim that *turbulence remains dissipative even at infinite Reynolds numbers.* Second, there is a consensus (but still only a belief) that solutions of NSE at *any* finite Re are not singular; for a discussion of these and similar issues, see chapter 9.

As mentioned, in his 1941a paper Kolmogorov explicitly treated only the second-order function, though all his hypotheses were formulated for statistical properties of velocity increments and thereby for structure functions of any order. Instead, the 1941a paper was followed by the most remarkable quantitative prediction, perhaps the only prediction in the theory of turbulence made so far, of his 4/5 law obtained as a direct consequence from the Navier–Stokes equations (Kolmogorov, 1941b) for the inertial range $L \gg r \gg \eta$

$$S_3^{\|}(r) = -4/5\langle \epsilon \rangle r, \tag{5.3}$$

in which the constant $C_3 = -4/5$. However, this relation was obtained for globally-, not for locally-isotropic turbulence. A number of justifications for

example is *Kolmogorov took a further step: based on his result for the third-order longitudinal structure function $S_{3,0}(r)$, he concluded that $U = O(\epsilon r^{1/3})$ and proposed a general law: $S_{n,0} \propto r^{n/3}$*, Yakhot (2003). Kolmogorov never handled structure functions of order higher than three, though the scaling $S_{n,0} \propto r^{n/3}$ follows directly from his second similarity hypothesis.

[7]See references in Monin and Yaglom (1975); Frisch (1995); Saddoughi (1997); Sreenivasan and Antonia (1997). In a recent experiment by Kholmyansky et al. (2001b) and Gulitski et al. (2007a) the 5/3 law spans over three decades for all the three velocity components.

various versions of locally-isotropic and/or locally- homogeneous turbulent flow was made by Danaila et al. (1999); Hill (1997); Lindborg (1996, 1999); Mann et al. (1999); see also references in these papers and in Kurien et al. (2004); Tatarskii (2005); Antonia and Burattini (2006); Takaoka et al. (2007); Kaneda et al. (2008); Mininni et al. (2008). A rigorous proof that $\lim_{r/L,\eta/r\to 0} S_3^{\|}(r)/\langle\epsilon\rangle r = -4/5$ was obtained by Nie and Tanveer (1999) without any assumptions on local homogeneity, isotropy and stationarity with $S_3^{\|}(r)$ defined by integrating in space/time and over all possible orientations of $\mathbf{r}$, though for a number of reasons they were unable to check their results numerically: low Reynolds number ($\mathrm{Re}_\lambda = 155$) and prohibitive computational expense to compute over all orientations of $\mathbf{r}$ (solid angle integrations), which is necessary due to possible anisotropy of the flow, see Taylor et al. (2003), and also Duchon and Robert (2000) for a local version of the 4/5 law.

The Kolmogorov papers (1941a,b) raised a number of basic issues which have kept the turbulence community quite busy until now. We address briefly three of them.

The first issue is about the validity of the 4/5 law. Since the 4/5 law is a consequence of the Navier–Stokes equations, it should be possible, at least in principle, to obtain it from experiments (laboratory, DNS or even field) in a rather clean way. However, even at rather high Reynolds numbers, the results exhibit large variability for the range of r in which $K = S_p^{||}(r)/(-4/5\varepsilon r) \approx 1$. For example, in the experiments by Gagne (1987) in the S1 wind tunnel in Modane at $\mathrm{Re}_\lambda \sim 3000$, this range was less than one decade, whereas in similar experiments by Malecot (1998) at $\mathrm{Re}_\lambda \sim 2500$ and in experiments by Mydlarski and Warhaft (1996) with the turbulent grid flow even at $\mathrm{Re}_\lambda \sim 450$ there was almost no such range at all. In experiments by Praskovsky (1998) at similar Re_λ the range of r in which $K \approx 1$ was about one and a half decades for two flows (mixing layer and return channel) in the large wind tunnel of TSAGI (near Moscow). Even at $\mathrm{Re}_\lambda \approx 10^4$ in experiments by Sreenivasan and Dhruva (1998); Kholmyansky et al. (2001b); Gulitski et al. (2007a); Kholmyansky and Tsinober (2008) in the atmospheric surface layer, this range was less than three decades. However, attempting to define the scaling range in a more precise manner via local slopes of K causes this range to become much shorter and in some experiments even to disappear. In some cases when the large scales are approximately isotropic, the 4/5 law is observed even at $\mathrm{Re}_\lambda \sim 220$, though in a rather limited range slightly more than half a decade, as in the DNS by Chen and Cao (1997).

Among the possible causes for deviations from the 4/5 law, Frisch (1995, p. 129) lists the following: *lack of asymptoticity (e.g., contamination by the dissipation range), lack of homogeneity and/or isotropy, violations of Taylor* *hypothesis, violation of the hypothesis ... of the finiteness of energy dissipation, inaccurate determination of the dissipation rate, and poor quality of the data.*

It is noteworthy that these and/or possibly other causes (see e.g., Moisy et al., 1999; Danaila et al., 1999 and references therein) result in considerable deviations from the 4/5 law, whereas these same causes, whatever they are, have little (but not negligible) effect on the 2/3 (5/3) law. This seems surprising, since the 4/5 law is a consequence of Navier–Stokes equations, while the 2/3 law is only a consequence of dimensional arguments (dimensional necessity) supplemented by the above mentioned hypotheses. So far there is no clear answer to this issue. One of the possible reasons is the lack of isotropy in two meanings. First, the 4/5 law applies strictly to globally-isotropic flows. Second, flows even with very large Reynolds number may lack local isotropy[8].

There were many attempts to test the hypothesis of local isotropy in a variety of ways, all of which exploit some consequences of this hypothesis. These include testing the kinematic relations between various quantities both in physical and in Fourier space, and many other such approaches. The issue has a long history and cannot be reviewed here in full; for a partial list of references, see Ferchichi and Tavoularis (2000); Saddoughi (1997); Shen and Warhaft (2000); Tsinober (1993, 1998a); Yeung et al. (1995) and Zhou and Antonia (2000).

We mention here only the turbulent shear flows as those for which considerable evidence has accumulated since the early fifties *against* the hypothesis of local isotropy at rather high Reynolds numbers. The recent ones are the experimental results obtained in an approximately homogeneous shear flow by Garg and Warhaft (1998) and Shen and Warhaft (2000) for $Re_\lambda \sim 10^3$ (see also Ferchichi and Tavoularis, 2000)[9]. On a qualitative

[8]The difficulty in a clean experimental confirmation of the 4/5 law at large Reynolds numbers is two-fold. First, it is extremely difficult, if not impossible, to set up experimentally a large Reynolds number flow which is isotropic in large scales.

Second, if one gives up the isotropy in the large scales, then one has to use the result obtained by Nie and Tanveer (1999). That is in order to determine the third-order structure function it is necessary to perform both space/time and solid angle averaging, i.e. over all possible orientations of the separation vector $\mathbf{r}$. With the existing techniques this is impossible in high-Reynolds-number physical experiments where the averages are performed over time only. As for direct numerical simulations, that may be possible for moderate Reynolds numbers with rather large scale of computations as was done by Taylor et al. (2003). Until then *we can hold strong opinions either way* (Feynmann, 1963), though there is little doubt about the validity of the 4/5 law in a globally-isotropic turbulent flow.

As concerned the higher-order statistics (e.g., structure functions) the problems are even more serious, as one is forced to use very long time records to achieve statistical convergence. This means that very large scales (which hardly are isotropic) are involved in the process.

[9]Ferchichi and Tavoularis (2000) draw contrasting conclusions to those of Shen and

level their results show that local isotropy holds for second-order statistics, but is violated for higher-order statistics[10] in agreement with the results mentioned above regarding the deviations from the 4/5 law. One of their quantitative results is that the skewness, $\langle(\Delta u_{12})^3\rangle/\langle(\Delta u_{12})^2\rangle^{3/2}$, of the longitudinal velocity increments, $\Delta u_{12} = u_i(x_1,x_2+r,x_3) - u_1(x_1,x_2,x_3)$, along the mean velocity gradient, is $\sim$0.5 in the inertial range, whereas it should be close to zero if the flow is locally-isotropic in this range. This effect is much stronger for the super-skewness $\langle(\Delta u_{12})^5\rangle/\langle(\Delta u_{12})^2\rangle^{5/2}$. These results, compared to those for flows without mean shear, imply that there is a *direct* influence of mean shear on the small scales – an effect inferred already by Townsend (1954) among others.

Such an effect is possible due to the permanent bias of the mean shear to which the field of fluctuations is exposed due to its very large *residence* time in the mean shear. More generally, the anisotropy in large scales seems to be felt down to the smallest scales in the inertial range and may be (but may also not be, see chapter 6) 'forgotten' only in the range of scales comparable with the Kolmogorov scale, η. In this sense the 'cascade' (see next section) is not an information-losing process. In other words, there is quite reliable experimental evidence, that, at least in turbulent shear flows, the hypothesis of local isotropy is violated[11], and this seems to be the case with many other (but not all) flows, which are anisotropic in the large scales[12].

One of the main possible reasons for the violation of local isotropy (and the so-called anomalous scaling) even at the largest accessible Reynolds numbers is the direct and bidirectional coupling between large and small scales when the large scales are anisotropic – one of the manifestations of nonlocality of turbulent flows. This matter is taken up in chapter 6 in the section on nonlocality. We add here that violation of local isotropy means violation of the basic hypothesis of the so-called fully-developed turbulence, that all the symmetries of the Navier–Stokes equations are restored in the statistical sense locally in time and space (see, e.g., Frisch, 1995).

The second issue is about the behaviour of dissipation of turbulent flows at very large Reynolds numbers. It is widely thought that the Kolmogorov similarity hypotheses imply that the mean dissipation, $\langle\epsilon\rangle$, remains *finite/*

Warhaft (2000). This seems to be a matter of interpretation: closer inspection of the results shows that both support the conclusions of Shen and Warhaft (2000).

[10] Anisotropy of velocity derivatives at the level of second-order statistics was observed in jet flows in the form of locally axisymmetric turbulence; see Hussein (1994) and references therein.

[11] Similar evidence exists since Stewart (1969) for passive scalars in the presence of a mean gradient of passive scalar; see references in Warhaft (2000) and Villermaux et al. (2001).

[12] This is one of the reasons we chose to discuss the 4/5 law in this section.

non-vanishing as $\mathrm{Re} \to \infty$. More precisely the issue is whether the normalized mean dissipation $\varepsilon = U^3 L^{-1}\langle\epsilon\rangle$ tends really to a finite limit as $\mathrm{Re} \to \infty$ ($\nu \to 0$), or is Re dependent even at very large Reynolds numbers[13]. There is much speculation about this subject, while the experimental and recent evidence from DNS favouring the former is still limited (see references in Pearson et al., 2004; Burattini et al., 2005 and Ishihara et al., 2009) though in engineering practice this fact was recognized long ago in a great variety of flow configurations (see e.g., Idelchik, 1996 and also figure 1.8). Recent results obtained using glycerol, water and low-temperature helium gas (Cadot et al., 1997; see also references therein), show that $\varepsilon = const$ within the range of Reynolds numbers varying over more than three decades, $3 \cdot 10^3 < \mathrm{Re} < 7 \cdot 10^6$ (see figure 5.1) when the flow is forced by very 'rough' moving boundaries.

However, when the moving boundaries were smooth, $\varepsilon(\mathrm{Re}) \neq const$, and was a decreasing function of Reynolds number similar to such Re-dependence in other configurations, e.g., in pipes with smooth walls. Nevertheless, the *bulk* of the flow exhibited clear Re-independent behaviour, indicating that the main difference between the two cases is due to the dissimilarity in the coupling between the boundaries and the bulk of the flow, or more generally due to the difference in the mechanisms of turbulence production[14].

An interesting 'exception' is the observation by Seoud and Vassilicos (2007) on dissipation and decay of turbulence generated by space-filling fractal square grids. In these experiments the normalized dissipation is reported to behave as $100\mathrm{Re}_\lambda^{-1}$ reaching the value below 0.15 as contrasted to the 'usual' value $\sim$0.5. This together with a kind of 'counterexample' by Ohkitani (2008) raises the question whether $\varepsilon(\mathrm{Re})$ remains always finite at large enough Re. There is limited evidence that the dissipation of helicity tends to remain finite as well, see chapter 6, Galanti and Tsinober (2006) and references therein.

It is noteworthy that rigorous upper bounds of ε are independent of the Reynolds number at large Re (see Doering, 2009 and references therein) and thereby are consistent with the experimental results[15]. The Reynolds-number-independent behaviour of some global characteristics of turbulent flows at large Re, such as the total dissipation in the above example, the drag of bluff bodies (figure 1.8) and resistance in many other configurations, comprises one of the *quantitative* universal properties of turbulence at large

[13] The scaling U^3L^{-1} for the mean dissipation was obtained by Taylor (1935).

[14] It should be emphasized that the independence of some parameter of viscosity at large Reynolds numbers does not mean that viscosity is unimportant. It means *only* that the (cumulative) effect of viscosity is Reynolds number independent.

[15] Unfortunately the *lower* bounds are not as good, since they correspond just to the values for laminar flows.

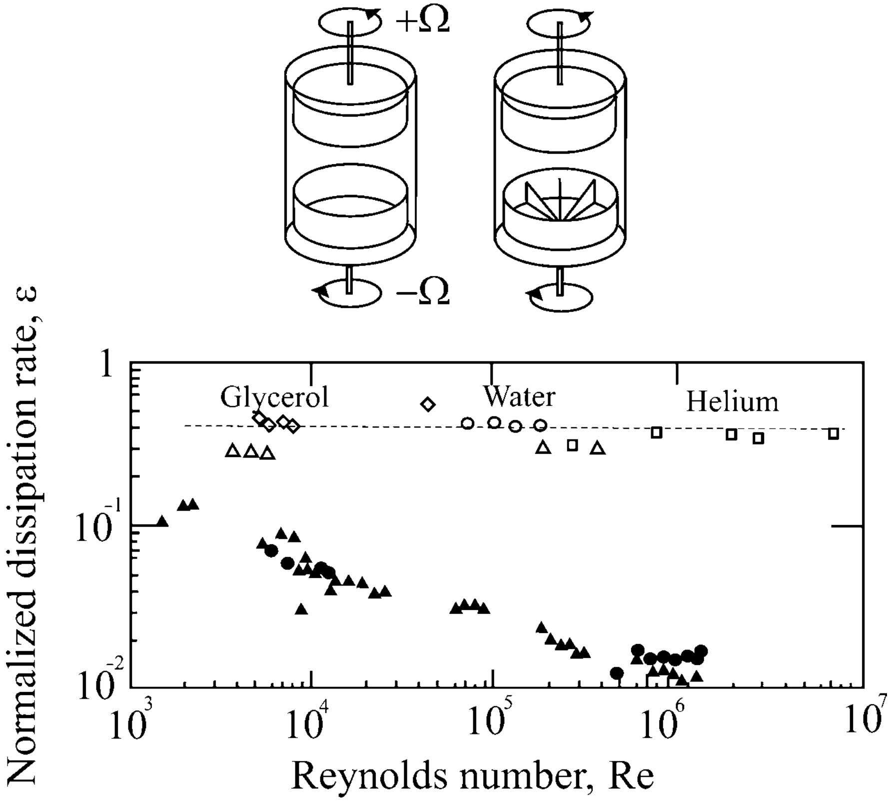

Figure 5.1. Reynolds number dependence of normalized dissipation rate of energy ε in a turbulent flow in a circular tank forced by counter rotating top and bottom. The schematic of the water/glycerol and low-temperature helium gas facilities are shown in the upper part of the figure. Adapted from the Ph.D. Thesis by O. Cadot et al. (1995), Laboratoire de Physique Statistique de l'Ecole Normale Supérieure, Université Paris VII. Similar results were obtained in the Couette–Taylor flow, on both see Cadot et al. (1997)

Re. This is distinct from *some possibly* universal (mostly scaling) properties of small-scale turbulence.

Since the finite limit of the mean dissipation $\varepsilon = U^3 L^{-1} \langle \epsilon \rangle$ at large Re defines a unique scaling exponent in the 2/3 law (5.1) and since the 4/5 law (5.3) is a consequence of the Navier–Stokes equations, the two scaling exponents, $\zeta_2 = 2/3$ and $\zeta_3 = 1$, possess a special status.

The third issue is about the scaling exponents, ζ_p, of order higher than 3, $p > 3$ *in the inertial range* of separations r, defined as $\eta \ll r \ll L$.

5.3. Anomalous scaling

> *It is plausible but not certain that there are intermittency corrections to the K41 theory of the inertial range.* (Frisch, 1995).
> *The presence of a finite injection scale, L, irrespective of its large value, is felt throughout the inertial range precisely via the anomalies $< (\Delta u)^p > \propto r^{p/3}(L/r)^{p/3-\sigma_p}$.* (Falkovich et al., 2001).
> . . . *the dissipative structure is responsible for the anomalous scaling of the structure functions of order larger than 5.* (Nakano et al., 2003).
> *there are a variety of models of higher statistics that have meager or nonexistent deductive support from the NS equations but can be made to give good fits to experimental measurements* . . . (Goto and Kraichnan, 2004).

Attempts of experimental verification of the scaling exponents for the higher-order structure functions showed that instead of 'normal' scaling

$$S_p^{\|}(r) \propto r^{p/3}, \tag{5.4}$$

one has the 'anomalous' one (see figure 8.8 in Frisch, 1995 and figure 7.1)

$$S_p^{\|}(r) \propto r^{\zeta_p}, \tag{5.5}$$

with $\zeta_p = p/3 - \mu_p < p/3$, which means that in order to make the latter relation dimensionally correct it is necessary to extend the list of governing parameters beyond the mean dissipation rate, $\langle\epsilon\rangle$ (e.g., Kuznetsov et al., 1992 and references therein).

For example, such a parameter may arise due to the lack of local isotropy discussed above, an effect which is not 'felt' at low-order scaling exponents. In such a case, an additional parameter may be the degree of anisotropy and/or some characteristic large scale, so that 'simple' dimensional analysis is not simple anymore. Another commonly believed alternative is that the observed deviation of the scaling exponents in relations of the type (5.5) for structure functions $S_p^{\|}(r)$ for $p > 3$ from the values implied by the Kolmogorov theory (i.e., 'anomalous scaling') is a manifestation of the so-called small-scale intermittency (in the *inertial range!*) which frequently is understood as synonymous with the 'anomalous scaling'[16] as displayed by (5.5). This is directly related to what is called inertial (sub)range (and how inertial it is) and attempts to formulate various 'explanations' such as the so-called multi-fractal formalism.

[16]This matter is addressed in chapter 7, which deals with some aspects of intermittency and the turbulence structure(s).

5.3.1. INERTIAL RANGE. IS IT A WELL-DEFINED CONCEPT?

> The second hypothesis of similarity. *If the moduli of the vectors $y^{(k)}$ and of their differences $y^{(k)} - y^{(k')}$ (where $k \neq k'$) are large in comparison with λ, then the distribution laws F_n are uniquely determined by the quantity $\bar{\epsilon}$ and do not depend on ν* (Kolmogorov, 1941a).
>
> *... the mechanism of turbulent energy transport is not affected by the viscosity ... the nonlinear terms are not affected by the viscosity* (Kovasznay, 1948).
>
> *The corresponding subranges of r and τ ... are usually referred to as the inertial subranges (since inertial forces play the main role as far as the energy balance of the corresponding disturbances is concerned)* (Monin and Yaglom, 1975, p. 351).
>
> *We therefore conclude that, for the large eddies which are the basis of any turbulent flow, the viscosity is unimportant and may be equated to zero, so that the motion of these eddies obeys Euler's equation ... we may say that none of the quantities pertaining to the eddies of sizes $r \gg \eta$ can depend on ν (more exactly, these quantities cannot be changed if ν varies but other conditions of the motion are unchanged)* (Landau and Lifshits, 1944, pp. 118–119 in English edition).
>
> *In the inertial range, the viscosity plays in principle no role (Ruelle, 1984).*
>
> *In the inertial range the velocity structure functions are Re-independent; that is, if the displacement r belongs to the interval $\eta \ll r \ll L$,* then *$S_{n,m}(r)$ do not involve any information about the dissipation scale* (Yakhot and Sreenivasan, 2005).
>
> *If the Reynolds number is large enough, turbulence is expected to exhibit scale invariance in an intermediate (inertial) range of wave numbers, as shown by power-law behaviour of the energy spectrum and also by a constant rate of energy transfer through wave number ... The inertial range of wave numbers is defined as being where the time derivative and the viscous term are negligible* (McComb, 2008).

The first citation above is with the original notations of Kolmogorov (the English text is from *Proc. Roy. Soc.*, **434**, p. 12). It is the formulation of the second hypothesis of similarity stating that the statistics of the velocity increments $\Delta u_i \equiv u_i(\mathbf{x} + \mathbf{r}q) - u_i(\mathbf{x})$ depends on the mean dissipation only and is independent of viscosity as long as the separation r is much larger than the Kolmogorov scale η. This is the original definition of inertial range (IR). Note the caveat by Monin and Yaglom *as far as the energy balance of the corresponding disturbances is concerned.* This is a hint that the inertial range makes sense for low-order statistics in which strong events do not seem to play any significant role. In this sense *for* (most of) *the large eddies… the viscosity is unimportant* (though they still do not obey the Euler equations!). However, there is a non-negligible number of eddies-outliers/very strong events (comprising a significant subset mainly of the tails of the PDF of $\Delta u_i(r)$ in the *nominally defined inertial range* $\eta \ll r \ll L$) for which viscosity/dissipation is of utmost importance (see below) at whatever large Reynolds number. In other words the inertial range is ill-defined in the sense that not all but almost all statistics of $\Delta u_i(r)$ is independent of viscosity. As long as one deals with low-order statistics of $\Delta u_i(r)$ (as Kolmogorov did) this is of little (but not always negligible) importance. However, these events contribute significantly to the higher-order structure functions and thereby a nonnegligible contribution to the higher-order structure functions is dominated by viscosity/dissipation. Thus it is meaningless to speak about inertial-range behavior of higher-order structure functions in contrast to the views that each structure function is characterized by its own dissipation scale (i.e., crossover scale from inertial to dissipative behavior) which is a decreasing function of the order of the structure function (Paladin and Vulpiani, 1987a,b; Frisch and Vergassola, 1991; Lvov and Procaccia, 1996; Fairhall et al., 1998; Yakhot, 2003). In other words, the 'anomalous scaling' as exhibited by the behaviour of higher-order structure functions is due to significant contribution of viscosity/dissipation in the *inertial range*. The higher the order of the structure function the stronger is the contribution due to viscosity (i.e., from the tails of the PDFs of $\Delta u_i(r)$) and the weaker is the 'inertial' contribution (i.e., from the core of the PDFs of $\Delta u_i(r)$) to the structure function. Our claim that it occurs at least at *any* $\eta \ll r \ll L$ at whatever high Reynolds number[17], so that it is not just (but also) a finite-Reynolds-number effect, as proposed in a number of papers (Barenblatt et al., 1999; Lundgren, 2008; Lvov and Procaccia, 1995; Nakano et al., 2003; Qian, 1998). This is due to the fact that dissipative 'events', the outliers, are important at any, however large, Reynolds numbers. The support for this view comes from a recent analysis of large Reynolds number data in field experiments (Kholmyansky and Tsinober, 2009 and references therein). Two examples are shown in figure 5.2.

The first example is with scaling exponents of structure functions $S_p^{||}(r)$ up to order 8 corresponding to the full data and the same data in which the strong dissipative events with different thresholds were removed (top-left). By an event we mean a velocity increment, $\Delta u_i \equiv u_i(x+r) - u_i(x)$. It is qualified as a strong dissipative event if at least at one of its ends

[17] In order to cope properly with the issue on what happens at scales of order η and smaller one needs data at sub-Kolmogorov resolution for large Reynolds numbers.

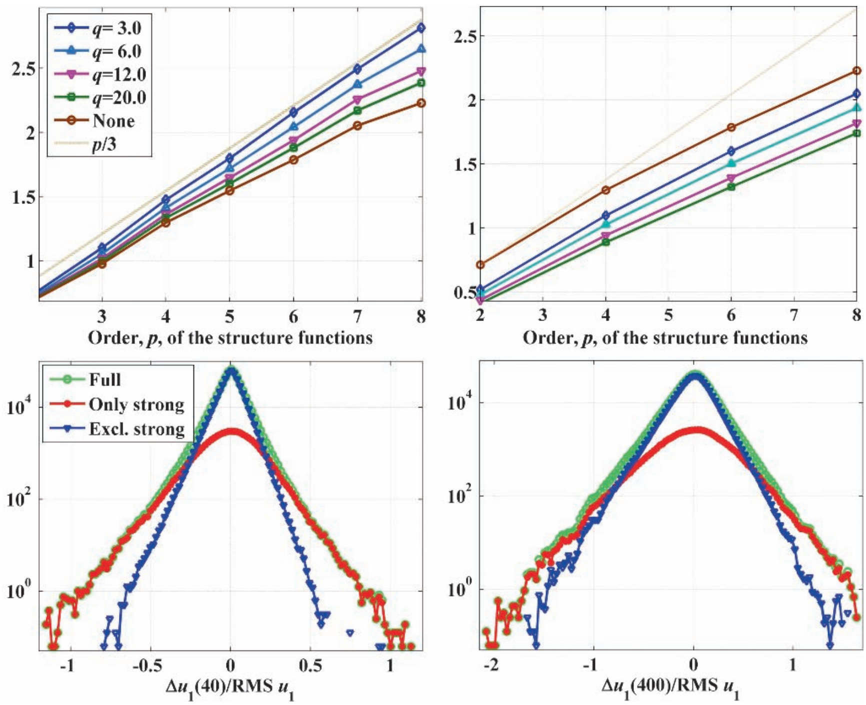

Figure 5.2. Top – exponents of structure functions for the longitudinal velocity component for the full data (left) and the same data in which the strong dissipative events with different thresholds were removed (right). Bottom – two examples of histograms of the increments of the longitudinal velocity component for the same data as above: $r/\eta = 40$ correspond to the lower edge of the inertial range (left) and $r/\eta = 400$ is deep in the inertial range (right). Kholmyansky and Tsinober (2009)

$(x, x + r)$ the instantaneous dissipation $\epsilon > k\langle\epsilon\rangle$ for $k > 1$. We have chosen $k = 3, 6, 12$ and 20. This corresponds to the instantaneous Kolmogorov-like scales 0.76, 0.64, $0, 54$, and 0.47 of the conventional Kolmogorov scale η based on the mean dissipation $\langle\epsilon\rangle$. It is seen that removal of the strong dissipative events results in an increase of the exponents ζ_p. For example, with removal of the dissipative events between the threshold $3\langle\epsilon\rangle$ (0.76η) and $6\langle\epsilon\rangle$ (0.64η) the dependence of ζ_p on p becomes pretty close to the Kolmogorov $p/3$. The strong events/outlilers themselves have different scaling properties (top-right). The second example shows that indeed removal of the strong dissipative events results in narrowing of the tails in the PDFs of $\Delta u_1(r)$. The effect of the removal of the dissipative events is obviously much stronger for higher- order structure functions. For example, there are only 5% of dissipative events for $k = 6$ sitting mostly at tails of the PDF of $\Delta u_i(r)$ for $r/\eta = 400$ (i.e., deep in the 'inertial' range), which contribute about 38% to the total dissipation. These events contribute about 45% to the value of $S_8^{\|}(r)$ at $\mathrm{Re}_\lambda \sim 10^4$. These same events change the $S_2^{\|}(r)$ by about 10%, but contribute about 7% to $S_3^{\|}(r)$ (see below). It is noteworthy that the data used in Kholmyansky and Tsinober, 2009 was somewhat spatially under-resolved, $1 \div 3\eta$. This means that the conclusions are to some extent qualitative. However, with properly resolved data the lost dissipative events in the under-resolved ones would enhance the tendencies just described above. Additional support comes from the fact that essentially the same results are obtained using the same data smoothed over $2 - 4\eta$. Finally, using enstrophy ω^2 and/or the surrogate $(\partial u_1/\partial x_1)^2$ as a criterion for the threshold instead of dissipation gives the same qualitative (but not quantitative) results. Indeed, essentially the same results are obtained using the same data smoothed over $2 - 4\eta$.

The above evidence (for more see Kholmyansky and Tsinober, 2009) and arguments indicate that the results and the suggestion by Schumacher et al. (2007) *that the asymptotic state of turbulence is attained for the velocity gradients at far lower Reynolds numbers than those required for the inertial range* are related to (and possibly can be explained by) the dominance of viscous effects as described above. This is qualitatively different from the common view/belief that the *nonlinear dependence of the algebraic scaling exponents ζ_n on the moment order n is a manifestation of the inertial-range intermittency,* see, e.g., page 2 in Schumacher et al. (2007). Similarly the possibility put forward by Schumacher et al. (2007) *that the magnitudes of inertial-range exponents* ($\mathrm{Re} \to \infty$) *are prescribed by the matching conditions on the ultra-violet cut-offs formed in the low-Reynolds-number regimes* follows directly from the above arguments[18].

Other effects which are expected to contribute to 'anomalous scaling' include nonlocality both of kinematical and dynamical nature. The former include the fact that the velocity field is defined by the fields of vorticity or strain, and there are purely kinematic correlations between large- and small-scale quantities. An example of the latter is the so-called Tennekes

[18] A related result of interest is by Kurien and Sreenivasan (2001a). They compared the relative scaling exponents (i.e., normalized on ζ_3) of structure functions up to the 10th order in high-Re_λ ($\sim$19,000) field experiment with those from DNS of NSE by Cao et al. (1996) at rather low $\mathrm{Re}_\lambda \sim 180$. Both sets appear to be practically identical both for *low-* and higher-order structure functions. It may be easy to understand the latter, but not so the former. The observed Re- independence (if true) of the scaling exponents of low-order SF and of higher-order SF is of different nature since the former are due to the inertial effects, whereas the latter are due to viscosity. So one has to understand why the higher-order SF do depend on viscosity in such a way so that their scaling is Re-independent. Solution of these and related issues requires information at high Reynolds numbers with sub-Kolmogorov resolution down to, say, 0.1η. There are other important reasons and motivation for much better resolution such as a variety of issues associated with the field of velocity derivatives, accelerations, modified helicity and other quantities possessing the property of being pointwise inviscid Lagrangian invariants.

and Lumley balance, see chapter 6. Again addressing such issues is a matter of far more precise and well-controlled experiments.

A special remark is about the contribution of the dissipative events as defined/described above to the low-order statistics and in particular to the 4/5 law. These events *do* contribute to the 4/5 law and removing them leads to an increase of the scaling exponent above unity, see figure 5.2 left. An important point here is that the neglected viscous term in the Karman–Howarth equation does not contain *all* the viscous contributions. Those which are present in the structure function S_3 itself remain and keep the 4/5 law precise. In this sense this law is not a pure inertial law.

It is the right place to mention that velocity increments (let alone structure functions and their scaling if such exists) are not the only objects of interest and *do not constitute a representation basis for a flow* (Goto and Kraichnan, 2004). This along with the ill-posedness of the inertial range as described above clearly shows that structure functions are not the best choice in several aspects/issues. This is especially acute in the issue of asymptotics at large Reynolds numbers.

5.3.2. ON THE MULTI-FRACTAL MODELS

> *One might hope, and even expect, that eventually a theoretical underpinning – like that of Kenneth Wilson's renormalization approach – will be developed to anchor this subject. Without that underpinning much of the work on fractals seems somewhat superficial and even slightly pointless. It is easy, too easy, to perform computer simulations upon all kinds of models and to compare the results with each other and with real-world outcomes. But without organizing principles, the field tends to decay into a zoology of interesting specimens and facile classifications. Despite the beauty and elegance of the phenomenological observations upon which the field is based, the physics of fractals is, in many ways, a subject waiting to be born.* (Kadanoff, 1986).
>
> *The multi-fractal model describes intermittency that increases with decrease of scale size. Multi-fractal models of turbulence have not been derived from the NS equation, but they are supported by theoretical arguments and their parameters can be tuned to agree well with a variety of experimental measurements … Multi-fractal models may or may not express well the cascade physics at large but finite Reynolds numbers.* (Goto and Kraichnan, 2004).

The Navier–Stokes equations obey the scaling transformation for $h = -1$ only (see appendix 3 (C.29)). However, it is a common belief that it *may be justified at very high Reynolds number … that there are infinitely many scaling groups, labelled by their scaling exponent h, which can be* *any real number*, i.e., *in the inviscid limit, the Navier–Stokes equation is invariant under infinitely many scaling groups, labelled by an arbitrary real scaling exponent h* (Frisch, 1995, p. 18, 144) just as in the case of the Euler equation. Along with the experimental results shown in figure 5.2 the above is one of the main premises for introducing the multi-fractal phenomenological model(s) (MFM)[19]. There are three problematic points of conceptual nature.

First, it is not at all clear why one can ignore the singular nature of the limit $Re \to \infty$ ($\nu \to 0$) when handling the issue of scaling exponents and/or related matters[20]. In fact there are two singular limits: one is the dissipation scale $\to 0$ (i.e., $\nu \to 0$) and the other some outer scale (e.g., forcing scale) $L \to \infty$ (Lvov et al., 1996).

Second, even if(!) *the Navier–Stokes equation is invariant under infinitely many scaling groups, labelled by an arbitrary real scaling exponent* h in the limit $Re \to \infty$ ($\nu \to 0$), the very existence of scaling exponents in a *statistical sense* (as, e.g., for various structure functions or corresponding PDFs, etc.) which is taken for granted, is a problem by itself. Namely, the existence of the scaling symmetry as any other symmetries of the Euler,

[19]The main ingredient of this model is the $\mathbf{H_{mf}}$ hypothesis replacing the Kolmogorov second similarity hypothesis (Frisch, 1995, p. 144):

$\mathbf{H_{mf}}$: Under the same assumptions as in H_1 the turbulent flow is assumed to possess a range of scaling exponents $I = (h_{\min}, h_{\max})$. For each h in this range, there is a set $\mathcal{J}_h \subset \mathbb{R}^3$ of Hausdorff dimension $D(h)$ such that as $\ell \to 0$,

$$\frac{\delta v_\ell(r)}{v_0} \sim \left(\frac{\ell}{\ell_0}\right), \; r \in \mathcal{J}_h, \tag{8.39}$$

For more details, references and other versions see Frisch, 1995, chapter 8. The main feature of the multi-fractal hypothesis/model is that here its authors are back with universality, postulating a whole range of exponents and a function. Unfortunately, in contrast with K41, the multi-fractal model (like many other intermittency models) is an arbitrary construction in the sense that it lacks dynamical motivations in general and with respect to the postulated multi-fractal universality in particular. Having that much assumed (i.e., a whole range of exponents and a function) it is really easy to fit to this frame almost any experimental/DNS data and whatever. In other words, though multi-fractality was designed to 'explain' the anomalous scaling, intermittency, etc. it is in fact a very intelligent fitting.

There are claims of using the first principles in obtaining the multi-scaling *à la* MFM (Belinicher et al., 1998; Yakhot and Sreenivasan, 2004 and references therein), which is true in part as all of them use *closures.*

[20]It is important to stress that we deal, in fact, with a 'pseudo-limit' $Re \gg 1$ ($\nu \ll 1$). This can make a qualitative difference for at least two reasons. First, as mentioned, the true limit $Re \to \infty$ ($\nu \to 0$) is singular, so one cannot exclude the possibility that what is observed at whatever large but finite Re can be very much different from what happens in the limit $Re \to \infty$ ($\nu \to 0$) if there is a way to define this limit properly. Second, there is a consensus (but still only a belief) that solutions of NSE at *any* finite Re are not singular, for a discussion of these and similar issues, see chapter 9. In other words it is not clear at all why one should forget that at any finite Re there is no such freedom as there is for a pure Euler case.

and *presumably* of the Navier–Stokes, equations at very large Reynolds numbers does not guarantee that various *statistical* characteristics should also possess such symmetries. Their restoring in the statistical sense is only a hypothesis. The experimental support of the existence of scaling ranges from the evidence available is rather weak, sometimes marginal, though there are continuing efforts on determination of varieties of scaling exponents (see references in Biferale and Procaccia, 2005). There are many difficulties in determining inertial ranges and the corresponding scaling exponents (see, for example, figures 6, 10 and 11 in Anselmet et al., 1984 and figure 8.6 in Frisch, 1995; also, Praskovsky and Onsely, 1997; Praskovsky, 1998; Sreenivasan and Dhruva, 1998 and references therein). As mentioned above, attempting to define the scaling range in a more precise manner via local slopes and/or compensated plots causes this range to become much shorter and in some experiments even to disappear. In a work based on the experimental data of Gagne (1987) and Malecot (1998), Arneodo et al. (1999) concluded that *because of the scale invariance breaking, the notion of inertial range is not well defined*[21]. Third, there is one more difficulty associated with the empirical basis of the MFM as follows. As mentioned above one of the consequences of the multi-fractal model is that each structure function is characterized by its own dissipation scale (i.e., crossover scale from inertial to dissipative behavior) which is a *decreasing* function of the order of the structure function (Paladin and Vulpiani, 1987a,b; Frisch and Vergassola, 1991; Lvov and Procaccia, 1996; Fairhall et al., 1998; Yakhot, 2003) and which becomes much smaller than the 'conventional' Kolmogorov scale for higher-order structure functions. That is according to MFM in order to adequately compute/measure the higher-order structure functions it is necessary to have resolution much finer than η. However, all the experimental results, which served as a basis for the MFM and similar models, were obtained with resolution at best of the order (typically a bit exceeding) of the Kolmogorov scale η. This means that, in fact, there is no direct experimental evidence on the multi-fractal structure of turbulent flows, as

[21] A more general difficulty in the experimental context is associated with the direct large/small-scale coupling, see chapter 6. As the Reynolds number increases the characteristic scale between (!) the regions of strong small-scale activity ('intermittency') is increasing, whereas the characteristic scale of these regions themselves is decreasing. On the other hand when looking at higher- order structure functions $S_p \sim r^{\zeta_p}$ one is forced to use very long time records to achieve statistical convergence. This means that very large scales (which are anisotropic) are involved in the process. In other words, as the Reynolds number and the order of the structure function increase so (most likely) does the 'deviation' from $p/3$. This means that experimentally (!) it may be not realistic to get a reasonable and reliable estimate of the exponents ζ_p if such exist. This becomes less realistic as the order of the structure function and the Reynolds number are increasing. All the above provided that the experimental errors are small. Here again the larger the order of the structure function the larger is the influence of the errors.

by the above argument these experiments cannot accurately predict the properties of the violent structures of turbulence, i.e., the anomalous scaling of higher-order SF. The bottom line is that references on the existing experimental results as a basis for MFM and similar models (see Eyink, 2008; Frisch, 1995; Yakhot and Sreenivasan, 2005 and references therein) are hardly justified: as mentioned, one needs far better resolution than η *at high Reynolds numbers* (!). This does not mean, however, that such experiments will provide better basis for the MFM in view of ill-posedness of the concept of the inertial range at the outset as described above.

5.4. Cascade

> *One gets an impression of little, randomly structured and distributed whirls in the fluid, with the cascade process consisting of the fission of the whirls into smaller ones, after the fashion of the Richardson poem. This picture seems to be drastically in conflict with what can be inferred about the qualitative structure of high-Reynolds-number turbulence from laboratory visualization techniques and from plausible application of Kelvin circulation theorem* (Kraichan, 1974).
> ... *the idea of conservative inertial cascade local in scale size is consistent prima facie, provided that the actual statistics do not differ strongly from Gaussian.* (Kraichan, 1991)
> *The notion that turbulent flows are hierarchical and involve entities ... of varying sizes is a common idea ... This common notion underlies the concept of cascade, the third key element of turbulence theory* (Frisch and Orszag, 1990).
> *All this cascade in Fourier space is a dream of linearized physicists.* (Betchov, 1993).
>
> *The tree examples* [jet, boundary layer, and wake] ... *show that there is something wrong with this idea (the Richardson poem). In each case turbulence begins at small scales and grows larger: not the other way around* (Gibson, 1996).
> *The conceptual picture is that of a cascade organized by wall distance and by eddy size, where energy is transferred to smaller scales at any given location, and to larger ones away from the wall* (Jimenez, 1999).
> *This suggests that the Kolmogorov cascade process is basically incorrect, albeit an excellent approximation* (Shen and Warhaft, 2000).

5.4.1. INTRODUCTION

The cascade[22] picture of turbulent flows takes its origin from Richardson (1922, p. 66): ... *we find that convectional motions are hindered by the formation of small eddies resembling those due to dynamical instability. Thus C.K.M. Douglas writing of observations from aeroplanes remarks: "The upward currents of large cumuli give rise to much turbulence within, below, and around the clouds, and the structure of the clouds is often very complex". One gets a similar impression when making a drawing of a rising cumulus from a fixed point; the details change before the sketch is completed. We realize thus that: big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity—in the molecular sense*[23].

The last sentence became very popular, though Richardson makes no further use of this 'cascade' picture at any time. Instead Richardson proposes to use the eddy viscosity approach, since he realized the inherent difficulty of the most problematic issue of decomposition of a turbulent field (see chapter 3): *Thus, because it is not possible to separate eddies into clearly defined classes according to the source of their energy ... therefore a single coefficient is used to represent the effect produced by eddies of all sizes and descriptions. We have then to study the variations of this coefficient.*

As mentioned above, the cascade picture supplemented by the assumption about the chaotic nature of cascade was used by Kolmogorov (1941a) in a 2/3 page footnote as a *qualitative* justification of his hypothesis on the local isotropy of turbulent flows for very large Reynolds numbers.

The cascade picture is based on the intuitive notion that turbulent flows possess a hierarchical structure consisting of 'eddies' (Richardson's 'whirls', Kolmogorov's 'pulsations', etc.) as a result of successive instabilities. The essence of this picture is in its successive hierarchical process, and in this sense it is the same as the Landau–Hopf picture of transition to turbulence as a 'cascade' of successive instabilities. The difference is that the Richardson-Kolmogorov cascade refers to a process at some fixed Reynolds number, whereas the Landau–Hopf picture describes the process of changes occurring as the Reynolds number increases (see chapter 2).

The cascade picture of turbulence is more a reflection of the hierarchic structure of various *models* of turbulent flows rather than reality. Most of these models have no connection with Navier–Stokes equations (e.g., see

[22]The term 'cascade' comes from Onsager (1945, 1949).

[23]Note that this observation was made by looking at the structure of clouds, i.e., condensed water vapour, at the interface between laminar and turbulent flows in their bulk, which do not necessarily reflect the structure of the underlying velocity field; see chapter 4.

Jimenez, 2000 and references therein). Hence the term 'phenomenological' models.

5.4.2. IS THERE CASCADE IN PHYSICAL SPACE?

The Richardson–Kolmogorov cascade picture was formulated in *physical* space and is used frequently without much distinction both in physical and Fourier space, as well as some others. However, it was Neumann (1949) (see also Onsager, 1949) who recognized that this process occurs not in physical space, but in Fourier space: ... *the system is "open" at both ends, energy is being supplied as well dissipated. The two "ends" do not, however, lie in ordinary space, but in its Fourier-transform. More specifically: The supply of energy occurs at the macroscopic end—it originates in the forced motions of macroscopic (bounding) bodies, or in the forced maintenance of (again macroscopic) pressure gradients. The dissipation, on the other hand, occurs mainly at the microscopic end, since it is ultimately due to molecular friction, and this is most effective in flow-patterns with high velocity gradients, that is, in small eddies ... Thus the statistical aspect of turbulence is essentially that of transport phenomenon (of energy)—transport in the Fourier-transform space.* That is, the nonlinear term in the Navier–Stokes equation redistributes energy among the *Fourier modes*[24] not scales as is frequently claimed, unless the 'scale' is defined just as an inverse of the magnitude of the wave-number of a Fourier mode, which is not easy for everybody to swallow. A natural question is then what does the nonlinear term in physical space do? Is energy transferred from large to small scales in physical space? The answer to the last question depends on the definition of what is a 'scale' in physical space. First, we recall that here is no contribution from the nonlinear term in the total energy balance equation (and in a homogeneous/periodic flow it's contribution is null in both the total and the mean), since the nonlinear term in the energy equation (C.8) has the form of a spatial flux, $\partial\{\ \dots\ \}/\partial x_j$. In other words the nonlinear term redistributes the energy in physical space, but does it do more than that? It is straightforward to see that in a statistically homogeneous turbulent flow the mean energy of volume of *any scale* (Lagrangian and/or Eulerian)[25] is changing due to external forcing and dissipation only – there is no contribution in the mean of the nonlinear term, which includes the term with the pressure. That is, if one chooses to define a 'scale', l, in physical space as a fluid (or a fixed) volume, say, of order $\sim l^3$, then in a statistically homogeneous flow there is no cascade in physical space in the sense that, in the

[24]See section 2.4 and 6.2.4 in Frisch (1995) for a demonstration of this process.

[25]Following a Lagrangian volume for a reasonable time in turbulent flows is not a simple matter, since Lagrangian volumes 'lose their identity' very fast in turbulent flows.

mean, there is no energy exchange between different scales. This happens because the nonlinear term in the energy equation (C.8) has the form of a spatial flux, $\partial\{\ \ldots\ \}/\partial x_j$, i.e., there is conservation of energy by non linear terms. In other words, the nonlinear term redistributes the energy in physical space if the flow is statistically nonhomogeneous. So, generally, it is a misconception to interpret this or any other process involving spatial fluxes, $\partial\{\ \ldots\ \}/\partial x_j$ (e.g., momentum flux), as a 'cascade' in physical space[26].

On the other hand, in a statistically stationary state $\int \mathbf{F}\cdot\mathbf{u}d\tau = \int \epsilon d\tau$, i.e., energy input, which is associated with large scales, equals dissipation, which occurs mostly in the small scales. Is there a contradiction? Does the equality $\int \mathbf{F}\cdot\mathbf{u}d\tau = \int \epsilon d\tau$ (or similarly, see appendix C, the text following the equation (C.49)) mean that the energy should be somehow 'transferred' from large to small scales via some multi-step process? Not necessarily – for example, two big neighboring eddies can dissipate energy directly through encounters with each other at small scale – much smaller than their own scales. Such a process still will look in Fourier space as continuous energy transfer from modes with small to modes with large wave numbers. The resolution of the apparent contradiction lies in clarifying the meaning of the term 'scale', which mostly is understood as an inverse of the wave-number magnitude in a Fourier representation, and what is the meaning of 'transfer of energy (or whatever) from large to small scales' in physical space. This issue is directly related to the decomposition/representation of the turbulent flow field. Indeed, the reason for the above result on the absence of energy exchange between different scales in physical space is because no decomposition is involved in the above 'definition of scale'.

Any decomposition (be it in physical space, Fourier or any other) brings the 'cascade' back to life. For example, there are various ways of filtering the flow field widely used in large eddy simulations. However, one of the problems with decompositions is that the nonlinear term redistributes the energy among the components of a particular decomposition in a *different* way for *different* decompositions, i.e., the energy exchange/transfer is decomposition dependent[27]. Therefore one may ask whether quantities like energy flux are well defined. In other words the term 'cascade' corresponds to a process of interaction/exchange of (not necessarily only) energy between components of some *particular* decomposition/representation of a turbulent field associated with the nonlinearity and the nonlocality of the turbulence phenomenon, two of the three N's: **n**onlinearity, **n**onlocality and **n**onintegrability, which make the problem so impossibly difficult (chapter 1).

[26]See Appendix D for a collection of some other major misconceptions.

[27]For instance, Fourier, wavelets, POD, filtering and so on: Frick and Zimin (1993); Germano (1999); Holmes et al.(1996); Mahrt and Howell (1994); Meneveau (1991); Sirovich (1997); Borue and Orszag (1998).

On the other hand, energy transfer, just like any physical process, should be invariant of particular decompositions/representations of a turbulent field. In this sense Kolmogorov's choice of dissipation (and energy input) are well defined and decomposition independent quantities, whereas the energy flux is (generally) not, since it is decomposition dependent. After all Nature may and likely does not know about *our* decompositions.

It is noteworthy that the 'cascade' arising from a decomposition of the flow field viewed as a process of exchange of energy, momentum, etc. between the components of this decomposition is a dynamical process. This should be distinguished from 'cascading processes' resulting from a decomposition of some quantity, e.g., dissipation, usually of its surrogate $(\partial u_1/\partial x_1)^2$, obtained from experimental signals (for recent examples see Frederiksen et al., 1998; Arneodo et al., 1999; Renner et al., 2001; Chen et al., 2003a; Cleve, et al., 2004; Davidson and Pearson, 2005 and references therein). The former is a *dynamical process*, whereas the latter is a *representation* characterizing some aspects of the *spatial and/or temporal structure of some flow characteristics.* In other words, 'structure' is not synonymous with 'process': it is the result of a process. Therefore, generally it is impossible to draw conclusions about the former from information about the latter, though this is done quite frequently. For example, simple chaotic systems with few degrees of freedom only (e.g., three as in the Lorenz (1963) system or four in the forced spherical pendulum, Miles (1984), also Mullin (1993)), produce also 'fine structure', e.g., continuous spectrum, but there is no 'cascade' whatsoever, though, of course, the signal with the continuous spectrum can be cast in a multiplicative representation.

Another example, is the complicated structure of a passive object[28] arising in a simple fluid flow via a single instability only(!) (Ott, 1999). This is true also of the vorticity field resulting from a linear instability of such a flow (see also chapter 4). One more example is represented by the phenomenon of entrainment in a broad class of partly-turbulent flows (see chapter 9) and transitional regimes such as pipe flows and turbulent spots. In all of these the characteristic feature is the abrupt transition of fluid particles from the laminar to turbulent state when passing across the laminar/turbulent 'interface' in 'no time' (on a time scale of the order of Kolmogorov scale) again without any 'cascade'. This is an indication that (just as in Eulerian representation) one can hardly speak about the Lagrangian nature of the cascade as the fluid particles are purely Lagrangian objects, see also section 3.6.

[28]Such structure with power law spectrum, (multi-)fractality and significant variations down to very small scale can be produced by a single instability at much larger scale without any 'cascade' of successive instabilities.

5.4.3. WHAT ARE THE 'SMALL SCALES' IN TURBULENT FLOWS?

We have seen that there is an ambiguity in defining the meaning of the term 'small scales' (or more generally 'scales' or 'eddies', see appendix C) and consequently the meaning of the term 'cascade'. The specific meaning of this term and associated inter-scale energy exchange/'cascade' (e.g., spectral energy transfer) is essentially decomposition/representation dependent[29]. Perhaps, the only common thing in all decompositions/representations (D/R) is that the small scales are *always* associated with the field of velocity derivatives[30]. Therefore, it is natural to look at this field as the one objectively (i.e., D/R independent) representing the small scales. Indeed, the dissipation is associated precisely with the symmetric part of the velocity derivative tensor $\partial u_i/\partial x_j$ – the rate of strain field s_{ij} both in Newtonian and non-Newtonian fluids, whereas vorticity $\omega_i = \epsilon_{ijk}\partial u_j/\partial x_k$ is, in fact, its anti-symmetric part. Before proceeding with the small scales let us mention that the large scales are naturally characterized by the velocity field itself, $\mathbf{u}$. This is justified also by the fact that sustaining turbulent flows requires energy input into the flow, e.g., in case of a prescribed force, $\mathbf{F}$, the power input is associated with this force is $\int \mathbf{F} \cdot \mathbf{u} dV$, i.e., with the velocity field, $\mathbf{u}$.

The advantage of the above 'definition' of small scales can be seen from the following.

While the mean contribution of the nonlinear term in the energy balance is vanishing, the nonlinearity definitely is producing vorticity and strain in physical space, since the mean enstrophy and strain production are strictly positive[31]. As mentioned above it is natural and justified from the physical point of view to associate the field of velocity derivatives with small scales. It is immediately seen that 3-D turbulent flows have a natural tendency to create small scales[32]. Namely, the velocity field (and its energy) arising in the process of (self-) production of the field of velocity derivatives is the one which is associated with small scales. This process is what can be called as

[29]Indeed, the meaning of (*small*) *scales* is different for different representations: it is not the same for *Fourier* ('regular' and helical) and similar (Littlewood–Paley), *Wavelets* (wavepackets, solitons), *POD, LES*. It is also different in various heuristic representations, e.g., 'two-fluid' (Organized/Incoherent, Deterministic/Random and some other two-fluid models); intermittency-prompted (breakdown coefficients/multipliers, fractals); Moffat's 'smart decomposition', Moffatt (1990b) and the 'punctuated' conservative dynamics.

[30]Differentiation is a kind of high-pass filtering, so one can use also higher-order derivatives, especially those appearing in the NSE and their consequences, such as velocity Laplacian, etc.

[31]This question involving also the relation between 'cascade' and the process of the self-production of the field of velocity derivatives is addressed in more detail in chapters 6 and 7, see also appendix C.

[32]This is why the cascade is usually associated with vortex stretching and enstrophy production. We shall see in chapter 6 that this is far from being a precise statement.

energy (and not only energy) transfer from large to small scales in physical space[33]. The latter are not necessarily created via a stepwise turbulent 'cascade': it can be bypassed, and most probably is so in turbulent flows, for example via broad-band instabilities with highest growth rate at short wavelengths (Pierrehumbert and Widnall, 1982; Smith and Wei, 1994) or some other approximately single-step process (Betchov, 1976; Douady et al., 1991; Ott, 1999; Shen and Warhaft, 2000; Vincent and Meneguzzi, 1994). The problem goes back to Townsend (1951): ... *the postulated process differs from the ordinary type of turbulent energy transfer being fundamentally a single process* (see also Corrsin, 1962a and Tennekes, 1968). Indeed, as mentioned two large neighboring eddies can dissipate energy directly by encountering each other on a very small scale. Such a view goes back to the observation made by Batchelor and Townsend in 1949: *the mean separation of the visible activated regions is comparable with the integral scale of the turbulence, i.e., with the size of the energy-containing eddies* (p. 253)[34].

The process of vorticity and strain production is not just creation of the field of velocity derivatives. It literally involves creation of small-scale *structure* in the following sense. Namely, an inevitable concomitant process to vortex stretching is tilting and folding of vorticity due to the energy constraint (Orszag, 1977; Chorin, 1982; Tsinober, 1998a). Simultaneously, the strain field is built up at the same rate too (see chapter 6). This together with limitations on the volume scale leads to formation of fine small-scale structure.

The above 'definition' of small scales has a variety of consequences. For example, since the whole flow field (including velocity, which is mostly a large-scale object) is determined entirely by the field of vorticity, i.e., the velocity field is a functional of vorticity $\mathbf{u} = F\{\omega(\mathbf{x}, t)\}$[35], the production of vorticity 'reacts back' in creating the corresponding velocity field[36], i.e., the small scales are not just 'swept' by the large ones. Similarly, the velocity field is a functional of the strain tensor, $\mathbf{u} = G\{s_{ij}(\mathbf{x}, t)\}$[37], so that

[33]We still insist on the above 'definition' of small and large scales and the processes associated with the creation of the small scales as given in the first edition of this book, Tsinober (2001, pp. 77–80). A very similar (but not identical) 'definition' of both was given by Sagaut and Cambon (2008, pp. 110 and 112).

[34]Note also the quite convincing 'anti-cascade' arguments by Chorin (1994, p. 56) in Fourier space. He makes an attempt to 'rescue' the cascade picture, but the reader is invited to see whether the arguments given really help.

[35]This is seen from the equation $\nabla^2\mathbf{u} = -curl\omega$, which together with boundary conditions defines uniquely the velocity field. In case of an infinite flow domain this results in the Biot–Savart relation $\mathbf{u} = 1/(4\pi)\int(\omega\times\mathbf{r})/r^3 dr$ (see appendix C).

[36]This essential process is fully ignored in (quasi-) linear approaches like rapid distortion theory (RDT) (Savill, 1987; Hunt and Carruthers, 1990).

[37]This is seen from the equation $\nabla^2 u = 2\partial s_{ik}/\partial x_k$, which together with boundary conditions defines uniquely the velocity field, see equation (C.14′) in appendix C. Of course, the vorticity and strain are not independent and are functionals of each other.

production of strain 'reacts back' on the velocity field as well. Therefore from the physical point of view it seems incorrect to treat the small scales as a kind of passive objects swept by the large scales or just 'slaved' to them (Novikov, 2000). Similarly it seems impossible to 'eliminate' the small scales (as is done in many theories) reducing their reaction back to some eddy viscosity or similar things only. It is noteworthy that due to nonlocality of the relations $\mathbf{u} = F\{\omega(\mathbf{x}, t)\}$, $\mathbf{u} = G\{s_{ij}(\mathbf{x}, t)\}$ mostly small-scale vorticity and strain are, generally, creating also some large-scale velocity. This and other aspects of nonlocality (see chapters 1 and 6) contradict the idea of cascade in physical space, which is local by definition (e.g., see Frisch, 1995, p. 104). In particular, the frequently assumed statistical independence of large scales, such as structure functions $S_p^{\|}(r) = \langle(\Delta u_{\|})^p\rangle$, $\Delta u_{\|} \equiv [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})]\cdot\mathbf{r}/r$, in the inertial range on the (nature) of dissipation, i.e., strain, stands in contradiction with the relation $\mathbf{u} = G\{s_{ij}(\mathbf{x}, t)\}$ together with the process of self-production of strain in turbulent flows, see chapter 6.

In view of the above arguments it seems that in physical space the energy is dissipated not necessarily via a multi-step cascade-like process[38]. Instead, there is an exchange of energy (and everything else) in both directions, whereas the dissipation occurs in 'small scales'.

Therefore, it is quite possible and quite plausible that, insofar as physical space is concerned, Richardson's famous verse mentioned above (1922, p. 66) on the hierarchy of 'whirls'

Big whirls have little whirls,
Which feed on their velocity.
And little whirls have lesser whirls
And so on to viscosity –
In the molecular sense

should be replaced by Betchov's (1976, p. 845)

Big whirls lack smaller whirls,
To feed on their velocity.
They crash and form the finest curls
Permitted by viscosity.

The notion of cascade is still widely used (see e.g., Eyink, 2008 and references therein), but some previous users admit that *however attractive the*

See chapter 6, section 6.6 on nonlocality and appendix C, equations (C.15) and the following text.

[38] As mentioned, linear stability analysis of a vorticity field in a smooth velocity field has shown that a power law and fractality are produced by a single instability. In other words, significant variations down to very small scale can be produced by a single instability at much larger scale without any 'cascade' of successive instabilities (Ott, 1999). This shows also that nonlinearity in the Lagrangian representation cannot be interpreted in terms of some cascade.

notion of energy cascades, though, it must be taken mostly as pedagogical imagery (Falkovich and Sreenivasan, 2006); and that… *the small-scale dynamics are strongly coupled to the large-scale phenomena. This may be a reason for a serious reexamination of the very concept of the turbulence energy cascade which, within the framework of the present development, seem neither possible nor needed.* (Yakhot, 2006), but see Falkovich (2009).

5.4.4. IS CASCADE LAGRANGIAN OR EULERIAN, IN SOME DECOMPOSITION, PHASE SPACE OR WHATEVER? CASCADE OF PASSIVE OBJECTS?

As mentioned in chapter 3, in the Lagrangian description the fluid particle acceleration is linear in the fluid particle displacement (see equation C.62) and the 'inertial' effects are manifested only by the term containing pressure – there are no terms like the advective terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in a pure Eulerian setting $(\mathbf{u} \cdot \nabla)\mathbf{u}$. Therefore, the nonlinearity in the Lagrangian representation cannot be interpreted in terms of some cascade (as it cannot be maintained by pressure gradient alone) and it is far less clear (if at all) how one can employ decompositions even at the problematic level as done in pure Eulerian setting.

It is natural to look at a 'cascade' of a passive object in the pure Lagrangian setting. However, here too there is no advective term like $(\mathbf{u} \cdot \nabla)C$ in the pure Eulerian setting, so it is not clear how meaningful is 'cascade' of a passive object in Lagrangian setting as well, though some proposals are reviewed in Falkovich (2009).

On conceptual level there are problems in the Eulerian setting as well. It is rather common, since Obukhov (1949) and Corrsin (1951), to speak about cascade in case of a passive scalar and more recently passive vector [39]. The main argument is from *some* analogy. Indeed, as mentioned in chapter 4, for instance in any random isotropic flow the rate of production of 'dissipation' (i.e., corresponding field of derivatives) of both passive scalars and passive vectors is essentially positive (see equations C.35, C.38 in appendix C), which can be interpreted as a sort of 'cascade'. However, the equations describing the behaviour of passive objects are *linear.* Hence, there is no interaction between modes of whatever decomposition of the field of a passive object: the principle of superposition is valid in case of passive objects[40]. Therefore, it seems more appropriate to describe the process in

[39] See references in, e.g., Warhaft (2000); Falkovich et al. (2001).

[40] Here by 'mode' is meant as a solution of the appropriate equation, e.g., of the advection-diffusion equation (C.30). Of course, there are many ways to use 'modes' that are *not* solutions of this equation, such as Fourier modes. In this case the Fourier modes do interact, since one of the coefficients of the advection-diffusion equation, the velocity field, is not constant. This interaction is interpreted frequently as a 'cascade' of passive objects. But, as mentioned, this interaction is decomposition dependent, and therefore is

terms of production of the field of derivatives of the passive object, which is performed by the *velocity* straining field, just as it is proposed above for the velocity field. Hence the extension of Kolmogorov arguments and phenomenology to passive objects seems to be much less justified (if at all)[41]. No wonder that the phenomenological paradigms for the velocity field failed in most cases when applied to passive objects[42]. We are reminded that the 'analogy' between the passive objects and the active variables is, at best, very limited for several reasons, the main ones of which are the linear nature of 'passive' turbulence, Lagrangian chaos, the 'irreversible' effect of the randomness of the velocity field on passive objects independent of the nature of this randomness, e.g., even a Gaussian one, and the one-way interaction between the velocity field and the field of a passive object (see chapters 4, 6 [e.g., section 6.8 p. 194], 7 and 9).

5.4.5. ON 'ANOMALOUS SCALING' FOR PASSIVE OBJECTS AND RELATED ISSUES

Passive objects exhibit also 'anomalous scaling', see e.g., figure 7.1 and numerous references in Falkovich et al., 2001; see also Falkovich, 2009. As in the case of velocity field the experimental evidence is obtained at finite Reynolds numbers, i.e., for *smooth velocity fields*. On the other hand the popular explanation of the anomalous scaling of passive objects is based on the existence of the so-called 'zero modes' and 'statistical conservation laws' so that *anomalous scaling arises only from nonsmooth velocity fields with power-law correlations in space ... in the inertial range* (Falkovich and Sreenivasan, 2006; see also Falkovich et al., 2001). That is, the question is again how justified/relevant is the above explanation of the observations. The problem with the *zero modes* becomes more serious and problematic as the behavior of structure functions for temperature becomes less anomalous with employing the same procedure (i.e., removing strongly dissipative

not appropriate for description of physical processes, which are invariant of our decompositions. There is a point concerning the behavior of an individual solution. Namely, the evolution of its energy spectrum is expected to exhibit positive energy transfer to higher wave numbers as a consequence of production of the field of derivatives of the passive field. Can one see this as a kind of 'cascade'? Even if the answer were affirmative it is a very different kind of cascade, if at all.

[41]As mentioned, this analogy was initiated by Obukhov (1949) and Corrsin (1951). It is noteworthy that the dimensional reasoning in this case is far 'less clean' as there are more parameters than in the case of a velocity field. One has additional dissipation and diffusivity, so that some saving tricks are necessary to arrive at 'analogous' results, see, e.g., Monin and Yaglom (1975, pp. 377–387); Tennekes and Lumley (1972, pp. 281–286).

[42]For example, experiments by Villermaux et al. (2001) clearly show that this is the case. The behaviour of passive scalar in their experiments is distinctly nonlocal in the sense that the main mechanism responsible for mixing involves direct interaction between large and small scales 'bypassing' the (nonexistent) cascade.

events) as described for velocity increments in section 5.3, Kholmyansky and Tsinober, 2009. Among the consequences is also ill-posedness of the 'inertial range' of a passive scalar.

A similar question concerns things like "spontaneous stochasticity" and "breakdown of Lagrangian flow" (Bernard et al., 1998; for other references and discussion see Falkovich et al., 2001 and Eyink, 2008) which arise in some pretty freely manipulated limits such as Re $\to \infty$, if there is a way to define the singular limit properly[43]. In particular, non-uniqueness and stochasticity of Lagrangian trajectories arise for an individual velocity field realization which is assumed to be rough in a manner convenient for analytical treatment (and in very special unphysical settings), but not necessarily physically meaningful and/or corresponding to the behaviour of velocity field at very large Reynolds numbers, which is smooth. Indeed, there is a consensus (but still only a belief) that solutions of NSE at *any finite* Re are not singular. Using rough velocity fields is a nice mathematical exercise, but it is far from clear how (if at all) it is relevant to observations in real turbulent flows at whatever large but finite Reynolds numbers. With rough velocity fields – as used in the above mentioned papers – the uniqueness of the solutions of equation (C.64) is violated. Thus using rough velocity fields does not seem to be justified. It is obscuring the physics rather than clarifying it. Our main premise is that the flow field is smooth. In such flows "phenomena" like "spontaneous stochasticity" and "breakdown of Lagrangian flow" do not arise and one has to look at different more realistic possibilities. The simple alternative given above in the context of velocity structure functions seems to be valid here too.

A final remark is in the form of a question: what about "cascade" in Lagrangian chaotic/Eulerian Laminar flows?

5.5. Summary

Kolmogorov *a priori* phenomenological hypotheses include the hypothesis of local isotropy and two similarity hypotheses. The hypothesis of local isotropy states that at large Reynolds numbers all the symmetries of

[43] The claim in Bernard et al. (1998) is that *the nondeterministic behavior of Lagrangian trajectories at high Reynolds number* (is) *caused by the sensitive dependence on initial conditions within the viscous range where the velocity fields are more regular.* That is, if one stays with finite, however large, Re and looks at the **whole** range of scales (not just the inertial range in the "limit $\nu \to 0$") there is no problem with *the nondeterministic behavior of Lagrangian trajectories at high Reynolds number* or *spontaneous stochasticity and breakdown of Lagrangian flow.* This is related to an important conceptual issue on whether the inertial range is a well-defined concept as discussed above. It is a good notion for some practical treatment especially of experimental data, but it is doubtful that it can be employed in clean theoretical approaches, especially those claiming mathematical rigor.

the Navier–Stokes are restored in the statistical sense (more precisely all the statistical properties of velocity increments) locally in time and space in regions far enough from the boundaries of any turbulent flow or its other special regions. That is, high-Reynolds-number turbulent flows are locally homogeneous, isotropic and stationary. It appears that turbulent shear flows do not conform with this hypothesis, so that regions with mean shear should be considered as 'special' in the sense of Kolmogorov. However, the problem seems to be more serious, since there is also evidence that other kinds of anisotropy in the large scales can result in anisotropy in the small scales. The most likely reason is the nonlocality of turbulence, which, among other things is manifested in direct and bidirectional coupling between large and small scales. Consequently, problems arise also with the similarity hypotheses, which state that all statistical properties of velocity increments for small separations in space/time and large Reynolds numbers are independent of the large-scale properties of turbulent flows, except for the mean energy dissipation or energy input. Thus, there are serious reasons to doubt the restoring of all the symmetries (including the scale invariance; see chapter 7) of the Navier–Stokes equations locally in time in space in the statistical sense.

The concept of inertial range is not well defined, e.g., in the context of 'anomalous' scaling behavior of higher-order structure functions in the nominally defined inertial range. This is due to the contamination of the inertial range by strong dissipative events at whatever large Reynolds numbers. One of the consequences is that the 'decomposition' into inertial and dissipative ranges is not that nice and that the anomalous scaling is not the attribute of the inertial range. Along with the fact that velocity increments (let alone structure functions and their scaling if such exists) are not the only objects of interest and *do not constitute a representation basis for a flow* (Goto and Kraichnan, 2004) they are not a good object to define a perfect IR which is especially acute in the issue of asymptotics at large Reynolds numbers. Such a definition seems to be not possible in principle due to a variety of nonlocal effects understood in a broad sense as direct and bidirectional coupling/interaction between large and small scales, see chapter 6. *How a theoretical attack on the inertial-range problem should proceed is far from clear* Kraichnan (1974).

A special remark is about the contribution of the dissipative events as defined/described above to the low-order statistics and in particular to the 4/5 law. These events *do* contribute to the 4/5 law and removing them leads to an increase of the scaling exponent above unity, see figure 5.3 left. An important point here is that the neglected viscous term in the Karman–Howarth equation does not contain *all* the viscous contributions. Those which are present in the structure function S_3 itself remain and keep the

4/5 law precise. In this sense this law is not a pure inertial law.

The multi-fractal and similar models are not based on adequately resolved experimental data.

The hypothesis of the finite dissipation limit with increasing Reynolds number seems to have reasonable experimental support.

The notion that turbulent flows are hierarchical, which underlies the concept of the cascade, though convenient, is more a refection of the unavoidable (due to the nonlinear nature of the problem) hierarchical structure of models of turbulence and/or decompositions rather than reality. The concept of cascade as occurring through the inertial range (which is not well defined) is not well defined as well. The class of flows called partly-turbulent comprises a strong counter-example to the cascade and does not comply with the concept of cascade. In all such flows the fluid becomes turbulent in 'no time' when passing across the laminar/turbulent interface without any cascade whatsoever. As mentioned in chapter 3 there is a conceptual necessity to handle turbulence as an undecomposable whole. The ill-posedness of the cascade concept is emphasized in the case of passive objects, whose evolution is governed by linear equations, with the velocity field entering multiplicatively in these equations, thus making them 'statistically nonlinear'.

DYNAMICS

With the emphasis on the rotational and dissipative nature of turbulence

6.1. Introduction

The true problem of turbulence dynamics is the problem of its origin(s) and successive development from some initial conditions and at some boundary conditions to an ultimate (statistical) state[1]. However, since this route is extremely complicated and involved, a second approach is used quite frequently. Namely, turbulent flows are studied 'as they are' disregarding their origin[2], and without looking into some of the details of the mechanisms of turbulence production and sustainment. In both approaches, at least some aspects of the time evolution are of central importance, since turbulence dynamics is a process.

One of the simplest examples of the second approach is when turbulence is produced numerically in a box with periodic boundary conditions by some forcing in the right hand-side of the Navier–Stokes equations. If the Reynolds number is not too small almost any forcing will do – random and deterministic – the flow will be turbulent. In both cases it is possible to produce a turbulent flow which is approximately (only) homogeneous and isotropic, and if the forcing is statistically stationary or just time independent the resulting turbulent flow is statistically stationary. In such a case the overall energy balance is described by equation similar to (C.49), and in this particular case the total turbulent energy balance for the whole flow domain is

$$\frac{d\mathcal{E}_T}{dt} = \mathcal{W}_f - \mathcal{D} \tag{C.49a}$$

[1]With the hope (based on rich experimental data) that, at least with statistically stationary (which includes time independent) boundary conditions and forcing, such a state does exist. This does not mean that it is impossible to define a kind of 'ultimate state' for statistically nonstationary turbulent flows. The simplest example is represented by periodically forced flows (e.g., blood flow in the human body) allowing us to employ the so-called phase averaging assuming *precise* periodicity.

[2]As mentioned in chapter 2 that once a flow becomes turbulent, it seems impossible to find its origin.

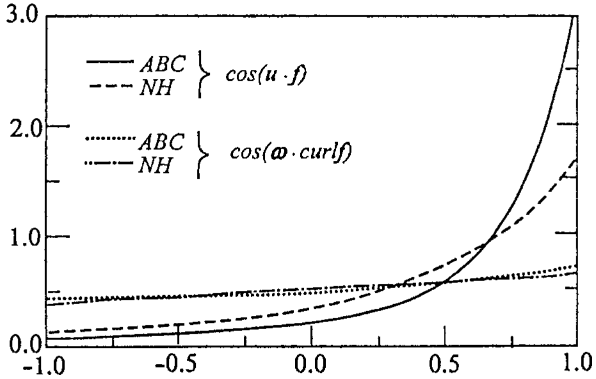

Figure 6.1. PDFs of the cosine of the angle between velocity and force, $\cos(\mathbf{u}, \mathbf{f})$ and between vorticity and the *curl* of the force, $\cos(\omega, curl\mathbf{f})$

where $\mathcal{E}_T = \int e_T dV$ is the total kinetic energy of turbulent fluctuations, $\mathcal{W}_f = \int u_i f_i dV$ is the total rate of production of energy of turbulent fluctuations by external forces, and $\mathcal{D} = 2\nu \int s_{ij} s_{ij} dV$ is the total rate of dissipation (simply dissipation) of energy of turbulent fluctuations by viscosity. If the flow is statistically stationary the production equals dissipation $\mathcal{W}_f = \mathcal{D}$, i.e., $\mathcal{W}_f > 0$. This implies that there should be a tendency for alignment between the velocity vector, $\mathbf{u}$, and the force, $\mathbf{f}$. Indeed, this is what is observed (see figure 6.1). These results (Galanti and Tsinober, 2000) were obtained with a deterministic forcing corresponding to the so-called ABC flow, $\mathbf{f} = f\{A \sin z + C \cos y,\ B \sin x + A \cos z,\ C \sin y + B \cos x\}$, $A = B = C$. This choice was mainly due to the strong instability of ABC flows guaranteeing fast transition to turbulence at a rather low Reynolds number (Galanti et al., 1992; Podvigina and Pouquet, 1994 and references therein). It should be emphasized that the results are practically the same in cases where the coefficients A, B, C are random functions of time. The ABC forcing is strongly helical, $curl\mathbf{f} \parallel \mathbf{f}$, and therefore along with kinetic energy such a forcing makes an input of helicity into the flow. Very similar results were obtained also for other kinds of forcing, e.g., with a force in the form $\mathbf{f} = f\{A \cos z \cos y,\ B \cos x \cos z,\ C \cos y \cos x\}$, $A = B = C$. This forcing, denoted in the sequel as NH, is nonhelical, $\mathbf{f} \cdot curl\mathbf{f} = 0$, and a mixed random forcing was also used consisting of weighted sums of the ABC and NH forcings. In all the above-mentioned cases the Taylor microscale Reynolds number $\mathrm{Re}_\lambda \approx 110$. In addition some runs were made at different Reynolds numbers in order to have a qualitative impression of the Reynolds number effects.

The results shown in figure 6.1 are not unexpected: there should be some alignment between velocity and force in order to have the work performed

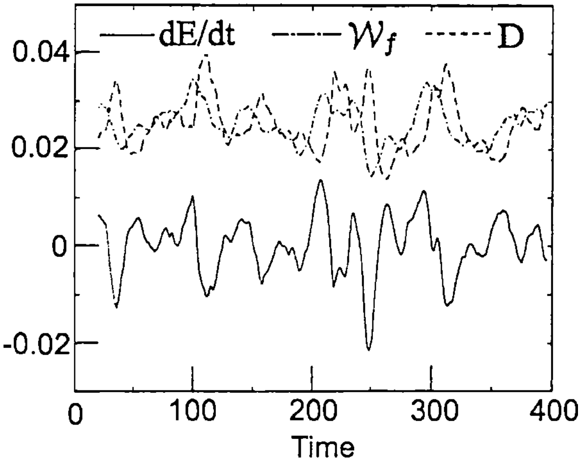

Figure 6.2. Time behaviour of the energy input, $\mathcal{W}_F = V^{-1} \int u_i F_i dV$, and dissipation, $\mathcal{D} = 2\nu V^{-1} \int s_{ij} s_{ij} dV$ for the case of ABC forcing at the resolution 64^3, corresponding to $\mathrm{Re}_\lambda \approx 60$. Time is measured in turnover times

by the force, $\mathbf{f}$, positive. A more interesting result is that both the energy input, $\mathcal{W}_f = \int u_i f_i dV$, and dissipation, $\mathcal{D} = 2\nu \int s_{ij} s_{ij} dV$, are far from being instantaneously equal, and exhibit quite large fluctuations in time, up to 40% of their long time averages (see figure 6.2)[3]. The latter are equal within less then 0.1%. The equality $\int_0^T \mathcal{W}_f dV = \int_0^T \mathcal{D} dV$ occurs at large times, $T \approx 500$, even at rather low Reynolds number[4]. The result shown in figure 6.2 is similar to the one obtained by Pinton et al. (1999) in an experiment in a 'French washing machine' such as the one mentioned in chapter 5, figure 5.1. Note the time lag that occurs from the large-scale energy injection to the fine-scale energy dissipation, which was observed also by Pearson et al. (2004). This should be compared with almost no delay when one looks at the enstrophy production and its dissipation and similarly for strain, see section on the Tennekes and Lumley balance below. Thus the lag seen in figure 6.1 cannot be explained by some kind of 'cascade' as 1) enstrophy and strain are dissipated 'immediately' and 2) the flow field is fully defined by the field of vorticity and/of strain at each time moment. A similar lag is observed also in the case of a passive scalar. But the lag is very small between its gradient production and dissipation just as in case of magnetic field.

[3]With prescribed force the energy input and consequently the dissipation depend on the velocity field. Therefore, generally, they cannot be prescribed independently.

[4]The energy input and dissipation, which are equal in the mean in statistically stationary flows, are two different quantities. Using the (pretty ambiguous) language of "scales" one is a large-scale quantity and the other is a small-scale one.

Among the most important attributes of turbulence are its rotational and dissipative nature. This dictated the main emphasis of this chapter.

6.2. Why velocity derivatives?

Velocity derivatives, $A_{ij} = \partial u_i/\partial x_j$, play an outstanding role in the dynamics of turbulence for a number of reasons. Their importance has become especially clear since the papers by Taylor (1937, 1938a)[5] and Kolmogorov (1941a,b). Taylor emphasized the role of vorticity as a manifestation of the rotational nature of turbulence, i.e., the antisymmetric part of the velocity gradient tensor $A_{ij} = \partial u_i/\partial x_j$, whereas Kolmogorov stressed the importance of the dissipative nature of turbulence, and thereby strain, i.e., the symmetric part of the velocity gradient tensor.

It is noteworthy that the whole (incompressible) flow field is fully determined by the fields of vorticity *or* strain with appropriate boundary conditions, see appendix C.

Apart from vorticity and strain/dissipation, there are many other reasons for special interest in the characteristics of the field of velocity derivatives, $A_{ij} = \partial u_i/\partial x_j$, in turbulent flows. For example,

• – The field of velocity derivatives is much more sensitive to the non-Gaussian nature of turbulence or more generally to its structure, and hence reflects more of its physics (see Tsinober, 2000, and references therein).

• – The possibility of singularities being generated by the Euler and the Navier–Stokes equations (NSE) and possible breakdown of NSE are intimately related to the field of velocity derivatives (Constantin, 1996; Doering, 2009; Doering and Gibbon, 1995).

• – In the Lagrangian description of fluid flow in a frame following a fluid particle, each point is a critical one, i.e., the direction of velocity is not determined. So everything happening in its proximity is characterized by the velocity gradient tensor $A_{ij} = \partial u_i/\partial x_j$. For instance, local geometry topology is naturally described in terms of critical points terminology[6] (see Chacin and Cantwell, 2000; Chertkov et al., 1999; Martin et al., 1998; Ooi et al., 1998 and references therein).

[5]Taylor (1937, 1938a) was motivated by the assumption of von Karman (1937) *that the expression* $\sum_i \sum_k \omega_i \omega_k \frac{\partial u_i}{\partial u_k}$ (i.e., enstrophy production, see equation (C.16)) *is zero* in the mean and that he (vK) *cannot see any physical reason for such a correlation.* Taylor (1937) conjectured that there is *a strong correlation between* ω_3^2 *and* $\frac{\partial u_3}{\partial u_3}$ *so that* (the mean of) $\omega_3^2 \frac{\partial u_3}{\partial u_3}$ *is not equal to zero* (x_3 is directed along vorticity) and showed that this is really the case, Taylor (1938a). He also expressed the view that *stretching of vortex filaments must be regarded as the principal mechanical cause of the higher rate of dissipation which is associated with turbulent motion.*

[6]This approach was initiated by Perry and Fairlie (1974).

• – There is a generic ambiguity in defining the meaning of the term *small scales* (or more generally scales) and consequently the meaning of the term *cascade* in turbulence research. As mentioned in chapter 5, the specific meaning of this term and associated inter-scale energy exchange/'cascade' (e.g., spectral energy transfer) is essentially decomposition/representation dependent. Perhaps, the only common element in all decompositions/representations (D/R) is that the small scales are associated with the field of velocity derivatives. Therefore, it is natural to look at this field as the one *objectively* (i.e., D/R independent) representing the small scales. Indeed, the dissipation is associated precisely with the strain field, s_{ij}, both in Newtonian and non-Newtonian fluids.

There is a number of more specific reasons why studying the field of velocity derivatives is so important in the dynamics of turbulence. This is one of the main themes of this chapter. Additional emphasis is given to several relatively new aspects, such as geometrical statistics.

6.2.1. VORTEX STRETCHING AND ENSTROPHY PRODUCTION

> *Vorticity amplification is a result of the kinematics of turbulence.* (Tennekes and Lumley, 1972).
> *The physics of vortex stretching is well understood.* (Siggia, 1977).
> *... amplification of the vorticity by vortex stretching, a well-understood mechanism in 3D Euler flow.* (Pomeau and Sciamarella, 2005).

One of the most basic phenomena and distinctive features of three-dimensional turbulence is the predominant vortex stretching, which is manifested in positive net enstrophy production, $\langle \omega_i \omega_j s_{ij} \rangle > 0$. This was discovered by Taylor (1938)[7], and was confirmed subsequently both experimentally and numerically in a number of investigations (see references in Tsinober, 1998ab, 2000).

The nonlinear terms $\omega_j s_{ij}$ and $\omega_i \omega_j s_{ij}$ are responsible for the predominant vortex stretching (VS) and enstrophy production. In other words, part (but not all) of the essential dynamics of 3D-turbulence is contained in the *interaction* between vorticity, ω, and the rate of strain tensor, s_{ij}. Both $\omega_j s_{ij}$ and $\omega_i \omega_j s_{ij}$ vanish identically for 2-D flows.

So far, no theoretical arguments in favor of positiveness of $\langle \omega_i \omega_j s_{ij} \rangle$ have been given. The argument that the reason is the (approximate) balance between the enstrophy production and enstrophy dissipation is misleading and puts the consequences before the reasons, since it is known that, for Euler

[7]Taylor, 1938 addressed the mean $\langle \omega_i \omega_j s_{ij} \rangle$. The positively skewed PDF of $\omega_i \omega_j s_{ij}$ was first observed by Betchov (1975) in a numerical simulation of 'Euler' equations.

equations, the enstrophy production increases with time very rapidly, apparently without limit (see references in Tsinober, 1998ab, 2000). Another rather common view that the prevalence of vortex stretching is due to the predominance of stretching of material lines is – at best – only partially true, since there exist several *qualitative* differences between the two processes, we discuss these differences in more detail in chapter 9.

As mentioned above, $\langle \omega_i \omega_j s_{ij} \rangle$ is an essentially positive quantity in 3-D turbulence – the PDF of $\omega_i \omega_j s_{ij}$ is strongly positively skewed (see figure 6.3). This fact reflects one of the most basic *specific* properties of three-dimensional turbulent flows – the prevalence of the vortex stretching process. The enstrophy production $\omega_i \omega_j s_{ij}$ is an outstanding nonzero *odd* moment of utmost dynamical importance in turbulence. Indeed, in the hypothetical case of absence of vortex stretching and enstrophy generation or even in cases in which only $\langle \omega_i \omega_j s_{ij} \rangle = 0$ – as assumed by von Karman (1938) – the three-dimensional turbulence, as we observe it, would not exist.

6.2.2. WHY STRAIN TOO?

It is to be stressed that along with vortex stretching and enstrophy production, of special interest is the production of strain. There are several reasons for this. First, though formally all the flow field is determined entirely by the field of vorticity, the relation between the strain and vorticity is strongly nonlocal (Constantin, 1994; Novikov, 1967; Ohkitani, 1994). In many cases, they are only weakly correlated (statistically) or not correlated at all. Second, energy dissipation is directly associated with strain and not with vorticity. Third, vortex stretching is essentially a process of interaction of vorticity and strain. Four, strain dominated regions appear to be the most active/nonlinear in a number of aspects (see section 6.4.2). The energy cascade (whatever this means) and its final result – dissipation, are associated with predominant self-amplification of the rate of strain/production and vortex compression rather than with vortex stretching. This means that another nonzero *odd* moment $s_{ij} s_{jk} s_{ki}$ (responsible for the production of strain, see below) is not less important than the enstrophy generation, Tsinober (2000). Finally, regions of major nonlinear activity are associated with large strain and its production rather than with regions of concentrated vorticity with lower dissipation. This is clearly seen from the 'tear drop' plots in a variety of flows, see for example figure 10.1.

Production of strain. Is turbulent dissipation due to vortex stretching?
The appropriate level of dissipation moderating the growth of turbulent energy is achieved by the build up of strain of sufficient magnitude which

is described by the equations (C.17, C.18). It is seen from the latter equation that in the mean the only term contributing positively to the production of strain/dissipation, s^2, is the term $-s_{ij}s_{jk}s_{ki} = -(\Lambda_1^3 + \Lambda_2^3 + \Lambda_3^3) = -3\Lambda_1\Lambda_2\Lambda_3$, since $\langle s_{ij}s_{jk}s_{ki}\rangle = -3/4\langle\omega_i\omega_j s_{ij}\rangle$, and $\langle s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}\rangle = 0$ due to homogeneity and incompressibility. Moreover, since, $\Lambda_1 > 0$ and Λ_2 is positively skewed, i.e., $\left\langle\Lambda_2^3\right\rangle > 0$[8], the positiveness of $-\left\langle s_{ij}s_{jk}s_{ki}\right\rangle$ comes only from the term $-\left\langle\Lambda_3^3\right\rangle$. In other words, Λ_3 is responsible for most of the 'cascade', at least, one of the final results of the 'cascade' – dissipation of energy, which is directly associated with s_{ij} and not with ω_i. An example of ratios between $\left\langle\Lambda_i^3\right\rangle$ is given in table 6.1. Hence, the 'cascade' is

TABLE 6.1. The ratios between the $\left\langle\Lambda_i^3\right\rangle$. The values for $Re_\lambda = 75$ are given for a computation in a box with periodic boundary conditions, but very similar results as well as many others were obtained in a grid turbulence experiment (Tsinober et al., 1997). The values for $Re_\lambda = 10^4$ are given for a field experiment in the atmospheric surface layer at the height $10m$ (Kholmyansky et al., 2001b)

	Re_λ	$\left\langle\Lambda_1^3\right\rangle$	$\left\langle\Lambda_2^3\right\rangle$	$\left\langle\Lambda_3^3\right\rangle$
DNS	75	1.2	0.05	−2.25
Field	10^4	1.62	0.05	−2.67

directly associated with compressing/squeezing of fluid elements and not with (vortex) stretching. It is noteworthy that this idea is not entirely new: '*It is clear, therefore, that production of vorticity is associated essentially with Λ_3 and production of ω_1 and ω_2. This suggests that the most important processes associated with production of vorticity and energy transfer resemble a jet collision and not the swirling of a contracting jet* (Betchov, 1956). Betchov arrived at this conclusion analyzing the means $\langle\omega_i\omega_j s_{ij}\rangle$ and $-\langle s_{ij}s_{jk}s_{ki}\rangle$, and assuming both of them positive. Looking at equation (C.18), it is seen that the above conclusion is also true of production of strain, which is associated with Λ_3, and with the 'jet collision' regions such as sheet-like structures as observed in the laboratory (Frederiksen et al., 1997; Schwarz, 1990) and numerical experiments (Brachet et al., 1992; Boratav and Pelz, 1997; Chen and Cao, 1997; Flohr, 1999). As for enstrophy production we shall see in the sequel that it is true in part: roughly two thirds of its positive contribution occurs in the 'jet collision' regions, the remaining third happens in the 'swirling of a contraction jet' regions. Also

[8]See for example figure 6.10. This was first discovered by Ashurst et al. (1987) and confirmed by She et al. (1991); Su and Dahm (1996); Tsinober et al. (1989, 1992). In a Gaussian velocity field the PDF of Λ_2 is strictly symmetric.

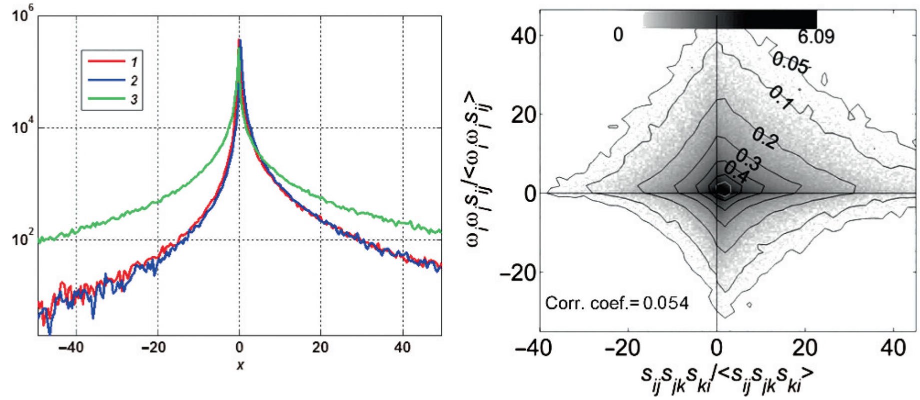

Figure 6.3. Left – PDFs of $\frac{3}{4}\omega_i\omega_j s_{ij}$, $-s_{ij}s_{jk}s_{ki}$, and $-17.5(\partial u_1/\partial x_1)^3$ normalized on their means, $\mathrm{Re}_\lambda = 10^4$. Right – Joint PDF and scatter plot of $\frac{3}{4}\omega_i\omega_j s_{ij}$ versus $-s_{ij}s_{jk}s_{ki}$, normalized on their means, $Re_\lambda = 10^4$ (Gulitski et al., 2007a)

it is noteworthy that production of ω^2 requires two partners ω_i and s_{ij}, and interaction between the two, but production of s_{ij} is in some sense (locally) less dependent on ω_i, though without vorticity it is impossible. Indeed enstrophy production is due to the term $\omega_i\omega_j s_{ij}$ containing both vorticity and strain, whereas production of strain is due to the term $s_{ij}s_{jk}s_{ki}$ containing strain only. In this sense strain production is more self-production and is a *local* process, whereas $\omega_i\omega_j s_{ij}$ is *nonlocal.* Among other things, the difference is manifested in correlation coefficients shown in table 6.2 for the field experiment mentioned above (Kholmyansky et al., 2001b). The main feature is that strain production is much less correlated with enstrophy than with strain, whereas enstrophy production is equally correlated with both, but its rate is more correlated with strain. This feature is better seen in joint PDFs/scatter plots (figure 6.4).

TABLE 6.2. Correlation coefficients between production terms versus enstrophy and strain

	$\omega_i\omega_i s_{ij}$	$-(4/3)s_{ij}s_{jk}s_{ki}$	$\omega_i\omega_i s_{ij}/\omega^2$	$-(4/3)s_{ij}s_{jk}s_{ki}/s^2$
ω^2	0.35	0.16	0.14	0.11
s^2	0.31	0.41	0.24	0.28

The next important point is that the enstrophy production $\omega_i\omega_j s_{ij}$ appears in the equation (C.18) with the negative sign, so that the vortex stretching is *opposing* the production of dissipation/strain: all instantaneous positive values of $\omega_i\omega_j s_{ij}$ make a negative contribution to the right-hand side of (C.18), i.e., enstrophy production $\omega_i\omega_j s_{ij}$ has an additional role as drain of "energy" of strain (i.e., s^2)[9]. Since $\omega_i\omega_j s_{ij}$ is essentially a positively skewed quantity, its mean contribution to strain production is negative. In other words, the energy cascade (whatever this means) is associated primarily with the quantity $-s_{ij}s_{jk}s_{ki}$, rather than with the enstrophy production $\omega_i\omega_j s_{ij}$ and that vortex stretching suppresses the cascade and does not aid it[10], at least in a *direct* manner (Tsinober et al., 1999; Tsinober, 2000). On the contrary, it is the vortex *compression,* i.e., $\omega_i\omega_j s_{ij} < 0$, that aids the production of strain/dissipation and, in this sense, the 'cascade'. Negative enstrophy production is associated with strong tilting of the vorticity vector and large curvature of vortex lines (see section 6.4.2), which in turn are associated with large magnitudes of the negative eigenvalue, Λ_3, of the rate of strain tensor (Kholmyansky et al., 2001b; Tsinober, 2000). This is in full conformity with the above mentioned fact that Λ_3 is responsible for most of the 'cascade'. i.e., predominant compressing rather than stretching!

One does not have to be confused that $\langle s_{ij}s_{jk}s_{ki}\rangle = -3/4\langle\omega_i\omega_j s_{ij}\rangle$ (due to Betchov, 1956) or even by similarity of their PDFs (figure 6.3, left), since their pointwise relation is strongly nonlocal due to the nonlocal relation between vorticity and strain (Constantin, 1994; Novikov, 1967; Ohkitani, 1994). Consequently, locally they are very different, as can be seen from their joint PDF and scatter plots (figure 6.3, right): they are only weakly correlated and there are a great many points with small $\omega_i\omega_j s_{ij}$ and large $-s_{ij}s_{jk}s_{ki}$ and vice versa[11]. More details can be found in Tsinober (2000a), Kholmyansky et al. (2001b) and Gulitski et al. (2007a).

[9]This is consistent with recent results on a Lagrangian experiment using a 3D particle tracking velocimetry with access to velocity derivatives (Lüthi et al., 2005; Guala et al., 2006). Namely, it was found that the statistical evolution of strain and enstrophy can be interpreted as a kind of a life-cycle for strain and enstrophy and can be summarized in a sequence of processes starting with the strain self-amplification in low strain low enstrophy regions. This is followed by enstrophy production and growth, leading to the formation of high strain high enstrophy regions. The depletion of both strain and its production in parallel to the growth of enstrophy is related to the evolution of these regions into high enstrophy low strain regions, i.e. to the evolution of vortex sheets (shear layers) into vortex filaments. These regions evolve into weak enstrophy - strain regions since the enstrophy production, in presence of low strain and preferential alignment between ω and λ_2, cannot oppose the viscous destruction of enstrophy. This cyclic sequence consists of local and non-local processes of different Lagrangian time scales which governs the dynamics of small-scale turbulence.

[10]In contrast to the most common belief: *It seems that the stretching of vortex filaments must be regarded as the principal mechanical cause of the high rate of dissipation which is associated with turbulent motion* (Taylor, 1938a).

[11]The same is true of ω^2 and s^2 which obey $2\left\langle s^2\right\rangle = \langle\omega^2\rangle$. In order to illustrate the limitations of looking at means only it is instructive to have a look at the equation of

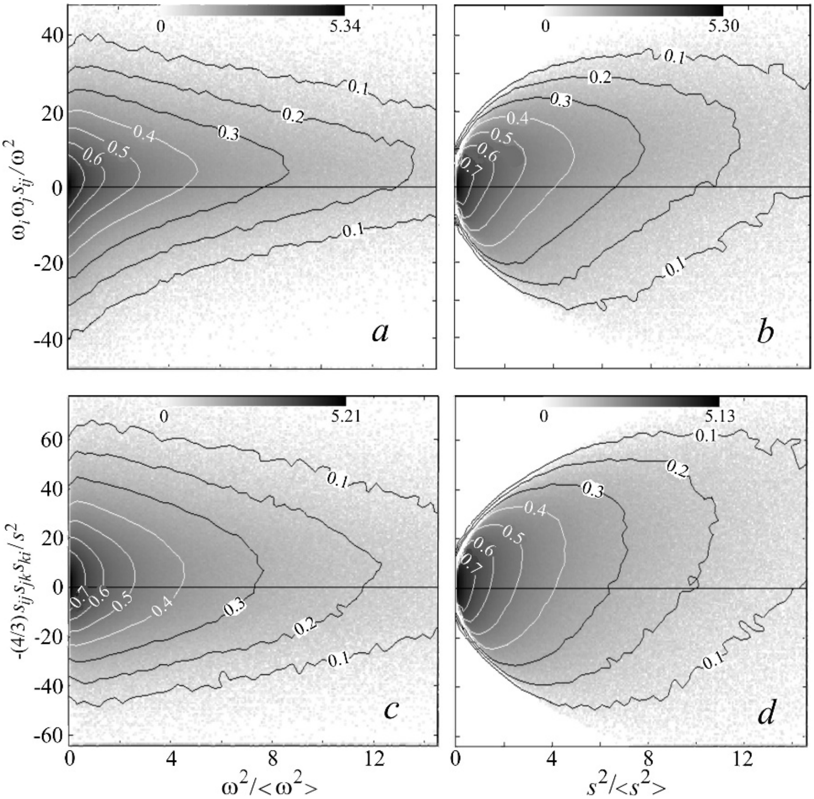

Figure 6.4. Joint PDFs/scatter plots of $\omega_i\omega_i s_{ij}/\omega^2$ and $-(4/3)s_{ij}s_{jk}s_{ki}/s^2$ versus ω^2 and s^2 from the field experiment, Kholmyansky et al. (2001b)

Apart from *nonlinear* interaction between vorticity and strain there is a conceptually and qualitatively different phenomenon concerning the field of strain in genuine turbulent flows. It is the self-amplification of the field of strain which is a specific feature of the dynamics of turbulence having no counterpart (more precisely analogous – not more) in the behaviour of

evolution of the quantity $\varepsilon = \frac{1}{2}\left(\omega^2 + 4s^2\right)$,

$$\frac{D\varepsilon}{Dt} = -4s_{ij}s_{jk}s_{ki} - 4s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j} + 4\nu s_{ij}\nabla^2 s_{ij} + 4s_{ij}F_{ij} + \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j} \qquad (C.18a)$$

which does not contain enstrophy production at all and its only production mechanism is due to $-4s_{ij}s_{jk}s_{ki}$. Though $\langle\varepsilon\rangle = 3\left\langle s^2\right\rangle = \frac{3}{2}\langle\omega^2\rangle$ the above equation governs neither the evolution of s^2 nor of ω^2. Another example is an equation for the second invariant, $Q = \frac{1}{4}(\omega^2 - 2s^2)$, of the velocity gradient tensor. For a homogeneous flow the RHS of this equation does contain production terms involving neither $\langle\omega_i\omega_j s_{ij}\rangle$ nor $\langle s_{ij}s_{jk}s_{ki}\rangle$.

passive and also active objects. This process (i.e., $-s_{ij}s_{jk}s_{ki}$) is *local* in contrast to $\omega_i\omega_j s_{ij}$, see section on nonlocality.

There are several other (see the examples below) important processes in which predominant compressing is the main player (Corrsin, 1953a): there is no reason to push stretching everywhere, especially vortex stretching.

Stretching or compressing?
Of course, there is no stretching without compressing, but their roles are quite different in different process. For example, that the predominant vorticity stretching occurs roughly in 2/3 of the fluid flow domain and is associated both with the positive and intermediate eigenvalue/vectors of the rate of strain, see section on geometrical statistics.

TKE production in turbulent shear flows The turbulent energy production in a turbulent shear flow is known to be represented by the term $-(\langle u_i u_k\rangle S_{ik}) > 0$ with u_i being the components of velocity fluctuations, and S_{ik} the mean rate of strain. Using the representation $-\langle u_i u_k\rangle S_{ik} = -\langle u^2\Lambda_k^S cos^2(\mathbf{u}, \lambda_k^S)\rangle$ one can see that the positiveness of $-\langle u_i u_k\rangle S_{ik})$ is due compression rather than stretching. Here, $u^2 = u_i u_i$, Λ_i^S are the eigenvalues and λ_1^S are the corresponding eigenvectors of the mean rate of strain tensor S_{ik}, $\Lambda_1^S > 0$, $\Lambda_2^S < 0$ and $\Lambda_3^S \lesseqgtr 0$ and is positively skewed.

A similar statement is true for the energy flux $\Pi_\ell = -\tau_{ij}^{SGS} \prec S_{ij} \succ$ at scale ℓ for the filtered quantities in the filtering approach. Namely, recalling that $\tau_{ij}^{SGS} = \prec u_i u_j \succ - \prec u_i \succ \prec u_j \succ$ and passing over to the eigenframe of $\prec S_{ij} \succ$. Here $\prec \cdots \succ$ means the filtering operation (see appendix 3) and u_i and are the instantaneous velocity components in, and $\Lambda_i^{\prec\succ}$ are the eigenvalues of, the filtered rate of strain tensor $\prec S_{ij} \succ$ and $\Lambda_1^{\prec\succ} > \Lambda_2^{\prec\succ} > \Lambda_3^{\prec\succ}$ and $\Lambda_1^{\prec\succ} > 0, \Lambda_3^{\prec\succ} < 0$ and $\Lambda_2^{\prec\succ} \lesseqgtr 0$ with positively skewed distribution.

For more details see chapter 8.

Passive objects The production of gradients of passive scalar $-G_iG_js_{ij} = -G^2\Lambda_k\cos^2(\mathbf{G},\ \lambda_k)$ is positive in the mean again due to the contribution associated with the alignment of the gradient $\mathbf{G}$ and the eigenvector, λ_3, corresponding to the compressive eigenvalue, Λ_3: it is due to $-G^2\Lambda_3\cos^2(\mathbf{G},\ \lambda_3)$. A similar statement is true for the production term $-A_iA_js_{ij}$ for the vector potential $\mathbf{A}$ of magnetic field ($\mathbf{B} = rot\mathbf{A}$) see Tsinober and Galanti (2003).

Production of vorticity gradients in two-dimensional turbulence This has the form $-\xi_i\xi_k s_{ik}$, with $\xi_i = \partial\omega/\partial x_i$ and it easy to see again that $-\langle\xi_i\xi_k s_{ik}\rangle > 0$ due to the compression rather than stretching.

Evolution of the disturbances in genuine and 'passive' turbulence In this case one looks at the evolution of the disturbance Δ_i^u of some flow realization u_i in a statistically stationary state and similarly for other quantities (active or passive, see again Tsinober and Galanti, 2003). The process of evolution and amplification of disturbances – both in genuine and 'passive' turbulence – is dominated by the strain field of the basic flow. For example, the energy production of the disturbance Δ_i^u has the form $-\Delta_i^u\Delta_j^u s_{ij}$, see equation (C.59). It is seen again that $-\langle\Delta_i^u\Delta_j^u s_{ij}\rangle > 0$ is due to a dominant contribution from compression rather than stretching. That is, just like the mean, $-\langle u_i u_j\rangle S_{ij}$, is positive in turbulent shear flows, the integral of the production of the energy of error, $P_{\boldsymbol{\Delta}^u} = -\int \Delta_i^u\Delta_J^u s_{ij} dV$, over the flow domain at any time moment is positive. There is a strong tendency of alignment between the error vector, $\boldsymbol{\Delta}^u$, and the eigenvector of the rate of strain tensor of the instantaneous velocity field, s_{ij}, corresponding to its negative eigenvalue. Concomitantly $\boldsymbol{\Delta}^u$ tends to be normal to the two other eigenvectors of s_{ij}.

Role of strain in the evolution of flows with polymer additives In this case an additional term appearing in the NSE is $\frac{\partial\tau_{ij}}{\partial x_j}$ with the stress τ_{ij} expressed in a variety of models[12] via the so-called conformation tensor R_{ij} as $\tau_{ij} = \frac{\nu_p}{\tau_p}\left\{f(\mathbf{x},t)\rho_0^{-2}R_{ij} - \mathbf{I}\right\}$, $f(\mathbf{x},t) = \frac{\rho_m^2-\rho_0^2}{\rho_m^2-R_{kk}(\mathbf{x},t)}$. The evolution of the tensor R_{ij} is governed by

$$\frac{DR_{ij}}{Dt} = A_{ik}R_{kj} + A_{kj}R_{ik} - \frac{1}{\tau_p}\left\{f(\mathbf{x})R_{ij} - \rho_0^2\delta_{ij}\right\}, A_{ij} = \partial u_i/\partial x_j$$

The only point to be stressed here is that the two terms $A_{ik}R_{kj} + A_{kj}R_{ik}$ in this equation, in fact, contain the strain only since $A_{ik}R_{kj} + A_{kj}R_{ik} \equiv s_{ik}R_{kj} + R_{ik}s_{kj}$. This latter allows us to see immediately that the stress tensor τ_{ij} is indeed a functional of the strain tensor s_{ij} only as it should be. This does not contradict the observations of vortex inhibition (Gadd, 1968, Chiou and Gordon, 1976; Latorre et al., 2004) as it occurred in the potential flow (which is vorticity free!) surrounding the vortex core and similar phenomena in vortex streets (Cressman et al., 2001).

Polymer solutions represent an example of active additives which react back on the flow field (including the field of strain) and modify it not only in the small, but in all dynamically relevant scales.

We would like to recapitulate the qualitative differences between the enstrophy and strain production. It is the strain production (rather than vortex stretching) that is directly responsible for the enhanced dissipation

[12]Whatever the detailed mechanism (individual molecules, aggregates/clusters, etc.) the process of interaction of diluted polymers and turbulence is manifested in stretching/compressing of material elements and their reaction back on turbulence – in this case the material elements are not passive anymore as in pure water.

of turbulent flows and it is a local process with predominant compressing whereas the enstrophy production is a nonlocal process with predominant stretching.

6.3. The Tennekes and Lumley (TL) balance and self-amplification of the field of velocity derivatives

6.3.1. VELOCITY DERIVATIVES

It is commonly believed that, at least at large Reynolds numbers, the vortex stretching is a process of self-amplification *because ... the deformation* [i.e. rate of strain]*tensor, responsible for amplification, is expressed in terms of local characteristics* (Novikov, 1993a), or self-sustaining, since it *does not require a large-scale mean flow* (Tennekes, 1989).

This belief is based on the order of magnitude estimates (Tennekes and Lumley, 1972; Novikov, 1993b), for the *mean* quantities entering the equation (C.51) for the balance of the mean enstrophy $\langle\omega^2\rangle$. In case of homogeneous turbulence with external force δ-correlated in time, this estimate shows (Novikov, 1993b), that the ratio of the term associated with the external forcing to $\langle\omega_i\omega_j s_{ij}\rangle$ is of order $\mathrm{Re}^{-3/2}$. So the approximate balance is

$$\langle\omega_i\omega_j s_{ij}\rangle \approx -\nu\langle\omega_i\nabla^2\omega_i\rangle.$$

This balance appears to be valid in different meanings (not only in the mean) as follows. This is seen from figure 6.5. The first feature that the TL balance holds at Re_λ as low as $\approx$60. Second, it holds pointwise in time, i.e., the integrals over the flow domain of the enstrophy production and of its viscous destruction are approximately balanced at any time moment, $\int\omega_i\omega_j s_{ij}dV \approx -\nu\int\omega_i\nabla\omega_i dV$ (see equation (C.20)) . Consequently, the time derivative of the overall enstrophy $d\left(\int\frac{1}{2}\omega^2 dV\right)/dt$ is at least an order of magnitude smaller than both $\int\omega_i\omega_j s_{ij}dV$ and $\nu\int\omega_i\nabla\omega_i dV$,, i.e., in this respect the process is quasi-stationary. The spatial integral of the corresponding forcing term $C_\omega = \int\omega_i(curl f)_i dV$ is much smaller than all the three integrals just mentioned.

Computations at larger Re_λ showed that – as expected – the above balance becomes more precise. We shall use the example of the ABC and NH forced turbulent flows mentioned above in order to make more precise the meaning of the term *self-amplification.* Namely, we will see that the process of self-amplification refers to the whole field of velocity derivatives, i.e., both vorticity and strain, since in fact there exist *two* nonlocally interconnected and weakly correlated processes: along with predominant vortex stretching/enstrophy production associated with the positiveness of $\langle\omega_i\omega_j s_{ij}\rangle > 0$, there exist a concomitant predominant self-amplification of

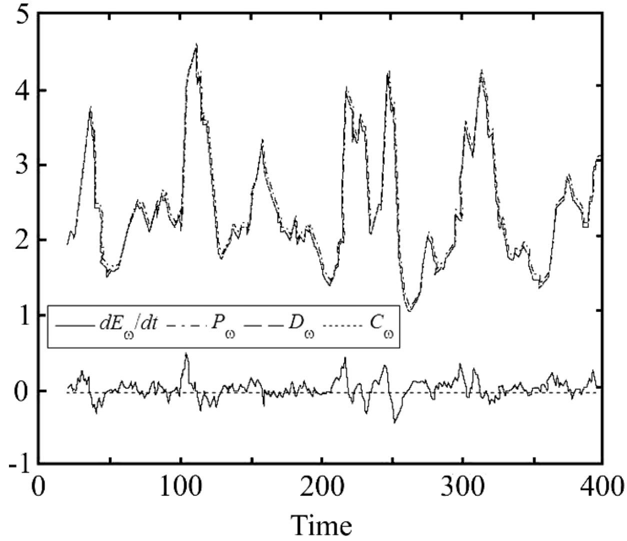

Figure 6.5. Time behaviour of dE_ω/dt, $E_\omega = V^{-1}\int \omega^2 dV$; $P\omega = V^{-1}\int \omega_i\omega_j s_{ij} dV$; $D\omega = \frac{\nu}{V}\int \omega_i \nabla^2 \omega_i dV$ and $C_\omega = V^{-1}\int \omega_i (curl\mathbf{f})_i dV$ for the case of ABC forcing at the resolution 64^3, corresponding to $\mathrm{Re}_\lambda \approx 60$. Time is measured in turnover times. Note that the graphs for the quantities associated with the strain production, $P_s = -\frac{4}{3}V^{-1}\int s_{ij}s_{jk}s_{ki}d\mathbf{x}$; $D_s = 2\nu V^{-1}\int s_{ij}\nabla^2 s_{ij} dV$ and $C_s = 2V^{-1}\int s_{ij}f_{ij}dx$, are precisely the same, since for periodical boundary conditions $P\omega = P_s$, $D_\omega = D_s$ and $C_\omega = C_s$ (see equation (C.21))

the rate of strain/production of total strain, $s^2 \equiv s_{ij}s_{ij}$, associated with the positiveness of $\langle -s_{ij}s_{jk}s_{ki}\rangle > 0$ (Tsinober, 2000).

The key result as obtained in the above computations relates to the comparison of the terms corresponding to the self-amplification of the field of velocity derivatives with the forcing terms in the equations (C.9) and (C.17) for the evolution of vorticity and strain, and in the equations (C.16) and (C.18) for the enstrophy, ω^2, and the total strain, s^2. Namely, it appears that the quantities $\{\omega_j s_{ij}\}^2$, $\{s_{ik}s_{kj}\}^2$, in the equations for ω_i and s_{ij}, and the quantities $\omega_i\omega_j s_{ij}$, and $-s_{ij}s_{jk}s_{ki}$ in the equations for ω^2 and s^2 are three orders of magnitude larger than the corresponding terms associated with forcing, $\{curl\mathbf{f}\}^2$ and $f_{ij}f_{ij}$, and $\omega{\cdot}curl\mathbf{f}$ and $s_{ij}f_{ij}$ respectively. This is true not only of the mean values, but is much stronger, since the same difference in values is observed for their $max\ |\cdot|$ (see tables 6.3, 6.4). Moreover, the essential dominance of the self-amplification of the velocity derivatives over the forcing occurs not only in the mean and in respect of their extremal values, but almost pointwise throughout the flow field. This was checked by looking at the volume fractions (relative number of points) of the whole flow domain, $\rho_K = V_K(|\omega\cdot curl\mathbf{f}|, |\omega_i\omega_j s_{ij}|)$, in which

TABLE 6.3. Comparison of the mean square values and the maxima of the squares of the terms $\omega_j s_{ij}$ and $(curl\mathbf{f})_i$, and the terms $-s_{ik}s_{kj}$ and $f_{ij} \equiv \frac{1}{2}\left\{\frac{\partial f_i}{\partial x_j} + \frac{\partial f_i}{\partial x_j}\right\}$ in the equations (2) and (3)

Forcing	A B C mean	A B C max	Nonhelical mean	Nonhelical max
$\{\omega_j s_{ij}\}^2$	8.3	$4.5 \cdot 10^4$	4.6	$7.3 \cdot 10^3$
$\{curl\mathbf{f}\}^2$	$4.3 \cdot 10^{-4}$	$1.7 \cdot 10^{-3}$	$2.3 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$
$\{s_{ij}s_{jk}\}^2$	18.5	$2.5 \cdot 10^4$	7.7	$9.8 \cdot 10^3$
$\{f_{ij}\}^2$	$8.6 \cdot 10^{-4}$	$3.4 \cdot 10^{-3}$	$4.7 \cdot 10^{-4}$	$2.4 \cdot 10^{-3}$

TABLE 6.4. Comparison of the mean values and the *max* $|\cdot|$ of the terms $\omega_i\omega_j s_{ij}$ and $\omega_i(curl\mathbf{f})_i$, and $-s_{ij}s_{jk}s_{ki}$ and $s_{ij}f_{ij}$ in the equations (4) and (5)

Forcing	A B C mean	A B C max$\lvert\cdot\rvert$	Nonhelical mean	Nonhelical max$\lvert\cdot\rvert$
$\omega_i\omega_j s_{ij}$	3.2	$4.7 \cdot 10^3$	1.5	$1.5 \cdot 10^3$
$\omega \cdot curl\mathbf{f}$	$7.6 \cdot 10^{-3}$	1.1	$2.7 \cdot 10^{-3}$	0.64
$-s_{ij}s_{jk}s_{ki}$	2.4	$1.5 \cdot 10^3$	1.1	$7.8 \cdot 10^2$
$f_{ij}s_{ij}$	$7.6 \cdot 10^{-3}$	0.8	$2.7 \cdot 10^{-3}$	0.4

$|\omega \cdot curl\mathbf{f}| > K\ |\omega_i\omega_j s_{ij}|$. It appears that for $K < 100$ there are no such points at all, for $K = 100$ this fraction $\rho_K < 10^{-8}$, and $\rho_K \sim 10^{-4}$ even for $K = 10^3$. Similar results are true for the volume fractions for the rest of the quantities discussed above. In other words the process of production of the field of velocity derivatives – both vorticity and strain – is a spatially local self-amplification process in the sense that the forcing does not play any role in the production of velocity derivatives locally in space. It is noteworthy that the Reynolds number, $\mathrm{Re}_\lambda \approx 110$, in the above simulations was very moderate. Selected runs for $\mathrm{Re}_\lambda \approx 250$ showed that the difference between the quantities responsible for the self-amplification of velocity derivatives ($\{\omega_j s_{ij}\}^2$, $\{s_{ik}s_{kj}\}^2$, $\omega_i\omega_j s_{ij}$, and $-s_{ij}s_{jk}s_{ki}$) and the quantities associated with the external forcing ($\{curl\mathbf{f}\}^2$, $f_{ij}f_{ij}$, $\omega \cdot curl\mathbf{f}$ and $s_{ij}f_{ij}$) becomes much larger, so that the former are about four orders of magnitude larger than the latter. It is interesting that even at $\mathrm{Re}_\lambda \approx 35$, the difference is still two orders of magnitude. There are clear indications that the predominance of the self-production of the velocity derivatives has a universal character

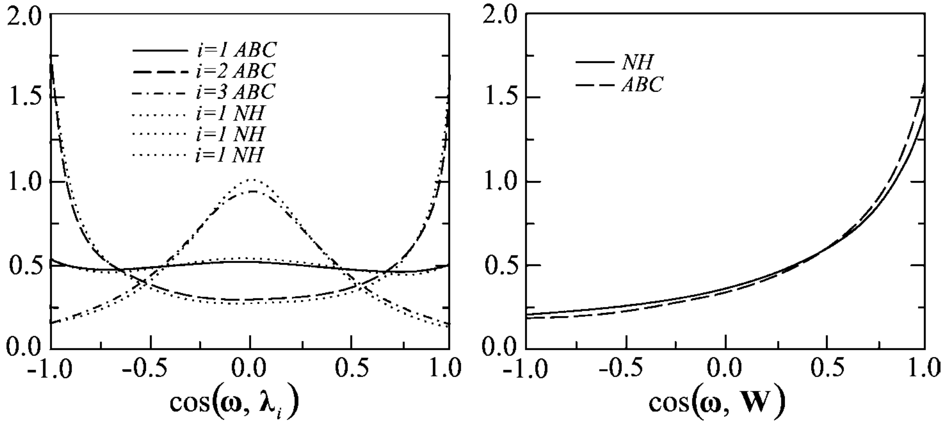

Figure 6.6. PDFs of the cosine of the angle between vorticity, ω, and the vortex stretching vector $W_i \equiv \omega_j s_{ij}$, $\cos(\omega, \mathbf{W})$ (right), and of the cosines of the angles between vorticity, ω, and the eigenframe λ_i of the rate of strain tensor, s_{ij} (left)

(Sandham and Tsinober, 2000; Kholmyansky et al., 2001b; Gulitski et al., 2007a; see chapter 8).

We return to figure 6.1. While there is a strong tendency for alignments between velocity, $\mathbf{u}$, and force, $\mathbf{f}$, the PDF of the cosine of the angle between vorticity and the *curl* of the force, $\cos(\omega, curl\mathbf{f})$ is practically flat. Another feature is that these PDFs for the ABC and the NH forcings are only qualitatively similar.

On the contrary, the alignments associated with the self-amplification process are much closer quantitatively as can be seen from the two examples[13] shown in figure 6.7. Similar behaviour is observed also for two groups of other quantities. The first group contains the PDFs of quantities associated with the forcing which appear to be quite different for the ABC and NH forcings, $(curl\mathbf{f})^2, f_{ij}f_{ij}, \omega \cdot curl\mathbf{f}$ and $s_{ij}f_{ij}$.

The second group consists of the quantities $(\omega_j s_{ij})^2$, $(s_{ik}s_{kj})^2, \omega_i\omega_i s_{ij}$ and $s_{ij}s_{jk}s_{ki}$ responsible for the self amplification of the velocity derivatives.

Their statistical properties for both ABC and NH cases are very similar as shown in figure 6.6 and other quantities associated with the self-amplification process tend to be universal in the sense that they are weakly sensitive to the details of the forcing. Many other, such as other relevant

[13]We shall see in section 6.4 on geometrical statistics that alignments of vorticity, ω, and the vortex stretching vector, $\mathbf{W}$, $W_i = \omega_i s_{ij}$, and of vorticity, ω, and the eigenframe, λ_i of the rate of strain tensor, s_{ij}, are dynamically very important. This is seen from the simple relation between the enstrophy production and the above quantities $\omega_i\omega_j s_{ij} = \omega \cdot \mathbf{W} = \omega W \cos(\omega, \mathbf{W}) = \omega^2 \Lambda_i \cos(\omega, \lambda_i)$. Here Λ_i, are the eigenvalues of the rate of strain tensor, s_{ij}.

alignments and those associated with the terms $\frac{\partial^2 p}{\partial x_i \partial x_j}$ and $s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}$, exhibit the same tendency. The terms $\frac{\partial^2 p}{\partial x_i \partial x_j}$ in (C.17) and $s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}$ are respectively at least an order of magnitude smaller than $s_{ij}s_{jk}$ and $s_{ij}s_{jk}s_{ki}$.

Finally, we are reminded that self-amplification of strain is more 'self' and more local than that of vorticity. This is seen from the equations (C.17) and (C.18), which show that production of strain is associated with the terms $s_{ij}s_{jk}$ and $-s_{ij}s_{jk}s_{ki}$ containing strain only. On the other hand, production of vorticity is, in the first place, *interaction* of vorticity and strain, since it is associated with the terms $\omega_j s_{ij}$ and $\omega_i\omega_j s_{ij}$, as is seen from the equations (C.9) and (C.16). This does not mean that strain production is totally independent of vorticity: there is no strain production without presence of vorticity in the flow. For example, in a homogeneous field, $s_{ij}s_{jk}s_{ki} = -(4/3)\omega_i\omega_j s_{ij}$ (see also equations (C.15, C.17, C.18, C.15′)).

The features concerning the TL balance appear to be true for temporally modulated turbulent flows and for flows with hyperviscosity of different orders, $h = 2, 4, 8$ ($h = 1$ corresponds to Newtonian fluid). This was observed in DNS of NSE for Re_λ up to 200.

6.3.2. PASSIVE OBJECTS

The DNS similar to those mentioned above reveal that the TL balance holds for the gradient of passive scalar G^2 and magnetic field B^2,

$$\langle G_iG_js_{ij}\rangle \approx \nu\langle G_i\nabla^2 G_i\rangle, \int G_iG_js_{ij}dV \approx \mathcal{D}\int dV G_i\nabla^2 G_i dV,$$

$$\langle B_iB_js_{ij}\rangle \approx -\nu\langle B_i\nabla^2 B_i\rangle, \int B_iB_js_{ij}dV \approx -\,\eta\int B_i\nabla^2 B_i dV.$$

There is, however, an important qualitative difference between the two. The balance for G^2 is similar to that of ω^2 in the sense that both quantities $-\langle G_iG_js_{ij}\rangle$ and $\nu\langle G_i\nabla^2 G_i\rangle$ reach a stationary mean; and both $-\int G_iG_js_{ij}dV \;\approx \mathcal{D}\int dV G_i\nabla^2 G_i dV$ oscillate around this mean. The quantities $\langle B_iB_js_{ij}\rangle$ and $-\eta\langle B_i\nabla^2 B_i\rangle$ with the means taken over short periods of time both grow exponentially in such a way that $\langle B_iB_js_{ij}\rangle$ is slightly larger than $\eta\langle B_i\nabla^2 B_i\rangle$. The integrals $\int B_iB_js_{ij}dV \approx -\eta\int B_i\nabla^2 B_i dV$ oscillate around these means with the former sitting a bit above the mean of the latter. The above behaviour for the spatial integrals can be characterized by the ratios r_ω, r_G, and r_B defined as $r_{\omega,G,B} = \frac{production+dissipation}{|production|+|dissiptation|}$. The ratios r_ω, r_G oscillate effectively around a zero ($<10^{-4}$) in the range ± 0.03, whereas r_B oscillates around a positive quantity of the order 10^{-2} in the range $-0.02 \div +0.05$. This is a reflection of the fact that $\frac{d}{dt}\int \omega^2 dV$ and $\frac{d}{dt}\int G^2 dV$ vanish in the mean, whereas $\frac{d}{dt}\int B^2 dV$ grows exponentially

(MHD-dynamo). The underlying reason for this difference is that there are (inviscid and nondiffusive) conservation laws for the kinetic energy u^2 and squared passive scalar θ^2, whereas there is no such law for the vector potential $\mathbf{A}$ of the magnetic field ($\mathbf{B} = curl\mathbf{A}$). Consequently there is no necessity to add forcing to the equation for $\mathbf{A}$ (or $\mathbf{B}$) as the field $\mathbf{A}$ is driven by the strain field s_{ij} via a term $-A_iA_js_{ij}$ (Galanti and Tsinober, 2003; see chapter 9).

It is important to stress that all the discussed balances are approximate (in some sense) and the small difference between production and dissipation plays a crucial role in the evolution of vorticity, strain, passive scalar gradient and magnetic field. It is the qualitative and fundamental dissimilarity between these differences which is one of the major factors responsible for the contrast in the evolution of vorticity, strain, passive scalar gradient and magnetic field.

The approximate balance as described above between the 'inertial' and diffusive processes shows that they are very far from being additive and point to strong mutual interaction. In particular, this is a strong indication that the nature of dissipation is important in the enstrophy production and the properties of the vorticity field. This in turn means that it should be important in the properties of the velocity too as the latter is fully determined by the field of vorticity. A final remark is that the T-L balance points to a problem in the definition of the Reynolds number: at the level of vorticity/strain the production terms are of the same order as the viscous ones, i.e., the Reynolds number defined via comparing the inviscid (production) and the viscous terms is always of order of unity.

6.3.3. A NOTE ON KELVIN/HELMHOLZ THEOREMS

One of the consequences of the TL balance for vorticity is that vortex lines are not frozen in the flow field at however large Reynolds number, i.e. at any however small viscosity $\nu \neq 0$, in contrast with the pure inviscid (Euler) case $\nu = 0$. In this latter case vorticity is frozen in the flow field (Helmholz theorem) and the Kelvin theorem on conservation of circulation over material loops holds:

$$\frac{d}{dt}\oint_{C(t)} \mathbf{u}(\mathbf{r},t)\cdot d\mathbf{r} = 0.$$

In the presence of viscosity this relation becomes

$$\frac{d}{dt}\oint_{C(t)} \mathbf{u}(\mathbf{r},t)\cdot d\mathbf{r} = -\nu \oint_{C(t)} curl\omega(\mathbf{r},t)\cdot d\mathbf{r}$$

and the TL balance tells us that the RHS in this equation is not small at any, however small, viscosity, thus being one of the manifestations of the singular nature of the limit $\nu \to 0$. Hence the Kelvin theorem does not hold at any however small viscosity either. Constantin and Iyer, 2008 derived a probabilistic representation of the deterministic 3D NSE based on stochastic Lagrangian paths with the particle trajectories given by a stochastic differential equations driven by a uniform Wiener process. In this formulation some identities for Euler equations are formally valid after averaging (over the Wiener measure) with $\nu \neq 0$. These include the stochastic versions of the Kelvin theorem and the Cauchy formula for the evolution of vorticity $\omega_i(\mathbf{X}, t) = \omega_k(\mathbf{a}, t_0)\partial X_i/\partial a_k$. It is tempting to interpret such results of Constantin and Iyer, 2008 as a kind of "frozenness" of (stochastic) vortex-lines to the ensemble of stochastic flows $\mathbf{X}(\mathbf{a}, t)$ which replace the classical Lagrangian trajectories[14]. However, though such relations *look* like those for the 'usual' Euler equations, the consequences are quite different. For example, Constantin and Iyer, 2008 prove that vorticity given by their analogue to the classical inviscid Cauchy formula satisfies the 'ordinary' Helmholz equation (C.10), as should be, with $\nu \neq 0$. This means that the 'conventional' Kelvin/Helmholz theorems do not hold and there is no frozenness of vorticity in the *physical* flow field, they are not material lines in any sense, and their behavior is essentially different from that of material lines, see chapter 9.

A final remark is about a similar phenomenon of self-production of superhelicity $H_s = \int \omega \cdot curl\omega d\mathbf{x}$. It is observed in both helically and non-helically forced flows along with approximate balance (like the TL balance discussed above) between its production $\int curl\omega \cdot curl(\mathbf{u} \times \omega) d\mathbf{x}$ and dissipation $H_h = \int curl\omega \cdot curl\omega d\mathbf{x}$ and irrelevance of forcing at this level, see Galanti and Tsinober, 2006 for more information, details and references.

6.4. Geometrical statistics

Geometrical invariant quantities (such as enstrophy and strain production, helicity, etc., see appendix C) and relations, such as alignments between various vectors – being independent of the frame of reference - are among the most appropriate for studying physical processes, their possible universal properties, and the characterization of the structure(s) of turbulent flows. Moreover, just like phase relations, these are the quantities and relations of

[14]One of the important ingredients of the approach of Constantin and Iyer (2008) is the replacement *of classical Lagrangian trajectories by stochastic flows driven by the velocity field. Averaging the stochastic trajectories produces the solution* of the NSE. That is instead of deterministic Lagrangian trajectories they use *noisy trajectories* of a corresponding auxiliary stochastic equation *and average the noise out.* These noisy trajectories (the stochastic flow) are not material in any sense and do not possess Lagrangian identity associated with the fluid flow. They are employed in order to reproduce the viscous term in NSE as was done first by Chorin (1973) who demonstrated the connection between stochastic evolution and the NSE in two-dimensional flows.

utmost dynamical significance. A number of subtle issues related to *quantitative aspects* of structure of turbulence and other questions[15], which are beyond phenomenology, can be effectively addressed via what is denoted in the sequel by the term *geometrical statistics*[16].

Examples which belong to geometrical statistics were already given in chapter 4 and figures 6.1 and 6.6. Here we bring two more general examples.

It is rather common to use 'surrogates' of the type $(\partial u_1/\partial x_1)^n$ to represent the 'true' quantities such as dissipation, enstrophy ($n = 2$), enstrophy production ($n = 3$), etc. However, this is true only of their means, whereas other properties of the surrogates' and of the true quantities are generally different. Their PDFs are essentially different even in the case of a random Gaussian velocity field (Tsinober et al., 1992 and Shtilman et al., 1993). Hence most of their statistical properties are different too, and so are their spectra and 'fractal' properties.

The widely known example of utmost importance of geometrical relations in turbulence is the *qualitative* difference between the dynamics of 3D and 2D turbulence. In the latter $\omega_j s_{ij}$ and $\omega_i \omega_j s_{ij} \equiv 0$ because in two-dimensional flows $\omega \perp \lambda_i$ ($i = 1, 2$). One of the essential aspects of dynamics of 3D-turbulence – the interaction between vorticity, ω, and the rate of strain tensor, s_{ij} – depends strongly on the geometry of the field of velocity derivatives. For instance, a usual phenomenological argument results in the estimate $\omega_i \omega_j s_{ij} \sim \omega^3$, whereas in reality it is only $\omega_i \omega_j s_{ij} \sim \omega^{7/3}$ in slots of ω, but $\omega_i \omega_j s_{ij} \sim \omega^3$ in slots of s (see below), showing the importance of taking into account the mutual *orientation* of vorticity, ω, and the eigenframe, λ_i, of the rate of strain tensor s_{ij}. In other words important dynamical aspects of 3D-turbulence contained in the interaction between vorticity, ω, and the rate of strain tensor, s_{ij}, depends strongly not only on the magnitude of vorticity and strain but also on the geometry of the field of velocity derivatives.

6.4.1. ALIGNMENTS

Various alignments comprise important simple *geometrical* characteristics and manifestation of the *dynamics* and *structure* of turbulence. For example, there is a distinct qualitative difference between the PDFs of $\cos(\omega, \lambda_i)$

[15]Such as active versus passive, weak versus strong, Gaussian versus non-Gaussian, structured versus nonstructured and some others.

[16]In a broader sense geometrical statistics has its beginning in the works of Buffon (1777). Most of the work on the subject has been summarized by Stoyan and Stoyan (1992). A number of random geometric problems suggested by turbulence were reviewed by Corrsin (1972b). Some topological aspects (partially related to turbulence) are treated in Moffatt and Tsinober (1990, 1992), Moffatt et al. (1992) and Ricca and Berger (1996). A number of specific aspects of geometrical statistics in turbulence were raised by Constantin (1994) and Tsinober et al. (1995); see references in Tsinober (1998a,b).

for a real turbulent flow and a random Gaussian velocity field. In the last case, *all* these PDFs are precisely flat. An example of special dynamical importance is the strict alignment between vorticity, ω_i, and the vortex stretching vector $W_i \equiv \omega_j s_{ij}$, since the enstrophy production is just their scalar product, $\omega_i\omega_j s_{ij} = \omega \cdot \mathbf{W}$. In real turbulent flows, the PDF of $\cos(\omega, \mathbf{W})$ is strongly asymmetric in full conformity with the prevalence of vortex stretching over vortex compressing, i.e., positiveness of $\langle \omega_i\omega_j s_{ij} \rangle$, whereas it is symmetric for a random Gaussian field (see figures 6.9, 6.10). Thus, the very existence of alignments such as mentioned above points to the presence of internal organization of flow at various scales, i.e., alignments belong to the rare *quantitative statistical* manifestation of the existence of structure in turbulence. They are the simplest representative of a much broader class of geometrical statistics in turbulent flows. It is noteworthy, that while the above mentioned (and some others) alignments are intimately related to the *dynamics* of turbulent flows, there are alignments which are mostly of kinematic nature, e.g. alignment between the Lamb vector $\omega \times \mathbf{u}$ and its potential part (pressure gradient), the alignment between velocity and the eigenvectors of rate of strain tensor and some others (Tsinober, 1996a, 1998a; see also sections 6.7 and 6.8).

Alignments, by their very definition, are suitable for events of *any* magnitude, since they do not contain the amplitude of the quantities involved. Finally, alignments are *invariant* in the sense that they are independent of the system of reference and therefore, along with other invariant quantities, are the most appropriate in studying of physical processes generally and in particular for characterization of the structural nature of turbulent flows.

Due to these properties, using of alignments enables us to answer in a simple and reliable way a number of questions on turbulence structure.

6.4.2. THE GEOMETRY OF VORTEX STRETCHING

In order to address this issue let us recall some simple relations for the key quantities of turbulence dynamics – the vortex stretching vector, $W_i = \omega_j s_{ij}$, and enstrophy production, $\omega_i\omega_j s_{ij}$, and some related quantities (see also appendix C).

$$\omega_i\omega_j s_{ij} = \omega_i^2 \Lambda_i \cos^2(\omega, \lambda_i) = \alpha\omega^2; \quad W^2 = \omega_i^2 \Lambda_i^2 \cos^2(\omega, \lambda_i), \tag{6.1}$$

Here $\alpha = \Lambda_i \cos(\omega, \lambda_i)$ is the rate of enstrophy production. It is seen from the relations (6.1) that indeed – as mentioned above – that part of dynamics of 3D-turbulence contained in the interaction between vorticity, ω, and the rate of strain tensor, s_{ij}, depends strongly not only on the magnitude of vorticity and strain but also on the geometry of the field of velocity derivatives, in particular on the mutual orientation of vorticity, ω, and the eigenframe, λ_i, of the rate of strain tensor, s_{ij}. This is true especially regarding the *rate* of enstrophy production, $\alpha = \omega_i\omega_j s_{ij}/\omega^2 = \Lambda_i \cos(\omega, \lambda_i)$, and a similar quantity for W^2, $W^2/\omega^2 = \Lambda_i^2 \cos(\omega, \lambda_i)$, both of which depend explicitly only on the orientation of vorticity and the shape of the strain tensor, but not on their magnitude.

In view of the importance of the predominant vortex stretching and positive net enstrophy production, i.e., $\langle\omega_i\omega_j s_{ij}\rangle > 0$, it is useful to introduce an angle between ω and $\mathbf{W}$, since $\omega_i\omega_j s_{ij} \equiv \omega \cdot \mathbf{W}$ (Tsinober et al., 1992). It is easy to see from the simple relation

$$\cos(\omega, \mathbf{W}) = \frac{\Lambda_i \cos^2(\omega, \lambda_i)}{\{\Lambda_i^2 \cos^2(\omega, \lambda_i)\}^{1/2}} \tag{6.2}$$

that the alignment between ω and $\mathbf{W}$ (i.e., positive $\omega_i\omega_j s_{ij}$) is realized in two situations: *i*) ω is aligned with λ_1 ($\Lambda_1 > 0$) and *ii*) ω is aligned with λ_2. Indeed, the contributions both to $\sigma \equiv \omega_i\omega_j s_{ij}$ and α associated with Λ_1 and Λ_2 are positive (see table 6.5). Most important is that the largest contribution to the enstrophy production and its rate comes from the regions associated with the *largest* eigenvalue[17], Λ_1, of the rate of strain tensor, s_{ij}, and not from the one associated with the *intermediate,* eigenvalue Λ_2, since it is known that there exists a strong alignment tendency between ω and λ_2, as shown in figure 6.8. This alignment was recognized by Siggia (1981) and discovered by Ashurst et al. (1987) (for subsequent references see Tsinober, 1998a). This apparent contradiction is resolved by noting that: *i*) the intermediate eigenvalue, Λ_2, assumes both positive and negative values thus reducing the terms $\omega^2\Lambda_2 \cos^2(\omega, \lambda_2)$ and $\Lambda_2 \cos^2(\omega, \lambda_2)$, whereas Λ_1 is positive; and *ii*) the magnitude of Λ_1 is much larger (see table 6.6).

The reason that the contribution to $\langle\omega_i\omega_j s_{ij}\rangle$ associated with Λ_2 is positive is because Λ_2 is positively skewed (see figure 6.7; Ashurst et al., 1987; She et al., 1991; Su and Dahm, 1996; Tsinober et al., 1989, 1992).

Meanwhile we note that the alignments between ω and λ_1, ω and λ_2 and between ω and λ_3 correspond to regions of turbulent flow that are, in several respects, qualitatively different (see below).

Turbulence background – not a stuctureless random sea

Use of alignments allowed us to show that – contrary to the common view – the so-called 'background' is strongly non-Gaussian, is dynamically not passive and is not structureless (figures 6.8–6.10).

[17]This is qualitatively different from the alignment of vorticity with the *largest* eigenvalue Λ_1 of the 'nonlocal' part of strain defined in different but similar ways. The common feature is the exclusion of the 'local' part of the strain in the proximity of some (any) point in space $\mathbf{x}$ such as obtained by integration over a relatively small domain around this point of (C.15), see Porter et al. (1998); Hamlington et al. (2008).

TABLE 6.5. Contribution to the total mean of enstrophy production $\langle \omega^2 \Lambda_i \cos^2(\omega, \lambda_i) \rangle$ from the terms corresponding to the eigenvalues Λ_i of the rate of strain tensor s_{ij}. Grid turbulence and DNS, $\mathrm{Re}_\lambda = 75$ and field experiment $\mathrm{Re}_\lambda = 10^4$

	Re_λ	$\langle \omega^2 \Lambda_1 \cos^2(\omega, \lambda_1) \rangle$	$\langle \omega^2 \Lambda_2 \cos^2(\omega, \lambda_2) \rangle$	$\langle \omega^2 \Lambda_3 \cos(\omega, \lambda_3) \rangle$
DNS	75	1.06	0.51	– 0.57
Grid	75	1.17	0.39	– 0.56
Field	10^4	1.44	0.47	– 0.97

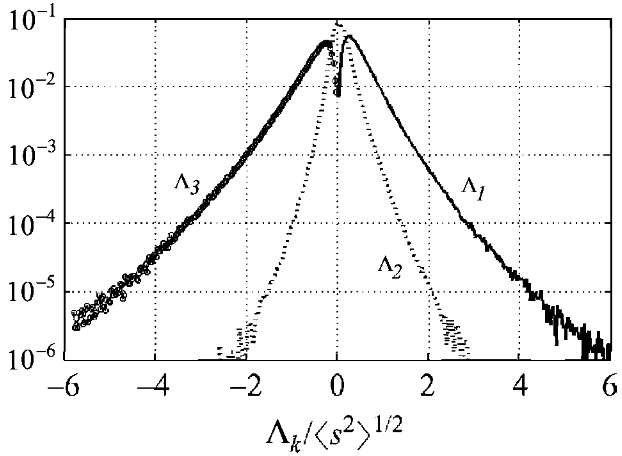

Figure 6.7. PDFs of the eigenvalues, Λ_i, of the rate of strain tensor, s_{ij}, in the field experiment at $\mathrm{Re}_{\lambda_i} = 10^4$. Note the skewed PDF of the intermediate eigenvalue, Λ_2. These PDFs are similar to those in the DNS and grid turbulence at $\mathrm{Re}_{\lambda_i} = 75$

TABLE 6.6. The ratios between the $\langle \Lambda_i \rangle$ and $\langle \Lambda_i^2 \rangle$ obtained in a field experiment in the atmospheric surface layer at the height $10m$, for $\mathrm{Re}_\lambda = 10^4$, see Kholmyansky et al. (2001b)

$\langle \Lambda_1 \rangle$	$\langle \Lambda_2 \rangle$	$\langle \Lambda_3 \rangle$
0.47	0.06	−0.53
$\langle \Lambda_1^2 \rangle$	$\langle \Lambda_2^2 \rangle$	$\langle \Lambda_3^2 \rangle$
0.41	0.04	0.55

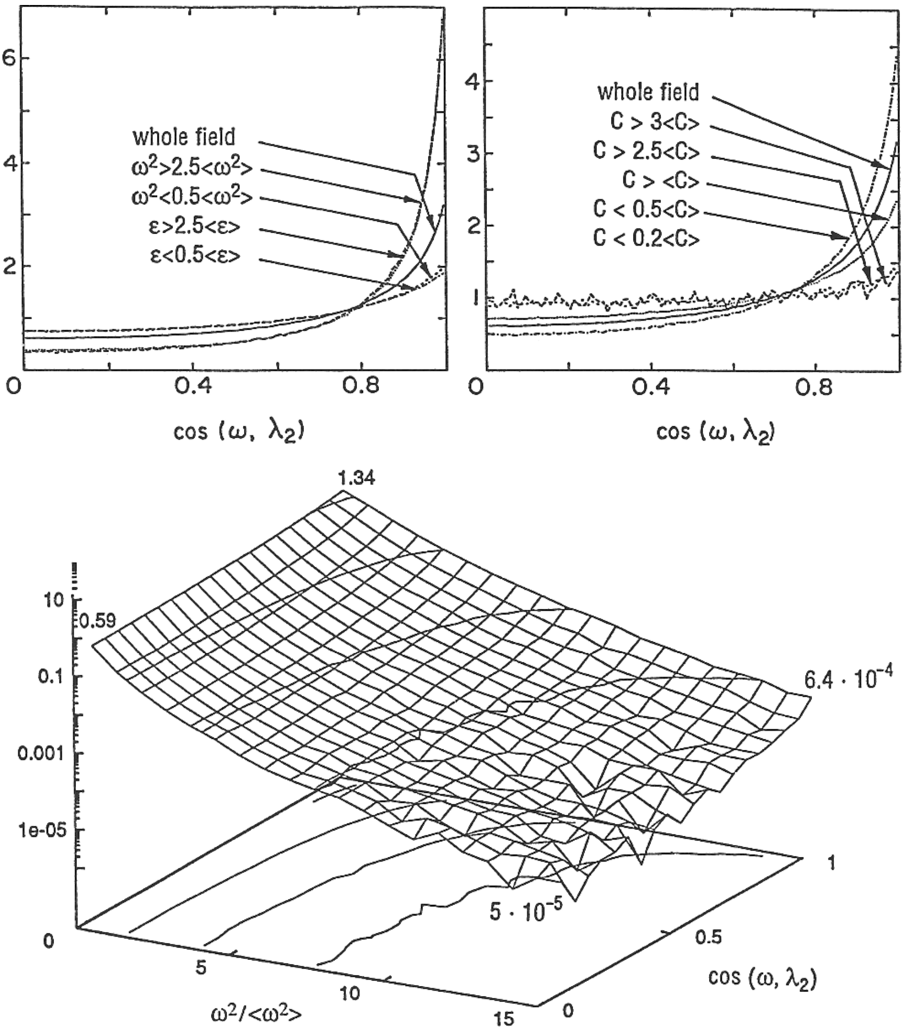

Figure 6.8. PDFs of $\cos(\omega, \lambda_2)$, DNS, $\mathrm{Re}_\lambda = 75$. Top left – conditioned on enstrophy ω^2 and s^2, top right – conditioned on curvature C of vortex lines. Note that the tendency for alignment between ω and λ_2 exists both in regions of large ω^2 and large s^2. For a Gaussian velocity field these PDFs are precisely flat. Bottom – joint PDF of $\cos(\omega, \lambda_2)$ and ω^2. The joint PDF of $\cos(\omega, \lambda_2)$ and s^2 is similar to the one shown in this figure. It is seen that the maximum of joint PDF of $\cos(\omega, \lambda_2)$ and ω^2 (and similarly of $\cos(\omega, \lambda_2)$ and s^2) takes place at $\cos(\omega, \lambda_2) \approx 1$ and $\omega^2 \approx 0$, i.e., at the points with *weakest* vorticity and *strongest* alignment between ω and λ_2

Though the strongest tendency for alignment between ω and λ_2 is observed for large ω^2, this alignment is still significant (see bottom of figure 6.8) in the 'background' (say $\omega^2 < \langle\omega^2\rangle$), especially taking into account that the background is occupying about 70% of the flow volume (*cf.* with the volume occupied by strong vorticity, say $\omega^2 > 3\langle\omega^2\rangle$, which is only about 6% of the flow volume). Note that this does not contradict the mostly known result about the tendency of alignment between ω and λ_2 in regions of concentrated vorticity; the regions with such an alignment are an order of magnitude larger than those with concentrated vorticity only.

Similar results are valid for the normalized enstrophy production $\omega_i\omega_j s_{ij}$ ω^{-1} $W^{-1} = \cos(\omega, \mathbf{W})$, figure 6.10. Just as in the case of $\cos(\omega, \lambda_2)$ the tendency for alignment between ω and $\mathbf{W}$ exists *both* in regions of large ω^2 and s^2. However, it is much stronger for large strain s^2, as are all the nonlinearities in these regions (see section 6.5). It has been seen that the maximum of joint PDF of $\cos(\omega, \mathbf{W})$ and ω^2 (and of $\cos(\omega, \mathbf{W})$ and s^2) takes place at $\cos(\omega, \mathbf{W}) \approx 1$ and $\omega^2 \approx 0$, i.e., at the points with *weakest* vorticity and *strongest* alignment between ω and $\mathbf{W}$.

Note also the strong asymmetry of the PDF of $\cos(\omega, \mathbf{W})$ for the background $\omega^2 < \langle\omega^2\rangle$, which is almost the same as for the whole field. This asymmetry remains significant, even for $\omega^2 < 0.1\langle\omega^2\rangle$, and becomes *stronger* for $\omega^2 < \langle\omega^2\rangle$ and $\cos(\omega, \lambda_2) > 0.9$ (not shown). Moreover, this asymmetry remains significant for *both* small ω^2 and s^2 (see figure 6.10) showing the significance of the background. We are reminded that, for a Gaussian velocity field, the PDF of $\cos(\omega, \mathbf{W})$ is symmetric. One can see from figures 6.8 and 6.9 that the maxima of the joint PDFs of both $\cos(\omega, \lambda_2)$ and $\cos(\omega, \mathbf{W})$ are located at *weakest* enstrophy and *strongest* alignment between ω and λ_2, and ω and $\mathbf{W}$. The same is true for a variety of joint PDFs of other quantities (see references in Tsinober, 1998a).

The above results show clearly that the background is strongly non-Gaussian, not structureless and not passive.

Strained vortical (Burgers-like) objects

Regions with concentrated vorticity constitute a subset of much larger regions in which there is a tendency for alignment between ω and λ_2. This is clearly seen from figure 6.8. Indeed, regions corresponding to $\cos(\omega, \lambda_2) > 0.9$ occupy about 20% of the total flow volume, whereas the set of points with concentrated vorticity, say, $\omega^2 > 3\langle\omega^2\rangle$, is comprised of less than 6% of the total flow volume.

The main feature and shortcoming of these objects (straight strained vortices) is that they possess *one*-dimensional vorticity and therefore zero curvature of vortex lines. Though the relation between vorticity and strain is essentially nonlocal, '*the presence of a strained vortex itself modifies the local strain field*' (Le Dizes et al., 1996) – after all both are composed of derivatives of the same velocity field. However, the special feature of the *straight* strained vortices is that they are impotent in the sense that they do not change that part of the strain by which they are strained themselves:

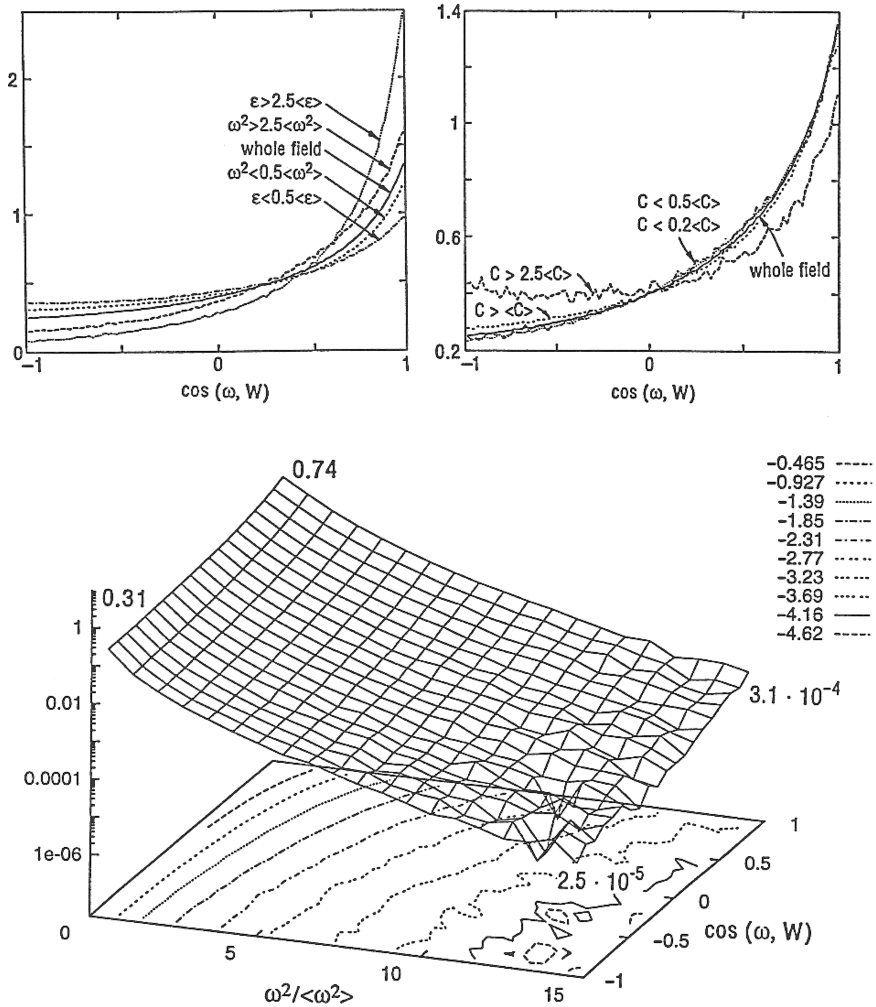

Figure 6.9. PDFs of $\cos(\omega, W)$, $W_i = \omega_i s_{ij}$; DNS, $\mathrm{Re}_\lambda = 75$. Top left – conditioned on enstrophy ω^2 and s^2, top right – conditioned on curvature of vortex lines; bottom – joint PDF of $\cos(\omega, W)$ and ω^2. The joint PDF of $\cos(\omega, W)$ and s^2 is similar to the one shown in this figure

this part of strain is prescribed *a priori,* i.e., it is independent decoupled from its vorticity. *These vortices do change only that part of the strain field which is not reacting back on their vorticity.* In other words, there is only one way interaction: the vorticity is strained by that part of strain which does not 'know' anything about the vorticity. In this sense such vortices are passive: the essential ingredient of nonlinearity, the main feature of true genuine nonlinear interaction – the *self-amplification* via *interaction*

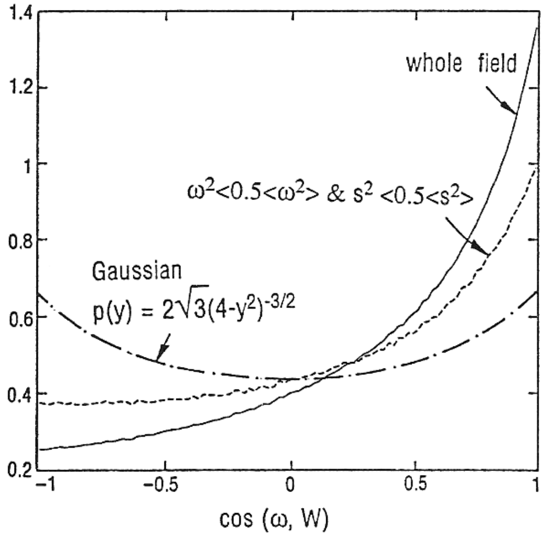

Figure 6.10. PDFs of $\cos(\omega, W)$ for the 'weakest' part of turbulent flow; DNS, $\mathrm{Re}_\lambda = 75$

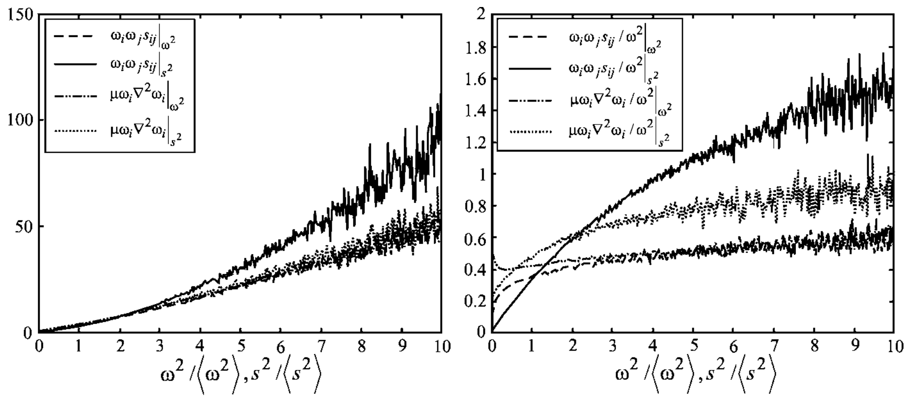

Figure 6.11. Comparison of enstrophy production (left) and its rate (right) with their viscous reduction in slots of ω^2 and s^2. DNS, $Re_\lambda = 100$

with strain – is absent in these objects. In this sense, the nonlinearity is reduced in these objects. This property is directly related to zero curvature of vortex lines in straight strained vortices – the genuine nonlinearity is present only in regions with nonvanishing curvature. This is what is observed when looking for (apparent) singularities of Euler equations and vortex reconnection (see references in Tsinober, 1998a). In other words regions with concentrated vorticity with small curvature in real turbulent flows seem to be mostly the result, the consequence rather than dominating factor of the turbulence dynamics. Possessing (almost) maximal enstrophy they are in an approximate equilibrium in the sense that their fairly large, (*but not largest*!, see next section) enstrophy production is approximately balanced by the viscous reduction, and in this sense, they are less active than the strain dominated regions possessing much larger (apparently maximal) enstrophy production, which is considerably larger than its viscous reduction. This is seen from the comparison of the rate enstrophy production $\alpha \equiv \omega_i\omega_k s_{ik}/\omega^2$ and its viscous reduction $\nu\omega_i\nabla^2\omega_i/\omega^2$ in slots of ω and s as shown in figure 6.11. Indeed, the imbalance between stretching and viscous terms in slots of s is much *larger* than in slots of ω. This difference is especially large at large values of ω and s. This means that the time scale estimated from the imbalance of stretching and viscous terms $\omega^2\{D_t(\omega^2/2)\}^{-1} \approx \{\omega_i\omega_k s_{ik}/\omega^2 + \nu\omega_i\nabla^2\omega_i/\omega^2\}^{-1}$ in slots of ω is much larger than such a time scale in slots of s. In other words, the life time of regions with concentrated vorticity is large compared to that of regions with large strain, i.e., large rate of energy dissipation. This explains – at least in part – the observability of the regions with concentrated vorticity and the difficulties in observing the regions with large dissipation. It also points to the importance of studying more carefully such regions of turbulent flows as those with strong imbalance between vortex stretching and viscous destruction of vorticity. It is noteworthy that the Burgers-like objects in real turbulent flows possess small but not vanishing curvature, so that the self-amplification of their vorticity is not vanishing, as in the perfectly straight ones used in a great variety of models[18].

Regions of strongest vorticity/strain interaction

The important point is that at least in quasi-isotropic flows the largest contribution to the enstrophy production $\omega_i\omega_j s_{ij} = \omega_i^2\Lambda_i\cos^2(\omega,\lambda_i)$ comes from the regions associated with the *largest* eigenvalue Λ_1 of the rate of strain tensor s_{ij} and not from the ones associated with the *intermediate* eigenvalue Λ_2 to which mainly belong the regions of concentrated vorticity. Namely the ratio of $\langle\omega^2\Lambda_1\cos^2(\omega,\lambda_1)\rangle$ to $\langle\omega^2\Lambda_2\cos^2(\omega,\lambda_2)\rangle$ is roughly 2:1 or even more as in the field experiment mentioned several times before (see table 6.5). The same is true of other nonlinearities (see section 6.5).

This shows that there exist regions (intense and weak – both structured and dynamically active) other than concentrated vorticity regions, which

[18]See, for example, the review by Pullin and Saffman (1997) and the papers by Hosokawa (2000) and Kambe and Hatakeyama (2000). A number of arguments and facts were given in Tsinober (1998a) as to why regions of concentrated vorticity in turbulent flows are not as important as previously thought. Here we mention in addition some of the latest references supporting various aspects of this view: Chavanis and Sire (2000); Dernoncourt et al. (1998); Min et al. (1996); Roux et al. (1998) and Sain et al. (1998).

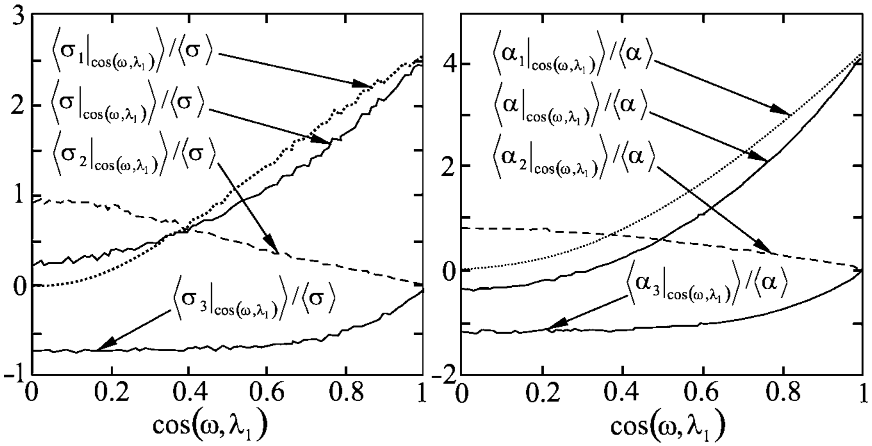

Figure 6.12. Conditional averages of enstrophy production σ (left) and its rate α (right) in slots of $\cos(\omega,\lambda_1)$. DNS, $\mathrm{Re}_\lambda = 75$

at least in the above sense are dynamically more important. These regions are associated mainly with largest strain rather than enstrophy, strong tendency of alignment between ω and λ_1 (see table 6.5 and figure 6.12) and fairly large curvature of vorticity lines. These regions are characterized by the largest, apparently maximal, enstrophy production and its rate (as shown in figure 6.14), which are much larger than their viscous reduction as discussed above. This is consistent with the PDFs of the rate of enstrophy production conditioned on ω and s (see figure 6.13) and with the results of Constantin et al. (1996) that the dominating contribution to $\omega_i\omega_j s_{ij}/\omega^2$ comes from the local (self) interaction of vorticity ω and strain s_{ij}, which is absent in Burgers-like objects. The behaviour of W^2 and W^2/ω^2 in slots of ω and s is essentially the same. These results also show that there is much vortex compression in regions with concentrated vorticity (see also Jimenez and Wray, 1994). Similarly the dependence of enstrophy production $\sigma \equiv \omega_i\omega_j s_{ij}$ and its rate $\alpha \equiv \Lambda_i \cos^2(\omega,\lambda_i)$ on ω and on $s \equiv (s_{ij}s_{ij})^{1/2}$ is qualitatively different for small and large curvature of vortex lines in such a way that the nonlinearity is manifested stronger in regions of large curvature. In particular, the disparity in the behaviour of σ and α in slots of ω and s becomes larger at small curvature, whereas at large curvature the dependence of σ and α on ω and s is very similar. This last fact is a reflection of stronger interaction of vorticity and strain in regions with *large* curvature and *positive* α and, consequently, with non-negligible vortex folding and tilting (see the next section). The regions just discussed comprise a *subset* of larger regions dominated by strain. Namely, these are the regions with large vortex lines curvature. There exist at least two other kinds of

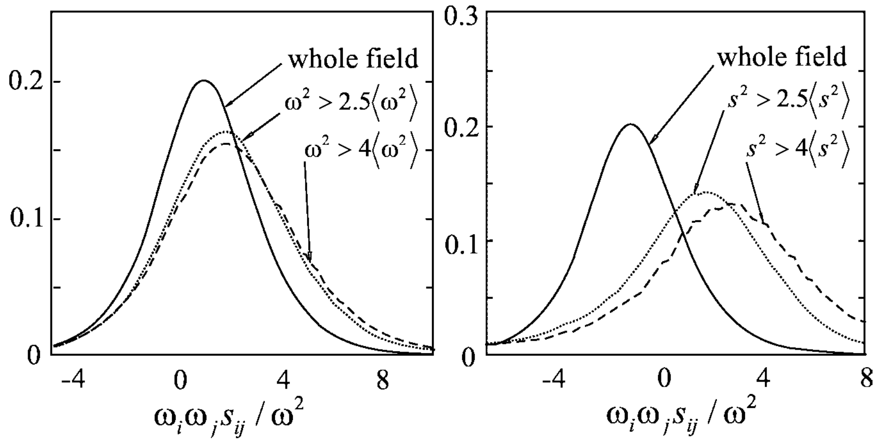

Figure 6.13. PDF's of the enstrophy production rate for the whole field and conditioned on ω^2 (left) and s^2 (right). DNS, $\mathrm{Re}_\lambda = 75$. Considerable regions with vortex compression exist also for large enstrophy (see also Jimenez and Wray, 1994), whereas in regions with large strain the rate of enstrophy production is mostly positive

strain dominated regions: those with small curvature of vortex lines, which wrap around the vorticity dominated regions (tubes/worms), and which contribute mostly to the alignment of ω and λ_2 as shown in figure 6.8. There are also regions with large magnitude of Λ_3 and large negative α, in which most of vortex *compressing, tilting and folding* occur.

Vortex compression, tilting, folding and curvature

The most basic phenomenon in turbulence – the predominant vortex stretching, i.e., predominant enstrophy production, $\sigma \equiv \omega_i\omega_j s_{ij}$, so that $\langle\omega_i\omega_j s_{ij}\rangle > 0$ – cannot occur in a finite volume and finite energy without its concomitants – vortex compressing ($\sigma < 0$) and folding (Chorin, 1982, 1994)[19]. Hence, the importance of looking at properties of turbulent flow in regions with large curvature and $\sigma < 0$, which typically occupy about 1/3 of the whole flow volume, and for the evidence and characterization of the vortex folding in three-dimensional turbulence. These regions play an important role in the dynamics of turbulence. For example, these regions make a positive contribution to the magnitude of the vortex stretching vector $W_i \equiv \omega_j s_{ij}$ in (C.9). Indeed, $W^2 = \omega^2\Lambda_i^2 \cos^2(\omega, \lambda_i)$ and $W^2/\omega^2 \equiv \Lambda_i^2 \cos^2(\omega, \lambda_i)$ are large for large $\Lambda_3^2 \cos^2(\omega, \lambda_3)$, for which the enstrophy production $\sigma \equiv \omega_i\omega_j s_{ij} = \omega^2\Lambda_i \cos^2(\omega, \lambda_i)$ and its rate $\alpha = \sigma\omega^{-2} =$

[19]The term folding was introduced by Reynolds in 1894 in the context of folding of material lines.

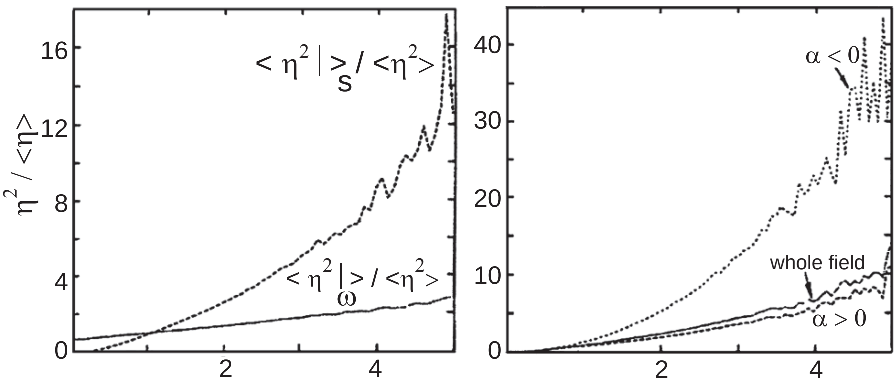

Figure 6.14. Conditional averages of the magnitude of the rate of change of vorticity direction $\eta^2 = W^2/\omega^2 - \alpha^2 = \Lambda_i^2 \cos^2(\omega, \lambda_i) - \{\Lambda_i \cos^2(\omega, \lambda_i)\}^2$. Left – in slots of ω and s, from which it is seen that the direction of vorticity is changing much stronger in strain dominated regions. Right – in slots of Λ_3, showing that this rate of change is (apparently) largest in (sub)regions of vortex compression with large magnitude of Λ_3. Similar increase of η^2 is observed in slots of Λ_1 and Λ_2 too, but at slower rates (not shown). Note that it is not so simple to separate the contributions to η^2 associated with the eigenvalues Λ_i

$\Lambda_i \cos^2(\omega, \lambda_i)$ are *negative* (see table 6.7). Similarly, enstrophy production (and α) can be small, whereas W^2 (and W^2/ω^2) can be large.

TABLE 6.7. Contribution to the total mean of the magnitude of vortex stretching vector $\langle W^2 \rangle \equiv \langle \omega^2 \Lambda_i^2 \cos^2(\omega, \lambda_i)\rangle$ from the terms corresponding to the eigenvalues Λ_i of the rate of strain tensor s_{ij}. DNS, $\mathrm{Re}_\lambda = 75$. Field experiment $\mathrm{Re}_\lambda = 10^4$

	Re_λ	$\langle \omega^2 \Lambda_1^2 \cos^2(\omega, \lambda_1)\rangle$	$\langle \omega^2 \Lambda_2^2 \cos^2(\omega, \lambda_2)\rangle$	$\langle \omega^2 \Lambda_3^2 \cos^2(\omega, \lambda_3)\rangle$
DNS	75	0.53	0.15	0.32
Field	10^4	0.52	0.12	0.36

A closely related process is the vortex tilting, which is characterized by the rate of change of direction of vorticity. This rate is obtained from the equations (6.3, 6.4) for the magnitude of vorticity ω and its unit vector $\varpi_i = \omega_i/\omega$, which are equivalent to the equations (C.9, C.16) without the forcing terms

$$D_t\omega = \alpha\omega + vt, \quad D_t\varpi_i = s_{ij}\varpi_j - \alpha\varpi_i + VT, \tag{6.3, 6.4}$$

where and VT stands for viscous terms. The vector $\eta_i = s_{ij}\varpi_j - \alpha\varpi_i = W_i/\omega - \alpha\omega_i/\omega$ is the inviscid rate of change of the unit vector ϖ along the direction of vorticity ω, and is responsible for the rate of change of its *direction*, Constantin (1994), and $\eta \perp \omega$, i.e., vector η is associated with inviscid vorticity *tilting*. Its magnitude is $\eta^2 = W^2/\omega^2 - \alpha^2 = \Lambda_i^2 \cos^2(\omega, \lambda_i) - \{\Lambda_i \cos^2(\omega, \lambda_i)\}^2$. From this it is seen that in regions with negative enstrophy production the rate of change η of the unit vector ϖ can be large, since, as mentioned, these regions make a positive contribution to the magnitude of the vortex stretching vector $W_i \equiv \omega_j s_{ij}$, so that $W^2/\omega^2 = \Lambda_i^2 \cos^2(\omega, \lambda_i)$ can be large and $\alpha^2 = \{\Lambda_i \cos^2(\omega, \lambda_i)\}^2$ can be small. This happens in regions associated with large magnitudes of Λ_3 as is seen from figure 6.14.

It is reasonable to associate the above process with large curvature of vortex lines and similar quantities, which should reflect their folding and tilting – at least the resulting aspect of these processes. Hence among the questions of interest are those about the properties of curvature and the relation between curvature and dynamically relevant quantities such as enstrophy ω^2, enstrophy production σ, rate of enstrophy generation $\alpha \equiv \sigma/\omega^2$ and relations such as various alignments. Of course, the ultimate clarification of such relations can be obtained from looking at global properties. One can hope that some insights can be gained from local analysis, i.e., from working with point quantities at a particular time moment.

In a simplified form, the logic is that strong stretching results in strong vorticity: indeed regions with strong vorticity are known to be tube-like with a small curvature, as observed visually in a number of numerical simulations. However, closer inspection shows that matters are much more complicated due to a number of qualitative differences between material and vortex lines (see chapter 9). The curvature decreases with magnitude of ω. It appears that this behaviour is practically the same for the whole field, for positive and for *negative* rate of enstrophy production[20]. This last fact, i.e., the behaviour of curvature C versus ω for *negative* rate of enstrophy production ($\alpha < 0$) and strong increase of curvature with strain[21] (see figure 6.15) undermines the simple analogy with the behaviour of material lines in turbulent flows.

Similarly, as is expected, the curvature of vortex lines is *increasing* with $|\alpha|$ for $\alpha < 0$ due to *folding* of vortex lines, but again, most interestingly the same behaviour of C is observed for $\alpha > 0$ due to self-induction, unlike the case of material lines. This is consistent with the results on the comparison of dependence of enstrophy generation $\omega_i \omega_k s_{ik}$ and its viscous reduction $\nu \omega_i \nabla^2 \omega_i$ on ω and s (figure 6.11) and also curvature C. Namely, the preferential alignment between ω and λ_2 is correlated with small curvature

[20]The reader is reminded again that typically regions with $\alpha > 0$ occupy about 2/3 of the turbulent flow field, and regions with $\alpha < 0$ comprise about 1/3 of the whole flow volume.

[21]Again for the whole field and both for $\alpha > 0$ and for $\alpha < 0$.

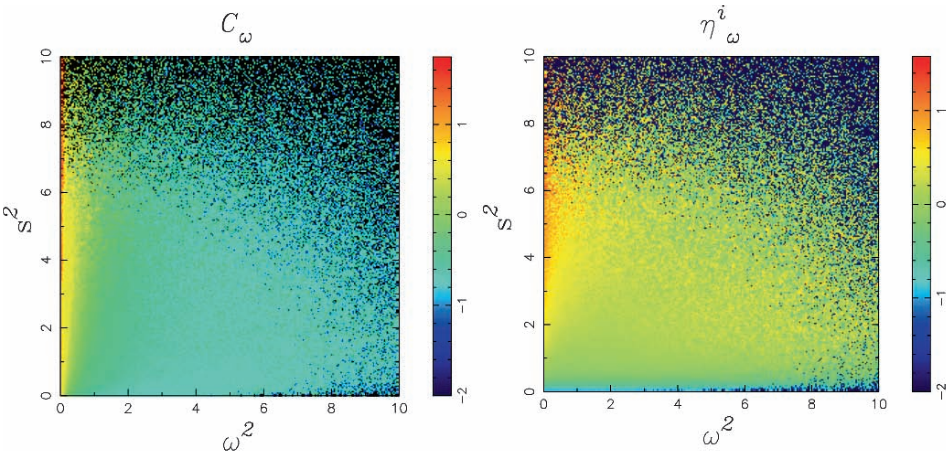

Figure 6.15. Conditional averages of curvature C_ω of vortex lines (left) and magnitude of the inviscid tilting η^i_ω of vorticity (right) on ω^2 and s^2 in a box DNS at $\mathrm{Re}_\lambda \approx 100$. Courtesy B. Galanti

and there is no preferential alignment between ω and λ_2 at large curvature (figure 6.8, top right).

The above shows that the 'most nonlinear' are the regions with large curvature, dissipation, i.e., strain, and preferable alignment between ω and λ_1, and not the regions of concentrated vorticity with small curvature and preferable alignment between ω and λ_2, such as the filaments observed in direct numerical simulations of the Navier–Stokes equations and laboratory experiments[22]. This brings us to next issue.

6.5. Depression of nonlinearity

The notion known as *depression of nonlinearity* was introduced by Kraichnan and Panda (1988). Since this paper, several aspects of this problem have been addressed (see references in Tsinober et al., 1999). Kraichnan and Panda suggested comparing the nonlinearities in real turbulent flows with their Gaussian counterparts. This is meaningful for even moments only, for example,

$\langle|\mathbf{u}\times\omega|^2\rangle/\langle|\mathbf{u}\times\omega|^2\rangle_G < 1$ (~ 0.8); $\left\langle W^2\right\rangle/\left\langle W^2\right\rangle_G < 1$ $(\sim 0.7 \div 0.8)$; $W_i \equiv \omega_j s_{ij}$.

$\langle|\mathbf{u}\times\omega - \nabla(p+\frac{1}{2}u^2)|\rangle/\langle|\mathbf{u}\times\omega - \nabla(p+\frac{1}{2}u^2)|\rangle_G < 1$ $(\sim 0.5 \div 0.6)$.

In this sense nonlinearity is reduced. However, as measured by odd moments the real nonlinearity is 'infinitely' larger, since for a Gaussian velocity

[22]It is noteworthy that regions of concentrated vorticity are not free of vortex compression in the same proportion as in the whole turbulent field, see below.

field the odd moments vanish identically. This includes the longitudinal velocity structure functions of odd order $S_{2n+1}(r) = \langle\{[\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{r}/r\}^{2n+1}\rangle$, enstrophy generation $\langle\omega_i\omega_k s_{ik}\rangle$, $\langle s_{ij}s_{jk}s_{ki}\rangle$ and many others. In other words, the build up of *odd* moments is an important *specific* manifestation of the nonlinearity of turbulence, and is, as well, a manifestation of its structure – there is no turbulence without odd moments; the nonzero $\langle\omega_i\omega_k s_{ik}\rangle$ is associated with the strict alignment between ω and $\mathbf{W}$. In this sense this alignment enhances the nonlinearity. We have seen above that this alignment is significant throughout *all* the regions of turbulent flow. On the other hand, the alignment between $\mathbf{u}$ and *rot* ω reduces $\langle\omega_i\omega_k s_{ik}\rangle$. Indeed, since $\langle\omega_i\omega_k s_{ik}\rangle = \langle\omega \cdot rot(\mathbf{u}\times\omega)\rangle = \langle rot\omega \cdot (\mathbf{u}\times\omega)\rangle = -\langle\omega \cdot (u \times rot\omega)\rangle$, and since $-\langle\mathbf{u} \cdot rot\ \omega\rangle \equiv \langle\omega\rangle^2 > 0$ there is a tendency of (anti-)alignment between $\mathbf{u}$ and *rot* ω reducing the magnitude of $\mathbf{u} \times rot\omega$ and thereby of $\langle\omega_i\omega_k s_{ik}\rangle$. Though this is a purely kinematic effect it is directly related to the dissipative and rotational nature of turbulent flows, since the mean dissipation $\langle\epsilon\rangle \simeq \nu\langle\omega\rangle^2$.

One more aspect of reduction of nonlinearity is related to the tendency of alignment between $\mathbf{u}$ and ω, and the so-called beltramization (see Moffatt and Tsinober, 1992; Tsinober, 1998a; and references therein). The alignment between $\mathbf{u}$ and ω implies reduction of the magnitude of the Lamb vector, $\omega \times \mathbf{u}$, (e.g. $\langle|\omega \times \mathbf{u}|^2\rangle$) and corresponding increase of a quantity called helicity density $h = \mathbf{u}\cdot\omega$ (e.g., $\langle|\mathbf{u}\cdot\omega|^2\rangle$). This is a very difficult and controversial issue for many reasons. First, $\mathbf{u}$ and ω are weakly correlated by their very nature – $\mathbf{u}$ is a large-scale quantity, and ω is a small-scale one. Second, the Lamb vector is neither potential nor solenoidal, and it has a large potential part (see section 6.6). The latter can be considered as a kind of reduction of nonlinearity, since only the solenoidal part of $\omega \times \mathbf{u}$ matters in the dynamics of vorticity. However, this is not the case with strain.

6.5.1. RELATIVE DEPRESSION OF NONLINEARITY IN REGIONS WITH CONCENTRATED VORTICITY

We are interested here in the behaviour of the key nonlinearities related to velocity gradients in flow regions dominated by enstrophy and strain. Typical examples are the magnitude of the vortex stretching vector $W \equiv |\omega_j s_{ij}|$, the enstrophy generation $\omega_i\omega_j s_{ij}$ and its rate $\omega_i\omega_j s_{ij}/\omega^2$, the production of dissipation, i.e., the inviscid terms in equation (C. 18), the inviscid terms in equations (C.23, C.24) and all physically meaningful nonlinearities involving velocity derivatives. The central result is that all of them – though increasing with ω (e.g., $\omega_i\omega_j s_{ij}$ increases as $\omega^{7/3}$, i.e. faster than ω^2, but slower than ω^3), are essentially reduced in regions dominated by enstrophy as compared to the strain dominated regions, see figures 6.11, 6.12, 6.14. Another manifestation of depression of nonlinearity is the decrease in the curvature of the vortex lines in the regions with concentrated vorticity, and enhanced rate of change of vorticity direction in the strain dominated regions (see figures 6.14, 6.15). Indeed, we have seen that enstrophy production and its rate are much larger in strain dominated regions (than that in enstrophy dominated ones) with finite curvature of vortex lines. Both are associated with the largest eigenvalue Λ_1 of the rate of strain tensor and alignment between ω and λ_1. The main contribution to vortex stretching in these regions comes from local effects associated with the (self) interaction of ω and s_{ij} (Constantin et al., 1996) in contrast with the enstrophy dominated regions in which the vortex stretching is sustained mostly by nonlocal effects. Similarly, other nonlinear dynamically relevant quantities, e.g., magnitude of the vortex stretching vector W^2 and its rate W^2/ω^2, and the quantity η^2 responsible for the vorticity tilting (see figure 6.14 left), eigen-contributions to the enstrophy production $\omega^2\Lambda_k \cos(\omega, \lambda_k)$ (no summation over k) are also strongly reduced (see figure 10. 2) in regions of concentrated vorticity, as compared to their values in strain dominated regions.[23]

Another aspect of reduction of nonlinearity is seen clearly from figure 6.16. Namely, the nonlinearity associated with the production of $\omega_i\omega_j s_{ij}$ (see equation (C.23)) occurs only in strain dominated regions. Note the *qualitatively* different behaviour of $\omega_i\omega_j \frac{\partial^2 p}{\partial x_i \partial x_j}$ (figure 6.16) in enstrophy dominated regions (decreasing with ω) as compared to that in strain dominated regions (increasing with s); for more see Tsinober et al. (1999) and Tsinober, 2001a.

6.5.2. ARE REGIONS OF CONCENTRATED VORTICITY QUASI-ONE-DIMENSIONAL?

One of the popular reasons for reduced nonlinearity in the regions of concentrated vorticity, i.e., mainly in the long, thin tubes-filaments-worms, is because these objects are believed to be in some (!) sense locally quasi-one-dimensional (Frisch, 1995), i.e., that nonlinearity is stronger *outside* of these structures. Hence the term depletion (expulsion) of nonlinearity. Following this line one would expect that in regions with strong alignment between vorticity ω and the intermediate eigenvector λ_2, vortex stretching and enstrophy generation should decrease as $|\cos(\omega, \lambda_2)|$ increases. Indeed W and its rate are decreasing but remain essentially finite. However, contrary to the above expectation the enstrophy generation and its rate *increase* in

[23] Though the above results are likely to be true at large Reynolds numbers, they cannot be seen as an indication that NSE may not develop a singularity in finite time, since these results reflect statistical tendencies. For example, there exist small regions with very large enstrophy, enstrophy production and alignment between ω and λ_1.

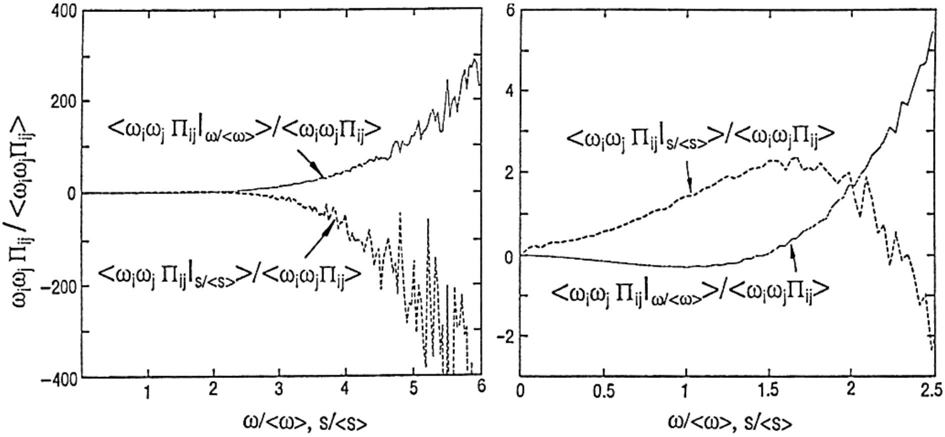

Figure 6.16. Conditional averages of the nonlocal term $\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}$ in slots of ω and s, DNS. $\mathrm{Re}_\lambda \approx 75$. (see equation (C.23))

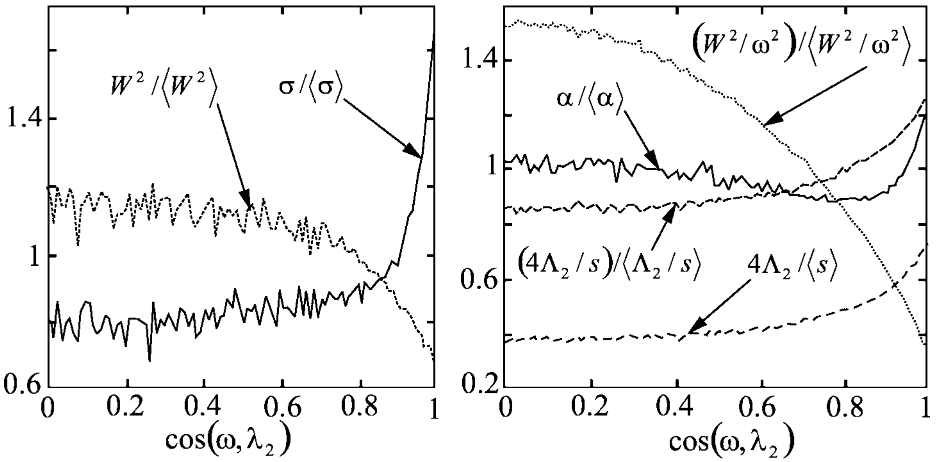

Figure 6.17. Conditional averages of: left – enstrophy production $\sigma \equiv \omega_i\omega_j s_{ij}$, vortex stretching W^2, right – rates of enstrophy production α, vortex stretching W^2/ω^2, intermediate eigenvalue of the rate of strain tensor Λ_2 and the ratio Λ_2/s in slots of $\cos(\omega, \lambda_2)$. DNS, $\mathrm{Re}_\lambda = 75$

slots of $|\cos(\omega, \lambda_2)|$ and become *maximal* at $|\cos(\omega, \lambda_2)| \sim 1$ (figure 6.17). In other words in these regions the rate of creation of enstrophy ω^2 is the largest and in this sense the nonlinearity is *stronger* and not weaker than in, at least, some of their background using $|\cos(\omega, \lambda_2)|$ as a criterion. The above tendencies are stronger in regions with strong vorticity and survive in the background, e.g., regions of weak enstrophy. Note that none of the quantities $\omega_i\omega_j s_{ij}$, W, Λ_2 and Λ_2/s become small for $|\cos(\omega, \lambda_2)| \sim 1$ indicating that *the flow does not become locally two-dimensional.* In particular, it is important that in these regions the intermediate strain (i.e., Λ_2) is positive and is increasing along with $|\cos(\omega, \lambda_2)|$, which corresponds to strong straining in these regions (*cf.* with pure two-dimensional flow in which $\Lambda_2 \equiv 0$). Thus one can speculate that there is a tendency to 'localization of nonlinearity' in space which, somewhat paradoxically, is sustained by nonlocal effects due to the nonlocal relation between strain and vorticity and due to pressure ('nonlocal localization'; see next section on nonlocality of turbulence). Note, that the claim to 'localization of nonlinearity' is supported by the behaviour of Λ_2/s in slots of $|\cos(\omega, \lambda_2)|$, which is similar to the one of $\Lambda_2/\langle s\rangle$ as shown in figure 6.17. In order to get more insight it is necessary to look into more subtle aspects of geometrical statistics than just single space/time point alignments. For the moment, it is clear that 'simple' structures in three-dimensional turbulence are qualitatively different from those in pure two-dimensional turbulence in which the nonlinearity is really depleted in such structures – however, in three-dimensional turbulence such structures do not seem to be the best candidates to look for depletion of nonlinearity in the absolute sense. Nevertheless, we have seen above that taking the enstrophy generation $\omega_i\omega_j s_{ij}$ as a measure of nonlinearity, the objects with strong alignment between ω and λ_2 appear to be not the most nonlinear, since their enstrophy generation $\omega_i\omega_j s_{ij}$ comes mostly from nonlocal effects and not from self-stretching.

The non-2D character of the regions with concentrated vorticity is seen also from figure 6.12 left and figure 6.18. In addition Jimenez and Wray, 1994 have shown that the statistics of the rate of enstrophy production is the same in the whole flow field and in the 'worms'. It is noteworthy that the features seen in figures 6.12, 6.17 and similar ones were, in fact, reported already by Ruetsch and Maxey, 1991.

6.5.3. ADDITIONAL ISSUES

Pressure Hessian

Equation (C.23) for the rate of change of enstrophy production contains two terms. The first term in (C.23) (which is just the squared magnitude of the vortex stretching vector) is strictly positive, $\omega_i s_{ij}\omega_k s_{ki} \equiv W^2 > 0$. This means that the nonlinear processes involving vortex stretching (or direct interaction of vorticity and strain) always act to increase even the *instantaneous* enstrophy production. This, however, does not explain why in turbulent flows $\langle\omega_i\omega_j s_{ij}\rangle > 0$, since the inviscid rate of change of enstrophy production contains also a second term reflecting the interaction (which has a nonlocal contribution) between vorticity and the pressure Hessian

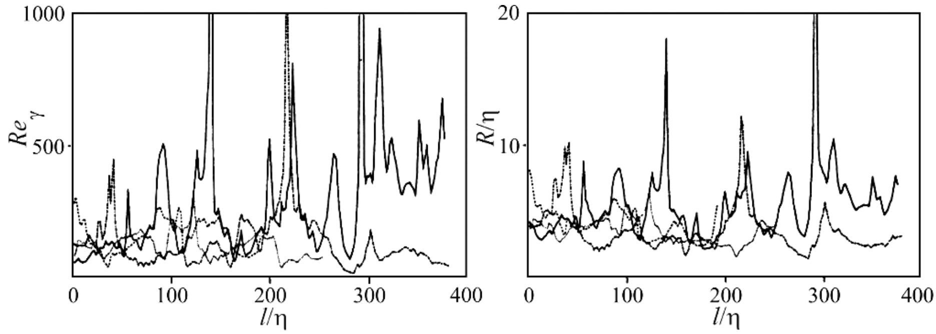

Figure 6.18. An example of distribution of normalized radius and circulation along worm axes, as a function of arclength for four worms chosen at random. $\mathrm{Re}_\lambda = 94.5$, Adapted from Jimenez et al., 1993

$\frac{\partial^2 p}{\partial x_i \partial x_j}$. This is the term $-\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}$. It appears (Tsinober et al., 1995) that $\langle\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}\rangle$ is positive and is about $\langle W^2\rangle/3$, i.e., in the mean, the nonlinearity in (C.23) is reduced by this nonlocal term, since for a Gaussian velocity field $\langle\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}\rangle \equiv 0$, see figures 6.19, 6.20.

Broadband forcing

Most turbulent flows are excited at large scales, so that in the rest of the scales the nonlinearity (mostly) does not have to 'cope' with forcing. This may not be the case with broadband forcing. If this forcing is strong enough not only in the large scales, it can balance the dissipative effects directly and thereby bypass the nonlinearity. That is in such a situation the nonlinearity may be reduced. Indications for this were obtained by Mazzi and Vassilicos (2004) in a DNS of a fractal forced flow with a bit 'unusual' forcing in Fourier space $\sim a_k k^\beta$, $\beta > 0$ and $a_k = 1$ for $k < K_F$ and vanishing otherwise. Namely, they observed reduced energy transfer in Fourier space and reduced velocity derivative skewness. Another interesting observation was made by Biferale et al. (2004). They used a random Gaussian forcing with a power-law spectrum, $E_f(k) \sim k^{3-y}$. They found that – judging by the scaling behavior – small-scale turbulent fluctuations change from a forcing independent, FI, (at $y > 4$) to a forcing dominated, FD (at $y < 4$). In the latter case, as the scale decreases, *the small-scale fluctuations get closer and closer to a Gaussian statistics and intermittency* (in the sense of scaling) *disappears.* As the forcing is Gaussian this is consistent with the scenario of bypassing (and reduction of) the nonlinearity in small scales[24]. This is

[24] A purely linear system with Gaussian forcing has Gaussian statistics.

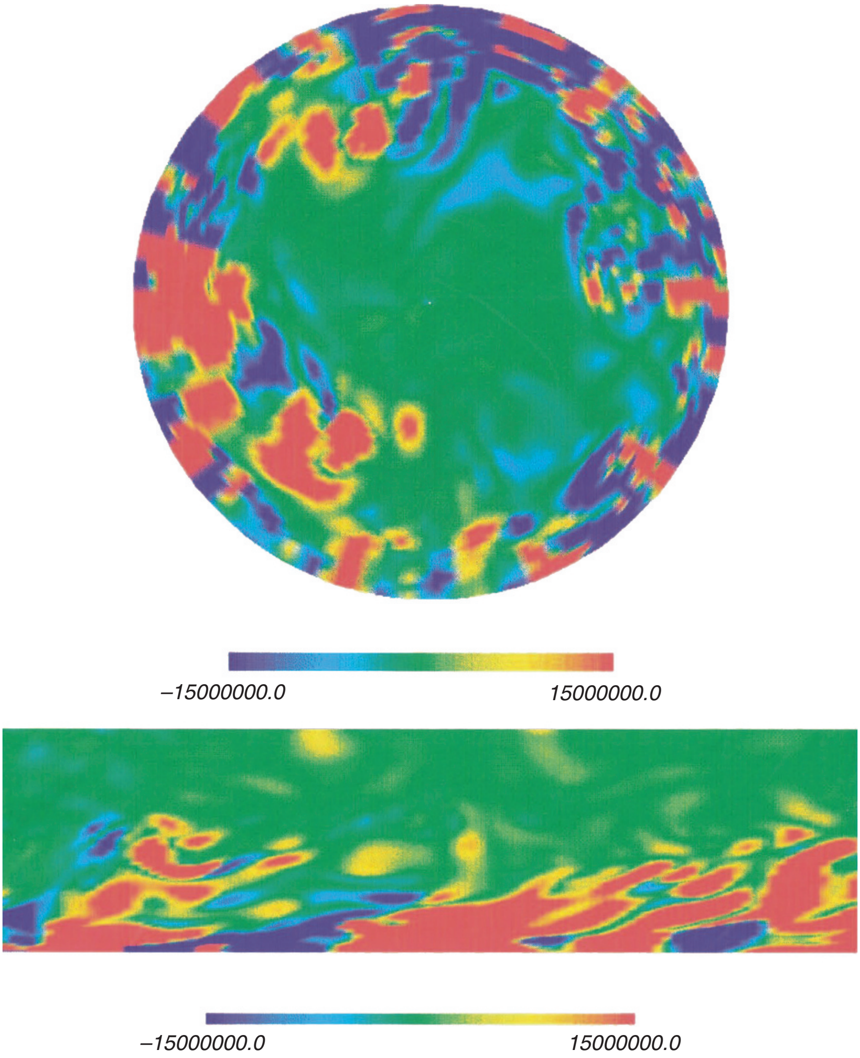

Figure 6.19. Visualization of the field of interaction of vorticity and pressure Hessian $\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}$ from the data in DNS of NSE in a circular pipe flow at Re $\approx$ 7000, performed by Eggels et al. (1994). Note large regions with strong interaction far away from the boundary. Courtesy of Professor F.T.M. Niewstadt and Dr. J.M.J. den Toonder

also supported by the observation of Biferale et al. (2004) that the tail in the PDF of the dissipation is much shorter in the forcing dominated (at $y = 3.5$) as compared to the forcing independent case (at $y = 6$).

It is natural to expect that in the FI regime the usual (Tennekes and Lumley) balance between $\omega_i\omega_j s_{ij}$ and $\nu\omega_i\nabla^2\omega_i$ should hold and the term associated with forcing will be small as compared to $\omega_i\omega_j s_{ij}$ and $\nu\omega_i\nabla^2\omega_i$.

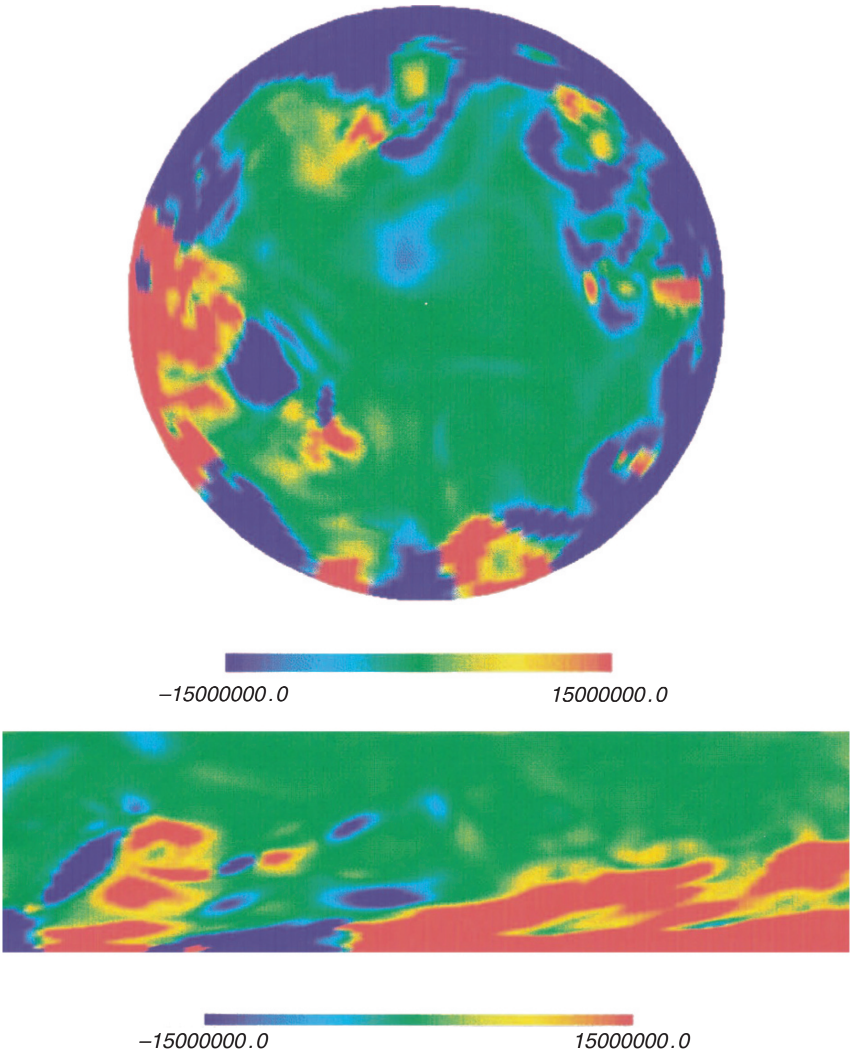

Figure 6.20. Visualization of the field of interaction of vorticity and pressure Hessian $s_{ik}s_{jk}\frac{\partial^2 p}{\partial x_i \partial x_j}$ from the data in DNS of NSE in a circular pipe flow at Re $\approx$ 7000, performed by Eggels et al. (1994). Note large regions with strong interaction far away from the boundary. Courtesy of Professor F.T.M. Niewstadt and Dr. J.M.J. den Toonder

In the FD regime the forcing term should come into play so that much of the balance will come from that between $\nu\omega_i\nabla^2\omega_i$ and $\varepsilon_{ijk}\omega_i\frac{\partial F_k}{\partial x_j}$. Similarly, in the equation for strain. In other words in the FD regime the forcing is expected to bypass the nonlinearity and to balance the dissipation directly. Hence one can expect that in the FD regime $\omega_i\omega_j s_{ij}$ and $s_{ij}s_{jk}s_{ki}$ will be strongly reduced as compared to the regime FI.

Along with the above difference one wonders what happens with passive scalar (and also vector) in these two cases. If one takes the case with linear mean gradient of a passive scalar the expectation is that there will be no "Reynolds analogy": in both cases, e.g. the production of the scalar gradient, $-G_i G_k s_{ik}$, will be of the same order and even larger in the FI regime.

6.6. Nonlocality

> *Kolmogorov related the poor predictability of flows with the influence of the higher-order harmonics on the basic, lower-order modes.* (Arnold, 1991).
> *... the local cascade ... seems to be actually rather diffuse.* (Kraichnan, 1968).
> *Are the small scales actually statistically independent of the large scales when $\nu \to 0$ and are they isotropic if the large scales are not?* (Saffman, 1978).
> *The mean strain rate in turbulent shear flow must tend to make the structure anisotropic in all parts of the spectrum.* (Corrsin, 1958).
> *... the energy transfer is rather nonlocal.* (Deissler, 1978).
> *The small and large scales are strongly coupled ... If cascade picture makes sense, one probably must have a complex interplay between distant shells.* (Chorin, 1994).
> *The experimental evidence shows that the large and the small scales are strongly coupled and that traditional cascade picture, which promotes universality, is a crude representation.* (Warhaft, 2000).
> *The large scales of turbulence are insensitive to viscosity at high enough Reynolds number.* (Mathieu and Scott, 2000).
> *... when there exists a range of scales (the inertial range) in which effects of viscosity, boundary conditions, and large-scale structures are not important ...* (Meneveau and Katz, 2000).

6.6.1. INTRODUCTION AND SIMPLE EXAMPLES

As mentioned in chapter 1, nonlocality is among the three main reasons[25] the problem of turbulence is so difficult.

The term *nonlocality* is used here in several related meanings which will become clear in the course of the discussion of the issues throughout this section (see also Tsinober, 2001a,b, 2003).

[25] The three N's: nonlinearity, non-integrability and nonlocality.

We start from a simple example. Taking the position that velocity fluctuations represent the large scales and the velocity derivatives represent the small scales, one can state that, in homogeneous (not necessarily isotropic) the large and the small scales do not correlate. This can be expressed quantitatively by a correlation between velocity and vorticity. For example, in a homogeneous turbulent flow the Lamb vector $\langle \omega \times \mathbf{u} \rangle = 0$ and also $\langle (\mathbf{u} \cdot \nabla)\mathbf{u} \rangle = 0$. If the flow is statistically reflectionally symmetric, then $\langle \omega \cdot \mathbf{u} \rangle = 0$ too. However, as mentioned in Chapter 1, vanishing correlations do not necessarily mean absence of dynamically important relations. Indeed, the quantities $(\mathbf{u} \cdot \nabla)\mathbf{u} \equiv \omega \times \mathbf{u} + \nabla(u^2/2)$ and $\omega \times \mathbf{u}$, are the main 'guilty parties' responsible for all we call turbulence. Both contain the large scales (velocity) and small scales (velocity derivatives, vorticity). So some kind of coupling between the two is unavoidable. Let us begin with the *kinematic* relation between velocity and vorticity, which is a mere consequence of the relation $\omega = curl\ \mathbf{u}$. Therefore any altering of ω results in its 'reacting back' on the velocity field. This point is not as trivial as may seem. Indeed, take a Helmholz decomposition of the most significant part of the nonlinear term in NSE, the Lamb vector, $\omega \times \mathbf{u}$ (Tsinober, 1990a)

$$\omega \times \mathbf{u} = \nabla\alpha + \nabla \times \beta.$$

Assuming that ω and $\mathbf{u}$ are random Gaussian and *unrelated*, i.e. $\omega \neq rot\mathbf{u}$, the result is that

$$\left\langle (\nabla\alpha)^2 \right\rangle = \left\langle (\nabla \times \beta)^2 \right\rangle.$$

However, if $\omega = rot\mathbf{u}$ and $\mathbf{u}$ is quasi-Gaussian, i.e., obeys the zero-forth-cumulant relation[26] then (Tsinober, 1990a)

$$\left\langle (\nabla\alpha)^2 \right\rangle \sim 2 \left\langle (\nabla \times \beta)^2 \right\rangle;$$

i.e. in this case the *rms* of the potential part of the Lamb vector is twice as large as its solenoidal part[27]. In other words ω 'reacts back' on velocity, and consequently on $\omega \times \mathbf{u}$, even for purely kinematic reasons.

More generally, vorticity does not just involve small scales. It is of special importance, since together with boundary conditions the whole flow field is determined entirely by the field of vorticity[28]. This, of course, includes

[26] The so-called Millionschikov hypothesis (Monin and Yaglom, 1975), suggested by A.N. Kolmogorov.

[27] In real turbulent flows this difference is even larger (Shtilman et al., 1993).

[28] As mentioned before this is seen from the equation $\nabla^2\mathbf{u} = -curl\omega$, which together with boundary conditions defines uniquely the velocity field. One can imagine a kind of "counter-argument" that the Biot–Savart law connects the (weak) large-scale vorticity to the large-scale velocity and small-scale vorticity to the (weak) small-scale velocity. The following simple counter-example shows that this is obviously incorrect. Take the potential vortex which – except at the origin (i.e., "almost nowhere") – has no vorticity, but has lots of velocity also far from the origin, and very large (!) velocity close to the origin.

the velocity field itself, and therefore the large scales are determined by the small scales. This is the simplest indication not only for direct interaction/coupling between large and small scales, but also that this interaction is bidirectional.

A natural question is what about the correlation(s) and coupling between velocity, u_i, and the strain, s_{ij}. First we recall that, just as in the case of vorticity, the whole flow field is determined entirely by the field of strain. This is seen from the equation $\nabla^2 u_i = 2\partial s_{ik}/\partial x_k$, which together with boundary conditions defines uniquely the velocity field[29]. Again in a homogeneous turbulent flow velocity, u_i, and the strain, s_{ij} do not correlate, $\langle u_i s_{ij}\rangle = 0$. However, velocity is correlated with small scales of 'higher order'. Namely, the correlation, $\left\langle u_i \nabla^2 u_i\right\rangle = -2\left\langle s_{ij}s_{ij}\right\rangle = -\left\langle \omega^2\right\rangle$, is essentially nonvanishing again for purely kinematic reasons. This and other aspects of "kinematic" nonlocality coupled with self-production of ω_i and s_{ij} have non-trivial dynamical consequences. It is of special dynamical significance as it is directly related to the dissipation of turbulent energy, $\langle \epsilon\rangle = 2\nu\left\langle s_{ij}s_{ij}\right\rangle$, which for dynamical reasons remains finite for very small ν. Therefore the correlation $\left\langle u_i \nabla^2 u_i\right\rangle$ becomes very large at small ν. However, the corresponding correlation coefficient $\frac{\left\langle u_i\nabla^2 u_i\right\rangle}{\langle \mathbf{u}^2\rangle^{1/2}\left\langle(\nabla^2\mathbf{u})^2\right\rangle^{1/2}} = \frac{\left\langle \omega^2\right\rangle}{\langle \mathbf{u}^2\rangle^{1/2}\left\langle(rot\mathbf{u})^2\right\rangle^{1/2}}$ at large Reynolds numbers is roughly of the order $\mathrm{Re}^{-1/4}$, i.e., becomes very small[30]. This, of course, does not mean that the coupling between $\mathbf{u}$ and $\nabla^2\mathbf{u}$ becomes unimportant at large Reynolds numbers. The correlation between $\mathbf{u}$ and $\nabla^2\mathbf{u}$ is directly related to the correlation between velocity, $\mathbf{u}$, and acceleration, $\mathbf{a}$, since, for example, in a homogeneous turbulent flow, $\langle \mathbf{u}\cdot\mathbf{a}\rangle = \nu\left\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\right\rangle$ (Mann et al., 1999). Hence again coupling between large ($\mathbf{u}$) and small ($\mathbf{a}$) scales.

One can use the above example with the Lamb vector to illustrate the dynamical nature of this coupling. For this we retreat from (quasi-) homogeneous isotropic flows and consider a unidirectional *in the mean* fully-developed turbulent shear flow, such as the flow in a plane channel in which all statistical properties depend on the coordinate normal to the channel

[29]This undermines the most common belief that turbulence is an inertial phenomenon in the sense that the precise nature of the dissipation mechanism does not affect the structure of the large (energy containing) scales, so that turbulence is statistically indistinguishable on energy-containing scales in gases, liquids, slurries, foams, and many non-Newtonian media.

[30]Indeed, $\frac{\left\langle u_i\nabla^2 u_i\right\rangle}{\left\langle \mathbf{u}^2\right\rangle^{1/2}\left\langle(\nabla^2\mathbf{u})^2\right\rangle^{1/2}} = \frac{\left\langle \omega^2\right\rangle}{\left\langle \mathbf{u}^2\right\rangle^{1/2}\left\langle(rot\omega)^2\right\rangle^{1/2}} = \frac{\langle\epsilon\rangle/\nu}{\left\langle \mathbf{u}^2\right\rangle^{1/2}\left\langle(rot\omega)^2\right\rangle^{1/2}} \sim \frac{\langle\epsilon\rangle/\nu}{U[(\epsilon/\nu)^{1/2}]/\eta} \sim Re^{1/2}\,(\eta/L)^{1/2} = \mathrm{Re}^{-1/4}$. We used here the standard order of magnitude phenomenological estimates (see Tennekes and Lumley, 1972). Namely, $\left\langle \mathbf{u}^2\right\rangle^{1/2} \sim U$, $\langle\epsilon\rangle \sim U^3/L$, and $\left\langle (rot\omega)^2\right\rangle^{1/2} \sim (\langle\epsilon\rangle/\nu)^{1/2}/\eta$ with U and L some integral scales of velocity and length.

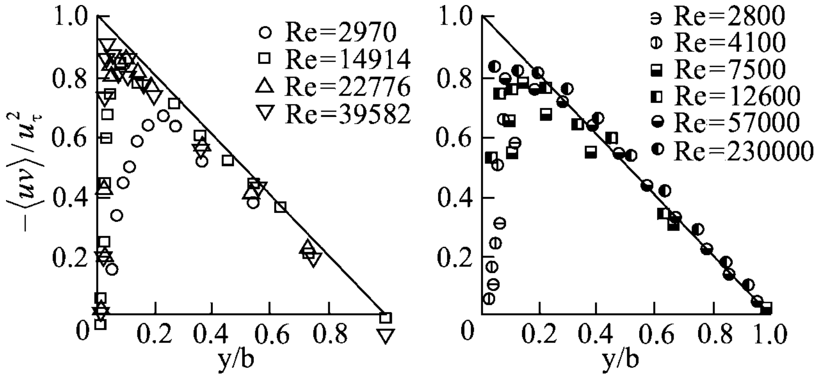

Figure 6.21. Dependence of the mean Reynolds stress $\langle u_1u_2\rangle$ on the distance from the wall in turbulent flows in channels of cross section with large aspect ratio. Adapted from Wei and Wilmarth (1989).

boundary, x_2, only. In such a flow, a simple precise kinematic relation is valid

$$d\langle u_1u_2\rangle/dx_2 \equiv \langle \omega \times \mathbf{u}\rangle_1 = \langle \omega_2u_3 - \omega_3u_2\rangle \neq 0, \tag{6.5}$$

which is just a consequence of the vector identity $(\mathbf{u}\cdot\nabla)\mathbf{u} \equiv \omega\times\mathbf{u}+\nabla(\frac{u^2}{2})$ in which incompressibility and $d\langle\cdots\rangle/dx_{1,3} = 0$ where used, and $\langle\cdots\rangle$ means an average in some sense (e.g., time or/and over the planes $x_2 = const$, etc.). The *dynamic* aspect is that in turbulent channel flows $d\langle u_1u_2\rangle/dx_2 \neq 0$ is essentially different from zero at *any arbitrarily large* Reynolds number (see figure 6.21). Therefore one can see from (6.5) that at least some correlations between velocity and vorticity in such flows are essentially different from zero.

Since vorticity is basically a small-scale quantity the relation (6.5) is a clear indication of a dynamically important statistical dependence between the large scales ($\mathbf{u}$) and small scales (ω). Without this dependence $d\langle u_1u_2\rangle/dx_2 \equiv 0$, which means that the mean flow would not 'know' about its turbulent part at all. It is noteworthy that both correlation coefficients $\frac{\langle\omega_2u_3\rangle}{\langle\omega_2^2\rangle^{1/2}\langle u_3^2\rangle^{1/2}}$, $\frac{\langle\omega_3u_2\rangle}{\langle\omega_3^2\rangle^{1/2}\langle u_2^2\rangle^{1/2}}$ (and many other statistical characteristics, e.g., some, but not all, measures of anisotropy) are of order 10^{-2} even at rather small Reynolds numbers. Nevertheless, as we have seen, in view of the dynamical importance of interaction between velocity and vorticity in turbulent shear flows[31] such 'small' correlations by no means imply absence

[31]The relation (6.5) is approximately valid in many important turbulent flows such as boundary layers, wakes, jets, etc. in which $d\langle\cdots\rangle/dx_{1,3} \ll d\langle\cdots\rangle/dx_2$. The argument here is based on nonvanishing *gradient* of the Reynolds stress $d\langle u_1u_2\rangle/dx_2$ and not the Reynolds stress itself. As noted, without this gradient the mean flow would not 'know'

of a dynamically important statistical dependence and a direct interaction between large and small scales. Indeed it is this interaction that results in drastic changes of the whole mean flow.

The direct interaction between large and small scales similar to the one in the above example may exist in a much broader class of turbulent flows and regions in these flows, e.g., with appropriate scale (in time and space) separation such as vorticity 'pancakes' (Brachet et al., 1992).

6.6.2. DIFFERENT ASPECTS OF NONLOCALITY

From the formal point of view a process is called local if all the terms in the governing equations are differential. If the governing equations contain integral terms, then the process is nonlocal. The Navier–Stokes equations are *integro-differential* for the velocity field in both physical and Fourier space (and any other). Therefore, generally, the Navier–Stokes equations describe nonlocal processes[32]. The problem is intimately related to the one of decompositions/representations, which was already discussed in chapter 5, but the relation between the two becomes more clear from what follows in the sequel.

The nonlocal nature of the Navier–Stokes equations in physical space is two-fold[33]. On one hand, it is due to pressure ('dynamic' nonlocality), since $\rho^{-1}\nabla^2 p = \omega^2 - 2s_{ij}s_{ij} = -\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j}$, so that pressure is nonlocal due to the nonlocality of the operator ∇^{-2} (see appendix C). This nonlocality is strongly associated with the essentially non-Lagrangian nature of pressure. For example, replacing in the Euler equations the pressure Hessian $\frac{\partial^2 p}{\partial x_i \partial x_j}$, which is both nonlocal and non-Lagrangian, by a local quantity $\frac{1}{3}\delta_{ij}\nabla^2 p = \frac{\rho}{6}\{\omega^2 - 2s_{ij}s_{ij}\}$ turns the problem into a local and integrable one and allows us to integrate the equations for the invariants of the tensor of velocity derivatives $\partial u_i/\partial x_j$ in terms of a Lagrangian system of coordinates moving with a particle (see Cantwell, 1992 and references therein. One of the reasons for the disappearance of turbulence (and formation of singularities in finite time) in such models, called restricted Euler models, is that the eigenframe of s_{ij} in these models is *fixed* in space (Novikov, 1990b), whereas in a real turbulent flow it is oriented randomly in space and time. This means that nonlocality due to pressure is essential for (self-)sustaining turbulence: no pressure Hessian – no turbulence. A related aspect is that the Lagrangian acceleration $D\mathbf{u}/Dt$ – a kind of small-scale quantity – is dominated by pressure gradient, ∇p (Vedula and Yeung, 1999; see next section).

about its turbulent part at all. Precisely this happens when a *shear* flow is assumed to be *homogeneous.* In such a flow the gradient $d\langle u_1 u_2\rangle/dx_2 \equiv 0$.

[32]This does not mean that processes described by pure differential equations are local. An example is a passive object (scalar, vector) in a random velocity field.

[33]On nonlocality of turbulence in Fourier space see, for example Deissler (1979).

Taking the *rot* of the NSE and getting rid of the pressure does not remove the nonlocality. Indeed, the equations for vorticity (C.9) and enstrophy (C.16) are nonlocal in vorticity, ω, since they contain the rate of strain tensor, s_{ij}, due to the nonlocal relation between vorticity, ω, and the rate of strain tensor, s_{ij} ('kinematic' nonlocality)[34]. The two aspects of nonlocality are related, but are not the same. For example, in compressible flows there is no such relatively simple relation between pressure and velocity gradient tensor as above (see equation C.13), but the vorticity-strain relation remains the same. It is noteworthy that while the production of enstrophy $\omega_i\omega_j s_{ij}$ is nonlocal, the main responsible for production of strain $-s_{ij}s_{jk}\ s_{ki}$ is local, though production of strain involves participation of nonlocal contributions from $-\frac{1}{4}\ \omega_i\omega_j s_{ij}$ and (in inhomogeneous flows) of $-s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}$, see equation (C.18).

Both aspects of nonlocality are reflected in equations (C.17) and (C.18) for the rate of strain tensor and total strain/dissipation, $s^2 \equiv s_{ij}s_{ij}$, and equations (C.23) and (C.24) for the third-order quantities. An important aspect is that equations (C.23) and (C.24) contain invariant quantities $\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}$ and $s_{ik}s_{kj}\frac{\partial^2 p}{\partial x_i \partial x_j}$ reflecting the nonlocal dynamical effects due to pressure and can be interpreted as interaction between vorticity and pressure and between strain and pressure. In particular, equation (C.23) for the rate of change of enstrophy production shows both aspects of nonlocality of the vortex stretching process (see also Ohkitani and Kishiba, 1995). The first term in (C.23) (which is just the squared magnitude of the vortex stretching vector) is strictly positive, $\omega_i s_{ij}\omega_k s_{ki} \equiv W^2 > 0$. This means that the nonlinear processes involving vortex stretching (or direct interaction of vorticity and strain) always tend to increase even the *instantaneous* enstrophy production. Here also the term $W_3^2 = \omega^2\lambda_3^2\cos^2(\omega, \lambda_3)$ associated with the negative eigenvector of the rate of strain tensor Λ_3, i.e., vortex compressing or negative enstrophy production, $\omega_3^2\Lambda_3\cos^2(\omega, \lambda_3)$, makes a positive (!) contribution to the rate of change of enstrophy generation, $W_i^2 = \omega_i^2\Lambda_3^2\cos^2(\omega, \lambda_i)$. However, the inviscid rate of change of enstrophy production contains also a second term reflecting the interaction between vorticity and the pressure Hessian $\frac{\partial^2 p}{\partial x_i \partial x_j}$. This is the term $-\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}$. Without this term the question why $\langle\omega_i\omega_j s_{ij}\rangle > 0$ would be immediately answered. It appears (Tsinober et al., 1995) that $\langle\omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j}\rangle$ is positive and is about $\langle W^2\rangle/3$, i.e., in the mean, the nonlinearity in (C.23) is reduced

[34]Nonlocality of the same kind is encountered in problems dealing with the behaviour of vortex filaments in an inviscid fluid. Its importance is manifested in the breakdown of the so-called localized induction approximation (LIA) as compared with the full Bio–Savart induction law, see Ricca et al. (1999).

by this nonlocal term, since for a Gaussian velocity field $\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \rangle \equiv 0$. The nonvanishing correlations $\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \rangle$, $\langle s_{ik} s_{kj} \frac{\partial^2 p}{\partial x_i \partial x_j} \rangle$ are also one of the manifestations of nonlocality, see figures 6.19 and 6.20.

Direct coupling between large and small scales

Nonlocality, in the sense as discussed above, is an indication of direct coupling between large and small scales. There exists massive evidence that this is really the case as there are many indications that this interaction is bidirectional, see also section 5.3.1[35]. We should first mention the well-known effective use of fine honeycombs and screens in reducing large-scale turbulence in various experimental facilities (Laws and Livesey, 1978; Tan-Attichat et al., 1982). The experimentally observed phenomenon of strong drag reduction in turbulent flows of dilute polymer solutions and other drag reducing additives (Gyr and Bewersdorff, 1995) is another example of such a 'reacting back' effect of small scales on the large scales (see chapter 8). Third, one can substantially increase the dissipation and the rate of mixing in a turbulent flow by *directly* exciting the small scales experimentally in a jet (Wiltse and Glezer, 1998)[36] and in DNS in a periodic box (Suzuki and Nagano, 1999). Before proceeding further, we should also mention a related phenomenon concerning the stability of spatially periodic flows. Such flows may destabilize *directly into small-scale* three-dimensional structures (Pierrehumbert and Widnall, 1982).

Anisotropy. One of the manifestations of direct interaction between large and small scales is the anisotropy in the small scales. Though local isotropy is believed to be one of the universal properties of high Re turbulent flows, it appears that it is not so universal: in many situations the small scales do not forget the anisotropy of the large ones. There exists considerable evidence for this point, which has a long history starting somewhere in the 1950s (see references in Biferale and Procaccia, 2005; Ferchichi and Tavoularis, 2000; Gylfason and Warhaft, 2004; Shen and Warhaft, 2000, 2002; Sreenivasan and Antonia, 1997; Staicu et al., 2003; Tsinober, 1993b, 1998b; Yeung et al., 1995; Warhaft, 2000; Warhaft and Shen, 2002). Along with other manifestations of direct interaction between large and small scales, the deviations from local isotropy seem to occur due to various external constraints like boundaries, initial conditions, forcing (e.g., as in DNS), mean shear/strain, centrifugal forces (rotation), buoyancy, magnetic field, etc., which usually act as an organizing factor, favoring the formation of coherent structures

[35] Note that in case of passive objects, there is no such a bidirectional relation – it is only one way.

[36] These authors claim also that *forcing induces coupling (or long-range interactions) between small and large scales within the flow*, whereas this coupling does exist independently of forcing, and in the presence of the latter is manifested in enhanced dissipation.

of different kinds (quasi-two-dimensional, helical, hairpins, etc.). These are as a rule, large-scale features which depend on the particularities of a given flow and thus are not universal. These structures, especially their edges seem to be responsible for the contamination of the small scales. This 'contamination' is unavoidable even in homogeneous and isotropic turbulence, since there are many ways to produce such a flow, e.g., many ways to produce the large scales. It is the difference in the mechanisms of large-scale production which 'contaminates' the small scales. Hence, non-universality.

Let us turn again to the 'simple' example above and look at the properties in the proximity of the midplane, $x_2 \approx 0$, of the turbulent channel flow. In this region $dU/dx_2 \approx 0$, but *the flow is neither homogeneous nor isotropic*, since though $\langle u_1 u_2 \rangle \approx 0$ in this region too, the gradient $d\langle u_1 u_2 \rangle / dx_2$ is essentially $\neq 0$ and is *finite independently of the Reynolds number*, as far as the data allow one to make such a claim (see figure 6.21). This is also a clear indication of nonlocality, since in the bulk of the flow, i.e., *far from the boundaries*, $dU/dx_2 \sim 0$. This is also a clear counterexample to the hypothesis of the local isotropy: even in the proximity of the centerline of the channel this hypothesis does not hold for any magnitude large Reynolds numbers[37].

The first experimental evidence on anisotropy in small scales at large Reynolds numbers was provided by the atmospheric boundary layer experiments. It was found that the skewness of the derivative of temperature fluctuations is not small and is of order 1 (Stewart, 1969), whereas for a locally isotropic flow it should be close to zero. This and related results were obtained later in a number of laboratory flows, field observations and in numerical simulations (see Biferale and Procaccia, 2005; Kurien and Sreenivasan, 2001b; Gylfason and Warhaft, 2004; Sreenivasan and Antonia, 1997 and Warhaft, 2000 for further references). An important feature of these flows is the presence of a mean gradient of the passive scalar – the rest is not so important: the phenomenon is observed for a Gaussian and two-dimensional velocity field (Holzer and Siggia, 1994; also Kraichnan and Kimura, 1994). This is related to the weak sensitivity of the passive-scalar field to the details of the velocity field and its Reynolds number (Kraichnan, 1968; Warhaft, 2000). On the other hand, passive objects exhibit chaotic behaviour and mixing in purely laminar flows – a phenomenon closely related to what is called Lagrangian chaos/chaotic advection. Therefore generally, the behaviour of a passive scalar may not reflect the structure of turbulent flows (chapter 4).

As mentioned in chapter 5, similar observations have, quite recently, been made for the velocity increments and velocity derivatives in the

[37] Those who like structure functions can see the derivative $d\langle u_1 u_2 \rangle / dx_2$ as the limit at small Δx_2 of a "structure function" $\langle u_1(\mathbf{x} + \Delta x_2 \mathbf{j}) u_2(\mathbf{x} + \Delta x_2 \mathbf{j}) - u_1(\mathbf{x}) u_2(\mathbf{x}) \rangle$.

direction of the mean shear both in numerical and laboratory experiments (see references in Biferale and Procaccia, 2005; Kurien and Sreenivasan, 2001b; Shen and Warhaft, 2000, 2002; Staicu et al., 2003 and Warhaft and Shen, 2002). It was found that the statistical properties of velocity increments and velocity derivatives in the direction of the mean shear do not conform with and do not confirm the hypothesis of local isotropy. *Moreover, our results imply that the large scales are directly coupled to the small scales. (The anisotropy disappears when the large-scale shear is removed;* Shen and Warhaft, 2000). More precisely these results imply that there is a *direct* influence of mean shear on the small scales due to the permanent bias of the mean shear to which the field of fluctuations is exposed on account of its large *residence* time in the mean shear[38]. Of course, one of the explanations of these results is *that the large scales are directly coupled to the small scales.* However, this does not mean that there is no such coupling *when the large-scale shear is removed.* This coupling is an intrinsic/generic property of turbulent flows and exists independently of the presence of mean shear or other external factors, but has different manifestations for different external factors (see below).

Recently attempts were made to employ the SO(3) decomposition (which is the tensorial generalization of the well-known procedure of decomposing a scalar function into components of different irreducible representations using spherical harmonics) to "separate" the anisotropic "part" of various tensorial objects/structure functions and to estimate their behaviour with scale (defined simply as the distance r between two points) Reynolds number and shear (see references in Biferale and Procaccia, 2005; Kurien and Sreenivasan, 2001b; Shen and Warhaft, 2000, 2002; Staicu et al., 2003 and Warhaft and Shen, 2002). The most effective way do so is to choose objects that are zero for purely isotropic turbulence, though this is good enough without making reference to the SO(3) decomposition[39]. Among other reasons the latter was necessary *assuming* that each component of this decomposition has its own ('universal') scaling exponent in r. The results seem to indicate that anisotropic exponents are consistently larger than those known for isotropic parts, which suggests that anisotropy effects decrease with decreasing scale, though more slowly than expected. The evidence is not uniform in the sense that even for the smallest accessible scales the anisotropic contribution appears to be of the same order of magnitude as the isotropic part (Kurien and Sreenivasan, 2001b; Staicu et al., 2003; Warhaft and Shen, 2002) because the *amplitudes* of the anisotropic contributions are not small as is assumed in the theories based on the SO(3)

[38]We remind a statement by Corrsin, 1958: *The mean strain rate in turbulent shear flow must tend to make the structure anisotropic in all parts of the spectrum.*

[39]As did first Stewart (1969); and Gibson et al. (1970, 1977) for passive scalars.

decomposition. Another problem concerns the moments (those which vanish for isotropic turbulent flow) composed of derivatives, which remain finite at all accessible parameters (Rosset et al., 2001; Stewart, 1969; Gibson et al., 1970, 1977; Schumacher, 2004 and references therein), and for which the above definition of "scale" just as the distance r between two points is obviously not good enough.

It should be mentioned that in the experiments by Shen and Warhaft, 2000 and Warhaft and Shen, 2002 the value of the Corrsin criterion $S_C^* = (dU/dx_2)(\nu/\langle\epsilon\rangle)^{1/2} \approx 2.4 \cdot 10^{-2}$. This criterion represents the ratio of the Kolmogorov time scale, $\tau_\eta = (\nu/\langle\epsilon\rangle)^{1/2}$, to the time scale, $(dU/dx_2)^{-1}$, associated with the mean shear, and it should be small enough in order to have isotropy in small scales (Corrsin, 1958). The main problem is how small. There is no agreement on this issue (see Saddoughi, 1997; Schumacher, 2004 and references therein), but there is evidence that in order to have one decade of isotropic inertial range in boundary layer flows (both simple and complex) at $\mathrm{Re}_\lambda \approx 1500$, it is necessary that $S_C^* < 10^{-2}$ (see Saddoughi, 1997 and references therein). This brings us to the next issue.

Statistical dependence of small and large scales. An important observation was made by Praskovsky et al. (1993), Sreenivasan and Dhruva (1998), Kholmyansky and Tsinober (2000) and Gulitski et al. (2007). The specific feature of these large-scale experiments is rather high Taylor microscale Reynolds number $\mathrm{Re}_\lambda \sim 10^4$. At these high Reynolds numbers, there is also clear evidence of strong coupling between large and small scales. In view of severe limitations on statistical convergence at such Reynolds numbers, the approach is different from the one undertaken by Shen and Warhaft (2000, 2002). Namely, if the large scales are *not* coupled directly to the small scales, there should be no dependence of conditional statistics of the small-scale quantities (e.g., velocity increments, enstrophy, total strain) conditioned on the large-scale ones (velocity). The results from Praskovsky et al. (1993); Sreenivasan and Dhruva (1998); Kholmyansky and Tsinober (2000) and Gulitski et al. (2007) show the opposite – there is such a dependence. Below we mention some results from Gulitski et al. (2007), since in their experiment the mean shear was rather small, less than $0.1s^{-1}$. This corresponds to the value of the Corrsin criterion $S_C^* = (dU/dx_2)(\nu/\langle\epsilon\rangle)^{1/2} \approx 2 \cdot 10^{-3}$, which is an order of magnitude smaller than in the experiments by Shen and Warhaft (2000) and five times lower than the value 0.01 mentioned above.

The results by Gulitski et al. (2007) similar to ones obtained by Praskovsky et al. (1993) exhibit the following tendencies. First, there is a clear tendency that the conditional averages of the structure functions increase with the *energy* of fluctuations and, second, such a tendency, that is the direct coupling, is observed also for the smallest distance of the order

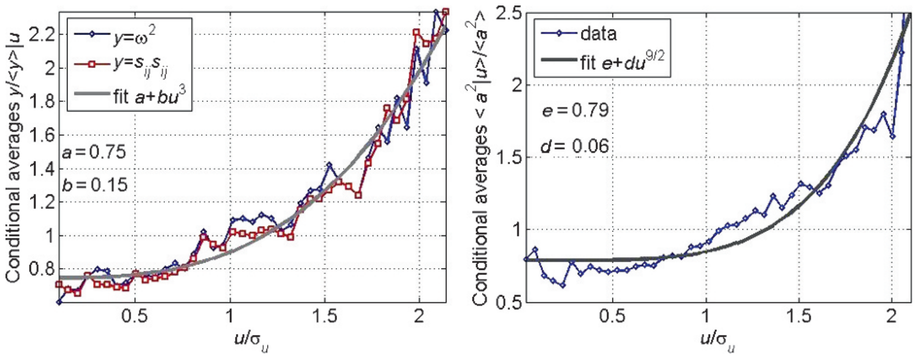

Figure 6.22. Left – conditional averages of enstrophy ω^2 and total strain $s_{ij}s_{ij}$ conditioned on magnitude of velocity fluctuations vector, u. The fit is in the spirit of the Kolmogorov refined similarity hypothesis, though it is a fit in the first place. This fit cannot be expected to be universal quantitatively and should at least have different coefficients a and b for flows with different large-scale properties in the spirit of the Landau remark. Right – conditional averages of squared acceleration magnitude a^2 on magnitude of velocity fluctuations vector, u. Gulitski et al. (2007a,b). See also Mordant et al. (2004)

of Kolmogorov scale $\sim\eta$, which was used for estimates of the derivatives. This is shown in figure 6.22 as conditional statistics for the enstrophy ω^2 and the total strain $s_{ij}s_{ij}$ and also of the squared acceleration a^2. An interesting observation, which seems to be related to the coupling between large and small scales, is that, in 3-D turbulence not only are $\langle\omega_i\omega_j s_{ij}\rangle$ and $-\langle s_{ij}s_{jk}s_{ki}\rangle$ essentially positive quantities but also *all* $\int_{V_L}\omega_i\omega_j s_{ij}dV_L$, $-\int_{V_L} s_{ij}s_{jk}s_{ki}dV_L$ over volumes of the order of integral scale are essentially positive (see figure 1 in Tsinober, 1998a and figure 6.5).

The observations on the coupling between large and small scales and the 'reaction back' of the small scales on the large ones by no means are exhausted by the references given above. As an example from atmospheric physics we bring a quotation of the first conclusion reached at the Symposium on the nature of the so-called CAT – clear air turbulence: *The energy dissipated at small scale by clear air turbulence influences the large-scale atmospheric motion* (Pao and Goldburg, 1969).

The next example concerns a set of pure kinematic exact relations for isotropic turbulence pointing to statistical dependence between large and small scales. Of special interest is the pure kinematic relation involving the third-order structure function (Hosokawa, 2007) $-\langle u_-^3\rangle = 3\langle u_+^2 u_-\rangle$, which together with the 4/5 law results in a relation equivalent to the 4/5 law

$$\left\langle u_+^2 u_-\right\rangle = \langle\epsilon\rangle\, r/30. \tag{6.6}$$

Here $2u_+ = u_1 + u_2$, $2u_- = u_2 - u_1 \equiv \Delta u$, $u_1 = u(x) = u_+ - u_-$, $u_2 = u(x+r) = u_+ + u$; $u(x)$ is the longitudinal velocity component, and $\langle\epsilon\rangle$

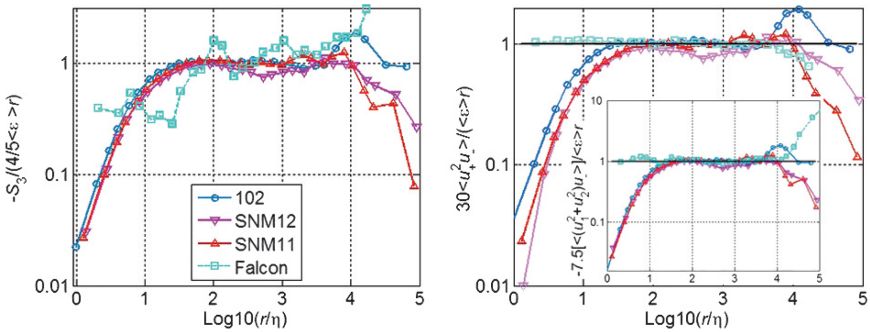

Figure 6.23. Left – conventional 4/5 law. Right – equation (6.6), Kholmyansky and Tsinober (2008) and a similar equation not involving u_+, Kholmyansky et al. (2008)

– is the mean kinetic energy dissipation. The relation (6.6), and thereby the 4/5 law, is a clear indication of absence of statistical independence between u_+ and u_-, i.e., between small and large scales. A second feature is that this relation has an important advantage for experimentalists: it is linear in velocity increments, while the 4/5 law is cubic. Therefore this relation holds much better than the 4/5 law, especially in the case of lower quality data, as in the airborne experiment, see figure 6.23, Kholmyansky and Tsinober (2008).

The role of kinematic relations in the issue of nonlocality goes far beyond their use in the nonlocal interpretation of the Kolmogorov 4/5 law. For example, it appears that structure functions of all orders are expressed via terms, all of which have the form of correlations between large- and small-scale quantities, Kholmyansky, Sabelnikov and Tsinober (2008). Thus, in the absence of nonlocal interactions – as manifested by correlations between large-scale (velocity) and small-scale (velocity increments) quantities – all structure functions vanish. There is no exaggeration in saying that without nonlocality (understood as direct and bidirectional coupling of large and small scales) there is no turbulence. Hence the utmost dynamical importance of purely kinematics relations. Another point is that all kinematic relations under consideration stand in contradiction with the so-called sweeping decorrelation hypothesis (SDH), understood as statistical independence between large and small scales. This is seen from many of the relations given by Kholmyansky, Sabelnikov and Tsinober (2008). For, example, as follows from the relation (6.6) the quantity $\langle u_+^2 u_- \rangle$ scales as r in the "inertial" range, whereas assuming SDH to be valid, it should scale as $r^{1/3}$. The simplest are the relations $\langle (\Delta u)^2 \rangle = -\langle u_1 \Delta u \rangle = \langle u_2 \Delta u \rangle$, the right-hand side of which vanishes assuming the sweeping decorrelation hypothesis to be valid, whereas in reality the left-hand side is well known to scale as $r^{2/3}$.

Nonlocality versus decompositions. As discussed in chapters 3 and 5 any decomposition of a turbulent flow field results in a process of interaction/exchange of (not necessarily only) energy between components of some *particular* decomposition/representation of a turbulent field associated with the nonlinearity of the problem. The general expectation is that this interaction should be nonlocal. This has been convincingly demonstrated by Laval et al. (2001) on an example of a particular decomposition of the flow field[40]. Laval et al. interpret their results as evidence *that nonlocal interactions are responsible for intermittency corrections in the statistical behaviour of 3-D turbulence* as well as for the *deviations from Gaussianity.* By *intermittency corrections* the authors mean *the anomalous corrections* in the scaling behaviour of the structure functions (and corresponding PDFs). In view of the existing evidence (see again 6.6) the claim about the importance of nonlocal interactions – understood in a broader sense as direct and bi-directional interaction/coupling of large and small scales – is definitely correct. However, *the anomalous corrections* in the scaling exponents can hardly be interpreted as necessary and reliable manifestations of intermittency of turbulence, nor are such corrections necessarily associated with or due to *thin vortices* as explained in the next chapter (see also Tsinober, 1998). Likewise, the non-Gaussian nature of turbulence – which is among the intrinsic/generic properties of turbulence – is far more than just a consequence of the nonlocality of turbulence and even it's intermittency, as follows, for example, from the 4/5 law.

[40]Laval et al. use a cutoff in Fourier space and define the large scales, U_i, and small scales, u_i, correspondingly as those below and above the cutoff; which results in essentially the same equations for U_i and u_i as (C.43, C.44) or (C.55, C.57). The authors interpret as *nonlocal terms, involving the product of a large-scale and a small-scale component, and a local term, involving two small-scale components.* This is not the same as the authors claim several lines before that *by nonlocal interactions,* they *mean interaction between well-separated scales (or highly elongated wave number triads),* since U_i and u_i are not that well separated, they are 'neighbors', since the smallest 'scales' of U_i are just of the same order as the largest 'scales' of u_i.

In order *to study the dynamical effect of these contributions* (i.e., the terms of the type uu called local, and terms of the type Uu called nonlocal) *at small scales,* the authors performed two types of numerical experiments along with the full DNS. The first type are those in which the local interactions were neglected in the equations for small scales (only), thus turning the entire problem into a linear one (the RDT simulation). Among other things this resulted in stronger deviations from the Kolmogorov-like scaling just as in other linear problems such as for passive objects. The second type are those in which local interactions are the only ones retained. *This resulted in even higher (than in RDT) level of small scales, but exhibited in much less intermittency* ... in the sense of deviations from the Kolmogorov-like scaling.

Along with the nonlocal nature of the interactions between components of some *particular* decomposition/representation of a turbulent field there are claims on *locality of cascades* (Eyink, 2005; Domaradzki and Carati, 2007 and references therein). The analysis of detailed interactions (Alexakis et al., 2005; Domaradzki and Carati, 2007) shows that the individual nonlocal contributions are always large, but significant cancellations lead to global/integrated quantities (energy transfer, the energy flux, and the SGS energy transfer) to be *asymptotically* dominated by local interactions. The key word is *asymptotically* as the results are based on pretty modest Re_λ slightly exceeding 200. One encounters a similar problem with the attempts to establish *sufficient conditions for locality of turbulent cascades, by an exact analysis of the fluid equations*, Eyink (2005). These are based on *the regularity that* is assumed *for Euler solutions in the high-Reynolds-number limit* to be *the Hölder type that was conjectured theoretically [1,2]* (Onsager, 1949 *and* Parisi and Frish, 1983) *and is observed (in a space-mean sense) experimentally [3]* (Anselmet et al., 1984). There are several problems with such an assumption. First, it is not at all clear how the properties of Euler(!) solutions can be observed experimentally (if at all). Second, all the experiments the authors refer to (and all others as well) are done at pretty low Reynolds numbers at which there are no very long inertial ranges to allow for local transfer to dominate, if this happens at all[41]. Thus the scaling exponents obtained from experiments such as in Anselment et al. (1984) and similar later ones (and, of course, numerical computations) are not necessarily those which can be expected at very large Reynolds numbers. Third, there is a problem concerning the relation of the exponents defined in the paper *to the usual (absolute) structure-function exponents*, which, in fact, are rather different objects.

The general belief that nonlocal interactions should weaken as the Reynolds number increases has rather moderate support such as via looking at the scaling with Re of the nonlocal energy fluxes (Mininni et al., 2008 and references therein). This, however, is a particular aspect of the whole issue of nonlocality in turbulence. Even if the nonlocal effects in the energy transfer do weaken at large Reynolds numbers, this does not mean that the same will happen with their impact on other aspects of nonlocality. An immediate example is the 'anomalous scaling' discussed in sections 5.3 and 5.4.5. It is likely that a variety of other aspects of nonlocality should exist at any large Reynolds number.

[41] A crude estimate is that one needs about six decades for this, i.e., Taylor microscale (!) Reynolds numbers exceeding 10^8, which cannot be reached in any experiment in the forseeable future if ever.

We recall also the problems concerning the observed scaling exponents, etc. as discussed in chapter 5.

One more example comes from the field of 'data assimilation' in meteorology and related to what is called 'determining modes', see Henshaw et al. (2003); Yoshida et al. (2005) and references therein. The main point is that one can reconstruct the small-scale spatial Fourier modes to a very good degree of accuracy by incorporating the time history of the first Fourier modes (the large scale) as known forcing into the equations governing the small-scale evolution[42], though there are situations when the small scale is uncoupled from the large scale and cannot be reconstructed. These results definitely suggest coupling between the large and small scales, but do not imply that *small eddies are subordinate to large eddies*, Yoshida et al. (2005), because the small scales are not passive and good reconstruction is achieved due to very special forcing known *a priori* which already "knows" about the small scales. In natural conditions the poor predictability is due to the reaction back of the small scales[43]. Moreover, as mentioned, the flow field can be modified substantially by *directly* exciting the small scales (Wiltse and Glezer, 1998; Suzuki and Nagano, 1999). Similar effects are observed with small-scale acoustic excitation.

Intermittency and structure(s). Batchelor and Townsend (1949) in their studies on small-scale intermittency (see chapter 7) wrote: *All the evidence is consistent with the inference that the fluctuations are small in the region of smallest wave-numbers of equilibrium range and become increasingly large at larger wave-numbers* (p. 252) and that ... *the mean separation of the visible activated regions is comparable with the integral scale of the turbulence, i.e., with the size of the energy-containing eddies* (p. 253). This latter observation is intimately related to the direct interaction/coupling of the large and small scales.

At least some of the structure(s) of turbulence reflects such a relation as well. The most popular 'structure' observed in various turbulent flows – the vortex filament/worm – has at least two essentially different scales: its length can be of the order of the integral scale, whereas its cross-section is of the order of the Kolmogorov scale. Similarly, the ramp-cliff fronts in the passive-scalar fields (and similar 'structures' associated with strain) have a

[42]This is done by updating in the second run the low k modes (which cover the integral scale and 'touch the inertial range') at each time step. Thus the second run is not the 'same' as there is a forcing which is absent in the first run.

[43]*Kolmogorov related the poor predictability of flows with the influence of the higher-order harmonics on the basic, lower-order modes. Suppose, he would say, that the velocity field be changed in every cubic kilometer without changing its average in this cube. We have to study what the time interval is beyond which this change will crucially affect the weather. It is clear that dynamic weather prediction is impossible for longer periods (and will remain impossible in spite of all the future progress in computer techniques). This reasoning, which Kolmogorov related to infinite-dimensional tori, is in fact independent of their conjectural existence: an attracting invariant torus of sufficiently high finite dimension, covered by quasi-periodic orbits. would lead to the same conclusions*, Arnold (1991).

thickness much smaller than the two other scales, see chapter 7. The issue dates back to the famous Landau remark stating that the *important part will be played by the manner of variation of ε over times of the order of the periods of large eddies (of size ℓ)*, Landau and Lifshits (1944, see 1987, p. 140).

Helicity. Helicity, $\int \omega \cdot ud\mathbf{x}$, and its density, $\omega \cdot u$, deserve here also special mention. The formal reason is that if $\langle \mathbf{u} \cdot \omega \rangle \neq 0$, this is a clear indication of direct coupling of large and small scales. So it is not surprising that, in flows with nonzero mean helicity, the direct coupling between small and large scales is stronger than otherwise. The stronger coupling between the large and small scales in flows with nonzero mean helicity $\langle \mathbf{u} \cdot \omega \rangle$ aids creation of large-scale structures out of small-scale turbulence (see Droegemeier et al., 1993 and references therein). This does not mean that in case $\langle \mathbf{u} \cdot \omega \rangle = 0$, or even $\mathbf{u} \cdot \omega = 0$ as in two-dimensional flows, such a coupling does not exist. We return to these matters below. The hypothesis of local isotropy (K41a) includes restoring of all the symmetries in small scales. Thus one expects restoring of reflection-invariance at small scales. However, to maintain finite helicity dissipation to balance the finite helicity input (in a statistically stationary turbulence) the tendency to restore reflection symmetry at small scales can not be realized. This is because helicity dissipation is associated with broken reflection symmetry at small scales, as helicity dissipation, $D_H = -\nu H_s$ which is just proportional to the superhelicity $H_s = \int \omega \cdot curl\omega d\mathbf{x}$, showing the lack of reflection symmetry of the small scales. The important point is that helicity dissipation is vanishing if reflectional symmetry holds in small scales. Moreover, if the helicity dissipation should remain finite as the Reynolds number increases (see figure 6.24) this lack of reflectional symmetry should increase since the dissipation of helicity ($D_H = -\nu H_s$) is proportional to viscosity.

A natural question is how it is possible that both energy and helicity dissipation (presumably) remain finite with increasing Reynolds number. A possible explanation is due to the imperfect alignment between vorticity ω and $curl\omega$ can make it possible that the 'singularities' arising from the finiteness of both energy and helicity dissipation can be matched (the correlation coefficient between ω and $curl\omega$ for the case shown in figure 6.24 is 0.1 only). This is also aided by the fact that $\omega \cdot curl\omega$ is not a positively defined quantity. Note that, nevertheless, the quantity $D_H = -\nu H_s$ acts as dissipation of helicity in flows with helical forcing: superhelicity H_s is single signed and has the same sign as helicity H, see Galanti and Tsinober, 2006 for more information, details and references.

Flows with additives (see references in Tsinober, 2003). The basic interaction of the carrier fluid flow with particulates occurs at the scale of the particle size, i.e., at small scales. However, a number of essential phenomena

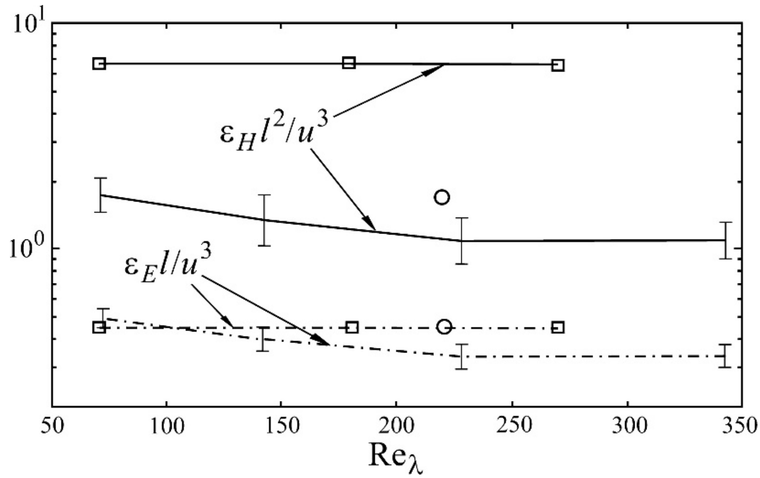

Figure 6.24. Dependence of normalized dissipation of helicity $\frac{D_H}{u'^3 l^{-2}}$ and energy $\frac{D_E}{u'^3 l^{-1}}$ on the Taylor microscale Reynolds number Re_λ. Error bars are shown. ○ – corresponds to the data from Chen et al. (2003b), □ – corresponds to the data from Kurien et al. (2004) and references therein

emerge at much larger scales in a variety of particulate flows: sedimenting suspensions, fluidized beds, formation of bedforms and their interaction with the carrier fluid, preferential concentration of particles/bubbles (clustering) in and modification of turbulent flows. These phenomena are treated in terms of large-scale instabilities, intrinsic convection in sedimenting suspensions, collective phenomena, long-range multibody hydrodynamic interactions/correlations, clusters. All these are essentially fluid mediated phenomena/interactions as contrasted with direct particle/particle interactions. Therefore, nonlocality is expected to be significant in these phenomena (see Tsinober, 2003 for more information and references). For example, an important process in the interaction of the carrier fluid flow with particles (or any other additives) is the production (or more generally modification) of velocity derivatives, i.e., vorticity and strain. The modified field of velocity derivatives reacts back in changing the large scales of the flow (both velocity and pressure). It is tempting to see this process as the one underlying the formation of the mentioned large-scale features, though the details in each case are different and in most cases are poorly understood. These processes are modified by specific features such as inertial bias, i.e. inertial response of particles to fluid accelerations and preferential concentration of particles(bubbles) in strain (vorticity) dominated regions. The latter may lead to enhanced bias of strain dominated regions (heavy particles), i.e., regions with large dissipation, or regions with strong enstrophy (bubbles).

Turbulent flows can be strongly modified by additives in even extremely small concentrations. The most spectacular changes occur with only few parts per million of flexible polymers added to the solvent. These changes are exhibited in a number of flow parameters, both in large and small scales. However, the direct interaction of the dissolved polymers with the carrier fluid flow is obviously in the small scales. Hence again nonlocality. The large-scale manifestations are represented in the first place by strong reduction of drag (up to 80%) in turbulent shear flows. Along with this effect, other global large-scale effects on turbulence structure are observed both experimentally and in simulations.

Effects of initial/inflow conditions. The far field statistical properties of free shear turbulent flows (mixing layers, wakes, jets and also boundary layers are known to possess strong memory ('nonlocality in time'): they are sensitive to the conditions at their 'start' (i.e., initial, inflow conditions and flow history) with some properties being not universal in Reynolds number and other aspects, Bevilaqua and Lykoudis (1978); Dimotakis (2005); George and Davidson (2004); George (2008) and references therein. These flows develop in space beginning with small scales into the large ones, in apparent contradiction to the Richardson–Kolmogorov cascade ideas. It is noteworthy that passive tracers in such flows possess even stronger memory, Cimbala et al. (1988), due to enhanced importance of Lagrangian aspects of their evolution. It should be kept in mind that many turbulent flows exhibiting memory effects are partly-turbulent and most probably have different *large-scale stability* properties for different inflow conditions not directly related to the turbulent nature of the flow within the turbulent region. This can make a contribution to the differences in observations.

The problem of predictability of turbulent flows involves nonlocality in time as well: a small-scale disturbance (both in time and space) perturb substantially the whole flow including the largest scales within time of the order of integral time scale. In this sense instability can be seen as nonlocality in time.

Other related issues. The nonlocality in the sense of concern here is especially strongly manifested in the atmospheric convective boundary layers in which the common downgradient approximation is not satisfactory due to countergradient heat fluxes, Zilitinkevich et al. (2006) and references therein. We mention also a similar phenomenon in stably stratified turbulent flows, the so called PCG, persistent countregradient fluxes. The essence of PCG is the countergradient transport of momentum and active scalar. It is observed at large scales when stratification is strong, but in small scales it is present with weak stratification as well. There is a class of flows with the so-called phenomenon of 'negative eddy viscosity', see chapter 8. Under certain conditions it occurs in the presence of an energy supply other than the mean velocity gradient. In such flows the turbulent transport of momentum occurs against the mean velocity gradient, i.e., from regions with low momentum to regions with high momentum (i.e., the Reynolds stresses as one of the agents of coupling the fluctuations with the mean flow act in such flows in the 'opposite' direction as compared to the usual turbulent shear flows). Concomitantly, kinetic energy moves in the 'opposite' direction too from fluctuations to the mean flow.

Flows with negative eddy viscosity are akin to nonturbulent but nonstationary flows in a fluid dominated by its fluctuating components and known (since Rayleigh, 1883) under the name (acoustic) steady streaming, Riley (2001), in the sense that in these flows a mean (time averaged) flow is induced and driven by the fluctuations. Recently turbulent flows of this kind were observed too, Scandura (2007).

There are examples of 'usual' turbulent flows with turbulence induced mean flows. The best known ones are flows in pipes with noncircular cross-section. In such flows a mean secondary flow is induced which is absent in purely laminar flow. For example, see Pettersson Reif and Andersson (2002) for references on such flows in a square duct. For earlier references, see Schlichting (1979).

On closures and constitutive relations. Nonlocality due to coupling between large and small scales is a concern in the problem of the relation between fluctuations and mean flow as in the Reynolds decomposition in the Reynolds averaged Navier–Stokes equations or resolved and unresolved scales in large eddy simulations (LES)[44]. Namely, this relation is a nonlinear *functional*, i.e., the field of fluctuations (the unresolved field in LES) at each time/space point depends on the mean (resolved) field in the whole time/space domain. Vice versa, the mean (resolved) flow at each time/space point depends on the field of fluctuations (unresolved scales) in the whole time/space domain. This means that, in turbulent flows, local 'constitutive' relations analogous to real material constitutive relations for fluids (such as stress/strain relations) can not exist, though the 'eddy viscosity' and 'eddy diffusivity' are used frequently as a crude approximation for taking into account the reaction back of fluctuations (unresolved scales) on the mean flows (resolved scales). The fact that the 'eddy viscosity' and 'eddy diffusivity' are flow (and space/time) dependent is just another expression of the strong coupling between the large and the small scales. The simplest version of this approach with a scalar eddy viscosity leads always to

[44]Or other low-dimensional representations with *modelling to account for neglected modes* (Holmes et al., 1996) ... *eliminating the small scales produces a stress whose dependence upon the large-scale velocity u is, in general, spatially non-local, history-dependent and stochastic (Lindenberg, West and Kottalam, 1987; Eyink, 1996). Thus, an exact constitutive relation for the turbulent stress is formally available, but it is quite unwieldy and not of direct practical use.* Eyink (2006). Our comment is that it is an illusion that small scales are really eliminated.

a positive subgrid dissipation (positive energy flux from the resolved to the unresolved scales), whereas *a priori* tests of data from real flows (experiments and DNS) show that there exist considerable regions in the flow with negative subgrid-scale dissipation (called backscatter). The exchange of 'information' between the resolved and unresolved scales is rich and is not limited by energy. Hence the qualification of large-scale (resolved) eddies as the most important ones is too subjective: all eddies are important in view of direct and bidirectional coupling of a great many 'eddies'[45].

6.7. Acceleration and related matters

> *If the velocity of the air stream which carries the eddies is very much greater than the turbulent velocity, one may assume that the sequence of changes in u at the fixed point are simply due to the passage of an unchanging pattern of turbulent motion over the point.* (Taylor, 1938b).
> *An underlying assumption of the Kolmogorov theory is that very large spatial scales of motion convect very small scales without directly causing significant internal distortion of the small scales. This assumption is considered to be consistent with, and to imply, statistical independence of small and large scales ... The physical picture is that large-scale motions should carry small eddies without distorting them. It is not obvious that this needs to be true, but the idea is certainly intuitively plausible.* (Kraichnan, 1964).

As the material derivative of the velocity vector, the fluid particle acceleration field in turbulent motion

$$\mathbf{a} \equiv \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

is among the most natural physical parameters of special interest in turbulence research for a variety of reasons, ranging from studies of small-scale intermittency to applications in Lagrangian modelling of dispersion (see references in Vedula and Yeung, 1999; Tsinober et al., 2001 and Gulitski et al., 2007b).

[45] There exist a number of attempts to make the filtering, LES and similar approaches 'rigorous' and to handle 'fundamental physics' by such methods (see Piomelli and Balaras, 2002; Eyink, 2006 and references therein). However, there is still a conceptual question whether these methods – as any other decompositions – are suitable in principle to address 'fundamental physics' of turbulence since a vitally important part of physics of turbulence resides in the small/unresolved scales.

6.7.1. ACCELERATION VARIANCE, DOES IT HAVE A KOLMOGOROV-LIKE SCALING, $\varepsilon^{3/2}\nu^{-1/2}$?

The acceleration variance, $\langle a^2 \rangle \equiv \langle a_k a_k \rangle$, is a key quantity in a number of issues. Its value and scaling with the Reynolds number are essential for stochastic Lagrangian models and for Lagrangian probability density function models of turbulent diffusion if these models are to incorporate finite-Reynolds-number effects. However, its scaling is a controversial issue. Following Yaglom (1949) (see Monin and Yaglom, 1975, pp. 368–369) it was mostly accepted that it scales as $\varepsilon^{3/2}\nu^{-1/2}$. The argument in favour of such a scaling goes as follows (see the above reference):

> ... in sufficiently small space-time regions the field $a(x,t)$ will be isotropic and its probability distribution will be stationary and dependent only on ε and ν.
>
> Consider the Lagrangian time correlation tensor for the acceleration field
> $$B_{0,ij}(\tau) = a_i(t|x_0)a_j(t+\tau|x_0), \tag{21.48}$$
> where $a(t|x_0) = a(X[x_0,t),t]$ is the acceleration of the fluid particle which occupies the point $x_0$at time t_0. In view of isotropy and similarity in the quasi-equilibrium range of t, this tensor should be of the form
> $$B_{0,ij}(\tau) = B_0(\tau)\delta_{ij}; B_0(\tau) = \frac{v_\eta^2}{\tau_\eta^2}\mathfrak{a}(\tau/\tau_\eta) = \varepsilon^{3/2}\nu^{-1/2}\mathfrak{a}(\varepsilon^{3/2}\nu^{-1/2}\tau), \tag{21.49}$$
> where $\mathfrak{a}(x)$ is a universal function ... The mean square of the turbulent acceleration is then given by
> $$\langle a^2 \rangle = K\varepsilon^{3/2}\nu^{-1/2}, \tag{21.51}$$
> where $K = 3\mathfrak{a}(0)$ is a universal numerical constant (Yaglom, 1949a).

The problematic aspect here is that one cannot take $x = 0$ in the "universal" function $\mathfrak{a}(x)$ as x is in "the quasi-equilibrium range", i.e., $x > 1$, at least. In other words, the above argument is not applicable at $\tau = 0$ and thus does not imply that acceleration variance does scale as $\varepsilon^{3/2}\nu^{-1/2}$ as widely claimed[46] A similar problem (i.e., the scaling $\varepsilon^{3/2}\nu^{-1/2}$) is with the pressure gradient variance as it makes a dominant contribution to the acceleration variance, though the contribution of the viscous term is of crucial importance[47]. Figure 6.25 shows a number of points obtained by Gulitski et al., 2007b. The main feature is that there seems to be no saturation in

[46]Note that in the quasi-equilibrium range, e.g., the Eulerian spatial correlations of accelerations should depend on $\langle\epsilon\rangle$ and r only, i.e., $\langle\epsilon\rangle^{4/3}\ r^{-2/3}$ though in many cases written as $\langle\epsilon\rangle^{3/2}\ \nu^{-1/2}(r/\eta)^{-2/3}$ thus making a misleading impression that viscosity is 'therein'.

[47]The viscous term is obviously extremely important as it is responsible for the solenoidal part of the field of acceleration and $curl\mathbf{a} = \partial\omega/\partial t + \omega\times\mathbf{u} = D\omega_i/Dt - \omega_i\omega_j s_{ij}$.

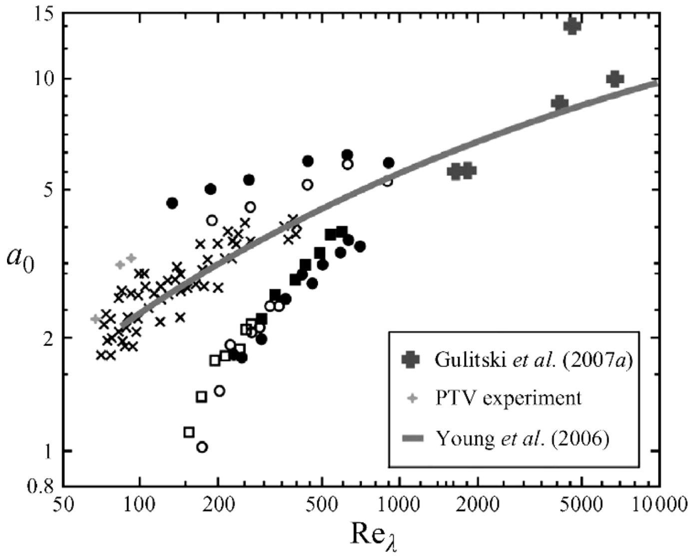

Figure 6.25. Normalized acceleration variance, $a_0 = (1/3)\langle a_k a_k \rangle \epsilon^{3/2} \nu^{-1/2}$, vs. Re_λ (from Gylfason et al., 2004) with added experimental data from field experiment (Gulitski et al., 2007) and from the PTV experiments (Lüthi et al., 2005).

the Re-dependence of the acceleration variance normalized on $\varepsilon^{3/2}\nu^{-1/2}$. This means that the scaling proposed by Yaglom (1949) is not 'perfect' and the acceleration variance is larger than that proposed by Yaglom. The trend seen in figure 6.25 may be contaminated by the imperfections of the method. The issue seems to be open and requires further, far more precise, measurements.

Another aspect of concern is that $a^2 \sim \varepsilon^{3/2}\nu^{-1/2} \sim u_0^{9/4} L^{-3/4} \nu^{1/4}$, i.e., the acceleration variance along with viscosity, ν, depends explicitly on the large-scale characteristics, u_0, of the flow contrary to the claim that *the turbulent acceleration is determined largely by the very small-scale motions* $l \lesssim \eta$ (Monin and Yaglom, 1971). Indeed, the observations show (see figure 6.22 right) that the conditional statistics of a^2 on u_0 show a significant statistical dependence, see also Biferale et al., 2005; Lüthi et al., 2008 and references therein. The bottom line is that *fluid particle acceleration variance does not* (seem to) *obey K41 scaling at any Reynolds number* Hill (2002).

6.7.2. THE LAGRANGIAN ACCELERATION VERSUS ITS EULERIAN COMPONENTS

Another basic issue concerns the relations between the Lagrangian acceleration, $\mathbf{a}$, and its various (Eulerian) 'components' $\mathbf{a} = \mathbf{a}_l + \mathbf{a}_c = \mathbf{a}_l + \mathbf{a}_L + \mathbf{a}_B = \mathbf{a}_{\|} + \mathbf{a}_{\perp} = \mathbf{a}_i + \mathbf{a}_s$. Here $\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}$ is the local acceleration, which expresses the unsteady rate of change of the velocity vector at a fixed point, and $\mathbf{a}_c = (\mathbf{u} \cdot \nabla)\mathbf{u}$ is the convective acceleration responsible for the velocity rate of change due to the spatial derivatives and also embodies nonlinearity effects. On the other hand $\mathbf{a}_i = -\frac{1}{\rho}\nabla p$ and $\mathbf{a}_s = \nu \nabla^2 \mathbf{u}$ represent respectively the irrotational and the solenoidal parts of $\mathbf{a}$. Further $\mathbf{a}_L = \omega \times \mathbf{u}$ is the Lamb vector, $\mathbf{a}_B = \nabla(\frac{1}{2}u^2)$, $\mathbf{a}_{\|} = (\mathbf{a} \cdot \widehat{\mathbf{u}})\widehat{\mathbf{u}}$ with $\widehat{\mathbf{u}} = \mathbf{u}/u$ and $\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\|}$. We make here some essential points. More details can be found in Tsinober, 2001a; Tsinober et al., 2001 and Gulitski et al. 2007b and references therein. The results in Tsinober et al. (2001) are obtained from a DNS database (Vedula and Yeung, 1999) for isotropic turbulence at ensemble-averaged Taylor microscale Reynolds numbers ranging from 38 to 240 using up to 512^3 grid points, whereas those in Gulitski et al. (2007b) are from a field experiment at large Re_λ of the order up to 10^4.

The relation between the total acceleration and its local and convective components

The familiar part of the issue is known as the random Taylor hypothesis or the sweeping decorrelation hypothesis. It takes its origin from the so-called Taylor hypothesis (Taylor, 1935) for computation of the spatial derivative in the direction of the mean flow, $\partial/\partial x_1$ via the time derivative $\partial/\partial t$ from the relation $\partial/\partial x_1 = -U^{-1}\partial/\partial t$ by assuming that (the grid) turbulence is transported by the mean velocity, U, without change. This approximation appears to be valid if the magnitude of turbulent fluctuations, say $\langle u^2 \rangle^{1/2}$, is small enough compared to U, i.e., $\langle u^2 \rangle^{1/2}/U \ll 1$. This is seen from a (thought) experiment in which the observer (probe) is moving through the turbulent flow with some velocity U, which can be (assumed to be) very large, for instance in a real experiment (as in Busen et al., 2002) when the probe is mounted on a research aircraft moving through a turbulent region in the atmosphere. For references on the Taylor hypothesis see Tsinober et al. (2001).

Tennekes (1975) suggested that *in turbulence with high Reynolds numbers ... the dissipative eddies flow past an Eulerian observer in a time much shorter than the time scale which characterizes their own dynamics*[48]. *This*

[48] That is $\tau_E/\tau_L \sim \mathrm{Re}^{-1/4}$. Hence in the Lagrangian setting the correlation times are expected to be much larger which is mostly – but not always – the case. Already at the

suggests that Taylor's 'frozen-turbulence' approximation should be valid for the analysis of the consequences of large-scale advection of the turbulent microstructure[49]. In fact, Tennekes' hypothesis consists of two ingredients. First, it is proposed that the Lagrangian acceleration of fluid particles, $\mathbf{a}$, is in some sense small, and in order to obtain estimates and comparison of Lagrangian and Eulerian time scales Tennekes put just $\mathbf{a} = 0$. It is noteworthy that this assumption is local pointwise in space/time and is not a statistical one. The second assumption made by Tennnekes is of statistical nature, namely, that *the microstructure is statistically independent of the energy containing eddies.*

The equality $\mathbf{a} = 0$ should not be understood literally[50]. The acceleration of fluid particles cannot be vanishing, since $\mathbf{a} = 0$ may be misinterpreted as the balance between the pressure gradient and the viscous term in the Navier–Stokes equations, which is obviously incorrect because $\mathbf{a}_i = -\frac{1}{\rho}\nabla p$ is irrotational and $\mathbf{a}_s = \nu\nabla^2\mathbf{u}$ is solenoidal. Also $\mathbf{a} = 0$ would mean that the flow is described by the equation $\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0$.

More precisely $\mathbf{a} \approx 0$ means that in some sense $\mathbf{a}$ is small compared both to $\mathbf{a}_l$ and $\mathbf{a}_c$, e.g., $\langle a^2\rangle/\langle a_l^2\rangle \ll 1$ and $\langle a^2\rangle/\langle a_c^2\rangle \ll 1$. This in turn is possible if there is mutual (statistical) cancellation between the local acceleration, $\mathbf{a}_l$, and convective acceleration, $\mathbf{a}_c$[51]. Since these quantities are vectors, the degree of this mutual cancellation should be studied both in terms of their magnitude and the geometry of vector alignments.

In terms of magnitude, it appears that the acceleration is much smaller than its local and convective components. The two ratios $\frac{\langle\mathbf{a}^2\rangle}{\langle\mathbf{a}_l^2\rangle}$, $\frac{\langle\mathbf{a}^2\rangle}{\langle\mathbf{a}_c^2\rangle}$ decrease, and the ratio $\frac{\langle\mathbf{a}_l^2\rangle}{\langle\mathbf{a}_c^2\rangle}$ tends to unity with increasing Reynolds number. At the highest accessible Reynolds number, $\mathrm{Re}_\lambda \approx 240$, the variance of the local acceleration, $\langle\mathbf{a}_l^2\rangle^{1/2}$, is about 0.9 of the magnitude of convective acceleration, $\langle\mathbf{a}_c^2\rangle^{1/2}$. At this Reynolds number the total acceleration is about $0.1\langle\mathbf{a}_c^2\rangle^{1/2}$, i.e., an order of magnitude smaller than both its components.

very beginning one encounters an ambiguity (not the only one) as the time scale (like many other things) of the Eulerian observer depends on the velocity of the Eulerian frame in which the Eulerian observer lives.

[49]In fact this is not so new: *An underlying assumption of Kolmogorov theory is that very large spatial scales of motion convect very small scales without directly causing significant internal distortion of the small scales. The assumption usually is considered to be consistent with, and to imply, statistical independence of small and large scales*, Kraichnan, 1964.

[50]Just as the microstructure is *not* statistically independent of the energy containing eddies (see previous section).

[51]A similar issue is encountered in other fields, e.g., in the Ohm's law in moving conductor $\mathbf{j} = rot\mathbf{H} = \sigma\{-\nabla\phi + \mathbf{u}\times\mathbf{B}\}$. On one hand the electrical current is $rot\mathbf{H}$ which is a small-scale quantity, but $\mathbf{u}\times\mathbf{B}$ is a large-scale quantity. There is no contradiction since there is a great deal of cancellation between the electrical $-\nabla\phi$ and $\mathbf{u}\times\mathbf{B}$. Note that both ($-\nabla\phi$ and $\mathbf{u}\times\mathbf{B}$) depend on the reference system, whereas $\mathbf{j}$ does not.

Similarly, the correlation coefficient between $\mathbf{a}_l$ and $\mathbf{a}_c$, is expected to be $\mathcal{O}(1)$ and tend to unity (with minus sign) with increasing Reynolds number. Indeed, it reaches the value -0.98 at $\mathrm{Re}_\lambda \approx 240$. It is noteworthy that $\mathbf{a}$ and $\mathbf{a}_l$ are practically decorrelated.

An important point is that though the magnitude of the total acceleration becomes small compared to both its local, $\mathbf{a}_l$, and convective, $\mathbf{a}_c$, components as the Reynolds number increases, the latter do not compensate each other totally. This cannot happen at *any* Reynolds number, since $\mathbf{a}_l$ is divergence-free, whereas $\mathbf{a}_c$ is not, and its irrotational part, which is equal to $\mathbf{a}_i = -\frac{1}{\rho}\nabla p$, contributes most to the magnitude of the total acceleration (Shtilman et al., 1993; Tsinober, 1990a; Vedula and Yeung, 1999). Since the solenoidal part of the total acceleration, $\mathbf{a}_s = \nu\nabla^2\mathbf{u}$ on the one hand, and $\mathbf{a}_s = \mathbf{a}_l + \mathbf{a}_{cs}$ ($\mathbf{a}_{cs}$ is the solenoidal part of $\mathbf{a}_c$) on the other, the only way $\mathbf{a}_s$ can become small is that $\mathbf{a}_l$ and $\mathbf{a}_{cs}$ compensate each other, i.e., they should be almost the same in magnitude but almost antiparallel. Indeed, the ratio $\frac{\langle \mathbf{a}_l^2\rangle}{\langle \mathbf{a}_{cs}^2\rangle} \approx 1$ similar to $\frac{\langle \mathbf{a}_l^2\rangle}{\langle \mathbf{a}_c^2\rangle}$ and their correlation coefficient is very close to -1 even at rather moderate Reynolds numbers.

The geometrical aspect of the above results is seen from the alignments between the vectors involved. Namely, if $\mathbf{a} = \mathbf{a}_l + \mathbf{a}_c$ is small compared to $\mathbf{a}_l$ and $\mathbf{a}_c$, then the last two vectors are expected to be (anti-) aligned, i.e. the cosine of the angle between $\mathbf{a}_l$ and $\mathbf{a}_c$, $\cos(\mathbf{a}_l,\mathbf{a}_c)$ should be negatively skewed, i.e., to have a maximum around -1. This, indeed, is observed in figure 6.26. This alignment exhibits an essential dependence on Reynolds number: the tendency of alignment between $\mathbf{a}_l$ and $\mathbf{a}_c$ is strongly enhanced with increasing Reynolds number. The result shown in figure 6.26 is for a DNS in a periodic box without a mean flow, Tsinober et al. (2001). The alignment between $\mathbf{a}_l$ and $\mathbf{a}_c$ as many other features is not Galilean invariant, i.e., depends on the reference system. This is seen from figure 6.27.

The 'true' dynamical effect is observed in the frame of reference moving with the mean velocity. The alignment between $\mathbf{a}_l$ and $\mathbf{a}_c$ is a consequence of even stronger alignment between $\mathbf{a}_l$ and $\mathbf{a}_{cs}$, with stronger Re-dependence as well. On the other hand both 'components' of $\mathbf{a}_s$, i.e., $\mathbf{a}_l$ and $\mathbf{a}_{cs}$, are about twenty times larger than $\mathbf{a}_s$ itself even at the smallest Reynolds number, and at the largest Re_λ, $\langle \mathbf{a}_s^2\rangle^2 \approx 3\cdot 10^{-3}$ of $\langle \mathbf{a}_{cs}^2\rangle^2$ and/or $\langle \mathbf{a}_l^2\rangle^2$. In other words, there is a tendency of compensation between the local acceleration, $\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}$, and the solenoidal part, $\mathbf{a}_{cs}$, of the convective acceleration, $\mathbf{a}_c = (\mathbf{u}\cdot\nabla)\mathbf{u}$. It is this tendency, which is increasing with Reynolds number, that makes the solenoidal part of the acceleration, $\mathbf{a}_s$, much smaller than the irrotational part of the acceleration, $\mathbf{a}_i = \mathbf{a}_{ci}$ ($\mathbf{a}_{ci}$ is the irrotational part of $\mathbf{a}_c$). It is natural to call this tendency the reduction of solenoidality of the total acceleration. In fact this tendency is seen in figure 1 of Vedula and Yeung (1999). We return to this matter in the next section.

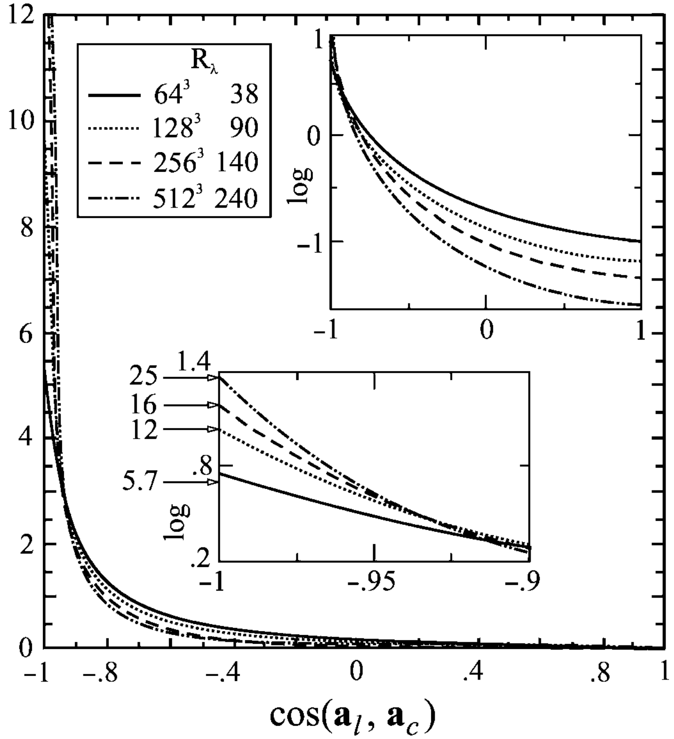

Figure 6.26. Alignments. PDFs of the cosine of the angle between $\mathbf{a}_l$ and $\mathbf{a}_c$. The insets show this dependence with the vertical in log and in the proximity of $\cos(\mathbf{a}_l, \mathbf{a}_c) \sim -1$. This alignment was observed in laboratory experiments (Lüthi et al., 2005) and in the atmospheric surface layer Gulitski et al. (2007b)

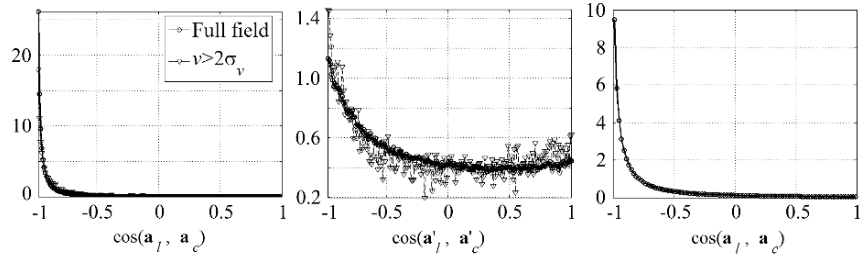

Figure 6.27. Alignments. PDFs of $cos(\mathbf{a}_l, \mathbf{a}_c)$ in (a) the frame attached to the ground and (b) in the frame moving with the mean velocity from the field experiment; (c) PDF of $cos(\mathbf{a}_l, \mathbf{a}_c)$ from PTV experiment. Gulitski et al. (2007b)

This tendency should be distinguished from an opposite tendency when comparing the solenoidal part, $\mathbf{a}_{cs}$, of the convective acceleration, i.e., of the nonlinearity $(\mathbf{u} \cdot \nabla)\mathbf{u}, \frac{\partial \mathbf{u}}{\partial t}$, (and also $\mathbf{a}_l$), and its potential part, $\mathbf{a}_{ci}$: as the Reynolds number increases, both $\mathbf{a}_l$ and $\mathbf{a}_{cs}$ become large compared to $\mathbf{a}_i$ (or $\mathbf{a}_{ci}$ which is the same). For example, the ratio $\frac{\langle \mathbf{a}_{ci}^2 \rangle}{\langle \mathbf{a}_{cs}^2 \rangle}$ decreases substantially with Reynolds number. In other words, along with the above-mentioned tendency of reduction of solenoidality of the total acceleration, there is a concomitant tendency of enhancement of solenoidality of its convective part, $\mathbf{a}_c$, i.e., the nonlinearity, $(\mathbf{u} \cdot \nabla)\mathbf{u}$, tends to become more solenoidal as the Reynolds number increases. It appears that this tendency is, to a large extent, of kinematical nature (see below Tsinober, 1990a and Pinsky et al., 2000). Namely, for a Gaussian velocity field the solenoidal part of the convective acceleration $\mathbf{a}_c \equiv (\mathbf{u} \cdot \nabla)\mathbf{u}$ is larger than its irrotational part.

The alignment as described above is of rather general nature in the sense that any two quantities of the type $\frac{\partial \mathbf{Q}}{\partial t}$ and $(\mathbf{Q} \cdot \nabla)\mathbf{Q}$ tend to align in the same manner as do $\frac{\partial \mathbf{Q}}{\partial t}$ and $(\mathbf{u} \cdot \nabla)\mathbf{u}$. Here $\mathbf{Q}$ can be a scalar, vector (gradient of temperature, magnetic field, vorticity) or tensor (rate of strain tensor), see Galanti et al. (2003).

The relation between total acceleration and its irrotational and solenoidal components

The relation between $\mathbf{a}$, $\mathbf{a}_i = -\frac{1}{\rho}\nabla p$ and $\mathbf{a}_s = \nu\nabla^2\mathbf{u}$ is qualitatively different from that considered in the previous section. The main contribution to the total acceleration variance comes from its irrotational part, $\mathbf{a}_i$ (Vedula and Yeung, 1999). Consequently, the correlation between $\mathbf{a}$ and $\mathbf{a}_i$ is close to unity and is $\sim$0.98 at $\mathrm{Re}_\lambda \approx 240$. The variance of the solenoidal part of the acceleration $\frac{\langle \mathbf{a}_s^2 \rangle}{\langle \varepsilon \rangle^{3/2} \nu^{-1/2}}$ is independent of Reynolds number in the range investigated (Vedula and Yeung, 1999), so that the increase of the ratios $\frac{\langle \mathbf{a}^2 \rangle}{\langle \mathbf{a}_S^2 \rangle}$ and $\frac{\langle \mathbf{a}_i^2 \rangle}{\langle \mathbf{a}_S^2 \rangle}$ with Reynolds number is due to the increase of the irrotational part of the acceleration.

Again the geometrical aspect is seen from the alignments between the vectors involved. Namely, the only strongly exhibited alignment is the one between $\mathbf{a}$ and $\mathbf{a}_i$. The alignment between $\mathbf{a}$ and $\mathbf{a}_s$ is weak, and there is no alignment between $\mathbf{a}_i$ and $\mathbf{a}_s$, since they are orthogonal in the sense that $\langle \mathbf{a}_i \cdot \mathbf{a}_s \rangle = 0$. All these alignments are weakly sensitive to the Reynolds number.

6.7.3. SCALE DEPENDENCE

The scale dependence can be seen by looking at one-dimensional spectra and/or the spectral analogues of the quantities addressed above, e.g., $\frac{S_a}{S_{a_l}}$, where S_a, S_{a_l} are the one-dimensional energy spectra of $\mathbf{a}$, $\mathbf{a}_l$.

The tendencies described above regarding the relations between $\mathbf{a}, \mathbf{a}_l$ and $\mathbf{a}_c$ are expected to be strongest at smaller scales. This is clearly seen

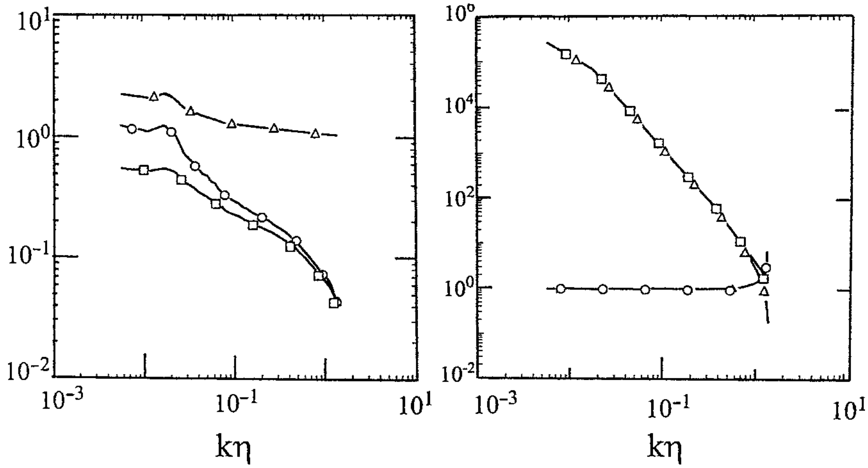

Figure 6.28. Ratios: left – $\frac{S_a}{S_{a_l}}(\square)$, $\frac{S_a}{S_{a_c}}(\circ)$ and $\frac{S_{a_c}}{S_{a_l}}(\triangle)$; right – $\frac{S_a}{S_{a_i}}(\circ)$, $\frac{S_a}{S_{a_s}}(\square)$ and $\frac{S_{a_i}}{S_{a_s}}(\triangle)$ as functions of the wave number at $Re_\lambda = 240$. Here η – is the Kolmogorov microscale

from figure 6.28, i.e., the ratio $\frac{S_{a_c}}{S_{a_l}}$ tends to unity, whereas the ratios $\frac{S_a}{S_{a_l}}$, $\frac{S_a}{S_{a_c}}$ decrease with $k\eta$ and become much smaller than their overall analogues $\frac{\langle \mathbf{a}^2 \rangle}{\langle \mathbf{a}_i^2 \rangle}$, $\frac{\langle \mathbf{a}^2 \rangle}{\langle \mathbf{a}_s^2 \rangle}$ at largest Reynolds numbers, i.e., these relations are quite sensitive to the Reynolds number.

A different picture is seen with respect the ratios $\frac{S_a}{S_{a_i}}$, $\frac{S_a}{S_{a_s}}$ and $\frac{S_{a_i}}{S_{a_s}}$. As expected the ratio $\frac{S_a}{S_{a_i}}$ is of order 1, except at the smallest scales, where it is larger due to the contribution from $\mathbf{a}_s$. Both ratios $\frac{S_a}{S_{a_s}}$ and $\frac{S_{a_i}}{S_{a_s}}$ are much larger than $\frac{S_a}{S_{a_i}}$, especially at large scales, and are quite similar except again in the small scales due to the same reason as above. The pattern is the same qualitatively at all four Reynolds numbers, but is quite different quantitatively: at large Reynolds numbers (the only one shown in figure 6.28) the differences become orders of magnitude larger than at small Reynolds numbers.

6.7.4. KINEMATICAL VERSUS DYNAMICAL EFFECTS

By kinematical we mean the effects which are exhibited in random Gaussian analogues of the real fields. In some aspects these effects can be dominant (Shtilman et al., 1993; Tsinober, 1990a). The reference random Gaussian fields are Gaussian in the sense of the velocity gradients being (artificially) Gaussian, but have the same form of the energy spectrum and Reynolds number corresponding to each grid resolution.

It appears that the behaviour of real fields and of their Gaussian counterparts is qualitatively the same in a number of aspects. This is true, for example, of the alignments discussed above and also of the joint PDFs of $\mathbf{a}_c$ and $\mathbf{a}_l$ and of $\mathbf{a}$ and $\mathbf{a}_i$. In other words, the discussed effects are mainly (but not entirely) of kinematical nature (see also figure 6.30).

Along with the significant qualitative resemblance between the real and the random Gaussian cases, there is a quantitative difference. This difference is exhibited in several ways. First, the Reynolds number dependence of the effects described above is stronger in the case of real flow field. Second, the essential difference between the real and the Gaussian fields is seen from the PDFs of the magnitudes of $\mathbf{a}$, $\mathbf{a}_c$, and $\mathbf{a}_l$, and $\mathbf{a}$, $\mathbf{a}_c$, and $\mathbf{a}_l$. All the PDFs for the real flow field are more intermittent than those for their Gaussian counterparts, in the sense that they have much higher flaring tails. The Reynolds number dependence of this effect is also stronger for the real field. For more details see Tsinober (2001a) and Tsinober et al. (2001).

It is noteworthy that Gaussian fields are by definition not intermittent: the non-Gaussian shape of the PDFs of quantities like a_i for a Gaussian velocity field is because a_i is nonlinear in velocity.

Thus the quantitative meaning of the smallness of the total acceleration, $\mathbf{a}$, is that it appears to be small *in comparison to* its local and convective components, $\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}$ and $\mathbf{a}_c = (\mathbf{u} \cdot \nabla)\mathbf{u}$. Already at $\mathrm{Re}_\lambda = 240$, the variance of $\mathbf{a}$ is more than an order of magnitude smaller than the variance of $\mathbf{a}_l$ and $\mathbf{a}_c$. At this Reynolds number the local and the convective accelerations are strongly (anti) correlated with a correlation coefficient exceeding $|-0.9|$. On the other hand, $\mathbf{a}$ is of the same order as its potential part $\mathbf{a}_i = -\frac{1}{\rho}\nabla p$, and both are much larger than the solenoidal part of the acceleration $\mathbf{a}_s = \nu\nabla^2\mathbf{u}$. The smallness of $\mathbf{a}_s$ is maintained by strong cancellation between local acceleration $\mathbf{a}_l$ (which is solenoidal), and the solenoidal part of the convective acceleration $\mathbf{a}_{cs}$ ($\mathbf{a}_c = \mathbf{a}_{cs} + \mathbf{a}_{ci}$), so that the irrotational part of $\mathbf{a}_c$, which is equal to the irrotational part of the total acceleration, $\mathbf{a}_{ci} = \mathbf{a}_i$, comprises the main contribution to the total acceleration, but is much smaller than both $\mathbf{a}_l$ and/or $\mathbf{a}_c$.

We summarize the above as

$$\langle \mathbf{a}_s^2 \rangle \ll \langle \mathbf{a}^2 \rangle \approx \langle \mathbf{a}_i^2 \rangle = \langle \mathbf{a}_{ci}^2 \rangle \ll \langle \mathbf{a}_c^2 \rangle \approx \langle \mathbf{a}_{cs}^2 \rangle \approx \langle \mathbf{a}_l^2 \rangle. \tag{6.8}$$

The meaning of $\ll$ in the above relation is (at least) 'an order of magnitude smaller than' at the largest accessible Reynolds numbers.

The relation between various components of acceleration as expressed by (6.8) becomes stronger in small scales. It is natural to expect that, at still larger Reynolds numbers the meaning of '$\ll$' in the above relation will become 'several orders of magnitude smaller than'. If true this fact provides some justification for the 'random sweeping decorrelation hypothesis'. We tend, however, to give a more limited interpretation to this hypothesis in the sense that the microstructure (whatever this means) is *statistically decorrelated* from the energy containing eddies. This is different from the original assumption made by Tennekes (1975), in which he held that *the microstructure is statistically independent of the energy containing eddies.* The large and small scales are statistically not independent, though they are practically decorrelated. Indeed, there is a variety of manifestations of direct and bidirectional impact/coupling of large and small scales as exposed in the previous section and section 5.3 of chapter 5.

It should be stressed that, though the 'components' $\mathbf{a}_l$ and $\mathbf{a}_{cs}$ (or $\mathbf{L}_s$) of the solenoidal part of the acceleration, $\mathbf{a}_s = \mathbf{a}_{cs} + \mathbf{a}_l$, mostly cancel each other, this fact does not mean that separately they are unimportant. Their importance is seen at the level of velocity derivatives. For example, the main contribution to the enstrophy production is associated with $\mathbf{a}_{cs}$ (or $\mathbf{L}_s$), but not with $\mathbf{a}_l$. Namely, the enstrophy production term is $curl\{\mathbf{a}_{cs} - (\mathbf{u} \cdot \nabla)\omega\}$ and is approximately balanced by the viscous term in the equation for the enstrophy, whereas $curl\{\mathbf{a}_l + (\mathbf{u} \cdot \nabla)\omega\}$ is much smaller than $curl\{\mathbf{a}_{cs} - (\mathbf{u} \cdot \nabla)\omega\}$ and is balanced by the sum of the enstrophy production and the viscous terms (see section 6.3). It is also noteworthy that the nonlinearity in NSE both the whole $(\mathbf{u} \cdot \nabla)u$ and the Lamb vector $\omega \times \mathbf{u}$ have a property of becoming more solenoidal as the Reynolds number increases.

A final remark concerns the nature of Kraichnan/Tennekes 'decomposition'. Namely, there are two main ingredients in the (Eulerian) decorrelation: i) – the sweeping of microstucture by the large-scale motions (and associated kinematic nonlocality), ii) – and the local straining (which is roughly pure Lagrangian). It appears that this kind of 'decomposition' is insufficient as it is missing an essential dynamical aspect – the interaction between the two. As we have seen the random Taylor hypothesis (and, of course, the conventional Taylor hypothesis) lack/discard this aspect at the outset (this does not mean that these hypotheses are useless): both are 'too kinematic'. A closely related issue is with the rather popular assumption that choosing an appropriate 'local' system of reference one can get rid (mostly) of the sweeping of the small scales by the large-scale motions. The underlying assumption is that small scales are 'passive' and just 'swept' by the large scales without any participation in the process, i.e., without any reaction back. This is a major misconception: we have seen that there is a rich direct an bidirectional coupling between SS and LS. The issue of sweeping is closely related to the comparative aspects of Lagrangian versus Eulerian descriptions – an issue of utmost importance and difficulty, see section 3.6 and appendix C.

6.8. Non-Gaussian nature of turbulence

> *The conclusion seems inescapable that the hypothesis of quasinormality has a good chance of success only outside Kolmogorov's "universal equilibrium range"*. (Hopf, 1962).
>
> ... *it would be a miracle if the usual procedure of imposing stationarity, truncating the resulting system of equations, and looking for a Gaussian solution, would lead to results much related to physics.* (Ruelle, 1976).

The non-Gaussian nature of turbulence is another 'N' contributing to the difficulty of turbulence – in addition to the three mentioned before nonlinearity, non-integrability and nonlocality.

A rather common view/assumption was that the field of turbulent fluctuations is – in some sense(s) – nearly-Gaussian. For example, there were many attempts to treat the problem as a perturbation of a Gaussian field[52] (see e.g., references in Saffman, 1978). More recently, it was proposed that some parts of a turbulent field are nearly-Gaussian, so that it can be decomposed into a (nearly-) Gaussian part and a (strongly-) non-Gaussian one (e.g., Chertkov et al., 1999; Farge et al., 1999; Katul et al., 1994; Lewalle et al., 2000; She et al., 1991). This is a tempting assumption, since it helps to simplify the problem. However, without entering into a comprehensive review of the whole issue, we shall argue that such an assumption is inadequate[53]. For this purpose, we give a number of (counter-) examples with emphasis on the dynamical aspects. Additional aspects are addressed in the next chapter.

6.8.1. ODD MOMENTS

First we mention that a purely Gaussian velocity field is dynamically impotent. Indeed, in such a field, all the odd moments vanish[54]. This contradicts the Kolmogorov 4/5 law, turns the Karman–Howarth equation into a linear one (see Monin and Yaglom, 1971), and prevents the production of enstrophy and strain (i.e., dissipation): in a Gaussian velocity field $\langle \omega_i \omega_j s_{ij} \rangle \equiv 0$ and $\langle s_{ij} s_{jk} s_{ki} \rangle \equiv 0$. Even if the flow field is initially Gaussian, the dynamics of turbulence makes it non-Gaussian with finite rate. This is seen by taking $\langle \dots \rangle$ from the equation (C.23) (dropping the viscous term)

$$\frac{D}{Dt}\langle \omega_i \omega_j s_{ij} \rangle = \langle \omega_j s_{ij} \omega_k s_{ik} \rangle - \left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle . \tag{6.9}$$

[52] Even Hopf (1962) was tempted by quasi-Gaussianity.

[53] Most frequently it is assumed that the 'unresolved' (i.e., mainly the small-scale) part of the turbulent field is nearly-Gaussian, in contradiction to observations showing that this part of flow is the most non-Gaussian (see also 6.4).

[54] Therefore they are so convenient in assessing the 'deviations' from Gaussianity.

For a Gaussian velocity field $\langle \omega_i \omega_j s_{ij} \rangle_G = 0$, $\left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle_G = 0$ and $\langle \omega_j s_{ij} \omega_k s_{ik} \rangle_G = \frac{1}{6} \langle \omega^2 \rangle^2 > 0$. Since the quantity $\omega_j s_{ij} \omega_k s_{ik} \equiv W^2, W_i = \omega_j s_{ij}$, it is positive pointwise for any vector field. Hence at $t = 0$,

$$\left\{ \frac{D}{Dt} \langle \omega_i \omega_j s_{ij} \rangle \right\}_{t=0} = \{ \langle \omega_j s_{ij} \omega_k s_{ik} \rangle \}_{t=0} > 0, \tag{6.10}$$

It follows from the equation (6.10) that, at least for a short time interval t, the mean enstrophy production will become positive. We recall that for later moments the vorticity–pressure Hessian correlation $\left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle$ becomes finite, and nothing is known rigorously. As follows from DNS of NSE in a periodic box, the correlation $\left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle$ is positive, but is smaller than $\langle \omega_j s_{ij} \omega_k s_{ik} \rangle \equiv \langle W^2 \rangle$. Namely, $\left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle \sim \frac{1}{3} \langle W^2 \rangle$, so that the RHS of (6.9) remains positive (Tsinober et al., 1995). The nonzero $\left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle$ is another manifestation of non-Gaussianity. It is noteworthy that the equation (6.9) with $\left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle = 0$ is precisely the one arising using the quasi-Gaussian approximation $\frac{D^2}{Dt} \langle \omega^2 \rangle = \frac{1}{3} \langle \omega^2 \rangle^2$ (Proudman and Reid, 1954; Orszag, 1977), since $\frac{D}{Dt} \langle \omega_i \omega_j s_{ij} \rangle = \frac{1}{2} \frac{D^2}{Dt} \langle \omega^2 \rangle$ and under quasi-Gaussian approximation $\langle \omega_j s_{ij} \omega_k s_{ik} \rangle = \frac{1}{6} \langle \omega^2 \rangle^2$ and $\left\langle \omega_i \omega_j \frac{\partial^2 p}{\partial x_i \partial x_j} \right\rangle = 0$. The essential point is that at $t = 0$, the relation (6.10) is precise due to freedom of the choice of initial condition. On the other hand, we have seen that, if the initial conditions are Gaussian, the flow ceases to be Gaussian with *finite rate*. In other words, it is seen directly from (6.10) that *turbulence cannot be Gaussian* (see also Novikov, 1967). In this sense, Gaussian initial conditions are not 'good', since no flow state existing in reality is Gaussian. In a similar way, one can see from the equation (C.24) that the mean rate of production of strain becomes positive at small times (at $t = 0$ it vanishes) for an initially Gaussian velocity field.

The next point is that the Kolmogorov 4/5 law, $S_3(r) = -\frac{4}{5} \langle \epsilon \rangle r$, $S_3(r) = \left\langle \left(\Delta u_{|||} \right)^3 \right\rangle$, is a clear demonstration of the non-Gaussian and dissipative nature of turbulence; $S_3(r)$ is essentially nonvanishing.

A special aspect of the non-Gaussian nature of turbulence as manifested in the 4/5 law is seen when one looks at the analogy between the 4/5 law and Yaglom's 4/3 law for fluctuations of a passive scalar, θ. The latter has the form $\langle \Delta u_{|||} (\Delta \theta)^2 \rangle = -\frac{4}{3} \langle \epsilon_\theta \rangle r$, where $\Delta u_{||} \equiv [\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{r}/r$, $\Delta \theta = \theta(\mathbf{x} + \mathbf{r}) - \theta(\mathbf{x})$, and $\epsilon_\theta = \mathcal{D} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_i}$ is the dissipation of the passive scalar. The analogy, though useful in some respects (Antonia et al., 1997)[55], is violated for a Gaussian velocity field. Namely, the 4/3 law remains valid for such (as any other isotropic) velocity field, whereas the 4/5 law is not, because $S_3(r) \equiv 0$ for a Gaussian velocity field. This points to serious limitations on analogies between the passive and active fields mentioned above, which are discussed in chapter 9.

More generally, the build-up of odd moments, such as $S_3(r)$, $\langle \omega_i \omega_k s_{ik} \rangle$, $\langle s_{ij} s_{jk} s_{ki} \rangle$ and many others is an important manifestation of the nonlinearity and non-Gaussianity of turbulence and is closely related to its irreversibility. It is one of the prominent and distinctive *specific* features of turbulent flows of utmost dynamical significance involving such processes as production of enstrophy and total strain (dissipation). Hence the particular emphasis on the *odd* moments. For this reason, the quantity $cos(\omega, \mathbf{W}) = \omega_i \omega_j S_{ij} |\omega|^{-1} |W|^{-1}$ proved quite useful in the diagnostics of the non-Gaussian nature of the 'random structureless' sea in turbulent flows, which appeared to be quite the opposite, i.e., not structureless, dynamically not passive and essentially non-Gaussian (see section 6.4). This is in contrast with various recent proposals that the 'weak', in some sense, part of the field is nearly-Gaussian. It seems that there exist no such part(s). More work is necessary to clarify the issue starting with the quasi-Gaussian manifestations of turbulent flows.

Before proceeding we mention that there were a number of attempts to 'explain' the origins of non-Gaussianity of fluid turbulence (e.g., Siggia, 1977; Betchov and Larsen, 1981 and references in Li and Meneveau, 2005), though it seems rather obvious: today it is pretty clear that Gaussianity and randomness/stochasticity are not synonymous and one has to explain why he would expect Gaussian behaviour in genuine turbulent flows and not the other way around.

It is important to reiterate, however, that the origins of non-Gaussian statistics in various nonlinear systems and genuine turbulence are generally quite different qualitatively. Therefore, it seems to be misleading to 'explain' such properties of genuine turbulence by analogy with non-Gaussian behaviour of, e.g., Burgers and/or restricted Euler or similar equations (e.g., Betchov and Larsen, 1981; Li and Meneveau, 2005): both pressure and dissipative effects are crucial for genuine turbulence at the very outset[56]. An important point is that these are integrable equations, and exhibit random behaviour only under random forcing and/or initial conditions, otherwise

[55] Antonia et al. (1997) looked at the analogy between the 4/3 law for the passive scalar and the 4/3 law for the velocity field in the form $\left\langle \Delta u_{\|} (\Delta \mathbf{u})^2 \right\rangle = -\frac{4}{3} \langle \epsilon \rangle r$, which turns into the 4/5 law by isotropy.

[56] A result very similar to that in Li and Meneveau (2005) is obtained from a trivial toy model based on elementary solution $u(t) = (t + u_0^{-1})^{-1}$ of the simplest Riccati equation $du/dt + u^2 = 0$ with random Gaussian initial condition $u_{t=0}$.

their solutions are not random. These examples represent the response of nonlinear systems to random forcing and which otherwise are not random, and should be distinguished from problems involving genuine turbulence. Navier–Stokes equations at not too small Reynolds number have the property of intrinsic stochasticity in the sense that they possess mechanisms of self-randomization.

6.8.2. QUASI-GAUSSIAN MANIFESTATIONS

Turbulence – being essentially non-Gaussian – is such a rich phenomenon that it can 'afford' a number of Gaussian-like manifestations, some of which are not obvious and even non-trivial.

Single-point moments of velocity fluctuations are known to be close to Gaussian[57], see for example, Van Atta and Chen (1969); Lu and Willmarth (1973). However, this does not mean that the field of velocity fluctuations is really Gaussian. For example, non-Gaussian behaviour of the field of velocity fluctuations has been observed in experiments by Frenkiel et al. (1979) on grid turbulence, see also Vedula et al. (2005) and references therein. They measured correlations of the type $R^{m,n}(\tau) = \frac{\langle u^m(t)u^m(t+\tau)\rangle}{\langle u^2(t)\rangle^{(m+n)/2}}$. The non-Gaussian behavior of the field of velocity fluctuations is seen clearly for $\tau \neq 0$: the odd-order correlations, $R^{2,1}(\tau)$, $R^{4,1}(\tau)$, $R^{3,2}(\tau)$, are essentially nonzero especially around $U\tau/M \sim 1$ (see an example shown in figure 6.29). It is noteworthy that at $h = 0$ the $R^{2,1}(\tau)$, $R^{4,1}(\tau)$, $R^{3,2}(\tau)$ are indistinguishable from zero, showing the importance of two-point correlations. The even moments up to the sixth order (not only numbers but the whole correlation functions) are very close to Gaussian too, see figure 6.29 left. This means, for example, that the fourth-order correlation shown in this figure is very well approximated by the Millionschikov hypothesis (zero-fourth cumulant) *as* for a Gaussian velocity field (see also Vedula et al., 2005). The velocity derivatives in the experiments of Frenkiel et al. (1979) exhibit clear non-Gaussian behaviour for all moments. There are more intricate examples of nearly-Gaussian manifestations of turbulent flows. Two examples are shown in figures 6.30 and 6.31. The example in figure 6.31 is interesting in that the Gaussian-like behaviour is exhibited by third-order quantities. For other examples, see Tsinober (1998a). The fourth example involves pressure. It was shown by Holzer and Siggia (1993) that for a Gaussian velocity field the PDF of pressure is strongly negatively skewed

[57]This is not true of strong fluctuations of velocity. It was observed in laboratory and numerical experiments that the single-point PDF of velocity at large amplitudes of velocity fluctuations is sub-Gaussian and recently was confirmed theoretically using the so called instanton formalism (see references in Tsinober, 1998b; for other quasi-Gaussian manifestations see Brun and Pumir, 2001 and references therein).

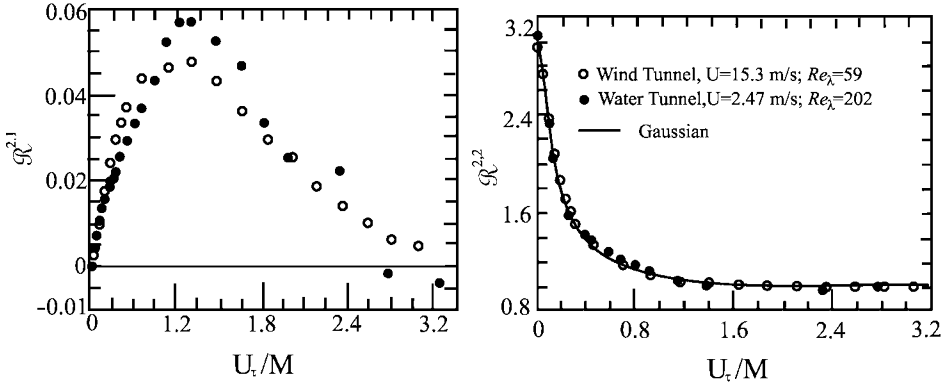

Figure 6.29. Third- (left) and fourth- (right) order time correlations for turbulent velocities. Adapted from Frenkiel et al. (1979). Here $\mathbf{R}^{m,n} = \frac{1}{2}\left[R^{m,n}(\tau) + R^{n,m}(\tau)\right]$. The correlation $\mathbf{R}^{m,n}$ was used instead of individual odd-order correlations $R^{m,n}(\tau)$, since the latter exhibited *large dispersions from sample recording to sample recording* – a well known problem to everybody tried to evaluate odd-order moments. Here M is the grid mesh and U is the mean velocity of the flow

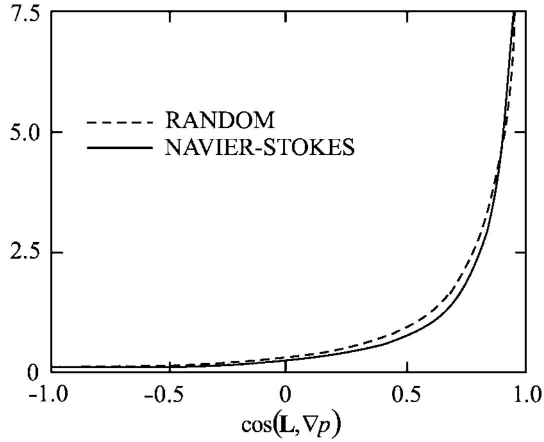

Figure 6.30. PDFs of the angle between the Lamb vector $\omega \times \mathbf{u}$ and its potential part $\nabla\alpha \equiv \nabla p$ for a numerically simulated turbulence and a random (approximately Gaussian) velocity field with the same energy spectrum (Shtilman et al., 1993). The similarity is obvious

and has exponential tails. This result is not unexpected, though it is not easy in this case to demonstrate it directly due to nonlocal relation between pressure and velocity fields. It is much easier to do this by looking at $\nabla^2 p$. Using the same method as in Shtilman et al. (1993), the PDF $\mathcal{P}(x)$, $x = \frac{\nabla^2 p}{\rho\langle\omega^2\rangle} = \frac{\omega^2 - 2s_{ij}s_{ij}}{2\langle\omega^2\rangle}$, is expressed in the following way (Spector, 1996,

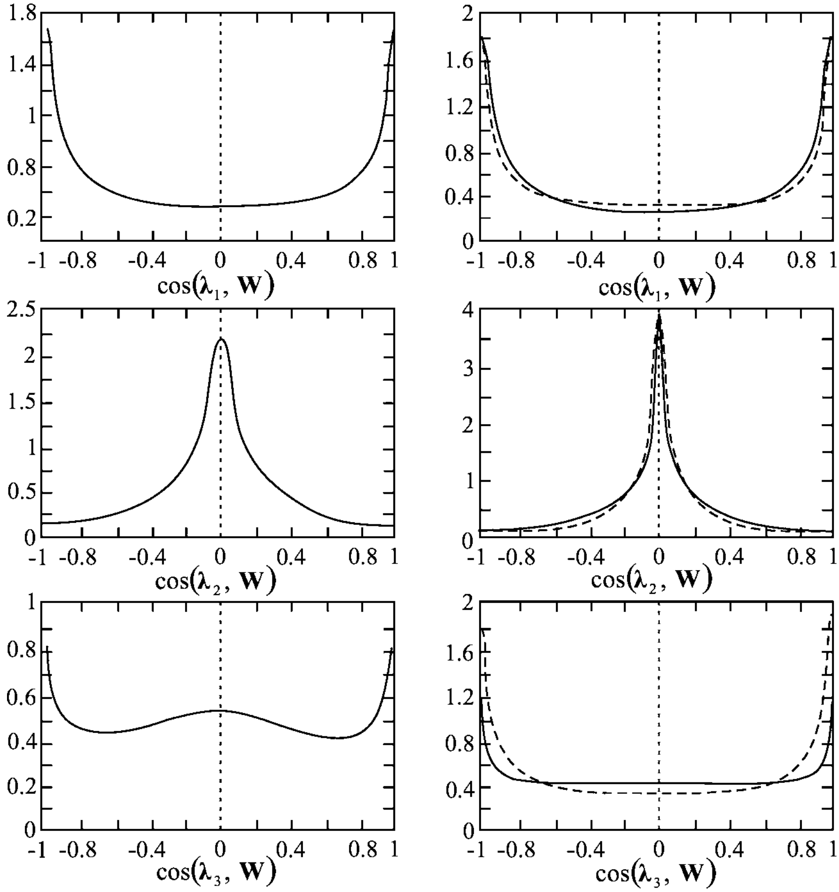

Figure 6.31. PDFs of the cosine of the angle between vortex stretching vector $W_i = \omega_i s_{ij}$, and the eigenvectors λ_i of the rate of strain tensor; left –grid turbulence, right – DNS and random Gaussian (– – – –). Re$_\lambda \approx 75$. The behaviour for λ_2 seems to contradict the ones shown in figures 6.8 and 6.9, since there is a tendency for alignment between ω and **W** and between ω and λ_2. However, closer inspection shows that the alignments shown in figure 6.31 and in figures 6.8, 6.9 are associated with different regions in the flow

private communication):

$$\mathcal{P}(x) = \{3^{1/2}5^{5/2}\}/(4\pi)\ x^2\ e^x\ [K_2(4x) - K_1(4x)], \qquad x < 0,$$

which for large $|x|$ has the asymptotics $\sim|x|^{1/2}\ e^{-3|x|}$, and

$$\mathcal{P}(x) = \{3^{1/2}5^{5/2}\}/(4\pi)\ x^2\ e^{-|x|}[K_2(4x) + K_1(4x)], \qquad x > 0,$$

which for large x has the asymptotics $\sim x^{3/2}\ e^{-5x}$. We see that the distribution of $\nabla^2 p$ for a Gaussian velocity field has exponential tails and is

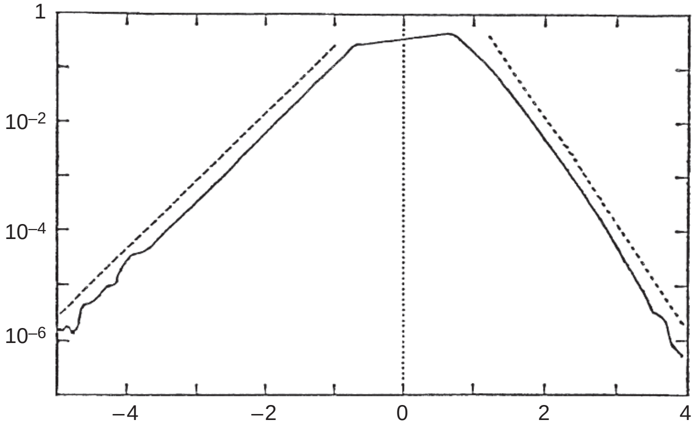

Figure 6.32. PDF of the Laplacian of pressure $x = \frac{\nabla^2 p}{\rho\langle\omega^2\rangle} = \frac{\omega^2 - 2s_{ij}s_{ij}}{2\langle\omega^2\rangle}$. ——— – grid turbulence, $\mathrm{Re}_\lambda \approx 75$; - - - - - - slope +4.5, — — — — – slope –2.9

negatively skewed. This result is in agreement with the ones from laboratory and DNS experiments (figure 6.32). It shows that these effects are mostly of kinematical nature, as many others like those given above usually are.

A related example is shown in figure 6.33. It is clear from this figure that the correlation function of the total acceleration, $\langle \mathbf{a}(\mathbf{x}+\mathbf{r})\mathbf{a}(\mathbf{x})\rangle$ is practically the same in real flows, as it is in laboratory grid turbulence (Hill and Thoroddsen, 1997) and in DNS (Vedula and Yeung, 1999), and in artificial velocity fields under zero forth-cumulant assumption – Millionschikov hypothesis (Pinsky et al., 2000), and/or for the assumption of joint Gaussian velocities (Hill and Thoroddsen, 1997). This behaviour is of the same nature as the one shown in figure 6.29, right. Similar observation can be seen in 1) Aringazin, 2004 for the PDFs of acceleration statistics in turbulent flows with Gaussian velocities, which is in good agreement with the experimental data by Mordant, Crawford and Bodenschatz (2004); and 2) experimental results by (Xu et al. 2007) on acceleration correlations and pressure structure functions with those by Obukhov and Yaglom (1951) in which it was assumed that velocity derivatives (!) at two spatial points have the joint Gaussian probability distribution. We recall the similarity in behaviour of real fields and of their Gaussian counterparts in a number of aspects as mentioned in the section on accelerations above.

The examples shown in figures 6.29–6.33 together with other quasi-Gaussian manifestations of turbulent flows should not be misinterpreted to conclude that turbulent flows are Gaussian. Indeed, we have seen that

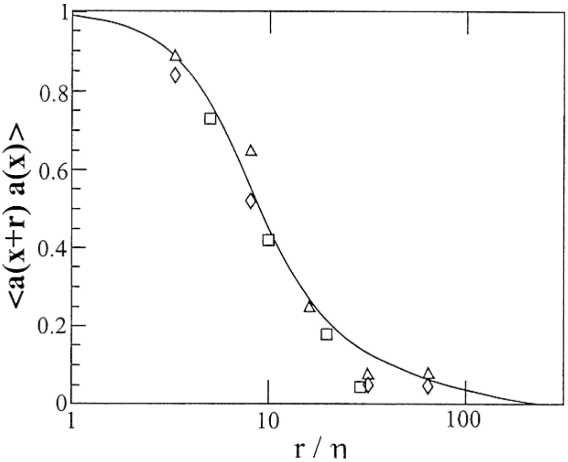

Figure 6.33. The correlation function of the total acceleration $\langle \mathbf{a}(\mathbf{x}+\mathbf{r})\mathbf{a}(\mathbf{x})\rangle$: $\diamond$ – grid turbulence (Hill and Thoroddsen, 1997); $\square$ – DNS (Vedula and Yeung, 1999); —— under zero forth-cumulant assumption – Millionschikov hypothesis (Pinsky et al., 2000); $\triangle$ – joint Gaussian velocities (Hill and Thoroddsen, 1997). The figure is from Pinsky et al. (2000)

there are many non-Gaussian manifestations of turbulent flows, which are of special dynamical significance. As mentioned, turbulence – being essentially non-Gaussian – is such a rich phenomenon that it can 'afford' a number of manifestations which are Gaussian-like. Non-Gaussianity adds to the list of n's together with non-lognormality and non-Markovianity.

6.9. Irreversibility of turbulence

The equations (6.9, 6.10) and similar ones for $\langle s_{ij}s_{jk}s_{ki}\rangle$ can be seen as one of the manifestations of the statistical irreversibility of turbulent flows (Betchov, 1974; Novikov, 1974). The corresponding dynamical *instantaneous* (inviscid) equations are reversible. Hence, the term *statistical.* The (apparent) randomness of turbulent flows is important even at the kinematical level, e.g., the predominant tendency of stretching of material lines and other passive objects (see chapter 4). There exist, at least, two different aspects of this problem. The first one is related to purely inertial behaviour governed by the Euler equations as mentioned above. Though these equations are reversible, it is (empirically) known that for Euler equations the enstrophy generation increases very rapidly with time – apparently without limit (see references in Tsinober, 1998b, 2000). This aspect is closely related to the (possible) formation of singularities in 3D Euler flows in finite or infinite time. The above example (equations (6. 9, 6.10) and similar ones for $\langle s_{ij}s_{jk}s_{ki}\rangle$) is closely related to this aspect. The 4/5 law can be seen also as a manifestation of statistical irreversibility of turbulent flows. The second aspect is associated with the dissipative nature of turbulent flows. Among other roles viscosity provides a sink of energy, enstrophy, etc. moderating their unbounded growth in the inviscid case. An important point is that turbulent flows are strongly (not slightly) dissipative at whatever large Reynolds numbers.

6.10. Summary

Turbulent flow in a box forced by a steady (or random) force exhibits large fluctuations in time of the overall quantities such as the energy input, dissipation, enstrophy, enstrophy production and others, of their long time averages. This process is similar to that observed experimentally in the 'French washing machine' at the level of energy dissipation and most probably occurs in almost all turbulent flows.

One of the intrinsic processes of turbulence dynamics is the process of self-amplification of velocity derivatives. It involves both vorticity and strain, i.e., this process is characteristic of the whole field of velocity derivatives. It consists of two strongly (but nonlocally) interconnected processes: predominant vortex stretching/enstrophy production and predominant self-amplification of the rate of strain/production of total strain. The enstrophy production is essentially a nonlocal process of interaction of vorticity and strain with predominant stretching. The production of strain (rather than vortex stretching, which resists the amplification of strain) is directly responsible for the enhanced dissipation of turbulent flows. It is more a self-amplification and is local with indirect (but essential) aid of vorticity with predominant compressing. The self-amplification of strain is a specific feature of the dynamics of three-dimensional genuine (as contrasted to 'passive') turbulence. The essential dominance of the self-amplification of the velocity derivatives over the external forcing (or other factors driving the flow at the level of velocity field) occurs already at very moderate Reynolds numbers and becomes stronger with increasing Reynolds number. A similar phenomenon is observed in a turbulent shear flow in a channel over most of its cross section except in the proximity of the wall, and in the atmospheric surface layer (see chapter 8). This dominance occurs not only in the mean, but practically pointwise throughout the whole flow field, i.e. the self-amplification of velocity derivatives by turbulence is a process which is local in space. The property of self-amplification is a universal one not only qualitatively, it possesses a number of quantitative universal properties which are independent of the details of forcing. The self-amplification is a quasi-stationary process in the sense that the integrals over the flow domain of the enstrophy (and strain) production and of its viscous destruction are approximately balanced at any moment in time. Hence the time derivative of the overall enstrophy (and strain) is much smaller than the overall enstrophy production and of its viscous destruction. It is not yet clear how to reconcile the self-amplification of velocity derivatives with the property of nonlocality of turbulent flows, especially the aspect of direct coupling/interaction between large and small scales.

Most of the enstrophy production, production of strain/dissipation and other nonlinear processes are associated with *i* – large strain, rather than with intense vorticity (large enstrophy), *ii* – alignment of vorticity with the largest eigenstrain (not with the intermediate one), *iii* – strong tilting of vorticity and finite curvature of vortex lines (not with Burgers-like objects with small curvature). The nonlinearities are an order of magnitude larger in the regions dominated by strain than in the enstrophy dominated regions. In this sense the enstrophy dominated regions are characterized by reduced nonlinearities. In other words the most intense nonlinear processes occur in the strain dominated regions. In particular, there is an approximate balance between the nonlinearities (e.g., vortex stretching) and the viscous terms (e.g., viscous destruction of vorticity) in the regions with concentrated vorticity, whereas in strain dominated regions the nonlinearities (e.g., the enstrophy production) are an order of magnitude larger than the viscous terms. All this also supports the view that regions of concentrated vorticity in turbulent flows are not as important as previously thought. The energy cascade (whatever this means) and its final result – dissipation are associated with the production of strain, i.e., with the local process of production of strain $-s_{ij}s_{jk}s_{ki}$, rather than with the nonlocal process of enstrophy production $\omega_i\omega_i s_{ij}$. The latter (i.e., vortex stretching) suppresses the 'cascade' (production of strain) and does not aid it, at least in a direct manner.

The primary effect of viscosity (apart from energy dissipation) is to balance the production of velocity derivatives, both strain and vorticity (the so-called Tennekes and Lumley balance). This balance holds at any, however large, Reynolds numbers and not only in the mean, and not only in statistically stationary flows. Viscosity allows what is called vortex reconnection. The role of this process in turbulence remains not fully clear. The conceptual difficulty is that with $\nu \neq 0$ (however small) vorticity is not frozen in the flow field (again at any, however, large Reynolds numbers). In other words, vorticity lines do not possess a Lagrangian identity so that there is no way to follow them unambiguously in time.

Turbulence is a nonlocal process. This nonlocality is manifested among other things in direct and bi-directional interaction/coupling of large and small scales. It is a generic internal property of turbulent flows and exists independently of the presence of mean shear or other external factors, but has different manifestations for different external factors. For example, in the presence of a mean shear the small scales become anisotropic, whereas if the small scales are artificiality excited, the overall dissipation and mixing rate of the turbulent flow increase substantially. The direct interaction/coupling of large and small scales is in full conformity and is the consequence of the generic property of Navier–Stokes equations, which are integro-differential due to the nonlocal relation between pressure and velocity fields. It appears that the Kolmogorov 4/5 law can be interpreted as one of the manifestations of nonlocality in the above sense. Nonlocality is associated also with 'kinematics' due to the nonlocal relations between, e.g., velocity and its increments and between vorticity and strain. The material velocity derivative – the fluid particle acceleration – is also related in a nonlocal manner to the velocity field, since the main contribution to the acceleration comes from the pressure gradient.

In view of the direct and bidirectional interaction/coupling of large and small scales the notion of an inertial range is not well defined. Consequently, there are conceptual problems on the nature, definition, distinction, properties of conventionally-defined inertial and dissipative ranges in turbulence at high Reynolds numbers. There is a conceptual necessity of studying turbulence as an undecomposable whole and sub-Kolmogorov resolution as a key means for coping with such problems, see chapter 5. This has serious implications for modelling such as 'eddy viscosity' representations of the subgrid scales, etc. Eddy viscosity does not explain the enhanced transfer rates. It is just a purely *empirical* way of accounting for such rates.

From the statistical point turbulence is irreversible and essentially non-Gaussian, but possesses a number of quasi-Gaussian manifestations.

CHAPTER 7

STRUCTURE(S) OF TURBULENT FLOWS

Is there turbulence without structure(s)?

Although symmetric causes must produce symmetric effects, nearly symmetric causes need not produce symmetric effects: a symmetry problem need have no stable symmetric solutions (Birkhoff, 1960).

Turbulence is and will remain the most difficult problem of fluid mechanics, and past experience suggests that a subsequent fall of interest in the coherent structures is more than likely. The resulting net gain in understanding of turbulence may be less than our high expectations of today but will certainly be positive (Liepmann, 1979).

I emphasized the concept of "broken symmetry", the ability of a large collection of simple objects to abandon its own symmetry as well as the symmetries of the forces governing it and to exhibit the "emergent property" of a new symmetry (Anderson, 1991).

Kolmogorov's work on fine-scale properties ignores any structure which may be present in the flow (Frisch, 1995).

At this stage, this alternative approach (i.e., the 'structural') *has not led to a generally applicable quantitative model, neither – for better or worse – has it a major impact on the statistical approaches* (S.B. Pope, 2000).

7.1. Introduction

The nature and characterization of the structure(s) of turbulent flows are among the most controversial issues in turbulence research with extreme views on many aspects of the problem – in words of Richard Feynmann (1963, p. 41–12), *holding strong opinions either way*. For example, as mentioned in chapter 3, it is common in the vast literature on turbulence to consider the terms *statistical* and *structural* as incompatible or even contradictory:

... *it became obvious that statistical averaging was in fact destroying the most interesting and important phenomena in turbulence – the formation, dynamics and persistence of vortex motion.*

Its blindness to these structural facts is precisely the disability of the statistical idea ... In place of theory without structure, the result to date has been structure without theory.

Following Lumley (1989), who provides many such examples, the references are suppressed to protect the guilty.

However, there are common points as well. For instance, it is mostly agreed that turbulence definitely possesses structure(s) – whatever this means – and that intermittency, which is addressed in the next section, is intimately related to *some* aspects of the structure(s) of turbulence.

It is argued in the section following the one on intermittency that it is a misconception to contrapose the *statistical* and the *structural* and that they represent different facets/aspects of the same problem, so that there is no gap between structure(s) and statistics[1]. Just as it seems impossible to separate the structure(s) from the so-called 'random structureless background' or the '*random processes from the nonrandom processes*' (Dryden, 1948) due to strong interaction (and nonlocality), both between individual structures, and between structures and the 'background'. In other words there is no turbulence *without* structure, every part of the turbulent field just like the whole possess structure[2]. Structureless turbulence or any of its part contradicts both the experimental evidence and the Navier–Stokes equations. It is noteworthy that the statement that turbulence has structure is in a sense trivial: *to say that turbulent flow is 'completely random' would define turbulence out of existence* (Tritton, 1988, p. 295) – after all turbulent flows seem to obey the Navier–Stokes equations.

7.2. Intermittency

> *At any instant the production of small scales is ... occurring vigorously in some places and only weakly in the others* (Tritton, 1988).
>
> *Typical distribution of scalar and vector fields is one in which there appear characteristic structures accompanied by high peaks or spikes with large intensity and small duration of spatial extent. The intervals between the spikes are characterized by small intensity and large extent* (Zeldovich et al., 1988).
>
> *Intermittency is a phenomenon where Nature spends little time, but acts vigorously* (Betchov, 1993).

[1]Both issues are intimately related to the non-Gaussian nature of turbulence (and *some* of its quasi-Gaussian manifestations) and the necessity and the only objective means to handle the issues of turbulence structure(s) via statistics.

[2]There are proposals to *scan out* the structure(s). In fact there is no way to do so, since *structure* is everywhere. Even the so-called 'simple' structures (worms) are 'renormalized' by the background.

The term intermittency is used in two distinct (but not independent) aspects of turbulent flows. The first one is the so-called external intermittency. It is associated with what is called here partly-turbulent flows, specifically with the strongly irregular and convoluted structure and random movement of the 'boundary' between the turbulent and nonturbulent fluid. This kind of intermittency was studied first by Townsend (1948). We will address this matter in chapter 8.

The second aspect is the so-called small-scale, internal or intrinsic intermittency. It is usually associated with the tendency to spatial and temporal localization of the 'fine' or small-scale structure(s) of turbulent flows.

Our concern here is with the intermittency of this second kind. It is noteworthy that in a broad sense intermittency is a ubiquitous phenomenon occurring in a great variety of qualitatively different systems; see chapter 8 in Zeldovich et al. (1990) for a lively exposition of a wide number of different systems exhibiting intermittency, and also Vassilicos (2001). The main common features of all of them are (space/time) randomness and localization (both spatial and temporal) of their 'fine' structure. However, this is not enough to define intermittency. For example, almost any nonlinear function or almost any nonlinear functional of a random Gaussian field is intermittent in the above sense, though random Gaussian fields by definition lack any intermittency.

7.2.1. WHAT IS SMALL-SCALE INTERMITTENCY?

The phenomenon of small-scale intermittency was discovered by Batchelor and Townsend (1949) in experiments with turbulent grid flows and in a wake past a circular cylinder[3]:

The basic observation which requires explanation is that activation of large wave-numbers is very unevenly distributed in space. These space variations in activation can be described as fluctuations in the spectrum at large wave-number ... As the wave-number is increased the fluctuations seem to tend to an approximate on-off, or intermittent variation. Whatever the reason for the occurrence of these fluctuations, they appear to be intrinsic to the equilibrium range of wave-numbers. All the evidence is consistent with the inference that the fluctuations are small in the region of smallest wave-numbers of equilibrium range and become increasingly large at larger wave-numbers ... the mean separation of the visible activated regions is comparable with the integral scale of the turbulence, i.e., with the size of the energy-containing eddies (pp. 252–253)

[3]The intermittent nature of the small-scale structure of turbulent flows was foreseen by Taylor (1938b): ... *the view frequently put forward by the author that the dissipation of energy is due chiefly to the formation of very small regions where vorticity is very high.* However, note that dissipation is high in regions where strain – not vorticity – is high (see section 6.2).

Batchelor and Townsend obtained some evidence that the deviation from Gaussianity is stronger as the Reynolds number is increased[4], which was confirmed by a number of subsequent experiments such as by Kuo and Corrsin (1971)[5].

An example of time records of the streamwise velocity component, and their derivatives obtained in a field experiment at $\mathrm{Re}_\lambda = 10^4$ is shown in figure 1.17. The increasingly intermittent behaviour of the signal with the derivative order is seen quite clearly. Also shown are records for the enstrophy ω^2, total strain $2s^2 \equiv 2s_{ij}s_{ij}$ and their surrogate $(\partial u_1/\partial x_1)^2$, and enstrophy production $\omega_i\omega_j s_{ij}$, $s_{ij}s_{jk}s_{ki}$ and their surrogate $(\partial u_1/\partial x_1)^3$. The experiment was performed in the atmospheric surface layer at a height $10m$ in approximately neutral (slightly unstable) conditions.

A qualitative summary is that small-scale intermittency of turbulence is associated with its spotty (spatio-temporal) *structure* which among other things is manifested as a *particular* kind of non-Gaussian behaviour of turbulent flows. This deviation from Gaussianity increases with both 1) increasing the Reynolds number and 2) decreasing the 'scale'. In other words, intermittency involves two (not independent) aspects of turbulent flows – their structure/geometry and statistics. These two aspects are reflected in attempts to 'define' intermittency. Two examples of such definitions are given below.

Structural/geometrical. A positively defined quantity μ (measure) is intermittent in space if for arbitrarily small μ_0, the fraction V_0/V of any volume V of fluid within which $\mu > \mu_0$ tends to zero as the Reynolds number $\mathrm{Re} \to \infty$, *i.e., $\mu(\mathbf{x}, t)$ becomes increasingly spiky, being concentrated almost entirely in this vanishingly small part V_0 of the volume V*. This is a modified version of Moffatt's (1988) definition of intermittency of dissipation. In fact, all known dynamically important quantities including those which are not positively defined (e.g., enstrophy and strain production) exhibit such a spiky behaviour. Note that the above definition refers to rather general property of turbulence structure – the 'increasingly spiky' structure of some variable can be realized in a great variety of ways: both the structure of these spiky regions and of the 'background' may be very different for the same V_0.

Statistical. A variable with zero mean will be called intermittent if it has a probability distribution such that extremely small and extremely large

[4]However, they did not appreciate this effect and claimed that the flatness factors *seem to vary a little with Reynolds number*, though this factor changed from 5 to 7 for the third-order derivative; see their figure 5 for the flatness factor of velocity derivatives of different orders and different Reynolds numbers.

[5]For an updated overview of the subsequent results see, e.g., Sreenivasan and Antonia (1997).

excursions are much more likely than in a normally distributed variable [i.e., Gaussian]. *Therefore, the kurtosis* [flatness] *of an intermittent variable with zero mean is large. Correspondingly, a non-negative variable is called intermittent if its variance is large compared to the square of its mean value* (Tennekes, 1973). This definition ignores Reynolds number dependence. For other definitions see Frisch (1995), Libby (1996).

It is important that intermittency implies non-Gaussianity, but not necessarily vice versa – practically any parameter can at most indicate the degree of intermittency of a flow *already* known to be intermittent (see next section).

7.2.2. MEASURES/MANIFESTATIONS OF INTERMITTENCY

Intermittency factor(s)
Loosely, an intermittency factor is defined as a fraction of volume (time) where the variable is 'active'. This is one of the most reliable *direct* measures of intermittency. The main deficiency is that intermittency factors depend on the choice of the threshold below which the variable is considered 'inactive' (Kuo and Corrsin, 1971; Kuznetsov et al., 1992 and references therein).

Flatness factor
Flatness factor or kurtosis of a variable is defined as

$$F(a) = \frac{\langle a^4 \rangle}{\langle a^2 \rangle^2}.$$

For a Gaussian field $F_G(a) = 3$, i.e., the flatness factor can be used as a measure of non-Gaussianity. The specific choice is justified by the fact that the inverse of flatness increases with and is roughly proportional to the fraction of volume/time where the variable is 'active'.

It has been established experimentally that the flatness increases with both the order of derivative (scale dependence) and the Reynolds number of the turbulence (Batchelor and Townsend, 1949; Kuo and Corrsin, 1971). Kuo and Corrsin interpreted this result (using also the intermittency factor) as a decrease in the volume fraction occupied by fine-structure both as the Reynolds number is increased and as the structure becomes finer.

As mentioned, a statistical measure such as flatness may deviate strongly from a Gaussian value without any intermittency in the flow field. The simplest example is the Gaussian field itself, which by definition lacks any intermittency. However, any nonlinear function (or functional) of a variable which is Gaussian, is non-Gaussian. For instance enstrophy, dissipation, pressure, etc. of a Gaussian velocity field possess *exponential* tails and their flatness is quite different from 3 (see chapter 6 and also figure 4 in Kennedy and Corrsin, 1961). For example, for a Gaussian velocity field $F_G(\omega^2) = \langle\omega^4\rangle/\langle\omega^2\rangle^2 = 5/3$ and $F_G(s^2) = \langle s^4\rangle/\langle s^2\rangle^2 = 7/5$. But this by no means indicates that, for a Gaussian velocity field, these quantities are intermittent. Moreover, the flatness of enstrophy is *larger* than that of total strain, $F_{\omega^2} - F_{s^2} = 4/15$. Does one have to conclude from the above result (as some authors did) that the enstrophy field is more intermittent than that of total strain in a *Gaussian* velocity field? Certainly not, since intermittency is, by definition, absent in any Gaussian field. This example shows that it is not sufficient to define intermittency as uneven distribution of enstrophy and dissipation in space, as is quite frequently done.

Similarly, the Reynolds stress u_iu_j exhibits 'intermittency'. The main contribution to this intermittency comes from the fact that u_iu_j is a product of two random variables both distributed close to Gaussian. For example, the PDF of the u_1u_2 of the strongly intermittent signal obtained by Lu and Willmarth (1973) in a turbulent boundary layer is strongly non-Gaussian. However, the PDF of u_1u_2 is approximated with high precision by assuming both u_1 and u_2 to be Gaussian with a correlation coefficient between them adjusted from the experiment (-0.44) (see chapter 8)[6].

Passive objects (scalars like heat, vectors like magnetic field) in a random velocity field (real or artificially prescribed) are nonlinear functionals of the velocity field and forcing. Therefore, even when both the velocity field and forcing are Gaussian the field of a passive object is expected to be strongly non-Gaussian as usually (but not always) is the case (Majda and Kramer, 1999). Such *kinematic* intermittency is observed in a great number of theoretical and some experimental works (for a partial list of references see chapter 4). The term 'kinematic' is used here in the sense that there is no relation to the dynamics of fluid motion, which does not enter in the problems in question, and the velocity field is prescribed and often assumed to be Gaussian.

It is noteworthy that both criteria – the intermittency and the flatness factors are of purely kinematic nature, i.e., they are not related – at least directly – to the dynamical aspects of turbulence. One would think that the situation is different with the so-called odd moments. However, this is not the case either.

Odd moments

Any odd moment of a Gaussian variable vanishes, for example skewness $S_G(a) \equiv \langle a^3\rangle/\langle a^2\rangle^{3/2} = 0$. Therefore, odd moments are very sensitive to

[6]The above examples may serve as a warning that multiplicative models enable us to produce intermittency for a purely nonintermittent field as is the Gaussian velocity field. See Zeldovich et al. (1990) on interesting observations on this and related matters.

deviations from Gaussianity, so that non-zero odd moments may be especially good indicators of intermittency. Build-up of odd moments is a result of both the (kinematic) evolution of a passive field in any random velocity field and the *dynamics* of turbulence itself. In the latter case, non-vanishing odd moments are the most important, dynamically significant manifestations of non-Gaussianity, i.e., they reflect *directly* the dynamic aspects of intermittency. The most prominent odd moments are the third-order structure function for longitudinal velocity increments $S_3^{||} = \langle\{[\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot (\mathbf{r}/r)\}^3\rangle$ entering the 4/5 law, the enstrophy production $\langle\omega_i\omega_k s_{ik}\rangle$ and the third-order moment of the strain tensor $\langle s_{ij}s_{jk}s_{ki}\rangle$ (see chapter 6). Note that all these and other odd moments vanish in a Gaussian velocity field. In contrast similar odd moments for passive objects, $\langle G_i G_k s_{ik}\rangle$, $\langle B_i B_k s_{ik}\rangle$ do not vanish (chapter 4). Hence, the passive objects in some respects are "more intermittent", e.g., see figure 7.1.

We remind that the non-Gaussian nature of genuine turbulent flows and of passive objects is qualitatively different (see chapter 9) just as is intermittency in physically different systems.

Scaling exponents and PDFs

It is commonly believed that among the manifestations of the small-scale intermittency[7] is the experimentally observed deviation of the scaling exponents for structure functions $S_p^{||} = \langle\{[\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot (r/r)\}^p\rangle$ for $p > 3$ from the values implied by the Kolmogorov theory (i.e., anomalous scaling), which in turn is due to rare strong events. Namely,

$$S_p^{||}(r) \propto r^{\zeta_p^{||}}, \tag{7.1}$$

where $\zeta_p^{||} = p/3 - \mu_p < p/3$ is a convex nonlinear function of p (see figure 7.1).

However, there are two major problems with scaling as follows, plus see sections 5.3 and 5.4.5.

First, there exists no one-to-one relation between simple statistical manifestations and the underlying structure(s) of turbulence[8]. An example is shown in figure 7.2. Moreover qualitatively different phenomena can possess the same set of scaling exponents (see appendix C, section C.2), so that one needs more subtle statistical characterizations of turbulence structure(s) and intermittency. For example, until recently one of the common

[7]A recent example is the paper by Seiwert et al. (2008) who studied *the decrease of intermittency in decaying rotating turbulence* via looking at the scaling of the longitudinal velocity structure functions, up to order $q = 8$. This decrease can be explained by suppression of strong dissipative events in the presence of rotation as proposed in section 5.3

[8]This issue is addressed in more detail in the next section.

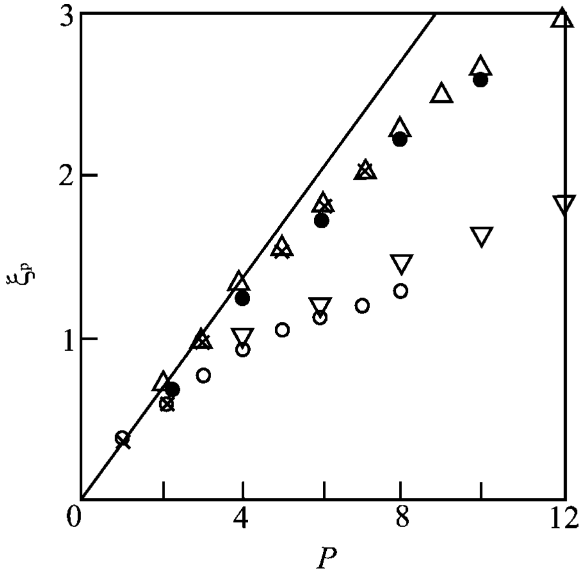

Figure 7.1. Exponents of structure functions for the longitudinal velocity component ($\triangle, \bullet, \times$) and temperature ($\triangledown, \circ$); $\triangle$ – Anselmet et al. (1984); $\triangledown$ – Antonia et al. (1984); $\circ$ – Ruiz-Chavaria et al. (1996); $\bullet$ – Vincent and Meneguzzi (1991). Exponents of structure functions for the transverse velocity component, $\times$ – Noullez et al. (1997). This figure is from Tsinober (1998b)

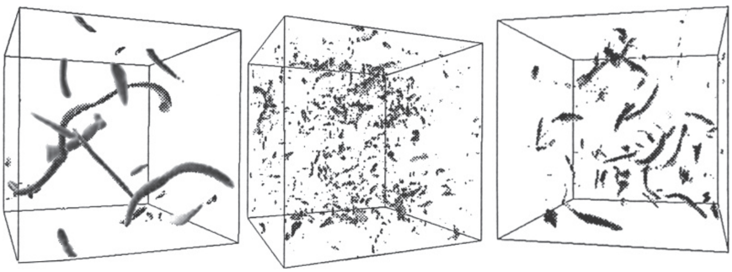

Figure 7.2. Iso-ω surfaces for three kinds of forcing of the RHS of DNS of NSE in a cubic box with periodic boundary conditions. Left – forcing in the lowest Fourier wave-numbers, middle – $\langle |f_k|^2 \rangle \sim k^{-3}$, and right – $\langle |f_k|^2 \rangle \sim k^{-5}$. In all cases the scaling properties are practically the same and are in agreement with other results (Sain et al., 1998).

beliefs was that the observed vortex filaments/worms are mainly responsible for the phenomenon of intermittency understood as anomalous scaling. However, it appears that this is not the case (see the evidence for that given in Tsinober, 1998a; and chapter 6). This was confirmed also by recent analysis by Roux et al. (1999) of the data from the experiments by Cadot et al. (1995), which *strongly suggests that the statistical contribution of vorticity filaments is not responsible for the intermittency phenomenon,* i.e., anomalous scaling. A similar result was obtained by Dernoncourt et al. (1998), (see also Chavanis and Sire, 2000; Min et al., 1996 and Sain et al., 1998)[9].

Likewise similar PDFs of *some* quantities can correspond to qualitatively different structure(s) and quantitatively different values of Reynolds number (Kraichnan and Kimura, 1994; Tsinober, 1998a,b). The emphasis is on *some* quantities like pressure or some other usually (but not necessarily) even-order quantities in velocities or their derivatives, since the PDFs of *other* appropriately chosen quantities are sensitive to structure (see below). Another example is represented by numerous models that attempted to reproduce the anomalous scaling (7.1) (a partial list of references is given in Sreenivasan and Antonia, 1997 and Tsinober, 1998b). These models followed the Kolmogorov (1962) refined similarity hypothesis (RSH) in which the mean dissipation $\langle \epsilon \rangle$ was replaced by 'local' dissipation ϵ_r averaged over a region of size r[10]. The scaling exponents obtained in all of these models are in good agreement with the experimental and numerical evidence, e.g., these models exhibit the same scaling properties (and some other such as PDFs) as in real turbulence. It is noteworthy that many of these models are based on *qualitatively different* premises/assumptions and with few exceptions have no direct bearing on the Navier–Stokes equations[11]. The most common justification for the preoccupation with such models is that they (at least some of them) share the same basic symmetries (perhaps also some hidden symmetries), conservation laws and some other general properties, etc. as the Navier–Stokes equations. The general belief is that this – along with the diversity of such systems (there are many having nothing to do with fluid dynamics, e.g., granular systems, financial markets, brain activity) – is the reason for the above mentioned agreement. However, this is not really the case, e.g., in Kraichnan (1974) a counter-example of a '*dynamical equation is exhibited which has the same essential invariances, symmetries, dimensionality and equilibrium statistical ensembles as the Navier–Stokes equations but which has radically different inertial-range behaviour*'! The

[9] Jimenez and Wray (1998) hold an opposite view, that *the filaments are responsible for most of the intermittency effects of the higher moments of the velocity derivatives* (p. 283).

[10] Kolmogorov proposed this hypothesis following the Landau objection to universality in the first Russian edition of *Fluid Mechanics* by Landau and Lifshitz (1944) about the role of large-scale fluctuations of energy dissipation rate, i.e., non-universality of both the scaling exponents ζ_p and the prefactors C_p in (7.1).

[11] Therefore the success of such models can hardly be evaluated on the basis of how well they agree with experiments. For example, there exist many theories which produce the $k^{-5/3}$ energy spectrum for qualitatively and/or physically different reasons. A recent example is a *suggestion that the spectrum of fully-developed turbulence is determined by the equilibrium statistics of the Euler equations and that a full description of turbulence requires only a perturbation, small in some appropriate metric, of a Gibbsian equilibrium*, (Chorin, 1996).

majority of models exhibit *temporal* chaos only. Therefore, such and most other models hardly can be associated with the intermittency of *real* fluid turbulence, which involves essentially *spatial* chaos as well. Again, for the above reasons the agreement between such models and experiments (both laboratory and numerical) cannot be used for evaluation of the success of such models. There are proposals to use two sets of independent exponents $\zeta_p^{\|}$ and $\zeta_p^{\perp}$ (Chen et al., 1997), and there exist other 'universality' proposals involving 'many more' scaling exponents (see e.g., Biferale and Procaccia, 2005; Frisch, 1995; Kurien and Sreenivasan, 2001b and references therein).

All the above is a clear indication that the question about the origins of intermittency (understood as anomalous scaling and/or in any other sense as above) in *real* turbulent flows remains open. Similarly open are the questions on universality of the intermittency manifestations (if such exist) in Navier–Stokes equations, though judging by the multitude of models of intermittency there is no universality whatsoever.

Phenomenology and models only will hardly be useful and convincing, since almost any dimensionally correct model, both right or wrong, will lead to correct scaling without appealing to NSE and/or elaborate physics. Scaling laws alone are not necessarily theories.[12] With all the importance of scaling, turbulence phenomena are infinitely richer than their manifestation in scaling and related matters. Most of these manifestations are beyond the reach of phenomenology. Phenomenology is inherently unable to handle the structure of turbulence in general, and phase and geometrical relations in particular, to say nothing of dynamical features such as build up of *odd* moments, interaction of vorticity and strain resulting in positive net enstrophy generation/predominant vortex stretching. It seems that there is little promise for progress in understanding the basic physics of turbulence in continuing to ask questions about scaling and related matters only (Badii and Talkner, 2001; Feigenbaum, 1997; Tsinober, 1996b), without looking into the structure and, where possible, basic mechanisms which are *specific* to turbulent flows. In fact, the main question of principle which should have been asked long ago is: Why on earth should we perform so many elaborate measurements of various scaling exponents without looking into the possible concomitant physics and/or without asking why and how more

[12] For example, the Kolmogorov theory and many subsequent models used dissipation as a basic quantity, i.e. intimately related to strain. Several later theories are based on hierarchies of vorticity-dominated structures. Most of both kinds of these theories agree with experimental results. However, while there is a basic reason, not only on dimensional grounds, for RSH $\Delta u_r^{\|} = \beta_1(r\epsilon_r)^{1/3}$, since it can be seen as a 'local' version of the 4/5 Kolmogorov law $\langle(\Delta u^{\|})^3\rangle = -4/5\langle\epsilon\rangle r$, a similar claim (Chen et al., 1997), that $\Delta u^{\perp} = \beta_2(r\Omega_r)^{1/3}$ ($\Omega = \nu\omega^2$) remains just one more dimensionally – but not necessarily physically – correct relation (note that $\langle(\Delta u^{\perp})^3\rangle \equiv 0$). Here, β_1, β_2 are stochastic variables independent of Re and r.

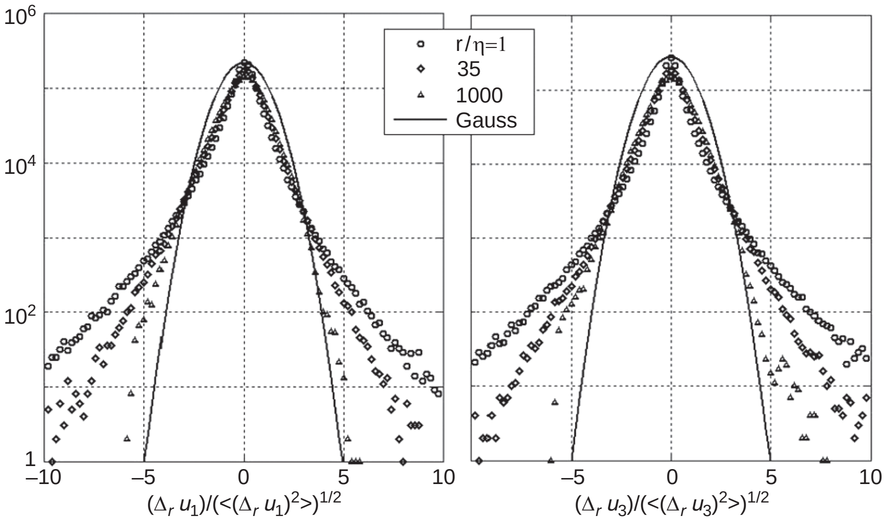

Figure 7.3. PDFs of the increments of longitudinal (a) and transverse (b) velocity fluctuations obtained in a field experiment at $\mathrm{Re}_\lambda = 10^4$ (Kholmyansky and Tsinober, 2000; Kholmyansky et al., 2001b). The increasing deviation from a Gaussian behaviour with decreasing r is manifested quite clearly. The curve for $\eta \approx 1$ is essentially a PDF of the longitudinal derivative. Note the asymmetry of the PDfs of the increments of longitudinal velocity fluctuations, which becomes largest at smallest separations

precise knowledge of such exponents, even assuming their existence, can aid our understanding of turbulent flows?

Second, as discussed in chapter 5 the very existence of scaling exponents in a statistical sense (as, e.g., for various structure functions or corresponding PDFs, etc.), which is taken for granted, is a problem by itself.

A similar question arises in respect with multi-fractality which was designed to ‘explain’ (but in fact it is another description of) the ‘anomalous’ scaling (7.1), since there is no direct experimental evidence on the multi-fractal structure of turbulent flows. So there is a possibility that multi-fractality in turbulence is an artifact (see Frisch, 1995, p. 190). Moreover, it is very likely that multi-fractality in fact is kind of finite Reynolds number effect at any large Re, see chapter 5.

The PDFs of an intermittent variable are quite useful, and they do not suffer from problems like scaling exponents. For example, using PDFs of velocity increments for different separations between two points it becomes clear that, the closer the two points, the more the PDFs deviate from a Gaussian distribution both in the center and at the tails. An example is shown in figure 7.3. However, PDFs also (both single-point and two-point PDFs) contain rather limited information. Namely, such PDFs carry the information showing that extremely small (the center anomaly) and extremely large values (tails of the PDF) are much more likely than for a Gaussian variable. However, they contain no information on the structure of the underlying weak and strong events, nor on the structure of the background field. Hence, the same PDFs can have qualitatively different underlying structure(s) of the flow, i.e., 'how the flow looks'. For example, the qualitative difference in the behaviour and properties of regions dominated by strain and those with large enstrophy cannot be captured by such means and other conventional measures of intermittency. Also the PDFs, like scaling exponents, do not allow us to infer much about the underlying dynamics. This, however, is true of 'conventional' PDFs like those of velocity increments, but not of any PDFs such as those directly associated with geometrical flow properties (see below section 7.3).

Note that the largest deviation from Gaussianity occurs at small scales (smallest distances between two points; see figure 7.3). In this sense, the field of velocity derivatives, $\partial u_i/\partial x_k$, is more intermittent than the field of velocity, u_i, itself. One of the possible reasons for this is in the different nature of nonlinearity at the level of velocity field, i.e., in the Navier–Stokes equations and, for example, in the equation for vorticity (C.9). Namely, the nonlinearity in the Navier–Stokes equations, $(\mathbf{u} \cdot \nabla)\mathbf{u} = \omega \times \mathbf{u} + \nabla(u^2/2) = \nabla(\alpha + u^2/2) + \nabla \times \beta$ contains a considerable irrotational part, $\nabla(\alpha + u^2/2)$ (see chapter 6, sections 6.5–6.7). This potential part can be included in the pressure term, i.e., the solenoidal part of nonlinearity is reduced. There is no such reduction of nonlinearity on the level of vorticity. Hence the difference, since the nonlinearity can be seen as one of the general reasons for intermittency of genuine turbulence.

On the 4/5 law – is it related to intermittency?

There are no 'corrections' to the scaling exponent in the 4/5 law – it is an exact consequence of NSE. However, as 1) it manifests the non-Gaussian nature of turbulence and 2) the PDFs of the longitudinal velocity increments especially at small r have flaring tails (hanging far above the Gaussian PDF) the 4/5 law should be considered as related to intermittency. This shows that 'intermittency corrections' are not that reliable as indicators of intermittency, if at all.

We turn now to the discussion of more specific possible reasons/origins of intermittency.

7.2.3. ON POSSIBLE ORIGINS OF SMALL-SCALE INTERMITTENCY

As one of the manifestations of turbulence structure(s), intermittency has its origins in the structure of turbulence (see next section). Therefore we briefly address here the issue on possible origins of intermittency. There are roughly two kinds of origins of intermittency: kinematic and dynamic.

Before proceeding we reiterate that non-Gaussianity and intermittency are not synonymous, just like the origins of non-Gaussian statistics in various systems and genuine turbulence are generally quite different even qualitatively. Therefore, it may be misleading to 'explain' such properties of genuine turbulence by analogy with non-Gaussian behaviour of, e.g., Burgers and/or restricted Euler equations. An important point is that these are integrable equations, and exhibit random behaviour only under random forcing and or initial conditions, otherwise their solutions are not random. These examples represent the response of nonlinear systems to random forcing and which otherwise are not random, and should be distinguished from problems involving genuine turbulence. Navier–Stokes equations at sufficiently large Reynolds number have the property of intrinsic stochasticity in the sense that they possess mechanisms of self-randomization.

Direct interaction/coupling between large and small scales

As discussed in chapter 6, direct interaction/coupling between large and small scales is one of the elements of the nonlocality of turbulence. It is both of kinematic and dynamic nature. The first recognized manifestation of such interaction is that the small scales do not forget the anisotropy of the large ones. There is a variety of mechanisms producing and influencing the large scales: various external constraints like boundaries with different boundary conditions, including the periodic ones, initial conditions, forcing (as in DNS), mean shear/strain, centrifugal forces (rotation), buoyancy, magnetic field, external intermittency in partially turbulent flows, etc. Most of these factors usually act as organizing elements, favouring the formation of coherent structures of different kinds (quasi-two-dimensional, helical, hairpins, etc.). These, as a rule, large- scale features depend on the particularities of a given flow that are not universal. Therefore the direct interaction between large and small scales leads to 'contamination' of small scales by the large ones, e.g., the edges of large-scale structures are believed to be responsible for such 'contamination'. This contamination seems to be unavoidable even in homogeneous and isotropic turbulence, since there are many ways to produce such a flow, i.e., many ways to produce the large scales. It is the difference in the mechanisms of large-scale production which 'contaminates' the small scales. Hence, non-universality.

The direct interaction/coupling of large and small scales seems to be a generic property of all turbulent flows and one of the main reasons for small-scale intermittency, non-universality, and quite modest manifestations of scaling. This dates back to the famous Landau remark stating that the *important part will be played by the manner of variation of* ε *over times* *of the order of the periods of large eddies (of size ℓ)*, Landau and Lifshits, 1944, see 1987, p. 140.

Near singularities
It is not known for sure whether Euler equations and/or Navier–Stokes equations at large Reynolds numbers develop a genuine singularity in finite time, though there is some evidence that, at least for Euler equations, this may be true. Whatever the real situation is, it seems a reasonable speculation that these 'near' singularities *trigger topological change and large dissipation events* (for Navier–Stokes equations)*; their presence is felt at the dissipation scales and is perhaps the source of small-scale intermittency* (Constantin, 1996). However, this does not help to understand the *inertial range* intermittency (if such exists[13]) without invoking the *reacting back* of the dissipation range on the inertial range. As mentioned, such reaction back is possible due to the direct coupling between the large and small scales and other nonlocal effects. The experimentally observed phenomenon of strong drag reduction and change of structure of turbulent flows of dilute polymer solutions and other drag reducing additives is an example of such a 'reacting back' effect (see section 6.6 for other examples).

Near singular objects associated with non-integer values of the energy spectrum scaling exponents[14] are thought to be closely related with some structure(s) and, consequently, with intermittency of turbulent flows (Vassilicos, 1996; see also Gibbon, 2009).

In any case, the 'near' singular objects *may* be among the origins of intermittency of a dynamical nature[15]. However, there is a problem with two-dimensional 'turbulence'. Namely, in this case everything is beautifully regular (Doering and Gibbon, 1995), but there is intermittency in the sense of the above definitions, with the exception of scaling exponents for velocity structure functions and corresponding quasi-Gaussian behaviour. However, non-Gaussianity is strong at the level of velocity derivatives of a second order (see chapter 8). Hence the possible formation of singularities in 3-D is not necessarily the underlying reason for intermittency in 3D turbulence. Another example relates to modified Navier–Stokes equations such as those using hyperviscosity replacing the Laplacian by a higher-order operator $(-1)^{h+1}\nabla^{2h}$ with $h > 1$ (see Borue and Orszag, 1998; Haugen and Brandenburg, 2004; Lamorgese et al., 2005 and references

[13] *It is plausible but not certain that there are intermittency corrections to the K41 theory of the inertial range* (Frisch, 1995).

[14] As mentioned the existence of such scaling exponents is an assumption and has some empirical foundation only as the data can be approximated in a different manner as well. In this sense there is a problem in a clean definition 'near' singularities.

[15] We mean singularities which appear at random in space and time and not in a strictly periodic (and fully coherent and mutually amplifying) fashion as in DNS with periodic boundary conditions.

therein) with the underlying assumption that this manipulation changes only the small scales. In this case also everything is beautifully regular for $h > 5/4$, (i.e., the solution remains regular for all times and any Reynolds number, Ladyzhenskaya, 1975; Lions, 1969) and *some* features of turbulence are reproduced well (such as the $k^{-5/3}$ spectrum) including intermittency, but its structure(s) appear quite different from those for true NSE.

Multiplicative noise, intermittency of passive objects in random media. It has been known for about thirty years that passive scalars exhibit 'anomalous scaling' behaviour (but see section 5.4.5) and other strong manifestations of intermittency (see figure 7.1) even in a pure Gaussian random velocity field (see Shraiman and Siggia, 1999, 2000; Sreenivasan and Antonia, 1997; Tsinober, 1998b; Warhaft, 2000; Zeldovich et al., 1988, 1990 and references therein). Similar behaviour is exhibited by passive vectors (Kraichnan and Kimura, 1994; Rogachevskii and Kleeorin, 1997; Vergassola, 1996). These are dynamically-linear systems, but they are of the kind which involve the so-called multiplicative 'noise', i.e., the coefficients in the equations that depend on the velocity field. Therefore, statistically they are 'nonlinear', since the field of passive objects is a nonlinear functional of the velocity field. Therefore, passive objects exhibit strong deviations from Gaussianity. In such systems, intermittency results either from external pumping (forcing term on RHS of the equations), or in systems without external forcing from instability (self-excitation) of a passive object in a random velocity field under certain conditions.

The velocity field does not 'know' about the passive objects. In this sense, problems involving passive objects are kinematic in respect with the velocity field in real fluid turbulence. They reflect the contribution of kinematic nature in real turbulent flows. It is noteworthy that some of the intermittency effects in such linear systems are stronger than in real fluid turbulence and exhibit anomalous scaling, which, generally, is non-universal (Falkovich et al., 2001; Shraiman and Siggia, 1999, 2000; Warhaft, 2000). In view of the recent progress in this field it was claimed that *investigation of the statistics of the passive-scalar field advected by random flow is interesting for the insight it offers into the origin of intermittency and anomalous scaling of turbulent fluctuations* (Pumir et al., 1997; see also Majda and Kramer, 1999; Shraiman and Siggia, 1999, 2000). More precisely it offers an insight into the origin of intermittency and anomalous scaling of fluctuations in random media generally and independently of the nature of the random motion (Zeldovich et al., 1988), i.e., it gives some insight into the contributions of kinematic nature, but does not offer much regarding the specific dynamical aspects of strong turbulence in fluids. Moreover, anomalous diffusion (including scaling) of passive objects occurs in purely laminar flows in the Eulerian sense (E-laminar flows) as a result of Lagrangian chaos (L-turbulent flows), i.e., intermittency of passive objects may have nothing to do with the random nature of fluid motion, see chapter 9.

Thus in real turbulent flows there are two contributions to the behaviour of passive objects, kinematic and dynamic. It seems hopeless to separate them in any sense. In a way, the problem of passive objects is more complicated than the dynamical one.

Summarizing, intermittency specifically in genuine fluid turbulence is associated mostly with *some* aspects of its spatio-temporal structure, especially the spatial one. Hence, the close relation between the origin(s) and meaning of intermittency and structure of turbulence. Just as there is no general agreement on the origin and meaning of the former, there is no consensus regarding what are the origin(s) and what turbulence structure(s) really mean. What is definite is that turbulent flows have lots of structure(s). The term structure(s) is used here deliberately in order to emphasize the duality (or even multiplicity) of the meaning of the underlying problem. The first is about how turbulence 'looks'. The second implies the existence of some entities. Objective treatment of both requires use of some statistical methods. It is thought that these methods alone may be insufficient to cope with the problem, but so far no satisfactory solution was found. One (but not the only) reason – as mentioned – is that it is not so clear what one is looking for: the objects seem to be still elusive. For example, some still are not sure that the concept of coherent structure is much different from the dress of the naked king.

7.3. What is (are) structure(s) of turbulent flows?

> ... *worms do not seem to play a special role in the overall dynamics of turbulent flows* (Jimenez et al., 1993).
>
> *Numerical experiment shows that turbulent flow is dominated by vortex tubes of small cross-section and bounded eccentricity* (Chorin, 1994).
>
> *Vortex filaments: the sinews of turbulence?* (Frisch, 1995).
>
> ... *it is now clear from direct numerical simulations that three-dimensional isotropic turbulence is composed of an ensemble of thin tubes of high vorticity* ... (Lesieur, 1997).
>
> ... *in turbulence one does not know yet what structures are key to our understanding the statistical properties of turbulent flows ... kinking or intertwining of tubes; the latter is known to provide building blocks of turbulent transfer of energy to small scales in 3D and as such the source of the multi-scale problem* (Pouquet et al., 2003).
>
> *What we see is real. The problem is interpretation.*

The difficulties of defining what the structure(s) of turbulence are (mean) are of the same nature as the question about what is turbulence itself. So first, and in order to 'see' or 'measure' the structure(s) of turbulence, one encounters the most difficult questions such as: what is a (say, dynamically relevant) structure?, Structure of what? Which quantities possess structure in turbulence? What is the relation between structure(s) and 'scales'? Can structure exist in 'structureless' (artificial) pure random Gaussian fields ? Which ones? All these – like many other issues – are intimately related to the skill/art of asking the right and correctly posed questions.

The meaning of structure(s) depends largely on what is meant by turbulence itself, and especially structure(s) of the particular field one is looking. For example, as discussed in chapter 4, the velocity field may have no structure, but the passive tracer may; simple laminar Eulerian velocity field (E-laminar) may create a complicated Lagrangian field (L-turbulent). A purely Gaussian, i.e., 'structureless' velocity field, creates structure in the field of passive objects. The structure(s) seen in the velocity field depend on the motion of the observer (see figure 7.4).

Finally, an example shown in figure 7.5 is what is called "coherent structures" or "organized motion", which has been rediscovered many times. This flow (mixing layer) represents the case in which the structures may be not directly related to the turbulent nature of the flow but are rather a result of large-scale instability of the flow as a 'whole' (zooming out).

7.3.1. ON THE ORIGINS OF STRUCTURE(S) OF/IN TURBULENCE

This question – in some sense – is a 'philosophical' one. But its importance is in direct relation to even more important questions about the origin of turbulence itself.

Instability

As mentioned in chapter 2, the most commonly accepted view on the origin of turbulence is flow instability. An additional factor is that instability is considered as one of the origins of structure(s) in/of turbulence. However, this latter view requires the assumption that turbulence has a pretty long 'memory' of or, alternatively, that the 'purely' turbulent flow regime (i.e., at large enough Reynolds numbers) has instability mechanisms similar to those existing in the process of transition from laminar to turbulent flow state. Tritton (1988) defines turbulence as *a state of continuous instability*. The problem is that speaking about (in)stability requires one to define the state of flow (in)stability of which is being considered, which is not a simple matter in the case of a turbulent flow.

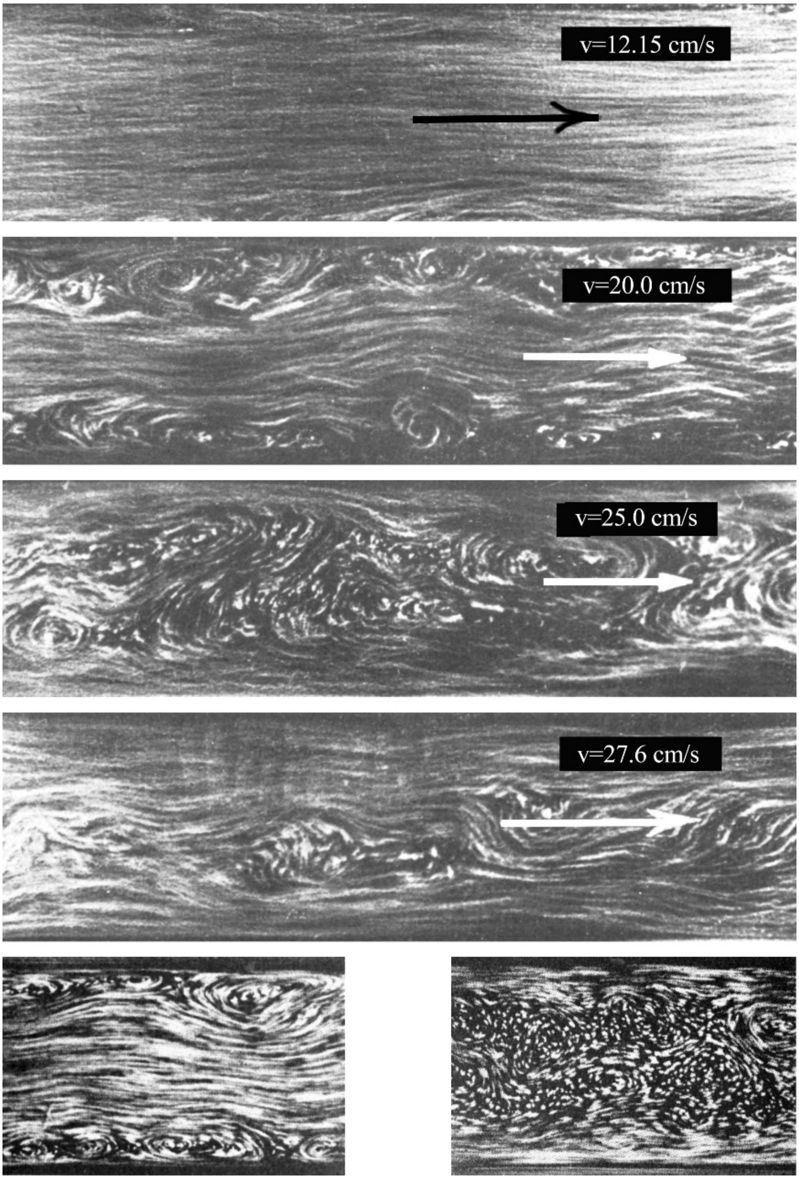

Figure 7.4. The four upper pictures, Tollmien (1931), correspond to the visualization of a turbulent water flow in an open 6 cm wide channel photographed by a moving camera at different speeds. The mean velocity of the flow is 16.7 cm/s. The two lower pictures are from Prandtl and Tietjens (1934). In the right-hand picture, the camera moves with the speed equal to the velocity of water in the center of the channel. In the left-hand picture, the speed of the camera is small and close to the velocity of the water near the walls

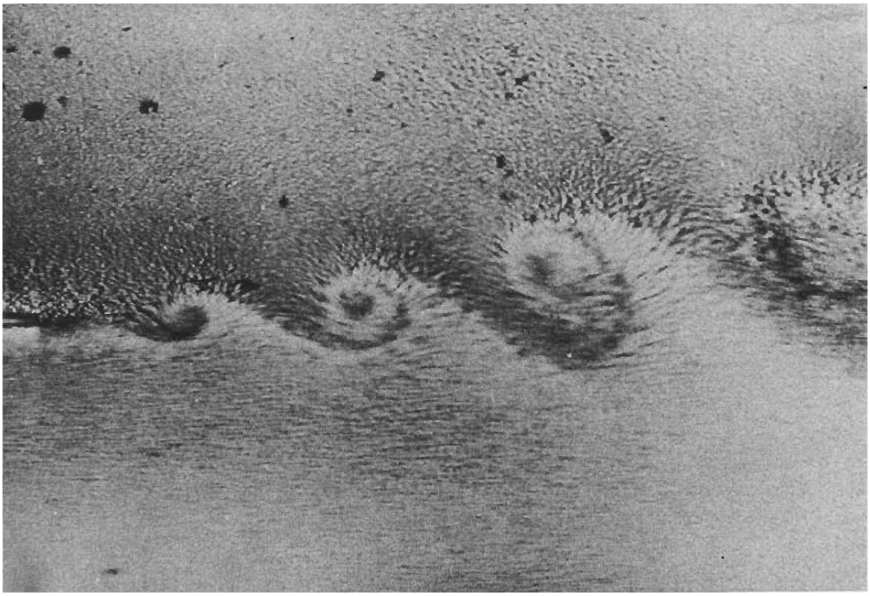

Figure 7.5. Coherent structures in a mixing layer flow (Michel, 1932)

Emergence
Another less known view holds that structure(s) emerge in large Reynolds number turbulence out of 'purely random structureless' background, e.g., via the so-called inverse cascades or negative eddy viscosity (see chapter 8). Among the spectacular examples, are the 'geophysical vortices' in the atmosphere, and ocean, as well as astrophysical objects. Another example is the emergence of coherent entities, such as vortex filaments/worms and other structure(s), out of an initially random Gaussian velocity field via the NSE dynamics[16]. An example of such structure(s) is shown in figure 1.16, for other examples see references in Tsinober (1998a,b).

It 'just exists', or do flows become or are they 'just' turbulent?
To the flows observed in the long run after the influence of the initial conditions has died down there correspond certain solutions of the Navier–Stokes equations. These solutions constitute a certain manifold $\mathcal{M} = \mathcal{M}(\mu)$ *(or* $\mathcal{M} = \mathcal{M}(Re)$*) in phase space invariant under phase flow* (Hopf, 1948).

[16]Recall P.W. Anderson (1971, 1995) who emphasizes *the concept of 'broken symmetry', the ability of a large collection of simple objects to abandon its own symmetry as well as the symmetries of the forces governing it and to exhibit the 'emergent property' of a new symmetry*. One of the difficulties in turbulence research is that no objects simple enough have been found so far such that a collection of these objects would *adequately* represent turbulent flows.

Kolmogorov's scenario was based on the complexity of the dynamics along the attractor rather than its stability (Arnold, 1991; see also Keefe, 1990a; Keefe et al., 1992).

This view is a reflection of one of the modern beliefs that the structure(s) of turbulence – as we observe in *physical space* – is (are) the manifestation of the generic structural properties of mathematical objects in *phase space*, which are called (strange) attractors and which are invariant in some sense. In other words here the structure(s) assumed to be 'built in' the turbulence independently of its origin (hence the tendency to universality)[17]. It is noteworthy that the assumed strange-attractor existence makes sense for statistically stationary turbulent flows. However, for flows which are not such, e.g., decaying turbulent flows past a grid or a DNS simulated flow in a box the attractor is trivial. Nevertheless, these flows possess many properties which are essentially the same as their statistically stationary counterparts provided that their Reynolds numbers are not too small ($\mathrm{Re}_\lambda \sim 10^2$).

The above refers to the dynamical aspects of real turbulent flows. We mention again here also the

Emergence of structures in passive objects in random media
in which the velocity field and the external forcing are prescribed. Whatever their nature – even Gaussian – structure is emerging in the field of passive objects (Zeldovich et al., 1988; Ott, 1999 and references therein). In other words structure(s) of passive objects emerges also in structureless (artificial) random Gaussian velocity fields.

7.3.2. HOW DOES THE STRUCTURE OF TURBULENCE 'LOOK'?

Until recently, very little was known about the nature of structure of turbulence and about the appearance of its structures (in physical space). The structure in question is the so-called fine structure and not the one which is promoted by various external factors and/or constraints like boundaries, mean shear, centrifugal forces (rotation), buoyancy, magnetic field, etc., which usually act as organizing factors, favouring the formation of coherent structures of different kinds (quasi-two-dimensional, helical, hairpins, etc.). These structures, are, as a rule, large-scale features which depend on the particularities of a given flow and thus are not universal. We will return to some of these mostly large-scale structures including what is called 'coherent structures' or 'organized motion' in chapter 8.

Since the first DNS simulations by Siggia (1981), a number of computations have been performed (see references in Tsinober, 1998a,b), which

[17] In the strange attractor theory, the experimental measurements are viewed as projections of these attractors onto low dimension that correspond to these measurements.

demonstrated clearly that even turbulence which is 'homogeneous' and 'isotropic' has structure(s), i.e., contains a variety of strongly localized events. The primary evidence is related to spatial localization of subregions with large enstrophy (i.e., intense vorticity) which are organized in long, thin tubes-filaments-worms. Such filaments were also directly observed in laboratory experiments (the ones mentioned in figure 5.1) employing the property of intense vorticity to be strongly correlated with regions of low pressure and using small air bubbles to visualize these regions (Douady et al., 1991; Villermaux et al., 1995; for more references see Tsinober, 1998a). This follows from the Poisson-like equation for pressure $2\nabla^2 p/\rho = \omega^2 - 2s_{ij}s_{ij}$. There is some evidence that in regions with moderate magnitude of vorticity it is organized in sheet-like structures. Much less is known about regions with large strain, $s_{ij}s_{ij}$, i.e., dissipation. They were tentatively identified as layered vortex sheets in Schwarz (1990), an observation that has not been confirmed by other observations or computations so far. Most common observations at Reynolds numbers accessible in DNS showed that isosurfaces of high strain are wrapped around the regions of strong enstrophy. However, in Tanaka and Kida (1993) and in recent computations by Boratav and Pelz (1997), the isosurfaces of large strain were observed as sheet-like objects with very sharp edges (razors/flakes). In fact such objects were observed already by Siggia (1981, figure 21). This does not mean that the vorticity field in these regions is simple and is necessarily sheet-like too, see Ishihara et al., 2009.

Some examples of the results mentioned are shown in figure 7.6. The relatively simple appearance of the observed structures as shown above prompted a rather popular view that turbulence structure(s) is (are) simple in some sense and that essential aspects of turbulence structure and its dynamics may be adequately represented by a random distribution of simple (weakly interacting) objects, such as straight strained (Burgers-like) vortices (see chapter 6 and references in Tsinober, 1998a). In particular, it is commonly believed that most of the structure of turbulence is associated with and is due to various strongly localized intense events/structures, e.g., mostly regions of concentrated vorticity so that '*turbulent flow is dominated by vortex tubes of small cross-section and bounded eccentricity*' (Chorin, 1994, p. 95) and that these events are mainly responsible for the phenomenon of intermittency (Belin et al., 1996; Frisch, 1995; Katul et al., 1994; Nelkin, 1995; Jimenez and Wray, 1998 and references therein). It is argued in Tsinober (1998a) that such views are inadequate (see chapter 6 for more details and latest references). It appears that – though important – these structures are not the most dynamically-important ones and are the consequence of the dynamics of turbulence rather than its dominating factor. Namely, regions *other* than those involving concentrated

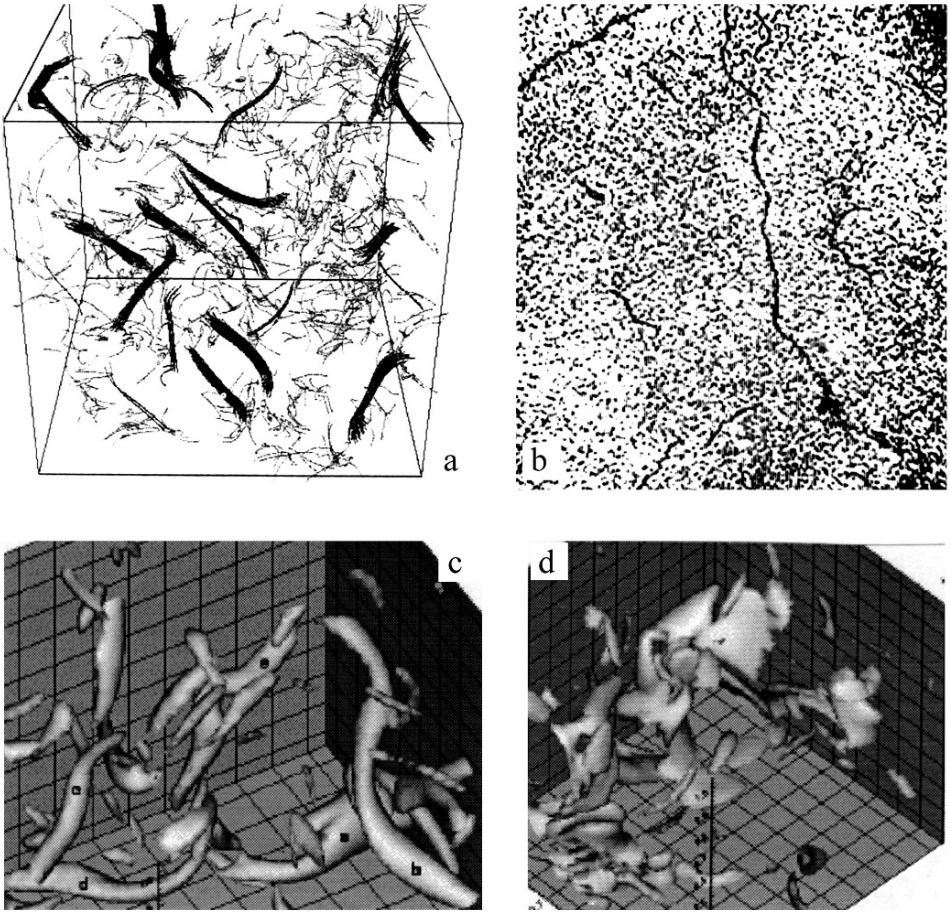

Figure 7.6. Vortex filaments in DNS (She et al., 1991; top left) and laboratory (Douady et al., 1991; top right). Isosurfaces of the second invariant of the velocity derivatives tensor $Q = \omega^2 - s_{ij}s_{ij}$ (bottom left) at 2 rms positive level, i.e., vorticity-dominated regions, and isosurfaces of strain $s_{ij}s_{ij}$ (bottom right) at 2 rms level, i.e., strain-dominated regions (Boratav and Pelz, 1997). The two bottom pictures were not included in Boratav and Pelz (1997), but are available at http://www.eng.uci.edu/˜boratav/ and are used here by permission. The figure is from Tsinober (1998a)

vorticity such as: *i* – 'structureless' background, *ii* – regions of strong vorticity/strain (self) interaction and largest enstrophy and strain production dominated by large strain rather than large enstrophy, and *iii* – regions with negative enstrophy production are all dynamically significant (in some important respects more significant than those with concentrated vorticity), strongly non-Gaussian, and possess structure. Due to the strong nonlocality of turbulence in physical space all the regions are in continuous interaction and are strongly coupled. A similar statement can be made regarding the so-called streamwise vortices observed in many turbulent flows (see chapter 8).

The above conclusions are the outcome of the use of quantitative manifestations of turbulence structure, which just like intermittency are in the first place of statistical nature.

7.3.3. STRUCTURE VERSUS STATISTICS

> *The statistical community ... strongly disputes the possibility of any coherence or order associated to turbulence.* (Lesieur, 1997).
>
> *The transition from laminar to turbulent flow is a nonequilibrium phase transition to a more organized motion.* (Klimontovich, 1996).
>
> *... to say that turbulent flow is completely random would define turbulence out of existence.* (Tritton, 1988).

On a qualitative level, it is widely recognized that fluid-dynamical turbulence (even 'homogeneous' and 'isotropic') has 'structure(s)', i.e. contains a variety of strongly localized events, which are believed to influence significantly the properties of turbulent flows. It is impossible to overestimate the observational information on the instantaneous structures of turbulent flows. Being extremely useful, the individual observations of such events/structures are inherently limited as compared to the statistical information, which requires us to employ the quantitative manifestations of turbulence structure. In order to proceed to the quantitative aspects of the problem it is not sufficient to look at pictures (however beautiful); one has to turn to numbers and quantitative relations such as in the abovementioned anomalous scaling, which is one of many other more specific quantitative manifestations of turbulence structure.

The question about what structure(s) of turbulence mean(s) can be answered via a statement of impotence: speaking about 'structure(s)' in turbulence the implication is that there exists something 'structureless', e.g., Gaussian random field as a representative of full/complete disorder. A gaussian field is appropriate/natural to represent the absence of structure in the statistical sense. Hence all non-Gaussian manifestations of turbulent flows can be seen as some statistical signature of turbulence structure(s)[18]. However, simple probability criteria are insufficient, since *one can find in statistical data irrelevant structures with high probability* (Lumley, 1981). In other words the structure(s) should be relevant/significant in some sense.

[18]This does not imply that an exactly Gaussian field does not necessarily possess any spatial or temporal structures, see, e.g., figure 3 in She et al. (1990) – any individual realization of a Gaussian field does have structures. However, an exactly Gaussian field does not possess *dynamically* relevant structure(s), it is dynamically impotent, see below.

For example, it should be *dynamically* relevant for a velocity field, and related quantities such as vorticity and strain. This does not mean that kinematical aspects of turbulence structure(s) are of no importance. For example, *anisotropy* is a typical *kinematic statistical* characteristic of turbulent flows which hardly can be applied to *individual* structures, e.g., a turbulent flow consisting mostly of ‘anisotropic’ individual structures can be statistically-isotropic. Among the first statistical treatments of turbulence structure is, of course, the first paper by Kolmogorov (1941a), the very title of which is *The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers.*

The advantage of such an approach is that it allows one to get insights into the *structure* of turbulence without the necessity of knowing much (if anything) about the actual appearance of its *structures*. This is especially important in view of numerous problems/ambiguities in definitions of *individual* structures in turbulent flows, their identification and statistical characterization as well as their incorporation in ‘theories’. The main reason is that there exist an intrinsic problem of both defining what the *relevant* structures are (see Bonnet, 1996 for references and a review of existing techniques) which all are based on statistics anyhow, and of defining extracting/educing and characterizing the so-called coherent structures. For a number of reasons, it is very difficult, if not impossible, to quantify the information on the *instantaneous* structures of turbulent flows into dynamically relevant/significant form. The observed individual structures are not simple, neither are they weakly interacting between themselves or with the background. Indeed, *you can find structures, essentially arbitrary, which have equal probability to the ones we have latched onto over the years: bursts, streaks, etc … If structures are defined as those objects which can be extracted by conditional sampling criteria, then they are everywhere one looks in turbulence* (Keefe, 1990b). For instance, looking at a snapshot of the enstrophy levels of a *purely Gaussian* velocity field in She et al. (1990), one can see a number of filaments (the irrelevant ones) like those observed in real turbulent flows, i.e., a pure Gaussian velocity field has some structure(s) too.

The next most difficult question is about the relevance/significance of some particular aspect of non-Gaussianity for a specific problem in question. It seems that here one enters the *subjective* realm: the criteria of significance (which is the matter of physics!) are decided by the researchers. However, the following examples show that objective choice of the structure sensitive statistics is dictated by general dynamical aspects of the problem. In the following, we will discuss the dynamical aspects of the problem. Various ‘kinematic’ issues, like the transport of passive objects (scalars, vectors, etc.), in which Gaussian or other *prescribed* velocity fields are used rather successfully, are beyond the scope of this section. We mention only that structure(s) of the field of passive objects can be treated in a similar way as the one described in this section.

For instance, the build up of *odd* moments is an important *specific* manifestation of structure of turbulence along with being the manifestation of its nonlinearity. The two most important examples are the third-order velocity structure function $S_3(r) = \langle\{[\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{r}/r\}^3\rangle$ and the mean enstrophy production $\langle\omega_i\omega_k s_{ik}\rangle$. The first one is associated with the $-4/5$ Kolmogorov law $S_3(r) = -4/5\langle\epsilon\rangle r$ (Kolmogorov, 1941b), which is the first strong indication of the presence of structure in the inertial range showing that both non-Gaussianity and the structure of turbulence are directly related to its dissipative nature. It is remarkable that the title of this paper by Kolmogorov is *Dissipation of energy in the locally-isotropic turbulence.* The $-4/5$ Kolmogorov law clearly overrules the claims that '*Kolmogorov's work on the fine-scale properties ignores any structure which may be present in the flow*' (Frisch, 1995, p. 182) and that it is associated with near-Gaussian statistics, (Chertkov et al., 1999; Farge and Guyon, 1999; Katul et al., 1994; She et al., 1991 and many others). It is noteworthy that – as shown by Hill (1997) – the 4/5 Kolmogorov law is more sensitive to the anisotropy, i.e., the third-order statistics (again odd moments), than the second-order statistics. Likewise the structure functions of higher odd orders $S_p^{||}(r) = \langle(\Delta u_{||})^p\rangle$ are essentially different from zero, see references in Betchov (1976), Sreenivasan and Antonia (1997), Tsinober (1998b).

The essentially positive value of the mean enstrophy generation $\langle\omega_i\omega_k s_{ik}\rangle$ discovered by Taylor (1938) is the first indication of the presence of structure in the small scales, where turbulence is particularly strongly non-Gaussian and intermittent (Kraichnan, 1967; Novikov, 1967; Sreenivasan and Antonia, 1997). The above two examples show that both the essential turbulence dynamics and its structure are associated with those aspects of its non-Gaussianity exhibited in the build up of odd moments, which among other things means phase and geometrical coherency, i.e., structure (see section on the non-Gaussian aspects of turbulence in chapter 6). Hence, the importance of odd moments as indicators of intermittency. It is to be noted that the non-Gaussianity found experimentally both in large and small scales is exhibited not only in the nonzero odd moments, but also in strong deviations of even moments from their Gaussian values. Thus both the large and small scales differ essentially from Gaussian indicating that both possess structure.

We return to the question about what kinds of statistics are most appropriate to characterize at least some aspects for turbulence structure. But first we must mention some

7.3.4. EXAMPLES OF STATISTICS WEAKLY SENSITIVE TO STRUCTURE(S)

The first examples of this kind are energy spectra in which the phase (and geometric) information is lost. Hence their weak sensitivity to the structure of turbulence. This insensitivity, in particular, is exhibited in the scaling exponents when/if such exist. For example, the famous $-5/3$ exponent can be obtained for a great variety of *qualitatively* different real systems – not necessarily fluid dynamical – and theoretical models. A partial list of references contains the papers by Biferale et al. (1994), Cheklov and Yakhot (1995), Chorin (1994, 1996), Kiya and Ishii (1991), Lundgren (1982), Moffatt (1993), Nore et al. (1997), Pullin and Saffman (1997), Taguchi (1995), Tsinober (1998b), Vassilicos and Brasseur (1996), Zakharov et al. (1993). Of course, one can also construct a set of purely-Gaussian velocity fields, i.e., lacking any dynamically-relevant structure(s), with any desired length of the $-5/3$ 'inertial' range (Elliott and Majda, 1995). An extreme example is a single sharp change in velocity. Represented in Fourier space it has an energy spectrum $\propto k^{-6/3}$ which is not so easy to distinguish from $k^{-5/3}$! Vice versa the spectral slope can change, but the structure remains essentially the same '*yet retaining all the phase information*' (Armi and Flament, 1987; see figure 7.7). Moreover, not only '*the spectral slope alone is inadequate to differentiate between theories*' (Armi and Flament, 1987), *alone* it does not correspond to any particular structure(s) in turbulence or it's absence : there is no one-to-one relation between scaling exponents and structure(s) of turbulence. This is true not only of exponents related to Fourier decomposition with its ambiguity (Tennekes, 1976), but of many other scaling exponents including those obtained in some wavelet space, SO(3) decomposition and in the physical space – a much overstressed aspect of turbulent flows. Likewise, similar PDFs of *some* quantities can correspond to qualitatively different structure(s) and quantitatively different values of Reynolds number (Kraichnan and Kimura, 1994; Tsinober, 1998a,b). The emphasis is on *some* quantities like pressure or some other usually (but not necessarily) even-order quantities in velocities or their derivatives, since the PDFs of *other* appropriately chosen quantities are sensitive to structure (see below).

7.3.5. STRUCTURE SENSITIVE STATISTICS

Use of odd-order structure functions

This is an example of how structure sensitive statistics can help in looking for the right reasons for measured spectra in the lower meso-scale range, Lindborg (1999). The procedure involves using the third-order structure functions which are generally positive in the two-dimensional case (contrary to the three-dimensional case). Calculations are based on wind data from

Figure 7.7. An example of spectral analysis of an infrared image of the ocean sea-surface temperature off the northern coast of California. The left figure corresponds to the spectral density of slope k^{-4}, the middle k^{-3}, and the right k^{-2}. An important point is that all the three correspond to the *same* original image and retain the phase information, *but place less emphasis on the observed spectrum*. Courtesy of Professor L. Armi; for more details see Armi and Flament (1987)

airplane flights, reported in the MOZAIC data set. It is argued that the k^{-3}-range is due to two-dimensional turbulence and can be interpreted as an enstrophy inertial range, while the $k^{-5/3}$-range is probably not due to two-dimensional turbulence and should not be interpreted as a two-dimensional energy inertial range. There is a competing hypothesis that the large-scale $-5/3$ range is the spectrum of weakly nonlinear internal gravity waves with a forward energy cascade (Van Zandt, 1982). A third claim is that the spectral slope in the enstrophy range is more shallow than -3 and is close to $-7/3$ (Tsinober, 1995a). This range and related anomalous diffusion is explained in terms of the phenomenon of spontaneous breaking of statistical isotropy (rotational and/or reflectional) symmetry – locally and/or globally.

Another example is the demonstration (mentioned in chapters 5 and 6) that the small-scale structure of a homogeneous turbulent shear flow is essentially *anisotropic* at Reynolds number up to $\mathrm{Re}_\lambda \approx 1000$ (Shen and Warhaft, 2000; see also Ferchichi and Tavoularis, 2000). In order to detect this anisotropy the authors measured the velocity structure functions of third and higher odd orders of both longitudinal and transverse velocity components and corresponding moments of velocity derivatives. In particular, they found a skewness of order 1 of the derivative of the longitudinal velocity in the direction of the mean gradient, which should be very small (or ideally vanish) for a locally-isotropic flow. Similar results were obtained in DNS (see references in Biferale and Procaccia, 2004; Kurien and Sreenivasan, 2001b; Shen and Warhaft, 2000, 2002; Warhaft, 2000 and Biferale and Procaccia, 2005; also Borue and Orszag, 1996 and Shebalin and Woodruff, 1997). We should recall that analogous 'misbehaviour' of large-Reynolds-number turbulence regarding the skewness of temperature fluctuations in the atmospheric boundary layer has been known since the late 1960s (Stewart, 1969; Gibson et al., 1970, 1977).

Geometrical statistics

This example shows how conditional sampling based on *geometrical* statistics can help to get insight into the nature of various regions of turbulent flow, e.g., those associated with strong/weak vorticity, strain, various alignments, and other aspects as described in the previous chapter. The first general aspect is the qualitative difference in the behaviour and properties of regions with large enstrophy from strain-dominated regions, which is also one of the manifestations of intermittency. Another example is the PDF of the cosine of the angle between vorticity, ω, and the vortex stretching vector, $W_i \equiv \omega_k s_{ik}$, $\cos(\omega, \mathbf{W})$. It is strictly symmetric for a Gaussian velocity field, whereas it is strongly positively skewed in real turbulent flows. It remains essentially positively skewed for any part of the turbulent field (see figure 6.10), e.g., in the 'weak background' (involving whatever definition based on enstrophy, strain, both and/or any other relevant quantity). Thus, contrary to common beliefs, the so-called 'background' is not structureless, dynamically not inactive and essentially non-Gaussian, just like the whole flow field or any part of it. The structure of the apparently random 'background' seems to be rather complicated. The previous qualitative observations (mostly from DNS) about the '*little apparent structure in the low intensity component*' or the '*bulk of the volume*' with '*no particular visible structure*' should be interpreted as meaning that no *simple visible* structure has been observed so far in the bulk of the volume in the flow. It is a reflection of our inability to 'see' more intricate aspects of turbulence structure: intricacy and 'randomness' are not synonyms for absence of structure.

Pressure Hessian

Recently, special attention has focused on the pressure Hessian $\frac{\partial^2 p}{\partial x_i \partial x_j}$. Among the general reasons for such an interest is that the pressure Hessian is intimately related to the nonlocality of turbulence in physical space (see chapter 6 and references in Tsinober, 1998a,b).

One of the quantities in the present context directly associated with the pressure Hessian is the scalar-invariant quantity $\omega_i \omega_k \frac{\partial^2 p}{\partial x_i \partial x_k}$. It is responsible for the nonlocal effects in the rate of change of enstrophy generation $\omega_i \omega_k s_{ik}$ (see equation [C.23] in appendix C). What is special about this quantity, which is of even order in velocity, is that for a Gaussian velocity field $\left\langle \omega_i \omega_k \frac{\partial^2 p}{\partial x_i \partial x_k} \right\rangle_G \equiv 0$, whereas in a real flow it is essentially positive and $\left\langle \omega_i \omega_k \frac{\partial^2 p}{\partial x_i \partial x_k} \right\rangle \sim \frac{1}{3} \left\langle W^2 \right\rangle$, where $W_i \equiv \omega_k s_{ik}$ is the vortex stretching vector.

Thus interaction between the pressure Hessian and the vorticity is one of the essential features of turbulence structure associated with its nonlocality. It is noteworthy that a similar quantity involving strain is non-vanishing for a Gaussian velocity field, $\langle s_{ik}s_{kj}\frac{\partial^2 p}{\partial x_i \partial x_j}\rangle_G = -\frac{1}{20}\langle\omega^2\rangle_G^2$, see figures 6.19 and 6.20.

7.4. Which quantities possess structure in turbulence and how one 'digs' them out?

We have seen that different quantities possess different structure(s) in the same flow the velocity field may have no structure, but the passive tracer may well have a pretty nontrivial one. A simple laminar Eulerian velocity field (E-laminar) creates complicated Lagrangian field (L-turbulent). A purely Gaussian, i.e., 'structureless' velocity field creates a structure in the field of passive objects. The structure(s) seen in the velocity field depend on the motion of the observer (see figure 7.4).

The most commonly used methods in looking at structure(s) are based on the so-called conditional sampling techniques, which employ some criteria to educe some structure(s). Among the simplest criteria is sampling based on equilevels of some function, e.g., enstrophy ω^2 (this is how the first evidence of concentrated vorticity/filaments/worms was obtained). More generally, this problem is related to pattern recognition and requires defining a conditional sampling scheme. This scheme is in turn based on what a particular investigator thinks are the most important physical processes, features, etc. This in turn opens a Pandora's box of possibilities and contains an inherent element of subjectivity and arbitrariness, since the physics of turbulence is not well understood. In this sense, the circle is closed: in order to objectively define and educe some structure, one needs clear understanding of the physics of turbulence, which, it is in turn believed, can be achieved via study of turbulence structure(s).

The most popular method is to look for structure(s) using a criterion based on one parameter only, e.g., enstrophy ω^2. Though such an approach is useful and 'easy', it is inherently limited and reflects the simplest aspects of the problem. For example, even for characterization of *some* aspects of the *local* (i.e., in a sense 'point'-wise) structure of the flow field in the frame following a fluid particle requires at least *two* parameters.[19] Therefore attempts to *adequately* characterize finite-scale structure(s) by one parameter only are unlikely to be successful, and one needs something

[19] $Q = 1/4(\omega^2 - 2s_{ij}s_{ij})$ and $R = -1/3(s_{ij}s_{jk}s_{ki} + 3/4\omega_i\omega_j s_{ij})$. Here Q is the second and R is the third invariant of the velocity gradient tensor $\partial u_i/\partial x_k$. The first is vanishing due to incompressibility (see Chacin and Cantwell, 2000; Ooi et al., 1998 and references therein). We return to this issue in chapters 8 and 9 in different contexts.

like pattern recognition based on some conditional sampling scheme involving more than two parameters. A similar problem arises when attempting to characterize structure(s) of turbulent flows using *two*-point information but based on a *single* velocity component only, e.g., longitudinal structure functions $S_n^{||}(\mathbf{r})$, since such an approach does not 'know' (almost) anything about the two other velocity components.

The 'not objective enough' nature of a variety of conditional sampling procedures resulted in a whole 'zoo' of 'structures' in different turbulent flows, which some people believe to be significant in some sense, but many do not. Among the reasons for such scepticism is some evidence that the attempts at representation of such a complicated phenomenon like turbulence as a collection of simple objects/structures only are unlikely to succeed (see chapters 5 and 6; and Tsinober, 1998a,b). As mentioned, until recently it was believed that concentrated vorticity/filaments is the dominating structure in turbulent flows in the sense that most of the structure of turbulence is associated with and is due to regions of concentrated vorticity. It appears that – though important – these structures are not the most dynamically important ones and are the consequence of the dynamics of turbulence rather than being its dominating factor.

Nevertheless, as shown above some 'objectiveness' can be achieved using quantities appearing in the NSE and/or the equations which are exact consequences of NSE.

More information and references about various attempts to define, educe and characterize various 'coherent structures' can be found in Bonnet (1996) and Holmes et al. (1996, 1997).

7.4.1. STRUCTURE(S) VERSUS SCALES AND DECOMPOSITIONS

It is natural to ask how meaningful it is to speak about different scales in the context of 'structure(s)' and in what sense, especially when looking at the 'instantaneous' structure(s) of/in turbulence. The known structures indeed possess quite different scales. Vortex filaments/worms have at least two essentially different scales; their length can be of the order of the integral scale, whereas their cross-section is of the order of the Kolmogorov scale. Similarly, the ramp-cliff fronts in the passive-scalar fields have a thickness much smaller than the two other scales. This fact is consistent with the observation by Batchelor and Townsend (1949), that *the mean separation of the visible activated regions is comparable with the integral scale of the turbulence, i.e., with the size of the energy-containing eddies.*

It is believed that appropriately chosen decompositions may represent structure(s) of turbulence, e.g., Holmes et al. (1996, 1997). Here again several notes are in order. First, this position depends strongly on what is meant by structure(s). Second, such a possibility is realistic when the flow is dominated by (usually large-scale) structures, when many, or practically any reasonable decompositions will do anyhow. And third, structure(s) (and related issues such as geometry) emerging in the 'simplest' case of turbulent flows, in a box with periodic boundary conditions, is(are) are inaccessible via Fourier decomposition, the most natural one in this case.

One of the popular 'decompositions' is into 'coherent structures' and random/dissipative 'background'[20]. This latter is generally considered as structureless and as a kind of passive sink of energy. As we have seen this is not true: the background is not passive at all, it is strongly coupled with the 'coherent structures', and possess lots of its 'own' structure(s). There are many problems of conceptual and technical nature with what is called 'coherent structures' starting from the very beginning of their definition and ending with their role in fluid flows both in Euler and Lagrange settings, see chapter 9. It is for this reason that *At this stage, this alternative approach* (i.e. the 'structural') *has not led to a generally applicable quantitative model, neither – for better or worse – has it a major impact on the statistical approaches. Consequently the deterministic viewpoint is neither emphasized nor systematically presented,* Pope (2000). This does not mean that there exists a "*generally* applicable quantitative model" based on statistical approaches. It appears that (so far) Liepmann was correct (but a bit over-optimistic) in his prediction: *Clearly, the exploration of the concept of coherent structure is still on the rise. Turbulence is and will remain the most difficult problem of fluid mechanics, and the past experience suggests that the subsequent fall of interest in the coherent structures is more than likely. The resulting net gain in understanding of turbulence may be less than our expectations of today but will certainly be positive*, Liepmann, 1979. Unfortunately, (so far) the *resulting net gain in understanding of turbulence* is far less than was expected in 1979 and on.

7.5. Summary

Small-scale intermittency of turbulence is associated with its spotty (spatio-temporal) *structure,* which among other things is manifested as a *particular* kind of non-Gaussian behaviour of turbulent flows (however,

[20]An example of a typical statement is represented by the following: *The emergence of collective modes in the form of coherent structures in turbulence amidst the randomness is an intriguing feature, somewhat reminiscent of the mix between the regular "islands" and the "chaotic sea" observed in chaotic, low-dimensional dynamical systems. The coherent structures themselves approximately form a deterministic, low-dimensional dynamical system. However, it seems impossible to eliminate all but finite number degrees of freedom in a turbulent flow–the modes not included form an essential, dissipative background, often referred as an eddy viscosity, that must be included in the description,* Newton and Aref (2003).

non-Gaussianity does not necessarily imply intermittency). This deviation from Gaussianity increases with both the *i)* Reynolds number and *ii)* as the 'scale' decreases. In other words intermittency involves two aspects of turbulent flows – their structure/geometry and statistics. Intermittency specifically in fluid turbulence is associated mostly with *some* aspects of its spatio-temporal structure. Hence, the close relation between the origin(s) and meaning of intermittency and the structure of turbulence.

There is no turbulence without structure(s). Every part (just as the whole) of the turbulent field – including the so-called 'structureless background' – possesses structure. Structureless turbulence (or any of its parts) contradicts both the experimental evidence and the Navier–Stokes equations. The qualitative observations on the *little apparent structure in the low intensity component* or the *bulk of the volume* with *no particular visible structure* should be interpreted as indicating that no *simple visible* structure has been observed so far in the bulk of the volume in the flow. It is a reflection of our inability to 'see' more intricate aspects of turbulence structure: intricacy and 'randomness' are not synonyms for absence of structure. This complexity of turbulence phenomena and its structure(s) makes it necessary and unavoidable to use statistical methods of description/characterization of turbulence phenomena and its structure(s). It is important to emphasize the distinction between statistics weakly dependent on structure(s) and structure sensitive statistics, which is intimately related to and underscores the non-Gaussian nature of turbulence. The latter allows one to obtain information on the structure of turbulence without knowing anything about its structures' actual appearance. Statistical description (not 'theories') is the only quantitative alternative to the visual qualitative methods. We remind the reader that 'statistical' means not only the 'traditional' things like means/averages and other simple means, but all kinds of statistics including rather exquisite ones such as conditional statistics can be, depending on the nature of problems in question and the ability/skill of the researcher to formulate such questions.

The view that turbulence structure(s) is(are) simple in some sense and that turbulence can be represented as a collection of simple objects only seems to be a nice illusion which, unfortunately, has little to do with reality. It seems somewhat wishfully naive to expect that such a complicated phenomenon like turbulence can merely be described in terms of collections of only such 'simple' and weakly interacting objects.

CHAPTER 8

TURBULENCE UNDER VARIOUS INFLUENCES AND PHYSICAL CIRCUMSTANCES

Closer to real-world turbulent flows

8.1. Introduction

As mentioned in chapter 1, there are many factors and influences which cause a real turbulent flow in nature and technology to deviate from the idealized homogeneous and isotropic state, sometimes strongly. In the latter case, turbulent flows may lose most of their resemblance to the three-dimensional homogeneous isotropic flow, but can be quite similar to the (quasi-) two-dimensional one. Here we are inevitably back also to the origins of turbulence but with a different focus: the source sustaining the turbulence, such as mean shear, buoyancy, electromagnetic forces, shock waves. In other words there are different kinds of influences. The 'simplest' influences are 'one way'[1]. They modify the turbulent flow in some way, but are neither influenced by the turbulent flow nor sustain it, e.g., rotation under some conditions. Another kind of influences are 'two way'. They are truly interactive in the sense that along with modifying the turbulent flow they are changed by the flow too, e.g., magnetic field, density stratification and other active scalars. Under certain circumstances these influences contribute also to production and sustaining of turbulent flows, such as in the case of turbulent convection. Turbulent flows with mean shear (strain) belong to this latter category.

What follows in the sequel is an overview of turbulent flows under various influences and physical circumstances, which include shear, buoyancy, rotation, (electro-) magnetic field, compressibility and additives. For obvious reasons the material of this chapter is limited by only the most important essential features, simple examples and qualitative aspects with references on comprehensive sources and some early and latest papers.

[1]These are 'one way' only in some sense since they act not as if the associated terms were additive and independent of, e.g., nonlinearities. The outcome is due to their interaction and mutual influence, which can change qualitatively the nature of and in some extreme cases suppress the nonlinearity. In this latter case the flow can hardly be qualified as turbulent. Any features in flows under some influence cannot be claimed to arise due to this influence solely as long as the flow is *turbulent*, see e.g., Staplehurst et al. (2008) and references therein.

For the same reasons, no summary is given at the end of this chapter[2].

The two main common features of turbulent flows under various influences and physical circumstances are anisotropy and inhomogeneity[3]. Along with other consequences anisotropy results in nonzero off-diagonal Reynolds stresses, $-\langle u_i u_j \rangle$, $i \neq j$, whereas inhomogeneity leads to nonzero gradients of the Reynolds stresses. Only in the latter case is there a two-way coupling between the mean flow and the field of fluctuations, since the Reynolds averaged equations (C.43) for the mean flow contain the *gradients* of the Reynolds stresses[4]. Note that anisotropy can have a great variety of manifestations. For example, the flow in the proximity of turbulent channel flow cannot be considered either approximately homogeneous or isotropic in spite of the fact that in this region both the gradient of the mean velocity dU_1/dx_2 and the Reynolds stress are small, because the gradient of the Reynolds stress, $d\{-\langle u_i u_j \rangle\}/dx_2$, is not small.

The Reynolds stresses, $-\langle u_i u_j \rangle$, react back on the mean flow. This reaction results – among other things – in enhanced (turbulent) momentum transport, and consequently turbulent kinetic energy production $-\langle u_i u_j \rangle S_{ij}$. The simplest manifestation is much larger than in the laminar counterpart resistance, i.e., large gradients of mean velocity on the wall, and a flat velocity profile far from the boundaries in the turbulent channel flow.

The total turbulent energy balance for the whole flow domain is

$$\frac{d\mathcal{E}_T}{dt} = \mathcal{P} + \mathcal{W}_F - \mathcal{D} \qquad (C.49)$$

where $\mathcal{E}_T = \int e_T dV$, $e_T = \frac{1}{2}\langle u^2 \rangle$, is the total kinetic energy of turbulent fluctuations, $\mathcal{P} = \int -\langle u_i u_j \rangle S_{ij} dV$ is the total rate of production/destruction of energy turbulent fluctuations by the mean strain, S_{ij}, (mean velocity gradients), $\mathcal{W}_F = \int \langle u_i F_i \rangle dV$ is the total rate of production of energy of turbulent fluctuations by the external forces and $\mathcal{D} = 2\nu \int \langle s_{ij} s_{ij} \rangle dV$ is the total rate of dissipation (simply dissipation) of energy of turbulent fluctuations by viscosity. If the flow is statistically stationary and is a pure shear

[2]For example, we do not include mostly empirical material such as the lively discussion whether the near-wall turbulence has complete similarity, leading to the *log* law for the mean velocity, or if it possesses incomplete similarity leading to a *power* law for the mean velocity. It seems that with appropriate 'tuning' both – being dimensionally correct – do well, and that without deeper physical foundation than just similarity and dimensional analysis the 'controversy' cannot be resolved. For discussion and references see Monkewitz et al. (2008).

[3]An inhomogeneous field is also anisotropic, but not vice versa.

[4]In case of a (hypothetical) homogeneous shear flow, the mean flow, which is just $\mathbf{U} = (Sx_2, 0, 0)$, does not 'know' about the turbulent fluctuations, since $d\langle u_1 u_2 \rangle/dx_2 = 0$. Therefore, without some additional sustaining mechanism turbulent fluctuations cannot be stationary and will decay. Moreover, it was shown by Harris et al. (1977) that such a flow as a whole (i.e., mean plus fluctuations) is impossible.

flow, i.e., $\mathcal{W}_F = 0$, the dissipation equals production $\mathcal{P} = \mathcal{D}$, i.e., $\mathcal{P} > 0$. If there is no mean flow to supply energy to the field of fluctuations, the statistically stationary state can be maintained by some external source (as, e.g., in thermal convection), and the energy balance equation takes the form $\mathcal{W}_F = \mathcal{D}$. It is noteworthy that in the presence of some energy supply *other* than the mean shear/strain the total rate of production of energy turbulent fluctuations by the mean strain, $\mathcal{P}$, does not have to be positive even in the case of statistically-stationary turbulent flow, since in this case the balance is $\mathcal{W}_F + \mathcal{P} - \mathcal{D} = 0$. This leads to the possibility that the field of turbulent fluctuations 'feeds' the mean flow, as in examples described in section 8.5.

8.2. Shear flows

We limit the following discussion to 'simple' turbulent shear flows, which have a (quasi-) one-dimensional mean $\mathbf{U} =(U\mathbf{i}, V\mathbf{j}, 0)$, $V \ll U$, with slow streamwise and no spanwise variations, $\partial\langle\cdots\rangle/\partial x_2 \ll \partial\langle\cdots\rangle/\partial x_1$, $\partial\langle\cdots\rangle/\partial x_3 = 0$, and axisymmetric analogues, where $\langle\cdots\rangle$ designates some mean of any quantity. These are turbulent flows in channels (pipes) and boundary layers (on weakly curved bodies), which are called wall-bounded turbulent flows, and turbulent flows in jets, plumes, wakes and mixing layers, which are called boundary-free (or simply free) turbulent shear flows. Moreover, for qualitative purposes it is sufficient to assume that $\mathbf{U} = (U\mathbf{i}, 0, 0)$, and $\partial\langle\cdots\rangle/\partial x_{1,3} = 0$ as in turbulent channel flows with x_1, x_2, x_3 for the streamwise, wall-normal and spanwise coordinates in which all statistical properties depend on the coordinate, x_2, normal to the channel boundary only. Such a flow possesses a mean vorticity $\Omega = (0, 0, dU/dx_2)$ having only a spanwise component, and a mean rate of strain, S_{ij}, with nonzero components $S_{12} = S_{21} = \frac{1}{2}dU/dx_2$, and with the eigenvectors along the axes inclined at $\pi/4$ and $3\pi/4$ to the streamwise direction, x_1, and corresponding eigenvalues $\Lambda_1^S = \frac{1}{2}dU/dx_2$, $\Lambda_2^S = -\frac{1}{2}dU/dx_2$ and $\Lambda_3^S = 0$.

The turbulent fluctuations are exposed to the persistent action of this mean strain, which leads to anisotropy of the Reynolds stress tensor of turbulent fluctuations $-\langle u_1u_2\rangle$ in such a way that the eigenframe of the instantaneous Reynolds stress tensor $-u_1u_2$ tends to be aligned with the eigenframe of S_{ij}, see left column (especially bottom) in figure 8.26. This alignment occurs in such a way that the term $-u_iu_jS_{ij}$ responsible for turbulent energy production (see equation [C.47]) is positively skewed, so that its mean is (usually) positive and so is the total rate of production/destruction of energy turbulent fluctuations by the mean strain, $\mathcal{P} = \int -\langle u_iu_j\rangle S_{ij}dV$. In fact, the above alignment is nothing but the tendency for alignment between vector u_i (more precisely its projection on the plane x, y) and the eigenvector of S_{ij} corresponding to its negative eigenvalue, which is seen from the relation $-u_i u_j S_{ij} = u^2 \Lambda_i^S \cos^2(\mathbf{u}, \lambda_i^S)$. It is noteworthy that this kind of behaviour is of more general nature and is observed in production of gradients of a passive scalar, energy of a disturbance/error and some other. In such situations the dominant process is compressing rather than stretching. Shear flows belong to this category.

TKE production in turbulent shear flows The turbulent energy production in a turbulent shear flow is known to be represented by the term $-\langle u_i u_k \rangle S_{ik}$, with u_i being the components of velocity fluctuations, and S_{ik} the mean rate of strain. In a turbulent flow which is two-dimensional in the mean (i.e., such that $\partial \langle \ldots \rangle / \partial x_3 = 0$) the production term can be represented as

$$-\langle u_i u_k \rangle S_{ik} = -\langle u^2 \Lambda_1^S \cos^2(\mathbf{u}, \lambda_1^S) \rangle - \langle u^2 \Lambda_2^S \cos^2(\mathbf{u}, \lambda_2^S) \rangle \qquad (8.1)$$

where $u^2 = {u_1}^2 + {u_2}^2$, Λ_i^S are the eigenvalues and λ_1^S are the corresponding eigenvectors of the mean rate of strain tensor S_{ik}, and $\Lambda_1^S > 0$, $\Lambda_2^S < 0$ ($\Lambda_3^S = 0$). Since the term associated with the stretching of material elements is negative, $-\langle u^2 \Lambda_1^S cos^2(\mathbf{u}, \lambda_1^S) \rangle < 0$, and the term associated with the compressing of material elements is positive, $-\langle u^2 \Lambda_2^S cos^2(\mathbf{u}, \lambda_2^S) \rangle > 0$, the production term $-\langle u_i u_k \rangle S_{ik}$ can be (and usually is) positive due to positiveness of the term associated with the compressive (negative) eigenvalue/eigenvector $[\Lambda_2^S, \lambda_2^S]$, of the mean strain S_{ik}. In this sense the turbulent energy production is due to the predominant compressing of material elements rather than stretching. One can see that $-\langle u_i u_k \rangle S_{ik} > 0$ in a general shear flow taking into account that Λ_3^S is a positively skewed quantity. The above was observed both experimentally in a turbulent boundary layer and numerically in a turbulent channel flow, Gurka et al. (2004), see figure 8.1.

A similar statement is true for the energy flux $\Pi_\ell = -\tau_{ij}^{SGS} \prec S_{ij} \succ$ at scale ℓ for the filtered quantities in the filtering approach. Namely, recalling that $\tau_{ij}^{SGS} = \prec u_i u_j \succ - \prec u_i \succ \prec u_j \succ$ and passing over to the eigenframe of $\prec S_{ij} \succ$ it is seen that

$$\Pi_\ell = -\{(\prec u_i^2 \succ - \prec u_i \succ^2) \Lambda_i^{\prec\succ}, \qquad (8.2)$$

where $\prec \ldots \succ$ means the filtering operation (see appendix 3), u_i are the instantaneous velocity components and $\Lambda_i^{\prec\succ}$ are the eigenvalues of the filtered rate of strain tensor $\prec S_{ij} \succ$, and $\Lambda_1^{\prec\succ} > \Lambda_2^{\prec\succ} > \Lambda_3^{\prec\succ}$ and $\Lambda_1^{\prec\succ} > 0, \Lambda_3^{\prec\succ} < 0$ and $\Lambda_2^{\prec\succ} \lesseqgtr 0$ with positively skewed distribution. It is easily seen that the energy flux can be positive if the term $-(\prec u_3^2 \succ - \prec u_3 \succ^2) \Lambda_3^{\prec\succ}$ (which is positive due to $\prec u_3^2 \succ - \prec u_3 \succ^2 > 0$ and $\Lambda_3^{\prec\succ} < 0$) corresponding to compression of material elements is dominating, i.e., the energy flux to small scales is due to such compressing events

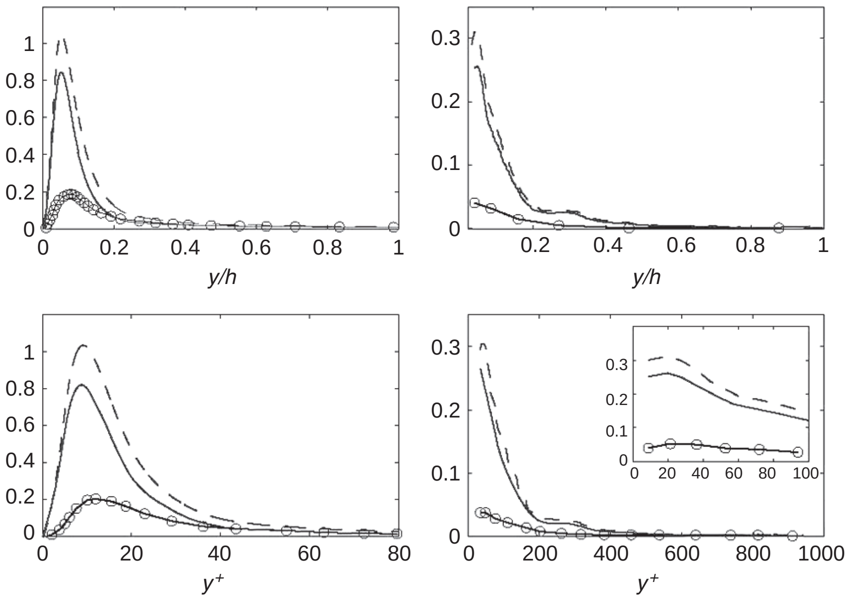

Figure 8.1. Eigencontributions $-u^2\Lambda_1^S \cos^2(\mathbf{u}, \lambda_1^S)$ (top) and $-u^2\Lambda_2^S \cos^2(\mathbf{u}, \lambda_2^S)$ (bottom) as functions of the distance from the wall. Left – DNS, channel flow, Re = 5600; Right – experiment, turbulent boundary layer, Re = 27000. In both the notations are as follows: Solid line, $-u^2\Lambda_1^S \cos^2(\mathbf{u}, \lambda_1^S)$; dashed line, $-u^2\Lambda_2^S \cos^2(\mathbf{u}, \lambda_2^S)$; dashed-dotted line, $-u^2\Lambda_1^S \cos^2(\mathbf{u}, \lambda_1^S)$ $-u^2\Lambda_2^S \cos^2(\mathbf{u}, \lambda_2^S)$; circles, $-2\langle u_1 u_2\rangle S_{12}$. As a check the total production term $-\langle u_i u_k\rangle S_{ik}$ was calculated both directly and via (8.1). Gurka et al. (2004)

rather than stretching. Moreover it is the "backscatter" which is associated with stretching of material elements. As the energy flux is known to be positively skewed (Cerutti and Meneveau, 1998; Chen et al., 2003c) the net flux is due to predominant compressing of material elements rather than stretching. Note that this statement is in terms of material elements rather than vorticity which is believed to be stretched in a similar way (Taylor, 1938 and on), though the physics of the two processes is essentially different (we return in more detail to this issue in chapter 9). In any case it is a clear indication that the widely accepted belief that 'vortex stretching plays the major role in energy cascade' (for one of the latest examples and references see Eyink, 2006, 2008) is quite a bit exaggerated and misinterpreted. On the contrary it is associated with the predominant compression[5] and is in conformity with the mentioned above conclusion by Betchov (1956) *that*

[5]of material elements which is qualitatively different from vortex stretching, see chapter 9.

the most important processes associated with production of vorticity and energy transfer resemble a jet collision and not the swirling of a contracting jet.

Similarly, vorticity (more precisely its projection on the plane x, y) tends to align in this region with the stretching eigenvector of the mean strain S_{ij}, which is inclined at the angle $\pi/4$ to the streamwise direction (Moin and Kim, 1985). This tendency is also of more general nature[6]. In wall-bounded flows, both alignments are influenced by the wall. Very close to the wall, the eigenframe of $-\langle u_1 u_2 \rangle$ tends to coincide with the x_1, x_2, x_3, so that the Reynolds stress $-\langle u_1 u_2 \rangle$ vanishes in the immediate proximity of the wall. Similarly vorticity, 'vortices' and other associated 'structures' tend to be closely aligned with the streamwise direction x_1, forming a pattern of 'streamwise vortices' with low speed 'streaks' in between. Among other things the importance of the streaks is in the bursts of intensity of the fluctuating motion, in which most turbulent kinetic energy is produced (see figure 8.2 and below). These bursts arise on the background of the streaks in the low speed regions with inflection points in the profile of the streamwise velocity component. The meaning/definition(s) of the quasi-streamwise vortices close to the wall and structures in the outer region such as horseshoe or hairpin vortices with two and/or one leg, asymmetric staggered vortices and many others, vary considerably among authors in the turbulent shear flows community (see Panton, 1997 and references therein; also Chacin and Cantwell, 2000). A recent conception of a hairpin vortex packet is promoted by Adrian et al. (2000). It is certain that all turbulent shear flows produce streamwise, ω_1 (and also wall-normal, ω_2) vorticity, in patterns possessing some structure with a number of robust features, which cannot be identified as purely random[7].

The quasi-streamwise vortices are believed to be the dynamically important feature of turbulent flows that is mostly responsible for the turbulent momentum transport, i.e., the Reynolds stress $-\langle u_1 u_2 \rangle$, and consequently turbulent kinetic energy production. However, this interpretation is not necessarily the correct one as is shown in the example of the fully-developed turbulent flow such as the flow in a plane channel considered in section 6.6.

[6]Namely, vorticity tends to align with the stretching eigenvector of the *large*-scale 'part' of the rate of strain tensor (Kevlahan and Hunt, 1997; Porter et al., 1998; Hamlington et al., 2008). This does not contradict the tendency of vorticity to align with the eigenvector corresponding to the intermediate eigenvalue of the instantaneous rate of strain (and its fluctuative part).

[7]Creation of streamwise vorticity is an inherent property of many flows possessing background spanwise vorticity with a primary instability that is two-dimensional. The secondary instability leads to the formation of streamwise 'vortices' (Brown, 1970; Pierrehumbert and Widnall, 1982; Phillips et al., 1996 and references therein).

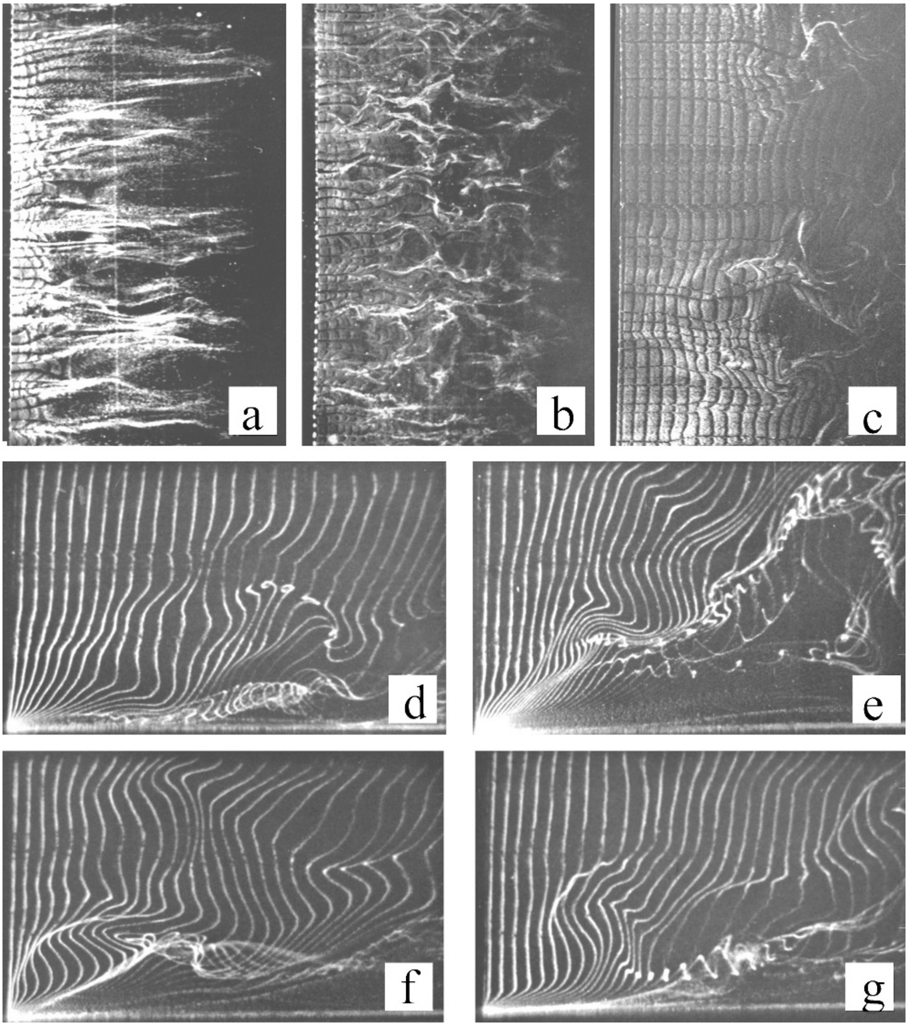

Figure 8.2. Turbulent boundary layer as visualized by hydrogen bubbles. Top view: a) $y^+ = 2.7$, b) $y^+ = 38$, c) $y^+ = 407$ (Kline et al., 1967). Side view: d), e) – selected frames showing formation and breaking of a streamwise vortex motion, f) – same for a transverse vortex motion, g) – showing a 'wavy mode' of bursting (Kim et al., 1971)

In such a flow

$$\langle u_1 u_2 \rangle \equiv \int_0^{x_2} \langle \omega_2 u_3 - \omega_3 u_2 \rangle dx_2, \tag{8.3}$$

i.e., the Reynolds stress is associated *directly* with the wall-normal and spanwise vorticity components ω_2 and ω_3, but not with the streamwise vorticity component ω_1. In fact, more important is the x_1-dependence, since a flow having also a ω_1-component, but lacking the x_1-dependence is impotent in the sense that, in such a flow, there is only one-way coupling between the flow in the cross-stream plane, x_2, x_3 and the streamwise flow. Namely, the flow in the cross-stream plane influences the streamwise flow, but the streamwise flow does not affect the flow in the cross-stream plane. Consequently there is no source of energy to sustain the flow in the cross-stream plane. Moreover, a pure two-dimensional 'turbulent' channel flow[8], possessing no streamwise vorticity at all (it has only spanwise vorticity, ω_3) *is* capable of producing considerable Reynolds stresses (see figure 8.22). In any case the relation (8.1) or the one from which it follows, $d\langle u_1u_2\rangle/dx_2 = \langle(\omega \times \mathbf{u})_1\rangle$, (6.5), show the importance of vorticity in maintaining the Reynolds stresses in turbulent flows (Tennekes and Lumley, 1972). We recall that in turbulent shear flows $\langle(\omega \times \mathbf{u})\rangle \neq 0$, whereas it is vanishing in homogeneous turbulent flows.

The origin of the (coherent) structures in turbulent shear flows is usually associated with some kind (not well defined) instability of imaginary flows with turbulence but without coherent structures (Lumley and Yaglom, 2000). This kind of instability is sometimes considered to be the driving instability of the underlying mean flow in fully-developed turbulence (Roshko, 1993). A constructive example of such an approach is given by Nikitin and Chernyshenko (1997). They looked at the instability of the mean flow in the near-wall region resulting from the action of a 'body force' $\partial Q/\partial x_2, Q = \langle u_3u_3\rangle - \langle u_2u_2\rangle$. The resulting spacing between the fastest growing modes is the same as experimentally observed between the low speed streaks. It is noteworthy that this agreement is achieved by using an *empirical* expression for Q by approximating the data from DNS. Another view is that coherent structures result from the preferential amplification of a particular class of perturbations (Farrell and Ioannou, 1998; Marasli et al., 1991). However, this approach is based on the *linear* stability theory, which means that turbulent flow should have quite long 'memory' and ability for selective amplification in the sea of broadband excitation (including direct excitation of small scales) occurring in naturally arising turbulent flows. Alternatively this may mean that at least in some flows (as mixing layers, wakes) the large-scale structure(s) result(s) from a large-scale instability not related directly to the turbulent nature of the flows under consideration.

The origin of coherent structures in the wall turbulent shear flows is also associated with the phenomenon of bursting. This name was given by Kline et al. (1967) to the sequence of events happening to the near-wall

[8]It is noteworthy that the interaction of fluctuations and the mean flow is essentially three-dimensional, i.e., it involves u_3 fluctuations. Therefore, it was believed that pure two-dimensional flows are incapable of developing appreciable Reynolds stressess (Tennekes and Lumley, 1972, p. 41).

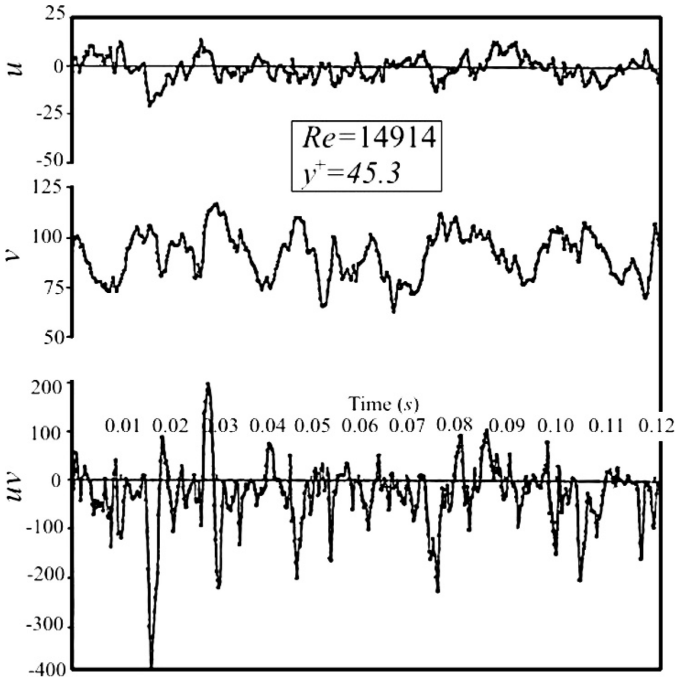

Figure 8.3. Time records of the streamwise, u, and normal, v, components of velocity fluctuations, and the Reynolds stress, uv, the latter exhibiting an intermittent behaviour (Wei and Willmarth, 1989)

structures: lift up, oscillation and break up (or down; see figure 8.2). Other people found an 'ejection sweep cycle' similar to bursting; see Cantwell (1990), McComb (1990) and Holmes et al. (1996) for an overview of what is called coherent structures and associated phenomena in turbulent shear flows. The first evidence was obtained from flow visualizations. These were interpreted as some sort of secondary instability producing a burst of instantaneous Reynolds stress, u_1u_2, mainly responsible for the turbulence production and maintaining in the wall-bounded flows. The bursty behaviour of u_1u_2 was observed directly in measurements, for example by Lu and Willmarth (1973; see figure 8.3) and many others. It is the right place to be reminded (see section 7.2.2) that the intermittent behaviour of u_1u_2 can be accounted for solely by the multiplicative nature of the Reynolds stress (as a product) assuming both u_1 and u_2 to be Gaussian with the correlation coefficient between them adjusted from the experiment (-0.44). In such a way the PDF of u_1u_2 is approximated with high precision (Lu and Willmarth, 1973). This means that dealing with such signals one has to be able to separate such 'false' intermittency from the one inherent to the flow field. Relatively simple low-dimensional systems also exhibit features

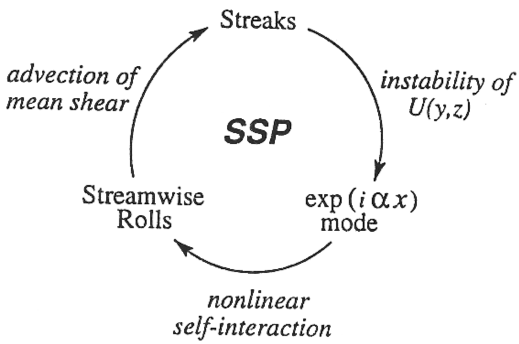

Figure 8.4. The self-sustaining process, courtesy of Professor F. Waleffe

like the bursting phenomenon (Holmes et al., 1996, 1997 and Knobloch and Moehlis, 2000). There seems to be little doubt that in real turbulence the bursting phenomenon is a result of its dynamics, but the above examples (see also Tsuji and Dhruva, 1999), show that such phenomena may arise not necessarily for the 'right' reasons. The bursting process involves most of the scales from those represented by the velocity field to those related to the velocity derivatives. In terms of time scales, the duration of the bursts is pretty short; it is a fast process. Therefore, it is likely that the bursting process is associated with strain-dominated regions, just as in the case of nonsheared turbulent flows the most intense nonlinear activity is associated with the regions dominated by strain (see chapter 6 and the discussion below at the end of this section).

The near-wall structure of wall-bounded turbulent flow is closely related to the process of production and self-sustaining of turbulence in such flows. Attempting to get an insight into the details on how energy is fed in the field of fluctuations in turbulent shear flows, such a self-sustaining mechanism was proposed by Waleffe (1990; see figure 8.4) the essence of which is a nonlinear mechanism consisting of creation, destruction and regeneration of streaks.

The proposal by Waleffe (1990) was followed by a convincing confirmation of such mechanisms in direct numerical simulations, though not without a variety of disagreements (Jimenez and Pinelli, 1999; Panton, 1997 and Waleffe and Kim, 1998). Waleffe (1990) also proposed that the streak spacing of $\sim$100 wall units, i.e., $x_2^+ = x_2 u^+/\nu$, $u^+ = (\nu dU/dx_2|_{x_2=0})^{1/2}$, should be considered as a critical Reynolds number for transition from laminar 1D flow to a 3D finite amplitude state in shear flows. The 100^+ spacing would then correspond to the smallest Reynolds number at which a flow can be maintained in a state different (not necessarily turbulent) from unidirectional laminar flow. This idea is based on the computations by Jimenez and Moin (1991) for different Reynolds numbers, the main result of which is that turbulent flow cannot be maintained in boxes which are narrower than 100 wall units in the spanwise direction. The self-sustaining processes are known to exist in other flows, e.g., in the near wake of a bluff body (see Huerre and Rossi, 1998).

The three-dimensional self-sustaining process (SSP) is reminiscent of enstrophy and strain self-production, discussed in section 6.3. The difference is that in the SSP energy is fed into the system directly, whereas, in case of self-amplification of the field of velocity derivatives, they are produced entirely by the fluctuative field itself, once created and supported by the velocity field. In case of turbulent shear flows, there are many terms contributing to production of enstrophy and strain (see equations [C.51, C.53] in appendix C). Well-known order of magnitude estimates (Tennekes and Lumley, 1972) show that at large Reynolds numbers production of enstrophy, $\frac{1}{2}\langle\omega^2\rangle$, is mainly associated with the term $\langle\omega_i\omega_k s_{ik}\rangle$, i.e., with the self-amplification of the field of vorticity/strain fluctuations. According to these estimates contributions to the enstrophy production associated with the mean velocity gradient, $\langle u_k\omega_i\rangle\partial\Omega_i/\partial x_k$, $\langle\omega_i\omega_k\rangle S_{ik}$, $\Omega_k\langle\omega_i s_{ik}\rangle$, i.e., due to presence of mean vorticity Ω_i and strain S_{ij}, are small compared to $\langle\omega_i\omega_k s_{ik}\rangle$. Similar estimates remain valid for the production of the total mean squared strain $s^2 \equiv \langle s_{ij}s_{ij}\rangle$ (see equation [C.53]). Namely, its production is mainly due to the term $-\langle s_{ij}s_{ki}s_{ki}\rangle$, whereas contributions to the strain production associated with the mean velocity gradient, $-\langle u_k s_{ij}\rangle\partial S_{ij}/\partial x_k$, $\langle s_{ij}s_{jk}\rangle S_{kj}$, are small compared to $-\langle s_{ij}s_{ki}s_{ki}\rangle$.

The field experiments by Kholmyansky et al. (2001b), Gulitski et al. (2007) at $\mathrm{Re}_\lambda \sim 10^4$ showed that this is really the case. The largest terms among the mentioned above, $\langle u_k\omega_i\rangle\partial\Omega_i/\partial x_k$ and $-\langle u_k s_{ij}\rangle\partial S_{ij}/\partial x_k$ are two orders of magnitude smaller than $\langle\omega_i\omega_k s_{ik}\rangle$ and $-\langle s_{ij}s_{ki}s_{ki}\rangle$.

It appears that the dominance of $\langle\omega_i\omega_k s_{ik}\rangle$ and $-\langle s_{ij}s_{ki}s_{ki}\rangle$ and 'smallness' of the RDT-like terms may occur already at rather moderate Reynolds numbers. Such an example is given by Sandham and Tsinober (2000) for a turbulent channel flow at the overall $\mathrm{Re} = 3300$, based on the half channel width and the mean velocity at the centreline (see figure 8.5). The main result, shown in 8.5, is that the terms $\langle\omega_i\omega_k s_{ik}\rangle$ and $-\langle s_{ij}s_{ki}s_{ki}\rangle$ are indeed the dominant ones, except in the proximity of the wall, $x_2^+ < 20$, $x_2^+ = x_2 u^+/\nu$, $u^+ = (\nu dU/dx_2|_{x2=0})^{1/2}$. In the region close to the wall the terms $\langle\omega_i\omega_k s_{ik}\rangle$ and $-\langle s_{ij}s_{ki}s_{ki}\rangle$ remain of the same order as some of the RDT-like terms. This result is consistent with the one obtained by Kim (1989) in his analysis of pressure fluctuations in simulated channel flow. Contrary to the common belief that the RDT-like contribution to pressure is the dominant component, Kim found that the pure nonlinear pressure is comparable near the wall and is larger away from the wall than the RDT-like contribution.

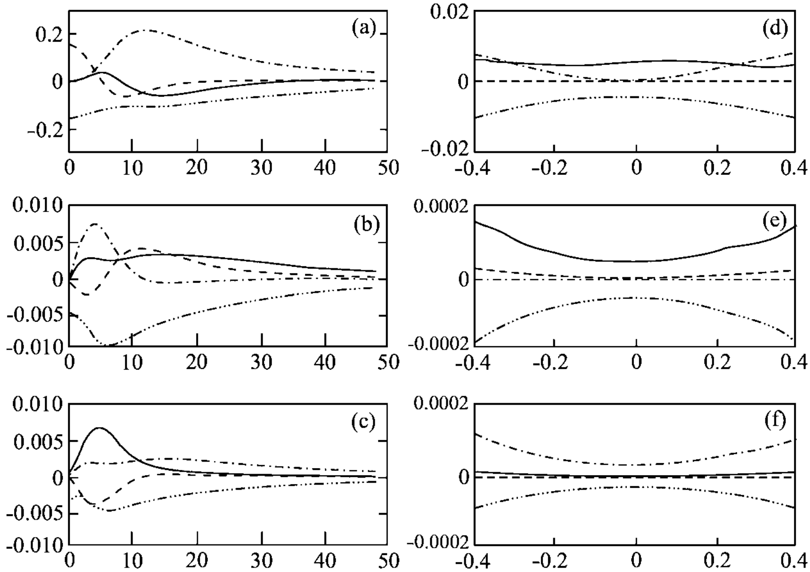

Figure 8.5. Budgets of from top to bottom kinetic energy, enstrophy and strain magnitude showing near-wall behaviour on the left and channel central region on the right. Solid, dashed, chain dot, and chain triple-dot stylelines refer respectively to terms 1–4 in the equations (C.48′), (C.51′) and (C.53′). The approximate sign means that other terms in the equations (C.48), (C.51) and (C.53) in appendix C turn out to be small in the channel flow. From Sandham and Tsinober (2000)

We mention also another result of importance for section 8.9. Namely, there is a strong correlation between vorticity and strain in the proximity of the wall, $x_2^+ < 20$, so that $\omega^2 = 2s^2$ instantaneously at $x_2^+ \leq 10$. Far away from the wall, they are decorrelated, as in homogeneous turbulence. In this latter case (see chapter 6), most enstrophy production, production of strain/dissipation and other nonlinear processes are associated with large strain, rather than with intense vorticity (large enstrophy). This was observed also by Kholmyansky et al. (2001b) at $\mathrm{Re}_\lambda = 10^4$ in the field experiments, and by Sandham and Tsinober (2000) in the turbulent channel flow.

$$\frac{D_U\langle e_T\rangle}{Dt} \approx -\frac{\partial}{\partial x_j}\left\{\langle u_j e_T\rangle - 2\nu\langle u_i s_{ij}\rangle\right\} - \langle u_i u_j\rangle S_{ij} - 2\nu\langle s^2\rangle. \qquad (C.48')$$

$$\frac{1}{2}\frac{D_U\langle\omega^2\rangle}{Dt} \approx \langle\omega_i\omega_j s_{ij}\rangle + \langle\omega_i\omega_j\rangle S_{ij} + \langle\omega_i s_{ij}\rangle\Omega_j + \langle\nu\omega_i\nabla^2\omega_i\rangle. \qquad (C.51')$$

$$\frac{D_U\frac{1}{2}\langle s_{ij}s_{ij}\rangle}{Dt} \approx -2\langle s_{ij}s_{ik}\rangle S_{kj} - \frac{1}{2}\langle\omega_i s_{ij}\rangle\Omega_j - \langle s_{ij}s_{jk}s_{ki}\rangle - \frac{1}{4}\langle\omega_i\omega_j s_{ij}\rangle + \nu s_{ij}\nabla^2 s_{ij}. \qquad (C.53')$$

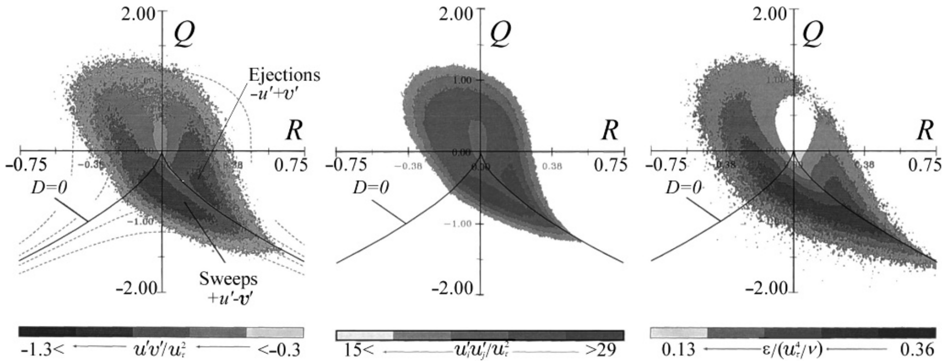

Figure 8.6. Time-averaged Reynolds stress (left), turbulent kinetic energy generating events (center) and dissipative events in the $R-Q$ plane. From Chacin and Cantwell (2000)

A similar phenomenon was observed in a recent analysis of flow in a turbulent boundary layer by Chacin et al. (1996), Chong et al. (1998) and Chacin and Cantwell (2000). These authors used directly the eigenvalues and invariants of the velocity gradient tensor $\partial u_i/\partial x_j$. In particular, they looked at various flow properties in association with the invariant map of the second invariant, $Q = \frac{1}{4}\omega^2 - \frac{1}{2}s_{ik}s_{ik}$ versus the third invariant, $R = -\frac{1}{3}s_{ik}s_{km}s_{mi} - \frac{1}{4}\omega_i\omega_k s_{ik}$ of the velocity gradient tensor. One of the most interesting (from our point of view) findings is that the main contribution to the shear stress, turbulent energy production, and dissipation comes from the regions with $Q < 0$ with larger contribution from the lower right quadrant, i.e., $Q < 0$ and $R > 0$, not only dominated by strain, but also by production of strain, $-s_{ik}s_{km}s_{mi}$, see figure 8.6 and also figures 5, 8, 11 and 14 in Chacin and Cantwell (2000). It should be emphasized that these regions are mainly not the ones corresponding to vortices (hairpins or whatever), which are located mostly in the regions with $Q > 0$, or a bit more precisely in regions with $D > 0$, where $D = \frac{27}{4}Q^3 + R^2$ is the discriminant of $\partial u_i/\partial x_j$. That is the regions of major nonlinear activity are really associated with large strain (mainly corresponding to what Chacin and Cantwell call 'blank' spaces) rather than with regions of concentrated vorticity with lower dissipation (see Tsinober, 2000 for similar results in quasi-isotropic turbulence). In other words, it seems that concentrated vorticity is not that important also in turbulent shear flows and that structure(s) associated with turbulence (not only its energy) production are mainly due to the large strain rather than large vorticity[9]. Structure(s) associated with the latter seem to be a consequence of the turbulent dynamics rather than its dominating factor[10].

In closing this section we mention that, in wall-bounded turbulent flows, there are two factors influencing turbulence: the mean shear and the boundary. Hence in order to 'isolate' the second factor, one can look at the shear-free turbulent flows near different boundaries, solid and free, permeable and not (see Aronson et al., 1997 and references therein).

8.3. Partly-turbulent flows – entrainment and phenomena in the proximity of interfaces

> *... the intermittent character of the disturbance. The disturbance would suddenly come on through a certain length of the tube and pass away and then come on again, giving the appearance of flashes, and these flashes would often commence at one point in the pipe ... This condition of flashing was quite as marked when the water in the tank was very steady, as when somewhat disturbed* (Reynolds, 1883).
>
> *In reality, however, every turbulent flow is bounded by fluid not in turbulent state* (Corrsin and Kistler, 1955).
>
> *One of the properties of the region of rotational turbulent flow is that the exchange of fluid between this region and the surrounding space can occur only in one direction. The fluid can enter this region from the region of potential flow, but can never leave it* (Landau and Lifshits, 1959).
>
> *... entrainment – the erosion by turbulence of the underlying non-turbulent fluid ...* (Phillips, 1966).

[9]This fact is, in a way, not so unexpected because it is the strain that is responsible for local deformation of fluid elements (and, consequently, dissipation in Newtonian and non-Newtonian fluids), whereas vorticity acts locally only as a rigid rotation. It is, therefore, likely that the turbulent strain is the cause of quite a fascinating phenomenon of bioluminescence (known, at least, since the fifteenth century, Harvey, 1952) in turbulent wakes of ships and dolphins in warm seas (Vasil'kov et al., 1992; Herring, 1998). Rohr et al. (1997) found that *turbulent flow is necessary to provide significant bioluminescence stimulation.* They suggested using bioluminescence as a flow diagnostic. This same idea was communicated to the author by Y. Couder and S. Douady in 1995 in Paris.

[10]The interpretation of the results by Chacin and Cantwell (2000) given here is not in full agreement with their conclusions, especially regarding the role of vortices and concentrated vorticity. This does not contradict (6.5), (8.1) stating the importance of vorticity in maintaining the Reynolds stress. First, these are relations for the mean quantities, and second, there is no turbulent flow without vorticity. However, important details of the relations between Reynolds stress, vorticity, strain and their production remain not clear enough.

Partly-turbulent flows have already been mentioned in chapters 1 and 2. The main special features of these flows are the coexistence of regions with laminar and turbulent states of flow and continuous transition of laminar flow into turbulent via the entrainment process through the boundary between the two. In fact, most turbulent flows are partly-turbulent: boundary layers, all free shear turbulent flows (jets, plumes, wakes, mixing layers), penetrative convection in the atmosphere and in the ocean, gravity currents, avalanches and other phenomena at the boundary between single phase fluid and fluid loaded by a sediment (which includes resuspension), clear air turbulence, and many others (e.g., combustion). Transitional flows consisting as a rule of turbulent regions growing in a laminar environment are also partly-turbulent flows.

The so-called external intermittency is associated with the coexistence of laminar and turbulent flow regions – an observer located in the proximity of (either side of) the 'mean' boundary between these regions observes intermittently laminar and turbulent flow in the form of a signal similar to that as, say ω^2, in figure 1.17, see also figure 21.4 in Tritton (1988), clearly demonstrating the external intermittency in the wake past a circular cylinder. Here we again encounter the question about what turbulence is. When we look at a flow like the one in figure 1.3 or figure 151 in van Dyke (1982), we clearly see what is turbulent and what is laminar. But the question is how one can say whether a *small* part of flow is turbulent. In other words if turbulence is to be identified by statistical means, then what is the meaning of 'turbulent' locally? This involves taking decisions about what is turbulent using some (conditional) criterion (see discussion and references in Kuznetsov et al., 1992).

This distinction starts with Reynolds (1883) in the form of such qualitative description of transitional phenomena as flashes of turbulence in the pipe, i.e., he makes a clear distinction between laminar and turbulent flow regions (quasi-) locally without invoking any statistical characteristics. The first physically qualitative distinction between turbulent and non-turbulent regions made by Corrsin (1943) and Corrsin and Kistler (1954, 1955), is that turbulent regions are rotational, whereas the non-turbulent ones are (practically) potential, thus employing one of the main differences between turbulent flow and its random irrotational counterpart on the 'other' side of the interface separating them. It is difficult to implement such a distinction, since it requires information on vorticity which until recently was not accessible, and no experiments are known to adequately employ vorticity in studying the properties of the laminar-turbulent interface and the entrainment across it[11]. The analysis by Bisset et al. (2001) of the data from direct numerical simulation of a temporally developing plane wake

[11] An attempt was made by Foss and Klewicki (1984) to measure and use for this purpose the spanwise vorticity component in a plane shear layer.

by Moser et al. (1998) confirmed Corrsin's approach. One can see a steep change of vorticity across the laminar-turbulent interface, and a much less sharp change in temperature.

Corrsin discovered that *i*) the boundary between the two regions is essentially a thin interface which he called the 'viscous superlayer', in which viscosity plays a dominant role, and *ii*) the 'effectiveness' of the entrainment process is strongly enhanced by the large-scale undulations of the interface due to large-scale motions which result in engulfment of irrotational fluid into the turbulent flow.

The main mechanism by which non-turbulent fluid becomes turbulent as it crosses the interface is believed to involve viscous diffusion of vorticity across the surface. As this process is associated with small scales it is thought to be the reason why the interface appears sharp compared to the scale of the whole flow.

However, at large Reynolds numbers, the entrainment rate and the propagation velocity of the interface relative to the fluid are known to be independent of viscosity (see for example Ricou and Spalding, 1961; Hinze, 1975; Townsend, 1976; Tritton, 1988 and Hunt et al., 2006 for more information and references). Therefore *the slow process of diffusion into the ambient fluid must be accelerated by interaction with velocity fields of eddies of all sizes, from viscous eddies to the energy-containing eddies so that the overall rate of entrainment is set by large-scale parameters of the flow* (Townsend, 1976). That is *although the spreading is brought about by small eddies* [viscosity] *its rate is governed by the larger eddies. The total area of the interface, over which the spreading is occurring at any instant, is determined by these larger eddies* (Tritton, 1988). This is analogous to independence of dissipation of viscosity in turbulent flows at large Reynolds numbers. In other words, small scales do the 'work', but the amount of work is fixed by the large scales[12] in such a way that the outcome is independent of viscosity. This shows that independence of some parameter of viscosity at large Reynolds numbers does not mean that viscosity is unimportant. It means *only* that the (cumulative) effect of viscosity is Reynolds number independent.

It is important that becoming rotational is only a necessary condition of becoming turbulent[13]. Once the irrotational fluid acquired some vorticity via viscous diffusion this vorticity is amplified by the process of predominant vortex stretching due to the random nature of the motion in the proximity

[12]In fact, there is strong bidirectional coupling between the large and small scales, i.e., the small scales are not passive as usually claimed.

[13]For example, becoming rotational in pure two-dimensional flow does not help much: entrainment is an essentially three-dimensional process – vorticity and strain cannot be amplified in two-dimensional flows.

of the interface, though there is no direct evidence that this really happens. This points to another possibility. Due to the random nature of the flow at both sides of the interface a small amount of 'seeding' of vorticity will bring into action the mechanism of the predominant vortex stretching in the 'irrotational' part of the flow. There is no evidence that this really happens either. Some indication comes from the equation (6.10, see chapter 6) showing that an initially Gaussian and *irrotational* velocity field with a small seeding of vorticity will produce – at least for a short time – an essentially positive enstrophy production (as well as production of strain) though strictly this is true for homogeneous turbulence. This mechanism can be effective with a sufficient level of seeding vorticity and only in the proximity of the interface where large strain should exist (e.g., Bisset et al., 2001), because the fluctuations attenuate exponentially with the distance from the interface in the irrotational part of the flow[14]. Since in real flows it is likely that some small amount of vorticity is always present in the 'potential' part of the flow, this second possibility would seem quite realistic. It appears, however, that the small-scale phenomena at the interface are more involved and the small-scale nibbling of the non-turbulent flow region by the turbulence at the interface consists not only of a nontrivial interplay of both diffusion of vorticity by viscosity and vorticity amplification via predominant vortex stretching in the proximity of the interface. An important additional process is associated with the strain production and viscous destruction. All these processes are enhanced due to the strongly convoluted shape of the interface caused by motions on a wide range of scales which includes what is sometimes called large-scale engulfment.

Typically, partly-turbulent flows involve shear and other influences as stratification, etc. However, in order to systematically address the physical processes in the proximity of turbulent/non-turbulent 'interface' (TNTI) it is logical to start from the simplest configuration without a mean shear or other influences with an emphasis on the "small-scale" processes associated with vorticity, strain and fluid particle accelerations. We recall (chapters 1–4, 9) that the continuous transition of laminar flow into turbulent flow in the proximity of the TNTI is of distinctly Lagrangian nature as it happens with fluid particles which are purely Lagrangian objects. Hence the necessity of the Lagrangian approach in studying this transition process. This approach was adopted in the study of flow properties in the proximity of a propagating interface produced by an oscillating grid (Holzner et al., 2007, 2008, 2009) via particle tracking velocimetry (PTV) and DNS in a similar configuration. One of the first results is a qualitative difference between the behavior of vorticity and total strain s^2 on the irrotational side of the TNTI[15] and on its turbulent side, see figure 8.7, and figures 8.8–8.10.

[14]See Landau and Lifshits (1959, p. 129) for a simple explanation of this attenuation.
[15]Note that the term "irrotational" is used for the region which is not all really irro-

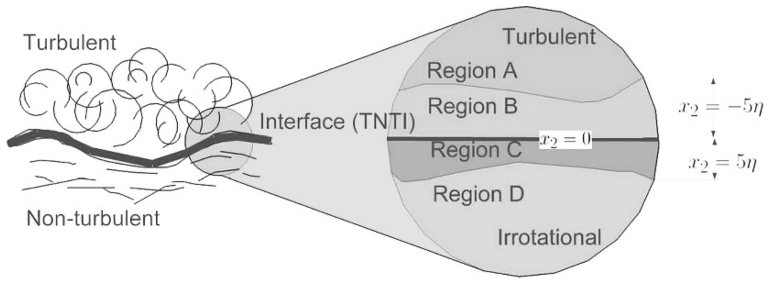

Figure 8.7. Schematic of the proximity of the TNTI. Courtesy A. Liberzon

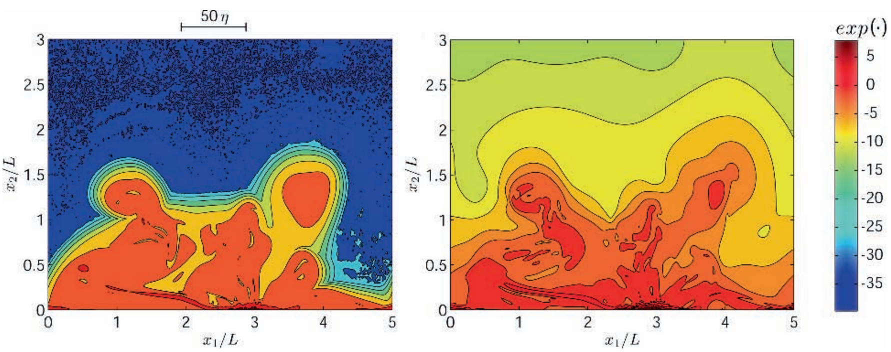

Figure 8.8. Logarithmic contour levels of ω^2 (left) and $2s^2$ (right) from a DNS snapshot at $x_3 = 2.5L$. The color bar refers to both, (a) and (b) and the axes are normalized by the integral length scale L. Holzner et al. (2009)

While vorticity is practically vanishing in all this region, the strain remains finite especially close to the TNTI and becomes (exponentially) small far away from the TNTI.

The initiation of the process of entrainment can be seen as started on the irrotational side of the TNTI via the production of strain due to its self production $-s_{ij}s_{jk}s_{ki}$ and the strain-pressure Hessian interaction $-s_{ij}p_{ij}$ ($p_{ij} \equiv \frac{\partial^2 p}{\partial x_i \partial x_j}$) see figures 8.10 right and 8.11.

Note that on the irrotational side both $-s_{ij}s_{jk}s_{ki}$ and $-s_{ij}p_{ij}$ have the form of a flux $\partial/\partial x_i\{\ldots\}$, i.e., their nonzero (as observed positive) means

tational. The very proximity of the region $x_2 < 0$ is rotational. This is due to the special choice of the origin of $x_2 = 0$, see below.

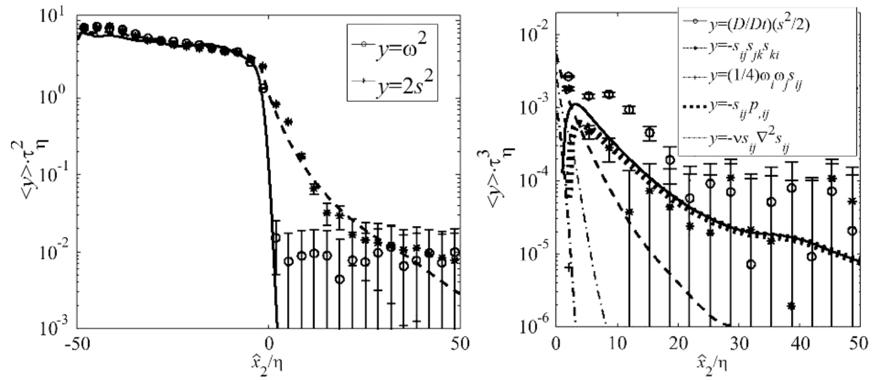

Figure 8.9. Average profiles of a variety of quantities from PTV (symbols), and DNS (lines) relative to the TNTI. Left – enstrophy and strain, Right – quantities involved in the strain balance equation. The axes are normalized by using the Kolmogorov length and time scales. The error bars represent the accuracy of the measurement. The symbol $\langle\cdots\rangle$ denotes the ensemble average of the respective quantity. Holzner et al. (2009)

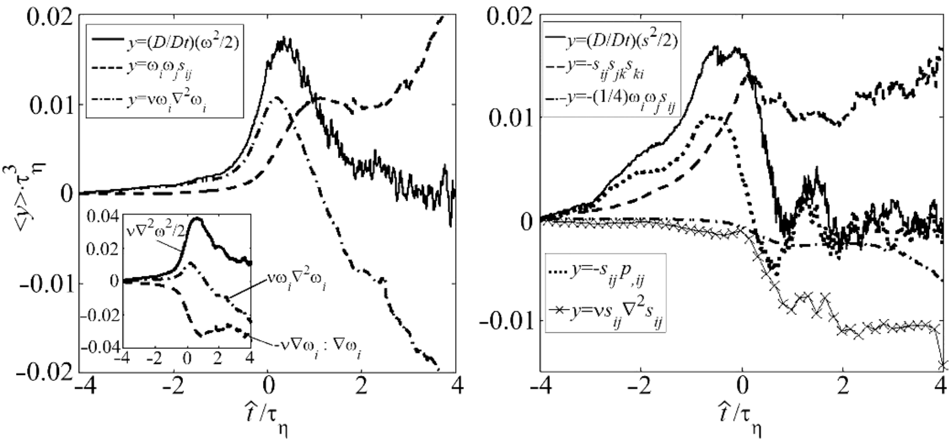

Figure 8.10. Conditionally-averaged Lagrangian evolution of quantities involved in the enstrophy (left) and strain (right) balance equation, DNS. Holzner et al. (2008)

are essentially due to the inhomogeneous nature of the flow in the proximity of TNTI. It is seen that the strain-pressure Hessian term, $-s_{ij}p_{ij}$, is more effective than the self production, $-s_{ij}s_{jk}s_{ki}$, in the strain production both in magnitude and at much farther distances. This is because $-s_{ij}p_{ij}$ is nonlocal, whereas $-s_{ij}s_{jk}s_{ki}$ is local.

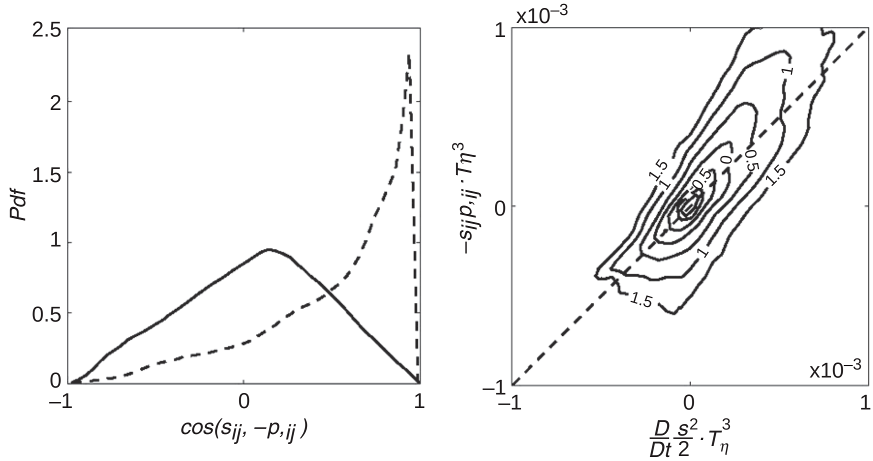

Figure 8.11. Left – PDF of the cosine between the rate of strain tensor and pressure Hessian for the turbulent (solid line) and irrotational (dashed line) regions. Right – Joint PDF of the rate of change of strain versus the product of the rate of strain tensor and pressure Hessian in the irrotational region, DNS. Holzner et al. (2009)

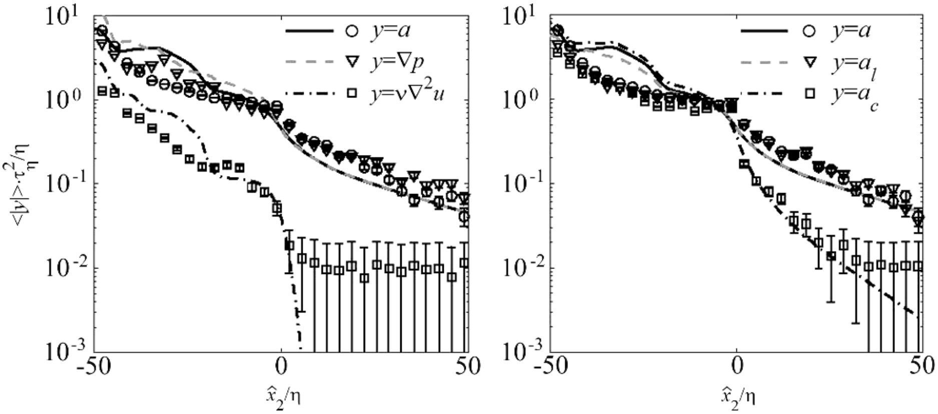

Figure 8.12. Average profiles of the modula of (left) acceleration, pressure gradient and viscous term and (right) Lagrangian acceleration and its Eulerian components. PTV – symbols, and DNS – lines. Note that the dominating contribution to the acceleration in the irrotational region is made by its local component a_l. Holzner et al. (2009)

The strain production on the irrotational side of the TNTI aids the enstrophy production closer to the TNTI, the latter occurring with some delay. Indeed, the peaks of Ds^2/Dt, $-s_{ij}s_{jk}s_{ki}$, and $-s_{ij}p_{ij}$ are located farther into the irrotational flow region than those of $D\omega^2/Dt$, $\omega_i\omega_j s_{ij}$ and especially $\nu\omega_i\nabla^2\omega_i$ (see figure 8.9) A feature of special interest is the behavior of the viscous term $\nu\omega_i\nabla^2\omega_i$: it has a distinct *positive maximum*. This is the reason for using the location of the maximum of the viscous term as an *objective* location of the 'interface', defined in a physically more appealing way than the threshold-dependent crossing. The main contribution to the enstrophy production close to the TNTI on the $x_2/\eta < 0$ side is mainly due to the viscous production ($\nu\omega_i\nabla^2\omega_i$ is positive in the mean) with a smaller contribution from the inviscid enstrophy production $\omega_i\omega_j s_{ij}$. This, in fact was conjectured by Corrsin. The viscous term remains positive and active in a small region $x_2 > 0$ (a bit larger than that for $x_2 < 0$), where it aids activation in full of the enstrophy production. At a short distance of the order of few Kolmogorov lengths on the $x_2 > 0$ side both terms become 'normal' as they are in the usual turbulent flow, i.e., in the mean $\omega_i\omega_j s_{ij}$ is positive and $\nu\omega_i\nabla^2\omega_i$ (changing sign) is negative and thus balance each other (T-L balance). The viscous term $\nu\nabla^2\omega_i$ in the equation for vorticity is usually interpreted as 'diffusion due to viscosity'. Though formally correct this interpretation hides an important conceptual aspect, since $\nu\nabla^2\omega = (1/\rho) curl \mathbf{F}^s$, where $F_i^s = 2\nu\partial s_{ik}/\partial x_k$ is the force due to viscous stresses which arise due to gradients of strain. Thus the term $\nu\omega_i\nabla^2\omega_i$ reflects the interaction between strain and vorticity due to viscosity. One more aspect concerning the term $\nu\omega_i\nabla\omega_i$ is that it can be positive in the mean (as observed in the proximity of the TNTI) due to the inhomogeneity of the flow. Indeed, $\omega_i\nabla^2\omega_i \equiv \nabla^2(\omega^2/2) - (\partial\omega_i/\partial x_k)(\partial\omega_i/\partial x_k)$ and only the purely diffusive term $\nabla^2(\omega^2/2)$ (which vanishes for a homogeneous flow) can make the viscous term positive in the mean, see inset in figure 8.9[16]. A noteworthy aspect is also the difference in the behavior of the vorticity and strain related quantities. In particular, the viscous term $\nu s_{ik}\nabla^2 s_{ik}$ is negative in the mean everywhere; i.e., it is not contributing to the production of s^2. As mentioned the strain production due to self production $-s_{ij}s_{jk}s_{ki}$ and the strain–pressure Hessian interaction $-s_{ij}p_{ij}$ ($p_{ij} \equiv \frac{\partial^2 p}{\partial x_i \partial x_j}$) is significant in the irrotational region in which no analogue exists for enstrophy. There are also differences as reflected in quantities like accelerations and pressure gradients. The acceleration of a fluid particle is precisely irrotational in the region $x_2 < 0$ (except very close to the origin) and $\mathbf{a}$ $(= -\nabla p) \approx \mathbf{a}_l$ ($\mathbf{a} = \mathbf{a}_l + \mathbf{a}_c$; $\mathbf{a}_l = \partial\mathbf{u}/\partial t$; $\mathbf{a}_c = (\mathbf{u}\cdot\nabla)\mathbf{u}$), since $\mathbf{a}_c$ is quadratic in velocity. This is similar, but qualitatively different from the region $x_2 > 0$ in which the acceleration is also dominated by the pressure

[16]Note that the above decomposition of $\nu\omega_i\nabla^2\omega_i$ – though useful – has a limitation since it is not unique and there is an infinite number of possibilities to represent it as a sum of a 'dissipation', i.e., a negatively defined quantity, and a flux term, i.e., as a divergence of some vector. For example, $\omega\cdot\nabla^2\omega = (\nabla\times\omega)^2 - \nabla\cdot\{\omega\times(\nabla\times\omega)\}$.

There is no way to define dissipation (i.e., to choose one among many purely negative expressions) of enstrophy as it is not an inviscidly-conserved quantity, unlike kinetic energy (Tsinober, 2001a).

gradients, i.e., in some sense $\mathbf{a} \approx -\nabla p$, but not precisely[17], i.e., $\mathbf{a} \neq -\nabla p$. Here the the relation $\mathbf{a} \approx -\nabla p$ is due to strong cancellation between $\mathbf{a}_l$ and $\mathbf{a}_c$ thus leading to eliminating most (but never all) of the solenoidal part in the sum $\mathbf{a}_l + \mathbf{a}_c$. These aspects are reflected in figure 8.12.

The above features reflect some of the basic aspects of turbulent processes such as entrainment in the proximity of the interfaces between turbulent and non-turbulent regions in partly-turbulent flows as concerns the 'small-scale' properties. It is hoped that they are universal, at least qualitatively. One cannot expect quantitative universality due to the non-universal nature of the large-scale properties in different flows. The small-scale processes can be considerably modified in the presence of shear, stratification and other influences, but also can react back on the large scales[18]. The real challenge is the simultaneous study of direct and bidirectional (i.e., local and nonlocal) interaction of the large- (e.g., velocity field) and small-scale (e.g., vorticity and strain) flow properties in the proximity of TNTI. An important issue of basic interest is whether the above described features are qualitatively the same in transitional flows such as in the proximity of the TNTI in puffs and slugs in transitional pipe flows and turbulent spots.

8.4. Variable density

There are several effects causing variations of fluid density. These are due to variations of temperature, presence of additives (salt, moisture, small particles) and compressibility. All of them can have a profound influence on the properties of turbulent flows and lead to a number of new effects. For example, in many cases (but not always) fluids with variable density can support waves, so that regions with turbulence can radiate such waves and can interact with waves radiated from other regions (see figure 8.15). Turbulence can be produced locally by the breaking of such waves, which also may transfer momentum and energy to the mean turbulent flow. Another important effect is anisotropy appearing, for example, in the presence

[17] The difference is essential. In the region $x_2 < 0$ the flow is irrotational ($\mathbf{a} = -\nabla p$), whereas in the region $x_2 > 0$ the flow field contains both irrotational and solenoidal components ($\mathbf{u} = \nabla\phi + \nabla \times \psi$; $\nabla^2\phi = 0$) in the proximity of TNTI and only far away becomes almost strictly solenoidal ($\mathbf{u} \approx \nabla \times \psi$) as in 'usual' turbulent flows; it is strictly solenoidal, that is $\mathbf{u} = \nabla \times \psi$, for example, in homogeneous turbulent flows or in flows with periodical boundary conditions. In this region $\mathbf{a} = -\nabla p + \nu\nabla^2\mathbf{u}$. The viscous term though small, e.g., in the sense of RMS (see chapter 6) it plays obviously an essential role: there is no turbulence without a dissipative term.

[18] There is an interesting issue associated with the irrotational nature of the flow on one side of the TNTI and solenoidal far enough from the TNTI on the other. Due to the 'engulfment' is it natural to expect existence of 'approximately' irrotational blobs in the turbulent side with weak vorticity – at least close to the the TNTI. One is curious to compare them with regions with the same level of vorticity (just voids of enstrophy) in the turbulent region far away from the TNTI as the latter are rotational.

of gravity. Finally, additional mechanisms of vorticity production and dissipation arise in some flows with variable fluid density.

8.4.1. CONVECTION

The term (free) *convection* refers usually to buoyancy-induced flows produced by inhomogeneity of fluid density in the presence of gravity. Thus, convection can be thermogravitational if the density variations are caused by changes of temperature, thermohaline (double-diffusive) if the density inhomogeneity is due to both variations in temperature and salt content (or salts with *different* diffusivities)[19], and compositional if the density variations are caused mainly by changes of composition such as those occurring in the presence of crystallization (ice, salt, magma) or resulting from dissolution/melting. Convection is a most typical example of a situation when turbulence is produced and driven by a mechanism totally different from mean shear. The simplest situation is when a heavier fluid is overlying lighter fluid, a state referred to as unstable stratification. Closely related are turbulent motions resulting from the so-called Rayleigh–Taylor instability (Dalziel et al., 1999) and Richtmyer–Meshkov instability (Prasad et al., 2000).

The term *convection* is used to emphasize the fact that the only reason for the motion is the spatial inhomogeneity of fluid properties, which in turn is changed by the fluid motion and which therefore contrast with passive objects, which are not felt by turbulent motions.

Turbulent convection sets in at large enough imposed gradients of, say, temperature or heat flux, reflected in large values of appropriately defined Rayleigh numbers, $\mathrm{Ra} = g\alpha\Delta T d^3/\nu\kappa$, – a non-dimensional imposed temperature difference ΔT, where g is the acceleration due to gravity, d is some external characteristic length (depth of the heated fluid layer), and α, ν and κ are the fluid properties, respectively coefficient of thermal expansion, kinematic viscosity and thermal diffusivity. The flow properties of both turbulent convection and its onset[20] depend on a further parameter – the Prandtl number, $\mathrm{Pr} = \nu/\kappa$. Systematic description and review (with plenty of references) of turbulent convection is found in Tritton (1988),

[19]In the case of several species contributing to fluid density, convection arises also when the fluid density is homogeneous and even statically stably-stratified, but the content of individual species is not homogeneous. The flow motion then is induced by the differences in diffusion properties of the species and subsequent instability leading to a release of the potential energy of the unstably (top-heavy) distributed component(s) (see Turner, 1985).

[20]The transitional regimes in thermal convection may exhibit an entire sequence of states of flows. For example, convection in a horizontal layer heated from below exhibits at least eight transitions judging by the behaviour of the dependence Nu(Ra) (Zimin and Frik, 1988; Siggia, 1994 and references therein).

Siggia (1994), Zimin and Frik (1988); see also Niemela et al. (2000) and Ahlers et al. (2008). Here, as usually in this chapter, we deal mostly with simple examples and qualitative aspects.

Several examples of convection are shown in figure 8.13. Just like in turbulent shear flows it was only recently recognized that there also exist in *turbulent* convection large-scale objects called thermal plumes, which are similar to those observed in transitional stages like those shown in figure 8.13c. These structures are believed to be important in heat transfer and interaction with the boundary layers and in some cases with the induced mean flow called 'turbulent wind' (see section 8.5). This recognition was mostly used in order to cope with the problem of the dependence of the Nusselt (and also Reynolds) number, $\mathrm{Nu} = Hd/(\rho C_p \kappa \Delta T)$, the nondimensional heat flux, H, on the Rayleigh number, Ra. This dependence is usually *assumed* as a power law, $\mathrm{Nu} \sim \mathrm{Ra}^{\beta}$, with the exponent β and a prefactor both depending on the Prandtl number, Pr. In this respect, turbulent convection is not different from some other fields in turbulence research: scaling behaviour to a large extent has monopolized researchers′ attention and resulted in several different theoretical explanations of the observed behaviour. These are described in Siggia (1994), Grossman and Lohse (2000), Niemela et al. (2000), Ahlers et al. (2009) and references therein. The only point we want to repeat here is that scaling laws by themselves are not sufficient for testing theories, since there is no one-to-one relation between scaling exponents and physical processes, which among other things is reflected in qualitatively different theories leading to the *same* scalings. As an example we mention the 2/7 law ($\mathrm{Nu} \sim \mathrm{Ra}^{2/7}$) corresponding to the so-called 'hard turbulence' at very large Ra. In reality this may be equally not a 2/7 exponent at all: as noted by Grossmann and Lohse (2000) the expression $\mathrm{Nu} = 0.27\mathrm{Ra}^{1/4} + 0.038\mathrm{Ra}^{1/3}$ (with the prefactors obtained from *experiment*) mimics the 2/7 power-law over ten orders of magnitude in Ra in the range $10^5 < \mathrm{Ra} < 10^{14}$ (for $\mathrm{Pr} \sim 1$), see their figure 4[21]. Similarly, Niemella et al. (2000) in their cryogenic helium gas experiments obtained a relation $\mathrm{Nu} = 0.124\mathrm{Ra}^{0.309}$ in the range $10^6 < \mathrm{Ra} < 10^{17}$ and Pr between 0.7 and 12, which is described equally well by another relation $\mathrm{Ra} = 0.0587(\mathrm{Ra}^{3/2} ln \mathrm{Ra}^{3/2})^{1/5}$. A third example refers to the so-called ultimate regime predicted by Kraichnan (1962a), in which $\mathrm{Nu} \sim \mathrm{Ra}^{1/2}$. This regime is characterized by the dominant contribution of turbulence (as compared to that by conduction) to the heat transport in the boundary layer on the heated (cooled) wall and is a dimensional necessity following from the assumption that the heat flux is independent of molecular properties, i.e., viscosity and thermal conductivity. The independence of heat flux

[21]High precision measurements by Xu et al. (2000) clearly indicate that there does not seem to exist any single exponent describing the Nu(Ra) relation.

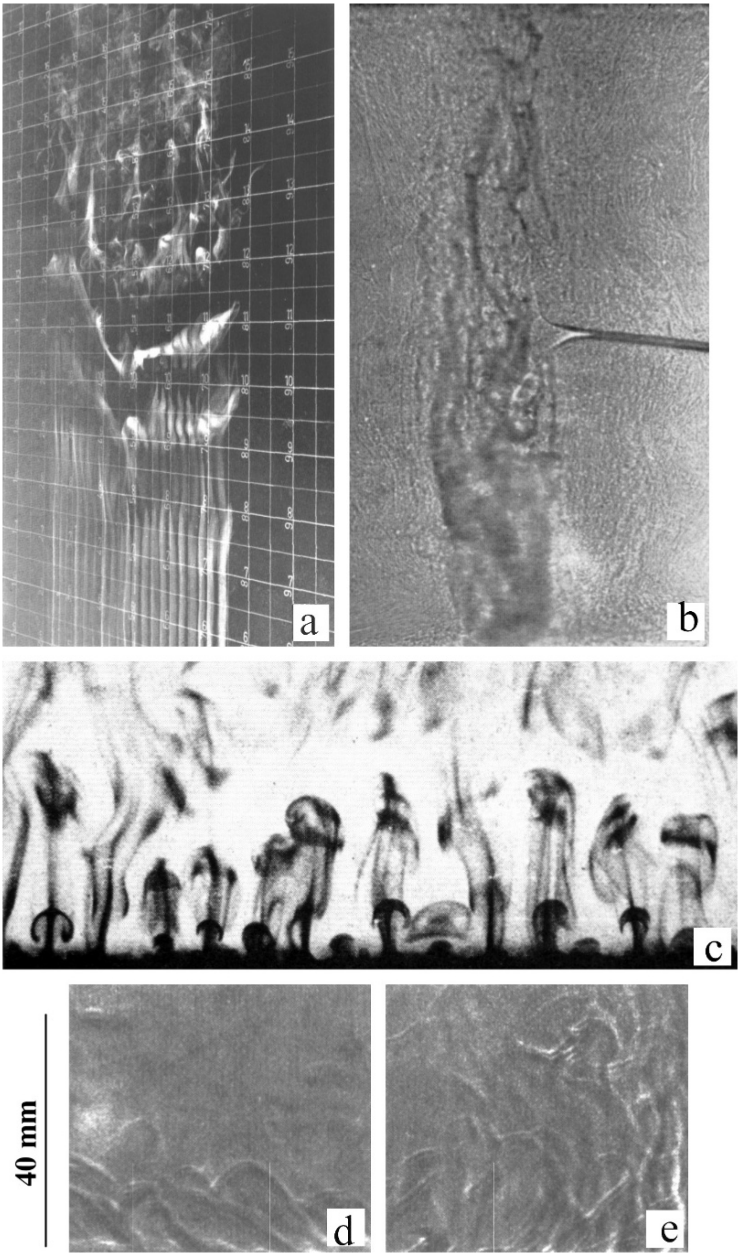

Figure 8.13. Examples of convective motions. a) Smoke visualization of transition in free convection boundary layer on a vertical heated plate (Čolak-Antić, 1964). b) Violent (double-diffusive) convective motion induced by injection of sugar solution into a salt solution of approximately the same density (Turner and Chen, 1974). c) Thermals rising in water from a heated horizontal surface (Sparrow et al., 1970). d) Thermals tilted by the 'thermal wind' along the bottom, and e) the same flow as in d) with the thermals prevented from tilting by vertical inserts (Ciliberto et al., 1996)

of molecular properties is expected to occur at very large Rayleigh numbers and indeed was observed in cryogenic experiments in cells with smooth and rough walls (Roche et al., 2001) at $\mathrm{Ra} > 10^{12}$ and $\mathrm{Pr} = 1.45 \div 4.9$ (see also figure 3 in Siggia, 1994). This regime was also observed in numerical simulations by Vincent and Yuen (2000) in the range $10^{10} < \mathrm{Ra} < 10^{12}$ at $\mathrm{Pr} = 1$, but for *two-dimensional* flow. In addition they observed the conventional relation $\mathrm{Nu} \sim \mathrm{Ra}^{1/3}$ in the range $10^{8} < \mathrm{Ra} < 10^{10}$. Does this mean that the three-dimensional nature of turbulence is unimportant to the overall heat transfer in turbulent convection? Since turbulent convection is essentially an interaction of hydrodynamic and thermal fields, the next question is what properties of the hydrodynamic field are important in this interaction. Whatever the answer, it seems fundamentally important to study the properties of real *three-dimensional* convective turbulence. This includes far more than just the existence (or not) of the 'ultimate state' of convection, which by some authors (Ahlers et al., 2009) is considered as *the most important challenge.*

The second important development in turbulent convection was recognition of the existence and importance of a 'turbulent wind' discovered by Krishnamurti and Howard (1981). This is a mean turbulent shear flow which also stirs the fluid in the bulk of the convective flow. The turbulent wind is fed by the thermal plumes, see section 8.6.

8.4.2. STABLE STRATIFICATION

Unstable stratification, as described above, is the cause itself of the turbulent flow. In case of stable stratification, i.e., lighter fluid overlying heavy fluid, the situation is drastically different. In the presence of stable stratification a fluid particle displaced from its equilibrium state experiences a restoring force due to buoyancy. This leads to two major effects. The first one is that stably-stratified fluids can support waves called internal waves. Internal waves at scales ranging from parts of meters to several kilometers are observed in various natural environments[22]. The second effect is that stable stratification limits the vertical motions, thereby tending to suppress turbulence (with an additional sink of kinetic energy by turning it into potential energy of the system in the process of mixing) and leading to a pancake-like anisotropy of turbulence. This is in contrast with cigar-like anisotropy in the case of unstable stratification. A more important difference is that stable stratification itself cannot be the cause of a (turbulent) motion, so that an additional factor such as mean shear is necessary to

[22]Internal waves play an important role in a variety of flows in technological, geophysical and astrophysical contexts, see, e.g., Staquet and Sommeria (1996). Internal waves in the atmosphere are thought to be used by some birds surfing on them while crossing the Atlantic, Mollo-Christensen (1980, private communication).

support turbulent flows in the presence of stable stratification. However, *due to* stable stratification, turbulence can be *produced* in a special way via breaking of the internal waves (see figure 8.15b and c)[23]. The so-called CAT (clear air turbulence; Pao and Goldburg, 1969) is believed to be associated mainly with the breaking of internal waves in the atmosphere. Similarly breaking waves both internal and surface may be important in production of turbulence in the ocean and air-sea interaction as well (see references in McIntyre, 1993; Melville, 1993 and Staquet and Sommeria, 1996). For review and references on turbulence in the presence of stable stratification see Hopfinger (1987), Riley and Lelong (2000), Thorpe (1987), Sagaut and Cambon (2008) and also Smyth and Moum (2000a,b), for some latest references and very useful expositions of the issues of scales and anisotropy in stably-stratified mixing layers. Finally, turbulence can be produced in double-diffusive systems with *statically stable* stratification created by, e.g., salinity and temperature together.

Examples of fluid phenomena in the presence of stable stratification are shown in figures 8.14 and 8.15.

Coexistence and especially interaction of turbulence and waves in stably-stratified fluids makes it difficult to distinguish between the two. The wave field is usually associated with that part of flow that propagates, whereas turbulence is identified with the nonpropagating part of the motion. However, it is not clear up to now how to separate *random* gravity-wave motion (which does not produce vertical transport) and genuine turbulence (which does) in a stably-stratified fluid (Stewart, 1959). This issue becomes more serious in cases of strongly nonlinear internal waves, and interaction between turbulence and waves, and, of course, when breaking of the latter produces turbulence, which in turn can radiate internal waves.

When the stratification is strong enough, turbulence becomes strongly anisotropic and sometimes is identified as quasi-two-dimensional due to strong suppression of the vertical velocity component. However, the similarity to two-dimensional flow, generally, ends at the level of velocity field. The vertical velocity gradient (i.e., horizontal vorticity) may be quite large, as observed by Fincham et al. (1996) and Spedding et al. (1996) in the form *of a complex 3D network of structures in which layers of eddies cannot evolve independently of one another*. Herring and Metais (1989), in their numerical experiments, observed formation of small scales in the vertical and other attributes of three-dimensionality at pretty strong stratification. Among others the reasons for such essentially three-dimensional nature are believed to be associated with instabilities specific for strongly-stratified flows[24].

[23]Breaking of internal waves leads to irreversible mixing. In that it is different from wave breaking at an interface between two immiscible fluids such as surface water waves.

[24]It is noteworthy that Phillips already in 1972 asked the question: *turbulence in strongly-stratified turbulence – is it unstable?* and pointed to *possible instability.*

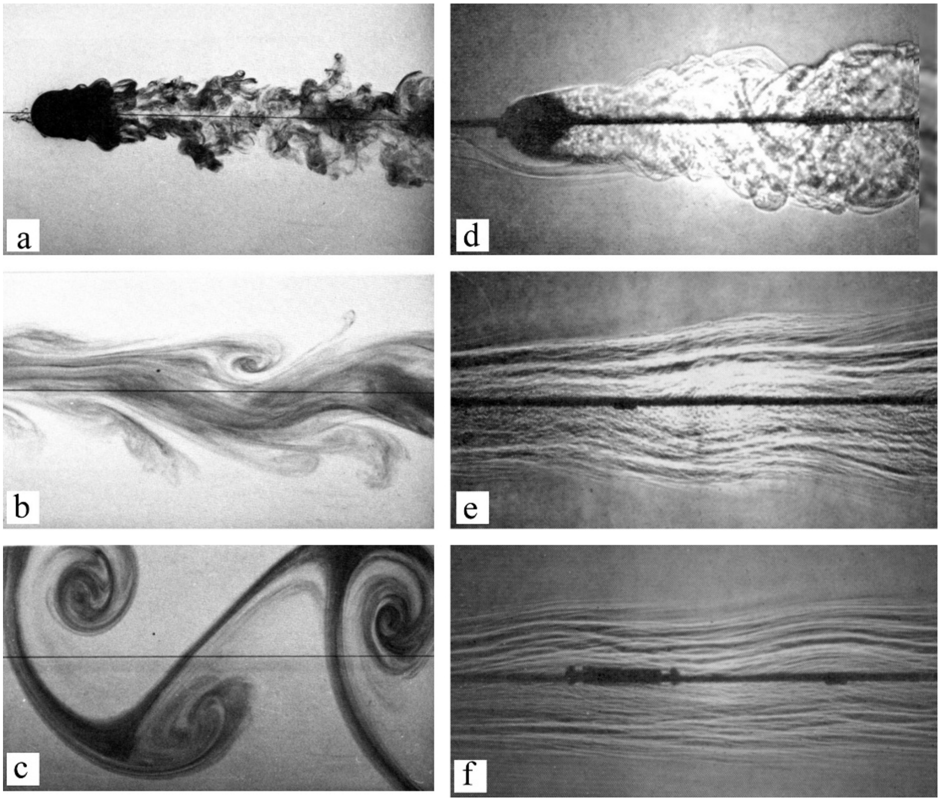

Figure 8.14. Turbulent wake past a sphere in stably-stratified (by salt) flow. Left – top view, Hopfinger (1997), right – side view Pao (1969). The distance from the sphere increases from top to bottom. It is seen that, farther from the sphere, the turbulence is suppressed (collapses) leaving striations in the vertical density structure due to incomplete mixing after multiple overturnings, and degenerates into a (quasi-) horizontal large-scale flow with vortices resembling those observed in a wake past a cylinder in a fluid with constant density. No internal wave radiation is discernible in these images, but see figure 4 in Spedding et al. (1996) in which the internal wave component is clearly seen, as it is 'separated' from the vortical one

A recent example is the so-called 'zigzag' instability (Billant and Chomaz, 2000). This instability was suspected to be one of generating mechanisms responsible for the multiple-layer phenomena observed in laboratory, field and numerical experiments in strongly- stratified flows as well as other 'unusual' phenomena such as 'pancake' eddies. In other words, due to the 'horizontal

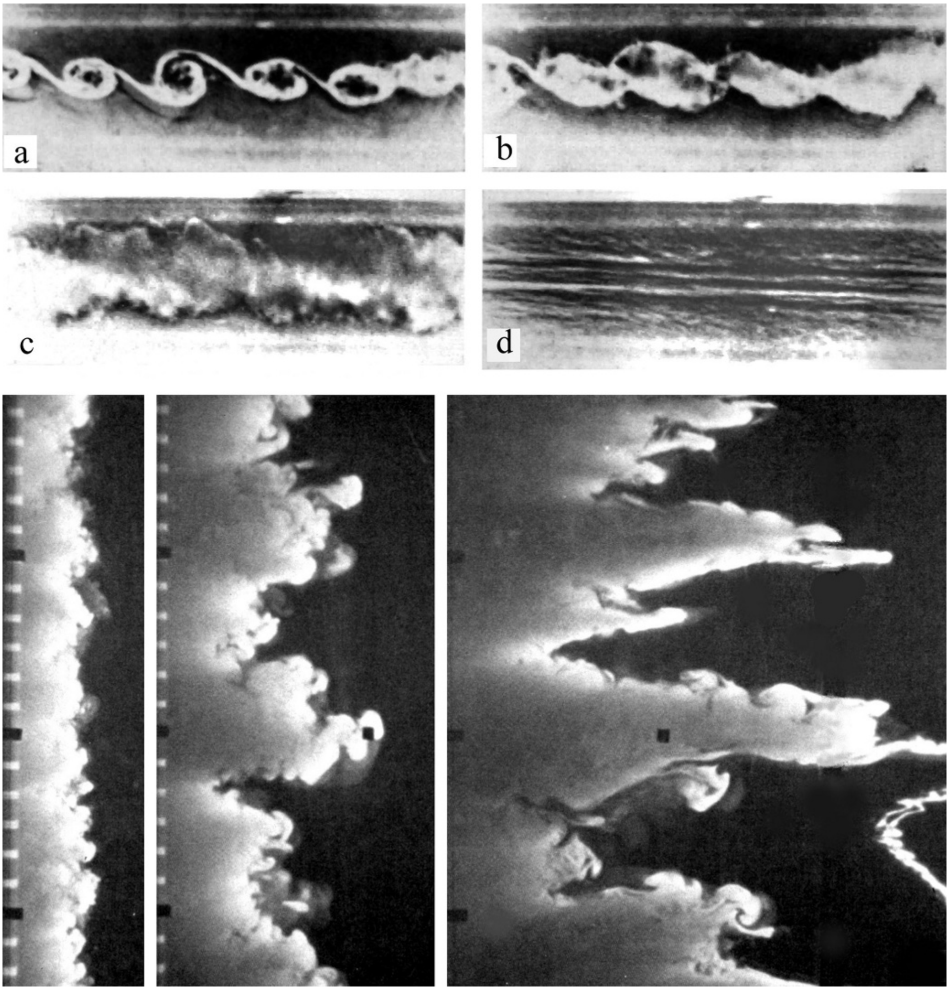

Figure 8.15. Turbulence suppression and collapse in a stably-stratified fluid. Top four figures – collapse of turbulence developed from Kelvin–Helmholz-type instability (Thorpe, 1971), time increases from left to right and from top to bottom (a $\rightarrow$ d). The three bottom figures show the time evolution (from left to right) of turbulence produced by a vertical (on the left of each frame) oscillating grid (Browand et al., 1987). One can see formation of turbulent intrusions due to limitation on the vertical scale of turbulent motions by the stratification and subsequent collapse of turbulence

freedom' and the internal waves mechanism, strongly-stratified fluids possess along with the 'intrinsic' stability caused by the restoring force, some specific instabilities leading to essentially three-dimensional structure of such flows especially in small scales. At even stronger stratification, active turbulence collapses into a state of nearly horizontal 'fossil' motion (figures 8.14 and 8.15). Later observations (see Brethouwer et al., 2007 and

Riley and Lindborg, 2008 and references therein) showed that strongly-stratified turbulent flows indeed can produce large gradients mainly in the vertical direction and maintain certain part of nonlinearities finite at no matter how strong the stratification. This and other related features are sometimes interpreted as a special kind of (anisotropic) downscale cascade (see Riley and Lindborg, 2008 and references therein), though as mentioned in chapter 5 the cascade/decomposition approach may appear of little help in understanding of the physics of turbulent flows. It seems that physical space would be more effective in this particular case.

Many complications and additional effects arise in the presence of boundaries (rigid or free) and other influences such as rotation.

8.4.3. COMPRESSIBLE FLOWS

Compressibility influences turbulence in several ways. For example, there are several mechanisms influencing the dissipation of turbulent energy. It appears that the so-called dilatational dissipation associated with compressibility is usually unimportant (see footnote after equation (C.19) in appendix C).

In shear flows, the dissipation is mainly reduced due to reduced level of turbulence production and not due to dilatational effects (Sarkar, 1995).

Compressibility can lead to an increase in dissipation as well. This happens in the presence of shock waves[25]. After such a wave vorticity increases, generally turbulence is amplified (Andreopulos et al., 2000), and most probably so does strain, so that the dissipation increases. As the Mach number of turbulence $M_t = u'/V_s$ increases (u' is the turbulence intensity and V_s is the speed of sound), eddy shocklets – shock waves associated with the turbulent motion – arise in the flow. They provide an additional mechanism for the dissipation of energy which can be important in astrophysical contexts.

Turbulence acts as an amplifier of almost any disturbance that it is subjected to, so no wonder that turbulence is a source of noise (see figure 8.16f). Finally, turbulence can be manipulated by acoustic excitation.

For review and references see Friedrich and Bertolotti (1997); Lele (1994, 1997); Andreopulos et al. (2000) and a special issue on aeroacoustics of *Theoretical and Computational Fluid Dynamics*, **6**, Nos. 5–6, October 1994, and Sagaut and Cambon (2008).

[25] But may also occur in subsonic flows (Briassulis et al., 2001).

8.5. Rotation

Rotating fluids have much in common with stably-stratified ones. They also can support waves called inertial waves[26] (see figure 8.16d,e), since the Coriolis force also acts as a restoring force, though somewhat differently than that of buoyancy[27]. Turbulence in the presence of rotation becomes anisotropic and tends to acquire in some sense a quasi-two-dimensional structure in the plane normal to the axis of rotation. Strong rotation in most cases leads to turbulence suppression and collapse. There are differences too. For example, strong rotation also reduces the gradients along the axis of rotation, so that turbulence acquires cigar-like anisotropy (in contrast with the pancake-like anisotropy of stably- stratified flows) and becomes close to two-dimensional at the level of velocity derivatives. The second difference involves generation of intense vortices in the presence of local forcing, for example at the boundaries (see figure 8.23b).

In the presence of rigid boundaries, shear and stratification lead to additional effects, resulting both in stabilizing and destabilizing influences. We mention here three examples. The first one is turbulent flow in a rotating pipe. Most observations show the stabilizing influence of rotation leading among other things to considerable drag reduction (see Orlandi, 1997 and references therein). On the other hand experiments by Nagib et al. (1971) clearly showed that under rotation the pipe flow becomes *less* stable, so that the critical Reynolds number drops from 2500 in the absence of rotation to 900 in the presence of rotation. The second example comes from the experiments of Ibbetson and Tritton (1975). Contrary to other observations, they observed *faster* decay of turbulence past a towed grid in the presence of rotation. One of the possible explanations is that the rotation was not strong enough, so that the additional dissipation in the so-called Eckmann layers on the walls normal to the axis of rotation won the competition with the effect of turbulence two-dimensionalization by rotation. The third example is provided by computations of Dritcshel et al. (1999). They show clearly that structure of rotating and stratified flows is intrinsically three-dimensional on small-to-intermediate scales, reflecting the competition between the pancake-like anisotropy of stratified flows with the cigar-like anisotropy of rotating flows. For review and references on turbulent rotating flows see Cambon (1994), Hopfinger (1989), Tritton (1985), Godeferd and Lollini (1999) and Sagaut and Cambon (2008). The examples shown in figure 8.16 do not exhaust the list of waves existing in fluids. For example, in addition to sound waves in compressible fluids, there

[26]The term is derived from Bjerknes et al. (1933).

[27]In the case of buoyancy, a fluid particle slightly displaced from its equilibrium state moves along a straight line, whereas in a rotating fluid it moves in a circle, since the Coriolis force acts in the direction normal to the direction of velocity.

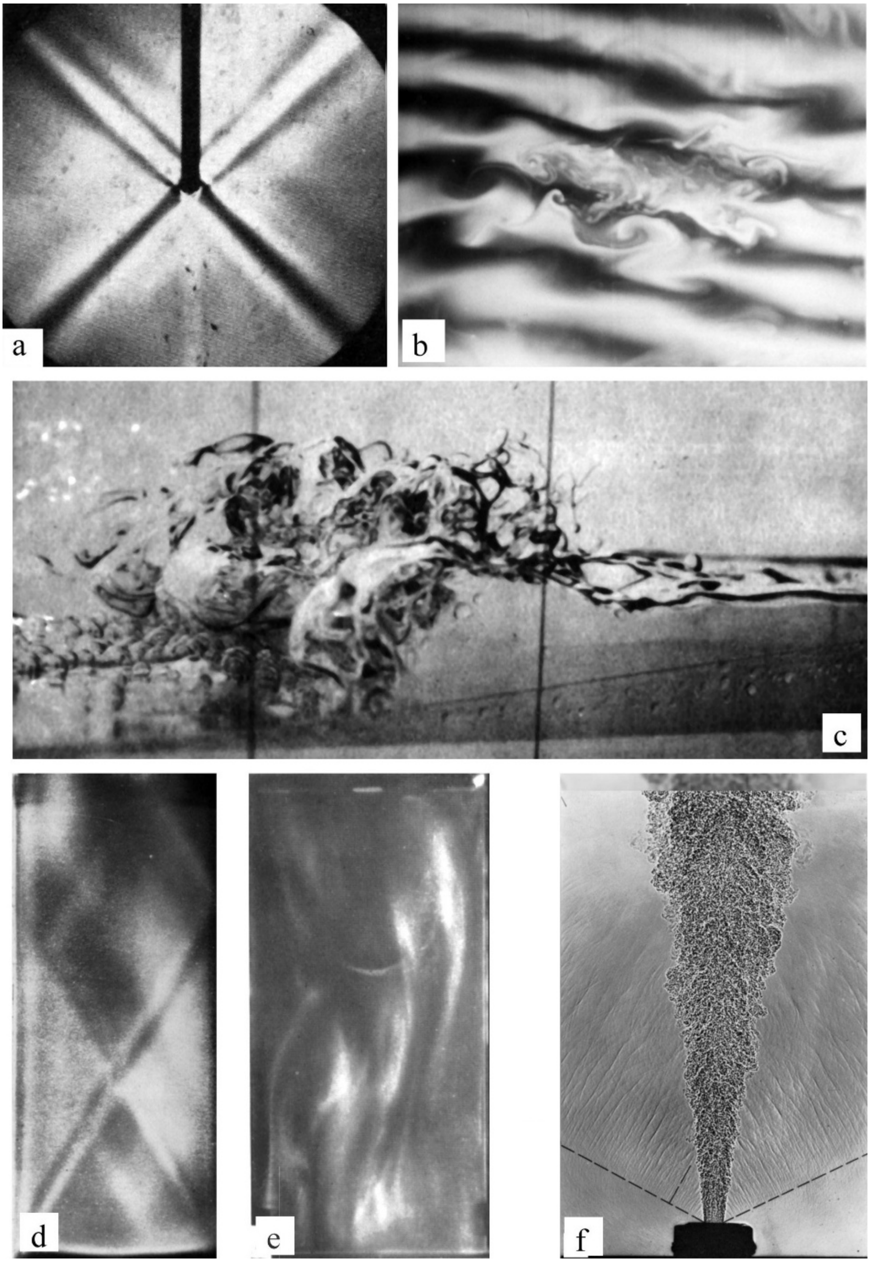

Figure 8.16. Turbulence and waves. a) – internal waves propagating from an oscillating body in a stratified fluid (Mowbray and Rarity, 1967). A similar pattern (d) is seen in a rotating fluid (inertial waves) which under certain conditions degenerates into state of disorder (e) due to the phenomenon of resonant collapse (McEwan, 1970). b) shows an internal wave breaking in a laboratory tank filled by a linearly-stratified fluid, courtesy of Dr. J. Sommeria. A similar phenomenon is shown in c) for the case when the density variation is confined in a relatively thin layer (Pao, 1969) which is analogous to the breaking waves on a water surface. f) shows a supersonic jet and sound waves radiated by turbulence in the jet (Tam, 1972)

are also shock waves. Apart from inertial waves in rotating fluids there exist Rossby waves arising in particular geometrical arrangements. A whole variety of waves exist also in flows of electrically conductive fluids in the presence of magnetic field (MHD flows).

8.5.1. HELICITY

Rotating fluids lack reflectional symmetry. As a consequence, the quantity called helicty[28], see section 2.2.1 in appendix C, is nonvanishing. Large helicity may lead to reduction of nonlinearity and consequently reduction of drag and dissipation. A recent example is the numerical study of turbulent flow in a rotating pipe by Orlandi (1997). He observed a clear positive correlation between increase in helicity and decrease in dissipation. More subtle issues include spontaneous breaking of reflectional symmetry and production of helicity 'out of nothing', and the role of helicity when its mean vanishes. For review and references on this issue and in general, on helicity in turbulent flows, see Droegemeier et al. (1993), Kholmyansky et al. (2001a), Kida and Takakoka (1994) and Moffatt and Tsinober (1992).

As noted in chapter 6, section, 6.6, nonzero mean helicity $\langle \mathbf{u} \cdot \omega \rangle$ is an indication of stronger coupling between large and small scales favouring creation of large-scale structures out of small-scale turbulence – a process which is frequently called 'inverse energy cascade'. A similar phenomenon is observed in the so-called turbulent dynamo when small-scale turbulent flows of an electrically conducting fluid are able to generate large-scale magnetic fields (Childress and Gilbert, 1995 and references therein). Such phenomena are akin to a class of fluid flows described in the next section.

8.6. Negative eddy viscosity phenomena

The term *negative eddy viscosity* is used to denote flow situations in which the turbulent transport of momentum occurs against the mean velocity gradient, i.e., from regions with low momentum to regions with high momentum, so that the Reynolds stress, $-\langle u_1 u_2 \rangle$, and the mean velocity gradient, dU/dx_2, are of *opposite sign*[29]. In other words, the role of the Reynolds stress as one of the agents of coupling the fluctuations with the mean flow

[28] The term 'helicity' in the fluid dynamic context was introduced by Betchov (1961) for the quantity $\epsilon_{ijk} \left\langle u_i(x) \frac{\partial u_j}{\partial x_k} \right\rangle \equiv \langle \mathbf{u} \cdot \omega \rangle = 6E(0)$, where $E(r)$ is the third scalar function defining the second-order correlation of homogeneous and 'semi-isotropic' turbulence, i.e., invariant under rotations but not reflectionally invariant: $\langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) \rangle = C(r)\delta_{ij} + S(r) r_i r_j + E(r)\epsilon_{ijk} r_k$.

[29] For the sake of simplicity, it is assumed here that the mean flow is one-dimensional, $\mathbf{U} = U\mathbf{i}$, and that the only nonvanishing gradient of the mean velocity is in the direction x_2.

is not necessarily one-directional, as in pure turbulent shear flows it may act also in the 'opposite' direction. Concomitantly, kinetic energy moves in the 'opposite' direction too – from fluctuations to the mean flow. It should be stressed that the above does not imply anything more than what is said, e.g., no eddy viscosity in the narrow sense or any other such thing.

Such a situation is possible in the presence of some energy supply other than the mean strain. Then the total rate of production/destruction of energy turbulent fluctuations by the mean strain, $\mathcal{P} = - \langle u_1 u_2 \rangle dU/dx_2$, does not have to be positive even in the case of statistically-stationary turbulent flows, since in such cases the balance is $\mathcal{W}_F + \mathcal{P} - \mathcal{D} = 0$. Such situations are observed both in laboratory experiments and in the large-scale flows of geo- and astrophysics. Selected examples and references are given below. Additional examples and references can be found in Maubach and Rehme (1972), Tsinober (1990b) and Paret and Tabeling (1998).

8.6.1. LABORATORY EXPERIMENTS

The popular example is the jet in which the turbulent energy production, $-\langle u_1 u_2 \rangle dU_1/dx_2 < 0$, is negative in the region between the locations in the flow cross section where $dU_1/dx_2 = 0$ and $-\langle u_1 u_2 \rangle = 0$. This region is rather narrow and the mean velocity gradient there is rather small. It was therefore argued that, though the mean velocity may gain energy locally, when integrated over the entire cross section, the effect will be a decrease in the total kinetic energy of the mean motion. Another example is the boundary layer at a convex wall. The effect of curvature on the outer part of the layer leads, in some cases, to a reversal of the turbulent stress.

In this connection, of special importance are experiments in highly coherently forced mixing layer (see Weisbrot and Wygnanski, 1988 and references therein). In these experiments, a whole region in the flow field was found to have turbulent shear opposing the mean velocity gradient over the entire flow cross section (see figure 8.17). It is noteworthy that, unlike other flows, this phenomenon is especially strong close to the middle of the mixing layer cross section, where the mean velocity gradient is maximal. Several simple kinematic explanations of the process were proposed in terms of orientation changes (tilting) of elliptically shaped vortices similar to the explanation given by Starr (1968) in a geophysical context (see also Busse, 1983). Though it is tempting to explain the above phenomenon in simple kinematic (two-dimensional) terms, or more generally in terms of (also dynamical) properties of two-dimensional turbulence (see section 8.7), it should be emphasized that the above phenomenon seems to be essentially three-dimensional. A clear indication of this is found in measurements by Oster and Wygnanski (1982). It follows from their results that, in the zone of opposing turbulent shear, the ratio $w/u \sim 0.4 \div 0.5$ $(v/u \approx 2)$, while for a regular (non-forced) mixing layers, it is $0.9 \div 1.0$ $(v/u \approx 1)$; here u, v, w – are the intensities of turbulent velocities correspondingly in the streamwise, lateral and spanwise directions. In other words, the forced mixing layer becomes *more anisotropic*, but still remains far from being even quasi-two-dimensional[30].

Krishnamurti and Howard (1981) discovered the so-called 'turbulent wind', i.e., a mean flow in experiments on turbulent convection in a horizontal layer of fluid heated from below and cooled from above. Namely, at high enough Rayleigh number, there exist a nonzero horizontally averaged velocity along the bottom in one direction and along the top in the opposite direction. The direction of this mean flow is random, as it would be in a symmetry breaking bifurcation. This, however, happens only when the flow is essentially turbulent and the direction of the momentum transport by the Reynolds stress is up and not down the gradient of this mean velocity, i.e., the Reynolds stress contributes to the maintenance of the mean flow. The main feature is that almost all plumes rising from the heated bottom were tilted away from the vertical in such a way as to aid the momentum transport to the mean flow. Concomitantly, the energy of the fluctuations is transferred to the mean flow. Later Krishnamurti and Howard (1983) performed a similar experiment in an annular cell in which they measured both the mean velocity and the Reynolds stress, which appeared to be of the sign opposite to that of the gradient of the mean velocity in most of the vertical positions from the heated bottom in conformity with the negative eddy viscosity. The opposite sign of the Reynolds stress to that of the mean velocity gradient is only a manifestation of the negative eddy viscosity phenomenon, but not an explanation of its existence[31].

The 'turbulent wind' was observed subsequently in other experiments (for references see Grossman and Lohse, 2009 and Siggia, 1994). For example, the tilted plumes observed in experiments by Ciliberto et al. (1996) are shown in figure 8.7d, whereas a number of statistical characteristics (including the mean horizontal velocity profiles) obtained by Burr et al. (2003) are shown in figure 8.18.

[30]The experiments by Hilberg and Fiedler, 1989 deserve special mention. They performed experiments on a mixing layer with lateral extent an order of magnitude larger that its spanwise width and observed that the coherence and energy contents of large-scale structures were highly above the values found in the normal shear layers flow. One can expect that, due to much higher anisotropy, a negative eddy viscosity phenomenon may exist in this kind of flow. Similar phenomena were observed in a shallow jet by Dracos et a1., 1992. We recall that in these flows (and many other partly-turbulent flows) large-scale instabilities not directly related to their turbulent flow nature may arise.

[31]Note that in the absence of a driving source, such as pressure gradient, the simplest form of the Reynolds equation is $\nu\ d^2U/dx_2^2 = d\langle u_1u_2\rangle/dx_2$, i.e., $\nu\ dU/dx = \langle u_1u_2\rangle + const$. This does not allow us to claim that the sign of $-\langle u_1u_2\rangle$ is opposite to that of dU/dx, since it is not a simple matter to find the *const*.

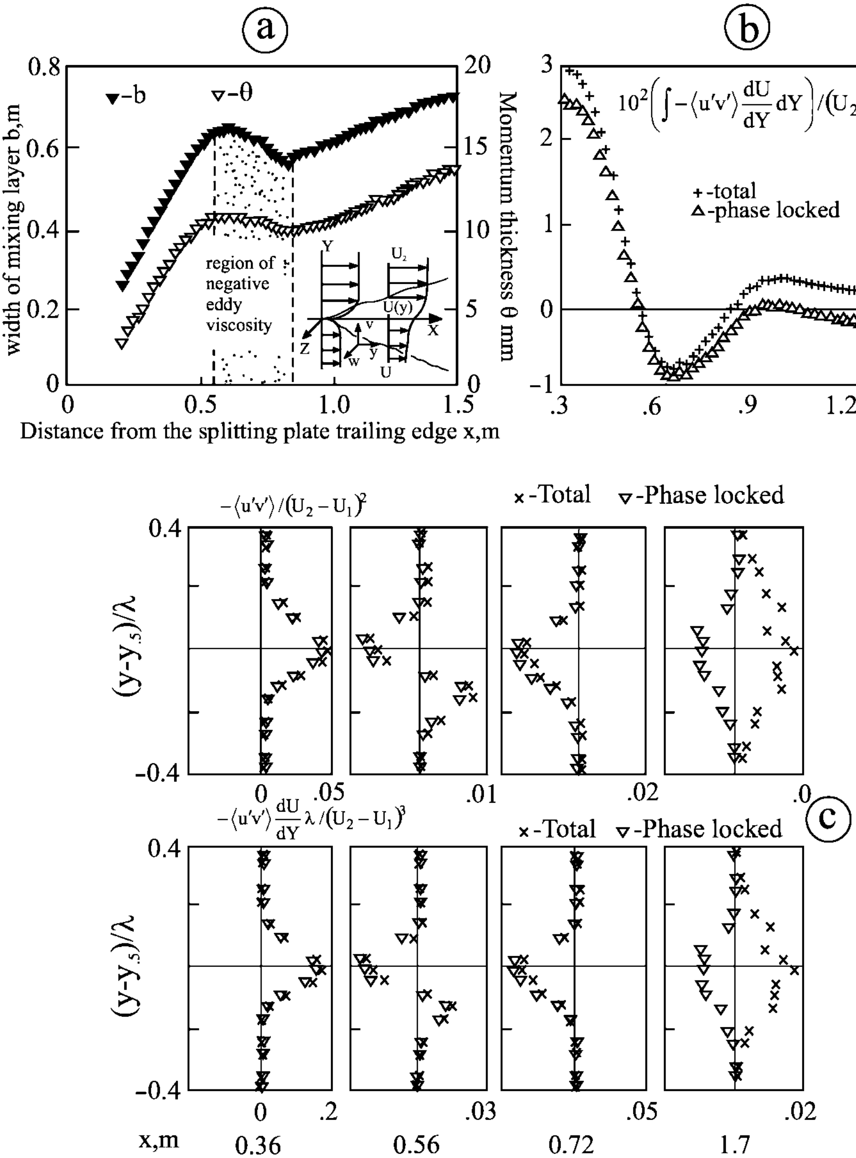

Figure 8.17. The negative eddy viscosity phenomenon in a forced mixing layer. a) – width and momentum thickness. b) – integrated turbulent energy production. Note the region $0.55 < x < 0.84m$ where the mixing layer becomes narrower and the Reynolds stress and the turbulent energy production reverse their signs. c) – profiles of Reynolds stresses and turbulent energy production at several streamwise locations. Adapted from Weisbrot and Wygnanski (1988). The figure is from Tsinober (1990b)

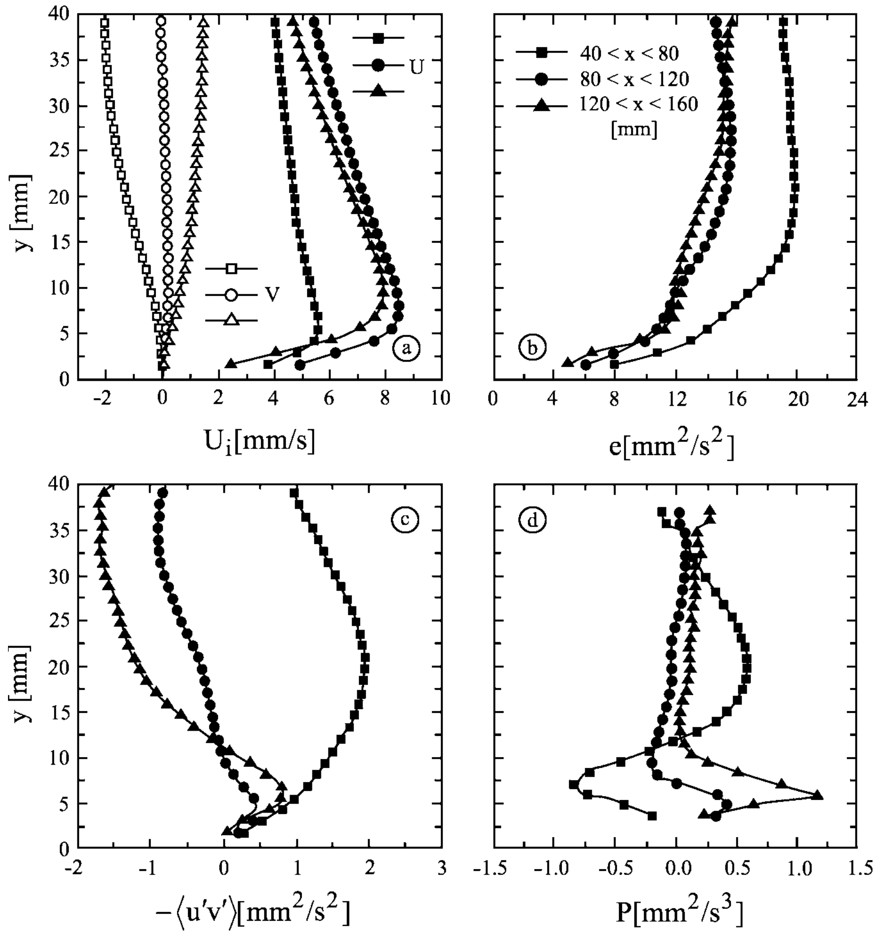

Figure 8.18. Distributions of flow quantities along the vertical direction y in the region adjacent to the bottom (heated wall). Averages are presented along the x-direction over three regions: the region $40\,mm \times 80\,mm$ where the mean flow is accelerating, the region $80\,mm \times 120\,mm$, where the vertical mean velocity is small and the region $120\,mm \times 160\,mm$, where the mean flow is decelerating. a) Mean velocity profiles U and V, b) turbulent kinetic energy $e = \langle(u^2 + v^2)\rangle$, c) Reynolds shear stresses $\langle uv \rangle$, d) turbulent energy production $P = -2\langle u_i u_j \rangle \partial U_i / \partial x_j; i, j = 1, 2$. Burr et al. (2003)

Systems in which small-scale flows are driven by electromagnetic forces in a particular way may develop a mean flow essentially in the same way (see references in Paret and Tabeling, 1998 and Tsinober, 1990b,c). Wei and Willmarth (1992) also observed negative Reynolds stresses and negative production of turbulent energy in the near-wall region in a turbulent channel flow with drag-reducing polymer injection (see section 8.9). Finally, we mention a related phenomenon in stably-stratified turbulent flows – the so-called PCG, persistent countergradient fluxes. The essence of PCG is the countergradient transport of momentum and active scalar. It is observed at large scales when stratification is strong, but in small scales it is present with weak stratification as well (see Gerz and Schumann, 1996; Komori and Nagata, 1996 and references therein).

8.6.2. EXAMPLES FROM GEOPHYSICS

There exists considerable evidence that a number of processes taking place in large-scale flows in geo- and astrophysics are associated with the phenomenon of the negative eddy viscosity. For example, the generation and maintenance of mean flows by large-scale fluctuations, such as jet flows in the atmosphere and in the ocean, as well as zonal circulation in the atmospheres of some planets of the solar system and of the sun (see Busse, 1983; Monin, 1987; Monin and Yaglom, 1996; Starr, 1968 and references therein). There are two main kinds of fluctuative motions feeding such mean flows. The first kind are the convective motions (similar to those producing the 'turbulent wind' in laboratory experiments), such as produced by the supply of solar energy to the Earths' atmosphere. The second kind are the so-called Rossby waves (see Tritton, 1988 for a simple explanation of what they are). Propagation and breaking of these waves are thought to be responsible for the generation of the large-scale mean motions mentioned above (Monin, 1987; McIntire, 1993)[32]. Along with turbulence production the wave propagation and breaking lead to transferring of momentum and energy to the mean flows. At least some of these phenomena are analogous to the so-called acoustic wind in which a mean flow is produced by acoustic waves (Lighthill, 1978). There is an essential difference too, since acoustic streaming is a (quasi-) linear phenomenon, whereas wave breaking, mostly responsible for the wave contribution to the negative eddy viscosity phenomenon, is essentially nonlinear. This makes it (almost) impossible to make a clear distinction between turbulence and waves – a difficulty noted already by Stewart in 1959 in the context of the issue of distinguishing between internal waves and turbulence in the atmospheric flows (see section 8.3.2).

8.6.3. POSSIBLE EXPLANATIONS

An explanation (at least qualitative) of the 'anomalous' negative eddy viscosity phenomena is usually given via properties and by analogy with two-dimensional turbulence (see section 8.7), which, under certain conditions,

[32]This idea dates back to the proposal by Rossby in 1947.

exhibit negative eddy viscosity and other 'anomalous' properties such as "inverse energy cascade". The first attempt of this kind was made by Lorenz (1953). He considered a two-dimensional (turbulent) flow consisting of mean and fluctuative components, both unsteady, and the interaction between the two. In the case when scales of the smaller motions of the mean flow and of the largest ones in the fluctuations overlap, the energy of the fluctuations can be transferred to the mean flow, i.e., the eddy viscosity becomes negative. For other two-dimensional examples, see Paret and Tabeling (1998) and Tsinober (1990b). The analogy, however, is qualitative only, and geophysical and other flows with negative eddy viscosity (and energy production reversal) at best can be considered as coexistence of quasi-two-dimensional structures (mostly in large scales) with more three-dimensional smaller scales. What seems to be certain is the fact that all negative eddy viscosity flow configurations are considerably anisotropic (more than 'normal' flows) due to some external influences (geometrical constraints, rotation, density stratification, magnetic field, etc.). There exist a number of theoretical models of three-dimensional flows in simple geometry (see references in Tsinober, 1990b). The common feature of these models is that the background small-scale motions should be in some sense *anisotropic* to be able to develop a large-scale instability. At present, it is not clear whether any of the existing approaches enable researchers to explain observations about negative eddy viscosity phenomena. In fact, no theoretical framework is available to describe the kinematical and dynamical features of these flows. *The more fundamental problem, to explain from first principles why there is a mean flow ... is beyond reach .... The effect is intrinsic and not understood* (Siggia, 1994; but see Malkus, 1996).

8.7. Magnetohydrodynamic flows

> *... it is extremely important to treat the deviations from hydrodynamics that are caused by the magnetic field* (Heisenberg, 1949).
>
> *A clearer understanding of turbulent hydromagnetic flows will result in greater insight into strictly hydrodynamic turbulence and into the mechanism of transition between laminar and turbulent flow regimes* (Harris, 1960).
>
> *Two-dimensional turbulence ... can occur in any uniform external magnetic field no matter how strong.... We may say that the two-dimensional flow "does not see" a uniform field. In a strong external field, the turbulence degenerates just into this two-dimensional form* (Landau and Lifshitz, 1984).

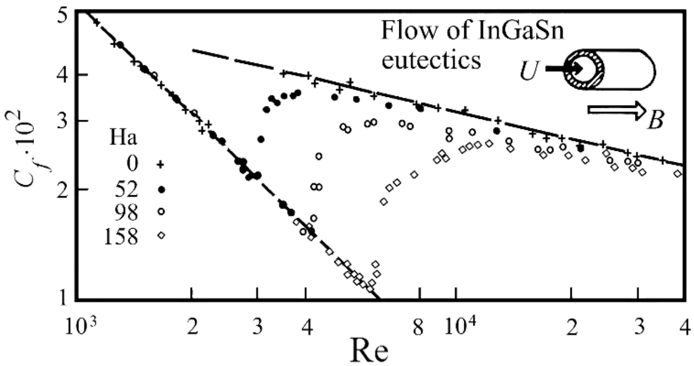

Figure 8.19. Drag reduction in a circular pipe by a longitudinal magnetic field, Krasilnikov et al. (1971). Here $\mathrm{Ha} = BL(\sigma/\rho\nu)^{1/2}$ is the non-dimensional parameter proportional to the intensity of the magnetic field B. The figure is from Tsinober (1990c)

An electrically conductive fluid flowing in the presence of an electromagnetic field experiences a (ponderomotive) body Lorenz force per unit mass $\mathbf{j} \times \mathbf{B}$, where $\mathbf{B}$ is the magnetic field induction (or simply magnetic field) and $\mathbf{j}$ is the electric current density induced by an externally applied electrical field and/or by the electrical field induced by the fluid motion in the presence of the magnetic field. The magnetic field is in turn changed by the electrical currents induced by the fluid flow and governed by an equation such as (C.36) without a forcing term. Concomitant to the interaction between the fluid flow and the electromagnetic field is the Joule dissipation, j^2/σ, σ is the fluid electrical conductivity, and corresponding losses. Nevertheless, the total losses (e.g., total drag and dissipation) can be smaller than in a flow without electromagnetic field by virtue of the dramatic changes in stability properties and/or structure of the fluid flow (figure 8.19).

There is a vast variety of possible configurations of MHD-flows even in the simplest geometries. We restrict ourselves in the following to examples in the situation when the changes of the magnetic field induced by the fluid flow are negligible (i.e., the case of small magnetic Reynolds number, $\mathrm{Re}_m = \sigma UL \ll 1)$[33] and when this magnetic field is homogeneous. For review and references, see Moreau et al. (2007) and Tsinober (1990c).

Our main purpose here is to illustrate some typical effects of turbulent flows arising in the presence of a magnetic field.

[33]The opposite situation, i.e., $\mathrm{Re}_m = \sigma UL \gg 1$, is typical in astrophysical contexts. In systems containing large amounts of liquid metal, such as fast reactors, Re_m can be of the order 10^2. It is many orders of magnitude larger in astrophysical objects. We address some specific aspects for finite Re_m in chapter 9.

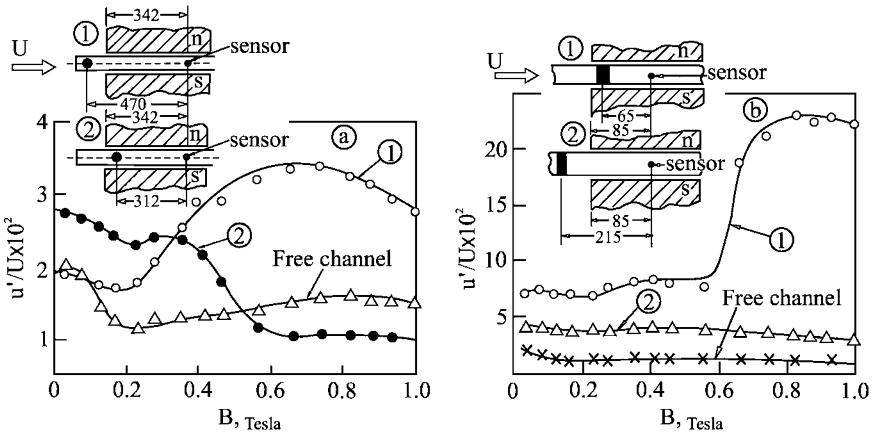

Figure 8.20. The effect of a magnetic field on the perturbations in the wake behind a cylinder. a) cylinder axis perpendicular to the magnetic field, b) cylinder axis parallel to the magnetic field (Kit et al., 1970)

One of the most prominent effects of a magnetic field on a turbulent flow of electrically conducting fluid is again anisotropy. Namely, in the presence of an externally imposed magnetic field, gradients of various flow properties in the direction of the magnetic field are reduced. Hence, there is a possibility of a quasi-two-dimensional flow when the magnetic field is strong enough, provided that boundaries (and boundary conditions) favour such a flow. Indeed, such flows have been observed in a number of cases (see references in Tsinober, 1990c; and Porthérat et al., 2000). As in some other quasi-two-dimensional flows, the MHD quasi-two-dimensional flows exhibit reduced dissipation, by analogy with purely two-dimensional flows which lack strain (i.e., dissipation) production. An example of such an effect is shown in figure 8.20. It indicates that in cases when the geometry of boundaries and the boundary conditions favour quasi-two-dimensional flows in the plane perpendicular to the magnetic field (the case of azimuthal magnetic field), i.e., when the axis of the cylinder is parallel to the magnetic field, the turbulence intensity in the wake of the cylinder is three times *larger* than in the absence of magnetic field. This happens because a quasi-two-dimensional turbulent flow is unable to dissipate the energy of the disturbances created in the proximity of the cylinder in the way in which a three-dimensional flow does in the absence of a magnetic field.

The above example is qualitatively different from many other MHD (and not only MHD) quasi-two-dimensional (Q2D) flows in which, e.g., boundaries normal to the direction of the magnetic field dominate the flow, such as the Hartmann flows. These flows, though Q2D in strong magnetic fields are inertialess and therefore can hardly be qualified as turbulent.

Another effect of quasi-two-dimensionalization of some MHD-flows is the anisotropic transport. Namely, in the presence of a magnetic field, one observes suppression of the transport of a passive scalar (e.g., indium dissolved in mercury) in the direction of the magnetic field and its enhancement in the plane perpendicular to the magnetic field (see figure 11b in Tsinober, 1990c). A related recent example is displayed in figure 8.21, which shows some results of convection of sodium-potassium alloy driven by a horizontal temperature gradient in the presence of a horizontal magnetic field. It is seen that a moderately strong magnetic field causes an increase in heat transport between the two vertical walls. The reduction of turbulent energy content in the small scales and its increase in the large scales is consistent with the tendency to quasi-two-dimensionalization. In larger magnetic fields, the flow is suppressed by the so-called Hartmann effect on the walls perpendicular to the magnetic field. In the absence of such a breaking effect, the enhancing influence of the magnetic field on the transport properties of the turbulent flow in the plane perpendicular to the magnetic field would be stronger.

There exist many turbulent MHD-flows with the influence of a magnetic field totally different from that shown in the above examples and that depends, e.g., on properties (geometry, electrical conductivity) of the flow boundaries and the nature of magnetic field (AC, DC, inhomogeneous).

8.8. Two-dimensional turbulence

> *Thus the non-ergodic conservation law which presumably blocks the development of turbulence in two dimensions is closely related to the conservation of vortex-points in two dimensions* (von Neumann, 1949).
>
> *This trend towards well-defined large-scale structures can make it questionable if the 2D flow should be described as 'turbulent' and it casts some doubts on the concept of inertial range and the relevance of energy spectra…. Random initial vorticity distribution quickly assumes a stringlike pattern, which persists as the flow simplifies into a few 'cyclones' or 'finite area vortex regions'* (Fornberg, 1977).

Two-dimensional turbulence (whatever this means) can be seen as one of the extreme anisotropic states of fluid flow. Though not strictly realizable (except in direct numerical simulations), two-dimensional turbulence is of interest for several reasons. First, it is believed that properties of pure two-dimensional turbulence *may* be useful in treating turbulence in a number of quasi-two-dimensional systems. Such flows include large-scale geophysical flows (Lindborg, 1999); flowing soap films (Rivera et al., 1998); some flows in rotating or/and stratified systems (Riley and Lelong, 2000),

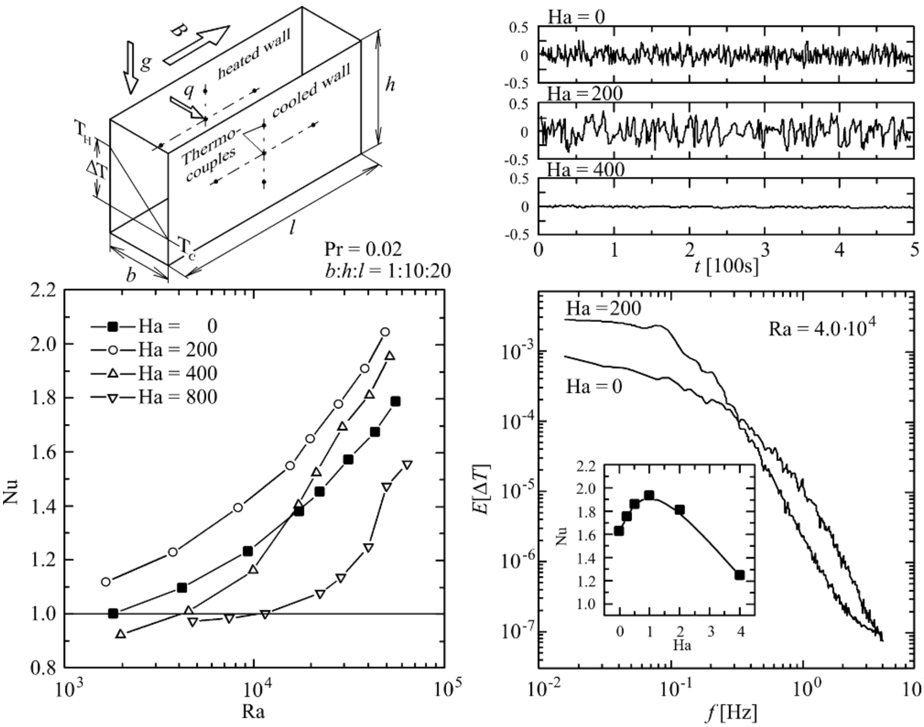

Figure 8.21. The effect of a horizontal magnetic field on convection of sodium-potassium alloy driven by a horizontal temperature gradient (Burr et al., 2000). Hartmann number, Ha, is defined as in figure 8.13

and some magnetohydrodynamic flows (Kraichnan and Montgomery, 1980; Porthérat et al., 2000). Second, pure two-dimensional turbulence is accessible, at least in part, by methods of statistical physics, contrary to the case of three-dimensional turbulence (Danilov and Gurarie, 2000; Kraichnan and Montgomery, 1980; Onsager, 1949; Pomeau, 1995; Sommeria, 2001 and references therein). Third, in two-dimensional turbulence, there exist a process which is to some extent analogous to vortex stretching in three-dimensional turbulent flows. Namely, this is the process of predominant stretching of the vorticity gradient $\zeta = \nabla\omega$ $(=\partial\omega/\partial x_i)$ or equivalently $\xi = rot\omega$ (see equation [C.22] in appendix C; Herring et al., 1974; Novikov, 1997; Weiss, 1991). For example, in numerical simulations of decaying two-dimensional turbulence the mean palinstrophy production $\langle\xi_i\xi_k s_{ik}\rangle$ becomes pretty large before starting to decay (see figure 8.22). The nonzero $\langle\xi_i\xi_k s_{ik}\rangle$ is a clear indication of non-Gaussian nature of two-dimensional turbulence, though at 'lower levels' (velocity, velocity increments for points at not too small distances), it is close to Gaussian (see Boffeta et al., 2000; Paret and Tabeling,

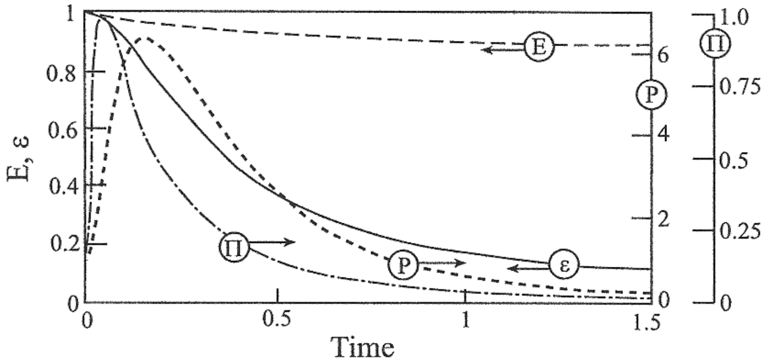

Figure 8.22. Time behaviour of total energy, $E = \frac{1}{2}\int u^2 dA$; enstrophy, $\mathcal{E} = \frac{1}{2}\int \omega^2 dA$; palinstrophy, $P = \frac{1}{2}\int \xi^2 dA$; and palinstrophy production, $\Pi = \int \xi_i \xi_k s_{ik} dA$, in 2D decaying turbulence from an initially random state with $\mathrm{Re}_\lambda = 1300$. Note the slow decay of energy and the increase of palinstrophy to six times its initial value before starting to decay. The palinstrophy production is normalized on $P\mathcal{E}^{1/2}$ at $t = 0$. The two-dimensional skewness $\Pi P^{-1}\mathcal{E}^{-1/2}$ reaches an approximately constant value after a relatively short transient time as in Herring et al. (1974). Courtesy of Dr. Y. Kimura

1998 and references therein). It is noteworthy that the predominant stretching of vorticity gradients and positiveness of palinstrophy production is a genuinely nonlinear (inviscid) process and not the consequence of the (approximate) balance between palinstrophy production and its viscous destruction.

Nevertheless, one of the main arguments against qualifying two-dimensional chaotic flows as turbulence is the absence of the vortex stretching process and strain production and generally of the process of self-amplification of the field of velocity derivatives. This qualitative difference leads to reduced ability of such flows to dissipate energy and a number of other important differences, see the next section. For example, in order to numerically simulate two-dimensional turbulence, it is common to do this by starting with *random* initial conditions for decaying turbulence and adding a *random* forcing in the right-hand side of the Navier–Stokes equation (and some dissipation at large scales too) when looking at/for a statistically stationary situation, just as in the case of the Burgers equation. In three-dimensional turbulence the results are not sensitive (at least qualitatively) to whether the forcing is random or deterministic or even time independent. This does not seem to be the case in two-dimensional turbulent flows. For instance, a two-dimensional flow in a plane channel driven by a constant pressure gradient or with fixed mean flow rate (Jimenez, 1990; Lomholt, 1996) exhibits a very different kind of chaotic behaviour than that which is observed in simulations with random forcing. Namely, the two-dimensional channel flow develops mostly large-scale and quite slow-in-time and organized-in-space variations, which hardly anybody will recognize as turbulent. This is a reflection of the fact that chaotic properties of two-dimensional flows are qualitatively different from those of three-dimensional turbulence.

The latter example is interesting also in the context of the similarities and differences between pure 2D and quasi-2D turbulent flows.

8.9. Pure two-dimensional versus quasi-two-dimensional

The common view was, until recently, that the essential aspects of quasi-two-dimensional turbulent flows (Q2D) can well be described by pure two-dimensional ones (P2D), both globally and locally, i.e., that ε in $Q2D = P2D + \varepsilon$ is small in some sense. Is it really the case that ε is always small? Not too long ago this question would seem rather strange since at least in some flows the factors causing the flow to become Q2D were considered just as stabilizing ('against' 3D-instability), e.g., such as strong stratification, rotation and magnetic field effects. There is a long history and long list of papers with claims that strongly anisotropic turbulence considered as quasi-2D is close to pure 2D turbulence (if such exists with its impotence without production of vorticity and strain and consequently weak dissipation); attempts to explain some properties of the former by invoking those of the latter; and even profound analogies between different Q2D states (stratified, rotating, MHD). Though anisotropy is extremely diverse, such claims and attempts were (and are being) made with respect to turbulence in strongly- (stably-)stratified environments, rotating frames, thin domains (e.g., in geophysical turbulence), MHD and even strong shears.

One of the popular beliefs is that locally this is true of Q2D regions with concentrated vorticity (vortex filaments). However, it appears that locally quasi-two-dimensional regions corresponding to large $\cos(\omega, \lambda_2)$ (see section 6.5.2), to which belong the regions of concentrated vorticity, are qualitatively different from purely two-dimensional ones, in that they possess essentially nonvanishing enstrophy generation $\omega_i \omega_j s_{ij}$ and intermediate eigenvalue Λ_2 of the rate of strain tensor, which are identically zero in P2D flows. Moreover, in these regions both $\omega_i \omega_j s_{ij}$ and Λ_2 are *larger* than in the whole field and in this sense ε is not small in $Q2D = P2D + \varepsilon$.

Another example relates to the behaviour of the wall-bounded turbulent shear flows in the wall proximity, namely, *that flow in the viscous sublayer is almost two-dimensional and two-component over fairly long periods of time* (Fischer et al., 2001 and references therein). However, this seems to be true (at best) of the level of the velocity field. The field of velocity derivatives and related quantities remains far from being quasi-two-dimensional even in the closest proximity of the wall (Tsinober et al., 1995).

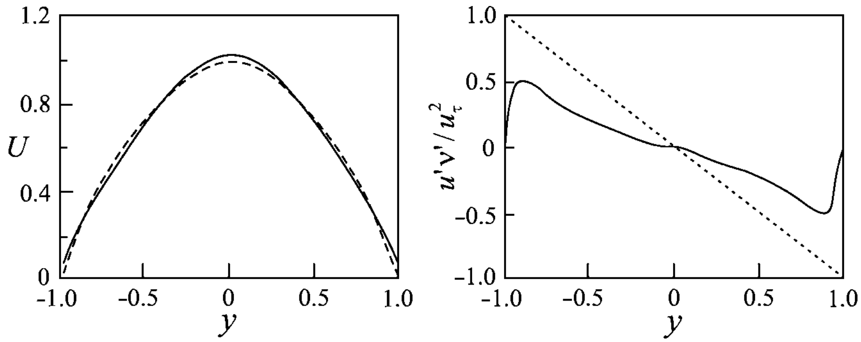

Figure 8.23. A two-dimensional 'turbulent' flow in a channel. Left – mean velocity, the continuous line corresponds to the actual flow, the dashed line is the Poiseulle parabolic velocity profile; note that, along with similarity of the two velocity profiles, there is a large velocity gradient at the wall. Right – the Reynolds stress (continuous line) and the total stress (dashed line). The figure is from Lomholt (1996). These results are practically the same as those obtained by Jimenez (1990)

In the case of globally Q2D turbulent flows, the matters seem to be even controversial. For example, the experimental (and recent numerical) results obtained for turbulent MHD flows in channels with large aspect ratio in the presence of an azimuthal magnetic field showed that, in such flows, at Re $\leq 10^4$ (which are Q2D) on the one hand, the drag is indistinguishable from its laminar value, and on the other hand, the level of turbulence may be substantially higher than that in the same flow without a magnetic field (see e.g., the review in Tsinober, 1990c). However, examination of the results of Jimenez (1990) concerning the DNS of NSE of a plane Poiseulle turbulent flow (which is P2D) at Re $\sim 10^4$ shows that its drag is about twice as large as the purely laminar value and is only twice as small as its value for the 3D turbulent flow. In other words, the P2D plane Poiseulle turbulent flow is not that low dissipative. Moreover, the Reynolds stresses in this flow are not small either (as had been expected) and contribute about a half to the total stress (see figure 8.23)[34].

Thus the problem of the relation(s) between Q2D and P2D turbulent flows is complicated further by the multiplicity of Q2D states: there exist several Q2D flows such as flows in rotating frames, flows with stable density stratification, MHD-flows and some others, which along with being

[34]It is noteworthy that since the common belief was that *any* two-dimensional turbulent flow should be low dissipative along with the results on really low dissipative nature of Q2D flows in MHD channels with spanwise magnetic field, the author of this book tended to think that the results by Jimenez (1990) were erroneous. It was, however, a calculation of the same flow undertaken by Sune Lomholt (1996) using an essentially different code that confirmed unequivocally the results of Jimenez (1990).

kinematically/geometrically similar are in many respects dynamically very different (see figure 8.23). As mentioned, Q2D MHD turbulent flows created in different conditions may be essentially qualitatively different, e.g., inertialess (i.e., not turbulent) and nonlinear and both different from pure two-dimensional flows. Another example is related to the MHD convection flow shown in figure 8.19. It appears that the P2D analogue to this flow studied numerically by Burr (1999, private communication) exhibits totally different behaviour from that of Q2D (see also Dolzhansky, 1999). Nevertheless, the importance of the two-dimensional configuration in MHD flows is believed to be in that it is a state to which tends any MHD flow in *some sense* depending on the specific configuration due to development of anisotropy (local and/or global) in such flows. The above mentioned belief is also extended to the MHD-dynamo problem. Namely, the common assumption is that in the saturated regime the flow field becomes quasi-two-dimensional *locally* with respect to the magnetic field (Schekochihin et al., 2004 and references therein). However, recent computations (Iskakov et al., 2009) showed that this effect is pretty mild. Instead there is a restructuring and mutual reorientation of both the fluid flow and the magnetic field in such a way that, in the saturated regime, the production of a magnetic field is balanced by Joule dissipation. This is different from strong reduction of the production of magnetic field $B_iB_is_{ij}$, which remains of the same order as in the kinematic regime just like the Joule dissipation. The flow field remains essentially three-dimensional becoming "just a bit more anisotropic" than in the kinematic regime locally with respect to the magnetic field[35]. For example, $b_ib_j\partial u_i/\partial x_j(\equiv B_iB_is_{ij}/B^2)$ (which is the gradient of the component of the velocity vector along the direction of magnetic field) is slightly reduced, but remains significant. However, the main attributes of three-dimensional turbulent flows are observed in the saturated regime. These include the PDFs of the eigenvalues of the rate of strain tensor, production of enstrophy and strain, the $R-Q$ plot and many others. Their statistics do not change much both *locally* with respect to the magnetic field and in a *fixed* frame.

An example of the difference in the *dynamical* nature as contrasted to *kinematical* similarity is exhibited in the above-mentioned zigzag instability in strongly stably-stratified flows (Billant and Chomaz, 2000). Recent developments in the study of strongly-stratified flows clearly show that they are far from being low dissipative, see Brethouwer et al., 2007; Riley and Lindborg, 2008 and references therein.

[35]An isotropic turbulent flow is locally anisotropic with respect to the magnetic field also in the kinematic regime due to special orientation of magnetic field as to enable the dynamo.

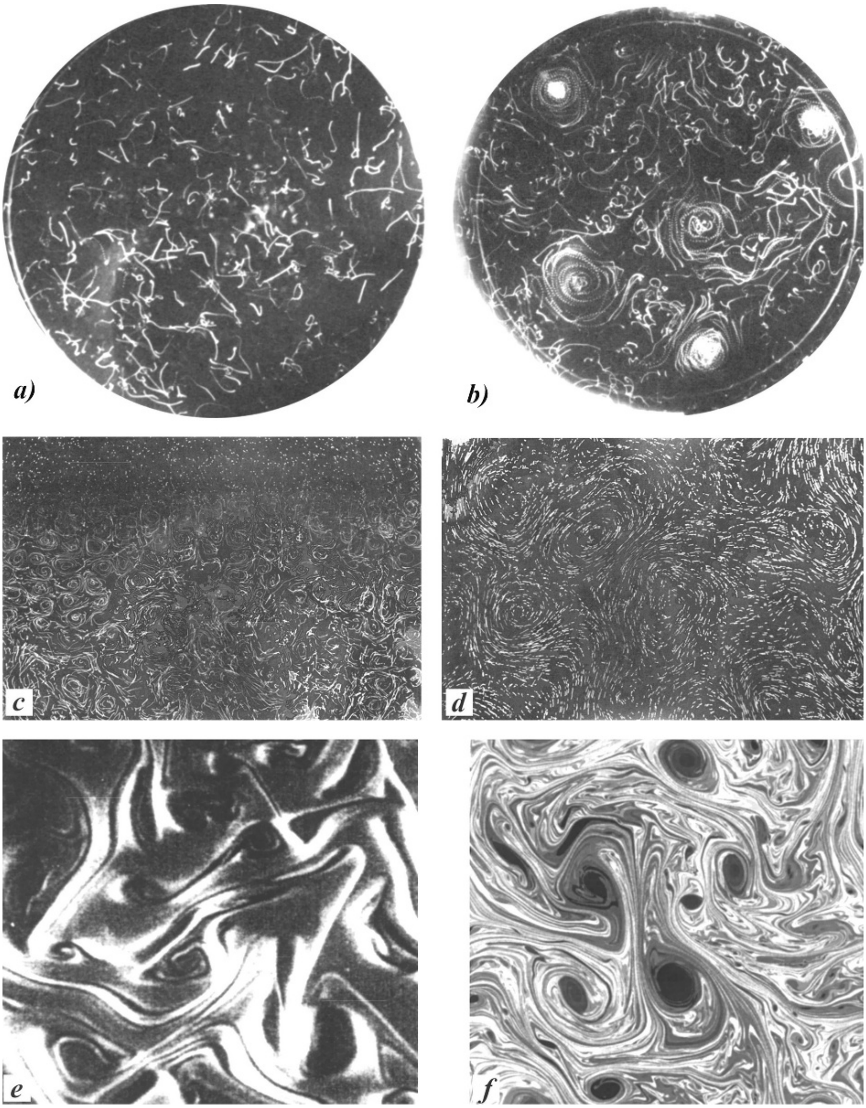

Figure 8.24. Two-dimensional and quasi-two-dimensional turbulence. a) three-dimensional and b) quasi-two-dimensional turbulence induced by excitation at the bottom, in b) the tank is rotating (McEwan, 1976); c) and d) quasi-two-dimensional turbulence (as seen from the top) past a grid towed in a stably-stratified fluid (Maxworthy et al., 1987). e) quasi-two-dimensional turbulence in a soap film (Rivera et al., 1998); f) numerically-simulated two-dimensional turbulence, courtesy of Dr. B.L. Hua

There is little doubt about the qualitative difference between Q2D states produced by physically different processes, e.g., the ones in MHD are of dissipative nature (Joule dissipation), whereas those with rotation are not. Strong anisotropy is a necessary condition only for Q2D and/or low dissipative behaviour, e.g., shear turbulent flows with strong shear are both strongly anisotropic and strongly dissipative. Similarly, strong correlations along some direction, i.e., Q2D behaviour, do not exclude the possibility of vorticity stretching in this direction. There is one more really profound difference between Q2D flows with rotation, stable stratification, rotation and P2D as the former possess an additional mechanism, the ability of sustaining waves, whereas the latter do not know anything of this kind of process.

A final remark concerns the limiting behaviour of Q2D flows which remain *turbulent* as some parameter grows without limit. We recall that one of the most popular and frustrating questions in 3D-turbulence is what happens when the Reynolds number is increasing. The question about the behaviour of ϵ in Quasi-2D = Pure 2D + ϵ as some parameter (associated with stratification, rotation, magnetic field and appropriately normalized) is increasing is in a way similar, e.g., in that here too one may expect phenomena like the "dissipative anomaly" in 3D turbulence so that our ϵ above may remain finite in the limit, at least in some cases.

8.9.1. SOME ADDITIONAL DIFFERENCES BETWEEN TWO-DIMENSIONAL AND THREE-DIMENSIONAL TURBULENCE

Apart from the above-mentioned differences between three-dimensional and two-dimensional turbulence, there are some additional ones which deserve special mention.

There is a positive net production of s^2 in the 3D case, whereas it is conserved (inviscidly) in the 2D case. Note that s^2 is not a pointwise Lagrangian invariant, as are vorticity and enstrophy in two-dimensional flows.

In 3D turbulence, vorticity, ω_i, and strain, s_{ij}, are equal partners, both are self-amplified, 'live' on the scales of the same order, and are related by a conjugation symmetric relation (Ohkitani, 1994). In 2D, it is $\xi = rot\omega$, or equivalently vorticity gradient, $\zeta = \nabla\omega$ $(=\partial\omega/\partial x_i)$ that is amplified via interaction with strain, so that the partners are not equal anymore: the strain is not amplified and the characteristic scales of ξ (and/or ζ) are much smaller than those of s_{ij}, which can be interpreted as a kind of scale separation. It is this scale separation and the absence of amplification of strain that makes the 2D problem more regular than in 3D, i.e., in 2D the nonlinearity is 'less nonlinear'. Indeed, it is known that the solution to the 2D Navier–Stokes equations with smooth initial and boundary conditions at any Re is smooth for all times, i.e., does not have any singularities (Doering and Gibbon, 1995). This is, of course, true of vorticity gradients. As mentioned, the reason for this is that the behaviour of strain (which plays a crucial role in amplification of vorticity gradients) is different in the 2D case than in the 3D case. Namely, in the 2D case the strain is an inviscid invariant, whereas in the 3D case it is not due to the predominance of its production due to term $-s_{ij}s_{jk}s_{ki}$ in the equation (C.18). This seems to be also the reason for some similarity in the behaviour of vorticity and passive scalar in two-dimensional turbulent flows along with some differences (Lapeyre et al., 2001 and references therein). The main reasons for such differences are that, just as in the three-dimensional case, the whole flow field is defined by vorticity with appropriate boundary conditions on velocity, and that the equation for a passive scalar is linear, whereas vorticity dynamics is governed by a nonlinear equation. Along with the absence of the self-amplification of velocity derivatives, this nonlinearity may lead to formation of large-scale structure(s) ('vortices') out of small-scale one(s) – a process which is usually called inverse cascade of energy (Bracco et al., 2000; Paret and Tabeling, 1998; Sommeria, 2001 and references therein). In this respect, the behaviour of the passive scalar is qualitatively different from that of vorticity; instead of 'vortices', sharp fronts are formed in the field of a passive scalar (e.g., Celani, 2001).

8.10. Additives[36]

Turbulent flows can be strongly modified by additives in even extremely small concentrations. The most spectacular changes occur with only a few parts per million of flexible polymers added to the solvent (see references in Cadot et al., 1998; Gyr and Bewersdodff, 1995; McComb, 1990 and Sreenivasan and White, 2000). These changes are exhibited in a number of flow parameters both large-scale and small-scale, though the direct action of the dissolved polymers is obviously in the small scales. The large-scale manifestations are represented in the first place by strong reduction of drag (up to 80%) in turbulent shear flows[37]. An example for smooth and rough pipes is shown in figure 8.25. Apart from drag reduction, the figure displays the phenomenon of maximum drag reduction or the maximum drag reduction asymptote (MDR). The essence of this phenomenon is that one cannot achieve drag reduction beyond MDR either by increasing the concentration of polymer or switching to another polymer. In other words, there is

[36]We address here only experimental aspects. The main reason is that there are no equations describing the flow of polymer solutions as reliable as do NSE for Newtonian fluids.

[37]There is evidence that the concentration of even 0.5 wt ppm of polyethileneoxide in water can reduce the drag up to 40% (McComb, 1990). There are polymers of extremely high molecular weight which lead to the same effect with only 0.05 wt ppm (Bewersdorff et al., 1993; Gyr and Bewersdorff, 1995). Along with turbulent momentum, transport of other quantities (heat, mass) is inhibited as well.

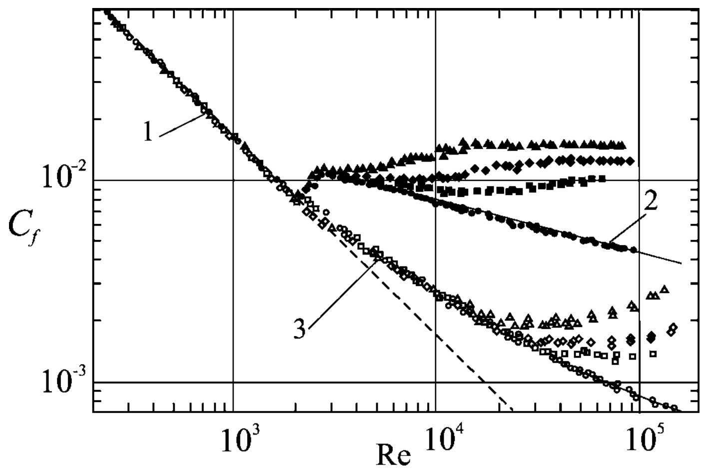

Figure 8.25. Friction factor for maximum drag reduction in smooth and rough pipes. Solid points refer to solvent, hollow points to polymer solutions yielding maximum drag reduction. (1) – laminar, Poiseuille's law, (2) – turbulent for smooth pipes, (3) – 'maximum' drag asymptote; Values of R/k – relative inverse roughness: △ – 14.6, ◇ – 22.8, □ – 35, circle – smooth (Virk, 1971). The dashed line is a continuation of (1) and it was added to indicate the difference between the minimal achievable drag with that of purely laminar flow. Note that the drag reducing effect is considerably diminished at large Reynolds numbers when the wall roughness is large enough. This effect was brought to an extreme by Cadot et al. (1998) in experiments with the facility schematically shown in figure 5.1: in the presence of baffles at the top and bottom of the rotating disks the polymer has no effect on the drag at all. This was also observed by Liberzon et al. (2005, 2006)

a saturation of drag reduction effect, which is limiting the drag reduction leaving the drag considerably larger than its purely laminar counterpart.

Another phenomenon is the threshold effect. Namely, the drag in flows of polymer solution follows the normal behaviour of the solvent until the deviation starts at some Reynolds number beyond which the drag reduction occurs. Sometimes this is thought to be connected with a threshold in the wall shear stress.

Along with global effects there are effects on turbulence structure. The first effect is directly related to drag reduction – it is a strong decrease of the Reynolds stresses (and turbulence production) as can be seen from an example given in figure 8.26.

However, the strong suppression of the Reynolds stresses occurs *without* substantial reduction of the energy of turbulent fluctuations. The suppression of the Reynolds stresses is due to decorrelation of the streamwise ($u \equiv u_1$) and wall-normal ($v \equiv u_2$) components of the velocity fluctuations (see figure 8.27).

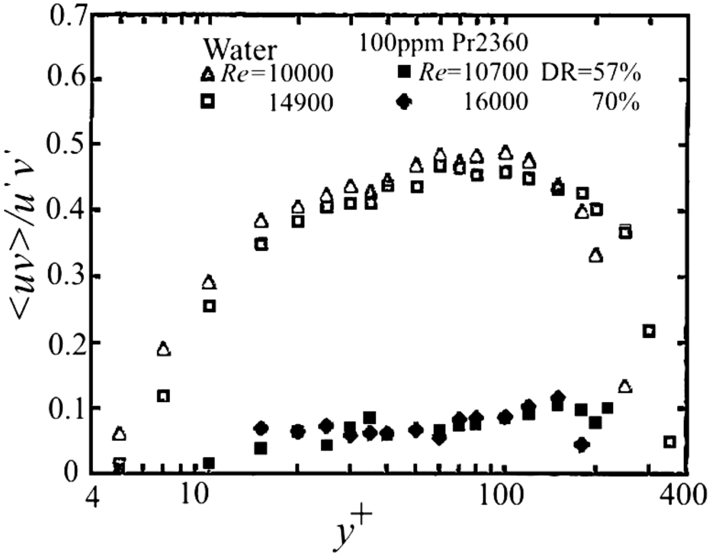

Figure 8.26. An example of suppression of the Reynolds stresses in a turbulent channel flow (Gampert and Yong, 1990; see also Warholic et al., 1999)

The turbulent energy in flows of drag reducing solutions may be somewhat smaller than in normal flows, but also may be increased (see references in Tsinober, 1990b). This is not in contradiction with substantial reduction of turbulent energy production, since concomitantly the dissipation is also strongly reduced as well. An important effect is the increased anisotropy: the wall-normal velocity fluctuations are considerably suppressed (McComb, 1990; Tsinober, 1990b; Tong et al., 1990; see also figure 8.27). Anisotropy was also observed in grid generated turbulence (Hibberd and Dohmann, 1988; Doorn et al., 1999; see also the references mentioned above). However, this latter anisotropy seems to be mostly associated with the effects in the process of turbulence production on and in the closest proximity of the grid. Some authors, the latest example being in Doorn et al. (1999), observe a reduced rate of decay of the grid generated turbulence[38], while others do not (e.g., Hibberd and Dohmann, 1988).

Visual observations by Cadot et al. (1998) show that polymers have an effect on the structure of the flow both when there is drag reduction and when such a reduction does not occur in the case of inertial forcing by baffles (see figures 8.25 and 5.1, see also Liberzon et al., 2005, 2006). In

[38]One of the possible reasons for the slower decay is the initial anisotropy. For example, a cigar-like turbulence was created past a honeycomb installed after a conventional grid (Hidenaru et al., 1988). The streamwise velocity component was considerably larger than the two other components. The rate of decay of this turbulence was observed to be substantially slower than that for quasi-isotropic turbulence.

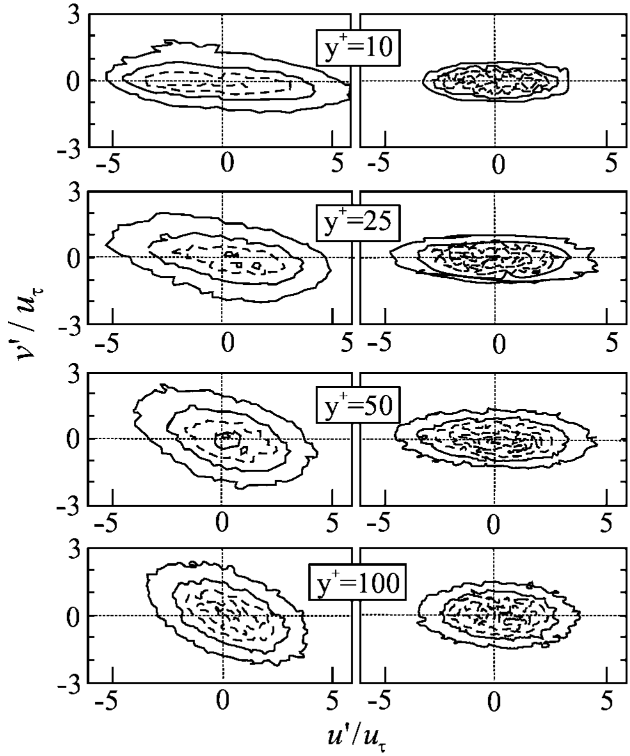

Figure 8.27. Joint PDFs of the streamwise and wall-normal velocity fluctuations in a turbulent channel flow. Left column – water; right column – polymer solution (Gampert and Yong, 1990)

both cases, the low pressure filaments (vortices) are smaller in numbers and larger in scale. The absence of drag reduction in the case of inertial forcing by baffles – as observed by Liberzon et al. (2005, 2006) – is consistent with rather small difference in the behavior of TKE production in this case. Namely, the difference in the TKE production between the pure water and polymer solution flows is much smaller in this case than in flow with smooth walls, see figure 8.28.

Cadot et al. (1998) also report changes on larger scales. This latter effect was observed in various forms in previous studies. An example is shown in figure 8.29 (see also figure 14.1 in McComb, 1990). The results in figure 8.28 show that changes on larger scales depend strongly on boundary conditions. Spectral and correlation measurements show that the energy content of turbulent flows is shifted to larger scales; the small scales carry much less energy than without polymers.

Thus, there are clear indications that the polymer drag reduction is not associated with suppression of turbulence, but with qualitative changes of

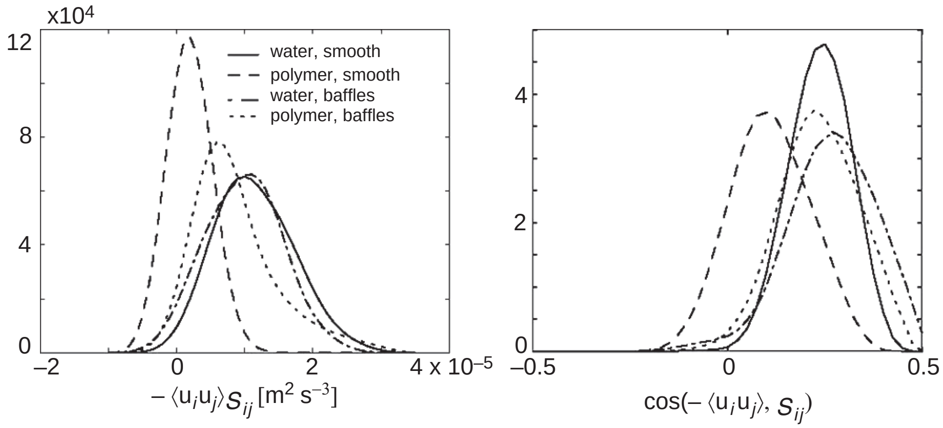

Figure 8.28. Left – PDFs of the turbulent kinetic energy production, P, average values are 11.5, 1.9, 9.8, and $8.6 \cdot 10^{-6} m^2 s^{-2}$, respectively. Right – PDFs of the cosine of the angle between the Reynolds and mean strain tensors. Average values are 24, 0.11, 0.26, and 0.23, respectively. Liberzon et al. (2006)

some of its structure and production. In other words, there exist turbulent flows with strongly reduced drag and consequently dissipation. This implies that in such turbulent flows the nonlinearities should be strongly reduced. Suppression of the Reynolds stresses is an effect of this kind. One can expect that such a reduction of nonlinear processes should occur also at the level of velocity derivatives, i.e., the process of (self-) production of velocity derivatives (vorticity and strain) should be suppressed in flows of drag reducing polymers.

A direct indication that this is the case was obtained by Gyr and Tsinober (1996). They compared the quantity $-\langle(\partial u/\partial x)^3\rangle$ in turbulent flows of polymer and surfactant solutions and of water in a pipe of square cross section. We are reminded that in isotropic turbulent flows $\langle\omega_i\omega_j s_{ij}\rangle = -\frac{4}{3}\langle s_{ij}s_{ik}s_{ki}\rangle = -\frac{35}{4}\langle(\partial u_1/\partial x_1)^3\rangle$, so that the latter can be used as a 'surrogate' of the enstrophy and strain production. The main result is that $-\langle(\partial u/\partial x)^3\rangle$ is an order of magnitude smaller in turbulent flows of polymer and surfactant solutions both in the flow bulk and in the near-wall region, thus indicating that enstrophy and strain production in drag reducing flows is strongly inhibited. As expected, a similar behaviour is observed for the 'surrogate' of the dissipation $\langle(\partial u/\partial x)^2\rangle$ in the near-wall region, but not in the bulk. This is in conformity with the view that the major contribution to drag reduction process comes from the near-wall region as was clearly shown in the experiments by Cadot et al. (1998).

There exist several attempts to explain the phenomenon in flows of drag reducing additives (see references in the above citations). In spite of

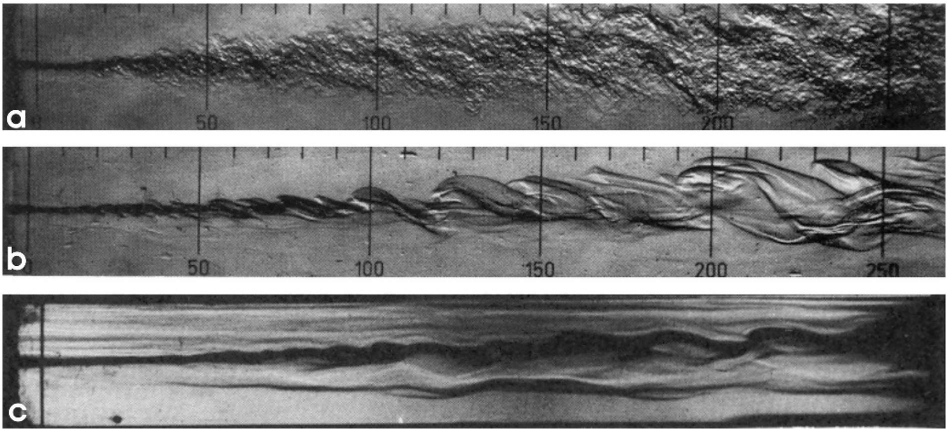

Figure 8.29. Schlieren images of mixing layer for a) water, b) 50 ppm Polyacrilamide, c) 900 ppm $C^{14)}TASal$ surfactant (Riediger, 1989)

considerable efforts, the physical mechanisms underlying the phenomenon remain poorly understood[39]. Perhaps, the common feature of all speculations is the belief that the effects observed are directly associated with extension of polymeric coils. This extension is caused by the field of strain, mostly by its fluctuative part. The existence of the threshold effect leads to a conjecture that only strain above some level is able to stretch the polymer coils. An important requirement is that the field of strain should be complicated enough, e.g., random in some sense, to be able to stretch the polymers effectively (see Chertkov, 2000; Groisman and Steinberg, 2000 and references therein). The reaction back of the polymer coils then is expected to change the field of strain in such a way (nobody seems to know/understand how precisely) as to cause the observed effects.

Most probably, the presence of polymer molecules resists large strain. Indeed, the PTV techniques allowed observation of this effect in the form of reduction of the rate of stretching of material lines, $l_i l_j s_{ij}/l^2$ in the presence of polymers, figure 8.30 by direct measurements of $l_i l_j s_{ij}/l^2$ and by estimating the first eigenvalue Υ_1 of the matrix $T_{km} = B_{ik} B_{jm} s_{ij}$, where B_{ij} is the well-known B-matrix (see, e.g., Girimaji and Pope, 1990 and Monin and Yaglom, 1975) along with reduction of strong strain and vorticity events. It appears that in isotropic flow $\langle l_i l_j s_{ij} \rangle = \langle \Upsilon_1 \rangle$, Liberzon et al. (2005), and as seen from figure 8.30 Υ_1 is substantially reduced in the turbulent flow of dilute polymer solutions as compared to that of pure water.

[39]This is not surprising, since the problem is a combination of two poorly understood problems, which in the words of McComb (1990) comprise a *possible candidate for the title of 'most difficult problem in physics'*.

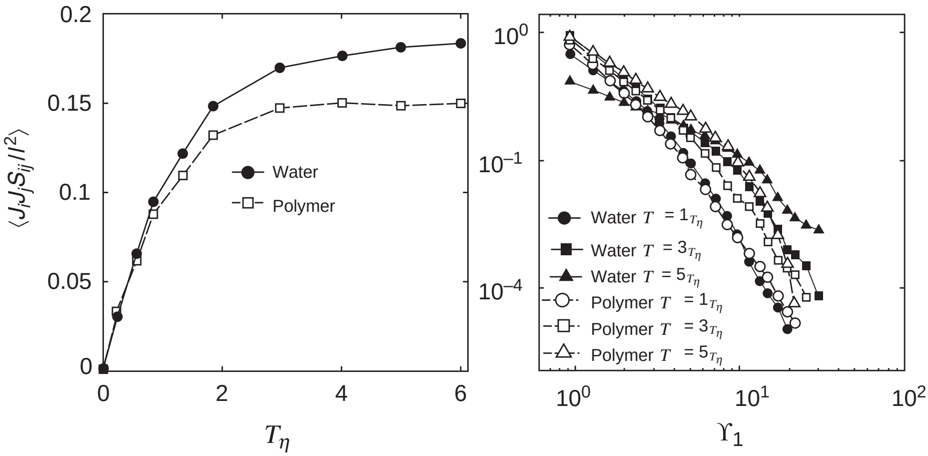

Figure 8.30. Left – time evolution of the mean stretching rate of material lines in the Kolmogorov time-scale units. Right – PDF of the first eigenvalue Υ_1 of the T-matrix for water (solid lines) and polymer solution (dashed lines) for different time moments. Liberzon et al. (2006)

The direct effect of polymers on a turbulent flow is the depletion of small-scale velocity derivatives, see figure 8.31. This effect is due to the additional mechanism of dissipation introduced by polymers and it is clearly observable in the turbulent bulk region. This is directly seen in the inertial forcing case with baffles, in which the amount of kinetic energy and its production remain unaltered in the water and the dilute polymer solution flow cases. Therefore, the observed reduction of the rate-of-strain[40] provides a direct estimate of the reduced viscous dissipation and of the added polymer induced dissipation.

This is consistent with the observed strong reduction of the occurrence of bursting events, and one can expect also reduction of small-scale intermittency in other turbulent flows.

In order to explain the effects with very low polymer concentrations, it is tempting to assume that the polymer molecules form associations which

[40]There is no contradiction between the primary role of the interaction of strain with polymer coils and the relation (8.3), showing the importance of vorticity in maintenance of the Reynolds stress. The answer is that in the near-wall region, vorticity and strain are strongly correlated, so that close to the wall ω^2 and $2s^2$ are (practically) instantaneously equal (e.g., Sandham and Tsinober, 1990). Hence suppression of strain (by whatever mechanism) results in suppression of vorticity and thereby of the Reynolds stresses, as follows from (8.3).

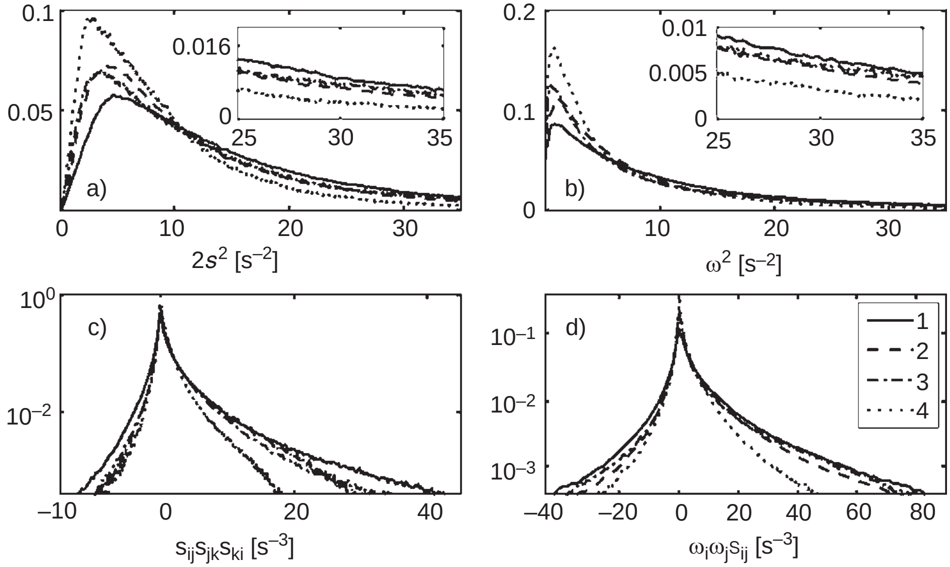

Figure 8.31. PDFs of a) total strain, b) enstrophy, c) strain production and d) enstrophy production for four flow cases: 1 – water flow forced by smooth disks, solid line; 2 – dilute polymer solution forced by smooth disks, dashed line, and the flows forced by baffles; 3 – water, chain line; and 4 – dilute polymer solution, dotted line. For the sake of comparison, the mean values extracted from PDFs are given for each figure: a) 8.5, 6.8, 7.3, and 5.4 s^{-2}; b) 18.7, 16.6, 17.9, and 12.0 s^{-2}; c) 2.6, 2.0, 1.5, and 0.4 s^{-3}; d) 6.3, 5.6, 6.5, and 2.9 s^{-3}. The Kolmogorov time scales are estimated as 0.3–0.4 s. Liberzon et al. (2006)

are preferentially located in regions of large strain, as happens with small particles possessing densities larger than that of the carrying fluid (Eaton and Fessler, 1994; Elperin et al., 2000; Vaillancourt and Yau, 2000 and references therein). This in turn results in the suppression of production of strain (and vorticity) in the 'hot spots', reduced dissipation and drag reduction, but not necessarily any suppression of turbulence. In such a case the fluid in a turbulent flow of dilute polymer solution is intermittently rheological just like, and because of, the turbulent flow is itself intermittent (Bewersdorff et al., 1993). The above is, however, just one more speculation among many. It seems, however, that in any case the intermittency, especially of strain and thereby of dissipation, is of central importance and hardly can be neglected.

Drag reduction and other modification effects on turbulence are observed in turbulent flows with other additives. Turbulent flows with added surfactants, very large aspect ratio fibres and some non-fibrous additives exhibit clear drag reducing effects (Gyr and Bewersdorff, 1995; McComb, 1990). However, much less is known about the structure of these flows. For example, some authors report suppression of turbulent energy, while others observed the contrary. A similar situation concerns particle and bubble loaded flows (Crowe, 2000; Poelma et al., 2007; Zaichik et al., 2008 and references therein).

It is noteworthy that the effects of additives on turbulence is one of the manifestations of nonlocality of turbulence: the primary effect is in the smallest scales which results in changes at all levels.

The overview presented in this chapter attempted to give an exposition of turbulent flows in a variety of physical situations. There are many more. See for example, the paper by Gibson (1996) for an interesting account of turbulent flows in geophysical and astrophysical contexts. Turbulent flow phenomena in liquid helium at very low temperatures, at which it is a mixture of normal and quantum fluids, comprise another most fascinating field (Donnelli, 1999; Liepmann and Laguna, 1984; Vinen and Niemela, 2002). All the factors influencing turbulent flows (those mentioned above and many others) can be used in attempting to control turbulence – a pretty bold and ambitious endeavor with a huge number of publications within a very short time. Apart of practical importance such an approach is definitely useful in using these influences in a systematic way to study the basic mechanisms of 'ordinary' turbulence.

CHAPTER 9

ANALOGIES, MISCONCEPTIONS AND ILL-DEFINED CONCEPTS

What is genuine turbulence?

Our understanding of the general character of the small-scale features of turbulent motion is very far from complete.... Very few theoretical or experimental results have been established so that for the most part we must proceed by analogy and plausible inference (Batchelor, 1956, p. 183).

9.1. Introduction

The above statement by Batchelor is true of far more than just what is called "small-scale" features of turbulence. In other words, analogies in turbulence research have a special status mainly due to unsatisfactory state of not only theory but also of hard evidence on the small-scale features/properties of turbulent flows. Most of these analogies are aimed to look at similarity between genuine turbulence and some "analogous" system such as evolution of some passive object (e.g., scalar, vector, etc.), polymers, and some other (see below) in a prescribed random (usually Gaussian) velocity field. This led in many cases to exaggerated and consequently misleading claims on analogous behaviour between the two and consequently to misconceptions even for systems with the same generic features, such as the same symmetries, conservation laws, etc., which do not guarantee similar behaviour, Kraichnan (1974). Hence the purpose of this chapter is twofold. First, the main emphasis is given to differences rather than similarities. The primary reason for this is that (at least some) understanding of differences is expected to aid better understanding of both systems and avoid misconceptions associated with extending the analogies too far. Second, dealing with conceptual aspects of turbulence research leads necessarily to addressing misconceptions which have arisen during more than a century of turbulence research attempting to achieve some physical understanding/picture of this enigmatic phenomenon. Apart from the critical aspects, the main constructive outcome from addressing a variety of misconceptions is the hope to achieve a deeper understanding of the problems to be encountered and coped with. This is the main aim and emphasis of this chapter.

The story starts with the ‘eddy viscosity’ of Boussinesq (1877) and the Reynolds analogy in 1874 on transport of momentum and heat (Reynolds, 1874)[1] and his proposal to study fluid motion by means of ‘color bands’ (Reynolds, 1894), which can be seen as the foundation of flow visualization. The next example concerns the frozenness of vorticity in the flow field in inviscid flows thereby indicating the analogy between vorticity and (infinitesimal) material lines (Helmholz, 1858; Kelvin, 1880). Taylor, 1938 used this analogy to justify the view that vorticity is amplified due to predominant stretching of material lines in a random flow and Batchelor, 1950 proposed a similar analogy between amplification of vorticity and magnetic field in a turbulent flow of conducting fluid. Recent statements (see below) are made in the same spirit.

The problem with misconceptions starts with the question What is turbulence? and the attempts to give a definition of what turbulence is. Are such attempts conceptually correct? In a mathematical theory the definition of the main object of the theory precedes the results. In physics, especially in new fields, it is vice versa. Usually it happens when one studies a new phenomenon and only at a later stage, after understanding it sufficiently, classifies it, finds its proper place in the existing theories and eventually the most reasonable definition is chosen. Though turbulence in not a new field the above is so much true of turbulence – there is no theory so far. This time has not yet come, and it may not come soon if at all – to quote A.N. Kolmogorov (1985): *I soon understood that there was little hope of developing a pure, closed theory, and because of absence of such a theory the investigation must be based on hypotheses obtained on processing experimental data*, see Tikhomirov (1991, p. 487).

The justification for using some ‘analogous’ system is usually based on the claim that it is ‘mimicking’ the real turbulent flow which is far from being synonymous to reflecting the real physical processes and in many cases (if not all) is just getting the ‘right’ result not necessarily for the right reasons. In particular, this approach is a widespread assumption (and a great variety and huge number of papers) that models represent physical processes in real turbulence. The list of models is a very long one indeed. It starts with the ‘eddy viscosity’ as the simplest version of the ‘solution’ of the ‘problem of closure’, low-dimensional representation and integrable systems and ends with most sophisticated versions of LES and similar (so far). Many of these models are based on qualitatively different (and even contradictory) premises/assumptions, but agree well with *some* experimental data and are claimed to “work well”.

[1]Reynolds postulated the existence of an analogy between wall shear and heat flux based on studies with fully-developed pipe flow and self-similar external boundary layers. Analogies in this spirit are pursued also at present, e.g., Abe and Antonia (2009).

9.2. Eddy viscosity, models

> *Thus, because it is not possible to separate eddies into clearly defined classes according to the source of their energy ; and as there is no object, for present purposes, in making a distinction based on size between cumulus eddies and eddies a few meters in diameter (since both are small compared with our coordinate chequer), therefore a single coefficient is used to represent the effect produced by eddies of all sizes and descriptions* (Richardson, 1922). Note that this citation from Richarson, 1922 follows right after his famous verse.
>
> *Gradient transport ideas (which have been around since the beginning) are understood to be wrong in principle, yet they are used daily with moderate success by industry. Understanding how this can be (it is thoroughly explained by Tennekes and Lumley, 1972, p. 57) sheds light on turbulence* (Kraichnan, 1976).
>
> *... the theoretical basics for the use of simple eddy viscosities to represent subsgrid scales is substantially insecure. Why then have they worked so well in practice? Apparently this is largely because the flow has built-in compensatory mechanisms. The effect of a crude and inaccurate term to represent the passage of energy or enstrophy through the boundary at k_m has the principal effect of distorting the flow in a relatively restricted wavenumber range below k_m* (Kraichnan, 1976).

One of the oldest and greatest analogies/misconceptions is the one on eddy viscosity in the sense that it 'explains the enhanced transfer rates', whereas it is just an empirical way of accounting for such rates but not at all an explanation in any sense. Similarly it is a too an optimistic claim, for example, that LES *of wall-bounded flows… resolve all the important eddies… has received increased attention, in recent years, as a tool to study the physics* (!) *of turbulence in flows at higher Reynolds number, or in more complex geometries, than DNS*, Piomelli and Balaras (2002). The qualification of large-scale (resolved) eddies as the most important ones is too subjective: unresolved eddies are not less important in view of direct and bidirectional coupling of essentially all eddies whatever the term 'eddies' means. It seems conceptually incorrect that LES or any other similar approach can be used as a tool to study the physics of turbulence, since a vitally important part of physics of turbulence resides in the unresolved scales. Nevertheless, most of the numerous models are in good agreement with the experimental and numerical evidence. This is not surprising since agreement with limited (by necessity) experimental evidence is not very much significant when one deals with such a highly-dimensional system as turbulence. The only exception seems to be the Navier–Stokes equations: *Perhaps the biggest fallacy about turbulence is that it can be reliably described (statistically) by a system of equations which is far easier to solve than the full time-dependent three-dimensional Navier–Stokes equations,* Bradshaw, 1994. In other words, it is doubtful that any model except (hopefully) the NSE can be used to adequately study the physics of turbulent flows which in the first place means its basic/fundamental and conceptual aspects.

9.3. Genuine turbulence versus passive "turbulence"

> *Passive contaminants are transported by turbulent motions in much the same way as momentum...Momentum is not a passive contaminant; "mixing" of mean momentum relates to the dynamics of turbulence, not merely its kinematics* (Tennekes and Lumley, 1972).
>
> *The advection-diffusion equation, in conjunction with a velocity field model with turbulent characteristics* (prescribed *a priori*), *... serves as a simplified prototype problem for developing theories for turbulence itself* (Majda and Kramer, 1999).
>
> *... the well-established phenomenological parallels between the statistical description of mixing and fluid turbulence itself suggest that progress on the latter front may follow from a better understanding of turbulent mixing* (Shraiman and Siggia, 2000).
>
> *An important progress has been achieved in the last decade in understanding some simpler systems exhibiting behaviors similar to developed turbulence. These include the so-called weak or wave turbulence, the advection of passive scalar and vector fields by random velocities that mimic the turbulent ones, and, to certain extent, the so-called burgulence, the phenomena described by the Burgers equation* (Gawedzki, et al. 2002).
>
> *The Kraichnan model... is a perfect paradigm for Onsagers vision of generalized "inviscid" solutions of PDEs that sustain turbulent dissipation.... It remains a huge challenge to carry over these important insights from the Kraichnan model to the incompressible Euler equation* (Eyink, 2008).

The differences are more than essential: the evolution of passive objects is not related to the dynamics of turbulence in the sense that the dynamics of fluid motion does not enter in the problems in question – the velocity field is prescribed *a priory* in all problems on evolution of passive objects. Consequently, the problems associated with the passive objects are linear (in Euler setting); whereas genuine turbulence is a strongly nonlinear problem – nonlinearity (along with other N's and n's) is in the heart of turbulent flows and is underlying the main manifestations of the differences between genuine and passive turbulence.

9.3.1. SELF-AMPLIFICATION OF VELOCITY DERIVATIVES

Nonlinearity of genuine turbulence is the reason for the *self-amplification* of the field of velocity derivatives, both vorticity and strain. In contrast there is no phenomenon of self-amplification in the evolution of passive objects (such as material lines, gradients of passive scalar and solenoidal passive vectors with finite diffusivity). We stress that *the process of self-amplification of strain is a specific feature of the dynamics of genuine turbulence having no counterpart in the behavior of passive objects.* In contrast, the process of self-amplification of vorticity, along with essential differences[2], has common features with analogous processes in passive vectors; in both, the main factor is their interaction with strain, whereas the production of strain is much more 'self' and (local).

A related important difference is absence of pressure in case of passive objects.

9.3.2. DIFFERENCES IN STRUCTURE(S)

Along with some common features, the mechanisms of formation of structure(s) are essentially different for passive objects and dynamical variables. Among the reasons is the presence of Lagrangian chaos, which is manifested as a rather complicated structure of passive objects even in very simple regular velocity fields. On the other hand, e.g., the ramp-cliff structures of a passive scalar are observed in a pure Gaussian 'structureless' random velocity field, just like those in a variety of real turbulent flows practically independently of the value of the Reynolds number. In other words the structure of passive objects in turbulent flows arises from two essentially inseparable contributions: one is kinematic due to the Lagrangian chaos and the other dynamic due to the random nature of the (Eulerian) velocity field itself, i.e., the behaviour of Lagrangian objects in E-turbulent flows is much more complicated than that of purely Eulerian objects. Therefore one cannot claim that statistical properties of this so-called 'passive-scalar' turbulence are decoupled from those of the underlying velocity field (Shraiman and Siggia, 2000), since the non-trivial statistical properties of scalars turn out to originate not only in the mixing process itself, but are inherited from the complexity of the turbulent velocity field as well. Study of passive-scalar

[2]We would like to stress again that vorticity is an active vector, since it 'reacts back' on the velocity (and thereby on strain) field. This is not the case with passive objects – the process here is 'one way': the velocity field does not 'know' anything about the passive object.

turbulence is therefore not decoupled from the still intractable problem of calculating the velocity statistics. Among other reasons are differences in sensitivity to initial (upstream) conditions (i.e., Lagrangian 'memory'), 'symmetries', e.g., the velocity field may be locally-isotropic, whereas the passive scalar may be not and some others (see references in Tsinober, 2001a). A recent result, Baig and Chernyshenko, 2005 for turbulent flow in a plane channel is an interesting addition to the list of these differences: although the vortical structure of the flow is the same, the scalar streak spacing varies by an order of magnitude depending on the mean profile of the scalar concentration. Moreover, passive-scalar streaks were observed even in an artificial "structureless" flow field.

One more issue is related to the so-called Lagrangian structure functions initiated by Haller in 2000 (see Mathur et al., 2007 and references therein). In this approach coherent structures in the Lagrangian (particle-based) frame are defined as distinguished sets of fluid particles (which are passive objects). These Lagrangian coherent structures (LCS) are claimed to have a decisive impact on fluid mixing[3] by their special stability properties. Direct Lyapunov exponents (DLE) are used to visualize the two kinds of LCSs (repelling and attracting) as local ridges of the DLE field which turn out to be close (!) to evolving material lines, i.e the fluid flux across them is claimed to be small at any time. *A fluid particle is subject to attraction to nearby blue curves and simultaneous repulsion by nearby red curves. The complex tangle formed by these two sets of curves is the underlying cause of turbulent particle motion, the Lagrangian skeleton of turbulence* (Mathur et al., 2007). It has to be stressed that the attraction/repulsion occurs not because these sets have the "power" to do so – it is just a reflection of the action of the underlying fluid flow (the Eulerian one) and the factors causing this flow. Therefore, the causal part of the above statement is overreaching as the authors deal (very nicely) with purely-kinematical aspects of the problem[4] whereas turbulence (particle motion included) is due to dynamical reasons/causes. In other words, what is observed is the reflection/description in terms of DLE, etc. of the real cause which is due to action of the underlying velocity field (especially strain) producing stretching/compressing of material *passive* objects closely related to repelling/attracting of fluid particles to the above mentioned sets. An important aspect is precisely in the same way one would observe a *Lagrangian skeleton* of, e.g., a pure Gaussian (structureless) velocity field as well. One of the claims is that these curves

[3]In this sense this is a kinematic aspect of LCS's as contrasted to the dynamics (i.e., Eulerian). In other words the most difficult question is about the relation/connection or importance of these distinguished sets of fluid particles for turbulence dynamics not just kinematics.

[4]That is they would observe a *Lagrangian skeleton* of pure Gaussian (structureless) velocity field as well.

turn out to be close to evolving material lines, i.e., the fluid flux across them is negligible at any time. An important point is that the two sets of curves are Lagrangian only approximately (i.e., their Lagrangian identity is not perfect). Therefore – though one can define them at any time moment – it is not clear how one can follow the *same* line in time. The LCS issues are a typical example of the issues associated with problems of the relation between the Lagrangian and Eulerian descriptions: there is a necessity *of an in-depth study of the connection between Lagrangian coherent structures and their dynamic signature...* (Salman, 2007). However, this is far more than being trivial as can be seen from the next subsection and mentioned in chapters 3 and 4.

9.3.3. SCALING EXPONENTS AND STATISTICALLY CONSERVED QUANTITIES

There is a number of publications insisting in some sense on a kind of essential linearization of genuine turbulence problem when this concerns scaling exponents (mainly of structure functions) and the role of statistically conserved quantities. The claims are summarized by arguing that *the mechanism leading to anomalous scaling in Navier–Stokes equations and other nonlinear models is identical to the one recently discovered for passively advected fields*, Angheluta et al. (2006).

If this is really true it means that this is just one more aspect – as in RDT – which can be treated via a linear model which in some cases enables us to handle some aspects of turbulent flows, but not their genuine nonlinear aspects: *One can thus speculate that the anomalous scaling for the genuine turbulence can also appear as a linear phenomenon in the following sense. Let us split the total velocity field into the two parts, the background field and the perturbation... linearize the original stochastic equation with respect to the latter, choose an appropriate statistics for the former.... Then the small-scale perturbation field will show anomalous scaling behavior with nontrivial exponents, which can be calculated systematically within a kind of ϵ-expansion model. In such a case the passive vector field can give the anomalous exponents for the NS velocity field exactly*, Antonov et al. (2003).

Similar statements are made with respect to so-called statistically conserved quantities/zero modes which have been discovered for passive objects, but not really for genuine NSE, see Falkovich and Sreenivasan (2006); and Eyink (2008) and references therein. The main point is that these authors (as some others) claim or/and strongly imply a fundamental analogy between results obtained for passive scalars for the Kraichnan model and genuine turbulence[5].

[5]The Kraichnan model employs what is sometimes called 'Kraichnan enesmble': a velocity field which is δ-correlated in time and Gaussian in space for fixed time, i.e., characterized by the two-point function $\langle u_i(t,\mathbf{r})u_j(t',\mathbf{r}')\rangle \propto \delta(t-t')D_{ij}(\mathbf{r}-\mathbf{r}')$.

One can now state with confidence that stochastic equations like those that describe aspects of turbulence (i.e., the Kraichnan model for passive-scalar and Burgers turbulence) *demonstrate the inadequacy of Kolmogorov dimensional reasoning. In particular, the community has learned that statistical conservation laws* (again for a passive scalar!) *play a fundamental role in establishing that inadequacy.... If anomalous scaling is to result, the advecting velocity field must not be smooth and it must generally possess power-law correlations in the inertial range,* Falkovich and Sreenivasan (2006); see also Falkovich (2009).

The Kraichnan model (21) is a perfect paradigm for Onsager's vision of generalized "inviscid" solutions of PDEs that sustain turbulent dissipation... It remains a huge challenge to carry over these important insights from the Kraichnan model to the incompressible Euler equation, Eyink (2008).

9.3.4. ISSUES ASSOCIATED WITH THE E-L RELATIONS. ANALOGY BETWEEN GENUINE TURBULENCE AND LAGRANGIAN CHAOS

The intricacy of the relation between the Eulerian and Lagrangian fields of the *same* fluid flow has a number of other important consequences. This is a part of a broader question. Namely, what can be learnt about the properties and especially dynamics of real turbulence from studies of passive objects (scalars, vectors)? In particular, what can be learnt about the velocity field and other dynamical variables in real turbulence from comparison of the behaviour of passive objects in real and some 'synthetic' turbulence?

The first issue includes flow visualization in which the relation between the field of velocity and that of a passive scalar (dye, very small particles) plays a crucial role, as does the interpretation of this relation. It appears that the meaning of 'seeing' turbulent flow is far from being trivial.

We have seen that the structure of a passive tracer can be (and usually is) very complicated (figure 4.8), whereas the corresponding velocity field is rather simple. Another example is shown in figure 9.1. After the two vortices merged, the structure of the passive markers is rather complicated as compared to the initial state, but the structure of the flow field is essentially the same as in each individual vortex before the merging.

One more example is shown in figure 9.2. It is seen that the passive tracer has structure at locations where the velocity field has none.

These examples show that flow visualizations used for studying the structure of dynamical fields (velocity, vorticity, etc.) of turbulent flows may be quite misleading. The general reason is that passive objects do not 'want' to follow the dynamical fields (velocity, vorticity, etc.)[6] – along

[6]Formally, it is obvious since passive objects and active fields obey *different* equations. In particular the problems associated with the former are essentially linear, whereas the problems involving the latter are genuinely nonlinear.

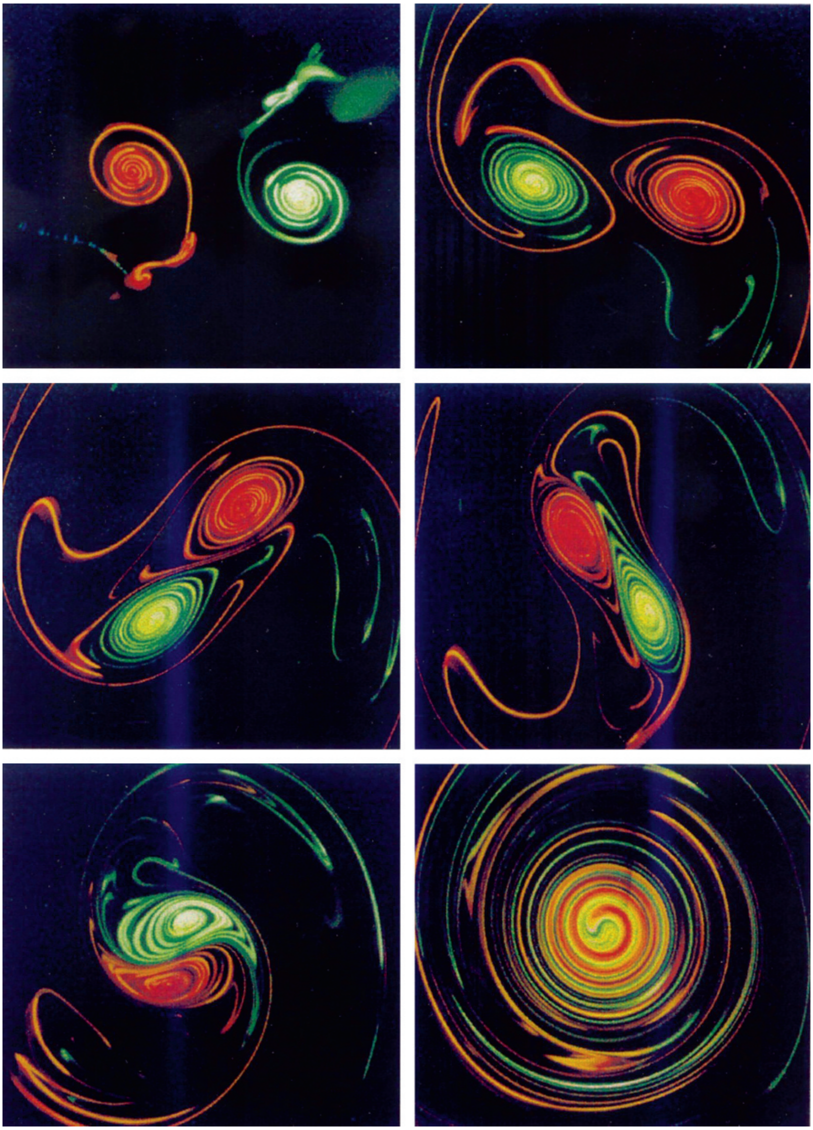

Figure 9.1. Dye visualization of two co-rotating vortices with the vorticity of the same sign. Time is increasing from left to right and from top to bottom. Note the resulting rather complicated pattern of dyes, whereas the corresponding fluid flow consists of just one 'simple' vortex. Courtesy of Dr. T. Leweke (see Meunier and Leweke, 2000)

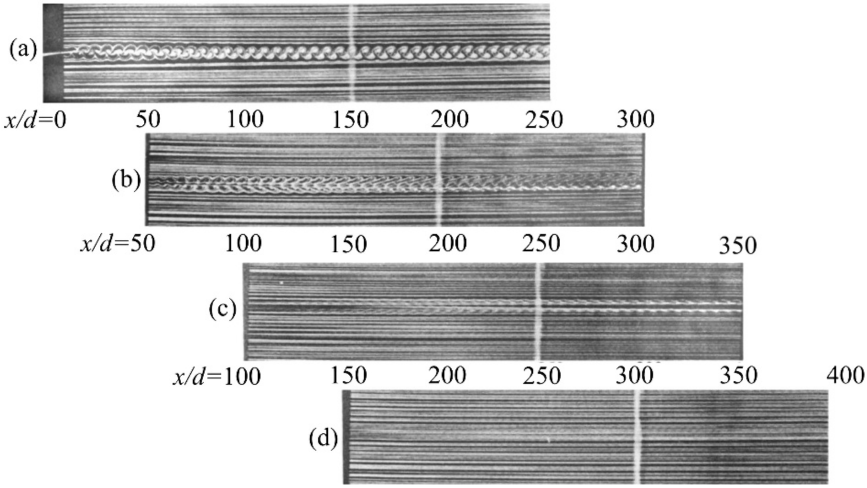

Figure 9.2. Same flow – not the same pattern. Smoke visualization of a wake past a circular cylinder at Re = 90 (Cimbala et al., 1988). The velocity field in figures a) – d) is the same. The difference is in the location of the smoke release. The figure d) clearly indicates that beyond the distance $x/d = 150$ there is no von Karman vortex street in the velocity field at all, whereas looking at the figure a) only would imply quite the opposite

with some common features, the mechanisms of formation of structure(s) are essentially different for the passive objects and dynamical variables. As mentioned, one of the reasons is the presence of Lagrangian chaos, which is manifested as a rather complicated structure of passive objects even in very simple regular velocity fields. On the other hand the ramp-cliff structures of a passive scalar are observed in a pure Gaussian 'structureless' random velocity field (Holzer and Siggia, 1994), just like those in a variety of real turbulent flows, practically independently of the value of the Reynolds number (Warhaft, 2000 and references therein). Therefore – as mentioned above and discussed in chapters 3 and 4 – one can expect that the structure of passive objects in turbulent flows arises from two (essentially inseparable) contributions: one of purely kinematic nature due to Lagrangian chaos and the other having a dynamical origin due to the random nature of the Eulerian velocity field itself. Among other reasons are the differences in sensitivity to initial (upstream) conditions, as in the case shown in figure 9.2, the difference in 'symmetries', e.g., the velocity field may be locally-isotropic, whereas the passive scalar may be not (see references in Celani et al., 2001; Villermaux et al., 2001 and Warhaft, 2000) and some others. This does not mean that qualitative study of fluid motion by means of colour bands (Reynolds, 1894) is impossible or necessarily erroneous. However, watching the dynamics of material 'coloured bands' in a flow may not reveal the nature of the underlying motion, and even in the case of correct qualitative observations the right result may come not necessarily for the right reasons. The famous verse by Richardson belongs to this kind of observation.

The second issue is how sensitive is, e.g., the field of a passive scalar to the properties of dynamical fields (velocity, vorticity, etc.). It appears that many, especially qualitative but also quantitative, characteristics of passive scalars are insensitive to the details of the velocity field (Kraichnan, 1968; Majda and Kramer, 1999; Warhaft, 2000), as long as the velocity field is random. For example, the 4/3 Richardson law is observed in a purely Gaussian and two-dimensional field (Elliott and Majda, 1996), and even *independent of the nature of the turbulent flow* (Ola, 2002). This is the reason that many properties of passive scalars are essentially the same for a Gaussian velocity field and for real turbulent flows (see Holzer and Siggia, 1994; Majda and Kramer, 1999; Warhaft, 2000; Shraiman and Siggia, 2000 and references therein). This is true not only of various statistical properties, but also of structural details such as formation of the mentioned above ramp-cliff structures – a name given to the sheetlike fronts (shocks) with sharp gradients of a passive scalar across them. They were observed both in numerical simulations of a (two-dimensional) passive scalar in a Gaussian velocity field (Holzer and Siggia, 1994), and in various experiments and field observations as well as in numerical simulations of the Navier–Stokes equations together with the advection-diffusion equation in a periodic box[7]. In other words nontrivial statistical behaviour of passive objects is expected for *any* random velocity field independently of the nature of this randomness. This is not very surprising, since – as mentioned – the Lagrangian field is an extremely complicated nonlinear functional of the Eulerian field. Still, it should be remembered that the phenomenon itself is a linear one in the Eulerian setting.

On the other hand there are properties of passive objects which do depend on the details of the velocity field. For example, the PDF of a passive scalar depends on a variety of factors, such as Reynolds number, the presence of mean shear flow and/or mean scalar gradient, and many others (see Majda and Kramer, 1999). Just these very properties can be effectively used to study the differences between real turbulent flows and artificial random fields. More precisely the essential differences in the behaviour of passive objects in a real and synthetic turbulence may be exploited in order to gain more insight into the dynamics of real turbulence. At present, however, the knowledge necessary for such a use is very far from being sufficient. With few exceptions it is not even clear what can be learnt about the dynamics of turbulence from studies of passive objects (scalars and vectors) in real

[7]For references and further details see Celani et al. (2001), Majda and Kramer (1999), Overholt and Pope (1996), Gawedzki and Vergassola (2000), Shraiman and Siggia (2000) and Warhaft (2000).

and ‘synthetic’ turbulence. This requires systematic comparative studies of both. An attempt of such a comparative study was made by Tsinober and Galanti (2003). This is a relatively small part of a much broader field of comparative study (in both Euler and Lagrange setting) of ‘passive’ turbulence (reflecting the kinematical aspects) and genuine turbulence representing both the dynamical processes and the kinematical aspects.

9.3.5. KOLMOGOROV 4/5 VERSUS YAGLOM 4/3 LAWS AND NON-GAUSSIAN NATURE OF GENUINE AND ‘PASSIVE’ TURBULENCE

The Kolmogorov and the Yaglom laws are respectively

$$S_3(r) \equiv \left\langle \left(\Delta u_{||}\right)^3 \right\rangle = -\frac{4}{5}\epsilon r \text{ and } \left\langle \Delta u_{||} \, (\Delta\theta)^2 \right\rangle = -\frac{4}{3}\varepsilon_\theta \mathrm{r} \qquad (9.1a, 9.1b)$$

where $\Delta u_{||} \equiv [\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}) \cdot \mathbf{r}/r$, $\Delta\theta = \theta(\mathbf{x}+\mathbf{r}) - \theta(\mathbf{x})$, ϵ is the rate of dissipation of kinetic energy and $\epsilon_\theta = D\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_i}$ is the rate of dissipation of fluctuations of a passive scalar. The analogy between these two laws[8], though useful in some respects (Antonia et al., 1997) is obviously violated for a Gaussian velocity field. Namely, the 4/3 law remains valid for such (as any other random isotropic) velocity field, whereas the 4/5 law is not, because $S_3(r) \equiv 0$ for a Gaussian velocity field[9]. This difference is one of the manifestations of the dynamical nature of the Kolmogorov law as contrasted to the kinematical nature of the Yaglom law. It reflects the difference between genuine turbulence as a dynamical phenomenon and ‘passive’ turbulence as a kinematical process.

The above underscores the essentially and qualitatively different origin of the non-Gaussian nature of genuine turbulence and various passive objects which sometimes is masked in the Lagrangian description. There are two main aspects here. The first one is seen in the pure Eulerian description in which the genuine turbulence is non-Gaussian due to the nonlinearity (see chapter 6), whereas the evolution of passive objects is an essentially linear process and its non-Gaussian properties arise due to the multiplicative manner in which velocity enters in the governing equations. Hence strong non-Gaussianity and nontrivial structure of passive objects appear even in a structureless purely-Gaussian[10] isotropic velocity field. Another aspect is the non-Gaussian behavior of L-turbulent passive objects (with no counterpart statistics) in purely E-laminar flows.

[8]The 4/5 Kolmogorov law follows by isotropy from the 4/3 law for the velocity field in the form $\left\langle \Delta u_{||}(\Delta \mathbf{u})^2 \right\rangle = -\frac{4}{3}\left\langle \epsilon \right\rangle r$.

[9]See below a law for the vector potential of a magnetic field similar to the 4/3 Yaglom law for passive scalar.

[10]Prescribed ‘by hand’ or in some other way as in an NS flow at Re $<$ 1 with a Gaussian forcing.

9.4. Vorticity versus passive vectors

9.4.1. MATERIAL LINES

> *Turbulent motion is found to be diffusive, so that particles which were originally neighbors move apart as motion proceeds. In a diffusive motion the average value of d^2/d_0^2 continually increases. It will be seen therefore... that the average value of ω^2/ω_0^2 continually increases* (Taylor, 1938).
> *...the interesting physical argument that $<\omega_i \omega_j s_{ij}>$ is positive because two particles on average move apart from each other and therefore vortex lines are on average stretched rather than compressed* (Hunt, 1973).
> *The relative diffusion of a pair of probe particles in grid turbulence at high Reynolds numbers is treated as the most clear-cut manifestation of vortex stretching* (Mori and Takayoshi, 1983).
> *When* Re *is large this vorticity is virtually frozen into the fluid*, p. 154. *However, since material-line stretching seems to be a norm for the broader class of kinematically-admissible fields, it should also be the norm for the narrower class of dynamically-admissible velocity fields, and so one should not be surprised that vortex-line stretching, like material-line stretching, is seen in practice*, p. 259 (Davidson, 2004).
> *...Vorticity amplification is a result of the kinematics of turbulence* (Tennekes and Lumley, 1972).
> *A major open problem, in particular, is how to relate turbulent dissipation of energy, precisely, to the inviscid motion of vortex-lines* (Eyink, 2008).

The above selection of citations[11] represent a rather common view and a major misconception that the prevalence of vortex stretching is due to the predominance of stretching of material lines[12]. Chorin (1994) points to the problematic aspect of such a view: *Vortex lines are special lines, and constitute a negligible fraction of all lines (there is one vortex direction at each point, but an infinite number of others). All arguments that involve averages with respect to a probability measure may fail to hold in a negligible fraction of cases, and thus one cannot conclude from (5.1)* (i.e., $d/dt\langle|\delta x(t)|^2\rangle$) *that vortex lines stretch, even in isotropic flow.* But he ends with the statement that *This conclusion is, however, eminently plausible.* Indeed, it is plausible, since it is observed in the laboratory and in numerical simulations.

[11] There is a multitude of similar statements in the literature.
[12] This view originates with Taylor (1938) who demonstrated experimentally the prevalence of vortex stretching in a turbulent grid flow.

But the underlying reasons/processes are still not understood, unlike in the case of passive material lines. The main problem with the view that the prevalence of vortex stretching is due to the predominance of stretching of material lines is that it is of *kinematic* nature (as employing the Richardson pair diffusion), whereas at the very outset one would expect that the underlying cause of vortex stretching in turbulent flows should be a *dynamical* one. So the main question is whether vorticity and material lines really are stretched in the same way and for the same reason, and what is the meaning of the "same way". Another important question is whether vortex lines are (approximately) frozen into the fluid at high Re. So the main problems with the kinematic view are as follows. First, the vortex lines are not frozen into the fluid at however high Reynolds number – otherwise how can the enstrophy production be approximately balanced by viscous terms (the Tennekes–Lumley balance see chapter 6) *at any however large a Reynolds number*[13], which is not the case with material lines: the production $l_i l_i s_{ij}$ is not balanced at all[14]. One of the consequences is that Kelvin/Helmoholz theorems do not hold – even approximately – at large Reynolds numbers[15]. Second, even if frozen vorticity is not a marker, it reacts back strongly: everybody knows the Biot–Savart law, or more generally $\nabla^2 \mathbf{u} = -curl\omega$. Third, even if frozen, those material lines coinciding with vorticity are special and not the other way around. Namely, the material line elements which initially and thereby consequently coincide with vorticity are special in the

[13]*... a material line which is initially coinciding with a vortex line continues to do so. It is thus possible and convenient to regard a vortex-line as having a continuing identity and as moving with the fluid. (In a viscous fluid it is, of course, possible to draw the pattern of vortex lines at any instant, but there is no way in which a particular vortex-line can be identified at different instants)*, Batchelor (1967, p. 274). In other words, at any Reynolds number a vortex line does not have a (Lagrangian) identity and it seems meaningless to speak about an "approximate" identity in view of the balance between enstrophy production and its destruction due to viscosity – we stress again – at any however large (but finite, so that the velocity field is smooth) Reynolds number. This lack of Lagrangian identity is also one of the difficulties in handling the phenomenon of reconnection. Thus it is clear that it is meaningless to look for the solution of the *major open problem... how to relate turbulent dissipation of energy, precisely, to the inviscid* (!) *motion of vortex-lines,* Eyink, 2008. Also we do recall that at any finite Reynolds number the causal relation is between dissipation and strain and its production rather than vorticity.

[14]Indeed, the equation for l^2,

$$\frac{Dl^2}{Dt} = l_i l_i s_{ij},$$

does not contain any diffusive term unlike the equation for the enstrophy (C.16).

[15]As discussed in chapter 6 the stochastic versions of the Kelvin theorem and the Cauchy formula for the evolution of vorticity for NSE (i.e., $\nu \neq 0$) by Constantin and Iyer, 2008 are just formal analogues and cannot be interpreted as any kind of "frozenness" of (stochastic) vortex-lines to the ensemble of stochastic flows $\mathbf{X}(\mathbf{a}, t)$ which replace the classical Lagrangian trajectories.

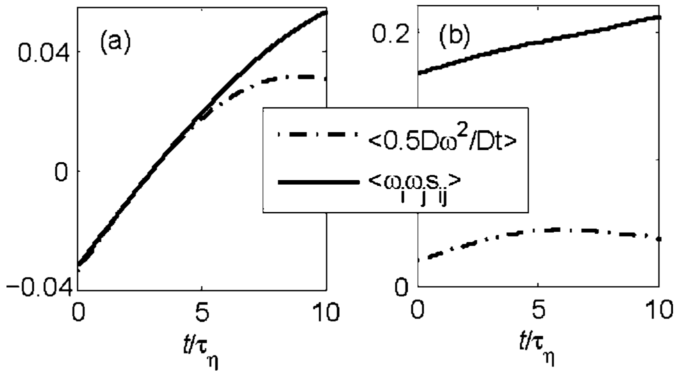

Figure 9.3. Comparison of time evolution of the $0.5D\omega^2/Dt$ (dashed line) and the enstrophy production $\omega_i\omega_j s_{ij}$ (solid line). (a) – trajectories originating at locations with balance of viscous and forcing terms; (b) – randomly chosen initial conditions. Galanti et al. (2008)

sense that they are not dynamically passive quantities anymore and react back on the flow precisely as does vorticity. In other words, the fact that vorticity is frozen in the inviscid flow field does not mean that vorticity behaves the same way as material lines, but the other way around: those material lines which coincide with vorticity behave like vorticity, because they are not passive anymore as are all the other material lines: a continuum of other choices. This is a different kind of "non-uniqueness": there is one vortex direction at each point, but an infinite number of others. This kind of behavior was observed in a low-Reynolds-number (!) numerical experiment ($\mathrm{Re}_\lambda = 50$). The idea was to look at the Lagrangian evolution of vorticity and material elements associated with fluid particles originating from locations where the forcing and the viscous terms are balancing each other. Due to persistency of Lagrangian evolution this (approximate) balance remains valid for about ten Kolmogorov time scales, thus allowing to observe locally in space/time 'purely' inviscid evolution, figure 9.3. Another feature is that the evolution of material lines and vorticity is very close only for those material lines which *i*) are initially identical to vorticity and *ii*) for trajectories originated at locations with balance of viscous and forcing terms, figure 9.4. Any other material lines behave differently, even those satisfying *ii*) but not *i*). For example, material lines initially identical to the compressing eigenvalue, λ_3, of the rate of strain tensor are strongly compressed during $10\tau_\eta$. More details are given in Galanti et al. (2008).

Other differences include the rate of stretching: $\langle l_i l_j s_{ij}/l^2\rangle$ is up to 1.5 times larger than $\langle \omega_i\omega_j s_{ij}/\omega^2\rangle$, Guala et al. (2005), Lüthi et al. (2005), alignment properties: material elements are preferentially aligned with λ_1

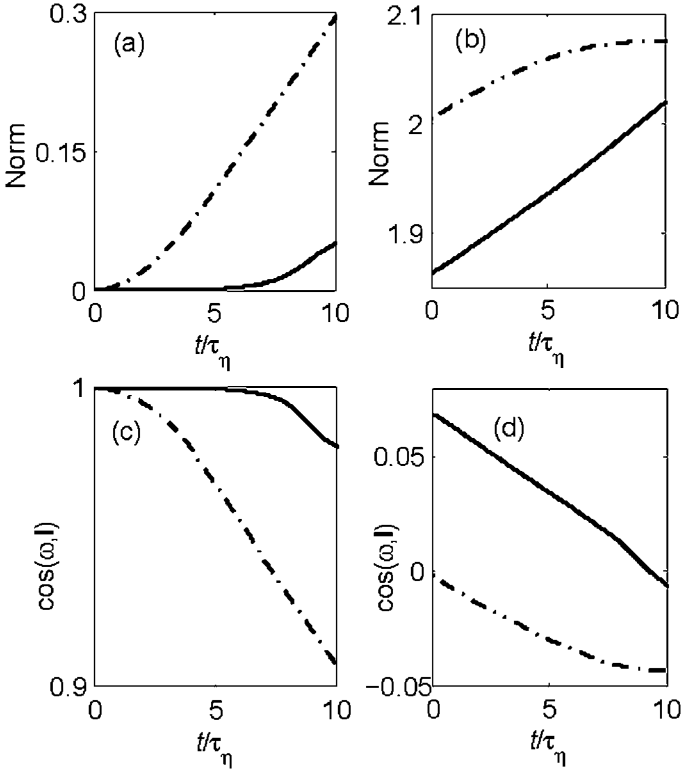

Figure 9.4. Comparison of the time evolution of vorticity and material lines. Upper panel: time evolution of the norm $2[\mathbf{l}/l_0 - \omega/\omega_0]^2[(l/l_0)^2 + (\omega/\omega_0)^2]$ for (a) material lines initially identical to vorticity for trajectories originated at locations with balance of viscous and forcing terms (solid line) and from randomly chosen initial conditions (dashed line); (b) material lines initially randomly oriented but for trajectories originated at locations with balance of viscous and forcing terms (solid line) and from randomly chosen regions (broken line); Bottom panel: evolution of the cosines between the vorticity vector and the material lines. (c) – material lines as in a); (d) material lines as in b). DNS data: resolution – 256^3, $\mathrm{Re}_\lambda = 50$. Based on average of 388 independent trajectories originated from regions with balance of viscous and forcing terms and 600 randomly selected trajectories. Galanti et al. (2008)

(Drummond, 1993; Lüthi et al., 2005), whereas vorticity aligns with λ_2 even in Euler flows (see references in Tsinober, 1998a,b). An additional difference is that a vorticity field is solenoidal, $div\omega \equiv 0$, whereas the field of material elements $\mathbf{l}$ is not: its divergence is a precise pointwise Lagrangian invariant $div\mathbf{l} \equiv const.$ It is vanishing only for an initially solenoidal field of $\mathbf{l}$, but, as mentioned, there is a continuum of other fields $\mathbf{l}$ with $div\mathbf{l} \not\equiv 0$. Finally, in two-dimensional turbulent flows (the Cocke, 1969 proof works in 2-D in the same way) and any Lagrangian chaotic flows (which are E-laminar) the material elements are predominantly stretched, whereas nothing of the kind happens with vorticity.

9.4.2. SOLENOIDAL VECTOR FIELDS WITH NONVANISHING DIFFUSIVITY

The usual comparison is based on looking at the equations for vorticity ω and the (solenoidal) passive vector, $\mathbf{B}$, e.g., magnetic field in electrically-conducting fluids (Batchelor, 1950),

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega) + \nu \nabla^2 \omega, \ \ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times B) + \eta \nabla^2 \mathbf{B} \qquad (9.2a, 9.2b)$$

Though a number of differences are known, they are hidden when one looks at the equations for ω and $\mathbf{B}$, which, as mentioned, *are identical in form*. However, a more 'fair' comparison should be made between the velocity field, $\mathbf{u}$, and the vector potential $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$, Tsinober and Galanti (2003). Such a comparison allows to see immediately one of the basic differences between the fields $\mathbf{u}$ and $\mathbf{A}$ (apart from the first being nonlinear and the second linear) which is not seen from the equations (9.2). Namely, the Euler equations conserve energy, since the scalar product of $\mathbf{u} \cdot (\omega \times \mathbf{u}) \equiv 0$ is identically vanishing. In contrast – unless initially and thereby subsequently $\mathbf{u} \equiv \ \mathbf{A}$ – the scalar product of $\mathbf{A} \cdot (\mathbf{u} \times \mathbf{B}) \not\equiv \mathbf{0}$.[16] It is this term $\mathbf{A} \cdot (\mathbf{u} \times \mathbf{B}) \equiv -A_i A_k s_{ik} + \partial/\partial x_k \{A_k A_l u_l - \frac{1}{2} u_k A^2\}$ which acts as a production term in the energy equation for $\mathbf{A}$. In other words, when the initial conditions for $\mathbf{u}$ and $\mathbf{A}$ are not identical, the field $\mathbf{A}$ (and $\mathbf{B}$), is sustained by the strain, s_{ik}, of the velocity field – in contrast to the field $\mathbf{u}$ which requires external forcing. The production term $-A_i A_k s_{ik}$ is positively skewed and $\langle -A_i A_k s_{ik} \rangle > 0$. A noteworthy feature is that an analogue of Kolmogorov 4/5 law is valid for the vector potential $\mathbf{A}$ (see e.g., Gomez et al., 1999 and references therein)

$$\left\langle \Delta u_{||} (\Delta \mathbf{A})^2 \right\rangle = -4/3 r \epsilon_A, \qquad (9.3)$$

where $\Delta u_{||} \equiv \Delta \mathbf{u} \cdot \mathbf{r}/r \equiv [\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{r}/r$, $\Delta \mathbf{A} = \mathbf{A}(\mathbf{x} + \mathbf{r}) - \mathbf{A}(\mathbf{x})$, and ϵ_A is the mean dissipation rate of the energy of $\mathbf{A}$. An important point is that the relation (9.3) holds for *any* random isotropic velocity field including the Gaussian one, which is not the case for the velocity field itself, since $\left\langle \Delta u_{||} (\Delta \mathbf{u})^2 \right\rangle \equiv 0$ for a Gaussian velocity field. Similarly, there are essential differences in the behaviour of vorticity, ω and $\mathbf{B}$. First, in a statistically stationary velocity field (NSE but not Gaussian) the enstrophy ω^2 saturates to some constant value, since vorticity is not a dynamically passive quantity. In contrast the energy of magnetic field B^2 grows exponentially without

[16] The corresponding equation for the vector potential $\mathbf{A}$ has the form

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{B} \times \mathbf{u} = -\nabla p_A + \eta \nabla^2 \mathbf{A}.$$

limit: in the kinematic regime the magnetic field is a passive vector and the fluid flow does not know anything about its presence. Second, growth of the magnetic field is insensitive to the particulars of the random flow, e.g., the velocity field can be artificial, such as Gaussian. In such a velocity field the production term $B_iB_ks_{ik}$ is also positively skewed and $\langle B_iB_ks_{ik}\rangle > 0$. This is not the case with vorticity: there is no amplification of vorticity in a Gaussian velocity field, the PDF of $\omega_i\omega_ks_{ik}$ is precisely symmetric and consequently $\langle \omega_i\omega_ks_{ik}\rangle \equiv 0$: to be amplified vorticity needs for this 'its own' genuine turbulent velocity field. For other results concerning differences between ω and $\mathbf{B}$ see Tsinober (2001, 2007), Tsinober and Galanti (2003) and references therein.

A note on the similarities and differences in production of enstrophy and its passive counterparts

In all the three cases the forcing is small compared to the production by strain, i.e. all the three $\omega_i\omega_js_{ij}, -G_iG_js_{ij}$ and $B_iB_js_{ij}$ are approximately balanced (mostly in the mean but not only) by the corresponding diffusive terms $\nu\omega_i\nabla^2\omega_j$, $\mathcal{D}G_i\nabla^2G_i$, and $\eta B_i\nabla^2B_i$. However, the details of this balance are important. For example, at later stages both the fields ω and $\mathbf{G}$ reach a statistically stationary state (i.e., there is a balance $\langle \omega_i\omega_js_{ij}\rangle = -\nu\langle \omega_i\nabla^2\omega_j\rangle$ and $\langle G_iG_js_{ij}\rangle = \mathcal{D}\langle G_i\nabla^2G_i\rangle$). This does not happen in case $\mathbf{B}$ which does not reach a statistically stationary state in the kinematic regime (the production $B_iB_is_{ij}$ is typically 'a bit' larger than $\eta B_i\nabla^2B_i$) and is growing all the time exponentially (dynamo) even without forcing in B (NSE should be forced). The origin of this difference is that while there are non-diffusive conservation laws for the velocity field $\mathbf{u}$ and the passive scalar θ, there is no such law for the vector potential $\mathbf{A}$ of the magnetic field. So a balance $\langle B_iB_js_{ij}\rangle = -\eta\langle B_i\nabla^2B_i\rangle$ is reached in the saturated regime, i.e., the dynamical regime via the reaction back of the magnetic field on the fluid flow field. The flow field in the saturated regime is different from the kinematic one, but is not close to a 2-D state as widely believed, since the production $B_iB_js_{ij}$ is not vanishing – it is just balanced more 'precisely' by $\eta B_i\nabla^2B_i$, see section 8.9.

Two-dimensional flows

In two dimensions (x, y) the differences between ω and $\mathbf{B}$ are even more drastic. First, a vorticity vector in this case has only a z-component and there is no stretching/amplification of vorticity as in three dimensions. A magnetic field can possess all three components and there is a process of stretching of the magnetic field in the plane (x, y). This process can lead to substantial transient growth of the magnetic field, which at later times is always overcome by the diffusion and consequent eventual decay.

However, this transient regime can be very long. This would mean that in the dynamical case the difference between the behaviour of ω and $\mathbf{B}$ becomes even larger than that at the kinematic level (see references in Tsinober, 2007).

Geometrical statistics

In a Gaussian velocity field all the production terms for passive objects ($l_i l_j s_{ij}, -G_i G_j s_{ij}$ and $B_i B_j s_{ij}$) are essentially positively skewed and similar to that in genuine turbulent flows, whereas $\omega_i \omega_j s_{ij}$ is precisely symmetric in a Gaussian velocity field. Therefore among other things the mean of the enstrophy production vanishes in a Gaussian velocity field, but the means of $l_i l_j s_{ij}, -G_i G_j s_{ij}$ and $B_i B_j s_{ij}$ are essentially positive, see figure 4.2. The alignment properties are directly related to the above. The $cos(\mathbf{l}, \lambda_i)$, $cos(\mathbf{G}, \lambda_i)$, $cos(\mathbf{B}, \lambda_i)$ and also $cos(\mathbf{W}^l, \mathbf{l})$, $cos(\mathbf{W}^G, \mathbf{G})$ and $cos(\mathbf{W}^B, \mathbf{B})$ are qualitatively the same for both a Gaussian velocity field and genuine turbulence. With a NSE velocity field there is alignment of $\mathbf{B}$ both with λ_1 and λ_2, whereas only alignment with λ_2 is observed in a Gaussian velocity field, figure 4.5. The $cos(\omega, \lambda_i)$ and $cos(\mathbf{W}^\omega, \omega)$ are qualitatively different: for a Gaussian velocity field all the $cos(\omega, \lambda_i)$ are flat and $cos(\mathbf{W}^\omega, \omega)$ is symmetric, and the PDF of Λ_2 is symmetric as well, whereas for genuine turbulence $cos(\omega, \lambda_i)$ (see figure 4.6) exhibits strong alignment with λ_2, and the PDFs of $cos(\mathbf{W}^\omega, \omega)$ and Λ_2 are essentially positively skewed.

9.4.3. EVOLUTION OF DISTURBANCES

Important aspects of the essential differences between the evolution of fields ω and $\mathbf{B}$ arising from the nonlinearity of the equation of ω and linearity of the equation for $\mathbf{B}$ are revealed when one looks at how these fields amplify disturbances, see figure 9.5. In other words, $\mathbf{B}$ and ω possess essentially different stability properties. The reason is that the equation for the disturbance of vorticity differs strongly from that for vorticity itself due to the nonlinearity of the equation for the undisturbed vorticity ω, whereas the equation for the evolution disturbance of the field $\mathbf{B}$ is the same as that for $\mathbf{B}$ itself due to the linearity of the equation for $\mathbf{B}$. Consequently, the evolution of disturbances of the fields ω and $\mathbf{B}$ is drastically different, figures 9.5 and 9.6. For example, in a statistically stationary velocity field the energy of the disturbance of $\mathbf{B}$ grows exponentially without limit (just like the energy of $\mathbf{B}$ itself), whereas the energy of vorticity disturbance grows much faster than that of $\mathbf{B}$ for some initial period, until it saturates at a value which is of order of the enstrophy of the undisturbed flow. It is noteworthy that much faster growth of the energy of disturbances of vorticity during the very initial (linear in the disturbance) regime is due to additional terms in the equation for the disturbance of vorticity, which have no counterpart in the case of passive vector **B**. Indeed, the equation for the disturbance of vorticity Δ_i^ω,

$$\frac{D\Delta_i^\omega}{Dt} = \Delta_j^\omega s_{ij} + \omega_j \Delta_{ij}^s + -\Delta_j^u \frac{\partial \omega_i}{\partial x_j} + \Delta_j^\omega \Delta_{ij}^s - \Delta_j^u \frac{\partial \Delta_i^\omega}{\partial x_j} + \nu \nabla^2 \Delta_i^\omega, \quad (C.60)$$

contains three terms $\Delta_j^\omega s_{ij}, \omega_j \Delta_{ij}^s, -\Delta_j^u \frac{\partial \omega_i}{\partial x_j}$ all *linear* in disturbance, whereas the equation for the disturbance of a magnetic field is just the same as that for the magnetic field itself. It is important to stress that these additional 'linear' terms $\left(\omega_j \Delta_{ij}^s \text{ and } - \Delta_j^u \frac{\partial \omega_i}{\partial x_j}\right)$ in (C.60) arise due to the nonlinearity of the equations for the undisturbed vorticity. In this sense the essential differences between evolution of the disturbances of vorticity and evolution of the disturbance of passive vector **B** with the same diffusivity can be seen as originating due to the nonlinear effects in genuine NSE turbulence even during the linear regime, see figure 9.6. Note the much faster growth of the energy of disturbances of active variables such as vorticity during the very initial (linear in the disturbance) regime and decay of disturbances associated with passive scalar. For more details and other results concerning differences between the evolution of disturbances of ω and **B** see Tsinober and Galanti (2003).

It is noteworthy that the gradient **G** of a passive scalar and the vector potential **A** of magnetic field **B** (=*rot***A**) both are frozen in the flows field in the sense that both are material surfaces in the purely nondiffusive case. But they are qualitatively different as the former is irrotational, whereas the latter is solenoidal. That is they belong to qualitatively different subsets of material surfaces and one should expect them to behave qualitatively differently as well. For example, in case of finite diffusivity shown in figure 9.5, the disturbance of **G** is decaying whereas that of **A** is growing exponentially. It has to be stressed that typically there is a single surface element passing through a point in space corresponding to the fields **G** or **A**, but there are infinitely many other (neither irrotational nor solenoidal) surface elements passing through this point, i.e., there are infinitely many fields **N** corresponding to surface elements. In other words, **G** and **A** represent small subsets (qualitatively different) of material surfaces.

The above concerns differences in the behavior of disturbances of vorticity and passive vectors in an Euler setting. These differences are not the same in the Lagrangian setting mainly due to the fact that in the Lagrangian setting that (unlike in the Euler approach) there is also a disturbance of strain s_{ij} too in the equation for passive objects, Galanti et al. (2008).

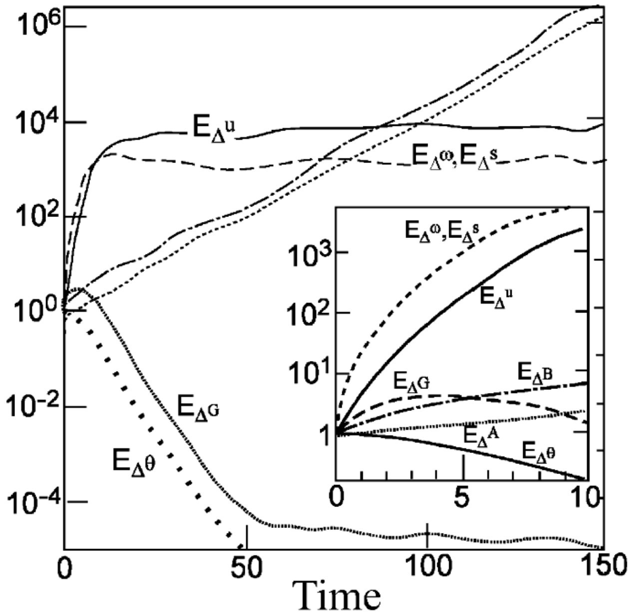

Figure 9.5. Time evolution of the energy of disturbances of velocity $E_{\Delta^u} = \frac{1}{2}\int(\Delta^u)^2 dV$, vorticity $E_{\Delta^\omega} = \frac{1}{2}\int(\Delta^\omega)^2 dV$, strain $E_{\Delta^s} = \frac{1}{2}\int(\Delta^s)^2 dV$, the vector potential $E_A = \frac{1}{2}\int A^2 dV$, passive vector $E_B = \frac{1}{2}\int B^2 dV$, passive scalar $E_\theta = \frac{1}{2}\int \theta^2 dV$, and its gradient $E_G = \frac{1}{2}\int G^2 dV$. All quantities are normalized on their initial values. Tsinober and Galanti (2003)

9.5. Summary

9.5.1. GENERAL REMARKS

Until recently the emphasis was on analogies between genuine and 'passive' turbulence. Most probably it started with the well known Reynolds analogy on transport of momentum and heat (Reynolds, 1874) and study of fluid motion by means of 'colour bands' (Reynolds, 1894). Since then such analogies were promoted in a number of papers as described in part in this chapter. However, the essential differences in the behaviour of passive and active fields including those described above point to serious limitations on analogies between the passive and active fields (and many others) and show that caution is necessary in promoting such analogies. They also serve as a warning that flow visualizations used for studying the structure of dynamical fields (velocity, vorticity, etc.) of turbulent flows may be quite misleading, making the question "what do we see?" extremely nontrivial. The general reason is that the passive objects may not 'want' to follow the dynamical fields (velocity, vorticity, etc.) due to the intricacy of the relation

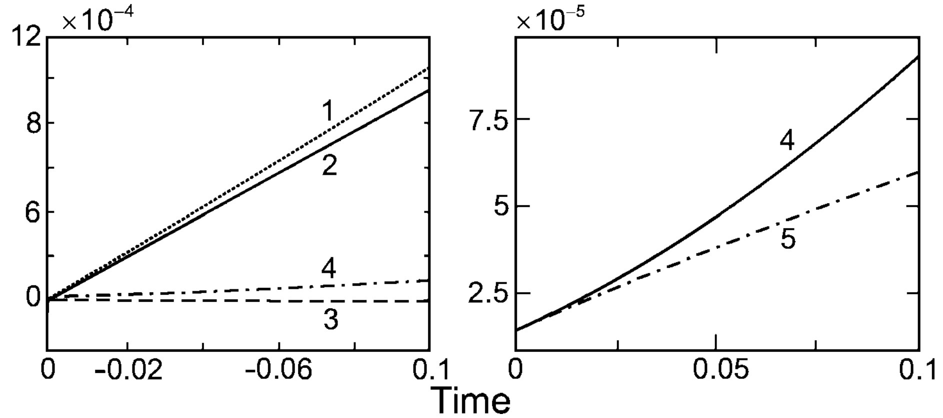

Figure 9.6. Left – time dependence of **1** – $\Sigma_{\Delta^\omega} = -\int \Delta^u_k \Delta^\omega_i \frac{\partial \omega_i}{\partial x_k} dV + \int \Delta^\omega_i \Delta^\omega_k s_{ik} dV + \int \Delta^\omega_i \Delta^s_{ik} \omega_k dV + \int \Delta^\omega_i \Delta^\omega_k \Delta^s_{ik} dV$ of the energy of disturbance Δ^ω in the equation (C.60), and contributions of separate terms to Σ_{Δ^ω} in the proximity of the origin $t = 0$. **2** – $-\int \Delta^u_k \Delta^\omega_i \frac{\partial \omega_i}{\partial x_k} dV$; **3** – $\int \Delta^\omega_i \Delta^s_{ik} \omega_k dV + \int \Delta^\omega_i \Delta^\omega_k \Delta^s_{ik} dV$; **4** – $\int \Delta^\omega_i \Delta^\omega_k s_{ik} dV$. Note that the main contribution is due to the vorticity gradients. Right – comparison of time dependence of the two analogous production terms **4** – $\int \Delta^\omega_i \Delta^\omega_k s_{ik} dV$ and **5** – $\int \Delta^B_i \Delta^B_k s_{ik} dV$. Tsinober and Galanti (2003)

between passive and active fields (in the same flow) and Lagrangian chaos, just as there is no one-to-one relation between the Lagrangian and Eulerian statistical properties in turbulent flows. This does not mean that qualitative and even quantitative study of fluid motion by means of 'colour bands' is always impossible or necessarily erroneous. However, watching the dynamics of material 'coloured bands' in a flow may not reveal the nature of the underlying motion, and even in the case of right qualitative observations the right result may come not necessarily for the right reasons. The famous verse by Richardson belongs to this kind of observation. It is the right place to recall the outstanding and specific property of genuine turbulence – *self*-amplification of the field of *strain*. This is underlying some of (but not all) main differences between genuine and passive turbulence since there is no counterpart to this process in the behaviour of passive objects. It is a reflection of a more general property of genuine turbulence possessing an intrinsic *dynamical* mechanism generating randomness (intrinsic stochasticity), whereas in the case of passive objects randomness is imposed by the velocity field and/or forcing. On the other hand, there are properties of passive objects which do depend on the details of the velocity field (see above, Tsinober, 2007; Tsinober and Galanti, 2003 and references therein). Just these very properties can be effectively used to study the differences between the real turbulent flows and the artificial random fields. More precisely the essential differences in the behaviour of passive objects in a real and synthetic turbulence may be exploited in order to gain more insight into the dynamics of real turbulence. At present, however, the knowledge necessary for such a use is very far from being sufficient. With few exceptions it is even not clear what can be learnt about the dynamics of turbulence from studies of passive objects (scalars and vectors) in real and 'synthetic' turbulence. This requires systematic comparative studies of both. An attempt of such a comparative study was made in Tsinober and Galanti (2003). This is a relatively-small part of a much broader field of comparative study of 'passive' turbulence reflecting the kinematical aspects and genuine turbulence representing also the dynamical processes. It seems that this branch of turbulence research is quite promising.

In the following subsections three lists are given as a kind of summaries of main accents of this chapter: 1) evolution of vorticity versus passive vectors, 2) other/more analogies, and 3) a list of misconceptions and ill-defined concepts, mentioned and not mentioned in this and previous chapters.

9.5.2. EVOLUTION OF VORTICITY VERSUS PASSIVE VECTORS

• – The equation for a material line element l is a linear one and the vector l is passive, i.e., the fluid flow does not 'know' anything whatsoever about l: the vector l (as any passive vector) does not exert any influence on the fluid flow. The material element is stretched (compressed) locally at an exponential rate proportional to the rate of strain along the direction of l, since the strain is independent of l.

• – On the contrary, the equation for vorticity is a nonlinear partial differential equation and the vector ω is an active one – it 'reacts back' on the fluid flow. The strain does depend in a nonlocal manner on ω and vice versa, i.e., the rate of vortex stretching is a nonlocal quantity, whereas the rate of stretching of material lines is a local one. Therefore the rate of vortex stretching (compressing) is different from the exponential one and is unknown. There are much 'fewer' vorticity lines than the material ones – at each point there is typically only one vortex line, but infinitely many material lines. This leads to differences in the statistical properties of the two fields. In the absence of viscosity, vortex lines are material lines, but they are special in the sense that they are not passive as are all the other material lines. Vorticity is divergence-free, whereas material elements are not, with the exception of the special ones.

• – Consequently while a material element l tends to be aligned with the eigenvector corresponding to the largest (positive) eigenvalue of s_{ij}, whereas vorticity ω tends to be aligned with the eigenvector corresponding to the intermediate (positively skewed) eigenvalue of s_{ij}: the eigenframe of s_{ij} rotates with an angular velocity Ω_s of the order of vorticity ω.

• – For a Gaussian isotropic velocity field the mean enstrophy generation vanishes identically, $\langle\omega_i\omega_j s_{ij}\rangle \equiv 0$ whereas the mean rate of stretching of material lines is essentially positive. The same is true of the mean rate of vortex stretching $\langle\omega_i\omega_j s_{ij}\rangle|\omega|^{-2}$ and for purely two-dimensional flows. In turbulent flows the mean growth rate of material lines is larger than that of vorticity. The nature of the vortex stretching process is dynamical and not a kinematic one as is the stretching of material lines and other passive vectors.

• – The curvature of vortex lines increases with strain and positive rate of vortex stretching, whereas the curvature of material lines decreases with strain and positive rate of material line stretching.

• – An additional difference due to viscosity is more than essential due to the Tennekes and Lumley balance at any large Reynolds number. Vortex reconnection is allowed by nonzero viscosity. No such phenomena exist for material lines. Reconnection is possible in 2D for magnetic field, it is not with vorticity.

• – Comparing vorticity with a passive vector in the presence of the same diffusivity as viscosity, the analogy is partial not just because the equation for vorticity is nonlinear, but also because in the case of vorticity the process is due to self-amplification of the field of velocity derivatives, whereas in case of a passive vector it is not.

• – In presence of diffusivity the energy of a passive vector under certain conditions grows without limit, i.e., not balanced by diffusivity, whereas growth of enstrophy is balanced by viscous effects.

• – Evolution of disturbances of vorticity is qualitatively different from that of passive vectors.

9.5.3. OTHER/MORE ANALOGIES

• 'Burgulence'[17].

• Dynamical systems, e.g., low-dimensional description. Truncated Euler and loop removals. Shell models.

• Analogy between the Navier–Stokes equations and Maxwell's equations: application to turbulence. Screening.

• Beyond the Navier–Stokes equations, e.g., analogy between Boltzmann kinetic theory of fluids and turbulence.

• Modelling nearly-incompressible turbulence with minimum Fisher information.

• Neural networks approach, the simulation and interpretation of free turbulence with a cognitive neural system.

• Analogy with statistical physics/mechanics. Variety of approaches from statistical physics/mechanics such as critical phenomena, Levy walks, Gibbsian hypothesis in turbulence, Tsalis non-extensive statistics, quantum kinetic models of turbulence, superfluid turbulence.

• Polymer analogies.

• Stock market dynamics and turbulence: parallel analysis of fluctuation phenomena.

There are more, but all (as the above) with modest success (if at all).

[17] Even Kraichnan and Sinai were tempted by this analogy: *In order to keep the formalism as simple as possible, we shall, work here with the one-dimensional scalar analog to the Navier–Stokes equation proposed by Burgers. In the method to be presented here, the true problem is replaced by models that lead, without approximation, to closed equations for correlation functions and averaged Green's functions… The treatment of Navier–Stokes equation for an incompressible fluid, which we shall discuss briefly, does not differ in essentials.* Kraichnan (1961).

Mathematical analysis will deal with several basic models. The simplest one is the 1D Burgers equation with random forcing. It displays several basic features of turbulence… 3D Navier–Stokes systems probably need completely new ideas. Sinai (1999).

9.5.4. MISCONCEPTIONS AND ILL-DEFINED CONCEPTS, MENTIONED AND NOT MENTIONED ABOVE

Some of the major misconceptions were mentioned above and in previous chapters. The latter include the concepts of inertial range and cascade. Here we mention some additional ones.

• – 'Statistical' and 'structural' contrapose each other.
• – Turbulence is nearly Gaussian and/or possesses a random (quasi-) Gaussian background.
• – Kolmogorov picture is structureless and quasi-Gaussian, i.e., K41 is consistent with near Gaussian statistics.
• – Large scales and small scales are statistically decoupled.
• – Turbulence can be described adequately by equations 'simpler' than the Navier–Stokes equations, e.g., by a low-dimensional system.
• – 'Eddy viscosity' and 'eddy diffusivity' explain the enhanced transfer rates of momentum, energy and passive objects.
• – Spatial fluxes represent 'cascade' in physical space.
• – At large Re the ratio of nonlinear and the viscous terms is large.
• – Vorticity amplification is a result of the kinematics of turbulence, i.e.,
• – Vortex lines are on average stretched rather than compressed, because two particles on average move apart from each other.
• – TKE production is the consequence of predominant vortex stretching.
• – When Re is large, vorticity is virtually frozen into the fluid.
• – The vorticity intensification process is the strongest where vorticity already happens to be large.
• – Vorticity is stretched only. Hence inadequate representation of turbulent field by a collection of purely stretched (or other 'simple') objects.
• – Concentrated vorticity dominates the flow and is quasi-two-dimensional.
• – Reynolds number represents the ratio of inertial to viscous forces.
• – For large Re the viscous interactions turn out to be quite weak.
• – Enhanced dissipation in turbulent flows is due to vortex stretching.
• – Strain rate in turbulent flows is irrotational.
• – Turbulent energy production is due to predominant (vortex and/or material elements) stretching.
• – The difference between quasi-two-dimensional and pure-two-dimensional turbulent flows is always small. Q2D and even P2D is always low dissipative.
• – For the very smallest eddies the motion is entirely laminar.
• – Efficient mixing requires random velocity field.
• – Well established phenomenological parallels between the statistical description of mixing and fluid turbulence itself and similar analogies between 'passive' and genuine turbulence.
• – The primary mechanism for production of scalar dissipation is the nonlinear amplification of scalar gradients by strain rate.
• – Richardson's energy cascade is a direct consequence of vortex stretching.
• – Spatial fluxes represent 'cascade' in physical space.
• – Ambiguity of language, 'definitions' of 'scale' or 'eddy'.

CONCLUSION/CLOSE

So what is important?

> *The problem of turbulence is not just to find more accurate formulae for various physical quantities associated with a turbulent fluid, but also to obtain a conceptually satisfactory theory based on first principles... In spite of satisfaction which one may have in writing rigorous inequalities originating from nontrivial linear theory, it must be said that the great difficulty which remains is to understand the nonlinear objects of turbulence* (Ruelle, 1990).
> *In spite of a huge number of papers and a large amount of research on turbulence, it remains an unsolved problem left for future generations* (Sinai, 1999).
> *...with all due respect for the coherent structures of the seventies, the insight gained from the chaos theory in the eighties, the achievements of Direct Numerical Simulations (DNS) and Large Eddy Simulation in the nineties, we all hope in our heart of hearts to see the great breakthrough in turbulence, stirring Sir Horace Lamb in his eternal sleep to bring him the long awaited revelation* (Wijngaarden, 2000).
> *Even after 100 years turbulence studies are still in their infancy. We do have a crude practical working understanding of many turbulence phenomena but certainly nothing approaching comprehensive theory and nothing that will provide predictions of an accuracy demanded by designers* (Lumley and Yaglom, 2001).
> *Turbulence nears a final answer* (Frisch, 1999).

The purpose of this last chapter is twofold. First, it aims to recapitulate some main points with somewhat different emphasis, and to discuss some issues of general nature not addressed above. The first issue is universality.

10.1. Universality

Since the Kolmogorov papers (1941a,b), there exists almost a religious belief in some universal properties of turbulence. This belief was strengthened by achievements in dynamical chaos, such as the discovery of some universal numbers by Feigenbaum, etc.

On the other hand, with the exception of 4/5 law, there appeared to exist no quantitative universality so far: the first doubt came from the famous remark by Landau in the first Russian edition of Fluid Mechanics by Landau and Lifshits about the fluctuations of energy dissipation rate. These were followed by various 'universal' corrections, which did not appear to be universal either. These corrections were followed by the (multi-)fractal approach using either the so-called $D(h)$ or $f(\alpha)$ formalisms, in which the *functions* $D(h)$ and/or $f(\alpha)$ are assumed to be universal. However, they do not seem to be universal either. There is quite solid evidence accumulated during the last 50 years against the most beautiful hypothesis on the restoring of the symmetries in the statistical sense of the Navier–Stokes equations locally in time and space, i.e., local isotropy together with scale invariance (see discussion and references in chapters 5, 6 and 7). And so people started to look for some universality in the anisotropic properties of turbulent flows (see references in Biferale and Procaccia, 2005; Kurien and Sreenivasan, 2001b). This involves the SO(3) decomposition of tensorial objects *assuming universal (!) scaling behaviour in* r *of each component of the decomposition.* Consequently there is no "simple" scaling of, say, structure functions in r, but rather the different terms of the SO(3) decomposition each with its own scaling exponent assumed to be universal. Thus all the attraction of simple scaling as in Kolmogorov 41 has gone.

The assumption of universality has no serious justification and is more a kind of a belief much weaker than the belief in the inertial range. Moreover, it is not clear at all why each "sector" of the irreducible representation is expected to have its own universal scaling exponent independently of the physical/dynamical nature/underlying mechanisms of anisotropy such as mean shear, strain, rotation, stratification (both stable and unstable), magnetic field, etc.? The expectation of universality is especially problematic in case of strong anisotropy (Q2D) in all the above cases. There is a claim that the amplitudes of the various contributions are non-universal and that it is possible to fit the experimental data by keeping fixed the scaling properties and adjusting only the prefactors[1]. This however, does not prove much regarding universality and may well be the "right result not necessarily for the right reason". One more difficulty may arise due to non-uniqueness of the SO(3) decomposition in the sense that there exists more than one possibility to choose its basis in the case when the SO(3) decomposition is applied to tensorial objects. There are also similar claims on universality related to passive objects. This kind of a claim is quite surprising, as passive objects are governed by linear equations and thus its

[1]It is also noteworthy that in determination of anisotropic scaling exponents one encounters the same kind of difficulties as those known from previous experience (see, for example, figures 6, 10 and 11 in Anselmet et al., 1984 and figure 8.6 in Frisch, 1995).

statistics and scaling exponents are expected to be sensitive to the statistics of the velocity field (see Falkovich et al., 2001; Majda and Kramer, 1999 and references therein).

In other words, it seems that our dream of quantitative universality of turbulence, i.e., universality of numbers, may never come true[2]. The main reason is the nonlocality leading to breaking of the symmetries embodied in the Navier–Stokes (and Euler) equations along with the ill-posedness of the concept of inertial range as discussed in section 5.3. However, though there may not exist such a thing as quantitative universality of turbulence (i.e., universality of numbers), there seems to exist a qualitative one. It is natural to include into the term 'qualitative universality' such general properties of turbulence as randomness, enhanced effective diffusivity and dissipation, rotational nature, and others as discussed in chapter 1. The question is whether there exist more *specific* qualitative universal properties of turbulent flows. The answer is positive. Moreover, these qualitative universal properties possess quantitative attributes, as will be seen in what follows. The likely reason for the qualitative universality is that the *non-linear terms ... remain active at surprisingly low Reynolds numbers,* as observed by Mansour and Wray (1994) in DNS of decaying turbulence at low Reynolds numbers. The resemblance of the flow patterns of turbulent flows in the same geometry, but at very different Reynolds numbers, also can be seen as one of the manifestations of the qualitative universality (see figures 1.9 and 1.12).

10.1.1. SELF-AMPLIFICATION OF VELOCITY DERIVATIVES

As discussed in chapter 6 there is some evidence that the process of self-amplification of velocity derivatives, both vorticity and strain, is a universal phenomenon which occurs at Reynolds numbers as low as $Re_\lambda \sim 60$. This seems to be one of the key physical processes in all turbulent flows. The details of this process are not well understood, and apart from geometrical statistics and similar information and tools, one needs much more. Since the whole flow is defined by the field of velocity derivatives (either vorticity or strain), proper understanding of the process of self-amplification of velocity derivatives in turbulent flows would bring considerable progress in the understanding of the physics of turbulence as a whole.

It should be stressed that there is an essential and qualitative difference between the process of self-amplification of strain and other similar processes. It is a specific feature of the dynamics of turbulence having no counterpart in the behaviour of passive objects. In contrast, the process of

[2]For other negative statements about universality see, for example, Saffman (1978, p. 216) and Hunt and Carruthers (1990, pp. 497, 498).

self-amplification of vorticity, along with essential differences, has a number of common features with analogous processes in passive vectors; in both the main factor is their interaction with strain, whereas the production of strain is much more 'self', i.e., local (section 6.3).

10.1.2. TENNEKES AND LUMLEY BALANCE

As discussed in chapter 6, this property is manifested in the approximate balance between enstrophy (strain) production and viscous destruction and similar (but qualitatively different) balance for passive objects. The main point here is that this property is observed for a variety of flows and for a wide range of Reynolds numbers starting with rather moderate ones.

10.1.3. ON UNIVERSAL ASPECTS OF TURBULENCE STRUCTURE

In dynamical systems, one looks for structure in the *phase space* (Shlesinger, 2000; Zaslavsky, 1999), since it is relatively 'easy' due to low-dimensional nature of the problems involved. In turbulence nothing is known about its properties in the corresponding infinite-dimensional phase space[3]. Therefore, it is common to look for structure in the *physical space* with the hope that the structure(s) of turbulence – as we observe it in *physical space* – is (are) the manifestation of the generic structural properties of mathematical objects (*in phase space*), which are called (strange) attractors and which are invariant in some sense. In other words, the structure(s) is (are) assumed to be 'built in' in the turbulence independently of its (their) origin – hence universality. However, as mentioned, the expectation for universal numbers seems to be unjustified. It is more natural to expect universal qualitative statistical features in the physical space rather than universal numbers. Indeed, some of such features have been already observed, which are common for very different – essentially all known – turbulent flows. These are not only the general qualitative features of turbulent flows as described in chapter 1, but rather specific ones.

We bring three examples with features which are essentially the same for all known incompressible flows such as grid-turbulent flow, periodic flow in a computational box, turbulent boundary layer and channel flow, mixing layer and compressible flows as well. Such features can be seen as universal statistical manifestations of the structure of turbulent flows.

The first example is the so-called 'tearing-drop' feature observed in the invariant map of the second invariant, $Q = \frac{1}{4}(\omega^2 - 2s_{ik}s_{ik})$, versus the third invariant $R = -\frac{1}{3}(s_{ik}s_{km}s_{mi} + \frac{3}{4}\omega_i\omega_k s_{ik})$ of the velocity gradient tensor

[3]Hopf (1948) conjectured that the underlying attractor is finite-dimensional due to presence of viscosity.

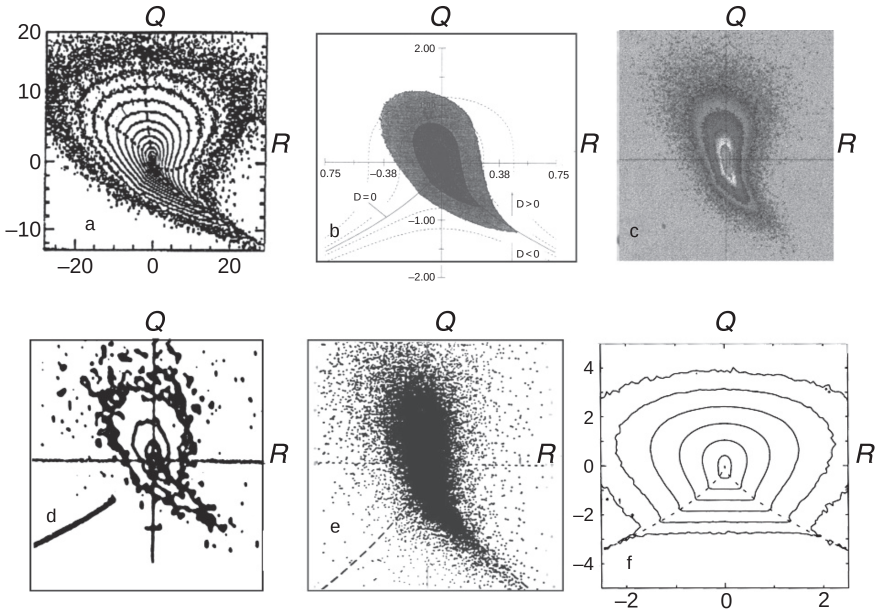

Figure 10.1. The 'tear-drop' pattern in the $Q - R$ plot, a) through e), in different turbulent flows and the symmetric pattern, and (f) for a Gaussian velocity field (Chertkov et al., 1999). a) – turbulent flow in a periodic box (Borue and Orszag, 1998); b) – turbulent boundary layer (Chong et al., 1998); c) – compressible flow, courtesy of A. Pouquet, P. Woodward and D. Porter; d) – mixing layer (Soria et al., 1994); e) – turbulent grid flow (Tsinober et al., 1997). Note that the two invariants are not describing all the aspects of the structure of the field of velocity derivatives, see Chacin and Cantwell (2000) and Tsinober (2000) for $Q - R$ plots with a variety of additional information

$\partial u_i/\partial x_k$. This feature appears to be essentially the same for a great variety of flows, some of which are shown in figure 9.1.

We draw attention to the 'tail' of the tear drop which is mainly located in the quadrant $Q < 0,\ R > 0$, in which most of turbulent activity happens in a variety of ways (Chacin and Cantwell, 2000; Chertkov et al., 1999; Tsinober, 2000). The important point is that this is the region dominated by strain as compared with enstrophy ($2s_{ik}s_{ik} > \omega^2$) and by *production* of strain as compared with production of enstrophy ($-s_{ik}s_{km}s_{mi} > \frac{3}{4}\omega_i\omega_k s_{ik}$). This is in full conformity with the behaviour of nonlinearities in these regions, see section 6.5 and figure 10.2 below).

The second example is related to depression of nonlinearity. We mention here one aspect of this problem, which seems to be universal in the sense that it is true for different flows and different Reynolds numbers, though the evidence is still quite limited. Namely, practically all nonlinearities appear to be much stronger in the strain-dominated regions rather than in regions

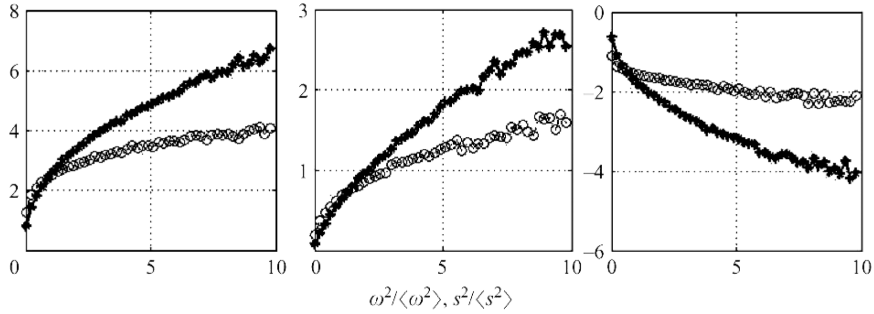

Figure 10.2. Examples of conditional averages of the eigencontributions to the enstrophy production showing the difference in the behaviour of nonlinearities in vorticity- (open symbols) and strain-dominated regions in an atmospheric surface turbulent boundary layer at $\mathrm{Re}_\lambda \sim 10^4$. Left – $\omega^2\Lambda_1\cos^2(\omega,\lambda_1)$, middle – $\omega^2\Lambda_2\cos^2(\omega,\lambda_2)$, right – $\omega^2\Lambda_3\cos^2(\omega,\lambda_3)$ (Gulitski et al., 2007a)

with concentrated vorticity, in contrast to the common expectation that, for example, the vorticity amplification process will be strongest where the vorticity already happens to be large. The regions with concentrated vorticity are in approximate equilibrium in the sense that the rate of enstrophy production is in approximate balance with the viscous destruction in these regions even at low Reynolds numbers, $\mathrm{Re}_\lambda \sim 80$. Therefore, their life time is considerably larger than the life time of the regions dominated by strain, which are in strong disequilibrium in the sense that the rate of enstrophy production is much larger than its destruction by viscosity in these regions. Here an example from the latest observations at large Reynolds numbers is shown in figure 9.4.

The third example is related to *geometrical statistics*. These are various alignments such as the alignments between vorticity and the eigenbasis of the rate of strain tensor, and between vorticity and the vortex stretching vector. It appears that these and many other similar properties are the same for all known flows and, moreover, for a broad range of Reynolds numbers. For example, the character of the above alignments is essentially the same at $\mathrm{Re}_\lambda \sim 10^2$ and $\mathrm{Re}_\lambda \sim 10^4$. An example for $\mathrm{Re}_\lambda \sim 10^4$ is shown in figure 9.3.

It is seen that the qualitative[4] behaviour of the above mentioned alignments is precisely the same as at $\mathrm{Re}_\lambda = 75$ shown in figure 6.7.

The same similarity was observed recently for a variety of alignments and other properties associated with fluid particle accelerations, Gulitski et al. (2007b). A variety of similar results were obtained for passive scalars,

[4]The quantitative difference is mostly due to the problems of underresolution in the case of $\mathrm{Re}_\lambda = 10^4$.

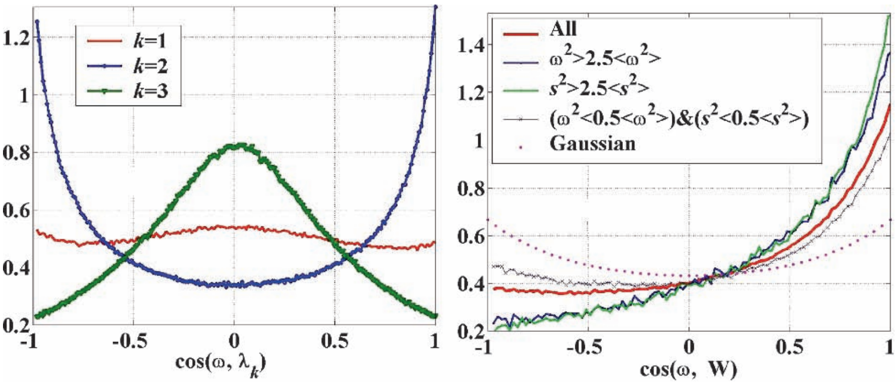

Figure 10.3. Alignments in a turbulent boundary layer at $\mathrm{Re}_\lambda = 10^4$. Right – PDFs of the cosine of the angle between vorticity, and the vortex stretching vector, $\cos(\omega, W)$. Left – PDFs of the cosine of the angle between vorticity, and the eigenframe of the rate of strain tensor, $\cos(\omega, \lambda_k)$ (Kholmyansky et al., 2001b)

Gulitski et al. (2007c), such as alignments of passive-scalar gradient, **G** with the eigenframe λ_i of the rate of strain tensor, conditional averages on ω^2 and s^2 of the production $-G_iG_js_{ij}$, tilting of **G** and some others.

This brings us to the issue of Re-dependence and the (possible) asymptotic state of turbulent flows at very large Reynolds numbers.

10.2. Reynolds-number dependence and the limit of vanishing viscosity

> *Theoreticians like to handle cases with very small or very large Reynolds numbers. Experimentalists and engineers encounter finite Reynolds numbers in real life* (Liepmann, 1962).
> *If no parameters in the world were very large or very small, science would reduce to an exhaustive list of everything* (Trefethen, 1998).
> *Does fully-developed turbulence exist? Reynolds-number independence versus asymptotic covariance* (Barenblatt and Goldenfeld, 1995).
> *Whatever the tools, it is crucial that the theory be able to describe what happens at finite Reynolds numbers, both to understand the limit of infinite Reynolds number and to interpret correctly the data from existing experiments and simulations* (Kraichnan, 1991).

Only *some* properties of turbulent flows become Reynolds-number independent as the Reynolds number becomes large enough. An example is given in figure 1.8 – the drag coefficient of a disc is independent of Reynolds number beyond $\mathrm{Re} \sim 10^3$. Another example is given in figure 5.1, showing the independence of dissipation of Reynolds number over three decades of

Re. There are many other related examples (Idelchik, 1996), just like the friction factor for pipes with rough walls possessing large enough roughness (Schlichting, 1979). It is this property that gives a special status to the scaling exponent 2/3. However, dissipation (energy input) or drag only are not sufficient to define the properties of a turbulent flow. For example, Bevilaqua and Lykoudis (1978) performed experiments on flows past a sphere and a porous disc with the same drag. However, other properties of these flows, even on the level of velocity fluctuations, were quite different; see also Wygnanski et al. (1986) who performed similar experiments with a larger variety of bodies with the same drag[5]. Similarly, many properties of turbulent flows with rough boundaries are not defined uniquely by their friction factor either (Krogstad and Antonia, 1999). Though the dissipation is known (empirically) to saturate to a nonzero limit as $\nu \to 0$, this however, does not mean that there exists a limit as $\nu \to 0$ in the sense (or any other sense) that all other flow characteristics do saturate as well.

There exists considerable evidence on Reynolds-number dependence of various properties in different turbulent flows; for a partial list of recent references, see Belin et al. (1997), Buschmann and Gad-el-Hak (2007), Ferchichi and Tavoularis (2000), Fisher et al. (2001), Kahaleras et al. (1998), McKeon (2007), Metzger et al. (2001), Moser et al. (1999), Shen and Warhaft (2000), Sreenivasan and Antonia (1997), Tsinober (1998b) and Zhou and Antonia (2000). For example, the flatness factor of the streamwise velocity derivative $\partial u_1/\partial x_1$ is increasing from $3 \div 4$ at $\mathrm{Re}_\lambda \sim 10$ to about 40 at $\mathrm{Re}_\lambda \sim 4 \cdot 10^4$, without showing any trend for saturation (see figure 6 in Sreenivasan and Antonia, 1997; Gulitski et al., 2007 and references therein). There is no understanding of the reasons for such a strong Reynolds-number dependence at large values of the Reynolds number. Another example is about the Reynolds-number dependence of the relation between the solenoidal and irrotational 'components' of the nonlinearity as represented by $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and Lamb vector $\omega \times \mathbf{u}$ shown in figure 10.4. There is a clear tendency of enhancement of solenoidality of the nonlinearity as the Reynolds number increases. Here too it is not clear what will happen when the Reynolds number will become very large.

The third example is the behavior of acceleration, see figure 6.22. Though the evidence is not conclusive, the indication is that there is no saturation of acceleration variance at large Reynolds numbers.

Finally, as discussed in section 5.3, the so-called 'anomalous scaling' in the inertial range is due to viscous/diffusive effects with two options. One is

[5]It should be kept in mind that in both cases the flow was partly-turbulent and most probably had different *large-scale stability* properties for different bodies not directly related to the turbulent nature of the flow within the wake. This may contribute too to the differences in the observations.

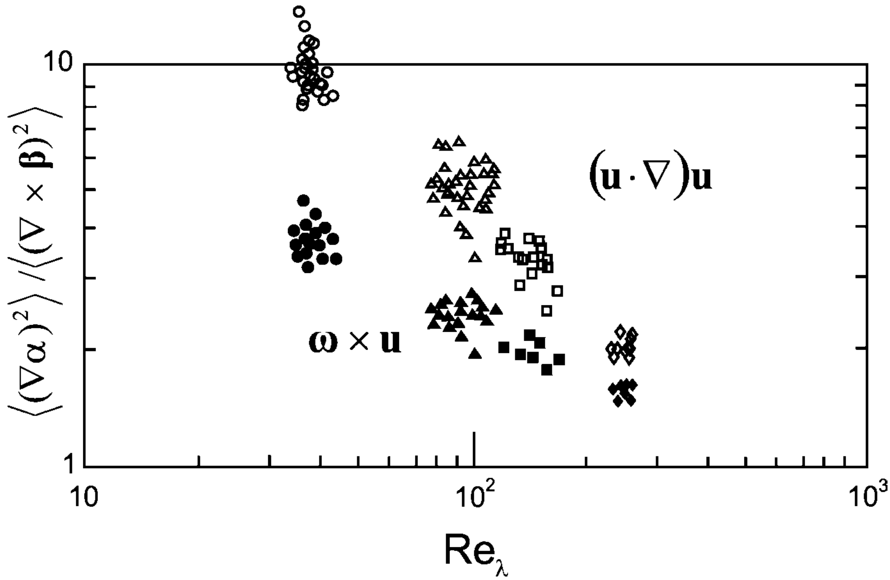

Figure 10.4. Reynolds-number dependence of the ratio of the variances of the irrotational and solenoidal parts of the nonlinear term $(\mathbf{u}\cdot\nabla)u$ and $\omega\times\mathbf{u}$ in a DNS simulation of quasi-isotropic turbulence (Tsinober et al., 2001)

just the finite-Reynolds-number effect. The other one is due to the influence of viscous/diffusive effects mostly (but not only) in the tails (i.e., strong events) of corresponding PDFs and is present at any Reynolds number. The evidence seems to favor the latter option.

These examples, along with other results, show that the issue of the asymptotic 'ultimate' regime/state of turbulent flows at very large Reynolds numbers remains and will remain open for quite a while.

10.3. Turbulence versus mathematics and vice versa

There are many mathematical publications which contain the term "turbulence" in some way or another[6].

So it is natural to ask the question: What really do we have from mathematics and mathematicians for understanding of turbulence? The claims are sometimes pretty strong. A recent example: *... there are enough firm results available assuring that many of the widely accepted experimental results are meaningful and in consonance with the theory of the Navier–Stokes*

[6]Here is a selection of publications with many more references in them: Bardos and Titi (2007); Constantin (2007); Doering (2009); Duchon and Robert (2000); Foias et al. (2001); Gibbon (2008); Hopf (1962); Ladyzhenskaya (1970, 1975, 2003); Leray (1933, 1934); Lions (1996); Ruelle (1983b, 1990); Shnirelman (2003); Smale (1977); Vishik and Fursikov (1988).

equations, Foias et al. (2001, p. 169). This is an example of the widespread view that what all experiments (both physical and numerical) should do is to validate a theory. Unfortunately, as mentioned in chapter 3, there is no theory in/of turbulence, at least as concerns its basic aspects. This is why Kolmogorov wrote in 1985: *I soon understood that there was little hope of developing a pure, closed theory, and because of absence of such a theory the investigation must be based on hypotheses obtained on processing experimental data* (see Tikhomirov, 1991, p. 487)[7].

All "consumers" of turbulence from those concerned with pure basic issues to people dealing with a great variety of applications believe that NSE are adequate for describing turbulent flows and would expect that NSE possess smooth solutions in a global sense, i.e., for all times and for any however large Reynolds numbers. Nevertheless, since Leray (1933, 1934) nobody has been able to prove this, so that this problem has been included by the Clay Mathematics Institute among seven major mathematical (not as a physical one) problems for the twenty-first century (Fefferman, 2000). If Navier–Stokes Equations are so nice, the question is again: what do the Navier–Stokes equations tell us about turbulence? (Foias, 1997). It appears, unfortunately, not so much (to put it mildly) as concerns mathematics and mathematicians.

At least since Kolmogorov (1941a,b) an enormous effort has been invested in attempts to study asymptotic properties of turbulent flows at vanishingly-small viscosity. Considerable evidence shows that these flows at however small viscosity possess nonvanishing dissipation (the so-called dissipation anomaly), i.e., such flows are very much unlike classical solutions of the Euler equations, which have the property of energy conservation for smooth velocity fields.

Less known (and less established) is that this seems to be the case for other dissipative mechanisms – most probably for a wide class of fluids (and even in other dissipative systems than fluid flows), e.g., fluids with hyperviscosity (but there are outstanding counter-examples). In other words this property is not specific of turbulence in the narrow sense and belongs to a number of manifestations of turbulence which are weakly (if at all) sensitive to the nature of dissipation at large Re (and even more generally to specific properties of the system as long as it is dissipative), e.g., things like 2/3, 4/5 and 4/15 laws, $k^{-5/3}$ spectrum and some others.

[7]This is also why this author (following the advice of Kolmogorov with quite a delay) switched to experiments after spending fifteen years on things like functional analysis and many other sophisticated things both in mathematics and theoretical physics at the very highest level all of which appeared to be of little help in turbulence research (so far). However, this was not in vain since, without a broad and deep corresponding theoretical background, it is impossible to properly plan, perform and especially to analyze the results of the experiments.

10.3.1. WEAK SOLUTIONS OF NAVIER–STOKES EQUATIONS

Duchon and Robert (2000) wrote a *local* energy balance for weak solutions of Navier–Stokes equations $\partial/\partial t(u^2/2) + \partial/\partial x_k\{u^2/2 + p)u_k\} - \nu\nabla^2(u^2/2) + \nu|\nabla u|^2 + D(u)$, where $D(u)$ is a distribution defined in terms of the local smoothness of velocity field $\mathbf{u}$. They found an explicit expression for $D(u)$ which makes the above equation identity in the sense of distributions. Thus $D(u)$ measures a possible dissipation (or production) of energy caused by a lack of smoothness in the velocity field $\mathbf{u}$ in the spirit of Onsager, 1949. For smooth solutions $D(u) \equiv 0$. *Thus* – they write – *the non-conservation of energy originates from two sources: viscous dissipation and a possible lack of smoothness in the solution*; and stress *that* $D(u)$ *measures a possible dissipation (or production) of energy caused by a lack of smoothness in the velocity field u, this term is by no means related to the presence or absence of viscosity.* The latter statement being formally/mathematically nice is problematic from the physical point of view. As long as one is speaking about NSE this looks definitely unphysical: so far no physical process is known that can bring an additional dissipation into operation that is formally described by the distribution $D(u)$. An important point is that if one looks at real turbulence at finite Reynolds numbers (however large) there seems to be no need for weak solutions at all.

10.3.2. THE LIMIT OF VANISHING VISCOSITY AND DISSIPATIVE SOLUTIONS OF EULER EQUATIONS

> *... There is still some doubt as to whether weak solutions of the Navier–Stokes equation, the uniqueness of which is unknown, or hypothetical weak solutions of the Euler equation, are relevant to the description of turbulent flows at high Reynolds numbers ...* (Duchon and Robert, 2000).
>
> *As for today, we have no weak solution (of the Euler equation) at hand which really describes a turbulent flow* (Shnirelman, 2003).
>
> *It may be less commonly appreciated that singular solutions of the incompressible Euler equations ... are a good candidate to describe turbulent flow in the asymptotic limit $Re \to \infty$, as first conjectured by Onsager* (Eyink, 2008).

One of the ‘natural’ conjectures in the mathematical community was that turbulent flows may be described asymptotically correctly by some sort of specially selected weak solutions of the Euler equations which are called “dissipative” – an approach which goes back to Onsager (1949). Indeed, examples of weak (or distributional) solutions have been constructed without energy conservation (Lions, 1996; Shnirelman, 2003; De Lellis and Sžekelyhidi, 2007; see Eyink, 2008 for more references). It appears that there exist very different kinds (examples only) of weak solutions, having little in common, and some of them are physically meaningless (with negative dissipation, i.e., energy creation), at least, in the context of turbulent flows. Moreover, there is no uniqueness of a weak solution: the phase space of Euler is too rich. In other words, one needs additional conditions to ensure physical meaning and uniqueness of solution. Simply stated the solution has to be dissipative in the first place. But this condition is not sufficient to guarantee the uniqueness and adequacy to real dissipation! Moreover, the dissipation is not the only issue either, even in finite Reynolds-number flows. Indeed, dissipation (energy input) or drag only are not sufficient to define the properties of a turbulent flow. As mentioned, Bevilaqua and Lykoudis (1978) and Wygnanski et al. (1986) performed experiments on flows past a sphere and a porous disc and other bodies with the same drag, i.e., the same dissipation (more examples can be found in George, 2008). However, other properties of these flows even on the level of velocity fluctuations were essentially different. Strongly-stratified flows are quasi-two-dimensional, but exhibit a $k^{-5/3}$ energy spectrum, a forward energy cascade and large dissipation comparable to non-stratified three-dimensional flows.

It is not clear whether there exist a set of criteria apart from positive dissipation enabling one to select the 'right' generalized/weak solutions[8] of the Euler equation adequately describing turbulent flows at $\mathrm{Re} \to \infty$. To quote Shnirelman, 2003: *as for today, we have no weak solution at hand which really describes a turbulent flow* (the observed one). If this is not a dream. Having a "good candidate" it would be an extremely difficult (if not impossible) task to decide whether it really describes a turbulent flow. Moreover, to find such a candidate seems to be as difficult as the "solution of the problem of turbulence" itself.

In view of the above difficulties (existence, non-uniqueness, etc., see Eyink, 2008, pp. 1960–1961) the possible justification for the view *that singular solutions of the incompressible Euler equations... are a good candidate to describe turbulent flow in the asymptotic limit,* $\mathrm{Re} \to \infty$, is sought in *physical considerations* and, in particular, in *that experiments and simulations of high-Reynolds-number turbulence show, that scaling laws hold* $\langle |\delta \mathbf{u}(\mathbf{r})|^p \rangle^{1/p} \sim r^{\sigma_p}$ *for all* $p \geqslant 1$ *in the inertial range* $\eta \ll r \ll L$. The general claim is that *If Onsager is correct, then inertial-range dynamics of turbulent flow are governed by singular solutions of the Euler fluid equations. Observational evidence and rigorous results are consistent with the idea* (Eyink, 2008). The issue of consistency (with) and relevance (to) the

[8]There are several problems with the issue of weak solutions, e.g. i – the formal definition of a weak solution is too wide, ii – there may exist other objects than distributions – quoting Shnirelman, 2005 (private communication): *I believe that in a viscous fluid with vanishing viscosity what is left of viscosity is some force which is not a distribution as well. This "immaterial" force results in quite material dissipation of the kinetic energy.*

available experimental and numerical evidence is quite problematic for the following reasons. First, as mentioned before it is not clear why results for finite Re (i.e., for NSE having no singularities or extremely 'intermittent' ones) are relevant for the limit (if such exists) Re $\to \infty$ (e.g., for Euler equation with space-filling singularities) let alone that all the results have been obtained for moderate finite Reynolds numbers. Second, there are several problems with the evidence itself even at rather moderate Reynolds numbers. For example, it is at best marginal in the context of the very existence of *scaling laws* and especially of multi-fractal scaling behaviour[9], see chapters 5 and 7. It is hard to accept claims like *the observations suggest that Euler solutions relevant to infinite-Reynolds turbulence have* $\mathbf{u} \in B_p^{\sigma_p}$... (see p. 1961 in Eyink, 2008). The simplest example with the most reliable data, concerns the Onsager conjecture directly. Among the first beliefs is that the Kolmogorov 2/3 law (or 5/3) implies that for the typical flows the velocity field is a Hölder function with the Hölder exponent close to 1/3. This means that the solutions of NSE are asymptotically, as Re goes to infinity, some sort of generalized, or weak, solutions of the Euler equations. However, (assuming the existence of) the corresponding solutions of Euler (following the conjecture by Onsager, 1949 and subsequent proofs in Besov spaces, see refs in Constantin, 2007; Eyink, 2008) conserve energy if the exponent is larger than 1/3 and are expected (but not more than that) to be dissipative otherwise. The experimental observations *however* show that the exponents are *never* equal to 2/3 ($-5/3$) but are *larger*(!) than 2/3 (smaller than $-5/3$), see figure 10.5. With such exponents there is energy conservation in Euler. So if one takes the position that experimental observations at finite (relatively large) Reynolds numbers can imply anything about the *Euler solutions relevant to infinite-Reynolds turbulence*, the implication is negative. This does not invalidate Onsager's conjecture about the possible existence of dissipative solutions of Euler, but there is a long (if not infinite) distance between this conjecture and the claim that they are relevant to turbulence. The claim that *Onsager's conjecture is not about an esoteric or unphysical mathematical problem but, rather, about the fluid dynamics of turbulence at high Reynolds numbers* (Eyink, 2008) is quite a bit an overshoot. One of the premises of the above speculations on the relevance of generalized/weak solutions of Euler to turbulence is the assumption that the properties of the inertial range (IR) of turbulence at large Reynolds numbers are independent of viscosity/nature of dissipation as long as it is associated with "small scales". In other words, all that remains in the limit Re $\to \infty$ ($\nu \to 0$) is the inertial range and therefore the hypothetical limit

[9]We recall that scaling alone is a too broad characteristic. There is no one-to-one relation between the scalings (of whatever nature) and various properties of specific systems such as PDFs, structure and many others, see chapter 7.

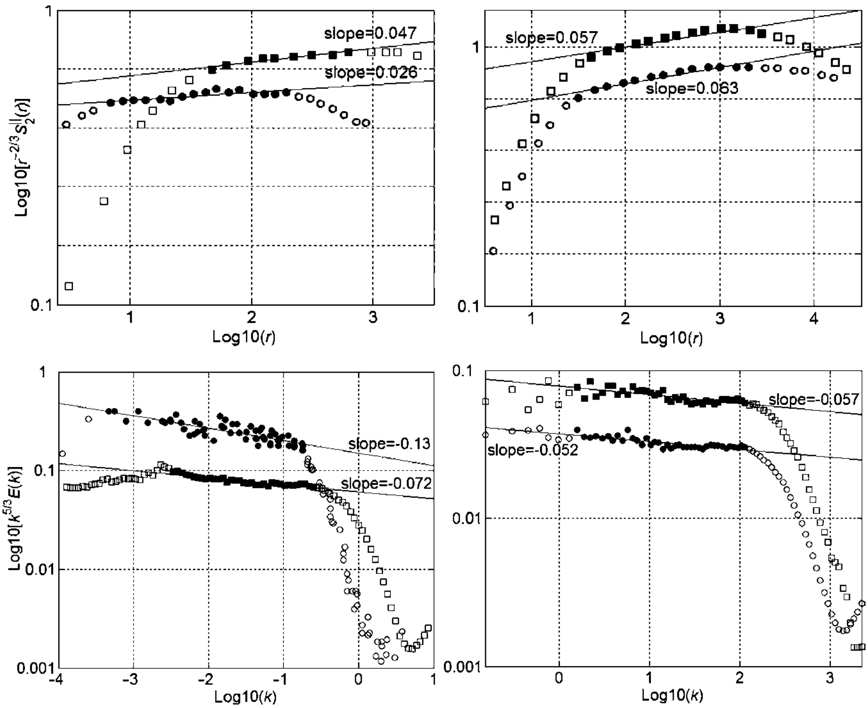

Figure 10.5. Examples of compensated second-order structure functions $r^{-2/3}S_2^{||}(r)$ (top panel) and energy spectra $k^{5/3}E(k)$ (bottom panel) in high-Reynolds-number experiments. Top left: squares – Chambers and Antonia, 1984; circles – Gagne, 1987. Top right: squares – Gulitski et al. (2007a); circles – Kholmyansky et al. (2001b). Bottom left: squares – Kurien and Sreenivasan (2001a); circles – Grant et al. (1962); Bottom right: squares – Gulitski et al. (2007a); circles – Kholmyansky et al. (2001b). The curves are slightly shifted

cannot be dependent on the nature of viscosity/nature of dissipation. However, there is hard evidence that at least some properties of IR do depend on the nature of dissipation at whatever large Re. It looks like an artificial trick to ascribe dissipation to an inviscid flow by imposing a very "rough" velocity field and moving to infinity the dissipative "tail"/sink of energy in a real flow (which is smooth) and thereby having a flux of energy to this sink. It is this flux which is interpreted as "dissipation". The premise is that the nature of this sink does not matter! However, the rough velocity field with space-filling dissipative singularities for Euler (but no singularities at all for Navier–Stokes) 'dissipates' energy 'everywhere', so where is the inertial range, which by definition is 'dissipationless', at least at any finite however large Reynolds number? All this makes the Euler dissipative business quite problematic regarding its relevance to turbulence.

10.3.3. NATURE OF DISSIPATION – IS IT (UN)IMPORTANT?

... the only connection between the equilibrium range and the remainder of the turbulence lies in the transfer of energy at a rate ϵ (Batchelor, 1953).

In any case, the dissipation processes, independently of their nature, serve only as energy sinks, which cut off the spectrum of turbulent fluctuations at small scales but do not affect the main turbulence scales (Biskamp, 2003).

In fact, turbulence is an inertial phenomenon. That is, turbulence is statistically indistinguishable on energy-containing scales in gases, liquids, slurries, foams, and many non-Newtonian media. These media have markedly different fine structures, and their mechanisms for dissipation of energy are quite different. This observation suggests that turbulence is an essentially inviscid, inertial phenomenon, and is uninfluenced by the precise nature of the viscous mechanism (Holmes, Berkooz and Lumley, 1996).

... there is nothing "irrelevant" in the (NSE) equation (except, maybe, as $\nu \to 0$, the precise nature of the dissipative term) (Frisch, 1984).

Causality is from large to small scale, and how the energy is dissipated in the latter does not influence the former, as long as the amount is correct ... (Jimenez, 2000).

The natural question is, therefore: Is it at all important that this subsidiary agent be viscosity? Might other dissipative, perturbing forces not do equally well? In planning for a test of this question, one might first think of investigating other forms of the law of viscosity, i.e., other equations of flow instead of those of Navier–Stokes, where viscosity might be described by a term other than $\nabla^2\mathbf{u}$ or by entirely different, nonlinear changes in the equations ... In any event, it would be interesting to determine, whether such modifications could lead to different forms of turbulence (in the pure limiting, i.e., $\nu \to 0$)... The whole character of the Kolmogorov–Onsager–Weizsacker theory would make one inclined to surmise that this is not the case (von Neuman, 1949).

We therefore conclude that, for the large eddies which are the basis of any turbulent flow, the viscosity is unimportant and may be equated to zero, so that the motion of these eddies obeys Euler's equation ... The viscosity of the fluid becomes important only for the smallest eddies, whose Reynolds number is comparable with unity ... (Landau and Lifshits, 1959).

Dissipation in real fluids is just the transfer of macroscopically organized (hydrodynamic) energy to molecular thermal energy. (Frisch et al., 2008).

Thus it is quite a common view that the precise nature of dissipation is mostly unimportant in high-Reynolds-number turbulence except for the smallest scales. This forms, e.g., the basis for what is called inertial range (IR), which properties are believed to be asymptotically independent of Re and/or the nature of dissipation. In view of various aspects of nonlocality the natural question is: what does it mean, what kind of quantities do not really depend on the nature of dissipation, why and in what sense, as well as many similar closely related questions, some of which were touched in chapters 5 and 6.

Is it obvious that the inviscid limit is always the "same" Euler independently of the nature of dissipation/viscosity? The $D(u)$ – as a distribution – may be (?) the same for different dissipation mechanisms, but the limiting "rough" velocity field may well depend on the nature of the dissipative processes. In other words the question is whether the hypothetical weak dissipative solutions of Euler (which are supposed to describe turbulence "adequately") depend on the nature of dissipation in the corresponding equations like NSE. Is the limit the same for any dissipative terms, be it Newtonian, hyper-viscous or whatever? This question is closely related to the issue of the influence of nature of dissipation and/or role of viscosity on the properties of turbulence for finite, however large $\mathrm{Re} \gg 1$, in general, and on what is called inertial range, in particular, see chapters 5 and 6. There is some evidence that the very concept of inertial range is ill-defined and that strong dissipative events (which appear to be not so rare) make a nonnegligible impact on the behaviour of traditionally-inertial characteristics such as structure functions and are at the origin of the so-called anomalous scaling, see section 5.3.

If the precise nature of dissipation is unimportant in high-Reynolds-number turbulence and if the nature of dissipation is not important either, why to work hard specifically on NSE instead of, e.g., taking some modified version of NSE (Leray, 1933, 1934[10]; Lions, 1969; Ladyzhenskaya, 1970, 1975; Friedlander and Pavlović, 2004 and references therein) or lattice gas hydrodynamics approximation (Chen and Doolen, 1998; Yu and Girimaji, 2005), which have regular solutions for any time and at any Reynolds numbers? And why does the Clay Mathematics Institute insist specifically on NSE? Is it really the case that the precise nature of dissipation is unimportant? Why not take such equations (which are all right at any Re and for all times) and use them to obtain the results which are claimed to be not achievable for NSE due to their "nasty" mathematical nature?

Is it really the case that the precise nature of dissipation is unimportant? This seems to be true (but mostly not proven) only in respect with

[10]The modification of NSE, introduced by Leray (1934), consists in mollifying the nonlinearity rather than changing the dissipative term as did the other authors.

a number of manifestations of turbulence which are really weakly-sensitive to the nature of dissipation at large Re (and even more generally to specific properties of the system as long as it is dissipative), e.g., things like 2/3, 4/5 and 4/15 laws, $k^{-5/3}$ spectrum and some others. The most convincing example is the 4/5 law showing that the third-order structure function is universal, i.e., it depends on the mean energy injection rate only. It is noteworthy that Duchon and Robert (2000) proved a local version of the 4/3 law with the mean energy dissipation rate replaced by an instantaneous mean energy dissipation rate over a local region of flow. Eyink, 2003 extended their result to the 4/5 law and pointed to the similarities and differences between this result and the Kolmogorov, 1962 refined similarity hypothesis. It should be related to the experimental observation that $\int \omega_i \omega_j s_{ij} dx$ and $-\int s_{ij} s_{jk} s_{ki} dx$ become all positive for finite volumes smaller than some integral scale of turbulence. However, there are many aspects which do depend on the nature of dissipation.

10.3.4. ROLES OF VISCOSITY/DISSIPATION

ii – destabilizing factor (at least in some flows); iii – modifies substantially, qualitatively the nonlinearity, i.e., it is not a passive sink of energy; iv – adds to Lagrangian acceleration a solenoidal part (absent in Euler); v – allows/is a cause of vortex reconnection, i.e., changes the topology of the vorticity field[11]; vi – prevents singularities, at least in modified versions of NSE and, most probably in NSE too; vii – turns the system into a finite-dimensional one, Hopf (1948) conjectured that the underlying attractor is finite-dimensional due to presence of viscosity, Foias et al. (2001) proved some aspects, but all starts with Kolmogorov–Landau; and viii – makes the flow ergodic (most probably, see Foias et al., 2001; Galanti and Tsinober, 2004 and chapter 3).

Is it possible that the nature of such a factor can be unimportant in turbulence? At least the points ii–vi indicate the opposite.

Of particular interest here is the point iii. Modification of nonlinearity by the dissipative term is clearly seen from looking at the equations for vorticity and enstrophy, e.g., at any Reynolds number the enstrophy production is approximately balanced by its viscous destruction, the forcing term at this level is irrelevant, Tennekes and Lumley (1972), Tsinober (2001, 2007), chapter 6. An essential point is that the evidence (and physical considerations) clearly indicate that this balance is a universal property and holds

[11] There seems to be no one-to-one relation between the reconnection of vortex lines and the realm of viscosity influence. In other words, viscosity reconnection events reflect only a part of events/locations where viscosity is important. For example, the strain field is influenced by viscosity as well, but in an essentially different manner. Almost nothing is known about this last process.

at whatever large Reynolds numbers – the larger the Reynolds number the more precise the balance. That is turbulence is an essential interaction between nonlinear and linear processes at any however-high Re and not just a simple cascade of energy or whatever down to smaller scales. For example, in case of modified equations such as those with hyperviscosity the enstrophy balance is quite different from that for NSE, so that the nonlinearity is quite different as well. Along with nonlocality (direct interaction of large and small scales irrespective of their separation, see references in Tsinober (2001, 2003) and broken scale invariance this means that the nature of dissipation is felt in large scales as well, see again section 5.3. Direct influence of viscosity on the inertial-range properties was observed also in "turbulence" shell models (Leveque and She, 1995; Schörghofer et al., 1995 and Gledzer, 2005).

One may say that vorticity and strain are "small-scale" quantities and so they are "irrelevant" to the inertial range. The whole point is that they are relevant very much for several reasons no matter how large the Reynolds numbers (see chapters 5 and 6, see e.g. figures 5.2 and 6.19). We mention here the most "trivial" one: the velocity field (and thereby velocity increments and whatever) is fully-defined by both the field of vorticity and/or strain.

10.3.5. POSSIBLE CONSEQUENCES FOR THE INVISCID LIMIT

The differences in the behaviour of systems with different dissipation at finite Reynolds numbers points to a possibility that the hypothetical limiting solution will depend on the kind of dissipation we have at finite Reynolds number (recall the above quotation by Neumann, 1949) even if their "inviscid dissipation" manifested in $D(U)$ would be the same. Of special interest is what happens in this limit with enstrophy/strain and their production and similar things. For example, assume that for a hyperviscous case the mean dissipation $\epsilon \to const$ (or just nonvanishing) as some viscosity ν_h goes to zero, then velocity derivatives (both vorticity and strain) grow on the average as $\nu^{1/2h}$, which compared to the Newtonian case $h = 2$ is pretty slow, if say, $h = 8$ as used in many simulations. Similarly, the enstrophy/strain production will be different and thereby the limit of the equations for the enstrophy and strain for the Newtonian case and of the modified equations (as e.g., with hyperviscosity) will be not the same. In fact, it is necessary to clarify the meaning of limits of quantities involving vorticity and strain. For example, let us look at the equation for enstrophy ω^2: $1/2D\omega^2/Dt = \omega_i\omega_j s_{ij} + VT + TF$. The viscous term here, VT, is different for NSE and for the modified equations. Since in some sense the enstrophy production $\omega_i\omega_j s_{ij}$ is mostly balanced by VT, this implies that $\omega_i\omega_j s_{ij}$ is different for NSE and for the modified equations. A similar statement is true in respect of strain production $-s_{ij}s_{jk}s_{ki}$. The question is what happens in the inviscid limit to the field of velocity derivatives (vorticity and strain) and their production, etc., which play a crucial role in turbulence and in formidable problems in mathematical issues. Will, for example, the limit of $\omega_i\omega_j s_{ij}$ be the same in some sense or will it be different for different viscous models, just like any quantity related to vorticity or the whole field of velocity derivatives? There should be an essential difference as enstrophy production $\omega_i\omega_j s_{ij}$ should be balanced by something in the limit too! What kind of objects are these quantities in the limit? Also what happens to the above equation itself. The bottom line is the question whether one can arrive at different limits for Euler moving along different "paths" or if any path would lead to the same limit[12]. A more general question is how does look, say, the equation for enstrophy production following from Euler, with singular solutions, etc. An important consequence of the balance between enstrophy production $\omega_i\omega_j s_{ij}$ and its viscous destruction at any however large Re is that there is no such thing as *inviscid motion of vortex-lines* at any however large Reynolds numbers (see chapter 9)[13] so it seems meaningless to tackle the problem (in the limit $\nu \to 0$) *how to relate turbulent dissipation of energy, precisely, to the inviscid motion of vortex-lines* (Eyink, 2008)[14].

Another set of questions about the Euler equation itself concerns the fluid particle acceleration $\mathbf{a}$. With finite viscosity there is a solenoidal part of acceleration: $curl\mathbf{a} = D\omega/Dt - (\omega \cdot \nabla)\mathbf{u}$, which contains the vortex stretching/compressing term, $(\omega \cdot \nabla)\mathbf{u}$, and plays an essential role in turbulence at any Re! What happens to the solenoidal part of the fluid particles acceleration as ν goes to zero? Formally in the limit one stays with the gradient of pressure only. So far there is no evidence to guide a guess/conjecture. The evidence as shown in figure 10.4 at moderate Reynolds numbers clearly indicates an *increase* of solenoidality of the nonlinearity, but is of little help here due to strong cancellation effect between, e.g., the local acceleration $\partial\mathbf{u}/\partial t$ and the solenoidal part of $(\mathbf{u} \cdot \nabla)\mathbf{u}$, see chapter 6.

It is noteworthy that in the two-dimensional case there are at least a number of examples in which the limit $\nu \to 0$ depends explicitly on the nature of the dissipative term (Kuksin, 2007, private communication).

[12]The issue dates back to Neumann in 1949: *the laws of non-viscous flow have a highly multiple infinite family of gliding and vortex motions (gliding is actually a limiting form of vortex motion) indeterminate. Any small amount of viscosity establishes a definite statistical pattern for these "extra" motions, and this pattern appears to be perfectly definite even in the limiting case of viscosity tending to zero* ($\nu \to 0$). So one wonders how presence of whatever *small amount of viscosity* aids the selection from the vast phase space of Euler, and as mentioned whether the nature of dissipation matters in this selection.

[13]This is like being slightly pregnant.

[14]The reader is reminded that the local causal relation is between dissipation and strain (and its production, see eq. (C.18)) rather than vorticity. See also chapters 6 and 9.

One more question concerns a number of issues in which the limit $\mathrm{Re} \to \infty$ (and other limits) is manipulated rather freely in the context of the behaviour of passive objects. A recent example is the so-called 'breakdown of the Lagrangian flow' or 'spontaneous stochasticity' (see references in Falkovich et al., 2001 and Eyink, 2008), which is roughly manifested in non-uniqueness and stochasticity of Lagrangian trajectories for an individual velocity field realization which is assumed to be rough in a manner convenient for analytical treatment but not necessarily physically-meaningful and/or corresponding to the behaviour of the velocity field at very large Reynolds numbers, which is smooth, see chapter 6.

10.3.6. WHAT ARE THE QUESTIONS?

> *. . . the essential mathematical complications of the subject were only disclosed by actual experience with the physical counterparts of these equations* (von Neumann, 1949).
>
> *. . . the observational material is so large, that it allows to foresee rather subtle mathematical results, which would be very interesting to prove* (Kolmogorov, 1978).
>
> *Sometimes experiments provide us with so beautiful and clear results that it is a shame on theorists that they cannot interpret them* (Yudovich, 2003).

This is one of the most difficult questions in turbulence research.

Some 'conventional' results

By working directly with the Navier–Stokes equations and stationary statistical solutions obtained through time averages, Foias et al. (2001b,c) prove in a mathematically rigorous manner, the existence of transfer of energy to higher modes and that the energy flux to higher modes is nearly equal to the energy dissipation rate throughout a certain range of wave numbers much smaller than the Taylor wave number. A number of estimates concerning characteristic numbers and non-dimensional numbers related to turbulent flows such as the number of degrees of freedom in the spirit of Kolmogorov dimensions of attractors and others were obtained by these and other authors, see references in Foias et al., (2001a,b,c, 2005) and Babin (2003). All these and similar ones are of interest for their own sakes, but from the point of view of theoretical study of turbulence (and its understanding) it is definitely not the first priority to reproduce in a (more or less) rigorous manner simple known results by complicated theory.

This seems to be true also of questions like *Do the Navier–Stokes equations on a 3-dimensional domain Ω have a unique smooth solution for all* *time?* the belief that *The solution of this problem might well be a fundamental step toward the very big problem of understanding turbulence* (Smale, 1998), and that *turbulence will be solved when well-formulated mathematical statements describing the properties of Navier–Stokes systems are proven* (Sinai, 1999).

It is quite a bit exaggerated to think that resolution of such and similar problems can aid in understanding. The majority of "practitioners" of turbulence (geo/astrophysicists, engineers and many others) cope with great many problems (not only 'practical') without having the solution of the above-mentioned problems and there seems to be no need for the solution of these problems in the above sense. Mathematics of this kind alone does not seem to help much (so far). Assume that the answer is known. So what does it add to understanding? Indeed, in the two-dimensional case the above problems are resolved (Doering and Gibbon, 1995), but this did not add much to the understanding of the basic physics of two-dimensional chaotic flows. *Non-uniqueness of the solutions per se thus cannot play a role in turbulence* (Lumley, 1970).

There are many deep mathematical ideas behind the dynamical systems approach to the study of nonlinear phenomena which are aimed at understanding the origins and structures of complicated behaviour, not merely describing it (Mullin, 1993, p. xvii). However, so far these ideas appear to be very useful in low-dimensional systems. It remains to hope that similar ideas will emerge in regard with highly-dimensional systems, as well as with the PDEs.

Singularities and intermittency

> *Real singularities, if they exist, provide a nice explanation for the scaling properties observed in fully-developed turbulence* (Frisch, 1984).
>
> ... *the main impediment to progress in the rigorous analysis of turbulence is the present lack of understanding of possible blow-up in individual solutions of the Euler and Navier–Stokes systems* (Constantin, 2000).
>
> ... *there is no chance of detecting a singularity of the 3D Euler equation by numerical simulations and the same observation holds for the Navier–Stokes equation with small viscosity* (Bardos, 2003).

As mentioned, there is reasonable evidence that the normalized mean dissipation $\varepsilon = U^3L^{-1}\langle\epsilon\rangle$ tends to a finite limit (not necessarily universal) as $\mathrm{Re} \to \infty$. Since $\langle\epsilon\rangle = 2\nu\langle s^2\rangle \approx \nu\langle\omega^2\rangle$ this means that at large Reynolds numbers both $\langle s^2\rangle, \langle\omega^2\rangle \approx \nu^{-1}$, i.e., the field of velocity derivatives is not only Reynolds-number dependent, but also becomes very large/singular in the limit $\nu \to 0$ ($\mathrm{Re} \to \infty$). Due to the intermittent nature of the field of velocity derivatives one can expect that the maximal values of s^2, ω^2 (or $\max(|\partial u_i/\partial x_k|)$) increase even faster with the Reynolds number. This possible unboundness of the field of velocity derivatives as Re $\to \infty$ has an implication that the Newtonian approximation can break down as $\nu \to 0$, since the linear stress/strain relation is only the first term in the gradient expansion. So far, however, there seems to be no evidence that the Navier–Stokes equations are inadequate for describing turbulent flows, i.e., breakdown of the hydrodynamic approximation[15]. Among the possible reasons that this (possible) violation is not so easy to detect is that, even if it happens, it will occur in rather large-Reynolds-numbers flows and small regions due to the strong intermittency of the field of velocity derivatives. So far there is no experimental evidence for this. This is also an indication of absence of breakdown of the NSE due to possible formation of singularities in finite time (Constantin, 1996).

Are singularities or their absence that important for turbulence, in general, and its understanding, in particular? Possible (real space) singularities of NSE fill a very slim set (see references in Doering and Gibbon, 1995; Gibbon and Doering, 2005), and they hardly exist at all. If we consider modified viscosity, there are no singularities at all for any small viscosity. Not many people believe in singularities for NSE either. As for Euler the set of possible singularities is enormous and one has to look for the "right" ones which possibly form as $\nu \to 0$. A number of authors relate intermittency and regularity issues in 3D Navier–Stokes turbulence (Constantin, 1996; Gibbon and Doering, 2005). This is typically based on the "definition" of intermittency as a state in which short, high-amplitude events associated with the fine structure are seen to be distinct from quieter, longer regions or a similar one (similarly in space).

But is it really necessary to associate intermittency and regularity? Not necessarily, as is seen readily from the four following examples: two-dimensional turbulence, modified Navier–Stokes equations such as those using hyperviscosity replacing the Laplacian by a higher-order operator, lattice gas hydrodynamics approximation and passive objects (scalars and vectors) in random velocity fields. In all these everything is beautifully regular, i.e., the solution remains regular for all times and any Reynolds number, but there is intermittency in the above sense. This does not necessarily mean that there is no relation between intermittency and regularity issues in genuine three-dimensional flows. This means only that understanding its intermittency in the above sense is too broad, e.g., any power of purely Gaussian field exhibit such kind of "intermittency", which is stronger for

[15]But see Ladyzhenskaya (1975) and McComb (1990, pp. 401–403), Friedlander and Pavlović (2004) on alternatives to NSE, and Tsinober (1993b) and references therein. It is safe to keep in mind that no equations are Nature.

higher powers (for other less trivial examples and discussion of related issues see chapter 7).

Some of **the** *questions*

Some of them were discussed above such as the ‘more chaotic’ nature of the Lagrangian setting, which is traced back to early Lagrangian simulations (Amsden and Harlow, 1964). It is doubtful that one has to accept the ‘natural expectation’ that nothing new can be expected in a pure Lagrangian setting? There are several reasons for caution. First, purely Lagrangian and Eulerian settings are different not only technically, but conceptually due to the ‘more chaotic’ nature of the Lagrangian setting, see chapter 3. The ergodic properties of both settings seem to be qualitatively different as well, though not much (if anything) is known on these as concerns *turbulent* flows. Finally, the issue of the limit $\nu \to 0$ has at least one additional nontrivial aspect concerning the so-called Cauchy invariant $\mathbf{\Omega}$ expressed in terms of vorticity and the Jacobi deformation matrix $\partial X_k(\mathbf{a}, t)/\partial a_j$. The peculiar feature is that the equation for the Cauchy invariant is a parabolic equation (C.66) which is linear in $\mathbf{\Omega}$ having a complicated viscous term with coefficients defined by the metrics of the Lagrangian trajectories $\mathbf{X}(\mathbf{a},t)$. Setting $\nu = 0$ results in $\mathbf{\Omega} = \mathbf{const}$ along fluid particle trajectories, i.e., it is an inviscid pointwise invariant. It is unlikely that the same result is arrived in the limit $\nu \to 0$?

One of *the* questions is the process of self-amplification of velocity derivatives (both vorticity and strain) which is intimately related to the main mathematical difficulties. Indeed, it is just the sort of quantities associated with this process that are involved when one is trying to find rigorous upper bounds to various norms when studying the well-posedness (in the large) of the Euler and Navier–Stokes problems. It is surprising that it has received relatively little attention. One of the reasons seems to be due to the misconception that it is ‘obvious’ due to the analogy with material lines behaviour, though there is no such an analogy for strain, see chapter 9. The main difficulty is that it is too trivial and crude to estimate quantities like $\omega_i\omega_j s_{ij}$ via global norms of the involved quantities. In this case $\omega_i\omega_j s_{ij}$ is estimated as $||\omega||^3$. It seems that an essential role is played by the geometrical relations between the quantities involved (such as alignments between ω and the eigenframe, λ_k of the tensor s_{ij}, see chapter 6). There are some empirical indications that $\omega_i\omega_j s_{ij}$ is estimated as $\sim|\omega|^{7/3}$, which is far ‘less’ than $|\omega|^3$, see Tsinober (1988) and references therein.

Another issue related to the above discussion is a set of questions related to the approximate balance between enstrophy production and its viscous destruction. This balance is not just something like mean or the L_2-norm. For example it holds (empirically) locally in time for the space integrals of the quantities involved. This feature is very robust and is observed in a variety of turbulent flows also at very modest values of Reynolds numbers.

There is some evidence that the (statistical) properties of some turbulent flows with the same geometry at very modest Reynolds numbers are invariant of the boundary and initial conditions (BC and IC). For example, typical DNS computations of NSE of turbulent flows (e.g., in a circular pipe and a plane channel, in a cubic box, etc.) involve extensive use of periodic BC. The results of these computations agree very well with those obtained in laboratory experiments, in which the BC have nothing to do with periodicity[16] and in which the IC were totally different from those in DNS (De Bruyn Kops and Riley, 1998; Nikitin, 1995, 2001 and references in Tsinober, 1998b). No explanation of this kind of invariance is known so far, but it is natural to expect that it is related to some kind of hidden symmetry(ies) of the NSE. If such exist, this may be the reason for the similarity of results obtained via DNS of NSE in, e.g., periodic boxes by various forcing (different deterministic, random/stochastic, etc.). This property, however, is not universal and there are many examples of long memory of turbulent flows which do 'remember', e.g., the inflow and initial conditions, see chapter 6.

10.4. On the goals of basic research in turbulence

> *Progress in numerical calculation brings not only great good but also awkward questions about the role of the human mind... The problem of formulating rules and extracting ideas from vast masses of computational or experimental results remains a matter for our brains, our minds* (Zeldovich, 1979).
>
> *Taking into account the absence of serious progress since the work of Leray and the fact that the NSE contains in itself turbulent phenomena, the reviewer conjectures that one should consider the problem the other way round and say that all that can be deduced from NSE by functional analysis is already done, and that in the absence of new progress in the qualitative understanding of turbulence there are very few chances that the problem of the regularity and uniqueness of the solution of 3D NSE will be solved* (Bardos, 2002).

[16]The correlation coefficient between two values of *any* quantity at the opposite ends of such boundaries (i.e., the points separated at *maximal distance* in the flow domain) will be precisely equal to unity and close to unity for the points in the proximity of such boundaries, whereas in any *real* flow the correlation coefficient becomes very small for points separated by a distance on the order of (and larger than) the integral scale of turbulent flow.

The phenomenon of turbulence is still one of the least understood ones, so that many issues in the 'problem' of turbulent flows are unsettled. As a consequence, different, strongly disparate and even contradictory views are not a rarity. As mentioned, there is no consensus even on what is(are) the problem(s), just as there is no agreement on what are the aims/goals of turbulence research/theories.

Our view has not changed since the first edition of this book, Tsinober (2001a). The first priority should be given to study of basic physical mechanisms of turbulence with the emphasis on qualitative aspects, keeping in mind a somewhat old-fashioned view that curiosity drives better science than 'strategies'[17]. This priority includes the study of turbulence itself (*per se*), rather than multitudes of its models. From the basic point of view, it seems not justified to put too much (often futile) effort into it's modelling which mostly is mimicking it without much understanding, as the former is not synonymous to the latter.

The only exception seems to be the Navier–Stokes equations. Since there is no simple mathematical aid for what is usually called understanding, it is naive to think that the 'problem of turbulence' would be resolved if one would have a super-hyper computer enabling one to 'solve' the NSE or whatever at any Reynolds number. Suppose one can do this and also to measure whatever one wants. The real problem is to know what to do next with the really huge amounts of data. Only this knowledge will aid in exposing the basic problems and will lead to real understanding and real 'solutions' and possibly will *produce a method of understanding the qualitative content of equations* [PDEs] (Feynmann, 1963). *Qualitative* is the key word, since there is a qualitative difference between being able to measure and/or compute/calculate all one wants and understanding. Perhaps the efforts of the turbulence community should be somewhat shifted to the qualitative aspects of the problem. Following the advice of Leonardo da Vinci: *Remember, when discoursing about water, to induce first experience, then reason* one has to put the emphasis on the physical aspects based in the first place on observations and empirical facts as distinct from intuitive conjectures, hypotheses and 'models'. The observational aspect is not that trivial in such a highly-dimensional system as turbulence: it is intimately related to the skill/art of asking the right and correctly-posed questions.

[17]Science pursued solely/mainly for the purpose of making better weapons and neater gadgets (and, of course, money) is destined to degeneration as science: *There are no such things as applied sciences, only applications of science,* Louis Pasteur (1872), Address 11 Sept. 1872, *Comptes Rendus des Travaux du Congress Viticole et Sericole de Lyon*, 9–14 Septembre, 1882, p. 49.

Nowadays scientists are forced to become literary businessmen rather than in-depth productive researchers. These trends of hunting money as a first priority (which were foreseen at least fifty years ago) may kill the remaining spots of basic research in turbulence with irreversible (or reversible at a very long time scale) consequences.

An important aspect is related to what is called here qualitative universality of turbulent flows. It implies that (contrary to the common views) it is not always necessary to 'hunt' very large values of Reynolds number in studying and trying to understand the basic physics of turbulence, and one needs neither very high Reynolds numbers nor precise determination of scaling exponents[18], etc. at such Reynolds numbers. This follows from a comparison of the results of the high-Reynolds-number experiments, $Re_\lambda \sim 10^4$ (Gulitski et al., 2007a,b,c and references therein) and low-Reynolds-numbers ($Re_\lambda \sim 10^2$), both from DNS and experiments. It appears that in many respects the basic physics of turbulent flow at high Reynolds numbers, at least qualitatively, is the same as at low ones. This is true of such basic processes as enstrophy and strain production, geometrical statistics, the role of concentrated vorticity and strain, and depression of nonlinearity, Tennekes and Lumley balance, a variety of non-local effects, and a number of issues concerning fluid particle accelerations and their Eulerian components and properties related to temperature and its gradient.

It is important to emphasize that our claim that "the basic physics of turbulent flow at high Reynolds number $Re_\lambda \sim 10^4$, at least qualitatively, is the same as at moderate Reynolds numbers, $Re_\lambda \sim 10^2$" does not mean that what is called 'Reynolds-number dependence' is unimportant. An immediate example comes from the indirect evaluation of the acceleration variance. There are clear indications that – if scaled as proposed by Yaglom (1949) – it exhibits a definite Re-dependence and does not saturate at least up to $Re_\lambda \sim 10^4$. Another well-known example is the behaviour of flatness of individual velocity derivatives and similar quantities based on vorticity and/or strain. Reynolds-number dependence is of extreme importance in a great variety of purely engineering problems and other applications. Reynolds-number dependence and the nature of dissipation is important in basic issues of asymptotic behaviour and limiting state(s) of turbulent flows as $Re \to \infty$. *It remains to classify and distinguish between Reynolds-number-dependent and Reynolds-number-independent quantities/phenomena in turbulence.* The existing modest evidence indicates that some of the former may never saturate as $Re \to \infty$. There is, of course, an issue with the 'inertial' range as an ill-defined concept. The resolution of this and similar issues requires both high Reynolds numbers and most importantly sub-Kolmogorov resolution.

[18] As mentioned in chapters 5 and 7, scaling and related matters have not proven very useful (so far) in understanding the basic physics of turbulence or in justifying, e.g., enormous efforts in accurately measuring exponents at very large values of Reynolds number. It is not at all clear why and to what extent accurate measurement and/or knowledge of exponents would aid understanding of turbulence. Moreover, the very existence of such exponents (with few exceptions) is quite problematic, e.g., Arneodo et al. (1999), Badii and Talkner (2001), Feigenbaum (1997), Tsinober (1996b). Recent evidence, section 5.3, indicates that scaling exponents and the concept of inertial range are not well-defined.

It is remarkable that in spite – or perhaps just because – of frustrated and unsuccessful attempts to construct a predictive theory of turbulent flows based on the first principles, the attraction of the turbulence problem is still very reasonable: curiosity drives better science than 'strategies' and bureaucratic 'planning'. However, this is not reflected anymore in continuing and increasing efforts in basic research in the field. So at present one cannot be optimistic that sometime soon, the basic aspects of the glorious enigma of turbulence as a physical phenomenon will be resolved (if at all). It seems that it will be a long time before the turbulence community will be no longer divided between experimentalists who observe what still cannot be explained and theoreticians who try to explain what can not be observed, and there will be no more attempts to replace explanation with mere description.

Presently the turbulence community is not that simple and is divided into more than just two groups. To quote Lumley: *Turbulence is rent by factionalism. Traditional approaches in the field are under attack, and one hears intemperate statements against long time averaging, Reynolds decomposition, and so forth. Some of these are reminiscent of the Einstein–Heisenberg controversy over quantum mechanics, and smack of a mistrust of any statistical approach. Coherent structure people sound like* The Emperors's new Clothes *when they say that* all *turbulent flows consist primarily of coherent structures, in the face of visual evidence to the contrary. Dynamical systems theory people are sure that turbulence is chaos. Simulators have convinced many that we will be able to compute anything within a decade... The card-carrying physicists dismiss everything that has been done on turbulence from Osborne Reynolds until the last decade. Cellular Automata were hailed on their appearance as the answer to a maidens prayer, so far as turbulence was concerned* (Lumley, 1990, as quoted by Cantwell, 1990). There are many more, see chapter 9.

APPENDIX A. WHAT IS TURBULENCE?

Attempts of definition(s)

This is a partial list. Other citations are given in the main text.

* – Turbulence is the name given to the imperfectly-understood class of chaotic solutions to the Navier–Stokes equation in which many degrees of freedom are excited.

H. Aref, 1999, Turbulent statistical dynamics of a system of point vortices, in A. Gyr, W. Kinzelbach and A. Tsinober, eds., Fundamental problematic issues in turbulence, Birkhäuser.

* – It is a well-known fact that under suitable conditions, which normally amount to a requirement that the kinematic viscosity ν be sufficiently small, some of these motions are such that the velocity at any given time and position in the fluid is not found to be the same when it is measured several times under seemingly identical conditions. In these motions the velocity takes random values which are not determined by the ostensible, or controllable, or, 'macroscopic' data of the flow, although we believe that the *average* properties of the motion are determined uniquely by the data. Fluctuating motions of this kind are said to be turbulent.

G.K. Batchelor, 1953, The theory of homogeneous turbulence, Cambridge University Press, p. 1.

* – Turbulence is a three-dimensional time-dependent motion in which vortex stretching causes velocity fluctuations to spread to all wavelengths between a minimum determined by viscous forces and a maximum determined by the boundary conditions of the flow. It is the usual state of fluid motion except at low Reynolds numbers.

P. Bradshaw, 1972, An introduction to turbulence and its measurement, Pergamon, p. 17.

* – The only short but satisfactory answer to the question "what is turbulence?" is that it is the general solution of the Navier–Stokes equation.

P. Bradshaw, 1972, The understanding and prediction of turbulent flow, Aeronaut. J., **76***, p. 406.*

* – The distinguishing feature of turbulent flow is that its velocity field appears to be random and varies unpredictably. The flow does, however,

satisfy a set of differential equations, the Navier–Stokes equations, which are not random. This contrast is the source of much of what is interesting in turbulence theory.

A.J. Chorin, 1975, Lectures on turbulence theory, Publish or Perish, Berkeley, p. 1.

* – Creation of small-scale activity and dissipation is the principle of turbulence. Classical fluid-dynamical instabilities play a role of the fuel, vortex stretching is the engine, and viscous dissipation is the breaks.

P. Constantin, 1994, Geometric statistics in turbulence, SIAM Review, **36***, p. 73.*

* – The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today, we cannot see whether Schrödinger's equation contains frogs, musical composers, or morality – or whether it does not. We cannot say whether something beyond it like God is needed, or not. And we can all hold strong opinions either way.

Richard P. Feynman, 1963, The Feynman lectures on physics, **II***, Addison–Wesley, p. 41–12.*

* – Turbulence with its limit of self-excitation, with the characteristic hysteresis in its appearance and disappearance as the velocity of flow producing it is increased or reduced and the primary role of nonlinearity in its developed (stationary) state, is, in fact, a *self-oscillation*. Its specific features are determined by the fact that it is self-oscillation of a continuous medium, i.e., a system with an infinite number of degrees of freedom.

G.S. Gorelik, 1956, as quoted by M.I. Rabinovich, Stochastic self-oscillations and turbulence, 1978, Sov. Phys. Uspekhi, **21***, p. 444.*

* – Das "Turbulenzproblem" der Hydrodynamik ist ein Problem der energetischen, nicht der dynamischen Stabilität.

W. Heisenberg, 1923, Über Stabilität und Turbulenz von Flüssigkeitströmen, Ph.D. Thesis, p. 37.

* – The following definition of turbulence can thus be tentatively proposed and may contribute to avoiding the somewhat semantic discussion on this matter:

a) Firstly, a turbulent flow must be unpredictable, in the sense that a small uncertainty as to its knowledge at a given initial time will amplify so as to render impossible a precise deterministic prediction of its evolution; b) Secondly, it has to satisfy the increased mixing property defined above; c) Thirdly, it must involve a wide range of spatial wave lengths.

M. Lesieur, 1997, Turbulence in fluids, Kluwer, p. 2.

* – Turbulence can be defined by a statement of impotence reminiscent of the second law of thermodynamics: flow at a sufficiently high Reynolds number cannot be decelerated to rest in a steady fashion. The deceleration always produces vorticity, and the resulting vortex interactions are apparently so sensitive to the initial conditions that the resulting flow pattern changes in time and usually in stochastic fashion.

H.W. Liepmann, 1979, The rise and fall of ideas in turbulence, American Scientist, 67, p. 221.

* – A body of fluid is a mechanical system with an infinite number of degrees of freedom. It may therefore be expected to execute a rather random motion comparable to that of the molecules in a gas. If one regards such a chaotic motion as analyzed into harmonic components of various scales, one recognizes that frictional forces tend to dissipate the small-scale oscillations and keep the motion more or less regular. Thus, when viscous forces are sufficiently strong, i.e., at sufficiently low Reynolds numbers, the motion will become laminar. On the other hand, at sufficiently high Reynolds numbers the motion will tend to become random fluctuating, even when external conditions are steady.

C.C. Lin and W.H. Reid, 1963, Turbulent flow, Theoretical aspects, in: Handbuch der Physik, Band VIII/2, Springer, p. 438.

* – Perhaps a satisfactory definition would be an ensemble of nonperiodic solutions of the Navier–Stokes equations. Ensembles of solutions of simplified or otherwise modified forms of the Navier–Stokes equations will not qualify as turbulence; we shall instead regard them as models of turbulence.

E.N. Lorenz, 1972, Investigating the predictability of turbulent motion, In: M. Rosenblatt and C. Van Atta, editors, Statistical models and turbulence, Springer, p. 195.

* – We have therefore defined turbulence as random fluctuations of the thermodynamic characteristics of *vortex* flows, thereby distinguishing it at the outset from any kind of whatever random irrotational, i.e., *potential* flows, ...

A.S. Monin, 1978, *On the nature of turbulence, Sov. Phys. Uspekhi, 21, p. 430.*

* – Definition of Randomness.

Of special interest to us here are the strange attractors, on which phase trajectories display the following properties of randomness:

(1) An extremely sensitive dependence on initial conditions, due to *exponential divergence* of trajectories which are initially close together (and

leading to their unpredictability for initial conditions which are given with arbitrarily-high (but finite) precision). (2) The everywhere-denseness at the attractor of almost all trajectories, i.e., their arbitrarily-close approach to any of the attractor's points (which implies that they return infinitely often to the attractor), and the property that any initial nonequilibrium probability distribution (measure) over the phase space (or, more precisely, over the region of attraction of the strange attractor) reduces to some limiting equilibrium distribution at the attractor (an invariant measure). (3) The mixing property: For any (measurable) subsets A and B of the attractor, the probability after emerging from A of arrival at B is proportional after a long time of measure of B:

$$\lim_{t\to\infty} P\{(F^t A)\cap B\} = P(A)\ P(B)$$

where the symbol $\cap$ denotes set intersection. A consequence of the mixing property is the fact that the time-averaged value $< \Phi[u(t)] >$ of any function $\Phi(u)$ defined on the strange attractor is independent of the initial conditions $\mathbf{u}_0$ (for almost all $\mathbf{u}_0$) and that this average value coincides with the average $\bar{\mathbf{\Phi}}(u)$ over the invariant measure (*ergodicity*):

$$< \Phi >\equiv \lim_{T\to\infty} T^{-1}\int_0^T \Phi[u(t)]dt = \int \Phi(u)Pd\mathbf{u} \equiv \bar{\mathbf{\Phi}}.$$

A characteristic of the mixing property is a rather *rapid decay of the correlation functions as* $\tau \to \infty$:

$$B^{jl}(\tau) =< [u^j(t)- < u^j >][u^l(t+\tau)- < u^l >] >,$$

which is to say *continuity* of their Fourier transforms with respect to τ, i.e., their *spectral functions.*

It appears expedient to have the term *turbulence* refer to the random evolution [in the sense of (1)–(3) above] of the flow of a (viscous) fluid which possesses *vorticity*. Stochastic *potential* flows of a fluid are by preference referred to as *random wave fields*, while for nonhydrodynamic systems one should preferably restrict oneself, where necessary, to the adjective *stochastic.*

A.S. Monin, 1978, Hydrodynamic instability, Sov. Phys. Uspekhi, 29, pp. 856–857.

* – Turbulence is a phenomenon which sets in in a viscous fluid for small values of the viscosity coefficient ν (reckoning ν in significant units, that is, as the reciprocal Reynolds' number $1/Re$), hence its purest, limiting form may be interpreted as the asymptotic, limiting behavior of a viscous fluid for $\nu \to 0\ldots$

The circumstances described above made it very plausible that turbulence is a phenomenon of instability . . .

A complete theory of the general solutions of the Navier–Stokes equations are called for . . . nothing less than a thorough understanding of the system of all their solutions would seem to be adequate to elucidate the phenomenon of turbulence.

Turbulence proper is tied . . . to 3-dimensionality.

John von Neumann, Recent theories of turbulence—A report to the office of Naval Research, 1949, in: Collected works, vol. 6, pp. 439, 441, 448, 462, ed. A.H. Taub, Pergamon.

* – One of the best definitions of turbulence is that it is a field of random chaotic vorticity.

P.G. Saffman, 1981, Vortex interactions and coherent structures in turbulence, in: Transition and turbulence, ed. R. Meyer.

* – . . . the turbulence syndrome includes the following symptoms: The velocity field is such a complicated function of space and time that a statistical description is easier than a detailed description; it is essentially three-dimensional, in the sense that the dynamical mechanism responsible for it (the stretching of vorticity by velocity gradients) can only take place in three dimensions; it is essentially nonlinear and rotational, for the same reasons; a system of partial differential equations exists, relating the instantaneous velocity field to itself at every time and place.

R.W. Stewart, as quoted by J.L. Lumley, 1972, Stochastic tools in turbulence, Academic Press, p. ix.

* – Turbulence is an irregular motion which, in general, makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or over one another.

G.I. Taylor and Th. von Karman, in: von Karman, Th., Twenty-fifth Wilbur Wright memorial lecture—Turbulence, 1937, J. Roy. Aeronaut. Soc., 41, p. 1109.

* – In many cases, it is allowable to deal with a system of finite dimensions as a model of a continuous fluid and this is particularly the case at the stage of generation of "turbulence", at which only a limited number of degrees of freedom of motion have been excited. The approximation of a fluid in terms of a model system of finite dimensions provides us with a powerful means of analysis and it is by this reason that the recent progress in the theory of "chaos" has enabled us to look straight at the fundamental mechanism of "turbulence".

On the other hand, it is generally recognized that "turbulence" in its Fully-developed state has a *singular* structure in space and time and that the singularity is closely connected with the peculiar property of "turbulence" such as the nonzero viscous dissipation in the limit of vanishing viscosity. Such a singular behaviour of the fluid cannot be described correctly by means of a model system of finite dimensions which remains *regular* in the inviscid limit. Thus, in this restricted area, "chaos in fluids" covers only a part of "turbulent" phenomena.

T. Tatsumi, 1984, Irregularity, regularity and singularity in turbulence, in: Turbulence and chaotic phenomena in fluids, ed. T. Tatsumi, Elsevier, p. 1.

* – Before 1970, I would not have dreamt of putting the words turbulence and predictability side by side, as in the title of this summer course. To me, turbulence was unpredictable by definition. Turbulence was the chaos that arises in fluids because of the innumerable instabilities associated with vortex stretching.

These days, I tend to think of turbulent flow as flow in which deterministic calculations become useless in a finite time interval.

H. Tennekes, 1985, A comparative pathology of atmospheric turbulence in two and three dimensions, in: Turbulence and predictability in geophysical fluid dynamics and climate dynamics, eds., M. Ghil, R. Benzi and G. Parisi, North-Holland, p. 45.

* – Everyone who, at one time or another, has observed the efflux from a smokestack has some idea about the nature of turbulent flow. However, it is very difficult to give a precise definition of turbulence. All one can do is list some of the characteristics of turbulence flows:
Irregularity... Diffusivity... Large Reynolds numbers... Three-dimensional vorticity fluctuations... Dissipation... Continuum... Turbulent flows are flows...

H. Tennekes and J.L. Lumley, 1972, A first course in turbulence, MIT Press, pp. 1–3.

APPENDIX B. ABOUT THE 'SNAGS' OF THE PROBLEM

◇ – *I had less difficulty in the discovery of the motion of heavenly bodies in spite of their astonishing distances, than in the investigation of the movement of flowing water before our very eyes.* Galileo as cited by R. Narasimha, 1983, The turbulence problem: a survey, *J. Indian Inst. Sci.,* **64(A)** Jan. p. 1. (1–59).

◇ – *As a doctorate I proposed to Heisenberg no theme from Spectroscopy but the difficult problem of Turbulence, in the hope, that WENN IRGENDEINER (if anybody), would solve this problem. However, the problem is until now not solved.* A. Sommerfeld, 1942, Scientia, Nov./Dez. 1942.

◇ – *The universal similarity theory of the small-scale components of the motion stands out in this rather grey picture as a valuable contribution, of which an increasing number of applications is being made...* G.K. Batchelor, 1962, The dynamics of homogeneous turbulence: introductory remarks, In: A. Favre, editor, Mécanique de la Turbulence, Colloques Internationaux du CNRS, No. 108 (Marseille, 28 Août–2 Septembre 1961), p. 96.

◇ – *It remains to call attention to the chief outstanding difficulty (i.e., turbulence) of our subject.* Sir Horace Lamb, 1927, Hydrodynamics, p. 651.

◇ – *I am an old man now, and when I die and go to Heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics and the other is the turbulent motion of fluids. And about the former I am rather optimistic.* Sir Horace Lamb, 1932, as quoted by S. Goldstein, 1969, *ARFM*, **1**, 23.

◇ – *I soon understood that there was little hope of developing a pure, closed theory*, Selected works of A.N. Kolmogorov, **I**, ed., V.M. Tikhomirov, p. 487, Kluwer, 1991.

◇ – *It is at this point that the study of turbulence does prove to be an exception: the applied physics involvement is almost completely absent. In view of the extraordinary practical importance of turbulence… this is* *quite astonishing. Yet the reason for such apparent neglect is easily found. Quite simply the fundamental problems of turbulence are still unresolved.* D. McComb, 1990, The physics of turbulence, Oxford Univ. Press, p. vii.

⋄ – *The entire experience with the subject indicates that the purely-analytical approach is beset with difficulties, which at this moment are still prohibitive. The reason for this is probably as was indicated above: That our intuitive relationship to the subject is still too loose – not having succeeded at anything like deep mathematical penetration in any part of the subject, we are still quite disoriented as to the relevant factors, and as to the proper analytical machinery to be used. Under these conditions there might be some hope to "break the deadlock" by extensive, but well-planned computational efforts.* J. von Neumann, 1949, Recent theories of turbulence—A report to Office of Naval Research. Collected works, 6 (1963), 469, ed. Taub., A.H., Pergamon.

⋄ – . . . *the absence of a sound theory is one of the most disturbing aspects of the turbulence syndrome.* R.W. Stewart, 1969, Turbulence Nat. Committee for Fluid Motion Films (dist. Encyclopedia Britannica Educational Corp.).

⋄ – . . . *we should not altogether neglect the possibility that there is no such thing as 'turbulence'. That is to say, it is not meaningful to talk of the properties of a turbulent flow independently of the physical situation in which it arises. In searching for a theory of turbulence, perhaps we are looking for a chimera,* P.G. Saffman, 1978, Problems and progress in the theory of turbulence, Lect. Notes in Phys., **76** (II), p. 276.

⋄ – . . . *I just cannot think of anything where a genuine prediction for the dynamics of turbulent flow has been confirmed by an experiment. So we have a big vast empty field.* P.G. Saffman, 1991, in: The Global Geometry of Turbulence, NATO ASI Ser., **B 268**, ed. J. Jimenez, p. 349, Plenum.

⋄ – *Turbulence was probably invented by the Devil on the seventh day of Creation when the Good Lord wasn't looking.* P. Bradshaw, 1994, Experiments in Fluids, **16**, p. 203.

⋄ – . . . *less is known about the fine-scale turbulence... than about the structure of atomic nuclei. Lack of basic knowledge about turbulence is holding back progress in fields as diverse as cosmology, meteorology, aeronautics and biomechanics.* U. Frisch and S. Orszag, 1990, *Phys. Today*, **43**, 32.

⋄ – ... *a fundamental theoretical understanding is still lacking*. M. Nelkin, 1994, Adv. Phys., **43**, 143.

⋄ – Turbulence is the last great unsolved problem of classical physics. *Remarks of this sort have been variously attributed to Sommerfeld, Einstein, and Feynman, although no one seems to know precise references, and searches of some likely sources have been unproductive. Of course, the allegation is a matter of fact, not much in need of support by a quotation from a distinguished author. However, it would be interesting to know when the matter was first recognized.* P.J. Holmes, G. Berkooz and J.L. Lumley, 1996, Turbulence, coherent structures, dynamical systems and symmetry, Cambridge University Press.

⋄ – *Turbulence is the graveyard of theories,* H.W. Liepmann, 1997, as cited by S.J. Kline, 1997, A brief history of boundary layer structure research, in Self-sustaining mechanisms of wall turbulence, ed. R.L. Panton, p. 4, Comp. Mech. Publ.

⋄ – *Every aspect of turbulence is controversial*, R. Salmon, 1998, *Lectures on geophysical fluid dynamics*, Oxford University Press.

APPENDIX C. GLOSSARY OF ESSENTIAL FLUID MECHANICS

Contains also some not broadly known facts

This appendix contains basic information on fluid mechanics, in general, and turbulent flows, in particular, with some specific relevant items. This includes flow kinematics, equations of motion, and some of their consequences for velocity derivatives, and basic relations for description of turbulent flows.

13.1. Kinematics

The evolution of vector, l_i, connecting two material points, $\mathbf{x}$ and $\mathbf{x}+\mathbf{l}$, follows the equation $\frac{Dl_i}{Dt} = u_i(\mathbf{x}+\mathbf{l}) - u_i(\mathbf{x})$. If the vector l_i is infinitesimal, this equation becomes

$$\frac{Dl_i}{Dt} = l_j \frac{\partial u_i}{\partial x_j} = l_j s_{ij} + l_j a_{ij} = l_j s_{ij} + \frac{1}{2}\varepsilon_{ijk}\omega_j l_k, \qquad \text{(C.1}a\text{)}$$

or in vector notation

$$\frac{D\mathbf{l}}{Dt} = (\nabla \cdot \mathbf{u})\mathbf{l} = \mathbf{s}\cdot\mathbf{l} + \frac{1}{2}\omega \times l, \qquad \text{(C.1}b\text{)}$$

where $s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the rate of strain tensor, $a_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ is the rotation tensor, and $a_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k$, where $\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \varepsilon_{ijk}a_{kj}$, $\omega = curl\mathbf{u}$ is the vorticity vector and $\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}$.

The equivalent to (C.1) statement is that the velocity field in a small region surrounding the position $\mathbf{x}$ consists, to the first order in the linear dimensions of this region, of the superposition of a uniform translation with velocity $\mathbf{u}(\mathbf{x})$, a pure straining motion characterized by the rate of strain tensor s_{ij}, and a rigid-body rotation with an angular velocity $\frac{1}{2}\omega$:

$$u_i(\mathbf{x}+\mathbf{dx}) = u_i(\mathbf{x}) + s_{ij}(\mathbf{x})dx_j + \frac{1}{2}\,\varepsilon_{ijk}\omega_j(\mathbf{x})dx_k. \qquad \text{(C.1}c\text{)}$$

The equation for infinitesimal area, identified by its vector normal, N_i, follows from the conservation of fluid volume, $N_i l_i$, i.e., $\frac{DN_i l_i}{Dt} = 0$ which together with (C.1) gives

$$\frac{DN_i}{Dt} = -N_k \frac{\partial u_k}{\partial x_i}, \tag{C.2}$$

In compressible fluids this should be replaced by $\frac{D\rho N_i l_i}{Dt} = 0$ with the resulting equation $\frac{DN_i}{Dt} = -N_k \frac{\partial u_k}{\partial x_i} + N_i \frac{\partial u_k}{\partial x_k}$, where ρ is the fluid density.

Note that, generally, neither is vector $\mathbf{l}$ solenoidal, nor vector $\mathbf{N}$ is potential. A useful relation concerns a rather special pointwise Lagrangian invariant which is $div\mathbf{l}$, i.e.,

$$\frac{D(div\mathbf{l})}{Dt} = 0, \tag{C.3}$$

13.2. Dynamics

13.2.1. BASIC EQUATIONS AND THEIR CONSEQUENCES

In the sequel $\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$ is the material (Lagrangian) derivative.

Incompressibility

$$div\ \mathbf{u} \equiv \frac{\partial u_i}{\partial x_i} = 0. \tag{C.4}$$

Euler Equations (EE)

$$\frac{Du_i}{Dt} = -\frac{\partial(p/\rho)}{\partial x_i}. \tag{C.5}$$

Navier–Stokes equations (NSE)

$$\frac{Du_i}{Dt} = -\frac{\partial(p/\rho)}{\partial x_i} + \nu\nabla^2\ u_i + F_i. \tag{C.6}$$

The Lamb's form of the NSE in vector notation

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} = \nabla\left(p/\rho + \frac{1}{2}u^2\right) + \nu\nabla^2\ \mathbf{u} + \mathbf{F}. \tag{C.7}$$

Three typical examples of real forces, $\mathbf{F}$, are as follows. The first one is the force due to buoyancy, $\mathbf{F}_b = \frac{\Delta\rho}{\rho_0}\mathbf{g}$, in fluids with density variations, represented by the difference $\Delta\rho = \rho - \rho_0$, with respect to some reference density ρ_0, where $\mathbf{g}$ is the gravitational acceleration. The second example is the Coriolis force in rotating systems, $\mathbf{F}_c = -2\boldsymbol{\Omega}_{syst} \times \mathbf{u}$, where $\boldsymbol{\Omega}_{syst}$ is the angular velocity of the system. Finally, the third example is the electromagnetic force in electrically-conductive fluids, $\mathbf{F}_{em} = \frac{1}{\rho}(\mathbf{j} \times \mathbf{B})$, where $\mathbf{j} = curl\mathbf{B} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$, $\mathbf{B}$ – magnetic field, $\mathbf{E}$ – electrical field.

It is important that the force $\mathbf{F}$ can be prescribed in any desirable way in direct numerical simulations of the Navier–Stokes equations. For example, one can realize the ‘unrealistic’ (quasi-) isotropic turbulent flow by choosing an appropriate force.

Note that NSE are *integro*-differential equations, since the pressure field at some point in space is defined by the velocity in the whole flow domain, due to the nonlocality of the inverse Laplace operator, because $\nabla^2 p = \{\rho(\omega^2/2 - s^2)\} \equiv 2Q$, $s^2 \equiv s_{ij}s_{ij}$. In an unbounded fluid flow $p(\mathbf{x}) = -(2\pi)^{-1}\int Q(\mathbf{x})\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}$. This can be represented as a sum of local and nonlocal terms, $p(\mathbf{x},t) = -\frac{1}{3}u^2(\mathbf{x},t) + N(\mathbf{x},t)$, $N(\mathbf{x},t) = \frac{1}{4\pi}\int_{P.V.}\{3\frac{y_iy_j}{y^2} - \delta_{ij}\}R_{ij}(\mathbf{x}-\mathbf{y},t)\frac{d\mathbf{y}}{|y|^3}$, $R_{ij} = u_iu_j$ (Constantin and Fefferman, 1994). Here $\int_{P.V.}$ stands for the Cauchy’s principal value.

Equation for the kinetic energy, $\frac{1}{2}u^2$

$$\frac{D}{Dt}\left(\frac{1}{2}u^2\right) = -\frac{\partial}{\partial x_j}\{u_jp/\rho - 2\nu u_is_{ij}\} - 2\nu s_{ij}s_{ij} + u_iF_i. \tag{C.8}$$

Note that – as follows from the NSE in Lamb’s form (C.7) – vorticity does not contribute *directly* to the local (i.e., without integration over the whole flow domain) energy balance/transfer, since $\mathbf{u}\cdot(\omega\times\mathbf{u}) \equiv 0$.

Equation(s) for vorticity, ω_i

$$\frac{D\omega_i}{Dt} = \omega_js_{ij} + \nu\nabla^2\omega_i + \varepsilon_{ijk}\frac{\partial F_k}{\partial x_j}, \tag{C.9}$$

or in vector notation

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\mathbf{u} + \nu\nabla^2\omega + curl\mathbf{F}. \tag{C.10}$$

The vortex-stretching vector $W_i \equiv \{(\omega\cdot\nabla)\mathbf{u}\}_i = \omega_j\frac{\partial u_i}{\partial x_j} = \omega_js_{ij}$ reflects the interaction between vorticity and rate of strain tensor and is responsible for stretching (compressing) and tilting of vorticity.

If the fluid density is not constant, the equation for vorticity becomes

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\mathbf{u} + \omega div\mathbf{u} + \frac{1}{\rho^2}\nabla\rho\times\nabla p + \nu\nabla^2\omega + curl\mathbf{F}. \tag{C.11}$$

This equation can be rewritten for the potential vorticity, ω/ρ, as

$$\frac{D\omega/\rho}{Dt} = \{(\omega/\rho)\cdot\nabla\}\mathbf{u} - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right)\times\nabla p + \frac{\nu}{\rho}\nabla^2\omega + \frac{1}{\rho}curl\mathbf{F}, \tag{C.12}$$

where use was made of mass conservation $div\mathbf{u} = -\frac{1}{\rho}\frac{D\rho}{Dt}$.

A useful equation for the dilatation, $\Theta = div\mathbf{u}$, is

$$\frac{D\Theta}{Dt} = \frac{1}{2}\omega^2 - s^2 - \frac{1}{\rho}\nabla^2 p + \frac{1}{\rho^2}\nabla\rho \cdot \nabla p + \nu\nabla^2\Theta + div\mathbf{F}. \tag{C.13}$$

Note, that the equation for vorticity is *integro*-differential, since the velocity field at some point in space is defined by the vorticity in the whole flow domain, for example, by the Biot–Savart law in case of the whole space,

$$u_i(\mathbf{x}, t) = \int \alpha_{ij}(\mathbf{r})\omega_j(\mathbf{y}, t)d1\mathbf{y},\ \alpha_{ij}(\mathbf{r}) = \frac{1}{4\pi}\varepsilon_{ijk}\frac{r_k}{r^3},\ r_i = x_i - y_i, \tag{C.14}$$

and consequently the rate of strain tensor

$$s_{ij}(\mathbf{x}) = \int_{P.V.} \beta_{ijk}(\mathbf{r})\omega_j(\mathbf{y},t)d^3\mathbf{y},\ \beta_{ijk} = -\frac{3}{8\pi}\frac{\varepsilon_{ijl}r_l r_k + \varepsilon_{kjl}r_l r_i}{r^5}, \tag{C.15}$$

where $\int_{P.V.}$ stands for the Cauchy principal value (Novikov, 1967). A similar expression for ω_i can be obtained in terms of strain, s_{ij}, and both relations can be cast in a symmetric form using the rotation tensor $a_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ (Ohkitani, 1994). It is important that this symmetry is only kinematic. Dynamically the two quantities are very different in many respects. For example, pressure is present in the equation for the rate of strain tensor, but it is absent in the equation for vorticity. Another difference is seen in the absence of viscosity. In this case vorticity is frozen in the fluid motion. There is no such property associated with the rate of strain.

In an arbitrary flow domain the velocity field is defined from a Poisson equation $\nabla^2\mathbf{u} = -curl\omega$, with appropriate boundary conditions. This is true also of the rate of strain tensor: just as in the case of vorticity the whole flow field is determined entirely by the field of strain. This is seen from the equation $\nabla^2 u_i = 2\partial s_{ik}/\partial x_k$, which together with boundary conditions defines uniquely the velocity field. In particular, it is straightforward to obtain an analogue of the Biot–Savart law for the whole space under the same conditions for the Biot–Savart law to be valid,

$$u_i(\mathbf{x}, t) = \int \gamma_j(\mathbf{r})s_{ij}(\mathbf{y}, t)d\mathbf{y},\ \gamma_j(\mathbf{r}) = -\frac{1}{2\pi}\frac{r_j}{r^3},\ r_i = x_i - y_i. \tag{C.14$'$}$$

We stress these simple purely-kinematic relations with the emphasis that the velocity field is completely defined by the field of vorticity or strain. Thus it is the first and the simplest indication that the 'small' scales (represented by velocity derivatives, i.e., vorticity and strain) and the 'large' scales (represented by velocity) are not that separate for whatever Reynolds

number. Note that the small scales, generally, are not 'integrated out' in (C.14) and (C.14′) due to the singular nature of the kernel.
Equation for enstrophy $\frac{1}{2}\omega^2$

$$\frac{1}{2}\frac{D\omega^2}{Dt} = \omega_i\omega_j s_{ij} + \nu\omega_i\nabla^2\omega_i + \varepsilon_{ijk}\omega_i\frac{\partial F_k}{\partial x_j}. \tag{C.16}$$

The term $\omega_i\omega_j s_{ij}$ is responsible for the enstrophy production.

The enstrophy production and its rate, $\alpha = \frac{\omega_i\omega_j s_{ij}}{\omega^2}$, can be expressed in terms of vorticity only for an infinite domain in the following nonlocal form (Constantin, 1994)

$$\alpha(\mathbf{x}) = \frac{3}{4\pi}\int_{P.V.} D\{\widetilde{\mathbf{y}},\widetilde{\omega}(\mathbf{x}+\mathbf{y},t),\widetilde{\omega}(\mathbf{x},t)\}|\omega(\mathbf{x}+\mathbf{y},t)|\frac{d\mathbf{y}}{|y|^3}, \tag{C.15′}$$

where $D\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\} = (\mathbf{a}_1\cdot\mathbf{a}_2)Det\{a_{ij}\}$ and $\widetilde{\mathbf{a}} = \mathbf{a}/a$.
Equation for the rate of strain tensor s_{ij}, (Yanitski, 1982)

$$\frac{Ds_{ij}}{Dt} = -s_{ik}s_{kj} - \frac{1}{4}(\omega_i\omega_j - \omega^2\delta_{ij}) - \frac{\partial^2 p}{\partial x_i\partial x_j} + \nu\nabla^2 s_{ij} + F_{ij}, \tag{C.17}$$

where $F_{ij} = \left(\frac{\partial F_i}{\partial x_j} + \frac{\partial F_j}{\partial x_i}\right)$.
Equation for the total strain, $s_{ij}s_{ij} \equiv s^2$

$$\frac{1}{2}\frac{Ds^2}{Dt} = -s_{ij}s_{jk}s_{ki} - \frac{1}{4}\omega_i\omega_j s_{ij} - s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j} + \nu s_{ij}\nabla^2 s_{ij} + s_{ij}F_{ij}. \tag{C.18}$$

Just like the term $\omega_i\omega_j s_{ij}$ in (C.16) is called enstrophy production, the term $-s_{ij}s_{jk}s_{ki} - \frac{1}{4}\omega_i\omega_j s_{ij} - s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}$ in (C.18) can be called (inviscid) production of total strain. Note that the mean $\langle -s_{ij}s_{jk}s_{ki} - \frac{1}{4}\omega_i\omega_j s_{ij}\rangle = \frac{1}{2}\langle\omega_i\omega_j s_{ij}\rangle$ is strictly positive in homogeneous flows and comprises the generation of strain in such (incompressible) flows, since $s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j} = \frac{\partial}{\partial x_k}\{\cdots\}$ and therefore $\langle s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}\rangle = 0$.

If the fluid density is not constant, the equation (C.18) for s^2 contains two more terms in its RHS: $-\frac{1}{\rho}s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j} + \frac{s_{ij}}{2\rho^2}\left\{\frac{\partial\rho}{\partial x_i}\frac{\partial p}{\partial x_j} + \frac{\partial\rho}{\partial x_j}\frac{\partial p}{\partial x_i}\right\}$.

There is a following *qualitative* difference between the equation for the kinetic energy, $\frac{1}{2}u^2$, and the equations for the enstrophy, $\frac{1}{2}\omega^2$, and total strain, $\frac{1}{2}s^2$. Integrate these over a domain with homogeneous, periodic or/and other boundary conditions causing the surface integrals to vanish. The nonlinear terms do not contribute to the equation for energy due to their conservative nature – they can be written in the form $\frac{\partial}{\partial x_j}(\cdots)$.

$$\frac{d}{dt}\int_V\left(\frac{1}{2}u^2\right)dV = -\int_V \epsilon dV + \int_V u_iF_i dV. \tag{C.19}$$

That is, kinetic energy is an inviscid invariant, and therefore it is meaningful to speak of its dissipation[1].

On the contrary the nonlinear term, $\int \omega_i \omega_j s_{ij} dV$, corresponding to the net enstrophy production is (empirically) known to be strictly positive both from laboratory and numerical experiments (Taylor, 1938a,b; Betchov, 1976; Tsinober et al., 1997), i.e., its contribution does not vanish,

$$\frac{1}{2}\frac{d}{dt}\int_V \omega^2 dV = \int_V \omega_i \omega_j s_{ij} dV - \int_V \epsilon_\omega dV + \int_V \varepsilon_{ijk}\omega_i \frac{\partial F_k}{\partial x_j} dV. \qquad \text{(C.20)}$$

That is, enstrophy is not an inviscid invariant and therefore the expression $\epsilon_\omega = -\nu \omega_i \nabla^2 \omega_i$, as any other, cannot be termed as dissipation of enstrophy, since there is no way to find a unique expression for ϵ_ω (as in the case of ϵ): it can be written as $(curl\omega)^2$, $\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j}$ and many other forms, all of them differing by terms in the form $\frac{\partial}{\partial x_j}(\cdots)$.

A similar equation can be written for the total strain,

$$\frac{1}{2}\frac{d}{dt}\int_V s^2 dV = -\int_V \left(s_{ij}s_{jk}s_{ki} + \frac{1}{4}\omega_i\omega_j s_{ij}\right) dV - \int_V \epsilon_s dV + \int_V s_{ij}F_{ij} dV, \qquad \text{(C.21)}$$

[1]See Loitsyanskii (1966), also Serrin (1959), for a simple demonstration that dissipation rate of kinetic energy, i.e., the rate at which mechanical energy is turned *locally* into heat in an incompressible Newtonian fluid, is $\epsilon = 2\nu s_{ij}s_{ij}$. It should be emphasized that this is the true local energy dissipation rate in incompressible flows. The stress is made here, since this expression can be written for example, as $\epsilon = 2\nu s_{ij}s_{ij} = \nu\omega_i\omega_i + \frac{\partial^2}{\partial x_i \partial x_j}\{u_i u_j\}$. That is with appropriate boundary conditions $2\nu \int s_{ij}s_{ij} dV = \nu \int \omega_i\omega_i dV$. Also for homogeneous turbulence $2\nu \langle s_{ij}s_{ij}\rangle = \nu\langle\omega_i\omega_i\rangle$, and at large Reynolds numbers $2\nu\langle s_{ij}s_{ij}\rangle \approx \nu\langle\omega_i\omega_i\rangle$. These are nothing more than *kinematic* relations. The *true physical causal* relation is between dissipation and strain both in Newtonian and non-Newtonian fluids. Therefore it is a misconception to associate dissipation directly with vorticity. Dissipation is a quantity essentially associated with strain.

For a viscous Newtonian compressible fluid,

$$\epsilon_c = 2\nu\left(s_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)\left(s_{ij}s_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \frac{\zeta}{\rho}\left\{\frac{\partial u_k}{\partial x_k}\right\}^2$$

where ζ is the so-called second or bulk viscosity. Even if $\zeta \approx 0$ (an assumption valid in many cases) compressibility effects reduce the dissipation.

It is customary to rearrange the terms in the expression for ϵ_c and to represent it as

$$\epsilon_c = \left(\frac{4}{3}\nu + \frac{\zeta}{\rho}\right)\left\{\frac{\partial u_k}{\partial x_k}\right\}^2 + \nu\omega_i\omega_i + \nu\frac{\partial}{\partial x_i}\left\{\frac{\partial u_i u_j}{\partial x_j} + u_j\frac{\partial u_i}{\partial x_j} - 3\frac{\partial}{\partial x_j}\left(u_j\frac{\partial u_k}{\partial x_k}\right)\cdots\right\}$$

with the first term called compressible or dilatational dissipation and the second term called solenoidal or incompressible dissipation (Lele, 1994; Friedrich and Bertolotti, 1997 and references therein). Though this is done for their means only, from the physical point the latter is still misleading.

where $\epsilon_s = -\nu s_{ij}\nabla^2 s_{ij}$, and the $-\int(s_{ij}s_{jk}s_{ki}+\frac{1}{4}\omega_i\omega_j s_{ij})dV = \frac{1}{2}\int \omega_i\omega_j s_{ij}dV$, corresponding to the net strain production, is also positive.

In summary, the nonlinearity does not contribute to the rate of change of energy, but only redistributes it in space. On the contrary, the nonlinearity makes a positive contribution to the rate of change of the total enstrophy and strain[2].

In two-dimensional flows the 'source' terms $\omega_i\omega_j s_{ij}$ and $s_{ij}s_{jk}s_{ki}$ vanish, and both enstrophy and total strain are inviscid invariants. Moreover, vorticity, and consequently enstrophy (but not the total strain) is a pointwise Lagrangian inviscid invariant, i.e., vorticity of any infinitesimal material fluid element does not change. However, the quantity of the next level, the palinstrophy, $\xi^2, \xi = curl\omega$, is not an inviscid invariant, since it obeys the equation

$$\frac{1}{2}\frac{D\xi^2}{Dt} = \xi_i\xi_j s_{ij} + \nu\xi_i\nabla^2\xi_i \tag{C.22}$$

and its net production $\int \xi_i\xi_j s_{ij}dV$ is again a positive quantity.

Equivalently, the equation for ζ^2, where $\zeta = \nabla\omega$ $(= \partial\omega/\partial x_i)$ is the vorticity gradient, takes the form

$$\frac{1}{2}\frac{D\zeta^2}{Dt} = -\zeta_i\zeta_j s_{ij} + \nu\zeta_i\nabla^2\zeta_i, \tag{C.22$'$}$$

and the net production of ζ^2 $(-\int \zeta_i\zeta_j s_{ij}dV = \int \xi_i\xi_j s_{ij}dV)$ is a positive quantity as well.

[2]This has interesting implications in the context of stability theory. Namely, the Reynolds–Orr equation for the total energy of a disturbance, u_i, of an undisturbed shear flow U_i (assumed to be a solution of NSE) does not contain cubic terms in the disturbance (corresponding to the nonlinear terms in NSE)

$$\frac{d}{dt}\int_V \left(\frac{1}{2}u^2\right) dV = -\int u_i u_j \frac{\partial U_i}{\partial x_j} dV - \int_V \epsilon dV.$$

This means that the rate of change of the energy of the disturbance $E^{-1}dE/dt$ does not depend on the disturbance amplitude, i.e. in some sense, is the same for infinitesimal and finite amplitude disturbances. This was interpreted (see references in Henningson, 1996) in the sense that the disturbance energy produced by linear mechanisms is the only disturbance energy available. In contrast to the energy equation, the corresponding equation for enstrophy (and a similar equation for strain)

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\omega^2\right) dV = \int \left(-\omega_i u_j \frac{\partial \Omega_i}{\partial x_j} + \omega_i\omega_j S_{ij} + \omega_i s_{ij}\Omega_j + \omega_i\omega_j s_{ij}\right) dV - \int_V \varepsilon_\omega dV.$$

does contain the cubic term, $\omega_i\omega_j s_{ij}$ corresponding to the self-amplification of vorticity. Hence the rate of change of the enstrophy of the disturbance $\mathcal{E}_\omega^{-1} d\mathcal{E}_\omega/dt$ does depend on the disturbance amplitude, and is different for infinitesimal and finite amplitude disturbances. A similar statement is true of the total strain.

This shows the advantages of using velocity derivatives (vorticity and strain) in elucidating the essential aspects of physics.

13.2.2. SOME ADDITIONAL CONSEQUENCES FROM THE NSE AND INVARIANT QUANTITIES

Equation for enstrophy production $\omega_i\omega_j s_{ij}$[3]

$$\frac{D}{Dt}\omega_i\omega_j s_{ij} = W^2 - \omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j} + VT. \tag{C.23}$$

Equation for $s_{ij}s_{jk}s_{ki}$

$$\frac{D}{Dt}s_{ij}s_{jk}s_{ki} = -3\left\{s^4 + \frac{1}{4}(W^2 - s^2\omega^2) + s_{ik}s_{kj}\frac{\partial^2 p}{\partial x_i \partial x_j}\right\} + VT, \tag{C.24}$$

where $W^2 = \omega_j s_{ij}\omega_k s_{ik}$, $s^4 = s_{ik}s_{kj}s_{il}s_{lj}$ and VT stands for viscous terms.

Writing such equations allows us to identify in a natural way the dynamically-significant geometrical invariant quantities and relations between such invariants of different order via dynamical equations, all of which are the consequence of the Navier–Stokes equations. Namely the quantities of the second order are ω^2 and s^2; the invariants of the third order are $\omega_i\omega_j s_{ij}$, $s_{ij}s_{jk}s_{ki}$ and $s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}$. All of them appear in the equations for enstrophy (C.16) and total strain (C.18). The invariants of the fourth order are

$$\mathcal{I}_1 = s_{ik}s_{kj}s_{il}s_{lj};\;\; \mathcal{I}_2 = \omega^2 s^2;\;\; \mathcal{I}_3 = \omega_i s_{ij}\omega_k s_{ik} \equiv W^2;\; \mathcal{I}_4 = \omega^4. \tag{C.25}$$

The right-hand side of the equations (C.23) and (C.24) contain three of the four invariants. The fourth invariant, $\mathcal{I}_4 = \omega^4$, appears in the equations for the higher-order quantities. Thus the invariants $\mathcal{I}_1 - \mathcal{I}_4$ are *pointwise* quantities of dynamical significance. Siggia (1981) identified the means $\langle \mathcal{I}_i \rangle$, $i = 1 - 4$, in the kinematical context as invariants determining all the 105 terms of the tensor of the fourth rank $T^{(4)}_{i,j,k,l...} = \langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} ... \rangle$ for an isotropic velocity field.

An important aspect is that the equations (C.18), (C.23) and (C.24) contain three additional invariant quantities containing the pressure Hessian, $h_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j}$: one of third order, $s_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}$ and two of fourth order,

$$\mathcal{I}_5 = \omega_i\omega_j\frac{\partial^2 p}{\partial x_i \partial x_j};\;\; \mathcal{I}_6 = s_{ik}s_{kj}\frac{\partial^2 p}{\partial x_i \partial x_j}. \tag{C.26}$$

All three reflect the nonlocal *dynamical* effects due to interaction of the pressure Hessian with vorticity and strain.

[3]In the equations of this subsection the terms associated with the external force, F_i, are dropped.

Geometrical invariants remain unchanged under the full group of rotations (i.e., rotations plus reflections)[4] in contradistinction with other *noninvariant* combinations of velocity derivatives. For this reason the geometrical invariants are mostly-appropriate for studying physical processes in (turbulent) fluid flows, their structure and universal properties (Tsinober, 1995, 1996a and references therein).

Another way of choosing invariant quantities is to look directly at the invariants of the velocity gradient tensor $\frac{\partial u_i}{\partial x_j}$: the first invariant $-P = \frac{\partial u_i}{\partial x_i}$, vanishing for incompressible flow; the second invariant $-Q = \frac{1}{4}(\omega^2 - 2s^2)$, and the third invariant $R = -\frac{1}{3}(s_{ij}s_{jk}s_{ki} + \frac{3}{4}\omega_i\omega_j s_{ij})$, both written for incompressible flows (see Chacin and Cantwell, 2000; Martin et al., 1998; Ooi et al., 1999 and references therein). These invariants arise naturally as coefficients in the characteristic equation for the eigenvalues of $\frac{\partial u_i}{\partial x_j}$. It is a convenient and useful way to study some of the local flow properties in the $R - Q$ plane. This analysis should be complemented by looking at the invariants of s_{ij} (e.g., its eigenvalues, Λ_i), the invariant quantities mentioned above as well as some other discussed below.

It is noteworthy that some of the invariant quantities allow useful geometrical interpretation. For example,

$$\omega_i\omega_j s_{ij} = \omega^2\Lambda_i \cos^2(\omega, \lambda_{\mathbf{i}}) \equiv \omega \cdot \mathbf{W} = \omega W \cos(\omega, \mathbf{W},) \tag{C.27}$$

where λ_i is the eigenframe of the rate of strain tensor, s_{ij}, and Λ_i are its eigenvalues. The vector $W_i = \omega_j s_{ij}$ is the vortex stretching vector. Another example, $\omega_i s_{ij}\omega_k s_{ik} \equiv W^2 = \omega^2\Lambda_i^2 \cos^2(\omega, \lambda_{\mathbf{i}})$.

Note a useful relation

$$\cos(\omega, \mathbf{W}) = \frac{\Lambda_i \cos^2(\omega, \lambda_{\mathbf{i}})}{\{\Lambda_i^2 \cos^2(\omega, \lambda_{\mathbf{i}})\}^{1/2}}, \tag{C.28}$$

which shows that $\cos(\omega, \mathbf{W})$ is independent both of the magnitude of vorticity and total strain. As expected it depends only on geometrical properties of the velocity gradients: the mutual orientation of vorticity vector, ω, and the eigenframe, λ_i, of the strain tensor, $s_{ij,}$ and of the shape of the latter, e.g., the ratios Λ_2/Λ_1 and Λ_3/Λ_1.

The cosines $\cos(\omega, \lambda_{\mathbf{i}})$ and $\cos(\omega, \mathbf{W})$ are examples of invariant quantities of the kind discussed here. As we shall see in chapters 6–9, they allow us to study an important aspect of the essential dynamics of 3-D turbulence associated with the geometrical properties of the field of velocity derivatives.

[4]They are also invariant under space/time translations and the Galilean transformation (see below).

Inviscid invariants
Here by invariants are meant quantities which remain unchanged under the inviscid dynamics, i.e., invariants of the Euler equations.
Kinetic energy

$$\frac{dE}{dt} = -\oint_A pu_n dA - \int_V \epsilon dV.$$

Helicity

$$\frac{d\mathcal{H}}{dt} = \oint_A \omega_n C \; dA - 2\nu \int_V \omega \cdot curl \; \omega \; dV,$$

where $E = \int_V \frac{1}{2}u^2 dV$, $\mathcal{H} = \int_V h dV$, $h = \omega \cdot \mathbf{u}$, $C = (1/2)\mathbf{u}^2 - p/\rho$, and A is the surface bounding some volume V, p – pressure, ρ – fluid density, and the fluid is assumed incompressible.

Thus in the absence of viscosity and external forces, and with vanishing surface integrals, both kinetic energy and helicity of a fluid (and space) volume are conserved. The essential difference is that – unlike kinetic energy – helicity is a non-positively defined quantity. This makes it more difficult to use this important quantity (see Moffatt and Tsinober, 1992 and Droegemeier et al., 1993). Another difference is that it is not invariant under reflections – it is pseudoscalar, so that it changes its sign under reflections. Finally we mention that one can define a modified helicity density as $\widetilde{h_\omega} \equiv \omega \cdot \mathbf{v}$ with $\mathbf{v} = \mathbf{u} + \nabla\phi$ and $\mathbf{u}$ being the fluid particle velocity. There exists a particular choice of ϕ such[5] that $\widetilde{h_\omega}$ is a pointwise inviscid Lagrangian invariant.

We did not mention here other invariants such as those linear in velocity (e.g., angular momentum) and several others (see Monin and Yaglom, 1971, 1975), as their application to turbulent flows is quite limited and is beyond the scope of this small book.

13.2.3. SYMMETRIES OF EULER AND NAVIER–STOKES EQUATIONS

The Euler and the Navier–Stokes equations are invariant under the following transformations (see Frisch, 1995; also Oberlack, 2002).

- Translations in space and time
- Full group of rotation including rotations and reflections
- Galilean transformation $\mathbf{u}(\mathbf{x}, t) \;\Rightarrow\; \mathbf{u}(\mathbf{x} - \mathbf{U}t, t) + \mathbf{U}, \qquad \mathbf{U} = \mathbf{const}$

The Euler equation is in addition invariant under

- Time reversal $t \;\Rightarrow\; -t, \quad \mathbf{u} \;\Rightarrow\; -\mathbf{u},\; p \;\Rightarrow\; p$
- Scaling transformation

$$\mathbf{r} \Rightarrow \lambda\mathbf{r};\;\; t \Rightarrow \lambda^{1-h}t;\;\; \mathbf{u} \Rightarrow \lambda^{h}\mathbf{u};\;\; p \Rightarrow \lambda^{2h}p,\;\; \lambda > 0 \text{ for any } h. \quad \text{(C.29)}$$

[5]The function ϕ should satisfy the equation $\frac{D\phi}{Dt} = p - u^2/2 + \nu\nabla^2\phi$, Oseledets (1989).

The Navier–Stokes equations obey the scaling transformation for $h = -1$ only. However, it is a common belief that it *may be justified at very high Reynolds number... that there are infinitely many scaling groups, labelled by their scaling exponent, h, which can be any real number, i.e., in the inviscid limit, the Navier–Stokes equation is invariant under infinitely many scaling groups, labelled by an arbitrary real scaling exponent h* (Frisch, 1995, p. 18, 144) just as in the case of the Euler equation. However, it is not at all clear why one can ignore the singular nature of the limit $\mathrm{Re} \to \infty (\nu \to 0)$ when handling the issue of scaling exponents and/or related matters.

13.3. Passive objects

13.3.1. PASSIVE SCALARS

The behavior of a passive scalar with concentration θ is described by the (deceptively simple, but still linear) advection-diffusion equation[6]

$$\frac{D\theta}{Dt} = \mathcal{D}\nabla^2\theta + \Phi, \tag{C.30}$$

where Φ is the external source/forcing[7]. The corresponding 'energy' equations are

$$\frac{D}{Dt}\left(\frac{1}{2}\theta^2\right) = -\mathcal{D}G^2 + \mathcal{D}\nabla^2\left(\frac{1}{2}\theta^2\right) + \Phi\theta, \tag{C.31}$$

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\theta^2\right) dV = -\mathcal{D}\int_V G^2(t)dV + \int_V \Phi\theta dV, \tag{C.32}$$

where $G^2 = G_iG_i, G_i = \frac{\partial\theta}{\partial x_i}$ and in the last equation the surface integrals are assumed vanishing.

13.3.2. PASSIVE VECTORS

There are roughly two kinds of passive vectors.

Gradient of a Passive Scalar
It is a passive vector, $G_i = \frac{\partial\theta}{\partial x_i}$, governed by the equation

[6]In compressible fluids, $\frac{\partial\theta}{\partial t} + u_k\frac{\partial\theta}{\partial x_k} + \theta\frac{\partial u_k}{\partial x_k} = \mathcal{D}\nabla^2\theta$.

[7]The external forcing can be roughly of two kinds. In the case of homogenous and isotropic flow, the only way to sustain a statistically-stationary state is to apply an isotropic forcing in the RHS of the corresponding equations. In the case of homogeneous flow, the forcing term may have its origin in the mean gradient. For example, one of the simplest cases – which is homogeneous, but not isotropic – is when a constant mean gradient, $\mathbf{g}$, is imposed on the scalar, i.e., $\Theta = \theta + \mathbf{g}\cdot\mathbf{x}$, and $\frac{\partial\theta}{\partial t} + u_k\frac{\partial\theta}{\partial x_k} = \chi\nabla^2\theta - u_kg_k$, so that the forcing is dependent on the velocity field.

$$\frac{\partial G_i}{\partial t} + u_k \frac{\partial G_i}{\partial x_k} = -G_k \frac{\partial u_k}{\partial x_i} + \mathcal{D}\nabla^2 G_i + \frac{\partial \Phi}{\partial x_k}, \tag{C.33}$$

with the energy equations in the form

$$\frac{D}{Dt}\left(\frac{1}{2}G^2\right) = -G_i G_k s_{ik} - \epsilon_G + \nabla^2\left(\frac{1}{2}G^2\right) + \frac{\partial \Phi}{\partial x_i} G_i, \tag{C.34}$$

$$\frac{d}{dt}\int_V \left(\frac{1}{2}G^2\right) dV = -\int_V G_i G_k s_{ik} dV - \int_V \epsilon_G dV + \int_V \frac{\partial \Phi}{\partial x_i} G_i dV, \tag{C.35}$$

where $\epsilon_G = \mathcal{D}\frac{\partial G_i}{\partial x_k}\frac{\partial G_i}{\partial x_k}$. In the absence of diffusivity, $\mathcal{D} = 0$, and external forcing, the vector G_i is proportional to the (infinitesimal) vector representing a material surface element (its normal), N_i, thus satisfying the equation

$$\frac{\partial N_i}{\partial t} + u_k \frac{\partial N_i}{\partial x_k} = -N_k \frac{\partial u_k}{\partial x_i}. \tag{C.2}$$

'Frozen' Passive Vectors
Such vectors obey the equation[8]

$$\frac{\partial B_i}{\partial t} + u_k \frac{\partial B_i}{\partial x_k} = B_k \frac{\partial u_i}{\partial x_k} + \chi\nabla^2 B_i + F_i, \tag{C.36}$$

with an 'energy' equation in the form

$$\frac{D}{Dt}\left(\frac{1}{2}B^2\right) = B_i B_k s_{ik} - \epsilon_B + \chi\nabla^2\left(\frac{1}{2}B^2\right) + F_i B_i, \tag{C.37}$$

$$\frac{d}{dt}\int_V \left(\frac{1}{2}B^2\right) dV = \int_V B_i B_k s_{ik} dV - \int_V \epsilon_B dV + \int_V F_i B_i dV, \tag{C.38}$$

where $\epsilon_B = \chi\frac{\partial B_i}{\partial x_k}\frac{\partial B_i}{\partial x_k}$.

Again such a vector, in the absence of molecular diffusive effects, $\chi = 0$, and external forcing is frozen into the fluid motion, i.e., its lines consist of the same material particles during the evolution, and it is proportional to the (infinitesimal) material line elements, l_i, thus satisfying the equation[9]

$$\frac{\partial l_i}{\partial t} + u_k \frac{\partial l_i}{\partial x_k} = l_k \frac{\partial u_i}{\partial x_k}. \tag{C.1}$$

[8]Provided that **B** is solenoidal, for example, a magnetic field. Otherwise an additional term $u_i \frac{\partial B_k}{\partial x_k}$ enters the RHS with one more term, $-B_i\frac{\partial u_k}{\partial x_k}$, in the case of compressible fluids.

[9]The equation for vorticity in an inviscid fluid has a *similar* form $\frac{\partial \omega_i}{\partial t} + u_k\frac{\partial \omega_i}{\partial x_k} = \omega_k\frac{\partial u_i}{\partial x_k}$. Moreover, vorticity is also frozen in the fluid motion. However, since $\omega_i = \epsilon_{ijk}\frac{\partial u_j}{\partial x_k}$, vorticity is *not* a passive vector – it 'reacts back', see Chapter 6.

Figure 13.1. Studies of an Old Man Seated and of Swirling Water, Leonardo da Vinci, Windsor, RL, No. 12, 579r, The Royal Collection ©2001, Her Majesty Queen Elizabeth II

Just as in the dynamical problem the quantities $G_iG_ks_{ik}$, $N_iN_ks_{ik}$, $B_iB_ks_{ik}$, $l_il_ks_{ik}$; $\cos(\mathbf{G}, \lambda_\mathbf{i})$, $\cos(\mathbf{N}, \lambda_\mathbf{i})$**,** $\cos(\mathbf{B}, \lambda_\mathbf{i})$, $\cos(\mathbf{l}, \lambda_\mathbf{i})$**;** $\cos(\mathbf{G}, \mathbf{W}^G)$**,** $\cos(\mathbf{G}, \mathbf{W}^N)$**,** $\cos(\mathbf{B}, \mathbf{W}^B)$**,** $\cos(\mathbf{l}, \mathbf{W}^l)$ all are geometrical invariants in the sense discussed above with $W_i^G = -G_js_{ij}, W_i^N = -N_js_{ij}, W_i^B = B_js_{ij}$, $W_i^l = l_js_{ij}$.

13.4. Some basic relations for the statistical description of turbulent flows

From now on (as in the main text) capital letters will be used for the mean/average quantities (if not indicated otherwise), whereas the lower case letters – for the fluctuations, i.e., the instantaneous value of some quantity, $\widetilde{z}$, is equal to $Z + z$. The mean/average in some sense (ensemble, time, space) will be denoted by $\langle \cdots \rangle$, i.e., $\langle \widetilde{z} \rangle = \langle Z + z \rangle = Z$, $\langle z \rangle = 0$, and where necessary specified. This is called Reynolds decomposition, which was described in words by Leonardo da Vinci on one of his drawings, figure 13.1 (Richter, 1970): *Observe the motion of the surface of the water which resembles that of hair, which has two motions, of which one depends on the weight of the hair, the other on direction of curls; thus the water forms eddying whirlpools, one part of which is due to the impetus of the principal current and the other of incidental motion and return flow.*

13.4.1. SCALING, SCALES AND RELATED MATTERS

> *Scaling laws are at the heart of turbulence research*
> (Tennekes and Lumley, 1972).
> *The wonderful thing about scaling is that you can get everything right without understanding anything*
> (Kraichnan, as cited by Kadanoff, 1990).

There is no contradiction between the two views[10], since the first is mainly about the parameters determining the order of magnitude of some quantity of interest, whereas the second concerns primarily the scaling in the sense of power laws and their scaling exponents. Both are closely related to the use of dimensional analysis, similarity and related matters[11].

However,

. . . it is clear that if a result can be derived by dimensional analysis alone . . . then it can be derived by almost any theory, right or wrong, which is dimensionally-correct and uses the right variables (Bradshaw, 1994). That is, the correspondence with the experimental results may occur for the wrong reasons, as happens from time to time in the field of turbulence.

Use of similarity and dimensional analysis in turbulence is more an art than science (Ya.B. Zel'dovich, 1971 as quoted by G.S. Golitsyn, 1999).

In other words a prerequisite for the use of dimensional analysis, similarity and symmetries (group theoretical methods) in turbulence is at least some minimal understanding of the basic physics of turbulent flows.

Scales and 'eddies'

> *What is meant by "scale" is hard to define precisely*
> (Chorin, 1994).
> *Fully-turbulent flows consist of a wide range of scales, which are classified somewhat loosely as either large or small scales*
> (Sreenivasan and Antonia, 1997).
> *An eddy eludes a precise definition, but it is conceived to be turbulent motion, localized within a region of size ℓ*
> (Pope, 2000).
> *. . . to a large extent we have failed. The key point is that we have not yet agreed on what we mean by an eddy*
> (Davidson, 2004, p. 412).

[10] There is a third view: *It is increasingly clear that deterministic chaos and universal scaling theories can explain everything* (Normal, 1993).

[11] See Corrsin (1953a) for a short and very instructive exposition of dimensional analysis and similarity in the fluid dynamical contexts including turbulence.

It is rather common to admit that the term 'scale' is not a well-defined concept. However, it is much more common to use the term scale(s) in a great variety of contexts and meanings in turbulence. For example, it is frequently used in physical space and in Fourier space without much distinction, though a narrow band in Fourier space involves a broad range of scales in physical space and vice versa due to the integral nature of the Fourier transform. This ambiguity of language is one (among many) of the reflections of the inherent problems in turbulence 'theories'[12]. One of the main difficulties is due to the nonlinearity and nonlocality of the turbulence phenomenon. That is, different 'scales' are not so separated as usually assumed, and therefore the division of turbulent flow into large and small scales is quite problematic, though useful from the technical point. Speaking about interaction, dependence, and coupling of scales, it is meant that some quantities 'residing' on these scales are involved and characterize such interaction. Therefore, it is important to specify these dynamically-relevant quantities, which adequately represent the 'scales'. We will use the term scale mainly in two meanings: i – in its simplest direct geometrical meaning in the physical space and in a similar way in time, and ii – implying the quantities representing these scales in some sense. For instance, velocity (fluctuations) are representative of large scales, whereas the field of velocity derivatives (vorticity and strain) is appropriate to represent the small scales (chapter 5). Note that using such a 'definition' of small scales does not specify the small scales uniquely: the small scales associated with vorticity are different from the scales related to strain.

Some characteristic statistical scales of turbulent flows

There are several useful scales used in turbulence, which belong to its simplest statistical characteristics, i.e., they are not some specific scales, but they are statistically-defined quantities. Some of these scales are given below, others that are more specific appear in the main text.

Kolmogorov scales. The Kolmogorov length scale is defined on dimensional grounds as

$$\eta = (\nu^3/\epsilon)^{1/4}, \tag{C.39}$$

where ν is the fluid kinematic viscosity and ϵ is the mean rate of energy being pumped into the system, which is equal to the mean dissipation in statistically-stationary flows (see below). Similarly, the corresponding time, velocity and acceleration scales are defined as $\tau_\eta = (\nu/\epsilon)^{1/2}$, $u_\eta = (\nu\epsilon)^{1/4}$ and $a = \epsilon^{3/2}\nu^{-1/2}$. The Kolmogorov scale, η, can be defined in various ways. For example, imposing the condition $\mathrm{Re} = \frac{u_\eta \eta}{\nu} \approx 1$ and

[12]The ambiguity of Fourier decomposition was addressed by Liepmann (1962) and Tennekes (1976); see also Lohse and Müller-Groeling (1996).

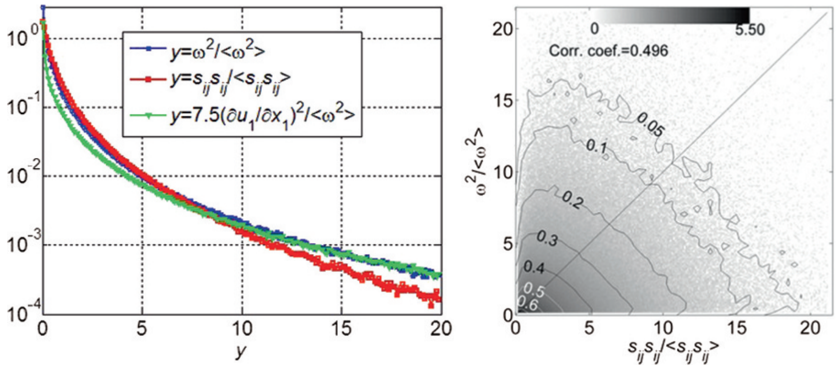

Figure 13.2. PDFs of ω^2 and s^2 (left) and their Joint PDF (right) in a field experiment at $\mathrm{Re}_\lambda \sim 10^4$ (Gulitski et al., 2007a)

writing $\epsilon \approx \nu(u_\eta/\eta)^2$ with $u_\eta \approx \nu/\eta$ one arrives again at (C.39). It is known that adequate space/time resolution (in laboratory, field or numerical experiments) is achieved if the smallest resolved scales are of the order of the space/time Kolmogorov scales as defined above. For instance, η is considered as the smallest spatial relevant scale in turbulence. As mentioned in Tsinober (2001a), this, however, is not obvious, since the instantaneous dissipation ε is not narrow-banded around its mean ϵ, but is distributed with a rather long tail, so that values as large as $10^2\epsilon$ are not that rare in laboratory experiments, Tsinober et al. (1992). In a field experiment, Gulitski et al. (2007a), the tail in the PDF of s^2 is longer (figure 13.2) and the instantaneous ε may reach values as high as $10^4 \langle\varepsilon\rangle$. This corresponds to scales an order of magnitude smaller than η.

Integral scales. These are in some sense the largest relevant scales of the system. Regions, separated by scales much larger than the integral ones, both in space and time, do not 'know' much about each other. For a homogeneous turbulent field this is defined, e.g., as

$$\mathcal{L} = \frac{1}{R_{uu}(0)} \int R_{uu}(\mathbf{r})d\mathbf{r}, \tag{C.40}$$

where $R_{uu}(\mathbf{r}) = \langle u_r(\mathbf{x}+\mathbf{r},t)u_r(\mathbf{x},t)\rangle$ is the longitudinal correlation function of the velocity component along $\mathbf{r}$. An integral scale which is associated with the transverse velocity correlation is of the same order of magnitude. Similarly the integral time scale $\mathcal{T}$ is defined via the analogous time correlation. One of the definitions of the integral velocity scale is $\mathcal{U} = \frac{1}{\mathcal{L}^3}\int R_{uu}(\mathbf{r})d\mathbf{r}$, which is of the order of the variance of velocity fluctuations $R_{uu}(0) = \langle u_r^2\rangle$, and the integral time scale $\mathcal{T} \sim \mathcal{L}/\mathcal{U}$. For practical purposes $\mathcal{L}$ and $\mathcal{T}$ are usually taken to be of the order of (but smaller than) the external scales of the system.

Taylor microscale. The Taylor microscale, λ, is defined from the relation[13]

$$\epsilon \sim \nu \frac{\mathcal{U}^2}{\lambda^2}, \tag{C.41}$$

where $\mathcal{U}$ is the integral scale of turbulent velocity fluctuations. That is, λ is a mixed hybrid scale, since $\mathcal{U}$ is an integral scale and ϵ occurs mostly in much smaller scales. The Taylor microscale, λ, is not the smallest relevant scale in turbulent flow, and it is different from the Kolmogorov scale. Though λ is not a physically-representative length scale, it is used as a convenient length scale in evaluating the Reynolds number, associated with the field of velocity derivatives.

Relation between the statistical scales. This is obtained using the estimate $\epsilon \sim \mathcal{U}^3/\mathcal{L}$, Taylor (1935). It follows that

$$\frac{\eta}{\mathcal{L}} \sim Re^{-3/4}, \ \frac{\eta}{\lambda} \sim Re^{-1/2}; \ \frac{\lambda}{\mathcal{L}} \sim Re^{-1/2};$$

$$\tau_\eta = \tau_\lambda = \left(\frac{\nu}{\epsilon}\right)^{1/2} \sim \mathcal{T} Re^{-1/2}. \tag{C.42}$$

These relations allow one to estimate the number of degrees of freedom as $(\mathcal{L}/\eta)^3 \sim Re^{9/4}$, see the first (!) edition of Landau and Lifshits (1959), since this estimate was removed from the subsequent editions.

It is noteworthy that all the above definitions make sense with the 'conventional' large-scale forcing. For example, the definition (C.39) of the Kolmogorov scale with the broadband forcing is less meaningful, since in the forcing-dominated regime (see section 6.5.3) the forcing is expected to bypass the nonlinearity and to balance the dissipation directly.

13.4.2. REYNOLDS-AVERAGED NAVIER–STOKES EQUATIONS AND RELATED

Introducing the quantities, decomposed into a sum of averages and fluctuations, into the Navier–Stokes equations and averaging, results in:

[13]This scale was originally defined by Taylor (1935) as $\epsilon \sim 15\nu \left\langle u_1^2 \right\rangle / \lambda^2$ where u_1 is the x_1-component of velocity fluctuations. The motivation for the coefficient 15 is that in isotropic turbulence $\epsilon = 15\nu \left\langle (\partial u_1/\partial x_1)^2 \right\rangle$, so that $\lambda^2 = \left\langle u_1^2 \right\rangle / \left\langle (\partial u_1/\partial x_1)^2 \right\rangle$ – the most frequently used definition. The Taylor microscale is associated with the correlation function $R_{uu}(\mathbf{r})$ and can be defined as $\lambda_T^2 = -R_{uu}(0)/2R''(0)$. It is a technical matter to show that $\lambda_T = 2\lambda$ in (C.41).

The Reynolds-averaged Navier–Stokes equations (RANS) for the mean flow[14]

$$\frac{D_U U_i}{Dt} = \frac{\partial}{\partial x_j}\left(-\frac{1}{\rho}P\delta_{ij} + 2\nu S_{ij} - \langle u_i u_j\rangle\right); \tag{C.43}$$

here $-\rho\langle u_i u_j\rangle = \tau_{ij}^{\text{Reynolds}}$ is the Reynolds stress tensor and $\frac{D_U}{Dt} = \frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x_j}$. Subtracting this equation from the NSE gives an equation for the field of fluctuations

$$\frac{\partial u_i}{\partial t} + U_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\{u_i u_j - \langle u_i u_j\rangle\} = \frac{\partial}{\partial x_j}\{-\frac{1}{\rho}p\delta_{ij} + 2\nu s_{ij}\}. \tag{C.44}$$

Using a similar equation for u_j, one can obtain an equation for the instantaneous tensor $u_i u_j$, which after averaging becomes:
The equation for the Reynolds stress tensor [15]

$$\begin{aligned} &\frac{D_U\langle u_i u_j\rangle}{Dt} + \langle u_j u_k\rangle\frac{\partial U_i}{\partial x_k} + \langle u_i u_k\rangle\frac{\partial U_j}{\partial x_k} \\ &= -\frac{\partial\langle u_i u_j u_k\rangle}{\partial x_k} - \frac{1}{\rho}\left\{\frac{\partial}{\partial x_i}\langle p u_j\rangle + \frac{\partial}{\partial x_j}\langle p u_i\rangle\right\} + \frac{2}{\rho}\langle p s_{ij}\rangle \\ &+ 2\nu\frac{\partial}{\partial x_k}\{\langle u_j s_{ik}\rangle + \langle u_i s_{jk}\rangle\} - 2\nu\left\{\left\langle s_{ik}\frac{\partial u_j}{\partial x_k}\right\rangle + \left\langle s_{jk}\frac{\partial u_i}{\partial x_k}\right\rangle\right\}. \end{aligned} \tag{C.45}$$

This equation contains third-order quantities (in the second line). This process can be continued, producing the (infinite) Keller–Friedmann (1925) chain of equations. Each finite subset of this chain has more unknowns than equations. Roughly, this comprises the essence of the famous 'closure' problem; see Kraichnan (1962b) for a detailed exposition of the closure problem. On the simplest level one tries to introduce some (statistical) hypotheses on the relation between the Reynolds stress tensor $\langle u_i u_j\rangle$ and the mean flow U_i. In order to do this in an intelligent way, one has to understand – at least qualitatively – the essential physical processes of turbulent flows under consideration. It should be emphasized that in the strict sense such a relation does not exist due to the nonlocal nature of the relation between the mean flow and the fluctuations: it is not a function, but a functional as can be seen from the equations (C.43), (C.44) and a similar equation for u_j, see section 6.6.

[14] We subsequently dropped the external force. It is explicitly noted when the force is present.

[15] Similar equations can be written for *two-point* correlations in space, such as $\langle u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})\rangle$, and in space/time $\langle u_i(\mathbf{x};t)u_j(\mathbf{x}+\mathbf{r};t+t')\rangle$ (see Hinze, 1975; Favre et al., 1976). Such quantities are particularly useful in the case of homogeneous and isotropic turbulence.

The equation for the kinetic energy of the mean flow $E_T = \frac{1}{2}U_iU_i$

$$\frac{D_U E_T}{Dt} = \frac{\partial}{\partial x_j}\left\{-\frac{1}{\rho}PU_j + 2\nu U_i S_{ij} - \langle u_i u_j\rangle U_i\right\} + \langle u_i u_j\rangle S_{ij} + 2\nu S^2. \quad \text{(C.46)}$$

The equation for turbulent kinetic energy $e_T = \frac{1}{2}u_iu_i$
Instantaneous

$$\frac{D_U e_T}{Dt} = u_i\frac{\partial\langle u_i u_j\rangle}{\partial x_j} - \frac{\partial}{\partial x_j}\left\{u_j e_T + \frac{1}{\rho}pu_j - 2\nu u_i s_{ij}\right\} - u_i u_j S_{ij} - 2\nu s^2. \quad (C.47)$$

Mean

$$\frac{D_U\langle e_T\rangle}{Dt} = -\frac{\partial}{\partial x_j}\left\{\langle u_j e_T\rangle - 2\nu\langle u_i s_{ij}\rangle + \frac{1}{\rho}\langle pu_j\rangle\right\} - \langle u_i u_j\rangle S_{ij} - 2\nu\langle s^2\rangle. \quad \text{(C.48)}$$

The total turbulent energy balance for the whole flow domain is

$$\frac{d\mathcal{E}_T}{dt} = \mathcal{P} - \mathcal{D} \quad \text{(C.49)}$$

where $\mathcal{E}_T = \int \langle e_T\rangle\, dV$ is the total kinetic energy of turbulent fluctuations, $\mathcal{P} = \int - < u_i u_j > S_{ij} dV$ is the total rate of production/destruction of energy turbulent fluctuations by the mean strain (velocity gradients), and $\mathcal{D} = 2\nu \int \langle s_{ij}s_{ij}\rangle\, dV$ is the total rate of dissipation (simply dissipation) of energy of turbulent fluctuations by viscosity. If the flow is statistically stationary, the dissipation equals production $\mathcal{P} = \mathcal{D}$, i.e., $\mathcal{P} > 0$.

If there is no mean shear (more precisely mean strain) to supply the energy to the field of fluctuations, the statistically-stationary state is still possible in the presence of some external forcing, as it happens in DNS of NSE. Then the energy balance equation takes the form $\mathcal{W}_F = \mathcal{D}$, where $\mathcal{W}_F = \int \langle u_i F_i\rangle\, dV$ is the total rate of production of energy of turbulent fluctuations by the external forces. It is noteworthy that, in the presence of some energy supply other than the mean strain, the total rate of production/destruction of energy turbulent fluctuations by the mean strain, $\mathcal{P}$, does not have to be positive even in the case of a statistically- stationary turbulent flow, since in this case the balance is $\mathcal{W}_F + \mathcal{P} - \mathcal{D} = 0$. For example, there is some evidence that in this way the fluctuative motions produced by the supply of solar energy to the Earth's atmosphere are feeding such a 'mean' flow as the famous jet stream (see chapter 8).

A similar procedure can be applied to the vorticity equation giving:
The equation for the enstrophy of the mean flow $\frac{1}{2}\Omega^2$

$$\frac{1}{2}\frac{D_U\Omega^2}{Dt} = -\frac{\partial}{\partial x_j}\left\{\Omega_i\langle\omega_i u_j\rangle\right\} + \langle u_j\omega_i\rangle\frac{\partial\Omega_i}{\partial x_j} + \langle\omega_i s_{ij}\rangle\Omega_j$$

$$+\Omega_i\Omega_j S_{ij} + \nu\Omega_i\nabla^2\Omega_i. \tag{C.50}$$

The equation for the mean enstrophy of the turbulent fluctuations $\frac{1}{2}\langle\omega^2\rangle$

$$\frac{1}{2}\frac{D_U\langle\omega^2\rangle}{Dt} = -\langle u_j\omega_i\rangle\frac{\partial\Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\left\{\langle u_j\omega_i\omega_i\rangle\right\} + \langle\omega_i\omega_j s_{ij}\rangle$$

$$+\langle\omega_i\omega_j\rangle S_{ij} + \langle\omega_i s_{ij}\rangle\Omega_j + \nu\langle\omega_i\nabla^2\omega_i\rangle. \tag{C.51}$$

One of the possible interpretations of the terms in this equation for turbulent enstrophy is given in Tennekes and Lumley (1972). Here we draw attention to the two kinds of terms associated with the production of enstrophy of fluctuations $\langle\omega^2\rangle$: i – the RDT[16]-type terms involving the mean velocity gradients, $-\langle u_j\omega_i\rangle\frac{\partial\Omega_i}{\partial x_j}$, $\langle\omega_i\omega_j\rangle S_{ij}$ and $\langle\omega_i s_{ij}\rangle\Omega_j$, and ii – the terms containing the fluctuative quantities only $\langle\omega_i\omega_j s_{ij}\rangle$ and $\nu\langle\omega_i\nabla^2\omega_i\rangle$. A similar procedure can be applied to the equation for strain, s_{ij}, giving:
The equation for the strain of the mean flow $\frac{1}{2}S_{ij}S_{ij}$

$$\begin{aligned}\frac{D_U\frac{1}{2}S_{ij}S_{ij}}{Dt} &= -\frac{\partial}{\partial x_k}\left\{S_{ij}\langle u_k s_{ij}\rangle\right\} + \langle u_k s_{ij}\rangle\frac{\partial S_{ij}}{\partial x_k} - S_{ij}S_{jk}S_{ki} \\ &\quad -\frac{1}{4}\left\{\langle\omega_i\omega_i\rangle S_{ij} + \Omega_i\Omega_j S_{ij}\right\} - S_{ij}\frac{\partial^2 P}{\partial x_i\partial x_j} + \nu S_{ij}\nabla^2 S_{ij}.\text{(C.52)}\end{aligned}$$

The equation for the mean total strain of the turbulent fluctuations $\langle s_{ij}s_{ij}\rangle$

$$\begin{aligned}\frac{D_U\frac{1}{2}\langle s_{ij}s_{ij}\rangle}{Dt} &= -\langle u_k s_{ij}\rangle\frac{\partial S_{ij}}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_k}\left\{\langle u_k s_{ij}s_{ij}\rangle\right\} - 2\langle s_{ij}s_{ik}\rangle S_{kj} \\ &\quad -\frac{1}{2}\langle\omega_i s_{ij}\rangle\Omega_j - \langle s_{ij}s_{jk}s_{ki}\rangle - \frac{1}{4}\langle\omega_i\omega_j s_{ij}\rangle - \left\langle s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}\right\rangle \\ &\quad +\nu\langle s_{ij}\nabla^2 s_{ij}\rangle.\text{(C.53)}\end{aligned}$$

As in (C.51), there are two kinds of terms in the equation (C.53) associated with the production of the total strain of fluctuations $\langle\omega^2\rangle$: i – the RDT-type terms involving the mean velocity gradients, $-\langle u_k s_{ij}\rangle\frac{\partial S_{ij}}{\partial x_k}$, $-2\langle s_{ij}s_{ik}\rangle S_{kj}$ and $-\frac{1}{2}\langle\omega_i s_{ij}\rangle\Omega_j$, and ii – the terms containing the fluctuative quantities only, $-\langle s_{ij}s_{jk}s_{ki}\rangle$, $-\frac{1}{4}\langle\omega_i\omega_j s_{ij}\rangle$, and $-\langle s_{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}\rangle$ and $\nu\langle s_{ij}\nabla^2 s_{ij}\rangle$.

Equations similar to those given above can be written for passive objects as well.

[16]Rapid distortion theory (Savill, 1987; Hunt and Carruthers, 1990).

13.4.3. FILTER DECOMPOSITION

In this case instead of some average, as above, one defines a large-scale or 'resolvable' component of each 'raw' variable, $\widetilde{f}(\mathbf{x}, \mathbf{t})$ as its convolution with some filter function[17]

$$\prec \widetilde{f}(\mathbf{x}, t) \succ \equiv F(\mathbf{x}, \mathbf{t}) \;=\; \int \mathcal{G}(\mathbf{x} - \mathbf{x}', t - t'; \Delta, \theta) f(\mathbf{x}', t') d\mathbf{x}' dt', \quad \text{(C.54)}$$

where Δ, θ are the widths of the spatial and temporal filters respectively, so that $\widetilde{f} = F + f$, where f represents the subgrid scale (SGS) or the unresolved part, which is an analogue of the fluctuations in the Reynolds decomposition.

The equations for the filtered ('resolved') quantities are found in a similar way as in the case of RANS applying the operation $\prec \cdots \succ$ to the NSE[18]

$$\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(-P\delta_{ij} - \tau_{ij}^{\mathrm{SGR}} + 2\rho\nu S_{ij} \right), \qquad \text{(C.55)}$$

where the *subgrid-scale* stress[19]

$$\tau_{ij}^{\mathrm{SGR}} = -\rho \left\{ \prec \widetilde{u_i} \widetilde{u_j} \succ - U_i U_j \right\} \qquad \text{(C.56)}$$

should be modelled in some way, i.e., expressed in terms of the filtered field $\prec u_i \succ$ on the basis of some hypothesis, though, again, in the strict sense such a relation does not exist for the same reason as in RANS.

Subtracting the equation (C.55) from the NSE gives the equation for the unresolved part of the flows, which is analogous to the field of fluctuations in the Reynolds decomposition

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \left\{ -p\delta_{ij} + 2\rho\nu s_{ij} - \rho u_i u_j - \tau_{ij}^{\mathrm{SGR}} \right\}. \quad \text{(C.57)}$$

The filter decomposition is formally more general than the Reynolds decomposition. However, the former is one among many decompositions, so to say, of a technical nature, whereas the latter is physically more natural provided that the means/averages do exist in some sense. It is noteworthy

[17] This approach is based on Leonard (1974); see also Germano (1999); Piomelli and Balaras (2002); Meneveau and Katz (2000) and references therein.

An example of a spatial filter function is $\mathcal{G}(\mathbf{x}; \Delta) = 6/(\pi\Delta^2)^{3/2} \;\; exp(-6\mathbf{x}^2/l^2)$; $\int \mathcal{G}(\mathbf{x};\ \Delta) d\mathbf{x} = 1$.

[18] That is, the quantities $\prec \widetilde{u_i} \succ$, etc., are analogous to U_i in RANS.

[19] Generally, $\tau_{ij}^{\mathrm{SGR}} = -\rho \left\{ U_i u_j + U_j u_i + \prec u_i u_j \succ \right\} \neq - \prec u_i u_j \succ$. Only for filters such that $\prec\prec f \succ\succ = \prec f \succ$ the subgrid stress takes the form of a usual Reynolds stress $\tau_{ij}^{\mathrm{SGR}} = -\rho \prec u_i u_j \succ$.

that equations such as (C.45) for correlations, or (C.55) for filtered quantities by their very nature (apart from the 'closure problem') contain much less information on the turbulent flow than the Navier–Stokes equations. The main problem is the lack of clarity about how much and which physics is retained in the RANS or LES equations.

13.4.4. EQUATIONS GOVERNING THE DYNAMICS OF 'ERROR'

Since in many cases one is unable to reproduce precisely the initial (and boundary) conditions, it is of interest to follow the dynamics of an 'error', e.g., in initial conditions[20]. That is, one looks at the behaviour of the difference Δ^u of some undisturbed flow realization $\mathbf{u}$ and the one with a disturbance $\mathbf{u} + \Delta^u$. This behaviour is governed by an equation similar to (C.44)

$$\frac{D\Delta_i^u}{Dt} = \frac{\partial}{\partial x_j}\left\{-\frac{1}{\rho}\Delta^p\delta_{ij} + 2\nu\Delta_{ij}^s - \Delta_i^u\Delta_j^u\right\} - \Delta_j^u\frac{\partial u_i}{\partial x_j}, \tag{C.58}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}$ and Δ^p, Δ_{ij}^s are the pressure and the rate of strain errors.

The corresponding equation for the energy of the error, $e_{\Delta^u} = \frac{1}{2}\Delta_i^u\Delta_i^u$, is analogous to the equation (C.47)

$$\frac{De_{\Delta^u}}{Dt} = -\frac{\partial}{\partial x_j}\left\{\Delta_j^u e_{\Delta^u} + \frac{1}{\rho}\Delta^p\Delta_j^u - 2\nu\Delta_i^u\Delta_{ij}^s\right\} - 2\nu\Delta_{ij}^s\Delta_{ij}^s - \Delta_i^u\Delta_j^u s_{ij}. \tag{C.59}$$

The 'source' term, $-\Delta_i^u\Delta_j^u s_{ij}$, in the RHS of (C.59) is analogous to the turbulent energy production term, $-u_iu_jS_{ij}$, in (C.47). The analogy is not only formal. Namely, just as the mean, $-\langle u_iu_j\rangle S_{ij}$, is positive in turbulent shear flows, the integral of the production of the energy of error, $P_{\Delta^u} = -\int \Delta_i^u\Delta_j^u s_{ij}dV$, over the flow domain at any time moment is positive, because the production of the error energy, $-\Delta_i^u\Delta_j^u s_{ij}$, appears to be a positively-skewed quantity (Tsinober and Galanti, 2003).

Similar equations can be easily written also for the errors of vorticity, Δ_i^ω, and strain, Δ_{ij}^s. For example,

$$\frac{D\Delta_i^\omega}{Dt} = \omega_j\Delta_{ij}^s + \Delta_j^\omega s_{ij} - \Delta_j^u\frac{\partial\omega_i}{\partial x_j} + \Delta_j^\omega\Delta_{ij}^s - \Delta_j^u\frac{\partial\Delta_i^\omega}{\partial x_j} + \nu\nabla^2\Delta_i^\omega, \tag{C.60}$$

$$\frac{De_{\Delta^\omega}}{Dt} = \Delta_i^\omega\Delta_{ij}^s\omega_j + \Delta_i^\omega\Delta_j^\omega s_{ij} - \Delta_j^u\Delta_i^\omega\frac{\partial\omega_i}{\partial x_j}$$

[20] This approach takes its beginning from the predictability problem in meteorology (Holloway and West, 1984; Lorenz, 1985; Novikov, 1959) but is of more general importance. See also Bohr et al. (1998) and Lesieur (1997) and references therein.

$$+\Delta_i^\omega \Delta_j^\omega \Delta_{ij}^s - \Delta_j^u \Delta_i^\omega \frac{\partial \Delta_i^\omega}{\partial x_j} + \nu \Delta_i^\omega \nabla^2 \Delta_i^\omega, \qquad \text{(C.61)}$$

where $e_{\Delta^\omega} = \frac{1}{2}\Delta_i^\omega \Delta_i^\omega$ is the energy of the vorticity error. It is seen from (C.61) that apart from $\Delta_i^\omega \Delta_j^\omega s_{ij}$, several other terms are involved in the production of e_{Δ^ω}. Equations similar to (C.60), (C.61) can be written for Δ_{ij}^s, see Tsinober and Galanti, 2003. Following evolution of an 'error' of some quantity along a fluid particle trajectory (i.e., in a Lagrangian fashion but in an Euler representation) results in somewhat-different equations, e.g., involving also quantities such as Δ_i^ω and Δ_{ij}^s in the equations for passive objects.

13.5. Pure Lagrangian description

The Lagrangian formulation of the Navier–Stokes equations (see Monin and Yaglom, 1971, ch. 9; Corrsin, 1962b).

Newton's Law

$$\partial^2 X_i/\partial^2 t = [X_j, X_k, p] + \nu\{[X_2, X_3, [X_2, X_3, \partial X_i/\partial t]] + [X_3, X_1, [X_3, X_1, \partial X_i/\partial t]] + [X_1, X_2, [X_1, X_2, \partial X_i/\partial t]]\} \qquad \text{(C.62)}$$

Incompressibility

$$[X_1, X_2, X_3] = 1. \qquad \text{(C.63)}$$

Here (i, j, k) means an even permutation of the indices $(1, 2, 3)$. The vector $\mathbf{X}(\mathbf{a},t)$ is the particle position vector for a particle labelled by $\mathbf{a}$. Usually $\mathbf{a} \equiv \mathbf{X}(\mathbf{a},t_0)$, i.e., the initial positions of fluid particles are used as their labels. The expression $[A, B, C] \equiv \frac{\partial(A,B,C)}{\partial(a_1,a_2,a_3)}$ is an abbreviation for the Jacobian of the variables A, B, C with respect to variables a_1, a_2, a_3. We denote $[X_1, X_2, X_3] \equiv J$.

The terms on the RHS of (C.62) are highly nonlinear, the viscous one being of fifth-order nonlinearity.

The relation between the Lagrangian and Eulerian setting is described by

$$\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t} = \mathbf{U}[\mathbf{X}(\mathbf{a},t); t], \qquad \text{(C.64)}$$

where $\mathbf{U}[\mathbf{x},t]$ is an Eulerian velocity field, i.e., $\mathbf{U}[\mathbf{X}(\mathbf{a},t);t] = \mathbf{V}(\mathbf{a}, t)$ is the velocity of the $\mathbf{a}$-particle at t.

Note that (in contrast to the NSE in the Euler setting) the acceleration term in the Lagrangian setting $\partial^2 X_i/\partial^2 t$ is linear, and the 'inertial' effects are manifested only by the term containing pressure.

The equation for vorticity retains a purely-inertial term – the interaction of vorticity and velocity gradients (strain)

$$\partial\omega_i/\partial t = \omega_m[X_{j,}X_k, \partial X_i/\partial t] + \nu\{[X_2, X_{3,}[X_2, X_{3,}\ \omega_i]] \\ +[X_{3,}X_{1,}[X_{3,}X_{1,}\ \omega_i]] + [X_{1,}X_{2,}[X_{1,}X_{2,}\omega_i]]\} \qquad \text{(C.65)}$$

with (m, j, k) meaning an even permutation of the indices $(1, 2, 3)$.

However, there is no 'conventional' nonlinearity in the equation (C.66) below for the Cauchy invariant vector $\Omega_i(\mathbf{a}, t) = \epsilon_{ilj}\frac{\partial v_k}{\partial a_l}\mathcal{J}_{kj}$ ($\mathcal{J}_{kj} = \frac{\partial X_k}{\partial a_j}$ is the Jacobi matrix), related to vorticity via $\omega = J^{-1}\{\mathbf{\Omega}(\mathbf{a},t)\cdot\nabla_a\}\mathbf{X}(\mathbf{a}, t)$ (here J is the determinant of the Jacobi matrix and $\mathbf{X}(\mathbf{a}, t)$ is the particle position vector for a particle labelled by $\mathbf{a}$, and $\mathcal{G} \equiv \mathcal{J}_{ki}\mathcal{J}_{kj}$ is the metric tensor, see Bennet, 2006; Kuznetsov, 2008; Yakubovich and Zenkovich, 2002 for more details and references):

$$\frac{\partial\mathbf{\Omega}}{\partial t} = -\nu curl_a\left\{\frac{\mathcal{G}}{J}curl_a\left(\frac{\mathcal{G}}{J}\mathbf{\Omega}\right)\right\}. \qquad (C.66)$$

Thus $\mathbf{\Omega}$ is indeed a pointwise Lagrangian inviscid invariant. Just as in equations (C.62) there is no "conventional" nonlinearity/advection, and equation (C.66) does not contain any analogue to the vortex stretching term $\omega_i s_{ij}$ in equation (C.9) for vorticity in the Euler setting and in the equation (C.65) for the Lagrangian setting. But note that the Jacobi matrix, $\mathcal{J}_{kj} = \frac{\partial X_k}{\partial a_j}$, plays an essential role in the relation between ω and $\mathbf{\Omega}$.

Similarly to (C.62) there is no term analogous to the advection term in the Euler setting in the equation for a passive scalar $\theta(\mathbf{a},t)$

$$\partial\theta/\partial t = \mathcal{D}\{[X_2, X_{3,}[X_2, X_{3,}\theta]] \\ + [X_{3,}X_{1,}[X_{3,}X, \theta]] + [X_{1,}X_{2,}[X_{1,}X_{2,}\theta]]\} \qquad \text{(C.67)}$$

i.e., θ does not depend on time when $\mathcal{D} = 0$ as any nondiffusive property is constant along a fluid particle trajectory.

APPENDIX D: GLOSSARY OF SOME TERMS

Anisotropy – Lack of isotropy. There are many kinds and different manifestations of anisotropy of turbulent flows, depending on quantities in question. These may be of different order, consist of velocity components, their derivatives of various orders taken at the same or different space/time points. See Monin and Yaglom (1971, 1975), Hinze (1975), Favre et al. (1976) and Lumley (1978).

Attractor – A set in the phase space that the system approaches at large times. Invariant under evolution (phase flow).

Bifurcation – Topological change of the phase flow as some parameter, say Reynolds number, changes.

Degrees of freedom – Proportional to (say, 1/2 of) the number of first-order ODEs, adequately describing the system in question. The effective number of degrees of freedom is smaller due to the conservation laws and constraints imposed on the system.

Ergodicity – For statistically stationary flows it is (roughly) equivalence of 'true' statistical properties (not only means/averages, but 'almost' all statistical properties) of an ensemble, to those obtained using time series in one very long realization. A similar property is defined in space by replacing time by space coordinate(s) in which the flow domain has an infinite extension, at least in one direction. Turbulent flows are known (empirically) to be ergodic. Other chaotic systems may be non-ergodic; see Shlesinger (2000), Zaslavsky (1999).

Homogeneity – Invariance of all statistical properties/parameters of turbulent flow to translations in space. No real flow is strictly homogeneous.

Isotropy – Invariance of all statistical properties/parameters of turbulent flow to rotations and reflections, i.e., to the full rotation group. This is essentially the definition given by Taylor (1935) and Kolmogorov (1941a). Note the *statistical* nature of isotropy, homogeneity and stationarity. Isotropic flows are necessarily homogeneous. No real flow is strictly isotropic.

Phase flow – All trajectories originating from all possible initial conditions in a given flow geometry (boundary conditions), i.e., all fluid flows in a given geometry for all possible initial conditions comprise the phase flow.

Phase space – Set of all possible (instantaneous) states of a system. For incompressible fluid flows, a ‘point’ in the phase space is a solenoidal velocity vector in the flow domain, satisfying the boundary conditions. A time-dependent fluid flow comprises a trajectory in the phase space.

Stationarity – Invariance of all statistical properties/characteristics of turbulent flow to translations in time.

A glossary of basic terms related to turbulence is found in Bradshaw (1971).

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