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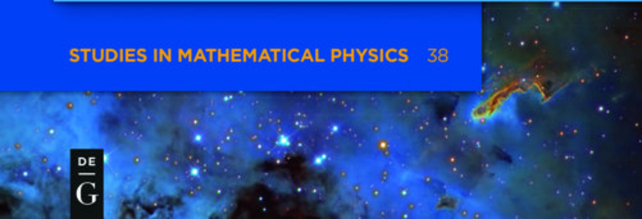
Alexander N. Petrov, Sergei M. Kopeikin, Robert R. Lompay, Bayram Tekin

METRIC THEORIES OF GRAVITY

PERTURBATIONS AND CONSERVATION LAWS

STUDIES IN MATHEMATICAL PHYSICS 38



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Authors
Prof. Dr. Alexander N. Petrov
Moscow State University
Sternberg Astronomical Institute
Universitetskii pr. 13
119992 Moscow, Russia

Prof. Dr. Sergei M. Kopeikin
University of Missouri
Department of Physics and Astronomy
322 Physics Building
Columbia MO 65211, USA

Prof. Dr. Robert R. Lompay
Uzhgorod National University
Department of Physics
Voloshyna str., 54
Uzhgorod 88000, Ucraine

Prof. Dr. Bayram Tekin
Middle East Technical University
Department of Physics
Ankara 06800, Turkey

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Preface

The year 2016 witnessed a tremendous scientific discovery which is surely to become perhaps one of the greatest discoveries of all time: a direct detection by Laser Interferometer Gravitational-Wave Observatory (LIGO) of the gravitational waves produced in the event of coalescence of binary black holes. One more time, general relativity triumphed after one hundred years of its profound foundations. Using general relativity the tiny fluctuations of test masses in the arms of the earth-based detectors were unambiguously identified as caused by the propagating curvature (and metric) fluctuations generated at the last few moments of in-spiraling black holes located more than a billion of light years away from us. As we are putting the final touches on this book, we feel lucky to have written it in this extremely exciting times in gravity research. Even though, this is not a book about gravity waves *per se*, the mathematical technique introduced over here is essentially similar to the one used for calculating gravitational waves in the weak-field region of their source. Besides that, we discuss the conservation laws, symmetries of spacetime, etc., which allow us to give a unique physical meaning to the integration parameters that appear in the exact or linearized solutions of Einstein's equations. Moreover, as we lay out the details below, we shall also consider these issues in generic gravity theories that modify Einstein's gravity at large and small distances.

Conservation of mechanical energy and linear momentum in various experiments had been discovered already by the great Galileo. Later on, it was realized that heat is also a form of energy which is always conserved. Those discoveries turned out to be the major achievements in physics of the nineteenth century that led to the substantial developments such as the formulation of the laws of thermodynamics, electrodynamics, etc. Solid theoretical foundation for these empirical observations was given in the series of remarkable papers by Emmy Noether published about the same time when general relativity was formulated by Albert Einstein. It is very hard to overestimate the importance of Noether's contribution as Feza Gursey noted *"before Noether's theorem the principle of conservation of energy was shrouded in mystery, leading to the obscure physical systems of Mach and Ostwald. Noether's simple and profound mathematical formulation did much to demystify physics."* Notably, the Noether's fascinating observation was that time translation of a physical system in the Minkowski space is associated with conservation of energy. In fact, the Noether's reasoning applies not only to a time translation but to all other spacetime's symmetries of the Minkowski space as well as to the intrinsic symmetries of a given theory.

In contrast to the spacetime of special relativity, a generic spacetime in general relativity or in any other metric theory of gravity is curved and, thus, lacks spacetime symmetries. For example, one of the simplest solutions of general relativity – the Schwarzschild black hole – does not have a global timelike symmetry due to the existence of the horizon and, therefore, it is not amenable to a straightforward definition of energy. It triggers the question: what does it mean to have the conservation of energy in time when there is no notion of a global time symmetry? Besides the problem of the global symmetries, there are others closely related to the local symmetries. More specifically, in a curved spacetime, freely-falling observers feel themselves as if they were in a flat spacetime due to the equivalence principle. It makes definitions of energy, momentum and angular momentum observer-dependent and, hence, ambiguous. Therefore, lively discussions on how to formulate the conservation laws in curved spacetime never stopped and have been sometimes highly heated. Particular problems have been resolved differently within diverse approaches. Apart from a significant theoretical interest in finding a universal definition of conserved quantities, this problem has ramifications on practical issues associated with the observations and measurements. For these reasons, various definitions of conserved quantities have shown up in the literature up to now.

Needless to say, over the past century, much has been published on the subject under discussion, and the accumulated material clearly requires a systematization and perhaps a collective treatment for further progress. So far, to the best of our knowledge, no monograph has been devoted to the discussion of the problem of construction of conservation laws in general relativity and in other metric theories even though one can find various valuable reviews and chapters in books expounding on this topic. With this monograph, we hope to fill this gap in this particular part of the history and the developments of the most popular approaches to the construction of conserved quantities. The book unifies various theoretical approaches, definitions and computations of conserved quantities in Einstein's theory of general relativity and its modifications. Therefore, we expect the book to be useful, first of all, for advanced researches working on these topics and in the adjacent fields of science. We have undertaken serious efforts to make the presentation of the material logically consistent and complete so that any diligent student of science engaged in gravity research might use the book for a detailed self-study. Some parts of the book have a valuable methodological value and can be definitely implemented by college instructors teaching an advanced level course on gravity.

Due to the non-linearity of the Einstein field equations, it is very difficult or even impossible to find exact solutions in the most cases of physical interest. Therefore, many theoretical studies and observational programs in general relativity are based on various perturbative approaches to a background spacetime often taken as an idealized highly-symmetric solution. Adequate formulation, derivation and analysis of the spacetime perturbations as well as the corresponding conservation laws for those perturbations, is a long-standing problem both in general relativity and other metric theories such as the multi-dimensional modifications of Einstein's theory of gravity. This book addresses these topics in detail. Among the multitude of perturbation methods used for solving the field equations, we single out the field-theoretical approach based on the application of the variational calculus on spacetime manifolds. For definition and discussion of the conserved quantities, we intensively use the canonical Noether's formalism along with the Belinfante procedure of symmetrization of the energy-momentum tensor also known as the Belinfante correcting method. The book presents the theoretical foundations of these ideas, discusses domain of their applicability and notes on the prospects for their further development and generalization both in general relativity and in other metric theories, such as the ones with arbitrary powers of curvature.

The field-theoretical treatment of perturbations in a metric theory of gravity is the most universal one among the other existing formalisms in that it allows one to reduce the geometric theory to the form of a standard field theory in a fixed spacetime manifold. In this formulation the full metric tensor is decomposed to a background value, whose geometric structure and time evolution are completely known, and a perturbation part which is considered as the dynamical field variable. Instead of the original intricate geometry, one then studies the properties of the field configuration propagating in the given background spacetime. This helps one to benefit from all the advantages of a field theory on the known (flat or curved) background: Lagrangian and Hamiltonian methods, covariance, gauge invariance, etc. The background spacetime is also used as a reference manifold whereon physical laws are introduced and conserved quantities are defined.

The Noether and Belinfante methods do not describe perturbations explicitly. Rather, they present a bi-metric description of a dynamical system with respect to a background solution which makes these methods very powerful and physically-meaningful tools in constructing conserved quantities. The canonical Noether method starts from the derivation of non-covariant Einstein pseudotensor and Freud superpotential of dynamical perturbations in flat spacetime. Subsequent developments include covariantization and generalization of the Noether formalism to curved backgrounds with the construction of various superpotentials and conserved currents in general relativity and in modified gravity theories. Belinfante pioneered a method of symmetrization of the canonical quantities in field theories in the Minkowski space. Later, the method was generalized to curved backgrounds where it furnishes conserved quantities in general relativity and in modified theories.

The notion "perturbation" is frequently associated with an approximation or a small quantity. This association remains also valid in the present book but we go further and extend it formally to the case of finite perturbations. Indeed, the very construction of the Noether and Belinfante methods allows us to treat perturbations exactly with no approximation involved. Being a reformulation of a metric theory, the field-theoretical formalism is exact as well. If necessary, the field-theoretical equations for the perturbations can be easily expanded into series and, thus, can be adjusted to describe approximate solutions and weak field disturbances.

The book presents the formalism of the field-theoretical approach to the metric perturbations and provides its developments both theoretically and in the light of applications by making use of the Noether and Belinfante approaches. A significant part of the book is devoted to explaining these methods in application to general relativity. In fact, the reader interested merely in general relativity can limit oneself to the first half of the book. In its second half, perturbation treatment of multi-dimensional modified theories is presented. Here, the book also contains numerous physical examples which elucidate the methods and practical settings used in the theory.

Let us lay out the contents of the chapters. The first chapter is a fairly elementary introduction written for students who want to understand the concept of conserved quantities in classical and relativistic mechanics as well as in field theories in the Minkowski space and for those researchers who want to be more informed on the classical development of this topic in general relativity. A particular attention is paid to the mathematical formulation, proof and applications of the Noether theorems. Conserved energy, linear and angular momenta are constructed in a wide class of field theories with the Lagrangian depending on the field variables and their first derivatives. Examples include a fundamental scalar field, ideal fluid and electrodynamics in the Minkowski space. The chapter, then, continues with a more specialized section focused on the history of constructing the conservation laws in general relativity. Herein, Einstein's arguments in support of the existence of the conserved quantities in general relativity are discussed. The chapter provides a short outline of classical pseudotensors and superpotentials in general relativity along with their systematization which appears after the exemplary of applications of the Noether theorems. The linearized approximation of general relativity is presented in the spirit of the field-theoretical approach in the Minkowski space. As a result, construction of the conserved quantities is provided and their connection with pseudotensors and superpotentials is established. Physical aspects of the linearized general relativity are illustrated with the examples of weak gravitational waves and isolated astronomical systems.

The second chapter is devoted to the discussion of the field-theoretical formulation of general relativity. We offer a historical description of earlier developments commenced from Deser's model of 1970. Then, we provide the Lagrangian-based treatment of the Einstein's theory considered as a theory of a tensor perturbation propagating in an arbitrary fixed (either curved or flat) spacetime. The field-theoretical equations are derived by making use of the least action principle. Differential conservation laws for the symmetric (metric) energy-momentum tensor of the perturbations are presented in the case of rather simple backgrounds taken either as the Minkowski space or the Einstein spaces. Though such a field theory is equivalent to general relativity in geometrical terms it allows us to apply the standard Noether formalism to find out many useful special properties of general relativity which are not easily seen in the standard geometrical formulation. One of these properties is the gauge invariance which reveals that the metric perturbations can be treated as the gauge (compensating) field. Based on this property and with the use of the localization technique, we provide a formulation of general relativity analogous to that of the standard gauge field theories of the Yang–Mills type. As an example of further developments of the field-theoretical formalism, a self-consistent formulation of gravity theory with massive gravitons is outlined.

In the third chapter, we develop the field-theoretical technique in general relativity to construct conserved quantities for isolated astronomical systems. We study these quantities for such systems at spatial infinity by making use of the weak-field approximation in the asymptotic regime. At the beginning of the chapter we introduce the standard Arnowitt-Deser-Misner (ADM) formalism and its modification by Regge and Teitelboim (RT) which have been developed mostly for the study of isolated gravitating systems. Then, we develop both the Lagrangian and Hamiltonian formulation of the isolated systems in the framework of the field-theoretical approach. We choose the Minkowski space as a background spacetime for the metric perturbations because it coincides with the flat spacetime at infinity After that we construct the conserved global integrals of motion and compare them with the standard ADM and RT integrals. Discussion of the permissible fall-off behavior for the radial dependence of the gravitational potentials that is weaker than the Newtonian one ($\sim 1/r$) for an isolated gravitating system, is given.

In the fourth chapter, we use the field-theoretical formulation to demonstrate how it can be applied to the description of exact solutions in general relativity with a field configuration defined on a flat spacetime and on a curved background. In case of the flat spacetime background, we pay particular attention to the treatment of the Schwarzschild solution and the closed Friedmann universe. Surprisingly, it turns out that it is possible to reformulate the Schwarzschild solution in terms of the field configurations defined on the flat background not only asymptotically at spatial infinity but also near (both outside and inside) the event horizon as well as in the singularity limit. Trajectories of test particles, distribution of the energy density and that of the total energy of the Schwarzschild black hole are discussed. The same technique applied to the closed Friedmann universe reveals that it can be also considered in terms of a field configuration residing in the Minkowski space. We describe the properties of its stereographic projection onto the flat spacetime and find out that all global conserved quantities vanish as expected by a topological argument. In the case of a curved background, we examine different field configurations propagating on the anti-de Sitter (AdS) spacetime, derive a generalized Abbott-Deser superpotential, and calculate the total mass of the Schwarzschild-AdS solution.

In the fifth chapter, we use the field-theoretical method to develop a theory of cosmological perturbations on a Friedmann–Lemaitre–Robertson–Walker (FLRW) background with three admissible spatial constant curvatures, $k = (-1, 0, +1)$. As a result we present the Lagrangian-based field theory of the cosmological perturbations of the metric tensor coupled to the perturbations of the ideal fluid considered as dark matter, and those of a massless scalar field considered as dark energy. All cosmological perturbations can be considered as generated by a primordial mechanism existed at the epoch of the very early universe or as induced by bare perturbations taken in the form of localized astronomical system formed after the recombination epoch. Gauge-invariant quantities for the perturbations and the equations governing their evolution in the universe are derived. We examine a special case of the perturbed field equations in a spatially-flat universe ($k = 0$) and demonstrate that the scalar modes of the perturbations can be completely decoupled from their vector and tensor modes. Mathematical technique developed in this chapter is useful to study gravitational waves created by distant astrophysical sources. Namely such gravitational waves have been detected recently by the LIGO interferometers.

The sixth chapter is devoted to constructing the conserved currents and related superpotentials in general relativity. Following Katz, Bičhák and Lynden-Bell (KBL) we use the canonical Noether method applied to the KBL bi-metric Lagrangian to construct the covariant conserved quantities on arbitrary curved backgrounds with making use of arbitrary displacement vectors instead of the Killing ones. Then, generalization of the Belinfante symmetrization procedure from the Minkowski space to curved backgrounds is worked out and applied to find out the Belinfante corrections to the KBL conserved quantities. The currents and superpotentials in the field-theoretical formulation are obtained after applying the Noether theorem. The KBL, KBL-Belinfante and field-theoretical definitions of the conserved quantities are compared. In particular, we show that the field-theoretical and Belinfante corrected quantities become equivalent if the dynamical variables obey the field equations. Criteria for choosing physically preferable currents and superpotentials are derived and discussed. Then, we demonstrate how the KBL, KBL-Belinfante and field-theoretical techniques work. We show that the FLRW solution can be viewed as a perturbation with respect to the de Sitter space in the framework of the KBL formalism. It helps us to define the energy of metric and matter perturbations in the Friedmann universe by making use of the Killing vectors of the de Sitter space. We also study the, so-called, integral constraints for the cosmological perturbations in a localized domain of space in the framework of the KBL-Belinfante formalism. These constraints tightly connect the magnitude of the matter perturbations inside the domain with the surface values of the metric perturbations at the boundary of the domain. These constraints are important to study the subtle features in the pattern of the Sachs-Wolfe effect in cosmic microwave background radiation.

The second half of the book is devoted to multi-dimensional modified metric theories of gravity. Chapter 7 is a theoretical foundation for constructing the conservation laws for perturbations in such theories. We consider arbitrary (not necessarily metric-based) multi-dimensional field theory by incorporating to spacetime as an auxiliary external metric so that the original fields of the theory are considered as (finite) perturbations with respect to the external metric. Such a presentation of the theory allows a straightforward application of the Noether and Noether-Belinfante procedures resulting in the related Noether identities which are covariant with respect to the external metric. These generic identities are used to construct various conserved currents and related superpotentials of the KBL, KBL-Belinfante and field-theoretical types for physical perturbations residing on the curved background in multi-dimensional theories.

In chapter eight, the theoretical results of the previous chapter are used to discuss the conservation laws in the Einstein-Gauss-Bonnet (EGB) gravity which is one of the most attractive modifications of general relativity as it appears in the low-energy limit of string-generated theories and so admits a supergravity extension. The new expressions for the EGB currents and superpotentials are derived in the most generic form. We use them to calculate the mass of the well known solution for the multi-dimensional Schwarzschild-AdS black hole and prove that it exactly coincides with the mass obtained by other independent methods. We also use our expressions of the EGB currents and superpotentials for giving physically meaningful interpretation to non-trivial Kaluza-Klein black hole solutions in the EGB gravity.

Chapter 9 is devoted to the significant extension of the work by Abbott and Deser (1982) where the Killing charges were introduced to study the perturbations on the de Sitter and AdS backgrounds in general relativity. We generalize this approach to higher-curvature theories with the generic Lagrangians depending on the Riemann tensor. Both de Sitter and AdS spacetimes play a significant role in modern physics and, therefore, proper formulations of perturbations and construction of the conserved charges on these backgrounds are vitally important. Gravity theories in three-dimensional (2+1) spacetime are used as theoretical labs to test some ideas of quantum gravity hence we provide detailed discussions of the charge construction for the topologically massive gravity (the dynamical theory of gravity in this dimensions) and apply the construction for studying the Banados-Teitelboim-Zanelli (BTZ) black hole solution. To look at different aspects of the problem under discussion we provide a second derivation of the Killing charges using the so called covariant phase-space formulation of the theory and apply our results to the recently discovered multi-dimensional rotating Kerr-AdS metric. A brief section at the end of the chapter is included for discussing conformal properties of the conserved charges in a gravity theory coupled non-minimally to scalar fields.

In the final chapter we focus on the problem of constructing the canonical conserved quantities in covariant field theories possessing the intrinsic (gauge) symmetries of the field Lagrangian. The Noether procedure implemented in such theories reveals that the conserved currents and superpotentials following from the gauge invariance are both covariant and gauge-invariant while those associated with the diffeomorphism invariance of the theory, do not possess such property. At present, there is no generally accepted method for constructing currents and superpotentials following from the diffeomorphism invariance which are simultaneously both covariant and gauge invariant. This chapter illustrates the essence of this problem with two particular examples: (1) a generally-covariant theory of the Yang–Mills field minimally-coupled to scalar fields evolving on a given geometrical manifold and possessing the intrinsic (gauge) symmetry with the group $SU(N)$; (2) a tetrad formulation of general relativity possessing the intrinsic freedom of the tetrad rotations in tangent space at each point of spacetime manifold. We show that the origin of the aforementioned difficulties is rather subtle and has a profound geometric nature related to the fact that in the field theories with intrinsic symmetries any local diffeomorphism induces a *family* of transformations for the field variables with arbitrary gauge parameters that brings about the ambiguity to the definition of the conserved currents. This differs drastically from the theories without the intrinsic symmetries where any diffeomorphism induces a *unique* transformation of the field variables. We prove that the modified Lie derivatives which are frequently used in attempt to remedy the problem of the current's ambiguity, do not resolve the problem. At last, we provide a brief review of the main results obtained by the leading groups of researchers actively working on the solution of this problem.

Appendix A is a brief, but hopefully a useful discussion of the main tensor operations. It also contains the introduction to shorthand (economic) index notations used throughout the book to conduct calculations in the most rational and concise way. The reader is encouraged to refer to this appendix when in doubt of some definitions, such as the derivatives of tensors, operation of permutation of indices, variations of geometric objects, etc. Appendix B deals with the retarded potentials relevant to the discussion in Chapter 5 and contains the proof of their Lorentz invariance. Appendix C summarizes the technical equations of the auxiliary fields used for derivation of the conserved currents and superpotentials in the Einstein-Gauss-Bonnet theory.

Acknowledgments

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Primary notations

Constants

- $c = 299792458\ \mathrm{m \cdot s^{-1}}$ – the speed of light in vacuum;
- $G = 6.67408 \times 10^{-11}\mathrm{m^3 \cdot s^{-2} \cdot kg^{-1}}$ – the Newtonian gravitational constant in 4-dimensional ($4D$) spacetime;
- $\kappa = 8\pi G/c^4$ – $4D$ Einstein's gravitational constant;
- $\kappa = \kappa_n = 2\Omega_{n-2}G_n$ – Einstein's gravitational constant in n-dimensional spacetime with Ω_{n-2} being the solid angle of the $(n-2)$-dimensional sphere, and in this expression $c = 1$.

Coordinates

- $\{x, y, z\}$ – the Cartesian spatial coordinates in 3-dimensional ($3D$) space;
- t – time coordinate in geometric units where $c = 1$;
- $\{t, x, y, z\}$ – the Lorentzian spacetime coordinates;
- $\{x^1, x^2, x^3\} = x^i$ – $3D$ curvilinear spatial or Cartesian coordinates; the Roman indices $i, j, k, \ldots$ take values $1, 2, 3$;
- $x^0 = ct$ – time coordinate in the system of units with the dimensional speed light $c = 299792458\ \mathrm{m \cdot s^{-1}}$;
- $\{x^0, x^1, x^2, x^3\} = x^\alpha$ – $4D$ curvilinear or Lorentzian coordinates; the Greek indices $\alpha, \beta, \gamma, \ldots$ run through values $0, 1, 2, 3$;
- $\{x^1, x^2, \ldots, x^{n-1}\} = x^i$ – curvilinear spatial or Cartesian coordinates in $(n-1)$-dimensional space; the Roman indices $i, j, k, \ldots$ take values $1, 2, \ldots, n-1$;
- $\{x^0, x^1, x^2, \ldots, x^{n-1}\} = x^\alpha$ – curvilinear or Lorentzian coordinates in n-dimensional spacetime; the Greek indices $\alpha, \beta, \gamma, \ldots$ run through values $0, 1, 2, \ldots, n-1$;
- repeated (dummy) indices obey the Einstein summation rule, for example, $P^\alpha Q_\alpha \equiv P^0Q_0 + P^1Q_1 + P^2Q_2 + P^3Q_3$ in $4D$ spacetime, and $P^\alpha Q_\alpha \equiv P^0Q_0 + P^1Q_1 + \ldots + P^{n-1}Q_{n-1}$ in n-dimensional spacetime, and so on.

Geometrical objects on a spacetime manifold

Similar notations are applied correspondingly to either 4-dimensional or to n-dimensional spacetime.

- $\bar{\mathscr{Q}}$ – the "bar" above $\mathscr{Q}$ means a background value of the geometric quantity $\mathscr{Q}$;
- $\eta_{\mu\nu} = \mathrm{diag}\{-1, +1, +1, +1\}$ or $\eta_{\mu\nu} = \mathrm{diag}\{-1, +1, +1, \ldots, +1\}$ – the Minkowski metric in the Lorentzian coordinates in 4-dimensional or n-dimensional spacetime;
- $\gamma_{\mu\nu}$ – the Minkowski metric in curvilinear coordinates;
- $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ – the dynamical and background metric, respectively, in an arbitrary curved spacetime;
- $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta}$ – the "Gothic" metric that is the metric density of weight +1;

- $\eta = \det \eta_{\mu\nu} = -1$ – the determinant of the Minkowski metric in the Lorentzian coordinates;
- $\gamma = \det \gamma_{\mu\nu}$ – the determinant of the Minkowski metric in curvilinear coordinates;
- $g = \det g_{\mu\nu}$ and $\bar{g} = \det \bar{g}_{\mu\nu}$ – the determinants of dynamical and background metrics, respectively;
- $C^\rho{}_{\alpha\beta}$ – the Christoffel symbols in the Minkowski space in curvilinear coordinates constructed from the metric $\gamma_{\mu\nu}$;
- $\Gamma^\rho{}_{\alpha\beta}$ and $\bar{\Gamma}^\rho{}_{\alpha\beta}$ – the Christoffel symbols constructed from the dynamical, $g_{\mu\nu}$ and background, $\bar{g}_{\mu\nu}$, metrics, respectively;
- $\Delta^\rho{}_{\alpha\beta} = \Gamma^\rho{}_{\alpha\beta} - \bar{\Gamma}^\rho{}_{\alpha\beta}$ – the tensor that is a difference between the dynamical and background Christoffel symbols;
- $R^\rho{}_{\alpha\sigma\beta}$ and $\bar{R}^\rho{}_{\alpha\sigma\beta}$ – the Riemann curvature tensors in physical and background spacetimes, respectively;
- $R_{\alpha\beta}$ and $\bar{R}_{\alpha\beta}$ – the Ricci tensors in physical and background spacetimes, respectively;
- R and $\bar{R}$ – the Ricci scalars (or scalar curvature) in physical and background spacetimes, respectively.

Fields on a spacetime manifold

- $\boldsymbol{P}, \boldsymbol{M}, \dots$ – the capital boldface letters denote 3-dimensional vectors;
- $\Phi^A,\ P_B, \dots$ – sets of tensor densities of arbitrary ranks and weights with the collective indices $A,\ B, \dots$ in a piggyback notation, see Appendix A.3;
- $\mathcal{Q}^A\big|^\alpha_\beta$ – a permutation linear operator depending on the transformation properties of the geometric object $\mathcal{Q}^A$ – a tensor density or set of tensor densities. Algebra of the operators $\mathcal{Q}^A\big|^\alpha_\beta$ along with their other useful properties is given in Appendix A.3;
- $\boldsymbol{\lambda}^\alpha,\ \boldsymbol{t}_\sigma{}^\mu, \boldsymbol{b}_\sigma{}^{\mu\nu}, \dots$ – notations in calligraphic boldface (sometimes in Gothic) for small letters, if they represent quantities of mathematic weight +1 (sometimes +2 or more). For example, $\boldsymbol{\lambda}^\alpha$ could be a density itself; $\boldsymbol{t}_\sigma{}^\mu$ could be a density expressed with the use of the tensor $t_\sigma{}^\mu$: $\boldsymbol{t}_\sigma{}^\mu = (-g)t_\sigma{}^\mu$, or $\boldsymbol{b}_\sigma{}^{\mu\nu} = \sqrt{-\bar{g}}b_\sigma{}^{\mu\nu}$ where $b_\sigma{}^{\mu\nu}$ is also a tensor, etc;
- $\mathcal{L}, \bar{\mathcal{R}}, \mathcal{U}_\sigma{}^\mu, \dots$ – the capital calligraphic letters denote geometric quantities of weight +1 (sometimes of other weights), analogous to previous item. For example, $\mathcal{L} = \sqrt{-\gamma}L,\ \ \bar{\mathcal{R}} = \sqrt{-\bar{g}}\bar{R},\ \ \mathcal{U}_\sigma{}^\mu = \sqrt{-g}U_\sigma{}^\mu$, etc.;
- ξ^α – arbitrary displacement vector in dynamical or fixed spacetimes;
- ξ^α_K – Killing vectors in the Minkowski space;
- $\bar{\xi}^\alpha$ – Killing vectors in an arbitrary curved background spacetimes;
- $\underset{\Omega}{\lambda}^\alpha$ – conformal Killing vectors in the FLRW background spacetime.

Derivatives. For more detail see Appendix A.2

- $\dfrac{\partial \mathcal{Q}^{\mathcal{A}}}{\partial x^\alpha} = \partial_\alpha \mathcal{Q}^A = \mathcal{Q}^A{}_{,\alpha}$ – the partial derivative;

- $\mathscr{Q}^{A}{}_{;\alpha}$ – the covariant derivative of $\mathscr{Q}^A$ compatible with $\gamma_{\mu\nu}$;
- $\bar{\nabla}_\alpha \mathscr{Q}^A$ – the covariant derivative of $\mathscr{Q}^A$ compatible with $\bar{g}_{\mu\nu}$;
- $\nabla_\alpha \mathscr{Q}^A$ – the covariant derivative of $\mathscr{Q}^A$ compatible with $g_{\mu\nu}$;
- $£_\xi \mathscr{Q}^A$ – the Lie derivative of the quantity $\mathscr{Q}^A$ along the vector field ξ^α:
- $\dfrac{\delta \mathscr{Q}^A}{\delta q}$ – the Lagrangian derivative of the quantity $\mathscr{Q}^A = \mathscr{Q}^A(q; q_{,\alpha}; q_{,\alpha\beta})$.

Metric perturbations

- $\varkappa_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$;
- $l^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$;
- $\mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}$, where $\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$;
- $h^{\mu\nu} = \mathfrak{h}^{\mu\nu} / \sqrt{-\bar{g}}$;
- the indices of tensor fields on the background manifold are lowered and raised with the background metric $\bar{g}_{\alpha\beta}$ and its inverse.

Remarks

- In Chapter 5 we use a number of additional specific notations which are given at the end of Section 5.1;
- In Chapter 9 the notations are mostly consistent with the rest of the book. One notable difference is that of the definition of a metric perturbation where we use $\epsilon h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$;
- Other notations will be introduced and explained as they appear in text.

1 Conservation laws in theoretical physics: A brief introduction

1.1 Conserved quantities in classical mechanics

A field theory is a mathematical formalism designed to describe fundamental forces and elementary particles in terms of a self-consistent theoretical framework. Forces can be described by fields that mediate interactions between separate objects. Historically, the basic definitions, conventions and concepts of field theories originated from the Lagrangian formalism of classical mechanics. Therefore, before discussing the field theory and its applications, it is reasonable to recall the Lagrangian formalism of the classical mechanics for the reader's convenience. This is the main goal of the present section.

1.1.1 Some basic notions of non-relativistic classical mechanics

One of the primary concepts of classical mechanics is that of a reference frame. A reference frame is introduced to describe the relative motion in a system of N bodies. The simplest reference frame consists of a spatial coordinate system attached to the bodies of reference (observers) which are endowed with the ideal clocks to measure the lapses of time. In classical mechanics, space and time are absolute entities being independent of the motion of observers. Moreover, the space is Euclidean and disconnected from time. The Euclidean space is homogeneous and isotropic. Time is uniform and has the same rate in all possible reference frames. Cartesian coordinates are considered as the most convenient coordinate system covering the entire Euclidean space although curvilinear coordinates, for example, spherical or polar coordinates, are equally mathematically admissible and sometimes more apt, depending on the symmetry of the problem.

Another important concept of classical mechanics is the notion of a point-like *test particle* defined as an idealized material object whose size and internal structure are inessential in the problem under consideration, and can be neglected. The test particle is characterized by its mass, m (usually considered to be constant, but it can depend on time $m = m(t)$ in some particular situations), and the position vector $\boldsymbol{r} = \{x, y, z\}$ where $\{x, y, z\}$ are the Cartesian coordinates of the particle which are assumed to be differentiable functions of time t:

$$\boldsymbol{r} = \boldsymbol{r}(t). \tag{1.1.1}$$

Here and everywhere else, vectors are denoted with boldface Roman letters. The coordinates $x = x(t)$, $y = y(t)$, $z = z(t)$ define three *degrees of freedom* of the test particle and are mutually-independent scalar functions in the absence of constraints imposed

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on the motion of the particle. *Velocity* and *acceleration* of the test particle are defined as time derivatives of its position vector:

$$\boldsymbol{v} \equiv \frac{d\mathbf{r}}{dt} \equiv \dot{\boldsymbol{r}}, \quad \boldsymbol{a} \equiv \frac{d\boldsymbol{v}}{dt} \equiv \frac{d^2\boldsymbol{r}}{dt^2} \equiv \dot{\boldsymbol{v}} \equiv \ddot{\boldsymbol{r}}. \tag{1.1.2}$$

In more general situations a mechanical system can consist of more than one particle and have n degrees of freedom defined by n functions:

$$q_i \equiv \{q_1(t), \dots, q_n(t)\}, \quad (i = 1, \dots, n), \tag{1.1.3}$$

which are called *generalized coordinates*. Masses of the particles may also depend on time. The set of the generalized coordinates along with time forms the so-called configuration space. We emphasize that the generalized coordinates are not necessarily position vectors. Particularly, in the case of N non-interacting point-like particles, a system without constraints possesses $n = 3N$ degrees of freedom. The corresponding *generalized velocities* of such a system are defined as

$$\dot{q}_i \equiv \frac{dq_i(t)}{dt}. \tag{1.1.4}$$

Generalized accelerations are defined as the first time derivative of the generalized velocities: $\ddot{q}_i(t) = d\dot{q}_i/dt$.

The *state* of a mechanical system is fully defined if the generalized coordinates and velocities are given at any moment of time.[1] How can one determine the state or possible states of a given system? To answer this question one has to solve the *equations of motion* of the particles that compose the mechanical system. The equations of motion connect the particles' accelerations with their coordinates and velocities through the second Newton's law which allow us to propagate the initial values of the particle's coordinates and velocities as time progresses. This is achieved by integrating the system of equations of motion with respect to time under the given initial conditions. Thus, the principal problem of mechanics which has to be solved is two-fold: (1) to derive the appropriate equations of motion for the generalized coordinates of the particles; (2) to integrate these equations to find the generalized coordinates as explicit functions of time. Solution of this problem determines the *real (actual) motion* of the system.

1.1.2 The least action principle

Soon after Newton had formulated his famous laws of mechanical motion it was discovered that the most economical and elegant way to derive the equations of motion

1 In the present section we consider the Lagrangian description in mechanics only. Necessary elements of the Hamiltonian formulation for a non-relativistic particle is given in Section 3.1.3 where the *state* of a system is defined by the generalized coordinates and momenta at any instant of time.

is based on making use of a certain function, called the *Lagrange function* or simply the *Lagrangian*, depending on generalized coordinates, velocities and time

$$L = L(q_i, \dot{q}_i, t). \tag{1.1.5}$$

The time dependence of the Lagrangian can be both implicit – through the coordinates and velocities, and explicit, for example, through variability of masses forming the mechanical system. If there is no explicit time-dependence of the Lagrangian, then, one simply writes

$$L = L(q_i, \dot{q}_i). \tag{1.1.6}$$

If the system is not subject to the influence of external forces it is called a *closed* system [283]. The system defined by the Lagrangian (1.1.5) is called an *open* system. It was established that the behavior of each mechanical system in the time interval from t_0 to t_1 is determined by a functional

$$S[q_i; t_0, t_1] = \int_{t_0}^{t_1} dt\, L(q_i, \dot{q}_i, t) \tag{1.1.7}$$

which is called the *action functional*, or simply – the *action*.

It is postulated that the mechanical system moves in accordance with the principle of a stationary action that tells us that in the case of an actual physical motion of the system between two fixed moments of time, t_0 and t_1, the action acquires an extremal value. Traditionally, "the extremal value" is understood as the least value (or the minimum) of the action, and, thus, the stationary action principle is more commonly known as *the least action principle*. Frequently, this principle is also called the *variational principle* because in order to calculate the lowest value of the action with given initial and final points, one has to resort to the calculus of variations.

Let the dynamical variables $q_i(t)$ describe an actual motion yet to be determined. Then, an infinitesimally disturbed motion, $q_i'(t)$, can be expressed with the help of an *instantaneous* variation, $\delta q_i(t)$, as follows: $q_i'(t) = q_i(t) + \delta q_i(t)$. The first order variation of the Lagrangian induced by the variation of the generalized coordinates, is

$$\delta L \equiv L(q_i', \dot{q}_i', t) - L(q_i, \dot{q}_i, t) = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i. \tag{1.1.8}$$

Employing the standard techniques of the variational calculus with respect to dynamical variables [283] one obtains for the first order variation of the action:

$$\delta S \equiv S[q_i'; t_0, t_1] - S[q_i; t_0, t_1] = \left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\Bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \tag{1.1.9}$$

where here and everywhere else we use the Einstein summation rule with respect to the repeated indices, $A^i B_i \equiv \sum_{i=1}^{n} A^i B_i$.

Let us assume that the initial and final end points of the system's trajectory are fixed and are not subject to the variation. It means that the end-point values of the infinitesimal variation $\delta q_i(t)$ vanish: $\delta q_i(t_0) = \delta q_i(t_1) = 0$. Generically, if there are no constraints, these are the only limitations on the variations $\delta q_i(t)$ which are otherwise arbitrary functions of time. These boundary conditions along with the principle of least action, $\delta S = 0$, demand the integrand of the integral in (1.1.9) to vanish, yielding

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{1.1.10}$$

These are the equations of motion which are often called the *Euler-Lagrange equations*; they present a system of n ordinary differential equations of the second order. Thus, solutions of these equations must contain $2n$ constants that are fixed by the initial conditions, $q_i(t_0)$ and $\dot{q}_i(t_0)$, which fully determine the time evolution of the system.

It is worth mentioning that the equations of motion are invariant under a non-degenerate transformation of the generalized coordinates and time:

$$q_i = f_i(q_i', t'), \quad t = f(t'), \tag{1.1.11}$$

which brings about the Lagrangian function to a new form

$$L'(q_i', \dot{q}_i', t') \equiv L\left(f_i(q_i', t'), \dot{f}_i(q_i', t'), f(t')\right), \tag{1.1.12}$$

where $\dot{q}'_i \equiv dq_i'/dt'$ is the generalized velocity in the new variables. Now, applying the least action principle in the transformed coordinates, it is straightforward to show that the Euler-Lagrange equations derived from the Lagrangian (1.1.12), are

$$\frac{\partial L'}{\partial q_i'} - \frac{d}{dt'}\frac{\partial L'}{\partial \dot{q}_i'} = 0. \tag{1.1.13}$$

These equations have exactly the same form as (1.1.10) and are equivalent to (1.1.10) that can be easily shown with the use of (1.1.11).

One more important fact is that the Euler-Lagrange equations (1.1.10) are the same for two Lagrangians that differ by a function which is a total derivative of time,

$$\tilde{L}(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{dK(q_i, t)}{dt}. \tag{1.1.14}$$

Transformation (1.1.14) is called a *gauge transformation* of the Lagrangian. It is straightforward to check by substituting $\tilde{L}$ to equations (1.1.10) that all terms depending on $K(q_i, t)$ cancel out leaving the equations (1.1.10) invariant. This result is also easy to understand from the integral formulation of the action (1.1.7). Indeed, integration of the total time derivative under the sign of the action integral (1.1.7) yields a constant term depending merely on the boundary values of the generalized coordinates but such a term does not, of course, affect the derivation of the equations of motion.

1.1.3 Noether's theorem in classical mechanics

Equalities of the type $I(q_i, \dot{q}_i, t) = \text{const}$ which follow from integrating the system of the Euler-Lagrange equations (1.1.10), are called the *first integrals of motion*. Among them, there exist integrals connected to the property of the action functional to remain invariant with respect to the group of transformations of the variables appearing as arguments of the Lagrangian, namely the generalized coordinates q_i and time t. This property forms the basis of the important *Noether's theorem* which associates the parameters of the group of the invariance of the Lagrangian to conservation laws and conserved quantities. Perhaps, the simplest formulation of the Noether theorem is as follows:

- *If the action functional is invariant with respect to a continuous group of transformations depending on k parameters, then, there exist k identities which, after using the Euler-Lagrange equations, yield k integrals of motion.*

Let us discuss in more detail the Noether theorem in the case of a mechanical system described by the action (1.1.7). It has been shown that the Euler-Lagrange equations (1.1.13) are form-invariant under the most general case of transformations (1.1.11) though they change the form of the Lagrangian ($L \to L'$) in accordance with (1.1.12) which means that the form of the action (1.1.7) changes correspondingly. It is not the case of the Noether theorem which is applicable only to a special case of the transformations that do not change the action functional. We prove the Noether theorem below.

For the sake of simplicity, let us consider a continuous group of transformations $\hat{T}(\varepsilon)$ whose action on the generalized coordinates and time depends on a single argument ε:

$$\{q_i(t); t\} \to \hat{T}(\varepsilon)\{q_i(t); t\} \equiv \{q_i'(t'); t'\} = \{f_i(q_i(t), t; \varepsilon); f(t; \varepsilon)\}. \tag{1.1.15}$$

The group product is defined by the rule $\hat{T}(\varepsilon_2)\hat{T}(\varepsilon_1) = \hat{T}\,(\varepsilon_3(\varepsilon_2, \varepsilon_1))$ which may be noncommutative in the most general case (the case of a non-abelian group). The existence of the unit element of the group, $\hat{T}(0) = 1$, suggests that functions $f_i(q_i(t), t; \varepsilon)$ and $f(t; \varepsilon)$ are such that

$$q_i(t) = f_i(q_i, t; 0). \quad t = f(t; 0). \tag{1.1.16}$$

Then, the linearized (with respect to a sufficiently small value of $\varepsilon \ll 1$) independent transformations for the generalized coordinates and time take the form:

$$q_i(t) \to q_i'(t') = q_i(t) + \varepsilon\delta' q_i(t, \varepsilon), \quad \delta' q_i = \frac{\partial f_i}{\partial \varepsilon} \tag{1.1.17}$$

$$t \to t' = t + \varepsilon\xi(t, \varepsilon), \quad \xi = \frac{\partial f}{\partial \varepsilon}, \tag{1.1.18}$$

where

$$\frac{\partial f_i}{\partial \varepsilon} \equiv \left.\frac{\partial f_i(t)}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad \frac{\partial f}{\partial \varepsilon} \equiv \left.\frac{\partial f(t)}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{1.1.19}$$

The quantities $\delta' q_i$ and ξ are called the infinitesimal *generators* of the group.

In what follows the *instantaneous* variation δq_i of the generalized coordinate,

$$\varepsilon \delta q_i = q_i'(t) - q_i(t), \tag{1.1.20}$$

plays an important role. The instantaneous variation can be singled out from the group generator $\delta' q_i$ after expanding $q_i'(t')$ in the Taylor series around time t and making use of (1.1.18):

$$\delta q_i = \delta' q_i - \xi \frac{dq_i}{dt}. \tag{1.1.21}$$

Before proving the Noether theorem we need to discuss some mathematical techniques of variational calculus in case of an arbitrary function $F[q_i(t), \dot{q}_i(t), t]$. The action of the group (1.1.15) transforms the function F to F':

$$F[q_i(t), \dot{q}_i(t), t] \to F'[q_i'(t'), \dot{q}_i'(t'), t'] = \hat{T}(\varepsilon) F[q_i(t), \dot{q}_i(t), t], \tag{1.1.22}$$

where the dot above function of time indicates a time derivative, for example, $\dot{q}_i(t) \equiv dq_i/dt$ and $\dot{q}_i'(t') \equiv dq_i'/dt'$. By definition

$$\hat{T}(\varepsilon) F = F[\hat{T}(\varepsilon) q_i(t), \hat{T}(\varepsilon)\dot{q}_i(t), \hat{T}(\varepsilon) t] = F[q_i'(t'), \dot{q}_i'(t'), t'], \tag{1.1.23}$$

so that

$$F'[q_i'(t'), \dot{q}_i'(t'), t'] = F[q_i'(t'), \dot{q}_i'(t'), t']. \tag{1.1.24}$$

In other words, the group operator $\hat{T}(\varepsilon)$ changes merely the arguments of the function F while its functional form does not change. Total variation of this function induced by the continuous group of transformation, is defined as

$$\varepsilon \delta' F = F[q_i'(t'), \dot{q}_i'(t'), t'] - F[q_i(t), \dot{q}_i(t), t]. \tag{1.1.25}$$

The instantaneous variation of F is the difference between the transformed value of the function $F[q_i'(t'), \dot{q}_i'(t'), t']$ taken at the instant of time t, and the value of the function $F[q_i(t), \dot{q}_i(t), t]$ taken at the same instant of time,

$$\varepsilon \delta F \equiv F[q_i'(t), \dot{q}_i'(t), t] - F[q_i(t), \dot{q}_i(t), t]. \tag{1.1.26}$$

Expanding $F[q_i'(t'), \dot{q}_i'(t'), t']$ on the right side of (1.1.25) in Taylor series with respect to the variation of time and making use of definition (1.1.26) we get

$$\delta F = \delta' F - \xi \frac{dF}{dt}, \tag{1.1.27}$$

where the last term is defined by the total derivative of F with respect to time:

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial F}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial F}{\partial t}. \tag{1.1.28}$$

It is worthwhile to make a few remarks with regard to the meaning of the variations of the function F:

(i) the relation (1.1.27) generalizes (1.1.21),

(ii) by definition (1.1.20) the variation (1.1.26) can be expanded in a Taylor series with respect to the variations of its arguments as

$$\delta F = \frac{\partial F}{\partial q_i}\delta q_i + \frac{\partial F}{\partial \dot{q}_i}\delta \dot{q}_i, \tag{1.1.29}$$

where the higher order terms have been truncated,

(iii) variation δF commutes with the total time derivative of F,

$$\delta\left(\frac{dF}{dt}\right) = \frac{d}{dt}(\delta F), \tag{1.1.30}$$

which makes it more advantageous compared with $\delta' F$ in calculations where both operations of taking the time derivative and variation are present. Formally it follows from the definition (1.1.26) where the unique time argument is considered. At the same time, the variation $\delta' F$ defined in (1.1.25) does not commute with total time derivative of F.

The technique presented above is used for proving the Noether theorem which demands that the action functional (1.1.7) is invariant with respect to the continuous group of transformations (1.1.15), (1.1.16) and the mechanical system obeys the equations of motion (1.1.10). We consider the total variation of the action (1.1.7) induced by the continuous group of transformation (1.1.15):

$$\varepsilon\delta' S = \int_{T'} dt' L[q_i'(t'), \dot{q}_i'(t'), t'] - \int_T dt \ L[q_i(t), \dot{q}_i(t), t], \tag{1.1.31}$$

where the integration is performed over a fixed time interval $T = [t_0, t_1]$ and a corresponding interval of time $T' = [t_0', t_1']$ that is connected with T by the time transformation.

We apply the variational technique (1.1.20–1.1.30) to the Lagrangian L along with the transformations (1.1.17) and (1.1.18). We restrict ourselves with the linearized, in regard to the group argument ε, approximation. Then, the variation of the action (1.1.31) is reduced to

$$\begin{aligned}\varepsilon\delta' S &= \int_T dt\left[\frac{dt'}{dt}\left(L+\varepsilon\delta' L\right)\right]-\int_T dt\ L(t)\\ &= \int_T dt\left[\left(1+\varepsilon\frac{d\xi}{dt}\right)\left(L+\varepsilon\delta L+\varepsilon\frac{dL}{dt}\xi\right)-L(t)\right]\\ &= \varepsilon\cdot\int_T dt\left[\delta L+\frac{d}{dt}(\xi L)\right],\end{aligned} \tag{1.1.32}$$

where $\delta' L$ and δL are defined in (1.1.25) and (1.1.26), respectively. Using (1.1.29) and applying the Leibniz rule to reshuffle terms, we can rewrite (1.1.32) as follows,

$$\delta' S = \int_T dt\left[\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i+\frac{d}{dt}\left(\xi L+\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)\right], \tag{1.1.33}$$

where the terms enclosed in the first brackets in the integrand constitute the left-hand side of the Euler-Lagrange equations (1.1.10). These terms appeared naturally in (1.1.33) but we don't demand at this point of the proof of the Noether theorem that the Euler-Lagrange equations are satisfied. The Noether theorem requires the invariance of the action functional under the group of transformation (1.1.15), that is the variation $\delta' S$ must vanish: $\delta' S = 0$. Then, the integrand in (1.1.33) must vanish as well, bringing the identity,

$$\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \equiv -\frac{d}{dt}\left(\xi L+\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right). \tag{1.1.34}$$

The last step is to use in the right side of (1.1.34) equation (1.1.21) expressing the variation δq_i in terms of $\delta' q_i$, and to make substitutions $\delta' q_i = \partial f_i/\partial\varepsilon$ and $\xi = \partial f/\partial\varepsilon$ in accordance with definitions (1.1.17) and (1.1.18) for $\delta' q_i$ and ξ, respectively. It yields,

$$\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \equiv \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i-L\right)\frac{\partial f}{\partial\varepsilon}-\frac{\partial L}{\partial \dot{q}_i}\frac{\partial f_i}{\partial\varepsilon}\right]. \tag{1.1.35}$$

The identity (1.1.35) is called the Noether identity. It is valid for arbitrary variation of the independent variables irrespectively whether the generalized coordinates q_i obey the equations of motion (1.1.10) or not. In case of an actual motion of the mechanical system, the equations of motion are satisfied making the left side of the Noether's identity (1.1.35) zero. It allows us to define the integrals of motion:

$$I(q_i,\dot{q}_i,t) = \left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i-L\right)\frac{\partial f}{\partial\varepsilon}-\frac{\partial L}{\partial \dot{q}_i}\frac{\partial f_i}{\partial\varepsilon} = \text{const}, \tag{1.1.36}$$

which are conserved quantities in the sense that $dI(q_i,\dot{q}_i,t)/dt = 0$ where $q_i = q_i(t)$ satisfy the Euler-Lagrange equations of motion.

This concludes the proof of the Noether theorem.

Integrals of motion and the intrinsic symmetries of the Lagrangian

The existence of the conserved quantities are intimately related to the transformation symmetries of the Lagrangian describing the mechanical system. In order to reveal the connection between the Lagrangian symmetries and the integrals of motion let us rewrite the variation of the action (1.1.33) in the form:

$$\delta' S = \int_T dt \left\{ \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) (\delta' q_i - \dot{q}_i \xi) + \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \delta' q_i + \left(L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \xi \right] \right\}, \quad (1.1.37)$$

where we have used (1.1.21) in order to replace δq_i with $\delta' q_i$. By a direct calculation without making use of the equations of motion, we can reduce the variation (1.1.37) to a more simple form

$$\delta' S = \int_T dt \left[\frac{\partial f}{\partial \varepsilon} \frac{\partial L}{\partial t} + \frac{\partial f_i}{\partial \varepsilon} \frac{\partial L}{\partial q_i} + \frac{\partial \dot{f}}{\partial \varepsilon} \left(L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial \dot{f}_i}{\partial \varepsilon} \frac{\partial L}{\partial \dot{q}_i} \right], \quad (1.1.38)$$

where we have made the substitutions $\xi = \partial f/\partial \varepsilon$, $\delta' q_i = \partial f_i/\partial \varepsilon$, and again the overdot denotes a total derivative with respect to time. Expression (1.1.38) allows us to connect the conservation laws with the functional structure of the Lagrangian and its internal symmetries with respect to the continuous group of transformations. Let us consider several important cases.

First, let us assume $\partial f/\partial \varepsilon = \text{const} \neq 0$ and $\partial f_i/\partial \varepsilon = 0$. Then, $\partial \dot{f}/\partial \varepsilon = \partial \dot{f}_i/\partial \varepsilon = 0$, and the variation of the action (1.1.38) becomes

$$\delta' S = \frac{\partial f}{\partial \varepsilon} \int_T dt \frac{\partial L}{\partial t}. \quad (1.1.39)$$

This expression points out that in the most general case of the Lagrangian depending explicitly on time, $L = L(q_i, \dot{q}_i, t)$, its partial derivative $\partial L/\partial t \neq 0$, and the variation $\delta' S$ of the action functional (1.1.7) does not vanish. It means that the Noether theorem cannot be applied and the integrals of motion associated with the time translations do not exist.[2]

On the other hand, if the Lagrangian does not depend on time explicitly, $L = L(q_i, \dot{q}_i)$, it stays invariant with respect to the (constant) time shifts. In this case we can use (1.1.36) for deriving one of the most important integrals of motion – energy, usually denoted as E. Taking transformations (1.1.17) and (1.1.18) in the form $f_i(t; \varepsilon) = 0$ and $f(t; \varepsilon) = \varepsilon$ yield $\partial f_i/\partial \varepsilon = 0$, $\partial f/\partial \varepsilon = 1$, and the corresponding conserved quantity (1.1.36) acquires the form:

$$E = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L. \quad (1.1.40)$$

It is called the *total energy* of the system. The systems with the conserved energy are called *conservative* [283]. In the case of an *open system* the Lagrangian depends

2 The same conclusion is extended, of course, for the more general case of function $f(\varepsilon, t)$ depending on time so that $\partial \dot{f}/\partial \varepsilon \neq 0$.

explicitly on time and the energy (1.1.40) is not conserved. Such a system is also called *non-conservative*. It can be easily checked by taking the total time derivative of E that in case of the Lagrangian depending explicitly on time, we have

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}. \tag{1.1.41}$$

Second, let us consider (1.1.38) for the case of transformations, $\partial f/\partial\varepsilon = 0$ and the constant $\partial f_i/\partial\varepsilon \neq 0$. Then, $\partial\dot{f}_i/\partial\varepsilon = 0$, and the variation (1.1.38) becomes

$$\delta' S = \frac{\partial f_i}{\partial\varepsilon}\int_T dt \frac{\partial L}{\partial q_i}. \tag{1.1.42}$$

Expression (1.1.42) points out that in the case of the Lagrangian depending explicitly on q_i the action functional (1.1.7) cannot be invariant with respect to the constant translations of the generalized coordinates q_i and the Noether theorem cannot be applied. Hence, in this case there are no integrals of motion associated with the constant translations of q_i. In the opposite case when the Lagrangian does not depend explicitly on the generalized coordinates (but may depend explicitly on time), the Noether theorem yields the integral of motion[3] corresponding to the constant translation of the coordinate q_i:

$$I_i(\dot{q}_i, t) = \frac{\partial L}{\partial\dot{q}_i} = \text{const}. \tag{1.1.43}$$

The coordinate q_i which drops out of the Lagrangian is called an *ignorable* coordinate.

We should draw attention of the reader to a subtlety associated to the integrals corresponding to the case of the constant translations of the generalized coordinates. Let us consider an example of a Lagrangian describing time evolution of a mechanical system consisting of two particles with generalized coordinates q_{1i} and q_{2i}. Let the Lagrangian depend explicitly on the difference of the coordinates: $L = L(q_{2i} - q_{1i}, \dot{q}_{1i}, \dot{q}_{2i})$. We consider the case of a constant translation such that both coordinates are shifted by the same amount: $\partial f_{2i}/\partial\varepsilon = \partial f_{1i}/\partial\varepsilon = \epsilon_i = \text{const}$. Variation of the action (1.1.38)

$$\delta' S = \epsilon_i \int_T dt \left[\frac{\partial L}{\partial q_{1i}} + \frac{\partial L}{\partial q_{2i}}\right] = 0, \tag{1.1.44}$$

and the Noether theorem can be applied in spite of the explicit dependence of the Lagrangian on the generalized coordinates.

Third, let us consider now the conservation laws for the case of time-dependent translation of time and/or generalized coordinates. In this case we operate with

3 Notice that we have chosen the sign of the integral (1.1.43) opposite to the sign of the corresponding quantity in (1.1.36). This convention is more suitable in the discussion of the integrals of motion of mechanical system of N particles given in the Section 1.1.4.

the group of transformations having the non-vanishing time-dependent generators: $\partial f/\partial\varepsilon \neq 0$, $\partial f_i/\partial\varepsilon \neq 0$. For this reason, the variation of the action (1.1.38) does not vanish even if the Lagrangian, $L = L(\dot{q}_i)$, that is it does not depend explicitly on time and the coordinate q_i is ignorable,

$$\delta' S = \int_T dt \left[\frac{\partial \dot{f}}{\partial \varepsilon} \left(L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial \dot{f}_i}{\partial \varepsilon} \frac{\partial L}{\partial \dot{q}_i} \right] \neq 0. \tag{1.1.45}$$

The Noether's theorem is not applied to such a situation in the most general case. Nonetheless, it may turn out that for the variation (1.1.45) of the action its integrand equals to a total time derivative

$$\frac{\partial \dot{f}}{\partial \varepsilon} \left(L - \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial \dot{f}_i}{\partial \varepsilon} \frac{\partial L}{\partial \dot{q}_i} = \frac{dK}{dt}, \tag{1.1.46}$$

where $K = K(\partial f/\partial\varepsilon, \partial f_i/\partial\varepsilon, q_i, t)$ is some function of time, generalized coordinates and the generators of the group of transformations.[4] In this particular case, the Noether theorem can be extended and the integrals of motion (1.1.36) are modified to

$$I(q_i, \dot{q}_i, t) = \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) \frac{\partial f}{\partial \varepsilon} - \frac{\partial L}{\partial \dot{q}_i} \frac{\partial f_i}{\partial \varepsilon} + K = \text{const}. \tag{1.1.47}$$

Let us consider an example of the time-dependent transformation: $\partial f/\partial\varepsilon = 0$ and $\partial f_i/\partial\varepsilon = \epsilon_i t$ with $\epsilon_i = \text{const}$. The function K entering (1.1.47) can be found from solving the equation (1.1.46) which takes, in the case under consideration, the following form

$$\epsilon_i \frac{\partial L}{\partial \dot{q}_i} = \frac{dK}{dt}, \tag{1.1.48}$$

if and only if, there exists a function K_i such that $K = \epsilon_i K_i$, and

$$\frac{dK_i}{dt} = \frac{\partial L}{\partial \dot{q}_i}. \tag{1.1.49}$$

The new integral associated with the above-mentioned symmetry is[5]

$$N_i = -K_i + t \frac{\partial L}{\partial \dot{q}_i}. \tag{1.1.50}$$

We discuss this integral in more detail in Section 1.1.4 – see equation (1.1.62).

4 It is crucial to emphasize that K is a function but not a functional (integral) of time.

5 Notice that for the sake of convenience we have chosen the sign of the integral (1.1.50) opposite to the sign of the corresponding quantity in (1.1.47).

1.1.4 Conserved quantities for a system of non-relativistic particles

In the framework of non-relativistic mechanics, the arena for describing physical phenomena is a 3-dimensional Euclidean space usually associated with the Cartesian coordinates $x^i = \{x^1, x^2, x^3\}$, or $x^i = \{x, y, z\}$. Time is considered separately and does not depend on space. The Newtonian spacetime is homogeneous and isotropic so that the infinitesimal group of transformations that leaves the Lagrangian of the Newtonian mechanical systems invariant, consists of a time shift, three independent spatial translations along the Cartesian coordinate axes and three rotations about them. According to the Noether theorem, these symmetries point out to the existence of 7 integrals of motion.

Besides these, the Euler-Lagrange equations of the Newtonian systems are invariant with respect to the, so-called, Galilean transformation which introduces the class of the *inertial* reference frames. The inertial frame is defined by the property of a free test particle to remain at rest or to move uniformly along a straight line with a constant velocity with respect to it. The Galilean transformations are defined by the following equations

$$t' = t; \quad x'^i = x^i + V^i t \tag{1.1.51}$$

where $V^i = \{V^1, V^2, V^3\}$ is a constant velocity. The Galilean transformation connects two arbitrary chosen, *inertial* frames with the Cartesian coordinates x^i and x'^i. Three components of the velocity V^i are additional three parameters entering the invariance group of the Newtonian mechanics. Thus, this group depends on 10 parameters and is called the Galilean group [11]. Euclidean space and time with the Galilean transformation (1.1.51) is called the Galilean spacetime which is the *physical* basis for applying the Noether theorem to obtain 10 of the first integrals for any conservative system.

We consider a system of N point-like particles with *constant* masses m_a ($a = 1, \dots, N$) embedded to 3-dimensional Euclidean space covered by Cartesian coordinates x^i which play the role of generalized coordinates q_i of the particles. The Lagrangian of the conservative system of N particles with constant masses does not depend on time explicitly and is of the type (1.1.6),

$$L = \frac{1}{2}\sum_a m_a \dot{x}^i_a \dot{x}^i_a - U(r_{ab}), \tag{1.1.52}$$

where $x^i_a = x^i_a(t)$ are coordinates of the particle a, $x^i_b = x^i_b(t)$ are coordinates of the particle b, the overdot denotes a time derivative, $\dot{x}^i_a = dx^i_a/dt$. The potential $U(r_{ab})$ describing the interactions between the particles, is a scalar function depending only on the relative distance $r_{ab} = |r^i_{ab}|$ between the particles $r^i_{ab} = x^i_a - x^i_b$ which makes it invariant with respect to translations and rotations of the coordinate system in space. It is more conventional to use the vector notation by denoting three-dimensional vectors with bold letters, $\boldsymbol{r}_a = \{x^1_a, x^2_a, x^3_a\}$, $\boldsymbol{v}_a = \dot{\boldsymbol{r}}_a = \{\dot{x}^1_a, \dot{x}^2_a, \dot{x}^3_a\}$. In the vector notation, the Lagrangian (1.1.52) reads

$$L = \frac{1}{2}\sum_a m_a \boldsymbol{v}_a^2 - U(r_{ab}). \tag{1.1.53}$$

The Lagrangian (1.1.53) does not depend on time explicitly and, hence, is invariant with respect to constant time translation. It gives the integral of energy that we have already defined in expression (1.1.40). Thus, substituting L from (1.1.53) into (1.1.40) one obtains

$$E = \frac{1}{2}\sum_a m_a \boldsymbol{v}_a^2 + U(r_{ab}), \tag{1.1.54}$$

which is a sum of two terms – the kinetic and potential energy of the system. It is worth mentioning that had we assumed the masses m_a of the particles as variable $m_a = m_a(t)$, it would make the Lagrangian explicitly dependent on time. In that case the total energy E would not be conserved but changing in accordance with (1.1.41) as follows [6]

$$\frac{dE}{dt} = -\frac{1}{2}\sum_a \frac{dm_a}{dt}\boldsymbol{v}_a^2. \tag{1.1.55}$$

The homogeneity of the Euclidean space is reflected in the invariance of the system with respect to three spatial translations. They are defined as $\boldsymbol{r}_a \to \boldsymbol{r}'_a = \boldsymbol{f}_a(\boldsymbol{r}_a, \boldsymbol{\varepsilon}) = \boldsymbol{r}_a + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a constant vector of the spatial translation, $\boldsymbol{\varepsilon} = \{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$. Then, $\partial \boldsymbol{f}_a/\partial \boldsymbol{\varepsilon} = \boldsymbol{\delta} = \{\delta_{ij}\}$ – the unit matrix, and (because the time is unchanged) $\partial f/\partial \boldsymbol{\varepsilon} = 0$. Hence, the corresponding conserved quantity (1.1.43) acquires the form:

$$\boldsymbol{P} = \sum_a \frac{\partial L}{\partial \boldsymbol{v}_a} = \sum_a \boldsymbol{p}_a = \sum_a m_a \boldsymbol{v}_a, . \tag{1.1.56}$$

It is called the total linear *momentum* of the system which is the sum of the momenta $\boldsymbol{p}_a = m_a \boldsymbol{v}_a$ of the individual particles. The integral of the linear momentum exists irrespectively of whether masses m_a depend on time or not.

Now, let us construct a conserved quantity and a conservation law, which follow from the isotropy of the Euclidean space which means that all directions in space are equivalent and the Lagrangian is invariant with respect to rotations. Under a rotation, the radius vector of a particle is transformed as $\boldsymbol{r}_a \to \boldsymbol{r}'_a = \boldsymbol{f}_a(\boldsymbol{r}_a, \boldsymbol{\omega}) = \boldsymbol{r}_a + [\boldsymbol{\omega} \times \boldsymbol{r}_a]$, where $\boldsymbol{\omega}$ is a constant vector of an infinitesimal rotation. Then, $\partial \boldsymbol{f}_a/\partial \boldsymbol{\omega} = \Lambda(\boldsymbol{r}_a)$ is the antisymmetric matrix with the components depending on the radius-vector of the particle, $\Lambda(\boldsymbol{r}_a) = \{\Lambda_{ij}\} = \epsilon_{ijk} r_a^k$ where ϵ_{ijk} is a fully-antisymmetric Levi-Civita symbol. The corresponding conserved quantity (1.1.36) applied to the Lagrangian (1.1.52) acquires the form:

$$\boldsymbol{M} = \sum_a \frac{\partial L}{\partial \boldsymbol{v}_a} \cdot \Lambda(\boldsymbol{r}_a) = \sum_a [\boldsymbol{r}_a \times \boldsymbol{p}_a] = \sum_a m_a [\boldsymbol{r}_a \times \boldsymbol{v}_a], \tag{1.1.57}$$

6 The potential $U(r_{ab})$ could also depend on masses and so it could also change, consider gravitational interaction as an example. In this case, formula (1.1.55) should include one more term.

where the signs "dot" and "cross" between two vectors denote the Euclidean dot and cross product of the vectors respectively [283]. The vector integral of motion $\boldsymbol{M}$ is called the total *angular momentum* of the system. The angular momentum is conserved even if masses m_a of the particles depend on time. Note that the appearance of the cross product in equation (1.1.57) shows that this formalism is restricted to 3 dimensions and the angular momentum is not a true vector but a pseudo vector. In more than 3 dimensions, one observes that the proper generalization of angular momentum is an anti-symmetric rank 2 tensor that is not reduced, in general, to a cross product between vectors.

At last, one has to analyze the invariance with respect to the infinitesimal Galilean transformations (1.1.51) with the constant velocity $\boldsymbol{V} = \boldsymbol{\varepsilon}$ considered as a small parameter of the group. Thus, the infinitesimal group of the Galilean transformation is given by equations

$$\boldsymbol{r}'_a = \boldsymbol{f}_a(\boldsymbol{r}_a, \boldsymbol{\varepsilon}) = \boldsymbol{r}_a + t\boldsymbol{\varepsilon}, \qquad \boldsymbol{v}'_a = \dot{\boldsymbol{f}}_a(\boldsymbol{r}_a, \boldsymbol{\varepsilon}) = \boldsymbol{v}_a + \boldsymbol{\varepsilon}, \tag{1.1.58}$$

from where one has for the group parameters,

$$\frac{\partial f}{\partial \boldsymbol{\varepsilon}} = 0, \qquad \frac{\partial \boldsymbol{f}_a}{\partial \boldsymbol{\varepsilon}} = t. \tag{1.1.59}$$

This is a special case of time-dependent transformations which we have considered above at the end of Section 1.1.3. Equation (1.1.49) for the Lagrangian (1.1.53) reads

$$\frac{d\boldsymbol{K}}{dt} = \sum_a m_a \boldsymbol{v}_a = \frac{d}{dt}\sum_a m_a \boldsymbol{r}_a. \tag{1.1.60}$$

Notice that the last term in (1.1.60) can be written in the form of the total derivative if and only if, masses m_a of the particles do not depend on time. Integration of (1.1.60) yields

$$\boldsymbol{K} = \sum_a m_a \boldsymbol{r}_a. \tag{1.1.61}$$

Then, equation (1.1.50) defines the conserved quantity which is called the integral of the *center of mass* of the system,

$$\boldsymbol{N} = -\sum_a m_a \boldsymbol{r}_a + t\sum_a m_a \boldsymbol{v}_a. \tag{1.1.62}$$

It is worth emphasizing that similar to the case of the integral of energy, the integral of the center of mass does not exist if masses m_a of the particles depend on time. This is because in such a case, equation (1.1.60) takes the form $d\boldsymbol{K}/dt = \sum_a m_a(t)\boldsymbol{v}_a$ whose right-hand side does not admit transformation to the total time derivative and makes it non-integrable.

The *center of mass* of the system is defined by a vector $\boldsymbol{R} = \boldsymbol{R}(t)$ which is introduced by the following identity

$$M\boldsymbol{R} \equiv \sum_a m_a \boldsymbol{r}_a \tag{1.1.63}$$

where

$$M = \sum_a m_a, \tag{1.1.64}$$

is a constant total mass of the system of particles. Then, the integral of the center of mass takes the traditional form

$$\boldsymbol{N} = -M\boldsymbol{R} + \boldsymbol{P}\,t, \tag{1.1.65}$$

where $\boldsymbol{P}$ is the conserved linear momentum of the system (1.1.56) which is directly related to the velocity of the center of mass

$$\boldsymbol{V} = \frac{\boldsymbol{P}}{M}. \tag{1.1.66}$$

It allows us to reformulate (1.1.65) in the following form

$$\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{V}t, \tag{1.1.67}$$

where $\boldsymbol{R}_0 = -\boldsymbol{N}/M$ is the constant radius-vector defining position of the center of mass at time $t = 0$. This equation tells us that the center of mass of the entire system of particles moves uniformly along a straight line with a constant velocity $\boldsymbol{V}$.

In many applications it is convenient to use an inertial frame whose total linear momentum vanishes, $\boldsymbol{P} = 0$, and its origin is placed at the center of mass, $\boldsymbol{R}_0 = 0$. Then, $\boldsymbol{R} = 0$ at any instant of time and the origin of the frame remains at rest at the center of mass of the system of particles.

1.1.5 The Minkowski space and the Poincaré group

Now, we turn to special relativity which corrects certain shortcomings of the classical mechanics by unifying the time and Euclidean 3-space into a fabric of spacetime continuum where space and time are no longer absolute but depend on the vantage point of the observer. Such a 4-dimensional spacetime is called the *Minkowski space* or spacetime. The Minkowski space admits the existence of the *inertial* frames of reference whose properties are similar to the ones in classical mechanics. Description of physical phenomena is invariant irrespective of the choice of the inertial frame. Therefore, transformations between the inertial frames are described by a 10-parameter group of transformations which is called the *Poincaré group*. It includes 4 translations along each spatial dimensions and time, 3 rotations in the Euclidean space, and 3 Lorentz boosts which generalize the Galilean transformations of classical mechanics and are called the *Lorentz transformations*. The Poincaré group forms a *geometrical* basis for applying the Noether theorem which yield 10 integrals of motion for a closed system as in classical mechanics. One can also think of the Poincaré group as the symmetry group of the Minkowski space which is a maximally symmetric flat spacetime.

The most convenient coordinates in the Minkowski space are the *Lorentzian coordinates* which add time to the set of the three-dimensional Cartesian coordinates. The time coordinate is defined as the product of time with the speed of light, c, and is denoted $x^0 = ct$. The Lorentzian coordinates are labeled with Greek indices, $x^\alpha \equiv \{ct, x, y, z\} \equiv \{x^0, x^1, x^2, x^3\}$; and the Greek indices take values from the set $0, 1, 2, 3$. Each set of the Lorentzian coordinates is associated with an inertial reference frame. Because two different inertial frames are connected by the elements of the Poincaré group, in the Minkowski space any two Lorentzian coordinate systems are connected by these transformations. A point in the Minkowski space is called a *world point* or *an event*. The distance between two world points in the Minkowski space is called an *interval* and is invariant with respect to coordinate transformations. The interval between two infinitesimally close events is denoted, ds, and ds^2 is expressed in terms of the infinitesimal increments of the Lorentzian coordinates as a *pseudo-Euclidean* quadratic form:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta. \tag{1.1.68}$$

The geometric object $\eta_{\alpha\beta} \equiv \mathrm{diag}\{-1, +1, +1, +1\}$ is known as the *Minkowski metric*. Here, we have taken the plus sign convention for the spatial part of the metric; the other choice, the minus sign convention is valid as well. Because the interval is invariant with respect to arbitrary coordinate transformations, the Minkowski metric is a tensor. Besides describing the metric properties of the Minkowski space, it is also used to raise and lower the tensor indices of other geometric objects, vectors, tensors etc. residing in the Minkowski space. Below, we use the units with the speed of light, $c = 1$.

The interval (1.1.68) in the Minkowski space can have either positive, or negative or a null values. Depending on its value the interval is termed either *spacelike*, $ds^2 > 0$, or *lightlike or null*, $ds^2 = 0$, or *timelike*, $ds^2 < 0$, interval. All events separated by a lightlike interval lie on a hypersurface in the Minkowski space, called the *light cone*. Vectors attached to a particular event and lying inside the light cone are called timelike, those lying outside of the light cone are called spacelike, and vectors lying on the light cone are called null vectors. A curve in the Minkowski space is called *spacelike*, *lightlike* or *timelike* if tangent vector taken at each point of such a curve is *spacelike*, *lightlike* or *timelike*, respectively. Trajectory of a point-like particle in the Minkowski space is called a *world line*. Because particles of matter cannot propagate faster than light, their world lines can be either *timelike* or *lightlike* but not *spacelike*. Each world line can be parametrized with a continuous parameter to identify the position of the particle in spacetime. The most convenient parameter for timelike world lines is called the *proper time*, τ, which is related to the infinitesimal interval ds between two events on the world line as follows:

$$(d\tau)^2 = -ds^2 = \left(1 - \boldsymbol{v}^2\right) dt^2, \tag{1.1.69}$$

where $\boldsymbol{v} = \{dx^i/dt\}$ is the three-dimensional, coordinate velocity of the particle. The proper time is a real quantity that can be measured by an ideal clock moving along the world line, attached to the particle, under consideration. Notice that for an observer at rest, $d\tau = dt$, that is the proper time τ and the coordinate time t coincide.

As we mentioned above, the Poincaré group is a 10-parameter group of transformations in the Minkowski space consisting of 4 translations and 6 rotations of the Lorentzian coordinates. The generators of this group can be found by studying the coordinate transformations that leave the metric intact. This boils down to studying the isometries of the spacetime. The group is generated by the Lie transformations along the curves whose tangent vectors are known as the Killing vectors, ξ^α_K; $K = 1, 2, \ldots, 10$. The Poincaré group leaves the Minkowski metric invariant which means that the Lie derivative, $\pounds$, (see Section 1.2.3 and Appendix A.2.3) of the Minkowski metric vanishes:

$$\pounds_{\xi_K} \eta_{\alpha\beta} = 0. \tag{1.1.70}$$

It leads to the first order Killing differential equation [154]

$$\xi_{K(\alpha,\beta)} = 0, \tag{1.1.71}$$

where the brackets denote symmetrization. The Killing equation has 10 linearly independent solutions consisting of four translations, $\xi^\alpha_K = \xi^\alpha_\beta$, and three rotations along with three boosts, $\xi^\alpha_K = \xi^\alpha_{[\beta\gamma]}$ where the square brackets denote anti-symmetrization. In the Lorentzian coordinates the set of 10 Killing vectors, ξ^α_K, of the Minkowski space is given by the following expresssions,

$$\xi^\alpha_0 = -\delta^\alpha_0, \quad \xi^\alpha_i = \delta^\alpha_i, \quad \xi^\alpha_{[\beta\gamma]} = \frac{1}{2}\left(\eta_{\rho\beta}\xi^\alpha_\gamma - \eta_{\rho\gamma}\xi^\alpha_\beta\right) x^\rho. \tag{1.1.72}$$

The “minus” sign for the timelike Killing vector agrees with the choice of the metric signature in this book (−, +, +, +). Partial derivatives of the Killing vectors with respect to a particular coordinate (denoted by a comma with an index after it indicating the corresponding coordinate) follow immediately from (1.1.72):

$$\xi^\alpha_{\beta\,,\rho} = 0, \qquad \xi^\alpha_{[\beta\gamma],\rho\sigma} = 0. \tag{1.1.73}$$

and they can be used to check the consistency of the Killing equation (1.1.71).

1.1.6 A point-like particle in special relativity

Let us extend the definitions and concepts related to the motion of a point-like particle from classical mechanics to the realm of special relativity. These notions will be used later on for the derivations of conservation laws and conserved quantities in the N-body system. It is convenient to parametrize the world line of the particle with its proper time τ which is invariant with respect to coordinate transformations.

An infinitesimal displacement along the world line of the particle is $dx^\alpha = u^\alpha d\tau$ where the 4-vector

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{dt}\frac{1}{\sqrt{1-\boldsymbol{v}^2}}, \tag{1.1.74}$$

is called the *4-velocity*, whereas $\boldsymbol{v}$ is a 3-dimensional velocity defined in (1.1.2). Because the parameter of the particle's world line was chosen as the proper time, the 4-velocity gets normalized, $u_\alpha u^\alpha = -1$. The *4-acceleration* of the particle is defined as the derivative of 4-velocity with respect to the proper time, $a^\alpha \equiv du^\alpha/d\tau$.

The motion of a free point-like particle follows from the principle of the least action with the action functional taken in the form:

$$S = m\int_{P_0}^{P_1} u_\alpha dx^\alpha = m\int_{P_0}^{P_1}\frac{dx_\alpha dx^\alpha}{d\tau} = -m\int_{P_0}^{P_1} d\tau, \tag{1.1.75}$$

where m is the rest mass of the particle (a constant), and the action is calculated between two *fixed* events P_0 and P_1.[7]

The action (1.1.75) can be easily reduced to its canonical form (1.1.7) if we choose the coordinate time t as the new parameter along the world line of the particle. This procedure turns the action (1.1.75) to

$$S = \int_{t_0}^{t_1} dt\ L(\boldsymbol{r},\boldsymbol{v}) \tag{1.1.76}$$

with the Lagrangian function

$$L = -m\frac{d\tau}{dt} = -m\sqrt{1-\boldsymbol{v}^2}, \tag{1.1.77}$$

and the instants t_0 and t_1 correspond to the points P_0 and P_1 on the particle's world line respectively. The action (1.1.76) is convenient for the 3-dimensional Lagrangian formalism but we are looking for its 4-dimensional version. To this end we notice that the action (1.1.76) is invariant under a re-parametrization of the world line of the particle: $\tau \to \lambda = \lambda(\tau)$,

$$S = \int_{\lambda_0}^{\lambda_1} d\lambda L(x^\alpha,\dot{x}^\alpha), \tag{1.1.78}$$

The Lagrangian $L = L(x^\alpha,\dot{x}^\alpha)$ with the new parameter, λ, takes the form

$$L = -m\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}, \tag{1.1.79}$$

where $\dot{x}^\alpha = dx^\alpha/d\lambda$ is a 4-dimensional dynamical variable generalizing the 3-dimensional velocity $\boldsymbol{v}$ of the particle.

7 In the case, when mass depends on time, $m = m(t)$, one cannot derive a conserved energy for the same reason as in non-relativistic mechanics.

The action, such as (1.1.78), is called a *parameterized* action, whereas the Lagrangian function (1.1.79) is called a *singular* Lagrangian, see [140, 193, 378] for more details. The idea behind the new parameter is that it makes all variables in the action dynamical variables, whereas the parameter λ serves merely as an argument of integration (external 'time') and plays an auxiliary role having no direct physical meaning. It can be replaced with another parameter,

$$\tilde{\lambda} = \tilde{\lambda}(\lambda). \tag{1.1.80}$$

The action (1.1.76) is invariant with respect to such replacements. The goal of introducing the new parameter into the action will become more transparent when we shall discuss the Hamiltonian formulation of the parameterized action for a point particle (see text from equations (3.1.105–3.1.112)).

Variation of (1.1.78) with respect to the coordinates $x^\alpha(\lambda)$, all of which are now considered as dynamical variables, is completely analogous to the variation of (1.1.7) in (1.1.9). This results in 4-dimensional equations of motion generalizing (1.1.10):

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} = 0. \tag{1.1.81}$$

The Lagrangian (1.1.79) does not depend on the coordinates x^α explicitly. Therefore, equations (1.1.81) tell us that there is a conserved quantity, $\partial L/\partial \dot{x}^\alpha$, called the 4-momentum of the particle:

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha} = m\frac{\eta_{\alpha\beta}\dot{x}^\beta}{\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}}. \tag{1.1.82}$$

It is straightforward to verify that after raising the index (1.1.82) can be written as:

$$p^\alpha = mu^\alpha. \tag{1.1.83}$$

Making use of the definition (1.1.74) of the 4-velocity we can write the components of 4-momentum (1.1.83) in terms of 3-dimensional coordinate velocity $\boldsymbol{v}$ of the particle:

$$p^0 = \frac{mc}{\sqrt{1-\boldsymbol{v}^2/c^2}}, \tag{1.1.84}$$

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2/c^2}}, \tag{1.1.85}$$

where we restored the speed of light c to find out the non-relativistic limit of the above expressions in case when $c \to \infty$. In this limit the time component p^0 of 4-momentum generalizes the classical expression for the kinetic energy of the particle: $m\boldsymbol{v}^2/2$. However, special relativity tells us that the particle has a non-vanishing energy even if its velocity vanishes. This is the rest energy of the particle $E_0 = mc^2$. The quantity (1.1.85) represents the relativistic 3-momentum generalizing the classical linear momentum of the particle: $\boldsymbol{p} = m\boldsymbol{v}$.

1.1.7 Conserved quantities for a system of relativistic particles

In order to describe a system of N non-interacting point-like particles in special relativity it is natural to follow the prescriptions of Section 1.1.6 and parameterize the world line of each particle with a single parameter, $x_a^\alpha = x_a^\alpha(\lambda)$. The Lagrangian, L_a, for each particle is

$$L_a = -m_a\sqrt{-\eta_{\alpha\beta}\dot{x}_a^\alpha\dot{x}_a^\beta}, \tag{1.1.86}$$

where m_a is a constant mass of the particle, and the overdot denotes differentiation with respect to the parameter λ. The Lagrangian of the entire system of the particles is the sum of the individual components: $L = \sum_a L_a$ and the action is

$$S = \int d\lambda \sum_a L_a. \tag{1.1.87}$$

Variation of (1.1.87) with respect to $4N$ generalized coordinates $x_a^\alpha(\lambda)$ leads to the equations of motion of the a-th particle. Because the Lagrangian (1.1.86) does not depend on $x_a^\alpha(\lambda)$ explicitly, the equations of motion have the form of the conservation law:

$$\frac{dp_\alpha^a}{d\lambda} = 0; \qquad p_\alpha^a \equiv \frac{\partial L_a}{\partial \dot{x}_a^\alpha}, \tag{1.1.88}$$

where p_a^α is the linear momentum of a-th particle.

The Lagrangian (1.1.86) is a particular case of (1.1.6). Thus, the Noether conserved quantities (1.1.36) and (1.1.47) are easily derived by applying the infinitesimal 10-parameter Poincaré group of motions generated by the Killing vectors (1.1.72). To apply the formalism of the Noether theorem of Section 1.1.3, we use the replacements: $t \to \lambda$, $q_i \to x_a^\alpha$, and $\dot{q}_i \to \dot{x}_a^\alpha$. We, first, consider transformation of coordinates x_a^α induced by the Killing vectors $\xi_K^\alpha = \xi_\beta^\alpha$, see (1.1.72), contracted with a constant parameters ε^β,

$$x_a^\alpha \to x_a'^\alpha = x_a^\alpha + \xi_\beta^\alpha\varepsilon^\beta. \tag{1.1.89}$$

This transformation does not change the Lagrangian, and leads to the conservation law of the linear momentum following directly from (1.1.36) where (in accordance with the transformation equation) we also use the replacements $f \to 0, f_i \to \xi_\beta^\alpha\varepsilon^\beta$,

$$\frac{d}{d\lambda}\sum_a \frac{\partial L_a}{\partial \dot{x}_a^\alpha}\xi_\beta^\alpha = \frac{d}{d\lambda}\sum_a p_\alpha^a\xi_\beta^\alpha = 0. \tag{1.1.90}$$

This law means the total 4-momentum of the system of particles,

$$P^\alpha = \sum_a p_a^\alpha = \sum_a m_a u_a^\alpha, \tag{1.1.91}$$

is conserved

$$\frac{dP^\alpha}{d\lambda} = 0. \tag{1.1.92}$$

The infinitesimal rotations in the Minkowski space about various axes of the Lorentzian coordinates are presented by 6 small parameters, $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$, multiplied with the Killing vectors $\xi^\alpha_K = \xi^\alpha_{[\beta\gamma]}$, see (1.1.72),

$$x^\alpha_a \to x'^\alpha_a = x^\alpha_a + \xi^\alpha_{[\beta\gamma]}\omega^{\beta\gamma} = x^\alpha_a + \frac{1}{2}\left(\eta_{\beta\rho}\xi^\alpha_\gamma - \eta_{\gamma\rho}\xi^\alpha_\beta\right) x^\rho_a \omega^{\beta\gamma}. \tag{1.1.93}$$

Then, substituting transformation (1.1.93) to (1.1.47) and replacing $t \to \lambda$, $q_i \to x^\alpha_a$, and $\dot{q}_i \to \dot{x}^\alpha_a$, $f \to 0$, $f_i \to \xi^\alpha_{[\beta\gamma]}\omega^{\beta\gamma}$, yields

$$\frac{d}{d\lambda}\sum_a \frac{\partial L_a}{\partial \dot{x}^\alpha_a}\xi^\alpha_{[\beta\gamma]} = \frac{d}{d\lambda}\sum_a p^\alpha_a \xi^\alpha_{[\beta\gamma]} = 0. \tag{1.1.94}$$

This law means the existence of the total *4-angular momentum* of the system of particles,

$$M^{\alpha\beta} = \sum_a \left(x^\alpha_a p^\beta_a - x^\beta_a p^\alpha_a\right), \tag{1.1.95}$$

which is conserved

$$\frac{dM_{\alpha\beta}}{d\lambda} = 0. \tag{1.1.96}$$

The total 4-angular momentum consists of three purely spatial components,

$$M^{ik} = \sum_a \left(x^i_a p^k_a - x^k_a p^i_a\right), \tag{1.1.97}$$

that represent a matrix of a *3-angular momentum* of the system. Note that, even though we have worked in four dimensions, this result is valid for any number of dimensions, for the angular momentum appears as an antisymmetric tensor (not just a pseudovector specific to 3 spatial dimensions as noted earlier). The other 3 components of (1.1.95) are a 3-vector,

$$N^i \equiv M^{0i} = \sum_a \left(x^0_a p^i_a - x^i_a p^0_a\right) \tag{1.1.98}$$

representing the, so-called, *Lorentzian momentum*.

If we take now the parameter $\lambda = t$ and set $x^0_a = t$, the components of the linear momentum of a particle takes the form of equations (1.1.84), (1.1.85). In this case the components of the conserved quantity (1.1.91) take a familiar form of the total energy, $E = P^0$, and the linear momentum, $\boldsymbol{P} = \{P^i\}$ of the system, respectively,

$$E = \sum_a \frac{m_a c^2}{\sqrt{1 - \mathbf{v}_a^2/c^2}}, \tag{1.1.99}$$

$$\boldsymbol{P} = \sum_a \frac{m_a \mathbf{v}_a}{\sqrt{1 - \mathbf{v}_a^2/c^2}}. \tag{1.1.100}$$

These relativistic expressions generalize the total energy (1.1.54) and the linear momentum (1.1.56) of classical mechanics in the case of non-interacting massive particles.

In the same parametrization the 3-angular momentum (1.1.97) is equivalent to a 3-vector

$$\boldsymbol{M} = \sum_a \frac{m_a [\boldsymbol{r}_a \times \mathbf{v}_a]}{\sqrt{1 - \mathbf{v}_a^2/c^2}}. \tag{1.1.101}$$

that generalizes the angular momentum of classical mechanics (1.1.57).

The conserved 3-vector (1.1.98) can be written more explicitly in the following form

$$\boldsymbol{N} = -\sum_a \frac{m_a \boldsymbol{r}_a}{\sqrt{1 - \mathbf{v}_a^2/c^2}} + t \sum_a \frac{m_a \mathbf{v}_a}{\sqrt{1 - \mathbf{v}_a^2/c^2}}. \tag{1.1.102}$$

One can see that (1.1.102) generalizes the integral of the center of mass (1.1.65) of classical mechanics. Dividing both sides of (1.1.102) by the total energy of the system, E, one obtains relativistic equation of motion of the center of mass of the system of N particles

$$\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{V} t, \tag{1.1.103}$$

where a constant vector $\boldsymbol{R}_0 = -\boldsymbol{N}/E$, a constant velocity of the center of mass, $\boldsymbol{V} = \boldsymbol{P}/E$, and the radius-vector $\boldsymbol{R}$ of the center of masses is defined by the identity

$$E\boldsymbol{R} = \sum_a \frac{m_a \boldsymbol{r}_a}{\sqrt{1 - \mathbf{v}_a^2/c^2}}, \tag{1.1.104}$$

which should be compared with its classical counterpart (1.1.63). Relativistic equation of motion of the center of mass (1.1.103) generalizes the analogous equation (1.1.67) of classical mechanics. Thus, for a conservative system of point-like particles, the conserved Lorentzian momentum states that there exists a center of mass defined by a radius-vector $\boldsymbol{R}$ which moves with a constant velocity $\boldsymbol{V}$.

1.2 Field theory in the Minkowski space

1.2.1 The action

Theory of physical fields in the Minkowski space is an important ingredient of modern particle physics. The starting point of the theory is the Lagrangian which governs the behavior of the fields and their interaction with the other fields and/or matter. Development of the Lagrangian formalism of *physical fields* (also called *matter fields*) in the Minkowski space is very similar to the general scheme of the Lagrangian formalism in mechanics which has been introduced in the previous section.

Each field is a tensor or tensor density whose components are smooth functions of the spacetime coordinates,

$$\phi^A \equiv \phi^A(x^\alpha). \tag{1.2.1}$$

The multi-index A stands for the tensor indices of the field (or a set of fields) which can be either covariant, or contravariant or a mixed type in a single piggyback notation. $x^\alpha = \{x^0, x^i\} = \{x^0, x^1, x^2, x^3\}$ are arbitrary coordinates in the Minkowski space with x^0 and x^i being the time and 3-dimensional spatial coordinates respectively. In what follows, we sometimes drop the coordinate index and use a simpler notation, $\phi^A = \phi^A(x)$ if it does not bring a confusion. A short description of tensors and tensor densities is given in Appendixes A.1 and A.3.

In the Lagrangian formalism of the field theory all four coordinates x^α are equivalent and play the role of independent arguments generalizing the independent argument of time t in the Lagrangian formalism of mechanics. The field components ϕ^A are treated as generalized coordinates and are genuine *dynamical variables*. The Lagrangian L of a physical field is, in most cases, a function of the dynamical variables and their first partial derivatives with respect to coordinates, $L = L(\phi^A, \phi^A{}_{,\alpha})$ where the comma with an index after it denotes a partial derivative with respect to the corresponding coordinate. In principle, the Lagrangian can also depend on coordinates x^α explicitly but we shall not consider this case because it complicates the mathematical formalism and it is not relevant to most of the physical cases. Dependence of the Lagrangian on higher derivatives of the field ϕ^A is allowed but in this section we don't consider this case.

The action functional of the field theory in the Minkowski space is originally defined in the Lorentzian coordinates

$$S = \int_\Omega d^4x L(\phi^A, \phi^A{}_{,\alpha}) \tag{1.2.2}$$

where $d^4x = dx^0 dx^1 dx^2 dx^3$ is the element of a coordinate volume, and Ω denotes a 4-dimensional domain of integration. Being a scalar, the action (1.2.2) has to be invariant under arbitrary coordinate transformations. Hence, any type of curvilinear coordinates in (1.2.2) are allowed and must be included to the Lagrangian formalism in a self-consistent way. This requires generalization of the concept of a partial derivative of tensor fields entering the action (1.2.2) since partial derivatives do not transform as tensors under arbitrary coordinate transformations. The derivative of a tensor field which transforms properly under arbitrary coordinate transformation is called the covariant derivative. The covariant derivative is an essential attribute of calculus on spacetime manifolds with curvature.

Let us introduce curvilinear coordinates $x'^{\alpha} = x'^{\alpha}(x^{\beta})$ which are functions of the Lorentzian coordinates x^{α}. The Minkowski interval (1.1.68) is invariant with respect to coordinate transformations

$$ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} = \gamma_{\alpha\beta} dx'^{\alpha} dx'^{\beta}, \tag{1.2.3}$$

where, here and everywhere else, $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(x')$ denotes the Minkowski metric in the curvilinear coordinates. Equation (1.2.3) defines the tensor law of transformation of the Minkowski metric tensor

$$\gamma_{\alpha\beta} = \eta_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x'^{\alpha}} \frac{\partial x^{\sigma}}{\partial x'^{\beta}}. \tag{1.2.4}$$

Coordinate transformations of other tensor quantities are explained in Appendix A.1.

For example, the Killing vectors (1.1.72) of the Minkowski space are transformed in accordance with the law of transformation of vectors:

$$\xi'^{\alpha}_{K} = \xi^{\rho}_{K} \frac{\partial x'^{\alpha}}{\partial x^{\rho}}. \tag{1.2.5}$$

The Killing vectors in curvilinear coordinates satisfy a covariant Killing vector equation which generalizes (1.1.70):

$$£_{\xi_K} \gamma_{\alpha\beta} = 0 \to \xi_{K\,(\alpha;\beta)} = 0 \tag{1.2.6}$$

where the indices are lowered or raised with the use of $\gamma_{\alpha\beta}$ and $\gamma^{\alpha\beta}$. The covariant version of equation (1.1.73) is, respectively,

$$\xi^{\alpha}_{\beta\,;\rho} = 0, \qquad \xi^{\alpha}_{[\beta\gamma]\,;\rho\sigma} = 0. \tag{1.2.7}$$

We note again that indices $\kappa = \{\beta, [\beta\gamma]\}$ numerate the Killing vectors, therefore the coordinate transformation does not act on them. In (1.2.7) and below, the semicolon with an index after it, denotes the covariant derivative in the Minkowski space with respect to the corresponding coordinate in accordance with notations and conventions adopted in Appendix A.3.

Briefly speaking, a covariant derivative of tensor density ϕ^A is defined as follows:

$$\phi^{A}{}_{;\mu} \equiv \phi^{A}{}_{,\mu} + C^{\pi}{}_{\mu\rho}\, \phi^{A}\big|^{\rho}_{\pi}, \tag{1.2.8}$$

where $C^{\pi}{}_{\mu\rho}$ denotes the Christoffel symbols constructed from the metric $\gamma_{\alpha\beta}$ and its partial derivatives:

$$C^{\pi}{}_{\mu\rho} = \frac{1}{2}\gamma^{\pi\sigma}\left(\gamma_{\sigma\mu,\rho} + \gamma_{\sigma\rho,\mu} - \gamma_{\mu\rho,\sigma}\right). \tag{1.2.9}$$

A permutation operator $\phi^{A}\big|^{\rho}_{\pi}$ is a complicated algebraic combination of the field ϕ^A contracted with the Kronecker symbols δ^{α}_{β} and originating from the transformation properties of the covariant derivative of the tensor field. For example, for a tensor density of the weight +1, $\phi^A \equiv \mathscr{M}_\sigma{}^{\alpha\beta}$, one has:

$$\mathscr{M}_\sigma{}^{\alpha\beta}\Big|^\rho_\pi = -\mathscr{M}_\sigma{}^{\alpha\beta}\delta^\rho_\pi - \mathscr{M}_\pi{}^{\alpha\beta}\delta^\rho_\sigma + \mathscr{M}_\sigma{}^{\rho\beta}\delta^\alpha_\pi + \mathscr{M}_\sigma{}^{\alpha\rho}\delta^\beta_\pi. \tag{1.2.10}$$

which should be used in (1.2.8) yielding

$$\mathscr{M}_\sigma{}^{\alpha\beta}{}_{;\mu} = \mathscr{M}_\sigma{}^{\alpha\beta}{}_{,\mu} - C^\pi{}_{\mu\pi}\mathscr{M}_\sigma{}^{\alpha\beta} - C^\pi{}_{\mu\sigma}\mathscr{M}_\pi{}^{\alpha\beta} + C^\alpha{}_{\mu\rho}\mathscr{M}_\sigma{}^{\rho\beta} + C^\beta{}_{\mu\rho}\mathscr{M}_\sigma{}^{\alpha\rho}. \tag{1.2.11}$$

The permutation operator $\phi^A\big|^\mu_\nu$ defined in (A.3.5) in the most general form, is quite universal and appears in all types of derivatives of tensor fields on geometric manifolds like the covariant derivative (A.3.7), Lie derivative (A.3.8), and an antisymmetric covariant derivative of tensor densities (A.3.9). The algebra of the permutation operator $\phi^A\big|^\mu_\nu$ is explained in Appendixes A.3.1 and A.3.2.

The action (1.2.2) depends on the infinitesimal volume of integration d^4x in the Lorentzian coordinates. We should establish its connection with the invariant measure of integration in curvilinear coordinates. To this end, let us consider two arbitrary coordinate charts in the Minkowski space, x^α and x'^α, connected by an invertible coordinate transformation, $x^\alpha = x^\alpha(x')$. The coordinate four-volumes, d^4x and d^4x', are related to each other by the determinant of the matrix of the transformation, $J = \det\left[\partial x'^\alpha/\partial x^\beta\right]$ also known as the *Jacobian* of the transformation,

$$d^4x' = Jd^4x. \tag{1.2.12}$$

Now let us consider the coordinate transformation of the metric tensor

$$\gamma_{\alpha\beta}(x) = \gamma'_{\mu\nu}(x')\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}. \tag{1.2.13}$$

Applying the rule of calculation of determinants from the product of matrices, yields

$$\gamma = J^2\gamma', \tag{1.2.14}$$

where $\gamma = \det \gamma_{\alpha\beta}(x)$ and $\gamma' = \det \gamma'_{\alpha\beta}(x')$. Equation (1.2.14) tells us that the sign of the determinant of the metric tensor is invariant under coordinate transformations. If we choose the Lorentzian coordinates the sign of the determinant of the metric tenor is $\det \eta_{\alpha\beta} = \eta = -1$. We conclude that the sign of the determinant of the metric tensor in arbitrary coordinates is always negative, $\gamma < 0$. Accounting for this fact, when extracting the root square from equation (1.2.14), and substituting the result into equation (1.2.12), brings about the equivalence

$$\sqrt{-\gamma'}d^4x' = \sqrt{-\gamma}d^4x. \tag{1.2.15}$$

It tells us that the invariant measure of integration on spacetime manifold is $\sqrt{-\gamma}d^4x$.

Now, we return to the formulation of action (1.2.2) in arbitrary coordinates. First of all, we notice that the measure of integration in the Lorentzian coordinates can be rewritten as $d^4x = \sqrt{-\eta}d^4x$. Second, the Lagrangian function is a scalar that depends on the tensor fields ϕ^A. To make it a scalar the tensor indices of the fields must be contracted with the help of the Minkowski metric. It means that the Lagrangian function includes the Minkowski metric explicitly: $L(\phi^A, \phi^A{}_{,\alpha}) = L(\phi^A, \phi^A{}_{,\alpha}, \eta_{\alpha\beta})$. As a result the action (1.2.2) is to be rewritten in the Lorentzian coordinates to the following form:

$$S = \int_\Omega d^4x\sqrt{-\eta}L\left(\phi^A, \phi^A{}_{,\alpha}, \eta_{\alpha\beta}\right). \tag{1.2.16}$$

Next, one transforms (1.2.16) to arbitrary curvilinear coordinates. In doing this transformation, the Minkowski metric, $\eta_{\alpha\beta}$, has to be replaced with its counterpart $\gamma_{\alpha\beta}$ in the curvilinear coordinates, and the partial derivatives of the dynamical variables, $\phi^A{}_{,\alpha}$, have to be replaced with the covariant ones, $\phi^A{}_{;\alpha}$. It brings the action (1.2.16) to the covariant form:

$$S = \int_\Omega d^4x\sqrt{-\gamma}L\left(\phi^A, \phi^A{}_{;\alpha}, \gamma_{\alpha\beta}\right) \equiv \int_\Omega d^4x\mathscr{L}\left(\phi^A, \phi^A{}_{;\alpha}, \gamma_{\alpha\beta}\right). \tag{1.2.17}$$

where $\mathscr{L} \equiv \sqrt{-\gamma}L$ is a scalar density of weight +1 which is more convenient to use in the applications of variational calculus on curved spacetime manifolds. In what follows, it is $\mathscr{L}$ which we call the *Lagrangian* instead of L. In the Lorentzian coordinates $\mathscr{L} = L$.

1.2.2 Variational field equations

The *principle of the least action* used in mechanics to obtain the equations of motion for particles can be also applied to a theory of continuous distribution of matter – *physical fields*, ϕ^A. The main idea of the principle is that among all virtually possible configurations of the fields under consideration only those are physically admissible (and stable) which correspond to a minimal value of the action S. Application of this principle gives us the field equations which are the analogs of the equations of motion of particles in mechanics.

We don't include in this section the metric tensor to the number of dynamical variables. Therefore, when applying the least action principle to S given in equation (1.2.17) one has to vary the fields ϕ^A but not the metric or coordinates. A corresponding variation of the fields is

$$\delta\phi^A \equiv \phi'^A(x) - \phi^A(x), \tag{1.2.18}$$

where the primed fields, ϕ'^A, are functions which are different from ϕ^A in the *most* general way.

Mathematically the variations of the type (1.2.18) are very convenient because, by definition, they commute with the operations of taking partial (but *not* covariant!) derivatives of the fields,

$$\delta\left(\partial_\alpha\phi^A\right) = \partial_\alpha\left(\delta\phi^A\right), \tag{1.2.19}$$

contraction of indices, etc.

The variation of the field variables in (1.2.18) leads to a variation of the action

$$\delta S = S' - S = \int_\Omega d^4x \mathscr{L}\left(\phi'^A, \phi'^A{}_{;\alpha}, \gamma_{\alpha\beta}\right) - \int_\Omega d^4x \mathscr{L}\left(\phi^A, \phi^A{}_{;\alpha}, \gamma_{\alpha\beta}\right). \tag{1.2.20}$$

The principle of the least action demands that physical system evolves along those trajectories of the variables, which conform to equation $\delta S = 0$. Recalling (1.2.19), expanding the Lagrangian in the first term of (1.2.20) into the Taylor series and keeping only the linear terms, the variation of the action takes the form

$$\delta S = \int_\Omega d^4x \left[\frac{\partial \mathscr{L}}{\partial \phi^A}\delta\phi^A + \frac{\partial \mathscr{L}}{\partial \phi^A{}_{,\alpha}}\delta\phi^A{}_{,\alpha}\right]. \tag{1.2.21}$$

Using partial derivatives permits us to make use of (1.2.19) to integrate the second term on the right side of (1.2.21) by parts. Then, we assume that variations of the field variables giving rise to variation δS vanish on the boundary $\partial\Omega$ of the domain of integration Ω in (1.2.20):

$$\delta\phi^A\big|_{\partial\Omega} = 0. \tag{1.2.22}$$

This assumption allows us to discard the surface terms as they vanish on the boundary $\partial\Omega$.

Finally, one arrives at

$$\delta S = \int_\Omega d^4x \frac{\delta \mathscr{L}}{\delta \phi^A}\delta\phi^A, \tag{1.2.23}$$

where the expression

$$\frac{\delta \mathscr{L}}{\delta \phi^A} \equiv \frac{\partial \mathscr{L}}{\partial \phi^A} - \frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathscr{L}}{\partial \phi^A{}_{,\alpha}}\right) \tag{1.2.24}$$

is called the *Lagrangian derivative* [266]. In the case, when the Lagrangian depends on the second derivatives of the field variable, the Lagrangian derivative is defined by equation (A.2.38) in Appendix A.2.4. The principle of the least action demands $\delta S = 0$ for an arbitrary variation $\delta\phi^A$. The only possible way to satisfy this principle, is to demand the vanishing of the Lagrangian derivative of $\mathscr{L}$,

$$\frac{\delta \mathscr{L}}{\delta \phi^A} = 0. \tag{1.2.25}$$

Equation (1.2.25) is the Euler-Lagrange equation generalizing the mechanical analog of this equation (1.1.10) to the field theory.

The Lagrangian derivative (1.2.24) is given in terms of the partial derivative of the Lagrangian with respect to the partial derivatives of the field. It can be re-formulated in terms of the covariant derivatives making it apparently covariant. Indeed, let us represent the Lagrangian $\mathscr{L}$ in the form where the partial derivative of ϕ^A appears explicitly:

$$\mathscr{L}\left(\phi^A, \phi^A{}_{;\alpha}, \gamma_{\alpha\beta}\right) = \mathscr{L}\left(\phi^A,\ \phi^A{}_{,\alpha} + C^\nu{}_{\alpha\mu}\ \phi^A\big|^\mu_\nu, \gamma_{\alpha\beta}\right), \tag{1.2.26}$$

where we have used the general definition (1.2.8) for covariant derivatives. Then, the first term in (1.2.24) has the following form:

$$\frac{\partial\mathscr{L}}{\partial\phi^A} = \frac{\partial^*\mathscr{L}}{\partial\phi^A} + \frac{\partial\mathscr{L}}{\partial\phi^B{}_{;\alpha}}\frac{\partial(C^\nu{}_{\alpha\mu}\ \phi^B\big|^\mu_\nu)}{\partial\phi^A} = \frac{\partial^*\mathscr{L}}{\partial\phi^A} - \left(\frac{\partial\mathscr{L}}{\partial\phi^A{}_{;\alpha}}\right)\bigg|^\mu_\nu C^\nu{}_{\alpha\mu}\ , \tag{1.2.27}$$

where the symbol $*$ means a partial derivative with respect to the filed ϕ^A with the covariant derivative of the filed $\phi^A{}_{;\alpha}$ being fixed; and the second term in (1.2.27) has been calculated with the use of the algebra of the permutation operators $|^\mu_\nu$ given in Appendix A.3.2 (we have used (A.3.16) and (A.3.23)). Equation (1.2.26) also points out that

$$\frac{\partial\mathscr{L}}{\partial\phi^A{}_{,\alpha}} = \frac{\partial\mathscr{L}}{\partial\phi^A{}_{;\alpha}}. \tag{1.2.28}$$

Then, after substituting (1.2.27) and (1.2.28) in the Lagrangian derivative (1.2.24) and taking into account the definition of the covariant derivative (1.2.8), one can recast (1.2.24) in terms of the covariant derivatives

$$\frac{\delta\mathscr{L}}{\delta\phi^A} \equiv \frac{\partial^*\mathscr{L}}{\partial\phi^A} - \left(\frac{\partial\mathscr{L}}{\partial\phi^A{}_{;\alpha}}\right)_{;\alpha}, \tag{1.2.29}$$

quod erat demonstrandum. It is worth emphasizing that, in fact, all equations of variational analysis in curvilinear coordinates of the Minkowski space and, more generally, on curved manifolds, can be written down in terms of covariant derivatives of the Lagrangian instead of its partial derivatives. The proof can be accomplished by direct calculations given, for example, in [316, 430].

Even though the derivation of the field equations for ϕ^A does not involve variation of the Lagrangian (1.2.26) with respect to the metric tensor $\gamma_{\alpha\beta}$, it will be useful in the calculations that follow. Therefore, we give it here. The Lagrangian derivative with respect to the metric tensor is defined similarly to (1.2.24):

$$\frac{\delta\mathscr{L}}{\delta\gamma_{\rho\sigma}} \equiv \frac{\partial\mathscr{L}}{\partial\gamma_{\rho\sigma}} - \frac{\partial}{\partial x^\alpha}\frac{\partial\mathscr{L}}{\partial\gamma_{\rho\sigma,\alpha}}. \tag{1.2.30}$$

We calculate the partial derivative of $\mathscr{L}$ with respect to the metric tensor with the technique being similar to that shown in equation (1.2.27). It brings (1.2.30) to the form

$$\frac{\delta\mathscr{L}}{\delta\gamma_{\rho\sigma}} = \frac{\partial^*\mathscr{L}}{\partial\gamma_{\rho\sigma}} + \frac{\partial\mathscr{L}}{\partial\phi^B{}_{;\alpha}}\frac{\partial(C^\nu{}_{\alpha\mu}\ \phi^B\big|^\mu_\nu)}{\partial\gamma_{\rho\sigma}} - \left(\frac{\partial\mathscr{L}}{\partial\phi^B{}_{;\beta}}\frac{\partial(C^\nu{}_{\beta\mu}\ \phi^B\big|^\mu_\nu)}{\partial\gamma_{\rho\sigma,\alpha}}\right)_{,\alpha}. \tag{1.2.31}$$

Then, after using the definition (1.2.9) to calculate the partial derivatives from the Christoffel symbols, we transform (1.2.30) to an explicitly covariant form:

$$\frac{\delta \mathscr{L}}{\delta \gamma_{\rho\sigma}} = \frac{\partial^* \mathscr{L}}{\partial \gamma_{\rho\sigma}} - \frac{1}{2}\left[\frac{\partial \mathscr{L}}{\partial \phi^B{}_{;\beta}} \phi^B\Big|^\mu_\nu \left(\gamma^{\nu(\rho}\delta^{\sigma)}_\beta \delta^\alpha_\mu + \gamma^{\nu(\rho}\delta^{\sigma)}_\mu \delta^\alpha_\beta - \gamma^{\alpha\nu}\delta^{(\rho}_\mu \delta^{\sigma)}_\beta\right)\right]_{;\alpha} . \tag{1.2.32}$$

This expression is a particular case of the Lagrangian derivative with respect to the metric given in Appendix A.3.3 (see (A.3.44)) where we use the notation $g_{\alpha\beta}$ for the general case of the metric tensor on manifold instead of $\gamma_{\alpha\beta}$ which is the Minkowski metric in curvilinear coordinates.

At last, it is important to remark that any covariant derivative in the Lagrangian which can be represented as a divergence, does not contribute to the Lagrangian derivative (1.2.24) and, consequently, does not change the field equations (1.2.25). One can see this, following a simple logic. Add to the Lagrangian $\mathscr{L}$ a divergence: $\mathscr{L} \to \mathscr{L} + \text{div}$, where $\text{div} = \mathscr{D}^\alpha{}_{,\alpha}$ from a vector density $\mathscr{D}^\alpha = \mathscr{D}^\alpha(\phi^A, \gamma_{\mu\nu})$. When $\mathscr{D}^\alpha$ vanishes on the boundary of integration, it preserves the numerical value of the action. The divergence of the vector field does not contain the second derivatives and is written as

$$\mathscr{D}^\alpha{}_{,\alpha} = \frac{\partial \mathscr{D}^\alpha}{\partial \phi^A} \phi^A{}_{,\alpha} + \frac{\partial \mathscr{D}^\alpha}{\partial \gamma_{\mu\nu}} \gamma_{\mu\nu,\alpha}. \tag{1.2.33}$$

Applying the operator of the Lagrangian derivative (1.2.24) to this expression, one easily obtains

$$\frac{\delta(\partial_\alpha \mathscr{D}^\alpha)}{\delta \phi^A} \equiv 0, \tag{1.2.34}$$

$$\frac{\delta(\partial_\alpha \mathscr{D}^\alpha)}{\delta \gamma_{\mu\nu}} \equiv 0. \tag{1.2.35}$$

The identities, like (1.2.34) and (1.2.35), are also valid in more general cases as explained in Appendix A.2.4 (see, for instance, formulae (A.2.40) and (A.2.41)).

1.2.3 The Noether theorems

We discuss in this section two important theorems on conserved laws in the field theory formulated and proved by Emmy Noether. These theorems find numerous applications and, for the sake of generality, we consider them in the case of the Lagrangian that depends not only on the first but also on the second derivatives of the field variables, thus, extending the Lagrangian (1.2.17):

$$S = \int_\Omega d^4x \mathscr{L}\left[\psi^A(x), \psi^A{}_{,\alpha}(x), \psi^A{}_{,\alpha\beta}(x)\right]. \tag{1.2.36}$$

where the set of the dynamical variables $\psi^A = \{\phi^B, m_{\alpha\beta}\}$ represents both the matter and metric fields. Here, the tensor $m_{\alpha\beta}$ can be either the Minkowski metric $\eta_{\alpha\beta}$, or the

Minkowski metric in curved coordinates $\gamma_{\alpha\beta}$, or a metric $\bar{g}_{\alpha\beta}$ of a background curved spacetime, or a dynamical metric $g_{\alpha\beta}$ on a curved spacetime manifold. We have shown the arguments of the Lagrangian as being chosen in the form of partial derivatives because they are more convenient for doing variational calculations due to the property (1.2.19). Of course, the Lagrangian is a covariant scalar density and depends in reality on the covariant derivatives of the fields.

Let us consider an arbitrary group of infinitesimal transformations of the coordinates and the field variables, generalizing the group (1.1.15) in mechanics. Restricting our consideration to a linear approximation, we represent the variations of the coordinates and the fields under such transformations as

$$\delta x^\alpha = x'^\alpha - x^\alpha, \tag{1.2.37}$$

$$\delta' \psi^A(x) = \psi'^A(x') - \psi^A(x), \tag{1.2.38}$$

which generalize (1.1.18) and (1.1.17), respectively. In the framework of the primed system, one can shift to the previous values of the coordinates, $x' \to x$,

$$\psi'^A(x + \delta x) = \psi'^A(x) + \frac{\partial \psi^A(x)}{\partial x^\alpha}\delta x^\alpha. \tag{1.2.39}$$

Now we introduce the other variation of the field variables:

$$\delta\psi^A(x) = \psi'^A(x) - \psi^A(x) = \delta'\psi^A(x) - \frac{\partial \psi^A(x)}{\partial x^\alpha}\delta x^\alpha \tag{1.2.40}$$

that is infinitesimal as well. The advantage of (1.2.40) with respect to the variation defined in (1.2.38) is that it commutes with the partial derivatives

$$\delta\left(\partial_\alpha \psi^A\right) = \partial_\alpha\left(\delta\psi^A\right), \tag{1.2.41}$$

as in (1.2.19) for arbitrary variations defined in (1.2.18). We emphasize, at the same time, the variation (1.2.38) does not commute with covariant derivatives.

Transformations (1.2.37), (1.2.38) induce a perturbation of the action (1.2.36), $\delta' S = S' - S$ which in linear approximation reads

$$\begin{aligned}\delta' S = {} & \int_{\Omega'} d^4x' \mathscr{L}' \left[\psi'^A(x'), \psi'^A{}_{,\alpha}(x'), \psi'^A{}_{,\alpha\beta}(x')\right] \\ & - \int_\Omega d^4x \mathscr{L}\left[\psi^A(x), \psi^A{}_{,\alpha}(x), \psi^A{}_{,\alpha\beta}(x)\right].\end{aligned} \tag{1.2.42}$$

As the coordinates x' are dummy arguments of integration, and since the change in the boundary Ω' is infinitesimal by assumption, the two integrals in (1.2.42) can be transformed to

$$\begin{aligned}\delta' S = & \int_\Omega d^4x \mathscr{L}' \left[\psi'^A(x), \psi'^A{}_{,\alpha}(x), \psi'^A{}_{,\alpha\beta}(x)\right] \\ & - \int_\Omega d^4x \mathscr{L} \left[\psi^A(x), \psi^A{}_{,\alpha}(x), \psi^A{}_{,\alpha\beta}(x)\right] \\ & + \oint_{\partial\Omega} ds_\mu \delta x^\mu \mathscr{L} \left[\psi^A(x), \psi^A{}_{,\alpha}(x), \psi^A{}_{,\alpha\beta}(x)\right],\end{aligned} \tag{1.2.43}$$

where $\partial\Omega$ is the boundary of the four-dimensional domain of integration, and ds_μ is a 3-dimensional element of the integration on the boundary. Equation (1.2.43) can be recast to the following form by making use of four-dimensional divergence theorem

$$\delta' S = \int_\Omega d^4x \left[\delta\mathscr{L} + \frac{\partial(\mathscr{L}\delta x^\alpha)}{\partial x^\alpha}\right]. \tag{1.2.44}$$

The variation of the Lagrangian

$$\delta\mathscr{L} = \mathscr{L}' \left[\psi'^A(x), \psi'^A{}_{,\alpha}(x), \psi'^A{}_{,\alpha\beta}(x)\right] - \mathscr{L} \left[\psi^A(x), \psi^A{}_{,\alpha}(x), \psi^A{}_{,\alpha\beta}(x)\right] \tag{1.2.45}$$

generalizes both (1.2.44) and (1.1.26), and we have introduced a shorthand notation for $\mathscr{L} \equiv \mathscr{L} \left[\psi^A(x), \psi^A{}_{,\alpha}(x), \psi^A{}_{,\alpha\beta}(x)\right]$ which we shall also use in the text that follows.

Now, we demand the invariance of the action (1.2.36) with respect to transformations of the group which means setting $\delta' S = 0$. It makes the integral on the right side of (1.2.44) vanish which means that its integrand must be zero. However, the right hand side can be amended with a divergence of a vector density $\mathscr{B}^\alpha$ that is chosen in the form of a solenoidal (divergenceless) field,

$$\delta\mathscr{L} + \frac{\partial(\mathscr{L}\delta x^\alpha)}{\partial x^\alpha} \equiv \frac{\partial\mathscr{B}^\alpha}{\partial x^\alpha}, \tag{1.2.46}$$

where, $\mathscr{B}^\alpha = \mathscr{B}^\alpha(\psi^A, \psi^A{}_{,\alpha})$ is a vector density of weight +1. Although the divergence in the right side of (1.2.46) vanishes identically it does not mean that the field $\mathscr{B}^\alpha$ is nil itself. Indeed, it can be always chosen in the form of a divergence $\mathscr{B}^\alpha \equiv \boldsymbol{b}^{\alpha\beta}{}_{,\beta}$ from a skew-symmetric tensor density $\boldsymbol{b}^{\alpha\beta} = \boldsymbol{b}^{[\alpha\beta]} \neq 0$. Such divergenceless vector fields play an essential role in constructing conserved quantities. For this reason, and for the sake of generality, we continue to consider the vector density $\mathscr{B}^\alpha$ explicitly. We have to note that $\mathscr{B}^\alpha$ can be either a part of or have no relation to the Lagrangian under consideration. Nonetheless, by using it, one can modify and correct the conserved quantities derived from the Lagrangian by the direct Noether's procedure.

It is necessary to remark the following. Some authors permit on the right hand side of the identity (1.2.46) a divergence that does not vanish identically,

$$\delta\mathscr{L} + \frac{\partial(\mathscr{L}\delta x^\alpha)}{\partial x^\alpha} \equiv \frac{\partial\mathscr{B}_1^\alpha}{\partial x^\alpha}, \tag{1.2.47}$$

see, for example, [240]. Indeed, sometimes symmetries under consideration lead just to an identity of the type (1.2.47), not to (1.2.46). However, because in this book we do not consider models which lead to the identities of the type (1.2.47) we do not use (1.2.47) in what follows.

The variation of the Lagrangian in (1.2.45), $\delta\mathscr{L}$, can be represented in terms of the variations of its arguments:

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\psi^A}\delta\psi^A + \frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu}}\delta(\psi^A{}_{,\mu}) + \frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu\nu}}\delta(\psi^A{}_{,\mu\nu}). \tag{1.2.48}$$

Because the operations of taking the δ-variation and the partial derivatives commute, the variation (1.2.48) of the Lagrangian can be reshuffled and presented in the following form:

$$\delta\mathscr{L} = \frac{\delta\mathscr{L}}{\delta\psi^A}\delta\psi^A + \partial_\mu\left[\frac{\delta\mathscr{L}}{\delta\psi^A{}_{,\mu}}\delta\psi^A + \frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu\nu}}\delta\psi^A{}_{,\nu}\right], \tag{1.2.49}$$

where the notation

$$\frac{\delta\mathscr{L}}{\delta\psi^A} = \frac{\partial\mathscr{L}}{\partial\psi^A} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu}}\right) + \partial_{\mu\nu}\left(\frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu\nu}}\right) \tag{1.2.50}$$

stands for the Lagrangian derivative generalizing the Lagrangian derivative (1.2.24) to the case of the Lagrangians depending on the second derivatives of the field variables (for more detail, see (A.2.38) and (A.2.42) in Appendix A.2.4), and we have used shorthand notations for the operators of partial derivatives $\partial_\nu \equiv \partial/\partial x^\nu$, $\partial_{\mu\nu} \equiv \partial_\mu\partial_\nu$. We have also defined a Lagrangian derivative with respect to the partial derivatives of the field variables,

$$\frac{\delta\mathscr{L}}{\delta\psi^A{}_{,\mu}} = \frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu}} - \partial_\nu\frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\nu\mu}}. \tag{1.2.51}$$

Finally, substituting (1.2.49) into the identity (1.2.46), one brings it to the following form,

$$\frac{\delta\mathscr{L}}{\delta\psi^A}\delta\psi^A \equiv \partial_\nu\left[-\frac{\delta\mathscr{L}}{\delta\psi^A{}_{,\nu}}\delta\psi^A - \frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu\nu}}\delta\psi^A{}_{,\mu} - \mathscr{L}\delta x^\nu + \mathscr{B}^\nu\right]. \tag{1.2.52}$$

This equation is customarily called the *main Noether's identity*.

Notice that deriving the identity (1.2.52) one does not impose any limitations on the field variables besides their differentiability. Furthermore, derivation of (1.2.52) does not set any constraints on the variations of the variables at the boundary of the integration domain Ω in (1.2.43). The vector density $\mathscr{B}^\alpha$ is divergenceless, $\partial_\alpha\mathscr{B}^\alpha = 0$, but otherwise remains arbitrary inside the domain of integration of the action. Its particular choice depends on the physical problem under consideration. For example, the procedure of symmetrization of the canonical energy-momentum tensor which will be discussed in Section 1.2.4 will lead to a specific choice of this vector field.

For particular applications of the Noether identity (1.2.52) we have to specify the transformation group (1.2.37), (1.2.38) of the dynamical variables and use it for calculating the variations entering (1.2.52). We consider this procedure in the next Sections.

The first Noether's theorem

The first Noether's theorem is formulated as follows.

– *If the action functional, S, is invariant with respect to transformations of a finite n-parameter (with n constant parameters ε^a) Lie group G_n, then, there exist n linearly independent identities relating the Lagrangian derivative to a divergence of a vector density field.*

To prove it we use the general identity (1.2.52). Transformations of coordinates and field variables (1.2.37) and (1.2.38) making up the finite group G_n, are linearly proportional to coordinates and field variables,

$$\delta x^\alpha = \varepsilon^a \hat{X}_a x^\alpha, \tag{1.2.53}$$

$$\delta' \psi^A = \varepsilon^a \hat{\Psi}_a \psi^A, \tag{1.2.54}$$

where ε^a are arbitrary *constant* parameters with the group index $a = 1, \dots, n$, while $\hat{X}_a$ and $\hat{\Psi}_a$ are operators generating the transformations of the group G_n but *not* depending on the fields ψ^A and/or their derivatives.

Let us now pick up the generators $\hat{X}_a$ in the form of partial derivatives, $\hat{X}_a = \xi^\mu_a \partial_\mu$, contracted with n vector fields, $\xi^\mu_a = \xi^\mu_a(x)$, of displacements. Then, the perturbations (1.2.53) and (1.2.40) which enter (1.2.52), take the form:

$$\delta x^\alpha = \varepsilon^a \xi^\alpha_a, \tag{1.2.55}$$

$$\delta \psi^A = \varepsilon^a \left(\hat{\Psi}_a \psi^A - \xi^\mu_a \partial_\mu \psi^A\right) . \tag{1.2.56}$$

Substituting (1.2.55) and (1.2.56) into (1.2.52), where we set for a divergenceless vector density $\mathscr{B}^\mu \equiv \varepsilon^a \mathscr{B}^\mu_a$, and dropping off the constant parameters ε^a, one obtains the identity

$$\left(\hat{\Psi}_a \psi^A - \xi^\mu_a \partial_\mu \psi^A\right) \frac{\delta \mathscr{L}}{\delta \psi^A} \equiv \partial_\mu \left(\mathscr{J}^\mu_a + \mathscr{B}^\mu_a\right). \tag{1.2.57}$$

The set of n quantities $\mathscr{J}^\mu_a$ presents the Noether currents:

$$\begin{aligned}\mathscr{J}^\mu_a = - \Bigg[& \frac{\delta \mathscr{L}}{\delta \psi^A{}_{,\mu}} \left(\hat{\Psi}_a \psi^A - \xi^\alpha_a \partial_\alpha \psi^A\right) \\ & + \frac{\partial \mathscr{L}}{\partial \psi^A{}_{,\mu\nu}} \left(\hat{\Psi}_a \psi^A{}_{,\nu} - \xi^\alpha_a \psi^A{}_{,\nu\alpha}\right) + \mathscr{L} \xi^\mu_a \Bigg] .\end{aligned} \tag{1.2.58}$$

Equations (1.2.57) *and* (1.2.58) *prove the first Noether's theorem.* There also exists the inverse theorem [266] but we do not consider it here.

A direct consequence of the first Noether's theorem is that if the action functional is invariant with respect to the group transformation and the Lagrangian derivative on the left side of (1.2.57) vanishes, then, the corresponding Noether currents $\mathscr{J}_a^\mu$ are conserved,

$$\partial_\mu \mathscr{J}_a^\mu = 0. \tag{1.2.59}$$

The corrected currents are also conserved:

$$\partial_\mu \left(\mathscr{J}_a^\mu + \mathscr{B}_a^\mu \right) = 0. \tag{1.2.60}$$

Now, we recall that $\psi^A = \{\phi^B, m_{\alpha\beta}\}$. Then, if the metric $m_{\alpha\beta}$ is a fixed (background) field, then the Lagrangian derivative $\delta\mathscr{L}/\delta\psi^A$ does not vanish because $\delta\mathscr{L}/\delta m_{\alpha\beta} \neq 0$. However, even in this case the conservation laws analogous to (1.2.59) or (1.2.60) can be established. Such a possibility will be shown, for example, in deriving the conservation law (1.2.99).

The second Noether's theorem

A local gauge group, an *infinite Lie group*, can be obtained from a finite group G_n by replacing the constant group's parameters ε^a by n continuous fields $\varepsilon^a(x)$, then $G_n \to G_{\infty n}$. Such a procedure is called the group's *localization*. For example, the group of coordinate transformations in spacetime is a Lie group $G_{\infty 4} := x'^\alpha = f^\alpha(x)$, where $f^\alpha(x)$ are smooth differentiable functions.

The infinite Lie groups are used to study the continuous symmetries of the action functional S. It turns out that these symmetries are directly associated with a set of certain differential equations. This is the essence of the *second Noether's theorem* which is formulated as follows:

– *If the action functional, S, is invariant with respect to transformations of a Lie group $G_{\infty n}$ parametrized by n differentiable fields and their derivatives up to the order k, then there exist n identical relations between the Lagrangian derivatives and derivatives from them up to the order k.*

Let us prove this theorem. Assume that the action (1.2.40) is invariant under the action of transformations of the group $G_{\infty n}$. Let δx^α and $\delta'\psi^A$ are linearly proportional to functions $\varepsilon^a(x)$ $(a = 1, 2, \ldots, n)$ and their derivatives. This extends the case of a finite group G_n to the generators which are continuously differentiable functions. For the sake of simplicity we consider here only the first derivatives of $\varepsilon^a(x)$ $(k = 1)$ as the case of the higher derivatives $(k > 1)$ is technically similar but more tedious. Thus, generalization of (1.2.55) and (1.2.56) to the case of the Lie group is:

$$\delta x^\alpha = \varepsilon^a(x)\xi_a^\alpha, \tag{1.2.61}$$

$$\delta\psi^A = \varepsilon^a(x)\omega_a^A(\psi, \psi_{,\alpha}, \psi_{,\alpha\beta}) + \frac{\partial\varepsilon^a(x)}{\partial x^\alpha}\omega_a^{A\alpha}(\psi, \psi_{,\alpha}, \psi_{,\alpha\beta}), \tag{1.2.62}$$

where again, $\xi_a^\mu = \xi_a^\mu(x)$, and we have included all terms depending on the derivatives of the field variables to functions ω_a^A and $\omega_a^{A\alpha}$ defining the group structure.

Substituting (1.2.61) and (1.2.62) into the main Noether's identity (1.2.52), where we set for a divergenceless vector density $\mathscr{B}^\mu \equiv \varepsilon^a(x)\mathscr{B}_a^\mu$, and making use of the Leibniz rule for differentiation by parts, result in:

$$\left[\omega_a^A \frac{\delta\mathscr{L}}{\delta\psi^A} - \partial_\mu\left(\omega_a^{A\mu}\frac{\delta\mathscr{L}}{\delta\psi^A}\right)\right]\varepsilon^a \equiv \partial_\nu\left(\mathscr{I}^\nu + \varepsilon^a\mathscr{B}_a^\nu\right). \tag{1.2.63}$$

Here the current $\mathscr{I}^\nu$ is defined as

$$\mathscr{I}^\nu \equiv -\left\{\frac{\delta\mathscr{L}}{\delta\psi^A{}_{,\nu}}\left[\varepsilon^a\omega_a^A + \frac{\partial\varepsilon^a}{\partial x^\alpha}\omega_a^{A\alpha}\right] + \frac{\partial\mathscr{L}}{\partial\psi^A{}_{,\mu\nu}}\partial_\mu\left[\varepsilon^a\omega_a^A + \frac{\partial\varepsilon^a}{\partial x^\alpha}\omega_a^{A\alpha}\right]\right.$$
$$\left. + \varepsilon^a\left(\mathscr{L}\xi_a^\nu + \omega_a^{A\nu}\frac{\delta\mathscr{L}}{\delta\psi^A}\right)\right\}. \tag{1.2.64}$$

Now, we integrate this identity, apply the Gauss's theorem, and assume that $\varepsilon^a(x)$ and their derivatives vanish at the boundary of the integration domain Ω. This yields the integral identity,

$$\int_\Omega d^4x\left[\omega_a^A\frac{\delta\mathscr{L}}{\delta\psi^A} - \partial_\mu\left(\omega_a^{A\mu}\frac{\delta\mathscr{L}}{\delta\psi^A}\right)\right]\varepsilon^a(x) \equiv 0. \tag{1.2.65}$$

Because $\varepsilon^a(x)$ are arbitrary functions inside the domain of integration, one obtains a set of n differential equations for the Lagrangian derivatives (of the first order in the case under consideration)

$$\partial_\mu\left(\omega_a^{A\mu}\frac{\delta\mathscr{L}}{\delta\psi^A}\right) - \omega_a^A\frac{\delta\mathscr{L}}{\delta\psi^A} \equiv 0. \tag{1.2.66}$$

This proves the second Noether's theorem. There is also an inverse theorem [266] but we do not consider it here.

Let us define in the space of the Lie group the inverse object ${}^{-1}\omega_{B\mu}^b$ such that ${}^{-1}\omega_{B\mu}^b\omega_b^{A\mu} \equiv \delta_B^A$, where δ_B^A is the tensor product of the Kronecker symbols with the indices belonging to the space of the field variables. Then, the differential equation (1.2.66) can be presented in a covariant form:

$$D_\mu\left(\omega_a^{A\mu}\frac{\delta\mathscr{L}}{\delta\psi^A}\right) \equiv 0\ ; \tag{1.2.67}$$

$$D_\mu \equiv \partial_\mu - G^b{}_{a\mu}, \quad G^b{}_{a\mu} \equiv {}^{-1}\omega_{B\mu}^b\omega_a^B$$

where D_μ means a covariant derivative constructed with the help of a generalized connection $G^b{}_{a\mu}$ introduced in the space of the Lie group.

The important point to notice is that equation (1.2.63) not only associates the Lagrangian derivatives with the set of identities (1.2.66), the same (1.2.67), but also states that the current (1.2.64) is conserved identically,

$$\partial_\nu \mathscr{I}^\nu \equiv 0. \tag{1.2.68}$$

The vector field $\mathscr{B}^\nu$ entering (1.2.63) is divergenceless, $\partial_\nu \mathscr{B}^\nu = 0$, on its own. Therefore the corrected current is conserved identically as well,

$$\partial_\nu \left[\mathscr{I}^\nu + \varepsilon^a(x)\mathscr{B}^\nu_a\right] \equiv 0. \tag{1.2.69}$$

Concluding the point, it is necessary to make some remarks.

First, setting in (1.2.61) and (1.2.62) $\varepsilon(x) = \varepsilon = \text{const}$, one finds that the identity (1.2.63) is simplified to the identity (1.2.57), current (1.2.64) is simplified to the currents (1.2.58).

Second, the identities of the general type (1.2.66) and (1.2.68) are the basis for constructing the Klein and the Klein-Noether systems of identities and constructing superpotentials, see Sections 1.4.1, 6.1.2 and 7.1.1.

Third, the currents in (1.2.68) and (1.2.69) are conserved identically, independently on whether the equations of motion are satisfied or not. If the equations of motion hold and the Lagrangian derivative $\delta\mathscr{L}/\delta\psi^A$ disappears from the expression (1.2.64) then the current, $\mathscr{I}^\mu$, transforms to the form $\mathscr{J}^\mu$ given in (1.2.58), and the identities (1.2.68) and (1.2.69) become physically sensible conservation laws:

$$\partial_\nu \mathscr{J}^\nu = 0, \tag{1.2.70}$$

$$\partial_\nu \left[\mathscr{J}^\nu + \varepsilon^a(x)\mathscr{B}^\nu_a\right] = 0. \tag{1.2.71}$$

Fourth, a possibility to include divergenceless vector density $\mathscr{B}^\mu = \varepsilon^a(x)\mathscr{B}^\mu_a$ into the current (1.2.64) allows us to develop the procedure of the Belinfante symmetrization of the canonical energy-momentum tensor of perturbations in general relativity and arbitrary metric theories, see Sections 6.2 and 7.2.3, respectively.

Diffeomorphisms and the Lie derivatives

Until now, we have not yet specified the group of transformations that leaves the action functional invariant. The corresponding variations of the field variables generated by the action of the group can be split in two categories:

- *Intrinsic variations*. They are generated by the gauge transformations of the dynamical field variables which change their functional form in the corresponding functional space. The intrinsic variations are not related to coordinate transformations *at all*, and do not change the values of the background fields.
- *Extrinsic variations*. They are generated by coordinate transformations. They change both the functional form of the dynamic variables and that of the background fields.

The intrinsic and extrinsic variations are associated with different symmetries of the physical system and should be clearly distinguished one from another. In this book we focus primarily to the metric theories of gravity and pay more attention to the symmetries and corresponding conserved quantities generated by the extrinsic variations. Our classification of the field variations coincide with the anatomy of variations in the Section 3.9.4.1 in the book [267].

Let us make an infinitesimally small deformation of coordinates in the spacetime manifold $\mathscr{M}$:

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x), \tag{1.2.72}$$

where a smooth vector field ξ^{α} defines a congruence of integral curves along which the coordinate grid is dragged. We demand that the transformation (1.2.72) preserves the local differentiable structure of the manifold $\mathscr{M}$. Such a differentiable mapping from the manifold $\mathscr{M}$ to a new manifold $\mathscr{M}'$ is called a *diffeomorphism*. In what follows, we assume that ξ^{α} vanishes on the boundary of integration of the action. Recall also that infinitesimal deformation (1.2.72) generalizes the time shift (1.1.18) in mechanics.

Under the action of the diffeomorphism (1.2.72) all geometrical objects, say $\psi^A(x)$, residing on the manifold $\mathscr{M}$ are mapped to the objects, $\psi'^A(x')$, residing on manifold $\mathscr{M}'$ in accordance with their transformation properties. In the linear approximation the change is given by

$$\psi'^A(x') = \psi^A(x) + \psi^A(x)\big|^{\mu}_{\nu}\,\partial_{\mu}\xi^{\nu}, \tag{1.2.73}$$

where the permutation operator $\psi^A(x)\big|^{\mu}_{\nu}$ is identical to that in (1.2.8) and defined by the transformation properties of $\psi^A(x)$ (see Appendices A.1 and A.3.1 for more detail). Diffeomorphism (1.2.72) and transformation (1.2.73) correspond to the infinitesimal perturbations (1.2.53) and (1.2.54) of coordinates and the field variables respectively:

$$\delta x^{\mu} = \xi^{\mu}(x), \tag{1.2.74}$$

$$\delta'\psi^A(x) = \psi^A(x)\big|^{\mu}_{\nu}\,\partial_{\mu}\xi^{\nu}. \tag{1.2.75}$$

The transformed fields $\psi'^A(x')$ reside on the deformed manifold $\mathscr{M}'$ and cannot be directly compared with the objects on the manifold $\mathscr{M}$. To compare the geometric objects in $\mathscr{M}'$ with those in $\mathscr{M}$, one has to pull them back from $\mathscr{M}'$ to $\mathscr{M}$ by making use of the, so-called, *Lie dragging* or *Lie displacement* along the integral curves of the vector field ξ^{α}. This procedure consists of two parts: first, we transform the geometric object ψ^A in accordance with equation (1.2.73), and, second, we shift the argument of $\psi'^A(x')$ from the point x' to the point x by making use of the Taylor expansion, so that

$$\psi'^A(x') = \psi'^A(x) + \xi^{\alpha}\partial_{\alpha}\psi^A(x) + O\left(\xi^2\right). \tag{1.2.76}$$

After that we drop off all terms which are non-linear in ξ^{α} and compare $\psi'^A(x)$ with $\psi^A(x)$ at the same point of $\mathscr{M}$. The difference

$$\delta\psi^A(x) = \psi'^A(x) - \psi^A(x), \tag{1.2.77}$$

between $\psi'^A(x)$ and $\psi^A(x)$ is the standard definition of the *Lie derivative* of $\psi^A(x)$, see e. g. [409],

$$\delta\psi^A \equiv \pounds_\xi \psi^A. \tag{1.2.78}$$

We would like to draw the attention of the reader to the fact that for historical reasons definition of the Lie derivative (1.2.78) which we use in this book, has an opposite sign with respect to the definition used in standard mathematical textbooks on differential geometry like [409]. It is also useful to point out that variation (1.2.77) has been already used in our calculations, see (1.1.26–1.1.27) and (1.2.40). In case of (1.2.78), this variation is fully extrinsic as it is induced by the *diffeomorphism* (1.2.72).

Explicit expression for the Lie derivative (1.2.78) can be obtained after combining (1.2.73) and (1.2.76–1.2.78) which yields:

$$\pounds_\xi \psi^A \equiv -\xi^\mu \partial_\mu \psi^A + \psi^A\big|^\mu_\nu \partial_\mu \xi^\nu. \tag{1.2.79}$$

This expression is given in terms of the partial derivatives but can be reformulated in an explicitly covariant form. To this end we use definition of the covariant derivative (1.2.8) to rewrite the partial derivatives in (1.2.79) as follows,

$$\partial_\mu \psi^A = \psi^A{}_{;\mu} - C^\alpha{}_{\mu\beta}\, \psi^A\big|^\beta_\alpha, \tag{1.2.80}$$

$$\partial_\mu \xi^\nu = \xi^\nu{}_{;\mu} - C^\nu{}_{\mu\beta}\xi^\beta. \tag{1.2.81}$$

Substituting (1.2.80), (1.2.81) into (1.2.79) and elaborating on terms depending on the Christoffel symbols, show that all such terms cancel out, yielding

$$\pounds_\xi \psi^A = -\xi^\mu \psi^A{}_{;\mu} + \psi^A\big|^\mu_\nu\, \xi^\nu{}_{;\mu}. \tag{1.2.82}$$

We have provided derivation of (1.2.82) in terms of the covariant derivatives of flat spacetime but, in fact, any curved spacetime manifold with a pseudo-Riemannian metric yields the same result (see, e. g., (A.3.27) in Appendix A.3.3). We discuss other important properties of the Lie derivatives in Appendix A.2.3.

1.2.4 Conserved quantities in field theories

The differential, integral and global conservation laws

It is very important for physical applications to find out the consequences which follow from the Noether theorems. The key quantities are currents, $\mathscr{J}^\mu$, which satisfy *a differential conservation law*,

$$\partial_\mu \mathscr{J}^\mu = 0, \tag{1.2.83}$$

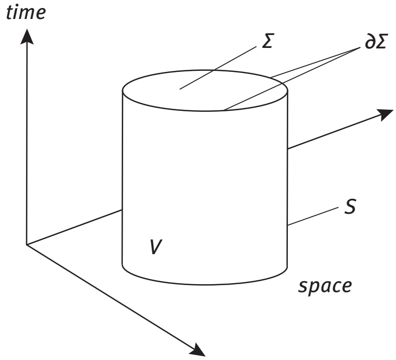

Figure 1.1: A volume V as a truncated cylinder in a n-dimensional spacetime manifold.

that can be interpreted as an equation of continuity,

$$\partial_0 \mathscr{J}^0 = -\partial_i \mathscr{J}^i. \tag{1.2.84}$$

Usually, $\mathscr{J}^\mu$ are vector densities of weight +1. If this is the case, the expression $\partial_\mu \mathscr{J}^\mu$ is a scalar density and the conservation law (1.2.83) can be rewritten in an equivalent covariant form:

$$D_\mu \mathscr{J}^\mu = 0, \tag{1.2.85}$$

where D_μ is a generalized notation for a covariant derivative that is interpreted depending on the problem under consideration either as a covariant derivative defined with the use of a flat metric in curvilinear coordinates, or as that of a pseudo-Riemannian metric of a fixed background manifold, or as one of a physical metric on a dynamical spacetime manifold, etc. For example, the transition from the partial to the covariant divergence of the vector density in case of the dynamical metric, see (A.2.12) in Appendix A.2.1. Also, the currents $\mathscr{J}^\mu$ and the conservation laws for them (1.2.83–1.2.85) are valid in case of an arbitrary n-dimensional pseudo-Riemannian spacetime manifolds with the metric signature $(-, +, +, \dots, +)$.

What are the global consequences of the covariant conservation law (1.2.85)? To answer this question it is more convenient to operate with the differential conservation law of the vector density written down in the form of the partial derivative (1.2.83). Let us consider an n-dimensional volume V in a spacetime whose boundary consists of an $(n-1)$-dimensional timelike cylinder S and two $(n-1)$-dimensional spacelike cross-sections, Σ_0 and Σ_1. For the sake of simplicity we assume that S is defined by the condition $x^1 = r =$ const in an appropriate coordinate frame; each of the cross-sections, Σ_0 and Σ_1 are defined by their own constant time: t_0 and t_1 respectively, see Figure 1.1. Each of the cross-sections Σ are restricted by the boundary $\partial\Sigma$ that is an intersection of Σ with S. Because equation (1.2.83) represents a scalar density of weight +1, it can be easily integrated over the volume V. Applying the Gauss (Stokes) theorem to (1.2.83), one gets,

$$\int_V d^n x \partial_\mu \mathscr{J}^\mu = \int_{\Sigma_1} d^{n-1}x \mathscr{J}^0 - \int_{\Sigma_0} d^{n-1}x \mathscr{J}^0 + \oint_S dt d^{n-2}x \mathscr{J}^1 = 0, \tag{1.2.86}$$

where we have employed our choice of the coordinate system for making simplification in the last term of (1.2.86), $d^n x = dt d^{n-1}x = dt dx^1 d^{n-2}x$, and $d^{n-1}x = dx^1 d^{n-2}x$ is the element of integration on the cross-sections Σ, while $d^{n-2}x$ is the element of integration on the boundary $\partial\Sigma$. Notice that the relation (1.2.86) contains the integral quantity

$$\mathscr{P} = \int_\Sigma d^{n-1}x \mathscr{J}^0 \tag{1.2.87}$$

where the integrand is the time component of the current, $\mathscr{J}^0$, that is interpreted as a density of $\mathscr{P}$. If one imposes a boundary condition on the last term in (1.2.86) such as

$$\oint_S dt d^{n-2}x \mathscr{J}^1 = \int_{t_0}^{t_1} dt \oint_{\partial\Sigma} d^{n-2}x \mathscr{J}^1 = 0, \tag{1.2.88}$$

then, (1.2.86) states that $\mathscr{P}$ is independent of the choice of the cross-section Σ, which means that it is conserved and does not depend on time. If the condition (1.2.88) does not hold, then $\mathscr{P}$ is not conserved.

Let us clarify the physical meaning of the boundary condition (1.2.88). Let us assume that the difference $\delta t = t_1 - t_0$ is infinitesimal. Then the equality (1.2.86) can be rewritten in the form:

$$\frac{d\mathscr{P}}{dt} = -\oint_{\partial\Sigma} d^{n-2}x \mathscr{J}^1. \tag{1.2.89}$$

The relation (1.2.89) tells us that if the flux of the vector field $\mathscr{J}^i$ through $\partial\Sigma$ is absent, then the quantity $\mathscr{P}$ does not depend on time; if the flux is not zero, then $\mathscr{P}$ is not conserved.

Generally, if the boundary condition (1.2.88) is satisfied, the quantity (1.2.87) is called an *integral* conserved quantity; in the case when the boundary $\partial\Sigma$ goes to infinity, the quantity (1.2.87) is called a *global* conserved quantity.

The canonical conserved quantities

The conserved quantities obtained on the basis of the Noether theorem outlined in the previous sections are called the *canonical* conserved quantities. In this section we demonstrate the principles of construction and properties of the canonical quantities by studying a simple example of a field theory in the Minkowski space in arbitrary curvilinear coordinates with an action taken in the covariant form (1.2.17). To shorten the formulae we use again a unified notation for the matter fields, ϕ^A, and the Minkowski metric, $\gamma_{\alpha\beta}$, by denoting $\psi^A = \{\phi^A, \gamma_{\alpha\beta}\}$. Then, the action (1.2.17) is rewritten as

$$S = \int_\Omega d^4x \mathscr{L}(\phi^B, \phi^B{}_{;\alpha}, \gamma_{\alpha\beta}) = \int_\Omega d^4x \mathscr{L}(\psi^A, \psi^A{}_{,\alpha}). \tag{1.2.90}$$

This is a simplified case of the action (1.2.36) without second derivatives of the fields that directly permits us to use the results of the Noether theorems.

We assume that the group transformations used in the first Noether's theorem are diffeomorphisms (1.2.72) transforming the fields as shown in (1.2.73) and inducing variations (1.2.74) and (1.2.75) of coordinates and fields. Comparing (1.2.75) with (1.2.54) we see that the operator $\hat{\Psi}_a$ entering (1.2.54) is

$$\hat{\Psi}_a \psi^A = \psi^A\Big|^{\alpha}_{\beta} \partial_\alpha \xi^\beta_a . \tag{1.2.91}$$

Replacing this expression for $\hat{\Psi}_a$ to both sides of equation (1.2.57) and recalling the definition (1.2.79) of the Lie derivative, we can recast the Noether identity (1.2.57) in the following form:

$$\frac{\delta \mathscr{L}}{\delta \psi^A} £_{\xi_a} \psi^A = \partial_\mu \left(\mathscr{J}^\mu_a + \mathscr{B}^\mu_a \right), \tag{1.2.92}$$

where the Noether current

$$\mathscr{J}^\mu_a = -\frac{\partial \mathscr{L}}{\partial \psi^A{}_{,\mu}} £_{\xi_a} \psi^A - \mathscr{L} \xi^\mu_a , \tag{1.2.93}$$

and the vector field density $\mathscr{B}^\mu_a$ is solenoidal

$$\mathscr{B}^\mu_{a,\mu} = 0. \tag{1.2.94}$$

Let us now return to the original variables, ϕ^B and $\gamma_{\alpha\beta}$, and write (1.2.92) and (1.2.93) in a more explicit form:

$$\frac{\delta \mathscr{L}}{\delta \phi^A} £_{\xi_a} \phi^A + \frac{\delta \mathscr{L}}{\delta \gamma_{\alpha\beta}} £_{\xi_a} \gamma_{\alpha\beta} = \partial_\mu \left(\mathscr{J}^\mu_a + \mathscr{B}^\mu_a \right), \tag{1.2.95}$$

$$\mathscr{J}^\mu_a = -\frac{\partial \mathscr{L}}{\partial \phi^A{}_{,\mu}} £_{\xi_a} \phi^A - \frac{\partial \mathscr{L}}{\partial \gamma_{\alpha\beta,\mu}} £_{\xi_a} \gamma_{\alpha\beta} - \mathscr{L} \xi^\mu_a . \tag{1.2.96}$$

We notice that the Lagrangian derivatives are covariant as shown in (1.2.29) and (1.2.31), the Lie derivative is covariant according to (1.2.82), and expressions $\partial\mathscr{L}/\partial\phi^A{}_{,\mu}$ and $\partial\mathscr{L}/\partial\gamma_{\alpha\beta,\mu}$ are covariant as well, see (1.2.28) and (1.2.31). This remark elucidates the covariant nature of expressions (1.2.95) and (1.2.96). In particular, current (1.2.96) is a covariant vector density of weight +1.

Now, we specify that the finite group of transformations (1.2.55) and (1.2.56) is the Poncaré group of motions of the Minkowski space with the displacement vectors ξ^α_a being ten Killing vectors, $\xi^\alpha_K = \{\xi^\alpha_\beta, \xi^\alpha_{[\beta\gamma]}\}$. Components of the Killing vectors are given in (1.1.72) and (1.2.5) in the Lorentzian and curvilinear coordinates, respectively. Because the Poincaré group does not change the Minkowski metric, the Lie derivative

$\pounds_{\xi_K}\gamma_{\alpha\beta} = 0$, see (1.2.6), and all terms being proportional to the Lie derivative of the metric tensor vanish. It reduces equation (1.2.95) to a simpler form

$$\frac{\delta\mathscr{L}}{\delta\phi^A}\pounds_{\xi_K}\phi^A = \partial_\mu\left(\mathscr{J}_C^\mu + \mathscr{B}_K^\mu\right), \tag{1.2.97}$$

while the Noether *canonical current* (1.2.96) reads

$$\begin{aligned}\mathscr{J}_C^\mu(\xi_K^\alpha) &= -\frac{\partial\mathscr{L}}{\partial\phi^A{}_{;\mu}}\pounds_{\xi_K}\phi^A - \mathscr{L}\xi_K^\mu \\ &= \frac{\partial\mathscr{L}}{\partial\phi^A{}_{;\mu}}\left(\xi_K^\alpha\phi^A{}_{;\alpha} - \phi^A\big|^\alpha_\beta\,\xi^\beta_{K;\alpha}\right) - \mathscr{L}\xi_K^\mu.\end{aligned} \tag{1.2.98}$$

Assuming that the field equations (1.2.25) hold, equations (1.2.97), (1.2.94) lead to a differential conservation law for the canonical current (1.2.98):

$$\partial_\mu\mathscr{J}_C^\mu = 0. \tag{1.2.99}$$

Ten canonical integral quantities (1.2.87) corresponding to the ten Killing vectors are:

$$\mathscr{P}_C(\xi_K^\alpha) = \int_\Sigma d^3x\,\mathscr{J}_C^0(\xi_K^\alpha). \tag{1.2.100}$$

They are analogous to the ten conserved quantities in mechanics of massive point particles, see (1.1.91) and (1.1.95).

To study physical properties of the canonical current (1.2.98) associated with its conservation, we notice that by employing the Killing equations (1.2.6) it can be written down in the following form:

$$\mathscr{J}_C^\mu = {}_C\boldsymbol{\theta}_\sigma{}^\mu\xi_K^\sigma + \boldsymbol{\sigma}^{\mu\beta\sigma}\xi_{K[\sigma;\beta]}, \tag{1.2.101}$$

where we have introduced the following notations,

$${}_C\boldsymbol{\theta}_\sigma{}^\mu \equiv \frac{\partial\mathscr{L}}{\partial\phi^A{}_{;\mu}}\phi^A{}_{;\sigma} - \mathscr{L}\delta^\mu_\sigma, \tag{1.2.102}$$

$$\boldsymbol{\sigma}^{\mu\beta}{}_\sigma \equiv -\frac{\partial\mathscr{L}}{\partial\left(\phi^A{}_{;\mu}\right)}\,\phi^A\big|^\beta_\sigma. \tag{1.2.103}$$

Here, the quantity ${}_C\boldsymbol{\theta}_\sigma{}^\mu$ is called the *canonical energy-momentum* tensor density[8] and $\boldsymbol{\sigma}^{\mu\beta}{}_\sigma$ is a *spin* (or *helicity*) tensor density both being of weight +1.

8 In some applications the canonical energy-momentum tensor ${}_C\theta_\sigma{}^\mu = {}_C\boldsymbol{\theta}_\sigma{}^\mu/\sqrt{-\gamma}$ is more useful instead of the tensor density ${}_C\boldsymbol{\theta}_\sigma{}^\mu$.

In many important cases the energy and momentum of a closed physical system are conserved. Therefore, we are interested in formulating the conservation law of the canonical energy-momentum tensor density ${}_C\boldsymbol{\theta}_\sigma{}^\mu$. They can be derived from the conservation law of the canonical current $\mathscr{J}_C^\mu$ associated with different Killing vectors. First, let us pick up the Killing vectors of translations in the Minkowski space, $\xi_K^\alpha = \xi_\beta^\alpha$, and substitute them to (1.2.101). Equation (1.2.7) tells us that the Killing vectors of translations are covariantly constant, $\xi^\alpha_{\beta;\rho} = 0$, so that the second term in the right side of (1.2.101) drops out, and the canonical current (1.2.101) corresponding to the translational symmetry, is reduced to

$$\mathscr{J}_C^\mu(\xi_\beta^\alpha) = {}_C\boldsymbol{\theta}_\sigma{}^\mu \xi_\beta^\sigma. \tag{1.2.104}$$

The conservation law (1.2.99) applied to (1.2.104) along with (1.2.7) yields

$${}_C\boldsymbol{\theta}_\sigma{}^\mu{}_{;\mu}\xi_\beta^\sigma = 0. \tag{1.2.105}$$

which tells us that the energy-momentum tensor density is conserved

$${}_C\boldsymbol{\theta}_\sigma{}^\mu{}_{;\mu} = 0. \tag{1.2.106}$$

The conservation law (1.2.106) is necessary but not sufficient for a number of applications. The fact of the matter is that the canonical energy-momentum tensor density ${}_C\boldsymbol{\theta}_{\alpha\beta}$ is not symmetric, ${}_C\boldsymbol{\theta}_{\alpha\beta} \neq {}_C\boldsymbol{\theta}_{\beta\alpha}$, excluding the simplest cases, and can not be used alone to describe the conservation law for angular momentum of the system.

To find out the skew-symmetric part ${}_C\boldsymbol{\theta}_{[\alpha\beta]}$ of the energy-momentum let us again turn to the canonical current (1.2.101) and use the conservation law (1.2.99) for it. But now we consider the Lorentzian rotations that are the infinitesimal spatial rotations and boosts in the Minkowski space generated by the corresponding Killing vectors, ξ_K^α. For these vectors the canonical current (1.2.101) takes the following form

$$\mathscr{J}_C^\mu(\xi_K^\alpha) = {}_C\boldsymbol{\theta}_\sigma{}^\mu \xi_K^\sigma + \boldsymbol{\sigma}^{\mu\rho\sigma}\xi_{K[\sigma;\rho]}, \tag{1.2.107}$$

$$\xi_K^\sigma \equiv \xi_{[\alpha\beta]}^\sigma, \tag{1.2.108}$$

where the term, ${}_C\boldsymbol{\theta}_\sigma{}^\mu \xi_{[\alpha\beta]}^\sigma$, is associated with the *orbital momentum* of the system and $\boldsymbol{\sigma}^{\mu\rho\sigma}\xi_{([\alpha\beta])[\sigma;\rho]}$ describes the *spin* or intrinsic angular momentum of the system. Applying the conservation law (1.2.99) to the canonical current (1.2.107) along with equations (1.2.7), (1.2.106) yields

$$\boldsymbol{\theta}_{[\alpha\beta]} = \boldsymbol{\sigma}^\mu{}_{[\alpha\beta];\mu}. \tag{1.2.109}$$

It tells us that in the most general case the canonical energy-momentum tensor density of a physical system is not symmetric, and its skew-symmetric part is associated with the divergence of the system's spin.

Fortunately, it is possible to build a symmetric tensor density of energy-momentum. A procedure was proposed by Belinfante [35] and is called the Belinfante symmetrization which we now discuss.

The Belinfante symmetrization

Belinfante [35] noticed that the conserved current in the right side of (1.2.97) is defined up to a solenoidal vector density $\mathscr{B}^{\alpha}_{K}$ added to the canonical current (1.2.101). The solenoidal vector field can always be chosen in the form of a divergence from a skew-symmetric tensor density $\boldsymbol{b}^{\alpha\beta\gamma} = \boldsymbol{b}^{[\alpha\beta]\gamma}$ of weight +1 such that

$$\mathscr{B}^{\alpha}_{K} \equiv \left(\boldsymbol{b}^{\alpha\beta}{}_{\gamma}\xi^{\gamma}_{K}\right)_{;\beta}. \tag{1.2.110}$$

Indeed, taking the divergence of both sides of (1.2.110) we get

$$\mathscr{B}^{\alpha}_{K;\alpha} \equiv 0. \tag{1.2.111}$$

Belinfante introduced a new conserved *symmetrized* current

$$\mathscr{J}^{\alpha}_{B} = \mathscr{J}^{\alpha}_{C} + \mathscr{B}^{\alpha}_{K}, \tag{1.2.112}$$

with $\mathscr{J}^{\alpha}_{C}$ defined in (1.2.101) and $\mathscr{B}^{\alpha}_{K}$ defined by (1.2.110) where

$$\boldsymbol{b}^{\alpha\beta\gamma} \equiv \boldsymbol{\sigma}^{\gamma[\alpha\beta]} + \boldsymbol{\sigma}^{\alpha[\gamma\beta]} - \boldsymbol{\sigma}^{\beta[\gamma\alpha]}. \tag{1.2.113}$$

The tensor density $\boldsymbol{b}^{\alpha\beta\gamma}$ is called the *Belinfante correction*.

The Belinfante energy-momentum tensor density is defined as a linear combination

$${}_{B}\boldsymbol{\theta}_{\alpha}{}^{\beta} = {}_{C}\boldsymbol{\theta}_{\alpha}{}^{\beta} + \boldsymbol{b}^{\beta\gamma}{}_{\alpha;\gamma}, \tag{1.2.114}$$

which is supposed to be symmetric, ${}_{B}\boldsymbol{\theta}_{\alpha\beta} = {}_{B}\boldsymbol{\theta}_{\beta\alpha}$. In order to prove that the Belinfante energy-momentum is indeed symmetric, let us consider the skew-symmetric part of (1.2.114). Because of (1.2.109), we have

$${}_{B}\boldsymbol{\theta}_{[\alpha\beta]} = \boldsymbol{\sigma}^{\gamma}{}_{[\alpha\beta];\gamma} + \boldsymbol{b}^{\gamma}{}_{[\alpha\beta];\gamma}. \tag{1.2.115}$$

Now we substitute the Belinfante correction $\boldsymbol{b}^{\alpha\beta\gamma}$ to (1.2.115) which immediately tells us that ${}_{B}\boldsymbol{\theta}_{[\alpha\beta]} = 0$, *q.e.d.* Thus, indeed, ${}_{B}\boldsymbol{\theta}_{\alpha\beta}$ is symmetrical in α and β. For this reason, it is called the *symmetrized energy-momentum*. Because the conservation law (1.2.99) was used in proving (1.2.115), and it assumes that the field equations are satisfied, one concludes that Belinfante's symmetrization procedure is also valid under the same condition.

Due to the antisymmetry of the Belinfante correction $\boldsymbol{b}^{\alpha\beta\gamma}$ in α and β in (1.2.113) and by the conservation law (1.2.106), the energy-momentum (1.2.114) is also conserved differentially,

$$ {}_B\boldsymbol{\theta}_\alpha{}^\beta{}_{;\beta} = 0, \tag{1.2.116}$$

of course, if the equations of motion hold as well.

Making use of the vector density (1.2.110) in the definition of the Belinfante current (1.2.112) and combining it with (1.2.101), we construct the current $\mathscr{J}_B^\beta$. We can see that it is expressed solely in terms of the Belinfante energy-momentum tensor density

$$ \mathscr{J}_B^\beta = {}_B\boldsymbol{\theta}_\alpha{}^\beta \xi_K^\alpha, \tag{1.2.117}$$

where all of the ten Killing vectors have been used. The symmetry of ${}_B\boldsymbol{\theta}_{\mu\nu}$ and its conservation law (1.2.116) lead to the conservation of the Belinfante current,

$$ \partial_\beta \mathscr{J}_B^\beta = {}_B\boldsymbol{\theta}^{\alpha\beta}\xi_{K(\alpha;\beta)} = 0. \tag{1.2.118}$$

This follows also after using (1.2.112), and taking into account the conservation of the canonical current (1.2.99) and the identity (1.2.111). Then, by the general recipe (1.2.87), one constructs ten integral quantities

$$ \mathscr{P}_B(\xi_K^\alpha) = \int_\Sigma d^3x\, \mathscr{J}_B^0(\xi_K^\alpha). \tag{1.2.119}$$

The metrical energy-momentum and conserved current

Now, let us turn to the second Noether's theorem to study the physical system (1.2.90) where we shall again separate the dynamical variables in the matter fields and the metric tensor, $\psi^A = \{\phi^A, \gamma_{\mu\nu}\}$. We assume that the variations of coordinates and variables (1.2.61) and (1.2.62) in the second Noether's theorem are induced by a diffeomorphism defined by (1.2.74) and (1.2.78) with (1.2.79) that are

$$ \delta x^\alpha = \xi^\alpha(x), \tag{1.2.120}$$

$$ \begin{aligned} \delta\phi^A &= \pounds_\xi \phi^A = -\xi^\alpha \phi^A{}_{,\alpha} + \phi^A\big|_\alpha^\beta \xi^\alpha{}_{,\beta} \\ &= -\xi^\alpha \phi^A{}_{;\alpha} + \phi^A\big|_\alpha^\beta \xi^\alpha{}_{;\beta}, \end{aligned} \tag{1.2.121}$$

$$ \delta\gamma^{\mu\nu} = \pounds_\xi \gamma^{\mu\nu} = -\xi^\alpha \gamma^{\mu\nu}{}_{,\alpha} + \gamma^{\mu\nu}\big|_\alpha^\beta \xi^\alpha{}_{,\beta} = \xi^{\mu;\nu} + \xi^{\nu;\mu}. \tag{1.2.122}$$

By comparing (1.2.120) and (1.2.121) with (1.2.61) and (1.2.62) we notice that the role of the continuous parameter $\varepsilon^a(x)\xi_a^\alpha(x)$ is played simply by $\xi^\alpha(x)$ and the following identifications are implied:

$$ \omega_a^A \to -\phi^A{}_{,\alpha}, \quad \omega_a^{A\beta} \to \phi^A\big|_\alpha^\beta, \tag{1.2.123}$$

along with

$$\omega_\alpha^{\mu\nu} \to -\gamma^{\mu\nu}{}_{,\alpha}, \quad \omega_\alpha^{\mu\nu\beta} \to \gamma^{\mu\nu}\big|_\alpha^\beta = \delta_\alpha^\mu \gamma^{\beta\nu} + \delta_\alpha^\nu \gamma^{\beta\mu}. \tag{1.2.124}$$

Then for the system (1.2.90) the current (1.2.64), that is conserved identically (1.2.68), together with divergenceless vector density (1.2.69), acquires the form:

$$\begin{aligned}\mathscr{I}_{\mathscr{B}}^\mu &= \mathscr{I}^\mu + \mathscr{B}^\mu \\ &= -\frac{\partial \mathscr{L}}{\partial \phi^A{}_{,\mu}} £_\xi \phi^A - \frac{\partial \mathscr{L}}{\partial \gamma^{\alpha\beta}{}_{,\mu}} £_\xi \gamma^{\alpha\beta} \\ &\quad -\left(\frac{\delta \mathscr{L}}{\delta \phi^A}\, \phi^A\big|_\nu^\mu + \frac{\delta \mathscr{L}}{\delta \gamma^{\alpha\beta}}\, \gamma^{\alpha\beta}\big|_\nu^\mu\right) \xi^\nu - \mathscr{L}\xi^\mu + \mathscr{B}^\mu. \end{aligned} \tag{1.2.125}$$

Using the same arguments, which have been applied for proving the covariance of expressions (1.2.95) and (1.2.96), one can prove that the current $\mathscr{I}^\mu$ is a covariant vector density of weight +1.

Now, we calculate the current, $\mathscr{I}^\mu$, by making use of the translation Killing vectors of the Minkowski space, $\xi^\alpha \to \xi_K^\alpha = \xi_\beta^\alpha$, see (1.1.72). In doing this we shall pick up $\mathscr{B}^\mu$ in (1.2.125) to be the Belinfante vector density $\mathscr{B}^\mu \to \mathscr{B}_K^\mu$ where $\mathscr{B}_K^\mu$ is defined in (1.2.110). We also use definitions (1.2.102), (1.2.103) along with the Belinfante energy-momentum ${}_B\boldsymbol{\theta}_\alpha^\beta$ introduced in (1.2.114). Taking into account the Killing equations for the metric tensor, $£_{\xi_K}\gamma_{\alpha\beta} = 0$, and assuming that the field equations, $\delta\mathscr{L}/\delta\phi^A = 0$, are satisfied, one obtains

$$\mathscr{I}_{\mathscr{B}}^\mu = \left({}_B\boldsymbol{\theta}_\nu{}^\mu - {}_S\boldsymbol{\theta}_\nu{}^\mu\right)\xi_K^\nu. \tag{1.2.126}$$

Here, a new quantity

$${}_S\boldsymbol{\theta}_{\alpha\beta} \equiv 2\frac{\delta \mathscr{L}}{\delta \gamma^{\alpha\beta}}, \tag{1.2.127}$$

is called the *metrical* energy-momentum tensor density which is just symmetric in α and β by the above-given definition, ${}_S\boldsymbol{\theta}_{\alpha\beta} = {}_S\boldsymbol{\theta}_{\beta\alpha}$. The metrical energy-momentum tensor ${}_S\theta_{\alpha\beta} = {}_S\boldsymbol{\theta}_{\alpha\beta}/\sqrt{-\gamma}$ is also used in physical applications.

Because both tensor densities in the right side of (1.2.126) are symmetric, and the current $\mathscr{I}_{\mathscr{B}}^\mu$ is conserved by (1.2.69), we conclude that

$${}_S\boldsymbol{\theta}_\nu{}^\mu{}_{;\mu} = {}_B\boldsymbol{\theta}_\nu{}^\mu{}_{;\mu}. \tag{1.2.128}$$

However, ${}_B\boldsymbol{\theta}_{\mu\nu}$ is conserved by (1.2.116). Thus, due to (1.2.128) the symmetrical energy-momentum density is conserved as well,

$${}_S\boldsymbol{\theta}_\sigma{}^\mu{}_{;\mu} = 0. \tag{1.2.129}$$

The equality (1.2.128) also assumes that the two tensor densities entering it are equal up to a divergence of a skew-symmetric tensor density of a third rank $\boldsymbol{o}^{\alpha\mu\nu}$. However, because of (1.2.128) and the symmetry of both tensor densities, ${}_S\boldsymbol{\theta}^{\mu\nu}$ and ${}_B\boldsymbol{\theta}^{\mu\nu}$, we conclude that $\boldsymbol{o}^{\alpha\mu\nu}$ must be fully skew-symmetric with respect to all three indices: $\boldsymbol{o}^{\alpha\mu\nu} = \boldsymbol{o}^{[\alpha\mu\nu]}$ which contradicts to the symmetry of the energy-momentum tensor densities. The only way to resolve this issue is to set $\boldsymbol{o}^{\alpha\mu\nu} \equiv 0$, which reveals that the two energy-momentum tensor densities are equal,

$$ {}_S\boldsymbol{\theta}^{\mu\nu} = {}_B\boldsymbol{\theta}^{\mu\nu}. \tag{1.2.130}$$

Effectively, it means that the current of the second Noether's theorem is zero under the conditions of the present consideration,

$$ \mathscr{I}^{\mu}_{\mathscr{B}} = 0. \tag{1.2.131}$$

Making use of the metrical energy-momentum, same as the Belinfante corrected energy-momentum, we can introduce the symmetrical current,

$$ \mathscr{I}^{\mu}_{S}(\xi^{\alpha}_{K}) \equiv {}_S\boldsymbol{\theta}_{\nu}{}^{\mu}\xi^{\nu}_{K}, \tag{1.2.132}$$

where all of the ten Killing vectors are used and which is differentially conserved:

$$ \partial_{\mu}\mathscr{I}^{\mu}_{S}(\xi^{\alpha}_{K}) = 0, \tag{1.2.133}$$

similar to the Noether's canonical current (1.2.99) and the Belinfante symmetrized current (1.2.118). It allows us to construct the corresponding integral quantity of the type (1.2.87):

$$ \mathscr{P}_{S}(\xi^{\alpha}_{K}) = \int_{\Sigma} d^3x\, \mathscr{I}^{0}_{S}(\xi^{\alpha}_{K}). \tag{1.2.134}$$

Discussion

Let us summarize the findings of this subsection. We have discovered the consequences of the first and second Noether's theorems in application to the conservation laws of the physical system described by the action functional (1.2.90) in the Minkowski space. The first Noether's theorem associates the global symmetries of the action of the physical system with the canonical current $\mathscr{I}^{\alpha}_{C}$ being composed of the canonical energy-momentum tensor density ${}_C\boldsymbol{\theta}^{\alpha}{}_{\beta}$ and spin $\boldsymbol{\sigma}^{\alpha\beta\gamma}$. The canonical tensor density ${}_C\boldsymbol{\theta}^{\alpha}{}_{\beta}$ is conserved but not symmetrical. The Belinfante symmetrization procedure introduces a new energy-momentum tensor density ${}_B\boldsymbol{\theta}^{\alpha}{}_{\beta}$ which is symmetrical and conserved. A corresponding symmetrized current $\mathscr{I}^{\alpha}_{B}$ can be build out of the Belinfante energy-momentum tensor density alone.

The second Noether's theorem associates the local symmetries of the action of the physical system with a conserved current $\mathscr{I}^{\alpha}$ and introduces the metrical energy-momentum tensor density ${}_S\boldsymbol{\theta}^{\alpha}{}_{\beta}$ which is symmetrical and conserved. A symmetrical current $\mathscr{J}_S^\alpha$ is build out of the metrical energy-momentum tensor density alone. The conservation law of the corrected current $\mathscr{I}_{\mathscr{B}}^\alpha$ establishes the equality of the Belinfante and metrical energy-momentum tensor densities which makes the current $\mathscr{I}_{\mathscr{B}}^\alpha = 0$ on the equations of motion of the physical fields and for translation Killing vectors.

It turns out that on the equations of motion, Noether's canonical, Belinfante's and metrical currents are tightly interrelated,

$$\mathscr{J}_B^\alpha = \mathscr{J}_S^\alpha = \mathscr{J}_C^\alpha + (\boldsymbol{b}^{\alpha\beta}{}_\sigma \xi_K^\sigma)_{;\beta}, \tag{1.2.135}$$

where $\boldsymbol{b}^{\alpha\beta}{}_\sigma$ is made of the spin tensor densities $\boldsymbol{\sigma}^{\alpha\beta}{}_\sigma$, and all the currents are defined for the ten Killing vectors of the Minkowski space, ξ_K^σ. Respectively, corresponding integral quantities, (1.2.100), (1.2.119) and (1.2.134), are connected as follows:

$$\mathscr{P}_B = \mathscr{P}_S = \mathscr{P}_C + \oint_{\partial\Sigma} ds_i \boldsymbol{b}^{0i}{}_\sigma \xi_K^\sigma, \tag{1.2.136}$$

where $\partial\Sigma$ is a two-dimensional boundary of the 3-dimensional volume Σ, and ds_i is a surface element of integration on $\partial\Sigma$.

One concludes that the values of $\mathscr{P}_B$ and $\mathscr{P}_S$ can coincide with that of $\mathscr{P}_C$, in case when the closed surface integral from $\boldsymbol{b}^{0i}{}_\sigma \xi_K^\sigma$ disappears. It can happen for various reasons: (1) the system can be closed so that the physical fields ϕ^A vanish on the boundary $\partial\Sigma$; (2) a fall-off behavior of the fields is rapid enough to make the surface integral nil in case when $\partial\Sigma \to \infty$; (3) the parity of the fields annihilates the surface integral.

It is important to notice the role of divergence, div $= \partial_\alpha \mathscr{B}^\alpha$, in the overall Noether's formalism of the conserved quantities. The divergence, being vanishing, does not contribute to the Noether identity but it linearly couples with the conserved currents which are, thus, defined with a certain degree of freedom of the solenodial vector field $\mathscr{B}^\alpha$. The condition of the symmetry imposed on the energy-momentum tensor density singles out the divergence in the form of the Belinfante correction (1.2.110) which connects the canonical Noether's current with the symmetrized Belinfante current in a unique way. The second Noether's theorem and the symmetry condition imposed on the energy-momentum tensor density equate the Belinfante and metrical energy-momentum tensor densities. It shows that the symmetric energy-momentum tensor density in the Minkowski space is unique.

1.2.5 Examples of field theories in the Minkowski space

It is useful to illustrate the applications of the Noether theorems with some examples of simple, but physically important field theories. We consider the Lagrangian of Maxwell's electromagnetic field, the ideal fluid and a free relativistic scalar field. We apply the Noether variational formalism to derive the field equations and to construct the corresponding conserved energy-momentum tensor densities for the fields under consideration.

Electromagnetic field

The independent dynamical variable of electromagnetic field is a vector-potential $A_\alpha = (A_0, A_i) = (\phi, \boldsymbol{A})$, which unites the components of electric and magnetic fields in the form of the electromagnetic tensor [285, 315]

$$F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}. \tag{1.2.137}$$

Covariant derivatives in $F_{\alpha\beta}$ can be replaced with the partial derivatives because, due to the antisymmetry of the electromagnetic tensor, all terms with the Christoffel symbols entering the covariant derivatives, are canceled out completely. Hence, the covariant derivatives in the expression for the electromagnetic tensor are irrelevant but they will be formally kept in the definition (1.2.137).

The Lagrangian for free electromagnetic field is

$$\mathscr{L}_{\rm em} = \frac{\sqrt{-\gamma}}{16\pi} F^{\mu\nu} F_{\mu\nu} = \frac{\sqrt{-\gamma}}{16\pi} \gamma^{\rho\sigma} \gamma^{\mu\nu} F_{\rho\mu} F_{\sigma\nu}. \tag{1.2.138}$$

The second form of the Lagrangian in (1.2.138) disentangles the metric tensor from the dynamical field variables which are A_α with the lower index, but not $A^\alpha = \gamma^{\alpha\beta} A_\beta$. Now, we substitute $\mathscr{L}_{\rm em}$ and $\phi^A \equiv A_\alpha$ into (1.2.29), take the variational derivative and obtain the equations of motion of free electromagnetic field in curvilinear coordinates:

$$\frac{\delta \mathscr{L}_{\rm em}}{\delta A_\alpha} = \left(\frac{\partial \mathscr{L}_{\rm em}}{\partial A_{\alpha;\beta}} \right)_{;\beta} = -\frac{1}{4\pi} \sqrt{-\gamma} F^{\alpha\beta}{}_{;\beta} = -\frac{1}{4\pi} \mathscr{F}^{\alpha\beta}{}_{,\beta} = 0, \tag{1.2.139}$$

where $\mathscr{F}^{\alpha\beta} \equiv \sqrt{-\gamma} F^{\alpha\beta}$.

Now, let us consider the intrinsic (gauge) transformation of the field variables:

$$A'_\alpha = A_\alpha + \partial_\alpha \varepsilon(x). \tag{1.2.140}$$

One can easily check that the electromagnetic tensor (1.2.137) is invariant under transformation (1.2.140). Hence, the Lagrangian (1.2.138) and the field equations (1.2.139) are invariant under (1.2.140) as well. Transformations (1.2.140) are the simplest example of the transformations (1.2.61) and (1.2.62) where $\delta x^\alpha = 0$, $\omega^A_a = 0$, and $\omega^{A\alpha}_a = 1$ with $\varepsilon^a(x) = \varepsilon(x)$. The identity (1.2.66) of the second Noether's theorem in application to electrodynamics is,

$$\frac{1}{4\pi} \partial_{\alpha\beta} \mathscr{F}^{\alpha\beta} \equiv 0, \tag{1.2.141}$$

which is apparently valid due to the antisymmetry of $\mathscr{F}^{\alpha\beta}$.

Now, let us construct the canonical energy-momentum tensor density (1.2.104) with using the Lagrangian (1.2.138):

$${}_C\boldsymbol{\theta}_\beta{}^\alpha = \frac{\partial \mathscr{L}_{\rm em}}{A_{\mu;\alpha}} A_{\mu;\beta} - \delta^\alpha_\beta \mathscr{L}_{\rm em} = \frac{\sqrt{-\gamma}}{4\pi} \left(F^{\alpha\rho} A_{\rho;\beta} - \delta^\alpha_\beta \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right). \tag{1.2.142}$$

This expression is neither symmetric nor gauge invariant with respect to transformation (1.2.140) but we can use the Belinfante symmetrization to fix it. The spin density (1.2.103) for the Lagrangian (1.2.138) is

$$\boldsymbol{\sigma}^{\mu\nu\rho} = -\frac{\sqrt{-\gamma}}{4\pi} F^{\mu\nu} A^{\rho}, \tag{1.2.143}$$

bringing about the corresponding Belinfane correction (1.2.113) in the form of

$$\boldsymbol{b}^{\mu\nu\rho} = \frac{\sqrt{-\gamma}}{4\pi} F^{\mu\nu} A^{\rho}. \tag{1.2.144}$$

Eventually, the Belinfante energy-momentum tensor density (1.2.114) acquires the form:

$${}_{B}\boldsymbol{\theta}_{\alpha\beta} = \frac{\sqrt{-\gamma}}{4\pi} \left(F_{\rho\alpha} F^{\rho}{}_{\beta} - \frac{1}{4} \gamma_{\alpha\beta} F^{\mu\nu} F_{\mu\nu} \right). \tag{1.2.145}$$

which is apparently symmetric and gauge invariant because it depends solely on the electromagnetic tensor in contrast to the canonical energy-momentum (1.2.143).

We can also obtain the metrical energy-momentum tensor density (1.2.127) by varying the Lagrangian (1.2.138) with respect to the metric tensor and to check by inspection that it coincides with the energy-momentum (1.2.145).

The ideal fluid

The ideal fluid is characterized by the following thermodynamical quantities: the particle rest-mass density ρ, the specific intrinsic energy of the fluid per particle Π, and the pressure p which obeys the equation of state, $p = p(\rho)$. The total energy density is $\epsilon = \rho\,(1 + \Pi)$. The dynamical description of the ideal fluid is based on the specific enthalpy of the fluid,

$$\mu = \frac{\epsilon + p}{\rho} = 1 + \Pi + \frac{p}{\rho}. \tag{1.2.146}$$

The primary dynamical variable is an auxiliary scalar field, so-called Clebsch potential Φ, which is also called the velocity potential. In the case of a single-component ideal fluid it is introduced by the following relationship [411]

$$\mu u_{\alpha} = -\Phi_{,\alpha} = -\Phi_{;\alpha} \tag{1.2.147}$$

where the four-velocity u^{α} of the fluid is normalized: $u^{\alpha} u_{\alpha} = -1$. Thus, the specific enthalpy can be expressed in the following form:

$$\mu = \sqrt{-\gamma^{\alpha\beta} \Phi_{;\alpha} \Phi_{;\beta}}. \tag{1.2.148}$$

The entropy of the ideal fluid remains constant which excludes it from consideration. The intrinsic energy of the ideal fluid, Π, is related to pressure, p, by the first law of thermodynamics,

$$d\Pi + pd\left(\frac{1}{\rho}\right) = 0. \tag{1.2.149}$$

It can be used to derive the following thermodynamic relationships,

$$dp = \rho d\mu, \tag{1.2.150}$$

$$d\epsilon = \mu d\rho, \tag{1.2.151}$$

which mean that all thermodynamic quantities are solely functions of the specific enthalpy μ, for example, $\rho = \rho(\mu)$, $\Pi = \Pi(\mu)$, etc.

Lagrangian description of the ideal fluid as a dynamical system on spacetime manifold is provided with the Lagrangian defined as $\mathscr{L}_{\mathrm{fl}} = -\sqrt{-\gamma}p$. Taking into account (1.2.146), it is more instructive to re-write the Lagrangian in the form:

$$\mathscr{L}_{\mathrm{fl}} = -\sqrt{-\gamma}p = \sqrt{-\gamma}\,(\epsilon - \rho\mu)\,, \tag{1.2.152}$$

with the specific enthalpy defined in terms of the Clebsch potential (1.2.148). It corresponds to the kinetic energy of the fluid. Now, substituting $\mathscr{L}_{\mathrm{fl}}$ and $\phi^A \equiv \Phi$ into (1.2.29), one obtains the equations of motion of the ideal fluid:

$$\frac{\delta\mathscr{L}_{\mathrm{fl}}}{\delta\Phi} = \left[\sqrt{-\gamma}\left(\frac{\partial\epsilon}{\partial\rho}\frac{\partial\rho}{\partial\mu} - \mu\frac{\partial\rho}{\partial\mu} - \rho\right)\frac{\Phi^{;\alpha}}{\mu}\right]_{;\alpha} = 0. \tag{1.2.153}$$

Combining (1.2.151) and (1.2.147), one can reduce the fields equations (1.2.153) to a simpler form

$$\frac{\delta\mathscr{L}_{\mathrm{fl}}}{\delta\Phi} = \left(\sqrt{-\gamma}\rho u^\alpha\right)_{;\alpha} = \left(\sqrt{-\gamma}\rho u^\alpha\right)_{,\alpha} = 0, \tag{1.2.154}$$

which is the continuity equation as expected.

Now, we derive the canonical energy-momentum (1.2.104) for the Lagrangian (1.2.152):

$$\begin{aligned} {}_C\boldsymbol{\theta}_\alpha{}^\beta = \frac{\partial\mathscr{L}_{\mathrm{fl}}}{\partial\Phi_{;\beta}}\Phi_{;\alpha} - \delta^\beta_\alpha\mathscr{L}_{\mathrm{fl}} &= \sqrt{-\gamma}\left(\frac{\rho}{\mu}\Phi^{;\beta}\Phi_{;\alpha} + \delta^\beta_\alpha p\right) \\ &= \sqrt{-\gamma}\left[(\epsilon + p)\,u^\beta u_\alpha + \delta^\beta_\alpha p\right] \end{aligned} \tag{1.2.155}$$

where the first equality in (1.2.155) was obtained with the help of (1.2.151) and definition (1.2.148), whereas the second equality in (1.2.155) was obtained with making use of (1.2.146) and definition (1.2.147). The canonical energy-momentum (1.2.155) turns out to coincide with a well-known energy-momentum tensor density of an ideal fluid [178, 315].

Lowering the index β in (1.2.155), we can easily see that ${}_C\boldsymbol{\theta}_{\alpha\beta}$ is symmetrical. Thus, there is no need in applying the Belinfante symmetrization. Indeed, the spin density (1.2.103) corresponding to the Lagrangian (1.2.152) is

$$\boldsymbol{\sigma}^{\alpha\beta}{}_{\sigma} = -\frac{\partial \mathscr{L}_{\text{fl}}}{\partial\left(\Phi_{;\alpha}\right)}\,\Phi|^{\beta}_{\sigma} = 0, \tag{1.2.156}$$

because $\Phi|^{\beta}_{\sigma} = 0$ for any scalar.

Finally, we notice that the expression ${}_{C}\boldsymbol{\theta}_{\alpha\beta}$ coincides with the metrical energy-momentum (1.2.127) obtained by variation of the Lagrangian $\mathscr{L}_{\text{fl}}$ with respect to $\gamma^{\alpha\beta}$:

$${}_{S}\boldsymbol{\theta}_{\alpha\beta} \equiv 2\frac{\delta \mathscr{L}_{\text{fl}}}{\delta\gamma^{\alpha\beta}} = {}_{C}\boldsymbol{\theta}_{\alpha\beta}. \tag{1.2.157}$$

In conclusion, we summarize that in the case of the ideal fluid all three types of the energy-momentum tensor densities coincide.

A scalar field

Lagrangian of a relativistic scalar field ϕ is

$$\mathscr{L}_{\phi} = \sqrt{-\gamma}\left[\frac{1}{2}\gamma^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta} - V(\phi)\right], \tag{1.2.158}$$

where $V(\phi)$ is the potential energy of the field. Again, a covariant derivative of a scalar field is simply a partial derivative, $\phi_{;\alpha} = \phi_{,\alpha}$. Now, substituting $\mathscr{L}_{\phi}$ and $\phi^{A} \equiv \phi$ into (1.2.29), one obtains the equations of motion for the scalar field in curvilinear coordinates:

$$\frac{\delta \mathscr{L}_{\phi}}{\delta\Phi} = -\sqrt{-\gamma}\left(\phi^{;\alpha}{}_{;\alpha} + \frac{\partial V(\phi)}{\partial\phi}\right) = -\left(\sqrt{-\gamma}\phi^{,\alpha}\right)_{,\alpha} - \sqrt{-\gamma}\frac{\partial V(\phi)}{\partial\phi} = 0. \tag{1.2.159}$$

The canonical energy-momentum tensor density is obtained after substituting the Lagrangian (1.2.158) to the general definition (1.2.104). It yields:

$${}_{C}\boldsymbol{\theta}_{\alpha}{}^{\beta} = \frac{\partial \mathscr{L}_{\phi}}{\partial\phi_{;\beta}}\phi_{;\alpha} - \delta^{\beta}_{\alpha}\mathscr{L}_{\phi} = \sqrt{-\gamma}\left(\phi^{;\beta}\phi_{;\alpha} - \frac{1}{2}\delta^{\beta}_{\alpha}\phi^{;\rho}\phi_{;\rho} + V(\phi)\right). \tag{1.2.160}$$

Lowering the index β in (1.2.160), one finds that ${}_{C}\boldsymbol{\theta}_{\alpha\beta}$ is symmetrical. Therefore, like in the case of the ideal fluid, there is no need for symmetrization, and all three types of the energy-momentum tensor densities coincide: ${}_{C}\boldsymbol{\theta}_{\alpha}{}^{\beta} = {}_{B}\boldsymbol{\theta}_{\alpha}{}^{\beta} = {}_{S}\boldsymbol{\theta}_{\alpha}{}^{\beta}$.

1.3 General relativity: fundamental mathematical relations

1.3.1 Lagrangians for the gravitational sector of general relativity

The main idea suggested by Einstein for constructing relativistic gravity theory which he called general relativity, was to identify the components of the pseudo-Riemannian metric, $g_{\alpha\beta}$, with the potentials of the gravitational field which also serve as ten independent *dynamical variables*. Hilbert got interested in this geometric approach and joined Einstein in the quest for the fundamental equations governing the gravitational field. While working on this problem, the two great minds were not competing but complementing each other. Hilbert was tackling the problem as a mathematician, using methods of variational calculus. Einstein was more physically intuitive, paying more attention to the conservation laws in the new theory, see discussion in the review [364].

Historical facts and ideas lying in the foundation of general relativity are pretty much well-known and can be found in a number of standard textbooks [285, 315]. Here, we discuss only the least action principle in general relativity and its applications to the derivation of the Einstein equations and the conservation laws. We start from the postulating the Lagrangian of a free gravitational field.

The Hilbert Lagrangian

Hilbert assumed that the equations of gravitational field are to be derived from the principle of the least action with the Lagrangian, $\mathscr{L}_H$, of the gravitational field which has to be a covariant geometric object made in the simplest way from the metric tensor and its derivatives – a Ricci scalar density,

$$\mathscr{L}_H = \sqrt{-g}R = \mathscr{R}. \tag{1.3.1}$$

The Ricci scalar is built out of the metric and its first and second derivatives. It may look like the application of the principle of the least action to the Lagrangian with the second derivatives of the dynamical variables gives rise to the field equations of a third or even a forth order. However, the Ricci scalar depends on the second derivatives of the metric linearly without coupling them with the first derivatives. Therefore, taking a variational derivative of such a Lagrangian can not bring about the higher-order derivatives of the metric tensor in the field equations, which remain the differential equations of the second order.

To find the Lagrangian (1.3.1) in explicit form we introduce the *curvature tensor* also known as the *Riemann tensor*:

$$R^\mu{}_{\alpha\nu\beta} = \partial_\nu\Gamma^\mu{}_{\alpha\beta} - \partial_\beta\Gamma^\mu{}_{\alpha\nu} + \Gamma^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\alpha\rho}\Gamma^\rho{}_{\beta\nu}. \tag{1.3.2}$$

Contracting this expression with respect to the indices μ and ν, one finds the *Ricci tensor*,

$$R_{\alpha\beta} = \partial_\nu\Gamma^\nu{}_{\alpha\beta} - \partial_\beta\Gamma^\nu{}_{\alpha\nu} + \Gamma^\rho{}_{\alpha\beta}\Gamma^\nu{}_{\rho\nu} - \Gamma^\nu{}_{\alpha\rho}\Gamma^\rho{}_{\beta\nu}. \tag{1.3.3}$$

The *Ricci* or *curvature* scalar that is used in the Lagrangian (1.3.1), is built out of the Ricci tensor by contracting it with the metric:

$$R = g^{\alpha\beta}R_{\alpha\beta}. \tag{1.3.4}$$

The important geometric object entering formulae (1.3.2)–(1.3.4) is called the Christoffel symbols

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} \left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\rho\beta} - \partial_\rho g_{\alpha\beta} \right), \tag{1.3.5}$$

which has 40 independent components in four dimensions. The Christoffel symbols are not forming a tensorial quantity as it compensates the non-covariant transformation law of partial derivatives. The covariant derivative, ∇_α, of a tensor density Φ^A is defined as follows:

$$\nabla_\alpha \Phi^A = \partial_\alpha \Phi^A + \Phi^A\big|^{\rho}_{\sigma} \Gamma^{\sigma}{}_{\alpha\rho}, \tag{1.3.6}$$

where the permutation operator $\Phi^A\big|^{\rho}_{\sigma}$ is defined in Appendix A.3.1 (see formula (A.3.7) for more detail). Necessary properties of the operation (1.3.6) with participation of quantities (1.3.5) are given in Appendix A.2.1.

The Einstein Lagrangian

Even though the Lagrangian (1.3.1) gives the field equations of the required second order, the presence of the second derivatives in $\mathscr{L}_H$ creates a problem in definition of the energy of weak gravitational waves [152]. Einstein had tried to fix it and proposed his own Lagrangian for the gravitational field that does not depend on the second derivatives of the metric tensor [150]. He noticed that the Hilbert Lagrangian is effectively split in two parts, one of which is a divergence,

$$\mathscr{L}_H = \mathscr{L}_E + \partial_\alpha \mathscr{W}^\alpha, \tag{1.3.7}$$

where

$$\begin{aligned} \mathscr{W}^\alpha &\equiv \sqrt{-g} \left(g^{\beta\gamma} \Gamma^{\alpha}_{\beta\gamma} - g^{\alpha\beta} \Gamma^{\gamma}_{\beta\gamma} \right) \\ &= \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \left(g_{\beta\mu,\nu} - g_{\mu\nu,\beta} \right). \end{aligned} \tag{1.3.8}$$

The divergence does not affect the field equations and can be dropped from the action. Note that currently we are not analyzing the conservation laws. The remaining part, $\mathscr{L}_E$, is the Einstein Lagrangian, [285]:

$$\mathscr{L}_E = \sqrt{-g} g^{\mu\nu} \left(\Gamma^{\alpha}{}_{\mu\beta} \Gamma^{\beta}{}_{\nu\alpha} - \Gamma^{\alpha}{}_{\alpha\beta} \Gamma^{\beta}{}_{\mu\nu} \right), \tag{1.3.9}$$

which does not contain the second derivatives of the metric. It is important to emphasize that the Einstein Lagrangian is not covariant because its coordinate transformation is not tensorial. Hence, it can be nullified at any given point of spacetime manifold by choosing at that point the normal Riemannian coordinates. For this reason, the Einstein Lagrangian makes sense only in the integral expression for the action functional.

The gravitational Lagrangians have dimension [cm^{-2}] while the action S has a dimension of [erg · sec]. The principle of correspondence with the Newtonian theory introduces the constant of proportionality $-c^3/16\pi G$ which appears explicitly in front of the gravitational part of the action. For the sake of simplicity, frequently we prefer to use the geometric system of units in which $c = G = 1$. Then, the Hilbert action has the form:

$$S_H = -\frac{1}{16\pi}\int_\Omega d^4x\mathscr{L}_H, \tag{1.3.10}$$

whereas the Einstein action is

$$S_E = -\frac{1}{16\pi}\int_\Omega d^4x\mathscr{L}_E, \tag{1.3.11}$$

where Ω is the 4-dimensional domain of integration.

Because $\mathscr{L}_H$ differs from $\mathscr{L}_E$ merely by a divergence, the Einstein action (1.3.11) differs from the Hilbert action (1.3.10) by a surface integral taken on the boundary $\partial\Omega$ of Ω from the quantity $\mathscr{W}^\alpha$ due to the Gauss's theorem,

$$S_E = S_H + \oint_{\partial\Omega} dS_\alpha \mathscr{W}^\alpha. \tag{1.3.12}$$

If we impose on the variations of the metric tensor and its first derivatives the boundary conditions

$$\delta g_{\alpha\beta}\big|_{\partial\Omega} = 0, \qquad \delta g_{\alpha\beta,\gamma}\big|_{\partial\Omega} = 0, \tag{1.3.13}$$

the surface integral in (1.3.12) vanishes and does not contribute to the variation of the action. Hence, from the point of view of variational calculus, both types of the action are equivalent for the purpose of derivation of the Einstein field equations. Notice that if we derive the Einstein equations from the action (1.3.11) the first condition in (1.3.13) is sufficient.

1.3.2 The Einstein equations

The source of the gravitational field of a physical system in general relativity is the energy-momentum tensor of matter composing the system. In order to describe the interaction of matter with gravitational field in general relativity the *principle of minimal coupling* of gravity with matter is employed. This principle establishes the simplest form of the coupling of gravity and matter and it is governed by the principle of equivalence, according to which any physical equation of special relativity can be turned into its general-relativistic counterpart by replacing the Minkowski metric, $\eta_{\alpha\beta}$ (or $\gamma_{\alpha\beta}$ in curved coordinates), with the relevant metric of the curved dynamical spacetime, $g_{\alpha\beta}$, and by replacing any partial derivative, ∂_α (or $(;\alpha)$ in curved coordinates), with a corresponding covariant derivative, ∇_α. The minimal coupling of matter to gravity leads to a natural appearance of the gravity field variable (the metric) and its first derivatives (the affine connection in the form of the Christoffel symbols) in the structure of the matter Lagrangian, $\mathscr{L}_M$, the original definition of which usually comes from special relativity, for simple cases see (1.2.17). Thus, the action functional of matter fields, Φ^A, is presented in general relativity as

$$S_M = \int_\Omega d^4x \mathscr{L}_M \left(\Phi^A, \nabla_\alpha \Phi^A, g_{\alpha\beta}\right). \tag{1.3.14}$$

Because the interaction of gravity with matter is incorporated to the matter Lagrangian, $\mathscr{L}_M$, the overall action S for gravitational field (we choose Hilbert's definition (1.3.10)) and matter interacting with gravitational field, (1.3.14), is a linear combination of two terms

$$S = \int_\Omega d^4x \mathscr{L}_{EH}, \tag{1.3.15}$$

where the total Lagrangian

$$\mathscr{L}_{EH} = -\frac{1}{16\pi}\mathscr{L}_H + \mathscr{L}_M = -\frac{1}{16\pi}\mathscr{R} + \mathscr{L}_M \tag{1.3.16}$$

is sometimes called the Einstein-Hilbert Lagrangian.

To obtain the Einstein equations one has to vary the action (1.3.15) with respect to the metric $g^{\alpha\beta}$, in a fashion similar to that used to calculate the variation of the Lagrangian with respect to the matter fields ϕ^A in the Minkowski space. One reminds that the variation of the metric and its first derivatives are chosen to be nil on the boundary of the integration domain in correspondence with (1.3.13). Calculating variation of the action, δS, by parts in (1.3.15) with respect to the metric variation, one obtains

$$\delta S = \int_\Omega d^4x \left(-\frac{1}{16\pi}\frac{\delta \mathscr{L}_H}{\delta g^{\alpha\beta}} + \frac{\delta \mathscr{L}_M}{\delta g^{\alpha\beta}}\right)\delta g^{\alpha\beta}, \tag{1.3.17}$$

where the Lagrangian derivatives taken with respect to the metric are

$$\frac{\delta \mathscr{L}_H}{\delta g^{\alpha\beta}} \equiv \frac{\partial \mathscr{L}_H}{\partial g^{\alpha\beta}} - \partial_\mu \left(\frac{\partial \mathscr{L}_H}{\partial g^{\alpha\beta}{}_{,\mu}}\right) + \partial_{\mu\nu}\left(\frac{\partial \mathscr{L}_H}{\partial g^{\alpha\beta}{}_{,\mu\nu}}\right), \tag{1.3.18}$$

$$\frac{\delta \mathscr{L}_M}{\delta g^{\alpha\beta}} \equiv \frac{\partial \mathscr{L}_M}{\partial g^{\alpha\beta}} - \partial_\mu \left(\frac{\partial \mathscr{L}_M}{\partial g^{\alpha\beta}{}_{,\mu}}\right). \tag{1.3.19}$$

Substituting into (1.3.18) the Hilbert Lagrangian $\mathscr{L}_H$ defined in (1.3.1), one gets the *Einstein tensor density*:

$$\mathscr{G}_{\alpha\beta} \equiv \frac{\delta \mathscr{L}_H}{\delta g^{\alpha\beta}} \equiv \frac{\delta \mathscr{R}}{\delta g^{\alpha\beta}} \equiv \mathscr{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathscr{R}, \tag{1.3.20}$$

where the covariant expression in the right side of (1.3.20) has been obtained with the help of (A.3.44) and (A.2.45) in Appendix A. The Lagrangian derivative (1.3.19) of the matter Lagrangian $\mathscr{L}_M$ defined in (1.3.14) yields by definition the *metrical* energy-momentum tensor density of matter:

$$\mathscr{T}_{\alpha\beta} \equiv 2\frac{\delta \mathscr{L}_M}{\delta g^{\alpha\beta}}. \tag{1.3.21}$$

The fact that this definition indeed leads to a covariant expression can be verified again by making use of (A.3.44) and (A.2.45) in Appendix A. The reader may notice that (1.3.21) is a generalization of the energy-momentum tensor density of matter (1.2.127) in a field theory in the Minkowski space.

The Einstein equations for the gravitational field are derived, as usual, from the principle of the least action that demands $\delta S = 0$ in (1.3.17). Then, a combination of (1.3.18–1.3.21) yields the variational equations for the gravitational field generated by matter:

$$\mathscr{G}_{\alpha\beta} = 8\pi \mathscr{T}_{\alpha\beta}. \tag{1.3.22}$$

The Einstein tensor density in the left side of this equation depends only on the metric tensor and its first and second partial derivatives while the energy-momentum tensor density of matter in the right side of (1.3.22) depends on the matter fields which are the source of gravity, as well as on the metric tensor and its first derivatives. After dividing both sides of (1.3.22) by $\sqrt{-g}$ we get the Einstein field equations in the usual form:

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{1.3.23}$$

where the *Einstein tensor*

$$G_{\alpha\beta} \equiv \frac{1}{\sqrt{-g}}\frac{\delta \mathscr{R}}{\delta g^{\alpha\beta}} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R, \tag{1.3.24}$$

and

$$T_{\alpha\beta} \equiv \frac{2}{\sqrt{-g}}\frac{\delta \mathscr{L}_M}{\delta g^{\alpha\beta}}, \tag{1.3.25}$$

is the *metrical* energy-momentum tensor.

Because the total set of dynamical variables of the system (1.3.15) consists of matter variables, Φ^A, along with the metric variables, one has to add the field equations for Φ^A to the system of gravitational field equations (1.3.23), and the total system of the equations has to be solved simultaneously. The matter field equations are derived by varying the total action (1.3.15) with respect to the matter variables Φ^A which yields,

$$\frac{\delta \mathscr{L}_M}{\delta \Phi^A} = \frac{\partial \mathscr{L}_M}{\partial \Phi^A} - \nabla_\alpha \frac{\partial \mathscr{L}_M}{\partial \nabla_\alpha \Phi^A} = 0. \tag{1.3.26}$$

This covariant equation is a particular case of a more general equation (A.3.37) derived in Appendix A.3.3. It is worth emphasizing some important points which we will explore below.

First, the Einstein equations in the form of equation (1.3.23) have been obtained by varying the action with respect to the contravariant metric components $g^{\alpha\beta}$ which are taken as independent dynamical variables of gravitational field. However, this is not the only possible choice of the dynamical variables in general relativity. In fact, an arbitrary metric density can be used as a dynamical variable, for more details see (2.2.114–2.2.117) in Section 2.2.6. One of the most commonly used choice of the dynamical variable is a tensor density with weight +1 which is denoted by the Gothic letter, $\mathfrak{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$. In this case, the field equations follow from the variational principle

$$\delta S = \int_\Omega d^4x \frac{\delta \mathscr{L}_{EH}}{\delta \mathfrak{g}^{\mu\nu}} \delta \mathfrak{g}^{\mu\nu} = 0, \tag{1.3.27}$$

where the Lagrangian derivative can be easily connected to $\delta\mathscr{L}_{EH}/\delta g^{\alpha\beta}$ used in derivation of (1.3.22) by the matrix of transformation

$$\frac{\partial g^{\alpha\beta}}{\partial \mathfrak{g}^{\mu\nu}} = \frac{1}{\sqrt{-g}} \left[\delta^{(\mu}_{\alpha} \delta^{\nu)}_{\beta} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \right]. \tag{1.3.28}$$

The Einstein equations following from the variational principle (1.3.27), takes on the form

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \tag{1.3.29}$$

where we denoted the trace $T \equiv g^{\alpha\beta}T_{\alpha\beta}$. One of the advantages of the variational principle (1.3.27) is that the Ricci tensor (1.3.3) is obtained directly by taking the variational derivative from $\mathscr{L}_H$ with respect to the Gothic metric as

$$R_{\mu\nu} \equiv \frac{\delta \mathscr{L}_H}{\delta \mathfrak{g}^{\mu\nu}} \equiv \frac{\delta \mathscr{R}}{\delta \mathfrak{g}^{\mu\nu}}. \tag{1.3.30}$$

Of course, the mathematical content of the Einstein equations in the form of (1.3.29) and (1.3.23) is the same.

Second important point to which we would like to bring attention of the reader is the application of the second Noether's theorem and the integral quantities following from the Hilbert action (1.3.10). In other words, we wish to elaborate on the identities (1.2.66) and (1.2.67) for the case corresponding to the Hilbert Lagrangian (1.3.1). Because the Lagrangian $\mathscr{L}_H$ is a covariant quantity, a scalar density of weight +1, the variation of S_H is invariant under one-parametric group of diffeomorphisms which is described by the variations (1.2.61) and (1.2.62). In our case the variations are represented as follows:

$$\delta x^{\mu} = \xi^{\mu}(x), \tag{1.3.31}$$

$$\delta g^{\mu\nu} = £_{\xi} g^{\mu\nu} = -\xi^{\alpha}\partial_{\alpha} g^{\mu\nu} + g^{\mu\nu}\big|^{\alpha}_{\beta}\,\partial_{\alpha}\xi^{\beta}. \tag{1.3.32}$$

Thus, comparing (1.2.61) and (1.2.62) with (1.3.31) and (1.3.32), respectively, one derives the group's structure coefficients

$$\omega^{A}_{a} = -\psi^{A}{}_{,\alpha} \to -g^{\mu\nu}{}_{,\alpha}, \tag{1.3.33}$$

$$\omega^{A\alpha}_{b} = \psi^{A}\big|^{\alpha}_{\beta} \to g^{\mu\nu}\big|^{\alpha}_{\beta}. \tag{1.3.34}$$

With the help of these coefficients we write the identity (1.2.66) for the Hilbert Lagrangian (1.3.1):

$$-g^{\mu\nu}{}_{,\beta}\frac{\delta\mathscr{L}_{H}}{\delta g^{\mu\nu}} \equiv \partial_{\alpha}\left(g^{\mu\nu}\big|^{\alpha}_{\beta}\frac{\delta\mathscr{L}_{H}}{\delta g^{\mu\nu}}\right). \tag{1.3.35}$$

Reshuffling terms in (1.3.5) with the use of simple relations, like $g_{\rho\alpha}\partial_{\beta}g^{\mu\rho} = -g^{\mu\rho}\partial_{\beta}g_{\rho\alpha}$, etc., one can express a partial derivative of the metric tensor in terms of the Christoffel symbols,

$$g^{\mu\nu}{}_{,\alpha} = \Gamma^{\mu}{}_{\alpha\rho}g^{\nu\rho} + \Gamma^{\nu}{}_{\alpha\rho}g^{\mu\rho}. \tag{1.3.36}$$

Substituting this equality into identity (1.3.35), one has

$$\partial_{\alpha}\left(g^{\mu\nu}\big|^{\alpha}_{\beta}\frac{\delta\mathscr{L}_{H}}{\delta g^{\mu\nu}}\right) + \Gamma^{\sigma}{}_{\alpha\beta}\,g^{\mu\nu}\big|^{\alpha}_{\sigma}\frac{\delta\mathscr{L}_{H}}{\delta g^{\mu\nu}} \equiv \nabla_{\alpha}\left(g^{\mu\nu}\big|^{\alpha}_{\beta}\frac{\delta\mathscr{L}_{H}}{\delta g^{\mu\nu}}\right) \equiv 0. \tag{1.3.37}$$

Thus, following definition (1.2.67) with the quantities (1.3.33) and (1.3.34), the Christoffel symbols can be interpreted as a gauge field

$$\Gamma^{\alpha}{}_{\beta\gamma} \equiv {}^{-1}\!\left(g^{\mu\nu}\big|^{\beta}_{\alpha}\right)\cdot\left(-g^{\mu\nu}{}_{,\gamma}\right). \tag{1.3.38}$$

The identity (1.3.37) is just the identity (1.2.67) formulated for the case of the Hilbert Lagrangian (1.3.1). Substituting to (1.3.37) the Lagrangian (1.3.1), one finally arrive to an interesting differential identity for the Einstein tensor defined above in (1.3.20) and (1.3.24),

$$2\nabla_{\alpha}\mathscr{G}_{\beta}{}^{\alpha} = 2\sqrt{-g}\nabla_{\alpha}G_{\beta}{}^{\alpha} \equiv 0 \tag{1.3.39}$$

which is known as the *Bianchi identity*. Thus, differentiating the Einstein equations (1.3.23) and keeping in mind (1.3.39), one obtains the differential conservation law for the matter metrical energy-momentum:

$$\nabla_{\alpha}T_{\beta}{}^{\alpha} = 0, \tag{1.3.40}$$

which is fully consistent with the matter equations of motion (1.3.21).

It is worth noticing that the Einstein way of derivation of the field equations was based on the requirement that the differential laws of motion of matter must be consistent with the field equations. Knowing the conservation law (1.3.40), Einstein was searching for an appropriate tensor in the left side of the field equations such that its divergence had to satisfy the identity of the type (1.3.39). Now, we know that for the Hilbert Lagrangian the only tensor of such type is the Einstein tensor $G_{\alpha\beta}$.

1.4 Classical conserved quantities in general relativity

1.4.1 The third Noether's theorem

Elaborating on the results of the first and second theorems, Noether has formulated the statement often referred as the *third Noether's theorem* [73, 74, 85, 266, 398] which is explicitly present in her paper's Section 6 "An Assertion of Hilbert" [278, 335], and formulates like this:

– *If the action S is invariant under an infinite continuous group of transformations $G_{\infty n}$, then a quantity constructed from the Lagrangian derivatives is expressed through a double divergence of a special quantity, the so-called superpotential.*

However, Noether had neither proved this statement nor provided a mathematical recipe for the superpotential construction. This is perhaps because an analogous statement had been proven by Klein (*Klein's boundary theorem*) in his work [258] that appeared prior to the Noether's paper. Furthermore, it was Klein who supplied the recipe for the superpotential construction.

In this subsection, we follow the book by Mitzkevich [316] to prove the third Noether's theorem. The procedure is applied directly to an arbitrary *covariant* field theory with the Lagrangian being a scalar density of the weight +1 to explore the invariance of the Lagrangian with respect to diffeomorphisms. Assuming further application of this study to the metric theories of gravity, we work with the Lagrangian which depends on the field variables, ψ^A, as well as on their first- and second-order derivatives. Such dependence is common to all metric theories which Lagrangians depend algebraically on the Riemann tensor[9].

Let us consider an arbitrary field theory with the action (1.2.36). Considering its invariance with respect to diffeomorfisms, we use the variations

$$\delta x^\alpha = \xi^\alpha(x), \tag{1.4.1}$$

$$\delta\psi^A = \pounds_\xi \psi^A = -\xi^\alpha \partial_\alpha \psi^A + \psi^A\big|^\alpha_\beta \partial_\alpha \xi^\beta \tag{1.4.2}$$

as a particular case of variations (1.2.61) and (1.2.62). Then, the main Noether's identity (1.2.52), now without the permissible term $\mathscr{B}^\nu$, is rewritten in the form:

9 This includes general relativity.

$$\frac{\delta \mathscr{L}}{\delta \psi^B} \pounds_\xi \psi^B + \partial_\alpha \left[\frac{\delta \mathscr{L}}{\delta \psi^B{}_{,\alpha}} \pounds_\xi \psi^B + \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\beta\alpha}} (\pounds_\xi \psi^B)_{,\beta} + \xi^\alpha \mathscr{L} \right] \equiv 0. \tag{1.4.3}$$

Here, we prefer to use the partial derivatives of the dynamical variables because in the metric theories the field variables are components of the metric tensor which covariant derivatives vanish identically, and the formalism may stall.

Substituting (1.4.2) into (1.4.3), reshuffling the terms by making use of the Leibniz rule, and converting the divergences, one can arrange terms in the series with respect to the vector field ξ^α, defining the diffeomorphism, and its first and second partial derivatives:

$$\begin{aligned} &- \left[\frac{\delta \mathscr{L}}{\delta \psi^B} \psi^B{}_{,\alpha} + \partial_\beta \left(\frac{\delta \mathscr{L}}{\delta \psi^B} \, \psi^B \big|^\beta_\alpha \right) \right] \xi^\alpha \\ &+ \partial_\alpha \left[\mathscr{U}_\sigma{}^\alpha \xi^\sigma + \mathscr{M}_\sigma{}^{\alpha\tau} \partial_\tau \xi^\sigma + \mathscr{N}_\sigma{}^{\alpha\tau\beta} \partial_{\beta\tau} \xi^\sigma \right] \equiv 0, \end{aligned} \tag{1.4.4}$$

where the (tensor) coefficients of the expansion are fully determined by the Lagrangian and its derivatives

$$\mathscr{U}_\sigma{}^\alpha \equiv \mathscr{L} \delta^\alpha_\sigma + \frac{\delta \mathscr{L}}{\delta \psi^B} \, \psi^B \big|^\alpha_\sigma - \frac{\delta \mathscr{L}}{\delta \psi^B{}_{,\alpha}} \partial_\sigma \psi^B - \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\beta\alpha}} \partial_{\beta\sigma} \psi^B \,, \tag{1.4.5}$$

$$\mathscr{M}_\sigma{}^{\alpha\tau} \equiv \frac{\delta \mathscr{L}}{\delta \psi^B{}_{,\alpha}} \, \psi^B \big|^\tau_\sigma - \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\tau\alpha}} \partial_\sigma \psi^B + \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\beta\alpha}} \partial_\beta (\psi^B \big|^\tau_\sigma) \,, \tag{1.4.6}$$

$$\mathscr{N}_\sigma{}^{\alpha\tau\beta} \equiv \frac{1}{2} \left[\frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\beta\alpha}} \, \psi^B \big|^\tau_\sigma + \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\tau\alpha}} \, \psi^B \big|^\beta_\sigma \right] . \tag{1.4.7}$$

Notice that in order to derive the coefficient (1.4.7) the symmetry property has been used, $\mathscr{N}_\sigma{}^{\alpha\tau\beta} = \mathscr{N}_\sigma{}^{\alpha\beta\tau}$, that follows directly from (1.4.4) due to the commutation property of the second partial derivatives.

Executing the operation of the partial derivative in the identity (1.4.4) and taking into account that the vector field ξ^σ, and all its partial derivatives are independent and arbitrary at each point of spacetime manifold, we come to the conclusion that all coefficients coupled with ξ^σ, and its partial derivatives must be separately equal to zero. It yields the system of identities:

$$\partial_\alpha \mathscr{U}_\sigma{}^\alpha \equiv \frac{\delta \mathscr{L}}{\delta \psi^B} \psi^B{}_{,\sigma} + \partial_\beta \left(\frac{\delta \mathscr{L}}{\delta \psi^B} \, \psi^B \big|^\beta_\sigma \right) , \tag{1.4.8}$$

$$\mathscr{U}_\sigma{}^\alpha + \partial_\lambda \mathscr{M}_\sigma{}^{\lambda\alpha} \equiv 0, \tag{1.4.9}$$

$$\mathscr{M}_\sigma{}^{(\alpha\beta)} + \partial_\lambda \mathscr{N}_\sigma{}^{\lambda(\alpha\beta)} \equiv 0, \tag{1.4.10}$$

$$\mathscr{N}_\sigma{}^{(\alpha\beta\gamma)} \equiv 0. \tag{1.4.11}$$

The system (1.4.8–1.4.11) was engineered by Klein [258]. Therefore, we shall refer to this system as the *Klein identities*. After differentiating (1.4.9) and using (1.4.10) and

(1.4.11) one obtains that $\partial_\alpha \mathscr{U}_\sigma{}^\alpha \equiv 0$. This means that the right hand side of (1.4.8) must be equal to zero identically as well,

$$\frac{\delta \mathscr{L}}{\delta \psi^B} \psi^B{}_{,\alpha} + \partial_\beta \left(\frac{\delta \mathscr{L}}{\delta \psi^B} \psi^B \Big|^\beta_\alpha \right) \equiv 0. \tag{1.4.12}$$

It repeats the claim (1.2.66) of the second Noether's theorem. Taking into account the historic development of the theory, we call the system (1.4.8) – (1.4.12) as the *Klein-Noether identities*.

The identity (1.4.12) suggests that instead of (1.4.4) one can use independently (1.4.12) and

$$\partial_\alpha \left[\mathscr{U}_\sigma{}^\alpha \xi^\sigma + \mathscr{M}_\sigma{}^{\alpha\tau} \partial_\tau \xi^\sigma + \mathscr{N}_\sigma{}^{\alpha\tau\beta} \partial_{\beta\tau} \xi^\sigma \right] \equiv 0. \tag{1.4.13}$$

The vector density entering under the divergence is the current

$$\mathscr{I}^\alpha(\xi) \equiv - \left[\mathscr{U}_\sigma{}^\alpha \xi^\sigma + \mathscr{M}_\sigma{}^{\alpha\tau} \partial_\tau \xi^\sigma + \mathscr{N}_\sigma{}^{\alpha\tau\beta} \partial_{\beta\tau} \xi^\sigma \right]. \tag{1.4.14}$$

The minus sign is chosen for making a correspondence with the minus sign in front of the gravitational (metric) action, like in (1.3.10) or in (1.3.11). Thus, the identity (1.4.13) is rewritten as

$$\partial_\alpha \mathscr{I}^\alpha(\xi) \equiv 0. \tag{1.4.15}$$

Because it is the identity, the current has to be expressed through a tensorial quantity (superpotential), $\mathscr{I}^\alpha(\xi) \equiv \partial_\beta \mathscr{I}^{\alpha\beta}(\xi)$, a double divergence of which has to be equal to zero identically: $\partial_{\alpha\beta} \mathscr{I}^{\alpha\beta}(\xi) \equiv 0$. Let us show this is indeed true. Due to the symmetry with respect to the last two indices in (1.4.7) and the identity (1.4.11), one has

$$\mathscr{N}_\sigma{}^{\alpha\tau\beta} + \mathscr{N}_\sigma{}^{\tau\beta\alpha} + \mathscr{N}_\sigma{}^{\beta\alpha\tau} \equiv 0. \tag{1.4.16}$$

Substituting (1.4.9) into (1.4.14), using (1.4.10) and (1.4.16), one obtains

$$\mathscr{I}^\alpha(\xi) \equiv \partial_\beta \left(\mathscr{M}_\sigma{}^{\beta\alpha} \xi^\sigma + 2\mathscr{N}_\sigma{}^{\beta\alpha\lambda} \partial_\lambda \xi^\sigma \right). \tag{1.4.17}$$

Due to (1.4.15), we should expect that a divergence of the right hand side of (1.4.17) would vanish. However, this is not obvious at the first glance. Nevertheless, there is a possibility to show this explicitly. Let us add the identical zero term, $\frac{4}{3}\partial_{\beta\lambda} \left(\mathscr{N}_\sigma{}^{[\lambda\beta]\alpha} \xi^\sigma \right) \equiv 0$, to the right side of (1.4.17). Then, after using (1.4.10) and (1.4.16), one gets

$$\mathscr{I}^\alpha(\xi) \equiv \partial_\beta \left(-\mathscr{M}_\sigma{}^{[\alpha\beta]} \xi^\sigma + \frac{2}{3} \partial_\lambda \mathscr{N}_\sigma{}^{[\alpha\beta]\lambda} \xi^\sigma - \frac{4}{3} \mathscr{N}_\sigma{}^{[\alpha\beta]\lambda} \partial_\lambda \xi^\sigma \right). \tag{1.4.18}$$

Now, the expression in brackets in the right side of (1.4.18) is explicitly antisymmetrical in α and β, and its divergence vanishes. Therefore, both expressions for the current, (1.4.17) and (1.4.18), can be rewritten in the form of the divergence,

$$\mathscr{I}^{\alpha}(\xi) \equiv \partial_{\beta}\mathscr{I}^{\alpha\beta}(\xi) \tag{1.4.19}$$

where

$$\mathscr{I}^{\alpha\beta}(\xi) = \mathscr{M}_{\sigma}{}^{\beta\alpha}\xi^{\sigma} + 2\mathscr{N}_{\sigma}{}^{\beta\alpha\lambda}\partial_{\lambda}\xi^{\sigma}, \tag{1.4.20}$$

corresponds to (1.4.17) and

$$\mathscr{I}^{\alpha\beta}(\xi) = -\left(\mathscr{M}_{\sigma}{}^{[\alpha\beta]}\xi^{\sigma} - \frac{2}{3}\partial_{\lambda}\mathscr{N}_{\sigma}{}^{[\alpha\beta]\lambda}\xi^{\sigma} + \frac{4}{3}\mathscr{N}_{\sigma}{}^{[\alpha\beta]\lambda}\partial_{\lambda}\xi^{\sigma}\right) \tag{1.4.21}$$

corresponds to (1.4.18) respectively. The quantities like $\mathscr{I}^{\alpha\beta}$ are called the *superpotentials*. In both cases $\partial_{\alpha\beta}\mathscr{I}^{\alpha\beta}(\xi) \equiv 0$. Thus, the identity (1.4.19) can be considered as one being equivalent to the conservation law (1.4.15) for the current.

Summing up the results, one rewrites the identity (1.4.4) in terms of the superpotential,

$$-\left[\frac{\delta\mathscr{L}}{\delta\psi^{B}}\psi^{B}{}_{,\alpha} + \partial_{\beta}\left(\frac{\delta\mathscr{L}}{\delta\psi^{B}}\,\psi^{B}\Big|^{\beta}_{\alpha}\right)\right]\xi^{\alpha} \equiv \partial_{\alpha\beta}\mathscr{I}^{\alpha\beta}(\xi). \tag{1.4.22}$$

This identity represents the statement of the third Noether's theorem while the identity (1.4.19) corresponds to the Klein boundary theorem mentioned above. Finally, repeating calculations from (1.2.83) to (1.2.87), one defines a conserved quantity, $\mathscr{P}(\xi)$,

$$\mathscr{P}(\xi) = \int_{\Sigma} d^{3}x\,\mathscr{I}^{0}(\xi). \tag{1.4.23}$$

which is effectively reduced to a surface (boundary) integral

$$\mathscr{P}(\xi) = \oint_{\partial\Sigma} ds_{i}\,\mathscr{I}^{0i}(\xi), \tag{1.4.24}$$

due to the identity (1.4.19) and definition (1.4.21). This integral relation supports Klein's assertion given in his boundary theorem related to the formulation of conserved quantities.

1.4.2 Pseudotensors and superpotentials

The metric tensor in general relativity (and other metric-based theories of gravity) plays a double role. From one side it describes the geometric properties of spacetime manifold but on the other side is a dynamical field. This double role of the metric tensor is a source of theoretical difficulties in the problem of construction of conserved quantities associated with gravitational field. For example, the *total* symmetric energy-momentum tensor for gravity and matter vanishes identically due to fact that the field equations hold. Indeed, in general relativity variation of the action (1.3.15) with the Lagrangian (1.3.16) with respect to $g^{\alpha\beta}$ gives the total symmetric energy-momentum tensor density[10]:

$$\mathcal{T}^{\text{tot}}_{\alpha\beta} \equiv 2\frac{\delta \mathcal{L}_{EH}}{\delta g^{\alpha\beta}} = -\frac{1}{\kappa}\left(\mathcal{G}_{\alpha\beta} - \kappa \mathcal{T}_{\alpha\beta}\right) \tag{1.4.25}$$

that disappears due to the Einstein equations (1.3.22). Therefore, the application of the Noether theorems to the case of the metric-based theories requires development of a more sophisticated procedure to constructing the canonical quantities corresponding the Noether currents. Einstein himself was the first who had suggested such a quantity – the *Einstein energy-momentum pseudotensor* for gravitational field. Einstein's idea was modified and used by other researchers who had suggested their own formulations of the pseudotensor of gravitational field and the corresponding superpotentials leading to the conserved quantities formulated in the form of a surface integral like that shown in (1.4.24).

It should be noticed that the pseudotensors and corresponding superpotentials, are only Lorentz covariant that is covariant under a linear transformation of coordinates but they are not transformed as tensors under general coordinate transformations. Besides, there is no unique recommendation for their construction. In this subsection, we outline the most interesting examples of the pseudotensors and superpotentials, which will be remarked later on in other chapters of the present book. Historically, various pseudotensors have been constructed by applying different (in some cases non-standard) approaches. However, we do not follow the historical development of the topic because it may be too confusing for the reader. To facilitate understanding of this subject, we *unify* the methods of construction of pseudotensors and superpotentials by making use of the Noether and Klein results as a powerful mathematical instrument.

Einstein's pseudotensor and Tolman's and Freud's superpotentials

The results of the third Noether's theorem are valid, if the action (1.2.36) of the theory is invariant with respect to diffeomorphisms. A question arises: can one apply these

10 Here we restore the dimensional Einstein constant, κ, useful for applications.

results to the action (1.3.11) with the non-covariant Einstein Lagrangian (1.3.9)? Generally speaking, the answer to this question is negative. However, let us assume that the transformation (1.4.1) is a simple coordinate shift:

$$\delta x^\alpha = \xi^\alpha = \mathrm{const}, \tag{1.4.26}$$

with ξ^α being a constant vector of translation along x^α. It is evident that under such a transformation the Lagrangian (1.3.9) is invariant and the Noether theorem is applied. Therefore, it makes sense to substitute (1.4.26) into the formulae of previous subsection to obtain physically sensible results. Because the vector field ξ^σ is constant all partial derivatives of ξ^σ vanish, and the current (1.4.14) takes the form:

$$\mathscr{I}^\alpha \to \mathscr{I}^\alpha_{(\sigma)} = \{-\mathscr{U}_\sigma{}^\alpha\}_E = -\frac{1}{2\kappa}\left(\frac{\partial \mathscr{L}_E}{\partial g^{\rho\lambda}{}_{,\alpha}}\partial_\sigma g^{\rho\lambda} - \delta^\alpha_\sigma \mathscr{L}_E - \frac{\delta \mathscr{L}_E}{\delta g^{\rho\lambda}}\, g^{\rho\lambda}\Big|^\alpha_\sigma\right). \tag{1.4.27}$$

We substitute the Einstein Lagrangian (1.3.9) to the right side of (1.4.27) and perform calculations. By simple inspection we can easily single out in the right side of (1.4.27) two terms:

$${}_E\boldsymbol{t}_\sigma{}^\alpha \equiv -\frac{1}{2\kappa}\left(\frac{\partial \mathscr{L}_E}{\partial g^{\rho\lambda}{}_{,\alpha}}\partial_\sigma g^{\rho\lambda} - \delta^\alpha_\sigma \mathscr{L}_E\right), \tag{1.4.28}$$

$$\mathscr{G}_\sigma{}^\alpha \equiv \frac{1}{2}\frac{\delta \mathscr{L}_E}{\delta g^{\rho\lambda}}\, g^{\rho\lambda}\Big|^\alpha_\sigma, \tag{1.4.29}$$

where ${}_E\boldsymbol{t}_\sigma{}^\alpha$ is the Einstein pseudotensor complex constructed merely of the metric tensor and its first partial derivatives, while $\mathscr{G}_\sigma{}^\alpha$ is the Einstein tensor density which we have already met in (1.3.20).

Explicit expression for the Einstein pseudotensor complex is

$$\begin{aligned}{}_E\boldsymbol{t}_\sigma{}^\alpha = \frac{1}{2\kappa}\Big[&\mathfrak{g}^{\rho\tau}\left(\Gamma^\lambda{}_{\rho\lambda}\Gamma^\alpha{}_{\tau\sigma} + \Gamma^\alpha{}_{\rho\tau}\Gamma^\lambda{}_{\lambda\sigma} - 2\Gamma^\alpha{}_{\rho\lambda}\Gamma^\lambda{}_{\tau\sigma}\right)\\ &-\mathfrak{g}^{\rho\tau}\left(\Gamma^\eta{}_{\rho\tau}\Gamma^\lambda{}_{\eta\lambda} - \Gamma^\eta{}_{\rho\lambda}\Gamma^\lambda{}_{\eta\tau}\right)\delta^\alpha_\sigma\\ &+\mathfrak{g}^{\alpha\lambda}\left(\Gamma^\tau{}_{\rho\tau}\Gamma^\rho{}_{\lambda\sigma} - \Gamma^\tau{}_{\lambda\tau}\Gamma^\rho{}_{\rho\sigma}\right)\Big].\end{aligned} \tag{1.4.30}$$

It was constructed by Einstein in 1918 [150–152] as the canonical energy-momentum for describing the energy carried out by the linearized gravitational waves.

Since the Einstein pseudotensor complex is a tensor analogue of the Noether's current, we can build a corresponding superpotential (1.4.20):

$$\mathscr{I}^{\alpha\beta} \to \mathscr{I}^{\alpha\beta}_{(\sigma)} = \left\{\mathscr{M}_\sigma{}^{\beta\alpha}\right\}_E = -\frac{1}{2\kappa}\frac{\partial \mathscr{L}_E}{\partial g^{\mu\nu}{}_{,\beta}}\, g^{\mu\nu}\big|^\alpha_\sigma. \tag{1.4.31}$$

Substituting the Einstein Lagrangian (1.3.9) into (1.4.31), one obtains the superpotential, $\left\{\mathscr{M}_\sigma{}^{\beta\alpha}\right\}_E = \mathscr{T}_\sigma{}^{\alpha\beta}$, in the explicit form:

$$\mathscr{T}_\sigma{}^{\alpha\beta} = \frac{\sqrt{-g}}{2\kappa}\left(2g^{\rho\alpha}\Gamma^\beta{}_{\rho\sigma} + 2\delta^{[\alpha}_\sigma g^{\beta]\pi}\Gamma^\rho{}_{\rho\pi} - g^{\alpha\beta}\Gamma^\rho{}_{\rho\sigma} - \delta^\alpha_\sigma\Gamma^\beta{}_{\mu\nu}g^{\mu\nu}\right). \tag{1.4.32}$$

Historically, it was the first superpotential in general relativity that has been constructed (by making use of a different approach) by Tolman [439] and carries his name.

Of course, the current (1.4.27) and the superpotential (1.4.31) satisfy the identities (1.4.13) and (1.4.19):

$$\partial_\alpha \mathscr{I}^\alpha_{(\sigma)} \equiv 0 \quad \to \quad \partial_\alpha \left\{-\mathscr{U}_\sigma{}^\alpha\right\}_E \equiv 0, \tag{1.4.33}$$

$$\mathscr{I}^\alpha_{(\sigma)} \equiv \partial_\beta \mathscr{I}^{\alpha\beta}_{(\sigma)} \quad \to \quad \left\{-\mathscr{U}_\sigma{}^\alpha\right\}_E \equiv \partial_\beta \left\{\mathscr{M}_\sigma{}^{\beta\alpha}\right\}_E. \tag{1.4.34}$$

Identities (1.4.33) and (1.4.34) are valid for arbitrary functional value of the metric tensor irrespectively whether they obey the field equations or not. To introduce a physical content one has to use the field equations governing the physical system. In the case of general relativity one has to use the Einstein equations (1.3.22), which allow us to rewrite the identity (1.4.33) in the form of a differential conservation law,

$$\partial_\alpha \left({}_E\boldsymbol{t}_\sigma{}^\alpha + \mathscr{T}_\sigma{}^\alpha\right) = 0, \tag{1.4.35}$$

for the sum of the Einstein pseudotensor, ${}_E\boldsymbol{t}_\sigma{}^\alpha$, and the energy-momentum tensor density of matter, $\mathscr{T}_\sigma{}^\alpha$. Note that in vacuum, $\mathscr{T}_\sigma{}^\alpha = 0$, equation (1.4.35) is transformed into the differential conservation law for the Einstein pseudotensor only:

$$\partial_\alpha \left({}_E\boldsymbol{t}_\sigma{}^\alpha\right) = 0. \tag{1.4.36}$$

In the same way the identity (1.4.34) transforms into the conservation law:

$${}_E\boldsymbol{t}_\sigma{}^\alpha + \mathscr{T}_\sigma{}^\alpha = \partial_\beta \mathscr{T}_\sigma{}^{\alpha\beta}. \tag{1.4.37}$$

Of course, by the general consideration given in the previous subsection, one has $\partial_{\alpha\beta}\mathscr{T}_\sigma{}^{\alpha\beta} \equiv 0$ although this identity is not apparent in (1.4.32). This might be seen easily if one could represent (1.4.32) explicitly as a skew-symmetric tensor density. This is usually achieved by making use of a trick used in derivation of (1.4.21) but it does not work in the case under consideration because $\left\{\mathscr{N}_\sigma{}^{\beta\alpha\gamma}\right\}_E \equiv 0$ for the Einstein Lagrangian. Fortunately, there is another way around found by Freud [179] who suggested to change the Tolman superpotential (1.4.32) as follows:

$$\mathscr{F}_\sigma{}^{\alpha\beta} = \mathscr{T}_\sigma{}^{\alpha\beta} + \frac{1}{\kappa}\partial_\lambda\left(\sqrt{-g}g^{\alpha[\beta}\delta^{\lambda]}_\sigma\right). \tag{1.4.38}$$

This change does not violate the right side of (1.4.37) because taking divergence from the second term in (1.4.38) yields an identical zero, $\partial_{\beta\lambda}\left(\sqrt{-g}g^{\alpha[\beta}\delta^{\lambda]}_{\sigma}\right) \equiv 0$. Straightforward calculation of the sum of two terms in (1.4.38) gives

$$\mathscr{F}_{\sigma}{}^{\alpha\beta} = \frac{\sqrt{-g}}{\kappa}\left(g^{\rho[\alpha}\Gamma^{\beta]}{}_{\rho\sigma} + \delta^{[\alpha}_{\sigma}g^{\beta]\pi}\Gamma^{\rho}{}_{\rho\pi} - \delta^{[\alpha}_{\sigma}\Gamma^{\beta]}{}_{\mu\nu}g^{\mu\nu}\right) \tag{1.4.39}$$

that is the famous Freud superpotential [179]. This superpotential is apparently skew-symmetric and allows one to write the conservation law (1.4.37) in the following form,

$${}_{E}\boldsymbol{t}_{\sigma}{}^{\alpha} + \mathscr{T}_{\sigma}{}^{\alpha} = \partial_{\beta}\mathscr{F}_{\sigma}{}^{\alpha\beta}, \tag{1.4.40}$$

where it is evident that $\partial_{\alpha\beta}\mathscr{F}_{\sigma}{}^{\alpha\beta} \equiv 0$ due to the antisymmetry of the Freud superpotential with respect to indices α and β.

The Mitzkevich and Møller conserved quantities

Now let us consider a pseudotensor and a superpotential corresponding to the action (1.3.10) with the Hilbert Lagrangian (1.3.1) and the diffeomorphism with the constant coordinate translation (1.4.26). The current (1.4.14) takes on the form:

$$\begin{aligned}\mathscr{I}^{\alpha} \to \mathscr{I}^{\alpha}_{(\sigma)} &= \{-\mathscr{U}_{\sigma}{}^{\alpha}\}_{H}\\ &= -\frac{1}{2\kappa}\left(\frac{\partial\mathscr{L}_H}{\partial g^{\mu\nu}{}_{,\beta\alpha}}\partial_{\beta\sigma}g^{\mu\nu} + \frac{\delta\mathscr{L}_H}{\delta g^{\mu\nu}{}_{,\alpha}}\partial_{\sigma}g^{\mu\nu} - \delta^{\alpha}_{\sigma}\mathscr{L}_H - \frac{\delta\mathscr{L}_H}{\delta g^{\mu\nu}}\, g^{\mu\nu}|^{\alpha}_{\sigma}\right),\end{aligned} \tag{1.4.41}$$

that again can be naturally split in two parts:

$${}_{H}\boldsymbol{t}_{\sigma}{}^{\alpha} \equiv -\frac{1}{2\kappa}\left(\frac{\partial\mathscr{L}_H}{\partial g^{\mu\nu}{}_{,\beta\alpha}}\partial_{\beta\sigma}g^{\mu\nu} + \frac{\delta\mathscr{L}_H}{\delta g^{\mu\nu}{}_{,\alpha}}\partial_{\sigma}g^{\mu\nu} - \delta^{\alpha}_{\sigma}\mathscr{L}_H\right), \tag{1.4.42}$$

$$\mathscr{G}_{\sigma}{}^{\alpha} \equiv \frac{1}{2}\frac{\delta\mathscr{L}_H}{\delta g^{\mu\nu}}\, g^{\mu\nu}|^{\alpha}_{\sigma}\,. \tag{1.4.43}$$

Here, ${}_{H}\boldsymbol{t}_{\sigma}{}^{\alpha}$ is the pseudotensor constructed with the use of the Hilbert Lagrangian (1.3.1)

$$\begin{aligned}{}_{H}\boldsymbol{t}_{\sigma}{}^{\alpha} = \frac{1}{\kappa}\Big(&\mathfrak{g}^{\rho(\alpha}\Gamma^{\beta)}{}_{\rho\tau}\Gamma^{\tau}{}_{\beta\sigma} - \mathfrak{g}^{\rho(\alpha}\Gamma^{\beta)}{}_{\rho\sigma}\Gamma^{\tau}{}_{\tau\beta} - \mathfrak{g}^{\rho\tau}\Gamma^{[\alpha}{}_{\beta\tau}\Gamma^{\beta]}{}_{\rho\sigma}\\ &- \mathfrak{g}^{\rho(\alpha}\Gamma^{\beta)}{}_{\rho\sigma,\beta} + \mathfrak{g}^{\alpha\rho}\Gamma^{\tau}{}_{\tau(\rho,\sigma)} + \frac{1}{2}\delta^{\alpha}_{\sigma}\mathscr{R}\Big)\end{aligned} \tag{1.4.44}$$

and the Einstein tensor density, $\mathscr{G}_{\sigma}{}^{\alpha}$, appears in (1.4.43), like in (1.4.29), because the Hilbert and Einstein Lagrangians differ by a total divergence whose variational derivative vanishes.

Now we consider the antisymmetric superpotential (1.4.21) that (for constant ξ^α) acquires the form:

$$\mathscr{I}^{\alpha\beta} \to \mathscr{I}^{\alpha\beta}_{(\sigma)} = \left\{-\mathscr{M}_\sigma{}^{[\alpha\beta]} + \frac{2}{3}\partial_\lambda \mathscr{N}_\sigma{}^{[\alpha\beta]\lambda}\right\}_H . \tag{1.4.45}$$

Using the formulae (1.4.6) and (1.4.7) for $\psi^B = \{g^{\mu\nu}\}$ with the Hilbert Lagrangian (1.3.1) one obtains

$$\left\{\mathscr{M}_\sigma{}^{\alpha\beta}\right\}_H = \frac{1}{2\kappa}\left(2\Gamma^\alpha{}_{\rho\sigma}\mathfrak{g}^{\beta\rho} - \mathfrak{g}^{\alpha\beta}\Gamma^\rho{}_{\rho\sigma} - \delta^\alpha_\sigma\Gamma^\beta{}_{\mu\nu}\mathfrak{g}^{\mu\nu}\right) , \tag{1.4.46}$$

$$\left\{\mathscr{N}_\sigma{}^{\alpha\beta\lambda}\right\}_H = \frac{1}{4\kappa}\left(2\delta^\alpha_\sigma\mathfrak{g}^{\beta\lambda} - \mathfrak{g}^{\alpha\beta}\delta^\lambda_\sigma - \mathfrak{g}^{\alpha\lambda}\delta^\beta_\sigma\right) . \tag{1.4.47}$$

Then, the superpotential (1.4.45) is reduced to a simple expression

$$\mathscr{I}^{\alpha\beta}_{(\sigma)} \equiv \mathscr{K}_\sigma{}^{\alpha\beta} = \frac{\sqrt{-g}}{\kappa} g^{\rho[\alpha}\Gamma^{\beta]}{}_{\rho\sigma}. \tag{1.4.48}$$

The quantities (1.4.44) and (1.4.48) have been suggested independently and almost at the same time by Mitzkevich [319] who followed the Noether procedure, and by Møller [321] who relied upon phenomenological arguments without applying the Noether theorems.

Of course, the quantities (1.4.41) and (1.4.45) satisfy the identities (1.4.13) and (1.4.19):

$$\partial_\alpha \mathscr{I}^\alpha_{(\sigma)} \equiv 0 \ \to \ \partial_\alpha \left\{-\mathscr{U}_\sigma{}^\alpha\right\}_H \equiv 0, \tag{1.4.49}$$

$$\mathscr{I}^\alpha_{(\sigma)} \equiv \partial_\beta \mathscr{I}^{\alpha\beta}_{(\sigma)} \ \to \ \left\{-\mathscr{U}_\sigma{}^\alpha\right\}_H \equiv \partial_\beta\left\{-\mathscr{M}_\sigma{}^{[\alpha\beta]} + \tfrac{2}{3}\partial_\lambda \mathscr{N}_\sigma{}^{[\alpha\beta]\lambda}\right\}_H . \tag{1.4.50}$$

Notice that the pseudotensor (1.4.44) has been obtained by direct calculation of derivatives in (1.4.42). A more economical way to get it is to use a combination of (1.4.41), (1.4.48) and the identity (1.4.50).

The relations, (1.4.49) and (1.4.50), are identities being valid for arbitrary metric tensor. To introduce a physical content to them one has to use the Einstein equations (1.3.22). Then the identity (1.4.49) becomes a differential conservation law,

$$\partial_\alpha \left({}_H\boldsymbol{t}_\sigma{}^\alpha + \mathscr{T}_\sigma{}^\alpha\right) = 0, \tag{1.4.51}$$

for the sum of the pseudotensor, ${}_H\boldsymbol{t}_\sigma{}^\alpha$, and the matter energy-momentum tensor density, $\mathscr{T}_\sigma{}^\alpha$. Again, in vacuum, $\mathscr{T}_\sigma{}^\alpha = 0$, and (1.4.51) is transformed into the differential conservation law for the pseudotensor only:

$$\partial_\alpha \left({}_H\boldsymbol{t}_\sigma{}^\alpha\right) = 0. \tag{1.4.52}$$

In a similar way the identity (1.4.34) transforms into a conservation law:

$${}_H\boldsymbol{t}_\sigma{}^\alpha + \mathscr{T}_\sigma{}^\alpha = \partial_\beta \mathscr{K}_\sigma{}^{\alpha\beta}. \tag{1.4.53}$$

Neither Einstein's nor the Mitzkevich-Møller pseudotensors and superpotentials are covariant. This restricts the range of applications of these quantities which are well defined only in the coordinates that are Lorentzian at infinity. The Einstein pseudotensor, like the Einstein Lagrangian itself, has some advantages in that it depends on the first derivatives only which permits to formulate the integration problem in a more economical way due to the Dirichlet boundary conditions imposed merely on the components of the metric tensor but not on its derivatives. On the other hand, the Mitzkevich-Møller pseudotensor (1.4.44) has its own advantage being "partially" covariant. Namely, the components ${}_H\boldsymbol{t}_\sigma{}^0$ of the pseudotensor are transformed as a 4-dimensional vector density under coordinate transformations of a particular type $x'^k = f^k(x^l)$, $x'^0 = x^0 + f^0(x^l)$ while the corresponding components of the Einstein pseudotensor do not. This property was one of the main requirements imposed by Møller on the conserved quantities in general relativity.

The above derivation of the Einstein and Mitzkevich-Møller conserved quantities can be generalized and applied to an arbitrary Lagrangian consisting of a linear superposition of the Einstein Lagrangian, $\mathscr{L}_E$, and a divergence,

$$\mathscr{L} = \mathscr{L}_E + \text{div}, \tag{1.4.54}$$

where the divergence (div) depends on the metric, $g_{\mu\nu}$, and its derivatives of an arbitrary order. The results of the third Noether's theorem are fully applicable to such a Lagrangian after a corresponding generalization of its derivation by including the terms containing the higher derivatives of the metric tensor. Making use of the group diffeomorphism (1.4.26) one can associate with the Lagrangian (1.4.54) its own canonical pseudotensor, ${}_C\boldsymbol{t}_\sigma{}^\alpha$, and superpotential. By choosing different expressions for the divergence we can get unlimited number of the pseudotensors and superpotentials. Analogously to (1.4.27) and (1.4.41) one can construct out of the Lagrangian (1.4.54), a current

$$\mathscr{I}^\alpha \to \mathscr{I}^\alpha_{(\sigma)} = \{-\mathscr{U}_\sigma{}^\alpha\} = {}_C\boldsymbol{t}_\sigma{}^\alpha + \frac{1}{\kappa}\mathscr{G}_\sigma{}^\alpha \tag{1.4.55}$$

depending on the canonical pseudotensor ${}_C\boldsymbol{t}_\sigma{}^\alpha$; and analogously to (1.4.31) and (1.4.45) one can construct a superpotential corresponding to (1.4.54):

$$\mathscr{I}^{\alpha\beta} \to \mathscr{I}^{\alpha\beta}_{(\sigma)} = \mathscr{S}_\sigma{}^{[\alpha\beta]}. \tag{1.4.56}$$

Papapetrou's symmetrization

Most of the pseudotensors discussed above have a serious problem with the definition of an angular momentum of an isolated gravitating system. For example, both the Einstein and Møller pseudotensors are non-symmetrical and cannot be directly used to describe the conservation law for the angular momentum. One needs to modify these expressions to make them symmetrical or to construct new symmetric pseudotensors as described in review [443]. This problem is similar to that we have with the canonical energy-momentum tensor of the field theories in the Minkowski space which are not symmetric due to the possible presence of spin, like in (1.2.109).

One way to resolve the problem is to construct a symmetric energy-momentum complex by the Belinfante procedure presented in (1.2.113–1.2.118). Such a procedure has been applied to symmetrization of the Einstein pseudotensor by Papapetrou [351]. However, his approach has a deficiency as it relies upon making use of the Minkowski metric for raising and lowering indices of tensors residing on a curved manifold. In other words, Papapetrou's procedure assumes that the dynamical metric, $g_{\mu\nu}$, is placed to the background Minkowski space with the Lorentzian coordinates which contradicts to the spirit of general relativity.

Nonetheless, Papapetrou calculated the spin density for the Einstein Lagrangian (1.4.6) by working with the Lorentzian coordinates and operating with the partial derivatives of the metric tensor in the equation (1.2.103) defining the spin density. Following Papapetrou, one obtains:

$$\begin{aligned} {}_E\boldsymbol{\sigma}^{\alpha\beta}{}_\sigma &= \frac{1}{2\kappa}\frac{\partial \mathscr{L}_E}{\partial\left(\mathfrak{g}^{\mu\nu}{}_{,\alpha}\right)}\,\mathfrak{g}^{\mu\nu}\big|^{\beta}_{\sigma} \\ &= \frac{1}{2\kappa}\left[\left(\mathfrak{g}^{\alpha\beta}\delta^\nu_\sigma + \delta^\alpha_\sigma\mathfrak{g}^{\beta\nu} - \mathfrak{g}^{\alpha\nu}\delta^\beta_\sigma\right)\Gamma^\rho{}_{\rho\nu}\right. \\ &\quad \left. -(\mathfrak{g}^{\mu\beta}\delta^\nu_\sigma + \delta^\mu_\sigma\mathfrak{g}^{\beta\nu} - \mathfrak{g}^{\mu\nu}\delta^\beta_\sigma)\Gamma^\alpha{}_{\mu\nu}\right]. \end{aligned} \tag{1.4.57}$$

Then by the rule (1.2.113) one constructs the corresponding Belinfante correction,

$${}_E\boldsymbol{b}^{\alpha\beta\gamma} = \frac{1}{\kappa}\left(\mathfrak{g}^{\mu[\alpha}\eta^{\beta]\nu}\Gamma^\gamma{}_{\mu\nu} + \mathfrak{g}^{\mu\gamma}\eta^{\nu[\alpha}\Gamma^{\beta]}{}_{\mu\nu} - \eta^{\mu\gamma}\mathfrak{g}^{\nu[\alpha}\Gamma^{\beta]}{}_{\mu\nu}\right), \tag{1.4.58}$$

which allows us to symmetrize the Einstein pseudotensor (1.4.30) by the Belinfante rule (1.2.114):

$${}_P\boldsymbol{t}_\sigma{}^\alpha = {}_E\boldsymbol{t}_\sigma{}^\alpha + \partial_\beta\left({}_E\boldsymbol{b}^{\alpha\beta\gamma}\eta_{\gamma\sigma}\right). \tag{1.4.59}$$

After straightforward but tedious calculations and raising the subscript index σ with the Minkowski metric, $\eta^{\sigma\beta}$, one obtains the Papapetrou energy-momentum complex

$$\begin{aligned} {}_P\boldsymbol{t}^{\alpha\beta} = \frac{1}{\kappa}\Big[&\frac{1}{2}\left(\mathfrak{g}^{\alpha\beta}\eta^{\rho\sigma} - \eta^{\alpha\beta}\mathfrak{g}^{\rho\sigma}\right)\Gamma^\lambda{}_{\rho\lambda,\sigma} + \left(\mathfrak{g}^{\rho\sigma}\eta^{\lambda(\alpha} - \eta^{\rho\sigma}\mathfrak{g}^{\lambda(\alpha}\right)\Gamma^{\beta)}{}_{\lambda\rho,\sigma} \\ &+ \eta^{\rho\sigma}\left(\mathfrak{g}^{\lambda\eta}\Gamma^{(\alpha}{}_{\lambda\rho}\Gamma^{\beta)}{}_{\eta\sigma} + \Gamma^\lambda{}_{\sigma\eta}\Gamma^{(\alpha}{}_{\lambda\rho}\mathfrak{g}^{\beta)\eta} - 2\Gamma^\lambda{}_{\sigma\lambda}\Gamma^{(\alpha}{}_{\eta\rho}\mathfrak{g}^{\beta)\eta}\right) \\ &+ \frac{1}{2}\left(\eta^{\rho\sigma}\mathfrak{g}^{\alpha\beta}\Gamma^\lambda{}_{\rho\lambda}\Gamma^\eta{}_{\sigma\eta} + \mathfrak{g}^{\lambda\eta}\eta^{\alpha\beta}\Gamma^\sigma{}_{\rho\lambda}\Gamma^\rho{}_{\sigma\eta}\right) \\ &+ \mathfrak{g}^{\lambda\eta}\left(\Gamma^\sigma{}_{\rho\sigma}\Gamma^{(\alpha}{}_{\lambda\eta} - \Gamma^\sigma{}_{\lambda\eta}\Gamma^{(\alpha}{}_{\rho\sigma} - \Gamma^\sigma{}_{\lambda\rho}\Gamma^{(\alpha}{}_{\eta\sigma}\right)\eta^{\beta)\rho}\Big]. \end{aligned} \tag{1.4.60}$$

The symmetry of this expression is the advantage with respect to the canonical energy-momentum complexes (1.4.30) or (1.4.44). On the other hand, it contains the second derivatives of the metric tensor that can be viewed as a disadvantage.

Application of the Belinfante correction modifies the Freud superpotential, $\mathscr{F}_\sigma{}^{\alpha\beta}$, as well. By adding $\partial_\beta({}_E\boldsymbol{b}^{\alpha\beta}{}_\sigma)$ to both sides of the conservation law (1.4.40) and using (1.4.59), one gets

$$ {}_P\boldsymbol{t}_\sigma{}^\alpha + \mathscr{T}_\sigma{}^\alpha = \partial_\beta \mathscr{P}_\sigma{}^{\alpha\beta} \tag{1.4.61}$$

where the quantity at the right side of this equation is known as the Papapetrou superpotential:

$$\mathscr{P}_\sigma{}^{\alpha\beta} = \mathscr{F}_\sigma{}^{\alpha\beta} + {}_E\boldsymbol{b}^{\alpha\beta}{}_\sigma = \frac{1}{\kappa}\partial_\lambda\left(\mathfrak{g}^{\rho[\alpha}\eta^{\beta]\lambda} - \mathfrak{g}^{\lambda[\alpha}\eta^{\beta]\rho}\right)\eta_{\rho\sigma}. \tag{1.4.62}$$

Let us make some remarks. First, the Papapetrou symmetrization applied to an arbitrary canonical pseudotensor in (1.4.55) and superpotential (1.4.56) related to the Lagrangian (1.4.54), yields the same results (1.4.60) and (1.4.62). This means that the divergence in the Lagrangian (1.4.54) is irrelevant to the result of the Belinfante-Papapetrou symmetrization. Later on, in Section 7.1.4, we will show it explicitly at a more general level.

Second, construction of the Belinfante correction in (1.4.57) and (1.4.58) was based on the application of the (external) Minkowski metric, $\eta_{\mu\nu}$, for raising and lowering indices. However, the Minkowski metric is a supplementary structure on the dynamical curved manifold. One may be tempted to use the dynamical metric $g_{\mu\nu}$ to rise and lower indices in the Papapetrou symmetrization procedure. However, this will make the Papapetrou pseudotensor exactly equal to the Einstein tensor with an opposite sign, thus, yielding

$$ {}_P\boldsymbol{t}_\sigma{}^\alpha + \frac{1}{\kappa}\mathscr{G}_\sigma{}^\alpha \longrightarrow -\frac{1}{\kappa}\mathscr{G}_\sigma{}^\alpha + \frac{1}{\kappa}\mathscr{G}_\sigma{}^\alpha \equiv 0. \tag{1.4.63}$$

In other words, the making use of the full metric in the Papapetrou symmetrization procedure leads to a degeneracy, that has been noticed and demonstrated for the first time by Szabados [426, 427], see also [25].

Generic approach to pseudotensors and superpotentials

Constructing canonical pseudotensors and superpotentials in (1.4.54) – (1.4.56) by making use of the Noether theorem offers a solid systematic approach to discussion of the conservation laws in general relativity. However, the way of constructing pseudotensors and superpotentials is broader than the standard canonical approach suggests. Indeed, in the canonical formalism the basic identity (1.4.19) with the current (1.4.55) and superpotential (1.4.56) has the form:

$$ {}_C\boldsymbol{t}_\sigma{}^\alpha + \frac{1}{\kappa}\mathscr{G}_\sigma{}^\alpha \equiv \partial_\beta \mathscr{S}_\sigma{}^{\alpha\beta}, \tag{1.4.64}$$

where $\mathscr{S}_\sigma{}^{\alpha\beta} = \mathscr{S}_\sigma{}^{[\alpha\beta]}$ is skew-symmetric with respect to the indices α and β. However, the identity like (1.4.64) can be always written down and used for defining a new pseudotensor by picking up a corresponding new superpotential even if it does not originate in the Noether formalism [364]. Let us demonstrate it.

To this end let us pick up a curved pseudo-Riemanian manifold with a metric and its partial derivatives defining its geometric structure. We construct an *arbitrary* anti-symmetric quantity $\mathscr{S}^{*\alpha\beta}_{\sigma} = \mathscr{S}^{*[\alpha\beta]}_{\sigma}$ out of the metric which automatically satisfies the differential identity, $\partial_{\alpha\beta}\mathscr{S}^{*\alpha\beta}_{\sigma} \equiv 0$ due to the commutation of the second partial derivatives. Next, we introduce a new quantity

$$\boldsymbol{t}^{*\alpha}_{\sigma} \equiv \partial_\beta \mathscr{S}^{*\alpha\beta}_{\sigma} - \frac{1}{\kappa}\mathscr{G}_\sigma{}^\alpha , \tag{1.4.65}$$

which is usually called an energy-momentum pseudotensor of gravitational field. Using the Einstein equations (1.3.22) to replace the Einstein tensor in (1.4.65), one obtains

$$\boldsymbol{t}^{*\alpha}_{\sigma} + \mathscr{T}_\sigma{}^\alpha = \partial_\beta \mathscr{S}^{*\alpha\beta}_{\sigma} . \tag{1.4.66}$$

The reader can easily recognize that (1.4.66) is nothing else but another form of the Einstein equations. The quantity $\mathscr{S}^{*\alpha\beta}_{\sigma}$ plays the role of the superpotential with a vanishing double divergence. Thus, taking divergence from both sides of (1.4.66) yields a differential conservation law

$$\partial_\alpha \left(\boldsymbol{t}^{*\alpha}_{\sigma} + \mathscr{T}_\sigma{}^\alpha\right) = 0 . \tag{1.4.67}$$

The above equations describe a generic formalism of construction of gravitational pseudotensors of weight +1. Landau and Lifshitz [285] had derived the gravitational pseudotensor of weight +2 which we discuss below in more detail. Goldberg [196] was able to further generalize the Landau-Lifshitz approach and suggested a whole family of symmetric pseudotensors and superpotentials with an arbitrary (integer) weight +n.

It is clear that formulae (1.4.65–1.4.67) offer unrestricted possibilities in constructing various pseudotensors and superpotentials. At the same time, they disclose a wide ambiguity in construction of conserved quantities in general relativity and show that there is no unique definition. The ambiguity can be restrained by making reasonable assumptions about physical and/or mathematical properties of pseudotensors like the absence of the metric tensor derivatives of higher order, symmetry, simplicity, physical meaningfulness, etc.

The Landau-Lifshitz pseudotensor

As an example of application of the generic formalism of pseudotensors we present the derivation of the famous Landau-Lifshitz pseudotensor and the corresponding superpotential [285]. The Landau-Lifshitz superpotential $\mathscr{S}^{\sigma\alpha\beta}_{LL}$ is defined in terms of the metric tensor density $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$ having weight +1 as a quadratic combination:

$$\mathscr{S}^{\sigma\alpha\beta}_{LL} = \frac{1}{2\kappa}\partial_\rho\left(\mathfrak{g}^{\sigma\alpha}\mathfrak{g}^{\rho\beta} - \mathfrak{g}^{\sigma\beta}\mathfrak{g}^{\alpha\rho}\right), \tag{1.4.68}$$

which apparently has weight +2.

The general equation (1.4.65) extended to the tensor density of weight +2, defines the Landau-Lifshitz pseudotensor:

$$\boldsymbol{t}_{LL}^{\alpha\beta} = \partial_\rho \mathscr{S}_{LL}^{\alpha\beta\rho} - \frac{1}{\kappa}\sqrt{-g}\mathscr{G}^{\alpha\beta}. \tag{1.4.69}$$

Direct calculation of (1.4.69) yields:

$$\begin{aligned}
\boldsymbol{t}_{LL}^{\alpha\beta} = \frac{1}{2\kappa}\Big[&\left(2\Gamma^{\sigma}{}_{\mu\nu}\Gamma^{\rho}{}_{\sigma\rho} - \Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} - \Gamma^{\sigma}{}_{\mu\sigma}\Gamma^{\rho}{}_{\nu\rho}\right)\left(\mathfrak{g}^{\alpha\mu}\mathfrak{g}^{\beta\nu} - \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu}\right)\\
&+ \mathfrak{g}^{\alpha\mu}\mathfrak{g}^{\nu\sigma}\left(\Gamma^{\beta}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} + \Gamma^{\beta}{}_{\nu\sigma}\Gamma^{\rho}{}_{\mu\rho} - \Gamma^{\beta}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\beta}{}_{\mu\nu}\Gamma^{\rho}{}_{\sigma\rho}\right)\\
&+ \mathfrak{g}^{\beta\mu}\mathfrak{g}^{\nu\sigma}\left(\Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} + \Gamma^{\alpha}{}_{\nu\sigma}\Gamma^{\rho}{}_{\mu\rho} - \Gamma^{\alpha}{}_{\sigma\rho}\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\rho}{}_{\sigma\rho}\right)\\
&+ \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\sigma\rho}\left(\Gamma^{\alpha}{}_{\mu\sigma}\Gamma^{\beta}{}_{\nu\rho} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\sigma\rho}\right)\Big].
\end{aligned} \tag{1.4.70}$$

The advantage of the Landau-Lifshitz pseudotensor is that it contains only the first derivatives of the metric tensor and, at the same time, is symmetric. Thus, it simultaneously bears the advantages of Einstein's (1.4.30) and Papapetrou's (1.4.60) pseudotensors. The difference of the Landau-Lifshitz pseudotensor (1.4.70) from other pseudotensors is that it has weight +2 while the other pseudotensors have weight +1. It is interesting to notice that Babak and Grishchuk [21] showed how to apply the Lagrangian formalism to derive a covariant analog of (1.4.70) with a more commonly used weight +1, see Sections 2.4.1 and 2.4.1.

Using the Einstein equations in equation (1.4.69), one obtains a conservation law:

$$\boldsymbol{t}_{LL}^{\mu\nu} + \sqrt{-g}\mathscr{T}^{\mu\nu} = \partial_\beta \mathscr{S}_{LL}^{\mu\nu\beta}, \tag{1.4.71}$$

which differential form is

$$\partial_\nu\left(\boldsymbol{t}_{LL}^{\mu\nu} + \sqrt{-g}\mathscr{T}^{\mu\nu}\right) = 0. \tag{1.4.72}$$

The covariance problem of pseudotensors

All pseudotensors are non-covariant - they are not transformed as tensors under arbitrary coordinate transformations which makes the physical interpretation of the conserved quantities in general relativity more difficult. The problem has been already known to Einstein who was trying to circumvent it by applying some reasonable arguments preventing the appearance of unphysical results. In particular, he noticed that in spite of the fact that the pseudotensors are not covariant the local conservation laws, like (1.4.67), are similar to the continuity equation which describe a local *balance* between the densities of energy and linear momentum of matter and gravitational field as well as between the density of the linear momentum and stresses of matter and gravitational field. To justify the physical meaningfulness of the pseudotensor, Einstein [151] appealed to an example of a physical system consisting of two point-like gravitating masses kept in an equilibrium by a rigid rod placed between them. The linear momentum of the whole system including the gravitational field is nil while the rod has a mechanical stress described by $\mathcal{T}_i^j$ in (1.4.67) which compensates the stress of the gravitational field described by $\boldsymbol{t}_i^{*j}$. Einstein had also suggested a simple method for "localization" of gravitational field and calculation of its energy in case of an isolated astronomical system [150] which is based on introduction of the global coordinates smoothly matching the "Galilean space" and "Galilean coordinates" at infinity. Finally, he had used his energy-momentum complex to describe physical properties of weak gravitational waves which he considered as the small perturbations with respect to the Minkowski metric [152].

Einstein's ideas have received further development in the mathematical approach based on introducing to the dynamical spacetime manifold with the metric tensor $g_{\mu\nu}$ a fixed Minkowskian background with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ in the Lorentzian coordinates. The full metric $g_{\mu\nu}$ is linearly decomposed into the Minkowski metric and a perturbation $l^{\mu\nu}$ which is considered as a tensor in the Minkowski space. In this approach one can operate with arbitrary curvilinear coordinates, x^α, introduced on the background Minkowskian manifold. So that the mentioned decomposition for description of the gravitational waves takes on the form

$$g^{\mu\nu} = \gamma^{\mu\nu} + l^{\mu\nu}, \tag{1.4.73}$$

where $\gamma_{\mu\nu}$ is the Minkowski metric in the curvilinear coordinates. All geometric objects residing on the background manifold can be made covariant by replacing the partial derivatives in the Lorentzian coordinates to the covariant derivatives with the Christoffel symbols $C^\alpha{}_{\mu\nu}$ constructed from the Minkowski metric $\gamma_{\mu\nu}$ in the curvilinear coordinates (see (1.2.9)).

The covariantization procedure described above allows us to construct the conserved quantities by making use of the Killing vectors ξ_K^σ of the background Minkowski space which are defined in the Lorentzian coordinates by (1.1.70). The left hand side in (1.4.66) is reformulated as the conserved vector density (current) defined as

$$\mathcal{J}^{*\alpha}(\xi_K) \equiv (\sqrt{-\gamma})^{1-n} \left(\boldsymbol{t}_\sigma^{*\alpha} + \mathcal{T}_\sigma^{*\alpha}\right) \xi_K^\sigma \,, \tag{1.4.74}$$

where we have generalized the pure gravitational energy-momentum complex in (1.4.66) to a pseudotensor $\boldsymbol{t}_\sigma^{*\alpha}$ of a generic weight $+n$ and a matter energy-momentum tensor density $\mathcal{T}_\sigma^{*\alpha}$ of a generic weight $+n$ also[11]. We emphasize that if $\boldsymbol{t}_\sigma^{*\alpha}$ is nonsymmetrical we can use only four translational Killing vectors to build four conserved currents while for the symmetric $\boldsymbol{t}_\sigma^{*\alpha}$ all ten Killing vectors can be used.

11 See, for example, the Landau-Lifshitz equation (1.4.71) with the related quantities $\boldsymbol{t}^*_{\mu\nu} = \boldsymbol{t}^{LL}_{\mu\nu}$ and $\mathcal{T}^*_{\mu\nu} = \sqrt{-g}\,\mathcal{T}_{\mu\nu}$ of weight +2.

The generalized conservation law (1.4.67) takes on the following form:

$$\partial_\alpha \mathcal{J}^{*\alpha}(\xi_K) = 0, \tag{1.4.75}$$

which can be integrated in a close analogy with (1.2.83). The corresponding integral conserved quantity is:

$$\mathcal{P}^*(\xi_K) = \int_\Sigma d^3x\, \mathcal{J}^{*0}(\xi_K). \tag{1.4.76}$$

The superpotential in the right side of (1.4.66) can be also covariantized:

$$\mathcal{J}^{*\alpha\beta}(\xi_K) = (\sqrt{-\gamma})^{1-n} \mathcal{S}_\sigma^{*\alpha\beta} \xi_K^\sigma, \tag{1.4.77}$$

where we have again generalized it to the weight $+n$. Finally, one can rewrite (1.4.66) in a fully covariant form:

$$\mathcal{J}^{*\alpha}(\xi_K) = \partial_\beta \mathcal{J}^{*\alpha\beta}(\xi_K), \tag{1.4.78}$$

which allows us to represent the conserved quantity (1.4.76) in the form of a surface integral:

$$\mathcal{P}^*(\xi_K) = \oint_{\partial\Sigma} ds_i \mathcal{J}^{*0i}(\xi_K) = \oint_{\partial\Sigma} ds_i (\sqrt{-\gamma})^{1-n} \mathcal{S}_\sigma^{*0i} \xi_K^\sigma. \tag{1.4.79}$$

The Komar superpotential

The covariantization of the classical pseudotensors and superpotentials by making use of the background Minkowski manifold is a useful mathematical device to better understanding the nature of the conservation laws in general relativity and the other field theories on curved manifolds. Nonetheless, this approach suffers from criticism that the background Minkowski space is not directly observable in general relativity. Furthermore, the conserved quantities crucially depend on the choice of the background manifold which brings about an uncertainty in their definition.

Komar [265] have found a genuine covariant definition of the conserved quantities in general relativity which does not depend on the splitting of the metric tensor in the background metric and a perturbation like in (1.4.73). Komar [265] worked with the Hilbert Lagrangian (1.3.1) which is generally covariant. In this case, constructing the conservation laws can done with the help of arbitrary displacement vectors $\xi^\mu(x)$, like in (1.4.1), instead of the constant coordinate shifts (1.4.26). Komar used the general expression for the current (1.4.14) and for the superpotential (1.4.21). After substituting the quantities (1.4.46), (1.4.47), and (1.4.50) into these expressions he obtained a covariant current and a famous Komar superpotential:

$$\mathcal{K}^{\alpha\beta} = \frac{\sqrt{-g}}{\kappa} \nabla^{[\alpha} \xi^{\beta]}. \tag{1.4.80}$$

Notice that if we chose ξ^μ = const in (1.4.80) it becomes equivalent to the Møller superpotential (1.4.48), and the Komar current goes to the Møller current (1.4.44). Unfortunately, in spite of the great advantage to be covariant the Komar superpotential has a major problem – it does not reproduce the correct angular momentum-to-mass ratio in the Kerr solution, see, e. g., [250, 316]. We discuss this problem in next section in more detail.

The problem of constructing various both not covariant and covariant approaches to pseudotensors and superpotentials has been tackled by a number of notable relativists including Bergman [38], Goldberg [196], Mitzkevich [316], Møller [323], and the others. A comprehensive review of early work on pseudotensors and superpotentials was given by Trautman [443].

1.5 Applications

It is instructive to apply the results of the previous sections to physical models describing isolated astronomical systems like a single or binary star, the solar system, globular stellar cluster, etc. We shall consider the case of slowly moving sources with a weak gravitational field when the relativistic effects of curved spacetime are small and a, so-called, linearized approximation of general relativity is sufficient. It means that the metric tensor perturbations are considered as functions describing small deviations from the Minkowski space which are found by solving the linearized Einstein equations with the help of the conventional methods of mathematical physics. We employ this approach in this section to discuss the problem of energy carried out by weak gravitational waves emitted by the isolated gravitating systems.

1.5.1 Linearized general relativity

The action and the field equations

Let us consider the Einstein equations in the form (1.3.23). They have been obtained by the variation of the gravitational Lagrangian with respect to the contravariant metric tensor $g^{\alpha\beta}$. We linearize (1.3.23) with respect to the Minkowski space by making use of the following decomposition,

$$g^{\alpha\beta} = \gamma^{\alpha\beta} + l^{\alpha\beta}, \tag{1.5.1}$$

and assuming that the absolute value of the metric tensor perturbations, $l^{\alpha\beta}$, are small compared with the components of the Minkowski metric $\gamma^{\alpha\beta}$ in curvilinear coordinates, $|l^{\alpha\beta}| \ll |\gamma^{\alpha\beta}|$.

The linearized approximation of the Chistoffel symbols (1.3.5) reads:

$$\Gamma^\alpha{}_{\mu\nu} = C^\alpha{}_{\mu\nu} + \Delta^\alpha{}_{\mu\nu}, \tag{1.5.2}$$

where $C^{\alpha}{}_{\mu\nu}$ are the Christoffel symbols (1.2.9) made out of the Minkowski metric in the curvilinear coordinates, and

$$\Delta^{\alpha}{}_{\mu\nu} = -\frac{1}{2}\left(l^{\alpha}{}_{\mu;\nu} + l^{\alpha}{}_{\nu;\mu} - l_{\nu\mu}{}^{;\alpha}\right), \tag{1.5.3}$$

is the linearized perturbation of the Christoffel symbols, and the semicolon denotes the covariant derivatives with respect to the metric $\gamma_{\alpha\beta}$.

We shall consider gravitational field in vacuum, thus, neglecting the energy-momentum tensor density of matter in (1.3.23). Then, keeping in mind (1.5.1) and (1.5.3), we calculate the linearized part of the Einstein tensor in (1.3.23) in the form,

$$G^L_{\alpha\beta} \equiv \frac{1}{2}\left(l_{\alpha\beta}{}^{;\rho}{}_{;\rho} + \gamma_{\alpha\beta} l^{\rho\sigma}{}_{;\rho\sigma} - l^{\rho}{}_{\alpha;\beta\rho} - l^{\rho}{}_{\beta;\alpha\rho} - \gamma_{\alpha\beta} l^{\pi}{}_{\pi}{}^{;\rho}{}_{;\rho} + l^{\pi}{}_{\pi;\alpha\beta}\right). \tag{1.5.4}$$

Linearized Einstein equations describing gravitational field in vacuum are:

$$G^L_{\alpha\beta} = 0, \tag{1.5.5}$$

which also implies that the linearized Ricci tensor $R^L_{\alpha\beta}$ vanishes, $R^L_{\alpha\beta} = 0$.

The field $l^{\alpha\beta}$ can be thought as a dynamical field propagating in the Minkowski space in accordance with the principle of the least action which variation with respect to $l^{\alpha\beta}$ leads to equations (1.5.5). The corresponding Lagrangian $\mathscr{L}^G$ of the gravitational action for the field $l^{\alpha\beta}$ can be taken as the Hilbert Lagrangian $\mathscr{L}_H$ in (1.3.9). We expand the Hilbert Lagrangian in the Taylor series with respect to $l^{\alpha\beta}$,

$$\mathscr{L}_H = \mathscr{L}^g_0 + \mathscr{L}^g_1 + \mathscr{L}^g_2 + \dots, \tag{1.5.6}$$

and take into account that the curvature of the Minkowski space is nil and the linearized Riemann tensor vanishes. It yields $\mathscr{L}^g_0 = 0$ and $\mathscr{L}^g_1 = 0$ correspondingly. The remaining quadratic term $\mathscr{L}^g_2$ can be simplified by discarding all terms which form the divergence. After long and tedious calculations it results in

$$\mathscr{L}^g_2 = \frac{1}{2}\sqrt{-\gamma}\left(l^{\rho\sigma}{}_{;\pi} l_{\rho}{}^{\pi}{}_{;\sigma} - l^{\rho\sigma}{}_{;\sigma} l_{\pi}{}^{\pi}{}_{;\rho} + \frac{1}{2} l_{\pi}{}^{\pi}{}_{;\rho} l_{\tau}{}^{\tau;\rho} - \frac{1}{2} l^{\rho\sigma;\pi} l_{\rho\sigma;\pi}\right). \tag{1.5.7}$$

The reader can check by inspection that (1.5.7) coincides with the Einstein Lagrangian (1.3.9) if one discards all terms of the third and higher order with respect to $l^{\alpha\beta}$. Eventually, the action for free gravitational field $l^{\rho\sigma}$ is

$$S_2 = -\frac{1}{2\kappa c}\int d^4x \mathscr{L}^g_2, \tag{1.5.8}$$

where $\kappa = 8\pi G/c^4$. One can easily check that by varying the action (1.5.8) with respect to $l^{\alpha\beta}$ yields expression (1.5.4),

$$\frac{\delta \mathscr{L}_2^g}{\delta l^{\alpha\beta}} = \mathscr{G}^L_{\alpha\beta} = \sqrt{-\gamma} G^L_{\alpha\beta}, \tag{1.5.9}$$

and, consequently, the least action principle leads to equations (1.5.5).

Energy-momentum tensor of a weak gravitational field

Formulae (1.5.4–1.5.9) represent the linearized general relativity in vacuum as a covariant theory of a weak gravitational field $l^{\mu\nu}$ in the Minkowski space. Therefore, one can apply the formalism developed in the Section 1.2 to derive the energy-momentum tensor of the gravitational field.

First, we calculate the canonical energy-momentum given by formula (1.2.104). Substituting the Lagrangian (1.5.7) to this formula, yields

$$\begin{aligned} {}_C\boldsymbol{\theta}_{\boldsymbol{\sigma}}{}^{\boldsymbol{\alpha}} &= -\frac{1}{2\kappa}\left[\frac{\partial \mathscr{L}_2^g}{\partial l^{\mu\nu}{}_{;\alpha}} l^{\mu\nu}{}_{;\sigma} - \delta^\alpha_\sigma \mathscr{L}_2^g\right] \\ &= -\frac{\sqrt{-\gamma}}{4\kappa}\Big[2l^\alpha{}_{\mu;\nu} l^{\mu\nu}{}_{;\sigma} - l^{\mu\nu;\alpha} l_{\mu\nu;\sigma} - l^\rho{}_{\rho;\mu} l^{\mu\alpha}{}_{;\sigma} - l^\rho{}_{\rho;\sigma} l^{\mu\alpha}{}_{;\mu} + l^\rho{}_{\rho;\sigma} l^\pi{}_\pi{}^{;\alpha} \\ &\quad - \delta^\alpha_\sigma \left(l^{\rho\tau}{}_{;\pi} l_\rho{}^\pi{}_{;\tau} - l^{\rho\tau}{}_{;\tau} l_\pi{}^\pi{}_{;\rho} + \frac{1}{2} l_\pi{}^\pi{}_{;\rho} l_\tau{}^{\tau;\rho} - \frac{1}{2} l^{\rho\tau;\pi} l_{\rho\tau;\pi}\right)\Big]. \end{aligned} \tag{1.5.10}$$

Second, we calculate the Belinfante corrected energy-momentum related to the theory (1.5.8). Formula (1.2.103) yields the spin density:

$$\begin{aligned} \boldsymbol{\sigma}^{\alpha\beta}{}_\sigma &= \frac{1}{2\kappa}\frac{\partial \mathscr{L}_2^g}{\partial\left(l^{\mu\nu}{}_{;\alpha}\right)} l^{\mu\nu}\Big|^\beta_\sigma = \frac{\sqrt{-\gamma}}{2\kappa}\Big[\left(l^\alpha{}_{\rho;\sigma} + l^\alpha{}_{\sigma;\rho} - l_{\rho\sigma}{}^{;\alpha}\right) l^{\beta\rho} \\ &\quad + \left(l^\rho{}_\rho{}^{;\alpha} - l^{\alpha\rho}{}_{;\rho}\right) l^\beta{}_\sigma - \frac{1}{2}\delta^\alpha_\sigma l^{\beta\tau} l^\rho{}_{\rho;\tau} - \frac{1}{2} l^{\alpha\beta} l^\rho{}_{\rho;\sigma}\Big]. \end{aligned} \tag{1.5.11}$$

The corresponding Belinfante correction is calculated with the use of the definition (1.2.113):

$$\begin{aligned} \boldsymbol{b}^{\alpha\beta\gamma} &= \boldsymbol{\sigma}^{\gamma[\alpha\beta]} + \boldsymbol{\sigma}^{\alpha[\gamma\beta]} - \boldsymbol{\sigma}^{\beta[\gamma\alpha]} \\ &= \frac{\sqrt{-\gamma}}{\kappa}\left(l^{\rho[\alpha} l^{\beta]\gamma}{}_{;\rho} + l^{\rho[\alpha;\beta]} l^\gamma{}_\rho + \frac{1}{2}\gamma^{\gamma[\alpha} l^{\beta]\rho} l^\pi{}_{\pi;\rho} + \frac{1}{2} l^\pi{}_\pi{}^{[;\alpha} l^{\beta]\gamma}\right). \end{aligned} \tag{1.5.12}$$

Finally, we use expression (1.5.12) to symmetrize the canonical energy-momentum by applying the rule (1.2.114). Thus, combining (1.5.10) and (1.5.12), we obtain

$$
\begin{aligned}
{}_B\boldsymbol{\theta}^{\sigma\alpha} &= {}_C\boldsymbol{\theta}^{\sigma\alpha} + \boldsymbol{b}^{\alpha\beta\sigma}{}_{;\beta} \\
&= \frac{\sqrt{-\gamma}}{2\kappa}\Big[l^{\rho\alpha}{}_{;\pi}l^{\sigma\pi}{}_{;\rho} - 2l^{\rho\pi(;\alpha}l^{\sigma)}{}_{\rho;\pi} + l^{\pi}{}_{\pi}{}^{(;\alpha}l^{\sigma)\rho}{}_{;\rho} + l^{\alpha\sigma}{}_{;\rho}l^{\rho\pi}{}_{;\pi} + l^{\alpha\pi;\rho}l^{\sigma}{}_{\pi;\rho} \\
&\quad + \frac{1}{2}l^{\pi\rho;\alpha}l_{\pi\rho}{}^{;\sigma} - \frac{1}{2}l^{\pi}{}_{\pi}{}^{;\alpha}l^{\rho}{}_{\rho}{}^{;\sigma} - \frac{1}{2}l^{\alpha\sigma}{}_{;\pi}l^{\rho}{}_{\rho}{}^{;\pi} \\
&\quad + \tfrac{1}{4}\gamma^{\alpha\sigma}\left(2l^{\rho\tau}{}_{;\pi}l_{\rho}{}^{\pi}{}_{;\tau} + l_{\pi}{}^{\pi}{}_{;\rho}l_{\pi}{}^{\pi;\rho} - l^{\rho\tau;\pi}l_{\rho\tau;\pi}\right) \\
&\quad + 2l^{\rho(\alpha}l^{\sigma)}{}_{\rho}{}^{;\pi}{}_{;\pi} - l^{\pi\rho}l^{\alpha\sigma}{}_{;\pi\rho} - l^{\pi\rho}{}_{;\rho}{}^{(;\alpha}l^{\sigma)}{}_{\pi} + l^{\rho}{}_{\rho;\pi}{}^{(;\alpha}l^{\sigma)\pi} \\
&\quad + \frac{1}{2}\gamma^{\alpha\sigma}l^{\pi\rho}l^{\tau}{}_{\tau;\pi\rho} + l^{\alpha\sigma}\left(l^{\pi\rho}{}_{;\pi\rho} - \frac{3}{2}l^{\rho}{}_{\rho}{}^{;\pi}{}_{;\pi}\right)\Big].
\end{aligned} \tag{1.5.13}
$$

It is important to remark that the symmetrized energy-momentum (1.5.13) coincides with the metrical energy-momentum defined following the rule (1.2.127) if the equations (1.5.5) are taken into account,

$$
{}_S\boldsymbol{\theta}_{\alpha\beta} = -\frac{1}{\kappa}\frac{\delta\mathscr{L}_2^g}{\delta\gamma^{\alpha\beta}} = {}_B\boldsymbol{\theta}_{\alpha\beta} \tag{1.5.14}
$$

that can be checked by a direct calculus.

Gauge invariance

Energy-momentum tensors of weak gravitational field (1.5.10), (1.5.13) and (1.5.14) are differentially conserved in accordance with equations (1.2.102), (1.2.14) and (1.2.129) of a field theory in the Minkowski space as a consequence of the invariance of the action (1.5.8) in the linearized general relativity with respect to the coordinate transformations in the Minkowski space. However, there is another type of invariance of the action which also leads to important physical results. This is a so-called gauge invariance. In order to understand this concept, let us consider transformation of the field variables $l^{\mu\nu}$ induced by an arbitrary vector field ξ^{α},

$$
l'^{\mu\nu} = l^{\mu\nu} + \xi^{\mu;\nu} + \xi^{\nu;\mu} \tag{1.5.15}
$$

without changing coordinates and the metric $\gamma_{\mu\nu}$. Transformation (1.5.15) can be interpreted as an intrinsic gauge transformation similar to the gauge transformations in electrodynamics (1.2.140) or in the Yang-Mills field theory [266]. Indeed, the linearized field equations (1.5.5) remain invariant under transformation (1.5.15). At the same time the Lagrangian (1.5.7) under the gauge transformation (1.5.15) is invariant up to a divergence that does not influence the field equations (1.5.5).

To study the consequences of the gauge invariance let us turn to the second Noether's theorem. We rewrite transformation (1.5.15) in the form of (1.2.61) and (1.2.62). Because the coordinates and the metric tensor are not changed we have, $\delta x^{\alpha} = 0$ and $\delta\gamma^{\mu\nu} = 0$. On the other hand, the field variables change,

$$\delta l^{\mu\nu} = \xi^\alpha \cdot 2C^{(\mu}{}_{\alpha\rho}\gamma^{\nu)\rho} + \xi^\alpha{}_{,\rho} \cdot 2\delta^{(\mu}_\alpha \gamma^{\nu)\rho} \tag{1.5.16}$$

where $C^\alpha{}_{\beta\gamma}$ are the Christoffel symbols constructed with the use of the Minkowski metric $\gamma_{\mu\nu}$, see (1.2.9). Thus, the coefficients in the transformation (1.2.62) of the field variable are as follows: $\varepsilon^a \to \xi^\alpha$, $\omega^A_a \to 2C^{(\mu}{}_{\alpha\rho}\gamma^{\nu)\rho}$ and $\omega^{A\alpha}_a \to 2\delta^{(\mu}_\alpha\gamma^{\nu)\rho}$. Then, the identity (1.2.66) is represented as

$$C^\mu{}_{\alpha\rho}\gamma^{\nu\rho}\frac{\delta\mathcal{L}^g_2}{\delta l^{\mu\nu}} \equiv \partial_\rho\left(\delta^\mu_\alpha\gamma^{\nu\rho}\frac{\delta\mathcal{L}^g_2}{\delta l^{\mu\nu}}\right). \tag{1.5.17}$$

Thus, keeping in mind (1.5.9), one finds easily that (1.5.17) transforms into the linearized Bianchi identity

$$\mathcal{G}^{L\ ;\rho}_{\alpha\rho} \equiv 0. \tag{1.5.18}$$

We remark that the identity (1.5.18) is quite analogous to the identity (1.2.141) in electrodynamics that follows from the gauge invariance of the electromagnetic potential.

1.5.2 Weak gravitational waves in general relativity

The Lorentz gauge

Solutions of equations (1.5.5) describe the metric perturbations with respect to the background Minkowski space. These perturbations are physically interpreted in general relativity as representing a weak gravitational field in vacuum. Any coordinates can be used to analyze the perturbations. For the sake of simplicity, we use the Lorentzian coordinates which allow to eliminate the background Christoffel symbols $C^\alpha{}_{\mu\nu}$. The invariance of (1.5.5) under the gauge transformation (1.5.15) means that there is a freedom in definition of the components $l^{\mu\nu}$ associated with the choice of the vector field ξ^α. The arbitrariness of the vector field allows us to impose four gauge conditions on the components of the metric perturbation $l^{\mu\nu}$.

One particular choice of the gauge conditions is the most useful. It is called the *Lorentz gauge* as it allows us to reduce equation (1.5.5) to the d'Alembert (wave) equation. Similar gauge is used in electrodynamics to describe the wave solutions of the Maxwell equations. Let us redefine the new field variables in the form of a linear combination of the old variables,

$$h^{\mu\nu} = l^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}l^\rho{}_\rho. \tag{1.5.19}$$

The Lorentz gauge is defined by the differential equation

$$\partial_\nu h^{\mu\nu} = 0, \tag{1.5.20}$$

that can be always fulfilled by making a relevant choice of ξ^α in (1.5.15). The Lorentz gauge allows us to cancel out a number of the gauge-dependent terms in equations (1.5.5) which is simplified and reduced to the wave equation

$$\Box h^{\mu\nu} \equiv h^{\mu\nu}{}_{,\rho}{}^{,\rho} = h^{\mu\nu}{}_{,i}{}^{,i} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}h^{\mu\nu} = 0, \tag{1.5.21}$$

where $\Box$ denotes the d'Alembert operator. Equation (1.5.21) still describes any weak gravitational field in vacuum but it is the most convenient for studying propagation of weak gravitational waves. According to general relativity the fundamental speed c entering the d'Alembert operator is equal to the speed of light in vacuum. It means that weak gravitational waves in general relativity propagate with the speed of light c.

The condition (1.5.20) does not completely fix the gauge freedom in the choice of the components $l^{\mu\nu}$ – a residual gauge freedom remains. Indeed, the reader can easily check that the gauge transformation (1.5.15) applied to the wave equation (1.5.21) does not change it if ξ^μ also satisfies the wave equation

$$\Box\xi^\mu = 0. \tag{1.5.22}$$

The residual gauge freedom can be fixed depending on the particular physical situation under consideration. In case of weak gravitational waves this is done by choosing a, so-called, *transverse-traceless (TT) gauge*.

TT gauge

Let us consider a weak gravitational wave propagating along a positive direction of $x^1 = x$ coordinate. In this case the wave equation (1.5.21) takes the following form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)h^{\mu\nu} = 0. \tag{1.5.23}$$

Solution of this equation, for the given direction of propagation of the wave, is an arbitrary function

$$h^{\mu\nu}(t,x) \equiv h^{\mu\nu}(ct - x). \tag{1.5.24}$$

The gauge condition (1.5.20) applied to this solution is, $\partial_0 h^{0\nu} + \partial_1 h^{1\nu} = \partial_0 h^{0\nu} - \partial_0 h^{1\nu} = 0$, where the partial derivative operator $\partial_0 = c^{-1}\partial_t$. After integrating this equation and choosing the constant of integration equal to zero which is always possible, we conclude that the Lorentz gauge condition (1.5.20) is equivalent to the following restriction:

$$h^{0\nu} = h^{1\nu}. \tag{1.5.25}$$

Now, we make use of the residual gauge freedom related to ξ^μ satisfying (1.5.22), and impose additional four restrictions on the components of the metric tensor perturbations. More specifically, we demand that the time-space components of the perturbation vanish,

$$h^{0i} = 0, \tag{1.5.26}$$

along with the trace of its spatial components

$$h^{11} + h^{22} + h^{33} = 0. \tag{1.5.27}$$

Combining (1.5.25) with (1.5.26), (1.5.27) one finds that among ten components of $h^{\mu\nu}$ merely two components, h^{23} and $h^{22} = -h^{33}$ are left non-zero. This fixes the gauge completely and makes it clear that a free gravitational field has only two degrees of freedom which characterize the state of *polarization* of a gravitational wave.

Physical meaning of the gauge condition (1.5.26) is associated with the transverse character of the gravitational wave perturbation. The wave vector, k^α, of a monochromatic gravitational wave is $k^\alpha = \omega n^\alpha$, where ω is a circular frequency of the wave, and n^α is a null vector $\eta_{\alpha\beta} n^\alpha n^\beta = 0$. In case of a plane wave propagating along a positive direction of x axis, the null vector $n^\alpha = (1, 1, 0, 0)$ in the Lorentzian coordinates. The condition (1.5.25) is equivalent to

$$n_\mu h^{\mu\nu} = 0, \tag{1.5.28}$$

which means that the metric perturbation is orthogonal (*transverse*) to the direction of propagation of the wave. The condition (1.5.26) means that the metric perturbation is purely spatial that is for an observer moving with a four velocity u^α the condition

$$u_\mu h^{\mu\nu} = 0, \tag{1.5.29}$$

holds. In case of the static observer, $u^\alpha = (1, 0, 0, 0)$ in the Lorentzian coordinates, and the condition (1.5.29) coincides exactly with (1.5.26). Finally, one can easily check that the remaining condition (1.5.27) is equivalent to

$$h^\mu{}_\mu = \eta_{\mu\nu} h^{\mu\nu} = 0. \tag{1.5.30}$$

which means that the physical components of the metric perturbation, in case of a weak gravitational wave, are *traceless*. The gauge satisfying the Lorentz condition (1.5.28) along with the two other conditions, (1.5.29) and (1.5.30), limiting the residual gauge freedom, is called the *transverse-traceless gauge* or simply *TT-gauge*. In conclusion, it is worth noting that the TT gauge condition (1.5.30) leads to the equality, $h^\mu{}_\nu = l^\mu{}_\nu$, which can be useful in practical calculations.

Energy-momentum tensor of a plane gravitational wave

It is instructive to study how the various energy-momentum complexes introduced in this section describe the energy and momentum carried out by a plane, weak gravitational wave. To derive the corresponding expression for the energy we use the TT gauge imposed by the conditions (1.5.28–1.5.30). The plane gravitational wave propagates in the direction of a null vector n^μ which is orthogonal to the wave front. Hence, the partial derivative of the metric tensor perturbation

$$l^{\mu\nu}{}_{,\alpha} = \frac{n_\alpha}{c}\dot{l}^{\mu\nu}, \tag{1.5.31}$$

where the overdot denotes a partial derivative with respect to time t.

At first, we consider the energy-momentum tensors (1.5.10) and (1.5.13), (1.5.14) of the *linearized general relativity* in the Lorentzian coordinates where $\sqrt{-\eta} = 1$. They contain a number of terms being quadratic with respect to the products of the metric tensor perturbation. However, a careful inspection of all terms reveals that most of the quadratic terms vanish in the TT gauge. For example, the quadartic terms like $l^{\mu\nu}{}_{,\sigma}l^{\alpha}{}_{\mu,\nu}$ vanish due to the gauge condition (1.5.28) and equation (1.5.31), the terms like $l^{\mu\nu,\rho}l_{\mu\nu,\rho} = 0$ due to the equation (1.5.31) and the condition that vector n^α is null, etc. It is straightforward to prove that all the energy-momentum tensors are reduced in the TT gauge to a rather simple and unique expression

$${}_C\boldsymbol{\theta}^{\alpha\beta} = {}_B\boldsymbol{\theta}^{\alpha\beta} = {}_S\boldsymbol{\theta}^{\alpha\beta} = \frac{1}{4\kappa}\,l^{\mu\nu,\alpha}l_{\mu\nu}{}^{,\beta} = \frac{n^\alpha n^\beta}{4\kappa}\dot{h}^{\mu\nu}\dot{h}_{\mu\nu}, \tag{1.5.32}$$

where we have used definition (1.5.19) and the fact that the trace of $l^{\mu\nu}$ is zero in the TT gauge, which yields $l^{\mu\nu} = h^{\mu\nu}$.

Now, let us calculate various pseudotensors in the quadratic approximation for a weak gravitational wave in the TT-gauge. We begin from the *Einstein pseudotensor*, ${}_E\boldsymbol{t}_\sigma{}^\alpha$, defined in (1.4.30). Recall that the Lagrangian (1.5.7) is a quadratic approximation of the Einstein Lagrangian (1.2.18). Therefore, one can expect that the energy-momentum tensor (1.5.10) coincides with the Einstein pseudotensor (1.2.53) in the quadratic approximation with respect to the metric tensor perturbation (1.5.1) if one uses the Lorentzian coordinates. Straightforward calculation shows that this expectation is true. Thus, the Einstein pseudotensor ${}_E\boldsymbol{t}_\sigma{}^\alpha$ calculated for the plane gravitational wave gives (1.5.32) in the TT gauge.

One might suppose that the *Papapetrou pseudotensor*, ${}_P\boldsymbol{t}_\sigma{}^\alpha$, given in (1.4.60) and the Belinfante corrected energy-momentum tensor (1.5.13) coincide with the Einstein pseudotensor in the quadratic approximation. This turns out to be not true. The reason is that the operations of taking the quadratic approximation and the Belinfante symmetrization procedure do not commute resulting in different approximate forms of the Belinfante correction, ${}_E\boldsymbol{b}^{\alpha\beta\sigma}$, in (1.4.58) and that, $\boldsymbol{b}^{\alpha\beta\sigma}$, in (5.1.12). Therefore, strictly speaking, the Papapetrou pseudotensor (1.4.60) differs in the quadratic approximation from that of the Einstein pseudotensor, ${}_E\boldsymbol{t}_\sigma{}^\alpha$, by a divergence of quantity, ${}_E\boldsymbol{b}^{\alpha\beta\sigma}$, given in (1.4.58). Nonetheless, one can easily prove by inspection that after imposing the TT-gauge condition the divergence $\partial_\beta({}_E\boldsymbol{b}^{\alpha\beta\sigma}) = 0$. Thus, the quadratic approximation of ${}_P\boldsymbol{t}_\sigma{}^\alpha$ is different from ${}_E\boldsymbol{t}_\sigma{}^\alpha$ merely by gauge terms which vanish in the TT gauge. We conclude that ${}_P\boldsymbol{t}_\sigma{}^\alpha$ also gives (1.5.32) for the plane gravitational waves in the TT gauge.

Considering the *Møller pseudotensor*, ${}_H\boldsymbol{t}_\sigma{}^\alpha$, in (1.4.44) one finds that only its part $-\kappa^{-1}\mathcal{G}_\sigma{}^\alpha$ is not zero under the TT-gauge conditions. The linear part of $-\kappa^{-1}\mathcal{G}_\sigma{}^\alpha$ is equal to zero by the wave equation (1.5.21). The quadratic part of $-\kappa^{-1}\mathcal{G}_\sigma{}^\alpha$ is just equal to (1.5.32) again.

Noticing that the *Komar pseudotensor* coincides with the Møller's one for the constant vector field ξ^μ = const entering its definition, we conclude that it gives the same result (1.5.32) for the energy-momentum tensor of gravitational waves in the quadratic approximation in the Minkowski space.

The *Landau-Lifshitz pseudotensor*, $\boldsymbol{t}_{LL}^{\alpha\beta}$, is given in (1.4.70). Using decomposition (1.5.1) with (1.5.3) in the Lorentzian coordinates, calculating the quadratic approximation, and applying again the TT-gauge condition, one finds that in case of the plane gravitational waves the Landau-Lifshitz pseudotensor yields formally the same expression (1.5.32).

The expression (1.5.32) can be interpreted as a tensor density of weight +1 in the Minkowski space which may be important in calculations in curvilinear (e.g., spherical) coordinates.

Now let us use expression (1.5.32) for describing the energy density $\mathscr{E}$ of a plane gravitational wave. It is given by projection of ${}_C\boldsymbol{\theta}^{\alpha\beta}$ on four-velocity of observer u^α: $\mathscr{E} = {}_C\boldsymbol{\theta}^{\alpha\beta}u_\alpha u_\beta$. Plane gravitational wave has two physical degrees of freedom (*polarizations*). In case of the wave propagating along x axis they are $h^{23} = h^{32} \equiv h_\times$ (*cross polarization*), and $h^{22} = -h^{33} \equiv h_+$ (*plus polarization*). Then, $u^\alpha = (1,0,0,0)$, $n^\alpha = (1,1,0,0)$, and one has for the energy density of the way:

$$\mathscr{E} = \frac{1}{2\kappa c^2}\left(\dot{h}_+^2 + \dot{h}_\times^2\right). \tag{1.5.33}$$

The energy flux is given by a vector quantity $\mathscr{S}^\alpha = {}_C\boldsymbol{\theta}^{\alpha\beta}u_\beta$ which is reduced for the wave propagating along x axis to $\mathscr{S}^\alpha = \mathscr{E}n^\alpha$.

The result (1.5.33) is unique and identical for all known pseudotensors. It describes the energy density carried out by a plane gravitational wave with two modes propagating in the Minkowski space. Because there is a large ambiguity in the definitions of conserved quantities in general relativity, expression (1.5.33) can be used as a *testbed* for checking applicability and self-consistency of new possible definitions of the conserved quantities which are suggested time to time in gravitational physics.

1.5.3 The energy of an isolated gravitating system in general relativity

Another model that is practically important in astrophysical applications especially those which are concerned with the emission of gravitational waves, is an isolated astronomical system. The simplest solution of Einstein's field equations corresponding to such a system is the Schwarzschild solution which describes gravitational field outside of a spherically-symmetric distribution of mass. The mass distribution can be static or commit a radial motion – it does not affect the external gravitational field due to the Birkhoff theorem. The metric tensor of the Schwarzschild solution can be written down in several forms depending on the choice of time and spherical coordinates. The most commonly used choice of the coordinates follows the original work of Schwarzschild that yields the metric tensor in the following form:

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)c^2dt^2 + \frac{1}{1 - \frac{r_g}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (1.5.34)$$

where a constant parameter r_g is called *gravitational radius* of the body that generates the gravitational field. The metric (1.5.34) is singular at the radial distance $r = r_g$. Therefore, it can describe gravitational field of a physical body (star, planet, etc.) if, and only if, the body has a radius R exceeding r_g. In case of $R \leq r_g$ the gravitational field becomes so strong that the body inevitably collapses to a black hole with the (spherical) *event horizon* located at the radial distance r_g.

The gravitational radius r_g is a constant of integration of the Einstein equations for the spherically-symmetric Schwarzschild solution which is directly associated with a total (Tolman or relativistic) mass m of the body by matching the Schwarzschild solution to the Newtonian gravitational field at spatial infinity in the limit of $r \to \infty$. It yields a unique relation,

$$r_g = \frac{2Gm}{c^2}. \qquad (1.5.35)$$

Because the mass m has a clear, unambiguous physical meaning, the Schwarzschild solution can be used as a testbed for checking the applicability of different pseudotensors and corresponding superpotentials for correct evaluation of the conserved quantities in various cases of astronomical isolated systems through the surface integration at spatial infinity like that shown in (1.4.79) with (1.4.77).

Let us discuss the conserved quantities given in the form of a surface integral (1.4.79) with the superpotential (1.4.77) which is the most convenient way for calculations because one needs to know only the asymptotic values of the metric tensor and other geometric quantities defining the superpotential on the surface of integration that can be taken in case of the Schwarzschild solution at spatial infinity. Because the Schwarzschild solution is spherically-symmetric the spherical coordinates are the most convenient coordinates for calculations. Therefore the integration requires a covariantization in a way of the integral (1.4.79) that has been constructed in any curvilinear coordinates.

We, first, pick up an arbitrary classical superpotential $\mathscr{S}_\sigma{}^{\alpha\beta}$ and rewrite the condition (1.4.77) in the form:

$$\mathscr{J}^{\alpha\beta}(\xi^\sigma_K) = (\sqrt{-\gamma})^{1-n}\mathscr{S}_\sigma{}^{\alpha\beta}\xi^\sigma_K = \sqrt{-\gamma}S_\sigma{}^{\alpha\beta}\xi^\sigma_K, \qquad (1.5.36)$$

where $S_\sigma{}^{\alpha\beta}$ is a tensor in the Minkowski space. The covariantization demands to replace the partial derivatives with the covariant ones in the Minkowski space, and to use the covariantized Christoffel symbols $\Delta^\mu{}_{\alpha\beta}$ instead of their non-covariant counterpart (1.3.5). It means that we have to make a replacement:

$$\Gamma^\mu{}_{\alpha\beta} \to \Delta^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\alpha;\beta} + g_{\rho\beta;\alpha} - g_{\alpha\beta;\rho} \right). \tag{1.5.37}$$

The fact that the covariantized Christoffel symbols $\Delta^\mu{}_{\alpha\beta}$ are a tensor of rank 3 in the Minskowski space can be also derived from the relation,

$$\Delta^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} - C^\mu{}_{\alpha\beta}, \tag{1.5.38}$$

between the dynamical Christoffel symbols (1.3.5) and the background Christoffel symbols constructed with the use of $\gamma_{\mu\nu}$, see (1.2.9).

Covariantization of the Freud superpotential (1.4.39) by making use of the procedure explained above, yields:

$$\mathscr{F}_\sigma{}^{\alpha\beta} \to F_\sigma{}^{\alpha\beta} = \frac{1}{\kappa} \frac{\sqrt{-g}}{\sqrt{-\gamma}} \left(g^{\rho[\alpha} \Delta^{\beta]}{}_{\rho\sigma} + \delta^{[\alpha}_\sigma g^{\beta]\pi} \Delta^\rho{}_{\rho\pi} - \delta^{[\alpha}_\sigma \Delta^{\beta]}{}_{\mu\nu} g^{\mu\nu} \right). \tag{1.5.39}$$

The covariant version of the Møller superpotential (1.4.48) is:

$$\mathscr{X}_\sigma{}^{\alpha\beta} \to X_\sigma{}^{\alpha\beta} = \frac{1}{\kappa} \frac{\sqrt{-g}}{\sqrt{-\gamma}} g^{\rho[\alpha} \Delta^{\beta]}{}_{\rho\sigma}. \tag{1.5.40}$$

In order to represent the Papapetrou (1.4.62) and Landau-Lifshitz (1.4.68) superpotentials into the covariant form we use the known formulae for the partial derivatives of the metric tensor:

$$\partial_\sigma g_{\mu\nu} = g_{\mu\rho} \Gamma^\rho{}_{\sigma\nu} + g_{\nu\rho} \Gamma^\rho{}_{\sigma\mu}, \qquad \partial_\sigma \sqrt{-g} = \sqrt{-g} \Gamma^\rho{}_{\rho\sigma}. \tag{1.5.41}$$

Then, the covariantized Papapetrou superpotential (1.4.62) is:

$$\begin{aligned} \mathscr{P}_\sigma{}^{\alpha\beta} \to P_\sigma{}^{\alpha\beta} = \frac{1}{\kappa} \frac{\sqrt{-g}}{\sqrt{-\gamma}} \Big[& \Delta^\mu{}_{\mu\lambda} \left(g^{\rho[\alpha} \gamma^{\beta]\lambda} - g^{\lambda[\alpha} \gamma^{\beta]\rho} \right) + \Delta^{[\alpha}{}_{\lambda\mu} \gamma^{\beta]\rho} g^{\lambda\mu} \\ & - \Delta^{[\alpha}{}_{\lambda\mu} \gamma^{\beta]\lambda} g^{\rho\mu} + \Delta^\rho{}_{\lambda\mu} \gamma^{\lambda[\alpha} g^{\beta]\mu} - \Delta^\lambda{}_{\lambda\mu} \gamma^{\rho[\alpha} g^{\beta]\mu} \Big] \gamma_{\rho\sigma}, \end{aligned} \tag{1.5.42}$$

whereas the covariantized Landau-Lifshitz superpotential (1.4.68) is given by:

$$
\begin{aligned}
{}_{LL}\mathscr{S}_\sigma{}^{\alpha\beta} \to {}_{LL}S_\sigma{}^{\alpha\beta} = \frac{1}{\kappa}\frac{\left(\sqrt{-g}\right)^2}{\left(\sqrt{-\gamma}\right)^2}\Big[&2\Delta^\mu{}_{\mu\lambda}\left(g^{\rho[\alpha}g^{\beta]\lambda} - g^{\lambda[\alpha}g^{\beta]\rho}\right) \\
&+ \Delta^{[\alpha}{}_{\lambda\mu}g^{\beta]\rho}g^{\lambda\mu} - \Delta^{[\alpha}{}_{\lambda\mu}g^{\beta]\lambda}g^{\rho\mu} \\
&+ \Delta^\rho{}_{\lambda\mu}g^{\lambda[\alpha}g^{\beta]\mu} - \Delta^\lambda{}_{\lambda\mu}g^{\rho[\alpha}g^{\beta]\mu}\Big]g_{\rho\sigma}.
\end{aligned}
\tag{1.5.43}
$$

The asymptotic form of the Schwarzschild solution (1.5.34) in spherical coordinates $x^\alpha = \{x^0, x^1, x^2, x^3\} = \{ct, r, \theta, \phi\}$, is the Minkowski metric

$$
ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.5.44}
$$

which determinant $\sqrt{-\gamma} = r^2\sin\theta$.

We apply the integration formula (1.4.79) for calculating the *total energy* $E = \mathscr{P}(\xi_0^\alpha)$ corresponding to the Killing vector $\xi_0^\alpha = (-1, 0, 0, 0)$. It yields

$$
E = \lim_{r\to\infty}\int_0^\pi d\theta\int_0^{2\pi} d\phi\, r^2\sin\theta\, S_\alpha{}^{01}\xi_0^\alpha \tag{1.5.45}
$$

where $S_0{}^{01}$ is one of the above given expressions (1.5.39), (1.5.40), (1.5.42), (1.5.43) for the covariantized superpotentials – Freud, Møller, Papapetrou, etc.

We use the metric tensor coefficients for the Schwarzschild solution (1.5.34):

$$
g^{00} = -\frac{1}{1-\frac{r_g}{r}}, \quad g^{11} = 1 - \frac{r_g}{r}, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2\sin^2\theta} \tag{1.5.46}
$$

and the Minkowski metric coefficients in the spherical coordinates (1.5.44):

$$
\gamma^{00} = -1, \quad \gamma^{11} = 1, \quad \gamma^{22} = \frac{1}{r^2}, \quad \gamma^{33} = \frac{1}{r^2\sin^2\theta}. \tag{1.5.47}
$$

The ordinary Christoffel symbols of the Schwarzschild solution (1.5.34):

$$
\begin{aligned}
&\Gamma^0{}_{10} = -\Gamma^1{}_{11} = \frac{r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}, \quad \Gamma^1{}_{00} = \frac{r_g}{2r^2}\left(1-\frac{r_g}{r}\right), \\
&\Gamma^1{}_{22} = -r\left(1-\frac{r_g}{r}\right), \quad \Gamma^1{}_{33} = -r\sin^2\theta\left(1-\frac{r_g}{r}\right), \\
&\Gamma^2{}_{21} = \Gamma^3{}_{31} = \frac{1}{r}, \quad \Gamma^3{}_{32} = \cot\theta, \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta,
\end{aligned}
\tag{1.5.48}
$$

whereas the Christoffel symbols of the Minkowski metric in the spherical coordinates (1.5.44) are:

$$C^2{}_{21} = C^3{}_{31} = \frac{1}{r}, \quad C^3{}_{32} = \cot\theta$$
$$C^1{}_{22} = -r, \quad C^1{}_{33} = -r\sin^2\theta, \quad C^2{}_{33} = -\sin\theta\cos\theta. \tag{1.5.49}$$

Using (1.5.48) and (1.5.49) one can easily obtain the covariant $\Delta^\alpha{}_{\mu\nu}$ by making use of equation (1.5.38).

To calculate the total energy (1.5.45) it is enough to find out the asymptotic behaviour of the component of the superpotentials $F_0{}^{01}$, $X_0{}^{01}$, etc. at spatial infinity for $r \to \infty$. Thus for the covariantized superpotentials by Freud, Papapetrou and Landau-Lifshitz we find:

$$F_0{}^{01} \sim P_0{}^{01} \sim L_0{}^{01} \sim -\frac{r_g}{\kappa}\frac{1}{r^2}, \tag{1.5.50}$$

but the Møller superpotential behaves differently,

$$X_0{}^{01} \sim -\frac{1}{2}\frac{r_g}{\kappa}\frac{1}{r^2}. \tag{1.5.51}$$

Recalling the definition of the gravitational radius r_g in (1.5.35) and that of the Einstein constant, $\kappa = 8\pi G/c^4$, one gets from the integration formula (1.5.45) for the cases (1.5.50):

$$E = mc^2, \tag{1.5.52}$$

whereas for the case of the Møller superpotential (1.5.51):

$$E = \frac{1}{2}mc^2. \tag{1.5.53}$$

We conclude that the Freud, Papapetrou and Landau-Lifshitz superpotentials are physically meaningful as they yield the correct value for the total energy of the body while the Møller superpotential has a problem.

It was discussed above that the Komar superpotential (1.4.80) is reduced to the Møller one if $\xi^\mu = \text{const}$. This means that the Komar superpotential also gives the problematic result (1.5.53) that is known as *Komar's anomaly*.

Similar calculations can be performed for the Kerr solution describing axially-symmetric gravitational field of a rotating star. The Kerr solution has another conserved quantity – the total angular momentum. The Papapetrou and Landau-Lifshitz superpotentials, being related to symmetrical pseudotensors, can be used for calculations of the total angular momentum and give physically meaningful result with a normal ratio of the mass to angular momentum. However, the Komar superpotential, being universal to using all possible Killing vectors, again reveals an anomaly in case of the Kerr solution. The anomaly can be cured, see, for example, the Katz work, [250] and the construction (6.1.24) in Section 6.1.2.

2 Field-theoretical formulation of general relativity: The theory

2.1 Development of the field-theoretical formulation

2.1.1 Geometrical formalism and field theories

In the framework of special relativity all physical systems (massive bodies, particles, radiation, fields of interaction, etc) exist in a fixed geometry – the Minkowski space that has no curvature. The *flat* spacetime (geometrical structure), being an arena for physical interactions, is a passive background for the physical systems, whereas the matter fields are active (dynamical) objects. We shall call such theories as *field theories*. In the modern field theories, the background geometry can be also chosen as a predetermined *curved* spacetime, such as an anti-de Sitter (AdS) space, FLRW cosmological geometries, black hole geometries, etc. (see, e. g., [50]).

General relativity was the first theory, where the spacetime acquired the status of a dynamical entity. This means that the metric, which describes the geometry of the spacetime, is a dynamical field like the other fields that reside in the spacetime. General relativity is not the only example of the theory with dynamical gravitational field, various theories of gravity, which generalize general relativity, have been suggested. Besides the metric, some other geometrical fields, like torsion, non-metricity, etc., can be also considered as dynamical fields. Collectively, we shall classify these theories as *geometrical theories*.

It is intuitively clear that the conceptual basis for constructing conservation laws in the framework of field theories is significantly simpler than that in geometrical theories where together with matter fields, one has to find and describe the dynamical evolution of geometry. A natural question arises. Can one transform a geometrical theory to the form of a field theory with a fixed background spacetime? The idea of reformulating a metric theory in the form of field theory (*field-theoretical* formulation) has a natural foundation. For example, in studying geometric perturbations in a metric theory, one chooses a background spacetime (be it curved or flat) and examines evolution of the perturbations with respect to the chosen background. The present chapter is devoted to the development of the principles of constructing the field-theoretical formulation of general relativity.

2.1.2 Earlier perturbative formulations of general relativity

Let us return to the general relativity action (1.3.15) with the Einstein-Hilbert Lagrangian (1.3.16), which represent a system of the metric field $g_{\mu\nu}$ and the matter fields Φ^A, a set of tensor densities, see Appendixes A.1 and A.3.1. Variation of (1.3.15) with

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respect to $g^{\alpha\beta}$ gives the Einstein's equations in the form (1.3.22), which can be rewritten in the more conventional form (1.3.23).

The study of perturbations in general relativity was started by Einstein himself. In 1918, in the framework of general relativity he [152] considered gravitational waves in the Minkowski space background. In the ensuing decades, perturbations in general relativity were considered regularly, mostly, in the Minkowski space. The history of developing the field-theoretical formulation of general relativity as a separate theoretical method began in 1940s–1950s see, for example, papers [84, 219, 220, 223, 279, 342, 390, 438, 455]. The work by Deser [120] stands out as the final benchmark in this series.[1]

Combining the results of the aforementioned works, we will show how the Einstein equations can be rewritten in the perturbative form by making use of a simple logic. In order to proceed in this way we turn to the results of Section 1.5.1 related to the linearized general relativity. Let us define the metric perturbations $l^{\mu\nu}$ on a flat background in the Lorentzian coordinates as

$$g^{\mu\nu} = \eta^{\mu\nu} + l^{\mu\nu} . \tag{2.1.1}$$

It is the type of the decomposition done in (1.5.1). Next, let us substitute (2.1.1) into the Einstein equations (1.3.23). Then, the terms linear in the metric perturbations at the left hand side are

$$G^L_{\mu\nu}(l) \equiv \tfrac{1}{2}\left(l_{\mu\nu}{}^{,\alpha}{}_{,\alpha} + \eta_{\mu\nu} l^{\alpha\beta}{}_{,\alpha\beta} - l^{\alpha}{}_{\mu,\nu\alpha} - l^{\alpha}{}_{\nu,\mu\alpha} - \eta_{\mu\nu} l^{\beta}{}_{\beta}{}^{,\alpha}{}_{,\alpha} + l^{\beta}{}_{\beta,\mu\nu}\right) , \tag{2.1.2}$$

in which the indices are raised and lowered by $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$. Observe that the expression (1.5.4) transforms to the expression (2.1.2) if the Lorentzian coordinates are used, thus (2.1.2) is the left hand side of the linearized general relativity equations (1.5.5). Next, let us transfer all the nonlinear terms to the right hand side of (1.3.23). Then, they, together with the matter energy-momentum tensor, are treated as an effective energy-momentum tensor, $t^{\text{eff}}_{\mu\nu}$. Thus, the Einstein equations (1.3.23) are rewritten in the equivalent form as

$$G^L_{\mu\nu}(l) = 8\pi\left[-(8\pi)^{-1}(G_{\mu\nu}(\eta + l) - G^L_{\mu\nu}(l)) + T_{\mu\nu}(\Phi^A, \eta + l)\right] \equiv 8\pi t^{\text{eff}}_{\mu\nu} . \tag{2.1.3}$$

The divergence of the left hand side of (2.1.3) is equal to zero identically, $\partial_\nu G^{\mu\nu}_L \equiv 0$, see (1.5.18), then

$$\partial_\nu t^{\mu\nu}_{\text{eff}} = 0 \tag{2.1.4}$$

which is a differential conservation law for the effective energy-momentum.

---

1 We must say that, the present book does not claim to be an authoritative and exhaustive work on the history of the developments in this context, but, rather, we take the pragmatic and pedagogical path and mention only the most pertinent works.

The above development can be interpreted as a *Lagrangian based* theory of the gravitational field $l^{\mu\nu}$ with self-interaction and the matter fields Φ^A in the Minkowski space. In this interpretation, the metric perturbations $l^{\mu\nu}$ play the role of a dynamical field, and $t^{\text{eff}}_{\mu\nu}$ is to be obtained by varying the action with respect to the background metric, as in the case of a symmetric energy-momentum (1.2.127). Now, following the introduction in work of Deser [120], we outline the main steps of constructing such a field theory that is fully equivalent to general relativity.

The first step. The principle of equivalence and theoretical considerations (see, e. g., textbook [315]) point out that the most preferable type of gravitational field is the *tensor field*, $l^{\mu\nu}$ describing spin 2. The linear equations of this field in the Minkowski space are

$$G^L_{\mu\nu}(l) = 0\,, \tag{2.1.5}$$

where the differential operator is defined in (2.1.2). They are the linearized equations of general relativity (1.5.5). Thus, they can be obtained by varying the corresponding quadratic Lagrangian $\mathscr{L}^g_2(l,\eta)$ defined in (1.5.7). The equations (2.1.5) are invariant with respect to the well known gauge transformation of the field:

$$l'^{\mu\nu} = l^{\mu\nu} + \xi^{\mu,\nu} + \xi^{\nu,\mu}\,, \tag{2.1.6}$$

see (1.5.15). By this invariance, first, the identity $\partial_\nu G^{\mu\nu}_L \equiv 0$ follows, see (1.5.18); second, the tensor field $l^{\mu\nu}$ has only two physical degrees of freedom, see Section 1.5.2.

The second step. Assuming that the *gravitational field is induced by matter fields*, one has to choose the symmetric energy-momentum tensor $T_{\mu\nu}(\Phi,\eta)$ of matter fields Φ^A as the source of $G^L_{\mu\nu}(l)$. Then, one obtains

$$G^L_{\mu\nu}(l) = 8\pi T_{\mu\nu}(\Phi,\eta)\,. \tag{2.1.7}$$

Due to the identity (1.5.18) one has $\partial_\nu T^{\mu\nu} = 0$. However, this contradicts the equations of motion of the fields Φ^A *interacting* with the gravitational field $l^{\mu\nu}$. How can one fix this disagreement? Recall that the equation (2.1.7) is assumed to be obtained by variation of the total Lagrangian with respect to $l^{\mu\nu}$, whereas the right hand side of (2.1.7) is assumed to be obtained by variation of the matter Lagrangian with respect to the background metric, like in (1.2.127), where the Minkowski metric is represented in curvilinear coordinates. In order to reconcile the two different variations, it is necessary to make the replacement $\mathscr{L}^M(\Phi,\gamma) \to \mathscr{L}^M(\Phi,\gamma + l)$ in the matter Lagrangian where the variation with respect to $\gamma^{\mu\nu}$ and $l^{\mu\nu}$ are equivalent. Then, automatically, the same exchange is provided in the matter energy-momentum tensor $T_{\mu\nu}(\Phi,\eta) \to T_{\mu\nu}(\Phi,\eta+l)$. This reflects the *universality of the gravitational interaction*, which is postulated from the beginning.

The third step. Now, one has to include the *gravitational self-interaction*. To this end, one adds the symmetric energy-momentum tensor of gravitational field ${}_2t^g_{\mu\nu}(l)$, corresponding to $\mathscr{L}^g_2$, to the right hand side of (2.1.7) together with $T_{\mu\nu}(\Phi,\eta + l)$.

It turns out that the tensor ${}_2t^g_{\mu\nu}(l)$ can be obtained from the variational principle only if a cubic term is added to the Lagrangian, $\mathscr{L}^g_2 + \mathscr{L}^g_3$. Then, one needs to consider the next iteration, and so on. As a result, one obtains the final variant of the gravitational equations:

$$G^L_{\mu\nu}(l) = 8\pi\left[\sum_{n=2}^{\infty} {}_nt^g_{\mu\nu}(l) + T_{\mu\nu}(\phi, \eta + l)\right] \equiv 8\pi\left(t^g_{\mu\nu} + t^m_{\mu\nu}\right) . \tag{2.1.8}$$

It turns out that the equations (2.1.8) are equivalent to the Einstein's equations (1.3.23). One can prove that this is, indeed, the case, if one identifies the sum of the Minkowski metric and the gravitational field perturbation with the full (effective) metric, see [342]:

$$\eta^{\mu\nu} + l^{\mu\nu} \equiv g^{\mu\nu} . \tag{2.1.9}$$

As a result, the background metric $\eta^{\mu\nu}$ and the field $l^{\mu\nu}$ disappear from the consideration completely, and the dynamical metric $g^{\mu\nu}$ is left alone, restoring general relativity in its original formulation.

2.1.3 Deser's field-theoretical model

Generalizing the earlier works, Deser [120] suggested the field-theoretical formulation of general relativity on the Minkowski background in a *closed* form without expansions in difference from (2.1.8). The basic principle is the same:

– *A consistent field theory of gravity in the Minkowski space is constructed as a theory of the spin-2 tensor field with self-interaction.*

The matter fields are coupled to gravity a universal way analogous to the case discussed above, therefore for the sake of simplicity we do not consider the matter fields in this subsection at all.

Unlike the previous case, Deser used the first order formalism where the *independent* dynamical variables are the components of two fields $\mathfrak{h}^{\mu\nu}$ and $\Delta^\alpha{}_{\mu\nu}$, where $\mathfrak{h}^{\mu\nu}$ is a symmetric tensor density of weight +1, $\Delta^\alpha{}_{\mu\nu}$ is a tensor of the rank 3. At the beginning, Deser considers the linear theory with the action

$$S = -\frac{1}{16\pi}\int d^4x \mathscr{L}^g_2(\mathfrak{h}, \Delta) , \tag{2.1.10}$$

where the Lagrangian is

$$\mathscr{L}^g_2(\mathfrak{h}, \Delta) \equiv \mathfrak{h}^{\mu\nu}\left(\Delta^\alpha{}_{\mu\nu;\alpha} - \Delta^\alpha{}_{\mu\alpha;\nu}\right) + \gamma^{\mu\nu}\left(\Delta^\alpha{}_{\mu\nu}\Delta^\beta{}_{\alpha\beta} - \Delta^\alpha{}_{\mu\beta}\Delta^\beta{}_{\alpha\nu}\right) , \tag{2.1.11}$$

and $\boldsymbol{\gamma}^{\mu\nu} = \sqrt{-\gamma}\gamma^{\mu\nu}$, $\gamma = \det \gamma_{\mu\nu}$. Since, later on, we will need variations with respect to the background metric we use curved coordinates in the Minkowski space. Thus we consider the Minkowski metric in the form of $\gamma_{\mu\nu}$ instead of $\eta_{\mu\nu}$. As a result we use the covariant derivatives $\{_{;\alpha}\}$ constructed with the help of the metric $\gamma_{\mu\nu}$ and defined in

(1.2.8) with the corresponding Christoffel symbols $C^{\alpha}{}_{\mu\nu}$, see (1.2.9). Of course, in the case of the Lorentzian coordinates, one has $\gamma_{\mu\nu} = \eta_{\mu\nu}$ and $C^{\alpha}{}_{\mu\nu} = 0$.

Varying (2.1.10) with respect to the dynamical variables one obtains the gravitational field equations of the first order:

$$ {}_2\mathscr{G}^{\mu\nu}{}_{\alpha} = -\mathfrak{h}^{\mu\nu}{}_{;\alpha} + \delta^{(\mu}_{\alpha}\mathfrak{h}^{\nu)\beta}{}_{;\beta} + \gamma^{\mu\nu}\Delta^{\beta}_{\alpha\beta} + \gamma^{\rho\beta}\Delta^{(\mu}_{\rho\beta}\delta^{\nu)}_{\alpha} - 2\gamma^{\beta(\mu}\Delta^{\nu)}_{\alpha\beta} = 0, \tag{2.1.12}$$

$$ {}_2G^{\Delta}_{\mu\nu} = \Delta^{\alpha}{}_{\mu\nu;\alpha} - \Delta^{\alpha}{}_{\alpha(\mu;\nu)} = 0\,. \tag{2.1.13}$$

Combining these equations, one easily obtains

$$ G^{L}_{\mu\nu}(h) \equiv \tfrac{1}{2}(h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + \gamma_{\mu\nu}h^{\alpha\beta}{}_{;\alpha\beta} - h^{\alpha}{}_{\mu;\nu\alpha} - h^{\alpha}{}_{\nu;\mu\alpha}) = 0\,, \tag{2.1.14}$$

where the notations $h^{\mu\nu} \equiv \mathfrak{h}^{\mu\nu}/\sqrt{-\gamma}$ are used. Substituting $h^{\mu\nu} \equiv l^{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}l^{\alpha}{}_{\alpha}$ into (2.1.14), one finds that it coincides with (2.1.5), see also (1.5.5). Thus, the action (2.1.10), indeed, describes a *spin 2 tensor field*.

The metrical energy-momentum tensor of the linear gravitational field $\mathfrak{h}^{\mu\nu}$ is

$$\begin{aligned} {}_2t^{g}_{\mu\nu} &= \frac{2}{\sqrt{-\gamma}}\left(-\frac{1}{16\pi}\frac{\delta\mathscr{L}^{g}_{2}(\mathfrak{h},\Delta)}{\delta\gamma^{\mu\nu}}\right) \\ &= -\frac{1}{8\pi}\left[\left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \tfrac{1}{2}\gamma_{\mu\nu}\gamma^{\rho\sigma}\right)\left(\Delta^{\alpha}{}_{\rho\sigma}\Delta^{\beta}{}_{\alpha\beta} - \Delta^{\alpha}{}_{\rho\beta}\Delta^{\beta}{}_{\alpha\sigma}\right) - Q^{\tau}{}_{\mu\nu;\tau}\right] \end{aligned} \tag{2.1.15}$$

with

$$\begin{aligned} 2Q^{\tau}_{\mu\nu} \equiv{} & -\gamma_{\mu\nu}h^{\alpha\beta}\Delta^{\tau}{}_{\alpha\beta} + h_{\mu\nu}\Delta^{\tau}{}_{\alpha\beta}\gamma^{\alpha\beta} - h^{\tau}{}_{\mu}\Delta^{\alpha}{}_{\nu\alpha} - h^{\tau}{}_{\nu}\Delta^{\alpha}{}_{\mu\alpha} \\ & + h^{\beta\tau}\left(\Delta^{\alpha}{}_{\mu\beta}\gamma_{\alpha\nu} + \Delta^{\alpha}{}_{\nu\beta}\gamma_{\alpha\mu}\right) \\ & + h^{\beta}{}_{\mu}\left(\Delta^{\tau}{}_{\nu\beta} - \Delta^{\alpha}{}_{\beta\rho}\gamma^{\rho\tau}\gamma_{\alpha\nu}\right) \\ & + h^{\beta}{}_{\nu}\left(\Delta^{\tau}{}_{\mu\beta} - \Delta^{\alpha}{}_{\beta\rho}\gamma^{\rho\tau}\gamma_{\alpha\mu}\right)\,. \end{aligned} \tag{2.1.16}$$

To account for the *self-interaction of the gravitational field* one has to require that the energy-momentum tensor (2.1.15) is the source of the gravitational field and appears in the right hand side of equations (2.1.14). Thus, one gets

$$ G^{L}_{\mu\nu}(h) = 8\pi\,{}_2t^{g}_{\mu\nu}(h,\Delta)\,. \tag{2.1.17}$$

But, here, there is a contradiction: this equation has to follow from a cubic in $\mathfrak{h}^{\mu\nu}$ and $\Delta^{\alpha}{}_{\mu\nu}$ Lagrangian, whereas (2.1.11) is only quadratic.

To include the self-interaction in a consistent way, Deser suggested a novel trick. Following the analogy with the theory of the Yang-Mills fields, he added the cubic term to the gravitational Lagrangian (2.1.11) as follows,

$$ \mathscr{L}^{g}(\mathfrak{h},\Delta) = \mathscr{L}^{g}_{2}(\mathfrak{h},\Delta) + \mathfrak{h}^{\mu\nu}\left(\Delta^{\alpha}{}_{\mu\nu}\Delta^{\beta}{}_{\alpha\beta} - \Delta^{\alpha}{}_{\mu\beta}\Delta^{\beta}{}_{\alpha\nu}\right)\,. \tag{2.1.18}$$

The important property of the additional term is that *it does not contain the metric at all*. Thus, it does not contribute to the energy-momentum tensor, that is formally

$$t^g_{\mu\nu}(h,\Delta) = {}_2t^g_{\mu\nu}(h,\Delta)\,. \tag{2.1.19}$$

On the other hand, instead of the first order equations (2.1.12) and (2.1.13) one has

$$\mathscr{G}^{\mu\nu}{}_{\alpha} = {}_2\mathscr{G}^{\mu\nu}{}_{\alpha} + \mathfrak{h}^{\mu\nu}\Delta^{\beta}{}_{\alpha\beta} + \mathfrak{h}^{\rho\beta}\Delta^{(\mu}{}_{\rho\beta}\delta^{\nu)}_{\alpha} - 2\mathfrak{h}^{\beta(\mu}\Delta^{\nu)}{}_{\alpha\beta} = 0\,, \tag{2.1.20}$$

$$G^{\Delta}_{\mu\nu} = {}_2G^{\Delta}_{\mu\nu} + \Delta^{\alpha}{}_{\mu\nu}\Delta^{\beta}{}_{\alpha\beta} - \Delta^{\alpha}{}_{\mu\beta}\Delta^{\beta}{}_{\alpha\nu} = 0\,. \tag{2.1.21}$$

It turns out, rather remarkably, that the combination of the new first order equations, (2.1.20) and (2.1.21), gives exactly:

$$G^{L}_{\mu\nu}(h) = 8\pi t^g_{\mu\nu}(h,\Delta)\,, \tag{2.1.22}$$

where the right hand side is the metric energy-momentum tensor (2.1.19) corresponding to (2.1.18)!

Recall that the goal of the above consideration is to construct a theory being equivalent to general relativity. Let us show how to reach this goal. After making use of the following identifications

$$\begin{aligned}\gamma^{\mu\nu} + \mathfrak{h}^{\mu\nu} &\equiv \mathfrak{g}^{\mu\nu}\,,\\ C^{\alpha}{}_{\mu\nu} + \Delta^{\alpha}{}_{\mu\nu} &\equiv \Gamma^{\alpha}{}_{\mu\nu}\end{aligned} \tag{2.1.23}$$

one finds that the equations (2.1.20) and (2.1.21) take on the following form:

$$\mathscr{G}^{\mu\nu}{}_{\alpha} = -\mathfrak{g}^{\mu\nu}{}_{,\alpha} + \delta^{(\mu}_{\alpha}\mathfrak{g}^{\nu)\beta}{}_{,\beta} + \mathfrak{g}^{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta} + \mathfrak{g}^{\rho\beta}\Gamma^{(\mu}{}_{\rho\beta}\delta^{\nu)}_{\alpha} - 2\mathfrak{g}^{\beta(\mu}\Gamma^{\nu)}{}_{\alpha\beta} = 0, \tag{2.1.24}$$

$$G^{\Delta}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\alpha(\mu,\nu)} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\alpha\nu} = 0\,. \tag{2.1.25}$$

These are the Einstein equations in the Palatini formulation, where the fields $\mathfrak{g}^{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$ are considered as independent variables. Let us show that the equations (2.1.24) and (2.1.25) represent the Einstein equations in the usual formulation. Keeping in mind $\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and using (A.2.9–A.2.11), one finds that (2.1.24) is equivalent to (1.3.5). This means that $G^{\Delta}_{\mu\nu} = R_{\mu\nu}$, see (1.3.5). Thus, the equations (2.1.25) become the vacuum Einstein equations in the standard form.

Let us compare also the Lagrangians. Adding the Lagrangian (2.1.18) by the terms $\gamma^{\mu\nu}R_{\mu\nu}(\gamma)$ and $\mathfrak{h}^{\mu\nu}R_{\mu\nu}(\gamma)$, which are equal to zero for the flat background, and substituting the identification (2.1.23), one finds easily that (2.1.18) is equivalent to the Hilbert Lagrangian in the Palatini form:

$$\mathscr{L}_H = \mathfrak{g}^{\mu\nu}\left(\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\alpha\mu,\nu} + \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta} - \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\alpha\nu}\right) \tag{2.1.26}$$

with $\mathfrak{g}^{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$ as independent variables. Varying (2.1.26) with respect to $\Gamma^{\alpha}{}_{\mu\nu}$ and $\mathfrak{g}^{\mu\nu}$, one obtains exactly (2.1.24) and (2.1.25). Thus, the system represented by (2.1.18–2.1.22) is equivalent to general relativity in vacuum.

The equations (2.1.22) were constructed in absence of matter sources. The matter is introduced to the theory appropriately by adding the matter energy-momentum tensor to (2.1.22), which takes on the form:

$$G^{L}_{\mu\nu}(h) = 8\pi\left(t^{g}_{\mu\nu} + t^{m}_{\mu\nu}\right) = 8\pi t^{\text{tot}}_{\mu\nu}\,. \qquad (2.1.27)$$

The divergence of the left hand side of (2.1.27) is equal to zero identically, then for the same reason as in (2.1.4) one has

$$t^{\mu\nu}_{\text{tot};\nu} = 0 \qquad (2.1.28)$$

that is the differential conservation law for the total energy-momentum.

We stress again that the use of the components of $\mathfrak{h}^{\mu\nu}$ in the role of dynamical variables (not $h^{\mu\nu}$, $l^{\mu\nu}$ or others) has permitted us to construct the theory of perturbations of gravitational field in the closed form being equivalent to general relativity without expansions in the Taylor series.

2.1.4 Various methods of the construction

Here, we discuss different approaches to represent general relativity in the field-theoretical form. The above recipes were based on the *relativistic* formulation of the theory from the very onset. They recommend to begin the construction in the Minkowski space. On the other hand, general relativity generalizes the Newtonian gravity which is a non-relativistic theory. It is important to overview the steps of this drastic conceptual transformation, *from gravistatic (Newtonian law) to gravidynamics (Einstein's equations)*. Following Grishchuk [203], the main points of this are as follow.

Let us begin from the Newtonian law $\Delta\varphi = -4\pi G\rho$ and reformulate it to make it compatible with special relativity. To satisfy the relativistic requirement: *first*, the mass density ρ has to be generalized to 10 components of the matter stress-energy tensor $T_{\mu\nu}(\Phi,\eta)$; *second*, the single component of the Newtonian potential φ should be replaced with 10 gravitational potentials; *third*, the Laplace operator should be replaced with the d 'Alembert operator $\Box$; thus the Newtonian equation is rewritten in the form:

$$\Box h_{\mu\nu} = 16\pi T_{\mu\nu}(\Phi,\eta)\,, \qquad (2.1.29)$$

where $\Box h_{\mu\nu} \equiv \eta^{\alpha\beta}h_{\mu\nu,\alpha\beta} \equiv h_{\mu\nu}{}^{,\alpha}{}_{,\alpha}$. *Fourth*, recalling the equations (1.5.21) with (1.5.20), one assumes that in (2.1.29) the gauge condition $\partial_\nu h^{\mu\nu} = 0$ is implied. Therefore to relax this assumption and to reestablish the full gauge invariance of (2.1.29) one has to add to its left hand side some additional terms. It turns out that the gauge invariant extension of (2.1.29) is

$$G^L_{\mu\nu} \equiv \frac{1}{2}\left(h_{\mu\nu}{}^{,\alpha}{}_{,\alpha} + \eta_{\mu\nu}h^{\alpha\beta}{}_{,\alpha\beta} - h^{\alpha}{}_{\mu,\nu\alpha} - h^{\alpha}{}_{\nu,\mu\alpha}\right) = 8\pi T_{\mu\nu}(\Phi, \eta) \tag{2.1.30}$$

which exactly coincides with (2.1.7), recall that $h^{\mu\nu} = l^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}l^{\alpha}{}_{\alpha}$. After taking into account both, *fifth*, the universality of the matter coupling and, *sixth*, gravitational self-interaction, one obtains the field-theoretical formulation of general relativity in the form (2.1.8).

Of course, from the theoretical point of view, it is very important to construct general relativity (or any other geometrical theory) in the field-theoretical form, by relying upon the principles of a field theory in the Minkowski space, as above. This approach has not only its own theoretical merit but can be highly efficient in practical calculations of approximate solutions of the Einstein equations by making use of their perturbative expansion with respect to a small parameter and doing successive iterations. It works as follows. The first (main) step is the *decomposition* of dynamical variables of general relativity into the background quantities and perturbations. The next step is the derivation of the field equations for the perturbations which are considered as *independent dynamical variables*. Usually the background spacetime (flat or curved) is determined by the problem under consideration and is taken as a known solution to the Einstein equations.

Let us demonstrate the decomposition method in the case of Deser's field approach to general relativity. We take the Hilbert Lagrangian in the Palatini form (2.1.26) with independent variables $\mathfrak{g}^{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$, make decomposition of the dynamical variables

$$\begin{aligned} \mathfrak{g}^{\mu\nu} &= \gamma^{\mu\nu} + \mathfrak{h}^{\mu\nu}, \\ \Gamma^{\alpha}{}_{\mu\nu} &= C^{\alpha}{}_{\mu\nu} + \Delta^{\alpha}{}_{\mu\nu}\,, \end{aligned} \tag{2.1.31}$$

take into account that for the flat background $R_{\mu\nu}(\gamma) = 0$ and treat the perturbations $\mathfrak{h}^{\mu\nu}$ and $\Delta^{\alpha}{}_{\mu\nu}$ as independent dynamical field variables. Then, the Palatini Lagrangian (2.1.26) transforms to the Deser's Lagrangian (2.1.26).

The advantages of the decomposition method are, first, it is rather straightforward and explicitly connected to the ordinary geometrical formulation of general relativity, and second, it is well adapted for constructing the field-theoretical formulation of an arbitrary metric theory. In Section 2.2, the decomposition method will be applied to reformulate general relativity as a field theory on an arbitrary curved background.

It is natural to assume that the field-theoretical formulation of general relativity can be constructed based on the gauge invariant properties of the field equations. Recall that the initial linear equations (2.1.5) are invariant under the transformations (2.1.6). One can check that this invariance is not preserved at the next steps of the iteration procedure, and must be extended in such a way that the final form of the field equations become invariant with respect to total diffeomorphisms in the *exact* sense. Keeping in mind all of these, one can try to construct the field-theoretical formulation of general relativity as a gauge theory. Such a construction is presented in Section 2.3. Using analogies with the gauge theories of the Yang-Mills type, we suggest a *non-standard* way of a localization that is postulated as a *“localization” of Killing vectors of the background spacetime instead of the usual localization of group parameters*.

In the modern theoretical models, instead of the Minkowski space, frequently one considers a fixed curved spacetime as a background, such as the anti-de Sitter space, cosmological models, black hole geometries, etc. Therefore, the field-theoretical formulation of general relativity is necessary for studying perturbations on arbitrary backgrounds. Such a generalization is presented in [206, 379] and is developed in Section 2.2 using the decomposition method. Although the construction in these works has been developed both in the first and in the second order formalisms, the only latter is used here, since it is more convenient and suitable. To the best of our knowledge, Barnebey [29] was the first one to suggest the use of the second order formalism for an exact (without expansions and approximations) description of perturbations in general relativity.

Elements of the field-theoretical approach in general relativity presented in [120, 206] appear independently in many studies. Recently, they have been actively developed by Babak and Grishchuk [21]. They require only the first derivatives of the metric perturbations in the total metric energy-momentum tensor. This has led to a *new* field-theoretical formulation of general relativity in the Minkowski space, where such a total energy-momentum tensor is the source in the field equations with a *non-linear* left hand side, unlike the formulation in [120, 206]. On the basis of the formulation of general relativity in [21], a promising variant of the gravitational theory with non-zero masses of gravitons has been developed in [22]. The interested reader can find the details of the Babak-Grishchuk theory in Section 2.4.

In presenting the developments of the field-theoretical method, it is worth mentioning the works by Pitts and Schive. In [374], a class of so-called “slightly bi-metric” gravitation theories has been constructed. In [375, 376], the behaviour of light cones in the Minkowski space and curved (physical) spacetimes has been examined. A special criterion (based on the causality principle) for constructing a field-theoretical model has been proposed as well. In [375], this criterion is used to show that if a spatially flat FLRW big bang model is considered as a field configuration on a flat background, then the cosmological singularity vanishes to past infinity in the Minkowski space. The references to earlier works on the theoretical foundation of the field-theoretical approach in general relativity can be found in papers [202, 203, 471]; the exhaustive bibliography has been provided in [374–376], see also [364].

2.2 The field-theoretical formulation of general relativity

In this section, the field-theoretical formulation of general relativity on arbitrary curved backgrounds is developed. The decomposition method derived above is applied in the second order formalism, the properties are examined and discussed.

2.2.1 A dynamical Lagrangian

Let us rewrite the Einstein-Hilbert action as

$$S = \int_\Omega d^4x \mathscr{L}_{EH} \equiv -\frac{1}{16\pi}\int_\Omega d^4x \mathscr{R}(\mathfrak{g}^{\mu\nu}) + \int_\Omega d^4x \mathscr{L}^M(\Phi^A, \mathfrak{g}^{\mu\nu})\,. \tag{2.2.1}$$

Following Deser's recipe of the previous section for constructing the field-theoretical formulation in general relativity, we note that for the role of the *dynamical* gravitational variables we choose the components of the Gothic metric $\mathfrak{g}^{\mu\nu}$:

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}\,. \tag{2.2.2}$$

The matter variables are components of a set of tensor densities Φ^A, see Appendix A.3; $\mathscr{L}^M$ depends on Φ^A and $\mathfrak{g}^{\mu\nu}$, and their derivatives up to the second order, although the consideration can be generalized easily to an arbitrary finite order.

Now, after varying the action (2.2.1) with respect to the dynamical variables, one obtains the gravitational and matter equations in the form:

$$\frac{\delta \mathscr{L}_{EH}}{\delta \mathfrak{g}^{\mu\nu}} = -\frac{1}{16\pi}\frac{\delta \mathscr{R}}{\delta \mathfrak{g}^{\mu\nu}} + \frac{\delta \mathscr{L}^M}{\delta \mathfrak{g}^{\mu\nu}} = 0\,, \tag{2.2.3}$$

$$\frac{\delta \mathscr{L}_{EH}}{\delta \Phi^A} = \frac{\delta \mathscr{L}^M}{\delta \Phi^A} = 0\,. \tag{2.2.4}$$

The equations (2.2.3) are the Einstein equations, the explicit form of which are

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T_\alpha{}^\alpha\right)\,, \tag{2.2.5}$$

see (1.3.29). Einstein's equations of the usual form (1.3.22) have been obtained by the variation with respect to $g^{\alpha\beta}$. These equivalent forms are connected by the multiplier

$$\frac{\partial g^{\alpha\beta}}{\partial \mathfrak{g}^{\mu\nu}} = \frac{1}{\sqrt{-g}}\left(\delta^\alpha_\mu \delta^\beta_\nu - \tfrac{1}{2}g^{\alpha\beta}g_{\mu\nu}\right)\,, \tag{2.2.6}$$

see (1.3.28).

Now, let us define the metric $\mathfrak{h}^{\mu\nu}$ and matter ϕ^A perturbations with respect to the background quantities $\bar{\mathfrak{g}}^{\mu\nu}$ and $\bar{\Phi}^A$ with the use of the decompositions

$$\mathfrak{g}^{\mu\nu} \equiv \bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu}\,, \tag{2.2.7}$$

$$\Phi^A \equiv \bar{\Phi}^A + \phi^A\,. \tag{2.2.8}$$

We stress that these are exact relations, there is no necessity for $\mathfrak{h}^{\mu\nu}$ and ϕ^A to be infinitesimal.

After that it is necessary to define a background (fixed) system. It is postulated that the background quantities are described by the action:

$$\bar{S} = \int d^4x \bar{\mathscr{L}}_{EH} \equiv -\frac{1}{16\pi} \int d^4x \bar{\mathscr{R}} + \int d^4x \bar{\mathscr{L}}^M , \tag{2.2.9}$$

where $\bar{\mathscr{R}} = \mathscr{R}(\bar{\mathfrak{g}}^{\mu\nu})$ and $\bar{\mathscr{L}}^M = \mathscr{L}^M(\bar{\mathfrak{g}}^{\mu\nu}, \bar{\Phi}^A)$. The corresponding background gravitational and matter equations have the form of the barred equations (2.2.3) and (2.2.4):

$$\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\mathfrak{g}}^{\mu\nu}} = -\frac{1}{16\pi} \frac{\delta \bar{\mathscr{R}}}{\delta \bar{\mathfrak{g}}^{\mu\nu}} + \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{\mathfrak{g}}^{\mu\nu}} = 0 , \tag{2.2.10}$$

$$\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^A} = \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{\Phi}^A} = 0 . \tag{2.2.11}$$

The explicit form of (2.2.10) is

$$\bar{R}_{\rho\sigma} - 8\pi \left(\bar{T}_{\rho\sigma} - \tfrac{1}{2} \bar{g}_{\rho\sigma} \bar{T} \right) = 0 . \tag{2.2.12}$$

A Ricci-flat (including flat) background is used in many applications, in this case the background equations (2.2.10) and (2.2.11) are transformed into the unique equation

$$\bar{R}_{\mu\nu} = 0. \tag{2.2.13}$$

Background quantities $\bar{\mathfrak{g}}^{\mu\nu}$ and $\bar{\Phi}^A$ are considered as known (although arbitrary) solutions to (2.2.10) and (2.2.11), therefore they are classified as given (fixed) quantities.

To transfer to the field-theoretical formulation, we interpret the perturbations $\mathfrak{h}^{\mu\nu}$ and ϕ^A as *independent dynamical* variables, representing the *field configuration* on the background of the system (2.2.9–2.2.11). To describe such a configuration in the framework of a field theory, one has to define the corresponding Lagrangian. From the start we turn to Deser's Lagrangian (2.1.18) rewriting it as

$$\mathscr{L}^g(\mathfrak{h}, \Delta) = \mathscr{R}(\bar{\mathfrak{g}} + \mathfrak{h}, C + \Delta) - \mathfrak{h}^{\mu\nu} R_{\mu\nu}(\gamma) - \mathscr{R}(\gamma) . \tag{2.2.14}$$

Here, $\mathscr{R}(\bar{\mathfrak{g}} + \mathfrak{h}, C + \Delta)$ is obtained from $\mathscr{R}(\mathfrak{g}, \Gamma) = \mathfrak{g}^{\mu\nu} R_{\mu\nu}(\Gamma)$ with $R_{\mu\nu}(\Gamma)$, see (1.3.5), after incorporating the decompositions (2.2.7), (2.2.8). Recall that the Lagrangian (2.2.14) is derived in the Minkowski space and for the vacuum case. Generalizing it for curved backgrounds with the presence of matter, we define the Lagrangian for the fields $\mathfrak{h}^{\mu\nu}$ and ϕ^A as

$$\begin{aligned} \mathscr{L}^{\text{dyn}} &= \mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h},\ \bar{\Phi} + \phi) - \mathfrak{h}^{\mu\nu} \frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\mathfrak{g}}^{\mu\nu}} - \phi^A \frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^A} - \bar{\mathscr{L}}_{EH} - \frac{1}{16\pi} \partial_\alpha \mathscr{D}^\alpha \\ &\equiv -\frac{1}{16\pi} \mathscr{L}^g + \mathscr{L}^m . \end{aligned} \tag{2.2.15}$$

Because it describes *independent dynamical* variables $\mathfrak{h}^{\mu\nu}$ and ϕ^A, it is called the *dynamical* Lagrangian [379].

Properties of the dynamical Lagrangian

The construction of (2.2.15) is as follows. The decompositions (2.2.7) and (2.2.8) are incorporated into the Lagrangian of the action (2.2.1); zeroth order and linear in $\mathfrak{h}^{\mu\nu}$ and ϕ^A terms in Taylor series are subtracted; a divergence may be added. Zeroth order term is the background Lagrangian, whereas the linear term is proportional to the left hand sides of the background equations (2.2.10) and (2.2.11). Before variation, however, one cannot take into account the fact that (2.2.10) and (2.2.11) hold. The explanation of this will be given in Section 2.2.3, see the explanation of the formula (2.2.46). The pure gravitational part is denoted as $\mathscr{L}^g$, whereas the Lagrangian for the matter sector interacting with the gravitational field is denoted $\mathscr{L}^m$. Now let us discuss the properties of the dynamical Lagrangian (2.2.15).

First, at least, the choice (2.2.15) satisfies the main requirement, namely: the variation with respect to $\mathfrak{h}^{\mu\nu}$ and ϕ^A gives the equations equivalent to the equations of the Einstein theory in the form (2.2.3) and (2.2.4). It is true because (a) by (2.2.7) and (2.2.8), the variation of the first term in (2.2.15) can be changed with the variation with respect to $\mathfrak{g}^{\mu\nu}$ and Φ^A; (b) it is assumed that the background equations (2.2.10) and (2.2.11) hold; (c) the divergence does not contribute into the equations.

Second, it seems that, unlike (2.2.15), a simple substitution of the decompositions (2.2.7) and (2.2.8) into the initial Lagrangian $\mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h},\ \bar{\Phi} + \phi)$ is the best variant for a dynamical Lagrangian. However, then the variation with respect to $\mathfrak{h}^{\mu\nu}$ can be changed by the variation with respect to $\bar{\mathfrak{g}}^{\mu\nu}$. But this means that the metric energy-momentum tensor obtained by variation with respect to $\bar{g}^{\mu\nu}$ disappears on the field equations. It is not a desirable situation because then the physical meaning of the metric energy-momentum tensor is not clear. This does not make the situation better than in the standard geometrical formulation in general relativity. Indeed, the total metric energy-momentum in general relativity is connected directly with the variation of (2.2.1) with respect to $\mathfrak{g}^{\mu\nu}$, but it disappears on the Einstein equations. The situation is improved by including the linear terms into (2.2.15): namely, the variation of them with respect to the background metric guaranties that metric energy-momentum is non-zero and has a physical meaning.

Third, up to a divergence the Lagrangian (2.2.15) disappears for a vanishing dynamical field configuration, it is natural and this property is guarantied by the subtraction of the background Lagrangian.

Fourth, in general, in (2.2.15) a vector density $\mathscr{D}^\alpha$ is not concreted. Its presence, on the one hand, does not have influence for deriving the field equations, on the other hand, it can modify boundary conditions under variations. In the simplest case, when $\mathscr{D}^\alpha = 0$ the pure gravitational Lagrangian is

$$
\begin{aligned}
\mathscr{L}^g &= \mathscr{R}(\bar{\mathfrak{g}} + \mathfrak{h}) - \mathfrak{h}^{\mu\nu}\bar{R}_{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}\bar{R}_{\mu\nu} \\
&= \mathfrak{h}^{\mu\nu}\left(\bar{\nabla}_\rho \Delta^\rho_{\mu\nu} - \bar{\nabla}_{(\mu}\Delta^\sigma_{\nu)\sigma}\right) \\
&\quad + (\bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu})\left(\Delta^\rho{}_{\mu\nu}\Delta^\sigma{}_{\rho\sigma} - \Delta^\rho{}_{\mu\sigma}\Delta^\sigma{}_{\rho\nu}\right) ,
\end{aligned}
\tag{2.2.16}
$$

where $\bar{\nabla}_\alpha$ is the covariant derivative constructed with the use of $\bar{g}_{\mu\nu}$.

The formula (2.2.16) has been obtained as follows. The components of the tensor $\Delta^\rho{}_{\mu\nu}$ are the perturbations of the Christoffel symbols

$$
\Delta^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha{}_{\mu\nu} - \overline{\Gamma}^\alpha{}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\rho}\left(\bar{\nabla}_\mu g_{\rho\nu} + \bar{\nabla}_\nu g_{\rho\mu} - \bar{\nabla}_\rho g_{\mu\nu}\right) ,
\tag{2.2.17}
$$

and depend on $\mathfrak{h}^{\mu\nu}$ through the decomposition (2.2.7): $g_{\mu\nu} = g_{\mu\nu}(\mathfrak{g}) = g_{\mu\nu}(\bar{\mathfrak{g}} + \mathfrak{h})$. The definition (2.2.17) generalizes (1.5.38) for the case of a flat background. Also, substituting $\Gamma^\alpha{}_{\mu\nu}$ from (2.2.17) into the Riemann tensor (1.3.2), one obtains

$$
R^\lambda{}_{\tau\rho\sigma} = \bar{\nabla}_\rho \Delta^\lambda{}_{\tau\sigma} - \bar{\nabla}_\sigma \Delta^\lambda{}_{\tau\rho} + \Delta^\lambda{}_{\rho\eta}\Delta^\eta{}_{\tau\sigma} - \Delta^\lambda{}_{\sigma\eta}\Delta^\eta{}_{\tau\rho} + \overline{R}^\lambda{}_{\tau\rho\sigma} .
\tag{2.2.18}
$$

The gravitational Lagrangian (2.2.16) coincides with Deser's Lagrangian (2.1.18) if the background metric $\bar{g}_{\mu\nu}$ is changed with the metric $\gamma_{\mu\nu}$ of the Minkowski space. To be convinced, one can check that the definition (2.2.17) and the equations of the first order (2.1.20) (plus (2.1.12)) coincide.

One of more popular choices [251] for $\mathscr{D}^\alpha$ is

$$
\mathscr{D}^\alpha \equiv \mathfrak{g}^{\alpha\nu}\Delta^\mu{}_{\mu\nu} - \mathfrak{g}^{\mu\nu}\Delta^\alpha{}_{\mu\nu} .
\tag{2.2.19}
$$

Then the pure gravitational Lagrangian is represented as

$$
\begin{aligned}
\mathscr{L}^g &= \mathscr{R}(\bar{\mathfrak{g}} + \mathfrak{h}) - \mathfrak{h}^{\mu\nu}\bar{R}_{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}\bar{R}_{\mu\nu} + \partial_\mu \mathscr{D}^\mu \\
&= -\left(\Delta^\rho{}_{\mu\nu} - \Delta^\sigma{}_{\mu\sigma}\delta^\rho_\nu\right)\bar{\nabla}_\rho \mathfrak{h}^{\mu\nu} \\
&\quad + (\bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu})\left(\Delta^\rho{}_{\mu\nu}\Delta^\sigma{}_{\rho\sigma} - \Delta^\rho{}_{\mu\sigma}\Delta^\sigma{}_{\rho\nu}\right) .
\end{aligned}
\tag{2.2.20}
$$

It depends only on the first derivatives of the gravitational variables $\mathfrak{h}^{\mu\nu}$. In the case of a flat background the Lagrangian (2.2.20) transfers to the covariant Lagrangian suggested by Rosen [390], which has been rediscovered in [250] and [204].

Fifth, the matter part of (2.2.15) is rewritten as

$$
\mathscr{L}^m = \mathscr{L}^M\left(\mathfrak{g} + \mathfrak{h}, \bar{\Phi} + \phi\right) - \mathfrak{h}^{\mu\nu}\frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{\mathfrak{g}}^{\mu\nu}} - \phi^A \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{\Phi}^A} - \bar{\mathscr{L}}^M .
\tag{2.2.21}
$$

Its a concrete form depends on the choice of $\mathscr{L}^M$.

2.2.2 The Einstein equations in the field-theoretical formulation

To obtain the equations for the gravitational field $\mathfrak{h}^{\alpha\beta}$, one has to vary the action (2.2.15) with respect to $\mathfrak{h}^{\alpha\beta}$:

$$\frac{\delta}{\delta\mathfrak{h}^{\alpha\beta}}\mathscr{L}^{\text{dyn}} = \frac{\delta}{\delta\mathfrak{h}^{\alpha\beta}}\mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h},\, \bar{\Phi} + \phi) - \frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} = 0\,. \tag{2.2.22}$$

In the first term, the variation with respect to $\mathfrak{h}^{\alpha\beta}$ can be changed with the variation with respect to $\mathfrak{g}^{\alpha\beta}$. Then, keeping in mind the background equations (2.2.10), one concludes that (2.2.22) are equivalent to the Einstein equations in the form (2.2.3).

Let us rewrite the equations (2.2.22) in a more appropriate form where the total energy-momentum is the source for the linear part of the field equations, see (2.1.27). Varying the action (2.2.15) with respect to $\bar{\mathfrak{g}}^{\alpha\beta}$ one gets

$$\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathscr{L}^{\text{dyn}} = \frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h},\, \Phi + \phi) - \frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A} + \bar{\mathscr{L}}_{EH}\right). \tag{2.2.23}$$

Because in the first term in (2.2.22) the variation with respect to $\mathfrak{h}^{\alpha\beta}$ can be changed with the variation with respect to $\bar{\mathfrak{g}}^{\alpha\beta}$ the first term in (2.2.23) vanishes on the background equations (2.2.10). As a result, (2.2.23) goes to

$$-\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A}\right) = \frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathscr{L}^{\text{dyn}} \tag{2.2.24}$$

which is the other form of the field equations (2.2.22). Recall that the total energy-momentum is related to the variation with respect to $\bar{g}^{\mu\nu}$, therefore let us contract the equation (2.2.24) with

$$\frac{\partial\bar{\mathfrak{g}}^{\alpha\beta}}{\partial\bar{g}^{\mu\nu}} = \sqrt{-\bar{g}}(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta})\,. \tag{2.2.25}$$

Dividing by $\sqrt{-\bar{g}}$ and multiplying by 16π, one gets the gravitational field equations in the form:

$$G^L_{\mu\nu} + \Phi^L_{\mu\nu} = 8\pi\left(t^g_{\mu\nu} + t^m_{\mu\nu}\right) \equiv 8\pi t^{\text{tot}}_{\mu\nu}\,. \tag{2.2.26}$$

The left hand side of (2.2.26) is linear in $\mathfrak{h}^{\mu\nu}$ and ϕ^A, the right hand side represents the total energy-momentum related to (2.2.15). The left hand side consists of the pure gravitational part and matter part:

$$G^L_{\mu\nu}(\mathfrak{h}) \equiv \frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{R}}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}$$
$$\equiv \tfrac{1}{2}\left(\bar{\nabla}_\rho{}^\rho h_{\mu\nu} + \bar{g}_{\mu\nu}\bar{\nabla}_{\rho\sigma}h^{\rho\sigma} - \bar{\nabla}_{\rho\nu}h^\rho_\mu - \bar{\nabla}_{\rho\mu}h^\rho_\nu\right), \tag{2.2.27}$$

$$\Phi^L_{\mu\nu}(\mathfrak{h},\phi) \equiv -\frac{16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \phi^A\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\Phi}^A}\right), \tag{2.2.28}$$

where $\bar{\nabla}_{\rho\sigma} = \bar{\nabla}_\rho\bar{\nabla}_\sigma$. The right hand side of (2.2.26) is simply the metric energy-momentum tensor related to the Lagrangian (2.2.15):

$$t^{\text{tot}}_{\mu\nu} \equiv \frac{2}{\sqrt{-\bar{g}}}\frac{\delta\mathscr{L}^{dyn}}{\delta\bar{g}^{\mu\nu}} \equiv \frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(-\frac{1}{16\pi}\mathscr{L}^g + \mathscr{L}^m\right) \equiv t^g_{\mu\nu} + t^m_{\mu\nu}. \tag{2.2.29}$$

The explicit form of the gravitational part is

$$t^g_{\mu\nu} = \frac{1}{8\pi}\left[\left(-\delta^\rho_\mu\delta^\sigma_\nu + \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\right)\left(\Delta^\alpha{}_{\rho\sigma}\Delta^\beta{}_{\alpha\beta} - \Delta^\alpha{}_{\rho\beta}\Delta^\beta{}_{\alpha\sigma}\right) + \bar{\nabla}_\tau Q^\tau{}_{\mu\nu}\right]; \tag{2.2.30}$$

$$\begin{aligned}2Q^\tau{}_{\mu\nu} \equiv{}& -\bar{g}_{\mu\nu}h^{\alpha\beta}\Delta^\tau{}_{\alpha\beta} + h_{\mu\nu}\Delta^\tau{}_{\alpha\beta}\bar{g}^{\alpha\beta} - h^\tau{}_\mu\Delta^\alpha{}_{\nu\alpha} - h^\tau{}_\nu\Delta^\alpha{}_{\mu\alpha}\\ &+h^{\beta\tau}\left(\Delta^\alpha{}_{\mu\beta}\bar{g}_{\alpha\nu} + \Delta^\alpha{}_{\nu\beta}\bar{g}_{\alpha\mu}\right)\\ &+h^\beta{}_\mu\left(\Delta^\tau{}_{\nu\beta} - \Delta^\alpha{}_{\beta\rho}\bar{g}^{\rho\tau}\bar{g}_{\alpha\nu}\right)\\ &+h^\beta{}_\nu\left(\Delta^\tau{}_{\mu\beta} - \Delta^\alpha{}_{\beta\rho}\bar{g}^{\rho\tau}\bar{g}_{\alpha\mu}\right).\end{aligned} \tag{2.2.31}$$

One can see that $t^g_{\mu\nu}$ is not less than quadratic in the gravitational variables. The matter part is expressed through the usual matter energy-momentum tensor $T_{\mu\nu}$ of the Einstein's equations (2.2.5) as

$$\begin{aligned}t^m_{\mu\nu} ={}& \left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\right)\left(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T_{\pi\rho}g^{\pi\rho}\right)\\ &-\frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \phi^A\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\Phi}^A}\right) - \bar{T}_{\mu\nu}.\end{aligned} \tag{2.2.32}$$

Now, let us compare the gravitational equations (2.2.26) defined on an arbitrary curved background with Deser's equations (2.1.27) defined on a flat background in curved coordinates.

First, the linear gravitational part of the equations (2.2.27) generalizes (2.1.14) on arbitrary curved backgrounds.

Second, the quantities (2.2.30), (2.2.31) generalize the energy-momentum tensor (2.1.15), (2.1.16) to arbitrary curved backgrounds. One can recall that $\Delta^\alpha{}_{\mu\nu}$ in (2.2.30), (2.2.31) depends on the gravitational variables through (2.2.17), whereas $\Delta^{\alpha}{}_{\mu\nu}$ in Deser's approach is the independent dynamic variable. However, the latter can be recalculated with the use of the equation of the first order (2.1.12) which is equivalent to (2.2.17).

Third, unlike (2.1.27), the equations (2.2.26) contain an additional linear expression (2.2.28); it is because the background system includes the "background matter". In the case of Ricci-flat backgrounds (2.2.13) the expression (2.2.28) disappears from (2.2.26) and it acquires the form of the equations (2.1.27).

Fourth, the matter energy-momentum, $t^m_{\mu\nu}$, in (2.1.27) is not concreted; the expression (2.2.32) represents it in a maximally general form on arbitrary curved backgrounds including the "background matter". Assuming expansions for (2.2.32) one finds that it is not less than the second order in dynamical variables, like the gravitational energy-momentum, $t^g_{\mu\nu}$.

Let us make other remarks. One can find that the expression, $\Phi^L_{\mu\nu}$, in (2.2.28) is included as a part into the matter energy-momentum in (2.2.32). This permits one to rewrite the equations (2.2.26) in the form:

$$G^L_{\mu\nu}(\mathfrak{h}) = 8\pi\left(t^g_{\mu\nu} + \delta t^M_{\mu\nu}\right) = 8\pi t^{\text{eff}}_{\mu\nu}\,; \tag{2.2.33}$$

$$\begin{aligned}\delta t^M_{\mu\nu} &\equiv t^M_{\mu\nu} - \bar{t}^M_{\mu\nu}\\ &= \left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\right)\left(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T_{\pi\rho}g^{\pi\rho}\right) - \bar{T}_{\mu\nu}\,.\end{aligned} \tag{2.2.34}$$

One can easily find that the equations (2.2.33) have the form of the equations (2.1.3) and, in fact, generalize them to arbitrary curved backgrounds. However, the effective source $t^{\text{eff}}_{\mu\nu}$ does not follow from the total Lagrangian (2.2.15) by the variational procedure. The matter part could be classified as a perturbation of

$$t^M_{\mu\nu} \equiv \frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\mathscr{L}^M\left(\bar{\mathfrak{g}} + \mathfrak{h}, \bar{\Phi} + \phi\right) \tag{2.2.35}$$

only.

At last, let us turn to the perturbed matter equations. They can be represented by the same way as the gravitational equations. Thus, indeed, they can be rewritten in the form:

$$\frac{\delta\mathscr{L}^{\text{dyn}}}{\delta\phi^A} = -\Phi^L_A + t^m_A = 0\,, \tag{2.2.36}$$

or

$$\Phi^L_A = t^m_A, \tag{2.2.37}$$

where the linear left hand side is defined as

$$\Phi_A^L(\mathfrak{h}, \phi) \equiv -\frac{\delta}{\delta\bar{\Phi}^A}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \phi^B\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\Phi}^B}\right), \tag{2.2.38}$$

whereas the source is represented by

$$t_A^m \equiv \frac{\delta\mathscr{L}^m}{\delta\bar{\Phi}^A}. \tag{2.2.39}$$

2.2.3 Functional expansions

The exact field-theoretical reformulation of the geometrical theory gives a possibility to construct an approximate scheme easily and in explicit expressions up to an arbitrary order in perturbations. In this subsection, it is demonstrated. Assuming smooth enough functions, let us expand the Lagrangian $\mathscr{L}_{HE}(\bar{\mathfrak{g}} + \mathfrak{h},\ \bar{\Phi} + \phi)$ as

$$\begin{aligned}\mathscr{L}_{EH} = \bar{\mathscr{L}}_{EH} &+ \mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \phi^B\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^B}\\ &+ \frac{1}{2!}\mathfrak{h}^{\alpha\beta}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \mathfrak{h}^{\rho\sigma}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}\phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A} + \frac{1}{2!}\phi^B\frac{\delta}{\delta\bar{\Phi}^B}\phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A}\\ &+ \ldots + \mathrm{div}\,. \end{aligned}\tag{2.2.40}$$

The expansions, like (2.2.40), are used in quantum field theories [138] and called as functional expansions. The relation can be proved, for example, by the method of mathematical induction after a prolonged calculation. One of main properties of such a calculation is that the Lagrangian derivatives (see (A.2.38) and (A.2.42) in Appendix A.2.4) commute up to a divergence, for example, as

$$\mathfrak{h}^{\rho\sigma}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}\phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A} = \phi^A\frac{\delta}{\delta\bar{\Phi}^A}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \mathrm{div}\,. \tag{2.2.41}$$

Also, one has to remember that the Lagrangian derivative of a divergence is identically equal to zero, see (A.2.40) in Appendix A.2.4.

The perturbative Lagrangian (2.2.40) can be represented in a compact form with the use of iterations,

$$\mathscr{L}_{EH} = \sum_{k=0}^{\infty}\mathscr{L}_k^{EH}, \tag{2.2.42}$$

where $\mathscr{L}_0^{HE} \equiv \bar{\mathscr{L}}_{HE}$. Here, for any $k \geq 1$,

$$\mathscr{L}_k^{HE} = \frac{1}{k}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\mathscr{L}_{k-1}^{EH}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi^A\frac{\delta\mathscr{L}_{k-1}^{M}}{\delta\bar{\Phi}^A}\right), \tag{2.2.43}$$

is the Lagrangian perturbation defined iteratively by taking Lagrangian derivatives from the Lagrangian perturbations of the previous iteration. In particular,

$$\mathscr{L}_1^{EH} = \mathfrak{h}^{\mu\nu}\frac{\delta\mathscr{L}_0^{EH}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi^A\frac{\delta\mathscr{L}_0^{M}}{\delta\bar{\Phi}^A}\,, \tag{2.2.44}$$

$$\mathscr{L}_2^{EH} = \tfrac{1}{2}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\mathscr{L}_1^{EH}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi^A\frac{\delta\mathscr{L}_1^{M}}{\delta\bar{\Phi}^A}\right), \tag{2.2.45}$$

and so on.

Now, substitute the series (2.2.40) into the dynamical Lagrangian (2.2.15). One can see that the zeroth order and linear terms have the opposite signs and, thus, are self-compensated. It is exactly the reason why in the linear terms of the Lagrangian (2.2.15), the background equations are not taken into account before variation. It is because they are absent, and, in fact, the dynamical Lagrangian is quadratic in the dynamical variables and has the form:

$$\begin{aligned}\mathscr{L}^{\text{dyn}} = {} & \frac{1}{2!}\mathfrak{h}^{\alpha\beta}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \mathfrak{h}^{\rho\sigma}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}\phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A} + \frac{1}{2!}\phi^B\frac{\delta}{\delta\bar{\Phi}^B}\phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A}\\ & + \frac{1}{3!}\mathfrak{h}^{\mu\nu}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\mu\nu}}\mathfrak{h}^{\alpha\beta}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \frac{1}{2!}\mathfrak{h}^{\mu\nu}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\mu\nu}}\mathfrak{h}^{\alpha\beta}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A}\\ & + \frac{1}{2!}\mathfrak{h}^{\mu\nu}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\mu\nu}}\phi^B\frac{\delta}{\delta\bar{\Phi}^B}\phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A} + \frac{1}{3!}\phi^C\frac{\delta}{\delta\bar{\Phi}^C}\phi^B\frac{\delta}{\delta\bar{\Phi}^B}\phi^A\frac{\delta\bar{\mathscr{L}}_{HE}}{\delta\bar{\Phi}^A} + \\ & + \ldots + \text{div}\,.\end{aligned} \tag{2.2.46}$$

Following (2.2.42), the same can be rewritten in the compact form:

$$\mathscr{L}^{\text{dyn}} = \sum_{k=2}^{\infty}\mathscr{L}_k^{EH}\,, \tag{2.2.47}$$

One has to stress that the remarkable structure of (2.2.46) permits one to represent the variation with respect to dynamical variables, $\mathfrak{h}^{\mu\nu}$ and ϕ^A, as the equations of the type

$$\frac{\delta\mathscr{L}^{\text{dyn}}}{\delta\mathfrak{h}^{\mu\nu}} = \frac{\delta}{\delta\bar{\mathfrak{g}}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A}\right) + \frac{\delta\mathscr{L}^{\text{dyn}}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} = 0, \tag{2.2.48}$$

$$\frac{\delta\mathscr{L}^{\text{dyn}}}{\delta\phi^A} = \frac{\delta}{\delta\bar{\Phi}^A}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \phi^B\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^B}\right) + \frac{\delta\mathscr{L}^{\text{dyn}}}{\delta\bar{\Phi}^A} = 0. \tag{2.2.49}$$

As is seen, they are exactly the equations (2.2.26) and (2.2.37), respectively, see (2.2.24) and (2.2.38), which are the main equations of the field-theoretical formulation.

Under necessary assumptions, the series (2.2.46) can be interrupted at the corresponding order. Then the approximate Lagrangian for the perturbed system can be obtained. Its variation gives both the approximate field equations and the energy-momentum tensor. For example, the quadratic approximation of (2.2.46) gives the possibility
(a) to construct the linear equations

$$\begin{aligned} & -\frac{\sqrt{-\bar{g}}}{2\kappa}\frac{\partial \bar{\mathfrak{g}}^{\rho\sigma}}{\partial \bar{\mathfrak{g}}^{\mu\nu}}\left(G^L_{\rho\sigma}(\mathfrak{h}) + \Phi^L_{\rho\sigma}(\mathfrak{h},\phi)\right) \\ & \equiv \frac{\delta}{\delta \bar{\mathfrak{g}}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\mathfrak{g}}^{\rho\sigma}} + \phi^A \frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^A}\right) = 0\,, \end{aligned} \tag{2.2.50}$$

$$-\Phi^L_A(\mathfrak{h},\phi) \equiv \frac{\delta}{\delta \bar{\Phi}^A}\left(\mathfrak{h}^{\rho\sigma}\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\mathfrak{g}}^{\rho\sigma}} + \phi^B \frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^B}\right) = 0\,, \tag{2.2.51}$$

(b) to construct the quadratic energy-momentum tensor:

$$\begin{aligned} t^{\text{tot}}_{\mu\nu} = {} & \frac{2}{\sqrt{-\bar{g}}}\frac{\delta}{\delta \bar{g}^{\mu\nu}}\left(\frac{1}{2!}\mathfrak{h}^{\alpha\beta}\frac{\delta}{\delta \bar{\mathfrak{g}}^{\alpha\beta}}\mathfrak{h}^{\rho\sigma}\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\mathfrak{g}}^{\rho\sigma}}\right. \\ & \left. + \mathfrak{h}^{\rho\sigma}\frac{\delta}{\delta \bar{\mathfrak{g}}^{\rho\sigma}}\phi^A\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^A} + \frac{1}{2!}\phi^B\frac{\delta}{\delta \bar{\Phi}^B}\phi^A\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^A}\right)\,. \end{aligned} \tag{2.2.52}$$

The cubic approximation of (2.2.46) gives the possibility (a) to construct the field equations including quadratic terms, which are related to the energy-momentum tensor (2.2.52), and (b) to construct the energy-momentum tensor, including quadratic (2.2.52) and *cubic* parts, etc. Thus, the representation of the dynamical Lagrangian in the form (2.2.46) gives, in fact, the algorithm for constructing an approximate system. Besides, such a structure of (2.2.46) explains simply the infinite series, for example, in the equations of the type (2.1.8).

2.2.4 Gauge transformations and their properties

Diffeomorphism invariance of a geometrical theory is connected with mapping a spacetime onto itself and is classified as an extrinsic symmetry. Gauge transformations and gauge invariance in general relativity (and other metric theories) are connected with the diffeomorphism invariance, see [154, 316] and (1.2.72–1.2.82) in Section 1.2.3. The gauge invariance properties of the field-theoretical formulation of general relativity follows from the properties of general relativity itself, and this kind of invariance is classified as intrinsic symmetry, as in electrodynamics. In this subsection, the definition of gauge transformations and their properties in the field-theoretical formulation of general relativity are described. The presentation is based on the exact theory of gauge transformations developed in the works [204, 206, 360, 361, 379].

At first, one has to consider differentiable coordinate transformations

$$x'^{\alpha} = f^{\alpha}(x^{\beta}), \qquad (2.2.53)$$

which can be connected with the smooth vector field ξ^{α}:

$$x'^{\alpha} = \exp\left(\xi^{\beta}(x)\frac{\partial}{\partial x^{\beta}}\right)x^{\alpha} = x^{\alpha} + \xi^{\alpha}(x) + \frac{1}{2!}\,\xi^{\beta}\xi^{\alpha}{}_{,\beta} + \frac{1}{3!}\,\xi^{\pi}(\xi^{\beta}\xi^{\alpha}{}_{,\beta})_{,\pi} + \dots . \qquad (2.2.54)$$

Here, the exponent is understood as an operator; the transformations (2.2.54) are understood as the exact, not infinitesimal, ones. Then the dynamical variables of the geometrical formulation of general relativity, metric density and matter fields, are transformed in the usual way, see Appendixes A.1 and A.3.1,

$$\mathfrak{g}^{\mu\nu}(x) \to \mathfrak{g}'^{\mu\nu}(x'), \qquad (2.2.55)$$

$$\Phi^{A}(x) \to \Phi'^{A}(x'). \qquad (2.2.56)$$

Now, let us provide the operation connected with the Lie displacement along the vector $\xi^{\alpha}(x)$ in (2.2.54). Already the definition for Lie derivatives is given in (1.2.77–1.2.82), and their properties are given in Appendix A.2.3. However, only the linear approximation in ξ^{α} was used. Here, the transformations (2.2.53–2.2.56) are not approximate. As a result, (2.2.55) and (2.2.56) lead to

$$\mathfrak{g}'^{\mu\nu}(x) = \exp £_{\xi}\, \mathfrak{g}^{\mu\nu}(x) = \mathfrak{g}^{\mu\nu}(x) + \sum_{k=1}^{\infty}\frac{1}{k!}\,£_{\xi}^{k}\mathfrak{g}^{\mu\nu}(x), \qquad (2.2.57)$$

$$\Phi'^{A}(x) = \exp £_{\xi}\, \Phi^{A}(x) = \Phi^{A}(x) + \sum_{k=1}^{\infty}\frac{1}{k!}\,£_{\xi}^{k}\Phi^{A}(x). \qquad (2.2.58)$$

The exponent is also an operator,

$$\exp £_{\xi} = 1 + £_{\xi} + \frac{1}{2!}£_{\xi}^{2} + \dots , \qquad (2.2.59)$$

and we will use the operator:

$$\sum_{k=1}^{\infty}\frac{1}{k!}\,£_{\xi}^{k} \to \left(\exp £_{\xi} - 1\right). \qquad (2.2.60)$$

to make formulae more compact.

Assume that components of the geometrical objects Ψ^{B} are differentiable functions of other geometrical objects ψ^{A} and their derivatives, but are not explicit functions of coordinates. Assume also that ψ^{A} are transformed analogously to (2.2.57) and (2.2.58),

$$\psi'^{A}(x) = \exp £_{\xi}\, \psi^{A}(x). \qquad (2.2.61)$$

Then a simple substitution gives $\Psi^B(\psi'^A) = \Psi'^B$ which means

$$\Psi^B(\exp \pounds_\xi \psi^A) = \exp \pounds_\xi \Psi^B(\psi^A). \qquad (2.2.62)$$

Taking into account this property, let us examine the invariance of the Einstein-Hilbert Lagrangian in (2.2.1) and the Einstein equations (2.2.3) and (2.2.4) with respect to the substitution of (2.2.57) and (2.2.58). One easily finds that the Lagrangian is invariant up to a divergence:

$$\begin{aligned}\mathscr{L}_{EH}(\mathfrak{g}', \Phi') &= \exp \pounds_\xi \mathscr{L}_{EH}(\mathfrak{g}, \Phi) = \mathscr{L}_{EH}(\mathfrak{g}, \Phi) + (\exp \pounds_\xi - 1)\mathscr{L}_{EH}(\mathfrak{g}, \Phi)\\ &= \mathscr{L}_{EH}(\mathfrak{g}, \Phi) + \text{div}. \end{aligned} \qquad (2.2.63)$$

The divergence has appeared because the Lagrangian is a scalar density of weight +1; and, as usual, the divergence does not influence the equations of motion. The variation of the Lagrangian (2.2.63) with respect to $\mathfrak{g}^{\mu\nu}$ and Φ^A gives again (2.2.3) and (2.2.4), respectively.

Substituting (2.2.57) and (2.2.58) into the operators of the equations (2.2.3) and (2.2.4) and keeping in mind the rule (2.2.62), one gets

$$\frac{\delta \mathscr{L}^{HE}(\mathfrak{g}', \Phi')}{\delta \mathfrak{g}'^{\mu\nu}} = \exp \pounds_\xi \frac{\delta \mathscr{L}^{HE}(\mathfrak{g}, \Phi)}{\delta \mathfrak{g}^{\mu\nu}}, \qquad (2.2.64)$$

$$\frac{\delta \mathscr{L}^{HE}(\mathfrak{g}', \Phi')}{\delta \Phi'^A} = \exp \pounds_\xi \frac{\delta \mathscr{L}^{HE}(\mathfrak{g}, \Phi)}{\delta \Phi^A}. \qquad (2.2.65)$$

Thus, if $\mathfrak{g}, \Phi$ satisfy (2.2.3) and (2.2.4) then $\mathfrak{g}', \Phi'$ satisfy them as well. To this end, transformations from (2.2.63) to (2.2.65) can be classified as a diffeomorphism invariance.

Let us outline how the above invariance is transformed to the gauge invariance in the framework of the field-theoretical formulation. From the beginning, one has to define the gauge transformations for the dynamical variables $\mathfrak{h}$ and ϕ. First, define the decompositions

$$\mathfrak{g}'^{\mu\nu}(x) \equiv \bar{\mathfrak{g}}^{\mu\nu}(x) + \mathfrak{h}'^{\mu\nu}(x), \qquad (2.2.66)$$

$$\Phi'^A(x) \equiv \bar{\Phi}^A(x) + \phi'^A(x) \qquad (2.2.67)$$

for the *primed* quantities at the left hand sides of (2.2.57) and (2.2.58). Notice that the background fields in (2.2.66) and (2.2.67), $\bar{\mathfrak{g}}^{\mu\nu}(x)$ and $\bar{\Phi}^A(x)$ are the *same* as in the decompositions (2.2.7) and (2.2.8). *We stress this point!* Second, substitute both (2.2.7), (2.2.8) and (2.2.66), (2.2.67) into (2.2.57) and (2.2.58). One easily obtains

$$\mathfrak{h}'^{\mu\nu}(x) = \mathfrak{h}^{\mu\nu}(x) + (\exp \pounds_\xi - 1)\left(\bar{\mathfrak{g}}^{\mu\nu}(x) + \mathfrak{h}^{\mu\nu}(x)\right), \qquad (2.2.68)$$

$$\phi'^A(x) = \phi^A(x) + (\exp \pounds_\xi - 1)\left(\bar{\Phi}^A(x) + \phi^A(x)\right), \qquad (2.2.69)$$

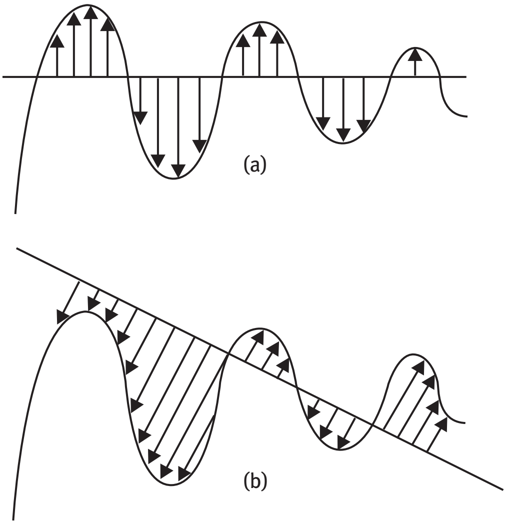

Figure 2.1: Perturbations in the cases (a) and (b) are shown by arrows. They look different but are tightly connected by the gauge transformations.

where $\bar{\mathfrak{g}}^{\mu\nu}(x)$ and $\bar{\Phi}^A(x)$ are canceled in both of the sides. The transformations (2.2.68) and (2.2.69) themselves can be interpreted outside of coordinate and diffeomorphism transformations. They influence the dynamical variables only, they do not influence both the coordinates and the background quantities. Formally they do not provide a mapping of the spacetime onto itself. Therefore such transformations are classified as intrinsic (gauge) ones, unlike diffeomorphisms (extrinsic transformations).

The transformations (2.2.68) and (2.2.69) can be symbolically illustrated by the Figure 2.1. In a geometrical theory both non-primed and primed quantities in (2.2.55) and (2.2.56) represent the *same* solution: it is symbolized by identical curves in both of the cases (a) and (b) at Figure 2.1. The choice of the *same* background fields, $\bar{\mathfrak{g}}^{\mu\nu}$ and $\bar{\Phi}^A$, both in the initial decomposition (2.2.7), (2.2.8) and in the diffeomorphism transformed decomposition (2.2.66), (2.2.67) is symbolized by different straight lines. For example, the Minkowski metric $\eta_{\mu\nu}$ defined in different, initial and mapped, manifolds are shifted one from the other. Then, of course, perturbations in the cases (a) and (b) are different, but they are connected by the gauge transformations of the type (2.2.68) and (2.2.69).

Now, substitute the transformations (2.2.68) and (2.2.69) into the dynamical Lagrangian (2.2.15). One finds for the first term in (2.2.15),

$$\begin{aligned}\mathscr{L}'_{EH}(\bar{\mathfrak{g}} + \mathfrak{h}, \bar{\Phi} + \phi) &= \mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h}', \bar{\Phi} + \phi') \\ &= \mathscr{L}_{EH}(\exp £_\xi(\bar{\mathfrak{g}} + \mathfrak{h}), \exp £_\xi(\bar{\Phi} + \phi)) \\ &= \exp £_\xi \mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h}, \bar{\Phi} + \phi)\,. \end{aligned} \tag{2.2.70}$$

Keeping this in mind and substituting the transformations (2.2.68) and (2.2.69) into the dynamical Lagrangian (2.2.15) one obtains finally:

$$\begin{aligned}\mathscr{L}'^{\text{dyn}} &= \mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h}, \bar{\Phi} + \phi) + (\exp £_\xi - 1)\mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h}, \bar{\Phi} + \phi) \\ &\quad - \mathfrak{h}'^{\mu\nu}\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\mathfrak{g}}^{\mu\nu}} - \phi'^A \frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^A} - \bar{\mathscr{L}}_{EH} - \frac{1}{16\pi}\partial_\alpha \mathscr{D}'^\alpha \,. \end{aligned} \tag{2.2.71}$$

Using the structure of the dynamical Lagrangian (2.2.15), it can be rewritten as

$$\begin{aligned}\mathscr{L}'^{\text{dyn}} &= \mathscr{L}^{\text{dyn}} - \left(\mathfrak{h}'^{\mu\nu} - \mathfrak{h}^{\mu\nu}\right)\frac{\delta \bar{\mathscr{L}}^{EH}}{\delta \bar{\mathfrak{g}}^{\mu\nu}} - \left(\phi'^A - \phi^A\right)\frac{\delta \bar{\mathscr{L}}_{EH}}{\delta \bar{\Phi}^A} \\ &\quad - \frac{1}{16\pi}\partial_\alpha\left(\mathscr{D}'^\alpha - \mathscr{D}^\alpha\right) + (\exp £_\xi - 1)\mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h}, \bar{\Phi} + \phi)\,. \end{aligned} \tag{2.2.72}$$

This means that the dynamical Lagrangian is gauge invariant on the background equations (2.2.10) and (2.2.11), and up to a divergence in the second line.

Now, let us examine the field equations (2.2.26) under the gauge transformations. Combining (2.2.23–2.2.26), the field equations operator of (2.2.26) can be rewritten as

$$G^L_{\mu\nu} + \Phi^L_{\mu\nu} - 8\pi t^{\text{tot}}_{\mu\nu} = -\frac{16\pi}{\sqrt{-\bar{g}}}\frac{\partial \bar{\mathfrak{g}}^{\alpha\beta}}{\partial \bar{g}^{\mu\nu}}\frac{\delta}{\delta \bar{\mathfrak{g}}^{\alpha\beta}}\left(\mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h},\ \Phi + \phi) - \bar{\mathscr{L}}_{EH}\right). \tag{2.2.73}$$

Then, substituting (2.2.58) and (2.2.59), one finds after simple but long transformations,

$$\begin{aligned} &G^L_{\mu\nu}(\mathfrak{h}') + \Phi^L_{\mu\nu}(\mathfrak{h}', \phi') - 8\pi t^{\text{tot}}_{\mu\nu}(\mathfrak{h}', \phi') \\ &= G^L_{\mu\nu}(\mathfrak{h}) + \Phi^L_{\mu\nu}(\mathfrak{h}, \phi) - 8\pi t^{\text{tot}}_{\mu\nu}(\mathfrak{h}, \phi) \\ &+ \frac{1}{\sqrt{-\bar{g}}}\frac{\partial \bar{\mathfrak{g}}^{\rho\sigma}}{\partial \bar{g}^{\mu\nu}}(\exp £_\xi - 1)\left[\sqrt{-\bar{g}}\frac{\partial \bar{g}^{\delta\pi}}{\partial \bar{\mathfrak{g}}^{\rho\sigma}}\left(G^L_{\delta\pi} + \Phi^L_{\delta\pi} - 8\pi t^{\text{tot}}_{\delta\pi}\right) - 16\pi\frac{\delta \bar{\mathscr{L}}^{HE}}{\delta \bar{\mathfrak{g}}^{\rho\sigma}}\right]. \end{aligned} \tag{2.2.74}$$

That means: if the background equation (2.2.10) hold and the fields $\mathfrak{h}, \phi$ are solutions to the field equations (2.2.26), then $\mathfrak{h}', \phi'$ are solutions to the same equations. Analogous conclusions are valid for the matter equations in the field-theoretical form (2.2.39).

The energy-momentum tensor is not gauge invariant even on the dynamical and background equations. Indeed, from (2.2.74) under the transformations (2.2.68) and (2.2.69), it follows

$$8\pi t'^{\text{tot}}_{\mu\nu} = 8\pi t^{\text{tot}}_{\mu\nu}(\mathfrak{h}', \phi') = 8\pi t^{\text{tot}}_{\mu\nu} + G^L_{\mu\nu}(\mathfrak{h}' - \mathfrak{h}) + \Phi^L_{\mu\nu}(\mathfrak{h}' - \mathfrak{h};\ \phi' - \phi)\,. \tag{2.2.75}$$

The mathematical reason is that the background equations in the gauge transformed Lagrangian (2.2.72) cannot be taken into account before the variation. In the case of a

Ricci-flat background (2.2.13) one has $\Phi^L_{\mu\nu} = 0$, therefore the energy-momentum $t^{\text{tot}}_{\mu\nu}$ is not gauge invariant up to $G^L_{\mu\nu}$ that is the covariant divergence (see (2.2.27)).

Let us turn also to the equivalent form (2.2.33) of the field equations with the effective source $t^{\text{eff}}_{\mu\nu}$. For the operator of the field equations $G^L_{\mu\nu} - 8\pi t^{\text{eff}}_{\mu\nu}$ the *form* of the transformations (2.2.74) can be applied without changing. Then

$$8\pi {t'}^{\text{eff}}_{\mu\nu} = 8\pi t^{\text{eff}}_{\mu\nu} + G^L_{\mu\nu}(\mathfrak{h}' - \mathfrak{h}) \tag{2.2.76}$$

on the dynamical and background equations. Thus, the energy-momentum $t^{\text{eff}}_{\mu\nu}$ is not gauge invariant up to a covariant divergence in the case of *arbitrary curved backgrounds*, whereas $t^{\text{tot}}_{\mu\nu}$ is not gauge invariant up to a covariant divergence on a *Ricci-flat background* only.

Such a gauge non-invariance of the energy-momentum reflects the fact that energy and other conserved quantities in general relativity are not localized. Unlike the other derivations, see the discussion in textbook [315], the formulae (2.2.75) and (2.2.76) of the field-theoretical formulation give a *quantitative* and *constructive* description of the non-localization.

In many applications, it is important to consider equations and gauge transformations in linear, quadratic and the leading approximations. Assume that perturbations and their derivatives are small: $\mathfrak{h}^{\mu\nu} \ll \bar{\mathfrak{g}}^{\mu\nu}$, $\phi^A \ll \bar{\Phi}^A$, $\mathfrak{h}^{\mu\nu} \approx \partial_\alpha \mathfrak{h}^{\mu\nu} \dots$ and $\phi^A \approx \partial_\alpha \phi^A \approx \dots$. Assuming that the background equations (2.2.12) give a connection $\bar{\mathfrak{g}}^{\mu\nu} \approx f(\kappa)\bar{\Phi}^A$ with a coefficient $f(\kappa)$ depending on the Einstein's constant, one can set $\mathfrak{h}^{\mu\nu} \approx f(\kappa)\phi^A$, etc. To present the main properties of the approximation scheme, let us rewrite the equations (2.2.26), say, up to the second order in perturbations:

$$G^L_{\mu\nu}(\mathfrak{h}) + \Phi^L_{\mu\nu}(\mathfrak{h},\ \phi) - 8\pi\, {}_2t^{\text{tot}}_{\mu\nu}(\mathfrak{h}\mathfrak{h},\ \mathfrak{h}\phi,\ \phi\phi) = 0\,. \tag{2.2.77}$$

Assuming iterations, the perturbations can be expanded as $\mathfrak{h}^{\mu\nu} = \mathfrak{h}^{\mu\nu}_1 + \mathfrak{h}^{\mu\nu}_2 + \dots$, and $\phi^A = \phi^A_1 + \phi^A_2 + \dots$. Then one can obtain a solution to the equations (2.2.26) step by step. Thus, to obtain the solution of (2.2.77) one has to find, firstly, $\mathfrak{h}_1$ and ϕ_1 and, secondly, $\mathfrak{h}_2$ and ϕ_2. Besides, assume $\xi^\mu = \xi^\mu_1 + \xi^\mu_2 + \dots$ with $\xi^\mu_1 \approx \partial_\alpha \xi^\mu_1 \approx \dots \approx \mathfrak{h}^{\mu\nu}_1 \approx f(\kappa)\phi^A_1$ and $\xi^\mu_2 \approx \partial_\alpha \xi^\mu_2 \approx \dots \approx \mathfrak{h}^{\mu\nu}_2 \approx f(\kappa)\phi^A_2$.

Now, let us present the linear version of the equations (2.2.77):

$$G^L_{\alpha\beta}(\mathfrak{h}_1) + \Phi^L_{\alpha\beta}(\mathfrak{h}_1,\ \phi_1) = 0\,. \tag{2.2.78}$$

In a linear approximation the transformations (2.2.68) and (2.2.69) have the simple form:

$$\mathfrak{h}'^{\mu\nu}_1 = \mathfrak{h}^{\mu\nu}_1 + \pounds_{\xi_1}\bar{\mathfrak{g}}^{\mu\nu} = \mathfrak{h}^{\mu\nu}_1 - \bar{\mathfrak{g}}^{\mu\nu}\,\bar{\nabla}_\rho \xi^\rho_1 + \sqrt{-\bar{g}}\left(\bar{\nabla}^\mu \xi^\nu_1 + \bar{\nabla}^\nu \xi^\mu_1\right), \tag{2.2.79}$$

$$\phi'^A_1 = \phi^A_1 + \pounds_{\xi_1}\bar{\Phi}^A. \tag{2.2.80}$$

Substituting (2.2.79) and (2.2.80) into (2.2.74) and saving the linear approximation, one has

$$\begin{aligned}\left[G^L_{\mu\nu}(\mathfrak{h}_1) + \Phi^L_{\mu\nu}(\mathfrak{h}_1,\ \phi_1)\right]' &= \left[G^L_{\mu\nu}(\mathfrak{h}_1) + \Phi^L_{\mu\nu}(\mathfrak{h}_1,\ \phi_1)\right] \\ &+ \left(\delta^\rho_\mu \delta^\sigma_\mu - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\right) £_{\xi_1} \left[\bar{R}_{\rho\sigma} - 8\pi\left(\bar{T}_{\rho\sigma} - \tfrac{1}{2}\bar{g}_{\rho\sigma}\bar{T}\right)\right]. \end{aligned} \tag{2.2.81}$$

Thus, the linear equations are gauge invariant on the background equations only, the fields $\mathfrak{h}_1$ and ϕ_1 could not satisfy to (2.2.78). In the simple case of the Ricci-flat background (2.2.13), the linear transformations have the form (2.2.79) only, without (2.2.80). Then the formula (2.2.81) transfers to the formula $G'^L_{\mu\nu} = G^L_{\mu\nu}$, which expresses the gauge invariance of the linear spin-2 field, see (2.1.5) and (2.1.6). All of these are the generalizations of the well known gauge invariance in the linear gravity under the transformations (1.5.15).

The quadratic version of the equations (2.2.77) has the form:

$$G^L_{\alpha\beta}(\mathfrak{h}_2) + \Phi^L_{\alpha\beta}(\mathfrak{h}_2,\ \phi_2) - 8\pi\left({}_2t^{\mathrm{g}}_{\alpha\beta}(\mathfrak{h}_1\mathfrak{h}_1) + {}_2t^{\mathrm{m}}_{\alpha\beta}(\mathfrak{h}_1\mathfrak{h}_1,\ \mathfrak{h}_1\phi_1,\ \phi_1\phi_1)\right) = 0\,. \tag{2.2.82}$$

The quadratic order of the gauge transformations (2.2.68) and (2.2.69) has the form:

$$\mathfrak{h}'^{\mu\nu}_2 = \mathfrak{h}^{\mu\nu}_2 + £_{\xi_2}\bar{\mathfrak{g}}^{\mu\nu} + \frac{1}{2!}£^2_{\xi_1}\bar{\mathfrak{g}}^{\mu\nu} + £_{\xi_1}\mathfrak{h}^{\mu\nu}_1 \tag{2.2.83}$$

$$\phi'^A_2 = \phi^A_2 + £_{\xi_2}\bar{\Phi}^A + \frac{1}{2!}£^2_{\xi_1}\bar{\Phi}^{\mu\nu} + £_{\xi_1}\phi^A_1\,. \tag{2.2.84}$$

Substituting (2.2.83) and (2.2.84) into (2.2.74) and saving the quadratic approximation, one gets

$$\begin{aligned}&\left[G^L_{\mu\nu}(\mathfrak{h}_2) + \Phi^L_{\mu\nu}(\mathfrak{h}_2,\ \phi_2) - 8\pi\,{}_2t^{\mathrm{tot}}_{\mu\nu}(\mathfrak{h}_1\mathfrak{h}_1,\ \mathfrak{h}_1\phi_1,\ \phi_1\phi_1)\right]' \\ &= \left[G^L_{\mu\nu}(\mathfrak{h}_2) + \Phi^L_{\mu\nu}(\mathfrak{h}_2,\ \phi_2) - 8\pi\,{}_2t^{\mathrm{tot}}_{\mu\nu}(\mathfrak{h}_1\mathfrak{h}_1,\ \mathfrak{h}_1\phi_1,\ \phi_1\phi_1)\right] \\ &+ \frac{1}{\sqrt{-\bar{g}}}\frac{\partial\bar{\mathfrak{g}}^{\rho\sigma}}{\partial\bar{g}^{\mu\nu}}\left(£_{\xi_2} + \frac{1}{2!}£^2_{\xi_1}\right)\left[\bar{R}_{\rho\sigma} - 8\pi\left(\bar{T}_{\rho\sigma} - \tfrac{1}{2}\bar{g}_{\rho\sigma}\bar{T}\right)\right] + \\ &+ \frac{1}{\sqrt{-\bar{g}}}\frac{\partial\bar{\mathfrak{g}}^{\rho\sigma}}{\partial\bar{g}^{\mu\nu}}£_{\xi_1}\left[\sqrt{-\bar{g}}\frac{\partial\bar{g}^{\delta\pi}}{\partial\bar{\mathfrak{g}}^{\rho\sigma}}\left[G^L_{\delta\pi}(\mathfrak{h}_1)) + \Phi^L_{\delta\pi}(\mathfrak{h}_1,\ \phi_1)\right]\right]. \end{aligned} \tag{2.2.85}$$

Thus, equations (2.2.82) are gauge invariant on the background equations (2.2.12) and on the linear equations (2.2.78). Of course, the procedure can be continued in the next orders.

2.2.5 Differential conservation laws

The energy-momentum tensor is the one of the important entities in a field theory in the Minkowski space. Its differential conservation together with symmetries of the Minkowski space permit one to construct integral conserved quantities. The field-theoretical formulation of general relativity in the Minkowski space has also the conserved energy-momentum with the same properties, see (2.1.4) and (2.1.28). Here, these results related to energy-momentum are generalized to curved backgrounds. More general conserved quantities, currents and superpotentials, on arbitrary curved backgrounds are constructed and studied in detail later in Chapter 6.

To construct conservation laws and conserved quantities we use both the invariance of the action

$$S = \int_\Omega dx^4 \mathscr{L}^{\text{dyn}} \tag{2.2.86}$$

with respect to a mapping of the spacetime onto itself and gauge transformations. In both the cases, the vector field ξ^α plays a crucial role. The main assumption below is that ξ^α and its derivatives are arbitrary and they vanish on the boundary $\partial\Omega$ of the volume of integration Ω.

The diffeomorphism invariance of S in (2.2.86) is evident, indeed

$$\delta_\xi S \equiv \int_\Omega dx^4 \pounds_\xi \mathscr{L}^{\text{dyn}} \equiv -\int_\Omega dx^4 \partial_\alpha \left(\xi^\alpha \mathscr{L}^{\text{dyn}}\right) \equiv -\int_{\partial\Omega} ds_\alpha \xi^\alpha \mathscr{L}^{\text{dyn}} \equiv 0\,. \tag{2.2.87}$$

On the other hand, let us consider the property (2.2.62) rewritten for $\mathscr{L}^{\text{dyn}}$ in the linear approximation and use (A.2.25) in Appendix A.2.3,

$$\begin{aligned}\pounds_\xi \mathscr{L}^{\text{dyn}} \equiv\ & \frac{\partial \mathscr{L}^{\text{dyn}}}{\partial \mathfrak{h}^{\mu\nu}} \pounds_\xi \mathfrak{h}^{\mu\nu} + \frac{\partial \mathscr{L}^{\text{dyn}}}{\partial \mathfrak{h}^{\mu\nu}{}_{,\alpha}} \pounds_\xi \mathfrak{h}^{\mu\nu}{}_{,\alpha} + \frac{\partial \mathscr{L}^{\text{dyn}}}{\partial \mathfrak{h}^{\mu\nu}{}_{,\alpha\beta}} \pounds_\xi \mathfrak{h}^{\mu\nu}{}_{,\alpha\beta} \\ & + \frac{\partial \mathscr{L}^{\text{dyn}}}{\partial \phi^A} \pounds_\xi \phi^A + \dots + \frac{\partial \mathscr{L}^{\text{dyn}}}{\partial \bar{\mathfrak{g}}^{\mu\nu}} \pounds_\xi \bar{\mathfrak{g}}^{\mu\nu} + \dots + \frac{\partial \mathscr{L}^{\text{dyn}}}{\partial \bar{\Phi}^A} \pounds_\xi \bar{\Phi}^A + \dots\,.\end{aligned} \tag{2.2.88}$$

Using here the formula (A.2.24) of Appendix A.2.3, one rewrites the identity (2.2.87) in another form:

$$\delta_\xi S \equiv \int_\Omega dx^4 \left[\frac{\delta \mathscr{L}^{\text{dyn}}}{\delta \mathfrak{h}^{\mu\nu}} \pounds_\xi \mathfrak{h}^{\mu\nu} + \frac{\delta \mathscr{L}^{\text{dyn}}}{\delta \phi^A} \pounds_\xi \phi^A + \frac{\delta \mathscr{L}^{\text{dyn}}}{\delta \bar{\mathfrak{g}}^{\mu\nu}} \pounds_\xi \bar{\mathfrak{g}}^{\mu\nu} + \frac{\delta \mathscr{L}^{\text{dyn}}}{\delta \bar{\Phi}^A} \pounds_\xi \bar{\Phi}^A \right] \equiv 0. \tag{2.2.89}$$

Here, surface terms have been suppressed because $\pounds_\xi$ and its derivatives, like ξ^α and its derivatives, disappear at $\partial\Omega$. Now, assuming that the field equations (2.2.22) and (2.2.37) hold, using the explicit expressions for the Lie derivatives,

$$\pounds_\xi \bar{\mathfrak{g}}^{\mu\nu} = 2\bar{\nabla}^{(\mu}\xi^{\nu)}\,, \tag{2.2.90}$$

$$\pounds_\xi \bar{\Phi}^A = -\xi^\mu \bar{\nabla}_\mu \bar{\Phi}^A + \bar{\Phi}^A\big|^\nu_\mu \bar{\nabla}_\nu \xi^\mu\,, \tag{2.2.91}$$

and removing surface terms in (2.2.89) one obtains the equality:

$$\sqrt{-\bar{g}}\bar{\nabla}^{\nu} t^{\mathrm{tot}}_{\mu\nu} + \bar{\nabla}_{\nu}\left(t^{m}_{A}\ \bar{\Phi}^{A}\big|^{\nu}_{\mu}\right) + t^{m}_{A}\bar{\nabla}_{\mu}\bar{\Phi}^{A} = 0\,, \tag{2.2.92}$$

where the definitions (2.2.29) and (2.2.39) are used. One can see that on arbitrary curved backgrounds the total energy-momentum is not conserved differentially. Rather, the relation (2.2.92) plays the role of the integrability conditions for the field equations (2.2.22) and (2.2.37).

The gauge invariance of S in (2.2.86) is evident due to the gauge transformation for the dynamical Lagrangian (2.2.72). Indeed, after taking into account the background equations and suppressing the surface terms, one has

$$\delta_{\xi} S \equiv \int_{\Omega} dx^{4}\left(\mathscr{L}'^{\mathrm{dyn}} - \mathscr{L}^{\mathrm{dyn}}\right) \equiv 0\,. \tag{2.2.93}$$

The same identity must be obtained after a direct substitution of the gauge transformations (2.2.57) and (2.2.58) into $\mathscr{L}^{\mathrm{dyn}}$. In the linear approximation in ξ^{α} and its derivatives, one has

$$\delta_{\xi} S \equiv \int dx^{4}\left[\frac{\delta\mathscr{L}^{\mathrm{dyn}}}{\delta\mathfrak{h}^{\mu\nu}}£_{\xi}\left(\bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu}\right) + \frac{\delta\mathscr{L}^{\mathrm{dyn}}}{\delta\phi^{A}}£_{\xi}\left(\bar{\Phi}^{A} + \phi^{A}\right)\right] \equiv 0, \tag{2.2.94}$$

where again the surface terms have been suppressed. Substituting the explicit expressions for the Lie derivatives,

$$£_{\xi}\left(\bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu}\right) = -\xi^{\alpha}\partial_{\alpha}\left(\bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu}\right) + \left(\bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu}\right)\big|^{\beta}_{\alpha}\partial_{\beta}\xi^{\alpha}\,, \tag{2.2.95}$$

$$£_{\xi}\left(\bar{\Phi}^{A} + \phi^{A}\right) = -\xi^{\alpha}\partial_{\alpha}\left(\bar{\Phi}^{A} + \phi^{A}\right) + \left(\bar{\Phi}^{A} + \phi^{A}\right)\big|^{\beta}_{\alpha}\partial_{\beta}\xi^{\alpha}\,, \tag{2.2.96}$$

into (2.2.94) and suppressing the surface terms, one obtains the identity that connects the Lagrangian derivatives (field equations operators) with their derivatives. It is just a conclusion of the second Noether's theorem, see Section 1.2.3.

Let us turn to the relation (2.2.92) that shows that the total energy-momentum tensor is not conserved $\bar{\nabla}^{\nu} t^{\mathrm{tot}}_{\mu\nu} \neq 0$ on an arbitrary curved background, unlike (2.1.28) on a flat background. This means that the divergence of the left hand side of (2.2.26) is not equal to zero identically, unlike (2.1.28). To analyze this situation, the gauge invariance can also be used. To examine the linear expressions at the left hand side, it is enough to consider the quadratic approximation of the dynamical Lagrangian (2.2.46) and use the corresponding approximation of the gauge transformation for the Lagrangian $\mathscr{L}^{\mathrm{dyn}}$ in the identity (2.2.72).

It is constructive to consider the gravitational and matter Lagrangians separately. The way of deriving (2.2.72) permits this. Thus,

$$\mathscr{L}_2^{\mathrm{g}} = \frac{1}{2!}\mathfrak{h}^{\alpha\beta}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{R}}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}, \tag{2.2.97}$$

$$\begin{aligned}\mathscr{L}_2^{\mathrm{m}} &= \frac{1}{2!}\mathfrak{h}^{\alpha\beta}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + \mathfrak{h}^{\rho\sigma}\frac{\delta}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}\phi^A\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\Phi}^A}\\ &+ \frac{1}{2!}\phi^B\frac{\delta}{\delta\bar{\Phi}^B}\phi^A\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\Phi}^A},\end{aligned} \tag{2.2.98}$$

Substituting the gauge transformation (2.2.57) for $\mathfrak{h}^{\alpha\beta}$ into (2.2.97), keeping quadratic in $\mathfrak{h}$ and in ξ terms, integrating over Ω and suppressing the surface terms at $\partial\Omega$, one has

$$\delta_\xi S \equiv \int_\Omega dx^4 \delta_\xi \mathscr{L}_2^{\mathrm{g}} \equiv \int_\Omega dx^4 \left(\frac{\delta}{\delta\bar{\mathfrak{g}}^{\mu\nu}}\mathfrak{h}^{\rho\sigma}\frac{\delta\bar{\mathscr{R}}}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}\right)\pounds_\xi\bar{\mathfrak{g}}^{\mu\nu}. \tag{2.2.99}$$

On the other hand, we integrate the identity (2.2.72), adopting it to $\mathscr{L}_2^{\mathrm{g}}$, again keeping quadratic in $\mathfrak{h}$ and in ξ terms and suppressing the surface terms at $\partial\Omega$, and taking into account the background equations,

$$\delta_\xi S \equiv -\int_\Omega dx^4 \frac{\delta\bar{\mathscr{R}}}{\delta\bar{\mathfrak{g}}^{\mu\nu}}\pounds_\xi\mathfrak{h}^{\mu\nu}. \tag{2.2.100}$$

Now, equating (2.2.99) and (2.2.100), using the definition (2.2.27) and suppressing the surface terms, one obtains

$$2\bar{\nabla}^\mu G^L_{\mu\nu}(h) \equiv h^{\mu\alpha}\bar{\nabla}_\nu\bar{R}_{\mu\alpha} - 2\bar{\nabla}_\alpha\left(h^{\mu\alpha}\bar{R}_{\mu\nu}\right). \tag{2.2.101}$$

This can be rechecked by a direct calculation applied to (2.2.27). Analogous steps provided with $\mathscr{L}_2^{\mathrm{m}}$ lead to the identity:

$$\begin{aligned}&\frac{\sqrt{-\bar{g}}}{8\pi}\bar{\nabla}^\mu\Phi^L_{\mu\nu} + \Phi^L_A\bar{\nabla}_\nu\bar{\Phi}^A + \bar{\nabla}_\beta\left(\Phi^L_A\bar{\Phi}^A\,|^\beta_\nu\right)\\ &\equiv -\tfrac{1}{2}\bar{\nabla}_\nu\left(\bar{T}_{\mu\alpha} - \tfrac{1}{2}\bar{g}_{\mu\alpha}\bar{T}\right)\mathfrak{h}^{\mu\alpha} + \bar{\nabla}_\alpha\left[\left(\bar{T}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)\mathfrak{h}^{\mu\alpha}\right].\end{aligned} \tag{2.2.102}$$

Combining the identities (2.2.101) and (2.2.102), and taking into account the background equations (2.2.12) one has

$$\frac{\sqrt{-\bar{g}}}{8\pi}\bar{\nabla}^\mu\left(G^L_{\mu\nu} + \Phi^L_{\mu\nu}\right) + \Phi^L_A\bar{\nabla}_\nu\bar{\Phi}^A + \bar{\nabla}_\beta\left(\Phi^L_A\bar{\Phi}^A\,|^\beta_\nu\right) \equiv 0. \tag{2.2.103}$$

Substituting here the field equations (2.2.26) and (2.2.37) one obtains again the relation (2.2.92).

As was remarked, it is very important to consider the case of the Ricci-flat background (2.2.13), when $\bar{\Phi}^A \equiv 0$ and $\bar{\mathscr{L}}^M \equiv 0$. The corresponding dynamical Lagrangian is simplified to

$$\mathscr{L}^{\mathrm{dyn}} = -\frac{1}{16\pi}\mathscr{L}^g + \mathscr{L}^m = -\frac{1}{16\pi}\mathscr{L}^g + \mathscr{L}^M\left(\bar{\mathfrak{g}} + \mathfrak{h}, \phi\right). \tag{2.2.104}$$

The field equations (2.2.26) transform to the form

$$G^L_{\mu\nu} = 8\pi\left(t^g_{\mu\nu} + t^m_{\mu\nu}\right) \equiv 8\pi t^{\mathrm{tot}}_{\mu\nu}. \tag{2.2.105}$$

Then, taking into account the identity $\bar{\nabla}^\mu G^L_{\mu\nu} \equiv 0$ for the Ricci-flat backgrounds, that follows also from (2.2.101), one has the differential conservation law:

$$\bar{\nabla}^\mu t^{\mathrm{tot}}_{\mu\nu} = 0. \tag{2.2.106}$$

The other important cases of backgrounds are presented by the Einstein spaces in Petrov's definition [372] that defines the background equations as

$$\bar{R}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}, \tag{2.2.107}$$

where Λ is a constant. Ricci-flat, de Sitter and AdS backgrounds are particular cases of the Einstein spaces. The Lagrangian of the background system has the form:

$$\bar{\mathscr{L}}^{HE} = -\frac{1}{16\pi}\bar{\mathscr{R}} + \bar{\mathscr{L}}^M = -\frac{1}{16\pi}\left(\bar{\mathscr{R}} - 2\Lambda\sqrt{-\bar{g}}\right). \tag{2.2.108}$$

Here, the constant Λ is interpreted as "degenerated" matter that is not varied. Then, the dynamical Lagrangian gets the form

$$\mathscr{L}^{\mathrm{dyn}} = -\frac{1}{16\pi}\mathscr{L}^g + \mathscr{L}^m = -\frac{1}{16\pi}\mathscr{L}^g + \left[\mathscr{L}^M\left(\bar{\mathfrak{g}} + \mathfrak{h}, \Lambda\right) - \mathfrak{h}^{\mu\nu}\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\mathfrak{g}}^{\mu\nu}} - \bar{\mathscr{L}}^M\right]. \tag{2.2.109}$$

Then $\Phi^L_{\mu\nu} = \Lambda h_{\mu\nu}$ and, thus, the field equations (2.2.26) acquire the simple form:

$$G^L_{\mu\nu} + \Lambda h_{\mu\nu} = 8\pi\left(t^g_{\mu\nu} + t^m_{\mu\nu}\right) \equiv 8\pi t^{\mathrm{tot}}_{\mu\nu}. \tag{2.2.110}$$

Then, because $\Phi^L_A \equiv 0$ for the Einstein space backgrounds (2.2.107), the identity (2.2.103) gives

$$\bar{\nabla}^\mu\left(G^L_{\mu\nu} + \Lambda h_{\mu\nu}\right) \equiv 0 \tag{2.2.111}$$

that leads to the conservation law:

$$\bar{\nabla}^\mu t^{\mathrm{tot}}_{\mu\nu} = 0. \tag{2.2.112}$$

In heuristic form the differential conservation law on AdS and de Sitter backgrounds was used in [1]; in the Lagrangian description it was shortly noted in [206]; and, in the paper [135], it was studied in more detail.

At last, it is interesting to consider the representation (2.2.33) that is valid on an arbitrary curved background. Due to (2.2.101), one concludes that the divergence of the left hand side of (2.2.33) is not equal to zero identically, thus in general $\bar{\nabla}^\mu t^{\text{eff}}_{\mu\nu} \neq 0$. However, for the Ricci-flat background (2.2.13), $t^{\text{eff}}_{\mu\nu} = t^{\text{tot}}_{\mu\nu}$. Then the conservation (2.2.106) can be rewritten for (2.2.33):

$$\bar{\nabla}^\mu t^{\text{eff}}_{\mu\nu} = 0 . \tag{2.2.113}$$

2.2.6 Different variants of the field-theoretical formulation in general relativity

In general relativity, components of each of different metrical variables,

$$\mathbf{g}^{\mu\nu}_a = \begin{cases} (\sqrt{-g})^m g^{\mu\nu} , \\ (\sqrt{-g})^n g_{\mu\nu} , \end{cases} \tag{2.2.114}$$

can be used as independent dynamical variables; $m, n \in \mathrm{R}$ and the index "a" corresponds to a concrete choice from the right hand side. Each choice of a in (2.2.114) gives a dynamical variable which is a one-to-one function of the others with $a \neq b$ in the sense that taking the Lagrangian derivative is a linear operation

$$\frac{\delta}{\delta \mathbf{g}^{\mu\nu}_a} = \frac{\partial \mathbf{g}^{\alpha\beta}_b}{\partial \mathbf{g}^{\mu\nu}_a} \frac{\delta}{\delta \mathbf{g}^{\alpha\beta}_b} , \tag{2.2.115}$$

which is not singular.

Thus the action of general relativity can be rewritten as

$$S = \int d^4x \mathcal{L}_{EH} \equiv -\frac{1}{16\pi} \int d^4x \mathcal{R}(\mathbf{g}_a) + \int d^4x \mathcal{L}^M(\mathbf{g}_a, \Phi) . \tag{2.2.116}$$

After variation with respect to the dynamical variables $\mathbf{g}^{\mu\nu}_a$, Einstein's equations acquire the generalized form:

$$-\frac{1}{16\pi} \frac{\delta \mathcal{R}}{\delta \mathbf{g}^{\mu\nu}_a} + \frac{\delta \mathcal{L}^M}{\delta \mathbf{g}^{\mu\nu}_a} = 0 . \tag{2.2.117}$$

Of course, by (2.2.115), all the variants are equivalent between themselves and are equivalent to (2.2.3). We do not show the formulation on matter variables here. The background action

$$\bar{S} = \int d^4x \bar{\mathcal{L}}_{EH} \equiv -\frac{1}{16\pi} \int d^4x \mathcal{R}(\bar{\mathbf{g}}_a) + \int d^4x \mathcal{L}^M(\bar{\mathbf{g}}_a, \bar{\Phi}) \tag{2.2.118}$$

and the background Einstein equations

$$-\frac{1}{16\pi}\frac{\delta\bar{\mathscr{R}}}{\delta\bar{\mathbf{g}}^{\mu\nu}_a} + \frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\mathbf{g}}^{\mu\nu}_a} = 0 \tag{2.2.119}$$

have the corresponding to (2.2.116) and (2.2.117) barred form, respectively.

The decomposition of (2.2.114) defines the corresponding perturbations as

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + \varkappa_{\mu\nu}, \\ g^{\mu\nu} &= \bar{g}^{\mu\nu} + l^{\mu\nu}, \\ \mathfrak{g}^{\mu\nu} &= \bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu}, \\ \ldots &= \ldots + \ldots . \end{aligned} \tag{2.2.120}$$

All of this can be presented in the unified form that induces the definition of perturbations for arbitrary real m and n in (2.2.114):

$$g^a = \bar{g}^a + h^a := \begin{cases} \mathfrak{h}^{\mu\nu}_{(m)} = (\sqrt{-g})^m g^{\mu\nu} - (\sqrt{-\bar{g}})^m \bar{g}^{\mu\nu}, \\ \mathfrak{h}_{(n)\mu\nu} = (\sqrt{-g})^n g_{\mu\nu} - (\sqrt{-\bar{g}})^n \bar{g}_{\mu\nu}. \end{cases} \tag{2.2.121}$$

To simplify the formulae below, we also simplify the notations with respect to (2.2.114) as g^a, being ensured that this will not lead to confusions. Of course, each of variables (2.2.114) can be presented as an algebraic function of another variable $g^a_1 = g^a_1(g^b_2)$. Then after the decomposition (2.2.121) applied to each of variables g^a_1 and g^b_2 and using the Taylor series one finds

$$h^a_1 = h^b_2\frac{\partial\bar{g}^a_1}{\partial\bar{g}^b_2} + \frac{1}{2!}h^b_2 h^c_2\frac{\partial^2\bar{g}^a_1}{\partial\bar{g}^b_2\partial\bar{g}^c_2} + \ldots . \tag{2.2.122}$$

Thus, all of the perturbations defined in (2.2.120), or (2.2.121), are different.

Following the rules used in constructing (2.2.15), one defines the generalized dynamical Lagrangian:

$$\mathscr{L}^{\mathrm{dyn}}_a = \mathscr{L}_{EH}\left(\bar{g}^a + h^a, \bar{\Phi} + \phi\right) - h^a\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{g}^a} - \phi^A\frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{\Phi}^A} - \bar{\mathscr{L}}_{EH} - \frac{1}{16\pi}\partial_\nu\mathscr{D}^\nu, \tag{2.2.123}$$

where h^a are independent dynamical gravitational variables. It is the basis for various variants of the field-theoretical formulations of general relativity. Thus, analogous to (2.2.22), the gravitational equations are obtained by varying the action (2.2.123) with respect to h^a:

$$\frac{\delta}{\delta h^a}\mathscr{L}^{\mathrm{dyn}}_a = \frac{\delta}{\delta h^a}\mathscr{L}_{EH}(\bar{g}^a + h^a,\, \bar{\Phi} + \phi) - \frac{\delta\bar{\mathscr{L}}_{EH}}{\delta\bar{g}^a} = 0\,. \tag{2.2.124}$$

Defining the total metric energy-momentum tensor as usual

$$t^{\text{tot}}_{\mu\nu}(h^a, \phi) \equiv \frac{2}{\sqrt{-\bar{g}}} \frac{\delta \mathscr{L}^{\text{dyn}}_a}{\delta \bar{g}^{\mu\nu}} \equiv \frac{2}{\sqrt{-\bar{g}}} \frac{\partial \bar{g}^a}{\partial \bar{g}^{\mu\nu}} \frac{\delta \mathscr{L}^{\text{dyn}}_a}{\delta \bar{g}^a}\,, \tag{2.2.125}$$

one finds

$$G^L_{\mu\nu}(h^a) + \Phi^L_{\mu\nu}(h^a, \phi) = 8\pi t^{\text{tot}}_{\mu\nu}(h^a, \phi)\,. \tag{2.2.126}$$

Here,

$$G^L_{\mu\nu}(h^a) \equiv \frac{1}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}^{\mu\nu}} h^a \frac{\delta \bar{\mathscr{R}}}{\delta \bar{g}^a} \equiv \frac{1}{\sqrt{-\bar{g}}} \frac{\partial \bar{g}^b}{\partial \bar{g}^{\mu\nu}} \frac{\delta}{\delta \bar{g}^b} h^a \frac{\delta \bar{\mathscr{R}}}{\delta \bar{g}^a}, \tag{2.2.127}$$

$$\begin{aligned}\Phi^L_{\mu\nu}(h^a, \phi) &\equiv -\frac{16\pi}{\sqrt{-\bar{g}}} \frac{\delta}{\delta \bar{g}^{\mu\nu}} \left(h^a \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{g}^a} + \phi^A \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{\Phi}^A} \right) \\ &\equiv -\frac{16\pi}{\sqrt{-\bar{g}}} \frac{\partial \bar{g}^b}{\partial \bar{g}^{\mu\nu}} \frac{\delta}{\delta \bar{g}^b} \left(h^a \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{g}^a} + \phi^A \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{\Phi}^A} \right).\end{aligned} \tag{2.2.128}$$

Compare with (2.2.27) and (2.2.28).

Similar to the above discussion, one can conclude that the theory (2.2.123–2.2.128) is equivalent to the Einstein's theory (2.2.116) and (2.2.117). The properties of the theory in the generalized representation (2.2.123–2.2.128) are the same as of the theory (2.2.15) studied in the previous sections. The question appears: is there a difference between the apparently different variants of the theory (2.2.123–2.2.128)?

To provide such a comparison one has to define the unified variables for (2.2.121):

$$\mathfrak{h}^{\mu\nu}_a = h^a \frac{\partial \bar{\mathfrak{g}}^{\mu\nu}}{\partial \bar{g}^a} := \begin{cases} \mathfrak{h}^{\mu\nu}_{(am)} = \frac{\sqrt{-\bar{g}}}{\sqrt{-\bar{g}}^m} \left(\mathfrak{h}^{\mu\nu}_{(m)} - \frac{1-m}{2} \bar{g}^{\mu\nu} \mathfrak{h}^\rho_{(m)\rho} \right), \\ \mathfrak{h}^{\mu\nu}_{(an)} = \frac{\sqrt{-\bar{g}}}{\sqrt{-\bar{g}}^n} \left(-\mathfrak{h}^{\mu\nu}_{(n)} + \frac{1-n}{2} \bar{g}^{\mu\nu} \mathfrak{h}^\rho_{(n)\rho} \right). \end{cases} \tag{2.2.129}$$

Then, taking into account the background equations (2.2.119) and the field equations (2.2.124), the equations (2.2.126) are rewritten in the equivalent form:

$$G^L_{\mu\nu}(\mathfrak{h}_a) + \Phi^L_{\mu\nu}(\mathfrak{h}_a, \phi) = 8\pi t^{\text{tot}}_{\mu\nu}(\mathfrak{h}_a, \phi)\,. \tag{2.2.130}$$

One finds that if $h^a = \mathfrak{h}^{\mu\nu}$ ($m = 1$), then the equations (2.2.130) acquire the form of (2.2.26). Of course, due to (2.2.122), a choice of two different arbitrary decompositions as $g^a_1 = \bar{g}^a_1 + h^a_1$ and $g^a_2 = \bar{g}^a_2 + h^a_2$ give the difference

$$\mathfrak{h}^{\mu\nu}_{a2} - \mathfrak{h}^{\mu\nu}_{a1} = {}_{12}\mathfrak{h}^{\mu\nu}\,, \tag{2.2.131}$$

which is not less than second order in perturbations. Because, by the comparison, the difference (2.2.131) enters the linear expressions of equation (2.2.130) the same difference exists between the energy-momentum tensors. Thus, in the case of the

Ricci-flat background,

$$
\begin{aligned}
& t_{\mu\nu}^{\rm tot}(\mathfrak{h}_{a1}, \phi) - t_{\mu\nu}^{\rm tot}(\mathfrak{h}_{a2}, \phi) \qquad (2.2.132)\\
&= \tfrac{1}{2}\left[\bar\nabla_\rho{}^\rho({}_{12}h_{\mu\nu}) + \bar g_{\mu\nu}\bar\nabla_{\rho\sigma}({}_{12}h^{\rho\sigma}) - \bar\nabla_{\rho\nu}({}_{12}h_\mu^\rho) - \bar\nabla_{\rho\mu}({}_{12}h_\nu^\rho)\right],
\end{aligned}
$$

where ${}_{12}h_{\mu\nu} = {}_{12}\mathfrak{h}_{\mu\nu}/\sqrt{-\bar g}$.

For the case of flat backgrounds, this ambiguity has been considered by Boulware and Deser [71]. Later it has been examined in [379] in the case of arbitrary curved backgrounds and in arbitrary metric theories. In Sections 6.3.2, 6.3.3 and 6.4.1, this ambiguity is resolved theoretically comparing with the Belinfante corrected quantities; as a result, the advantage of the third decomposition from (2.2.120) is founded. In the works [252, 369, 370] the same result is supported by comparing the BMS mass flux [61] for a radiating isolated system with the mass flux obtained with applying the Belinfante corrected and field-theoretical formulae.

2.2.7 The background as an auxiliary structure

The main property of the developed field-theoretical formulation of general relativity is its equivalence to general relativity in the standard geometric formulation. Let us demonstrate it explicitly. Firstly, let us turn to the gravitational equations. Combining the equations (2.2.22–2.2.26) by taking into account the background equations (2.2.10), one obtains for the field equation operators

$$
\begin{aligned}
& G^L_{\alpha\beta} + \Phi^L_{\alpha\beta} - 8\pi t^{\rm tot}_{\mu\nu}\\
&= \left(\delta^\mu_\alpha\delta^\nu_\beta - \tfrac{1}{2}\bar g_{\alpha\beta}\bar g^{\mu\nu}\right)\left[R_{\mu\nu} - 8\pi\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^\alpha_\alpha\right)\right]. \qquad (2.2.133)
\end{aligned}
$$

Secondly, let us turn to the matter equations. Keeping in mind (2.2.1) and (2.2.15), (2.2.4) and (2.2.37) by taking into account the background equation (2.2.11), it is easy to find

$$
\frac{\delta\mathscr{L}^{\rm dyn}}{\delta\phi^A} = \frac{\delta\mathscr{L}^{HE}}{\delta\Phi^A}. \qquad (2.2.134)
$$

How can one explain (2.2.133) and (2.2.134)? This means that if one substitutes

$$
\mathfrak{h}^{\mu\nu} \equiv \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}, \qquad (2.2.135)
$$

$$
\phi^A \equiv \Phi^A - \bar\Phi^A, \qquad (2.2.136)
$$

see (2.2.7) and (2.2.8), into the left hand sides of (2.2.133) and (2.2.134), then one finds that the background quantities $\bar{\mathfrak{g}}^{\mu\nu}$ and $\bar\Phi^A$ vanish. Therefore the background structures are not observed. Thus, the background spacetime and the background matter fields are *auxiliary* in character.

This means that the electromagnetic signal, for example, has no possibility to detect the background spacetime. Then, one might wonder if the gravitational waves have such a possibility? For the sake of simplicity consider the case of a flat background without matter at all, then the field equations (2.2.26) acquire the form:

$$G^L_{\alpha\beta} = 8\pi t^g_{\alpha\beta} \,. \tag{2.2.137}$$

Here, the left hand side, defined in (2.2.27), has the term with the wave operator $\Box h_{\alpha\beta} = \gamma^{\mu\nu} h_{\alpha\beta;\mu\nu}$. Namely this could define the null geodesics in the Minkowski space. However the right hand side of (2.2.137) defined in (2.2.29) has the terms, like $h^{\mu\nu} h_{\alpha\beta;\mu\nu}$. This means that the Minkowski wave operator in (2.2.137) really is “curved”, $\Box' h_{\alpha\beta} = (\gamma^{\mu\nu} + p h^{\mu\nu}) h_{\alpha\beta;\mu\nu}$ where p is any constant, therefore it is impossible to define the null geodesics in the Minkowski space and, consequently, the Minkowski space itself.

It is very useful to illustrate an auxiliary character of the background spacetime considering how the gauge transformations influence the formulation of a test particle’s motion. It is enough to consider the flat background. Then the action functional can be defined by the Lagrangian of the type (2.2.104):

$$S = S^g + S^m = -\frac{1}{16\pi} \int d^4x \mathscr{L}^g + \int d^4x \mathscr{L}^m \,, \tag{2.2.138}$$

where $\mathscr{L}^m = \mathscr{L}^M(\gamma + \mathfrak{h}, \phi)$. On the action for a free matter point in the Minkowski space see Section 1.1.7. The field $\mathfrak{h}^{\mu\nu}$, the background metric density $\gamma^{\mu\nu}$ and the world coordinates x^α are involved in

$$S^m = -m \int d\tau \,, \tag{2.2.139}$$

in the form $d\tau^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu$, where with the use of (2.2.7) one can express $g_{\mu\nu} = g_{\mu\nu}(\gamma^{\alpha\beta}, \mathfrak{h}^{\rho\sigma})$. The variation of S^m with respect to the coordinates gives the equations of motion for a test particle. It is assumed that their solutions exist and are the vector components of the particle “4-velocity”: $u^\alpha \equiv dx^\alpha/d\tau$.

Let us present S^m in a more suitable form:

$$S^m = \int d^4x \sqrt{-g} \rho g_{\mu\nu} u^\mu u^\nu; \qquad \rho \equiv \frac{m\delta(\vec{r} - \vec{r}_0)}{\sqrt{-\overset{3}{g} g_{00}}} \frac{d\tau}{dt} , \tag{2.2.140}$$

where $\delta(\vec{r} - \vec{r}_0)$ is the Dirac delta-function, g_{ab} is a spatial part of the tensor $g_{\alpha\beta}$ and $\overset{3}{g} \equiv \det g_{ab}$. Thus, matter fields in (2.2.140) are $\phi^A = \{\rho, u^\alpha\}$.

Of course, the theory with the action (2.2.138) with (2.2.140) has to be gauge invariant with respect the gauge transformations (2.2.68) and (2.2.69). In the case of the flat background the transformations (2.2.68) and (2.2.69) for all the variables in (2.2.138) with (2.2.140) are

$$\mathfrak{h}'^{\mu\nu}(x) = \mathfrak{h}^{\mu\nu}(x) + (\exp £_\xi - 1)\left(\gamma^{\mu\nu}(x) + \mathfrak{h}^{\mu\nu}(x)\right), \tag{2.2.141}$$

$$\rho'(x) = \rho(x) + (\exp £_\xi - 1)\rho(x), \tag{2.2.142}$$

$$u'^{\alpha}(x) = u^{\alpha}(x) + (\exp £_\xi - 1)u^{\alpha}(x). \tag{2.2.143}$$

Of course, both the set $\mathfrak{h}^{\mu\nu}(x)$, $\rho(x)$, $u^{\alpha}(x)$ and the set $\mathfrak{h}'^{\mu\nu}(x)$, $\rho'(x)$, $u'^{\alpha}(x)$ satisfy the equations of the field-theoretical formulation of general relativity. However, in general, $u^{\alpha}(x)$ and $u'^{\alpha}(x)$ defines *different trajectories* in the same background spacetime. This conclusion again stresses the fact that a background spacetime has an auxiliary character.

2.3 Metric perturbations as compensating fields

2.3.1 "Localization" of background Killing vectors

Gravity theory as a gauge theory

In Section 2.2.4, it was shown that the gauge properties of the field-theoretical formulation of general relativity are close to gauge properties of standard gauge theories with intrinsic symmetries. Then, one expects that the field-theoretical formulation of general relativity can be constructed as a classical gauge theory of the Yang-Mills type. This section is devoted to such a way of construction.

The gauge approach in constructing general relativity starts from the pioneering works by Utiyama and Kibble [256, 445]. Discussion on the following developments in constructing gravity theories in the framework the gauge approach can be found in the reviews [227, 243, 377, 444] and the references therein. To reduce a gauge theory of gravity to general relativity one has to assume some additional assumptions [243]. For example, in Utiyama's work [445] torsion is suppressed "by hand". Therefore, general relativity is frequently classified as a pseudo-gauge gravitational theory [378].

Here, following [359], we suggest a way of constructing general relativity as a gauge theory, which is close to the standard gauge methods, although it differs from the standard localization. We assume an existence of a fixed spacetime, which has a group of motions with the corresponding Killing vectors, and where the bare physical fields propagate. Then one notices that the action of the bare fields is invariant (up to surface terms) with respect to adding the Lie derivative of these fields along the Killing vectors to the bare fields, whereas both coordinates and background fields do not change. A new method of "localization" consists of the exchange of the Killing vectors with *arbitrary* vectors. After that, the initial invariance of the action integral is destructed. Then, one requires to restore the initial invariance. As is seen, such a technique of "localization" is close to the standard way of localization, when *constants* of a group of transformations are exchanged with arbitrary functions. However, conceptually it is not the same because finally one can repeat the new construction even without the Killing vectors. Therefore we use quotation marks in the word "localization".

As a result of the "localization", a compensating (gauge) field appears. Then one requires it to have a universal character, which means that the compensating field has the features of the gravitational field.

The simplest case

Let us consider the action integral for a covariant theory of matter fields ϕ^A propagating in a fixed spacetime with the fixed metric, $\bar{g}_{\mu\nu}$, of the simplest form:

$$S = \int d^4x \mathscr{L}_c^\phi(\phi^A; \bar{\nabla}_\alpha \phi^A \mid \bar{g}^{\mu\nu}) . \tag{2.3.1}$$

For our goal it is more convenient to use partial derivatives

$$S = \int d^4x \mathscr{L}^\phi(\phi^A; \phi^A{}_{,\alpha} \mid \bar{g}^{\mu\nu}; \bar{g}^{\mu\nu}{}_{,\alpha}) . \tag{2.3.2}$$

For the sake of simplicity, we assume that the background spacetime is Ricci-flat (2.2.13). The generalization to more complicated backgrounds is possible and discussed below. Besides, it is more economical to choose a background metric with upper indices, $\bar{g}^{\mu\nu}$, although it can be chosen arbitrarily from the set of barred variables (2.2.114); this problem is discussed below as well. At last, we assume that the background spacetime has Killing vectors (or, although one Killing vector), which we denote as $\bar{\xi}^\alpha$.

After all the above, we define the transformation for the dynamical variables in the form:

$$\phi'^A = \phi^A + \delta_{\bar{\xi}} \phi^A = \phi^A + £_{\bar{\xi}} \phi^A , \tag{2.3.3}$$

where the definition of Lie derivative is given above in (1.2.79) and (1.2.82), see also (A.2.23) and (A.3.8) in Appendix A. After substituting (2.3.3) into the Lagrangian of the action (2.3.2), taking into account the Killing equations, $£_{\bar{\xi}} \bar{g}^{\mu\nu} = 0$, and using the property (A.2.25), one has

$$\mathscr{L}'^\phi = \mathscr{L}^\phi + \delta_{\bar{\xi}} \mathscr{L}^\phi = \mathscr{L}^\phi + £_{\bar{\xi}} \mathscr{L}^\phi = \mathscr{L}^\phi - \partial_\alpha(\bar{\xi}^\alpha \mathscr{L}^\phi). \tag{2.3.4}$$

Thus the action (2.3.2) is invariant with respect to the transformation (2.3.3) up to a surface term.

Now, we "localize" the transformation (2.3.3), changing the Killing vector $\bar{\xi}^\alpha$ with an arbitrary vector ξ^α. Then the property (2.3.4) is destructed. To save it, we include a new (compensating) field l^J and define a new Lagrangian $\mathscr{L}^{\phi l}$ as

$$\mathscr{L}^{\phi l} = \mathscr{L}^{\phi l}(\phi^A; \phi^A{}_{,\alpha} \mid l^J; l^J{}_{,\alpha} \mid \bar{g}^{\mu\nu}; \bar{g}^{\mu\nu}{}_{,\alpha}) . \tag{2.3.5}$$

Both the set $\{J\}$ of indices and the dependence of $\mathscr{L}^{\phi l}$ on arguments are not determined at the moment. Of course, for the Lagrangian $\mathscr{L}^{\phi l}$ we require also

$$\mathscr{L}^{\phi l}\Big|_{l=0} = \mathscr{L}^{\phi}\,. \tag{2.3.6}$$

To represent the transformation of the Lagrangian (2.3.5) in the form of (2.3.4) one has to determine the field l^J and to find transformations for it analogous to (2.3.3). We will find such a transformation in the form:

$$l'^J = l^J + \delta_\xi l^J = l^J + \xi^\alpha(A^J{}_\alpha + l^J{}_{,\alpha}) + \xi^\alpha{}_{,\beta}(B^J{}_\alpha{}^\beta + l^J|^\beta_\alpha)\,, \tag{2.3.7}$$

where, in the general case, the coefficients $A^J{}_\alpha$ and $B^J{}_\alpha{}^\beta$ depend on ϕ^A, l^J and $\bar{g}^{\mu\nu}$ and their first derivatives. Thus, the functions l^J, $\mathscr{L}^{\phi l}$, $A^J{}_\alpha$ and $B^J{}_\alpha{}^\beta$ are unknown, and our goal is to determine them.

From the start, substituting (2.3.3), with $\bar{\xi}^\alpha \to \xi^\alpha$, and (2.3.7) into (2.3.5), and requiring the analogy with (2.3.4), one has

$$\begin{aligned}
\mathscr{L}'^{\phi l} &= \mathscr{L}^{\phi l} + \delta_\xi \mathscr{L}^{\phi l} \\
&= \mathscr{L}^{\phi l} + \frac{\partial \mathscr{L}^{\phi l}}{\partial l^J}\delta_\xi l^J + \frac{\partial \mathscr{L}^{\phi l}}{\partial l^J{}_{,\alpha}}(\delta_\xi l^J)_{,\alpha} + \frac{\partial \mathscr{L}^{\phi l}}{\partial \phi^A}\delta_\xi \phi^A + \frac{\partial \mathscr{L}^{\phi l}}{\partial \phi^A{}_{,\alpha}}(\delta_\xi \phi^A)_{,\alpha} \\
&= \mathscr{L}^{\phi l} + \pounds_\xi \mathscr{L}^{\phi l} = \mathscr{L}^{\phi l} - \partial_\alpha(\xi^\alpha \mathscr{L}^{\phi l})\,.
\end{aligned} \tag{2.3.8}$$

Thus we require

$$\delta_\xi \mathscr{L}^{\phi l} = \pounds_\xi \mathscr{L}^{\phi l}\,. \tag{2.3.9}$$

We substitute here $\delta_\xi l^J$ from (2.3.7) and open $\pounds_\xi \mathscr{L}^{\phi l}$ by using (A.2.25),

$$\begin{aligned}
\pounds_\xi \mathscr{L}^{\phi l} = {} & \frac{\partial \mathscr{L}^{\phi l}}{\partial l^J}\pounds_\xi l^J + \frac{\partial \mathscr{L}^{\phi l}}{\partial l^J{}_{,\alpha}}(\pounds_\xi l^J)_{,\alpha} + \frac{\partial \mathscr{L}^{\phi l}}{\partial \phi^A}\pounds_\xi \phi^A + \frac{\partial \mathscr{L}^{\phi l}}{\partial \phi^A{}_{,\alpha}}(\pounds_\xi \phi^A)_{,\alpha} \\
& + \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}}\pounds_\xi \bar{g}^{\mu\nu} + \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}{}_{,\alpha}}(\pounds_\xi \bar{g}^{\mu\nu})_{,\alpha}\,.
\end{aligned} \tag{2.3.10}$$

Because all the components ξ^α and their derivatives are arbitrary at each world point, coefficients at ξ^α, $\xi^\alpha{}_{,\beta}$ and $\xi^\alpha{}_{,\beta\gamma}$ have to be equal to zero. As a result, the equation (2.3.9) gives the system:

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^J}A^J{}_\alpha + \frac{\partial \mathscr{L}^{\phi l}}{\partial l^J{}_{,\beta}}A^J{}_{\alpha,\beta} = -\frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}}\bar{g}^{\mu\nu}{}_{,\alpha} - \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}{}_{,\beta}}\bar{g}^{\mu\nu}{}_{,\alpha\beta}\,, \tag{2.3.11}$$

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^J}B^J{}_\alpha{}^\beta + \frac{\partial \mathscr{L}^{\phi l}}{\partial l^J{}_{,\gamma}}(A^J{}_\alpha \delta^\beta_\gamma + B^J{}_\alpha{}^\beta{}_{,\gamma})$$

$$= \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}} \bar{g}^{\mu\nu}|^{\beta}_{\alpha} + \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}{}_{,\gamma}} \left[-\bar{g}^{\mu\nu}{}_{,\alpha} \delta^{\beta}_{\gamma} + \left(\bar{g}^{\mu\nu}|^{\beta}_{\alpha} \right)_{,\gamma} \right], \tag{2.3.12}$$

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^{J}{}_{,\gamma}} B^{J}{}_{\alpha}{}^{\beta} = \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}{}_{,\gamma}} \bar{g}^{\mu\nu}|^{\beta}_{\alpha}. \tag{2.3.13}$$

Consider the two cases, *first*, when $\mathscr{L}^{\phi l}$ depends on the derivatives $\bar{g}^{\mu\nu}{}_{,\alpha}$:

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}{}_{,\gamma}} \neq 0, \tag{2.3.14}$$

and, *second*, when $\mathscr{L}^{\phi l}$ does not depend on $\bar{g}^{\mu\nu}{}_{,\alpha}$:

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}{}_{,\gamma}} = 0. \tag{2.3.15}$$

Because we would like to represent a *universal* compensating field then for each of the two above cases we have to obtain the *same* field l^J with the transformation (2.3.7).

Let us turn to the equation (2.3.13), which, in fact, represents a system of four ($\gamma = 0, 1, 2, 3$) independent tensor equations. For the *first case,* the right hand side of (2.3.13) is not equal to zero. Keeping in mind the universality of the interaction, the matrix $B^{J}{}_{\alpha}{}^{\beta}$ has to have an inverse matrix ${}^{-1}B_{J}{}^{\alpha}{}_{\beta}$ satisfying the relations

$$B^{J}{}_{\alpha}{}^{\beta} \cdot {}^{-1}B_{K}{}^{\alpha}{}_{\beta} = \delta^{J}_{K}, \qquad B^{J}{}_{\alpha}{}^{\beta} \cdot {}^{-1}B_{J}{}^{\mu}{}_{\nu} = \delta^{\mu}_{\alpha} \delta^{\beta}_{\nu}. \tag{2.3.16}$$

For the *second case* (2.3.15), the equation (2.3.13) is simplified to the equation

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^{J}{}_{,\gamma}} B^{J}{}_{\alpha}{}^{\beta} = 0. \tag{2.3.17}$$

Because the relation (2.3.16) is universal for the *second case*, one gets

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^{J}{}_{,\alpha}} = 0. \tag{2.3.18}$$

After that the equations (2.3.11) and (2.3.12) are simplified to the equations:

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^{J}} A^{J}{}_{\alpha} = -\frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}} \bar{g}^{\mu\nu}{}_{,\alpha}, \tag{2.3.19}$$

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^{J}} B^{J}{}_{\alpha}{}^{\beta} = \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}} \bar{g}^{\mu\nu}|^{\beta}_{\alpha}. \tag{2.3.20}$$

The latter, with the use of the equation (2.3.16), is transformed to

$$\frac{\partial \mathscr{L}^{\phi l}}{\partial l^J} = \frac{\partial \mathscr{L}^{\phi l}}{\partial \bar{g}^{\mu\nu}}\, \bar{g}^{\mu\nu}|^{\beta}_{\alpha} \cdot {}^{-1}B_J{}^{\alpha}{}_{\beta}\,. \tag{2.3.21}$$

Here, notice the proportionality of derivatives with respect to l^J to derivatives with respect to $\bar{g}^{\mu\nu}$. Because one requires the universality of the interaction, one concludes that a concrete form for the search Lagrangian (2.3.5) has to be derived as

$$\mathscr{L}^{\phi l} = \mathscr{L}^{\phi l}(\phi^A; \phi^A{}_{,\alpha} \,|\, \bar{g}^{\mu\nu} + m^{\mu\nu}) \tag{2.3.22}$$

for the *second case* (2.3.15), where an addition $m^{\mu\nu}$ to the background metric satisfies the equation:

$$\frac{\partial m^{\mu\nu}}{\partial l^J} = \bar{g}^{\mu\nu}|^{\beta}_{\alpha} \cdot {}^{-1}B_J{}^{\alpha}{}_{\beta}\,. \tag{2.3.23}$$

Besides, the field $m^{\mu\nu}$ has the same transformation properties as the background metric $\bar{g}^{\mu\nu}$, besides it does not depend on $\bar{g}^{\mu\nu}$, ϕ^A and their derivatives; due to (2.3.18), $m^{\mu\nu}$ does not depend on derivatives of $l^J{}_{,\alpha}$ as well. To satisfy the requirement (2.3.6) one has to restrict $m^{\mu\nu}$ by the relation, $m^{\mu\nu}|_{l=0} = 0$. Thus, one has to find a smooth function

$$m^{\mu\nu} = m^{\mu\nu}(l^J)\,. \tag{2.3.24}$$

Because the operator on the right hand side of (2.3.23) has an inverse one there is a one-to-one correspondence between l^J and $m^{\mu\nu}$, and the relation (2.3.24) can be converted into $l^J = l^{(\pi\rho)}(m^{\mu\nu})$. Thus, without losing the generality, one can set $l^J \equiv l^{\mu\nu} \equiv m^{\mu\nu}$. Then, one easily finds from (2.3.19) and (2.3.20) the expressions

$$A^J{}_{\alpha} \to A^{(\mu\nu)}{}_{\alpha} = -\bar{g}^{\mu\nu}{}_{,\alpha}\,, \qquad B^J{}_{\alpha}{}^{\beta} \to B^{\mu\nu}{}_{\alpha}{}^{\beta} = \bar{g}^{\mu\nu}|^{\beta}_{\alpha} \tag{2.3.25}$$

symmetric in μ and ν. Substituting these expressions into the formula (2.3.7), one obtains the transformations for $l^{\pi\rho}$ in the form:

$$l'^{\pi\rho} = l^{\pi\rho} + \delta_\xi l^{\pi\rho} = l^{\pi\rho} + £_\xi l^{\pi\rho} + £_\xi \bar{g}^{\pi\rho}\,. \tag{2.3.26}$$

Now, the above results have to be generalized to the more complicated *first case* (2.3.14). Once again, relying on the requirement of universality of couplings of the field $l^{\pi\rho}$, one has to conclude that the transformation (2.3.26) could be safe or restricted, but it cannot be expanded. Thus, using (2.3.26), or the coefficients (2.3.25), in the system (2.3.11–2.3.13) one finds easily step by step,

$$\mathscr{L}^{\phi l} = \mathscr{L}^{\phi}(\phi^A; \phi^A_{,\alpha} \,|\, \bar{g}^{\mu\nu} + l^{\mu\nu}; (\bar{g}^{\mu\nu} + l^{\mu\nu})_{,\alpha}) = \mathscr{L}^{\phi}_c(\phi^A; \nabla_\alpha \phi^A \,|\, g^{\mu\nu})\,, \tag{2.3.27}$$

where we have defined

$$g^{\mu\nu} \equiv \bar{g}^{\mu\nu} + l^{\mu\nu}\,. \tag{2.3.28}$$

It is the generalization of the identification (2.1.9), also this can be interpreted as a decomposition in the second line in (2.2.120); ∇_α is a covariant derivative constructed with the use of the effective metric $g^{\mu\nu}$. Thus, for the *first case* (2.3.14), the same compensating field $l^{\pi\rho}$ and the transformations for them (2.3.26) are safe as well.

Higher derivative Lagrangians

Let us generalize the above results for the more general case, when instead of the action (2.3.2) we start from the action

$$\begin{aligned} S &= \int dx^4 \mathscr{L}^\phi \left(\phi^A; \phi^A{}_{,\alpha}; \ldots; \phi^A{}_{,\alpha,\beta\ldots\gamma} \,|\, \bar{g}^{\mu\nu}; \bar{g}^{\mu\nu}{}_{,\alpha}; \ldots; \bar{g}^{\mu\nu}{}_{,\alpha\beta\ldots\gamma}\right) \\ &= \int dx^4 \mathscr{L}^\phi_c \left(\phi^A; \bar{\nabla}_\alpha \phi^A; \ldots; \bar{\nabla}_{\gamma\ldots\beta\alpha}\phi^A \,|\, \bar{g}^{\mu\nu}\right) \end{aligned} \tag{2.3.29}$$

where $\bar{\nabla}_{\gamma\ldots\beta\alpha} \equiv \bar{\nabla}_\gamma \ldots \bar{\nabla}_\beta \bar{\nabla}_\alpha$. Thus, instead of (2.3.5) we search for the Lagrangian in the form

$$\mathscr{L}^{\phi l} = \mathscr{L}^{\phi l}\left(\phi^A; \ldots; \phi^A{}_{,\alpha,\beta\ldots\gamma} \,|\, l^J; \ldots; l^J{}_{,\tau\sigma\ldots\psi} \,|\, \bar{g}^{\mu\nu}; \ldots; \bar{g}^{\mu\nu}{}_{,\pi\rho\ldots\phi}\right)\,. \tag{2.3.30}$$

We further transform (2.3.30) by making use of the assumptions and technique used from (2.3.16) to (2.3.27), employing transformation (2.3.26) and applying the principles of mathematical induction. One obtains,

$$\begin{aligned} \mathscr{L}^{\phi l} &= \mathscr{L}^{\phi l}(\phi^A; \phi^A{}_{,\alpha}; \ldots \phi^A{}_{,\alpha,\beta\ldots\gamma} \,|\, \bar{g}^{\mu\nu} + l^{\mu\nu}; \ldots; (\bar{g}^{\mu\nu} + l^{\mu\nu})_{,\alpha\beta\ldots\gamma}) \\ &= \mathscr{L}^{\phi l}_c \left(\phi^A; \nabla_\alpha \phi^A; \ldots; \nabla_{\gamma\ldots\beta\alpha}\phi^A \,|\, g^{\mu\nu}\right)\,, \end{aligned} \tag{2.3.31}$$

where $\nabla_{\gamma\ldots\beta\alpha} \equiv \nabla_\gamma \ldots \nabla_\beta \nabla_\alpha$.

Thus the gauge field $l^{\mu\nu}$ is coupled to matter fields in a *universal* way, therefore, by the equivalence principle, $l^{\mu\nu}$ can be called the gravitational field.

2.3.2 The total action

Now, let us construct the *total* action for the fields, $l^{\mu\nu}$ and ϕ^A, propagating on the fixed background with the metric $\bar{g}^{\mu\nu}$. Then Lagrangian

$$\mathscr{L}^l = \mathscr{L}^l(l^{\mu\nu}; l^{\mu\nu}{}_{,\alpha}; \ldots \,|\, \bar{g}^{\mu\nu}; \bar{g}^{\mu\nu}{}_{,\alpha}; \ldots) \tag{2.3.32}$$

for free gravitational field $l^{\mu\nu}$ has to be constructed. Following the recipe (2.3.8), after substituting the transformations (2.3.26) into (2.3.32), we require

$$\mathscr{L}'^{l} = \mathscr{L}^{l} + \delta\mathscr{L}^{l} = \mathscr{L}^{l} + £_{\xi}\mathscr{L}^{l} = \mathscr{L}^{l} - \partial_{\alpha}(\xi^{\alpha}\mathscr{L}^{l}) . \tag{2.3.33}$$

Then, step by step, one easily finds that

$$\mathscr{L}^{l} = \mathscr{L}^{l}\,(l^{\mu\nu} + \bar{g}^{\mu\nu}; (l^{\mu\nu} + \bar{g}^{\mu\nu})_{,\alpha}; \dots) , \tag{2.3.34}$$

analogously to the construction of (2.3.27).

The *simplest* way to represent a *covariant* Lagrangian with the use of the "effective" metric $g^{\mu\nu}$ defined in (2.3.28) is to construct the related curvature scalar, see [285],

$$\mathscr{L}^{l} = \sqrt{-g}R(\bar{g}^{\mu\nu} + l^{\mu\nu}) = \mathscr{R}(g^{\mu\nu}) . \tag{2.3.35}$$

Here, we consider such a choice only, although there are unrestricted possibilities to construct a covariant Lagrangian, see Chapter 7, for the free gravitational field with the use of the effective metric, but all of them is more complicated than (2.3.35).

Thus, the total Lagrangian acquires the form of the Einstein-Hilbert one with the effective metric and sources fields:

$$\mathscr{L}_{EH} = -\frac{1}{16\pi}\mathscr{L}^{l}(\bar{g} + l) + \mathscr{L}^{\phi l}\,(\bar{g} + l,\, \phi) . \tag{2.3.36}$$

The variation with respect to ϕ^{A} and $l^{\mu\nu}$ gives the equations, which coincide with the general relativity equations. Returning to the Section 2.2.6, one constructs a dynamical action for dynamical variables ϕ^{A} and $l^{\mu\nu}$ on a Ricci flat background:

$$S = \int d^4x\mathscr{L}^{\rm dyn} = -\frac{1}{16\pi}\int d^4x\mathscr{L}^{g} + \int d^4x\mathscr{L}^{\phi l} \tag{2.3.37}$$

in the field-theoretical formulation of general relativity. Notice that, here, $\mathscr{L}^{g}$ is constructed by the universal rule in (2.2.16) or (2.2.20) for the second decomposition in (2.2.120).

2.3.3 Discussion of the results

Different definitions of gravitational variables

The choice of the background metric in the form of the contravariant components $\bar{g}^{\mu\nu}$ in the bare Lagrangian (2.3.2) is not unique. Any other choice from the barred set $\bar{g}_{\mu\nu}$, $\bar{\mathfrak{g}}^{\mu\nu}$, …, leads to a sum (2.2.121) represented by the set (2.2.120). Each of the choice leads to the corresponding variant of the field-theoretical formulations, which have been presented and analyzed in Section 2.2.6.

Exact (non-infinitesimal) gauge transformations

Let us replace the linear in ξ^α gauge transformations with the full transformations (2.2.68) and (2.2.69) that can be represented in the compact form:

$$\phi'^A = \phi^A + (\exp £_\xi - 1)\phi^A, \qquad l'^{\mu\nu} = l^{\mu\nu} + (\exp £_\xi - 1)(\bar{g}^{\mu\nu} + l^{\mu\nu}). \tag{2.3.38}$$

Then each of the Lagrangians in (2.3.36) is invariant under transformations (2.3.38) up to a divergence:

$$\begin{aligned} \mathscr{L}'^{\phi l} &= \exp £_\xi \mathscr{L}^{\phi l} = \mathscr{L}^{\phi l} + (\exp £_\xi - 1)\mathscr{L}^{\phi l} = \mathscr{L}^{\phi l} + \mathrm{div}\,, \\ \mathscr{L}'^{l} &= \exp £_\xi \mathscr{L}^{l} = \mathscr{L}^{l} + (\exp £_\xi - 1)\mathscr{L}^{l} = \mathscr{L}^{l} + \mathrm{div}\,. \end{aligned} \tag{2.3.39}$$

The Lagrangian $\mathscr{L}^{\mathrm{dyn}}$ in the action (2.3.37) is invariant with respect to (2.3.26) up to a divergence on the background equations, the same as in Section 2.2.4.

Arbitrary curved backgrounds

Up to now, we have worked on the Ricci-flat background (2.2.13). This means that the background system defined by (2.2.10) and (2.2.11) is vacuum without matter fields $\bar{\Phi}^A$. How can one generalize the above construction to a non-Ricci-flat background satisfying (2.2.12) with the presence of $\bar{\Phi}^A$? Instead of the action (2.3.2) one has to use

$$S = \int dx^4 \mathscr{L}^\phi = \int dx^4 \mathscr{L}^\phi(\bar{\Phi}^A + \phi^A; (\bar{\Phi}^A + \phi^A)_{,\alpha} \,|\, \bar{g}^{\mu\nu}; \bar{g}^{\mu\nu}{}_{,\alpha}) \tag{2.3.40}$$

as a bare action. Instead of the initial transformation (2.3.3) one has to use the transformations:

$$\phi'^A = \phi^A + £_{\bar{\xi}}(\bar{\Phi}^A + \phi^A) \tag{2.3.41}$$

that is "localized" to

$$\phi'^A = \phi^A + £_\xi(\bar{\Phi}^A + \phi^A)\,. \tag{2.3.42}$$

The next steps are analogous to the above ones.

No group of motion

Notice that the background spacetime can have no Killing vectors. In this case, one considers the search Lagrangian in the form:

$$\mathscr{L}^{\phi l} = \mathscr{L}^{\phi l}(\bar{\Phi}^A + \phi^A; (\bar{\Phi}^A + \phi^A)_{,\alpha} \,|\, l^J; l^J{}_{,\alpha} \,|\, \bar{g}^{\mu\nu}; \bar{g}^{\mu\nu}{}_{,\alpha})\,. \tag{2.3.43}$$

To repeat the presentation in this section, it is enough to require the transformation of the type (2.3.8) for the Lagrangian (2.3.43) with respect to the transformations (2.3.7) and (2.3.42).

What is the gauge field?

The field $l^{\mu\nu}$ is not a connection and does not transform derivatives to gauge invariant derivatives, like in the standard Yang-Mills type theories [266]. In this sense the field $l^{\mu\nu}$ can be called as the compensating field only. One can see, for example, that in (2.3.31) the tensor field,

$$\Delta^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu}(\bar{g} + l) - \bar{\Gamma}^{\alpha}{}_{\mu\nu}(\bar{g}), \tag{2.3.44}$$

transforms the background derivatives $\bar{\nabla}_\alpha$ in (2.3.29) to the dynamical derivatives ∇_α; $\Gamma^{\alpha}{}_{\mu\nu}$ are the Christoffel symbols constructed with the use of the effective metric (2.3.28), whereas $\bar{\Gamma}^{\alpha}{}_{\mu\nu}$ are the Christoffel symbols constructed with the use of the background metric. Thus, the role of the gauge field is played by the quantity $\Delta^{\alpha}{}_{\mu\nu}$. The question is: how $\Delta^{\alpha}{}_{\mu\nu}$ can be obtained by the standard gauge methods?

Let us rewrite the transformations (2.3.26) in the form

$$l'^{\mu\nu} = l^{\mu\nu} + (A^{\mu\nu}{}_{\alpha} - l^{\mu\nu}{}_{,\alpha})\,\xi^{\alpha} + \left(B^{\mu\nu}{}_{\alpha}{}^{\beta} + l^{\mu\nu}|^{\beta}_{\alpha}\right)\xi^{\alpha}{}_{,\beta}\,, \tag{2.3.45}$$

where $A^{\mu\nu}{}_{\alpha}$ and $B^{\mu\nu}{}_{\alpha}{}^{\beta}$ are defined in (2.3.25). Now, rewrite the coefficients (1.3.33) and (1.3.34) relatively to (2.3.45):

$$\omega^{A}_{a} \to -(\bar{g}^{\mu\nu} + l^{\mu\nu})_{,\alpha} = -g^{\mu\nu}{}_{,\alpha}\,, \tag{2.3.46}$$

$$\omega^{A\alpha}_{b} \to (\bar{g}^{\mu\nu} + l^{\mu\nu})|^{\beta}_{\alpha} = g^{\mu\nu}|^{\beta}_{\alpha}\,. \tag{2.3.47}$$

One easily finds that the coefficients (2.3.46) and (2.3.47) coincide with the coefficients (1.3.33) and (1.3.34). However, let us rewrite the transformations (2.3.45) in the covariantized form:

$$l'^{\mu\nu} = l^{\mu\nu} - \bar{\nabla}_\alpha g^{\mu\nu}\xi^{\alpha} + g^{\mu\nu}|^{\beta}_{\alpha}\,\bar{\nabla}_\beta\xi^{\alpha}\,. \tag{2.3.48}$$

This permits to derive covariantized coefficients analogous to (2.3.46) and (2.3.47),

$$\omega^{A}_{a} \to -\bar{\nabla}_\alpha(\bar{g}^{\mu\nu} + l^{\mu\nu}) = -\bar{\nabla}_\alpha g^{\mu\nu}\,, \tag{2.3.49}$$

$$\omega^{A\alpha}_{b} \to (\bar{g}^{\mu\nu} + l^{\mu\nu})|^{\beta}_{\alpha} = g^{\mu\nu}|^{\beta}_{\alpha}\,. \tag{2.3.50}$$

To construct the connection related to (2.3.48) one has to follow the recipe of (1.2.67) and (1.3.38). However, now, instead of (1.3.36) one has

$$\bar{\nabla}_\alpha g^{\mu\nu} = \Delta^{\mu}{}_{\alpha\rho}g^{\nu\rho} + \Delta^{\nu}{}_{\alpha\rho}g^{\mu\rho}\,, \tag{2.3.51}$$

where $\Delta^{\mu}{}_{\alpha\rho}$ is defined in (2.3.44). Then, instead of (1.3.38) one has

$$G^{b}{}_{a\mu} \to {}^{-1}\!\left(g^{\mu\nu}|^{\beta}_{\alpha}\right)\cdot\left(-\bar{\nabla}_\gamma g^{\mu\nu}\right) = \Delta^{\alpha}{}_{\beta\gamma}\,. \tag{2.3.52}$$

Thus, indeed, $\Delta^{\alpha}{}_{\beta\gamma}$ could be called the gauge field.

2.4 The Babak-Grishchuk gravity with a non-zero graviton mass

One of desirable properties of a physical theory is that the energy-momentum tensor could be free of the second (highest) derivatives of the field variables. The energy-momentum tensor of the gravitational field (2.2.29) and (2.2.30) in the field-theoretical formulation of general relativity does not satisfy this requirement. Babak and Grishchuk have analyzed this situation [21], and have improved it, reformulating the original field-theoretical approach suggested in [206].

There are many fundamental results, which show that general relativity could be (has to be) modified by adding “mass terms” into the Lagrangian. Babak-Grishchuk modification [21] turns out to be useful in constructing such a gravitational theory, and they suggested it in [22]. All the local weak-field predictions of their massive theory are in an agreement with the experimental data. Otherwise, the exact non-linear equations of the new theory eliminate the black hole event horizons and replace a permanent power-law expansion of the homogeneous isotropic universe with an oscillator behaviour. One variant of the massive theory allows “an accelerated expansion” of the universe. The Babak-Grishchuk theories are described, analyzed and discussed in the present section.

Concerning the gravitational theories with massive gravitons in general, they constitute a separate branch of the modern research in gravitational physics, see, for example, chronologically [96, 136, 137, 238, 388, 393, 452] and references therein. There are many applications of the theoretical models, see, for example, [23, 218, 476]. One of the problems in this way is, for example, the presence of ghosts. Such problems step by step are solved. Here, we do not consider such theories, and do not analyze them. Only, the Babak-Grishchuk variant of the massive gravity is given as an illustration of that how the field-theoretical approach could be fruitful in constructing non-standard gravitational theories.

2.4.1 Second derivatives in the energy-momentum tensor

We present the results by Babak and Grishchuk [21, 22], which are based on the calculations (2.2.15–2.2.32) in Section 2.2. Consider the Minkowski space as a background spacetime, when

$$\bar{R}_{\alpha\rho\beta\sigma} = 0. \tag{2.4.1}$$

Then, in fact, one can use the formulae (2.2.104–2.2.106) presented for the case of the Ricci-flat background satisfying (2.2.13). Because a generalization to curved backgrounds is possible we use here for a background metric and for a background covariant derivatives the notations $\bar{g}_{\mu\nu}$ and $\bar{\nabla}_\alpha$ (not $\gamma_{\mu\nu}$ and “$_{;\alpha}$” as this corresponds to a flat background).

Here, it is more convenient to use $\mathfrak{h}^{\mu\nu}$ defined in (2.2.7) as independent gravitational variables, or corresponding quantities $h^{\mu\nu} = \mathfrak{h}^{\mu\nu}/\sqrt{-\bar{g}}$ in formulae. Then, we rewrite the formula (2.2.17) in these terms:

$$\Delta^\lambda_{\mu\nu} \equiv \frac{1}{2}\left[g_{\mu\rho}\bar{\nabla}_\nu h^{\lambda\rho} + g_{\nu\rho}\bar{\nabla}_\mu h^{\lambda\rho} - g_{\mu\alpha}g_{\nu\beta}g^{\lambda\rho}\bar{\nabla}_\rho h^{\alpha\beta} + \tfrac{1}{2}\left(g_{\alpha\beta}\delta^\lambda_\mu\bar{\nabla}_\nu h^{\alpha\beta} + g_{\alpha\beta}\delta^\lambda_\nu\bar{\nabla}_\mu h^{\alpha\beta} - g_{\alpha\beta}g_{\mu\nu}g^{\lambda\rho}\bar{\nabla}_\rho h^{\alpha\beta}\right)\right], \tag{2.4.2}$$

where $g_{\mu\nu}$ and $g^{\mu\nu}$, being functions of $\mathfrak{g}^{\mu\nu}$, with the use of (2.2.7) are thought as functions of $\bar{g}^{\mu\nu}$ and $\mathfrak{h}^{\mu\nu}$. Substituting (2.4.2) into (2.2.30) and (2.2.31) one finds that the energy-momentum tensor of the gravitational field, $t^g_{\mu\nu}$, depends on the second derivatives of $h^{\mu\nu}$. Attempts to exclude terms with second derivatives with the use of the field-theoretical equations (2.2.105) lead to the energy-momentum, $t'^g_{\mu\nu}$, at the right hand side of (2.2.105)

$$t'^g_{\mu\nu} = t^{\rm red}_{\mu\nu} + Q^{\alpha\beta}{}_{\mu\nu}\left(t^m_{\alpha\beta} - \tfrac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\gamma\rho}t^m_{\gamma\rho}\right) + \frac{1}{16\pi}\bar{\nabla}_{\alpha\beta}(h^\alpha{}_{(\mu}h^\beta{}_{\nu)} - h_{\mu\nu}h^{\alpha\beta}), \tag{2.4.3}$$

where the reduced part of the energy-momentum tensor of the gravitational field, $t^{\rm red}_{\mu\nu}$, indeed, has the first derivatives only:

$$\begin{aligned} t^{\rm red}_{\mu\nu} = \frac{1}{32\pi}\Big[&2\bar{\nabla}_\rho h_{\mu\nu}\bar{\nabla}_\sigma h^{\rho\sigma} - 2\bar{\nabla}_\alpha h_\mu{}^\alpha\bar{\nabla}_\beta h_\nu{}^\beta \\ &+ g_{\alpha\beta}\left(2g^{\rho\sigma}\bar{\nabla}_\rho h_\mu{}^\alpha\bar{\nabla}_\sigma h_\nu{}^\beta + g_{\mu\nu}\bar{\nabla}_\sigma h^{\alpha\rho}\bar{\nabla}_\rho h^{\beta\sigma}\right) \\ &- 4g_{\beta\rho}g^{\alpha\gamma}\bar{g}_{\gamma(\mu}\bar{\nabla}_\sigma h_{\nu)}{}^\beta\bar{\nabla}_\alpha h^{\rho\sigma} \\ &+ \tfrac{1}{4}(2\delta^\tau_\mu\delta^\omega_\nu - g_{\mu\nu}g^{\omega\tau})(2g_{\rho\alpha}g_{\sigma\beta} - g_{\alpha\beta}g_{\rho\sigma})\bar{\nabla}_\tau h^{\rho\sigma}\bar{\nabla}_\omega h^{\alpha\beta}\Big]. \end{aligned} \tag{2.4.4}$$

Concerning the matter part in (2.4.3), it is defined by the matter energy-momentum in (2.2.105) and by the expression:

$$Q^{\alpha\beta}{}_{\mu\nu} \equiv h^\alpha{}_{(\mu}\bar{g}_{\nu)}{}^\beta + h^\beta{}_{(\mu}\bar{g}_{\nu)}{}^\alpha + h^\alpha{}_{(\mu}h_{\nu)}{}^\beta - \tfrac{1}{2}\bar{g}_{\mu\nu}h^{\alpha\beta} - \tfrac{1}{2}h_{\mu\nu}\left(\bar{g}^{\alpha\beta} + h^{\alpha\beta}\right). \tag{2.4.5}$$

As a result, the second derivatives of the gravitational variables $h^{\mu\nu}$ participate anyway in (2.4.3), and there are no possibilities to remove them "by hand". Nevertheless, can one reformulate the field-theoretical approach to exclude the second derivatives in (2.4.3), preserving the equivalence to general relativity? The answer is given in Section 2.4.2.

2.4.2 Modified Lagrangian and equations

Let us consider the gravitational Lagrangian

$$\mathscr{L}^{g^\dagger} = \mathscr{L}^g + \Lambda^{\alpha\beta\rho\sigma}\bar{R}_{\alpha\rho\beta\sigma} \tag{2.4.6}$$

instead of the Lagrangian (2.2.20). This is a typical way of incorporating constraints (here, they are (2.4.1)) with the use of undetermined Lagrange multipliers (here, they are the components of $\Lambda^{\alpha\beta\rho\sigma}$). It is assumed that the tensor density $\Lambda^{\alpha\beta\rho\sigma}$ depends on $\bar{g}_{\mu\nu}$ and $\mathfrak{h}^{\mu\nu}$ only, without their derivatives,

$$\Lambda^{\alpha\beta\rho\sigma} = \Lambda^{\alpha\beta\rho\sigma}(\mathfrak{h}^{\mu\nu}, \bar{g}_{\mu\nu}) , \tag{2.4.7}$$

and satisfies the symmetries of the Riemann tensor: $\Lambda^{\alpha\beta\rho\sigma} = -\Lambda^{\rho\beta\alpha\sigma} = -\Lambda^{\alpha\sigma\rho\beta} = \Lambda^{\beta\alpha\sigma\rho}$.

Then, the dynamical Lagrangian (2.2.104) transforms into

$$\mathscr{L}^{\mathrm{dyn}^\dagger} = -\frac{1}{16\pi}\mathscr{L}^{g^\dagger} + \mathscr{L}^{m} = -\frac{1}{16\pi}\mathscr{L}^{g^\dagger} + \mathscr{L}^{M}\left(\bar{\mathfrak{g}} + \mathfrak{h}, \phi\right) . \tag{2.4.8}$$

The field equations, which are obtained by the variation of (2.4.8) with respect to $\mathfrak{h}^{\mu\nu}$ are equivalent to (2.2.105). Indeed, the variation of the Lagrangian multipliers in (2.4.8), see (2.4.6), do not contribute to the equations by the condition (2.4.1). Thus, repeating the steps (2.2.22–2.2.29) the equations following from (2.4.8) acquire the form:

$$G^{L^\dagger}_{\mu\nu} - 8\pi t^{\mathrm{tot}^\dagger}_{\mu\nu} \equiv G^{L}_{\mu\nu} - 8\pi t^{\mathrm{tot}}_{\mu\nu} = 0 . \tag{2.4.9}$$

Then, where is the difference between (2.4.9) and (2.2.105)?

The energy-momentum tensor, $t^{\mathrm{tot}^\dagger}_{\mu\nu}$, in (2.4.9) is defined by the standard variation of (2.4.8):

$$t^{\mathrm{tot}^\dagger}_{\mu\nu} \equiv \frac{2}{\sqrt{-\bar{g}}}\frac{\delta\mathscr{L}^{\mathrm{dyn}^\dagger}}{\delta\bar{g}^{\mu\nu}} = t^{g^\dagger}_{\mu\nu} + t^{m}_{\mu\nu} , \tag{2.4.10}$$

where the modified gravitational energy-momentum tensor is

$$8\pi t^{g^\dagger}_{\mu\nu} = 8\pi t^{g}_{\mu\nu} - (\sqrt{-\bar{g}})^{-1}\bar{\nabla}_{\alpha\beta}\left(\Lambda_{\mu\nu}{}^{\alpha\beta} + \Lambda_{\nu\mu}{}^{\alpha\beta}\right) \tag{2.4.11}$$

instead of (2.2.30).

In general, Lagrange multipliers are determined after the solution to the total system of equations corresponding to the Lagrangian, like (2.4.6). However, the restriction (2.4.7) does not allow us to define components $\Lambda^{\mu\nu\alpha\beta}$ which are left undetermined. Their choice is free. We choose them in such a way that the second derivatives in (2.4.3) are suppressed by the modification (2.4.11). The unique possibility is

$$\Lambda^{\mu\nu\alpha\beta} = \frac{1}{4\sqrt{-\bar{g}}}\left(\mathfrak{h}^{\alpha\nu}\mathfrak{h}^{\beta\mu} - \mathfrak{h}^{\alpha\beta}\mathfrak{h}^{\mu\nu}\right) . \tag{2.4.12}$$

Combining the definition (2.2.27), the equivalence of the expressions in (2.4.9) and the relation (2.4.11), one concludes for the left hand sides in (2.4.9):

$$\begin{aligned} G^{L^\dagger}_{\mu\nu}(h) &\equiv G^L_{\mu\nu}(h) - \frac{2}{\sqrt{-\bar g}}\bar\nabla_{\alpha\beta}\Lambda_{\mu\nu}{}^{\alpha\beta} \\ &\equiv \frac{1}{2}\bar\nabla_{\alpha\beta}\left[(\bar g^{\mu\nu} + h^{\mu\nu})(\bar g^{\alpha\beta} + h^{\alpha\beta}) - (\bar g^{\mu\alpha} + h^{\mu\alpha})(\bar g^{\nu\beta} + h^{\nu\beta})\right] \\ &\equiv \frac{1}{2(-\bar g)}\bar\nabla_{\alpha\beta}\left[(\bar{\mathfrak g}^{\mu\nu} + \mathfrak h^{\mu\nu})(\bar{\mathfrak g}^{\alpha\beta} + \mathfrak h^{\alpha\beta}) - (\bar{\mathfrak g}^{\mu\alpha} + \mathfrak h^{\mu\alpha})(\bar{\mathfrak g}^{\nu\beta} + \mathfrak h^{\nu\beta})\right] . \end{aligned} \tag{2.4.13}$$

It is non-linear in $h^{\mu\nu}$, and this is a price for the requirement to have $t^{\text{red}}_{\mu\nu}$ without second derivatives. Combining (2.4.3) and (2.4.11), one finds for the right hand side of (2.4.9):

$$t^{\text{tot}^\dagger}_{\mu\nu} = t^{\text{red}}_{\mu\nu} + Q^{\alpha\beta}{}_{\mu\nu}(t^m_{\alpha\beta} - \tfrac{1}{2}\bar g_{\alpha\beta}t^{m\rho}_\rho) + t^m_{\mu\nu} , \tag{2.4.14}$$

where the pure gravitational part does not depend on second derivatives. Finally, keeping in mind (2.2.101) and (2.4.12), one finds for the flat background (2.4.1),

$$\bar\nabla^\mu G^{L^\dagger}_{\mu\nu} \equiv 0 . \tag{2.4.15}$$

Then, from (2.4.9) the conservation law,

$$\bar\nabla^\mu t^{\text{tot}^\dagger}_{\mu\nu} = 0 , \tag{2.4.16}$$

follows.

Let us show explicitly that the dragged equations (2.4.9) are equivalent to the Einstein equations in the usual form. Substituting the expressions (2.4.13) and (2.4.14) into (2.4.9), multiplying it by $(-\bar g)$, using the definitions (2.4.4), (2.4.5) and (2.2.32), taking into account the decomposition (2.2.7), and, at last, applying the Lorentzian coordinates in the background spacetime, one gets

$$\tfrac{1}{2}\partial_{\alpha\beta}\left[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\right] = 8\pi(-g)\left(t^{\mu\nu}_{LL} + T^{\mu\nu}\right) . \tag{2.4.17}$$

Here, $(-g)t^{\mu\nu}_{LL} = \boldsymbol{t}^{\mu\nu}_{LL}$ is the Landau-Lifshitz's pseudotensor presented in (1.4.70), whereas the equality (2.4.17) itself is the same as (1.4.71) with (1.4.68). Thus, indeed, (2.4.17) are the Einstein's equations rewritten in the other form only. Besides, such a representation (1.4.12) evidently shows that the tensor density $(-\bar g)t^{\mu\nu}_{\text{red}}$ defined by (2.4.4) can be interpreted as the covariantized pseudotensor $\boldsymbol{t}^{\mu\nu}_{LL}$.

2.4.3 Non-zero masses of gravitons

A new variant of massive gravity

The technique of including the Lagrange multipliers in (2.4.6) and (2.4.8) turns out very useful in constructing the massive variant of the gravitational theory. From this point of view, an additional term similar to the term (2.4.12) could be included into the Lagrangian (2.4.6). Let us describe a way of construction. Initially one considers the curvature tensor $\widetilde{R}_{\alpha\rho\beta\sigma}$ of an *abstract* spacetime of a constant non-zero curvature: $\widetilde{R}_{\alpha\rho\beta\sigma} = K\left(\widetilde{g}_{\alpha\beta}\widetilde{g}_{\rho\sigma} - \widetilde{g}_{\alpha\sigma}\widetilde{g}_{\rho\beta}\right)$, where K is with the dimensionality of $[length]^{-2}$. Next, one adds

$$\widetilde{\Lambda} \equiv \widetilde{\Lambda}^{\alpha\beta\rho\sigma}\widetilde{R}_{\alpha\rho\beta\sigma} = \frac{1}{4\sqrt{-\widetilde{g}}}\left(\mathfrak{h}^{\alpha\sigma}\mathfrak{h}^{\beta\rho} - \mathfrak{h}^{\alpha\beta}\mathfrak{h}^{\rho\sigma}\right)\widetilde{R}_{\alpha\rho\beta\sigma} \tag{2.4.18}$$

to the Lagrangian (2.4.6). Then, one changes $\widetilde{g}^{\mu\nu} \to \bar{g}^{\mu\nu}$, and a new additional term in the Lagrangian (2.4.6) acquires the form

$$\widetilde{\Lambda} = \tfrac{1}{2}\sqrt{-\bar{g}}K\left(h^{\alpha\beta}h_{\alpha\beta} - h_\alpha{}^\alpha h_\beta{}^\beta\right) \tag{2.4.19}$$

in that one easily recognizes the Fierz-Pauli mass-term [175]. Developing this construction, Babak and Gishchuk suggested to consider two independent quadratic combinations of $h^{\mu\nu}$ in (2.4.19) separately. As a result, a two-parameter family of theories with the additional mass terms in the gravitational Lagrangian (2.4.6):

$$\mathscr{L}^{g^\ddagger} = \mathscr{L}^{g^\dagger} + (\sqrt{-\bar{g}})^{-1}\left[k_1(\mathfrak{h}^{\alpha\beta}\mathfrak{h}_{\alpha\beta}) + k_2(\mathfrak{h}_\alpha{}^\alpha\mathfrak{h}_\beta{}^\beta)\right], \tag{2.4.20}$$

can be studied, where k_1 and k_2 have a dimensionality of $[length]^{-2}$, like K.

Finally, instead of the total dynamical Lagrangian (2.4.8), one has

$$\mathscr{L}^{\mathrm{dyn}^\ddagger} = -\frac{1}{16\pi}\mathscr{L}^{g^\ddagger} + \mathscr{L}^m = -\frac{1}{16\pi}\mathscr{L}^{g^\ddagger} + \mathscr{L}^M(\bar{\mathfrak{g}} + \mathfrak{h}, \phi). \tag{2.4.21}$$

To obtain the related field equations one has to vary it with respect to $\mathfrak{h}^{\mu\nu}$. Then one has to apply the technique used in (2.2.22–2.2.26). In the result, instead of (2.4.9), one easily gets:

$$G^{L^\dagger}_{\mu\nu} + 2k_1h_{\mu\nu} - (k_1 + 2k_2)\bar{g}_{\mu\nu}h_\alpha{}^\alpha = 8\pi t^{\mathrm{tot}^\dagger}_{\mu\nu}. \tag{2.4.22}$$

To give a physical interpretation of k_1 and k_2, following to the technique by Ogievetsky and Polubarinov [342], and by van Dam and Veltman [449], one considers the linear approximation of (2.4.22):

$$\tfrac{1}{2}\left(\bar{\Box}h_{\mu\nu} + \bar{g}_{\mu\nu}\bar{\nabla}_{\rho\sigma}h^{\rho\sigma} - \bar{\nabla}_{\rho\nu}h_\mu{}^\rho - \bar{\nabla}_{\rho\mu}h_\nu{}^\rho\right) + 2k_1h_{\mu\nu} - (k_1 + 2k_2)\bar{g}_{\mu\nu}h_\alpha{}^\alpha = 0. \tag{2.4.23}$$

where $\bar{\Box} \equiv \bar{g}^{\alpha\beta}\bar{\nabla}_{\alpha\beta}$. The divergence of this expression

$$\bar{\nabla}_\nu\left[2k_1h^{\mu\nu} - (k_1 + 2k_2)\bar{g}^{\mu\nu}h_\alpha{}^\alpha\right] = 0, \tag{2.4.24}$$

can be considered as a constraint that has to be satisfied, when solutions to (2.4.23) are searched.

The case $k_1 + k_2 \neq 0$

For this case, the system (2.4.23) and (2.4.24) is equivalent to

$$\bar{\Box} H^{\mu\nu} + \alpha^2 H^{\mu\nu} = 0, \tag{2.4.25}$$

$$\bar{\Box} h_\alpha{}^\alpha + \beta^2 h_\alpha{}^\alpha = 0. \tag{2.4.26}$$

Here, the quantity

$$H^{\mu\nu} \equiv h^{\mu\nu} - \frac{k_1 + k_2}{3k_1} \bar{g}^{\mu\nu} h_\alpha{}^\alpha - \frac{k_1 + k_2}{6k_1^2} \bar{\nabla}^{\mu\nu} h_\alpha{}^\alpha + \frac{k_1 + k_2}{12k_1^2} \bar{g}^{\mu\nu} \bar{\Box} h_\alpha{}^\alpha \tag{2.4.27}$$

satisfies the conditions $\bar{g}_{\mu\nu} H^{\mu\nu} = 0$ and $\bar{\nabla}_\nu H^{\mu\nu} = 0$. The parameters in the wave-like equations (2.4.25) and (2.4.26) are

$$\alpha^2 = 4k_1, \qquad \beta^2 = -\frac{2k_1(k_1 + 4k_2)}{k_1 + k_2}. \tag{2.4.28}$$

They can be thought as inverse Compton wavelengths of the *spin-2* graviton with the mass $m_2 = \alpha\hbar/c$ associated with the field $H^{\mu\nu}$ and of *spin-0* graviton with mass $m_0 = \beta\hbar/c$ associated with the scalar field $h_\alpha{}^\alpha$.

Studying the weak gravitational waves in the massive gravity, one finds certain modifications of general relativity. Thus the spin-0 gravitational waves represented by the trace $h_\alpha{}^\alpha$ and the polarization state of the spin-2 graviton represented by the spatial trace $H^{ik}\eta_{ik}$, both, unlike in general relativity, become essential. They provide additional contributions to the energy-momentum flux carried by the gravitational wave, and the extra components of motion of the test particles. However, gravitational wave solutions, their energy-momentum characteristics and observational predictions of general relativity are fully recovered in the massless limit $\alpha \to 0, \beta \to 0$.

The case $k_1 + k_2 = 0$

This variant of the theory represents the Fierz-Pauli type massive gravity, which corresponds to $\beta \to \infty$, see (2.4.28). The system (2.4.23) and (2.4.24) becomes equivalent to

$$h_\alpha{}^\alpha = 0, \qquad \bar{\nabla}_\nu h^{\mu\nu} = 0, \qquad \bar{\Box} h^{\mu\nu} + 4k_1 h^{\mu\nu} = 0. \tag{2.4.29}$$

However, even in the limit $\alpha \to 0$, there remains a non-vanishing "comoving mode" motion of test particles in the plane of the wave front. The extra component of the motion is accounted for the corresponding additional flux of energy from the source, typically, of the same order of magnitude as the general relativity flux. This, at least, is in conflict with the indirect gravitational-wave observations of binary pulsars [434]. Such theories probably have to be rejected.

2.4.4 Black holes and cosmology in massive gravity

To obtain solutions in the massive gravity and to provide a direct comparison with general relativity effects, it is more convenient to represent the equations (2.4.22) of the field-theoretical theory in a quasi-geometrical form with an effective metric $g_{\mu\nu}$ close to general relativity. Using the invariance (2.4.9), let us rewrite (2.4.22) as

$$\begin{aligned} & G^{L^\dagger}_{\rho\sigma} - 8\pi t^{\text{tot}^\dagger}_{\rho\sigma} + 2k_1 h_{\rho\sigma} - (k_1 + 2k_2)\bar{g}_{\rho\sigma} h_\alpha{}^\alpha \\ &= G^{L}_{\rho\sigma} - 8\pi t^{\text{tot}}_{\rho\sigma} + 2k_1 h_{\rho\sigma} - (k_1 + 2k_2)\bar{g}_{\rho\sigma} h_\alpha{}^\alpha = 0\,. \end{aligned} \tag{2.4.30}$$

Now, adopting (2.2.133) for the flat background case (2.4.1), one easily obtains from (2.4.30):

$$\begin{aligned} & \left(\delta^\rho_\alpha \delta^\sigma_\beta - \tfrac{1}{2}\bar{g}_{\alpha\beta}\bar{g}^{\rho\sigma}\right)\left(G^{L^\dagger}_{\rho\sigma} - 8\pi t^{\text{tot}^\dagger}_{\rho\sigma} + 2k_1 h_{\rho\sigma} - (k_1 + 2k_2)\bar{g}_{\rho\sigma} h_\tau{}^\tau\right) \\ &= R_{\alpha\beta} - 8\pi\left(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T_\tau{}^\tau\right) + 2(k_1 h_{\alpha\beta} + k_2 \bar{g}_{\alpha\beta} h_\tau{}^\tau) = 0\,. \end{aligned} \tag{2.4.31}$$

At last, multiplying (2.4.31) by $\left(\delta^\alpha_\mu \delta^\beta_\nu - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta}\right)$ one obtains that the equations (2.4.22) are equivalent to

$$G_{\mu\nu} + M_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.4.32}$$

where the Einstein's equations are added by the massive term

$$M_{\mu\nu} \equiv \left(2\delta^\alpha_\mu \delta^\beta_\nu - g^{\alpha\beta} g_{\mu\nu}\right)(k_1 h_{\alpha\beta} + k_2 \bar{g}_{\alpha\beta} h_\rho{}^\rho). \tag{2.4.33}$$

Of course, the Bianchi identity $\nabla^\nu G_{\mu\nu} \equiv 0$ is valid an effective spacetime. Besides, the matter equations determine the conservation $\nabla^\nu T_{\mu\nu} = 0$, as usual. Thus, after the differentiation of (2.4.33) one obtains

$$\nabla^\nu M_{\mu\nu} = 0, \tag{2.4.34}$$

which must be considered as constraints for (2.4.32). Recall that $g_{\mu\nu}$ in (2.4.33) and (2.4.34) are functions of $\bar{g}^{\alpha\beta}$ and $\mathfrak{h}_{\alpha\beta}$, then one recognizes that (2.4.24) is a linear approximation of (2.4.34).

Thus, to find a solution to the new theory one has to search for the components $h^{\mu\nu}$, which satisfy the system (2.4.32) and (2.4.34). This consists of several steps: first, one has to select appropriate coordinates in the flat background spacetime; second, one has to use the connection (2.2.7) between the gravitational variables and the effective metric in order to substitute it in (2.4.32) and (2.4.34) and solve them; third, after determining $h^{\mu\nu}$, one again has to use the connection (2.2.7) for comparison with solutions in general relativity.

Thus, let us illustrate how a search for static spherically symmetric solutions in vacuum can be provided. Then, it is naturally to use the spherical coordinates, and the metric of the background spacetime in the form:

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.4.35}$$

In these coordinates, non-zero components of the gravitational field $h^{\mu\nu}$ are written as

$$h^{00} = -A(r), \qquad h^{11} = B(r), \qquad h^{22} = C(r), \qquad h^{33} = \frac{C(r)}{\sin^2\theta}, \tag{2.4.36}$$

where the functions $A(r)$, $B(r)$ and $C(r)$ have to be found from the equations (2.4.32) in vacuum, $T_{\mu\nu} = 0$, by taking into account the constraints (2.4.34). Then three independent equations among (2.4.32) and (2.4.34) survive only. For a comparison, in general relativity two independent equations survive only, when spherically symmetric static solutions in vacuum are searched. The consideration is simplified if one assumes $\alpha = \beta$, however all the qualitative conclusions remain valid for $\alpha \neq \beta$.

Combining analytical and numerical techniques, Babak and Grishchuk demonstrated that the solution of the massive theory is practically indistinguishable from that of general relativity for all $r_g \ll r \ll 1/\alpha$, where r and r_g are the radial coordinate and the gravitational radius of the Schwarzschild solution, see (1.5.34) (1.5.35). For r larger than $1/\alpha$ the solution takes the form of the Yukawa-type potentials; therefore they call this massive theory as the finite-range gravity.

The solution of the new theory also deviates strongly from that of general relativity in the vicinity of $r = r_g$, which is the location of the globally defined event horizon of the Schwarzschild black hole in general relativity. In the massive gravity the event horizon *does not form* at all, and the solution smoothly continues to the region $r < r_g$ and terminates at $r = 0$ where the curvature singularity develops. Since the αr_g can be extremely small, the redshift of the photon emitted at $r = r_g$ can be extremely large, but it remains finite in contrast with general relativity solutions. Infinite redshift is reached only at the singularity $r = 0$.

In the astrophysical sense, all conclusions that rely specifically on the existence of the black hole event horizon, are likely to be abandoned. It is very remarkable and surprising that the phenomena of black hole should be so unstable with respect to the inclusion of the tiny mass-terms, whose Compton wavelength can exceed, say, the present-day Hubble radius.

Homogeneous isotropic solutions were also considered in the framework of the massive gravity. Now one has to solve the system (2.4.32) and (2.4.34) with a non-zero matter energy-momentum tensor. The metric of the flat background spacetime is considered to be in the form:

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2, \quad (2.4.37)$$

then the gravitational field components have to depend on time only and to have a diagonal form:

$$h^{00} = -A(t), \qquad h^{11} = h^{22} = h^{33} = B(t). \quad (2.4.38)$$

The matter sources are described by a perfect fluid model

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - pg^{\mu\nu}. \quad (2.4.39)$$

The conservation law $\nabla^\nu T_{\mu\nu} = 0$ is reduced to the equation

$$\varepsilon' + 3\frac{a'}{a}(\varepsilon + p) = 0. \quad (2.4.40)$$

At last, the final simplification is the use of the equation of state in the form: $p(t) = q\varepsilon(t)$ with the constant $-1 < q < 1$.

As a result of the above assumptions there are two independent field equations from the set (2.4.32) and (2.4.34), unlike the case of general relativity where there is only one. First, if the mass of the *spin-0* graviton is zero, $\beta^2 = 0$, the cosmological solutions are exactly the same as those of general relativity, independently of the mass of the *spin-2* graviton, i.e., independently of the value of α^2. This result is expected due to the highest spatial symmetry: the *spin-2* degrees of freedom have no chance to reveal themselves.

Then, for $\beta^2 \neq 0$ the technically simpler case, $4\beta^2 = \alpha^2$, was studied in full detail. Qualitative results are valid for $4\beta^2 \neq \alpha^2$. Thus, by *combining analytical approximations and numerical calculations* it has been demonstrated that the massive solution has a long interval of evolution where it is practically indistinguishable from the Friedmann solution of general relativity. The deviation from general relativity is dramatic at the very early times and very late times. The unlimited expansion is being replaced by a regular maximum of the scale factor, whereas the singularity is being replaced by a regular minimum. Smaller β values induce the higher maximum and the deeper minimum, that is an arbitrary small mass term in the Lagrangian gives rise to the oscillatory behaviour of the cosmological scale factor.

Following the logic of interpretation that α^2 and β^2 define the masses, they are thought as positive. However, the general structure of the Lagrangian (2.4.21), see (2.4.20), does not imply this. Then, if one allows α^2 and β^2 to be negative, the late time evolution of the scale factor gives an "accelerated expansion" that is similar to the one governed by a cosmological Λ-term. The development of this point could be useful in the light of the modern cosmological observational data [100].

2.4.5 Gauge invariance in the Babak-Grishchuk modifications

Here, we study the gauge invariance properties of the Babak-Grishchuk formulation of general relativity given in Section 2.4.2 and of their variant of massive gravity described in 2.4.3. From the start, we adopt the gauge transformations (2.2.68) and (2.2.69) for the flat background (2.4.1). They acquire the form:

$$\mathfrak{h}'^{\mu\nu}(x) = \mathfrak{h}^{\mu\nu}(x) + \left(\exp \pounds_\xi - 1\right)\left(\bar{\mathfrak{g}}^{\mu\nu}(x) + \mathfrak{h}^{\mu\nu}(x)\right), \tag{2.4.41}$$

$$\phi'^A(x) = \phi^A(x) + \left(\exp \pounds_\xi - 1\right)\phi^A(x). \tag{2.4.42}$$

It is easy to state that the gauge invariance properties of the modification in Section 2.4.2 are similar to those described in Section 2.2.4 in the framework of the field-theoretical formulation of general relativity. Indeed, the Lagrangian (2.4.8) with (2.4.6) differs from the Lagrangian (2.2.15), or (2.2.104) for the case of a Ricci flat background, by the presence of the additional term proportional to the *background curvature tensor*, see (2.4.1). Then, adopting (2.2.72) to the case of the Babak-Grishchuk formulation of general relativity, one finds that the Lagrangian (2.4.8) is transformed under the transformations (2.4.41) and (2.4.42) as

$$\begin{aligned}\mathscr{L}'^{\text{dyn}^\dagger} = \mathscr{L}^{\text{dyn}^\dagger} &+ \frac{1}{16\pi}\left(\mathfrak{h}'^{\mu\nu} - \mathfrak{h}^{\mu\nu}\right)\bar{R}_{\mu\nu} + \frac{1}{16\pi}\left(\Lambda'^{\alpha\beta\rho\sigma} - \Lambda^{\alpha\beta\rho\sigma}\right)\bar{R}_{\alpha\rho\beta\sigma}\\ &+ \left(\exp \pounds_\xi - 1\right)\mathscr{L}_{EH}(\bar{\mathfrak{g}} + \mathfrak{h}, \phi).\end{aligned} \tag{2.4.43}$$

The second and third terms disappear by the condition (2.4.1), whereas the last term is a divergence and does not contribute into the equations of motion. Thus Lagrangian (2.4.8) is gauge invariant in the sense, like the Lagrangian $\mathscr{L}^{\text{dyn}}$ in (2.2.72).

Adopting the formula (2.2.74) to the Babak-Grishchuk formulation of general relativity and substituting (2.4.41) and (2.4.42) into (2.4.9), one finds

$$\begin{aligned}\left[G^{L^\dagger}_{\mu\nu}(\mathfrak{h}) - 8\pi t^{\text{tot}^\dagger}_{\mu\nu}(\mathfrak{h}, \phi)\right]' &= G^{L^\dagger}_{\mu\nu}(\mathfrak{h}) - 8\pi t^{\text{tot}^\dagger}_{\mu\nu}(\mathfrak{h}, \phi)\\ &+ \frac{1}{\sqrt{-\bar{g}}}\frac{\partial \bar{\mathfrak{g}}^{\rho\sigma}}{\partial \bar{g}^{\mu\nu}}\left(\exp \pounds_\xi - 1\right)\left[\sqrt{-\bar{g}}\frac{\partial \bar{g}^{\delta\pi}}{\partial \bar{\mathfrak{g}}^{\rho\sigma}}\left(G^{L^\dagger}_{\delta\pi} - 8\pi t^{\text{tot}^\dagger}_{\delta\pi}\right) + \bar{R}_{\rho\sigma}\right].\end{aligned} \tag{2.4.44}$$

If the background equations (2.4.1) hold, and if $\mathfrak{h}, \phi$ is the solution to the field equations (2.4.9) then $\mathfrak{h}', \phi'$ is the solution to the same equations. Analogous conclusions are valid for the matter equations in the Babak-Grishchuk formulation of general relativity field-theoretical form, as in (2.2.39).

The energy-momentum tensor (2.4.10), see also (2.4.11), is not gauge invariant even if the field-theoretical equations hold, the same as in (2.2.75). Indeed, keeping in mind the transformation (2.4.44) and the expression (2.4.13), one has

$$8\pi t'^{\text{tot}^\dagger}_{\mu\nu}(\mathfrak{h},\phi) = 8\pi t^{\text{tot}^\dagger}_{\mu\nu}(\mathfrak{h},\phi) + (-\bar{g})^{-1}(\exp \pounds_\xi - 1)\left[(-\bar{g})G^{L^\dagger}_{\mu\nu}(\mathfrak{h})\right]. \tag{2.4.45}$$

Following the arguments in Section 2.2.7, based on the gauge invariance properties, one concludes that the background spacetime has an auxiliary character. Indeed, the gauge invariance properties follow from the fact that the background metric disappears from the consideration. To show this it is easy to adopt (2.2.133) for the equations (2.4.9):

$$(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta})\left[G^{L^\dagger}_{\alpha\beta} - 8\pi t^{\text{tot}^\dagger}_{\alpha\beta}\right] = R_{\mu\nu} - 8\pi\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^\alpha_\alpha\right) = 0\,. \tag{2.4.46}$$

This means that if one substitutes (2.2.135) and (2.2.136) into the left hand side of (2.4.46) one can determine only the effective metric $g_{\mu\nu}$. As a consequence, conserved quantities are not localized, see (2.4.45), and the trajectories of test particles are gauge dependent.

Now, let us turn to the massive gravitational theory in Section 2.4.3. This theory is not gauge invariant. The reason is in the additional term in the Lagrangian (2.4.21), see (2.4.20); as a consequence, the additional term appears in the equations (2.4.22). Unlike (2.4.9), where the background metric is *absent really*, see (2.4.46), the equations (2.4.22) it *contain really* the background metric that one can see from the equivalent equations (2.4.32) with (2.4.33). However, because the gravitational field $\mathfrak{h}^{\mu\nu}$ is included into the matter Lagrangian in (2.4.46) in a *universal way* one cannot to determine the Minkowski space with the use of matter fields, like electromagnetic signals. However, characteristics of the background Minkowski space must be observable by the gravitational waves due to the presence of the term (2.4.33) in (2.4.32). As a result, in the case of observations conducted in the Minkowski space the energy-momentum tensor in equations (2.4.22) has to be localized, similarly to an arbitrary field theory in the Minkowski space, see Section 1.2.

3 Asymptotically flat spacetime in the field-theoretical formulation

In the present chapter, considering an asymptotically flat spacetime at spatial infinity, we develop the field-theoretical methods in general relativity laid out in the previous chapter. Historically, asymptotically flat spacetime was studied more intensively in the framework of the Hamiltonian formulation of general relativity. Therefore, in Section 3.1, we review the Arnowitt-Deser-Misner (ADM) formulation of general relativity in a reader-friendly form, we present its essential details and its modification introduced by Regge and Teitelboim. In Sections 3.2 and 3.3, we elaborate on both the Lagrangian and Hamiltonian forms of the field-theoretical description of the asymptotically flat spacetime.

3.1 The Arnowitt-Deser-Misner formulation of general relativity

3.1.1 The ADM action principle

The (3 + 1)-decomposition of a spacetime

This topic is pretty much standard by now and we follow the textbook [315]. Let a spatial hypersurface Σ_0 be marked by the time coordinate t = const. Then, a "later" spatial section Σ_1 is defined by $t + dt$ = const with an infinitesimal dt. We require that such hypersurfaces do not intersect, technically, the spacetime is globally hyperbolic and admits a spacelike foliation. Each spacelike hypersurface has its own space coordinates (corresponding to space coordinates in spacetime) and a 3-dimensional positive definite metric. For example, Σ_0 has the metric $g_{ij}(t, x, y, z)$ and Σ_1 has the metric $g_{ij}(t + dt, x, y, z)$. However, separate, independent definition of Σ_0 and Σ_1 is not enough to get the structure of a 4-dimensional spacetime. To resolve this problem one has to connect Σ_0 with Σ_1. First, let us define a proper distance between Σ_0 and Σ_1 related to every point (x, y, z) on Σ_0 in the orthogonal direction:

$$d\tau = N(t, x, y, z)dt\,, \tag{3.1.1}$$

where $N(t, x, y, z)$ is called the *lapse* function. Second, points on Σ_1 can be shifted with respect to the points on Σ_0:

$$x_1^i = x_0^i - N^i(t, x, y, z)dt\,, \tag{3.1.2}$$

where $N^i(t, x, y, z)$ is called the *shift* (3-dimensional) vector. Now, we are in a position to derive the interval between the world points $x^\alpha = (t, x^i)$ and $x^\alpha + dx^\alpha = (t + dt, x^i + dx^i)$, the line element reads

$$ds^2 = -(Ndt)^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right). \tag{3.1.3}$$

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Rewriting this in a 4-dimensional form, one finally has

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \,, \tag{3.1.4}$$

where, in the matrix form, we have

$$\begin{Vmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{Vmatrix} = \begin{Vmatrix} \left(N_s N^s - N^2\right) & N_j \\ N_i & \overset{3}{g}_{ij} \end{Vmatrix} \,. \tag{3.1.5}$$

Notice that the spatial components of the full 4-dimensional metric g_{ij}, are equal to those of 3-dimensional metric $\overset{3}{g}_{ij}$, thus in (3.1.4) the components of the shift vector are defined as $N_i = \overset{3}{g}_{ij} N^j = g_{ij} N^j$. The components of the inverse matrix $g^{\alpha\beta}$ are obtained without difficulty as

$$\begin{Vmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{Vmatrix} = \begin{Vmatrix} -\left(1/N^2\right) & N^j/N^2 \\ N_i/N^2 & \left(\overset{3}{g}{}^{ij} - N^i N^j/N^2\right) \end{Vmatrix} \,. \tag{3.1.6}$$

Now, it is easy to calculate the determinant of $g_{\alpha\beta}$ in terms of the 3-dimensional quantities:

$$g = \det g_{\alpha\beta} = -N^2 \overset{3}{g} \,, \tag{3.1.7}$$

where $\overset{3}{g} = \det g_{ij} = \det \overset{3}{g}_{ij}$. In addition, it is necessary to define the components of the unique vector n^α being normal to Σ . Turning to the definition of the proper distance (3.1.1), one has

$$n_\alpha = (-N, 0, 0, 0) \,. \tag{3.1.8}$$

Making use of (3.1.6), one finds the contravariant form of the normal vector as

$$n^\alpha = \left(1/N, -N^i/N\right) \,. \tag{3.1.9}$$

Then, indeed, one has $n^2 = n_\alpha n^\alpha = -1$.

It is important to express the 4-dimensional curvature in terms of the above (3 + 1)-decomposition. Concerning the 3-dimensional (intrinsic) curvature of a hypersurface Σ, one uses simply all the notions given in Section 1.3.1 for the case of four dimensions, by replacing the 4-metric $g_{\alpha\beta}$ with the 3-metric $\overset{3}{g}_{ij} = g_{ij}$. The Christoffel symbols of the hypersurface $\overset{3}{\Gamma}{}^i{}_{jk}$ are constructed with $\overset{3}{g}_{ij}$ by the formula (1.3.5). Thus, the Riemann tensor (1.3.2) is replaced with the 3-dimensional one, $\overset{3}{R}{}^i{}_{kjl}$; the Ricci tensor (1.3.3) is replaced with the 3-dimensional one, $\overset{3}{R}_{kl}$; and the scalar curvature (1.3.4) is replaced with the 3-dimensional one, $\overset{3}{R}$.

However, one must note that the 3-dimensional curvature of Σ is its intrinsic curvature. Therefore, to have a full description, it is necessary to add an extrinsic curvature, which describes the embedding of Σ into the 4-dimensional spacetime. For this purpose, let us consider the vector $n^\alpha(x^i)$ on Σ taken at the point with the local coordinates (x^i). Now we displace it along Σ by making use of the parallel transport in 4-dimensional spacetime to the point with the local coordinates $(x^i + dx^i)$ and compare it with $n^\alpha(x^i + dx^i)$. It is evident that the difference exists: it is proportional to dx^i and depends on the parallel transport of the normal vector,

$$(dn)_j = \nabla_i n_j dx^i \,, \tag{3.1.10}$$

where ∇_k is the 4-dimensional covariant derivative with a space index only. A 3-dimensional rank-two tensor defined as

$$K_{ij} = -\nabla_i n_j = -N\Gamma^0{}_{ij} \tag{3.1.11}$$

is called the *extrinsic curvature* of the hypersurfaces Σ embedded in the spacetime. To derive (3.1.11) we have used the components (3.1.8), and the components $\Gamma^0{}_{ij}$ of the 4-dimensional Christoffel symbols (1.3.5). Recalculating (3.1.11) with the help of (3.1.5) and (3.1.6), one can rewrite the extrinsic curvature in the form:

$$K_{ij} = \frac{1}{2N}\left(\overset{3}{\nabla}_j N_i + \overset{3}{\nabla}_i N_j - \frac{\partial g_{ij}}{\partial t} \right) , \tag{3.1.12}$$

where $\overset{3}{\nabla}_k$ is the 3-dimensional covariant derivative compatible with the 3-dimensional metric. Another useful form of the extrinsic curvature (3.1.12) is

$$K_{ij} = \tfrac{1}{2}£_n g_{ij} \,, \tag{3.1.13}$$

where $£_n$ represents the Lie derivative along the normal vector n^α. This equivalence can be shown by a direct calculation using the definition of the Lie derivative, see above (1.2.82).[1]

In the next part, we connect the 3-dimensional intrinsic curvature, constructed with the use of $\overset{3}{g}_{ij}$, and the extrinsic curvature, K_{ij}, with the full 4-dimensional curvature.

The projection technique in the (3 + 1)-decomposition

In the previous part, we have used the technique of foliation of spacetime into separate spatial hypersurfaces Σ. However, the technique of projecting geometrical objects (physical fields) onto the spacelike hypersurfaces is also needed to have a complete (3 + 1)-decomposition. We present this now, following the book [378].

Let us take a set of the spacelike non-intersecting hypersurfaces Σ in the parameterized form. This means that each of the hypersurfaces Σ is numerated by its own parameter t. We suppose also that each local domain in spacetime is covered with its

1 Notice that formula (3.1.13) has another sign with respect to the analogous formula in [315]. The reason is that in the present book we use the definition of the Lie derivative with the opposite sign for $£$, see (1.2.82).

own local coordinates $x = (x^\alpha)$. Then the parameter t can be represented as a scalar function f of the spacetime coordinates:

$$t = f(x)\,. \tag{3.1.14}$$

Next, let the hypersurfaces Σ be covered with their own local coordinates $\mathbf{x} = (x^i)$. Then, a connection between the 4-dimensional coordinates, from one side, and the 3-dimensional coordinates with the parameter t, from the other side, can be provided by the following relation,

$$x^\alpha = e^\alpha(\mathbf{x}, t)\,, \tag{3.1.15}$$

with four functions e^α. By the construction, the use of (3.1.15) in the equality (3.1.14) satisfies the latter identically.

After that we define a normalized basis of four vectors associated with the hypersurface Σ, which consists of the unit normal vector n_α and a triad of tangential vectors $e^\alpha{}_j$ defined as

$$e^\alpha{}_j \equiv \partial_j e^\alpha; \quad n_\alpha e^\alpha{}_j = 0; \quad n_\alpha n^\alpha = -1\,. \tag{3.1.16}$$

One easily finds that vector n^α is identified with the one given in (3.1.9). A 3-dimensional metric on Σ is induced from the 4 dimensional metric as

$$\overset{3}{g}_{ij} = g_{\alpha\beta}(e(\mathbf{x}, t))e^\alpha{}_i e^\beta{}_j\,. \tag{3.1.17}$$

If, in (3.1.15), $x^\alpha = (t, \mathbf{x})$ with $x^0 = t$ one finds $\overset{3}{g}_{ij} = g_{ij}$ that is consistent, of course, with (3.1.5). Defining $e_\alpha{}^j \equiv g_{\alpha\beta}\overset{3}{g}{}^{ij} e^\beta{}_i$, we find the identities

$$e_\alpha{}^i e^\alpha{}_j \equiv \delta^i_j; \quad e_\beta{}^i e^\alpha{}_i \equiv \delta^\alpha_\beta + n_\beta n^\alpha\,. \tag{3.1.18}$$

Now we are in a position to represent an arbitrary 4-vector ϱ^α in terms of the basis vectors (3.1.16) as

$$\varrho^\alpha = n^\alpha \varrho_\perp + e^\alpha{}_j \varrho^j\,, \tag{3.1.19}$$

where

$$\varrho_\perp = -n_\alpha \varrho^\alpha; \quad \varrho^j = e_\alpha{}^j \varrho^\alpha\,. \tag{3.1.20}$$

To define the 3-dimensional connection, $\overset{3}{\Gamma}{}^i{}_{jl}$, on Σ we use the same technique of projections in (3.1.19) and (3.1.20). Thus, the 3-dimensional covariant derivative can be derived as

$$\overset{3}{\nabla}_i \varrho^j = \nabla_\beta\left(\varrho^l e^\alpha{}_l\right) e_\alpha{}^j e^\beta{}_i\,. \tag{3.1.21}$$

Then, from here one easily finds

$$\overset{3}{\Gamma}{}^i{}_{jl} = (\nabla_l e^\alpha{}_j) e_\alpha{}^i\,, \tag{3.1.22}$$

from which the 3-dimensional Riemann tensor (intrinsic curvature of Σ) can be constructed. We define the following *notations* for the projections of the 4-covariant derivatives:

$$\begin{aligned}
\nabla_j \varrho^\alpha &\equiv e^\beta{}_j \nabla_\beta \varrho^\alpha\,,\\
\nabla_j \varrho_i &\equiv e^\alpha{}_i e^\beta{}_j \nabla_\beta \varrho_\alpha\,,\\
\nabla_j \varrho_\perp &\equiv -n^\beta e^\alpha{}_j \nabla_\alpha \varrho_\beta\,,\\
\nabla_\perp \varrho_j &\equiv -n^\alpha e^\beta{}_j \nabla_\alpha \varrho_\beta\,,\\
\nabla_\perp \varrho_\perp &\equiv n^\alpha n^\beta \nabla_\alpha \varrho_\beta\,.
\end{aligned} \tag{3.1.23}$$

For a constructive use of these definitions they have to be represented through 3-dimensional quantities at the spatial hypersurface Σ.

The extrinsic curvature is defined exactly as in (3.1.11). Then, by one of the equalities in (3.1.16), it can be recast to the form:

$$K_{ij} = n_\alpha \nabla_j e^\alpha{}_i\,. \tag{3.1.24}$$

One can see, then, that K_{ij} and $\overset{3}{\Gamma}{}^i{}_{jl}$ are projections of $\nabla_j e^\alpha{}_i$ onto the normal basis (3.1.16), thus

$$\nabla_j e^\alpha{}_i = -K_{ij} n^\alpha + \overset{3}{\Gamma}{}^l{}_{ij} e^\alpha{}_l\,. \tag{3.1.25}$$

Now introduce the vector

$$N^\alpha \equiv \frac{de^\alpha}{dt} \tag{3.1.26}$$

and its projections onto the normal basis (3.1.16), $N = -n_\alpha N^\alpha$ and $N^i = e_\alpha{}^i N^\alpha$. Then (3.1.26) can be rewritten as

$$N^\alpha = n^\alpha N + e^\alpha{}_i N^i\,. \tag{3.1.27}$$

It is clear that, here, N and N^i can be identified with the lapse function in (3.1.1) and the shift vector in (3.1.2), respectively. It is also evident that the covariant derivative along N^α has to be identified with the time derivative:

$$\frac{d}{dt} \equiv N^\beta \nabla_\beta \equiv \nabla_N\,. \tag{3.1.28}$$

To re-express definitions (3.1.23) in terms of the 3-dimensional quantities, one has to find the deformations of the normal basis in spacetime, when the basis vectors are displaced along the vector N^α. Keeping in mind the identity $N^\beta \partial_\beta e^\alpha{}_j \equiv \partial_j N^\alpha$ that follows from the definitions of $e^\alpha{}_j$ in (3.1.16) and N^α in (3.1.26), using the rules of projections (3.1.18–3.1.20) and the formulae (3.1.24) and (3.1.25), one obtains

$$\nabla_N e^\alpha{}_j = \left(\overset{3}{\nabla}_j N - K_{ij} N^i\right) n^\alpha + \left(\overset{3}{\nabla}_j N^i - N K^i_j\right) e^\alpha{}_i\,. \tag{3.1.29}$$

With the use of (3.1.16) this equality leads to the following:

$$\nabla_N n^\alpha = \left(\overset{3}{\nabla}_j N - K_{ij} N^i\right) e^{\alpha j} . \tag{3.1.30}$$

At last, combining (3.1.17) and (3.1.29), one gets

$$\frac{d\overset{3}{g}_{ij}}{dt} = \nabla_N \overset{3}{g}_{ij} = -2NK_{ij} + 2\overset{3}{\nabla}_{(i} N_{j)} , \tag{3.1.31}$$

which is nothing else but formula (3.1.12); the reason why in (3.1.31) we have a total time derivative, unlike (3.1.12) with a partial time derivative, is that in (3.1.31) $\overset{3}{g}_{ij}$ is defined by the general formula (3.1.17), whereas in (3.1.12) it is simply $\overset{3}{g}_{ij} = g_{ij}(x^i, t)$.

Next, it is necessary to introduce the variation along the normal component of the vector (3.1.27):

$$\delta_\perp \equiv N n^\alpha \nabla_\alpha \tag{3.1.32}$$

and the variation along the tangential component of the vector (3.1.27):

$$\delta_{||} \equiv -\pounds_{N^i} . \tag{3.1.33}$$

Now, consider an arbitrary vector field ϱ^α and its projections ϱ^i and $\varrho_\perp$. From the beginning, let us apply (3.1.32) to $\varrho_\perp = -n^\alpha \varrho_\alpha$. After simple manipulations using the above technique one arrives at

$$\delta_\perp \varrho_\perp = N n^\alpha \nabla_\alpha \varrho_\perp = -\varrho^i \overset{3}{\nabla}_i N - N \nabla_\perp \varrho_\perp . \tag{3.1.34}$$

From here one obtains

$$N \nabla_\perp \varrho_\perp = -\left(\delta_\perp \varrho_\perp + \varrho^i \overset{3}{\nabla}_i N\right) . \tag{3.1.35}$$

Analogously, after application of (3.1.32) to $\varrho^i = e_\alpha{}^i \varrho^\alpha$ one obtains

$$N \nabla_\perp \varrho_i = -\left(\delta_\perp \varrho_i + \varrho_\perp \overset{3}{\nabla}_i N + N K_{ij} \varrho^j\right) . \tag{3.1.36}$$

Now, let us apply the operator (3.1.33) to ϱ_i:

$$\delta_{||} \varrho_i = -\pounds_{N^j} \varrho_i = N^j \overset{3}{\nabla}_i \varrho_j + \varrho_j \overset{3}{\nabla}_i N^j . \tag{3.1.37}$$

On the other hand, since ϱ_i is a set of 3 scalars in 4 dimensions this can be rewritten in the form:

$$\delta_{||} \varrho_i = \nabla_N \varrho_i \,|_{N^\alpha = e^\alpha{}_j N^j} = \varrho_\perp N^j K_{ij} + \varrho_j \overset{3}{\nabla}_i N^j + N^j \nabla_j \varrho_i , \tag{3.1.38}$$

where (3.1.29) has been used. Comparing (3.1.37) and (3.1.38) one has

$$\nabla_j \varrho_i = \overset{3}{\nabla}_j \varrho_i - \varrho_\perp K_{ij} . \tag{3.1.39}$$

Analogously one obtains

$$\nabla_j \varrho_\perp = \overset{3}{\nabla}_j \varrho_\perp - \varrho^i K_{ij} . \tag{3.1.40}$$

The technique of projections (3.1.18–3.1.40) can easily be extended for a tensor of arbitrary rank and for derivatives of arbitrary order.

Applying the above rules for projecting the identity

$$\left(\nabla_{\nu\mu} - \nabla_{\mu\nu}\right) \varrho_\alpha \equiv \varrho_\lambda R^\lambda{}_{\alpha\mu\nu} , \tag{3.1.41}$$

one obtains the well known [153] Gauss-Codazzi equations:

$$R_{\perp kij} = \overset{3}{\nabla}_j K_{ki} - \overset{3}{\nabla}_i K_{kj} , \tag{3.1.42}$$

$$R_{klij} = \overset{3}{R}_{klij} + K_{ki}K_{lj} - K_{kj}K_{li} . \tag{3.1.43}$$

Also, one obtains the equality

$$R_{\perp i \perp j} = \delta_\perp K_{ij} + NK_{ik}K_j{}^k + \overset{3}{\nabla}_{ji} N . \tag{3.1.44}$$

We have made a rather long tour of projections but it is clear that the equalities (3.1.42–3.1.44) are necessary to recast the Hilbert Lagrangian in the terms of the (3 + 1)-decomposition. In addition to these, one has to use the following relations,

$$\begin{aligned} \delta_\perp \overset{3}{g}{}^{1/2} &= N \overset{3}{g}{}^{1/2} K_i{}^i , \\ \delta_\perp \left(\overset{3}{g}{}^{1/2} K_i{}^i\right) &= \nabla_N \left(\overset{3}{g}{}^{1/2} K_i{}^i\right) - \delta_{||} \left(\overset{3}{g}{}^{1/2} K_i{}^i\right) , \\ \delta_{||} \left(\overset{3}{g}{}^{1/2} K_i{}^i\right) &= -\pounds_{N^i} \left(\overset{3}{g}{}^{1/2} K_i{}^i\right) = \overset{3}{\nabla}_j \left(\overset{3}{g}{}^{1/2} N^j K_i{}^i\right) . \end{aligned} \tag{3.1.45}$$

Now, combining (3.1.7) and (3.1.42–3.1.45) one obtains the Hilbert Lagrangian in terms of (3 + 1)-decomposition as

$$\begin{aligned} \sqrt{-g}R = N \overset{3}{g}{}^{1/2} R = N \overset{3}{g}{}^{1/2} \left[\overset{3}{R} + K_{ij}K^{ij} + (K_i{}^i)^2\right] \\ -2\nabla_N \left(\overset{3}{g}{}^{1/2} K_i{}^i\right) + 2\overset{3}{g}{}^{1/2} \overset{3}{\nabla}_j \left(K_i{}^i N^j - \overset{3}{\nabla}{}^j N\right) . \end{aligned} \tag{3.1.46}$$

Canonical action and the equations in general relativity

To obtain the canonical action of general relativity, let us integrate (3.1.46) over a 4-dimensional volume V restricted by a timelike cylinder S and the spacelike sections Σ_0 and Σ_1, see Figure 2.1. Thus, (3 + 1)-version of the Hilbert action acquires the form:

$$S = \int_V d^4x\sqrt{-g}R = \int_{t_0}^{t_1} dt\left(L_0 - H_S - \frac{d}{dt}H_\Sigma\right). \tag{3.1.47}$$

Here, we follow the standard notations adopted in [315, 378] and drop off, for simplicity, the coefficient "1/16π" that is used in these books in the definition of the Hilbert action. Thus, in (3.1.47),

$$\begin{aligned}
L_0 &\equiv \int_\Sigma d^3x\mathscr{L}_0\,, \quad \mathscr{L}_0 \equiv N\overset{3}{g}{}^{1/2}\left[\overset{3}{R} + K_{ij}K^{ij} - (K_i{}^i)^2\right]\,; \\
H_S &\equiv 2\oint_S ds_j\overset{3}{g}{}^{1/2}\left(\overset{3}{\nabla}{}^jN - K_i{}^iN^j\right)\,; \\
H_\Sigma &\equiv 2\int_\Sigma d^3x\overset{3}{g}{}^{1/2}K_i{}^i\,.
\end{aligned} \tag{3.1.48}$$

As usual, $v^i = x^i/r$ and $ds_i = d^2\Omega r^2 v_i$ is the infinitesimal element of integration on the 2-dimensional surface $\partial\Sigma$ surrounding the isolated system. To derive the last quantity the relation (3.1.28) has been used.

Now, picking up the metric tensor components $g_{ij} = \overset{3}{g}_{ij}$ at Σ as generalized coordinates, we define their time derivatives

$$\dot{g}_{ij} \equiv \frac{dg_{ij}}{dt} \tag{3.1.49}$$

as generalized velocities. Then, varying (3.1.47) with respect to $\dot{g}_{ij}$, ignoring the surface integrals and using (3.1.12), we define the canonical momenta as usual

$$\pi^{ij} \equiv \frac{\partial\mathscr{L}_0}{\partial\dot{g}_{ij}} = G^{ijkl}K_{kl}\,. \tag{3.1.50}$$

Here the four-index object

$$G^{ijkl} \equiv \tfrac{1}{2}\overset{3}{g}{}^{1/2}\left(2\overset{3}{g}{}^{ij}\overset{3}{g}{}^{kl} - \overset{3}{g}{}^{ik}\overset{3}{g}{}^{jl} - \overset{3}{g}{}^{il}\overset{3}{g}{}^{kj}\right) \tag{3.1.51}$$

is the De Witt supermetric (which is sometimes called as the Wheeler-DeWitt superspace metric) [139] whose inverse reads

$$G_{ijkl} = \tfrac{1}{2}\overset{3}{g}{}^{-1/2}\left(g_{ij}g_{kl} - g_{ik}g_{jl} - g_{il}g_{jk}\right)\,, \tag{3.1.52}$$

which is obtained via the condition $G_{ijkl}G^{klmn} = \delta^m_{(i}\delta^n_{j)}$.

Finally, we are in a position to define the Hamiltonian of the gravitational field by the standard way:

$$H = \int_\Sigma d^3x\pi^{ij}\dot{g}_{ij} - L_0 + H_S + \frac{d}{dt}H_\Sigma\,. \tag{3.1.53}$$

Here, one has to exchange the generalized velocities with the generalized momenta. First, we use the relation (3.1.31) and combine (3.1.50–3.1.52)

$$K_{ij} = G_{ijkl}\pi^{kl} . \tag{3.1.54}$$

Second, substituting all of these into (3.1.53), we obtain the Hamiltonian function related to the Lagrangian function in (3.1.47):

$$H[g_{ij}, \pi^{ij}] = H_0 + \tilde{H}_S + \frac{d}{dt}H_\Sigma , \tag{3.1.55}$$

where K_{ij} is expressed through π^{kl} with the use of (3.1.54). The space integral in (3.1.55) is

$$H_0 = \int_\Sigma d^3x \mathscr{H}_0 , \quad \mathscr{H}_0 \equiv N^{(\mu)} \mathscr{H}_\mu . \tag{3.1.56}$$

We denote the set of $N^\perp \equiv N$ and N^i in (3.1.27) as

$$N^{(\mu)} \equiv \left\{N^\perp, N^i\right\} . \tag{3.1.57}$$

Of course, the quantities $N^{(\mu)}$, being projections of N^μ onto the normalized basis defined on Σ, have to be distinguished from the components of N^μ themselves in the spacetime. We introduce the following short-hand notations:

$$\mathscr{H}_\mu \equiv \left\{\mathscr{H}_\perp, \mathscr{H}_j\right\} ; \tag{3.1.58}$$

$$\mathscr{H}_\perp \equiv G_{ijkl}\pi^{ij}\pi^{kl} - \overset{3}{g}{}^{1/2}\overset{3}{R} , \tag{3.1.59}$$

$$\mathscr{H}_i \equiv -g_{ij}\overset{3}{\nabla}_k\pi^{jk} . \tag{3.1.60}$$

The last two expressions can be rewritten as

$$\mathscr{H}_\perp = -2\overset{3}{g}{}^{1/2}G_{\perp\perp} , \tag{3.1.61}$$

$$\mathscr{H}_i = 2\overset{3}{g}{}^{1/2}G_{i\perp} , \tag{3.1.62}$$

where $G_{\perp\perp}$ and $G_{i\perp}$ are projections of the components of the Einstein tensor onto Σ. The surface integral in (3.1.55) is

$$\tilde{H}_S \equiv 2\oint ds_j \left[\overset{3}{\nabla}{}^j\left(\overset{3}{g}{}^{1/2}N\right) - \pi_i{}^{(i}N^{j)}\right] . \tag{3.1.63}$$

At last, ignoring the term $H_\Sigma|_{t_0}^{t_1}$ to represent *fixed* initial and final states, one derives the canonical action with the Hamiltonian (3.1.55):

$$S\left[g_{ij}, \pi^{ij}, N^{(\mu)}\right] = \int_{t_0}^{t_1} dt \int_\Sigma d^3x \left(\pi^{ij}\dot{g}_{ij} - \mathscr{H}_0\right) - \int_{t_0}^{t_1} dt\tilde{H}_S ; \tag{3.1.64}$$

Varying the action (3.1.64) with respect to the generalized momenta and the generalized coordinates by the standard way, one obtains the field equations

$$\dot{g}_{ij} = \frac{\delta \mathscr{H}_0}{\delta \pi^{ij}} \equiv 2\frac{N}{\overset{3}{g}{}^{1/2}}\left(\pi_{ij} - \tfrac{1}{2}g_{ij}\pi^k{}_k\right) + 2\overset{3}{\nabla}_{(i}N_{j)}\,, \tag{3.1.65}$$

$$\begin{aligned}\dot{\pi}^{ij} = -\frac{\delta \mathscr{H}_0}{\delta g_{ij}} &\equiv -\overset{3}{g}{}^{1/2}\left[N\left(\overset{3}{R}{}^{ij} - \tfrac{1}{2}g^{ij}\overset{3}{R}{}^k{}_k\right) - \left(\overset{3}{\nabla}{}^{ij}N - g^{ij}\overset{3}{\nabla}_k{}^kN\right)\right]\\ &+\frac{N}{\overset{3}{g}{}^{1/2}}\left[\tfrac{1}{2}g^{ij}\left(\pi^k{}_l\pi^l{}_k - \tfrac{1}{2}(\pi^k{}_k)^2\right) - 2\left(\pi^{ik}\pi_k{}^j - \tfrac{1}{2}\pi^{ij}\pi^k{}_k\right)\right]\\ &+\overset{3}{\nabla}_k\left(N^k\pi^{ij}\right) - 2\pi^{k(i}\overset{3}{\nabla}_kN^{j)}\,. \end{aligned}\tag{3.1.66}$$

Here, the symbol δ/δ is the 3-dimensional Lagrangian derivative, see Appendix A.2.4; of course, the surface terms in (3.1.64) do not contribute to the result. Variation of (3.1.64) with respect to the Lagrange multipliers N and N^i leads to the equations, which are interpreted as constraints:

$$\mathscr{H}_\perp = 0\,, \tag{3.1.67}$$

$$\mathscr{H}_i = 0\,. \tag{3.1.68}$$

The canonical equations (3.1.65) and (3.1.66) together with the constraints (3.1.67) and (3.1.68) are equivalent to the vacuum Einstein equations in the standard 4-dimensional covariant formulation.

Including the matter sector

The natural question is how to incorporate the matter sector into the canonical action (3.1.64)? Without giving details, only to show the construction, we consider a simple example of fields denoted as ϕ^A with the action:

$$S = \int d^4x \mathscr{L}^m\left(\phi^A, \nabla_\mu\phi^A\right) = \int d^4x\sqrt{-g}L^m\left(\phi^A, \nabla_\mu\phi^A\right)\,. \tag{3.1.69}$$

As we did before, let us project both the fields $\phi^A \to \phi^{\mathsf{A}}$ and the metric, and their derivatives onto the hypersurface Σ, see conventions (3.1.23) and definitions (3.1.35), (3.1.36), (3.1.39) and (3.1.40). Now, we interpret ϕ^{A} as generalized coordinates and define the corresponding generalized momenta as

$$p_{\mathsf{A}} \equiv \frac{\partial \mathscr{L}^m}{\partial \dot{\phi}^{\mathsf{A}}}\,. \tag{3.1.70}$$

Then, the canonical action corresponding to (3.1.69) can be derived as

$$S = \int_{t_0}^{t_1} dt \int_\Sigma d^3x\left(p_{\mathsf{A}}(\mathbf{x})\dot{\phi}^{\mathsf{A}}(\mathbf{x}) - \mathscr{H}_0^m\right) - \int_{t_0}^{t_1} dtH_S^m\,, \tag{3.1.71}$$

where

$$\mathcal{H}_0^m \equiv N^{(\mu)} \mathcal{H}_\mu^m \left[\phi^{\mathrm{A}}, p_{\mathrm{A}}; g_{ij}, \partial_\mu g_{ij}\right] . \tag{3.1.72}$$

We notice that $\mathcal{H}_\mu^m$ does not depend on the lapse function and the shift vector, and H_S^m is a surface integral over S, see Figure 2.1, corresponding to the Lagrangian in (3.1.69).

To combine the matter action (3.1.71) with the gravitational action (3.1.64) one has to redefine the notion (3.1.50) of the gravitational generalized momenta

$$p^{ij} = \pi^{ij} + P^{ij} , \tag{3.1.73}$$

$$P^{ij} = \frac{\partial \mathcal{L}^m}{\partial \dot{g}_{ij}} . \tag{3.1.74}$$

Next, it is more convenient to rewrite a matter super-Hamiltonian as

$$\mathcal{H}_\perp^m = \mathcal{H}^m + 2K_{ij}P^{ij} . \tag{3.1.75}$$

Then, the total canonical action for a gravitating system acquires the form:

$$\begin{aligned} S\left[g_{ij}, p^{ij}; \phi^{\mathrm{A}}, p_{\mathrm{A}}; N^{(\mu)}\right] &= \int_{t_0}^{t_1} dt \int_\Sigma d^3x \left(p^{ij}\dot{g}_{ij} + p_{\mathrm{A}}\dot{\phi}^{\mathrm{A}} - N^{(\mu)}\mathcal{T}_\mu\right) \\ &\quad - \int_{t_0}^{t_1} dt \left(\tilde{H}_S + \tilde{H}_S^m\right) , \end{aligned} \tag{3.1.76}$$

where $\tilde{H}_S^m$ is the surface integral revised after substituting (3.1.75). Keeping in mind that

$$K_{ij} = G_{ijkl}\left(p^{kl} - P^{kl}\right) , \tag{3.1.77}$$

one derives

$$\mathcal{T}_\mu = \{\mathcal{T}_\perp, \mathcal{T}_i\} , \tag{3.1.78}$$

$$\mathcal{T}_\perp = \mathcal{H}^m + \mathcal{H}_\perp\big|_{\pi^{ij}=p^{ij}-P^{ij}} , \tag{3.1.79}$$

$$\mathcal{T}_i = \mathcal{H}_i^m - 2g_{ij}\overset{3}{\nabla}_k p^{jk} . \tag{3.1.80}$$

Analogous to (3.1.61) and (3.1.62), one can represent the last two expressions as

$$\mathcal{T}_\perp = -2\overset{3}{g}{}^{1/2}\left(G_{\perp\perp} - \tfrac{1}{2}T_{\perp\perp}\right) , \tag{3.1.81}$$

$$\mathcal{T}_i = 2\overset{3}{g}{}^{1/2}\left(G_{i\perp} - \tfrac{1}{2}T_{i\perp}\right) . \tag{3.1.82}$$

Here, one easily finds that the expressions at the right hand sides are the projections of the field operator in the Einstein equations (1.3.22) onto Σ if one restores the coefficient "$1/16\pi$" in front of the Einstein tensor.

Varying the action (3.1.76) with respect to the generalized momenta and the generalized coordinates by the standard way one gets

$$\dot{g}_{ij} = \frac{\delta N^{(\mu)} \mathscr{T}_\mu}{\delta p^{ij}}, \tag{3.1.83}$$

$$\dot{\pi}^{ij} = -\frac{\delta N^{(\mu)} \mathscr{T}_\mu}{\delta g_{ij}}. \tag{3.1.84}$$

Varying the action (3.1.76) with respect to the Lagrange multipliers N and N^i, one obtains the equations

$$\mathscr{T}_\perp = 0, \tag{3.1.85}$$

$$\mathscr{T}_i = 0, \tag{3.1.86}$$

which are interpreted as constraints. The canonical equations (3.1.83) and (3.1.84) together with the constraints (3.1.85) and (3.1.86) are equivalent to the Einstein equations with matter sources in the standard 4-dimensional covariant formulation. Let us remark once again that the surface terms in (3.1.76) do not participate in the construction of (3.1.83–3.1.86).

3.1.2 Asymptotically flat spacetime at spatial infinity in general relativity

A concept of the asymptotically flat spacetime in general is *usually* used to model the gravitational field of a *real* isolated gravitating system. Such a model has played an important role in gravitational physics: see, for example, [16, 19, 33, 180, 226, 356, 357, 406, 419] and the numerous references therein. The asymptotically flat spacetime is studied in two regimes: at spatial infinity and at null infinity, see the textbook [315]. In the present chapter, we pay attention to the case of the spatial infinity only.

What makes the asymptotically flat spacetime so interesting? First, in spite of its apparent simplicity, the model opens the possibility of studying the fundamental properties of gravitational field. As an example, the proof of the positive energy theorem for an isolated system has been provided in the works by Yau and Schoen, Witten and Nester [333, 407, 408, 463] specifically in asymptotically flat spacetimes. In its own turn, this remarkable result stimulated a farther growth of keen interest to the model, see, for example, [18, 31, 51, 52, 102, 103, 160, 186, 254, 347, 387, 413, 418, 428, 454].

Second, the model of the asymptotically flat spacetime is useful from the astronomical point of view. Indeed, most of the astrophysical objects are isolated systems up to a good approximation. In this regard, for example, in the work [328], based on the concept of the asymptotically flat spacetime, a special variational principle has been developed. With the use of such a principle one can construct mathematical models both for the stationary rotating stars and for non-stationary collapsing stars.

Third, the study of asymptotically flat spacetimes can be useful from the methodological point of view because a large class of exact solutions in general relativity represents such models. Then, some fundamental problems or properties of the gravity theory can be analyzed or illustrated on the examples of the exact solutions.

When considering a real isolated system, one assumes that all of the physical fields, including gravitational waves, are *effectively* occupying a restricted domain of space. This means that the leading term in the Taylor series of the metric tensor expansion with respect to a small parameter has to coincide with the Newtonian potential. Therefore, solution of the Einstein equations for an isolated system has to acquire the form of the metric coefficients of the Schwarzschild solution very far away from the system because the constant of integration in the latter is matched to the Newtonian potential [385]. Besides, there are no coordinates where the fall-off could be stronger than the one in the Schwarzschild metric.

To define the asymptotically flat spacetime one has to postulate the asymptotic behavior of the fields at infinity. As the, perhaps, simplest definition, we use the one given by Faddeev [160]:

(i) At the spatial infinity, world points are parameterized in a one-to-one correspondence by the coordinates $\{x^\alpha\}$ such that $-\infty < x^\alpha < \infty$.

(ii) Among all such possible coordinate systems one can find a chart where the metric has the asymptotic behavior:

$$g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1}), \quad g_{\mu\nu,\alpha} = O(r^{-2}) \tag{3.1.87}$$

for $r \to \infty$; $r^2 \equiv \eta_{ik}x^i x^k$. Such coordinate systems are called asymptotically Lorentzian systems. The notation $Q = O(r^\omega)$ corresponds to $\lim_{r\to\infty}[Q/r^\omega] =$ const.

(iii) There are no coordinate charts where the fall-off of the metric coefficients can be stronger than in (3.1.87). Also, one excludes a possible asymptotic behavior of the metric tensor as $\sim \ln r$, see, for example, [418].

(iv) A condition of effective localization of matter is defined by the requirement for the asymptotic behaviour of the matter energy-momentum tensor in the Einstein equations as

$$T_{\mu\nu} = O(r^{-3-\alpha}), \quad \alpha > 0\,. \tag{3.1.88}$$

This requirement corresponds to the asymptotic behaviour of the Lagrangian for the matter sources:

$$\mathscr{L}^M = O(r^{-3-\alpha}), \quad \alpha > 0\,. \tag{3.1.89}$$

The above conditions (i–iv) in the Lagrangian formulation can be re-expressed in the framework of the Hamiltonian formulation given in Section 3.1.1. The behaviour of the metric coefficients (3.1.87) are reformulated as the behaviour of generalized coordinates g_{ik} and generalized momenta π^{ik} on the spacelike hypersurfaces, Σ. So,

$$g_{ik} = \eta_{ik} + O(r^{-1}), \quad g_{ik,l} = O(r^{-2})\,. \tag{3.1.90}$$

Spatial coordinates $\{x^k\}$ on Σ, for which the behaviour (3.1.90) takes place, are called the asymptotically Cartesian coordinates. The requirement to preserve the behaviour (3.1.90) under asymptotic Poincaré transformations and using the Hamiltonian equations of general relativity (3.1.65) and (3.1.66) lead to a conclusion that generalized momenta, π^{ik}, have the asymptotic behaviour:

$$\pi^{ik} = O(r^{-2}) \,. \tag{3.1.91}$$

In order to preserve (3.1.90) and (3.1.91) under asymptotic deformations of Σ one applies simple restrictions on the asymptotic behaviour of the lapse function, N, and the shift vector, N^i:

$$N = 1 + O(r^{-1}), \quad N_{,l} = O(r^{-2}), \tag{3.1.92}$$

$$N^i = O(r^{-1}), \quad N^i{}_{,l} = O(r^{-2}). \tag{3.1.93}$$

The definitions (3.1.90–3.1.93) in the Hamiltonian formulation are equivalent to (3.1.87) in the Lagrangian formulation.

3.1.3 The ADM definition of conserved quantities

Since the surface integrals in (3.1.76) do not contribute to the Hamiltonian equations of motion they usually are not considered. In this case, a Hamiltonian function of the system is defined by the terms $\mathscr{T}_\mu$ only. Therefore, in fact, the Hamiltonian function is equal to zero due to the constraints (3.1.85) and (3.1.86):

$$H = \int_\Sigma d^3x N^{(\mu)} \mathscr{T}_\mu = 0 \,. \tag{3.1.94}$$

Of course, such a Hamiltonian function cannot describe the energy of the system in the common sense. Lagrangians, which lead to such type of the Hamiltonians are called as *singular* [140, 193, 378], or *parameterized* Lagrangians. Below, we present a case with a singular Lagrangian using the simplest example of a point particle which was also given in the ADM paper.

Parameterized action for a point particle

Let us recall the usual way of constructing the Hamiltonian formulation of the system (1.1.7) with the Lagrangian function (1.1.5):

$$S = \int dt L(q_i, \dot{q}_i); \quad i = 1, \ldots, n \,. \tag{3.1.95}$$

Define the generalized momenta as

$$p^i \equiv \frac{\partial L}{\partial \dot{q}_i} \,. \tag{3.1.96}$$

Now derive the total differential of the Lagrangian function in (3.1.95):

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \,, \tag{3.1.97}$$

where summation over the repeated indices from 1 to n is implied. Due to the definition (3.1.96) and the Lagrangian equation (1.1.10), this can be rewritten as

$$dL = \dot{p}^i dq_i + p^i d\dot{q}_i \to d\left(p^i \dot{q}_i - L\right) = -\dot{p}^i dq_i + \dot{q}_i dp^i \,. \tag{3.1.98}$$

One can see that the quantity

$$H(p, q) = p^i \dot{q}_i - L \tag{3.1.99}$$

depends on q_i and p^i only, and is called the *Hamiltonian function*. Comparing with the definition of energy (1.1.40) one recognizes that H has to define the energy as well. Thus, the action (3.1.95) is rewritten in the Hamiltonian form:

$$S = \int dt \left(p^i \dot{q}_i - H\right) \,. \tag{3.1.100}$$

Varying (3.1.100) and taking into account (3.1.98), one obtains the (first order) Hamiltonian equations:

$$\dot{q}_i = \frac{\partial H}{\partial p^i} \,, \quad \dot{p}^i = -\frac{\partial H}{\partial q_i} \,. \tag{3.1.101}$$

The total derivative of H with respect to time is

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p^i} \dot{p}^i \,. \tag{3.1.102}$$

Keeping in mind (3.1.101), one can see that H is conserved

$$\frac{dH}{dt} = 0 \,. \tag{3.1.103}$$

In fact, it is another form of the conservation law given previously in (1.1.40). At last, the Hamiltonian action (3.1.100) provides the so-called [13] generating function:

$$G(t) = p^i \delta q_i - H \delta t \,, \tag{3.1.104}$$

which generates time translations.

Now, let us assume that the time coordinate is another dynamical variable, $t \equiv q_{n+1}$, whereas a new parameter τ plays the role of time. Then the action (3.1.100) is rewritten in the form:

$$S = \int d\tau p^j q_j' \,, \quad j = 1, \dots, n, n+1 \,, \tag{3.1.105}$$

where $q' = dq/d\tau$ and the constraint

$$H_c = p^{n+1} + H(p,q) = 0 \tag{3.1.106}$$

should be imposed. The constraint can be incorporated into (3.1.105) with the use of a Lagrange multiplier λ:

$$S = \int d\tau \left(p^j q'_j - \lambda H_c\right) . \tag{3.1.107}$$

The Hamiltonian function in (3.1.107) is zero,

$$H_\tau \equiv \lambda H_c = 0 , \tag{3.1.108}$$

like the general relativistic Hamiltonian in (3.1.94). The generic problem for both of the cases (3.1.94) and (3.1.108) is how can one transfer from the parameterized action to the action in the usual form with a non-zero Hamiltonian? We show this using the simplest example of the action (3.1.107).

If one substitutes solution of the constraint equation (3.1.106) into the action (3.1.107) one finds

$$S = \int dq_{n+1} \left(p^i \frac{dq_i}{dq_{n+1}} - H(p,q)\right) . \tag{3.1.109}$$

Expressions for the actions (3.1.107) and (3.1.109) show that q_{n+1} plays the role of “an intrinsic coordinate”. On the other hand, equation for q_{n+1} exists and is given as

$$q'_{n+1} = \lambda \frac{dH_c}{dp^{n+1}} . \tag{3.1.110}$$

Then, keeping in mind the constraint (3.1.106), one concludes that λ, being dynamically arbitrary, guaranties that q_{n+1} is dynamically arbitrary as well. This means that we can choose q_{n+1} in an arbitrary way $q_{n+1} = q_{n+1}(\tau)$. The transfer from (3.1.107) to (3.1.109) as $q_{n+1} = \tau$ means that q_{n+1} is the *intrinsic coordinate* under this coordinate condition. Then (3.1.110) defines λ. The formulation (3.1.107) is τ-invariant that is invariant with respect to replacements, $\tilde{\tau} = \tilde{\tau}(\tau)$. This is evident because (3.1.109) does not depend on τ at all. Thus τ can be classified as an *extrinsic coordinate*. At last, one notes that (3.1.107) and (3.1.109) provide the following two generating functions, respectively,

$$G(\tau) = p^j \delta q_j , \tag{3.1.111}$$

$$G(q_{n+1}) = p^i \delta q_i - H\delta q_{n+1} . \tag{3.1.112}$$

In (3.1.111) the constraint has been used. Besides, not only q_{n+1} can be used as an intrinsic coordinate, in fact, it can be another variable q_i or combination of the variables.

To conclude this part, we note that, in order to transfer from a parameterized action to the usual Hamiltonian action one has *to solve the constraint equation and to impose a coordinate condition.*

The ADM way of reconstructing a parameterized action in general relativity

In the rest of this subsection we follow the presentation in the key paper by Arnowitt, Deser and Misner [13]. For the sake of simplicity we consider the vacuum case of general relativity with the action (3.1.64), (3.1.56), without the surface terms,

$$S\left[g_{ij}, \pi^{ij}, N^{(\mu)}\right] = \int_{t_0}^{t_1} dt \int_\Sigma d^3x \left(\pi^{ij}\dot{g}_{ij} - \mathscr{H}_0\right) \tag{3.1.113}$$

with the Hamiltonian equations (3.1.65) and (3.1.66). One can see that (3.1.113) with the constraints (3.1.67) and (3.1.68) is in the same form of (3.1.107) with the constraint (3.1.106). The generating function corresponding to (3.1.113), analogously to (3.1.111), after taking into account the constraints, is

$$G = \int_\Sigma d^3x \pi^{ij} \delta g_{ij} \tag{3.1.114}$$

with 12 phase space variables. To transfer to the standard action with a nonzero Hamiltonian one has, first, to solve 4 constraints (3.1.67) and (3.1.68), and, second, to set 4 coordinate conditions. As a result, one has to obtain a generating function, analogous to (3.1.112), in the form:

$$G = \int_\Sigma d^3x \left[\pi^A \delta g_A - \mathscr{H}_0{}^0(\pi^A, g_A)\delta t + \mathscr{H}_i^0(\pi^A, g_A)\delta x^i\right] . \tag{3.1.115}$$

Here, $A = 1, 2$, π^A and g_A are rest of the phase space variables, and the last items in (3.1.115) represent the generating functions for the translations δt and δx^i. The Hamiltonian action, corresponding to (3.1.115) is

$$S = \int dt \int_\Sigma d^3x \left[\pi^A \dot{g}_A - \mathscr{H}_0{}^0(\pi^A, g_A)\right] \tag{3.1.116}$$

with the Hamiltonian $\mathscr{H}_0{}^0$. Below we provide this program.

The linearized theory

Following ADM [13], to present the above declared transformations we apply the program to an asymptotically flat system described in Section 3.1.2. In the present discussion, using this assumption, the linearized theory is developed. Let us rewrite the constraints (3.1.67) and (3.1.68) in an approximate form,

$$g_{ij,ij} - g_{ii,jj} = \mathfrak{P}_2{}^0(g_{ij}, \pi^{ij}) , \tag{3.1.117}$$

$$-2\pi^{ij}{}_{,j} = \mathfrak{P}_2{}^i(g_{ij}, \pi^{ij}) . \tag{3.1.118}$$

On the left hand sides, one finds purely linear in g_{ij} and π^{ij} expressions; the place of the indices (upper or lower) is not important because we use the behaviour (3.1.90) and (3.1.91), and the 3-dimensional metric $\eta_{ij} = \delta_{ij}$ is used as the background one. On the right hand sides, $\mathfrak{P}_2{}^0$ and $\mathfrak{P}_2{}^i$ are quadratic in g_{ij} and π^{ij}.

To solve equations (3.1.117) and (3.1.118) one has to make the usual linear orthogonal decomposition of g_{ij} and π^{ij}. We illustrate it on the example of any symmetric tensor $\varrho_{ij} = \varrho_{ji}$. Thus,

$$\varrho_{ij} = \varrho_{ij}{}^{TT} + \varrho_{ij}{}^{T} + \varrho_{i,j} + \varrho_{j,i}\,, \tag{3.1.119}$$

where each of the quantities on the right hand side can be expressed uniquely as a linear function of ϱ_{ij}. First, the quantities $\varrho_{ij}{}^{TT}$ represent the transverse traceless components of ϱ_{ij}: $\varrho_{ij}{}^{TT}{}_{,j} = 0$ and $\varrho_{ii}{}^{TT} = 0$. Second, the quantities $\varrho_{ij}{}^{T}$ represent the transverse components of ϱ_{ij}: $\varrho_{ij}{}^{T}{}_{,j} = 0$, and are defined uniquely by the trace $\varrho^T = \varrho_{ii}{}^T$:

$$\varrho_{ij}{}^{T} = \tfrac{1}{2}\left[\delta_{ij}\varrho^T - (1/\nabla^2)\varrho^T{}_{,ij}\right]\,, \tag{3.1.120}$$

where the operator $1/\nabla^2$ is the inverse of the flat Laplacian with the appropriate boundary conditions. Third, the longitudinal parts of ϱ_{ij} reside in the remaining part of (3.1.119): $\varrho_{i,j} + \varrho_{j,i}$. Decomposing ϱ_i into its transverse and longitudinal (curl-less) parts, one has $\varrho_i = \varrho_i{}^T + \frac{1}{2}\varrho^L{}_{,i}$. Thus, inversely, each of six quantities on the right hand side of (3.1.119) may be expressed through ϱ_{ij}:

$$\varrho_i = (1/\nabla^2)\left[\varrho_{ij,j} - \tfrac{1}{2}(1/\nabla^2)\varrho_{kj,kji}\right]\,, \tag{3.1.121}$$

$$\varrho^T = \varrho_{ii} - (1/\nabla^2)\varrho_{ij,ij}\,, \tag{3.1.122}$$

$$\varrho_{ij}{}^{TT} = \varrho_{ij} - \varrho_{ij}{}^{T}[\varrho_{mn}] - \left(\varrho_{i,j}[\varrho_{mn}] + \varrho_{j,i}[\varrho_{mn}]\right)\,. \tag{3.1.123}$$

Here, $\varrho_{ij}{}^T[\varrho_{mn}]$ and $\varrho_{i,j}[\varrho_{mn}]$, are obtained with the use of (3.1.121) and (3.1.122).

Now, let us turn to the constraints (3.1.117) and (3.1.118) and substitute there the decomposition (3.1.119). Keeping in mind (3.1.120–3.1.123), one finds

$$-\nabla^2 g^T = \mathfrak{P}_2{}^0\,, \tag{3.1.124}$$

$$-2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) = \mathfrak{P}_2{}^i\,. \tag{3.1.125}$$

According to the Section 3.1.2, we use the boundary conditions in such a way that g^T and π^i vanish asymptotically. Then, because the structures of (3.1.124) and (3.1.125) begin at the second order these equations can be written as

$$g^T = -(1/\nabla^2)\overset{lin}{\mathfrak{P}}{}^0\,, \tag{3.1.126}$$

$$-2(\pi^{iT} + \pi^L{}_{,i}) = (1/\nabla^2)\overset{lin}{\mathfrak{P}}{}^i\,, \tag{3.1.127}$$

where $\overset{lin}{\mathfrak{P}}{}^0$ and $\overset{lin}{\mathfrak{P}}{}^i$ are obtained from $\mathfrak{P}_2{}^0$ and $\mathfrak{P}_2{}^0$ by setting g^T and π^i equal to zero there. In fact, we have shown that constraint equations are solved for g^T and π^i in terms of other variables as for the four *extra* momenta, just analogously to p^{n+1} as the extra momentum in (3.1.106).

To see explicitly that $\overset{lin}{\mathfrak{P}}{}^0$ and $\overset{lin}{\mathfrak{P}}{}^i$ generate the appropriate time and space translations, one must return to the generating function (3.1.114). Inserting the decomposition (3.1.119) for both g_{ij} and π^{ij}, one obtains

$$G = \int_\Sigma d^3x \left[\pi^{ijTT}\delta g_{ij}{}^{TT} + \pi^{ijT}\delta g_{ij}{}^{T} + 2(\pi^i{}_{,j} + \pi^j{}_{,i})\delta g_{i,j}\right] . \tag{3.1.128}$$

The cross terms in (3.1.128) have vanished due to properties of the orthogonal decomposition. Integrating by parts and adding total variations, one transforms (3.1.128) into

$$\begin{aligned} G &= \int_\Sigma d^3x \left[\pi^{ijTT}\delta g_{ij}{}^{TT} - \nabla^2 g^T \delta(1/2\nabla^2)\pi^T - 2\nabla^2(\pi^{iT} + \pi^L{}_{,i})\delta g_i\right] \\ &= \int_\Sigma d^3x \left[\pi^{ijTT}\delta g_{ij}{}^{TT} - \mathfrak{P}_2{}^0\delta[-(1/2\nabla^2)\pi^T] + \mathfrak{P}_2{}^i\delta g_i\right] . \end{aligned} \tag{3.1.129}$$

The form of (3.1.129) represents the form (3.1.115), only the final step in reduction to the standard (non-singular) canonical form is to be imposed by coordinate conditions

$$t = -(1/2\nabla^2)\pi^T , \tag{3.1.130}$$

$$x^i = g_i . \tag{3.1.131}$$

Keeping in mind the relations of the orthogonal decomposition (3.1.121) and (3.1.122), the conditions (3.1.130) and (3.1.131) can be rewritten in an alternative form:

$$\pi^{ii}{}_{,jj} - \pi^{ij}{}_{,ij} = 0 , \tag{3.1.132}$$

$$g_{ij,j} = 0 . \tag{3.1.133}$$

Next, the linear part of the Hamiltonian equation (3.1.65) gives after the orthogonal decomposition

$$\frac{d}{dt}(g_{ij} + g_{ji}) = N_{i,j} + N_{j,i} . \tag{3.1.134}$$

Recall that Lagrange multipliers $N_i = g_{0i}$ are functions determined only when coordinate conditions are imposed and must vanish at spatial infinity where the space is flat, see (3.1.93). Inserting (3.1.131) into (3.1.134), one obtains that $N_i = 0$ everywhere. This is consistent with the boundary conditions. Similarly, from (3.1.66) after the linear approximation and the decomposition one obtains

$$\frac{d}{dt}\left[-(1/2\nabla^2)\pi^T\right] = N . \tag{3.1.135}$$

Then the condition (3.1.130) implies $N = (-g_{00})^{-1/2} = 1$. This is consistent with the required asymptotic limit , see (3.1.92).

Reading inversely (3.1.126) and (3.1.127), one finds that

$$\overset{lin}{\mathfrak{P}}{}^0(\pi^{ijTT}, g_{ij}{}^{TT}) = -\nabla^2 g^T , \tag{3.1.136}$$

$$\overset{lin}{\mathfrak{P}}{}^i(\pi^{ijTT}, g_{ij}{}^{TT}) = -2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) \tag{3.1.137}$$

are the linearized theory's Hamiltonian and momentum densities and so their coefficients in the generating function (3.1.129) must be δt and δx^i, see (3.1.130) and (3.1.131). Finally, following (3.1.116), one constructs the canonical action for the linearized theory:

$$S = \int dt \int_\Sigma d^3x \left[\pi^{ijTT}\dot{g}_{ij}{}^{TT} - \overset{lin}{\mathfrak{P}}{}^0(\pi^{ijTT}, g_{ij}{}^{TT})\right] . \tag{3.1.138}$$

The full theory

Now, the full theory can be easily put into the canonical form. Turning to the initial constraints (3.1.67) and (3.1.68), one can rewrite them in the form of (3.1.124) and (3.1.125), although now they are fully exact and nonlinear. One finds

$$-\nabla^2 g^T = \mathfrak{P}^0(g_{ij}{}^{TT}, \pi^{ijTT}; g^T, \pi^i; g_i, \pi^T) , \tag{3.1.139}$$

$$-2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) = \mathfrak{P}^i(g_{ij}{}^{TT}, \pi^{ijTT}; g^T, \pi^i; g_i, \pi^T) . \tag{3.1.140}$$

Here, $\mathfrak{P}^\mu$ are non-linear functions of g_{ij} and π^{ij}. In any case, one can solve these coupled equations for g^T and π^i in principle by a perturbation-iteration expansion. Thus, one can again choose $-\nabla^2 g^T$ and $-2\nabla^2(\pi^{iT} + \pi^L{}_{,i})$ as the four extra momenta to be eliminated. We denote symbolically the solutions to equations (3.1.139) and (3.1.140) by

$$-\nabla^2 g^T = \mathfrak{T}_0{}^0(g_{ij}{}^{TT}, \pi^{ijTT}; g_i, \pi^T) , \tag{3.1.141}$$

$$-2\nabla^2(\pi^{iT} + \pi^L{}_{,i}) = \mathfrak{T}_0{}^i(g_{ij}{}^{TT}, \pi^{ijTT}; g_i, \pi^T) . \tag{3.1.142}$$

These equations are the counterpart of the equation (3.1.106) in the particle case.

We now impose the same coordinate conditions (3.1.130) and (3.1.131) which determine g_i and π^T. Then, the $\dot{g}_i$ and $\dot{\pi}^T$ equations become the determining equations of N and N^i. In the full theory they are not equal to 1 and 0, respectively, but now become specific functions of $g_{ij}{}^{TT}$ and π^{ijTT}, which could be calculated explicitly, in principle. In the last four equations, then, N and N^i may be eliminated in principle, leaving a system of four equations involving only $g_{ij}{}^{TT}$ and π^{ijTT}, and linear in their time derivatives.

The generating function, which generalizes (3.1.129) for the full theory, acquires the form

$$G = \int_\Sigma d^3x \left(\pi^{ijTT}\delta g_{ij}{}^{TT} - \mathfrak{T}_0{}^0\delta t + \mathfrak{T}_0{}^i\delta x^i\right) , \tag{3.1.143}$$

whereas the corresponding canonical action has the form:

$$S = \int dt \int_\Sigma d^3x \left[\pi^{ijTT}\dot{g}_{ij}{}^{TT} - \mathfrak{T}_0{}^0(\pi^{ijTT}, g_{ij}{}^{TT})\right] . \tag{3.1.144}$$

Energy and momentum expressions for an isolated system

The above formalism has properties close to the usual Lorentz covariant field theories. As a consequence, the physical interpretation of the gravitational field maybe carried out in terms of energy, momentum, etc., as well.

The energy E of the gravitational field yelds the numerical value of Hamiltonian, as usual, for a particular solution of the field equations. In obtaining this numerical value, the form of the Hamiltonian $\mathfrak{T}_0{}^0$ as a function of the canonical variables is irrelevant, but one may use the equation (3.1.141) to express E as a surface integral. It should be emphasized that, while *energy* and *momentum* densities are indeed divergences, the integrands in the generating function (3.1.143) $\mathfrak{T}_0{}^\mu$ are not divergences when expressed as functions of the canonical variables. Thus, for the total energy one has

$$P^0 \equiv E = \int_\Sigma d^3x\mathfrak{T}^0{}_0 = -\int_\Sigma d^3x\nabla^2 g^T = -\oint_\infty ds_i g^T{}_{,i} = \oint_\infty ds_i\left(g_{ij,j} - g_{jj,i}\right) , \tag{3.1.145}$$

where the notation $\oint_\infty \equiv \lim_{r\to\infty}\oint_{\partial\Sigma}$ is used. Similarly, the total momentum P_i with the use of (3.1.142) may be written as

$$P^i = -2\oint_\infty ds_j\left(\pi^i{}_{,j} + \pi^j{}_{,i}\right) = -2\oint_\infty ds_j\pi^{ij} . \tag{3.1.146}$$

In (3.1.145) and (3.1.146), we have assumed that the coordinates are asymptotically Cartesian at Σ, the spacetime becomes flat at spatial infinity. These requirements lead to the conclusion that $P^\mu = \{P^0, P^i\}$ is a Lorentz invariant vector. Of course, having P^μ and using the asymptotic flatness property, one could easily construct the total angular momentum and the integral of the center of mass by the standard way. However, one meets serious difficulties with the behaviour (3.1.90) and (3.1.91), when one tries to construct the total angular momentum and integral of the center of mass with the use of integrands in (3.1.145) and (3.1.146). They become divergent. It is called as the *super-translation* ambiguity at spatial infinity.

3.1.4 The Regge-Teitelboim modification

Let us now look at the classical work of Regge and Teitelboim [385]. Above, we stated that variation of the Hamiltonian action (3.1.113) leads to the Hamiltonian equations (3.1.65) and (3.1.66) without taking into account any surface terms. The main result of Regge and Teitelboim is that, in the case of asymptotically flat spacetime, one must include into the Hamiltonian function additional surface terms. They were based on the *Hamilton variation principle* that has been applied more carefully than is usually done. Thus, let us vary (3.1.56):

$$\delta H_0 = \int_\Sigma d^3x \left[\frac{\delta \mathcal{H}_0}{\delta g_{ij}} \delta g_{ij}(x) + \frac{\delta \mathcal{H}_0}{\delta \pi^{ij}} \delta \pi^{ij}(x) \right]$$
$$- \oint_\infty ds_l G^{ijkl} \left[N \overset{3}{\nabla}_k (\delta g_{ij}) - \delta g_{ij} \overset{3}{\nabla}_k N \right]$$
$$- \oint_\infty ds_l \left[2N_k \delta \pi^{kl} + \left(2N^k \pi^{jl} - N^l \pi^{jk} \right) \delta g_{jk} \right] . \tag{3.1.147}$$

The canonical equations (3.1.65) and (3.1.66) have to follow from (3.1.147), thus the surface terms in (3.1.147) have to be neglected. The *Regge-Teitelboim variation principle* is as follows. Variations on hypersurfaces, Σ_0 at initial time, and Σ_1 at final time, are to be equal to zero identically. On the other hand, the variations δg_{ij} and $\delta \pi^{ij}$ at the space boundary, $\partial \Sigma$, do not vanish identically. Only, for δg_{ij} and $\delta \pi^{ij}$ they require the *same fall-off* as the fields g_{ij} and π^{ij} themselves. But, in this case, surface terms in (3.1.147) are not equal to zero!

To save the situation additional compensating surface terms are to be added in (3.1.147). From the start, Regge and Teitelboim use the initial simplest behaviour (3.1.90) and (3.1.91). Then, it is necessary to add only

$$P^\perp = \oint_\infty ds_i \left(g_{ij,j} - g_{jj,i} \right) \tag{3.1.148}$$

to the integral (3.1.56). One easily recognizes that (3.1.148) is exactly the ADM energy integral (3.1.145).

However the principle has to be more universal, it has to be invariant with respect to the asymptotic Poincaré transformations. To include them into the consideration one has to change the behaviour of the lapse function (3.1.92) and the shift vector (3.1.93) as

$$N = \alpha^\perp + \beta^\perp{}_r x^r + O^-(1) + O(r^{-1}) , \tag{3.1.149}$$

$$N^i = \alpha^i + \beta^i{}_r x^r + O^-(1) + O(r^{-1}) . \tag{3.1.150}$$

Here and below, (+) and (−) mean even and odd parity functions with respect to the sign-change of the 3-vector: $v^k = x^k/r$; it is the main assumption in the Regge and Teitelboim approach. Then, keeping in mind (3.1.149) and (3.1.150), the Hamiltonian function (3.1.56) has to be augmented by the integrals:

$$P^r = -2 \oint_\infty ds_l \, \pi^{rl} , \tag{3.1.151}$$

$$M^{rs} = -2 \oint_\infty ds_l \left(x^r \pi^{ls} - x^s \pi^{lr} \right) , \tag{3.1.152}$$

$$M_{\perp r} = \oint_\infty ds_l \left[x^r \left(g_{sl,s} - g_{ss,l} \right) - g_{rl} + \eta_{rl} g_{ss} \right] \tag{3.1.153}$$

together with (3.1.148). They can be identified with the 3-dimensional momentum, angular momentum and the center of mass integral, respectively.

To have the integrals (3.1.151–3.1.153) well defined (finite) one has to modify the behaviour of the phase variables (3.1.90) and (3.1.91) as

$$g_{ij} = \delta_{ij} + {}^1g_{ij}(v^k)r^{-1} + {}^2g_{ij}(v^k)r^{-2} + O(r^{-2-\alpha})\,, \tag{3.1.154}$$

$$\pi^{ij} = {}^2\pi^{ij}(v^k)r^{-2} + {}^3\pi^{ij}(v^k)r^{-3} + O(r^{-3-\alpha}), \quad \alpha > 0\,, \tag{3.1.155}$$

where ${}^1g_{ij}(v^k)$ and ${}^2\pi^{ij}(v^k)$, respectively, even and odd functions of the angular arguments v^k are of the order $O(1)$, the other terms have undetermined parity.

Finally, Regge and Teitelboim suggested the Hamiltonian function

$$H_{RT} = H_0 - \alpha^{\perp}P_{\perp} - \alpha^r P_r + \beta^{\perp r}M_{\perp r} + \tfrac{1}{2}\beta^{rs}M_{rs}\,, \tag{3.1.156}$$

where $P_{\perp} = -P^{\perp}$. Variation of (3.1.156) with respect to phase varibles and with taking into account the Regge-Teitelboim corrected fall-off does not lead to the surface terms, unlike (3.1.147).

Together with a permissible fall-off, Regge and Teitelboim discussed permissible asymptotic deformations of the spacelike hypersurfaces Σ represented as

$$N = \zeta(x)\,, \quad N^i = \zeta^i(x)\,, \tag{3.1.157}$$

which do not change the asymptotics (3.1.154) and (3.1.155). Substituting (3.1.157) into the equations (3.1.65) and (3.1.66), and requiring that perturbations $\delta g_{ij} \sim \dot{g}_{ij}$, $\delta(g_{ij,k)} \sim \dot{g}_{ij,k}$ and $\delta\pi^{ij} \sim \dot{\pi}^{ij}$ do not disturb the behaviour (3.1.154) and (3.1.155), Regge and Teitelboim found the conditions

$$\zeta\,, \zeta^i = O^-(r^{-1-\varepsilon}) + O^+(r^{-1-\delta})\,, \quad \varepsilon \ge 1\,, \quad \delta \ge 2\,. \tag{3.1.158}$$

3.2 An isolated system in the Lagrangian description

Usually, the study of an asymptotically flat spacetime is carried out in the framework of the geometrical formulation of general relativity. Many powerful and elegant mathematical methods have been developed, see in Section 3.1.2 the cited works and references therein. On the other hand, it has been noted that it is also natural to analyze an asymptotically flat spacetime at infinity with the use of an auxiliary flat spacetime, see, for example, [17, 467]. Therefore, the field-theoretical technique developed in the previous chapter in Section 2.2 could be a valuable instrument in this context and hence we apply it here.

Recall some properties of the field-theoretical formulation of general relativity which will be used in the present section. First, it is exact (not approximate, not asymptotic) theory of perturbations, which are considered as independent fields –*field configurations*– propagating in a fixed spacetime. The field-theoretical formulation is equivalent to general relativity in the standard geometrical formulation. Second, a choice of a background is defined by the problem under consideration. In the case of an asymptotically flat spacetime it is natural to choose a flat background, as the whole spacetime, corresponding to the Minkowski space at infinity. Third, the global integrals of motion for the field configuration are defined exactly, like in an arbitrary field theory in the Minkowski space, see Section 1.2.4. They are defined on flat spacelike hypersurfaces by using the energy-momentum tensor and the Killing vectors of the Minkowski space. Fourth, the global integrals of motion are converted to surface integrals that is easily provided in the Minkowski space.

Advantages of the field-theoretical approach in studying asymptotically flat solutions in general relativity are as follow. One obtains coordinate independent well defined expressions. In the framework of the traditional geometrical derivation, as a rule, one has to consider simultaneously the problem of a permissible asymptotic behaviour for coordinates, the problem of permissible deformations of spacelike hypersurfaces at infinity, etc. At the same time, when the field-theoretical approach is used, all the aforementioned problems are considered as a *united* problem of a permissible asymptotic behaviour for gauge transformations, see Section 2.2.4, in surface global integrals.

Thus, in the present chapter, using the formalism of the Section 2.2 and following the presentation in the works [360, 361], we study the asymptotically flat spacetimes. We use both the Lagrangian (in the present section) and Hamiltonian (in the next section) formulations. The following assumptions are used:

(i) We assume that a manifold, which supports a physical metric, has to support a background flat metric as well.

(ii) We assume that an asymptotically flat spacetime corresponds to a *real* isolated system, see Section 3.1.2.

(iii) In the present chapter, only a spatial region at infinity is studied, thus we consider systems without gravitational radiation.

In the field-theoretical formulation it is not necessary to use the assumption (i), however then in the case of a complicated solution one needs to resort to exotic interpretations. To avoid this we introduce the assumption (i) that means that the same manifold is supplied by two metrics, $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$. The assumptions (ii) and (iii) are natural.

3.2.1 Asymptotically flat spacetime as a field configuration

Now we reformulate the definition of asymptotically flat spacetime given in the points (i–iv) in Section 3.1.2 in the framework of the field-theoretical approach. We choose a background spacetime as the Minkowski space. Asymptotic field configuration, $\mathfrak{h}^{\mu\nu}$, is defined by the decomposition (2.2.7). For *convenience*, to have an evident (explicit) fall-off of potentials one has to use the Lorentzian (Cartesian) coordinates in the

Minkowski space, although, owing to the covariant formulation, arbitrary curvilinear coordinates can also be used. In the Lorentzian coordinates, $\sqrt{-\bar{g}} = \sqrt{-\eta} = 1$, and, consequently, one finds from (2.2.7):

$$h^{\mu\nu} = \mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}\,. \tag{3.2.1}$$

Thus, the field-theoretical formulation of the definition (i–iv) in subsection (3.1.2) is as follows.

(i) Keeping in mind a one-to-one correspondence between world points of physical spacetime and world points of the Minkowski space, one parameterizes the latter with the Lorentzian coordinates $\{x^\alpha\}$.

(ii) There is a gauge fixing issue which we deal with by demanding the components of the field configuration to have the behaviour:

$$h^{\mu\nu} = O(r^{-1})\,, \quad h^{\mu\nu}{}_{,\alpha} = O(r^{-2})\,, \tag{3.2.2}$$

where $r^2 = \eta_{ij}x^ix^j$ with Cartesian coordinates $\{x^k\}$ in Euclidean space on sections Σ defined as $x^0 = t = \text{const}$.

(iii) There is no a gauge fixing, when the fall-off of gravitational potentials could be stronger than in (3.2.2).

(iv) To obtain the condition of effective localization of matter sources one has to use the restrictions (3.1.88) and (3.1.89), and the definition of the matter energy-momentum in the field-theoretical formulation, $t^m_{\mu\nu}$, in (2.2.32) adopted to a background Minkowski space, then it acquires the form:

$$t^m_{\mu\nu} = O(r^{-3-\alpha})\,, \quad \alpha > 0\,. \tag{3.2.3}$$

Below, a naive definition of the field configuration for an asymptotically flat spacetime given above in items (i–iv), will be elaborated to be more precise.

A model of a real isolated system has to be invariant under asymptotic Poincaré transformations. Sometimes, a spacetime with such a property is called an *asymptotically Minkowskian spacetime* [19]. Then, a field-theoretical model of a real isolated system has to be invariant under Poincaré transformations in the Minkowski space. Therefore, this is related to the behaviour (3.2.2). It is an important statement and we analyze it in more detail. We consider the behaviour of potentials and fields at $r \to \infty$. The requirement of Poincaré invariance means that the fall-off (3.2.2) is conserved at $r' \to \infty$ on the sections Σ' defined as $x'^0 = t' = \text{const}$ for a frame $\{x'^\alpha\}$ connected with the initial one $\{x^\alpha\}$ by the Poincaré transformations:

$$x'^\alpha = x^\alpha + a^\alpha + b_\beta{}^\alpha x^\beta\,, \tag{3.2.4}$$

where a^α are constant components of the 4-dimensional translation vector, whereas $b_\beta{}^\alpha$ are constant components of the Lorentzian matrix.

Let us choose a slice in the form: $x^0 = \text{const} = a$ on that a radius-vector, $\mathbf{r}$, is defined by two points, x^α_1 and x^α_2. Its origin has the coordinates $x^\alpha_1 = \{a, 0, 0, 0\}$, whereas its peak has the coordinates in the Lorentzian system $x^\alpha_2 = \{a, x^1_2, x^2_2, x^3_2\}$. The interval between these points is

$$(\Delta s)^2 = \eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta = \eta_{ik}x_2^i x_2^k = r^2\,, \tag{3.2.5}$$

where $\Delta x^\alpha = x_2^\alpha - x_1^\alpha$.

Now, apply the transformations (3.2.4), then the interval (3.2.5) is transformed as

$$(\Delta s')^2 = \eta_{\alpha\beta}\Delta x'^\alpha \Delta x'^\beta = -(\Delta x'^0)^2 + \eta_{ik}\Delta x'^i \Delta x'^k, \tag{3.2.6}$$

where

$$\Delta x'^0 = b_k{}^0 \Delta x^k \equiv \boldsymbol{b}\cdot\boldsymbol{r}\,, \tag{3.2.7}$$

$$\Delta x'^k = b_i{}^k \Delta x^i \tag{3.2.8}$$

with the constant vector $\boldsymbol{b}$. The left hand side of (3.2.8) can be written in the form:

$$\Delta x'^k = x_2'^k - x_1'^k = x_2'^k - (a^k + a b_0{}^k) \equiv x_2'^k + c^k\,, \tag{3.2.9}$$

where c^k are components of a constant vector $\boldsymbol{c}$. We define the radius-vector $\boldsymbol{r}'$ by the same way as in (3.2.5). It is the radius vector of the point with coordinates $x_2'^k$ with respect to the origin on the transformed flat slice Σ'. Thus, now $r'^2 = \eta_{ik}x_2'^i x_2'^k$. Substituting (3.2.7) and (3.2.9) into the interval expression (3.2.6) and equalizing it to (3.2.5), one obtains

$$r^2 = -(\boldsymbol{b}\cdot\boldsymbol{r})^2 + r'^2 + 2\boldsymbol{r}'\cdot\boldsymbol{c} + c^2\,. \tag{3.2.10}$$

As is seen, when $r \to \infty$ one has $r' \to \infty$, and conversely. It is the main statement necessary for our goal. More concretely, from (3.2.10) one has

$$r\cdot\left[1 + (b_k{}^0 v^k)^2\right]^{1/2} = r'\cdot\left(1 + \frac{2v'_k c^k}{r'} + \frac{c^2}{r'^2}\right)^{1/2} = r' + O(1) \tag{3.2.11}$$

where $v^k = x^k/r$ and $v'^k = x'^k/r'$, and we generalize an observation: $x_2^k \to x^k$ and $x_2'^k \to x'^k$. Combining (3.2.4) and (3.2.11) it is not difficult to find

$$\left[1 + (b_l{}^0 v^l)^2\right]^{-1/2} b_j{}^k v^j = v'^k + O(r'^{-1})\,. \tag{3.2.12}$$

The conclusion that follows from (3.2.11) and (3.2.12) is that, up to constant coefficients, the behaviour at infinity both r and r', and, v^k and v'^k, are the same.

Now let us apply the Poincaré transformations (3.2.4) to the gravitational potentials $h^{\mu\nu}$:

$$h'^{\mu\nu} = (\delta^\mu_\alpha + b_\alpha{}^\mu)(\delta^\nu_\beta + b_\beta{}^\nu)h^{\alpha\beta}\Big|_{\substack{r=r(r')\\ v=v(v')}}\,. \tag{3.2.13}$$

Then, keeping in mind the relations (3.2.11) and (3.2.12) and that ${b_\beta}^\alpha$ have constant components, one finds from (3.2.13) after taking into account (3.2.2):

$$h'^{\alpha\beta} = O(r'^{-1}), \quad {h'^{\alpha\beta}}_{,\gamma} = O(r'^{-2}). \tag{3.2.14}$$

Thus the behaviour (3.2.2) is Poincaré invariant.

Let us make some remarks.

First, if the requirement of the Poincaré invariance is satisfied, then, instead of the behaviour (3.2.2) one can use only

$$h^{ik} = O(r^{-1}), \quad {h^{ik}}_{,l} = O(r^{-2}). \tag{3.2.15}$$

Second, the behaviour of the gravitational potentials in (3.2.2) determines the behaviour of the gravitational Lagrangian

$$\mathscr{L}^g = O(r^{-4}). \tag{3.2.16}$$

To derive this the general definition (2.2.20) was used for the field-theoretical formulation of general relativity on the Ricci flat (including flat) background (2.2.104–2.2.106).

Third, the requirement of the effective localization in (3.2.3) determines the behaviour of the matter Lagrangian in (2.2.104):

$$\mathscr{L}^m = O(r^{-3-\alpha}), \quad \alpha > 0. \tag{3.2.17}$$

Thus, the total dynamical Lagrangian (2.2.104) defines a finite action functional for an asymptotically flat spacetime and for finite time intervals.

3.2.2 Global conserved quantities

Global conserved quantities play a crucial role in describing the model of an isolated system. It was already noted that the technique of defining such quantities in the framework of the field-theoretical approach with the Minkowski background is the same as in an arbitrary field theory in the Minkowski space. Then we can turn to the formulae (1.2.83–1.2.88).

To define the main quantity in these formulae, the current $\mathscr{I}^\mu_S$, we use the total energy-momentum, $\boldsymbol{t}^{\text{tot}}_{\mu\nu}$, on the right hand side in (2.2.105), with the differential conservation law (2.2.106). Because $\boldsymbol{t}^{\text{tot}}_{\mu\nu}$ is a symmetric energy-momentum tensor, we define the current with the use of the formula (1.2.132):

$$\mathscr{I}^\mu_S = \boldsymbol{t}^{\mu\nu}_{\text{tot}}\xi^K_\nu, \tag{3.2.18}$$

where the Killing vectors of the Minkowski space, ξ^ν_K, in the Lorentzian coordinates are defined in (1.1.72). Then, due to the conservation law of the type (2.2.106) one obtains the conservation law (1.2.133) for the current (3.2.18):

$$\mathscr{J}^{\mu}_{S\ ;\mu} = \partial_\mu \mathscr{J}^{\mu}_{S} = 0\,. \tag{3.2.19}$$

It is a realization of the conservation law (1.2.83) in the general form. Then, one can construct the quantity of the type (1.2.87):

$$\mathscr{P}_S(\xi_K) = \int_\Sigma d^3x\, \mathscr{J}^0_S(\xi_K) = \int_\Sigma d^3x \boldsymbol{t}^{0\nu}_{\rm tot}\xi^K_\nu\,. \tag{3.2.20}$$

The integration is carried out over the whole space Σ. The quantities (3.2.20) are conserved if boundary conditions (fall-off behaviour) satisfy

$$\lim_{r\to\infty} \oint_{\partial\Sigma} ds_k\, \boldsymbol{t}^{k\mu}_{tot}\, \xi^K_\mu = \oint_\infty ds_k\, \boldsymbol{t}^{k\mu}_{tot}\, \xi^K_\mu = 0\,, \tag{3.2.21}$$

see (1.2.88).

In fact, it is difficultly to use the energy-momentum tensor $\boldsymbol{t}^{\rm tot}_{\mu\nu}$ since the gravitational part is quite cumbersome, whereas we did not define the matter part at all. However, we keep in mind that the Einstein equations hold, and we can use them in the form (2.2.105). We rewrite them in the form:

$$t^{\mu\nu}_{\rm tot} = \frac{1}{8\pi}\, G^{\mu\nu}_L(h) = \frac{1}{8\pi}\, G^{\mu\nu\beta}_{L\ \ \ ,\beta}\,, \tag{3.2.22}$$

where we define

$$G^{\mu\nu\beta}_L \equiv \frac{1}{2}\,(h^{\mu\nu,\beta} + \eta^{\mu\nu} h^{\beta\alpha}{}_{,\alpha} - h^{\mu\beta,\nu} - \eta^{\mu\beta} h^{\nu\alpha}{}_{,\alpha})\,, \tag{3.2.23}$$

for which $G^{\mu\nu\beta}_L = -G^{\mu\beta\nu}_L$. Thus, in constructing conserved integrals with the use of (3.2.22) and (3.2.23) we need not know how matter falls-off at all. Such a behaviour is taken into account by the gravitational field equations automatically.

Substituting (3.2.22) into (3.2.20) and applying the Gauss' theorem, one obtains for each of the Killing vectors (1.1.72) rewritten in the Lorentzian coordinates:

$$\begin{aligned}\mathscr{P}^{(\alpha)}_S &= \frac{1}{8\pi}\int_\Sigma dx^3 \left[G^{\alpha 0 i}_{L\ \ \ ,i}\right]\\ &= \frac{1}{16\pi}\oint_\infty ds_i\, \left[h^{\alpha 0,i} + \eta^{\alpha 0} h^{i\beta}{}_{,\beta} - h^{\alpha i,0} - \eta^{\alpha i} h^{0\beta}{}_{,\beta}\right]\end{aligned} \tag{3.2.24}$$

for the total 4-momentum that has been obtained with the time translation $\xi^\alpha_K = -\delta^\alpha_0$ and space translations $\xi^\alpha_K = \delta^\alpha_k$;

$$\begin{aligned}\mathscr{P}^{([mn])}_S &= \frac{1}{16\pi}\int_\Sigma d^3x \left[(G^{n0i}_L x^m - G^{m0i}_L x^n)_{,i} + G^{m0n}_L - G^{n0m}_L\right]\\ &= \frac{1}{32\pi}\oint_\infty ds_i\, \left[(h^{n0,i} - h^{ni,0} - \delta^{ni} h^{0\alpha}{}_{,\alpha})x^m + \delta^{ni} h^{m0}\right.\\ &\quad \left. -\ (h^{m0,i} - h^{mi,0} - \delta^{mi} h^{0\alpha}{}_{,\alpha})x^n - \delta^{mi} h^{n0}\right]\end{aligned} \tag{3.2.25}$$

for the total angular momentum with the space rotations $\xi^\alpha_K = \xi^\alpha_{[mn]}$; and

$$\begin{aligned}\mathscr{P}^{([m0])}_S &= \frac{1}{16\pi}\int_\Sigma dx^3\left[(G^{00i}_L x^m - G^{m0i}_L x^0)_{,i} - G^{00m}_L\right]\\ &= \frac{1}{32\pi}\oint_\infty ds_i\,\left[h^{00,i} - h^{ik}{}_{,k})x^m\right.\\ &\quad\left. - (h^{m0,i} - h^{mi,0} - \delta^{mi}h^{0\alpha}{}_{,\alpha})x^0 - \delta^{mi}h^{00} + h^{mi}\right]\end{aligned} \quad (3.2.26)$$

for the total Lorentz momentum with the Lorentzian rotations $\xi^\alpha_K = \xi^\alpha_{[m0]}$.

The integrals (3.2.24–3.2.26) have been derived in the Lorentzian coordinates. It is necessary to analyze *explicitly* a permissible fall-off for fields at spatial infinity. However, it could be better to use the advantage of the field-theoretical approach, namely, the covariant formulation. Conserving the flat sections Σ in the initial definitions of (3.2.24–3.2.26), one can use arbitrary curved *space* coordinates on Σ. After this, partial space derivatives transform to covariant ones: $\partial_i Q \to Q_{;i}$. Using the relations for the Killing vectors in curved coordinates (1.2.6) and (1.2.7) one finds that all the ten integrals (3.2.24–3.2.26) are united into the form:

$$\begin{aligned}\mathscr{P}_S(\xi^\nu_K) = \frac{1}{16\pi}\oint_\infty ds_i\,&\left[\left(\gamma^{ij}h^0{}_\nu + \delta^0_\nu h^{ij} - \delta^j_\nu h^{0i}\right)_{;j}\xi^\nu_K -\right.\\ &\left.- \left(\gamma^{ij}h^0{}_\nu + \delta^0_\nu h^{ij} - \delta^i_\nu h^{0j}\right)\xi^\nu_{K;j}\right].\end{aligned} \quad (3.2.27)$$

Here, among curved coordinates, usually spherical coordinates are used.

3.2.3 The parity conditions

Substituting potentials with the behaviour (3.2.2) into integrals (3.2.24–3.2.26) one finds that the integrals (3.2.24) have finite values, whereas the integrals (3.2.25) and (3.2.26) diverge. But a real isolated system has to have a finite total angular and Lorentz momenta as well. The first researchers who clarified this problem were Regge and Teitelboim [385], they have resolved the problem suggesting the conditions (3.1.154) and (3.1.155). Here, we follow their strategy in the framework of the field-theoretical approach. Considering in detail the integrands in (3.2.25) and (3.2.26) one finds that it is necessary to restrict the behaviour of $h^{\mu\nu}$, $h^{\mu\nu}{}_{,k}$, $h^{ik}{}_{,0}$ and $h^{00}{}_{,0}$. The requirement of the Poincaré invariance leads to a necessity of analogous restriction for the rest of the components $h^{0k}{}_{,0}$, which are absent in (3.2.25) and (3.2.26).

To obtain well defined (finite) total angular momentum (3.2.25) and Lorentzian momentum (3.2.26) integrals we introduce the behaviour

$$\begin{aligned}h^{\mu\nu} &= O^+(r^{-1}) + O^-(r^{-\beta}),\\ h^{\mu\nu}{}_{,\pi} &= O^-(r^{-2}) + O^+(r^{-1-\beta}) \quad \beta \ge 2\end{aligned} \quad (3.2.28)$$

instead of (3.2.2). The index notations (+) and (−) mean even and odd parity functions with respect to changing the sign of the 3-vector: $v^k = x^k/r$. The definitions (3.2.28) assume that, in both lines, between the first and the second terms; other terms with the parity of the first terms could appear. Besides, conditions (3.2.28) introduced by us do not coincide with the Regge-Teitelboim conditions (3.1.154) and (3.1.155). Indeed, transforming (3.2.28) into the Hamiltonian formulation, one obtains

$$\begin{aligned} g_{ij} &= \eta_{ij} + O^+(r^{-1}) + O(r^{-\beta}), \\ g_{ij,k} &= O^-(r^{-2}) + O(r^{-1-\beta}), \\ \pi^{ij} &= O^-(r^{-2}) + O(r^{-1-\beta}), \ \beta \geq 2 \end{aligned} \tag{3.2.29}$$

that are weaker than (3.1.154) and (3.1.155). Also, the conditions (3.2.28) mean that there is a special gauge fixing, however they can be made *weaker*. Here, relying on the gauge invariance properties of the field-theoretical formulation, we study this problem.

Let us analyze an arbitrary asymptotic behaviour from the point of view of the Poincaré invariance requirement. Now, we consider the Poincaré transformations (3.2.4) in the perturbed form:

$$x'^\alpha = \delta a^\alpha + (\delta^\alpha_\beta + \delta b_\beta{}^\alpha)x^\beta \sim x^\alpha + \delta x^\alpha \tag{3.2.30}$$

with infinitesimal δa^α and $\delta b_\beta{}^\alpha$. Besides, keeping in mind (3.2.7) and (3.2.8), one has

$$\delta x^\alpha = O^-(r^1). \tag{3.2.31}$$

As a generic example, we consider a quantity (or a set of quantities), $\mathcal{Q}$, with the asymptotic behaviour as $r \to \infty$,

$$\mathcal{Q} = O^\pm(r^{-\gamma}). \tag{3.2.32}$$

After that we require the asymptotic behaviour of the perturbation, $\delta\mathcal{Q}$, induced by (3.2.30) to be the same as (3.2.32):

$$\delta\mathcal{Q} \sim \mathcal{Q}_{,\alpha}\delta x^\alpha = O^\pm(r^{-\gamma}). \tag{3.2.33}$$

Such a requirement is a Lagrangian analog of the requirement of the asymptotic Poincaré invariance in the Hamiltonian description [385].

Combining (3.2.31–3.2.33), one finds

$$\mathcal{Q}_{,\alpha} = O^\mp\left(r^{-1-\gamma}\right). \tag{3.2.34}$$

Lowering the order in r under differentiation with respect to space (Cartesian) coordinates looks evident, whereas lowering the order in r under differentiation with respect to time coordinate, x^0, looks quite unusual. However, there is no contradiction. As an example, one can consider a Lorentz-transformed Schwarzschild metric [225], where the coordinate x^0 is included only with the combination x^0/r. Then a differentiation with respect to x^0 is in correspondence with (3.2.34). Next one requires the Poincaré invariance of the behaviour (3.2.34). With the use of the same reasoning one obtains

$$\mathcal{Q}_{,\alpha\beta} = O^{\pm}\left(r^{-2-\gamma}\right), \tag{3.2.35}$$

and so on. Now, applying the above described logic of the Poincaré invariance to (3.2.28), one gets step by step,

$$\begin{aligned} h^{\mu\nu}{}_{,\pi\rho} &= O^{+}(r^{-3}) + O^{-}(r^{-2-\beta}), \\ h^{\mu\nu}{}_{,\pi\rho\sigma} &= O^{-}(r^{-4}) + O^{+}(r^{-3-\beta}), \\ \dots\dots\dots &= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\,. \end{aligned} \tag{3.2.36}$$

Substituting (3.2.30) and (3.2.36) into (2.2.30) and combining it with (3.2.3) one gets the behaviour for the total energy-momentum:

$$t^{\text{tot}}_{\mu\nu} = O(r^{-3-\alpha}), \quad \alpha > 0 \tag{3.2.37}$$

that satisfies (3.2.21), leading to the claim that global motion integrals (3.2.24–3.2.26) are conserved in time.

3.2.4 Gauge invariance of the motion integrals

To define the fall-off conditions weaker than in (3.2.28), conserving the values of (3.2.24–3.2.26), we use gauge invariance properties of the field-theoretical formulation of general relativity. Let us turn to the gauge transformations of the total energy-momentum (2.2.75). Then for the flat background in Lorentzian coordinates one has

$$t'^{\text{tot}}_{\mu\nu} = t^{\text{tot}}_{\mu\nu} + \frac{1}{8\pi} G^{L}_{\mu\nu}\left[\sum_{k=1}^{\infty} \frac{1}{k!}\, £^{k}_{\xi}\left(\boldsymbol{\eta}^{\mu\nu} + \mathfrak{h}^{\mu\nu}\right)\right], \tag{3.2.38}$$

where as before $\boldsymbol{\eta}^{\mu\nu} = \sqrt{-\eta}\eta^{\mu\nu}$ and $\mathfrak{h}^{\mu\nu} = \sqrt{-\eta}h^{\mu\nu}$ with $\sqrt{-\eta} = 1$. By the definition (2.2.27) for $G^{L}_{\mu\nu}(\mathfrak{h})$, the last term here is a divergence, therefore under the gauge transformations the integrals of motion (3.2.24–3.2.26) acquire surface terms.

A natural requirement is that values of the globally conserved quantities (3.2.24–3.2.26) must be unchanged. In other words, the gauge induced terms in (3.2.38) must not contribute to surface integrals (3.2.24–3.2.26). Below to satisfy this requirement we search for the weakest asymptotic behaviour for functions ξ^{α} and their derivatives as possible.

At first, one has to recall that $G^{L}_{\mu\nu}(\mathfrak{h})$ is invariant under the transformation

$$\mathfrak{h}'^{\mu\nu} = \mathfrak{h}^{\mu\nu} + £_{\xi}\boldsymbol{\eta}^{\mu\nu} = \mathfrak{h}^{\mu\nu} - \xi^{\alpha}\boldsymbol{\eta}^{\mu\nu}{}_{,\alpha} - \xi^{\alpha}{}_{,\alpha}\boldsymbol{\eta}^{\mu\nu} + 2\boldsymbol{\eta}^{\alpha(\mu}\xi^{\nu)}_{,\alpha} \tag{3.2.39}$$

that means:

$$G^L_{\mu\nu}(\pounds_\xi \boldsymbol{\eta}^{\mu\nu}) \equiv 0\,, \tag{3.2.40}$$

which disappears from (3.2.38). It is equivalent to the invariance of (1.5.5) with respect to (1.5.15). Recalling the use of Lorentzian coordinates, one can rewrite (3.2.39) as

$$h'^{\mu\nu} = h^{\mu\nu} - \xi^\alpha{}_{,\alpha}\eta^{\mu\nu} + 2\xi^{(\mu,\nu)}\,. \tag{3.2.41}$$

For arbitrary curved backgrounds such a kind of invariance is presented by the formulae (2.2.78–2.2.81). Thus, anyway, one need not consider the part (3.2.39) (or (3.2.41)) in (3.2.38).

Concerning other terms in (3.2.38), we use the following assumptions and properties:

(i) Dynamical fields $\mathfrak{h}^{\mu\nu}$ and ϕ^A satisfy the Einstein equations, besides they have the *initial* asymptotic behaviour (3.2.28), (3.2.36) and (3.2.37).

(ii) Each of the initial components of the gauge field and their derivatives ξ^α, $\xi^\alpha{}_{,\beta}$, $\xi^\alpha{}_{,\beta\gamma}, \dots$ are arbitrary quantities at every point of the Minkowski space. We use also a symmetry in partial derivatives in order 2 and more, like $\xi^\alpha{}_{,\beta\gamma} = \xi^\alpha{}_{,\gamma\beta}$.

(iii) The quantities ξ^α, $\xi^\alpha{}_{,\beta}$, $\xi^\alpha{}_{,\beta\gamma}, \dots$ transform as tensors under the Poincaré transformations.

(iv) Functions ξ^α are of the class C^∞. Also, we require the Poincaré invariance for the behaviour of the gauge transformed $\mathfrak{h}'^{\mu\nu}$. Then, the behaviour of derivatives of $\mathfrak{h}'^{\mu\nu}$ has to follow the behaviour (3.2.36).

Now we adopt the transformation (2.2.68) for the case of a flat background in Lorentzian coordinates, exclude the terms (3.2.39), owing to the invariance (3.2.40), and present the rest terms in the form:

$$\begin{aligned} h'^{\mu\nu} &= h^{\mu\nu} + \xi\xi_{,\alpha\beta} + \xi_{,\alpha}\xi_{,\beta} + \dots + \xi h_{,\alpha} + \xi_{,\alpha}h \\ &\quad + \xi\xi h_{,\alpha\beta} + \xi\xi_{,\alpha}h_{,\beta} + \xi\xi_{,\alpha\beta}h + \xi_{,\alpha}\xi_{,\beta}h + \dots \\ &\equiv h^{\mu\nu} + \delta_\xi h^{\mu\nu}\,. \end{aligned} \tag{3.2.42}$$

Here, and frequently below, we do not use all the indices, it is permissible owing to the behaviour (3.2.28) and (3.2.36) and to the above requirements (iii) and (iv). After differentiating (3.2.42) one has

$$\begin{aligned} h'^{\mu\nu}{}_{,\alpha} &= h^{\mu\nu}{}_{,\alpha} + \xi\xi_{,\alpha\beta\gamma} + \xi_{,\alpha}\xi_{,\beta\gamma} + \dots \\ &\quad + \xi h_{,\alpha\beta} + \xi_{,\alpha}h_{,\beta} + \xi_{,\alpha\beta}h + \xi\xi h_{,\alpha\beta\gamma} + \xi\xi_{,\alpha}h_{,\beta\gamma} \\ &\quad + \xi\xi_{,\alpha\beta}h_{,\gamma} + \xi_{,\alpha}\xi_{,\beta}h_{,\gamma} + \xi_{,\alpha}\xi_{,\beta\gamma}h + \xi\xi_{,\alpha\beta\gamma}h + \dots \\ &\equiv h^{\mu\nu}{}_{,\alpha} + \left(\delta_\xi h^{\mu\nu}\right)_{,\alpha}\,. \end{aligned} \tag{3.2.43}$$

Next, let us assume a general form for the fall-off of the components of ξ^α:

$$\xi^\alpha = O^-(r^{1-\varepsilon}) + O^+(r^{1-\delta}), \tag{3.2.44}$$

where at the moment ε and δ are not determined. To satisfy the requirement (iv), the behaviour (3.2.44) has to be added by

$$\begin{aligned} \xi^\alpha{}_{,\beta} &= O^+(r^{-\varepsilon}) + O^-(r^{-\delta}), \\ \xi^\alpha{}_{,\beta\gamma} &= O^-(r^{-1-\varepsilon}) + O^-(r^{-1-\delta}), \\ \dots\dots\dots &= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \end{aligned} \tag{3.2.45}$$

From the start, we require that the fall-off (3.2.28) and (3.2.36) are unchanged after the transformations (3.2.42) and (3.2.43) with (3.2.44) and (3.2.45). Thus,

$$h^{\mu\nu} + \delta_\xi h^{\mu\nu} \le O^+(r^{-1}) + O^-(r^{-\beta}), \qquad \beta \ge 2, \tag{3.2.46}$$

from where, keeping in mind the above point (ii), using the inequality (3.2.46), and (3.2.44) with (3.2.45), one obtains

$$\varepsilon \ge 1, \quad \delta \ge \beta \ge 2. \tag{3.2.47}$$

Thus, one concludes that with (3.2.47) the components of $h'^{\mu\nu}$ satisfy (3.2.28) and (3.2.36).

Now, we try to find a maximally weak asymptotic condition for gauge transformations , under which the value of 4-momentum (3.2.24), $\mathscr{P}(\xi^\alpha_K = \xi^\alpha_\beta)$, does not change. It is necessary to assume that the odd part of the gauge variation in integrands decays stronger than r^{-2}, that is

$$O^-\left(\partial_\alpha \delta_\xi h^{\mu\nu}\right) < O^-\left(r^{-2}\right). \tag{3.2.48}$$

Keeping in mind the requirement (ii), consider all the terms of the type $\xi\xi_{,\alpha\beta\gamma}$ in (3.2.48) as independent ones. Then, with (3.2.44) and (3.2.45) owing to the requirement (iv), one has the asymptotic behaviour,

$$\xi\xi_{,\alpha\beta\gamma} = O^-(r^{-1-2\varepsilon}) + O^-(r^{-1-2\delta}) + O^-(r^{-1-\varepsilon-\delta}), \tag{3.2.49}$$

using (3.2.48) gives the restriction:

$$\varepsilon > \tfrac{1}{2}, \quad \delta > \tfrac{1}{2}. \tag{3.2.50}$$

Using (3.2.50), one concludes that all the gauge terms in (3.2.43) do not contribute to $\mathscr{P}(\xi^\alpha_K = \xi^\alpha_\beta)$ either.

Lastly, we will find a maximally weak asymptotic behaviour for gauge transformations, under which the value of 4-angular momentum (3.2.25) and (3.2.26), $\mathscr{P}(\xi^\alpha_K = \xi^\alpha_{[\alpha\beta]})$, does not change. We require once again that the odd part of gauge variation in integrands of (3.2.25) and (3.2.26) falls stronger than r^{-2}. Firstly, we analyze the contribution from the terms without $h^{\mu\nu}$ and their derivatives in the items $(\delta_\xi h^{\mu\nu})_{,\alpha}x^k$ and $\delta_\xi h^{\mu\nu}$. Consider, as independent quantities, the terms of the types $x^k\xi\xi_{,\alpha\beta\gamma}$, $x^k\xi\xi_{,\alpha\beta\gamma\delta}$, ... and $\xi\xi_{,\alpha\beta}$, $\xi\xi_{,\alpha\beta\gamma}$, ... Finally, one obtains the restrictions

$$\varepsilon + \delta > 2\,, \quad \varepsilon \geq 0\,, \quad \delta > \tfrac{2}{3}\,. \tag{3.2.51}$$

With these restrictions all the other terms without $h^{\mu\nu}$ and their derivatives in the expressions of the type $(\delta_\xi h^{\mu\nu})_{,\alpha}x^k$ and $\delta_\xi h^{\mu\nu}$ do not contribute to $\mathscr{P}(\xi^\alpha_K = \xi^\alpha_{[\alpha\beta]})$ either. Next, requirements (i) and (ii) applied to the terms of the type $x^k\xi h_{,\alpha}$ and $x^k\xi h_{,\alpha\beta}$ lead to the restrictions $\delta > 1$, $\varepsilon > 2-\beta$, $\beta \geq 2$, combination of which with (3.2.51) gives

$$\varepsilon + \delta > 2\,, \quad \delta > 1\,, \quad \varepsilon \geq 0\,, \quad \beta > 2 \quad (\text{or } \varepsilon > 0\,, \ \beta = 2)\,. \tag{3.2.52}$$

Under these restrictions all the other terms in gauge transformations with $h^{\mu\nu}$ and their derivatives do not contribute to $\mathscr{P}(\xi^\alpha_K = \xi^\alpha_{[\alpha\beta]})$ either. Concluding, we note that for analyzing the gauge invariance of the integrals of motion, it was enough to consider the gauge transformations up to the second order in ξ^α only.

Combining (3.2.50) and (3.2.52), one obtains the unified restriction on the behaviour (3.2.44):

$$\varepsilon + \delta > 2\,, \quad 1 \geq \varepsilon > \tfrac{1}{2}\,, \quad \delta > 1\,. \tag{3.2.53}$$

Finally, one has the transformed behaviour

$$h'^{\mu\nu} = O^+(r^{-\varepsilon}) + O^-(r^{-\delta}) \tag{3.2.54}$$

instead of (3.2.28), with a corresponding behaviour for derivatives satisfying the Poincaré invariance.

Let us make some remarks.

First, the condition $\varepsilon \leq 1$ expresses the fact that in a real isolated system the fall-off of the gravitational potentials in (3.2.54) cannot be stronger than the Newtonian one.

Second, the gauge transformations (3.2.42) with the restrictions (3.2.53), $h''^{\mu\nu} = h'^{\mu\nu} + \delta_\xi h'^{\mu\nu}$, do not change the behaviour (3.2.54).

Third, the results (3.2.54) with (3.2.53) obtained in the framework of the field-theoretical approach makes the previous results [18, 31, 51, 52, 102, 103, 160, 186, 254, 347, 387, 413, 418, 428, 454] more precise. Only the result in the Soloviev work [418] almost coincides with the result (3.2.54) plus (3.2.53), although it has been obtained, in principle, in another way. The difference is that in [418] the result (3.2.54) plus (3.2.53) is augmented by the condition $|\varepsilon - \delta| \leq 1$. It is not correct. Indeed, for the usual Schwarzschild solution one has $\varepsilon = 1$, $\delta = \infty$ that contradicts to the additional condition.

3.2.5 Concluding remarks

First, it is not difficult to show that the definitions of the integrals of motion (3.2.24) and (3.2.25) coincide with the corresponding definitions (3.1.148), (3.1.151) and (3.1.152) introduced by Regge and Teitelboim. At the same time, the integral (3.2.26) differs from the Regge-Teitelboim integral (3.1.153). The reason is that (3.2.26) is the Lorentzian integral, whereas (3.1.153) is the center of mass integral only. How could one reconcile the difference between them? For this, let us define the shift vector in a more general (more complete) form:

$$N_*^i = \alpha^i + \beta^i{}_r x^r + \beta^i{}_\perp x^0 + O^-(1) + O(r^{-1}) \tag{3.2.55}$$

instead of (3.1.150). Then, following to the Regge and Teitelboim technique one obtains

$$M^*_{\perp r} = \oint_\infty ds_l \left[x^r \left(g_{sl,s} - g_{ss,l}\right) - g_{rl} + \eta_{rl} g_{ss} + 2x^0 \pi_{rl}\right] \tag{3.2.56}$$

instead of (3.1.153), and that coincides with (3.2.26). The difference between (3.1.153) and (3.2.56) is essential because one can show that $\dot{M}^*_{\perp r} = 0$ on the field equations, whereas $\dot{M}_{\perp r} \neq 0$. A special attention has been paid to this problem in [429], where a necessity to include a time dependent term, like in (3.2.55), is discussed.

Second, deformations (3.1.157) with (3.1.158) are defined by coordinate transformations and transferring from slices $x^0 = \text{const}$ in an initial frame to slices $x'^0 = \text{const}$ in a final frame. Then, the use of the interpretation of coordinate transformations as gauge transformations in the field-theoretical formulation permits one to conclude that deformations and gauge transformations are in one-to-one correspondence.

Third, in the framework of the field-theoretical approach we need not work in the Regge-Teitelboim variational principle. Indeed, varying the *dynamical* action with Lagrangians, the asymptotic behaviour defined by (3.2.16) and (3.2.17), one obtains the Einstein equations *without* additional requirements.

Fourth, in the Regge and Teitelboim derivation it is not so simple to show that (3.1.148) and (3.1.151) is a 4-vector, and (3.1.152) and (3.1.153) is a 4-tensor under asymptotic Poincaré transformations. The field-theoretical approach significantly simplifies the situation: the integrals (3.2.24–3.2.26) are tensors under the Poincaré transformations by the construction.

3.3 An isolated system in the Hamiltonian description

In the present section, based on the results of the previous section, we develop the Hamiltonian formulation of an asymptotically flat spacetime in the framework of the field-theoretical formulation of general relativity. We demonstrate the advantages of the field-theoretical approach; besides, we present some original results related to important outstanding problems such as the super-translation invariance. We follow the works [205, 361], although, in the book [193] one can find similar elements.

3.3.1 The difference between the canonical and symmetric currents

Because we study *real* isolated systems their conserved characteristics have to be the same in various formulations. Usually, for real physical systems, canonical and Hamiltonian conserved quantities are identical, see also below (3.3.28) and (3.3.29). Therefore, to compare the Hamiltonian conserved quantities with the symmetric ones, one has to compare symmetric and canonical currents, which differ in divergences. As a result, global conserved quantities differ one from another by surface integrals. To obtain the same values for global conserved quantities in both the cases one has to restrict an asymptotic behaviour of field potentials guaranteeing the disappearance of the additional surface integrals.

To provide the comparison we rely on the formulae of a field theory with a Lagrangian $\mathscr{L} = \mathscr{L}(\psi)$ in the Minkowski space derived in Section 1.2. The difference between the two types of the currents is presented in (1.2.135) where the canonical and symmetrical currents are defined in (1.2.101) and (1.2.132), respectively. The difference between corresponding conserved global quantities is given in (1.2.136):

$$\Delta\mathscr{P}(\xi_K) = \mathscr{P}_C(\xi_K) - \mathscr{P}_S(\xi_K) = -\oint_\infty ds_i \boldsymbol{b}^{0i}{}_\sigma \xi^\sigma_K, \tag{3.3.1}$$

where $\boldsymbol{b}^{\alpha\beta}{}_\sigma$ is defined in (1.2.113) with (1.2.103):

$$\boldsymbol{b}^{\alpha\beta\gamma} = \boldsymbol{\sigma}^{\gamma[\alpha\beta]} + \boldsymbol{\sigma}^{\alpha[\gamma\beta]} - \boldsymbol{\sigma}^{\beta[\gamma\alpha]}, \tag{3.3.2}$$

$$\boldsymbol{\sigma}^{\mu\beta}{}_\sigma \equiv -\frac{\partial\mathscr{L}}{\partial\left(\psi^A{}_{,\mu}\right)} \psi^A\Big|^\beta_\sigma, \tag{3.3.3}$$

and the Killing vectors, ξ^α_K, are given in (1.1.72). Thus, global conserved symmetrical and canonical integrals are equal, if integral (3.3.1) vanishes.

Let the role of the field theory with the Lagrangian $\mathscr{L} = \mathscr{L}(\psi)$ be played by the field-theoretical formulation of general relativity in the Minkowski space. For the convenience of presentation, let us reconsider the dynamical Lagrangian in the generalized variables ψ^A:

$$\mathscr{L}^{\text{dyn}} = -\frac{1}{16\pi}\mathscr{L}^g + \mathscr{L}^m \equiv \mathscr{L}(\psi), \tag{3.3.4}$$

where now $\psi^A = \{\gamma^{\mu\nu}, \phi^B, \mathfrak{h}^{\mu\nu}\}$. We consider a pure gravitational Lagrangian, $\mathscr{L}^g$, in a more preferable form (2.2.20):

$$\mathscr{L}^g = -(\Delta^\rho{}_{\mu\nu} - \Delta^\sigma{}_{\mu\sigma}\delta^\rho_\nu)\mathfrak{h}^{\mu\nu}{}_{;\rho} + (\gamma^{\mu\nu} + \mathfrak{h}^{\mu\nu})\left(\Delta^\rho{}_{\mu\nu}\Delta^\sigma{}_{\rho\sigma} - \Delta^\rho{}_{\mu\sigma}\Delta^\sigma{}_{\rho\nu}\right) \tag{3.3.5}$$

with only the *first* derivatives of $\gamma^{\mu\nu}$ and $\mathfrak{h}^{\mu\nu}$. We consider Lagrangian of the matter sources (2.2.21) in the simplest form (without derivatives of $(\gamma^{\mu\nu} + \mathfrak{h}^{\mu\nu})_{,\alpha}$) in the Minkowski background:

$$\mathscr{L}^m = \mathscr{L}^m(\gamma^{\mu\nu} + \mathfrak{h}^{\mu\nu}; \phi^A, \phi^A{}_{,\alpha}) . \tag{3.3.6}$$

In the other case, additional problems with defining generalized momenta appear, and we do not discuss them here.

Then, the spin density (3.3.3) is separated into the gravitational and the matter parts,

$${}^g\boldsymbol{\sigma}_\sigma{}^{\alpha\beta} = \frac{1}{16\pi}\left(\frac{\partial\mathscr{L}^g}{\partial\mathfrak{h}^{\mu\nu}{}_{,\alpha}}\,\mathfrak{h}^{\mu\nu}\big|^\beta_\sigma + \frac{\partial\mathscr{L}^g}{\partial\gamma^{\mu\nu}{}_{,\alpha}}\,\gamma^{\mu\nu}\big|^\beta_\sigma\right)\bigg|_{\gamma^{\mu\nu}\to\eta^{\mu\nu}} , \tag{3.3.7}$$

$${}^m\boldsymbol{\sigma}_\sigma{}^{\alpha\beta} = \left(-\frac{\partial\mathscr{L}^m}{\partial\phi^A{}_{,\alpha}}\,\phi^A\big|^\beta_\sigma\right)\bigg|_{\gamma^{\mu\nu}\to\eta^{\mu\nu}} , \tag{3.3.8}$$

in the Lorentzian coordinates. After a direct calculation for the gravitational Lagrangian (3.3.5) one has

$$\begin{aligned}{}^g\boldsymbol{\sigma}_\sigma{}^{\alpha\beta} = \frac{\sqrt{-\eta}}{8\pi}\Big(&\Delta^{[\alpha}{}_{\rho\mu}\eta^{\beta]\rho}h^\mu_\sigma - \Delta^{[\alpha}{}_{\sigma\mu}h^{\beta]\mu}\\ &+ \Delta^\pi{}_{\rho\mu}h^{\mu[\alpha}\eta^{\beta]\rho}\eta_{\pi\sigma} - \Delta^\mu{}_{\rho\mu}h^{\rho[\alpha}\delta^{\beta]}_\sigma - \Delta^\mu{}_{\rho\mu}h^{[\alpha}_\sigma\eta^{\beta]\rho}\Big) .\end{aligned} \tag{3.3.9}$$

When the Lorentzian coordinates are used one has $\Delta^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} \sim h_{,\alpha}$ asymptotically. Then, taking into account the weakest fall-off for the field potentials (3.2.54) with (3.2.53), one finds for the asymptotic behaviour of the gravitational part of (3.3.2):

$${}^g\boldsymbol{b}^{\alpha\beta}{}_\sigma = O^-\left(r^{-1-2\varepsilon}\right) + O^+\left(r^{-1-\varepsilon-\delta}\right) . \tag{3.3.10}$$

It is easy to check that for all the 10 Killing vectors (1.1.72) of the Minkowski space the quantity (3.3.10) does not contribute to integral (3.3.1).

Unlike the gravitational part, the situation with the matter part of (3.3.2) requires additional restrictions. Already, analogous problems were analyzed in [51, 413, 429]. Let us turn to the matter spin density (3.3.8) and construct the corresponding part, ${}^m\boldsymbol{b}^{\alpha\beta}{}_\sigma$, of the quantity (3.3.2). Requiring the Poincaré invariance and applying general results (3.2.30–3.2.34), one recalls that every differentiation makes stronger the fall-off by the factor r^{-1} with an opposite parity. Then, for the matter Lagrangian with the fall-off (3.2.3) one has

$${}^m\boldsymbol{b}^{\alpha\beta}{}_\sigma = O\left(r^{-2-\alpha}\right), \quad \alpha > 0 . \tag{3.3.11}$$

As is seen, for the rotational Killing vectors in (1.1.72) the behaviour (3.3.11) does not lead to vanishing the difference (3.3.1). Analyzing the integral (3.3.1), one finds a necessary (stronger) fall-off for its vanishing:

$$ {}^{m}\boldsymbol{b}^{\alpha\beta}{}_{\sigma} = O^{-}\left(r^{-2-\alpha}\right) + O^{+}\left(r^{-3-\alpha}\right), \quad \alpha > 0\,. \tag{3.3.12}$$

By inverse exercises, one finds that a corresponding fall-off for the matter Lagrangian has to be

$$ \mathscr{L}^{m} = O^{+}\left(r^{-3-\alpha}\right) + O^{-}\left(r^{-4-\alpha}\right), \quad \alpha > 0\,. \tag{3.3.13}$$

We stress that a specification (3.3.13) with respect to the behaviour (3.2.17) of the matter Lagrangian is induced by the requirement to have a vanishing difference between canonical and symmetrical integrals of motion. Of course, if the matter is absent at infinity, one needs no restriction in (3.3.13) at all.

3.3.2 Phase variables and their asymptotic behaviour

To transfer from the Lagrangian formulation to the Hamiltonian formulation one has to start from the (3 + 1)-splitting of a flat background spacetime. First, we define Σ in the Lorentzian coordinates; second, we project fields $\mathfrak{h}^{\mu\nu}$ and ϕ^{A} onto Σ by the usual way, see (3.1.14–3.1.46). Thus,

$$ \mathfrak{h}^{\mu\nu} \to \mathfrak{h}^{\mathrm{ab}} = \{\mathfrak{h}^{\perp\perp},\, \mathfrak{h}^{\perp i},\, \mathfrak{h}^{ij}\}\,; \quad \phi^{A} \to \phi^{\mathrm{A}}\,, \tag{3.3.14}$$

where $\mathfrak{h}^{\mathrm{ab}}$ are the 3-dimensional densities of weight +1 on Σ: $\mathfrak{h}^{\mathrm{ab}} = \sqrt{-\eta}h^{\mathrm{ab}}$ After transformations (3.3.14), the Lagrangian (3.3.4) is transformed into the (3 + 1)-splitting form also: $\mathscr{L}^{g} \to \mathscr{L}^{g}(\boldsymbol{q}^{ij}, q^{\mathrm{a}})$ and $\mathscr{L}^{m} \to \mathscr{L}^{m}(\boldsymbol{q}^{ij}, q^{\mathrm{a}}, \phi^{\mathrm{A}})$. After that, for the convenience, we redefine variables:

$$ \boldsymbol{q}^{ij} \equiv \mathfrak{h}^{\perp i}\mathfrak{h}^{\perp j} - \left(\boldsymbol{\eta}^{\perp\perp} + \mathfrak{h}^{\perp\perp}\right)\left(\boldsymbol{\eta}^{ij} + \mathfrak{h}^{ij}\right), \tag{3.3.15}$$

$$ q^{\mathrm{a}} \equiv \frac{\mathfrak{h}^{\perp\mathrm{a}}}{\boldsymbol{\eta}^{\perp\perp} + \mathfrak{h}^{\perp\perp}}\,, \tag{3.3.16}$$

where $\boldsymbol{q}^{ij} = (-\eta)q^{ij}$ is the 3-dimensional tensor density of weight +2. Below, we will frequently use the notations h^{ab} and q^{ij} instead of $\mathfrak{h}^{\mathrm{ab}}$ and $\boldsymbol{q}^{ij}$ because the Lorentzian coordinates are used.

To transfer to the Hamiltonian formulation let us consider the variables $\boldsymbol{q}^{ij}$, q^{a}, and ϕ^{A} as *generalized coordinates*. Then, non-zero *generalized momenta* are defined as usual

$$ \mathscr{K}_{ij} \equiv -\frac{1}{16\pi}\frac{\delta\mathscr{L}^{g}(\boldsymbol{q}^{ij}, q^{\mathrm{a}})}{\delta\dot{\boldsymbol{q}}^{ij}}\,, \quad \pi_{\mathrm{A}} \equiv \frac{\delta\mathscr{L}^{m}(\phi^{\mathrm{A}})}{\delta\dot{\phi}^{\mathrm{A}}} \tag{3.3.17}$$

where $\mathscr{K}_{ij}$ is a 3-dimensional tensor density of weight −1. One can easily check that

$$\mathbf{q}^{ij} \equiv (-g)\overset{3}{g}{}^{ij}\,, \tag{3.3.18}$$

$$\mathscr{K}_{ij} \equiv -\frac{1}{16\pi}\frac{1}{\sqrt{-g}}K_{ij}\,, \tag{3.3.19}$$

where $\overset{3}{g}_{ij}$ and K_{ij} are the intrinsic metric and extrinsic curvature in the framework of the standard ADM (3 + 1)-decomposition, respectively, see Section 3.1.

Now, exchange generalized velocities $\dot{\mathbf{q}}^{ij}$ and $\dot{\phi}^{\mathsf{A}}$ by the generalized momenta in the (3 + 1)-redefined Lagrangian

$$\mathscr{L}^{\text{dyn}}(\mathbf{q}, \phi^{\mathsf{A}}) = -\frac{1}{16\pi}\mathscr{L}^{g}(\mathbf{q}^{ij}, q^{\mathsf{a}}) + \mathscr{L}^{m}(\mathbf{q}^{ij}, q^{\mathsf{a}}, \phi^{\mathsf{A}})\,. \tag{3.3.20}$$

Then, by the standard way, one achieves the Hamiltonian action:

$$S = \int dt \int d^3x \left\{\mathscr{K}_{ij}\dot{\mathbf{q}}^{ij} + \pi_{\mathsf{A}}\dot{\phi}^{\mathsf{A}} \right.$$
$$\left. - \{1\}\cdot\left[(q^{\perp}-1)(\mathscr{H}^{g}+\mathscr{H}^{m}) + q^{i}(\mathscr{H}^{g}_{i}+\mathscr{H}^{m}_{i})\right] - \mathscr{D}^{i}{}_{,i}\right\}\,, \tag{3.3.21}$$

where

$$\mathscr{H}^{g} \equiv \frac{16\pi}{\sqrt{-\eta}}\left[\mathbf{q}^{ij}\mathbf{q}^{kl}\left(\mathscr{K}_{ik}\mathscr{K}_{jl} - \mathscr{K}_{ij}\mathscr{K}_{kl}\right) - \frac{1}{16\pi}\mathbf{q}^{ij}\overset{3}{R}_{ij}\right]\,, \tag{3.3.22}$$

$$\mathscr{H}^{g}_{i} \equiv -\mathbf{q}^{kl}{}_{,i}\mathscr{K}_{kl} - 2\left(\mathbf{q}^{kl}\mathscr{K}_{ik}\right)_{,l} + 2\left(\mathbf{q}^{kl}\mathscr{K}_{kl}\right)_{,i}\,, \tag{3.3.23}$$

$\overset{3}{R}_{ij}$ is the 3-dimensional Ricci tensor constructed with the use of the 3-dimensional metric density $\mathbf{q}^{ij}$. Keeping in mind (3.3.18) and (3.3.19), one finds easily that (3.3.22) and (3.3.23) coincide exactly with (3.1.79) together with (3.1.59), and (3.1.80) together with (3.1.60), respectively.

As is seen, the quantities q^{a} in the action (3.3.21) have the sense of *Lagrangian multipliers*, variations with respect to which give the constraints:

$$\mathscr{H}^{g+m} = \mathscr{H}^{g} + \mathscr{H}^{m} = 0\,, \tag{3.3.24}$$

$$\mathscr{H}^{g+m}_{i} = \mathscr{H}^{g}_{i} + \mathscr{H}^{m}_{i} = 0\,. \tag{3.3.25}$$

Taking into account (3.3.18) and (3.3.19), one finds that (3.3.24) and (3.3.25) coincide with (3.1.85) and (3.1.86). However, observe that $q^{\perp}$ and q^{i} are not the lapse function and the shift vector. In the present discussion, when Σ is defined by $x^0 = t =$ const in the Lorentzian coordinates, the lapse function $\bar{N} = 1$ and the shift vector $\bar{N}^i = 0$ are constant quantities only. There is no variation with respect to $\bar{N}$ and $\bar{N}^i$ that is expressed by a special unit in front of the square brackets in (3.3.21). Quantities $q^{\perp}$ and q^{i} are analogous to the component A^0 that is the Lagrangian multiplier in the Hamiltonian description of electrodynamics [315]. Thus, the Lagrangian in (3.3.21) is a singular Lagrangian in the sense of the Dirac constraint algebra [140] in an arbitrary field theory. Thus, all of Dirac's relevant techniques can be applied to the system (3.3.21) directly.

The transformations (3.3.14–3.3.17), of course, can be interpreted as a simple redefinition of variables [379]. Then, the fall-off of the phase variables is fully determined by (3.2.54) with (3.2.53). As a result one has

$$\begin{aligned} q^{ij} &= \eta^{ij} + O^+\left(r^{-\varepsilon}\right) + O^-\left(r^{-\delta}\right), \\ q^{ij}{}_{,\alpha} &= O^-\left(r^{-1-\varepsilon}\right) + O^+\left(r^{-1-\delta}\right), \\ q^{\mathsf{a}} &= O^+\left(r^{-\varepsilon}\right) + O^-\left(r^{-\delta}\right), \\ q^{\mathsf{a}}{}_{,\alpha} &= O^-\left(r^{-1-\varepsilon}\right) + O^+\left(r^{-1-\delta}\right). \end{aligned} \tag{3.3.26}$$

$$\begin{aligned} \mathscr{K}_{ij} &= O^-\left(r^{-1-\varepsilon}\right) + O^+\left(r^{-1-\delta}\right), \\ \mathscr{K}_{ij,\alpha} &= O^+\left(r^{-2-\varepsilon}\right) + O^-\left(r^{-2-\delta}\right). \end{aligned} \tag{3.3.27}$$

Recall that every differentiation makes the fall-off stronger by the factor r^{-1} with an opposite parity.

3.3.3 Global conserved integrals

Now we construct conserved integrals in the Hamiltonian formulation. Here, it is convenient to rewrite the canonical current (1.2.101) with the Lagrangian (3.3.4) in the convenient form:

$$\mathscr{I}_C^\mu(\xi_K) = {}_C\boldsymbol{\theta}_\alpha{}^\mu \xi_K^\alpha + \boldsymbol{\sigma}^{\mu\beta}{}_\sigma \partial_\beta \xi_K^\sigma = -\frac{\delta\mathscr{L}(\psi,\eta)}{\delta\psi^A{}_{,\mu}} £_{\xi_K}\psi^A - \mathscr{L}(\psi,\eta)\xi_K^\mu. \tag{3.3.28}$$

After a one-to-one redefinition (3.3.14–3.3.16) we derive the zero component (the only necessary in the integration, see (1.2.87)) of (3.3.28):

$$\mathscr{I}_C^0(q,\ \phi^{\mathsf{A}};\ \xi_K) = \frac{1}{16\pi}\frac{\delta\mathscr{L}^g(q)}{\delta\dot{\boldsymbol{q}}^{ij}} £_{\xi_K}\boldsymbol{q}^{ij} - \frac{\delta\mathscr{L}^m(\phi^{\mathsf{A}})}{\delta\dot{\phi}^{\mathsf{A}}} £_{\xi_K}\phi^{\mathsf{A}} - \mathscr{L}^{dyn}(q,\ \phi^{\mathsf{A}})\xi_K^0\,. \tag{3.3.29}$$

At last, changing generalized velocities by the generalized momenta (3.3.16), one rewrites (3.3.29) through the phase variables of the Hamiltonian action (3.3.21):

$$\mathscr{I}_C^0(q,\mathscr{K};\ \phi^{\mathsf{A}},\pi_{\mathsf{A}};\ \xi_K) = -\mathscr{K}_{ij} £_{\xi_K}\mathbf{q}^{ij} - \pi_{\mathsf{A}} £_{\xi_K}\phi^{\mathsf{A}} - \mathscr{L}^{dyn}(q,\mathscr{K};\ \phi^{\mathsf{A}},\pi_{\mathsf{A}})\xi_K^0\,. \tag{3.3.30}$$

The same as in (1.2.87), one constructs the conserved integrals based on (3.3.30)

$$\mathscr{P}(\xi_K) = \int_\Sigma d^3x\, \mathscr{I}_C^0(q,\mathscr{K};\ \phi^{\mathsf{A}},\pi_{\mathsf{A}};\ \xi_K)\,. \tag{3.3.31}$$

Due to the behaviour (3.3.13) for the matter Lagrangian fall-off one concludes that for a real isolated system integrals (3.3.31) coincide with the integrals (3.2.20) in the Lagrangian formulation.

Substituting concrete Killing vectors from (1.1.72) into (3.3.30) one obtains a density for calculating the corresponding conserved integral. Thus, for the timelike Killing vector $\xi^\alpha_K = -\delta^\alpha_0$ one gets

$$\mathscr{J}^0(\xi^\alpha_0) = -\left[\mathscr{K}_{ij}\dot{\boldsymbol{q}}^{ij} + \pi_{\mathrm{A}}\dot{\phi}^{\mathrm{A}} - \mathscr{L}^{dyn}\right] = -\left[(q^\perp - 1)\left(\mathscr{H}^g + \mathscr{H}^m\right) + q^i\left(\mathscr{H}^g_i + \mathscr{H}^m_i\right) + \partial_i\mathscr{D}^i\right], \tag{3.3.32}$$

which is exactly the Hamiltonian in the action (3.3.21). By the constraints (3.3.24) and (3.3.25), only the divergence in (3.3.32) can contribute to (3.3.30). It is decomposed into a pure gravitational and matter parts, thus, $\mathscr{D}^i \equiv {}^g\mathscr{D}^i + {}^m\mathscr{D}^i$. Using the asymptotic behaviour (3.3.13) one finds for the matter part,

$${}^m\mathscr{D}^i = O^-\left(r^{-2-\alpha}\right) + O^+\left(r^{-3-\alpha}\right), \quad \alpha > 0\,, \tag{3.3.33}$$

from where it follows that $\partial_i({}^m\mathscr{D}^i)$ does not contribute to (3.3.31). The gravitational part,

$${}^g\mathscr{D}^i = -\frac{1}{16\pi}\left[(\sqrt{-\eta})^{-1}\left(q^\perp - 1\right)\boldsymbol{q}^{ij}{}_{,j} + \sqrt{-\eta}\left(q^i q^j{}_{,j} - q^j q^i{}_{,j} + \frac{q^i\dot{q}^\perp}{-1+q^\perp}\right)\right] + 2\mathscr{K}_{jk}\left(q^j\boldsymbol{q}^{ik} - q^i\boldsymbol{q}^{jk}\right), \tag{3.3.34}$$

requires more scrutiny. Taking into account (3.3.26) and (3.3.27), one finds that (3.3.31) gives the total energy:

$$E = \mathscr{P}^{(0)}_H = \int_\Sigma d^3x\,\mathscr{J}^0(\xi^\alpha_0) = -\frac{1}{16\pi}\oint_\infty ds_i\,(\sqrt{-\eta})^{-1}\boldsymbol{q}^{ij}{}_{,j}\,. \tag{3.3.35}$$

Restoring "$1/16\pi$" ahead of the ADM energy integral (3.1.145), recalling the definitions (3.3.15) and (3.3.18) for $\boldsymbol{q}^{ij}$ and providing a careful comparison with the use of relations like $g_{mn,k} = -g_{mi}g_{mj}\overset{3}{g}{}^{ij}{}_{,k}$, one finds that the ADM energy integral E in (3.1.145) and the field-theoretical energy integral E in (3.3.35) coincide.

Now, we describe Hamiltonian 3-dimensional momentum and angular momentum. After excluding $\xi^0_K = \xi^0_0$ and $\xi^\alpha_K = \xi^\alpha_{[0m]}$ from (1.1.72) the rest of the Killing vectors are denoted as

$$\xi^{*\alpha}_K = \xi^\alpha_k,\ \xi^\alpha_{[mn]}\,. \tag{3.3.36}$$

We do not consider the integral of the center of mass here because nuances appear (they are discussed at the end of the previous section), and the analysis becomes very cumbersome. The reader can refer to the paper [32] for a more detail. Substituting (3.3.36) into (3.3.30) one obtains the density corresponding to its own Killing vector:

$$\begin{aligned}\mathscr{I}_C^0(q,\mathscr{K};\ \phi^{\mathsf{A}},\pi_{\mathsf{A}};\ \xi_K^*) &= -\mathscr{K}_{ij}\pounds_{\xi_K^*}q^{ij} - \pi_{\mathsf{A}}\pounds_{\xi_K^*}\phi^{\mathsf{A}}\\ &= \left[\ \mathscr{K}_{ij}q^{ij}{}_{,k} - 2\left(\mathscr{K}_{ij}q^{ij}\right)_{,k} + 2\left(\mathscr{K}_{ik}q^{ij}\right)_{,j} + \pi_{\mathsf{A}}\phi^{\mathsf{A}}{}_{,k} + \left(\pi_{\mathsf{A}}\ \phi^{\mathsf{A}}\big|^i_k\right)_{,i}\right]\xi_K^{*k}\\ &\quad + \left[2\mathscr{K}_{ij}q^{ij}\xi_K^{*k} - 2\mathscr{K}_{ij}q^{ik}\xi_K^{*j} - \pi_{\mathsf{A}}\ \phi^{\mathsf{A}}\big|^k_i\ \xi_K^{*i}\right]_{,k}.\end{aligned} \tag{3.3.37}$$

Now let us turn to the asymptotic behaviour of the matter Lagrangian (3.3.13) and find

$$\begin{aligned}\pi_{\mathsf{A}}\ \phi^{\mathsf{A}}\big|^i_j &= O^-\left(r^{-2-\alpha}\right) + O^+\left(r^{-3-\alpha}\right), \quad \alpha > 0\\ \pi_{\mathsf{A}}\phi^{\mathsf{A}}{}_{,k} &= O^+\left(r^{-3-\alpha}\right) + O^-\left(r^{-4-\alpha}\right).\end{aligned} \tag{3.3.38}$$

Keeping this in mind and taking into account the constraint (3.3.25), one finds that second line in (3.3.37) does not contribute to integral (3.3.31). Now, one can assign a role to the behaviour (3.3.13). It is also quite important for deriving well-defined conserved integrals themselves in the Hamiltonian formulations.

Finally, substituting (3.3.37) into (3.3.31), one obtains for the total Hamiltonian 3-dimensional momentum and angular momentum:

$$\mathscr{P}_H^{(k)} = \oint_\infty ds_i P^{ik}, \tag{3.3.39}$$

$$\mathscr{P}_H^{([mn])} = \frac{1}{2}\oint_\infty ds_i\left(x^m P^{in} - x^n P^{im}\right), \tag{3.3.40}$$

where

$$P^l{}_k \equiv 2\left(\delta^l_k\mathscr{K}_{ij}\boldsymbol{q}^{ij} - \mathscr{K}_{ik}\boldsymbol{q}^{il}\right). \tag{3.3.41}$$

Restoring “1/16π” ahead of π^{ij} in (3.1.50), recalling the definition (3.3.19) for $\mathscr{K}^{ij}$, one can easily recognize that integrals (3.3.39) and (3.3.40) coincide with the corresponding ADM integrals (3.1.151) and (3.1.152)[2] derived in Section 3.1.

Here, we use the Faddeev [160] “geometrical” phase derivatives q^{ij} and $\mathscr{K}_{ij}$. Nevertheless, there is a principal difference between the status of q^{ij} and $\mathscr{K}_{ij}$ in various derivations, like geometrical and field-theoretical ones. In the first case, they are the metric on spacelike sections and the corresponding external curvature tensor. In the

2 Notice that, in fact, there exists a difference in the coefficient “1/2” (3.3.40) and (3.1.152). It is a question of a convention, namely, the coefficient “1/2” is an external one in (3.1.156), whereas it is included in (3.3.40) with the rotational Killing vector. Therefore, in our opinion, the choice (3.3.40) is more acceptable.

second case, q^{ij} and $\mathscr{K}_{ij}$, are fields propagating in space of flat hypersurfaces Σ as in an arbitrary field theory.

Now, let us compare *explicitly* the global conserved quantities obtained in the Hamiltonian formulation (3.3.35), (3.3.39) and (3.3.40) with the corresponding Lagrangian global integrals (3.2.24) and (3.2.25). For the sake of convenience, at the moment we rewrite the latter in the form:

$$\mathscr{P}_S^{(0)} = \frac{1}{16\pi}\oint_\infty ds_i P^{(0)i}\,, \tag{3.3.42}$$

$$\mathscr{P}_S^{(k)} = \frac{1}{16\pi}\oint_\infty ds_i P^{(k)i}\,, \tag{3.3.43}$$

$$\mathscr{P}_S^{([mn])} = \frac{1}{32\pi}\oint_\infty ds_i P^{[mn]i}\,. \tag{3.3.44}$$

Comparing the integrand in (3.3.35), (3.3.39) and (3.3.40) with those in (3.3.42), (3.3.43) and (3.3.44) one finds, respectively,

$$-q^{ij}{}_{,j} = P^{(0)i} + O^-\left(r^{-1-2\varepsilon}\right) + O^+\left(r^{-1-\varepsilon-\delta}\right), \tag{3.3.45}$$

$$16\pi\eta^{kl}P^i{}_l = P^{(k)i} + 2\left(\eta^{k[i}h_0^{j]}\right)_{,j} + O^-\left(r^{-1-2\varepsilon}\right) + O^+\left(r^{-1-\varepsilon-\delta}\right), \tag{3.3.46}$$

$$32\pi x^{[m}\eta^{n]l}P^i{}_l = P^{([mn])i} + 2\left(x^m\eta^{n[i}h_0^{j]} - x^n\eta^{m[i}h_0^{j]}\right)_{,j} + O^-\left(r^{-\varepsilon-\delta}\right) + O^+\left(r^{-2\varepsilon}\right)\,. \tag{3.3.47}$$

By the asymptotic behaviour in (3.3.45–3.3.47), and, using the Stokes theorem, one is convinced that, indeed, the Hamiltonian integrals (3.3.35), (3.3.39) and (3.3.40) are equal to the Lagrangian integrals (3.3.42), (3.3.43) and (3.3.44), respectively.

3.3.4 Gauge invariance of global integrals

To obtain gauge transformations for the phase space variables we use the gauge transformations of the field variables in the Lagrangian description. At first, using projections, like (3.3.14) and redefinitions (3.3.15) and (3.3.16), we transform the 4-dimensional components linear in ξ^α and quadratic in ξ^α and $\mathfrak{h}^{\mu\nu}$ in (3.2.42) and (3.2.43) into 3-dimensional components on Σ:

$$\boldsymbol{q}'^{ij} = \boldsymbol{q}^{ij} + \delta_\xi \boldsymbol{q}^{ij} = \boldsymbol{q}^{ij} + \pounds_{\xi^k}\boldsymbol{q}^{ij} + \xi^\perp\delta_\perp\boldsymbol{q}^{ij} + 2\xi^\perp{}_{,k}\left(q^k\boldsymbol{q}^{ij} - \boldsymbol{q}^{k(i}q^{j)}\right)\,. \tag{3.3.48}$$

Here, $\xi^a = \{\xi^\perp, \xi^k\}$ are gauge functions; $\delta_\perp$ is the Lie derivative along the unique normal to flat sections Σ [315], see (3.1.32). After the standard variation of the Hamiltonian action (3.3.21) with respect to $\mathscr{K}_{ij}$ one obtains the Hamiltonian equations,

$$\delta_{\perp} q^{ij} = -\frac{\delta}{\delta \mathscr{K}_{ij}} \left[((q^{\perp} - 1)\mathscr{H}^{g+m} + q^{k} \mathscr{H}_{k}^{g+m} \right] . \tag{3.3.49}$$

After substituting (3.3.49) into (3.3.48) one obtains a more compact expression:

$$\delta_{\mathrm{Y}} q^{ij} = 32\pi \left(\sqrt{-\eta} \right)^{-1} \mathscr{K}_{kl} \left(q^{ik} q^{jl} - q^{ij} q^{kl} \right) \mathrm{Y}^{\perp} + \pounds_{\mathrm{Y}^k} q^{ij} , \tag{3.3.50}$$

where redefined gauge functions Y^{a} are expressed as

$$\mathrm{Y}^{\perp} \equiv (1 - q^{\perp})\xi^{\perp} , \quad \mathrm{Y}^{i} \equiv -q^{i}\xi^{\perp} + \xi^{i} \tag{3.3.51}$$

with the asymptotic behaviour

$$\mathrm{Y}^{\mathrm{a}} = O^{-}(r^{1-\varepsilon}) + O^{+}(r^{1-\delta}) , \tag{3.3.52}$$

which follows from (3.2.44) and (3.3.26) with (3.2.53).

It is easy to show that (3.2.49) interpreted as gauge transformations can be obtained in the standard way, see [140], with the use of the Poisson brackets:

$$\delta_{\mathrm{Y}} q^{ij} = \left\{ \boldsymbol{q}^{ij}, \mathscr{H}(\mathrm{Y}) \right\} \tag{3.3.53}$$

with the Hamiltonian generator of gauge transformations,

$$\mathscr{H}(\mathrm{Y}) \equiv \mathrm{Y}^{\perp} \mathscr{H}^{g+m} + \mathrm{Y}^{i} \mathscr{H}_{i}^{g+m} . \tag{3.3.54}$$

Analogous to (3.3.53), one obtains gauge transformations for the conjugated momenta:

$$\begin{aligned} \delta_{\mathrm{Y}} \mathscr{K}_{ij} &= \left\{ \mathscr{K}_{ij}, \mathscr{H}(\mathrm{Y}) \right\} \\ &= \left[32\pi (\sqrt{-\eta})^{-1} \boldsymbol{q}^{kl} \left(\mathscr{K}_{ij} \mathscr{K}_{kl} - \mathscr{K}_{ik} \mathscr{K}_{jl} \right) + \frac{(\sqrt{-\eta})^{-1}}{16\pi} \overset{3}{R}_{ij} - \frac{\partial \mathscr{H}^{m}}{\partial \boldsymbol{q}^{ij}} \right] \mathrm{Y}^{\perp} \\ &\quad - \frac{(\sqrt{-\eta})^{-1}}{16\pi} \left(\det \boldsymbol{q}^{kl} \right)^{-1/4} \left\{ \partial_{ij} \left[\mathrm{Y}^{\perp} \left(\det \boldsymbol{q}^{kl} \right)^{1/4} \right] \right. \\ &\quad \left. - \partial_{m} \left[\mathrm{Y}^{\perp} \left(\det \boldsymbol{q}^{kl} \right)^{1/4} \right] \overset{3}{\Gamma}{}^{m}{}_{ij} \right\} + \pounds_{\mathrm{Y}^k} \mathscr{K}_{ij} , \end{aligned} \tag{3.3.55}$$

where $\overset{3}{\Gamma}{}^{m}{}_{ij}$ are the 3-dimensional Christoffel symbols constructed with the use of the 3-dimensional metric $\overset{3}{g}{}^{ij} = \boldsymbol{q}^{ij}/(-g)$.

Taking into account the asymptotic (3.3.26) and (3.3.27) for phase space variables, and (3.3.52) for gauge functions, one finds the asymptotic behaviour for various terms in (3.3.50) and (3.3.55) in a symbolic form:

$$
\begin{aligned}
\mathscr{K} qq\mathrm{Y}^{\perp} &= O^{-}\left(r^{-\varepsilon-\delta}\right) + O^{+}\left(r^{-2\varepsilon}\right) , \\
£_{\mathrm{Y}^k} q &= 2\left(\mathrm{Y}^{(i,j)} - \eta^{ij}\mathrm{Y}^{k}{}_{,k}\right) + O^{-}\left(r^{-\varepsilon-\delta}\right) + O^{+}\left(r^{-2\varepsilon}\right) , \\
q\mathscr{K}\mathscr{K}\mathrm{Y}^{\perp} &= O^{-}\left(r^{-1-3\varepsilon}\right) + O^{+}\left(r^{-1-2\varepsilon-\delta}\right) , \\
\overset{3}{R}\mathrm{Y}^{\perp} &= O^{-}\left(r^{-1-2\varepsilon}\right) + O^{+}\left(r^{-1-\varepsilon-\delta}\right) , \\
\frac{\partial \mathscr{H}^{m}}{\partial q}\mathrm{Y}^{\perp} &= O^{-}\left(r^{-2-\varepsilon-\alpha}\right) + O^{+}\left(r^{-2-\delta-\alpha}\right) , \\
(\det q)^{-1/4}\,[\ldots] &= -\frac{1}{2\kappa}\mathrm{Y}^{\perp}_{,ij} + O^{-}\left(r^{-1-2\varepsilon}\right) + O^{+}\left(r^{-1-\varepsilon-\delta}\right) , \\
£_{\mathrm{Y}^k} K &= O^{-}\left(r^{-1-2\varepsilon}\right) + O^{+}\left(r^{-1-\varepsilon-\delta}\right) .
\end{aligned}
\tag{3.3.56}
$$

Summing (3.3.56) in (3.3.50) and (3.3.55), one has

$$
\delta_{\mathrm{Y}}\boldsymbol{q}^{ij} = 2\left[\mathrm{Y}^{(i,j)} - \eta^{ij}\mathrm{Y}^{k}{}_{,k}\right] + O^{-}\left(r^{-\varepsilon-\delta}\right) + O^{+}\left(r^{-2\varepsilon}\right) , \tag{3.3.57}
$$

$$
\delta_{\mathrm{Y}}\mathscr{K}_{ij} = -\frac{1}{16\pi}\mathrm{Y}^{\perp}_{,ij} + O^{-}\left(r^{-1-2\varepsilon}\right) + O^{+}\left(r^{-1-\varepsilon-\delta}\right) . \tag{3.3.58}
$$

Now, substituting (3.3.57) and (3.3.58) into the expressions (3.3.35), (3.3.39) and (3.3.40), one obtains the gauge variations of energy, momentum and angular momentum defined in the Hamiltonian description:

$$
\delta_{\mathrm{Y}}\,\mathscr{P}^{(0)} = \frac{1}{8\pi}\oint_{\infty} ds_i\left[\mathrm{Y}^{[j,i]}{}_{,j} + O^{-}\left(r^{-1-2\varepsilon}\right)\right] , \tag{3.3.59}
$$

$$
\delta_{\mathrm{Y}}\,\mathscr{P}^{(k)} = \frac{1}{4\pi}\oint_{\infty} ds_i\left[\left(\mathrm{Y}^{\perp,[i}\eta^{j]k}\right)_{,j} + O^{-}\left(r^{-1-2\varepsilon}\right)\right] , \tag{3.3.60}
$$

$$
\begin{aligned}
\delta_{\mathrm{Y}}\,\mathscr{P}^{([mn])} = \frac{1}{8\pi}\oint_{\infty} ds_i\Big[&(x^{m}\mathrm{Y}^{\perp,[i}\eta^{j]n} - x^{n}\mathrm{Y}^{\perp,[i}\eta^{j]m} + \eta^{m[i}\eta^{j]n}\mathrm{Y}^{\perp})_{,j} \\
&+O^{-}\left(r^{-1-\varepsilon-\delta}\right)\Big] .
\end{aligned}
\tag{3.3.61}
$$

Due to the Stokes theorem and the restrictions (3.2.53) for ε and δ one finds that all the integrals (3.3.59–3.3.61) are equal to zero, thus the Hamiltonian energy, momentum and angular momentum in the field theoretical description are gauge invariant with respect to the variations (3.3.57) and (3.3.58).

In conclusion, let us discuss the problem of a super-translation invariance. Repeating the York arguments [467], we consider the Regge-Teitelboim variables (geometrical formulation) redefined in the form, $g_{ij}, \pi^{kl} \to \boldsymbol{q}^{ij}, \mathscr{K}_{ij}$,

$$
\boldsymbol{q}^{ij} = \boldsymbol{\eta}^{ij} + O^{+}\left(r^{-1}\right) + O^{-}\left(r^{-2}\right) , \tag{3.3.62}
$$

$$
\mathscr{K}_{ij} = O^{-}\left(r^{-2}\right) + O^{+}\left(r^{-3}\right) . \tag{3.3.63}
$$

Terms of the order $O^+\left(r^{-1}\right)$ in (3.3.62) contribute to the Hamiltonian integral (3.3.35), $\mathscr{P}^{(0)}$. Terms of the order $O^-\left(r^{-2}\right)$ and $O^+\left(r^{-3}\right)$ in (3.3.63) contribute to (3.3.39) and (3.3.41), $\mathscr{P}^{(k)}$ and $\mathscr{P}^{([mn])}$, respectively. Namely, the part $O^+(r^{-3})$ contributes to an angular momentum.

Generators of the gauge transformations in [467] have the same form (3.3.54), *but* with the asymptotic behaviour

$$\mathrm{Y}^{\mathrm{a}} = O(1)\,,\, \mathrm{Y}^{\mathrm{a}}{}_{,i} = O(r^{-1})\,, \tag{3.3.64}$$

instead of (3.3.52). Then, for (3.3.62–3.3.64) one has

$$\delta_{\mathrm{Y}}\boldsymbol{q}^{ij} = \left\{\boldsymbol{q}^{ij},\, \mathscr{H}(\mathrm{Y})\right\} = 2\left(\mathrm{Y}^{(i,j)} - \eta^{ij}\mathrm{Y}^{k}{}_{,k}\right) + O\left(r^{-2}\right)\,, \tag{3.3.65}$$

$$\delta_{\mathrm{Y}}\mathscr{K}_{ij} = \left\{\mathscr{K}_{ij},\, \mathscr{H}(\mathrm{Y})\right\} = -\frac{1}{2\kappa}\mathrm{Y}^{\perp}{}_{,ij} + O\left(r^{-3}\right)\,. \tag{3.3.66}$$

The first terms in (3.3.65) and (3.3.66) do not contribute to the global integrals by the Stokes theorem, the terms of the order $O\left(r^{-2}\right)$ in (3.3.65) do not contribute either. However, the terms of the order $O(r^{-3})$ *change* the integrals $\mathscr{P}^{([mn])}$. For a *real* isolated system this is not permissible. Sometimes this problem is called the problem of super-translations at spatial infinity [467]. What is suggested by the field-theoretical approach? After deriving the weakest fall-off we have stated the fall-off for the gauge functions as in (3.3.52) instead of (3.3.64). Thus, all the integrals (3.3.59–3.3.61), including angular momentum, are gauge invariant. Thus, the problem of super-translations invariance does not appear at all in the field-theoretical formulation. More formally, it is because in (3.2.53) and (3.3.52) $\delta > 0$.

The problem of super-translations invariance of angular momentum at null infinity for a radiating isolated system described by Bondi, Metzner and Sachs (BMS) [59, 61] is significantly more complicated. Many attempts have been provided to resolve it. An interesting one has been suggested by Helfer [229, 230]. It is certainly true that any physically reasonable definition of angular momentum in this case must be BMS covariant – that is, be natural as far as the asymptotic structure is concerned. But this is not the same as saying the physically preferred angular momenta must be conjugate to BMS generators (as they would be in the case of Noether's construction). For our well knowledge, the Helfer approach to angular momentum is the one of more fruitful methods which seems to be satisfactory and to resolve these difficulties. It does require twistor theory.

4 Exact solutions of general relativity in the field-theoretical formalism

It is important to describe exact solutions in general relativity in terms of the field-theoretical formalism. First, this has advantages in constructing conserved quantities. Second, this gives a physical understanding of the features of this approach, such as the non-observability of a background spacetime or the non-invariance of trajectories of test particles with respect to the background. Third, this presents a physically reasonable interpretation of properties (sometimes exotic) of exact solutions. Fourth, describing concrete models, one illustrates the power of the field-theoretical approach in mathematical treatment of various problems of theoretical physics.

One of the most famous and physically relevant solutions in gravity is the Schwarzschild solution. In Section 4.1, it is considered in great detail, including various interesting and unexpected properties of the corresponding field-theoretical configuration and possibilities for constructing conserved quantities. In Section 4.2, in the framework of the field-theoretical formalism, we examine the closed Friedmann model and calculate the energy characteristics for the Schwarzschild-AdS black hole.

In the present chapter, it is more useful and pedagogical to retain the constants c and G explicitly in all equations which we shall do.

4.1 The Schwarzschild solution

Here, we concentrate on the spherically symmetric, exact solution of general relativity, which is the Schwarzschild black hole. We emphasize, first of all, that it is the simplest yet most relevant solution in general relativity, and its properties (in the framework of the geometrical description) are well known. Second, it represents asymptotically flat spacetimes, a feature which is especially important. Third, it is surprising, but even a non-trivial intrinsic structure of the Schwarzschild black hole can be described in the framework of the field-theoretical approach.

4.1.1 The total energy

Classical black hole solutions in general relativity (without a cosmological constant) represent asymptotically flat spacetimes. Therefore, it is instructive to illustrate the results of the previous chapter with regard to the computation of global conserved quantities on example of these solutions. In the present subsection, we calculate the total energy of the Schwarzschild black hole.

Being asymptotically flat spacetimes, the black hole solutions asymptotically (at infinity) admit a flat metric. Therefore, we choose the flat metric as a background metric which matches the full metric in the asymptotic regime. In spherical coordinates, the metric is

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$$ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.1.1}$$

where notations for the coordinates are: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. Besides, we recall that the background metric of the Minkowski space in curved coordinates is denoted as $\bar{g}_{\mu\nu} = \gamma_{\mu\nu}$. Non-zero components of the Christoffel symbols corresponding to the metric (4.1.1) are

$$\begin{aligned} &C^1{}_{22} = -r, \;\; C^1{}_{33} = -r\sin^2\theta, \;\; C^2{}_{12} = C^3{}_{13} = \frac{1}{r}, \\ &C^2{}_{33} = -\sin\theta\cos\theta, \;\; C^3{}_{23} = \cot\theta. \end{aligned} \tag{4.1.2}$$

We consider the Schwarzschild solution in two different forms. First, we use the line element (1.5.34):

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)c^2dt^2 + \frac{1}{1-(r_g/r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{4.1.3}$$

where $r_g \equiv 2mG/c^2$. This is the form of the metric in the usual Schwarzschild coordinates. Second, we represent the Schwarzschild solution in the so-called isotropic coordinates [285]:

$$ds^2 = -\frac{\left(1 - r_g/4r\right)^2}{\left(1 + r_g/4r\right)^2}c^2dt^2 + \left(1 + \frac{r_g}{4r}\right)^4\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{4.1.4}$$

Of course, the coordinate "r" here is not the same as the coordinate "r" in (4.1.3). This is because one and the same background metric in the form (4.1.1) is used to derive the field configuration in both the field cases. Now, let us derive the field configurations corresponding to geometrical solutions (4.1.3) and (4.1.4), respectively. We use the decomposition (2.2.7) adapted to these solutions:

$$\mathfrak{g}^{\mu\nu} \equiv \bar{\mathfrak{g}}^{\mu\nu} + \mathfrak{h}^{\mu\nu} = \sqrt{-\gamma}\,(\gamma^{\mu\nu} + h^{\mu\nu}) \tag{4.1.5}$$

where $\sqrt{-\gamma} = r^2\sin\theta$. Then, the field configuration for the solution (4.1.3) takes on the form:

$$h^{00} = -\frac{r_g}{r}\frac{1}{1-(r_g/r)}, \quad h^{11} = -\frac{r_g}{r}, \tag{4.1.6}$$

whereas for the solution (4.1.4), the field configuration is

$$h'^{00} = 1 - \frac{\left(1 + r_g/4r\right)^7}{\left(1 - r_g/4r\right)}, \quad h'^{11} = h'^{22} = h'^{33} = -\left(\frac{r_g}{4r}\right)^2. \tag{4.1.7}$$

Recall that the metric elements (4.1.3) and (4.1.4) represent the same physical solution which is written down in two different coordinates. Thus, the field configurations (4.1.6) and (4.1.7) have to be connected by the gauge transformations (2.2.68). Symbolically this situation is illustrated in Figure 2.1.

To show explicitly that the field configurations represent the asymptotically flat spacetime, one has to use the Cartesian coordinates instead of the spherical ones. In the Cartesian coordinates the decomposition of the metric (4.1.5) reads

$$\mathrm{g}^{\mu\nu} \equiv \sqrt{-\eta}\,(\eta^{\mu\nu} + h^{\mu\nu})\,. \tag{4.1.8}$$

Equations (4.1.6) and (4.1.7) are replaced in the Cartesian coordinates with

$$h^{00} = -\frac{r_g}{r}\frac{1}{1-(r_g/r)}, \quad h^{ik} = -\frac{r_g}{r}\frac{x^i x^k}{r^2}, \tag{4.1.9}$$

and

$$h'^{00} = 1 - \frac{\left(1 + r_g/4r\right)^7}{\left(1 - r_g/4r\right)}, \quad h'^{ik} = -\delta^{ik}\left(\frac{r_g}{4r}\right)^2, \tag{4.1.10}$$

where $x^1 = x$, $x^2 = y$ and $x^3 = z$. Both of the field configurations, indeed, satisfy the condition (3.2.2) defining the asymptotically flat spacetime.

One can see that the asymptotic behavior of the field variables (4.1.10) and (4.1.9) satisfies the restrictions (3.2.53) and (3.2.54). Thus, both configurations have to give the same unique values for global conserved quantities. By a symmetry of the solution, it is evident that only the total energy can be non-zero among all of the ten global conserved quantities. To calculate the energy it is more convenient to use the general formula (3.2.27) derived for curved coordinates in a flat spacetime. Adapted to the spherical coordinates with the corresponding Killing vector, $\xi^0_K = -\delta^0_0$, the total energy is

$$\mathscr{P}^{(0)}_S = \frac{1}{2\kappa}\oint_\infty d\theta d\phi \sqrt{-\gamma}\left(h_0{}^0\gamma^{j1} + \delta^0_0 h^{j1}\right)_{;j}\xi^0_K \tag{4.1.11}$$

Simple calculations for both cases (4.1.6) and (4.1.7) give

$$E = \mathscr{P}^{(0)}_S = mc^2 \tag{4.1.12}$$

that is quite a natural result for the total energy of the black hole.

4.1.2 The energy distribution for the Schwarzschild black hole

Problems in the interpretation of the Schwarzschild solution

The Schwarzschild solution, being a far non-trivial solution, has the well known problems with regard to its interpretation in the geometric language. These difficulties have been outlined by Narlikar [331]. Here, we follow his presentation and show how such problems can be resolved in framework of the field-theoretical formalism.

Let us derive the spherically symmetric line element for a static system consisting of matter and gravitational field in the most general form:

$$ds^2 = -e^{\nu}c^2dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.1.13}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. The Einstein equations (1.3.23) for metric (4.1.13) read

$$\kappa T_0^{\ 0} = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2}, \tag{4.1.14}$$

$$\kappa T_1^{\ 1} = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) - \frac{1}{r^2}, \tag{4.1.15}$$

$$\kappa T_2^{\ 2} = \kappa T_3^{\ 3} = \frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right). \tag{4.1.16}$$

Assuming that matter is confined in a finite volume of space, one finds

$$\nu + \lambda = 0, \quad e^{-\lambda} = 1 - C/r. \tag{4.1.17}$$

In order to be consistent with the Newtonian gravity in the field-weak limit at spatial infinity one chooses the constant of integration C as

$$C = \frac{2mG}{c^2} \equiv r_g, \tag{4.1.18}$$

where m is interpreted as the Newtonian mass measured at infinity. On the other hand, the same mass appears as the total mass of the Schwarzschild solution (4.1.12). In the latter case, it has to be interpreted as the mass defined by matter together with gravitational field, see the transformation from (3.2.20) to (3.2.27). Thus, in the framework of the field-theoretical formalism, one illustrates explicitly the contribution of the self-interaction of gravitational field to the total energy of the spherically-symmetric system in general relativity.

Let us desribe the *first* problem discussed in [331] related to interpretation of the defect mass on an example of spherically-symmetric system in general relativity. Following the book [285], rewrite equation (4.1.14) in the form[1]:

$$-\frac{d}{dr}\left[r\left(1 - e^{-\lambda}\right)\right] = \kappa r^2 c^2 T_0^{\ 0} = -8\pi G r^2 \rho, \tag{4.1.19}$$

where $\rho = \rho(r)$ is the mass density of a spherically symmetrical body with the radius, $r = r_S$, of its surface boundary. Let us integrate (4.1.19)

$$m \equiv m(r_S) = 4\pi \int_0^{r_S} r^2 \rho(r) dr. \tag{4.1.20}$$

This apparently innocent definition of the gravitational mass is not so natural as it looks. In fact, it is the result of a formal integration only. Note that for the line element

1 Signs in (4.1.19) correspond to signature in (4.1.13) for which $T_0^{\ 0} \le 0$.

(4.1.13) the physical volume element on a spacelike hypersurface $x^0 = \text{const}$ is not $4\pi r^2 dr$ but $4\pi r^2 e^{\lambda/2} dr$. Work [285] explains this fact as a gravitational defect of mass but without providing more detailed discussion. The other work [315] is trying to make the definition (4.1.20) look more natural by splitting it in a linear combination of three pieces

$$m = 4\pi \int_0^{r_S} r^2 e^{\lambda/2} \rho_N dr + 4\pi \int_0^{r_S} r^2 e^{\lambda/2} (\rho - \rho_N) dr$$
$$+ 4\pi \int_0^{r_S} r^2 e^{\lambda/2} \rho (e^{-\lambda/2} - 1) dr \equiv m_N + \frac{U}{c^2} + \frac{\Omega}{c^2}. \tag{4.1.21}$$

Here, m_N is the nucleonic mass of the body made of the rest mass density ρ_N of all of the N particles. The quantity U is the intrinsic energy accounting for the density difference $\rho - \rho_N$, while Ω is the gravitational potential energy. In the weak field approximation one has

$$\Omega = -4\pi \int_0^{r_S} r^2 \rho \frac{Gm(r)}{r} dr \tag{4.1.22}$$

which is in agreement with the definition of the Newtonian potential energy. However, the modified formula (4.1.21) has also a problem: Bondi [60] noted that the term m_N is not relativistically invariant. Also, both in (4.1.20) and in (4.1.21), the integration is naturally performed only over the matter distribution up to $r = r_S$. It makes an impression that the total mass m is defined only by the contribution of matter and gravitational field inside the volume of the body limited by its radius. However, the total m in (4.1.18) has been defined by an observer at a very large (asymptotically-infinite) distance from the system, so that one can wonder what happens with the contribution of the gravitational field to the total mass from the exterior domain located outside of the body's volume.

The *second* problem considered by Narlikar [331] is related to the concept of the point mass in general relativity. This problem also exists in the Newtonian gravity but it is resolved in a simple way by assuming that the mass distribution has the form $\rho(r) = m\delta(r)$ where δ-function satisfies the ordinary Poisson equation

$$\nabla^2 \left(\frac{1}{r}\right) = \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\frac{1}{r} = -4\pi\delta(r). \tag{4.1.23}$$

Then, both for a regular distribution $\rho(r)$ and for a point mass $\rho(r) = m\delta(r)$, the total mass of the gravitating system is calculated with the use of the *same* integral:

$$m = \int_\Sigma dx^3 \rho(r). \tag{4.1.24}$$

Thus, the massive point particle located at the origin, $r = 0$, is included into consideration in the Newtonian gravity by making use of the mathematical apparatus of distributions (generalized functions).

If one tries to use the Schwarzschild solution in order to describe a point mass in general relativity a conceptual difficulty arises. If we try to employ the Newtonian concept of the point mass and assume that the gravitational potential defined by (4.1.17) and (4.1.18) holds formally in the whole *empty* spacetime, including the world-line $r = 0$, the matter distribution (due to the Einstein equations (4.1.14–4.1.16)) must have the energy-momentum tensor in the following form:

$$T_0^{\ 0} = T_1^{\ 1} = 0, \quad T_2^{\ 2} = T_3^{\ 3} = -\frac{mc^2}{2}\,\delta(\mathbf{r}). \tag{4.1.25}$$

It is quite easy to check by a direct inspection that it is impossible to obtain the correct total mass for this distribution by performing the ordinary volume integration, as in (4.1.24) because the time-time component of the energy-momentum tensor, which characterizes the mass density distribution, is equal to zero. The situation cannot be saved even if one remembers that the time coordinate and the radial coordinate exchange roles inside the event horizon.

In spite of the above difficulties, modeling of a true singularity in general relativity by a δ-function looks plausible, see, for example, [86, 198, 298]. Below we show how to circumvent the above problems using the field-theoretical formalism. To this end, we again resort to the measurements made by the infinitely distant observer, where the spacetime is asymptotically flat. It is surprising, but in this case a black hole geometry can be interpreted as the usual field configuration defined at the event horizon or behind it down to the true physical singularity at $r = 0$. Then, both of the above-mentioned problematic issues are resolved by defining the total mass of the field configuration as the integral (3.2.20):

$$\mathscr{P}^{(0)} = \int_\Sigma d^3x\sqrt{-\gamma}t^{00}_{\text{tot}}\xi^K_0 = \int_\Sigma d^3x\sqrt{-\gamma}t^{00}_{\text{tot}} \tag{4.1.26}$$

performed over the whole Minkowski space including $r = 0$ with the energy density (energy distribution) t^{00}_{tot}, defined in Chapter 3. Notice that the volume element of integration in (4.1.26) is $d^3x\sqrt{-\gamma}$ that is a real geometric volume element in the Minkowski space. The massive point particle case is included to the integral (4.1.26), if the δ-function representing the singularity is included into t^{00}_{tot} in a mathematically self-consistent way which we shall explain below in more detail. Thus, (4.1.26), in fact, generalizes the Newtonian formula (4.1.24). In what follows, we elaborate on this prescription in detail.

Regular spherically symmetrical static body

The Schwarzschild solution can be obtained in two ways: either as an *external* field of a spherically symmetrical static body or as a gravitational field in empty spacetime representing a black hole. In the present discussion, we rely upon the first interpretation. Proceeding in this way, we do not consider the interior solution for λ and ν in (4.1.13) explicitly, but we assume that the body has a physically-admissible equation of state. Then, the functions λ and ν can be thought as smooth and physically-admissible. As a result, we conclude that the volume integration,

$$E^S = \int_0^{r_S} d^3x\sqrt{-\gamma}t^{00}_{\text{tot}}, \tag{4.1.27}$$

is well-defined. This is enough to achieve our goal, so that, we do not provide the integration (4.1.27) explicitly. Instead of that, to obtain the energy E^S in the domain enclosed by the surface, $r = r_S$, we will use the surface integral (4.1.11) at $r = r_S$. In order such an integration to be well-defined one has to make a natural assumption that the values of the components $h_{-}^{\mu\nu}$ inside the surface are matched smoothly with the values of $h_{+}^{\mu\nu}$ outside the surface of the body. As a result, one has

$$E^S = mc^2 \left[\frac{1}{2} \frac{r_g \left(2r_S - r_g\right)}{\left(r_S - r_g\right)^2} + 1 \right]. \tag{4.1.28}$$

To calculate the distribution of the total energy outside the body, we use the explicit expression for the component t_{tot}^{00}. We assume that the field equations (2.2.105) hold, and we use the left hand side of them given by (2.2.27), or in the Lorentzian coordinates by (3.2.22),

$$t_{\mu\nu}^{\text{tot}} = t_{\mu\nu}^{g} + t_{\mu\nu}^{m} = \frac{1}{\kappa} G_{\mu\nu}^{L} = \frac{1}{2\kappa} \left(h_{\mu\nu}{}^{;\alpha}{}_{;\alpha} + \gamma_{\mu\nu} h^{\alpha\beta}{}_{;\alpha\beta} - h^{\alpha}{}_{\mu;\nu\alpha} - h^{\alpha}{}_{\nu;\mu\alpha} \right). \tag{4.1.29}$$

Substituting here, for example, the configuration (4.1.6) that is defined outside r_S, one has

$$t_{tot}^{00} = t_{00}^{g} = -\frac{r_g^2}{\kappa r^4} \frac{1}{\left(1 - \frac{r_g}{r}\right)^3}. \tag{4.1.30}$$

Then, using (4.1.26), one obtains

$$E^{\text{out}} = \int_{r_S}^{\infty} d^3x \sqrt{-\gamma}\, t_{\text{tot}}^{00} = -\frac{mc^2}{2} \frac{r_g \left(2r_S - r_g\right)}{\left(r_S - r_g\right)^2}. \tag{4.1.31}$$

Summing up (4.1.28) and (4.1.31), one gets $E^S + E^{\text{out}} = mc^2$ which is in a perfect agreement with the previous result (4.1.12).

Interpretation of the above results is as follows.

First, the energy of the gravitational field outside the body (4.1.31) is *negative*. This coincides with the classical results on the value of the binding energy of gravitational field which is always negative. Indeed, in order to break apart a binary system comprised of two massive stars, one has to inject into the system an additional positive energy. Therefore, the gravitational binding energy has to be negative. Another argument in support of the negative value of the energy of gravitational field [285] is that the total energy of the closed Friedmann universe is equal to zero. This means that the positive energy of the matter sources is compensated exactly by the negative energy of the gravitational field. These considerations are in a total agreement with the observation that the potential energy of the gravitational field in (4.1.22) is negative.

Second, we note that the total energy within the body (4.1.28) exceeds the value of mc^2. This result seems to be in conflict with the negative value of the potential energy of gravitational field which is expected to be subtracted from the positive value of the total mass inside the body, thus, making the total mass of matter and gravitational field less than mc^2. The “paradox” is solved by noticing that in (4.1.28) we consider the energy of the body without the energy of the external gravitational field. Looking more carefully at this situation one can understand better how the gravitational defect of mass emerges in our calculations in the following way. Let the body of the total mass m consist of two pieces which are bound together by gravitational forces. In order to completely separate the two pieces from each other and to put them to infinity we have to inject some positive energy. Then, the total mass of the separated pieces of the body measured at spatial infinity by an observer must exceed the mass of the initial configuration. We conclude that a sum of the total energies of these pieces (*together with the energies of their own gravitational fields*) is more than mc^2. So, the total mass of a self-gravitating system measured by a distant observer is less than the sum of the masses of its constituent particles which proves the existence of the gravitational defect of mass.

The Schwarzschild black hole

Let us consider the Schwarzschild black hole represented by the solution (4.1.3) with the field configuration (4.1.6) in the *whole* the Minkowski space. One assumes also that the Einstein equations (2.2.105) hold in *all the points* in Minkowski space including $r = 0$. Then it is necessary to use the techniques of the generalized functions [185, 273]. Calculations by taking into account the special point $r = 0$ require an specific approach if one follows to the idea of [185]. The main principle requires a correspondence between the volume integration and the surface integration. For this purpose, it is important to define $\nabla^2(1/r^{k+1})$ with integer $k \geq 0$. Already for $k = 0$ we have derived the well known formula (4.1.23). Using the differentiation rules for the generalized functions [185], one obtains [363]:

$$\nabla^2 \frac{1}{r^{k+1}} = (k+1)\left[\frac{k}{r^{k+3}} - \frac{4\pi}{r^k}\delta(\mathbf{r})\right]. \tag{4.1.32}$$

Many results here are obtained with the use of the formula (4.1.32), although we will not refer to it below.

Now, with the use of (4.1.29) we calculate the 00-component of the total energy momentum for the field configuration (4.1.6):

$$t_{00}^{tot} = \frac{mc^2}{2}\delta(r)\left[1 - \frac{1}{\left(1-(r_g/r)\right)^2}\right] - \frac{{r_g}^2}{\kappa r^4}\frac{1}{\left(1-(r_g/r)\right)^3} \tag{4.1.33}$$

which represents the energy distribution and is depicted in Figure 4.1.

It is natural to see that the total energy obtained after substituting (4.1.33) into (4.1.26) and integrating over the whole space is exactly $E = mc^2$. If one calculates the energy outside the horizon only, one obtains $-\infty$; the energy inside the horizon is equal to $+\infty$. However the infinite contributions near horizon are compensated. One can find also that the contribution into E from the δ-function is equal to $mc^2/2$, while the contribution from the free gravitational field outside $r = 0$ is also equal to $mc^2/2$.

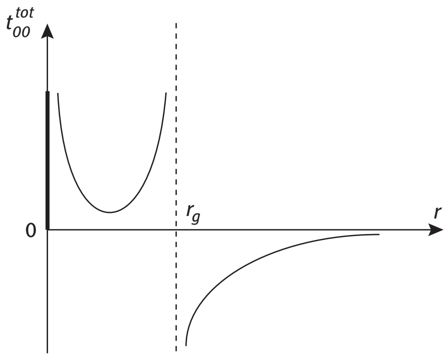

Figure 4.1: The plot of the energy density of the gravitational field of the Schwarzschild solution in the frame of a distant observer in the Minkowski space.

In fact, we extend the concept of the Minkowski space from spatial infinity up to the horizon $r = r_g$, and even below the horizon including the worldline $r = 0$, see Figure 4.1. However, in reality the distant observer cannot see the space within the horizon. Therefore, it is more pedagogical to consider the situation outside the horizon only. Naively this picture can be explained as follows. When the test particle moves closer to horizon then it is more difficult to escape the black hole. From the point of view of the distant observer this can be interpreted by the way that the negative density of the gravitational energy (and, consequently, the attraction) is stronger near the horizon. The infinite negative density for the gravitational energy exactly at the horizon, $t_{00}^{\text{tot}} = t_{00}^{g} = -\infty$, signals to the observer on impossibility to escape the black hole at all.

It is interesting to examine the contribution into (4.1.33) from the matter source (that is “localized” now at $r = 0$ only) and from the free gravitational field, separately. The most economical way is as follows. We use the formula (2.2.32) connecting the matter energy-momentum tensor in the field-theoretical and geometrical formulations of general relativity. Let us rewrite it for the Minkowski background:

$$t_{\mu\nu}^{m} = \left(\delta_\mu^\alpha \delta_\nu^\beta - \tfrac{1}{2}\gamma_{\mu\nu}\gamma^{\alpha\beta}\right)\left(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T_{\pi\rho}g^{\pi\rho}\right), \tag{4.1.34}$$

where $g_{\alpha\beta}$ is defined in (4.1.3). Then, using (4.1.25) in (4.1.34), one obtains

$$t_{00}^{m} = -\frac{mc^2}{4}\delta(r)\left[1 - \frac{r_g}{r} - \frac{1}{1-(r_g/r)}\right], \tag{4.1.35}$$

that is localized at the point $r = 0$. Then, subtracting (4.1.35) from the total quantity (4.1.33), one obtains for the gravitational part:

$$t_{00}^{g} = -\frac{mc^2}{4}\,\delta(r)\frac{r_g}{r}\left[1 + \frac{3}{1-(r_g/r)} + \frac{2}{\left(1-(r_g/r)\right)^2}\right] - \frac{r_g{}^2}{\kappa r^4}\frac{1}{\left(1-(r_g/r)\right)^3}. \tag{4.1.36}$$

One can see that separately the δ-functions in (4.1.35) and (4.1.36) make $(-\infty)$-contribution and $(+\infty)$-contribution, respectively, to the total energy. The infinite contributions cancel each other. However, anyway, it is in the spirit of general relativity that $t^m_{\mu\nu}$ *cannot* be considered separately from $t^g_{\mu\nu}$.

Let us also derive the other components $t^m_{\mu\nu}$ and $t^{tot}_{\mu\nu}$; their differences give $t^g_{\mu\nu}$. Thus the other non-zero components for the matter energy-momentum are

$$\begin{aligned} t^m_{11} &= -\gamma_{11}\frac{mc^2}{4}\delta(r)\left(1-\frac{r_g}{r}-\frac{1}{1-(r_g/r)}\right), \\ t^m_{AB} &= -\gamma_{AB}\frac{mc^2}{4}\delta(r)\left(1-\frac{r_g}{r}+\frac{1}{1-(r_g/r)}\right), \end{aligned} \tag{4.1.37}$$

where $A, B = 2, 3$, and non-zero components for the total energy-momentum are

$$t^{tot}_{AB} = \gamma_{AB}\left(\frac{mc^2}{2}\,\delta(r)-\frac{2r_g}{\kappa r^3}\right). \tag{4.1.38}$$

The main assumption made above is that the field-theoretical equations are valid at all points of the Minkowski space, including $r = 0$. As is seen, see (4.1.35) and (4.1.36), the situation is more comprehensive than for the point mass in the Newtonian gravity, where the δ-function enters the matter energy density only. Nevertheless, we can use the volume integration over the whole Minkowski space (4.1.26). Thus, the problem of the point mass is indeed resolved with the use of the field-theoretical formulation, unlike the case in the geometrical formulation.

4.1.3 The Schwarzschild black hole as a point particle

In spite of its advantages, the interpretation of the point mass in the previous subsection has some questions. At $r = r_g$ both the gravitational potentials and the energy density have discontinuities. This highlights the fact that in the standard formulation of general relativity, one has a coordinate singularity at $r = r_g$ in the Schwarzschild coordinates. It is not a real singularity in the field-theoretical formulation either, where this break-down is interpreted as a "bad" fixing of gauge freedom which needs to be improved. In other words, the break-down at $r = r_g$ has to be countered with the use of an appropriate choice of a flat background, which is determined by related coordinates for the Schwarzschild solution. The present subsection is devoted to this problem.

The use of the coordinates without singularities at the horizon, like Novikov's, Kruskal-Szekeres's, etc., coordinates [285, 315], could resolve the problem locally in the neighborhood of $r = r_g$. Together with this, we restrict ourself by the following. First, we represent a point particle at rest in the whole Minkowski space. Therefore it has to be natural to describe the true singularity by the world line $r = 0$. Second, the Schwarzschild solution in appropriate coordinates has to be asymptotically flat. Third, we require a fulfillment of a so-called "η-causality" — property, when the physical light cone is inside the flat light cone at all the points of the Minkowski space. It is necessary to avoid interpretation difficulties under the field-theoretical presentation of general relativity. By this requirement, all the causally connected events in the physical (dynamical) spacetime are described by the right causal structure of the Minkowski space. The related position of the light cones is not gauge invariant. Properties of the η-causality and gauge transformations conserving it were studied in detail by Pitts and Schieve [376].

We consider the third requirement *only* for convenience in applications and interpretation for the field configuration. To avoid ambiguities we stress again that, unlike Pitts and Schieve who give a real physical sense to the background, we use it as an *auxiliary* construction. Thus, we agree with the assertion by Grishchuk [202, 203] that, changing the mutual disposition of the light cones, one cannot change the physical properties of the solution. The requirement of the η-causality can be strengthened by the requirement of the so-called "stable η-causality" [374–376]. The latter means that the physical light cone has to be *strictly* inside the flat light cone. This could be important, when quantization problems are under consideration. Indeed, in the case of tangency, a field is on the verge of η-causality violation [376]. Returning to the representation in the Schwarzschild coordinates in the previous subsection, we note that it does not satisfy the third requirement.

More appropriate coordinates, satisfying the above requirements, are, first, the *stationary* (not static) coordinates presented in [358], and recently improved in [374–376]; second, contracting Eddington-Finkelstein coordinates in a stationary form [315]. These coordinate systems can be generalized to a parameterized family, where all the systems satisfy all the above requirements as well.

Except for a pure theoretical interest, the models of black holes in the form of point particles could be also interesting and useful for experimental gravity problems. Recently, gravitational wave detectors of the LIGO type have discovered gravitational waves from coalescing binary systems comprising of compact relativistic objects. Therefore it is necessary to derive equations of motion of such components, e. g., two black holes. As a rule, at an *initial* step the black holes are modeled by point-like particles represented by Dirac δ-functions. Then consequent post-Newtonian approximations are used, see the works with excellent mathematical rigor [116, 404] and references therein. However this approach meets difficulties related to the non-linear nature of the Einstein equations. Different regularization methods have been suggested to bypass them. However, in spite of a significant progress, so far the problem of motion of the black holes in general relativity has many open questions [116, 404]. The way of definition of a point-like source in general relativity in the present subsection is different. Not making *initial* assumptions on its structure, one uses the Schwarzschild solution itself from the start. The resulting field configuration, including a description of the true singularity in the form of a point-like particle, is easy for applications. This allows to reproduce the Schwarzschild solution without approximations, with a correctly-defined position of the horizon, etc.

A point particle with an external distribution of energy

Thus, let us turn to the standard Schwarzschild line element (4.1.3). Following [358] and [374] - [376], we only change the time coordinate

$$ct \to ct - r_g \ln\left|1 - \frac{r_g}{r}\right|, \tag{4.1.39}$$

whereas the other coordinates $\{r,\ \theta,\ \phi\}$ are not changed. As a result one has

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)c^2dt^2 + 2\frac{r_g^2}{r^2}\,c\,dt\,dr + \left(1 + \frac{r_g}{r}\right)\left(1 + \frac{r_g^2}{r^2}\right)dr^2$$
$$+ r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.1.40}$$

The important properties of this solution are that a falling test particle approaches the horizon $r = r_g$ in finite coordinate time t, below the horizon, it is always falling towards the singularity, it gets arbitrarily close to it, but only hits it at $t = \infty$.

Using the decomposition (4.1.5) for the solution (4.1.40) one obtains the corresponding field configuration:

$$h^{00} = -\left(\frac{r_g}{r} + \frac{r_g^2}{r^2} + \frac{r_g^3}{r^3}\right), \quad h^{01} = \frac{r_g^2}{r^2}, \quad h^{11} = -\frac{r_g}{r}. \tag{4.1.41}$$

Calculating $t_{\mu\nu}^{tot}$ for the configuration (4.1.41), we use again the expression (4.1.29). The non-zero components are

$$\begin{aligned} t_{00}^{\text{tot}} &= mc^2\delta(\mathbf{r}) + mc^2\frac{r_g}{r}\left(1 + \frac{3}{2}\frac{r_g}{r}\right)\delta(\mathbf{r}) - \frac{mc^2}{4\pi}\frac{r_g}{r^4}\left(1 + 3\frac{r_g}{r}\right), \\ t_{11}^{tot} &= -mc^2\delta(\mathbf{r}), \\ t_{AB}^{tot} &= -\frac{1}{2}\gamma_{AB}\, mc^2\delta(\mathbf{r}). \end{aligned} \tag{4.1.42}$$

One can calculate the components $t_{\mu\nu}^{m}$ of the matter energy-momentum using the formula (4.1.34) once again. One has to use $T_\alpha{}^\beta$ defined in (4.1.25) this time as well, because the transformation (4.1.39) has been applied. Besides, in (4.1.34), one has to use $g_{\mu\nu}$ defined in (4.1.40). As a result, one obtains that $t_{\mu\nu}^{m}$ is concentrated at $r = 0$ by the delta-function. However, the spirit of general relativity that $t_{\mu\nu}^{m}$ *cannot* be considered separately from $t_{\mu\nu}^{g}$ therefore we do not derive components of $t_{\mu\nu}^{m}$ here. It is important to note that outside $r = 0$ the gravitational energy-momentum $t_{\mu\nu}^{g}$ coincides with the total energy-momentum.

Let us discuss the properties of the field-theoretical representation of the solution (4.1.40). First, as usual, in the Minkowski space, the energy density distribution is described by the 00-component of the energy-momentum tensor, see Figure 4.2. Then, the total energy of the system is calculated by a substitution of t_{00}^{tot} from (4.1.42) into (4.1.26). Again, one obtains $E = mc^2$! This is defined only by the first term $mc^2\,\delta(\mathbf{r})$ in t_{00}^{tot}. The other contributions into E from the δ-functions in t_{00}^{tot} are infinite, but they

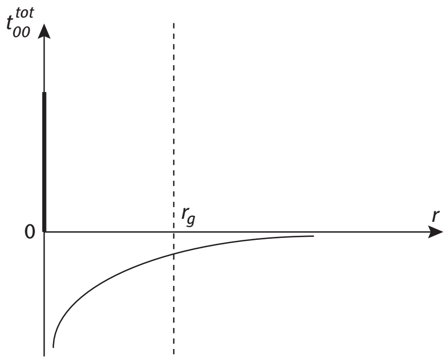

Figure 4.2: The plot of the energy density of the gravitational field of the Schwarzschild solution generated by a point source with an external distribution of energy in the Minkowski space.

are compensated by the energy distribution without the δ-functions. The other components t_{11}^{tot} and t_{AB}^{tot} in (4.1.42) formally could be interpreted as related to the "inner" properties of the point. Indeed, they are proportional to $\delta(\mathbf{r})$ only and, thus, describe the "intrinsic radial" and "intrinsic tangent" stresses.

Second, after the transformation from the spherical coordinates to the Cartesian coordinates, one can see that the configuration (4.1.41) is asymptotically flat with the $1/r$-like fall-off at spatial infinity. Therefore the result $E = mc^2$ is not surprising.

Third, the metric (4.1.40) with the background metric (4.1.1) satisfies the requirement of the *stable η-causality* at all points of the Minkowski space down to the true singularity at $r = 0$. Thus, all the requirements are satisfied.

At least, the picture derived above resolves the problem of the break-down at the horizon. The field configuration (4.1.41) is continuous at *all* points of the Minkowski space except at the true singularity $r = 0$, which is natural. A falling test particle approaches and intersects the horizon $r = r_g$ in a finite Minkowski time t. The components t_{00}^{tot} and t_{00}^{g} have no breakdowns outside $r = 0$, and all the other energy-momentum components in (4.1.42) are defined only by the δ-function. Besides, as in Newtonian gravity, the problem of point mass is resolved simply. The energy-momentum tensor (4.1.42) contain δ-functions at $r = 0$, and, like in the Newtonian case, the volume integration over the whole space gives a satisfactory total energy.

However, the energy distribution in the Newtonian case (4.1.22) is represented by the δ-function only, whereas in (4.1.42) there is an external energy distribution, see Figure 4.2. The question arises; is there a possibility to represent the energy distribution for the Schwarzschild solution with the use of the δ-function only? We answer this question in the next subsection.

A point particle without an external distribution of energy

Let us examine the contracting Eddington-Finkelstein metric for the Schwarzschild geometry [315]. But, we make a transformation from the null coordinate $\tilde{V}$ to the time coordinate t: $ct = c\tilde{V} - r$, after that one has

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) c^2 dt^2 + 2\frac{r_g}{r}\, c\, dt\, dr + \left(1 + \frac{r_g}{r}\right) dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \qquad (4.1.43)$$

Then the corresponding gravitational field configuration is

$$h^{00} = -\frac{r_g}{r}, \quad h^{01} = \frac{r_g}{r}, \quad h^{11} = -\frac{r_g}{r}. \tag{4.1.44}$$

The properties of the solutions (4.1.40) and (4.1.43) are very close. Both metrics are stationary and asymptotically flat. In the whole Minkowski space they induce asymptotically flat and continuous (except at $r = 0$) configurations (4.1.41) and (4.1.44). Falling test particles intersect the horizon $r = r_g$ in finite times t, but in the case (4.1.43) test particles reach the true singularity in a finite time t. This is the result of the time transformation for the Schwarzschild time:

$$ct \to ct - r_g \ln\left|\frac{r}{r_g} - 1\right|. \tag{4.1.45}$$

After making use of the expression (4.1.29) the components of the total energy-momentum tensor for the configuration (4.1.44) are calculated as

$$\begin{aligned} t^{\text{tot}}_{00} &= mc^2\delta(\mathbf{r}), \\ t^{\text{tot}}_{11} &= -mc^2\delta(\mathbf{r}), \\ t^{\text{tot}}_{AB} &= -\frac{1}{2}\gamma_{AB}\, mc^2\delta(\mathbf{r}). \end{aligned} \tag{4.1.46}$$

All these energy-momentum components, unlike (4.1.42), are concentrated *only* at $r = 0$, see for the energy distribution Figure 4.3. The volume integration (4.1.26) of t^{tot}_{00} from (4.1.46) again gives $E = mc^2$. Of course, the surface integration (4.1.11) with the configuration (4.1.44) gives $E = mc^2$ as well. However, this result follows with an arbitrary radius, r_0, of 2-sphere in a surface integration, it is not necessary to set $r_0 \to \infty$. This is an exact analog for calculating the electric charge in electrodynamics, or calculating the point mass in Newtonian gravity.

A family of point-like representations

The transformation

$$ct \to ct - r_g \ln\left|\frac{r/r_g - 1}{(r_g/r)^\alpha}\right| \tag{4.1.47}$$

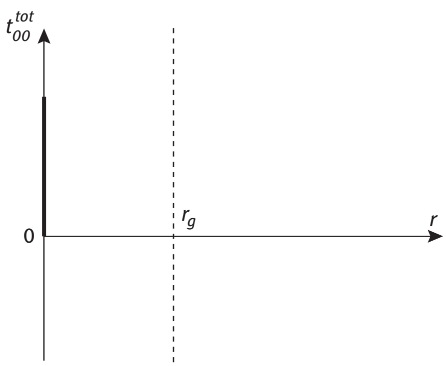

Figure 4.3: The plot of the energy density of the gravitational field of the Schwarzschild solution generated by a point source without an external distribution of energy in the Minkowski space.

gives a family of metrics parameterized by $\alpha \in [0, 2]$, all of which satisfy all our requirements set in the beginning of the subsection. The cases $\alpha = 0$ and $\alpha = 1$ correspond to (4.1.43) and (4.1.40), respectively. However, the requirement of the *stable* η-causality *is not* satisfied with $\alpha = 0$ at $0 \leq r \leq \infty$. Properties of the field configurations corresponding to $\alpha \in (0, 2]$ qualitatively are the same as for $\alpha = 1$. In terms of the field-theoretical approach all the field configurations for $\alpha \in [0, 2]$ are connected by gauge transformations and are physically equivalent. Thus, inside this family, the *η-causal* description with (4.1.44) can be converted into a *stable η-causal* description explicitly. Note also that the technique of infinitesimal gauge transformations developed in [374–376] permits to do this conversion approximately without relating to this family.

4.1.4 The Schwarzschild solution and the harmonic gauge fixing

On harmonic coordinates in general relativity

To simplify the Einstein equations, one frequently makes an appropriate choice of coordinates. Probably, the harmonic coordinates are the most popular ones. The related bibliography is very wide, therefore it is hard to list the papers here. Fock [178] developed the applications of these coordinates. For example, he suggested the harmonic coordinates for the Schwarzschild solution. The other applications of the harmonic coordinates are as follows. They are used in elaborating theoretical problems [36, 142, 329, 394], for studying the detailed structure of the gravitational filed outside of isolated systems [42], for constructing relativistic theory of frames in Solar system [78, 268].

In the present subsection, new harmonic coordinates for the Schwarzschild solution are constructed. What is their advantage compared to the Fock harmonic coordinates [178]? The latter, like the Schwarzschild coordinates, are singular at the horizon. Many coordinate systems without this defect are known, but they are not harmonic. The new coordinates that we shall present are both harmonic and regular at the horizon.

Here, developing applications of the field-theoretical methods, we interpret the transition from the Fock coordinates to the new harmonic coordinates in terms of gauge transformations. In both of the gauge fixings, we consider trajectories of test particles falling into the Schwarzschild black hole. We find that trajectories in the Minkowski space are gauge dependent, see Section 2.2.7. Because gauge transformations do not change the physical picture, we confirm that the background Minkowski space is an auxiliary structure. Thus, a break-down in the trajectories at the horizon for the field configuration in the Fock picture is interpreted as non-physical. Indeed, such a break-down is canceled for the field configuration corresponding to the new harmonic coordinates. These problems, of course, are resolved clearly in the framework of the usual geometrical formalism of general relativity. However, here, we provide the below exercises to illustrate useful properties of the field-theoretical technique.

New harmonic coordinates for the Schwarzschild solution
Let us derive the Schwarzschild metric in the Fock coordinates:

$$ds^2 = -\frac{r-\alpha}{r+\alpha}c^2dt^2 + \frac{r+\alpha}{r-\alpha}dr^2 + (r+\alpha)^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{4.1.48}$$

where in the Fock notations, $\alpha = r_g/2$. Transferring to the asymptotically Cartesian coordinates in the standard way, one finds that for the solution (4.1.48) the harmonic (de Donder) conditions

$$\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\right) = 0 \tag{4.1.49}$$

hold.

To simplify the presentation we consider a test particle, falling radially into a black hole. Besides, we restrict ourselves to the "parabolic orbit" case, when a particle begins its motion from the rest at the infinity $r = \infty$. Then, the equation of motion of a test particle has the form:

$$\begin{aligned} ct = -2\alpha \Bigg[&\frac{2}{3}\left(\frac{r+\alpha}{2\alpha}\right)^{3/2} + 2\left(\frac{r+\alpha}{2\alpha}\right)^{1/2} + \ln\left|\frac{r}{\alpha} - 1\right| \\ &- 2\ln\left|\left(\frac{r+\alpha}{2\alpha}\right)^{1/2} + 1\right|\Bigg] + \text{const.} \end{aligned} \tag{4.1.50}$$

The existence of the term $-2\alpha\ln|r/\alpha - 1|$ leads to the situation, when a particle falls to the event horizon $r = \alpha$ infinitely long in the coordinate time t. Radially falling photons have the same qualitative behaviour.

Let us construct the new harmonic coordinates. The general system of equations conserving the conditions (4.1.49) is given by Fock [178]. We derive these equations for the new time, τ, and radial, ρ, coordinates, saving the spherical symmetry:

$$\begin{aligned} \left(r^2-\alpha^2\right)\tau'' + 2r\tau' - \frac{(r+\alpha)^3}{r-\alpha}\frac{\partial}{c\partial t}\tau &= 0, \\ \left(r^2-\alpha^2\right)\rho'' + 2r\rho' - 2\rho - \frac{(r+\alpha)^3}{r-\alpha}\frac{\partial}{c\partial t}\rho &= 0, \end{aligned} \tag{4.1.51}$$

here again $(') = \partial/\partial r$. Requiring that the new metric will not depend on the new time coordinate τ, one finds that it is only possible if $\tau = A_1 t + A_2 + R(r)$, $\rho = \rho(r)$. Then the system (4.1.51) admits the solution:

$$\tau = A_1 t + A_2 + B_1\left(\ln\left|\frac{r-\alpha}{r+\alpha}\right| + B_2\right), \tag{4.1.52}$$

$$\rho = C_1 r + C_2\left(\frac{1}{2\alpha}\ln\left|\frac{r-\alpha}{r+\alpha}\right| + 1\right), \tag{4.1.53}$$

where A_1, A_2, B_1, B_2, C_1, C_2 are constants. Without losing the generality, one sets $A_2 = B_2 = 0$. The requirement to have the Minkowski metric at $r \to \infty$ after transferring to the Cartesian coordinates leads to $A_1 = C_1 = 1$. Because for $C_2 \neq 0$, there is no one-to-one correspondence between the world points and the points of harmonic coordinates even outside the horizon, we choose $C_2 = 0$. Requiring to have a finite coordinate time, when a test particle approaches horizon, one sets $B_1 = 2\alpha/c$. By this, the aforementioned logarithmic term in (4.1.50) disappears. Finally the transformations (4.1.52) and (4.1.53) acquire the form:

$$c\tau = ct + 2\alpha \ln\left|\frac{r-\alpha}{r+\alpha}\right|, \quad r = r, \quad \theta = \theta, \quad \phi = \phi. \tag{4.1.54}$$

Thus, applying the transformations (4.1.54), one obtains the Schwarzschild solution in the form:

$$\begin{aligned} ds^2 &= -\frac{r-\alpha}{r+\alpha}c^2 d\tau^2 + 2\left(\frac{2\alpha}{r+\alpha}\right)^2 c\,d\tau dr \\ &+ \left[1 + \frac{2\alpha}{r+\alpha} + \left(\frac{2\alpha}{r+\alpha}\right)^2 + \left(\frac{2\alpha}{r+\alpha}\right)^3\right] dr^2 \\ &+ (r+\alpha)^2(d\theta^2 + \sin^2\theta\, d\phi^2), \end{aligned} \tag{4.1.55}$$

instead of (4.1.48), in the new harmonic coordinates. Notice that with the use of the shift $r \to r - \alpha$ in the transformation (4.1.54) and the metric (4.1.55), they go to (4.1.39) and (4.1.40), respectively. One can check that after transferring to asymptotically Cartesian coordinates, the metric (4.1.55) satisfies (4.1.49) as well. At last, the metric coefficients in (4.1.55) are finite everywhere except of the true singularity $r = -\alpha$.

The equation of the "parabolic orbit" acquires the form:

$$\begin{aligned} c\tau = -2\alpha &\left[\frac{2}{3}\left(\frac{r+\alpha}{2\alpha}\right)^{3/2} + 2\left(\frac{r+\alpha}{2\alpha}\right)^{1/2} + \ln\left|\frac{r}{\alpha}+1\right|\right. \\ &\left. - 2\ln\left|\left(\frac{r+\alpha}{2\alpha}\right)^{1/2} + 1\right|\right] + \text{const}, \end{aligned} \tag{4.1.56}$$

where, unlike (4.1.50), there is no divergent logarithmic term. Hence, in the coordinate system (τ, r), a falling particle trajectory without break-downs goes through the Schwarzschild sphere.

Both the form of the metric (4.1.55) and the structure of the light cones

$$c\frac{d\tau}{dr}\bigg|_1 = \frac{(r+\alpha)^2 + (2\alpha)^2}{r^2 - \alpha^2}, \qquad c\frac{d\tau}{dr}\bigg|_2 = -\frac{r+3\alpha}{r+\alpha} \tag{4.1.57}$$

clearly show the following. In the domain $r < \alpha$ both r and τ become spacelike, like in the Finkelstein coordinates [176]. It is permissible, because the metric signature in the domain $r < \alpha$ remains correct, as we have seen above. However, when $r < \alpha$ the description of the particle motion is somewhat unusual: evolution of the spacelike coordinate r is considered in terms of another spacelike coordinate τ .

It follows also from the above that the sections τ = const are spacelike both outside and inside the horizon. If some events belong to the surface τ = const, then in this sense, one can speak about their simultaneity outside the event horizon, on it and inside it. It may be useful for investigations using (3 + 1)-decomposition, see Section 3.1.2.

Particle trajectories and gauge transformations

Because gauge transformations act on the gravitational variables (2.2.68) together with the matter variables (2.2.69), they have to act also on the particle trajectories, see Section 2.2.7. Thus, trajectories in a fixed background spacetime are not gauge invariant, see (2.2.143), which in the weak field approximation has been studied first by Mashhoon and Grishchuk [310]. This part is devoted to this problem related to "parabolic orbits" for the Schwarzschild solution in harmonic coordinates: both in (4.1.48) and in (4.1.55). We consider the *exact* transformations, without using the ξ^μ-vector, that is after the infinite sum in (2.2.68) and (2.2.69) leading to the closed expressions. The gauge transformation from a one field configuration to another is described in Section 2.2.4, illustrated at Figure 2.1 and interpreted by a different choice of the same background.

From the beginning, we construct the field configurations related to the solutions (4.1.48) and (4.1.55). For the latter we make a mapping $\tau \to t$. After that for each of the solutions we choose the unique background metric in the form (4.1.1). By this, we exclude from the consideration the domain $-\alpha \le r < 0$. It is permissible here because we consider the trajectories in the neighborhood of the event horizon only. Thus, using the decomposition (4.1.5), one finds the field configuration for the solution (4.1.48):

$$h^{00} = 1 - \frac{(1+\alpha/r)^3}{1-\alpha/r}, \quad h^{11} = -\frac{\alpha^2}{r^2}, \tag{4.1.58}$$

and the field configuration for the solution (4.1.55):

$$\begin{aligned} h'^{00} &= 1 - \left(1+\frac{\alpha}{r}\right)^2 \left[1 + \frac{2\alpha}{r+\alpha} + \left(\frac{2\alpha}{r+\alpha}\right)^2 + \left(\frac{2\alpha}{r+\alpha}\right)^3\right], \\ h'^{01} &= \frac{4\alpha^2}{r^2}, \quad h'^{11} = -\frac{\alpha^2}{r^2}. \end{aligned} \tag{4.1.59}$$

Returning to the Section 2.2.4, we conclude that the above configurations are connected by gauge transformations induced by the coordinate transformations (4.1.54). Thus, they describe the same physical reality.

Both the configurations (4.1.58) and (4.1.59) have many similar properties. First, they do not depend on time t. Second, both of them represent asymptotically flat spacetime. Then, third, it is not surprising that the total energy calculated for both of the cases is $E = mc^2$. At last, the condition (4.1.49) transforms into

$$h^{\mu\nu}{}_{;\nu} = 0, \quad \text{and} \quad h'^{\mu\nu}{}_{;\nu} = 0 \tag{4.1.60}$$

for both of the configurations.

Now let us discuss the trajectories of test particles. To obtain them one has to vary the action (2.2.140) with respect to the coordinates. At the end, one obtains the equations for 4-velocities u^α and u'^α, formally they are the equations for the geodesics. Recall also that we consider only the "parabolic trajectories". Thus for the configuration (4.1.58) one has:

$$u^0 = \frac{r+\alpha}{r-\alpha}, \quad u^1 = -\left(\frac{2\alpha}{r+\alpha}\right)^{1/2}, \quad u^2 = u^3 = 0. \tag{4.1.61}$$

After integration of $cdt = (u^0/u^1)dr$ one obtains the equation (4.1.50). Thus, now the particle approaches the event horizon, $r = \alpha$, for infinitely long time t. On the other hand, for the field configuration (4.1.59) we have

$$u'^0 = \frac{1}{1+\left(\frac{2\alpha}{r+\alpha}\right)^{1/2}}\left[1+\left(\frac{2\alpha}{r+\alpha}\right)^{1/2} + \frac{2\alpha}{r+\alpha} + \left(\frac{2\alpha}{r+\alpha}\right)^{3/2} + \left(\frac{2\alpha}{r+\alpha}\right)^{2}\right],$$
$$u'^1 = -\left(\frac{2\alpha}{r+\alpha}\right)^{1/2}, \quad u'^2 = u'^3 = 0. \tag{4.1.62}$$

Now, integrating $cdt = (u'^0/u'^1)dr$, one obtains the equation (4.1.56) by replacing τ with t. Unlike (4.1.61), now the particle approaches the event horizon and penetrates it at a time t. Thus, by a gauge transformation, trajectories are saved from a "catastrophic" discontinuity at the event horizon. Or, on the contrary, an initially continuous trajectory can be "broken" also by a gauge transformation.

4.2 Other exact solutions of general relativity

4.2.1 The Friedmann solution for a closed universe

It looks natural that asymptotically flat solutions, including the Schwarzschild solution, can be described in terms of field-theoretical formalism. However the formalism is more powerful. It turns out that even closed Friedmann model can be represented in a consistent way as a field-theoretical configuration. This is illustrated below.

Let us derive the metric of the closed universe in the isotropic coordinates:

$$ds^2 = -c^2dt^2 + \frac{a^2(t)}{1+r^2/4}\eta_{ij}dx^i dx^j. \tag{4.2.1}$$

Recall that this solution has been obtained with the use of the so-called stereographic projection visualized in Figure 4.4. The bottom ("South") pole of the sphere corresponds to the origin of the coordinate frame chosen arbitrary, whereas the top ("North")

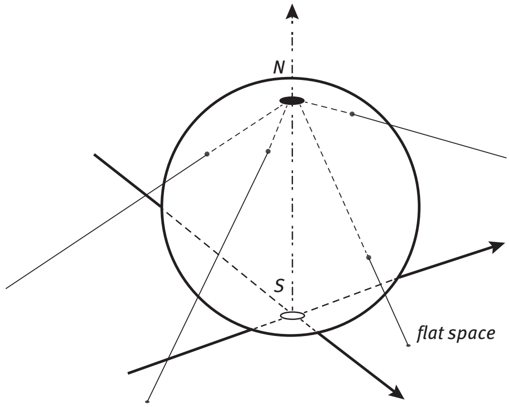

Figure 4.4: A symbolic projection of a 3-sphere onto an Euclidean flat 3-space.

pole is knocked out and identified with the point at spatial infinity in the Minkowski space covered by the isotropic coordinates.

In order to represent (4.2.1) in the field-theoretical formalism we choose a background metric as the Minkowski metric in the Lorentzian coordinates, which are identified with the isotropic coordinates used in (4.2.1). Next, after the decomposition (4.1.8) one obtains for the solution (4.2.1) the field potentials in the form:

$$h^{00} = 1 - \left(\frac{a^2}{1 + r^2/4}\right)^3, \quad h^{11} = h^{22} = h^{33} = -1 + \frac{a^2}{1 + r^2/4}, \tag{4.2.2}$$

which are to be interpreted as a tensor field residing in the Minkowski space. Formally, the field (4.2.2) is defined in the infinite volume of the Minkowski space with the spatial infinity excluded. The fact that the spatial infinity is to be excluded is seen from the stereographic projection and can be also realized if physically-reasonable measurements are done as described below.

Let us conduct the following *gedanken* experiment. Let an observer be placed in the Minkowski space occupied with the field (4.2.2). The observer will perceive especially strange metric relationships when approaching to infinity $r \to \infty$ because the full metric (4.2.1) degenerates in this limit. Let us introduce the spherical coordinates (r, θ, ϕ) and consider a ray traveling in a fixed plane with the angle $\theta = \theta_0 = \text{const}$, along a circle with a fixed radial distance from the observer, $r = r_0 = \text{const}$. The reader should understand that although we consider propagation of light in the Minkowski space, the trajectory of the light ray is subject to the influence of the field (4.2.2) and, hence, the motion of the ray is governed by the equation $ds = 0$ with the interval ds given in (4.2.1). This is interpretation of the light propagation corresponds to the field-theoretical approach to gravity theory.

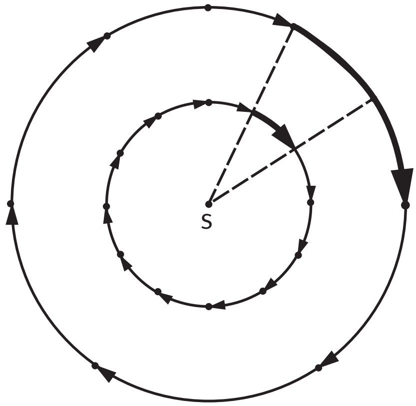

Figure 4.5: Light rays traveling around a circle in the Minkowski space occupied by the gravitational field configuration corresponding to the closed Friedmann universe.

The light cone condition, $ds = 0$, derived with (4.2.1) yields a differential equation for the trajectory of the ray:

$$\frac{d\phi}{dt} = \pm\frac{c}{a}\frac{1 + r_0^2/4}{r_0 \sin\theta_0}, \tag{4.2.3}$$

which corresponds to a uniform motion along the circle. Of course, light rays cannot propagate around circles in the empty Minkowski space if they are not guided by an infinite chain of mirrors forming the circle, see Figure 4.5. It is the gravitational field which plays the role of such mirrors in the field-theoretical interpretation of the motion of light rays around circles in the closed Friedmann universe. As is seen, after integrating dt from (4.2.3) for a full circle with ϕ changing from 0 to 2π, a time interval $\Delta t \to 0$ when $r_0 \to \infty$. This happens in the neighborhood of a point that corresponds to the North pole, see Figure 4.4. Such behavior of light rays in the Minkowski space filled up with the gravitational field illustrates an auxiliary character of the background spacetime in the field-theoretical formulation of general relativity discussed in Section 2.2.7 in detail. Considering the metric relations established through physical measurements and employing (4.2.1) the observer will also infer that space is homogeneous which means that the result of measurements performed by observers located at different points of the Minkowski space are equivalent despite of the *explicit* dependence on the radial coordinate r of the gravitational field potentials (4.2.2).

It is also interesting to construct conserved quantities for the configuration (4.2.2). To achieve this goal it is more convenient to use formula (4.1.29). It is simple to obtain:

$$t_{00}^{\text{tot}} = \frac{3a^3}{4\kappa}\frac{3 - 5r^2/4}{(1 + r^2/4)^5} + \frac{a}{4\kappa}\frac{3 - r^2/4}{(1 + r^2/4)^3}. \tag{4.2.4}$$

When r is not too large, this quantity will be positive; otherwise, however, it will become negative, tending to zero as $r \to \infty$. At times t when $a(t) = 0$ the value of

t_{00}^{tot} will vanish everywhere. Substituting (4.2.4) into (4.1.26), one obtains for the total energy:

$$E = 0. \tag{4.2.5}$$

It is not surprising because the fall-off (4.2.2), $\sim 1/r^2$, significantly stronger than the permissible fall-off for an isolated gravitating system $\sim 1/r$. All the other 9 integrals of motion are equal to zero by the symmetry of the solution. Such zero results, like (4.2.5), coincide with the accepted formulation of the closed universe [285], only, here they are represented in the field-theoretical terms.

The configuration that we have described may be treated as a microuniverse with a Planck-size physical volume. Such a configuration with zero energy, momentum and angular momentum, is exactly the same as it would be for the Minkowski vacuum characterized by the total absence of classical fields and particles, bringing us back to the conjecture of a quantum birth of the universe [201].

The replacement of the topological properties of the manifold (in our case, it is a 3-sphere) by exterior potentials (fields) is encountered in resolving many problems. Thus, Rubakov and Shaposhnikov [392] showed that scalar particles, being not too energetic, could become effectively trapped in a potential wall even in a topologically trivial universe, although non-trivial classical solutions would have to be present to play the role of the external field.

4.2.2 The Abbott-Deser superpotential and its generalizations

In the previous applications we considered various physical models in the framework of the field-theoretical approach with using a flat background only. However, possibilities of the method are clearly wider because arbitrary curved solutions to general relativity can be used as background solutions for describing perturbations. The most popular curved backgrounds in applications are probably the FLRW cosmological solution, see the next chapter, and (anti)-de Sitter ((A)dS) spaces. Both of them are frequently used for describing various kinds of perturbations, see, e. g., [295, 326] and many references there in. Besides that, the AdS space is probably the most interesting geometry in modern fundamental researches, see, e. g., [412] and references there in. The last two subsections of this section are devoted to applications of the field-theoretical methods for describing perturbations in the AdS background.

The AdS and dS solutions have received a lot of attention in the last decades. They are used in describing both the stage of early universe (so-called inflation scenario, see [294] and references therein) and the stage of accelerated expansion. Also, the (A)dS solution is an irreplaceable part of modern theoretical theories and conjectures. For example (A)dS spaces are known to be dual to conformal field theories in one less dimension, dubbed as the AdS/CFT conjecture or Maldacena conjecture, see, e. g., [26, 257, 307, 464, 465].

Here, we consider conserved quantities constructed for perturbations on AdS backgrounds. However, we restrict ourselves to the global conserved quantities constructed with the use of surface integrals only. In the present subsection, the main properties of the AdS solution is outlined, also we construct a family of superpotentials, among of which the famous Abbott-Deser superpotential [1] is presented. Namely, with the use of these superpotentials, conserved surface integrals are derived. In Section 4.2.3, we use the constructed superpotentials in order to calculate the total mass of the Schwarzschild-AdS (S-AdS) black hole. More details on these ideas and their generalization can be found in the later chapters of this book.

The AdS space is the covering space of the 4-dimensional surface, in a flat 5-dimensional space with the signature $\{-,+,+,+,-\}$, described by the equality (here we are considering the negative cosmological constant case, not the de Sitter case):

$$-z_0^2 + z_1^2 + z_2^2 + z_3^2 - z_4^2 = 1/\Lambda \equiv -l^2. \tag{4.2.6}$$

The symmetries of this space are the ten rotations and boosts of the initial 5-dimensional embedding space. Thus, the 4-dimensional AdS space, like the Minkowski space, has a maximal set of Killing vectors. We are only interested in the timelike Killing vector as it is necessary to construct the total mass of the system. Such a global timelike Killing vector mixes z_0 and z_4, its components are $\bar{\boldsymbol{\xi}}_5 = \{-z_4, 0, 0, 0, z_0\}$ for which $\bar{\boldsymbol{\xi}}_5^2 = -z_4^2 - z_0^2 < 0$. By (4.2.6), $\bar{\boldsymbol{\xi}}_5$ is timelike everywhere excluding the point $z_4 = z_0 = 0$. Various coordinates can be used on the surface (4.2.6), one of more popular metrics for the AdS space is

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) c^2 dt^2 + \frac{1}{1 + (r^2/l^2)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.2.7}$$

For such a representation, the timelike Killing vector has a form:

$$\bar{\xi}^\mu = \{-1, \mathbf{0}\}. \tag{4.2.8}$$

All the Killing vectors for the solution (4.2.7) satisfy the standard Killing equation, see (1.1.70) and (1.2.6):

$$\bar{\nabla}_{(\mu}\bar{\xi}_{\nu)} = 0. \tag{4.2.9}$$

To make our presentation more universal we consider a field-theoretical formalism for various types of metric perturbations, see (2.2.120) and (2.2.121). Then, the general relativity equations in the field-theoretical form are presented in (2.2.130) where dynamical variables are represented by the generalized form (2.2.129) for perturbations. Let us adopt this derivation for the AdS background with the metric (4.2.7) and satisfying the background Einstein equations (2.2.107). Let us represent them again,

$$\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}. \tag{4.2.10}$$

We rewrite the equations (2.2.130) for the background (4.2.10). For an arbitrary choice of perturbations the form (2.2.110) is generalized to

$$G^L_{\mu\nu}(\mathfrak{h}^a) + \Lambda\mathfrak{h}^a_{\mu\nu} = \kappa \boldsymbol{t}^{\rm tot}_{\mu\nu}. \tag{4.2.11}$$

Here, $\mathfrak{h}^a_{\mu\nu}$ is defined in (2.2.129):

$$\mathfrak{h}^{\mu\nu}_a \equiv h^a \frac{\partial \bar{\mathfrak{g}}^{\mu\nu}}{\partial \bar{\mathfrak{g}}^a}, \tag{4.2.12}$$

and the linear operator in (4.2.11) generalizes (2.2.27):

$$G^L_{\mu\nu}(\mathfrak{h}^a) \equiv \frac{1}{2}\left(\bar\nabla_\rho{}^\rho \mathfrak{h}^a_{\mu\nu} + \bar g_{\mu\nu}\bar\nabla_{\rho\sigma}\mathfrak{h}^{(a)\rho\sigma} - \bar\nabla_{\rho\nu}\mathfrak{h}^{(a)\rho}_\mu - \bar\nabla_{\rho\mu}h^{(a)\rho}_\nu\right). \tag{4.2.13}$$

For the left hand side of (4.2.11) one has the identity that is based on (4.2.10), analogous to (2.2.111),

$$\bar\nabla^\nu\left(G^L_{\mu\nu}(\mathfrak{h}^a) + \Lambda\mathfrak{h}^a_{\mu\nu}\right) \equiv 0, \tag{4.2.14}$$

and, consequently, one has the conservation law analogous to (2.2.112):

$$\bar\nabla^\nu \boldsymbol{t}^{\rm tot}_{\mu\nu} = 0. \tag{4.2.15}$$

Above, we have shown that in the case of a flat background a desire to construct global (integral) conserved quantities leads to the possibility to construct surface integrals, see (3.2.24–3.2.26), or, more generally in covariant form (3.2.27). To derive such integrals it was necessary to assume that the gravitational equations to general relativity in the field-theoretical form (2.2.105) hold and to contract them with the Killing vectors of the Minkowski space. It turns out that an analogous procedure takes place in the case of the AdS background (4.2.7) satisfying (4.2.10). Below, developing the Abbott-Deser approach [1], we reproduce this program.

Let us contract the equations (4.2.11) with one of Killing vectors, $\bar\xi^\nu$, of AdS background. Then, using (4.2.9) and (4.2.10), one obtains

$$\left(G^{\mu\nu}_L(\mathfrak{h}^a) + \Lambda\mathfrak{h}^{\mu\nu}_a\right)\bar\xi_\nu = \bar\nabla_\nu\left(\bar\xi^{[\mu}\bar\nabla_\rho\mathfrak{h}^{\nu]\rho}_a - \bar\xi^\rho\bar\nabla^{[\mu}\mathfrak{h}^{\nu]}_{(a)\rho} + \mathfrak{h}^{\rho[\mu}_a\bar\nabla_\rho\bar\xi^{\nu]}\right) = \kappa \boldsymbol{t}^{\mu\nu}_{\rm tot}\bar\xi_\nu. \tag{4.2.16}$$

Combining the conservation law (4.2.15), the equality (4.2.9) and fact that $\boldsymbol{t}^{\mu\nu}_{\rm tot}$ is symmetrical, one easily obtains

$$\bar\nabla_\nu\left(\boldsymbol{t}^{\mu\nu}_{\rm tot}\bar\xi_\mu\right) = \partial_\nu\left(\boldsymbol{t}^{\mu\nu}_{\rm tot}\bar\xi_\mu\right) = 0. \tag{4.2.17}$$

Then, following the way of constructing (1.2.87), one obtains a conserved quantity as

$$\mathscr{P}(\bar\xi) = \int_\Sigma d^3x \boldsymbol{t}^{0\nu}_{\rm tot}\bar\xi_\nu. \tag{4.2.18}$$

Here, Σ is derived as a spacelike hypersurface defined as $x^0 = ct = \text{const}$ for the solution (4.2.7).

Now, let us turn to the equation (4.2.16). The expression under a covariant divergence is an antisymmetric tensor density of the weight +1:

$$\mathscr{J}^{\mu\nu}(\mathfrak{h}^a) = \frac{1}{\kappa}\left(\bar{\xi}^{[\mu}\bar{\nabla}_\rho \mathfrak{h}_a^{\nu]\rho} - \bar{\xi}^\rho \bar{\nabla}^{[\mu}\mathfrak{h}^{\nu]}_{(a)\rho} + \mathfrak{h}_a^{\rho[\mu}\bar{\nabla}_\rho \bar{\xi}^{\nu]}\right), \tag{4.2.19}$$

which is called a superpotential. Then, the covariant divergence in (4.2.16) can be exchanged with a partial divergence, and the conserved quantity (4.2.18) can be expressed through a surface integral:

$$\mathscr{P}(\bar{\xi}) = \oint_{\partial\Sigma} ds_i \mathscr{J}^{0i}(\mathfrak{h}^a) \tag{4.2.20}$$

where ds_i is a coordinate element of integration at the boundary $\partial\Sigma$, as usual.

In fact, the equation (4.2.19) represents a family of superpotentials for each of possible definitions of perturbations, $\mathfrak{h}_a^{\mu\nu}$, in (2.2.120), or (2.2.121) for various a. Let us rewrite the difference for metric perturbations (2.2.131):

$${}_{12}\mathfrak{h}^{\mu\nu} = \mathfrak{h}_{a2}^{\mu\nu} - \mathfrak{h}_{a1}^{\mu\nu}. \tag{4.2.21}$$

Because superpotentials in (4.2.19) are linear in perturbations, the difference between them is linear in ${}_{12}\mathfrak{h}^{\mu\nu}$ also:

$$\Delta\mathscr{J}^{\mu\nu} = \mathscr{J}^{\mu\nu}(\mathfrak{h}_{a2}) - \mathscr{J}^{\mu\nu}(\mathfrak{h}_{a1}) = \mathscr{J}^{\mu\nu}({}_{12}\mathfrak{h}). \tag{4.2.22}$$

Consequently, the difference of the integrals (4.2.20) is calculated as

$$\Delta\mathscr{P}(\bar{\xi}) = \oint_{\partial\Sigma} ds_i \mathscr{J}^{0i}({}_{12}\mathfrak{h}). \tag{4.2.23}$$

We derive two more interesting superpotentials from the set (4.2.19). One of them corresponds to the main definition for perturbations in the book,

$$\mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}, \tag{4.2.24}$$

Then, with the use of (4.2.12) one obtains

$$\mathfrak{h}_{a1}^{\mu\nu} = \mathfrak{h}^{\mu\nu}, \tag{4.2.25}$$

and the superpotential (4.2.19) acquires the form:

$$\mathscr{J}^{\mu\nu}(h, \bar{\xi}) = \frac{\sqrt{-\bar{g}}}{\kappa}\left(\bar{\xi}^{[\mu}\bar{\nabla}_\rho h^{\nu]\rho} - \bar{\xi}^\rho \bar{\nabla}^{[\mu}h^{\nu]}{}_\rho + h^{\rho[\mu}\bar{\nabla}_\rho \bar{\xi}^{\nu]}\right), \tag{4.2.26}$$

where $\mathfrak{h}^{\mu\nu} = \sqrt{-\bar{g}}h^{\mu\nu}$. For the other popular definition,

$$\varkappa_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}, \tag{4.2.27}$$

one has from (4.2.12),

$$\mathfrak{h}^{\mu\nu}_{a2} = -\sqrt{-\bar{g}}\left(\varkappa^{\mu\nu} - \tfrac{1}{2}\bar{g}^{\mu\nu}\varkappa^{\rho}{}_{\rho}\right), \tag{4.2.28}$$

and the superpotential (4.2.19) acquires the form:

$$\mathscr{I}^{\mu\nu}_{AD}(\varkappa, \bar{\xi}) = \frac{\sqrt{-\bar{g}}}{\kappa}\left(\bar{\xi}^{\rho}\bar{\nabla}^{[\mu}\varkappa^{\nu]}{}_{\rho} - \bar{\xi}^{[\mu}\bar{\nabla}_{\rho}\varkappa^{\nu]\rho} - \varkappa^{\rho[\mu}\bar{\nabla}_{\rho}\bar{\xi}^{\nu]} \right.$$
$$\left. + \bar{\xi}^{[\mu}\bar{\nabla}^{\nu]}\varkappa^{\rho}{}_{\rho} + \tfrac{1}{2}\varkappa^{\rho}{}_{\rho}\bar{\nabla}^{[\mu}\bar{\xi}^{\nu]}\right). \tag{4.2.29}$$

The last superpotential is merely the famous Abbott-Deser superpotential [1]. Such superpotentials can be useful in various researches, for example, Abbott and Deser with the use of (4.2.29) and supergravity techniques have proved that the AdS space is stable.

Let us make some remarks.

First, the existence of the family (4.2.19) of superpotentials means that there is an ambiguity (4.2.22) in their definition and, possibly, an ambiguity (4.2.23) in the definition of integral conserved quantities for any systems. By construction, we find easily that all of these follow from an ambiguity in definition of energy-momentum (2.2.133) that, in the case of the flat background, has been considered by Boulware and Deser [71]. What can be the criteria for the choice of more preferable ones? The criteria can be purely fundamental, when a theoretical foundation can help for a choice. In Sections 6.3.2, 6.3.3 and 6.4.1, the choice (4.2.24) is recognized as more preferable. Criteria could be found in applications, when known classical solutions are studied, but not all the members of the family (4.2.19) give acceptable results, and, those have to be discarded. In the works [252, 369, 370], calculating the Bondi mass flux at null infinity, the superpotential (4.2.26) with perturbations (4.2.24) has been chosen as a preference.

Second, in spite of the background equations (4.2.10) contain the cosmological constant Λ, an expression (4.2.19) does not contain it at all. Besides, for obtaining (4.2.19) the equations (4.2.10) have been used only, solutions of which are Einstein spaces, not only AdS ones. Also, one easily finds that (4.2.19) holds even if $\Lambda = 0$ in (4.2.10). Thus, finally, the expression for a family of superpotentials (4.2.19) holds for more general background spacetimes: Ricci-flat ones and Einstein spaces.

4.2.3 The total mass of the Schwarzschild-AdS black hole

One of the most known solutions of the vacuum Einstein equations with the cosmological constant is the Schwarzschild-AdS (S-AdS) solution:

$$ds^2 = -\left(1 + \frac{r^2}{l^2} - \frac{r_g}{r}\right)c^2dt^2 + \frac{1}{1 + r^2/l^2 - r_g/r}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.2.30}$$

First, this solution is asymptotically AdS; second, one finds easily that in limit $l^2 \to \infty$ the solution (4.2.23) reduces to the usual Schwarzschild solution (4.1.3).

Thus, to describe the solution (4.2.23) in the field-theoretical terms it is natural to choose the AdS space with the metric (4.2.7) as the background. The non-vanishing Christoffel symbols for the metric (4.2.7) are

$$\bar{\Gamma}^1{}_{22} = -r\left(1+\frac{r^2}{l^2}\right), \quad \bar{\Gamma}^1{}_{33} = -r\sin^2\theta\left(1+\frac{r^2}{l^2}\right), \quad \bar{\Gamma}^2{}_{12} = \bar{\Gamma}^3{}_{13} = \frac{1}{r},$$
$$\bar{\Gamma}^0{}_{01} = -\bar{\Gamma}^1{}_{11} = \frac{1}{r}\frac{1}{\left(1+\frac{l^2}{r^2}\right)}, \quad \bar{\Gamma}^1{}_{00} = \frac{r}{l^2}\left(1+\frac{r^2}{l^2}\right),$$
$$\bar{\Gamma}^2{}_{33} = -\sin\theta\cos\theta, \quad \bar{\Gamma}^3{}_{23} = \cot\theta. \tag{4.2.31}$$

The next step is to define field configurations. Here, the definitions (4.2.24) and (4.2.27) are more relevant. For the first case one obtains

$$h^{00} = -\frac{r_g}{r}\frac{1}{(1+r^2/l^2)\left(1+r^2/l^2-r_g/r\right)},$$
$$h^{11} = -\frac{r_g}{r}, \tag{4.2.32}$$

whereas for the definition (4.2.27) one has

$$\varkappa_{00} = \frac{r_g}{r},$$
$$\varkappa_{11} = \frac{r_g}{r}\frac{1}{(1+r^2/l^2)\left(1+r^2/l^2-r_g/r\right)}. \tag{4.2.33}$$

To calculate the mass of the S-AdS solution one could substitute (4.2.32) into (4.2.26), or substitute (4.2.33) into (4.2.29), using the Killing vector (4.2.8) and covariant derivatives constructed with the Christoffel symbols (4.2.31). However, here, we suggest a more general calculation. Using (4.2.32) and (4.2.33), we can introduce more general perturbations defined in (2.2.121). Because for (4.2.7) and (4.2.30), $\sqrt{-g} = \sqrt{-\bar{g}} = r^2\sin\theta$, one has $\mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu} = \sqrt{-\bar{g}}h^{\mu\nu}$ with $h^{\mu\nu} = g^{\mu\nu} - \bar{g}^{\mu\nu}$. Then the generalized perturbations (2.2.121) can be rewritten as

$$\mathfrak{h}^{\mu\nu}_{(m)} = (\sqrt{-\bar{g}})^m\left(g^{\mu\nu} - \bar{g}^{\mu\nu}\right) = (\sqrt{-\bar{g}})^m h^{\mu\nu}, \tag{4.2.34}$$

$$\mathfrak{h}_{(n)\mu\nu} = (\sqrt{-\bar{g}})^n\left(g_{\mu\nu} - \bar{g}_{\mu\nu}\right) = (\sqrt{-\bar{g}})^n \varkappa_{\mu\nu}, \tag{4.2.35}$$

where $h^{\mu\nu}$ and $\varkappa_{\mu\nu}$ are defined in (2.2.120). Concretely, for the S-AdS solution the perturbations (4.2.34) and (4.2.35) are defined through (4.2.32) and (4.2.33). Then we transform (4.2.34) and (4.2.35) to the form (4.2.26):

$$\mathfrak{h}^{\mu\nu}_{(am)} = \sqrt{-\bar{g}}\left(h^{\mu\nu}_{(m)} - \frac{1-m}{2}\bar{g}^{\mu\nu}h^{\rho}_{(m)\rho}\right)$$
$$= \sqrt{-\bar{g}}\left(h^{\mu\nu} - \frac{1-m}{2}\bar{g}^{\mu\nu}h^{\rho}{}_{\rho}\right), \tag{4.2.36}$$

$$\mathfrak{h}^{\mu\nu}_{(an)} = \sqrt{-\bar{g}}\left(-h^{\mu\nu}_{(n)} + \frac{1-n}{2}\bar{g}^{\mu\nu}h^{\rho}_{(n)\rho}\right)$$
$$= \sqrt{-\bar{g}}\left(-\varkappa^{\mu\nu} + \frac{1-n}{2}\bar{g}^{\mu\nu}\varkappa^{\rho}{}_{\rho}\right), \qquad (4.2.37)$$

where $h^{\mu\nu}_{(m)} = \mathfrak{h}^{\mu\nu}_{(m)}/(\sqrt{-\bar{g}})^m$ and $h^{\mu\nu}_{(n)} = \mathfrak{h}^{\mu\nu}_{(n)}/(\sqrt{-\bar{g}})^n$.

Now, calculate the total mass with the use of (4.2.20) and (4.2.19) at $r \to \infty$ and with the Killing vector (4.2.8). As a field configuration we use the general definitions (4.2.36) or (4.2.37) with (4.2.32) or (4.2.33). All the cases for arbitrary m and n give

$$\mathscr{P}(\bar{\xi}) = mc^2 \qquad (4.2.38)$$

which is quite natural and acceptable. A confusion between m (number) and m (mass) has no to be appeared.

Of course, among all the superpotentials of the family (4.2.19), the superpotentials (4.2.26) and (4.2.29) have been used for obtaining (4.2.38). Thus, first, the result (4.2.38) demonstrates the power of the field-theoretical methods. Second, a calculation of the total mass (4.2.38) for the S-AdS solution cannot resolve the ambiguity in the definition of superpotentials in (4.2.19). Third, returning to the Schwarzschild solution (setting here $l^2 \to \infty$), one easily finds that all the different definitions of variables (4.2.34–4.2.37) give the same result (4.1.12) for the total energy.

5 Field-theoretical derivation of cosmological perturbations

5.1 Introduction: Post-Newtonian, post-Minkowskian and post-Friedmanninan approximations in cosmology

Post-Newtonian celestial mechanics is a branch of fundamental gravitational physics [77, 79, 267, 417] that deals with the theoretical concepts and experimental methods of measuring gravitational fields and testing general relativity both in the solar system and beyond [80, 462]. In particular, the relativistic celestial mechanics of binary pulsars (see [297], and references therein) was instrumental in providing conclusive evidences for the existence of gravitational radiation as predicted by Einstein's theory of relativity [421, 458].

Over the last few decades, various groups within the International Astronomical Union (IAU) have been active in exploring the application of general relativity to the modelling and interpretation of high-accuracy astrometric observations in the solar system. A Working Group on Relativity in Celestial Mechanics and Astrometry was formed in 1994 to define and implement a relativistic theory of reference frames and time scales. This task was successfully completed with the adoption of a series of resolutions on astronomical reference systems, time scales, and Earth rotation models by 24-th General Assembly of the IAU, held in Manchester, UK, in 2000. The IAU resolutions are based on the first post-Newtonian approximation of general relativity which is a conceptual basis of the fundamental astronomy in the solar system [416].

The mathematical formalism of the Post-Newtonian approximations is getting progressively complicated as one goes from the Newtonian to higher orders [115, 405]. For this reason the theory has been primarily developed for an isolated astronomical systems with a matter distribution having a compact support and under simplifying assumptions that gravitational field perturbation is weak everywhere, decays rapidly enough at infinity, and the background spacetime is asymptotically flat. Mathematically, it means that the full spacetime metric, $g_{\alpha\beta}$, is decomposed around the background Minkowskian metric, $\eta_{\alpha\beta}$, into a linear combination

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \kappa_{\alpha\beta}, \tag{5.1.1}$$

where the perturbation $\kappa_{\alpha\beta}$ is represented as the post-Minkowski[1] series decomposition with respect to the powers of the universal gravitational constant,

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1 The term "post-Minkowskian" was introduced by Damour and Blanchet [55] to emphasize that the metric tensor $g_{\alpha\beta}$ is built as a perturbative series around the Minkowski metric $\eta_{\alpha\beta}$, and it does not assume any limitation on the velocity of matter generating gravitational field.

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$$\kappa_{\alpha\beta} = G\underset{(1)}{\kappa_{\alpha\beta}} + G^2\underset{(2)}{\kappa_{\alpha\beta}} + G^3\underset{(3)}{\kappa_{\alpha\beta}} + \dots, \tag{5.1.2}$$

where each term, $\underset{(k)}{\kappa_{\alpha\beta}}$, $(k = 1, 2, 3, \dots)$ of the post-Minkowskian series is decomposed into the post-Newtonian series

$$\underset{(k)}{\kappa_{\alpha\beta}} = c^{-2}\underset{(k)}{\kappa^{[2]}_{\alpha\beta}} + c^{-3}\underset{(k)}{\kappa^{[3]}_{\alpha\beta}} + c^{-4}\underset{(k)}{\kappa^{[4]}_{\alpha\beta}} + \dots, \tag{5.1.3}$$

with respect to the powers of $1/c$, where c is the speed of gravity in general relativity.[2] Post-Minkowskian series (5.1.2) is analytic with respect to the parameter G while the post-Newtonian series (5.1.3) loses analyticity at higher-order approximations where the backreaction of gravitational radiation becomes important [56].

Post-Newtonian approximations suggest that there exists a method to determine $\kappa_{\alpha\beta}$ by doing successive iterations of Einstein's field equations with the tensor of energy-momentum of matter field θ of the localized astronomical system, $T^{\rm p}_{\alpha\beta}(\Theta, g_{\alpha\beta})$, taken as a source of the gravitational field perturbation $\kappa_{\alpha\beta}$. The iterations start from $\kappa_{\alpha\beta} = 0$ which is inserted to the expression for $T^{\rm p}_{\alpha\beta}$ which becomes a well-defined function of the matter variables θ. Einstein's equations are solved at the first iteration yielding $\underset{(1)}{k_{\alpha\beta}}$. This solution is substituted back to the tensor $T^{\rm p}_{\alpha\beta}$ which is used to find $\underset{(2)}{\kappa_{\alpha\beta}}$, and so on. The post-Minkowskian solution for the metric perturbations $\underset{(k)}{\kappa_{\alpha\beta}}$ naturally depend on the retarded time $s = t - r/c$ which accounts for the finite speed of propagation of gravity passing the distance r from the mass emitting gravitational radiation. The post-Newtonian decomposition (5.1.3) of the metric tensor perturbation represents an additional expansion of the retarded functions around the time event t. Thus, the post-Newtonian expansion assumes $r \ll \lambda$, where λ is a characteristic wavelength of gravitational radiation. It means that the post-Newtonian series (5.1.3) is valid only in the near zone of the isolated astronomical system.

The solution of the field equations and the equations of motion of the astronomical bodies are derived in some coordinates $r^\alpha = \{ct, \boldsymbol{r}\}$ where t is the coordinate time, and $\boldsymbol{r} = \{x, y, z\}$ are spatial coordinates. The post-Newtonian theory in asymptotically flat spacetime has a well-defined Newtonian limit determined by:

(1) solution of Poisson's equation for the Newtonian potential, $U \equiv \underset{(1)}{\kappa^{[2]}_{00}}/2$,

$$U(t, \boldsymbol{r}) = \int_{\mathscr{V}} \frac{\rho(t, \boldsymbol{r}')d^3r'}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{5.1.4}$$

where $T^{\rm p}_{00}(\Theta, \eta_{\alpha\beta})$, is the mass density of matter producing the gravitational field,

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2 A common convention is to call c the speed of light irrespectively of the nature of the fundamental interaction under consideration [462] but it may lead to confusion and misinterpretation of gravitational experiments and astronomical observations [158, 270].

(2) equation of motion for massive particles

$$\ddot{\boldsymbol{r}} = \nabla U, \tag{5.1.5}$$

where $\nabla = \{\partial_x, \partial_y, \partial_z\}$ is the operator of gradient, $\boldsymbol{r} = \boldsymbol{r}(t)$ is time-dependent position of a particle (worldline of the particle), and the dot denotes a total derivative with respect to time t,

(3) equation of motion for light (massless particles)

$$\ddot{\boldsymbol{r}} = 0. \tag{5.1.6}$$

These equations are foundational for creation of astronomical ephemerides of celestial bodies in the solar system [79, 267] and in any other localized system of self-gravitating bodies like a binary pulsar [297]. In all practical cases they have to be extended to take into account the post-Newtonian corrections sometimes up to the 3-d post-Newtonian order of magnitude [461]. It is important to notice that in the Newtonian limit the coordinate time t of the gravitational equations of motion (5.1.5), (5.1.6) coincides with the proper time of observer τ that is practically measured with an atomic clock.

So far, the post-Newtonian theory was mathematically successful and passed through numerous experimental tests with a flying colour. Nevertheless, it hides several pitfalls. The first one is the problem of convergence of the post-Newtonian series and regularization of divergent integrals that appear in the post-Newtonian calculations at higher post-Newtonian orders [405]. The second problem is that the background manifold is not asymptotically flat Minkowskian spacetime but the FLRW metric, $\bar{g}_{\alpha\beta}$. We live in the expanding universe which rate of expansion is determined by the present value H_0 of the Hubble parameter $H = H(t)$ depending on time. Therefore, the right thing would be to replace the post-Newtonian decomposition (5.1.1) with a more adequate post-Friedmannian series [435]

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \varkappa_{\alpha\beta}, \tag{5.1.7}$$

where

$$\varkappa_{\alpha\beta} = \varkappa^{\{0\}}_{\alpha\beta} + H\varkappa^{\{1\}}_{\alpha\beta} + H^2\varkappa^{\{2\}}_{\alpha\beta} + \dots, \tag{5.1.8}$$

is the metric perturbation around the cosmological background represented as a series with respect to the Hubble parameter, H. Each term of the series has its own expansion into the post Minkowskian/Newtonian series like (5.1.2) and (5.1.3). For example, $\varkappa^{\{0\}}_{\alpha\beta} = \kappa_{\alpha\beta}$, and there is no asymptotically flat spacetime analogue to $\varkappa^{\{1\}}_{\alpha\beta}, \varkappa^{\{2\}}_{\alpha\beta}$, etc. Generalization of the theory of Post-Newtonian approximations from the Minkowski space to that of the expanding universe is important for extending the applicability of the post-Newtonian celestial dynamics to testing cosmological effects, for more deep understanding of the process of formation of large and small scale structure in the universe and gravitational interaction between pairs of galaxies and their clusters.

Whether cosmological expansion affects gravitational dynamics of bodies inside a localized astronomical system was a matter of considerable efforts of many researchers [62, 93, 148, 149, 280, 311, 313, 410]. Most of the previous works on celestial dynamics in cosmology assumed spherical symmetry of matter distribution and gravitational field which allowed to use exact spherically-symmetric solutions of Einstein's equations approximating the Schwarzschild solution near the body and a cosmological solution far outside of it. Matching of the two solutions in the intermediate zone was achieved in several different ways but all of them suggest some kind of a fine tuning of the size of the matching zone to the cosmological parameters and the mass of the central body. This fine-tuning is physically unrealistic. Furthermore, real astronomical systems in cosmology (galaxies, clusters, filaments, etc.) have no spherical symmetry. McVittie's solution [313] is perhaps the most successful mathematically among the spherically-symmetric approaches but yet lacks a clear physical interpretation [93].

Cosmological observations are now performed so accurately that we need a precise mathematical formulation of the post-Newtonian theory for interpretation of these observations. This theory is not to be limited by the assumption of the spherical symmetry of the isolated astronomical system which must be coupled to the time-dependent background geometry through the gravitational interaction. Theoretical description of the post-Newtonian dynamics of a localized astronomical system in expanding universe should correspond in the limit of vanishing H to the post-Newtonian dynamics in the asymptotically flat spacetime. Such a description will allow us to directly compare the equations of the standard post-Newtonian celestial dynamics with its cosmological counterpart. Therefore, the task is to derive a set of the post-Newtonian equations in cosmology in some coordinates introduced on the background manifold, and to map them onto the set of the Newtonian equations (5.1.4–5.1.6) in asymptotically flat spacetime. The post-Newtonian celestial dynamics would be of a paramount importance for extending the tools of experimental gravitational physics to the field of cosmology, for example, to properly formulate the cosmological extension of the PPN formalism [460]. The present chapter discusses the main ideas and principal results of such a theoretical approach in the linearized approximation with respect to the gravitational perturbations of the cosmological background caused by the presence of a localized astronomical system. The formalism of the present chapter has been employed in [271] to check the theoretical consistency of equations (5.1.4–5.1.6) on expanding cosmological background and to analyse the outcome of some experiments like the excessive Doppler effect discovered by Anderson et al. [9, 10] in the hyperbolic motion of Pioneer 10 and 11 space probes in the solar system.

The original goal in developing the theory of cosmological perturbations was to relate the physics of the early universe to CMB anisotropy and to explain the formation and growth of large-scale structure from a primordial spectrum. The ultimate goal of this theory is to establish a mathematical link between the fundamental physical laws at the Planck epoch and the output of the gravitational wave detectors which are the only experimental devices being capable to measure the parameters and the state of the universe at that time [263]. Originally, two basic approximation schemes for calculating cosmological perturbations have been invented by Lifshitz with his collaborators [292, 293] and, later on, by Bardeen [27]. Lifshitz [292] worked out a coordinate-dependent theory of cosmological perturbations in a synchronous gauge while Bardeen [27] concentrated on finding the gauge-invariant combinations for perturbed quantities and derivation of a perturbation technique based on gauge-invariant field equations. At the same time, Lukash [301] had suggested an original approach for deriving the gauge-invariant scalar equations based on the thermodynamic theory of the Clebsch potential [411] also known in cosmology as the scalar velocity potential [282, 411] or the Taub potential [433]. It turns out that the variational principle with a Lagrangian of cosmological matter formulated in terms of the Clebsch potential, is the most useful mathematical device for developing the theory of relativistic celestial dynamics of localized astronomical systems embedded in expanding cosmological manifold [272].

In the years that followed, the gauge-invariant formalism was refined and improved by Durrer and Straumann [144, 145], Ellis et al. [155–157] and, especially, by Mukhanov et al. [326, 327]. Irrespectively of the approach a specific gauge must be fixed in order to solve equations for cosmological perturbations. Any gauge is allowed and its particular choice is simply a matter of mathematical convenience. Imposing a gauge condition eliminates four degrees of freedom in the cosmological metric pertrubations and brings the differential equations for them to a solvable form. Nonetheless, the residual gauge freedom associated with the tensor nature of the gravitational field remains. This residual gauge freedom leads to appearance of spurious perturbations which must be disentangled from the physical modes. Lifshitz's theory of cosmological perturbations [292, 293] is worked out in a synchronous gauge and contains the spurious modes but they are easily isolated from the physical perturbations and suppressed [200]. The other gauges are described in Bardeen's article [27] and used in cosmological perturbation theory as well. Among them, the longitudinal (conformal or Newtonian) gauge is one of the most common. This gauge is advocated by Mukhanov [327] because it removes spurious coordinate degrees of freedom in scalar perturbations. Detailed comparison of the cosmological perturbation theory in the synchronous and conformal gauges was given by Ma and Bertschinger [302].

Unfortunately, none of the previously known cosmological gauges can be applied for analysis of the cosmological perturbations caused by localized matter distributions like an isolated astronomical system which can be a single star, a planetary system, a galaxy, or even a cluster of galaxies. The reason is that the synchronous gauge has no Newtonian limit and is applicable only for freely falling test particles while the longitudinal gauge separates the scalar, vector and tensor modes in the metric tensor perturbation in the way that is incompatible with the technique of the post-Newtonian approximation schemes having been worked out in asymptotically flat spacetime [267]. We also notice that standard cosmological perturbation technique often operates with harmonic (Fourier) decomposition of both the metric tensor and matter perturbations when one is interested in statistical statements based on the cosmological principle. This technique is unsuitable and must be avoided in sub-horizon approximation for working out the post-Newtonian celestial dynamics of self-gravitating isolated systems.

Current paradigm is that the cosmological generalization of the Newtonian field equations of an isolated gravitating system like the solar system or a galaxy or a cluster of galaxies can be easily obtained by simply making use of the linear principle of superposition with a simple algebraic addition of the local system to the tensor of energy momentum of the background matter. It is assumed that the superposition procedure is equivalent to operating with the Newtonian equations of motion derived in asymptotically flat spacetime and adding to them ("by hands") the tidal force due to the presence of the external universe (see, for example, [311]). Though such a procedure may look pretty obvious it lacks a rigorous mathematical analysis of the perturbations induced on the background cosmological manifold by the local system. This analysis should be done in the way that embeds cosmological variables to the field equations of standard Post-Newtonian approximations not by 'hands' but by precise mathematical technique which is the goal of the present chapter. The variational calculus on manifolds is the most convenient for joining the standard theory of cosmological perturbations with the Post-Newtonian approximations in asymptotically flat spacetime. It allows us to track down the rich interplay between the perturbations of the background manifold with the dynamic variables of the local system which cause these perturbations. The output is the system of the post-Newtonian field equations with the cosmological effects incorporated to them in a physically-transparent and mathematically-rigorous way. This system can be used to solve a variety of physical problems starting from celestial dynamics of localised systems in cosmology to gravitational wave astronomy in expanding universe that can be useful for deeper exploration on scientific capability of such missions as LISA and Big Bang Observer (BBO) [110].

In fact, the problem of whether the cosmological expansion affects the long-term evolution of an isolated N-body system (galaxy, solar system, binary system, etc.) had a long controversial history. The reason is that there was no an adequate mathematical formalism for describing the cosmological perturbations caused by an isolated system so that different authors have arrived to opposite opinions. It seems that McVittie [313] was first who had considered the influence of the expansion of the universe on the dynamics of test particles orbiting around a massive point-like body immersed to the cosmological background. He found an exact solution of the Einstein equations in his model which assumed that the mass of the central body is not constant but decreases as the universe expands. Einstein and Straus [148, 149] suggested a different approach to discuss motion of particles in gravitationally self-interacting systems residing on the expanding background. They showed that a Schwarzschild solution could be smoothly matched to the Friedman universe on a spherical surface separating the two solutions. Inside the surace ("vacuole") the motion of the test particles is totally unaffected by the expansion. Thus, Einstein and Straus [148, 149] concluded that the cosmic expansion is irrelevant for the Solar system. Bonnor [62] generalized the Einstein-Straus vacuole and matched the Schwarzschild region to the inhomogeneous Lemaître-Tolman-Bondi model thus, making the average energy density inside the vacuole be independent of the exterior energy density while in the Einstein-Straus model they must be equal. Bonnor [62] concluded that the local systems expand but at a rate which is negligible compared with the general cosmic expansion. Similar conclusion was reached by Mashhoon et al. [311] who analysed the tidal dynamics of test particles in the Fermi coordinates.

The vacuole solutions are not appropriate for adequate physical solution of the N-body problem in the expanding universe. There are several reasons for it. First, the vacuole is spherically-symmetric while majority of real astronomical systems are not. Second, the vacuole solution imposes physically unrealistic boundary conditions on the matching surface that relates the central mass to the size of the surface and to the cosmic energy density. Third, the vacuole is unstable against small perturbations. In order to overcome these difficulties a realistic approach based on the approximate analytic solution of the Einstein equations for the N-body problem immersed to the cosmological background, is required. In the case of a flat spacetime there are two the most advanced techniques for finding approximate solution of the Einstein equations describing gravitational field of an isolated astronomical system – the post-Newtonian and Post-Minkowskian approximations [115] that have been briefly discussed in introduction. The post-Newtonian approximation technique is applicable to the systems with weak gravitational field and slow motion of matter. The Post-Minkowskian approximations also assume that the field is weak but does not imply any limitation on the speed of matter. The post-Newtonian iterations are based on solving the elliptic-type Poisson equations while the post-Minkowskain approach operates with the hyperbolic-type (wave) D'Alembert equations. The Post-Minkowskian approximations naturally include description of the gravitational radiation emitted by the isolated system while the post-Newtonian scheme has to use additional mathematical methods to describe generation of the gravitational waves [97]. In the present chapter we concentrate on the development of a generic scheme for calculation of cosmological perturbations caused by a localized distribution of matter (small-scale structure) which preserves many advantages of the post-Minkowskian approximation scheme. The cosmological Post-Newtonian approximations are derived from the post-Minkowskian perturbation scheme by making use of the slow-motion expansion with respect to a small parameter v/c where v is the characteristic velocity of matter in the N-body system and c is the fundamental speed.

There were several attempts to work out a physically-adequate and mathematically-rigorous approximation schemes in general relativity in order to construct and to adequately describe dynamics of small-scale structures in the universe. The most notable work in this direction has been done by Kurskov and Ozernoy [281], Futamase et al. [43, 182, 183, 431], Buchert and Ehlers [82, 147], Mukhanov et al. [2, 325–327], Zalaletdinov [469]. These approximation schemes have been designed to track the temporal evolution of the cosmological perturbations from a very large down to a small scale up to the epoch when the perturbation becomes isolated from the expanding cosmological background. These approaches looked hardly connected between each other until recent works by Clarkson et al. [106, 107], Li and Schwarz [288, 289], Räsänen [383], Buchert and Räsänen [83], Wiegand and Schwarz [459]. In particular, Wiegand and Schwarz [459] have shown that the idea of cosmic variance (that is a standard way of thinking) is closely related to the cosmic averages defined by Buchert and Ehlers [82, 147]. All researchers agree that the Post-Newtonian approximations are important to understand the backreaction of the cosmological perturbations on the expansion rate of the universe [2, 182, 241, 242, 326, 469]).

Development of observational cosmology and gravitational wave astronomy demands to extend the linearized theory of cosmological perturbations to second and higher orders of approximation. A fair number of works have been devoted to solving this problem. Non-linear perturbations of the metric tensor and matter affect evolution of the universe and this backreaction of the perturbations should be taken into account. This requires derivation of the effective stress-energy tensor for cosmological perturbations formed by freely-propagating gravitational waves and scalar field [2, 325–327]. The conservation laws for the effective stress-energy tensor are important for derivation of the post-Newtonian equations of motion of the isolated astronomical system.

In the present chapter we construct a non-linear theory of successive cosmological perturbations for isolated systems which generalizes the post-Minkowskian approximation scheme in asymptotically flat spacetime. As a mathematical foundation we use the Lagrangian-based theory of dynamical perturbations of gravitational field on a curved background, see the field-theoretical formulation of general relativity developed in Section 2.2. Let us list its specific advantages more important for the study in the present chapter:

(i) Lagrangian-based approach is covariant and can be implemented for any curved background spacetime that is a solution of the Einstein gravity field equations;
(ii) the system of the partial differential equations describing dynamics of the perturbations is determined by a dynamic Lagrangian $\mathscr{L}^{\rm dyn}$ which is derived from the total Lagrangian of general relativity, $\mathscr{L}_{HE}$, defined in (1.3.16). The presentation is exact, but one can use Taylor expansions with respect to the perturbations and accounting for the background field equations.
(iii) The dynamic Lagrangian $\mathscr{L}^{\rm dyn}$ defines the conserved currents for the perturbations. Energy, angular momentum, etc. can be constructed, if the symmetries of the background manifold exist;
(iv) the dynamic Lagrangian $\mathscr{L}^{\rm dyn}$ and the corresponding field equations for the perturbations are gauge-invariant in any order of the perturbation theory.

Gauge transformations map the background manifold onto itself and are associated with arbitrary (analytic) coordinate transformations on the background spacetime;

(v) the entire perturbation theory is self-reproductive and is extended to the next perturbative order out of a previous iteration by making use of the same equations with a corresponding substitution of quantities from the previous iteration. The linearized approximation is the basic starting point of the theory.

Perhaps, it would be more appropriate to call the perturbative technique explained in this chapter as the post-Friedmannian approximations – the term proposed by Tegmark [435]. However, we shall continue to use the conventional name of post-Newtonian approximations to emphasize that it is applicable not only to large-scale perturbations but also to the discussion of formation and dynamics of small-scale structures in cosmology – the topic being intimately related to relativistic celestial mechanics.

Because the chapter is quite complicated we present now its organization. Section 5.2 introduces the Lagrangian of gravitational field and matter of the background cosmological model as well as the Lagrangian of an isolated astronomical system which perturbs the background cosmological manifold. Section 5.3 describes the geometric structure of the background spacetime manifold of the cosmological model and the corresponding equations of motion of the matter and field variables. Section 5.4 introduces the reader to the theory of the Lagrangian perturbations of the cosmological manifold and the dynamic variables. Section 5.5 makes use of the preceding sections in order to derive the field equations in the gauge-invariant form. Beginning from Section 5.6 we focus on the spatially-flat universe in order to derive the post-Newtonian field equations that generalize the post-Newtonian equations in the asymptotically flat spacetime. These equations are coupled in the scalar sector of the proposed theory. Therefore, we consider in Section 5.7 a few particular cases when the equations can be fully decoupled one from another, and solved in terms of the retarded potentials.

The present chapter is very rich in notations, but one can be convinced that there is no confusion with the other notations in the book. Also, every time, when there is no confusion about the system of units, we shall choose, as usual here, a geometrized system of units such that $G = c = 1$.

Notations

- $\bar{g}_{\alpha\beta}$ is the FLRW metric on the background spacetime manifold;
- $\mathtt{f}_{\alpha\beta}$ is the metric on the conformal spacetime manifold: $\bar{g}_{\alpha\beta} = a(\eta)\mathtt{f}_{\alpha\beta}$;
- T and $X^i \equiv \{X, Y, Z\}$ are the coordinate time and isotropic spatial coordinates on the background FLRW manifold;
- $X^\alpha \equiv \{X^0, X^i\} = \{c\eta, X^i\}$ are the conformal coordinates with η being a conformal time;
- a prime $\mathscr{Q}' \equiv d\mathscr{Q}/d\eta$ denotes a total derivative with respect to the conformal time η;

- a dot $\dot{\mathscr{Q}} \equiv d\mathscr{Q}/dT$ denotes a total derivative with respect to the cosmic time T;
- a vertical bar, $\mathscr{Q}_{|\alpha}$ denotes a covariant derivative of a tensor quantity $\mathscr{Q}$ with respect to the background metric $\bar{g}_{\alpha\beta}$;
- a semicolon, $\mathscr{Q}_{;\alpha}$ denotes a covariant derivative of a tensor quantity $\mathscr{Q}$ with respect to the conformal metric $\mathtt{f}_{\alpha\beta}$; it will be no confusion with a covariant derivative with respect to a flat spacetime metric $\eta_{\alpha\beta}$;
- the tensor indices of geometric objects on the conformal spacetime are raised and lowered with the conformal metric $\mathtt{f}_{\alpha\beta}$;
- the scale factor of the FLRW metric is denoted as $R \equiv R(T)$, or as $a \equiv a(\eta) = R[T(\eta)]$; it will be no confusion with a scalar curvature in a dynamic spacetime in general relativity;
- the Hubble parameter, $H \equiv \dot{R}/R$, and the conformal Hubble parameter, $\mathscr{H} = a'/a$.

5.2 Lagrangian and field variables

Basing on general relativity, we consider a universe filled up with matter consisting of three components. The first two components are: (1) an ideal fluid composed of particles of one type with transmutations excluded; (2) a scalar field; and (3) a matter of the localized astronomical system. The ideal fluid consists of baryons and cold dark matter, while the scalar field describes dark energy [7]. We assume that these two components do not interact with each other directly, and are the source of the *Friedmann-Lemître-Robertson-Walker* (FLRW) geometry. There is no dissipation in the ideal fluid and in the scalar field so that they can only interact through the gravitational field. It means that the equations of motion for the fluid and the scalar field are decoupled in the main approximation, and we can calculate their evolution separately. Mathematically, it means that the Lagrangian of the ideal fluid and that of the scalar field depend only on their own field variables and the metric tensor.

The tensor of energy-momentum of matter of the localized astronomical system is not specified in agreement with the approach adopted in the post-Newtonian approximation scheme developed in the asymptotically flat spacetime [115, 269]. This allows us to generate all possible types of cosmological perturbations: scalar, vector and tensor modes. We are the most interested in developing our formalism for application to the astronomical system of massive bodies bound together by intrinsic gravitational forces like the solar system, galaxy, or a cluster of galaxies. It means that our approach admits a large density contrast between the background matter and the matter of the localized system. The localized system perturbs the background matter and gravitational field of FLRW metric locally but it is not included to the matter source of the background geometry, at least, in the approximation being linearized with respect to the metric tensor perturbation. Our goal is to study how the perturbations of the background matter and gravitational field are incorporated to the gravitational field perturbations of the standard post-Newtonian theory of relativistic celestial dynamics.

Let us now consider the action functional and the Lagrangian of each component.

5.2.1 Action functional

We consider a theory with the Hilbert-Einstein action (2.2.1), let us rewrite it:

$$S = \int_{\Omega} d^4x \mathscr{L}^{HE} = -\frac{1}{16\pi}\int_{\Omega} d^4x \mathscr{L}^{H} + \int_{\Omega} d^4x \mathscr{L}^{M}. \tag{5.2.1}$$

The Lagrangian, here, is specified in a more concrete form. Thus, for matter sources:

$$\mathscr{L}^M = \mathscr{L}^{\mathrm{m}} + \mathscr{L}^{\mathrm{q}} + \mathscr{L}^{\mathrm{p}}, \tag{5.2.2}$$

where $\mathscr{L}^{\mathrm{m}}$, $\mathscr{L}^{\mathrm{q}}$ are Lagrangians of the dark matter, and the scalar field that governs the accelerated expansion of the universe [197], and $\mathscr{L}^{\mathrm{p}}$ is the Lagrangian describing the source of the cosmological perturbations. Gravitational field Lagrangian is defined by the Hilbert Lagrangian (1.3.1):

$$\mathscr{L}^H = \sqrt{-g}R = \mathscr{R}. \tag{5.2.3}$$

Correct choice of the matter variables is a key element in the development of the Lagrangian theory of the post-Newtonian perturbations of the cosmological manifold caused by a localized astronomical system. Already the Lagrangian treatment of ideal fluid and scalar field in the Minkowski space has been given in Section 1.2.5, where the accent is related to constructing conserved quantities. Below we provide this treatment in a curved spacetime of general relativity with the action (5.2.1).

5.2.2 Lagrangian of the ideal fluid

The ideal fluid is characterized by the following thermodynamic parameters: the rest-mass density ρ_{m}, the specific internal energy Π_{m} (per unit of mass), pressure p_{m}, and entropy s_{m} where the sub-index "m" stands for "matter". We shall assume that the entropy of the ideal fluid remains constant, that excludes it from further consideration. The standard approach to the theory of cosmological perturbations preassumes that the constant entropy excludes rotational (vector) perturbations of the fluid component from the start, and only scalar (adiabatic) perturbations are generated [7, 327, 456, 457]. However, the present chapter deals with the cosmological perturbations that are generated by a localized astronomical system described by its own Lagrangian (see Section 5.2.4) which is left as general as possible. This leads to the tensor of energy-momentum of the matter of the localized system that incorporates the rotational motion of matter which is the source of the rotational perturbations of the background ideal fluid. This extrapolates the concept of the gravitomagnetic field of the post-Newtonian dynamics of localized systems in the asymptotically flat spacetime [79, 105, 267] to cosmology. Further details regarding the vector perturbations are given in Section 5.5.5 of the present chapter.

The total energy density of the fluid

$$\epsilon_m = \rho_m(1 + \Pi_m). \tag{5.2.4}$$

One more thermodynamic parameter is the specific enthalpy of the fluid defined as

$$\mu_m = \frac{\epsilon_m + p_m}{\rho_m} = 1 + \Pi_m + \frac{p_m}{\rho_m}. \tag{5.2.5}$$

In the most general case, the thermodynamic equation of state of the fluid is given by equation $p_m = p_m(\rho_m, \Pi_m)$, where the specific internal energy Π_m is related to pressure by the first law of thermodynamics.

Since the entropy has been assumed to be constant, the first law of thermodynamics reads

$$d\Pi_m + p_m d\left(\frac{1}{\rho_m}\right) = 0. \tag{5.2.6}$$

It can be used to derive the following thermodynamic relationships

$$dp_m = \rho_m d\mu_m, \tag{5.2.7}$$

$$d\epsilon_m = \mu_m d\rho_m, \tag{5.2.8}$$

which means that all thermodynamic quantities are solely functions of the specific enthalpy μ_m, for example, $\rho_m = \rho_m(\mu_m)$, $\Pi_m = \Pi_m(\mu_m)$, etc. The equation of state is also a function of the variable μ_m, that is

$$p_m = p_m(\mu_m). \tag{5.2.9}$$

Derivatives of the thermodynamic quantities with respect to μ_m can be calculated by making use of equations (5.2.7) and (5.2.8), and the definition of the (adiabatic) speed of sound v_s of the fluid

$$\frac{\partial p_m}{\partial \epsilon_m} = \frac{v_s^2}{c^2}, \tag{5.2.10}$$

where the partial derivative is taken under a condition that the entropy, s_m, of the fluid does not change. Then, the derivatives of the thermodynamic quantities take on the following form

$$\frac{\partial p_m}{\partial \mu_m} = \rho_m, \qquad \frac{\partial \epsilon_m}{\partial \mu_m} = \frac{c^2}{v_s^2}\rho_m, \qquad \frac{\partial \rho_m}{\partial \mu_m} = \frac{c^2}{v_s^2}\frac{\rho_m}{\mu_m}, \tag{5.2.11}$$

where all partial derivatives are performed under the same condition of constant entropy.

The Lagrangian of the ideal fluid is usually taken in the form of the total energy density, $\tilde{\mathscr{L}}^{\mathrm{m}} = \sqrt{-g}\epsilon_{\mathrm{m}}$ [315]. However, this form is less convenient for applying the variational calculus on manifolds. The above thermodynamic relationships and the integration by parts of the action (5.2.1) allows us to recast the Lagrangian, $\tilde{\mathscr{L}}^{\mathrm{m}}$, to the form of pressure, $\mathscr{L}^{\mathrm{m}} = -\sqrt{-g}p_{\mathrm{m}}$, so that the Lagrangian density becomes

$$\mathscr{L}^{\mathrm{m}} = -\sqrt{-g}p_{\mathrm{m}} = \sqrt{-g}\left(\epsilon_{\mathrm{m}} - \rho_{\mathrm{m}}\mu_{\mathrm{m}}\right). \tag{5.2.12}$$

Theoretical description of the ideal fluid as a dynamic system on spacetime manifold is given the most conveniently in terms of the Clebsch potential, Φ which is also called the velocity potential [411]. In the case of a single-component ideal fluid the Clebsch potential is introduced by the following relationship

$$\mu_{\mathrm{m}} w_\alpha = -\Phi_{,\alpha}. \tag{5.2.13}$$

In fact, equation (5.2.13) is a solution of relativistic equations of motion of the ideal fluid [282].

The Clebsch potential is a primary field variable in the Lagrangian description of the isentropic ideal fluid. The four-velocity is normalized to $w^\alpha w_\alpha = g_{\alpha\beta}w^\alpha w^\beta = -1$, so that the specific enthalpy can be expressed in the following form

$$\mu_{\mathrm{m}} = \sqrt{-g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}}. \tag{5.2.14}$$

One may also notice that

$$\mu_{\mathrm{m}} = w^\alpha \Phi_{,\alpha}. \tag{5.2.15}$$

It is important to notice that the Clebsch potential Φ has no direct physical meaning as it can be changed to another value $\Phi \to \Phi' = \Phi + \tilde{\Phi}$ such that the gauge function, $\tilde{\Phi}$, is constant along the worldlines of the fluid: $w^\alpha \tilde{\Phi}_{,\alpha} = 0$.

In terms of the Clebsch potential the Lagrangian (5.2.12) of the ideal fluid is

$$\mathscr{L}^{\mathrm{m}} = \sqrt{-g}\left(\epsilon_{\mathrm{m}} - \rho_{\mathrm{m}}\sqrt{-g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}}\right). \tag{5.2.16}$$

Metrical tensor of energy-momentum of the ideal fluid is obtained by taking a variational derivative of the Lagrangian (5.2.16) with respect to the metric tensor,

$$T^{\mathrm{m}}_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}^{\mathrm{m}}}{\delta g^{\alpha\beta}}. \tag{5.2.17}$$

Calculation yields

$$T^{\mathrm{m}}_{\alpha\beta} = \left(\epsilon_{\mathrm{m}} + p_{\mathrm{m}}\right) w_\alpha w_\beta + p_{\mathrm{m}} g_{\alpha\beta}, \tag{5.2.18}$$

where $w^\alpha = dx^\alpha/d\tau$ is the four-velocity of the fluid, and τ is the proper time of the fluid element taken along its worldline. This is a standard form of the tensor of energy-momentum of the ideal fluid [315]. Because the Lagrangian (5.2.16) is expressed in terms of the dynamical variable Φ, the Noether approach based on taking the variational derivative of the Lagrangian with respect to the field variable, can be applied to derive the canonical tensor of the energy-momentum of the ideal fluid. This calculation has been done in [267, pp. 334–335] and it leads to the expression (5.2.18). It could be expected because we assumed that the ideal fluid consists of bosons. The metrical and canonical tensors of energy-momentum for the liquid differ, if and only if, the liquid's particles are fermions (see [267, pp. 331–332] for more detail). We do not consider the fermionic liquids in the present chapter.

5.2.3 Lagrangian of scalar field

The Lagrangian of the scalar field Ψ is given by

$$\mathscr{L}^{\mathrm{q}} = \sqrt{-g}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\Psi\partial_\beta\Psi + W\right), \tag{5.2.19}$$

where $W \equiv W(\Psi)$ is a potential of the scalar field. We assume that there is no direct coupling between the scalar field and the matter of the ideal fluid. They can interact only through the gravitational field. Many different potentials of the scalar field are used in cosmology [7]. At this step, we do not chose a specific form of the potential which will be selected later.

Metrical tensor of energy-momentum of the scalar field is obtained by taking a variational derivative

$$T^{\mathrm{q}}_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}^{\mathrm{q}}}{\delta g^{\alpha\beta}}, \tag{5.2.20}$$

that yields

$$T^{\mathrm{q}}_{\alpha\beta} = \partial_\alpha\Psi\partial_\beta\Psi - g_{\alpha\beta}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi + W(\Psi)\right]. \tag{5.2.21}$$

The canonical tensor of energy-momentum of the scalar field is obtained by applying the Noether theorem and leads to the same expression (5.2.21).

One can *formally* reduce the tensor (5.2.21) to the form similar to that of the ideal fluid by making use of the following procedure. First, we define the analogue of the specific enthalpy of the scalar field "fluid"

$$\mu_{\mathrm{q}} = \sqrt{-g^{\sigma\nu}\partial_\sigma\Psi\partial_\nu\Psi}, \tag{5.2.22}$$

and the effective four-velocity, v^α, of the "fluid"

$$\mu_{\mathrm{q}} v_\alpha = -\partial_\alpha\Psi. \tag{5.2.23}$$

The four-velocity v^α is normalized to $v_\alpha v^\alpha = -1$. Therefore, the scalar field enthalpy μ_q can be expressed in terms of the partial derivative from the scalar field

$$\mu_q = v^\alpha \partial_\alpha \Psi. \tag{5.2.24}$$

Then, we introduce the analogue of the rest mass density ρ_q of the scalar field "fluid" by defining,

$$\rho_q \equiv \mu_q = v^\alpha \partial_\alpha \Psi = \sqrt{-g^{\sigma\nu}\partial_\sigma \Psi \partial_\nu \Psi}. \tag{5.2.25}$$

As a consequence of the above definitions, the energy density, ϵ_q and pressure p_q of the scalar field "fluid" can be introduced as follows

$$\epsilon_q \equiv -\frac{1}{2} g^{\sigma\nu}\partial_\sigma \Psi \partial_\nu \Psi + W(\Psi) = \frac{1}{2}\rho_q \mu_q + W(\Psi), \tag{5.2.26}$$

$$p_q \equiv -\frac{1}{2} g^{\sigma\nu}\partial_\sigma \Psi \partial_\nu \Psi - W(\Psi) = \frac{1}{2}\rho_q \mu_q - W(\Psi). \tag{5.2.27}$$

One notices that a relationship

$$\mu_q = \frac{\epsilon_q + p_q}{\rho_q}, \tag{5.2.28}$$

between the specific enthalpy μ_q, the density ρ_q, the pressure p_q and the energy density ϵ_q, of the scalar field "fluid" formally holds on the same form (5.2.5) as in the case of the barotropic ideal fluid.

After applying the above-given definitions in equation (5.2.21), it is formally reduced to the tensor of energy-momentum of an ideal fluid

$$T^q_{\alpha\beta} = \left(\epsilon_q + p_q\right) v_\alpha v_\beta + p_q g_{\alpha\beta}. \tag{5.2.29}$$

It is worth emphasizing that the analogy between the tensor of energy-momentum (5.2.29) of the scalar field "fluid" with that of the barotropic ideal fluid (5.2.18) is rather formal since the scalar field, in the most general case, does not satisfy *all* required thermodynamic equations because of the presence of the potential $W = W(\Psi)$ in the energy density ϵ_q, and pressure p_q of the scalar field. For example, equation of continuity (5.3.66) for scalar field differs from that for the ideal fluid (5.3.58) if the potential $W(\Psi) \neq 0$.

5.2.4 Lagrangian of a localized astronomical system

The Lagrangian $\mathscr{L}^p$ of matter of a localized astronomical system (a small-scale structure inhomogeneity) which perturbs the geometry of the background manifold of the FLRW metric, can be chosen arbitrary. We shall call the perturbation of the background manifold that is induced by $\mathscr{L}^p$, the *bare* perturbation. We assume that the matter of the bare perturbation is described by a (multi-component) field variable,

Θ, which physical meaning depends on a specific problem we want to solve. The Lagrangian density of the bare perturbation is given by $\mathscr{L}^{\rm p} = \sqrt{-g}L^{\rm p}\left(\Theta, g_{\alpha\beta}\right)$. Tensor of energy-momentum of the matter of the bare perturbation, $T^{\rm p}_{\alpha\beta}$, is obtained by taking a variational derivative

$$T^{\rm p}_{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta \mathscr{L}^{\rm p}}{\delta g^{\alpha\beta}}. \tag{5.2.30}$$

Tensor $T^{\rm p}_{\alpha\beta}$ is a source of the small-scale gravitational perturbation of the background manifold that is associated with a particular solution of the linearised Einstein equations which will be derived in next sections.

5.3 Background manifold

5.3.1 Hubble flow

We shall consider the background universe as described by the Friedmann-Lemître-Robertson-Walker (FLRW) metric. The functional form of the metric depends on the coordinates introduced on the manifold. Because the FLRW metric describes homogeneous and isotropic spacetime there is a preferred class of coordinates which clearly reveal these properties of the background manifold. These coordinates materialize a special set of freely falling observers, called comoving observers. These observers are following with the flow of the expanding universe and have constant values of spatial coordinates. The proper distance between the comoving observers increases in proportion to the scale factor $R(T)$. In the preferred cosmological coordinates, the time coordinate of the FLRW metric is just the proper time as measured by the comoving observers. A particle moving relative to the local comoving observers has a peculiar velocity with respect to the Hubble flow. An observer with a non-zero peculiar velocity does not see the universe as isotropic.

For example, the peculiar velocity of the solar system implies the dipole anisotropy of cosmic microwave background (CMBR) radiation corresponding to $|\boldsymbol{v}_\odot| = 369.0 \pm 0.9$ km·s^{-1}, towards a point with the galactic coordinates $(l, b) = (264^\circ, 48^\circ)$ [234, 246]. Such a solar system's velocity implies a velocity $|\boldsymbol{v}_{LG}| = 627 \pm 22$ km·s^{-1} toward $(l, b) = (276^\circ, 30^\circ)$ for our Galaxy and the Local Group of galaxies relative to the CMBR [177, 262]. The existence of the preferred frame in cosmology should not be understood as a violation of the Einstein principle of relativity. Indeed, any coordinate chart can be used in order to describe the FLRW metric. A preferred frame exists merely because the FLRW metric admits only six-parametric group (3 spatial translations and 3 spatial rotations) as contrasted with the ten-parametric group of Minkowski (or De Sitter) spacetime which includes the time translation and three Lorentz boosts as well. The metric of FLRW does not remain invariant with respect to the time translation and the Lorentz transformations because its expansion makes different spacelike hypersurfaces non-equivalent. It may lead to some interesting observational predictions of cosmological effects within the solar system [271].

5.3.2 Friedmann-Lemître-Robertson-Walker metric

In what follows, we shall consider the problem of calculation of the post-Newtonian perturbations in the expanding universe described by the FLRW class of models. The FLRW metric is an exact solution of Einstein's field equations of general relativity that describes a homogeneous, isotropically expanding or contracting universe. The general form of the metric follows from the geometric properties of homogeneity and isotropy of the manifold [456, 457]. Einstein's equations are only needed to derive the scale factor of the universe as a function of time.

The most general form of the FLRW metric is given by

$$ds^2 = -dT^2 + R^2\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2\left(d^2\vartheta + \sin^2\vartheta d^2\upsilon\right)\right], \tag{5.3.1}$$

where T is the coordinate time, $\{\rho, \vartheta, \upsilon\}$ are spherical coordinates, $R = R(T)$ is the scale factor depending on time and characterizing the size of the universe compared to the present value of $R = 1$. The time T has a physical meaning of the proper time of a comoving observer that is being at rest with respect to the cosmological frame of reference. The present epoch corresponds to the value of the time $T = T_0$. The constant k can take on three different values $k = \{-1, 0, +1\}$, where $k = -1$ corresponds to the spatial hyperbolic geometry, $k = 0$ does the spatially flat FLRW model, and $k = +1$ does the spatially closed world [315].

The Hubble parameter H characterizes the rate of the temporal evolution of the universe. It is defined by

$$H \equiv \frac{\dot{R}}{R} = \frac{1}{R}\frac{dR}{dT}. \tag{5.3.2}$$

For mathematical reasons, it is convenient to introduce a conformal time, η, via differential equation

$$d\eta = \frac{dT}{R(T)}. \tag{5.3.3}$$

If the time dependence of the scale factor is known, the equation (5.3.3) can be solved, thus, yielding $T = T(\eta)$. It allows us to re-express the scale factor $R(T)$ in terms of the conformal time, $R\left(T(\eta)\right) \equiv a(\eta)$. The conformal Hubble parameter is, then, defined as

$$\mathscr{H} \equiv \frac{a'}{a} = \frac{1}{a}\frac{da}{d\eta}. \tag{5.3.4}$$

The two expressions for the Hubble parameters are related by means of equation

$$H = \frac{\mathcal{H}}{a}, \tag{5.3.5}$$

that allows us to link their time derivatives

$$a^2\dot{H} = \mathcal{H}' - \mathcal{H}^2, \tag{5.3.6}$$

$$a^3\ddot{H} = \mathcal{H}'' - 4\mathcal{H}\mathcal{H}' + 2\mathcal{H}^3, \tag{5.3.7}$$

and so on.

It is also convenient to introduce the isotropic Cartesian coordinates $X^i = \{X, Y, Z\}$, by transforming the radial coordinate

$$\rho = \frac{r}{1 + \frac{k}{4}r^2}, \tag{5.3.8}$$

and defining $r^2 = X^2 + Y^2 + Z^2 = \delta_{ij}X^iX^j$. In the isotropic coordinates the interval (5.3.1) takes on the following form

$$ds^2 = \bar{g}_{\alpha\beta}dX^\alpha dX^\beta, \tag{5.3.9}$$

where the coordinates $X^\alpha = \{X^0, X^1, X^2, X^3\} = \{\eta, X, Y, Z\}$, and the metric has a conformal form

$$\bar{g}_{\alpha\beta} = a^2(\eta)\mathrm{g}_{\alpha\beta} \tag{5.3.10}$$

$$\mathrm{g}_{00} = -1, \qquad \mathrm{g}_{0i} = 0, \qquad \mathrm{g}_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{k}{4}r^2\right)^2}. \tag{5.3.11}$$

The spacetime interval (5.3.9) in the isotropic Cartesian coordinates reads

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{\delta_{ij}dX^idX^j}{\left(1 + \frac{k}{4}r^2\right)^2}\right]. \tag{5.3.12}$$

The distinctive property of the isotropic coordinates in the FLRW metric is that the radial coordinate r is defined in such a way that the three-dimensional space looks exactly Euclidean and null cones appear in it as round spheres irrespectively of the value of the space curvature k. The isotropic coordinates do not represent proper distances on the sphere, nor does the radial coordinate r represents a proper radial distance measured with the help of radar astronomy technique. The proper spatial distance in the isotropic coordinates is $(1 + kr^2/4)^{-1}ar$ [456].

The FLRW metric presented in the conformal form by equation (5.3.12) singles out a preferred cosmological reference frame defined by the congruence of worldlines of the fiducial test particles being at rest with respect to the spatial coordinates X^i. Four-velocity of a fiducial particle is denoted as $\bar{U}^\alpha = dX^\alpha/d\tau$, where $d\tau = -ds$ is the proper time on the worldline of the particle. In the isotropic conformal coordinates, $\bar{U}^\alpha = (1/a, 0, 0, 0)$. The four-velocity is a unit vector, $\bar{U}^\alpha\bar{U}_\alpha = \bar{g}_{\alpha\beta}\bar{U}^\alpha\bar{U}^\beta = -1$. It implies that the covariant components of the four-velocity are $\bar{U}_\alpha = (-a, 0, 0, 0)$. In the preferred frame the universe looks homogeneous and isotropic. The choice of the isotropic Cartesian coordinates reflects these fundamental properties explicitly in the symmetric form of the metric (5.3.10). However, the set of the fiducial particles is a mathematical idealization. In reality, any isolated astronomical systems (galaxy, binary star, the solar system, etc.) have a peculiar velocity with respect to the preferred cosmological frame formed by the Hubble flow. We have to introduce a locally-inertial coordinate chart which is associated with the isolated system and moves along with it. Transformation from the preferred cosmological frame to the local chart must include the Lorentz boost and a geometric part due to the expansion and curvature of cosmological spacetime. It can take on multiple forms which originate from certain geometric and/or experimental requirements [93, 101, 237, 259].

We do not impose specific limitations on the choice of coordinates on the background manifold and keep the overall formalism of the Post-Newtonian approximations, covariant. The arbitrary coordinates are denoted as $x^\alpha = (x^0, x^i)$ and they are related to the preferred isotropic coordinates $X^\alpha = (\eta, X^i)$ by the coordinate transformation $x^\alpha = x^\alpha\left(X^\beta\right)$. This transformation has inverse $X^\alpha = X^\alpha\left(x^\beta\right)$, at least in some local domain of the background manifold. In this domain, the matrices of the coordinate transformations

$$\Lambda^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial X^\beta}, \qquad\qquad \mathrm{M}^\alpha{}_\beta = \frac{\partial X^\alpha}{\partial x^\beta}, \tag{5.3.13}$$

and they satisfy to the apparent equalities $\Lambda^\alpha{}_\gamma \mathrm{M}^\gamma{}_\beta = \delta^\alpha_\beta$ and $\mathrm{M}^\alpha{}_\gamma\Lambda^\gamma{}_\beta = \delta^\alpha_\beta$.

Four-velocity of the Hubble observers written in the arbitrary coordinates has the following form

$$\bar{u}^\alpha = \Lambda^\alpha{}_\beta\bar{U}^\beta = a^{-1}\Lambda^\alpha{}_0, \qquad \bar{u}_\alpha = \mathrm{M}^\beta{}_\alpha U_\beta = -a\mathrm{M}^0{}_\alpha. \tag{5.3.14}$$

The background FLRW metric (5.3.10) written down in the arbitrary coordinates, x^α, takes on the following form

$$\bar{g}_{\alpha\beta}(x^\alpha) = a^2\bar{\mathfrak{f}}_{\alpha\beta}(x^\alpha). \tag{5.3.15}$$

Here the scalar function $a(x^\alpha) \equiv a\left[\eta(x^\alpha)\right]$, and the conformal metric

$$\bar{\mathfrak{f}}_{\alpha\beta}(x^\alpha) = \mathrm{M}^\mu{}_\alpha\mathrm{M}^\nu{}_\beta\mathfrak{g}_{\mu\nu}(X^i). \tag{5.3.16}$$

Any metric admits 1+3 decomposition with respect to a congruence of a timelike vector field [315]. FLRW metric admits a privileged congruence formed by the four-velocity $\bar{u}^\alpha$ of the Hubble observers which is a physically privileged vector field. The 1+3 decomposition of the FLRW metric is applied in arbitrary coordinates and has the following form

$$\bar{g}_{\alpha\beta} = -\bar{u}_\alpha \bar{u}_\beta + \bar{P}_{\alpha\beta}, \tag{5.3.17}$$

where the tensor

$$\bar{P}_{\alpha\beta} = a^2 \mathrm{M}^i{}_\alpha \mathrm{M}^j{}_\beta \mathfrak{g}_{ij}, \tag{5.3.18}$$

describes the metric on the spacelike hypersurface being everywhere orthogonal to the four-velocity $\bar{u}^\alpha$ of the Hubble flow. Tensor $\bar{P}_{\alpha\beta}$ is the operator of projection on this hypersuface. It can be also interpreted as a metric on the hypersurace of orthogonality to the Hubble vector flow. Equation (5.3.17) can be used in order to prove that $\bar{P}_{\alpha\beta}$ satisfies the following relationship

$$\bar{P}^{\beta\mu}\bar{P}_\beta{}^\nu = \bar{P}^{\mu\nu}, \tag{5.3.19}$$

which can be confirmed by inspection. The trace $\bar{P}^\alpha{}_\alpha = \bar{g}^{\alpha\beta}\bar{P}_{\alpha\beta} = \bar{P}^{\alpha\beta}\bar{P}_{\alpha\beta} = 3$.

Now, we consider how to express the partial derivatives of any scalar function $F = F(\eta)$, which depends only on the conformal time $\eta = \eta(x^\alpha)$, in terms of the four-velocity $\bar{u}^\alpha$ of the Hubble flow. Taking into account that $\eta = x^0$ and applying equation (5.3.14), we obtain

$$F_{,\alpha} = \frac{\partial F}{\partial x^\alpha} = \frac{dF}{d\eta}\frac{\partial \eta}{\partial x^\alpha} = F' \mathrm{M}^0{}_\alpha = -\frac{F'}{a}\bar{u}_\alpha = -\dot{F}\bar{u}_\alpha. \tag{5.3.20}$$

In particular, the partial derivative from the scale factor, $a_{,\alpha} = -\dot{a}\bar{u}_\alpha = -\mathscr{H}\bar{u}_\alpha$, and the partial derivative from the Hubble parameter $\mathscr{H}_{,\alpha} = -\dot{\mathscr{H}}\bar{u}_\alpha$.

5.3.3 Christoffel symbols and covariant derivatives

In the following sections of the chapter we will need to calculate the covariant derivatives from various geometric objects on the background cosmological manifold covered by an arbitrary coordinate chart $x^\alpha = (x^0, x^i)$. The calculation engages the affine connection $\bar{\Gamma}^\alpha{}_{\beta\gamma}$ (the Christoffel symbols) of the background manifold which is decomposed into an algebraic sum of two parts because of the conformal structure of the FLRW metric [453]. By definition (see, for example, (A.2.2) for a dynamic metric),

$$\bar{\Gamma}^\alpha{}_{\beta\gamma} = \frac{1}{2}\bar{g}^{\alpha\nu}\left(\bar{g}_{\nu\beta,\gamma} + \bar{g}_{\nu\gamma,\beta} - \bar{g}_{\beta\gamma,\nu}\right), \tag{5.3.21}$$

where, in our specific consideration,

$$\bar{g}_{\alpha\beta,\gamma} = -2H\bar{g}_{\alpha\beta}\bar{u}_\gamma + a^2\bar{\mathfrak{f}}_{\alpha\beta,\gamma}. \tag{5.3.22}$$

Separating terms at the right side side of (5.3.21) yields

$$\bar{\Gamma}^\alpha{}_{\beta\gamma} = \bar{A}^\alpha{}_{\beta\gamma} + \bar{B}^\alpha{}_{\beta\gamma}, \tag{5.3.23}$$

where

$$\bar{A}^\alpha{}_{\beta\gamma} = -H\left(\delta^\alpha_\beta\bar{u}_\gamma + \delta^\alpha_\gamma\bar{u}_\beta - \bar{u}^\alpha\bar{g}_{\beta\gamma}\right) \tag{5.3.24}$$

is a tensor with transformation properties (A.1.16), whereas

$$\bar{B}^\alpha{}_{\beta\gamma} = \frac{1}{2}\bar{\mathfrak{f}}^{\alpha\mu}\left(\bar{\mathfrak{f}}_{\mu\beta,\gamma} + \bar{\mathfrak{f}}_{\mu\gamma,\beta} - \bar{\mathfrak{f}}_{\beta\gamma,\mu}\right) \tag{5.3.25}$$

is left a connection with transformation properties (A.2.3). The non-vanishing components of (5.3.24) and (5.3.25) are given in the isotropic Cartesian coordinates X^α by

$$\bar{A}^\alpha{}_{0\beta} = \mathscr{H}\delta^\alpha_\beta, \qquad \bar{A}^0{}_{ij} = \mathscr{H}\mathfrak{g}_{ij}, \qquad \bar{B}^i{}_{pq} = -\frac{k}{2}\frac{\delta^i_p X_q + \delta^i_q X_p - \delta_{pq}X^i}{1+\frac{k}{4}r^2}, \tag{5.3.26}$$

where $X_q \equiv \delta_{qj}X^j$, and all the other components vanish.

A covariant derivative of a tensor density in a dynamic spacetime is given in (A.2.5). In this chapter, a covariant derivative on a curved background, $\bar{\nabla}_\beta$, is only a covariant derivative on the FLRW background manifold. It is defined, of course, by the same way, however here, for the sake of simplification in the writing it is denoted with a vertical bar. For example, the covariant derivative of a vector field F^α is

$$F^\alpha{}_{|\beta} = F^\alpha{}_{,\beta} + \bar{\Gamma}^\alpha{}_{\beta\gamma}F^\gamma. \tag{5.3.27}$$

Equation (5.3.27) can be brought to yet another form if we denote the covariant derivative of the affine connection $\bar{B}^\alpha{}_{\beta\gamma}$ with a semicolon; in this chapter there is no a confusion with the cases in other chapters of the book where a semicolon means a covariant derivatives in the Minkowski space in curved coordinates. Making use of (5.3.23) in equation (5.3.27) transforms it to the following form

$$F^\alpha{}_{|\beta} = F^\alpha{}_{;\beta} + \bar{A}^\alpha{}_{\beta\gamma}F^\gamma. \tag{5.3.28}$$

The covariant derivative of a covector F_α is defined in a similar way,

$$F_{\alpha|\beta} = F_{\alpha,\beta} - \bar{\Gamma}^\gamma{}_{\alpha\beta}F_\gamma \tag{5.3.29}$$

which is equivalent to

$$F_{\alpha|\beta} = F_{\alpha;\beta} - \bar{A}^{\gamma}{}_{\alpha\beta}F_{\gamma}, \tag{5.3.30a}$$

$$F_{\alpha;\beta} = F_{\alpha,\beta} - \bar{B}^{\gamma}{}_{\alpha\beta}F_{\gamma}. \tag{5.3.30b}$$

Equations for tensors of higher rank can be presented in a similar way. Of course, the covariant derivative of a scalar field F always coincides with its covariant derivative by definition,

$$F_{|\alpha} = F_{;\alpha} = F_{,\alpha}. \tag{5.3.31}$$

We also provide an equation for the covariant derivative of the four-velocity of the Hubble flow. Doing calculations in the isotropic coordinates X^{α} for the four-velocity $\bar{U}^{\alpha}$, and applying the tensor law of transformation to arbitrary coordinates x^{α}, results in

$$\bar{u}_{\alpha|\beta} = H\bar{P}_{\alpha\beta}, \qquad \bar{u}^{\alpha}{}_{|\beta} = H\left(\delta^{\alpha}_{\beta} + \bar{u}^{\alpha}\bar{u}_{\beta}\right), \qquad \bar{u}^{\alpha|\beta} = H\bar{P}^{\alpha\beta}, \tag{5.3.32}$$

where the tensor indices are raised and lowered with the metric $\bar{g}_{\alpha\beta}$.

5.3.4 Riemann tensor

The Riemann tensor for the FLRW background is defined, as usual (A.2.15), by

$$\bar{R}^{\alpha}{}_{\beta\mu\nu} = \bar{\Gamma}^{\alpha}{}_{\beta\nu,\mu} - \bar{\Gamma}^{\alpha}{}_{\beta\mu,\nu} + \bar{\Gamma}^{\alpha}{}_{\mu\gamma}\bar{\Gamma}^{\gamma}{}_{\beta\nu} - \bar{\Gamma}^{\alpha}{}_{\nu\gamma}\bar{\Gamma}^{\gamma}{}_{\beta\mu} \tag{5.3.33}$$

and can be calculated directly from this equation. We prefer a slightly different way by making use of the algebraic decomposition of the Riemann tensor into the irreducible parts

$$\begin{aligned}\bar{R}_{\alpha\beta\mu\nu} = {} & \bar{C}_{\alpha\beta\mu\nu} + \frac{1}{2}\left(\bar{S}_{\alpha\mu}\bar{g}_{\beta\nu} + \bar{S}_{\beta\nu}\bar{g}_{\alpha\mu} - \bar{S}_{\alpha\nu}\bar{g}_{\beta\mu} - \bar{S}_{\beta\mu}\bar{g}_{\alpha\nu}\right) \\ & + \frac{\bar{R}}{12}\left(\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu} - \bar{g}_{\alpha\nu}\bar{g}_{\beta\mu}\right),\end{aligned} \tag{5.3.34}$$

where $\bar{C}_{\alpha\beta\mu\nu}$ is the Weyl tensor,

$$\bar{S}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{4}\bar{R}\bar{g}_{\mu\nu}, \tag{5.3.35}$$

$\bar{R}_{\mu\nu} = \bar{g}^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu}$ is the Ricci tensor, and $\bar{R} = \bar{g}^{\alpha\beta}\bar{R}_{\alpha\beta}$ is the Ricci scalar. The Weyl tensor of a conformally-flat spacetime vanishes identically,

$$\bar{C}_{\alpha\beta\mu\nu} \equiv 0. \tag{5.3.36}$$

Therefore, FLRW cosmological metric (5.3.1) has a remarkable property – it can be always brought up to the conformally-flat form by applying an appropriate coordinate transformation [239].

Direct evaluation of other tensors entering (5.3.34) by making use of the FLRW metric (5.3.10), (5.3.11) yields

$$\bar{R}_{\mu\nu} = \frac{1}{a^2}\left[\mathscr{H}'\left(\bar{g}_{\mu\nu} - 2\bar{u}_\mu\bar{u}_\nu\right) + 2\left(\mathscr{H}^2 + k\right)\left(\bar{g}_{\mu\nu} + \bar{u}_\mu\bar{u}_\nu\right)\right], \tag{5.3.37}$$

$$\bar{S}_{\mu\nu} = \frac{2}{a^2}\left(-\mathscr{H}' + \mathscr{H}^2 + k\right)\left(\bar{u}_\mu\bar{u}_\nu + \frac{1}{4}\bar{g}_{\mu\nu}\right), \tag{5.3.38}$$

$$\bar{R} = \frac{6}{a^2}\left(\mathscr{H}' + \mathscr{H}^2 + k\right). \tag{5.3.39}$$

Making use of equations (5.3.36–5.3.39) in the decomposition (5.3.34) of the Riemann tensor, yields the following result

$$\bar{R}_{\alpha\beta\mu\nu} = \frac{1}{a^2}\left[\mathscr{H}'\left(\bar{g}_{\alpha\mu}\bar{g}_{\beta\nu} - \bar{g}_{\alpha\nu}\bar{g}_{\beta\mu}\right) - \left(\mathscr{H}' - \mathscr{H}^2 - k\right)\left(\bar{P}_{\alpha\mu}\bar{P}_{\beta\nu} - \bar{P}_{\alpha\nu}\bar{P}_{\beta\mu}\right)\right], \tag{5.3.40}$$

where

$$\bar{P}_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{u}_\alpha\bar{u}_\beta, \tag{5.3.41}$$

is the operator of projection that was introduced earlier in (5.3.18).

5.3.5 The Friedmann equations

The Einstein tensor $\bar{G}_{\alpha\beta} \equiv \bar{R}_{\alpha\beta} - \bar{g}_{\alpha\beta}\bar{R}/2$ of the FLRW cosmological model is derived from equations (5.3.37) and (5.3.39). It reads

$$\bar{G}_{\alpha\beta} = -\frac{1}{a^2}\left[2\left(\mathscr{H}' - \mathscr{H}^2 - k\right)\bar{P}_{\alpha\beta} + 3\left(\mathscr{H}^2 + k\right)\bar{g}_{\alpha\beta}\right]. \tag{5.3.42}$$

Einstein's field equations on the background spacetime takes on the following form

$$\bar{G}_{\alpha\beta} = 8\pi\bar{T}_{\alpha\beta}, \tag{5.3.43}$$

where the tensor of energy-momentum of the background spacetime manifold includes the background matter and the scalar field

$$\bar{T}_{\alpha\beta} = \bar{T}^{\rm m}_{\alpha\beta} + \bar{T}^{\rm q}_{\alpha\beta}. \tag{5.3.44}$$

Here, tensors of energy-momentum in the right side of Einstein's equations are derived from the Lagrangians (5.2.16) and (5.2.19), and represent an algebraic sum of tensors

(5.2.18) and (5.2.22). Each tensor of energy-momentum, $\bar{T}^{\mathrm{m}}_{\alpha\beta}$ and $\bar{T}^{\mathrm{q}}_{\alpha\beta}$, is Lie-invariant with respect to the group of symmetry of the background FLRW metric independently, and each of them has the form of the tensor of energy-momentum of the perfect (ideal) fluid. Hence, the tensor of energy-momentum $\bar{T}_{\alpha\beta}$ in the right side of (5.3.43) has the form of a perfect fluid as well,

$$\bar{T}_{\alpha\beta} = (\bar{\epsilon} + \bar{p})\,\bar{u}_\alpha \bar{u}_\beta + \bar{p}\,\bar{g}_{\alpha\beta}. \tag{5.3.45}$$

It imposes a certain restriction on the effective energy density $\bar{\epsilon}$ and pressure $\bar{p}$ which must obey Dalton's law for a partial energy density and pressure of the background matter and the scalar field components

$$\bar{\epsilon} = \bar{\epsilon}_{\mathrm{m}} + \bar{\epsilon}_{\mathrm{q}}, \tag{5.3.46}$$

$$\bar{p} = \bar{p}_{\mathrm{m}} + \bar{p}_{\mathrm{q}}. \tag{5.3.47}$$

Here, $\bar{\epsilon}_{\mathrm{m}}$ and $\bar{p}_{\mathrm{m}}$ are the energy density and pressure of the ideal fluid, and $\bar{\epsilon}_{\mathrm{q}}$ and $\bar{p}_{\mathrm{q}}$ are the energy density and pressure of the scalar field which are related to the time derivative $\dot{\bar{\Psi}}$ of the scalar field and its potential $\bar{W} = \bar{W}(\bar{\Psi})$ by equations (5.2.26), (5.2.27). On the background spacetime these equations takes on the following form

$$\bar{\epsilon}_{\mathrm{q}} = \frac{1}{2}\bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}} + \bar{W}, \tag{5.3.48}$$

$$\bar{p}_{\mathrm{q}} = \frac{1}{2}\bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}} - \bar{W}, \tag{5.3.49}$$

where $\bar{\mu}_{\mathrm{q}}$ is the background specific enthalpy of the scalar field defined by (5.2.22), and $\bar{\rho}_{\mathrm{q}} = \bar{\mu}_{\mathrm{q}}$ is the background density of the scalar field "fluid". It is worthwhile to remind to the reader that due to the homogeneity and isotropy of the FLRW metric, all matter variables on the background manifold are functions of the conformal time η only when being expressed in the isotropic Cartesian coordinates.

Einstein's equation (5.3.43) can be projected on the direction of the background four-velocity of matter and on the spatial hypersurface being orthogonal to it. It yields two Friedmann equations for the evolution of the scale factor a,

$$H^2 = \frac{8\pi}{3}\bar{\epsilon} - \frac{k}{a^2}, \tag{5.3.50}$$

$$2\dot{H} + 3H^2 = -8\pi\bar{p} - \frac{k}{a^2}, \tag{5.3.51}$$

where $\bar{\epsilon}$ and $\bar{p}$ are the effective energy density and pressure of the mixture of matter and scalar field as defined above.

A consequence of the Friedmann equations (5.3.50), (5.3.51) is an equation

$$\dot{H} = -4\pi\,(\bar{\epsilon} + \bar{p}) + \frac{k}{a^2}, \tag{5.3.52}$$

relating the time derivative of the Hubble parameter with the sum of the overall energy density and pressure, which can be expressed in terms of the density and specific enthalpy of the background components of matter,

$$\bar{\epsilon} + \bar{p} = \bar{\rho}_{\rm m}\bar{\mu}_{\rm m} + \bar{\rho}_{\rm q}\bar{\mu}_{\rm q}. \tag{5.3.53}$$

In order to solve the Friedmann equations (5.3.50), (5.3.51) we have to employ the equation of state of matter. Customarily, it is assumed that each matter component obeys its own cosmological equation of state,

$$\bar{p}_{\rm m} = w_{\rm m}\bar{\epsilon}_{\rm m}, \qquad \bar{p}_{\rm q} = w_{\rm q}\bar{\epsilon}_{\rm q}, \tag{5.3.54}$$

where $w_{\rm m}$ and $w_{\rm q}$ are parameters lying in the range from −1 to +1. In the most simple cosmological models, parameters $w_{\rm m}$ and $w_{\rm q}$ are fixed. More realistic models admit that the parameters of the equation of state may change in the course of the cosmological expansion, that is they may depend on time. The equation of state does not close the system of the Friedmann equations, which have to be complemented with the equations of motion of the scalar field and of the ideal fluid in order to make the system of differential equations for the gravitational and matter field variables complete.

5.3.6 Hydrodynamic equations of the ideal fluid

The background value of the Clebsch potential of the ideal fluid, $\bar{\Phi}$, depends only on the conformal time η of the FLRW metric. The partial derivative of the potential, taken in arbitrary coordinate chart on the background manifold, can be expressed following equation (5.2.14) in terms of the background four-velocity $\bar{u}^\alpha$ as follows

$$\bar{\Phi}_{|\alpha} = -\bar{\mu}_{\rm m}\bar{u}_\alpha, \tag{5.3.55}$$

where the background value of the specific enthalpy is

$$\bar{\mu}_{\rm m} = \sqrt{-\bar{g}^{\alpha\beta}\bar{\Phi}_{,\alpha}\bar{\Phi}_{,\beta}} \tag{5.3.56}$$

in accordance with definition (5.2.14). It allows us to write down the specific enthalpy of the ideal fluid in terms of a derivative from the Clebsch potential $\bar{\Phi}$. Multiplying both sides of (5.3.55) with $\bar{u}_\alpha$, and accounting for $\bar{u}^\alpha\bar{u}_\alpha = -1$, we obtain

$$\bar{\mu}_{\rm m} \equiv \bar{u}^\alpha\bar{\Phi}_{|\alpha} = \dot{\bar{\Phi}}. \tag{5.3.57}$$

The background equation of continuity for the rest mass density $\bar{\rho}_{\rm m}$ of the ideal fluid is

$$\left(\bar{\rho}_{\rm m}\bar{u}^\alpha\right)_{|\alpha} = 0, \tag{5.3.58}$$

that is equivalent to

$$\bar{p}_{\mathrm{m}|\alpha} - 3H\bar{p}_{\mathrm{m}}\bar{u}_\alpha = 0, \tag{5.3.59}$$

where we have used (5.3.32). The background equation of conservation of energy is

$$\bar{\epsilon}_{\mathrm{m}|\alpha} - 3H\left(\bar{\epsilon}_{\mathrm{m}} + \bar{p}_{\mathrm{m}}\right)\bar{u}_\alpha = 0, \tag{5.3.60}$$

where we have employed definition of the energy (5.2.4), and equation (5.3.59) along with (5.2.6).

5.3.7 Scalar field equations

Background equation for the scalar field $\bar{\Psi}$ is derived from the action (5.2.1) by taking variational derivatives with respect to $\bar{\Psi}$. It yields

$$\bar{g}^{\alpha\beta}\bar{\Psi}_{|\alpha\beta} - \frac{\partial\bar{W}}{\partial\bar{\Psi}} = 0. \tag{5.3.61}$$

In terms of the time derivatives with respect to the Hubble time T, equation (5.3.61) reads

$$\ddot{\bar{\Psi}} + 3H\dot{\bar{\Psi}} + \frac{\partial\bar{W}}{\partial\bar{\Psi}} = 0. \tag{5.3.62}$$

Here, we have taken into account that the background value of the scalar field, $\bar{\Psi}$, depends only on time $T = T(\eta)$, and its derivative with respect to T (denoted with a dot) is proportional to the background four-velocity

$$\bar{\Psi}_{|\alpha} = -\dot{\bar{\Psi}}\bar{u}_\alpha, \tag{5.3.63}$$

which follows directly (5.3.20). If we use the definition of the background enthalpy of the scalar field

$$\bar{\mu}_{\mathrm{q}} \equiv \bar{u}^\alpha\bar{\Psi}_{|\alpha} = \dot{\bar{\Psi}}, \tag{5.3.64}$$

and account for definition (5.2.26) of the specific energy ϵ_{q} of the scalar field, equation (5.3.62) will become

$$\bar{\epsilon}_{\mathrm{q}|\alpha} - 3H\left(\bar{\epsilon}_{\mathrm{q}} + \bar{p}_{\mathrm{q}}\right)\bar{u}_\alpha = 0 \tag{5.3.65}$$

that looks similar to the hydrodynamic equation (5.3.59) of conservation of energy of the ideal fluid. Because of this similarity, the second Friedmann equation (5.3.51) can be derived from the first Friedmann equation (5.3.50) by taking a time derivative and applying the energy conservation equations (5.3.60) and (5.3.65).

The background density $\bar{\rho}_{\rm q}$ of the scalar filed "fluid" is $\bar{\rho}_{\rm q} = \bar{\mu}_{\rm q}$ in accordance with (5.2.25). The equation of continuity for the density $\bar{\rho}_{\rm q}$ of the ideal fluid is obtained by differentiating definition of $\bar{\rho}_{\rm q}$, and making use of (5.3.62). It yields

$$\left(\bar{\rho}_{\rm q}\bar{u}^{\alpha}\right)_{|\alpha} = -\frac{\partial \bar{W}}{\partial \bar{\Psi}}, \tag{5.3.66}$$

or, equivalently,

$$\bar{\rho}_{{\rm q}|\alpha} - 3H\bar{\rho}_{\rm q}\bar{u}_{\alpha} = \frac{\partial \bar{W}}{\partial \bar{\Psi}}\bar{u}_{\alpha}, \tag{5.3.67}$$

which shows that the 'density' $\bar{\rho}_{\rm q}$ of the scalar field 'fluid' is not conserved in the most general case of an arbitrary potential function $\bar{W}(\bar{\Psi})$. We emphasize that there is no any violation of physical laws, since (5.3.67) is simply another way of writing equation (5.3.61), and the scalar field is not thermodynamically equivalent to the ideal fluid. Equation (5.3.67) is convenient in the calculations that follow in next sections.

5.3.8 Equations of motion of matter of the localized astronomical system

Matter of the localized astronomical system is described by the tensor of energy-momentum $T^{\rm p}_{\alpha\beta}$ defined in (5.2.30) in terms of the Lagrangian derivative. It can be given explicitly as a function of field variables after we chose a specific form of matter, for example, gas, liquid, solid, or something else. We do not restrict ourselves with a particular form of this tensor, and shall develop a more generic approach that is applicable to any kind of matter comprising the localized astronomical system.

Background equation of motion of matter of the astronomical system is given by the conservation law

$$T^{\alpha\beta}_{\rm p}{}_{|\beta} = 0. \tag{5.3.68}$$

It can be also written down in terms of a covariant derivative of the conformal metric

$$\left(\sqrt{-\bar{g}}T^{\alpha\beta}_{\rm p}\right)_{;\beta} + \sqrt{-\bar{g}}\bar{A}^{\alpha}{}_{\beta\gamma}T^{\beta\gamma}_{\rm p} = 0, \tag{5.3.69}$$

where the connection $\bar{A}^{\alpha}{}_{\beta\gamma}$ is defined in (5.3.24). Equation (5.3.68) tells us that the matter of the small-scale perturbation follows geodesics of the background manifold. This is the starting point for doing the Post-Newtonian approximations in cosmology. In the geodesic approximation the matter of the isolated astronomical system has no self-interaction through its own gravitational field. The self-interaction appears at the next step of the post-Newtonian iteration procedure.

It is natural to write down equation (5.3.68) in 1+3 form by projecting it on the direction of 4-velocity of the Hubble flow, $\bar{u}^\alpha$, and on the hypersurface being orthogonal to it. This is achieved by introducing the following projections

$$\sigma \equiv \bar{u}^\mu \bar{u}^\nu T^{\mathrm{p}}_{\mu\nu}, \tag{5.3.70a}$$

$$\tau \equiv \bar{P}^{\mu\nu} T^{\mathrm{p}}_{\mu\nu}, \tag{5.3.70b}$$

$$\tau_\alpha \equiv -\bar{P}_\alpha{}^\mu \bar{u}^\nu T^{\mathrm{p}}_{\mu\nu}, \tag{5.3.70c}$$

$$\tau_{\alpha\beta} \equiv \bar{P}_\alpha{}^\mu \bar{P}_\beta{}^\nu T^{\mathrm{p}}_{\mu\nu}, \tag{5.3.70d}$$

which corresponds to the kinemetric-invariant decomposition[3] of $T^{\mathrm{p}}_{\mu\nu}$ introduced by Zelmanov [473, 474]. Quantity σ is the energy density of matter of the localized system, τ_α is a density of a linear momentum of the matter, and $\tau_{\alpha\beta}$ is the stress tensor of the matter (the reader should not confuse τ in (5.3.70b) with the proper time).

Equations of motion (5.3.68) of the localized matter can be rewritten in terms of the chronometric quantities as follows,

$$(\sigma\bar{u}^\alpha + \tau^\alpha)_{|\alpha} = -H\tau, \tag{5.3.71a}$$

$$\left(\tau^{\alpha\beta} + \bar{u}^\beta \tau^\alpha\right)_{|\beta} = -H\left(\tau^\alpha - \bar{u}^\alpha \tau\right), \tag{5.3.71b}$$

where $\tau^\alpha \equiv \bar{g}^{\alpha\beta}\tau_\beta$ and $\tau^{\alpha\beta} \equiv \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\tau_{\mu\nu}$. Equation (5.3.71a) is equivalent to the law of conservation of energy of matter of the localized system. Equation (5.3.71b) is analogues to the Euler equation of motion of fluid or the equation of the force balance in case of solids.

5.4 Lagrangian perturbations of FLRW manifold

5.4.1 The concept of perturbations

Recall that FLRW background manifold is defined by the metric $\bar{g}_{\alpha\beta}$ which dynamics is governed by background matter fields - the Clebsch potential $\bar{\Phi}$ of the ideal fluid and the scalar field $\bar{\Psi}$. We assume that the background metric and the background values of the fields are perturbed by a localized astronomical system which is considered as a *bare* perturbation associated with a field variable Θ.

3 This decomposition is also known as a threading approach or 1 + 3 orthonormal frame approach [450]. It is different from 3 + 1 decomposition ADM technique considered in Section 3.1, see discussion later.

Various possibilities for definition of metric perturbations are discussed in Section 2.2.6. Here, we consider perturbations of the metric, $\varkappa_{\alpha\beta}$, defined by the splitting,

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \varkappa_{\alpha\beta}, \tag{5.4.1}$$

as a more popular ones, and the perturbations, $\mathfrak{h}^{\alpha\beta}$, defined in (5.4.6), as more convenient ones, for doing variational calculus see below. The matter fields caused by the *bare* perturbations can be split in their backgrounds values and the corresponding perturbations,

$$\Phi = \bar{\Phi} + \phi, \quad \Psi = \bar{\Psi} + \psi. \tag{5.4.2}$$

These equations are exact. We emphasize that all functions entering equation (5.4.1) and (5.4.2) are taken at one and the same point of the background manifold. The *bare* perturbation does not remain the same in the presence of the perturbations of the metric and the matter fields. Therefore, the field variable Θ corresponding to the *bare* perturbation, is also can be presented in the perturbed form:

$$\Theta = \bar{\Theta} + \theta. \tag{5.4.3}$$

Thus, for the goals of the present chapter, (5.4.2) and (5.4.3) are considered as a particular example of the decomposition (2.2.8) in Section 2.2. Although (5.4.1–5.4.3) are exact, we consider the perturbations of the metric - $\varkappa_{\alpha\beta}$, the Clebsch potential - ϕ, and the scalar field - ψ as being small with respect to their corresponding background values $\bar{g}_{\alpha\beta}$, $\bar{\Phi}$, and $\bar{\Psi}$, which dynamics is governed by the background equations that have been explained in Section 5.3. Because the field variable Θ is the source of the *bare* perturbation, we postulate that its background value is equal to zero: $\bar{\Theta} = 0$. The perturbations $\varkappa_{\alpha\beta}$, ϕ, and ψ and their derivatives have the same order of magnitude as θ.

Perturbation of the contravariant component of the metric is determined from the condition $g_{\alpha\gamma}g^{\gamma\beta} = \bar{g}_{\alpha\gamma}\bar{g}^{\gamma\beta} = \delta^{\beta}_{\alpha}$, and is given by

$$g^{\alpha\beta} = \bar{g}^{\alpha\beta} - \varkappa^{\alpha\beta} + \varkappa^{\alpha}{}_{\gamma}\varkappa^{\gamma\beta} + \dots, \tag{5.4.4}$$

where the ellipses denote terms of the higher order.

Here, we refer to the perturbations $\varkappa_{\alpha\beta}$ in (5.4.1) because in literature they are very popular, see e.g., the textbook [283]. However, as it was discovered in Section 2.2, in the framework of the field-theoretical derivation a more convenient field variable of the gravitational field in the theory of Lagrangian perturbations of curved manifolds, is a contravariant metric density,

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g}\,g^{\alpha\beta}, \tag{5.4.5}$$

that we call the Gothic metric. The convenience of the Gothic metric stems from the fact that it enters the de Donder (harmonic) gauge conditions which significantly simplifies the Einstein equations [285, 453]. Making use of the Gothic metric allows us to significantly reduce the amount of algebra in taking the first and second variational derivatives from the Hilbert Lagrangian and the Lagrangian of the background matter in FLRW metric as explains in the rest of this section.

Thus, following (2.2.7), we accept that $\mathfrak{g}^{\alpha\beta}$ is expanded around its background value, $\bar{\mathfrak{g}}^{\alpha\beta} = \sqrt{-\bar{g}}\bar{g}^{\alpha\beta}$, as follows

$$\mathfrak{g}^{\alpha\beta} = \bar{\mathfrak{g}}^{\alpha\beta} + \mathfrak{h}^{\alpha\beta}, \tag{5.4.6}$$

which is an exact equation and where $\mathfrak{h}^{\alpha\beta}$ is also a tensor density of the weight +1. Further calculations prompt that it is more suitable to operate with a variable quantity

$$h^{\alpha\beta} \equiv \frac{\mathfrak{h}^{\alpha\beta}}{\sqrt{-\bar{g}}} \tag{5.4.7}$$

that is a tensor. This variable splits the dynamic degrees of freedom of the gravitational perturbations from those of the background manifold which evolves in according with the unperturbed Friedmann equations. Tensor indices of $h^{\alpha\beta}$ are raised and lowered with the help of the background metric, for example, $h_{\alpha\beta} \equiv \bar{g}_{\alpha\mu}\bar{g}_{\beta\nu}h^{\mu\nu}$. The field variable $h^{\alpha\beta}$ relates to the perturbation $\varkappa_{\alpha\beta}$ of the metric tensor. To establish this relationship, we start from (5.4.5), substitute equation (5.4.6) to its left side, and expand its right side in the Taylor series with respect to $\varkappa_{\alpha\beta}$. It results in

$$\mathfrak{h}^{\alpha\beta} = \frac{\partial\bar{\mathfrak{g}}^{\alpha\beta}}{\partial\bar{g}_{\mu\nu}}\varkappa_{\mu\nu} + \frac{1}{2}\frac{\partial^2\bar{\mathfrak{g}}^{\alpha\beta}}{\partial\bar{g}_{\mu\nu}\partial\bar{g}_{\rho\sigma}}\varkappa_{\mu\nu}\varkappa_{\rho\sigma} + \dots \tag{5.4.8}$$

that is a particular case of (2.2.122). The partial derivatives in (5.4.8) are calculated by successive application of the following rules

$$\frac{\partial\bar{\mathfrak{g}}^{\alpha\beta}}{\partial\bar{g}_{\mu\nu}} = -\frac{1}{2}\sqrt{-\bar{g}}\left(\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu} + \bar{g}^{\alpha\nu}\bar{g}^{\beta\mu} - \bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\right), \tag{5.4.9a}$$

$$\frac{\partial\bar{g}^{\alpha\beta}}{\partial\bar{g}_{\mu\nu}} = -\frac{1}{2}\left(\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu} + \bar{g}^{\alpha\nu}\bar{g}^{\beta\mu}\right), \tag{5.4.9b}$$

$$\frac{\partial\sqrt{-\bar{g}}}{\partial\bar{g}_{\mu\nu}} = +\frac{1}{2}\sqrt{-\bar{g}}\bar{g}^{\mu\nu}, \tag{5.4.9c}$$

which can be easily confirmed by inspection and which are particular cases of the coefficients in (2.2.115). Replacing the partial derivatives in (5.4.8) and making use of the definition (5.4.7), yields the relationship between $h^{\alpha\beta}$ and $\varkappa^{\alpha\beta}$ as follows

$$h^{\alpha\beta} = -\varkappa^{\alpha\beta} + \frac{1}{2}\bar{g}^{\alpha\beta}\varkappa + \varkappa^{\mu(\alpha}\varkappa^{\beta)}{}_{\mu} - \frac{1}{2}\varkappa^{\alpha\beta}\varkappa - \frac{1}{4}\bar{g}^{\alpha\beta}\left(\varkappa^{\mu\nu}\varkappa_{\mu\nu} - \frac{1}{2}\varkappa^2\right) + \dots, \tag{5.4.10}$$

where $\varkappa \equiv \varkappa^{\sigma}{}_{\sigma} = \bar{g}^{\rho\sigma}\varkappa_{\rho\sigma}$, and ellipses denote terms of the cubic and higher order in $\varkappa_{\alpha\beta}$.

Perturbations of four-velocities, w^α and v^α, entering definitions of the energy-momentum tensors (5.2.18), (5.2.29), are fully determined by the perturbations of the metric and the potentials of the matter fields. Indeed, according to definitions (5.2.13) and (5.2.25) the four-velocities are defined by the following equations

$$w_\alpha = -\frac{\Phi_{,\alpha}}{\mu_{\rm m}}, \qquad v_\alpha = -\frac{\Psi_{,\alpha}}{\mu_{\rm q}}, \tag{5.4.11}$$

where $\mu_{\rm m} = \sqrt{-g^{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta}}$ and $\mu_{\rm q} = \sqrt{-g^{\alpha\beta}\Psi_{,\alpha}\Psi_{,\beta}}$ in accordance with (5.2.14) and (5.2.22) respectively. We define perturbation of the covariant components of the four-velocities as follows

$$w_\alpha = \bar{u}_\alpha + \delta w_\alpha, \qquad v_\alpha = \bar{u}_\alpha + \delta v_\alpha, \tag{5.4.12}$$

where the unperturbed values of the four-velocities coincide and are equal to the four-velocity of the Hubble flow due to the requirement of the homogeneity and isotropy of the background FLRW metric. Substituting these expansions to the left side of definitions (5.4.11), and expanding its right side by making use of the expansions (5.4.2) and (5.4.4) of the scalar fields and the metric, yields

$$\delta w_\alpha = -\frac{1}{\bar{\mu}_{\rm m}}\bar{P}^\beta{}_\alpha \phi_{|\beta} - \frac{1}{2}\mathfrak{q}\bar{u}_\alpha, \qquad \delta v_\alpha = -\frac{1}{\bar{\mu}_{\rm q}}\bar{P}^\beta{}_\alpha \psi_{|\beta} - \frac{1}{2}\mathfrak{q}\bar{u}_\alpha, \tag{5.4.13}$$

where we have introduced a new notation

$$\mathfrak{q} \equiv -\bar{u}^\alpha \bar{u}^\beta \varkappa_{\alpha\beta}, \tag{5.4.14}$$

for the gravitational perturbation of the metric tensor projected on the background four-velocity of the Hubble flow. Making use of $h_{\alpha\beta}$, the previous equation can be recast to

$$\mathfrak{q} \equiv \bar{u}^\alpha \bar{u}^\beta h_{\alpha\beta} + \frac{h}{2}, \tag{5.4.15}$$

where $h \equiv h^\alpha{}_\alpha = \bar{g}^{\alpha\beta}h_{\alpha\beta}$. Remembering that $\bar{g}^{\alpha\beta} = \bar{P}^{\alpha\beta} - \bar{u}^\alpha\bar{u}^\beta$, we can put equation (5.4.15) yet to another form

$$\mathfrak{q} \equiv \frac{1}{2}\left(\bar{u}^\alpha\bar{u}^\beta + \bar{P}^{\alpha\beta}\right) h_{\alpha\beta}, \tag{5.4.16}$$

which is useful in the calculations that follow.

5.4.2 The background field equations

The action of the unperturbed FLRW metric is a functional

$$\bar{S} = \int_\Omega d^4x \bar{\mathscr{L}}^{HE}, \tag{5.4.17}$$

depending on the unperturbed Lagrangian

$$\bar{\mathscr{L}}^{HE} = -\frac{1}{16\pi}\bar{\mathscr{L}}^H + \bar{\mathscr{L}}^{\rm m} + \bar{\mathscr{L}}^{\rm q}, \tag{5.4.18}$$

where the action (5.2.1) and the Lagrangian (5.2.2) are taken on the background values of the field variables $\bar{g}_{\alpha\beta}$, $\bar{\Phi}$, and $\bar{\Psi}$. Thus, $\bar{\mathscr{L}}^H = \bar{\mathscr{R}}$, $\bar{\mathscr{L}}^{\rm m} = \mathscr{L}^{\rm m}\left(\bar{\Phi}, \bar{\mathfrak{g}}^{\alpha\beta}\right)$, $\bar{\mathscr{L}}^{\rm q} = \mathscr{L}^{\rm q}\left(\bar{\Psi}, \bar{\mathfrak{g}}^{\alpha\beta}\right)$.

Dynamics of the background universe is governed exclusively by the background matter. The background equations corresponding to (5.4.17) with (5.4.18) are

$$-\frac{1}{16\pi}\frac{\delta\bar{\mathscr{L}}^H}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} + \frac{\delta\bar{\mathscr{L}}^{\rm m}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} + \frac{\delta\bar{\mathscr{L}}^{\rm q}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} = 0, \tag{5.4.19a}$$

$$\frac{\delta\bar{\mathscr{L}}^{\rm m}}{\delta\bar{\Phi}} = 0, \tag{5.4.19b}$$

$$\frac{\delta\bar{\mathscr{L}}^{\rm q}}{\delta\bar{\Psi}} = 0. \tag{5.4.19c}$$

These equations, for the goals of the present chapter, make concrete the background system (2.2.10) and (2.2.11) in the Section 2.2.

These equations have been thoroughly discussed also in Section 5.3. Solution of these equations depends on the equation of state of the background matter. We assume that the solution exists and that the time dependence of the FLRW metric $\bar{g}_{\alpha\beta} = \bar{g}_{\alpha\beta}(\eta)$, the Clebsch potential $\bar{\Phi} = \bar{\Phi}(\eta)$, and the scalar field $\bar{\Psi} = \bar{\Psi}(\eta)$ is explicitly known.

5.4.3 The dynamic Lagrangian for perturbations

The presence of a localized astronomical system perturbs the spacetime manifold and the background values of the field variables described in previous subsection. The perturbed Lagrangian becomes an algebraic sum of four terms, as it was defined in Section 5.2,

$$\mathscr{L}^{HE} = -\frac{1}{16\pi}\mathscr{L}^H + \mathscr{L}^{\rm m} + \mathscr{L}^{\rm q} + \mathscr{L}^{\rm p}, \tag{5.4.20}$$

where the Lagrangian $\mathscr{L}^{\mathrm{p}} = \mathscr{L}^{\mathrm{p}}(\Theta, g_{\alpha\beta})$ describes the *bare* perturbation, the Hilbert Lagrangian of the gravitational field is $\mathscr{L}^{H} = \sqrt{-g}R = \mathscr{R}$, where R is the Ricci scalar, the Lagrangian density of matter is $\mathscr{L}^{\mathrm{m}} = \mathscr{L}^{\mathrm{m}}(\Phi, g_{\alpha\beta})$, and the Lagrangian density of the scalar field $\mathscr{L}^{\mathrm{q}} = \mathscr{L}^{\mathrm{q}}(\Psi, g_{\alpha\beta})$.

The field equations are obtained by taking the variational derivatives from the perturbed action with the Lagrangian (5.4.20) with respect to various variables subject to the least action principle. In accordance with this principle, we obtain the Einstein equations, and equations for the matter fields in the form:

$$-\frac{1}{16\pi}\frac{\delta\mathscr{L}^{H}}{\delta\mathfrak{g}^{\alpha\beta}} + \frac{\delta\mathscr{L}^{\mathrm{m}}}{\delta\mathfrak{g}^{\alpha\beta}} + \frac{\delta\mathscr{L}^{\mathrm{q}}}{\delta\mathfrak{g}^{\alpha\beta}} = -\frac{\delta\mathscr{L}^{\mathrm{p}}}{\delta\mathfrak{g}^{\alpha\beta}}, \tag{5.4.21a}$$

$$\frac{\delta\mathscr{L}^{\mathrm{m}}}{\delta\Phi} = 0, \tag{5.4.21b}$$

$$\frac{\delta\mathscr{L}^{\mathrm{q}}}{\delta\Psi} = 0 \tag{5.4.21c}$$

$$\frac{\delta\mathscr{L}^{\mathrm{p}}}{\delta\Theta} = 0. \tag{5.4.21d}$$

The Post-Newtonian approximations of these equations can be rendered directly by a separation of the background values from their perturbed values and the use of Taylor decompositions. One has to assume also that gravitational dynamics of the unperturbed FLRW metric obeys the background field equations shown in Section 5.4.2. Then, the perturbed part of the equations represent a series of the post-Newtonian equations of the first, second, third, etc. order, which can be solved by successive iterations.

However, here we use more elegant and economical field-theoretical approach to general relativity developed in Section 2.2. From the one hand it is exact; from the other hand, it permits also to represent equations in series of the post-Newtonian equations of the first, second, third, etc. order. In this chapter we restrict ourselves with the linearized approximation of the first order with respect to the perturbations. It generalizes the first post-Newtonian field equations in asymptotically flat spacetime to the case of the expanding universe.

Thus, for the Lagrangian (5.4.20) we derive the dynamical Lagrangian (2.2.15) for the perturbations $\mathfrak{h}^{\mu\nu}$, ϕ, ψ and θ as dynamic variables on the FLRW background:

$$\mathscr{L}^{\mathrm{dyn}} = \mathscr{L}^{HE} - \mathfrak{h}^{\mu\nu}\frac{\delta\bar{\mathscr{L}}^{HE}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} - \phi\frac{\delta\bar{\mathscr{L}}^{m}}{\delta\bar{\Phi}} - \psi\frac{\delta\bar{\mathscr{L}}^{q}}{\delta\bar{\Psi}} - \bar{\mathscr{L}}^{HE} + \mathrm{div}\,. \tag{5.4.22}$$

One has to notice that this Lagrangian is defined up to a divergence, which can be important in the discussion of the boundary conditions but it does not enter equations of motion of fields which represent a system of the differential equations in partial derivatives for the perturbations. The variation of (5.4.22) with respect to the metric perturbations $\mathfrak{h}^{\mu\nu}$, and to matter perturbations ϕ, ψ and θ gives the field-theoretical equations, which are equivalent to the full set of the equations (5.4.21a),(5.4.21b), (5.4.21c) and (5.4.21d), respectively. These field-theoretical equations represent a particular case of the equations (2.2.26) and (2.2.38) in the field-theoretical derivation of general relativity, and we discuss them below in detail.

The dynamical Lagrangian (5.4.22) can be represented in the expansion form. Following (2.2.47), one has for the perturbations on the FLRW background:

$$\mathscr{L}^{\mathrm{dyn}} = \mathscr{L}^{EH} - \mathscr{L}_1 - \mathscr{L}_0 = \sum_{n=2}^{\infty} \mathscr{L}_n + \mathscr{L}^{\mathrm{p}}, \tag{5.4.23}$$

where $\mathscr{L}_0 \equiv \bar{\mathscr{L}}^{HE}$ is the Lagrangian describing the dynamic properties of the background manifold, and $\mathscr{L}^{\mathrm{p}}$ is the Lagrangian of the bare perturbation. For any $n \leq 1$,

$$\mathscr{L}_n = \frac{1}{n}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\mathscr{L}_{n-1}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi\frac{\delta\mathscr{L}_{n-1}}{\delta\bar{\Phi}} + \psi\frac{\delta\mathscr{L}_{n-1}}{\delta\bar{\Psi}}\right), \tag{5.4.24}$$

is the Lagrangian perturbation defined iteratively by taking variational derivatives from the Lagrangian perturbations of the previous iteration. In particular,

$$\mathscr{L}_1 = \mathfrak{h}^{\mu\nu}\frac{\delta\bar{\mathscr{L}}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi\,\frac{\delta\bar{\mathscr{L}}}{\delta\bar{\Phi}} + \psi\,\frac{\delta\bar{\mathscr{L}}}{\delta\bar{\Psi}}, \tag{5.4.25a}$$

$$\mathscr{L}_2 = \frac{1}{2}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\mathscr{L}_1}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi\frac{\delta\mathscr{L}_1}{\delta\bar{\Phi}} + \psi\frac{\delta\mathscr{L}_1}{\delta\bar{\Psi}}\right), \tag{5.4.25b}$$

and so on.

5.4.4 The Lagrangian equations for gravitational field perturbations

Varying the Lagrangian (5.4.23) with respect to $\mathfrak{h}^{\alpha\beta}$ one obtains differential equations for the metric (gravitational) perturbations. Contracting them with

$$\frac{\partial\mathfrak{g}^{\alpha\beta}}{\partial g^{\mu\nu}} = \sqrt{-g}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \tfrac{1}{2}g^{\alpha\beta}g_{\mu\nu}\right), \tag{5.4.26}$$

see (A.2.46) in Appendix A.2.4, one obtains

$$G^L_{\mu\nu} + \Phi^L_{\mu\nu} = 8\pi t^{\mathrm{gen}}_{\mu\nu}. \tag{5.4.27}$$

This concretizes the field-theoretical equations (2.2.26) for the goals of the present chapter. One can notice that (5.4.27) generalizes the Einstein field equations in asymptotically flat spacetime to the case of the expanding FLRW metric.

The left hand side of (5.4.27) is linear in perturbations. Recall that $G^L_{\mu\nu}$ is defined in (2.2.27),

$$G^L_{\mu\nu} \equiv \frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}^H}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\right). \tag{5.4.28}$$

Tensor $\Phi^L_{\mu\nu}$ is an algebraic superposition

$$\Phi^L_{\mu\nu} \equiv F^{\mathrm{m}}_{\mu\nu} + F^{\mathrm{q}}_{\mu\nu}, \tag{5.4.29}$$

where the linear operators are defined through the Lagrangian derivatives as follows,

$$F^{\mathrm{m}}_{\mu\nu} \equiv -\frac{16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}^{\mathrm{m}}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} + \phi\frac{\delta\bar{\mathscr{L}}^{\mathrm{m}}}{\delta\bar{\Phi}}\right), \tag{5.4.30a}$$

$$F^{\mathrm{q}}_{\mu\nu} \equiv -\frac{16\pi}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}^{\mathrm{q}}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} + \psi\frac{\delta\bar{\mathscr{L}}^{\mathrm{q}}}{\delta\bar{\Psi}}\right). \tag{5.4.30b}$$

These concretize (2.2.28) for the model of the present chapter.

The right hand side of equation (5.4.27) is the generic metric energy-momentum tensor for the system (5.4.22) (the same (5.4.23)) defined by the general rule in (2.2.29):

$$t^{\mathrm{gen}}_{\mu\nu} = \frac{2}{\sqrt{-\bar{g}}}\frac{\delta\mathscr{L}^{\mathrm{dyn}}}{\delta\bar{g}^{\mu\nu}} = t^{\mathrm{tot}}_{\mu\nu} + T^{\mathrm{p}}_{\mu\nu}. \tag{5.4.31}$$

The last term here is the energy-momentum tensor of the *bare* gravitational perturbation which is generated by the matter of the localized astronomical system and is associated with the last term in (5.4.23). Because $T^{\mathrm{p}}_{\mu\nu}$ has a *special* meaning of a bare perturbation we separate it from $t^{\mathrm{tot}}_{\mu\nu}$ defined in field-theoretical equations (2.2.26). The first term at the right hand side of (5.4.31) is the energy-momentum tensor of the non-linear corrections of the second and higher order of magnitude. Corresponding to (2.2.29) and (5.4.23), they are given by

$$t^{\mathrm{tot}}_{\mu\nu} = \frac{2}{\sqrt{-\bar{g}}}\left(\frac{\delta\mathscr{L}_2}{\delta\bar{g}^{\mu\nu}} + \frac{\delta\mathscr{L}_3}{\delta\bar{g}^{\mu\nu}} + \dots\right). \tag{5.4.32}$$

Tensor $t^{\mathrm{tot}}_{\mu\nu}$ can be split in three algebraically-independent parts

$$t^{\mathrm{tot}}_{\mu\nu} = t^{g}_{\mu\nu} + t^{\mathrm{m}}_{\mu\nu} + t^{\mathrm{q}}_{\mu\nu}, \tag{5.4.33}$$

where $t^{g}_{\mu\nu}$ is the stress-energy tensor of pure gravitational perturbations $\mathfrak{h}^{\mu\nu}$ while and $t^{\mathrm{m}}_{\mu\nu}$ and $t^{\mathrm{q}}_{\mu\nu}$ are the stress-energy tensors characterizing gravitational coupling of the matter fields ϕ and ψ with the gravitational perturbations $\mathfrak{h}^{\mu\nu}$. The exact expression for the tensor $t^{g}_{\mu\nu}$ is given by (2.2.30) with (2.2.31) in Section 2.2.

If we restrict ourselves only with the second order non-linear corrections, the corresponding stress-energy tensors are given by variational derivatives

$$t^{g}_{\mu\nu} = -\frac{1}{16\pi\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}G^{L}_{\rho\sigma} - \frac{1}{2}\mathfrak{h}\bar{g}^{\rho\sigma}G^{L}_{\rho\sigma}\right), \tag{5.4.34}$$

$$t^{\rm m}_{\mu\nu} = -\frac{1}{16\pi\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}F^{\rm m}_{\rho\sigma} - \frac{1}{2}\mathfrak{h}g^{\rho\sigma}F^{\rm m}_{\rho\sigma} + \sqrt{-\bar{g}}\phi F^{\rm m}_{\Phi}\right), \tag{5.4.35}$$

$$t^{\rm q}_{\mu\nu} = -\frac{1}{16\pi\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\rho\sigma}F^{\rm q}_{\rho\sigma} - \frac{1}{2}\mathfrak{h}g^{\rho\sigma}F^{\rm q}_{\rho\sigma} + \sqrt{-\bar{g}}\psi F^{\rm q}_{\Psi}\right), \tag{5.4.36}$$

where $F^{\rm m}_{\Phi}$ and $F^{\rm q}_{\Psi}$ are defined below in (5.4.47), (5.4.53).

Notice that contribution of $t^{\rm tot}_{\mu\nu}$ to the linearized field equations should be neglected as it is of the higher order as compared with other terms in (5.4.27). The differential operator, $G^{L}_{\mu\nu}$, represents a linearized perturbation of the Ricci tensor and is given in (2.2.27),

$$G^{L}_{\mu\nu} = \frac{1}{2}\left(h_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}h^{\alpha\beta}{}_{|\alpha\beta} - h_{\alpha\mu|\nu}{}^{|\alpha} - h_{\alpha\nu|\mu}{}^{|\alpha}\right), \tag{5.4.37}$$

where each vertical bar denotes a covariant derivative with respect to the background metric $\bar{g}_{\mu\nu}$ of the FLRW background.

Operators $F^{\rm m}_{\mu\nu}$ and $F^{\rm q}_{\mu\nu}$ depend essentially on a particular choice of the Lagrangian of matter and scalar field, and take on different forms depending on the specific analytic dependence of $\mathscr{L}^{\rm m}$ and $\mathscr{L}^{\rm q}$ on the field variables. In the particular case of the ideal fluid, the term embraced in the round parentheses in the right side of equation (5.4.30a) is

$$\mathfrak{h}^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}^{\rm m}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} + \phi\frac{\delta\bar{\mathscr{L}}^{\rm m}}{\delta\bar{\Phi}} = \frac{1}{2}\mathfrak{h}^{\alpha\beta}\left(\bar{T}^{\rm m}_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{T}^{\rm m}\right) + \phi\,\partial_\alpha\left(\bar{\rho}_{\rm m}\sqrt{-\bar{g}}\bar{u}^{\alpha}\right), \tag{5.4.38}$$

where $\bar{u}^{\alpha} \equiv -\bar{g}^{\alpha\beta}\bar{\Phi}_{,\beta}/\bar{\mu}_{\rm m}$, and $\bar{T}^{\rm m}_{\alpha\beta}$ is given in (5.2.18). We emphasize that though the ideal fluid satisfies the equation of continuity (5.3.58), it should not be immediately implemented in (5.4.38) because this expression is to be further differentiated with respect to the metric tensor according to (5.4.30a).

For the scalar field, the term enclosed to the round parentheses in the right side of (5.4.30b) is

$$\mathfrak{h}^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}^{\rm q}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} + \psi\frac{\delta\bar{\mathscr{L}}^{\rm q}}{\delta\bar{\Psi}} = \frac{1}{2}\mathfrak{h}^{\alpha\beta}\left(\bar{T}^{\rm q}_{\alpha\beta} - \frac{1}{2}\bar{g}_{\alpha\beta}\bar{T}^{\rm q}\right) + \psi\left[\sqrt{-\bar{g}}\frac{\partial\bar{W}}{\partial\bar{\Psi}} + \partial_\alpha\left(\bar{\rho}_{\rm q}\sqrt{-\bar{g}}\bar{u}^{\alpha}\right)\right], \tag{5.4.39}$$

where $\bar{u}^{\alpha} \equiv -\bar{g}^{\alpha\beta}\bar{\Psi}_{,\beta}/\bar{\mu}_{\rm q}$, $\bar{\rho}_{\rm q} = \bar{\mu}_{\rm q}$, $\bar{T}^{\rm q}_{\alpha\beta}$ is given in (5.2.29), and the equation of continuity for the scalar field (5.3.66) should not be implemented until differentiation with respect to the metric tensor (5.4.30b) is completed.

Taking the variational derivatives with respect to $\bar{g}^{\mu\nu}$ from the expressions (5.4.38) and (5.4.39), and applying thermodynamic equations (5.2.11), allows us to write down the right sides of equations (5.4.30a), (5.4.30b) in a more explicit form as follows,

$$F^{\rm m}_{\mu\nu} = -4\pi\left[(\bar{p}_{\rm m} - \bar{\epsilon}_{\rm m})l_{\mu\nu} + \left(1 - \frac{c^2}{v_{\rm s}^2}\right)(\bar{\epsilon}_{\rm m} + \bar{p}_{\rm m}){\rm q}\bar{u}_\mu\bar{u}_\nu\right] \tag{5.4.40}$$

$$+ 8\pi\bar{\rho}_{\rm m}\left\{\bar{u}_\mu\phi_{,\nu} + \bar{u}_\nu\phi_{,\mu} + \left[\left(1 - \frac{c^2}{v_{\rm s}^2}\right)\bar{u}_\mu\bar{u}_\nu - \bar{g}_{\mu\nu}\right]\bar{u}^\alpha\phi_{,\alpha}\right\},$$

$$F^{\rm q}_{\mu\nu} = -4\pi\left[\left(p_{\rm q} - \epsilon_{\rm q}\right)l_{\mu\nu} - 2\bar{g}_{\mu\nu}\frac{\partial\bar{W}}{\partial\bar{\Psi}}\psi\right] \tag{5.4.41}$$

$$+ 8\pi\bar{\rho}_{\rm q}\left(\bar{u}_\mu\psi_{,\nu} + \bar{u}_\nu\psi_{,\mu} - \bar{g}_{\mu\nu}\bar{u}^\alpha\psi_{,\alpha}\right),$$

where $\bar{\rho}_{\rm q} \equiv \dot{\bar{\Psi}} = \bar{\mu}_{\rm q}$ in accordance with definition (5.2.25). The potential energy of the scalar field, $\bar{W} = \bar{W}(\bar{\Psi})$, is kept arbitrary.

It is important to emphasize that in the most general case the ratio $v_{\rm s}^2/c^2$ of the speed of sound in fluid to the fundamental speed c, may be not equal to the parameter $w_{\rm m}$ of the equation of state (5.3.54), that is there are physical equations of state such that $w_{\rm m} \neq (v_{\rm s}/c)^2$. Indeed, the speed of sound is defined as a partial derivative of pressure $p_{\rm m}$ with respect to the energy density $\epsilon_{\rm m}$ taken under the condition of a constant entropy $s_{\rm m}$,

$$\frac{v_{\rm s}^2}{c^2} = \left(\frac{\partial p_{\rm m}}{\partial\epsilon_{\rm m}}\right)_{s_{\rm m}={\rm const.}}. \tag{5.4.42}$$

This equation is equivalent to the following relation

$$\frac{v_{\rm s}^2}{c^2} = \frac{(\partial p_{\rm m}/\partial\mu_{\rm m})_{s_{\rm m}={\rm const.}}}{(\partial\epsilon_{\rm m}/\partial\mu_{\rm m})_{s_{\rm m}={\rm const.}}}, \tag{5.4.43}$$

which is a consequence of thermodynamic relations and a definition of the partial derivative. The ratio of the partial derivatives in (5.4.43) is not reduced to $w_{\rm m}$ in case when $w_{\rm m}$ depends on some other thermodynamic parameters which are functions of the specific enthalpy. For example, in case of an ideal gas the equation of state $p_{\rm m} = w_{\rm m}\epsilon_{\rm m}$, where $w_{\rm m} = kT/mc^2$, k is the Boltzmann constant, m - mass of a particle of the ideal fluid, and T is the fluid temperature. The speed of sound $v_{\rm s}^2 = c^2(\partial p_{\rm m}/\partial\epsilon_{\rm m})_{s_{\rm m}={\rm const.}} = \Gamma w_{\rm m} > w_{\rm m} = p_{\rm m}/\epsilon_{\rm m}$, where $\Gamma > 1$ is the ratio of the heat capacities of the gas taken for the constant pressure and the constant volume respectively [284].

The scalar field with the potential function $W(\Psi) \neq 0$ does not bear all thermodynamic properties of an ideal fluid. Nevertheless, we can formally define the speed of "sound" c_s propagating in the scalar field "fluid", by equation being similar to (5.4.43). More specifically,

$$\frac{c_\mathrm{s}^2}{c^2} = \frac{\left(\partial p_\mathrm{q}/\partial \mu_\mathrm{q}\right)_{\Psi=\mathrm{const.}}}{\left(\partial \epsilon_\mathrm{q}/\partial \mu_\mathrm{q}\right)_{\Psi=\mathrm{const.}}}. \tag{5.4.44}$$

Simple calculation reveals that the speed of "sound" for scalar field is always equal to the fundamental speed

$$c_\mathrm{s} = c, \tag{5.4.45}$$

irrespectively of the value of the potential function $W(\Psi)$. It explains why the terms being proportional to the factor $1 - c^2/c_\mathrm{s}^2$, do not appear in the expression (5.4.41) as contrasted with (5.4.40).

5.4.5 The Lagrangian equations for dark matter perturbations

The perturbed field equations for the dark matter which is modelled by the ideal fluid, are obtained by taking the variational derivatives with respect to the field Φ from the Lagrangian (5.4.20) – it corresponds to the middle equation in (5.4.21a). Taking into account the background equation (5.4.19b) yields the equation of sound waves propagating in the fluid as small perturbations of the potential ϕ,

$$F_\Phi^\mathrm{m} = 8\pi\Sigma^\mathrm{m}, \tag{5.4.46}$$

where the linear differential operator

$$F_\Phi^\mathrm{m} \equiv -\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\Phi}}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\bar{\mathscr{L}}^\mathrm{m}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \phi\frac{\delta\bar{\mathscr{L}}^\mathrm{m}}{\delta\bar{\Phi}}\right), \tag{5.4.47}$$

and the source term

$$\Sigma^\mathrm{m} \equiv \frac{1}{8\pi\sqrt{-\bar{g}}}\left(\frac{\delta\mathscr{L}_2^\mathrm{m}}{\delta\bar{\Phi}} + \frac{\delta\mathscr{L}_3^\mathrm{m}}{\delta\bar{\Phi}} + \dots\right). \tag{5.4.48}$$

Thus, the equation (5.4.46) concretizes the general equation (2.2.37) for the case of the dynamic field ϕ that is a perturbation of the background Clebsch potential $\bar{\Phi}$.

According to equation (5.2.12), the Lagrangian of the ideal fluid can be rewritten as $\mathscr{L}^\mathrm{m} = -\sqrt{-g}p_\mathrm{m}$ which is further transformed to (5.2.16). In the case of a single-component ideal fluid, the Lagrangian (5.2.16) depends merely on the derivative of the Clebsch potential Φ and on the metric tensor. Therefore, the explicit form of the linear operator F_Φ^m is reduced to a covariant divergence

$$F_\Phi^\mathrm{m} = Y^\alpha{}_{|\alpha}, \tag{5.4.49}$$

where a vector field

$$Y^\alpha \equiv \frac{\partial}{\partial \bar{\Phi}_{,\alpha}}\left[\left(h^{\mu\nu} - \frac{1}{2}h\bar{g}^{\mu\nu}\right)\left(\frac{\partial \bar{L}^{\mathrm{m}}}{\partial \bar{g}^{\mu\nu}} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{L}^{\mathrm{m}}\right) + \phi_{,\beta}\frac{\partial \bar{L}^{\mathrm{m}}}{\partial \bar{\Phi}_{,\beta}}\right], \tag{5.4.50}$$

$h \equiv \bar{g}^{\alpha\beta}h_{\alpha\beta}$ and the partial derivatives are taken from the Lagrangian $L^{\mathrm{m}} = -p_{\mathrm{m}}$. More specifically, calculations yield

$$Y^\alpha \equiv \frac{\bar{\rho}_{\mathrm{m}}}{\bar{\mu}_{\mathrm{m}}}\phi^{|\alpha} - \bar{\rho}_{\mathrm{m}}h^{\alpha\beta}\bar{u}_\beta + \left(1 - \frac{c^2}{v_{\mathrm{s}}^2}\right)\left(\frac{\bar{\rho}_{\mathrm{m}}}{\bar{\mu}_{\mathrm{m}}}\bar{u}^\alpha\bar{u}^\beta\phi_{|\beta} - \frac{1}{2}\bar{\rho}_{\mathrm{m}}\bar{u}^\alpha \mathfrak{q}\right). \tag{5.4.51}$$

Similar expression was derived by Lukash [301] who used the variational method to analyze the production and quantization of sound waves in the early universe.

5.4.6 The Lagrangian equations for dark energy perturbations

Equations for the perturbations ψ of dark energy, which is modelled by a scalar field Ψ, are derived by taking the variational derivative from the Lagrangian (5.4.20) with respect to the field variable Ψ – see equation (5.4.21c). Subtracting the background equation (5.4.19c) from (5.4.21c) and making use of the Lagrangian decomposition in the Taylor (post-Newtonian) series leads to

$$F_\Psi^{\mathrm{q}} = 8\pi\Sigma^{\mathrm{q}}, \tag{5.4.52}$$

where the linear differential operator

$$F_\Psi^{\mathrm{q}} \equiv -\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{\Psi}}\left(\mathfrak{h}^{\mu\nu}\frac{\delta\bar{\mathscr{L}}^{\mathrm{q}}}{\delta\bar{\mathfrak{g}}^{\mu\nu}} + \psi\frac{\delta\bar{\mathscr{L}}^{\mathrm{q}}}{\delta\bar{\Psi}}\right), \tag{5.4.53}$$

and the source term

$$\Sigma^{\mathrm{q}} \equiv \frac{1}{8\pi\sqrt{-\bar{g}}}\left(\frac{\delta\mathscr{L}_2^{\mathrm{q}}}{\delta\bar{\Psi}} + \frac{\delta\mathscr{L}_3^{\mathrm{q}}}{\delta\bar{\Psi}} + \dots\right). \tag{5.4.54}$$

Equation (5.4.52) concretizes the general equation (2.2.37) for the case of the dynamic field ψ that is a perturbation of the background scalar field $\bar{\Psi}$.

According to equation (5.2.19), the Lagrangian of the scalar field can be rewritten as $\mathscr{L}^{\mathrm{q}} = \sqrt{-g}L^{\mathrm{q}}$ and depends on both the field Ψ and its first derivative, $\Psi_{,\alpha}$. For this reason, the differential operator F^{q} is not reduced to the covariant derivative from a vector field as the partial derivative of the Lagrangian with respect to Ψ does not vanish. We have

$$F_\Psi^{\mathrm{q}} \equiv Z^\alpha{}_{|\alpha} - \frac{h}{2}\frac{\partial\bar{W}}{\partial\bar{\Psi}} - \psi\frac{\partial^2\bar{W}}{\partial\bar{\Psi}^2}, \tag{5.4.55}$$

where vector field

$$Z^\alpha \equiv \frac{\partial}{\partial \bar{\Psi}_{,\alpha}}\left[\left(h^{\mu\nu} - \frac{1}{2}h\bar{g}^{\mu\nu}\right)\left(\frac{\partial \bar{L}^{\mathrm{q}}}{\partial \bar{g}^{\mu\nu}} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{L}^{\mathrm{q}}\right) + \psi_{,\beta}\frac{\partial \bar{L}^{\mathrm{q}}}{\partial \bar{\Psi}_{,\beta}}\right]. \tag{5.4.56}$$

Performing the partial derivatives in equation (5.4.56), yields a rather simple expression

$$Z^\alpha \equiv \psi^{|\alpha} - \bar{\rho}_{\mathrm{q}} h^{\alpha\beta}\bar{u}_\beta, \tag{5.4.57}$$

where we have used equation $\bar{\Psi}_{|\alpha} = -\bar{u}^\beta \bar{\Psi}_{|\beta}\bar{u}_\alpha = -\bar{\rho}_{\mathrm{q}}\bar{u}_\alpha$. The reader is invited to compare equation (5.4.57) with (5.4.51) to observe the differences between the Lagrangian perturbations of the ideal fluid and the scalar field. One may observe that (5.4.51) becomes identical with (5.4.57) in the limit $v_{\mathrm{s}} \to c$, and $\bar{\rho}_{\mathrm{m}} \to \bar{\mu}_{\mathrm{m}}$. This corresponds to the case of an extremely stiff equation of state $w_{\mathrm{m}} = 1$ in equation (5.3.54). According to the discussion following equations (5.4.44), (5.4.45) the speed of "sound" c_{s} in the scalar field "fluid" is always equal to c. However, it does not assume that the parameter w_{q} of the equation of state of the scalar field, $\bar{p}_{\mathrm{q}} = w_{\mathrm{q}}\bar{\epsilon}_{\mathrm{q}}$, in (5.3.54) is equal to unity. This is because the scalar field is not completely equivalent to the ideal fluid in the sense of thermodynamic [7].

5.4.7 Linearized post-Newtonian equations for field variables

Equations for the metric tensor perturbations

Linearized field equations for gravitational field variables, $h_{\mu\nu}$, are obtained from (5.4.27) after neglecting in its right side the non-linear source $t^{\mathrm{tot}}_{\mu\nu}$, and making a series of transformations to sort out similar terms. First, let us make use of equations (5.4.40) and (5.4.41) to find out

$$\begin{aligned} F^{\mathrm{m}}_{\mu\nu} + F^{\mathrm{q}}_{\mu\nu} &= 4\pi\left(\bar{\epsilon} - \bar{p}\right) h_{\mu\nu} \\ &\quad + 8\pi\bar{\rho}_{\mathrm{m}}\left[\bar{u}_\mu\phi_{,\nu} + \bar{u}_\nu\phi_{,\mu} - \bar{g}_{\mu\nu}u^\alpha\phi_{,\alpha} + \left(1 - \frac{c^2}{v_{\mathrm{s}}^2}\right)\right. \\ &\quad \left.\times\left(\bar{u}^\alpha\phi_{,\alpha} - \frac{1}{2}\bar{\mu}_{\mathrm{m}}\mathrm{q}\right)\bar{u}_\mu\bar{u}_\nu\right] \\ &\quad + 8\pi\bar{\rho}_{\mathrm{q}}\left[\bar{u}_\mu\psi_{,\nu} + \bar{u}_\nu\psi_{,\mu} - \bar{g}_{\mu\nu}u^\alpha\psi_{,\alpha} + \bar{g}_{\mu\nu}\frac{\partial \bar{W}(\bar{\Psi})}{\partial \bar{\Psi}}\frac{\psi}{\bar{\mu}_{\mathrm{q}}}\right], \end{aligned} \tag{5.4.58}$$

where we used the superposition $\bar{\epsilon} = \bar{\epsilon}_{\mathrm{m}} + \bar{\epsilon}_{\mathrm{q}}$, $\bar{p} = \bar{p}_{\mathrm{m}} + \bar{p}_{\mathrm{q}}$. Second step is to transform the linear differential operator $F^{\mathrm{g}}_{\mu\nu}$ in (5.4.37) to a more convenient form that will allow us to single out the gauge-dependent vector denoted by

$$A^\mu \equiv h^{\mu\nu}{}_{|\nu}. \tag{5.4.59}$$

Changing the order of the covariant derivatives in (5.4.37) and taking into account that the commutator of the second covariant derivatives is proportional to the Riemann tensor, we recast (5.4.37) to the following form,

$$G^L_{\mu\nu} \equiv \frac{1}{2}\left(h_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha} - A_{\mu|\nu} - A_{\nu|\mu}\right) - \bar{R}^{\alpha}{}_{(\mu}h_{\nu)\alpha} - \bar{R}_{\mu\alpha\beta\nu}h^{\alpha\beta}, \tag{5.4.60}$$

where the round brackets around indices denote symmetrization. The terms with the Ricci and Riemann tensors can be expressed in terms of the total background energy and pressure of the ideal fluid and scalar field by making use of equations (5.3.34), (5.3.36) and Einstein's equations (5.3.43). It yields

$$\begin{aligned} R^{\alpha}{}_{(\mu}h_{\nu)\alpha} + \bar{R}_{\mu\alpha\beta\nu}h^{\alpha\beta} = 4\pi\Bigg[&\left(\frac{5\bar{\epsilon}}{3} - \bar{p}\right)h_{\mu\nu} + \frac{h}{2}\left(\bar{p} - \frac{\bar{\epsilon}}{3}\right)\bar{g}_{\mu\nu} \\ &+ (\bar{\epsilon} + \bar{p})\left(2\bar{u}^{\alpha}\bar{u}_{\mu}h_{\nu\alpha} + 2\bar{u}^{\alpha}\bar{u}_{\nu}h_{\mu\alpha} - \bar{u}_{\mu}\bar{u}_{\nu}h - \bar{g}_{\mu\nu}\mathfrak{q}\right)\Bigg]. \end{aligned} \tag{5.4.61}$$

Finally, substituting equations (5.4.58), (5.4.60) and (5.4.61) to (5.4.27) results in

$$\begin{aligned} &h_{\mu\nu}{}^{|\alpha}{}_{|\alpha} + \bar{g}_{\mu\nu}A^{\alpha}{}_{|\alpha} - A_{\mu|\nu} - A_{\nu|\mu} \\ &- 16\pi\left[\frac{\bar{\epsilon}}{3}h_{\mu\nu} + \frac{h}{4}\left(\bar{p} - \frac{\bar{\epsilon}}{3}\right)\bar{g}_{\mu\nu}\right. \\ &+ \left.(\bar{\epsilon} + \bar{p})\left(\frac{1}{2}\bar{u}^{\alpha}\bar{u}_{(\mu}h_{\nu)\alpha} - \frac{1}{2}\bar{u}_{\mu}\bar{u}_{\nu}h - \frac{1}{2}\bar{g}_{\mu\nu}\mathfrak{q}\right)\right] \\ &+ 16\pi\bar{\rho}_{\rm m}\left[\bar{u}_{\mu}\phi_{,\nu} + \bar{u}_{\nu}\phi_{,\mu} - \bar{g}_{\mu\nu}u^{\alpha}\phi_{,\alpha} + \left(1 - \frac{c^2}{v_{\rm s}^2}\right)\left(\bar{u}^{\alpha}\phi_{,\alpha} - \frac{1}{2}\bar{\mu}_{\rm m}\mathfrak{q}\right)\bar{u}_{\mu}\bar{u}_{\nu}\right] \\ &+ 16\pi\bar{\rho}_{\rm q}\left[\bar{u}_{\mu}\psi_{,\nu} + \bar{u}_{\nu}\psi_{,\mu} - \bar{g}_{\mu\nu}u^{\alpha}\psi_{,\alpha} + \bar{g}_{\mu\nu}\frac{\partial\bar{W}(\bar{\Psi})}{\partial\bar{\Psi}}\frac{\psi}{\bar{\mu}_{\rm q}}\right] \\ &= 16\pi T^{\rm p}_{\mu\nu}, \end{aligned} \tag{5.4.62}$$

where the non-linear term, $t^{\rm tot}_{\mu\nu}$, was neglected in the right side of (5.4.62).

The very first term in (5.4.62) is a tensorial Laplace-Beltrami operator, $h_{\mu\nu}{}^{|\alpha}{}_{|\alpha} \equiv \bar{g}^{\alpha\beta}h_{\mu\nu|\alpha\beta}$, that is a rather complicated geometric object. Its explicit expression can be developed by making use of the Christoffel symbols given in (5.3.23). Tedious but straightforward calculation yields [272]

$$\begin{aligned} h_{\mu\nu}{}^{|\alpha}{}_{|\alpha} = \bar{g}^{\alpha\beta}h_{\mu\nu;\alpha\beta} &+ 2H\bar{u}^{\alpha}h_{\mu\nu;\alpha} - 2\left(H\bar{u}^{\alpha}h_{\alpha\mu}\right)_{|\nu} - 2\left(H\bar{u}^{\alpha}h_{\alpha\nu}\right)_{|\mu} \\ &+ 2H\left(\bar{u}_{\mu}A_{\nu} + \bar{u}_{\nu}A_{\mu}\right) + 2\dot{H}\left(h_{\mu\nu} - \bar{u}^{\alpha}\bar{u}_{\mu}h_{\nu\alpha} - \bar{u}^{\alpha}\bar{u}_{\nu}h_{\mu\alpha}\right) \\ &+ 2H^2\left(2h_{\mu\nu} + 3\bar{u}_{\mu}\bar{u}^{\alpha}h_{\alpha\nu} + 3\bar{u}_{\nu}\bar{u}^{\alpha}h_{\alpha\mu} - \bar{g}_{\mu\nu}\bar{u}^{\alpha}\bar{u}^{\beta}h_{\alpha\beta} - \bar{u}_{\mu}\bar{u}_{\nu}h\right), \end{aligned} \tag{5.4.63}$$

where the semicolon denotes a covariant derivative that is calculated with the Christoffel symbols $B^{\alpha}{}_{\mu\nu}$ like in (5.3.30b).

Further derivation of the differential post-Newtonian field equation for the linearized metric tensor perturbations can be significantly simplified if we choose the gauge function, A^α, in the following form

$$A^\alpha = -2Hh^{\alpha\beta}\bar{u}_\beta + 16\pi\left(\bar{\rho}_{\rm m}\phi + \bar{\rho}_{\rm q}\psi\right)\bar{u}^\alpha + B^\alpha, \qquad (5.4.64)$$

where B^α is an arbitrary gauge vector field. This choice of the gauge function A^α allows us to eliminate two terms in equation (5.4.63) which depend on the first covariant derivatives with respect to the background metric $\bar{g}_{\alpha\beta}$. Moreover, it allows to eliminate a number of terms depending on the first derivatives of the fields ϕ and ψ in equation (5.4.62). Since we keep the gauge function B^α arbitrary, the equation (5.4.64) does not fix any gauge. The choice of the gauge is controlled by the gauge function B^α.

One substitutes the gauge function (5.4.64) to equations (5.4.63) and (5.4.62) and make use of the background Friedmann equations (5.3.50), (5.3.51) to replace the background values of the energy density, $\bar{\epsilon}$, and pressure, $\bar{p}$, with the Hubble parameter H and its time derivative $\dot{H}$. It brings about equation (5.4.62) to the following form

$$\begin{aligned}
&\bar{g}^{\alpha\beta}h_{\mu\nu;\alpha\beta} + 2H\bar{u}^\alpha h_{\mu\nu;\alpha} \qquad (5.4.65)\\
&+ 2\left(\dot{H} + H^2\right)\left(h_{\mu\nu} + \bar{u}_\mu\bar{u}^\alpha h_{\alpha\nu} + \bar{u}_\nu\bar{u}^\alpha h_{\alpha\mu} - h\bar{u}_\mu\bar{u}_\nu\right)\\
&- \frac{2k}{a^2}\left[h_{\mu\nu} + 2\bar{u}_\mu\bar{u}^\alpha h_{\alpha\nu} + 2\bar{u}_\nu\bar{u}^\alpha h_{\alpha\mu} - h\bar{u}_\mu\bar{u}_\nu - \left(\mathrm{q} + \frac{h}{2}\right)\bar{g}_{\mu\nu}\right]\\
&+ 16\pi\bar{u}_\mu\bar{u}_\nu\left[\bar{\rho}_{\rm m}\left(1 - \frac{c^2}{v_{\rm s}^2}\right)\left(\bar{u}^\alpha\phi_{,\alpha} - \frac{1}{2}\bar{\mu}_{\rm m}\mathrm{q}\right) - 2\frac{\partial\bar{W}}{\partial\bar{\Psi}}\psi - 4H\left(\bar{\rho}_{\rm m}\phi + \bar{\rho}_{\rm q}\psi\right)\right]\\
&+ \bar{g}_{\mu\nu}B^\alpha{}_{|\alpha} - B_{\mu|\nu} - B_{\nu|\mu} + 2H\left(\bar{u}_\mu B_\nu + \bar{u}_\nu B_\mu - \bar{g}_{\mu\nu}\bar{u}^\alpha B_\alpha\right)\\
&= 16\pi T^{\rm p}_{\mu\nu}.
\end{aligned}$$

This equation is fully covariant and is valid in any gauge and/or coordinate chart.

Now, let us fix the gauge by selecting a specific gauge function B^α in (5.4.64). The task is to decouple the linearised field equations for h_{00}, h_{0i} and h_{ij} components of the metric tensor perturbations. For this purpose, let us work in the isotropic coordinates associated with the Hubble flow, where $\bar{u}^\alpha = (1/a, 0, 0, 0)$ and choose the gauge condition, $B^\alpha = 0$. It brings equation (5.4.65) for different components of the metric perturbations to the form

$$\Box q + 2\mathscr{H}q_{;0} + 4kq - 4\pi\left(1 - \frac{c^2}{v_{\rm s}^2}\right)\bar{\rho}_{\rm m}\bar{\mu}_{\rm m}q = 8\pi a^2\left(T^{\rm p}_{00} + T^{\rm p}_{kk}\right) - \qquad (5.4.66a)$$

$$8\pi a^3\left[\left(1 - \frac{c^2}{v_{\rm s}^2}\right)\bar{\rho}_{\rm m}\phi_{,0} - 2a\frac{\partial\bar{W}}{\partial\bar{\Psi}}\psi - \mathscr{H}\left(\bar{\rho}_{\rm m}\phi + \bar{\rho}_{\rm q}\psi\right)\right], \qquad (5.4.66b)$$

$$\Box h_{0i} + 2\mathscr{H} h_{0i;0} + 2k h_{0i} = 16\pi a^2 T^{\rm p}_{0i}, \tag{5.4.66c}$$

$$\Box h_{<ij>} + 2\mathscr{H} h_{<ij>;0} + 2\left(\mathscr{H}' - k\right) h_{<ij>} = 16\pi a^2 T^{\rm p}_{<ij>}, \tag{5.4.66d}$$

$$\Box h_{kk} + 2\mathscr{H} h_{kk;0} + 2\left(\mathscr{H}' + 2k\right) h_{kk} = 16\pi a^2 T^{\rm p}_{kk}. \tag{5.4.66e}$$

Here, $\mathscr{H}$ is a conformal Hubble parameter (5.3.4), a prime means a derivative with respect to η, and we denoted $\Box h_{\mu\nu} \equiv \bar{\mathfrak{f}}^{\alpha\beta} h_{\mu\nu;\alpha\beta}$, $q \equiv \left(h_{00} + h_{kk}\right)/2$, $h_{kk} \equiv h_{11} + h_{22} + h_{33}$, $h_{<ij>} \equiv h_{ij} - (1/3)\delta_{ij} h_{kk}$, and the same index notations are applied to the tensor of energy-momentum $T^{\rm p}_{ij}$ of the localized astronomical system, $T^{\rm p}_{<ij>} = T^{\rm p}_{ij} - (1/3)\delta_{ij} T^{\rm p}_{kk}$. These equations are clearly decoupled from one another, thus, demonstrating the advantage of the gauge condition $B^\alpha = 0$.

Equations (5.4.66c–5.4.66e) can be solved independently if the initial and boundary conditions are known, and the tensor of energy-momentum of the localized astronomical system is well-defined. Equation (5.4.66a) for a scalar q demands besides knowledge of $T^{\rm p}_{\alpha\beta}$, knowing the scalar field perturbations, ϕ and ψ, that contribute to the source of the field equation for q in the right side of (5.4.66a). Equations for these perturbations are obtained below.

Equations for the dark matter perturbations

The dark matter perturbations, ϕ, evolve in accordance with the Lagrangian equation (5.4.46). In the linear approximation we can neglect the non-linear source term $\Sigma^{\rm m}$ in its right side. The covariant derivative in the definition of the linear operator $F^{\rm m}$ given in the right side of (5.4.49), can be explicitly performed, thus, yielding equation for the Clebsch potential

$$\begin{aligned}\phi^{|\alpha}{}_{|\alpha} - 2\bar{\mu}_{\rm m} H {\rm q} + 16\pi\bar{\mu}_{\rm m}\left(\bar{\rho}_{\rm m}\phi + \bar{\rho}_{\rm q}\psi\right)&\\ + \left(1 - \frac{c^2}{v_{\rm s}^2}\right)\left(\bar{u}^\alpha \bar{u}^\beta \phi_{|\alpha\beta} - \frac{1}{2}\bar{\mu}_{\rm m}\bar{u}^\alpha {\rm q}_{,\alpha}\right) &= \bar{\mu}_{\rm m}\bar{u}^\alpha B_\alpha,\end{aligned} \tag{5.4.67}$$

where equation (5.4.64) has been used. The gauge B^α remains yet unspecified so that equation (5.4.67) is covariant and is valid in any coordinate chart. To make it compatible with equations (5.4.66a)–(5.4.66e) for the metric tensor perturbations, we have to choose $B^\alpha = 0$.

Equations for the dark energy perturbations

Linearized equation for the dark energy perturbations, ψ, is obtained from the Lagrangian equation (5.4.52) after neglecting the (non-linear) source term $\Sigma^{\rm q}$. After performing the covariant differentiation in equation (5.4.55), we conclude that the dark energy perturbation obeys the following equation

$$\psi^{|\alpha}{}_{|\alpha} - \left(2\bar{\mu}_{\rm q} H + \frac{\partial \bar{W}}{\partial \bar{\Psi}}\right){\rm q} + 16\pi\bar{\mu}_{\rm q}\left(\bar{\rho}_{\rm m}\phi + \bar{\rho}_{\rm q}\psi\right) - \frac{\partial^2 \bar{W}}{\partial \bar{\Psi}^2}\psi = \bar{\mu}_{\rm q}\bar{u}^\alpha B_\alpha, \tag{5.4.68}$$

where equation (5.4.64) has been used along with the equality $\bar{\rho}_q = \bar{\mu}_q$. The gauge function B^α is kept unspecified so that equation (5.4.68) is covariant and is valid in any coordinates. To make it compatible with equations (5.4.66a–5.4.66e) for the metric tensor perturbations, we have to choose $B^\alpha = 0$.

5.5 Gauge-invariant scalars and field equations in 1+3 threading formalism

5.5.1 Threading decomposition of the metric perturbations

We have derived the system of coupled differential equations (5.4.65), (5.4.67), (5.4.68) for the field variables $h_{\alpha\beta}$, ϕ and ψ, describing perturbations of the gravitational field, dark matter and dark energy respectively. These equations are gauge-invariant and written down in arbitrary coordinates on the background manifold. Nonethelees, they operate with the field variables which are not gauge-invariant in themselves. Therefore, solutions of equations (5.4.65), (5.4.67), (5.4.68) that are found in a particular gauge has no direct physical interpretation and must be connected to physical observables to match theory with observations. Another way around is to find out some gauge-invariant geometric objects built out of $h_{\alpha\beta}$, ϕ and ψ which will not depend on a particular choice of gauge and coordinates. This program was initiated by Bardeen [27] who proposed to split the perturbations of the metric tensor in scalar, vector, and tensor components by making use of 3+1 spacetime slicing ADM technique [12], and to build gauge-invariant cosmological variables out of these elements. Gauge-invariant scalars are the most important quantities in cosmology as they describe the structure formation in the universe. Ellis and Bruni [155] pointed out that Bardin's variables are not directly related to the density fluctuations but to it second derivatives which makes them less useful in relativistic calculations of structure formation. They proposed their own gauge-invariant variables that are build out of gradients of the geometric objects which vanish on the background manifold so that only their perturbations make physical sense.

In this section we propose even more direct approach to the definition of the gauge-invariant scalars by making use of the scalar potentials Φ and Ψ for description of the dark matter and dark energy. In this way we shall find out the gauge-invariant scalars that are equivalent to the matter density fluctuation itself but not to its gradient or a second order derivative. We shall employ 1+3 threading (it is not 3+1 splitting) approach to split four-dimensional tensors into scalar, vector, and three-dimensional tensors. The original idea was proposed by Zelmanov [475] who called the elements of the tensorial decomposition the chronometric invariants. Later on, the theory of chronometric invariants was reinvented by a number of researchers. The central ingredient of the theory is a congruence of worldlines threading spacetime. In FLRW cosmology, this congruence is naturally associated with the Hubble flow and the Hubble velocity $\bar{u}^\alpha$. Threading (chronometric) decomposition is achieved with the invariant operator of projection $\bar{P}_{\alpha\beta}$ onto a hypersurface being orthogonal to the congruence of world lines of the Hubble flow,

$$\bar{P}_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{u}_\alpha \bar{u}_\beta, \tag{5.5.1}$$

where $\bar{g}_{\alpha\beta}$ is FLRW background metric. The operator $\bar{P}_{\alpha\beta}$ can be considered as a metric on the spatial hypersurface of the background FLRW manifold.

The post-Newtonian theory under development admits four, algebraically-independent scalar perturbations. Two of them are the Clebsch potential of the ideal fluid ϕ and the scalar field ψ. The two other scalars characterize the scalar perturbations of the gravitational field. They can be chosen, for example, as a projection of the metric tensor perturbation on the direction of the background four-velocity field, $\bar{u}^\alpha \bar{u}^\beta h_{\alpha\beta}$, and the trace of the metric tensor perturbation, $h = \bar{g}^{\alpha\beta} h_{\alpha\beta}$. However, it is more convenient to work with two other scalars, defined as their linear combinations,

$$\mathfrak{q} \equiv \frac{1}{2}\left(\bar{u}^\alpha \bar{u}^\beta + \bar{P}^{\alpha\beta}\right) h_{\alpha\beta}, \tag{5.5.2a}$$

$$\mathfrak{p} \equiv \bar{P}^{\alpha\beta} h_{\alpha\beta}. \tag{5.5.2b}$$

Notice that the scalar $\mathfrak{q}$ has been introduced earlier in (5.4.16). The scalar $\mathfrak{p}$ is, in fact, a projection of $h_{\alpha\beta}$ onto the space-like hypersurface being orthogonal everywhere to the worldlines of Hubble observers.

Vectorial chronometric perturbations are defined by a spacial-temporal projection

$$\mathfrak{p}_\alpha \equiv -\bar{P}_\alpha{}^\beta \bar{u}^\gamma h_{\beta\gamma}, \tag{5.5.3}$$

where minus sign was taken for the sake of mathematical convenience. Due to its definition, vector $\mathfrak{p}^\alpha = \bar{g}^{\alpha\beta}\mathfrak{p}_\beta$ is orthogonal to the four-velocity $\bar{u}^\alpha$, that is $\bar{u}^\alpha \mathfrak{p}_\alpha = 0$. Hence, it describes a space-like vector-like gravitational perturbations with three algebraically-independent components.

Tensorial chronometric perturbations are associated with the projection

$$\mathfrak{p}^{\mathrm{T}}_{\alpha\beta} \equiv \mathfrak{p}_{\alpha\beta} - \frac{1}{3}\bar{P}_{\alpha\beta}\mathfrak{p}, \tag{5.5.4}$$

where

$$\mathfrak{p}_{\alpha\beta} \equiv \bar{P}_\alpha{}^\mu \bar{P}_\beta{}^\nu h_{\mu\nu}. \tag{5.5.5}$$

Here, the tensor $\mathfrak{p}_{\alpha\beta}$ is a double projection of $h_{\alpha\beta}$ onto space-like hypersurface being orthogonal to the worldlines of Hubble observers. The trace of this tensor coincides with the scalar $\mathfrak{p}$. Indeed,

$$\bar{g}^{\alpha\beta}\mathfrak{p}_{\alpha\beta} = \bar{g}^{\alpha\beta}\bar{P}_\alpha{}^\mu \bar{P}_\beta{}^\nu h_{\mu\nu} = \bar{P}^{\beta\mu}\bar{P}_\beta{}^\nu h_{\mu\nu} = \bar{P}^{\mu\nu} h_{\mu\nu} = \mathfrak{p}, \tag{5.5.6}$$

where the property of the projection tensor $\bar{P}^{\beta\mu}\bar{P}_{\beta}{}^{\nu} = \bar{P}^{\mu\nu}$ has been used. Equation (5.5.6) makes it clear that tensor $\mathfrak{p}^{\top}_{\alpha\beta}$ is traceless, that is $\bar{g}^{\alpha\beta}\mathfrak{p}^{\top}_{\alpha\beta} = 0$. Because of this property, and four orthogonality conditions, $\bar{u}^{\alpha}\mathfrak{p}^{\top}_{\alpha\beta} = 0$, the symmetric tensor $\mathfrak{p}^{\top}_{\alpha\beta}$ has only five, algebraically-independent components.

Gravitational perturbation $h_{\alpha\beta}$ can be decomposed into the algebraically-irreducible scalar, vector and tensor parts as follows

$$h_{\alpha\beta} = \mathfrak{p}^{\top}_{\alpha\beta} + \bar{u}_{\alpha}\mathfrak{p}_{\beta} + \bar{u}_{\beta}\mathfrak{p}_{\alpha} + \left(\bar{u}_{\alpha}\bar{u}_{\beta} + \frac{1}{3}\bar{P}_{\alpha\beta}\right)\mathfrak{p} + 2\bar{u}_{\alpha}\bar{u}_{\beta}\left(\mathfrak{q} - \mathfrak{p}\right). \tag{5.5.7}$$

One should not confuse the pure algebraic (threading) decomposition of the metric tensor perturbation with its functional (slicing) decomposition. The slicing (or kinemetric, according to Zelmanov [472]) decomposition was pioneered by Arnowitt et al. [12] Misner et al. [315], see Section 3.1. It is commonly used in the research on the relativistic theory of formation of the large-scale structure in the universe. The ADM decomposition of the metric tensor perturbations is done by foliating spacetime [27, 260] with a set of spacelike hypersurfaces and making use of three dimensional Helmholtz theorem [24] which singles out the longitudinal (L), transversal (T) and transverse-traceless (TT) parts of the perturbations. In other words, the slicing decomposition make vector $\mathfrak{p}_{\alpha}$ and tensor parts of the gravitational perturbation, $\mathfrak{p}^{\top}_{\alpha\beta}$, are further decomposed in the functionally-irreducible components which include two more scalars, and two transverse spatial vectors each having only two (out of three) independent components. The remaining part of the tensor perturbations, $\mathfrak{p}^{\top}_{\alpha\beta}$, is transverse-trackless and has only two functionally-independent components denoted as $\mathfrak{p}^{\mathrm{TT}}_{\alpha\beta}$. The ADM decomposition of the metric tensor is a powerful technique in the theory of gauge-invariant cosmological perturbations [28, 327]. However, it is not convenient in the development of the systematic post-Newtonian approximations and celestial dynamics of inhomogeneities in cosmology. Thus, we do not use it in the present chapter.

Our next step is a to find the gauge-invariant scalars directly reproducing the density fluctuation and to derive the post-Newtonian field equations for the algebraically-irreducible components of matter and gravitational field. We, first, discuss the gauge transformations of the corresponding field variables.

5.5.2 Gauge transformation of the field variables

We discuss physical perturbations, tensor $h_{\alpha\beta}$, and scalars Φ, Ψ in the framework of general relativity. Their gauge transformation is generated by a flow of an arbitrary vector (gauge) field ξ^{α} that maps the manifold into itself, Section 2.2.4 in detail. Generic gauge transformation of the fields on a curved manifold is associated with their Lie transport along the vector flow ξ^{α} [285, 456] while an infinitesimal gauge transformation is a Lie derivative of the field taken at the value of the parameter on the curves of the vector flow ξ^{α}.

Let us repeat results of Section 2.2.4 in brief. Consider a mapping of spacetime manifold onto itself induced by a vector flow, $\xi^\alpha = \xi^\alpha(x^\beta)$. This means that each point of the manifold with coordinates x^α is mapped to another point with coordinates $x'^\alpha = \ldots$ in transformations (2.2.54). In linear approximation it is

$$x'^\alpha = x^\alpha + \xi^\alpha(x). \tag{5.5.8}$$

This mapping of the manifold onto itself can be interpreted as a local diffeomorphism which transforms the field variables in accordance to their tensor properties. The transformed value of the field variable is pulled back to the point of the manifold having the original coordinates x^α, and is compared with the original value of the field at this point. The difference between the transformed and the original value of the field, generated by the diffeomorphism (5.5.8) is the gauge transformation of the field that is given by the Lie derivative taken along the vector flow ξ^α at the point of the manifold with coordinates x^α, for details see Section 2.2.4.

We denote the transformed values of the field variables with a prime, like in (5.5.8). In the linearized perturbation theory the gauge transformations of the field variables are given in (2.2.79) and (2.2.80). For the present case of the cosmological manifold the metric tensor perturbations $\varkappa_{\alpha\beta}$ (or $\mathfrak{h}^{\alpha\beta}$), the scalar field ϕ and ψ are given by equations

$$\varkappa'_{\alpha\beta} = \varkappa_{\alpha\beta} + \pounds_\xi \bar{g}_{\alpha\beta} = \varkappa_{\alpha\beta} - \xi_{\alpha|\beta} - \xi_{\beta|\alpha}, \tag{5.5.9a}$$

$$\mathfrak{h}'^{\alpha\beta} = \mathfrak{h}^{\alpha\beta} + \pounds_\xi \bar{\mathfrak{g}}^{\alpha\beta} = \mathfrak{h}^{\alpha\beta} + \sqrt{-\bar{g}}\left(\xi^{\alpha|\beta} + \xi^{\beta|\alpha} - \bar{g}^{\alpha\beta}\xi^\gamma{}_{|\gamma}\right), \tag{5.5.9b}$$

or the same

$$h'_{\alpha\beta} = h_{\alpha\beta} + \xi_{\alpha|\beta} + \xi_{\beta|\alpha} - \bar{g}_{\alpha\beta}\xi^\gamma{}_{|\gamma}, \tag{5.5.9c}$$

$$\phi' = \phi + \pounds_\xi \bar{\Phi} = \phi - \bar{\Phi}_{|\alpha}\xi^\alpha, \tag{5.5.9d}$$

$$\psi' = \psi + \pounds_\xi \bar{\Psi} = \psi - \bar{\Psi}_{|\alpha}\xi^\alpha, \tag{5.5.9e}$$

where the prime above each symbol denotes a new value of the field variable after applying the gauge transformation (5.5.8), and all functions are calculated at the same value of coordinates x^α. The gauge transformations of the field variables are expressed in terms of the covariant derivatives on the manifold. With the use of the relation (5.4.10) connecting $\varkappa_{\alpha\beta}$ and $h_{\alpha\beta}$, one can show that equation (5.5.9b) can derived from the Lie transformation (5.5.9a).

Gauge invariance of the Lagrangian perturbation theory of geometric manifolds means that the gauge transformations of the field variables can not change the content of the theory. In other words, the equations for the field variables must be invariant with respect to the gauge transformations (5.5.9a–5.5.9e), see Section 2.2.4. However, direct inspection of equations (5.4.65), (5.4.67), (5.4.68) shows that they do depend on the choice of the gauge in the form of the gauge function B^α introduced in equation (5.4.64). To find out the gauge-invariant content of the theory one should search for the *gauge-invariant field variables* and to derive the *gauge-invariant equations* for them. This program has been completed by Bardeen [28] who used the functional 3+1 slicing decomposition of the metric tensor perturbations and the vector field ξ^α to build the gauge-invariant variables out of the various projections of the metric tensor components on space an time. Modifications of Bardeen's approach can be found in [81, 147, 155, 157, 301, 326] and in the book by Mukhanov [327]. We use algebraic 1+3 threading decomposition of the metric tensor perturbations (5.5.7) that allows us to build gauge-invariant scalars. Vector and tensor perturbations remain gauge-dependent in the threading approach. In order to suppress the gauge degrees of freedom in these variables we impose a particular gauge condition $B^\alpha = 0$ in equation (5.4.64). This limits the freedom of the gauge field ξ^α by a particular set of differential equations which are discussed in Section (5.5.7).

5.5.3 Gauge-invariant scalars

The existence of the preferred four-velocity, $\bar{u}^\alpha$, of the Hubble flow in the expanding universe provides a natural way of separating the perturbations of the field variables in scalar, vector, and tensor components. This section discusses how to build the gauge-invariant scalars. Vector and tensor perturbations are discussed afterwards.

The gauge-invariant scalar perturbations can be build from the perturbation of the Clebsch potential, ϕ, the perturbation of the scalar field ψ, and a scalar q defined in (5.5.2a). To build the first gauge-invariant scalar, we introduce the scalar perturbations

$$\chi_\mathrm{m} \equiv \frac{\phi}{\bar{\mu}_\mathrm{m}}, \qquad \chi_\mathrm{q} \equiv \frac{\psi}{\bar{\mu}_\mathrm{q}}, \tag{5.5.10}$$

that normalize perturbations of the Clebsch potential ϕ and that of the scalar field ψ to the corresponding background values of the specific enthalpy, $\bar{\mu}_\mathrm{m}$ and $\bar{\mu}_\mathrm{q}$. The gauge transformations for the three scalars q, χ_m, and χ_q are obtained from (5.5.9b–5.5.9e), and read

$$\mathrm{q}' = \mathrm{q} + 2\bar{u}^\alpha\bar{u}^\beta\xi_{\alpha|\beta}, \tag{5.5.11a}$$

$$\chi'_\mathrm{m} = \chi_\mathrm{m} + \bar{u}_\alpha\xi^\alpha, \tag{5.5.11b}$$

$$\chi'_\mathrm{q} = \chi_\mathrm{q} + \bar{u}_\alpha\xi^\alpha, \tag{5.5.11c}$$

where we have used the definition of the background four-velocity,

$$\bar{u}^\alpha = -\frac{\bar{\Phi}_{|\alpha}}{\bar{\mu}_\mathrm{m}} = -\frac{\bar{\Psi}_{|\alpha}}{\bar{\mu}_\mathrm{q}}, \tag{5.5.12}$$

in terms of the partial derivatives of the background values of the scalar fields Φ and Ψ. Equations (5.5.11b), (5.5.11c) immediately reveal that the linear combination

$$\chi \equiv \chi_\mathrm{m} - \chi_\mathrm{q}, \tag{5.5.13}$$

is gauge-invariant, $\hat{\chi} = \chi$, that is the diffeomorphism (5.5.8) does not change the value of the scalar variable χ.

Two other gauge-invariant scalars are defined by the following equations,

$$V_{\mathrm{m}} \equiv \bar{u}^{\alpha}\chi_{\mathrm{m}|\alpha} - \frac{\mathfrak{q}}{2}, \tag{5.5.14a}$$

$$V_{\mathrm{q}} \equiv \bar{u}^{\alpha}\chi_{\mathrm{q}|\alpha} - \frac{\mathfrak{q}}{2}, \tag{5.5.14b}$$

or, more explicitly,

$$V_{\mathrm{m}} = \frac{1}{\bar{\mu}_{\mathrm{m}}}\bar{u}^{\alpha}\phi_{|\alpha} - \frac{\mathfrak{q}}{2} + 3\frac{v_{\mathrm{s}}^2}{c^2}H\chi_{\mathrm{m}}, \tag{5.5.15a}$$

$$V_{\mathrm{q}} = \frac{1}{\bar{\mu}_{\mathrm{q}}}\bar{u}^{\alpha}\psi_{|\alpha} - \frac{\mathfrak{q}}{2} + 3H\chi_{\mathrm{q}} + \frac{\chi_{\mathrm{q}}}{\mu_{\mathrm{q}}}\frac{\partial \bar{W}}{\partial \bar{\Psi}}, \tag{5.5.15b}$$

where the last terms in the right side of these equations were obtained by making use of thermodynamic relationships (5.2.11), the equality $\bar{\rho}_{\mathrm{q}} = \bar{\mu}_{\mathrm{q}}$, and the equations of continuity (5.3.59) and (5.3.67) for the density of the ideal fluid, $\bar{\rho}_{\mathrm{m}}$, and that of the scalar field, $\bar{\rho}_{\mathrm{q}}$, respectively.

One can easily check that both scalars, V_{m} and V_{q} remain unchanged after making the infinitesimal coordinate transformation (5.5.8). Indeed, the gauge transformation of the derivatives

$$\chi'_{\mathrm{m}|\alpha} = \chi_{\mathrm{m}|\alpha} + H\bar{P}_{\alpha\beta}\xi^{\beta} + \bar{u}_{\beta}\xi^{\beta}{}_{|\alpha}, \tag{5.5.16a}$$

$$\chi'_{\mathrm{q}|\alpha} = \chi_{\mathrm{q}|\alpha} + H\bar{P}_{\alpha\beta}\xi^{\beta} + \bar{u}_{\beta}\xi^{\beta}{}_{|\alpha}, \tag{5.5.16b}$$

where $\bar{P}_{\alpha\beta} = \bar{g}_{\alpha\beta} + \bar{u}_{\alpha}\bar{u}_{\beta}$ is the operator of projection on the hypersurface being orthogonal to the Hubble flow of four-velocity $\bar{u}^{\alpha}$. After performing the gauge transformation (5.5.8), and substituting the gauge transformations of functions $\mathfrak{q}$, χ_{m} and χ_{q} to the definitions of V_{m} and V_{q}, we find out

$$V'_{\mathrm{m}} = V_{\mathrm{m}}, \qquad V'_{\mathrm{q}} = V_{\mathrm{q}}, \tag{5.5.17}$$

that proves the gauge-invariant property of the scalars V_{m} and V_{q}.

Physical meaning of the gauge-invariant quantity V_{m} can be understood as follows. We consider the perturbation of the specific enthalpy μ_{m} defined in equation (5.2.14). Substituting the decomposition (5.4.2) of the field variables to equation (5.2.14) and expanding, we obtain

$$\mu_{\mathrm{m}} = \bar{\mu}_{\mathrm{m}} + \delta\mu_{\mathrm{m}}, \tag{5.5.18}$$

where the perturbation $\delta\mu_{\mathrm{m}}$ of the specific enthalpy is defined (in the linearized order) by

$$\delta\mu_{\rm m} = \bar{u}^{\alpha}\phi_{|\alpha} - \frac{1}{2}\bar{\mu}_{\rm m}\mathfrak{q}. \tag{5.5.19}$$

It helps us to recognize that

$$V_{\rm m} = \frac{\delta\mu_{\rm m}}{\bar{\mu}_{\rm m}} + 3\frac{v_{\rm s}^2}{c^2}H\chi_{\rm m}. \tag{5.5.20}$$

Fractional perturbation of the specific enthalpy can be re-written with the help of thermodynamic equations (5.2.11) in terms of the perturbation $\delta\epsilon_{\rm m}$ of the energy density of the ideal fluid,

$$\frac{\delta\mu_{\rm m}}{\bar{\mu}_{\rm m}} = \frac{v_{\rm s}^2}{c^2}\frac{\delta\epsilon_{\rm m}}{\bar{\epsilon}_{\rm m} + \bar{p}_{\rm m}}, \tag{5.5.21}$$

or, by making use of equation (5.2.8), in terms of the perturbation $\delta\rho_{\rm m}$ of the density of the ideal fluid

$$\frac{\delta\mu_{\rm m}}{\bar{\mu}_{\rm m}} = \frac{v_{\rm s}^2}{c^2}\frac{\delta\rho_{\rm m}}{\bar{\rho}_{\rm m}}. \tag{5.5.22}$$

This allows us to write down equation (5.5.20) as follows

$$V_{\rm m} = \frac{v_{\rm s}^2}{c^2}\left(\frac{\delta\rho_{\rm m}}{\bar{\rho}_{\rm m}} + 3H\chi_{\rm m}\right), \tag{5.5.23}$$

which elucidates the relationship between the gauge-invariant variable $V_{\rm m}$ and the perturbation $\delta\rho_{\rm m}$ of the rest mass density of the dark matter. More specifically, $V_{\rm m}$ is an algebraic sum of two scalar functions, $\delta\rho_{\rm m}$ and $\chi_{\rm m}$ neither of each is gauge-invariant. The gauge transformation of the dark matter density perturbation is

$$\delta\rho'_{\rm m} = \delta\rho_{\rm m} + \bar{\rho}_{{\rm m}|\alpha}\xi^{\alpha} = \delta\rho_{\rm m} - 3H\bar{\rho}_{\rm m}\bar{u}_{\alpha}\xi^{\alpha}, \tag{5.5.24}$$

and the gauge transformation of the variable $\chi_{\rm m}$ is given by (5.5.11b). Their algebraic sum in equation (5.5.23) does not change under the diffeomorphism (5.5.8) showing that $V_{\rm m}$ is the gauge-invariant density fluctuation that does not depend on a particular choice of coordinates on spacetime manifold.

Similar considerations, applied to function $V_{\rm q}$ reveals that it can be represented as an algebraic sum of the perturbation, $\delta\rho_{\rm q}$, of the density of the dark energy, and function $\chi_{\rm q}$,

$$V_{\rm q} = \frac{\delta\rho_{\rm q}}{\bar{\rho}_{\rm q}} + 3H\chi_{\rm q}. \tag{5.5.25}$$

It is easy to check out that each term in the right side of this equation taken separately, is not gauge-invariant but their linear combination does. It is worth emphasizing that standard textbooks on cosmological theory (see, for example, [296, 353, 456, 457]) derive equations for the density perturbations $\delta\rho/\bar{\rho}$ but those equations are not gauge-invariant and, hence, their solutions have no direct physical meaning and should be interpreted with care.

5.5.4 Field equations for the gauge-invariant scalar perturbations

Equation for a scalar $\mathfrak{q}$

Function $\mathfrak{q}$ was defined in (5.5.2a). In order to derive a differential equation for $\mathfrak{q}$, we apply the covariant Laplace-Beltrami operator to $\mathfrak{q}$, and make use of the covariant equations (5.4.62) and (5.4.64). Straightforward but fairly long calculation yields

$$\begin{aligned}
&\mathfrak{q}^{|\alpha}{}_{|\alpha} - 2\left(\dot{H} + H^2 - \frac{2k}{a^2}\right)\mathfrak{q} + 8\pi\bar{\rho}_{\rm m}\bar{\mu}_{\rm m}\left[\left(1 - \frac{c^2}{v_{\rm s}^2}\right)V_{\rm m} - \left(1 + 3\frac{v_{\rm s}^2}{c^2}\right)H\chi_{\rm m}\right] \\
&- 16\pi\bar{\rho}_{\rm q}\left(\frac{\partial\bar{W}}{\partial\bar{\Psi}} + 2H\bar{\mu}_{\rm q}\right)\chi_{\rm q} - 2\bar{u}^\alpha\bar{u}^\beta B_{\alpha|\beta} - 4H\bar{u}^\alpha B_\alpha \\
&= 8\pi\left(\sigma + \tau\right),
\end{aligned} \tag{5.5.26}$$

where the source density $\sigma + \tau$ for the field $\mathfrak{q}$ is

$$\sigma + \tau = \left(\bar{u}^\alpha\bar{u}^\beta + \bar{P}^{\alpha\beta}\right)T^{\rm p}_{\alpha\beta} \tag{5.5.27}$$

in accordance with the definitions introduced in (5.3.70a), (5.3.70b). The reader should notice that equation (5.5.26) depends on the gauge function B^α which remains arbitrary so far.

Equation for a scalar $\mathfrak{p}$

Function $\mathfrak{p}$ was defined in (5.5.2b). In order to derive equation for $\mathfrak{p}$, we apply the covariant Laplace-Beltrami operator to the definition of $\mathfrak{p}$, and make use of the covariant equations (5.4.62) and (5.4.64). It results in a wave equation

$$\mathfrak{p}^{|\alpha}{}_{|\alpha} + \frac{4k}{a^2}\mathfrak{p} + B^\alpha{}_{|\alpha} - 2\bar{u}^\alpha\bar{u}^\beta B_{\alpha|\beta} - 6H\bar{u}^\alpha B_\alpha = 16\pi\tau, \tag{5.5.28}$$

where the source density τ has been defined in (5.3.70b). Equation (5.5.28) depends on the arbitrary gauge function B^α.

Equation for a scalar χ

Equation for the gauge-invariant scalar, $\chi = \chi_{\rm m} - \chi_{\rm q}$, is derived from the definitions (5.5.10) and the field equations (5.4.67), (5.4.68). Replacing ϕ and ψ in those equations with $\chi_{\rm m}$ and $\chi_{\rm q}$, and making use of equations (5.3.52), (5.3.53) for reshuffling some terms, yields

$$\chi_{\mathrm{m}|\alpha}^{|\alpha} + 2H\bar{u}^{\alpha}\chi_{\mathrm{m}|\alpha} - \left(\dot{H} - \frac{4k}{a^2}\right)\chi_{\mathrm{m}} \tag{5.5.29a}$$
$$+ 4HV_{\mathrm{m}} + \left(1 - \frac{c^2}{v_{\mathrm{s}}^2}\right)\bar{u}^{\alpha}V_{\mathrm{m}|\alpha} - 16\pi\bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}}\chi = \bar{u}^{\alpha}B_{\alpha},$$

$$\chi_{\mathrm{q}\ |\alpha}^{|\alpha} + 2H\bar{u}^{\alpha}\chi_{\mathrm{q}|\alpha} - \left(\dot{H} - \frac{4k}{a^2}\right)\chi_{\mathrm{q}} \tag{5.5.29b}$$
$$+4HV_{\mathrm{q}} + \frac{2}{\bar{\mu}_{\mathrm{q}}}\frac{\partial\bar{W}}{\partial\bar{\Psi}}V_{\mathrm{q}} + 16\pi\bar{\rho}_{\mathrm{m}}\bar{\mu}_{\mathrm{m}}\chi = \bar{u}^{\alpha}B_{\alpha}.$$

Subtracting (5.5.29b) from (5.5.29a) cancels the gauge-dependent term, $\bar{u}^{\alpha}B_{\alpha}$, and brings about the field equation for χ,

$$\chi^{|\alpha}{}_{|\alpha} + 6H\bar{u}^{\alpha}\chi_{|\alpha} + 3\dot{H}\chi = \frac{2}{\bar{\mu}_{\mathrm{q}}}\frac{\partial\bar{W}}{\partial\bar{\Psi}}V_{\mathrm{q}} - \left(1 - \frac{c^2}{v_{\mathrm{s}}^2}\right)\bar{u}^{\alpha}V_{\mathrm{m}|\alpha}. \tag{5.5.30}$$

This equation is apparently gauge-invariant since any dependence on the arbitrary gauge function B^{α} disappeared. It is also covariant that is valid in any coordinates.

Equation (5.5.30) can be recast to the form of an inhomogeneous wave equation:

$$(\bar{\rho}_{\mathrm{m}}\chi)^{|\alpha}{}_{|\alpha} = 2\frac{\bar{\rho}_{\mathrm{m}}}{\bar{\rho}_{\mathrm{q}}}\frac{\partial\bar{W}}{\partial\bar{\Psi}}V_{\mathrm{q}} - \left(1 - \frac{c^2}{v_{\mathrm{s}}^2}\right)\bar{\rho}_{\mathrm{m}}\bar{u}^{\alpha}V_{\mathrm{m}|\alpha}. \tag{5.5.31}$$

Yet another form of equation (5.5.30) is obtained in terms of the variable $\psi = \bar{\rho}_{\mathrm{q}}\chi = \bar{\mu}_{\mathrm{q}}\chi$. By simple inspection we can check that equation (5.5.30) is transformed to

$$\psi^{|\alpha}{}_{|\alpha} - m_{\psi}^2\psi = 2\frac{\partial\bar{W}}{\partial\bar{\Psi}}V_{\mathrm{m}} - \left(1 - \frac{c^2}{v_{\mathrm{s}}^2}\right)\bar{\rho}_{\mathrm{q}}\bar{u}^{\alpha}V_{\mathrm{m}|\alpha}, \tag{5.5.32}$$

where we introduced notation $m_{\psi} \equiv \sqrt{\partial^2\bar{W}/\partial\bar{\Psi}^2}$. This is an inhomogeneous Klein-Gordon equation for the field ψ governed by V_{m}. The "mass" m_{ψ} of the scalar field excitation, ψ, depends on the second derivative of the potential function $\bar{W}$ which defines the "coefficient of elasticity" of the background scalar field $\bar{\Psi}$.

Inhomogeneous equations (5.5.30), (5.5.31), (5.5.32) have the source terms that is determined by variables V_{m} and V_{q}. We derive differential equations for these field variables in the next sections.

Equation for a scalar V_{m}

Equation for the field variable V_{m} is derived from the equations for functions χ_{m} and q that enter its definition (5.5.14a). By applying the Laplace-Beltrami operator to function V_{m} we get

$$V_{\mathrm{m}|\alpha}^{|\alpha} = \bar{u}^{\beta}\left(\chi_{\mathrm{m}|\alpha}^{|\alpha}\right)_{|\beta} + 2H\chi_{\mathrm{m}|\alpha}^{|\alpha} - \frac{1}{2}\mathfrak{q}^{|\alpha}{}_{|\alpha} + \bar{u}^{\beta}\bar{R}^{\alpha}{}_{\beta}\chi_{\mathrm{m}|\alpha} + 2H\bar{u}^{\alpha}\left(V_{\mathrm{m}} + \frac{1}{2}\mathfrak{q}\right)_{|\alpha} + 3H^2\left(V_{\mathrm{m}} + \frac{1}{2}\mathfrak{q}\right). \tag{5.5.33}$$

The Laplace-Beltrami operator for function χ_{m} is given in equation (5.5.29a) which is not gauge-invariant. Taking the covariant derivative from this equation and contracting it with $\bar{u}^{\alpha}$ brings about the first term in the right side of equation (5.5.33),

$$\begin{aligned}\bar{u}^{\beta}\left(\chi_{\mathrm{m}|\alpha}^{|\alpha}\right)_{|\beta} = &-\left(1-\frac{c^2}{v_{\mathrm{s}}^2}\right)\bar{u}^{\alpha}\bar{u}^{\beta}V_{\mathrm{m}|\alpha\beta} - 6H\bar{u}^{\alpha}V_{\mathrm{m}|\alpha} \\ &-\left(5\dot{H} + \frac{4k}{a^2}\right)V_{\mathrm{m}} - H\bar{u}^{\alpha}\mathfrak{q}_{|\alpha} - \left(\frac{1}{2}\dot{H} + \frac{2k}{a^2}\right)\mathfrak{q} \\ &-3H\left[\left(1+\frac{v_{\mathrm{s}}^2}{c^2}\right)\dot{H} - \left(3+\frac{v_{\mathrm{s}}^2}{c^2}\right)\frac{k}{a^2}\right]\chi_{\mathrm{m}} \\ &+8\pi\bar{\rho}_{\mathrm{q}}\frac{\partial\bar{W}}{\partial\bar{\Psi}}\left(4\chi_{\mathrm{q}} - 3\chi_{\mathrm{m}}\right) \\ &+16\pi\bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}}\left[\bar{u}^{\alpha}\chi_{|\alpha} - 6H\chi + \frac{3}{4}H\left(1-\frac{v_{\mathrm{s}}^2}{c^2}\right)\chi_{\mathrm{m}}\right] + \bar{u}^{\alpha}\bar{u}^{\beta}B_{\alpha|\beta}.\end{aligned} \tag{5.5.34}$$

The Laplace-Beltrami operator for function $\mathfrak{q}$ has been derived in (5.5.26). Now, we make use of equations (5.5.26), (5.5.29a), (5.5.34) in calculating the right side of (5.5.33). After a significant amount of algebra, we find out that all terms explicitly depending on q and the gauge functions B^{α} cancel out, so that equation for V_{m} becomes

$$\begin{aligned}V_{\mathrm{m}|\alpha}^{|\alpha} &+ \left(1-\frac{c^2}{v_{\mathrm{s}}^2}\right)\bar{u}^{\alpha}\bar{u}^{\beta}V_{\mathrm{m}|\alpha\beta} + 2\left(3-\frac{c^2}{v_{\mathrm{s}}^2}\right)H\bar{u}^{\alpha}V_{\mathrm{m}|\alpha} \\ &+\left[2\left(\dot{H} + 3H^2 + \frac{2k}{a^2}\right) - 4\pi\bar{\rho}_{\mathrm{m}}\bar{\mu}_{\mathrm{m}}\left(1-\frac{c^2}{v_{\mathrm{s}}^2}\right)\right]V_{\mathrm{m}} \\ &-16\pi\bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}}\left[\bar{u}^{\alpha}\chi_{|\alpha} - 3\left(H + \frac{1}{2\bar{\mu}_{\mathrm{q}}}\frac{\partial\bar{W}}{\partial\bar{\Psi}}\right)\chi\right] = -4\pi\left(\sigma+\tau\right).\end{aligned} \tag{5.5.35}$$

Second-order covariant derivatives in this equation read

$$\left[\bar{g}^{\alpha\beta} + \left(1-\frac{c^2}{v_{\mathrm{s}}^2}\right)\bar{u}^{\alpha}\bar{u}^{\beta}\right]V_{\mathrm{m}|\alpha\beta} \equiv \left(-\frac{c^2}{v_{\mathrm{s}}^2}\bar{u}^{\alpha}\bar{u}^{\beta} + \bar{P}^{\alpha\beta}\right)V_{\mathrm{m}|\alpha\beta}, \tag{5.5.36}$$

and they form a hyperbolic-type operator describing propagation of sound waves in the expanding universe from the source of the sound waves towards the field point with the constant velocity v_{s}^2. Additional terms in the left side of equation (5.5.35) depend on the Hubble parameter H, and take into account the expansion of the universe.

Equation (5.5.35) contains only gauge-invariant scalars, V_m and χ. Moreover, it does not depend on the choice of coordinates on the background manifold. It also becomes clear that the field variables V_m and χ are coupled through the differential equations (5.5.32) and (5.5.35) which should be solved simultaneously in order to determine these variables. Solution of the coupled system of differential equations is a very complicated task which cannot be rendered analytically in the most general case. Only in some simple cases, the equations can be decoupled. We discuss such cases in Section 5.7.

Equation for a scalar V_q

The field variable V_q is not independent since it relates to V_m and χ by a simple relationship

$$V_\mathrm{q} = V_\mathrm{m} - \bar{u}^\alpha \chi_{|\alpha}, \tag{5.5.37}$$

which is obtained after subtraction of equation (5.5.14a) from (5.5.14b). Equation for V_q is derived directly from (5.5.37) and equations (5.5.35) and (5.5.30) for V_m and χ respectively. We obtain,

$$\begin{aligned} &V_{\mathrm{q}\;|\alpha}^{|\alpha} + 4\left(H + \frac{1}{2\bar{\mu}_\mathrm{q}}\frac{\partial \bar{W}}{\partial \bar{\Psi}}\right)\bar{u}^\alpha V_{\mathrm{q}|\alpha} \\ &+ \left[2\left(\dot{H} + 3H^2 + \frac{2k}{a^2}\right) - 4\pi\bar{\rho}_\mathrm{m}\bar{\mu}_\mathrm{m}\left(1 - \frac{c^2}{v_s^2}\right)\right. \\ &+ \left.\frac{2}{\bar{\mu}_\mathrm{q}}\left(5H + \frac{1}{\bar{\mu}_\mathrm{q}}\frac{\partial \bar{W}}{\partial \bar{\Psi}}\right)\frac{\partial \bar{W}}{\partial \bar{\Psi}} + 2\frac{\partial^2 \bar{W}}{\partial \bar{\Psi}^2}\right] V_\mathrm{q} \\ &+ 4\pi\bar{\rho}_\mathrm{m}\bar{\mu}_\mathrm{m}\left(3 + \frac{c^2}{v_\mathrm{s}^2}\right)\left(\bar{u}^\alpha \chi_{|\alpha} - 3\frac{v_\mathrm{s}^2}{c^2}H\chi\right) = -4\pi\,(\sigma + \tau)\,. \end{aligned} \tag{5.5.38}$$

This equation can be also derived by the procedure being similar to that used in the previous subsection in deriving equation for V_m. We followed this procedure and confirm that it leads to (5.5.38) as expected. Equation (5.5.38) is clearly gauge-invariant. Besides V_q it depends on variable χ and should be solved along with equation (5.5.30).

5.5.5 Field equations for vector perturbations

Vector perturbations of the ideal fluid and scalar field are gradients, $\phi_{|\alpha}$ and $\psi_{|\alpha}$. However, they are insufficient to build a gauge-invariant vector perturbation out of the vector perturbation of the metric tensor $\mathfrak{p}_\alpha$. Field equations for vector $\mathfrak{p}_\alpha$ can be derived by applying the covariant Laplace-Beltrami operator to both sides of definition (5.5.3) and making use of equation (5.4.65). After performing the covariant differentiation and a significant amount of algebra, we derive the field equation

$$\mathfrak{p}_\alpha{}^{|\beta}{}_{|\beta} - 2H\bar{u}_\alpha \mathfrak{p}_\beta{}^{|\beta} - \left(2\dot{H} + 3H^2 - \frac{2k}{a^2}\right)\mathfrak{p}_\alpha + \bar{P}_\alpha{}^\beta \bar{u}^\gamma \left(B_{\beta|\gamma} + B_{\gamma|\beta} + 2H\bar{u}_\gamma B_\beta\right) = 16\pi\tau_\alpha, \tag{5.5.39}$$

where the matter current τ_α is defined in (5.3.70c). This equation is apparently gauge-dependent as shown by the appearance of the gauge function B^α. This equation reduces to a much simpler form

$$\mathfrak{p}_\alpha{}^{|\beta}{}_{|\beta} - 2H\bar{u}_\alpha \mathfrak{p}_\beta{}^{|\beta} - \left(2\dot{H} + 3H^2 - \frac{2k}{a^2}\right)\mathfrak{p}_\alpha = 16\pi\tau_\alpha, \tag{5.5.40}$$

in a special gauge B^α=0 which imposes a restriction on the divergence of the metric tensor perturbation in equation (5.4.64).

Equation (5.5.39) points out that the vector perturbations are generated by the current of matter τ_α existing in the localized astronomical system which physical origin may be a relict of the primordial perturbations. We do not discuss this interesting scenario in the present chapter as it would require a non-conservation of entropy and non-isentropic background fluid – the case which we have intentionally excluded in order to focus on derivation of cosmological generalization of the post-Newtonian equations of relativistic celestial dynamics [267].

5.5.6 Field equations for tensor perturbations

Field equations for traceless tensor $\mathfrak{p}^{\mathrm{T}}_{\alpha\beta}$ can be derived by applying the covariant Laplace-Beltrami operator to the definition (5.5.4) and making use of equation (5.4.65) along with a tedious algebraic transformations. This yields the following equation

$$\mathfrak{p}^{\mathrm{T}}_{\alpha\beta}{}^{|\gamma}{}_{|\gamma} - 2H\left(\bar{u}_\alpha \mathfrak{p}^{\mathrm{T}}_{\beta\gamma}{}^{|\gamma} + \bar{u}_\beta \mathfrak{p}^{\mathrm{T}}_{\alpha\gamma}{}^{|\gamma}\right) - 2\left(H^2 + \frac{k}{a^2}\right)\mathfrak{p}^{\mathrm{T}}_{\alpha\beta} - \bar{P}_\alpha{}^\mu \bar{P}_\beta{}^\nu \left(B_{\mu|\nu} + B_{\nu|\mu}\right) + \frac{2}{3}\bar{P}_{\alpha\beta}\bar{P}^{\mu\nu}B_{\mu|\nu} = 16\pi\tau^{\mathrm{T}}_{\alpha\beta}. \tag{5.5.41}$$

Here the transverse and traceless tensor source of the tensor perturbations is

$$\tau^{\mathrm{T}}_{\alpha\beta} \equiv \tau_{\alpha\beta} - \frac{1}{3}\bar{P}_{\alpha\beta}\,\tau, \tag{5.5.42}$$

where $\tau_{\alpha\beta}$ has been introduced in (5.3.70d), and $\tau = \bar{P}^{\alpha\beta}\tau_{\alpha\beta}$ in accordance with equation (5.3.70b). Tensor $\tau^{\mathrm{T}}_{\alpha\beta}$ is traceless, that is $\bar{g}^{\alpha\beta}\tau^{\mathrm{T}}_{\alpha\beta} = \bar{P}^{\alpha\beta}\tau^{\mathrm{T}}_{\alpha\beta} = 0$.

Equation (5.5.41) is gauge-dependent. The gauge freedom is significantly reduced by imposing the gauge condition $B^\alpha = 0$ which brings equation (5.5.41) to the following form,

$$\mathfrak{p}^{\mathrm{T}}_{\alpha\beta}{}^{|\gamma}{}_{|\gamma} - 2H\left(\bar{u}_\alpha \mathfrak{p}^{\mathrm{T}}_{\beta\gamma}{}^{|\gamma} + \bar{u}_\beta \mathfrak{p}^{\mathrm{T}}_{\alpha\gamma}{}^{|\gamma}\right) - 2\left(H^2 + \frac{k}{a^2}\right)\mathfrak{p}^{\mathrm{T}}_{\alpha\beta} = 16\pi\tau^{\mathrm{T}}_{\alpha\beta}. \tag{5.5.43}$$

5.5.7 Residual gauge freedom

The gauge freedom of the theory under discussion is associated with the gauge function B^α appearing in equation (5.4.64). The most favourable choice of the gauge condition is

$$B^\alpha = 0, \tag{5.5.44}$$

which drastically simplifies the above equations for vector and tensor gravitational perturbations. The gauge (5.5.44) is a generalization of the harmonic (de Donder) gauge condition used in the gravitational wave astronomy and in the post-Newtonian dynamics of extended bodies. This choice of the gauge establishes differential relationships between the algebraically-independent metric tensor components introduced in Section 5.5.1. Indeed, substituting the algebraic decomposition (5.5.7) of the metric tensor perturbations to equation (5.4.64) and imposing the condition (5.5.44) yields

$$\begin{aligned}\mathfrak{p}^{\mathrm{T}}_{\alpha\beta}{}^{|\beta} + \bar{u}_\alpha \mathfrak{p}_\beta{}^{|\beta} + \bar{u}_\beta \mathfrak{p}_\alpha{}^{|\beta} - \left(\bar{u}_\alpha \bar{u}_\beta - \frac{1}{3}\bar{P}_{\alpha\beta}\right)\mathfrak{p}^{|\beta} + 2H\mathfrak{p}_\alpha \\ +2\bar{u}_\alpha \bar{u}_\beta \mathfrak{q}^{|\beta} + 2H\mathfrak{q}\bar{u}_\alpha = 16\pi\left(\bar{\rho}_{\mathrm{m}}\bar{\mu}_{\mathrm{m}}\chi_{\mathrm{m}} + \bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}}\chi_{\mathrm{q}}\right)\bar{u}_\alpha .\end{aligned} \tag{5.5.45}$$

Projecting this relationship on the direction of the background 4-velocity, $\bar{u}^\alpha$, and on the hypersurface being orthogonal to it, we derive two algebraically-independent equations between the perturbations of metric tensor components and of the matter variables. They are

$$\mathfrak{p}_\beta{}^{|\beta} + \bar{u}_\beta\left(2\mathfrak{q} - \mathfrak{p}\right)^{|\beta} + 2H\mathfrak{q} = 16\pi\left(\bar{\rho}_{\mathrm{m}}\bar{\mu}_{\mathrm{m}}\chi_{\mathrm{m}} + \bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}}\chi_{\mathrm{q}}\right), \tag{5.5.46a}$$

$$\mathfrak{p}^{\mathrm{T}}_{\alpha\beta}{}^{|\beta} + \bar{u}_\beta \mathfrak{p}_\alpha{}^{|\beta} + \frac{1}{3}\bar{P}_{\alpha\beta}\mathfrak{p}^{|\beta} + 2H\mathfrak{p}_\alpha = 0. \tag{5.5.46b}$$

The gauge (5.5.44) does not fix the gauge function ξ^α uniquely. The residual gauge freedom is described by the gauge transformations that preserve equations (5.5.46a), (5.5.46b). Substituting the gauge transformation (5.5.9b) of the gravitational field perturbation $h_{\alpha\beta}$ to equation (5.4.64) and holding on the gauge condition (5.5.44), yields the differential equation for the vector function ξ^α

$$\begin{aligned}\xi^{\alpha|\beta}{}_{|\beta} + \bar{g}^{\alpha\gamma}\left(\xi^\beta{}_{|\gamma\beta} - \xi^\beta{}_{|\beta\gamma}\right) + 2H\left(\xi^{\alpha|\beta}\bar{u}_\beta + \xi^{\beta|\alpha}\bar{u}_\beta - \xi^\beta{}_{|\beta}\bar{u}^\alpha\right) \\ -16\pi\left(\bar{\rho}_{\mathrm{m}}\bar{\mu}_{\mathrm{m}} + \bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}}\right)\xi^\beta \bar{u}_\beta \bar{u}^\alpha = 0,\end{aligned} \tag{5.5.47}$$

which can be further recast to

$$\begin{aligned}\xi^{\alpha|\beta}{}_{|\beta} + 2H\left(\xi^{\alpha|\beta}\bar{u}_\beta + \xi^{\beta|\alpha}\bar{u}_\beta - \xi^\beta{}_{|\beta}\bar{u}^\alpha\right) \\ + 2\left(\dot{H} - \frac{k}{a^2}\right)\xi^\beta \bar{u}_\beta \bar{u}^\alpha + \left(\dot{H} + 3H^2 + \frac{2k}{a^2}\right)\xi^\alpha = 0.\end{aligned} \tag{5.5.48}$$

The gauge function ξ^α can be decomposed in time-like, $\xi \equiv -\xi^\beta \bar{u}_\beta$, and space-like, $\zeta^\alpha \equiv \bar{P}^\alpha{}_\beta \xi^\beta$, components,

$$\xi^\alpha = \zeta^\alpha + \bar{u}^\alpha \xi. \tag{5.5.49}$$

Calculating covariant derivatives from ξ and ζ^α and making use of equation (5.5.48), yield equations

$$\xi^{|\beta}{}_{|\beta} + 2H\bar{u}^\beta \xi_{|\beta} - \left(\dot{H} - \frac{4k}{a^2}\right)\xi = 0, \tag{5.5.50a}$$

$$\zeta^{\alpha|\beta}{}_{|\beta} + 2H\left(\bar{u}^\beta \zeta^\alpha{}_{|\beta} - \bar{u}^\alpha \zeta^\beta{}_{|\beta}\right) + \left(\dot{H} + H^2 + \frac{2k}{a^2}\right)\zeta^\alpha = 0. \tag{5.5.50b}$$

These equations have non-trivial solutions which describe the residual gauge freedom in choosing the coordinates on the background manifold subject to the gauge condition (5.5.44). It is remarkable that equations (5.5.50a), (5.5.50b) are decoupled and can be solved separately. It means that the residual gauge transformations along the worldlines of the Hubble flow are functionally independent of those performed on the hypersurface being orthogonal to the Hubble flow. Equations (5.5.50a–5.5.50b) of the residual gauge freedom in the cosmological setting given in this subsection generalise equations of the residual gauge freedom in harmonic coordinates of asymptotically flat spacetime [56, 115].

5.6 Post-Newtonian field equations in a spatially-flat universe

5.6.1 Cosmological parameters and scalar field potential

Linearized equations of the field perturbations given in the previous section are valid for a wide class of matter models of the FLRW metric. They neither specify the equation of state of dark matter, nor that of dark energy. We also keep the parameter of the space curvature k free. By choosing a specific model of matter and picking up a value of $k = -1, 0, +1$, we can solve, at least, in principle the field equations governing the time evolution of the background cosmological manifold. Realistic models of the cosmological dark matter and dark energy are rather sophisticated and, as a rule, include several components. It leads to the system of coupled field equations which can be solved only numerically [7]. However, the large scale structure of the universe is formed at rather late stages of the cosmological evolution being fairly close to the present epoch. Therefore, the study of the impact of cosmological expansion on the post-Newtonian dynamics of isolated astronomical systems is based on recent and present equation of state of matter in the universe.

Precise radiometric observations of the relic CMB radiation and photometry of type Ia supernova explosions reveal that at the present epoch the space curvature of the universe, $k = 0$, and the evolution of the universe is primarily governed by the dark energy and dark matter, which make up to 73 % and 23 % of the total energy density of the universe respectively, while 4 % of the energy density of the universe belongs to visible matter (baryons), and a tiny fraction of the energy density occupies by the CMBR radiation [177, 234, 246, 262]. It means that we can neglect the effects of the baryonic matter and CMB radiation field in consideration of the post-Newtonian dynamics of astronomical systems in the expanding universe.

We model dark matter by an ideal fluid and dark energy is represented by a scalar field with a potential function $\bar{W}$ which structure has not yet been specified. We also follow the discussion given in by assuming that the spatial curvature $k = 0$, and the potential, $\bar{W}$, of the scalar field relates to its derivative by a simple equation

$$\frac{\partial \bar{W}}{\partial \bar{\Psi}} = -\sqrt{8\pi}\lambda \bar{W}, \tag{5.6.1}$$

where the time-dependent parameter, $\lambda = \lambda(\bar{\Psi})$, characterizes the slope of the field potential $\bar{W}$. The time evolution of the background universe can be described in terms of the parameter λ and two other parameters, $\mathrm{x}_1 = \mathrm{x}_1(\bar{\Psi})$ and $\mathrm{x}_2 = \mathrm{x}_2(\bar{\Psi})$, which are functions of the density, $\bar{\rho}_{\mathrm{q}} = \bar{\mu}_{\mathrm{q}} = \dot{\bar{\Psi}}$, of the background scalar field, and the potential, $\bar{W}$, scaled to the Hubble parameter, H. These parameters are defined as follows,

$$\bar{\rho}_{\mathrm{q}}^2 = \frac{3H^2}{4\pi}\,\mathrm{x}_1, \tag{5.6.2}$$

$$\bar{W} = \frac{3H^2}{8\pi}\,\mathrm{x}_2. \tag{5.6.3}$$

The energy density of the scalar field, $\bar{\epsilon}_{\mathrm{q}}$, is expressed in terms of the parameters x_1 and x_2 and the parameter $\Omega_{\mathrm{q}} \equiv 8\pi\bar{\epsilon}_{\mathrm{q}}/3H^2$, by a simple relationship

$$\Omega_{\mathrm{q}} = \mathrm{x}_1 + \mathrm{x}_2. \tag{5.6.4}$$

Time evolution of the parameters x_1 and x_2 is given by a system of two ordinary differential equations which are obtained by differentiating definitions (5.6.2), (5.6.3) and making use of equations (5.3.67) taken along with the Friedmann equation (5.3.52) with $k = 0$. It yields

$$\frac{d\mathrm{x}_1}{d\omega} = -6\mathrm{x}_1 + \lambda\sqrt{6\mathrm{x}_1}\mathrm{x}_2 + 3\mathrm{x}_1\left[(1-w_{\mathrm{m}})\,\mathrm{x}_1 + (1+w_{\mathrm{m}})\,(1-\mathrm{x}_2)\right], \tag{5.6.5a}$$

$$\frac{d\mathrm{x}_2}{d\omega} = -\lambda\sqrt{6\mathrm{x}_1}\mathrm{x}_2 + 3\mathrm{x}_2\left[(1-w_{\mathrm{m}})\,\mathrm{x}_1 + (1+w_{\mathrm{m}})\,(1-\mathrm{x}_2)\right], \tag{5.6.5b}$$

where $\omega \equiv \ln a$ is the logarithmic scale factor characterizing the number of e-folding of the universe, w_{m} is the parameter entering the hydrodynamic equation of state

(5.3.54), and the parameters x_1 and x_2 are restricted by the condition imposed by the Friedmann equation (5.3.50), that is $\Omega_q + \Omega_m = 1$, or

$$x_1 + x_2 = 1 - \Omega_m, \tag{5.6.6}$$

where $\Omega_m \equiv 8\pi\bar{\epsilon}_m/3H^2$.

The parameter λ obeys the following equation

$$\frac{d\lambda}{d\omega} = -\sqrt{6x_1}\lambda^2\left(\Gamma_q - 1\right), \tag{5.6.7}$$

where

$$\Gamma_q = \frac{\partial^2\bar{W}/\partial\bar{\Psi}^2}{\left(\partial\bar{W}/\partial\bar{\Psi}\right)^2}\bar{W}, \tag{5.6.8}$$

If $\Gamma_q = 1$, the parameter λ is constant, and equation (5.6.1) can be integrated yielding an exponential potential

$$\bar{W}(\bar{\Psi}) = \bar{W}_0 \exp(-\sqrt{8\pi}\lambda\bar{\Psi}). \tag{5.6.9}$$

In this case, and under assumption that, $w_m = \text{const.}$, the system of two differential equations (5.6.5a), (5.6.5b) is closed. If $\Gamma_q \neq 1$, three equations (5.6.5a), (5.6.5b), (5.6.7) must be solved together in order to describe temporal evolution of the background cosmological manifold.

In the general case, derivatives of the potential $\bar{W}$ are expressed in terms of the parameters under discussion. Namely,

$$\frac{\partial\bar{W}}{\partial\bar{\Psi}} = -\frac{3\lambda}{\sqrt{8\pi}}H^2x_2, \qquad \frac{\partial^2\bar{W}}{\partial\bar{\Psi}^2} = 3\Gamma_q\lambda^2H^2x_2. \tag{5.6.10}$$

It is also useful to express the products $\bar{\rho}_q\bar{\mu}_q$ and $\bar{\rho}_m\bar{\mu}_m$ in terms of the parameters x_1 and x_2. For $\bar{\mu}_q = \bar{\rho}_q$, one can use definition (5.6.2) to obtain

$$\bar{\rho}_q\bar{\mu}_q = \frac{3H^2}{4\pi}x_1. \tag{5.6.11}$$

The product $\bar{\rho}_m\bar{\mu}_m = \bar{\epsilon}_m + \bar{p}_m$, so that making use of the matter equation of state, $\bar{p}_m = w_m\bar{\epsilon}_m$, and equation (5.6.6), we derive

$$\bar{\rho}_m\bar{\mu}_m = \frac{3H^2}{8\pi}(1 + w_m)\Omega_m, \tag{5.6.12}$$

where $\Omega_m = 1 - x_1 - x_2$. These equations allow us to recast equation (5.3.52) for the time derivative of the Hubble parameter to the following form

$$\dot{H} = -\frac{3}{2}\left(1 + w_{\text{eff}}\right)H^2, \tag{5.6.13}$$

where

$$w_{\text{eff}} \equiv w_{\text{m}} + (1 - w_{\text{m}})\text{x}_1 - (1 + w_{\text{m}})\text{x}_2, \tag{5.6.14}$$

is the (time-dependent) parameter of the effective equation of state of the mixture of the ideal fluid and the scalar field.

5.6.2 Conformal cosmological perturbations

The FLRW metric (5.3.15) is a product of the scale factor a and a conformal metric $\bar{\mathfrak{f}}_{\alpha\beta}$. The conformal spacetime is comoving with the Hubble flow and is not globally expanding. In case of the flat spatial curvature, $k = 0$, the conformal spacetime becomes equivalent to the Minkowski space which is used as a starting point in the standard theory of the Post-Newtonian approximations [115]. Therefore, it is mathematically instructive to formulate the field equations for cosmological perturbations in the conformal spacetime. It also allows us to simplify the differential operators in the left side of the equations for perturbations (see Section 5.6.3 below). Nonetheless, the reader must keep in mind that the conformal spacetime is unphysical and additional scale transformations of coordinates are required to convert mathematical results from the conformal spacetime to a real physical world [271].

Let us associate the cosmological perturbation, $\kappa_{\alpha\beta}$, of gravitational field in the conformal spacetime with the background metric $\bar{\mathfrak{f}}_{\alpha\beta}$ with physical perturbation $\varkappa_{\alpha\beta}$ of the metric as follows

$$\varkappa_{\alpha\beta} = a^2(\eta)\kappa_{\alpha\beta}, \tag{5.6.15}$$

where perturbation $\varkappa_{\alpha\beta}$ has been defined in (5.4.2) and $a(\eta)$ is the scale factor of the FLRW metric. Gravitational perturbation $h_{\alpha\beta}$ relates to $\varkappa_{\alpha\beta}$ by equation (5.4.10), and can be also represented in the conformal form

$$h_{\alpha\beta} = a^2(\eta)\mathrm{h}_{\alpha\beta}, \tag{5.6.16}$$

where

$$\mathrm{h}_{\alpha\beta} = -\kappa_{\alpha\beta} + \frac{1}{2}\bar{\mathfrak{f}}_{\alpha\beta}\kappa, \tag{5.6.17}$$

with $\kappa \equiv \bar{\mathfrak{f}}^{\alpha\beta}\kappa_{\alpha\beta}$. In what follows, tensor indices of geometric objects in the conformal spacetime are raised and lowered with the help of the conformal metric $\bar{\mathfrak{f}}_{\alpha\beta}$.

We assume that the scale factor a of the universe remains unperturbed. This assumption is justified since we can always include the perturbation of the scale factor to the perturbation $\kappa_{\alpha\beta}$ of the conformal metric. Thus, the perturbed physical spacetime interval, ds, of the FLRW metric relates to the perturbed conformal spacetime interval, $d\tilde{s}$, by the conformal transformation

$$ds^2 = a^2(\eta)d\tilde{s}^2. \tag{5.6.18}$$

Here, the perturbed conformal spacetime interval reads

$$d\tilde{s}^2 = f_{\alpha\beta}dx^\alpha dx^\beta, \tag{5.6.19}$$

where

$$f_{\alpha\beta} = \bar{\mathfrak{f}}_{\alpha\beta} + \kappa_{\alpha\beta}, \tag{5.6.20}$$

is the perturbed conformal metric. Here, $\bar{\mathfrak{f}}_{\alpha\beta}$ is the unperturbed conformal metric defined in (5.3.16), $\kappa_{\alpha\beta}$ is the perturbation of the conformal metric, and $x^\alpha = (x^0, x^i)$ are arbitrary coordinates which are the same as in the physical spacetime manifold in correspondence with the definition of the conformal metric transformation [314].

It is worth emphasizing that in case of the space curvature $k = 0$, the background conformal metric, $\mathfrak{g}_{\alpha\beta}(\eta, X^i)$, expressed in the isotropic Cartesian coordinates (η, X^i), is the diagonal Minkowski metric, $\mathfrak{g}_{\alpha\beta}(\eta, X^i) = \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. Therefore, in this case the background metric $\bar{\mathfrak{f}}_{\alpha\beta}$ remains the Minkowski metric with the components expressed in arbitrary coordinates by means of tensor transformation

$$\bar{\mathfrak{f}}_{\alpha\beta} = \mathrm{M}^\mu{}_\alpha \mathrm{M}^\nu{}_\beta \eta_{\mu\nu}, \tag{5.6.21}$$

where the matrix of transformation has been defined in (5.3.13). If the matrix of transformation, $\mathrm{M}^\mu{}_\alpha$, is the Lorentz boost, the conformal metric, $\bar{\mathfrak{f}}_{\alpha\beta}$, remains flat, $\bar{\mathfrak{f}}_{\alpha\beta} = \eta_{\alpha\beta}$. It is worth noticing that, in general, the unperturbed conformal metric can be chosen flat even in case of $k = -1, +1$ [239]. Hence, all equations given above will remain intact which means that, in fact, our formalism is applicable to FLRW metric with any space curvature. The only change will be in the conformal factor which, in the case of $k = \pm 1$, is not merely the scale factor $a(\eta)$ of the FLRW metric but a more complicated function, $a(\eta, x^a)$, of time and spatial coordinates [239]. Though it is not difficult to handle all three cases of $k = -1, 0, +1$ on the same footing but it burdens equations for the field perturbations with a number of terms being proportional to k. Moreover, consideration of the dark energy equations with $k = \pm 1$ given in the preceding section gets complicated [7]. For this reason, we restrict ourselves with the case of the spatially-flat universe with $k = 0$ which is an excellent approximation in treating cosmological observations [246].

Similarly to (5.5.7) the conformal metric perturbation, $\mathrm{h}_{\alpha\beta}$, can be split in 1+3 algebraically-irreducible components

$$\mathrm{h}_{\alpha\beta} = p^{\mathrm{T}}_{\alpha\beta} + \bar{\nabla}_\alpha p_\beta + \bar{\nabla}_\beta p_\alpha + \left(\bar{\nabla}_\alpha \bar{\nabla}_\beta + \frac{1}{3}\bar{\pi}_{\alpha\beta}\right) p + 2\bar{\nabla}_\alpha \bar{\nabla}_\beta (q - p), \tag{5.6.22}$$

where the four-velocity $\bar{\mathrm{v}}^\alpha = a\bar{u}^\alpha$, $\bar{\mathrm{v}}_\alpha = \bar{\mathrm{f}}_{\alpha\beta}\bar{\mathrm{v}}^\beta = a^{-1}\bar{g}_{\alpha\beta}\bar{u}^\beta = a^{-1}\bar{u}_\alpha$, and

$$\bar{\pi}_{\alpha\beta} = \bar{\mathrm{f}}_{\alpha\beta} + \bar{\mathrm{v}}_\alpha\bar{\mathrm{v}}_\beta, \tag{5.6.23}$$

is the operator of projection on the conformal space which represents a hypersurface being everywhere orthogonal to the congruence of worldlines of four-velocity $\bar{\mathrm{v}}^\alpha$. Four-velocity $\bar{\mathrm{v}}^\alpha$ is an analogue of the Hubble flow in the conformal spacetime. We also notice that $\bar{P}_{\alpha\beta} = a^2\bar{\pi}_{\alpha\beta}$.

Different pieces of the conformal metric perturbation, $\mathrm{h}_{\alpha\beta}$, are related to those of the physical metric perturbation, $h_{\alpha\beta}$, by the powers of the scale factor,

$$\mathfrak{p}^{\top}_{\alpha\beta} = a^2 p^{\top}_{\alpha\beta}, \qquad \mathfrak{p}_\alpha = a p_\alpha, \qquad \mathfrak{p} = p, \qquad \mathfrak{q} = q. \tag{5.6.24}$$

More specifically,

$$q = \frac{1}{2}\left(\bar{\mathrm{v}}^\mu\bar{\mathrm{v}}^\nu + \bar{\pi}^{\mu\nu}\right)\mathrm{h}_{\mu\nu}, \tag{5.6.25a}$$

$$p = \bar{\pi}^{\mu\nu}\mathrm{h}_{\mu\nu}, \tag{5.6.25b}$$

$$p_\alpha = -\bar{\pi}_\alpha{}^\beta\bar{\mathrm{v}}^\gamma\mathrm{h}_{\beta\gamma}, \tag{5.6.25c}$$

$$p^{\top}_{\alpha\beta} = p_{\alpha\beta} - \frac{1}{3}\bar{\pi}_{\alpha\beta}p, \tag{5.6.25d}$$

where

$$p_{\alpha\beta} = \bar{\pi}_\alpha{}^\mu\bar{\pi}_\beta{}^\nu\mathrm{h}_{\mu\nu}. \tag{5.6.26}$$

The trace of the gravitational perturbation, $\mathrm{h} \equiv \bar{\mathrm{f}}^{\alpha\beta}\mathrm{h}_{\alpha\beta} = 2(p-q)$. The components $\kappa_{\alpha\beta} = -\mathrm{h}_{\alpha\beta} + \bar{\mathrm{f}}_{\alpha\beta}\mathrm{h}/2$ are used in calculating dynamical behavior of particles and light in the conformal spacetime as well as for matching theory with observables. The components of $\kappa_{\alpha\beta}$ are

$$\kappa_{\alpha\beta} = -p^{\top}_{\alpha\beta} - \bar{\mathrm{v}}_\alpha p_\beta - \bar{\mathrm{v}}_\beta p_\alpha + \frac{2}{3}\bar{\pi}_{\alpha\beta}p - \left(\bar{\mathrm{v}}_\alpha\bar{\mathrm{v}}_\beta + \bar{\pi}_{\alpha\beta}\right)q, \tag{5.6.27}$$

and $\kappa \equiv \bar{\mathrm{f}}^{\alpha\beta}\kappa_{\alpha\beta} = 2(p-q) = \mathrm{h}$.

It turns out that the conformal Hubble parameter, $\mathscr{H} = a'/a$ is more convenient in the conformal spacetime than the "canonical" Hubble parameter, $H = \dot{R}/R = R^{-1}dR/dT$, where T is the cosmological time (see Section 5.3.2). Relations between $\mathscr{H}$ and H, and their derivatives are shown in equations (5.3.5–5.3.7). These relations are employed along with equations (5.3.6) and (5.6.13) in order to express the time derivative, $\mathscr{H}'$, of the conformal Hubble parameter in terms of $\mathscr{H}^2$ and the parameter w_{eff} of the effective equation of state

$$\mathscr{H}' = -\frac{1}{2}(1 + 3w_{\mathrm{eff}})\mathscr{H}^2. \tag{5.6.28}$$

We shall use this expression in the calculations that follow.

5.6.3 Post-Newtonian field equations in conformal spacetime

The set of the post-Newtonian field equations in cosmology consists of equations for the perturbations of the background dark matter, dark energy and gravitational field. Perturbations of dark matter and dark energy are described by four scalars, $V_{\rm m}$, $V_{\rm q}$, $\chi_{\rm m}$ and $\chi_{\rm q}$ but only three of them are functionally-independent because of equality (5.5.37), that is

$$V_{\rm m} - V_{\rm q} = \bar{u}^{\alpha}\left(\chi_{\rm m} - \chi_{\rm q}\right)_{|\alpha} . \tag{5.6.29}$$

Depending on a particular situation, any of the three scalars can be taken as independent variables in description of scalar perturbations.

The gravitational field perturbations are q, p, p_{α}, $p^{\rm T}_{\alpha\beta}$ but among them the scalar q is not independent and can be expressed in terms of $\chi_{\rm m}$ and $V_{\rm m}$ in accordance with (5.5.14a),

$$q = -2\left(V_{\rm m} - \bar{u}^{\alpha}\chi_{{\rm m},\alpha}\right), \tag{5.6.30}$$

where we have also used the equality $\mathrm{q} = q$ as follows from (5.6.24). The scalar q can be also expressed in terms of $\chi_{\rm q}$ and $V_{\rm q}$ in accordance with (5.5.14b). Hence, as soon as the pairs, $V_{\rm m}$ and $\chi_{\rm m}$ or $V_{\rm q}$ and $\chi_{\rm q}$ are known, the scalar gravitational perturbation q can be easily calculated from (5.6.30). Functions p, p_{α}, $p^{\rm T}_{\alpha\beta}$ are independent and decouple both from each other and from the other perturbations. Thus, the most difficult part of the perturbation theory is to find out solutions of the scalar perturbations which are coupled one to another.

The post-Newtonian field equations in the conformal spacetime for variables $\chi_{\rm m}$, $\chi_{\rm q}$, $V_{\rm m}$ and for p, p_{α}, $p^{\rm T}_{\alpha\beta}$ are derived from the equations of the previous section by transforming all functions and operators from physical to conformal spacetime. The important part of the transformation technique is based on formulas converting the covariant Laplace-Beltrami wave operators, defined on the background spacetime manifold, to their conformal spacetime counterparts.

Laplace-Beltrami operator in conformal spacetime

Let F be an arbitrary scalar, F_{α} – an arbitrary covector, and $F_{\alpha\beta}$ – an arbitrary covariant tensor of the second rank. We have three different types of the Laplace-Beltrami operators on the curved background manifold: a scalar – $F^{|\mu}{}_{|\mu}$, a vector – $F_{\alpha}{}^{|\mu}{}_{|\mu}$, and a tensor – $F_{\alpha\beta}{}^{|\mu}{}_{|\mu}$ where the covariant derivatives are taken with the help of the affine connection $\bar{\Gamma}^{\alpha}{}_{\beta\gamma}$ being compatible with the metric $\bar{g}_{\alpha\beta}$ as shown in (5.3.21). Covariant derivatives gives the invariant description of differential equations of mathematical physics on curved manifolds. However, for handling a more pragmatic purpose of finding solution of a differential equation, the covariant operators must be expressed in terms of partial derivatives with respect to the coordinates chosen for solving the equation.

Transformation of the covariant Laplace-Beltrami operators to the partial derivatives is achieved after writing down the covariant derivatives for a scalar F, a vector f_α, and a tensor $F_{\alpha\beta}$ in explicit form by making use of the Christoffel symbols given in (5.3.23–5.3.25). Tedious but straightforward calculations of the covariant derivatives yield the scalar, vector and tensor Laplace-Beltrami operators in the following form [272]

$$F^{|\mu}{}_{|\mu} = \frac{1}{a^2}\left[\Box F - 2\mathscr{H}\bar{\nabla}^\mu F_{;\mu}\right], \tag{5.6.31a}$$

$$F_\alpha{}^{|\mu}{}_{|\mu} = \frac{1}{a^2}\Big[\Box F_\alpha - 2\mathscr{H}\bar{\nabla}^\mu F_{\mu;\alpha} + 2\mathscr{H}\bar{\nabla}_\alpha \bar{\mathfrak{f}}^{\mu\nu} F_{\mu;\nu} \tag{5.6.31b}$$
$$+ \left(\mathscr{H}' + 2\mathscr{H}^2\right) F_\alpha - 2\mathscr{H}^2 \bar{\nabla}_\alpha \bar{\nabla}^\mu F_\mu\Big],$$

$$F_{\alpha\beta}{}^{|\mu}{}_{|\mu} = \frac{1}{a^2}\Big[\Box F_{\alpha\beta} + 2\mathscr{H}\bar{\nabla}^\mu F_{\alpha\beta;\mu} - 2\mathscr{H}\bar{\nabla}^\mu F_{\mu\alpha;\beta} - 2\mathscr{H}\bar{\nabla}^\mu F_{\mu\beta;\alpha} \tag{5.6.31c}$$
$$+ 2\mathscr{H}\bar{\mathfrak{f}}^{\mu\nu}\left(\bar{\nabla}_\alpha F_{\beta\mu;\nu} + \bar{\nabla}_\beta F_{\alpha\mu;\nu}\right) + 2\left(\mathscr{H}' + \mathscr{H}^2\right) F_{\alpha\beta}$$
$$- 4\mathscr{H}^2\Big(\bar{\nabla}^\mu \bar{\nabla}_\alpha F_{\beta\mu} + \bar{\nabla}^\mu \bar{\nabla}_\beta F_{\alpha\mu} - \frac{1}{2}\bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\mathfrak{f}}^{\mu\nu} F_{\mu\nu} - \frac{1}{2}\bar{\mathfrak{f}}_{\alpha\beta}\bar{\nabla}^\mu \bar{\nabla}^\nu F_{\mu\nu}\Big)\Big],$$

where we have introduced notations

$$\Box F \equiv \bar{\mathfrak{f}}^{\mu\nu} F_{;\mu\nu}, \qquad \Box F_\alpha \equiv \bar{\mathfrak{f}}^{\mu\nu} F_{\alpha;\mu\nu}, \qquad \Box F_{\alpha\beta} \equiv \bar{\mathfrak{f}}^{\mu\nu} F_{\alpha\beta;\mu\nu}, \tag{5.6.32}$$

of the wave operators for the scalar, vector and tensor fields in the conformal spacetime and in arbitrary coordinates. Notice that although the conformal spacetime coincides, in case of $k = 0$, with the Minkowski space, the metric $\bar{\mathfrak{f}}_{\alpha\beta}$ is not the diagonal Minkowski metric $\eta_{\alpha\beta}$ unless the coordinates are Cartesian. Of course, the covariant derivative from a scalar must be understood as a partial derivative, that is $F_{;\alpha} = F_{,\alpha}$.

We will need several other equations to complete the transformation of the Laplace-Beltrami operators to the conformal spacetime since the wave operator $\Box$ acts on functions like those shown in (5.6.24), which are made of a product of the scale factor, $a = a(\eta)$ in some power n (may be not an integer), with a geometric object, $\mathcal{F} = \mathcal{F}(x^\alpha)$, which can be a scalar, a vector or a tensor of the second rank (we have suppressed the tensor indices of $\mathcal{F}$ since they do not interfere with the derivation of the equations which follow). These equations are

$$\left(a^n \mathcal{F}\right)_{;\mu} = a^n\left(\mathcal{F}_{;\mu} - n\mathscr{H}\bar{\nabla}_\mu \mathcal{F}\right), \tag{5.6.33a}$$

$$\left(a^n \mathcal{F}\right)_{;\mu\nu} = a^n\Big[\mathcal{F}_{;\mu\nu} - n\mathscr{H}\left(\bar{\nabla}_\mu \mathcal{F}_{;\nu} + \bar{\nabla}_\nu \mathcal{F}_{;\mu}\right) \tag{5.6.33b}$$
$$+ n\left(\mathscr{H}' + n\mathscr{H}^2\right)\bar{\nabla}_\mu \bar{\nabla}_\nu\Big],$$

and they allow us to write down the wave operator from the product of a^n and $\mathcal{F}$ in the following form

$$\Box\left(a^n\mathcal{F}\right) = a^n\left[\Box\mathcal{F} - 2n\mathscr{H}\bar{v}^\mu\mathcal{F}_{;\mu} - n\left(\mathscr{H}' + n\mathscr{H}^2\right)\mathcal{F}\right], \tag{5.6.34}$$

It is easy to confirm that contraction of (5.6.33b) with the conformal four-velocity, $\bar{v}^\alpha$, brings about another differential operator

$$\bar{v}^\mu\bar{v}^\nu\left(a^n\mathcal{F}\right)_{;\mu\nu} = a^n\left[\bar{v}^\mu\bar{v}^\nu\mathcal{F}_{;\mu\nu} + 2n\mathscr{H}\bar{v}^\mu\mathcal{F}_{;\mu} + n\left(\mathscr{H}' + n\mathscr{H}^2\right)\mathcal{F}\right]. \tag{5.6.35}$$

We remind that if the object $\mathcal{F}$ is a scalar, the covariant derivative $\mathcal{F}_{;\alpha} = \mathcal{F}_{,\alpha}$ is reduced to a partial derivative. In case, when $\mathcal{F}$ is either a vector or a tensor, the covariant derivative must be calculated with taking into account the affine connection $\bar{B}^\alpha{}_{\beta\gamma}$ defined in (5.3.25).

It is also interesting to notice that in the expanding universe the conformal Laplace operator, $\Delta\mathcal{F} \equiv \bar{\pi}^{\mu\nu}\mathcal{F}_{;\mu\nu}$ is the scale invariant in the sense that

$$\Delta\left(a^n\mathcal{F}\right) = a^n\Delta\mathcal{F}, \tag{5.6.36}$$

where $\mathcal{F}$ is a tensor of an arbitrary rank. Equation (5.6.36) can be proven by adding up (5.6.34) and (5.6.35), and accounting for definition (5.6.23) of the projection operator on the hypersurface being orthogonal to $\bar{v}^\alpha$.

Now, we are ready to formulate the field equations for cosmological perturbations in the conformal spacetime.

Equations for perturbations of dark matter and dark energy

Dark matter and dark energy are described by scalar fields Φ and Ψ. The fields themselves are not gauge-invariant. Therefore, physical meaning have only the equations for the gauge-invariant perturbations of these fields which are V_m, V_q, and χ. We consider, first, equation (5.5.35) for the gauge-invariant scalar V_m. We convert the covariant derivatives taken with respect to the background metric, $\bar{g}_{\alpha\beta}$, to the partial derivatives of the conformally-flat metric, $\bar{\mathfrak{f}}_{\alpha\beta}$ and use equation (5.6.31a) for the Laplace-Beltrami operator along with the expressions for various cosmological parameters given in Section 5.6.2. After arranging terms with respect to the powers of the Hubble parameter $\mathscr{H}$, we obtain the scalar equation for function V_m describing the perturbations of dark matter,

$$\begin{aligned}\Box V_\mathrm{m} &+ \left(1 - \frac{c^2}{v_\mathrm{s}^2}\right)\bar{v}^\alpha\bar{v}^\beta V_{\mathrm{m};\alpha\beta} + \left(3 - \frac{c^2}{v_\mathrm{s}^2}\right)\mathscr{H}\bar{v}^\alpha V_{\mathrm{m},\alpha} \\ &+ 3\left[1 - w_\mathrm{eff} - \frac{1}{2}(1 + w_\mathrm{m})\left(1 - \frac{c^2}{v_\mathrm{s}^2}\right)\Omega_\mathrm{m}\right]\mathscr{H}^2 V_\mathrm{m} \\ &+ 12\mathscr{H}^2\left[\bar{v}^\alpha\chi_{,\alpha} - 3\left(1 - \sqrt{\frac{3}{8\mathrm{x}_1}}\lambda\mathrm{x}_2\right)\mathscr{H}\chi\right]\frac{\mathrm{x}_1}{a} = -4\pi a^2(\sigma + \tau)\,.\end{aligned} \tag{5.6.37}$$

This is a wave equation with the speed of sound v_s which determines the speed of propagation of the scalar perturbations in the dark matter considered as an ideal fluid. These perturbations can be interpreted as acoustic or sound waves of different wavelengths propagating in spacetime. Solution of homogeneous equation (5.6.37) describes the propagation of primordial scalar perturbations of dark matter. A particular solution of the inhomogeneous equation (5.6.37) tells us how the perturbation of dark matter caused by the isolated astronomical system propagate.

Similar procedure is applied to equation (5.5.38) and leads to a wave equation for function V_q describing propagation of perturbations of dark energy considered as a scalar field,

$$\begin{aligned}\Box V_q &+ 2\left(1-\sqrt{\frac{3}{2x_1}}\lambda x_2\right)\mathscr{H}\bar{\nabla}^\mu V_{q,\mu} \\ &+ 3\left[1-w_{\text{eff}}-\frac{1}{2}(1+w_m)\left(1-\frac{c^2}{v_s^2}\right)\Omega_m\right]\mathscr{H}^2 V_q \\ &+ \lambda x_2\left[3\lambda\left(2\Gamma_q+\frac{x_2}{x_1}\right)-5\sqrt{\frac{6}{x_1}}\right]\mathscr{H}^2 V_q \\ &+ \frac{3}{2}\mathscr{H}^2(1+w_m)\left(3+\frac{c^2}{v_s^2}\right)\left[\bar{\nabla}^\mu\chi_{,\mu}-3\frac{v_s^2}{c^2}\mathscr{H}\chi\right]\frac{\Omega_m}{a} = -4\pi a^2(\sigma+\tau)\,.\end{aligned} \tag{5.6.38}$$

The speed of propagation of dark energy is naturally equal to the fundamental speed c as contrasted with dark matter. Dark matter has an intrinsic elasticity associated with the bulk modulus $K = \epsilon(dp/d\epsilon)$ that is proportional to pressure p, and where ϵ is the energy density of the fluid. The speed of sound $v_s = \sqrt{K/\epsilon} < c$ for a fluid because in this case $K < \epsilon$. However, in case of the scalar field $K = |\epsilon|$, and $v_s = c$.

Equations (5.6.37) and (5.6.38) depend on the scalar function χ which obeys equation (5.5.30). Making use of the same transformations as applied to derivation of (5.6.37) and (5.6.38), we can recast (5.5.30) to a wave equation for χ,

$$\begin{aligned}\Box\chi + 4\mathscr{H}\left(1-\sqrt{\frac{3}{8x_1}}\lambda x_2\right)\bar{\nabla}^\alpha\chi_{,\alpha} - \frac{9}{2}(1+w_{\text{eff}})\mathscr{H}^2\chi \\ = -a\left[\sqrt{\frac{6}{x_1}}\lambda x_2\mathscr{H}V_m + \left(1-\frac{c^2}{v_s^2}\right)\bar{\nabla}^\alpha V_{m,\alpha}\right].\end{aligned} \tag{5.6.39}$$

We can observe that the speed of propagation of the field χ is equal to the fundamental speed c. Moreover, (5.6.39) depends on V_m and should be solved simultaneously with equation (5.6.37) for V_m after imposing certain boundary conditions. As soon as the gauge-invariant scalar χ is known, the potential, V_q, can be determined either as a particular solution of the inhomogeneous equation (5.6.38) or, more simple, from algebraic relation (5.5.37).

We also need equations for the normalized Clebsch and scalar potentials, $\chi_{\rm m}$ and $\chi_{\rm q}$. These potentials are required to determine the gravitational perturbation, q, with the help of (5.6.30) and/or to check on self-consistency of the solutions of the field equations in the matter sector of perturbation theory. Conformal-spacetime equations for $\chi_{\rm m}$ and $\chi_{\rm q}$ are derived from their definition (5.5.10) and the field equations (5.4.67) and (5.4.68). They are

$$\Box\chi_{\rm m} + \frac{3}{2}\left(1+w_{\rm eff}\right)\mathscr{H}^2\chi_{\rm m} \tag{5.6.40}$$

$$= 12\mathscr{H}^2 \mathrm{x}_1\chi - a\left[4\mathscr{H}V_{\rm m} + \left(1-\frac{c^2}{v_{\rm s}^2}\right)\bar{\nabla}^{\alpha}V_{{\rm m},\alpha}\right],$$

$$\Box\chi_{\rm q} + \frac{3}{2}\left(1+w_{\rm eff}\right)\mathscr{H}^2\chi_{\rm q} \tag{5.6.41}$$

$$= -6\mathscr{H}^2(1+w_{\rm m})\Omega_{\rm m}\chi - a\left(4-\sqrt{\frac{6}{\mathrm{x}_1}}\lambda\mathrm{x}_2\right)\mathscr{H}V_{\rm q}.$$

By subtracting one of these equations from another, we get back to equation (5.6.39). Notice that $\chi_{\rm m}$ and $\chi_{\rm q}$ are not gauge-invariant perturbations and, hence, the solutions of (5.6.40), (5.6.41) should be interpreted with care.

Equations for the metric perturbations

Post-Newtonian equations for gravitational perturbations in physical spacetime are (5.5.26), (5.5.28), (5.5.39) and (5.5.41). These equations are gauge-dependent. In order to fix the gauge we imposed the gauge conditions (5.4.64), (5.5.44). In this gauge, equations for the conformal metric tensor perturbations become

$$\Box q - 2\mathscr{H}\bar{\nabla}^{\alpha}q_{,\alpha} + \left(1+3w_{\rm eff}\right)\mathscr{H}^2 q = 8\pi a^2\left(\sigma+\tau\right)$$
$$-24\mathscr{H}^2\left[\sqrt{\frac{3\mathrm{x}_1}{8}}\lambda\mathrm{x}_2 - \mathscr{H}\mathrm{x}_1\right]\frac{\chi_{\rm q}}{a} - 3\left(1+w_{\rm eff}\right)\mathscr{H}^2\Omega_{\rm m}$$
$$\times\left[\left(1-\frac{c^2}{v_{\rm s}^2}\right)V_{\rm m} - \mathscr{H}\left(1+3\frac{v_{\rm s}^2}{c^2}\right)\frac{\chi_{\rm m}}{a}\right], \tag{5.6.42a}$$

$$\Box p - 2\mathscr{H}\bar{\nabla}^{\alpha}p_{,\alpha} = 16\pi a^2\tau, \tag{5.6.42b}$$

$$\Box p_{\alpha} - 2\mathscr{H}\bar{\nabla}^{\beta}p_{\alpha;\beta} + \left(1+3w_{\rm eff}\right)\mathscr{H}^2 p_{\alpha} = 16\pi a\tau_{\alpha}, \tag{5.6.42c}$$

$$\Box p^{\top}_{\alpha\beta} - 2\mathscr{H}\bar{\nabla}^{\gamma}p_{\alpha\beta;\gamma} = 16\pi\tau^{\top}_{\alpha\beta}. \tag{5.6.42d}$$

The reader can observe that equations (5.6.42a–5.6.42d) for linearized metric perturbations are decoupled from each other. Moreover, equations (5.6.42b–5.6.42d) are decoupled from the matter perturbations $V_{\rm m}$, $\chi_{\rm m}$, etc. Only equation (5.6.42a) for q is coupled with the matter perturbations governed by equations (5.6.37), (5.6.40), (5.6.41) so that these equations should be solved together. As we have mentioned above, function q is a linear combination of $V_{\rm m}$ and $\chi_{\rm m}$ according to (5.6.30). Hence, in order to determine q it is, in fact, sufficient to solve (5.6.37), (5.6.38) and (5.6.40). Nevertheless, it is convenient to present the differential equation (5.6.42a) for q explicitly for the sake of mathematical completeness and rigour. It can be used for independent validation of the solution of the system of equations (5.6.37), (5.6.40) and (5.6.38). Unfortunately, these equations are strongly coupled and cannot be solved analytically in the most general situation of a multi-component background universe governed by dark energy and dark matter. Solution of (5.6.37–5.6.41) would require a numerical integration.

It would be instrumental to get better insight to the post-Newtonian theory of cosmological perturbations by making some simplifying assumptions about the background model of the expanding universe in order to decouple the system of the post-Newtonian equations and to find their analytic solution explicitly. We discuss these assumptions and the corresponding system of the decoupled post-Newtonian equations in Section 5.7 below.

5.6.4 Residual gauge freedom in the conformal spacetime

The gauge conditions (5.4.64), (5.5.44) in physical space are given by equations (5.5.46a), (5.5.46b). After transforming to the conformal spacetime the equations for the gauge condition reads

$$p^{\beta}{}_{;\beta} + \bar{\mathrm{v}}^{\beta}(2q-p)_{,\beta} + 2\mathscr{H}q = 16\pi a\left(\bar{\rho}_{\mathrm{m}}\bar{\mu}_{\mathrm{m}}\chi_{\mathrm{m}} + \bar{\rho}_{\mathrm{q}}\bar{\mu}_{\mathrm{q}}\chi_{\mathrm{q}}\right), \tag{5.6.43a}$$

$$p^{\mathrm{T}\alpha\beta}{}_{;\beta} + \bar{\mathrm{v}}^{\beta}p^{\alpha}{}_{;\beta} + \frac{1}{3}\bar{\pi}^{\alpha\beta}p_{,\beta} + 2\mathscr{H}p^{\alpha} = 0. \tag{5.6.43b}$$

The residual gauge freedom in the conformal spacetime is described by two gauge functions, $\zeta \equiv \xi/a$ and ζ^{α}, where ξ and ζ^{α} have been defined in Section 5.5.7. Differential equations for ζ and ζ^{α} are obtained by making transformation of equations (5.5.50a), (5.5.50b) to the conformal spacetime. The calculation is straightforward and results in

$$\Box\zeta - 2\mathscr{H}\bar{\mathrm{v}}^{\beta}\zeta_{,\beta} + (1+3w_{\mathrm{eff}})\,\mathscr{H}^2\zeta = 0, \tag{5.6.44a}$$

$$\Box\zeta^{\alpha} - 2\mathscr{H}\bar{\mathrm{v}}^{\beta}\zeta^{\alpha}{}_{;\beta} = 0. \tag{5.6.44b}$$

Solutions of equations (5.6.42a–5.6.42d) are determined up to the gauge transformations

$$q' = q + 2\bar{\mathrm{v}}^{\alpha}\zeta_{,\alpha} + 2\mathscr{H}\zeta, \tag{5.6.45a}$$

$$p' = p + \zeta^{\alpha}{}_{;\alpha} + 3\bar{\mathrm{v}}^{\alpha}\zeta_{,\alpha} + 6\mathscr{H}\zeta, \tag{5.6.45b}$$

$$p'_{\alpha} = p_{\alpha} + \bar{\pi}_{\alpha\beta}\left(\bar{\mathrm{v}}^{\gamma}\zeta^{\beta}{}_{;\gamma} - \zeta^{,\beta} + 2\mathscr{H}\zeta^{\beta}\right), \tag{5.6.45c}$$

$$p'_{\alpha\beta} = p_{\alpha\beta} - \left(\bar{\pi}_{\mu\alpha}\bar{\pi}_{\beta}{}^{\nu} + \bar{\pi}_{\mu\beta}\bar{\pi}_{\alpha}{}^{\nu}\right)\zeta^{\mu}{}_{;\nu} + \bar{\pi}_{\alpha\beta}\left(\zeta^{\alpha}{}_{;\alpha} + \bar{\mathrm{v}}^{\alpha}\zeta_{,\alpha} + 2\mathscr{H}\zeta\right), \tag{5.6.45d}$$

where the gauge functions ζ, ζ^α are solutions of the differential equations (5.6.44a), (5.6.44b).

5.7 Decoupled system of the post-Newtonian field equations

5.7.1 The universe governed by dark matter and cosmological constant

Case 1: Arbitrary equation of state of dark matter

Let us consider a special case of the background value of dark energy represented by cosmological constant $\Lambda = 8\pi\bar{W}$. In this case, the equation of state of the scalar field is $w_q = -1$, and we have $\bar{\rho}_q\bar{\mu}_q = \bar{\epsilon}_q + \bar{p}_q = 0$. The parameter $x_1 = 0$, and $x_2 = \Lambda/(3H^2)$. It yields the parameter $\Omega_q = x_2$, and $\Omega_m = 1 - x_2$. Since the cosmological constant corresponds to a constant potential $\bar{W}$ of the scalar field, we get for its derivative $\partial\bar{W}/\partial\bar{\Psi} = 0$, and equation (5.6.1) points out that the parameter $\lambda = 0$.

In the universe governed by dark matter and cosmological constant the parameter of the effective equation of state of the dark matter is

$$w_{\text{eff}} = w_m - (1 + w_m)\frac{\Lambda}{3H^2}. \tag{5.7.1}$$

Hence, the time derivative of the Hubble parameter defined in (5.6.13), is reduced to a more simple expression,

$$\dot{H} = \frac{1}{2}(1 + w_m)\left(\Lambda - 3H^2\right). \tag{5.7.2}$$

On the other hand, equation (5.3.52) tells us that in this model of the universe the time derivative of the Hubble parameter is

$$\dot{H} = -4\pi\bar{\rho}_m\bar{\mu}_m. \tag{5.7.3}$$

The field equation (5.6.37) for scalar V_m is reduced to that describing the time evolution of the perturbation of the ideal fluid density, $\delta\rho_m$. Indeed, the scalar V_m defined by equation (5.5.14a), can be recast to the form given by equation (5.5.23), that is

$$V_m = \frac{v_s^2}{c^2}\,\delta_m, \tag{5.7.4}$$

where the gauge-invariant scalar perturbation

$$\delta_m \equiv \frac{\delta\rho_m}{\bar{\rho}_m} + 3H\chi_m, \tag{5.7.5}$$

is a linear combination of the perturbation of the mass density of the dark matter and the normalized Clebsch potential χ_m. Replacing expression (5.7.4) in equation (5.6.37), yields an exact equation for δ_m that is

$$\left(1-\frac{v_s^2}{c^2}\right)\bar{\mathrm{v}}^\alpha\bar{\mathrm{v}}^\beta\delta_{\mathrm{m};\alpha\beta}-\frac{v_s^2}{c^2}\Box\delta_{\mathrm{m}}+\left(1-3\frac{v_s^2}{c^2}\right)\mathscr{H}\bar{\mathrm{v}}^\alpha\delta_{\mathrm{m},\alpha}$$
$$-\frac{3}{2}\left[(1-3w_{\mathrm{m}})\frac{v_s^2}{c^2}+(1+w_{\mathrm{m}})\right]\mathscr{H}^2\delta_{\mathrm{m}}$$
$$+\frac{1}{2}(1+w_{\mathrm{m}})\left(1-3\frac{v_s^2}{c^2}\right)a^2\Lambda\delta_{\mathrm{m}}=4\pi a^2(\sigma+\tau)\,. \tag{5.7.6}$$

This equation describes propagation of the density perturbation of dark matter, δ_{m}, in the form of sound waves with velocity v_s. Equation (5.7.6) is decoupled from any other perturbation and can be solved separately after the boundary conditions are specified. For this reason we call (5.7.6) *master equation*.

Equation (5.6.39) for potential χ makes no sense since the normalized perturbation $\chi_{\mathrm{q}}=\psi/\bar{\mu}_{\mathrm{q}}$ of dark energy in the form of cosmological constant diverges due to the condition $\mu_{\mathrm{q}}=\rho_{\mathrm{q}}=0$. Equation for the perturbation of dark energy, ψ, itself is obtained from (5.4.68) and is reduced to a homogeneous wave equation

$$\Box\psi-2\mathscr{H}\bar{\mathrm{v}}^\mu\psi_{,\mu}=0. \tag{5.7.7}$$

Equation for the normalized Clebsch potential, χ_{m}, is derived from equation (5.6.40) and, in the case of the universe under consideration, reads

$$\Box\chi_{\mathrm{m}}+\frac{1}{2}(1+w_{\mathrm{m}})\left(3\mathscr{H}^2-a^2\Lambda\right)\chi_{\mathrm{m}}=\left(1-\frac{v_s^2}{c^2}\right)a\bar{\mathrm{v}}^\mu\delta_{\mathrm{m},\mu}-4a\mathscr{H}\frac{v_s^2}{c^2}\delta_{\mathrm{m}}. \tag{5.7.8}$$

This is an inhomogeneous equation that can be solved as soon as one knows δ_{m} from the master equation (5.7.6). The potential χ_{m} is necessary to determine the perturbation of the four velocity of dark matter. We also need it to find out the metric perturbation q.

Gravitational potential, q, can be determined directly from equation (5.6.30) after solving equations (5.7.6) and (5.7.8) or by solving equation (5.6.42a) which (in the dark matter+cosmological constant universe) takes on the following form,

$$\Box q-2\mathscr{H}\bar{\mathrm{v}}^\mu q_{,\mu}+\left[(1+3w_{\mathrm{m}})\mathscr{H}^2-(1+w_{\mathrm{m}})a^2\Lambda\right]q \tag{5.7.9a}$$
$$=8\pi a^2\left\{\sigma+\tau+\bar{\rho}_{\mathrm{m}}\bar{\mu}_{\mathrm{m}}\left[\left(1-\frac{v_s^2}{c^2}\right)\delta_{\mathrm{m}}+H\left(1+3\frac{v_s^2}{c^2}\right)\chi_{\mathrm{m}}\right]\right\}.$$

Equations for other components of the metric tensor perturbations are found from (5.6.42b–5.6.42d). In dark matter+cosmological constant universe they read

$$\Box p-2\mathscr{H}\bar{\mathrm{v}}^\mu p_{,\mu}=16\pi a^2\tau, \tag{5.7.9b}$$
$$\Box p_\alpha-2\mathscr{H}\bar{\mathrm{v}}^\mu p_{\alpha;\mu}+\left[(1+3w_{\mathrm{m}})\mathscr{H}^2-(1+w_{\mathrm{m}})a^2\Lambda\right]p_\alpha \tag{5.7.9c}$$
$$=16\pi a\tau_\alpha,$$
$$\Box p^{\mathrm{T}}_{\alpha\beta}-2\mathscr{H}\bar{\mathrm{v}}^\mu p^{\mathrm{T}}_{\alpha\beta;\mu}=16\pi\tau^{\mathrm{T}}_{\alpha\beta}. \tag{5.7.9d}$$

Equations given in this section are valid for arbitrary cosmological equation of state of dark matter, $\bar{p}_m = w_m \bar{\epsilon}_m$, that is physically reasonable and makes sense. The parameter w_m of the equation of state should not be replaced with the ratio of v_s^2/c^2 which characterizes the derivative of pressure, $\bar{p}_m$, with respect to the energy density, $\bar{\epsilon}$, of dark matter. This is because the parameter w_m can depend in the most general case on the other thermodynamic quantities (like enthropy, temperature, etc.) which may implicitly depend on $\bar{\epsilon}$. Equations (5.7.6–5.7.9d) are decoupled in the sense that all of them can be solved one after another starting from solving the master equation (5.7.6) for δ_m.

Case 2: Cold dark matter

Equations of the previous section can be further simplified for some particular equations of state of dark matter. For example, in the case of cold dark matter (CDM) we can think about it as being made out of collisionless dust. Background pressure of dust drops down to zero making the parameter of the cold dark matter equation of state $w_m = 0$. Sound waves do not propagate in dust. Hence, the speed of sound $v_s = 0$. For this reason all terms being proportional to v_s^2 and w_m vanish in equation (5.7.6). Moreover, dust has the specific enthalpy, $\mu_m = 1$ making the energy density of dust equal to its rest mass density $\bar{\epsilon}_m = \bar{\rho}_m$, and the normalized perturbation χ_m of the Clebsch potential of dust equal to the perturbation ϕ of the Clebsch potential itself, $\chi_m = \phi$. The Friedmann equation (5.3.50) (for $k = 0$) tells us that

$$\mathscr{H}^2 = \frac{a^2}{3}\left(8\pi\bar{\rho}_m + \Lambda\right). \tag{5.7.10}$$

Accounting for this result in the master equation (5.7.6), and neglecting all terms being proportional to the speed of sound, v_s and w_m, we obtain

$$\bar{v}^\alpha \bar{v}^\beta \delta_{m;\alpha\beta} + \mathscr{H}\bar{v}^\alpha \delta_{m,\alpha} - 4\pi a^2 \bar{\rho}_m \delta_m = 4\pi a^2 (\sigma + \tau), \tag{5.7.11}$$

where the terms depending on the cosmological constant, Λ, have cancelled out. This equation is more familiar when is written down in the preferred FLRW frame, where $\bar{v}^\alpha = (1, 0, 0, 0)$. Equation (5.7.11) assumes the following form

$$\delta_m'' + \mathscr{H}\delta_m' - 4\pi a^2 \bar{\rho}_m \delta_m = 4\pi a^2 (\sigma + \tau), \tag{5.7.12}$$

where the time derivatives (denoted with a prime) are taken with respect to the conformal time η. Converting the time derivatives in (5.7.12) from the conformal time η to the cosmic time T reduces it to a canonical form

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi\delta_m = 4\pi(\sigma + \tau) \tag{5.7.13}$$

which can be found in many textbooks on cosmology [296, 327, 353, 456, 457]. All textbooks always dropped off the source of the bare perturbation in the right side of (5.7.13) as they are concerned with the description of the formation of the large scale structure in the universe out of the primordial perturbations. However, omitting the bare perturbation in the right side of (5.7.13) is equivalent to neglecting the contribution of the small-scale density fluctuations in the early universe to the formation of the large scale structures – the process which can be physically significant in the cold dark matter scenario of galaxy formation [57, 58].

Equation (5.7.13) has been derived by previous researchers without resorting to the concept of the Clebsch potential of the ideal fluid. For this reason, the density contrast, $\delta_{\rm m}$, was interpreted as the ratio of the perturbation of the dust density to its background value, $\delta = \delta\rho_{\rm m}/\bar{\rho}_{\rm m}$, without taking into account the perturbation, ϕ, of the Clebsch potential. However, the quantity δ is not gauge-invariant which was considered as a drawback. The scrutiny analysis of the underlying principles of hydrodynamics in the expanding universe given in the present chapter, reveals that equation (5.7.12) is, in fact, valid for the gauge-invariant density perturbation $\delta_{\rm m}$ defined above in (5.7.5). Another distinctive feature of equation (5.7.12) is the presence of the source of a *bare* perturbation in its right side. The *bare* perturbation is caused by the effective density $\sigma+\tau$ of the matter which comprises the isolated astronomical system and initiates the growth of instability in the cosmological matter that, in its own turn, induces formation of the large scale structure of the universe [353, 457]. Standard approach to cosmological perturbation theory always set $\sigma + \tau = 0$ and operates with the spectrum of the primordial perturbation of the density $\delta\rho_{\rm m}/\rho_{\rm m}$ (but not with the spectrum for $\delta_{\rm m}$).

Equation (5.7.8) in case of dust reads,

$$\Box\chi_{\rm m} + \frac{1}{2}\left(3\mathscr{H}^2 - a^2\Lambda\right)\chi_{\rm m} = a\bar{v}^\alpha\delta_{{\rm m},\alpha}, \tag{5.7.14}$$

where $\chi_{\rm m} = \phi$ is reduced to the perturbation ϕ of the Clebsch potential Φ for the reason that in case of dust $\mu_{\rm m} = 1$. If equations (5.7.12) and (5.7.14) are solved, the gravitational perturbations can be found from equations (5.7.9a–5.7.9d), which take on the following form

$$\Box q - 2\mathscr{H}\bar{v}^\alpha q_{,\alpha} + \left(\mathscr{H}^2 - a^2\Lambda\right)q = \tag{5.7.15a}$$
$$8\pi a^2\left[\sigma + \tau + \bar{\rho}_{\rm m}\left(\delta_{\rm m} + H\chi_{\rm m}\right)\right],$$

$$\Box p - 2\mathscr{H}\bar{v}^\alpha p_{,\alpha} = 16\pi a^2\tau, \tag{5.7.15b}$$

$$\Box p_\alpha - 2\mathscr{H}\bar{v}^\beta p_{\alpha;\beta} + \left(\mathscr{H}^2 - a^2\Lambda\right)p_\alpha = 16\pi a\tau_\alpha, \tag{5.7.15c}$$

$$\Box p^\intercal_{\alpha\beta} - 2\mathscr{H}\bar{v}^\gamma p^\intercal_{\alpha\beta;\gamma} = 16\pi\tau^\intercal_{\alpha\beta}. \tag{5.7.15d}$$

It is interesting to notice that besides the *bare* density perturbation, $\sigma + \tau$, caused by an isolated astronomical system, the source for the scalar gravitational perturbation, q, contains in the right side of equation (5.7.15a) also the *induced* density perturbation $\bar{\rho}_{\rm m}\left(\delta_{\rm m} + H\chi_{\rm m}\right) = \delta\rho_{\rm m} + H\bar{\rho}_{\rm m}\phi$ of the background dark matter. This *induced* density perturbation depends on time and leads to a temporal change of the initial (bare) mass of the isolated astronomical system in the course of the Hubble expansion of the universe. Thus, our post-Newtonian approach to cosmology explains the origin of the time-dependence of the central, point-like mass in the cosmological solution found by McVittie [313] (see also discussion in [93]).

Case 3: Hot dark matter

Hot dark matter (HDM) is a hypothetical form of dark matter which consists of ultrarelativistic particles that travel with velocities being very close to the fundamental speed c. A plausible candidate for the hot dark matter is neutrino. Hot dark matter taken alone, cannot explain how individual galaxies were formed from the primordial perturbations. Therefore, hot dark matter is discussed only as part of a mixed dark matter theory [40]. Nonetheless, the case of the hot dark matter is interesting from mathematical point of view. Equation of state of the hot dark matter is approximated by the radiative equation of state, $p_{\rm m} = (1/3)\epsilon_{\rm m}$ which yields the parameter $w_{\rm m} = 1/3$. We assume that this parameter is constant and, hence, the speed of sound $v_{\rm s} = \sqrt{1/3}c$. This value of $v_{\rm s}$ is comparable with the fundamental speed c which means that we have to keep the terms with the speed of sound in the master equation (5.7.6). The values of $w_{\rm m}$ and $v_{\rm s}$ for the hot dark matter equation of state reduce the master equation for the gauge-invariant HDM density perturbation $\delta_{\rm m}$ to the following form

$$\Box_{\rm s}\delta_{\rm m} + 6\mathscr{H}^2\delta_{\rm m} = -12\pi a^2(\sigma+\tau)\,, \tag{5.7.16}$$

where

$$\Box_{\rm s} \equiv \left(-\frac{c^2}{v_{\rm s}^2}\bar{\rm v}^\alpha\bar{\rm v}^\beta + \bar{\pi}^{\alpha\beta}\right)\partial_{\alpha\beta}, \tag{5.7.17}$$

is the D'Alembert wave operator with the speed of propagation of sound waves $v_{\rm s} = c/\sqrt{3}$. Equation for the perturbation of the Clebsch potential of the hot dark matter is derived from (5.7.8) and reads

$$\Box\chi_{\rm m} + 2\left(\mathscr{H}^2 - \frac{1}{3}a^2\Lambda\right)\chi_{\rm m} = \frac{2a}{3}\left(\bar{\rm v}^\mu\delta_{{\rm m},\mu} - 2\mathscr{H}\delta_{\rm m}\right). \tag{5.7.18}$$

Equations for the gravitational perturbations are

$$\Box q - 2\mathscr{H}\bar{\rm v}^\mu q_{,\mu} + 2\left(\mathscr{H}^2 - \frac{2}{3}a^2\Lambda\right)q = 8\pi a^2\left[\sigma + \tau + \frac{2}{3}\bar{\rho}_{\rm m}\bar{\mu}_{\rm m}\left(\delta_{\rm m} + 3H\chi_{\rm m}\right)\right]. \tag{5.7.19a}$$

Equations for other components of the metric tensor perturbations are found from (5.6.42b–5.6.42d). In dark matter+cosmological constant universe they read

$$\Box p - 2\mathscr{H}\bar{\rm v}^\mu p_{,\mu} = 16\pi a^2\tau, \tag{5.7.19b}$$

$$\Box p_\alpha - 2\mathscr{H}\bar{\rm v}^\mu p_{\alpha;\mu} + 2\left(\mathscr{H}^2 - \frac{2}{3}a^2\Lambda\right)p_\alpha = 16\pi a\tau_\alpha, \tag{5.7.19c}$$

$$\Box p^{\rm T}_{\alpha\beta} - 2\mathscr{H}\bar{\rm v}^\mu p^{\rm T}_{\alpha\beta;\mu} = 16\pi\tau^{\rm T}_{\alpha\beta}. \tag{5.7.19d}$$

5.7.2 The universe governed by dark energy

In this section we explore the case of the universe governed primarily by a dark energy (scalar field Ψ) with dark matter constituent being unimportant. In this case, the time evolution of the background universe is defined exceptionally by equations (5.6.5a), (5.6.5b). The most general solution of (5.6.5a), (5.6.5b) is complicated and can not be achieved analytically. Numerical analysis shows that the solution evolves in the phase space of the two variables $\{x_1, x_2\}$ from an unstable to a stable fixed point by passing through a saddle point [7]. The cosmic acceleration is realized by the stable point with the values of $x_1 = \lambda^2/6$ and $x_2 = 1 - \lambda^2/6$, which is equivalent to the equations of state (5.3.54) with the values of the parameters, $w_m = 0$, and, $w_q = -1 + \lambda^2/3$. It also requires the energy density of the background matter $\bar{\epsilon}_m = 0$, that is $\Omega_m = 0$. In such a universe the derivatives of the potential of the scalar field are

$$\frac{1}{\bar{\mu}_q}\frac{\partial \bar{W}}{\partial \bar{\Psi}} = -\frac{3}{2}H\left(1 - w_q\right), \qquad \frac{\partial^2 \bar{W}}{\partial \bar{\Psi}^2} = \frac{9}{2}H^2\left(1 - w_q^2\right). \tag{5.7.20}$$

Moreover, because $\bar{\rho}_m\bar{\mu}_m = \bar{\epsilon}_m + \bar{p}_m = 0$, the time derivative of the Hubble parameter is

$$\dot{H} = -4\pi\bar{\rho}_q\bar{\mu}_q = -\frac{3}{2}H^2\left(1 + w_q\right). \tag{5.7.21}$$

In the point of the attractor of the scalar field, perturbations of the dark matter are fully suppressed that is the normalized value of the perturbed Clebsch potential of the dark matter, $\chi_m = 0$. It makes the function $V_m = q/2$, that is reduced to the perturbation of the scalar component of the gravitational field only. Perturbations of the scalar field are described by the scalar field variable, χ_q. In particular, after substituting the derivatives (5.7.20) of the scalar field potential along with the derivative (5.7.21) of the Hubble parameter, to (5.5.38), we obtain the post-Newtonian equation for function V_q,

$$\Box V_q - \left(1 - 3w_q\right)\mathscr{H}\bar{v}^\mu V_{q,\mu} + \frac{3}{2}\mathscr{H}^2\left(1 - w_q\right)\left(1 + 3w_q\right)V_q = -4\pi a^2(\sigma + \tau). \tag{5.7.22}$$

Field equation for the perturbation of the scalar field, χ_q, is reduced to

$$\Box\chi_q + \frac{1}{2}\left(1 + 3w_q\right)\mathscr{H}^2\chi_q = -\left(1 + 3w_q\right)a\mathscr{H}V_q. \tag{5.7.23}$$

Post-Newtonian equations for gravitational perturbations are (5.6.42a–5.6.42d). After substituting the values of the parameters x_1, x_2, w_{eff}, etc., corresponding to the model of the universe governed by the dark energy alone, the post-Newtonian equations for the metric perturbations become

$$\Box q - 2\mathscr{H}\bar{\nabla}^{\mu} q_{,\mu} + \left(1 + 3w_{\mathrm{q}}\right)\mathscr{H}^2 q = 8\pi a^2 (\sigma + \tau) + \frac{3}{a}\left(1 + w_{\mathrm{q}}\right)\left(1 + 3w_{\mathrm{q}}\right)\mathscr{H}^3 \chi_{\mathrm{q}}, \tag{5.7.24a}$$

$$\Box p - 2\mathscr{H}\bar{\nabla}^{\mu} p_{,\mu} = 16\pi a^2 \tau, \tag{5.7.24b}$$

$$\Box p_{\alpha} - 2\mathscr{H}\bar{\nabla}^{\mu} p_{\alpha;\mu} + \left(1 + 3w_{\mathrm{q}}\right)\mathscr{H}^2 p_{\alpha} = 16\pi a \tau_{\alpha}, \tag{5.7.24c}$$

$$\Box p^{\mathrm{T}}_{\alpha\beta} - 2\mathscr{H}\bar{\nabla}^{\mu} p^{\mathrm{T}}_{\alpha\beta;\mu} = 16\pi \tau^{\mathrm{T}}_{\alpha\beta}. \tag{5.7.24d}$$

One can see that the field equations for the perturbations of dark energy and gravitational field are decoupled, and can be solved separately starting from the master equation (5.7.22).

5.7.3 Post-Newtonian potentials in the linearized Hubble approximation

The metric tensor perturbations

The post-Newtonian equations for cosmological perturbations of gravitational and matter field variables crucially depend on the equation of state of the matter fields in the background universe. It determines the time evolution of the scale factor $a = a(\eta)$ and the Hubble parameter $\mathscr{H} = \mathscr{H}(\eta)$ which are described by the wide range of elementary and special functions of mathematical physics (see, for example, textbooks [7, 303, 425] and references therein). It is not the goal of the present chapter to provide the reader with an exhaustive list of the mathematical solutions of the perturbed equations which requires theoretical development of cosmological Green's function (see, for example, [224, 292, 293, 382]). We notice that solving the field equations of the post-Newtonian approximations in cosmology is more complicated than in case of the post-Newtonian theory in asymptotically flat spacetime. The reason is twofold: (1) the system of the post-Newtonian equations on cosmological background involves, besides equations for the metric tensor perturbations, also the equations for the perturbations of the matter that curves the background manifold and governs its temporal evolution; and (2) the post-Newtonian field equations in cosmology depend on the time dependent Hubble parameter that makes finding the Green functions of the field equations pretty difficult task. If we are interested in finding the far zone (radiative) solution for the gravitational field of an isolated astronomical system, we have to fulfil this task exactly. This problem has not yet been solved though it is very important for doing precise cosmology with gravitational wave astronomy. On the other hand, we can employ the post-Friedmannian approximations by looking for the solution of the cosmological post-Newtonian equations as a series with respect to the Hubble parameter. In this section we shall limit ourselves with the linearized Friedmann approximation. In other words, we shall take into account only the terms which are proportional to the Hubble parameter $\mathscr{H}$, and shall systematically neglect all terms which are quadratic, cubic, and higher-orders with respect to $\mathscr{H}$.

As we shall see, in the linearized Friedmann approximation the post-Newtonian equations for the field perturbations have identical mathematical structure so that they are not only decoupled from one another, but their generic solution can be found irrespectively of the equation of state governing the background universe. Indeed, if we neglect all quadratic with respect to $\mathscr{H}$ terms, the field equations for the conformal metric perturbations are reduced to the following set,

$$\Box q - 2\mathscr{H}\bar{\mathrm{v}}^{\alpha}q_{,\alpha} = 8\pi a^2(\sigma+\tau)\,, \tag{5.7.25a}$$

$$\Box p - 2\mathscr{H}\bar{\mathrm{v}}^{\alpha}p_{,\alpha} = 16\pi a^2\tau, \tag{5.7.25b}$$

$$\Box p_{\alpha} - 2\mathscr{H}\bar{\mathrm{v}}^{\beta}p_{\alpha;\beta} = 16\pi a\tau_{\alpha}, \tag{5.7.25c}$$

$$\Box p^{\mathrm{T}}_{\alpha\beta} - 2\mathscr{H}\bar{\mathrm{v}}^{\gamma}p^{\mathrm{T}}_{\alpha\beta;\gamma} = 16\pi\tau^{\mathrm{T}}_{\alpha\beta}, \tag{5.7.25d}$$

where the wave operator $\Box$ has been defined in (5.6.32), and the source of the *bare* perturbation is the tensor of energy-momentum of a localized astronomical system having a bounded matter support in space – see Section 5.3.8. The differential structure of the left side of equations (5.7.25a–5.7.25d) is the same for all functions. The equations differ from each other only in terms of the order of $\mathscr{H}^2$ which have been omitted.

In order to bring equations (5.7.25a–5.7.25d) to a solvable form, we resort to relation (5.6.34) which reveals that in the linearized Friedmann approximation, the post-Newtonian equations for metric perturbations can be reduced to the form of a wave equation

$$\Box(aq) = 8\pi a^3(\sigma+\tau)\,, \tag{5.7.26a}$$

$$\Box(ap) = 16\pi a^3\tau, \tag{5.7.26b}$$

$$\Box\left(ap_{\alpha}\right) = 16\pi a^2\tau_{\alpha}, \tag{5.7.26c}$$

$$\Box\left(ap^{\mathrm{T}}_{\alpha\beta}\right) = 16\pi a\tau^{\mathrm{T}}_{\alpha\beta}. \tag{5.7.26d}$$

So far, we did not impose any limitations on the curvature of space that can take three values: $k = \{-1, 0, +1\}$. Solution of wave equations (5.7.26a–5.7.26d) can be given in terms of special functions in case of the Riemann ($k = +1$) or the Lobachevsky ($k = -1$) geometry [292, 293]. The case of the spatial Euclidean geometry ($k = 0$) is more manageable, and will be discussed below.

If the FLRW metric is spatially-flat universe, $k = 0$, and we chose the Cartesian coordinates x^{α} related to the isotropic coordinates X^{α} of the FLRW metric by a Lorentz transformation, $X^{\alpha} = L^{\alpha}{}_{\beta}x^{\beta}$, where $L^{\alpha}{}_{\beta}$ is the matrix of the Lorentz boost. In these coordinates the operator $\Box$ becomes a wave operator in the Minkowski space,

$$\Box = \eta^{\mu\nu}\partial_{\mu\nu}. \tag{5.7.27}$$

and equations (5.7.26a–5.7.26d) are reduced to the inhomogeneous wave equations which solution depends essentially on the boundary conditions imposed on the metric tensor perturbations at conformal past-null infinity $\mathscr{J}^-$ of the cosmological manifold [315]. We shall assume a no-incoming radiation condition also known as Fock-Sommerfeld's condition [115, 178]

$$\lim_{\substack{r\to+\infty\\ t+r=\text{const.}}} n^\gamma\partial_\gamma\left[a(\eta)rh_{\alpha\beta}(x^\gamma)\right]=0, \tag{5.7.28}$$

where $x^\gamma=(x^0,x^i)$, $\eta\equiv X^0$ is the conformal time in isotropic coordinates connected to the coordinates x^α by a Lorentz boost $\eta=\eta(x^\gamma)=L^0{}_\beta x^\beta$, the null vector $n^\alpha=\{1,x^i/r\}$, and $r=\delta_{ij}x^ix^j$ is the radial distance. This condition ensures that there is no infalling gravitational radiation arriving to the localized astronomical system from the future null infinity $\mathscr{J}^+$. Effectively, it singles out the retarded solution of the wave equation. Whether the boundary condition (5.7.28) is valid or not, we do not know for sure because our knowledge of the universe is limited by the existence of the cosmological (also known as light or particle) horizon [296] that represents the boundary between the observable and the unobservable regions of the universe. Nonetheless, in case of spatially flat ($k=0$) universe, the condition (5.7.28) seems to be highly plausible.

A particular solution of the wave equations satisfying condition (5.7.28), is the retarded integral [285]

$$q(t,\boldsymbol{x})=-\frac{2}{a\left[\eta(t,\boldsymbol{x})\right]}\int_{\mathscr{V}}\frac{a^3\left[\eta\left(s,\boldsymbol{x}'\right)\right]\left[\sigma\left(s,\boldsymbol{x}'\right)+\tau\left(s,\boldsymbol{x}'\right)\right]d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{5.7.29a}$$

$$p(t,\boldsymbol{x})=-\frac{4}{a\left[\eta(t,\boldsymbol{x})\right]}\int_{\mathscr{V}}\frac{a^3\left[\eta(s,\boldsymbol{x}')\right]\tau\left(s,\boldsymbol{x}'\right)d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{5.7.29b}$$

$$p_\alpha(t,\boldsymbol{x})=-\frac{4}{a\left[\eta(t,\boldsymbol{x})\right]}\int_{\mathscr{V}}\frac{a^2\left[\eta(s,\boldsymbol{x}')\right]\tau_\alpha\left(s,\boldsymbol{x}'\right)d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{5.7.29c}$$

$$p^{\mathrm{T}}_{\alpha\beta}(t,\boldsymbol{x})=-\frac{4}{a\left[\eta(t,\boldsymbol{x})\right]}\int_{\mathscr{V}}\frac{a\left[\eta(s,\boldsymbol{x}')\right]\tau^{\mathrm{T}}_{\alpha\beta}\left(s,\boldsymbol{x}'\right)d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{5.7.29d}$$

where the scale factor a in front of the integrals depends on the coordinates of the field point $a\equiv a\left[\eta(t,\boldsymbol{x})\right]$, and the functions in the integrand depend on the retarded time

$$s=t-|\boldsymbol{x}-\boldsymbol{x}'|, \tag{5.7.30}$$

because gravity propagates with finite speed. Equation (5.7.30) describes characteristic of the null cone in the conformal Minkowski space that determines the causal nature of the gravitational field in the expanding universe with $k=0$. Solutions (5.7.29a–5.7.29d) are Lorentz-invariant as shown by calculations in Section B.1.

Integration in (5.7.29a–5.7.29d) is performed over the finite volume, $\mathscr{V}$, occupied by the matter of the localized astronomical system. In case of the system comprised of N massive bodies which are separated by distances being much larger than their characteristic size, the matter occupies the volumes of the bodies. In this case the integration in equations (5.7.29a–5.7.29d) is practically performed over the volumes of the bodies. It means that each post-Newtonian potential $q, p, p_\alpha, p^{\mathrm{T}}_{\alpha\beta}$ is split in the algebraic sum of N pieces

$$q = \sum_{A=1}^{N} q_A, \qquad p = \sum_{A=1}^{N} p_A, \qquad p_\alpha = \sum_{A=1}^{N} p_{A\alpha}, \qquad p^{\mathrm{T}}_{\alpha\beta} = \sum_{A=1}^{N} p^{\mathrm{T}}_{A\alpha\beta}, \tag{5.7.31}$$

where each function with sub-index A has the same form as one of the corresponding equations (5.7.29a–5.7.29d) with the integration performed over the volume, $\mathscr{V}_A$, of the body A. This confirms the principle of superposition in the linearised Friedmann approximation.

The gauge functions

The residual gauge freedom describes arbitrariness in adding a solution of homogeneous wave equations (5.7.29a–5.7.29d). It is described by two functions, $\zeta \equiv \xi/a$ and ζ^α as discussed in Section 5.6.4. Since we neglected the terms being quadratic with respect to the Hubble parameter, equations (5.6.44a), (5.6.44b) gets simpler, and read

$$\Box\zeta - 2\mathscr{H}\bar{v}^\beta\zeta_{,\beta} = 0, \tag{5.7.32a}$$

$$\Box\zeta^\alpha - 2\mathscr{H}\bar{v}^\beta\zeta^\alpha{}_{;\beta} = 0. \tag{5.7.32b}$$

They are equivalent to the homogeneous wave equations in the conformal flat spacetime

$$\Box(a\zeta) = 0, \qquad \Box(a\zeta^\alpha) = 0, \tag{5.7.33}$$

which point out that (in the approximation under consideration) the products, $a\zeta$ and $a\zeta^\alpha$, are the harmonic functions.

Potentials $q, p, p_\alpha, p^{\mathrm{T}}_{\alpha\beta}$ must satisfy the gauge conditions (5.6.43a), (5.6.43b). Neglecting terms being quadratic with respect to the Hubble parameter, the gauge conditions (5.6.43a), (5.6.43b) can be written down as follows

$$(ap^\alpha)_{,\alpha} + \bar{v}^\alpha(2aq - ap)_{,\alpha} + \mathscr{H}ap = 0, \tag{5.7.34a}$$

$$\left(ap^{\mathrm{T}\alpha\beta}\right)_{,\beta} + \bar{v}^\beta(ap^\alpha)_{,\beta} + \frac{1}{3}\bar{\pi}^{\alpha\beta}(ap)_{,\beta} + \mathscr{H}ap^\alpha = 0, \tag{5.7.34b}$$

where we have taken into account $a_{,\alpha} = -a\mathscr{H}\bar{v}_\alpha$, and $p^\alpha\bar{v}_\alpha = 0$, $p^{\mathrm{T}\alpha\beta}\bar{v}_\beta = 0$. The potentials p^α and $p^{\mathrm{T}\alpha\beta}$ are obtained from p_α and $p^{\mathrm{T}}_{\alpha\beta}$ by rising the indices with the Minkowski metric and taking into account that the indices of τ_α and $\tau^{\mathrm{T}}_{\alpha\beta}$ in the integrands of (5.7.29c) and (5.7.29d) should be raised with the full background metric $\bar{g}^{\alpha\beta} = a^{-2}\eta^{\alpha\beta}$ taken at the point of integration. This is because by convention having been adopted in Section 5.3.8, the notations $\tau^\alpha \equiv \bar{g}^{\alpha\beta}\tau_\beta$ and $\tau^{\alpha\beta} \equiv \bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\tau_{\mu\nu}$. It yields

$$p^{\alpha}(t,\boldsymbol{x}) = -\frac{4}{a\left[\eta(t,\boldsymbol{x})\right]}\int_{\mathscr{V}} \frac{a^4\left[\eta(s,\boldsymbol{x}')\right]\tau^{\alpha}\left(s,\boldsymbol{x}'\right)d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{5.7.35a}$$

$$p^{\mathrm{T}\alpha\beta}(t,\boldsymbol{x}) = -\frac{4}{a\left[\eta(t,\boldsymbol{x})\right]}\int_{\mathscr{V}} \frac{a^5\left[\eta(s,\boldsymbol{x}')\right]\tau^{\mathrm{T}\alpha\beta}\left(s,\boldsymbol{x}'\right)d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{5.7.35b}$$

It is instrumental to write down solutions for the products of the potentials p and $p^{\alpha} = \eta^{\alpha\beta}p_{\beta}$ with the Hubble parameter. Multiplying both sides of equations (5.7.26b), (5.7.26c) with the Hubble parameter $\mathscr{H}$, and neglecting the quadratic with respect to $\mathscr{H}$ terms, we obtain

$$\square(a\mathscr{H}p) = 16\pi a^3\mathscr{H}\tau, \qquad \square(a\mathscr{H}p^{\alpha}) = 16\pi a^4\mathscr{H}\tau^{\alpha}, \tag{5.7.36}$$

which solutions are the retarded potential

$$a\mathscr{H}p(t,\boldsymbol{x}) = -4\int_{\mathscr{V}} \frac{a^3\left[\eta(s,\boldsymbol{x}')\right]\mathscr{H}\left[\eta(s,\boldsymbol{x}')\right]\tau\left(s,\boldsymbol{x}'\right)d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{5.7.37a}$$

$$a\mathscr{H}p^{\alpha}(t,\boldsymbol{x}) = -4\int_{\mathscr{V}} \frac{a^4\left[\eta(s,\boldsymbol{x}')\right]\mathscr{H}\left[\eta(s,\boldsymbol{x}')\right]\tau^{\alpha}\left(s,\boldsymbol{x}'\right)d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{5.7.37b}$$

Substituting functions $q, p, p^{\alpha}, p^{\mathrm{T}\alpha\beta}$ and $a\mathscr{H}p$, $a\mathscr{H}p^{\alpha}$ to the gauge equations (5.7.34a), (5.7.34b), bring about the following integral equations

$$\int_{\mathscr{V}}\left[\left(a^4\tau^{\alpha} + \bar{\mathrm{v}}^{\alpha}a^3\sigma\right)_{,\alpha} + a^3\mathscr{H}\tau\right]\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} = 0, \tag{5.7.38a}$$

$$\int_{\mathscr{V}}\left[\left(a^5\tau^{\mathrm{T}\alpha\beta} + a^4\bar{\mathrm{v}}^{\beta}\tau^{\alpha} + \frac{1}{3}\bar{\pi}^{\alpha\beta}a^3\tau\right)_{,\beta} + a^4\mathscr{H}\tau^{\alpha}\right]\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} = 0, \tag{5.7.38b}$$

where all functions in the integrands are taken at the retarded time s and at the point $\boldsymbol{x}'$, for example, $a \equiv a[\eta(s,\boldsymbol{x}')]$, $\mathscr{H} \equiv \mathscr{H}[\eta(s,\boldsymbol{x}')]$, $\sigma \equiv \sigma[(s,\boldsymbol{x}')]$, and so on. These equations are satisfied by the equations of motion (5.3.71a), (5.3.71b) of the localized matter distribution. Indeed, divergences of any vector F^{α} and a symmetric tensor $F^{\alpha\beta}$ obey the following equalities

$$F^{\alpha}{}_{|\alpha} = \frac{1}{\sqrt{-\bar{g}}}\left(\sqrt{-\bar{g}}F^{\alpha}\right)_{,\alpha}, \tag{5.7.39}$$

$$F^{\alpha\beta}{}_{|\beta} = \frac{1}{\sqrt{-\bar{g}}}\left(\sqrt{-\bar{g}}F^{\alpha\beta}\right)_{,\beta} + \bar{\Gamma}^{\alpha}_{\beta\gamma}F^{\beta\gamma}. \tag{5.7.40}$$

Moreover, the root square of the determinant of the background metric tensor is expressed in terms of the scale factor, $\sqrt{-\bar{g}} = a^4$, while the four-velocity $\bar{u}^{\alpha} = \bar{\mathrm{v}}^{\alpha}/a$. Employing these expressions along with equations (5.7.39), (5.7.40) in equations of motion (5.3.71a), (5.3.71b), transform them to

$$\left(a^4\tau^\alpha + \bar{\nabla}^\alpha a^3\sigma\right)_{,\alpha} + a^3\mathscr{H}\tau = 0, \tag{5.7.41a}$$

$$\left(a^4\tau^{\alpha\beta} + a^3\bar{\nabla}^\beta\tau^\alpha\right)_{,\beta} + 2a^3\mathscr{H}\tau^\alpha = 0. \tag{5.7.41b}$$

Equation (5.7.41a) proves that the integral equation (5.7.38a) is valid. In order to prove the second integral equation (5.7.38b), we multiply equation (5.7.41b) with the scale factor a, and reshuffle its terms. It brings (5.7.41b) to the following form

$$\left(a^5\tau^{\alpha\beta} + a^4\bar{\nabla}^\beta\tau^\alpha\right)_{,\beta} + a^4\mathscr{H}\tau^\alpha = 0. \tag{5.7.42}$$

Substituting, $\tau^{\alpha\beta} = \tau^{\intercal\alpha\beta} + (1/3a^2)\bar{\pi}^{\alpha\beta}\tau$, to (5.7.42) and comparing with the integrand in (5.7.38b) makes it clear that (5.7.38b) is valid. We conclude that the retarded integrals (5.7.29a–5.7.29d) yield the complete solution of the linearized wave equations (5.7.26a–5.7.26d) in the sense that there is no residual gauge freedom since the gauge functions $\zeta = \zeta^\alpha = 0$.

Perturbations of dark matter and dark energy

What remains is to find out solutions for the scalar functions $V_{\rm m}$ and $V_{\rm q}$ and $\chi_{\rm m}$ and $\chi_{\rm q}$. In the linearised Friedmann approximation equation for $V_{\rm m}$ is obtained from (5.6.37) by discarding all terms of the order of $\mathscr{H}^2$. It yields

$$\Box V_{\rm m} + \left(1 - \frac{c^2}{v_{\rm s}^2}\right)\bar{\nabla}^\alpha\bar{\nabla}^\beta V_{{\rm m},\alpha\beta} + \left(3 - \frac{c^2}{v_{\rm s}^2}\right)\mathscr{H}\bar{\nabla}^\alpha V_{{\rm m},\alpha} = -4\pi a^2(\sigma + \tau). \tag{5.7.43}$$

Applying relations (5.6.34), (5.6.35) in equation (5.7.43) allows us to recast it to

$$\frac{1}{a^n}\left[\Box(a^n V_{\rm m}) + \left(1 - \frac{c^2}{v_{\rm s}^2}\right)\bar{\nabla}^\alpha\bar{\nabla}^\beta(a^n V_{\rm m})_{,\alpha\beta}\right] \tag{5.7.44}$$
$$+\left[3 + (2n-1)\frac{c^2}{v_{\rm s}^2}\right]\mathscr{H}\bar{\nabla}^\alpha V_{{\rm m},\alpha} = -4\pi a^2(\sigma + \tau),$$

where n is yet undetermined real number. Now, we postulate that the speed of sound $v_{\rm s}$ is constant. Then, choosing, $n \equiv n_{\rm s}$, with

$$n_{\rm s} = \frac{1}{2}\left(1 - 3\frac{v_{\rm s}^2}{c^2}\right), \tag{5.7.45}$$

annihilates the term being proportional to $\mathscr{H}$ in the left side of (5.7.44) and reduces it to

$$\Box(a^{n_{\rm s}} V_{\rm m}) + \left(1 - \frac{c^2}{v_{\rm s}^2}\right)\bar{\nabla}^\alpha\bar{\nabla}^\beta(a^{n_{\rm s}} V_{\rm m})_{,\alpha\beta} = -4\pi a^{2+n_{\rm s}}(\sigma + \tau). \tag{5.7.46}$$

This equation describes propagation of perturbation $V_{\rm m}$ with the speed of sound $v_{\rm s}$. Indeed, let us introduce the sound-wave Laplace-Beltrami operator (5.7.17). Then, equation (5.7.46) reads

$$\Box_{\rm s}\left(a^{n_{\rm s}}V_{\rm m}\right) = -4\pi a^{2+n_{\rm s}}\left(\sigma+\tau\right). \qquad (5.7.47)$$

This equation has a well-defined Green function with characteristics propagating with the speed of sound $v_{\rm s}$. We discard the advanced Green function because we assume that at infinity the function $V_{\rm m}$ and its first derivatives vanish. Solution of (5.7.47) is explained below in Appendix B.2, and has the following form

$$V_{\rm m}(t,\boldsymbol{x}) = \frac{1}{a^{n_{\rm s}}(t,\boldsymbol{x})}\int_{\mathscr{V}} \frac{a^{2+n_{\rm s}}\left(\varsigma,\boldsymbol{x}'\right)\left[\sigma(\varsigma,\boldsymbol{x}')+\tau(\varsigma,\boldsymbol{x}')\right]}{\sqrt{1+\gamma^2\left(1-\dfrac{c^2}{v_{\rm s}^2}\right)(\boldsymbol{\beta}\times\boldsymbol{n})^2}}\frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \qquad (5.7.48)$$

where the retarded time ς is given by equation (B.2.18), $\boldsymbol{\beta} = \beta^i = \bar{\rm v}^i/c$, $\gamma = 1/\sqrt{1-\boldsymbol{\beta}^2}$ is the Lorentz factor, and the unit vector $\boldsymbol{n} = (\boldsymbol{x}-\boldsymbol{x}')/|\boldsymbol{x}-\boldsymbol{x}'|$. The retardation in the solution (5.7.48) is due to the finite speed of propagation of acoustic (sound) waves in the ideal fluid that represents the dark matter.

Equation for $V_{\rm q}$ is obtained in the lineazised Friedmann approximation from (5.6.38) after discarding all terms being proportional to $\mathscr{H}^2$. It yields

$$\Box V_{\rm q} + 2\left(1-\sqrt{\frac{3}{2{\rm x}_1}}\lambda{\rm x}_2\right)\mathscr{H}\bar{\rm v}^{\mu}V_{{\rm q},\mu} = -4\pi a^2\left(\sigma+\tau\right). \qquad (5.7.49)$$

Applying relation (5.6.34) in (5.7.49) allows us to recast it to

$$\frac{1}{a^n}\Box\left(a^nV_{\rm q}\right) + 2\left(n+1-\sqrt{\frac{3}{2{\rm x}_1}}\lambda{\rm x}_2\right)\mathscr{H}\bar{\rm v}^{\alpha}V_{{\rm m},\alpha} = -4\pi a^2\left(\sigma+\tau\right). \qquad (5.7.50)$$

If, and only if, the ratio $\lambda{\rm x}_2/\sqrt{{\rm x}_1}$ is constant, we can choose, $n \equiv n_{\rm q} = -1+\sqrt{3/(2{\rm x}_1)}\lambda{\rm x}_2$, in order to eradicate the second term in the left side of (5.7.50). In those models of the universe where this condition is satisfied, the resulting equation for $V_{\rm q}$ is simplified and reads

$$\Box\left(a^{n_{\rm q}}V_{\rm q}\right) = -4\pi a^{2+n_{\rm q}}\left(\sigma+\tau\right). \qquad (5.7.51)$$

This is the wave equation in flat spacetime. We pick up the retarded solution as the most physical one,

$$V_{\rm q} = \frac{1}{a^{n_{\rm q}}(t,\boldsymbol{x})}\int_{\mathscr{V}}\frac{a^{2+n_{\rm q}}\left(s,\boldsymbol{x}'\right)\left[\sigma\left(s,\boldsymbol{x}'\right)+\tau\left(s,\boldsymbol{x}'\right)\right]d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}, \qquad (5.7.52)$$

where the retarded time s has been defined in (5.7.30).

Perturbations $\chi_{\rm m}$ and $\chi_{\rm q}$ can be found by integrating equations (5.5.14a) and (5.5.14b) that can be written as

$$\bar{v}^{\alpha}\chi_{{\rm m},\alpha} = a\left(V_{\rm m} + \frac{q}{2}\right), \qquad \bar{v}^{\alpha}\chi_{{\rm q},\alpha} = a\left(V_{\rm q} + \frac{q}{2}\right). \tag{5.7.53}$$

These are the ordinary differential equations of the first order. Their solutions are

$$\chi_{\rm m} = \int_{t_0}^{t} a[t, \boldsymbol{x}(t)]\{V_{\rm m}[t, \boldsymbol{x}(t)] + \frac{1}{2}q[t, \boldsymbol{x}(t)]\}dt, \tag{5.7.54a}$$

$$\chi_{\rm q} = \int_{t_0}^{t} a[t, \boldsymbol{x}(t)]\{V_{\rm q}[t, \boldsymbol{x}(t)] + \frac{1}{2}q[t, \boldsymbol{x}(t)]\}dt, . \tag{5.7.54b}$$

where t_0 is an initial epoch of integration, and the integration is performed along the Hubble flow of the background universe

$$\frac{dx^i}{dt} = \bar{v}^i(t, \boldsymbol{x}). \tag{5.7.55}$$

Therefore, the most simple way to integrate equations (5.7.53) would be to work in the preferred coordinate frame $X^\alpha = (\eta, X^i)$ where the velocity $\bar{v}^i = 0$, and the spatial coordinates $X^i = $ const. After the calculation in the rest frame of the Hubble flow is finished, the transformation to a moving frame of observer can be done with the help of the coordinate transformation between the two frames.

6 Currents and superpotentials on arbitrary backgrounds: Three approaches

Recall that classical pseudotensors and superpotentials help us to define the integral properties of physical systems in terms of the surface integration, see Section 1.4. To make such an integration meaningful, an auxiliary *flat* background metric is introduced; besides, calculations are performed with making use of the Killing vectors of the flat space, see (1.4.75–1.4.79). In the framework of the field-theoretical formulation of general relativity, Chapter 2, the curved background manifold and its Killing vectors are used for the surface integration in the definitions of the integral quantities. Such a form of integral conservation law follows directly from the differential form of conservation law, where the conserved current is expressed through a divergence of a superpotential. Let us give two examples.

First, for the study of perturbations on FLRW backgrounds the so-called *integral constraint* is introduced. It represents a powerful form of an integral conservation law that connects the volume integration over the matter perturbations only, with the surface integral of the metric perturbations. It turns out to be very important for resolving some problems in cosmology [441], e. g., to analyze the measurable effects of the cosmic microwave background radiation [442]. In the definition of the integral constraints *the integral constraint vectors*, not necessarily the Killing vectors, play a crucial role.

Second, in [451], a new conserved energy-momentum pseudotensor was found and used in an effort to integrate Einstein's equations with scalar perturbations and topological defects on FLRW backgrounds. In [446], it was realized that these conservation laws are associated with the *conformal* Killing vector of time translation, but not with the ordinary Killing vectors.

Thus, it is desirable that mathematical formalism for describing perturbations and conservation laws was constructed in the most general way including the possibility to use arbitrary displacement vectors on arbitrary curved backgrounds. Keeping in mind the above arguments and (1.4.75–1.4.79), the requirements for constructing such generalized conservation laws are formulated as follows:

(i) Mathematical exprsssions have to be covariant on a chosen curved background manifold with the metric which is a solution of the field equations of general relativity.
(ii) Conservation laws have to be based on the Lagrangian of the perturbed system.
(iii) The conserved currents must be vector densities, $\mathcal{I}^{\mu}(\xi)$, which are differentially conserved, $\partial_{\mu}\mathcal{I}^{\mu}(\xi) = 0$, on the equations of motion for perturbations.
(iv) The currents, $\mathcal{I}^{\mu}(\xi)$, have to be expressed through corresponding superpotentials that are antisymmetric tensor densities, $\mathcal{I}^{\mu\nu}$, in the form of a divergence $\mathcal{I}^{\mu}(\xi) = \partial_{\nu}\mathcal{I}^{\mu\nu}(\xi)$, where $\partial_{\mu\nu}\mathcal{I}^{\mu\nu}(\xi) \equiv 0$.

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(v) There has to be a possibility to use arbitrary displacement vectors, ξ^μ, not just the Killing vectors of the background.
(vi) Applications of the suggested conserved quantities and conservation laws have to satisfy the known results obtained for simple physical systems.

Below, in Sections 6.1–6.3, we present three different methods (canonical, Belinfante correcting and field-theoretical) of constructing such conservation laws and conserved quantities satisfying the above requirements (i–vi). After constructing the set of conserved quantities, one can ask: what quantity from the set is more preferable? Section 6.4 answers this question and proposes a number of criteria for making such a choice.

In the final two sections, we demonstrate how conserved currents and superpotentials can be used to study perturbations in the FLRW universe. It is well known that the FLRW geometry has only 6 Killing vectors (but not 10 that the maximally symmetric spaces only have), among which there is no timelike Killing vector. Thus, one has no possibility to construct *directly* the energy density and energy integral for perturbations on the FLRW backgrounds. In Section 6.5, by using the KBL method, we present a feasible approach to solving this problem.

In Section 6.6, using the Belinfante corrected quantities we construct the integral relations for perturbations in the FLRW universe. We describe how matter perturbations inside a restricted domain can be connected with the metric perturbations at its boundary.

In the present chapter many notations, which are used only in this chapter, are introduced. We hope that no confusion will arise. Again, we use the units with $G = c = 1$.

6.1 The Katz, Bičák and Lynden-Bell conservation laws

6.1.1 A bi-metric KBL Lagrangian

The first of these methods has been elaborated by Katz, Bičák and Lynden-Bell in the paper [251] (later we call as KBL). They, by using the standard Noether's canonical procedure, presented conservation laws for arbitrary (not only infinitesimal) perturbations.

The KBL strategy is bi-metric, it is based on using both dynamical metric and a background metric. Then, comparing them, one defines and describes perturbations with respect to the background solution. In Section 1.2.3, we provided main definitions and properties of diffeomorphisms and mapping a spacetime onto itself. The technique of a bi-metric derivation requires to recall and slightly reformulate some of the main notions of mapping a dynamical spacetime onto a background spacetime.

Let $g_{\mu\nu}(x)$ be the metric of a dynamical spacetime $\mathscr{M}^4$, and let $\bar{g}_{\mu\nu}(\bar{x})$ be the metric of a background spacetime $\bar{\mathscr{M}}^4$. Both are tensors with respect to arbitrary coordinate transformations. Once we have chosen a mapping so that points $\{p\}$ of $\mathscr{M}^4$ map into points $\{\bar{p}\}$ of $\bar{\mathscr{M}}^4$, then we can use the convention that $\{p\}$ and $\{\bar{p}\}$ shall always be represented by the same coordinates $x^\alpha = \bar{x}^\alpha$. This convention implies that a coordinate transformation on $\mathscr{M}^4$ inevitably induces a coordinate transformation with the same functions on $\bar{\mathscr{M}}^4$. With this convention the expressions, like $g_{\mu\nu}(x) - \bar{g}_{\mu\nu}(x)$, which are perturbations, become true tensors. If the particular coordinate presentation of the mapping is changed it must not violate the tensorial nature of the equations associated with the mapping. This property of the tensorial equation to preserve their form is known as the gauge invariance, and the freedom in choosing the mapping is called the gauge freedom. This freedom has been examined in detail in Section 2.2.4 in the framework of the field-theoretical approach.

The main KBL idea is to construct a relative bi-metric Lagrangian in a generalized form:

$$\mathscr{L}_{KBL} = \mathscr{L}^G - \bar{\mathscr{L}}^G - \frac{1}{16\pi}\partial_\alpha \mathscr{D}^\alpha\,, \tag{6.1.1}$$

where $\mathscr{L}^G$ is a pure gravitational Lagrangian of an arbitrary metric theory; a divergence is chosen by various criteria. In the case of general relativity, KBL choose the Hilbert Lagrangian (1.3.1) as the gravitational one, thus (6.1.1) acquires the form:

$$\mathscr{L}_{KBL} = -\frac{1}{16\pi}\left(\mathscr{R} - \bar{\mathscr{R}} + \partial_\alpha \boldsymbol{k}^\alpha\right)\,. \tag{6.1.2}$$

Here, the physical scalar curvature density, $\mathscr{R}$, is constructed with the use of $g_{\mu\nu}$, which satisfies the dynamical Einstein equations while the background one, $\bar{\mathscr{R}}$, is constructed with the use of $\bar{g}_{\mu\nu}$, which satisfies the background Einstein equations and corresponds to an arbitrary curved spacetime. With the use of (2.2.17) and (2.2.18) one obtains for the difference between the curvature scalar densities in (6.1.2):

$$\begin{aligned}\mathscr{R} - \bar{\mathscr{R}} &= \mathfrak{g}^{\tau\sigma}R_{\tau\sigma} - \bar{\mathfrak{g}}^{\tau\sigma}\bar{R}_{\tau\sigma}\\ &= \mathfrak{g}^{\tau\sigma}\left(\bar{\nabla}_\rho\Delta^\rho{}_{\tau\sigma} - \bar{\nabla}_\sigma\Delta^\rho{}_{\tau\rho} + \Delta^\rho{}_{\rho\eta}\Delta^\eta{}_{\tau\sigma} - \Delta^\rho{}_{\sigma\eta}\Delta^\eta{}_{\tau\rho}\right)\\ &\quad + \left(\mathfrak{g}^{\tau\sigma} - \bar{\mathfrak{g}}^{\tau\sigma}\right)\bar{R}_{\tau\sigma}\,.\end{aligned} \tag{6.1.3}$$

Here, as usual, $\Delta^\rho{}_{\tau\sigma} = \Gamma^\rho{}_{\tau\sigma} - \bar{\Gamma}^\rho{}_{\tau\sigma}$, but now the components $\Delta^\rho{}_{\tau\sigma}$ are *not* independent variables. KBL choose the divergence in (6.1.2) with the vector density (2.2.19) that first has been introduced by Katz [250]. It reads

$$\boldsymbol{k}^\alpha \equiv \mathfrak{g}^{\alpha\nu}\Delta^\mu{}_{\mu\nu} - \mathfrak{g}^{\mu\nu}\Delta^\alpha{}_{\mu\nu}\,. \tag{6.1.4}$$

Of course, $\bar{\boldsymbol{k}}^\alpha = 0$. Using $\partial_\alpha\boldsymbol{k}^\alpha = \bar{\nabla}_\alpha\boldsymbol{k}^\alpha$, one obtains the KBL Lagrangian (6.1.2) in the final form:

$$\mathscr{L}_{KBL} = -\frac{1}{16\pi}\mathfrak{g}^{\tau\sigma}\left(\Delta^\rho{}_{\rho\eta}\Delta^\eta{}_{\tau\sigma} - \Delta^\rho{}_{\sigma\eta}\Delta^\eta{}_{\tau\rho}\right) - \frac{1}{16\pi}\mathfrak{h}^{\tau\sigma}\bar{R}_{\tau\sigma}\,. \tag{6.1.5}$$

Here, as before, $\mathfrak{h}^{\tau\sigma} = \mathfrak{g}^{\tau\sigma} - \bar{\mathfrak{g}}^{\tau\sigma}$, but now the components $\mathfrak{h}^{\tau\sigma}$ are *not* considered as independent variables. One can see that (6.1.5) does not depend on the second derivatives of the metric. Therefore, the corresponding energy-momentum will not depend on the second derivatives either. This simplifies the Cauchy problem, when the action with the Lagrangian (6.1.5) is varied under the Dirichlet boundary condition which is imposed only on the components of the metric but not on its first derivatives.

In the case of a flat background in the Lorentzian coordinates, when $\Delta^{\rho}{}_{\sigma\eta} = \Gamma^{\rho}{}_{\sigma\eta} - C^{\rho}{}_{\sigma\eta} = \Gamma^{\rho}{}_{\sigma\eta}$, the Lagrangian (6.1.5) transfers to the Einstein Lagrangian (1.3.9). Also for the case $g_{\mu\nu} = \bar{g}_{\mu\nu}$, when a dynamical (perturbed) system coincides with the background one, the Lagrangian (6.1.5) vanishes.

6.1.2 KBL conserved quantities

Noether's canonical procedure

Now, let us analyze the Lagrangian (6.1.5). Because it is a scalar density, it has to satisfy the main Noether's identity related to diffeomorphisms. To provide a pure canonical procedure we take the original form (1.2.46) of this identity with the right side omitted

$$
£_{\xi}\mathscr{L}_{KBL} + \partial_{\mu}\left(\xi^{\mu}\mathscr{L}_{KBL}\right) \equiv 0\,. \tag{6.1.6}
$$

In the case of a generalized theory of fields ψ^A for arbitrary displacement vectors ξ^{μ}, the same identity has been studied from (1.4.3) to (1.4.13) in Section 1.4. Therefore we rewrite the identity (1.4.13), setting, $\psi^A = \{g_{\mu\nu}, \bar{g}_{\mu\nu}\}$,

$$
\partial_{\mu}\boldsymbol{j}^{\mu}_{C} \equiv \bar{\nabla}_{\mu}\boldsymbol{j}^{\mu}_{C} \equiv 0\,, \tag{6.1.7}
$$

where the current in (6.1.7) has the form of (1.4.14):

$$
\begin{aligned}
\boldsymbol{j}^{\mu}_{C} \equiv & \left[-\frac{\delta\mathscr{L}_{KBL}}{\delta g_{\rho\sigma}}\, g_{\rho\sigma}\Big|^{\mu}_{\nu} - \frac{\delta\mathscr{L}_{KBL}}{\delta\bar{g}_{\rho\sigma}}\, \bar{g}_{\rho\sigma}\Big|^{\mu}_{\nu} + \frac{\partial\mathscr{L}_{KBL}}{\partial\left(\bar{\nabla}_{\mu}g_{\rho\sigma}\right)}\bar{\nabla}_{\nu}g_{\rho\sigma} - \mathscr{L}_{KBL}\delta^{\mu}_{\nu}\right]\xi^{\nu} \\
& - \frac{\partial\mathscr{L}_{KBL}}{\partial\left(\bar{\nabla}_{\mu}g_{\rho\sigma}\right)}\, g_{\rho\sigma}\Big|^{\rho}_{\lambda}\, \bar{g}^{\lambda\sigma}\bar{\nabla}_{\rho}\xi_{\sigma} + \boldsymbol{z}^{\mu}\,.
\end{aligned} \tag{6.1.8}
$$

Notice that, here, the covariant derivatives have been introduced by applying the technique of Appendix A.3.3, the very last term is explained in (6.1.13), and indices are lowered and raised by $\bar{g}_{\mu\nu}$ and $\bar{g}^{\mu\nu}$. We explain the meaning of each term entering (6.1.8) in the text that follows.

To calculate the first two terms in (6.1.8) it is more constructive to use the (6.1.2) representation for the KBL Lagrangian. One easily obtains that they are expressed through dynamical, $\mathscr{G}_{\nu}{}^{\mu}$, and background, $\bar{\mathscr{G}}_{\nu}{}^{\mu}$, densities of the Einstein tensor:

$$
-\frac{\delta\mathscr{L}_{KBL}}{\delta g_{\rho\sigma}}\, g_{\rho\sigma}\Big|^{\mu}_{\nu} \equiv \frac{1}{8\pi}\mathscr{G}_{\nu}{}^{\mu}\,, \qquad -\frac{\delta\mathscr{L}_{KBL}}{\delta\bar{g}_{\rho\sigma}}\, \bar{g}_{\rho\sigma}\Big|^{\mu}_{\nu} \equiv -\frac{1}{8\pi}\bar{\mathscr{G}}_{\nu}{}^{\mu}\,. \tag{6.1.9}
$$

The other terms in square brackets in (6.1.8) have the form of a canonical energy-momentum (1.2.102) presented in a covariant form. Now, for calculational purposes, it is more effective to use the (6.1.5) representation for the KBL Lagrangian. As a result, one has

$$\frac{\partial \mathscr{L}_{KBL}}{\partial\left(\bar{\nabla}_\mu g_{\rho\sigma}\right)}\bar{\nabla}_\nu g_{\rho\sigma} - \mathscr{L}_{KBL}\delta^\mu_\nu \equiv \boldsymbol{t}_\nu{}^\mu + \frac{1}{16\pi}\mathfrak{h}^{\rho\sigma}\bar{R}_{\rho\sigma}\delta^\mu_\nu\,, \tag{6.1.10}$$

where

$$\begin{aligned}16\pi\boldsymbol{t}_\nu{}^\mu \equiv\ & \mathfrak{g}^{\rho\sigma}\left[\Delta^\lambda{}_{\rho\lambda}\Delta^\mu{}_{\sigma\nu} + \Delta^\mu{}_{\rho\sigma}\Delta^\lambda{}_{\lambda\nu} - 2\Delta^\mu{}_{\rho\lambda}\Delta^\lambda{}_{\sigma\nu}\right.\\ & \left. - \left(\Delta^\eta{}_{\rho\sigma}\Delta^\lambda{}_{\eta\lambda} - \Delta^\eta{}_{\rho\lambda}\Delta^\lambda{}_{\eta\sigma}\right)\delta^\mu_\nu\right]\\ & + \mathfrak{g}^{\mu\lambda}\left(\Delta^\sigma{}_{\rho\sigma}\Delta^\rho{}_{\lambda\nu} - \Delta^\sigma{}_{\lambda\sigma}\Delta^\rho{}_{\rho\nu}\right)\,,\end{aligned} \tag{6.1.11}$$

depends on the first derivatives $\bar{\nabla}_\mu g_{\rho\sigma}$, only, and is interpreted as the canonical energy-momentum tensor density for gravitational field. The last term in (6.1.10) reflects the interaction of the perturbed system with the curved background.

The important term in (6.1.8) is the spin tensor density:

$$\begin{aligned}16\pi\boldsymbol{\sigma}^{\mu\rho\sigma} &\equiv -16\pi\frac{\partial \mathscr{L}_{KBL}}{\partial\left(\bar{\nabla}_\mu g_{\rho\sigma}\right)}\, g_{\rho\sigma}\Big|^\rho_\lambda\, \bar{g}^{\lambda\sigma} \\ &= \left(2\mathfrak{g}^{\rho(\mu}\bar{g}^{\nu)\sigma} - \mathfrak{g}^{\mu\nu}\bar{g}^{\rho\sigma}\right)\Delta^\lambda{}_{\nu\lambda} - \left(2\mathfrak{g}^{\rho(\nu}\bar{g}^{\lambda)\sigma} - \mathfrak{g}^{\nu\lambda}\bar{g}^{\rho\sigma}\right)\Delta^\mu{}_{\nu\lambda}\,.\end{aligned} \tag{6.1.12}$$

After simplification to the Minkowski background in the Lorentzian coordinates one obtains from (6.1.12) the quantity presented by Papapetrou [351] to construct the angular momentum with the use of Einstein's pseudotensor. The last term in (6.1.8) has the form:

$$\boldsymbol{z}^\mu(\xi) = \frac{1}{16\pi}\left[\mathfrak{h}^{\mu\lambda}\partial_\lambda\zeta_\rho{}^\rho + \mathfrak{h}^{\rho\sigma}\left(\bar{\nabla}^\mu\zeta_{\rho\sigma} - 2\bar{\nabla}_\rho\zeta_\sigma{}^\mu\right)\right], \tag{6.1.13}$$

with the notation:

$$\zeta_{\rho\sigma} \equiv -\tfrac{1}{2}\pounds_\xi\bar{g}_{\rho\sigma} = \bar{\nabla}_{(\rho}\xi_{\sigma)}\,. \tag{6.1.14}$$

By definition, this quantity vanishes if ξ^σ is a Killing vector of the background spacetime.

Thus, finally, (6.1.8) can be rewritten in a short form:

$$\boldsymbol{j}^\mu_C(\xi) \equiv {}_C\boldsymbol{\tau}_\nu{}^\mu\xi^\nu + \boldsymbol{\sigma}^{\mu\rho\sigma}\bar{\nabla}_\rho\xi_\sigma + \boldsymbol{z}^\mu\,, \tag{6.1.15}$$

where

$${}_C\boldsymbol{\tau}_\nu{}^\mu \equiv \frac{1}{8\pi}\left(\mathscr{G}_\nu{}^\mu - \bar{\mathscr{G}}_\nu{}^\mu\right) + \boldsymbol{t}_\nu{}^\mu + \frac{1}{16\pi}\mathfrak{h}^{\rho\sigma}\bar{R}_{\rho\sigma}\delta^\mu_\nu \tag{6.1.16}$$

is a coefficient in front of ξ^ν, which can be interpreted as a generalized energy-momentum related to the KBL Lagrangian (6.1.2).

KBL superpotential

Because the equation (6.1.7) is the identity, the current, $\boldsymbol{j}^\mu_C(\xi)$, in (6.1.15) can be represented through a superpotential, $\boldsymbol{j}^{\mu\nu}_C(\xi)$ - an antisymmetric tensor density, in the form:

$$\boldsymbol{j}^\mu_C(\xi) \equiv \partial_\nu \boldsymbol{j}^{\mu\nu}_C(\xi) \equiv \bar{\nabla}_\nu \boldsymbol{j}^{\mu\nu}_C(\xi)\,. \tag{6.1.17}$$

How can one show this? The scheme is analogous to that used previously in Section 1.4. One has to construct the system of the Klein identities, like (1.4.8–1.4.11), with the use of which the identity (6.1.17), like (1.4.19), can be constructed. One can see that (6.1.7) contains derivatives of ξ^ν up to the third order. Then, since ξ^ν, $\partial_\alpha\xi^\nu$, $\partial_{\beta\alpha}\xi^\nu$ and $\partial_{\gamma\beta\alpha}\xi^\nu$ are arbitrary at every world point, one has to set the coefficients independently to zero and obtain the system of the Klein identities related to the KBL Lagrangian. Because this procedure must not depend on the choice of coordinates we can get a covariant form of the Klein identities by writing down the identity (6.1.7) in terms of the covariant derivatives

$$\partial_\mu \boldsymbol{j}^\mu_C \equiv \mathcal{O}_\nu\xi^\nu + \mathcal{O}_\nu{}^\mu\bar{\nabla}_\mu\xi^\nu + \mathcal{O}_\nu{}^{\rho\sigma}\bar{\nabla}_{(\rho\sigma)}\xi^\nu + \mathcal{O}_\nu{}^{\mu\rho\sigma}\bar{\nabla}_{(\mu\rho\sigma)}\xi^\nu \equiv 0 \tag{6.1.18}$$

and equate the coefficients at ξ^ν, $\bar{\nabla}_\mu\xi^\nu$, $\bar{\nabla}_{(\rho\sigma)}\xi^\nu$ and $\bar{\nabla}_{(\mu\rho\sigma)}\xi^\nu$ to zero. The resulting covariant system is equivalent to the initial non-covariant system because the set of the covariant identities is a non-degenerate linear combination of the identities of the initial system.

Thus, from (6.1.18) one has

$$\mathcal{O}_\nu = \bar{\nabla}_\mu\left({}_C\boldsymbol{\tau}_\nu{}^\mu\right) + \tfrac{1}{2}\boldsymbol{\sigma}^{\rho\sigma\mu}\bar{R}_{\mu\nu\rho\sigma} \tag{6.1.19}$$
$$+\frac{1}{16\pi}\left(\bar{R}^\mu{}_{\rho\sigma\nu}\bar{\nabla}_\mu\mathfrak{h}^{\rho\sigma} - \mathfrak{h}^{\rho\sigma}\bar{\nabla}_\nu\bar{R}_{\rho\sigma} - \tfrac{1}{2}\bar{R}_{\sigma\nu}\bar{\nabla}_\rho\mathfrak{h}^{\rho\sigma}\right) \equiv 0\,,$$

$$\mathcal{O}_\nu{}^\mu = {}_C\boldsymbol{\tau}_\nu{}^\mu + \bar{\nabla}_\rho\boldsymbol{\sigma}^{\rho\mu}{}_\nu - \frac{1}{8\pi}\mathfrak{h}^{\mu\rho}\bar{R}_{\nu\rho} \equiv 0\,, \tag{6.1.20}$$

$$\mathcal{O}_\nu{}^{(\rho\sigma)} = \boldsymbol{\sigma}^{(\rho\sigma)}{}_\nu + \bar{\nabla}_\mu\mathcal{O}_\nu{}^{\mu\rho\sigma} \equiv 0\,, \tag{6.1.21}$$

$$\mathcal{O}_\nu{}^{(\mu\rho\sigma)} \equiv 0\,, \tag{6.1.22}$$

where the quantity $\mathcal{O}_\nu{}^{\mu\rho\sigma}$ in (6.1.21) and (6.1.22) is

$$\mathcal{O}_\nu{}^{\mu\rho\sigma} = \frac{1}{32\pi}\left(\mathfrak{h}^{\mu\rho}\delta^\sigma_\nu + \mathfrak{h}^{\mu\sigma}\delta^\rho_\nu - 2\mathfrak{h}^{\rho\sigma}\delta^\mu_\nu\right)\,. \tag{6.1.23}$$

One easily recognizes that for such a quantity the last identity (6.1.22), indeed, holds. With using definitions (6.1.11–6.1.16) one can be convinced that the other identities (6.1.19–6.1.21) hold as well.

Using the Klein identities (6.1.19–6.1.22) in the expression for the current (6.1.15), one obtains the wanted identity (6.1.17), where the superpotential at the right hand side acquires the form:

$$\boldsymbol{j}_C^{\mu\nu}(\xi) = \frac{1}{8\pi}\left(\sqrt{-g}\nabla^{[\mu}\xi^{\nu]} - \sqrt{-\bar{g}}\bar{\nabla}^{[\mu}\xi^{\nu]} + \xi^{[\mu}\boldsymbol{k}^{\nu]}\right). \tag{6.1.24}$$

One finds that it is a relative (to a background system) Komar superpotential (1.4.80) added by a contribution from the divergence in (6.1.2). Also the expression (6.1.24) can be rewritten in the other form:

$$\boldsymbol{j}_C^{\mu\nu}(\xi) = \frac{1}{8\pi}\left(\mathfrak{h}^{\rho[\mu}\bar{\nabla}_\rho\xi^{\nu]} + \mathfrak{g}^{\rho[\mu}\Delta^{\nu]}{}_{\rho\lambda}\xi^\lambda + \xi^{[\mu}\mathfrak{g}^{\nu]\rho}\Delta^\lambda{}_{\rho\lambda} - \xi^{[\mu}\Delta^{\nu]}{}_{\rho\sigma}\mathfrak{g}^{\rho\sigma}\right). \tag{6.1.25}$$

This expression easily transforms to the Freud superpotential (1.4.39) if (6.1.25) is derived for a flat background with the Lorentzian coordinates, and the displacement vector is represented by the coordinate translations $\xi^\mu = \delta^\mu_\sigma$.

KBL conservation laws

The identity (6.1.17) itself is the algebraic equality only. It has no physical sense because the field equations (Einstein's equations) have not been used at all. Let us take them in the form (1.3.22) along with their background version and substitute both of them into the current (6.1.15). Besides, recall that z-term in (6.1.15) depends on the quantity, $\bar{\nabla}_{(\rho}\xi_{\sigma)}$, in the form of (6.1.14) only. Therefore, we reconstruct the second term in (6.1.15) and add the quantity $\boldsymbol{\sigma}^{\mu\rho\sigma}\bar{\nabla}_{(\rho}\xi_{\sigma)}$ to z-term. Finally, the current (6.1.15) transforms to a physically sensible one:

$$\mathscr{I}_C^\mu \equiv {}_C\boldsymbol{\theta}_\nu{}^\mu\xi^\nu + \boldsymbol{\sigma}^{\mu\rho\sigma}\bar{\nabla}_{[\rho}\xi_{\sigma]} + \boldsymbol{z}_C^\mu, \tag{6.1.26}$$

where the generalized canonical energy-momentum tensor density and z-term are

$${}_C\boldsymbol{\theta}_\nu{}^\mu = \left(\mathscr{T}_\nu{}^\mu - \bar{\mathscr{T}}_\nu{}^\mu\right) + \boldsymbol{t}_\nu{}^\mu + \frac{1}{16\pi}\mathfrak{h}^{\rho\sigma}\bar{R}_{\rho\sigma}\delta^\mu_\nu; \tag{6.1.27}$$

$$16\pi\boldsymbol{z}_C^\mu(\xi) = \left(\zeta_\rho{}^\mu\mathfrak{g}^{\rho\sigma} + 2\mathfrak{g}^{\mu[\rho}\zeta_\rho{}^{\sigma]}\right)\Delta^\lambda{}_{\sigma\lambda} + \left(\mathfrak{g}^{\rho\sigma}\zeta_\lambda{}^\lambda - 2\mathfrak{g}^{\rho\lambda}\zeta_\lambda{}^{\sigma]}\right)\Delta^\mu{}_{\rho\sigma}$$
$$+ \mathfrak{h}^{\mu\lambda}\partial_\lambda\zeta_\rho{}^\rho + \mathfrak{h}^{\rho\sigma}\left(\bar{\nabla}^\mu\zeta_{\rho\sigma} - 2\bar{\nabla}_\rho\zeta_\sigma{}^\mu\right). \tag{6.1.28}$$

Comparing the KBL current with the canonical current defined for a field theory in the Minkowski space, one finds that (6.1.26) generalizes (1.2.101). Indeed, for Killing vectors the term (6.1.28) disappears and the current (6.1.26) transforms to (1.2.101). More details on the structure of the energy-momentum (6.1.27): the first term is a difference between the dynamical and background matter energy-momentum tensor densities – its perturbation; the second term is the gravitational energy-momentum (6.1.11); the third term describes an interaction of a perturbed system with a complicated curved non-Ricci flat background.

Thus, the identity (6.1.7) transforms into a physically sensible conservation law for the current (6.1.26):

$$\partial_\mu\mathscr{I}_C^\mu \equiv \bar{\nabla}_\mu\mathscr{I}_C^\mu = 0. \tag{6.1.29}$$

Analogous to the current (1.2.101) in a field theory that is conserved on field equations, (1.2.83), the KBL current is conserved on the Einstein's equations (1.3.22). The other form of this conservation law is expressed with the use of the identity (6.1.17) with the superpotential represented in (6.1.24) or in (6.1.25):

$$\mathscr{I}_C^{\mu}(\xi) = \partial_\nu \mathscr{I}_C^{\mu\nu}(\xi) \equiv \bar{\nabla}_\nu \mathscr{I}_C^{\mu\nu}(\xi)\,. \tag{6.1.30}$$

Here, $\mathscr{I}_C^{\mu\nu}(\xi)$ formally coincides with $\boldsymbol{j}_C^{\mu\nu}(\xi)$ but, unlike $\boldsymbol{j}_C^{\mu\nu}(\xi)$, the metric coefficients in $\mathscr{I}_C^{\mu\nu}(\xi)$ are the solutions to the Einstein equations. The form (6.1.30) of conservation laws is generic. Recall that, by the conservation law of the type (6.1.29), one can construct the integral conserved quantities at hypersurfaces $x^0 = \text{const}$ following the recipe (1.2.83–1.2.87). The conservation law (6.1.30) permits to transform such integrals into surface ones, see, e. g., (1.4.76–1.4.79).

Let us compare the KBL expressions with the others in general relativity. For simplification let us reproduce them in a flat background covered with the Lorentzian coordinates, and let a displacement vector be represented by the coordinate translations $\xi^\mu = \delta^\mu_\sigma$. Then one has $\bar{g}_{\mu\nu} \to \eta_{\mu\nu}$, and the KBL gravitational energy-momentum, $\mathbf{t}_\nu{}^\mu$, defined in (6.1.11) transforms to Einstein's pseudotensor (1.4.30), ${}_E\boldsymbol{t}_\nu{}^\mu$. More generally, the KBL current, $\mathscr{I}_C^{\mu}(\xi)$, defined in (6.1.26) goes to ${}_E\boldsymbol{t}_\nu{}^\mu + \mathscr{T}_\nu{}^\mu$ in (1.4.40).

It is important to note that both the current and the superpotential in (6.1.30) cannot be obtained by a simple covariantization of the classical quantities in (1.4.40). This is a result of application of the Noether procedure to the KBL Lagrangian. Then, first, the KBL conserved quantities hold on arbitrary curved backgrounds, not only on flat backgrounds in curved coordinates. By this, the current, $\mathscr{I}_C^{\mu}$, includes the interaction term that cannot be found by the rule of thumbs. Second, $\mathscr{I}_C^{\mu}$ includes the spin term which plays a crucial role because it permits to take into account the Killing vectors corresponding to rotations in a consistent way and, thus, to obtain a reasonable definition of the angular momentum for rotating black holes, for example.

The KBL quantities were also checked from the point of view of the problem of uniqueness. Julia and Silva [247, 414], and independently Chen and Nester [99], stated that the KBL quantities are uniquely defined and unambiguously associated with the Dirichlet boundary conditions.

6.2 The Belinfante procedure

6.2.1 The Belinfante symmetrization in general relativity

Recall that the Belinfante method [34, 35] has been elaborated in a field theory in the Minkowski space to present the energy, momentum and angular momentum densities with the use of an *unique complex*. Such a construction, see (1.2.113–1.2.118), is not complicated because the background Minkowski space is used.

The use of the Belinfante method in general relativity is not evident. Szabados [426, 427] clearly has shown that if the method is based on the dynamical metric only, without using a background metric, then the Belinfante symmetrization of classical pseudotensors leads *uniquely* to the Einstein tensor. Thus, arbitrary vacuum solutions of general relativity, including wave solutions, have vanishing value of energy that is not permissible. On the other hand, the success of Papapetrou [351] in applying the Belinfante symmetrization to the Einstein pseudotensor, see (1.4.57–1.4.62), had been possible because the background Minkowski space has been used. For the same reason, the use of a flat background in a perturbed variant of general relativity has permitted Berezin [37] to construct an effective energy-momentum tensor of all the physical fields. It is also important to note the work by Borokhov [63], where he has generalized the Belinfante procedure for an arbitrary field theory on an arbitrary background geometry with the Killing vectors.

Let us return to the KBL model. If one adds different divergences to the Lagrangian, different expressions both for currents and for superpotentials appear by the canonical Noether's procedure. The freedom in the choice of different types of the divergence can be employed for studying a diversity of physical systems whose behavior is determined by different boundary conditions. Besides gravitational physics this freedom is widely used, for example, in thermodynamics [98]. Nonetheless, most of practically important physical problems in gravitational physics demand the expressions for conserved quantities being as much independent on the choice of the divergence in the Lagrangian as possible. Examples of such divergence-independent quantities are given by the symmetric energy-momentum tensor in classical electrodynamics and the Belinfante symmetrized energy-momentum (1.2.128) in a field theory.

In the present section, summing up the above, we develop the Belinfante method in general relativity for the cases, when arbitrary curved background manifolds are introduced. The KBL model seems like the most appropriate one for the application of the Belinfante procedure. The KBL current (6.1.26) with the Killing vectors has the structure similar to (1.2.101) and contains two *different* complexes: the generalized energy-momentum ${}_{c}\boldsymbol{\theta}_{\nu}{}^{\mu}$ and the spin term $\boldsymbol{\sigma}^{\mu\rho\sigma}$. It is anticipated that the application of the Belinfante procedure will lead to a *unique* complex. The presentation follows the papers [369, 370].

6.2.2 The Belinfante method applied to the KBL model

Identities

Return to the initial standard form of the Noether identity (1.2.46) rewritten for the diffeomorphisms in the KBL model:

$$£_{\xi}\mathscr{L}_{KBL} + \partial_{\mu}\left(\xi^{\mu}\mathscr{L}_{KBL}\right) \equiv \partial_{\mu}\mathscr{B}^{\mu}\,. \tag{6.2.1}$$

Unlike (6.1.6), we keep the divergence in the right hand side for that $\partial_{\mu}\mathscr{B}^{\mu} \equiv 0$. To construct the vector density $\mathscr{B}^{\mu}$ we turn to the classical definition of the Belinfante correction (1.2.113) in a field theory. Following such a recipe, we combine the components of the spin tensor density (6.1.12) and construct the quantity being antisymmetric in μ and ν:

$$\boldsymbol{b}^{\mu\nu\rho} = -\boldsymbol{b}^{\nu\mu\rho} = \boldsymbol{\sigma}^{\rho[\mu\nu]} + \boldsymbol{\sigma}^{\mu[\rho\nu]} - \boldsymbol{\sigma}^{\nu[\rho\mu]} \tag{6.2.2}$$

that is called the Belinfante correction. Using (6.2.2), we define the vector density as $\mathscr{B}^\mu \equiv \partial_\nu(\boldsymbol{b}^{\mu\nu\rho}\xi_\rho)$ for that indeed $\partial_\mu\mathscr{B}^\mu \equiv 0$. The canonical Noether current in the Noether identity in the KBL model (6.1.17) goes to the Belinfante symmetrized current $\boldsymbol{j}^\mu_B = \boldsymbol{j}^\mu_C + \mathscr{B}^\mu$. Thus the identity (6.1.17) is rewritten in the equivalent form:

$$\boldsymbol{j}^\mu_C + \partial_\nu\left(\boldsymbol{b}^{\mu\nu\rho}\xi_\rho\right) \equiv \partial_\nu\left(\boldsymbol{j}^{\mu\nu}_C + \boldsymbol{b}^{\mu\nu\rho}\xi_\rho\right) . \tag{6.2.3}$$

Renaming here the quantities with the Belinfante symmetrized current, $\boldsymbol{j}^\mu_B$, and the Belinfante corrected superpotential, $\boldsymbol{j}^{\mu\nu}_B$, one gets the identity

$$\boldsymbol{j}^\mu_B(\xi) \equiv \partial_\nu\boldsymbol{j}^{\mu\nu}_B(\xi) \equiv \bar{\nabla}_\nu\boldsymbol{j}^{\mu\nu}_B(\xi) \tag{6.2.4}$$

instead of (6.1.17).

The Belinfante corrected current

Let us derive the current in (6.2.4), using (6.2.2) and (6.1.11), along with (6.1.16–6.1.28),

$$\boldsymbol{j}^\mu_B(\xi) = \boldsymbol{j}^\mu_C(\xi) + \partial_\nu\left(\boldsymbol{b}^{\mu\nu\rho}\xi_\rho\right) = {}_B\boldsymbol{\tau}_\nu{}^\mu\xi^\nu + \boldsymbol{z}^\mu_B(\xi) \tag{6.2.5}$$

with $\boldsymbol{j}^\mu_C(\xi)$ defined in (6.1.15). The quantity ${}_B\boldsymbol{\tau}_\nu{}^\mu$ is a symmetrized (Belinfante corrected) energy-momentum:

$${}_B\boldsymbol{\tau}_\nu{}^\mu = \frac{1}{8\pi}\left(\mathscr{G}_\nu{}^\mu - \bar{\mathscr{G}}_\nu{}^\mu + \tfrac{1}{2}\mathfrak{h}^{\rho\sigma}\bar{R}_{\rho\sigma}\delta^\mu_\nu\right) + \boldsymbol{t}_\nu{}^\mu + \bar{\nabla}_\rho\boldsymbol{b}^{\mu\rho}{}_\nu . \tag{6.2.6}$$

Here, as before, $\mathfrak{h}^{\tau\sigma} = \mathfrak{g}^{\tau\sigma} - \bar{\mathfrak{g}}^{\tau\sigma}$, but the components $\mathfrak{h}^{\tau\sigma}$ are not independent variables, because it is a bi-metric formulation. The last term in the current (6.2.5), that we call z-term depends only on the quantity (6.1.14) by our convention. With the use of (6.1.12), (6.1.13) and (6.2.2) one obtains

$$\begin{aligned} 16\pi\boldsymbol{z}^\mu_B(\xi) &= 16\pi\left[\boldsymbol{z}^\mu_C + \left(\boldsymbol{b}^{\mu\rho\sigma} + \boldsymbol{\sigma}^{\mu\rho\sigma}\right)\bar{\nabla}_\rho\xi_\sigma\right] \\ &= 2\left(\zeta^{\rho\sigma}\bar{\nabla}_\rho\mathfrak{h}_\sigma{}^\mu - \mathfrak{h}^{\rho\sigma}\bar{\nabla}_\rho\zeta_\sigma{}^\mu\right) - \left(\zeta_{\rho\sigma}\bar{\nabla}^\mu\mathfrak{h}^{\rho\sigma} - \mathfrak{h}^{\rho\sigma}\bar{\nabla}^\mu\zeta_{\rho\sigma}\right) \\ &\quad + \left(\mathfrak{h}^{\mu\nu}\bar{\nabla}_\nu\zeta_\rho{}^\rho - \zeta_\rho{}^\rho\bar{\nabla}_\nu\mathfrak{h}^{\mu\nu}\right). \end{aligned} \tag{6.2.7}$$

The equality (6.2.4) is an identity. To make it physically meaningful one has to use the Einstein equations. To this end one has to work out the Einstein tensor densities in (6.2.6). Picking up a symmetric part and raising the lower index, we transform the related part as follows,

$$\frac{1}{8\pi}\left(\mathscr{G}_\rho{}^\mu - \bar{\mathscr{G}}_\rho{}^\mu\right)\bar{\mathfrak{g}}^{\rho\nu} = \mathscr{T}_\rho{}^{(\mu}\bar{\mathfrak{g}}^{\nu)\rho} - \bar{\mathscr{T}}^{\mu\nu} + \frac{1}{8\pi}\mathscr{G}_\rho{}^{[\mu}\bar{\mathfrak{g}}^{\nu]\rho} . \tag{6.2.8}$$

Then, the energy-momentum (6.2.6) acquires the final form:

$$ {}_B\boldsymbol{\theta}^{\mu\nu} = (\mathscr{T}_\rho{}^{(\mu}\bar{g}^{\nu)\rho} - \bar{\mathscr{T}}^{\mu\nu}) + \boldsymbol{t}_B^{\mu\nu} + \frac{1}{16\pi}\bar{g}^{\mu\nu}\mathfrak{h}^{\rho\sigma}\bar{R}_{\rho\sigma} + \frac{1}{8\pi}\mathfrak{h}^{\lambda[\mu}\bar{R}_\lambda^{\nu]} . \tag{6.2.9} $$

Here, the first term is a symmetrized perturbation of the matter energy-momentum, and the second term is the symmetric energy-momentum for free gravitational field:

$$ \begin{aligned} 8\pi\boldsymbol{t}_B^{\mu\nu} &= 8\pi\boldsymbol{t}^{\mu\nu} + 8\pi\bar{\nabla}_\rho\boldsymbol{b}^{\mu\rho\nu} \\ &= \tfrac{1}{2}\left(\mathfrak{h}^{\mu\nu}\bar{g}^{\rho\sigma} - \bar{g}^{\mu\nu}\mathfrak{h}^{\rho\sigma}\right)\bar{\nabla}_\sigma\Delta^\lambda{}_{\rho\lambda} \\ &\quad + \left(\mathfrak{h}^{\rho\sigma}\bar{g}^{\lambda(\mu} - \bar{g}^{\rho\sigma}\mathfrak{h}^{\lambda(\mu}\right)\bar{\nabla}_\sigma\Delta^{\nu)}{}_{\lambda\rho} \\ &\quad + \bar{g}^{\rho\sigma}\left(\tfrac{1}{2}\mathfrak{g}^{\mu\nu}\Delta^\lambda{}_{\rho\lambda}\Delta^\eta{}_{\sigma\eta} + \mathfrak{g}^{\lambda\eta}\Delta^{(\mu}{}_{\lambda\rho}\Delta^{\nu)}{}_{\eta\sigma}\right) \\ &\quad + \bar{g}^{\rho\sigma}\left(\Delta^\lambda{}_{\sigma\eta}\Delta^{(\mu}{}_{\lambda\rho}\mathfrak{g}^{\nu)\eta} - 2\Delta^\lambda{}_{\sigma\lambda}\Delta^{(\mu}{}_{\eta\rho}\mathfrak{g}^{\nu)\eta}\right) \\ &\quad + \tfrac{1}{2}\mathfrak{g}^{\lambda\eta}\bar{g}^{\mu\nu}\Delta^\sigma{}_{\rho\lambda}\Delta^\rho{}_{\sigma\eta} \\ &\quad + \mathfrak{g}^{\lambda\eta}\left(\Delta^\sigma{}_{\rho\sigma}\Delta^{(\mu}{}_{\lambda\eta} - \Delta^\sigma{}_{\lambda\eta}\Delta^{(\mu}{}_{\rho\sigma} - \Delta^\sigma{}_{\lambda\rho}\Delta^{(\mu}{}_{\eta\sigma}\right)\bar{g}^{\nu)\rho} . \end{aligned} \tag{6.2.10} $$

The third and fourth terms in (6.2.9) describe interactions with the background geometry, if it is non-Ricci flat.

Thus, the identically conserved current (6.2.5) is transformed to the Belinfante corrected current

$$ \mathscr{I}_B^\mu(\xi) = {}_B\boldsymbol{\theta}_\nu{}^\mu\xi^\nu + \boldsymbol{z}_B^\mu(\xi) \tag{6.2.11} $$

that is consistent with the equations of motion of the physical system under consideration, and satisfies the differential conservation law:

$$ \partial_\mu\mathscr{I}_B^\mu(\xi) = \bar{\nabla}_\mu\mathscr{I}_B^\mu(\xi) = 0 . \tag{6.2.12} $$

Let us set out the properties of the Belinfante corrected current (6.2.11).

(i) One sees that the current (6.2.11) does not contain the spin term. Moreover, if ξ^μ is a Killing vector of the background, $\bar{\xi}^\mu$, the current, $\mathscr{I}_B^{\mu\nu}(\bar{\xi})$, takes on the form of the Belinfante corrected current (1.2.117) in a field theory in the Minkowski space. The fact that the current is determined by the energy-momentum complex only, is a consequence of applying the classical Belinfante corrected procedure.

(ii) Let us turn to the fourth term in (6.2.9) that is the unique antisymmetric term. One can see that the energy-momentum is symmetric, ${}_B\boldsymbol{\theta}^{\mu\nu} = {}_B\boldsymbol{\theta}^{\nu\mu}$, if and only if $\bar{R}_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$, that is for the cases, when the backgrounds are the Einstein spaces in Petrov's classification [372]. As a result, one concludes that the Belinfante symmetrization in general relativity does not lead to a symmetric energy-momentum in more general cases of arbitrary curved backgrounds.

(iii) Let the curved background has a Killing vector $\bar{\xi}^{\nu}$, then the current (6.2.11) is defined, like in (1.2.117):

$$\mathscr{J}_B^{\mu}(\bar{\xi}) = {}_B\boldsymbol{\theta}_{\nu}{}^{\mu}\bar{\xi}^{\nu}\,, \tag{6.2.13}$$

and it is conserved, see (6.2.12),

$$\partial_{\mu}\left({}_B\boldsymbol{\theta}_{\nu}{}^{\mu}\bar{\xi}^{\nu}\right) = 0\,, \tag{6.2.14}$$

compare with (1.2.118). However, unlike (1.2.118), the energy-momentum ${}_B\boldsymbol{\theta}^{\mu\nu}$ is not symmetric in general, and the differential conservation law for the energy-momentum does not hold,

$$\bar{\nabla}_{\nu}\left({}_B\boldsymbol{\theta}^{\mu\nu}\right) \neq 0\,. \tag{6.2.15}$$

As a matter of principle, the conservation $\bar{\nabla}_{\nu}\left({}_B\boldsymbol{\theta}^{\mu\nu}\right) = 0$ is valid, if and only if the background is Einstein's space. Nonetheless, the conservation law (6.2.14) is useful, e. g., for constructing the angular momenta of relativistic astrophysical objects on the FLRW background, which is not Einstein's space, but which has the Killing vectors corresponding to spatial rotations.

(iv) Unlike to the KBL gravitational energy-momentum (6.1.11), the second derivatives of $g_{\mu\nu}$ appear in the Belinfante corrected energy-momentum (6.2.10). This needs some comments. The canonical KBL energy-momentum (6.1.11) is quadratic in the first order derivatives, and this is a normal behaviour for a conserved quantity related to the standard initial conditions. Consider the local quantities $\mathscr{J}_B^{\mu}$ in (6.2.11). Recall that the integral conserved quantities on hypersurfaces x^0 = const are defined by the integration of the only time component, $\mathscr{J}_B^0$, in (6.2.19), see formula (1.2.87). The initial conditions are generically defined only on such hypersurfaces. Then, it is sufficient to examine the initial conditions for the time component $\mathscr{J}_B^0 = \mathscr{J}_C^0 + \partial_k(\boldsymbol{b}^{0k\sigma}\xi_{\sigma})$ where we have taken into account that $\boldsymbol{b}^{\mu\nu\sigma}$ is anti-symmetric in the first two indices, see (1.2.1). Thus, since $\mathscr{J}_C^0$ and $\boldsymbol{b}^{0k\sigma}\xi_{\sigma}$ contain only the first order *time* derivatives, $\mathscr{J}_B^0$ itself contains only the first order *time* derivatives of the metric, and therefore, it does not require knowledge of the higher-order derivatives on the initial hypersurface.

The Belinfante corrected superpotential

To obtain an explicit expression for the new superpotential defined in (6.2.3) and derived in (6.2.4) we combine (6.1.24), or (6.1.25), with (6.2.2) and (6.1.12),

$$\boldsymbol{j}_B^{\mu\nu} = \boldsymbol{j}_C^{\mu\nu} + \boldsymbol{b}^{\mu\nu\rho}\xi_{\rho} = \frac{1}{8\pi}\mathfrak{h}^{\rho[\mu}\bar{\nabla}_{\rho}\xi^{\nu]} + \mathscr{P}_{\lambda}{}^{\mu\nu}\xi^{\lambda}\,. \tag{6.2.16}$$

It is antisymmetric in μ and ν because the quantity $\mathscr{P}^{\rho\nu\mu}$ is defined as

$$\mathscr{P}^{\rho\mu\nu} = \frac{1}{16\pi}\bar{\nabla}_{\sigma}\left(\bar{g}^{\rho\mu}\mathfrak{h}^{\nu\sigma} - \bar{g}^{\rho\nu}\mathfrak{h}^{\mu\sigma} - \bar{g}^{\sigma\mu}\mathfrak{h}^{\nu\rho} + \bar{g}^{\sigma\nu}\mathfrak{h}^{\mu\rho}\right)\,. \tag{6.2.17}$$

If one sets $\bar{g}^{\mu\nu} = \eta^{\mu\nu}$ the quantity $\mathscr{P}^{\rho\mu\nu}$ transforms to the Papapetrou [351] superpotential (1.4.62), which means that (6.2.16) generalizes the Papapetrou superpotential to arbitrary curved backgrounds for arbitrary displacement vectors ξ^α.

It is important to note that the new superpotential (6.2.16) depends *linearly* on perturbations of the metric densities, $\mathfrak{h}^{\mu\nu} \equiv \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}$. On the other hand, it is not an approximate formulation, it is exact. There is another useful form of the superpotential (6.2.16):

$$\boldsymbol{j}_B^{\mu\nu} = \frac{1}{8\pi}\left(\xi^{[\mu}\bar{\nabla}_\sigma\mathfrak{h}^{\nu]\sigma} - \xi^\sigma\bar{\nabla}^{[\mu}\mathfrak{h}^{\nu]}_\sigma + \mathfrak{h}^{\rho[\mu}\bar{\nabla}_\rho\xi^{\nu]}\right) . \qquad (6.2.18)$$

One can see that it generalizes the superpotential (4.2.26) derived for the Killing vectors on the AdS background, as well as for arbitrary displacement vectors ξ^α on arbitrary curved backgrounds. Continuing, one can tell that the superpotential (6.2.18) generalizes the Abbott-Deser superpotential (4.2.29) to arbitrary displacement vectors ξ^α, arbitrary curved backgrounds and is also valid for an alternative definition of perturbations, like $\mathfrak{h}^{\mu\nu}$.

The Belinfante corrected conservation laws

Finally, as a result of using the Einstein's equations, the identity (6.2.4) transforms to the physically meaningful conservation law

$$\mathscr{J}_B^\mu(\xi) = \partial_\nu\mathscr{J}_B^{\mu\nu}(\xi) \equiv \bar{\nabla}_\nu\mathscr{J}_B^{\mu\nu}(\xi) \qquad (6.2.19)$$

instead of the canonical one (6.1.30). As in the canonical derivation, formally $\mathscr{J}_B^{\mu\nu}(\xi)$ coincides with $\boldsymbol{j}_B^{\mu\nu}(\xi)$ but, unlike $\boldsymbol{j}_B^{\mu\nu}(\xi)$, the coefficients of the metric tensor in $\mathscr{J}_B^{\mu\nu}(\xi)$ are solutions of the Einstein equations.

Concluding, we repeat that a divergence in the Lagrangian has no contribution to the Belinfante corrected quantities. This will be proved in detail on a more general ground, when the Belinfante procedure will be described in the framework of multidimensional metric theories, see Section 7.1.4 from (7.1.85) to (7.1.87). Now, we only note that this property resolves the problem of the KBL model, where with making a choice between divergences in the Lagrangian one changes conserved quantities.

6.3 Currents and superpotentials in the field-theoretical formulation

6.3.1 Noether's procedure applied to the field-theoretical model

We already know that in the framework of the field-theoretical formulation, the integral conserved quantities can be expressed through the surface integrals. However such integrals have been constructed only on backgrounds represented by the Einstein spaces. Besides, the existence of the Killing vectors on these backgrounds was crucial. Let us recall those results. In the case of a flat background, the conservation law

(2.2.106) has been used to construct a conserved current (3.2.18) and the correspondent conserved surface integrals in the asymptotically flat spacetimes (3.2.24–3.2.26), which have been formulated in a consolidated form (3.2.27). In the case of the Einstein spaces, the conservation law (2.2.112) has been used to construct the conserved current of the same form (3.2.18) along with a family of corresponding superpotentials (4.2.19).

In the case of the field-theoretical formulation, the question arises: can one construct the conservation laws, like (6.1.30) and (6.2.19), on arbitrary curved backgrounds and with arbitrary displacement vectors? The question has far going consequences for the development of the theory because in the generic case of a curved background one cannot construct a conservation law of the form (2.2.106) or (2.2.112). A formal reason is that, in general, there is no conservation law for the energy-momentum standing in the right hand side of the field equations. This follows from the relations (2.2.101) and (2.2.103) which reveal that:

$$\bar{\nabla}^{\nu}\left(G^{L}_{\mu\nu} + \Phi^{L}_{\mu\nu}\right) \neq 0\,, \tag{6.3.1}$$

$$\bar{\nabla}^{\nu} G^{L}_{\mu\nu} \neq 0\,. \tag{6.3.2}$$

A physical reason of the conclusions (6.3.1) and (6.3.2) is that the perturbed system interacts with a complicated non-Ricci flat background that contains a background matter represented by fields $\bar{\Phi}^{A}$. In the framework both of the KBL approach, Section 6.1, and of the Belinfante corrected procedure, Section 6.2, such a kind of interaction has been taken into account in a construction of conserved quantities by applying the Noether procedure that automatically includes the background Ricci tensor into consideration. Therefore it is expected that the *standard* Noether methods will be valid for applying in the field-theoretical approach.

To make the Noether technique applicable in such a model it is necessary to convert the field-theoretical Lagrangian defined for perturbations, say the Lagrangian, $\mathscr{L}^{g}$ in (2.2.20), into a bi-metric form and make a replacement $\mathfrak{h}^{\mu\nu} \to \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}$, which does not influence the result of the Noether procedure.

It is expedient to use directly the technique worked out in Section 6.1 and applied it to the KBL Lagrangian, $\mathscr{L}_{KBL}$ in (6.1.5). Then, one has to relate the field-theoretical gravitational Lagrangian, $\mathscr{L}^{g}$, in (2.2.20) to $\mathscr{L}_{KBL}$. In both cases we use the same divergence defined by (2.2.19) and (6.1.4), respectively. After making the replacement $\mathfrak{h}^{\mu\nu} \to \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}$ we denote the Lagrangian obtained as

$$\mathscr{L}_{G2} \equiv -\frac{1}{16\pi}\mathscr{L}^{g}\,. \tag{6.3.3}$$

It can be easily shown that

$$\mathscr{L}_{G2} = \mathscr{L}_{KBL} - \mathscr{L}_{G1} = -\frac{1}{16\pi}\mathfrak{g}^{\mu\nu}\left(\Delta^{\rho}_{\mu\nu}\Delta^{\sigma}_{\rho\sigma} - \Delta^{\rho}_{\mu\sigma}\Delta^{\sigma}_{\rho\nu}\right)\,, \tag{6.3.4}$$

where

$$\mathscr{L}_{G1} \equiv -\frac{1}{16\pi}\left(\mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}\right)\bar{R}_{\mu\nu}\,. \tag{6.3.5}$$

Comparing the Lagrangians $\mathscr{L}_{G2}$ and $\mathscr{L}_{KBL}$ with the Rosen Lagrangian $\mathscr{L}_R$ [390, 391], we note that $\mathscr{L}_{G2}$ is a direct generalization of $\mathscr{L}_R$ to arbitrary backgrounds, whereas $\mathscr{L}_{KBL}$ is reduced to $\mathscr{L}_R$ for the Ricci-flat backgrounds only. Notice also that, unlike the KBL Lagrangian (6.1.5), the Lagrangian $\mathscr{L}_{G2}$ depends only on the first derivatives of the *background metric*. This simplifies the Noether procedure significantly.

To efficiently apply the technique of the Section 6.1 it is fruitful to consider the Lagrangian (6.3.4) in the form:

$$\mathscr{L}_{G2} = -\frac{1}{16\pi}\left(\mathscr{R} - \mathfrak{g}^{\mu\nu}\bar{R}_{\mu\nu} + \partial_\alpha \boldsymbol{k}^\alpha\right) , \tag{6.3.6}$$

compare with (6.1.2). The Lagrangian (6.3.6) is used to derive the main Noether's identity of the type (6.1.6),

$$£_\xi \mathscr{L}_{G2} + \partial_\mu(\xi^\mu \mathscr{L}_{G2}) \equiv 0 . \tag{6.3.7}$$

It is directly transformed to the identity

$$\partial_\mu \boldsymbol{j}_2^\mu \equiv \bar{\nabla}_\mu \boldsymbol{j}_2^\mu \equiv 0 \tag{6.3.8}$$

that is analogous to the KBL identity (6.1.7). The identically conserved current in (6.3.8) is

$$\begin{aligned}\boldsymbol{j}_2^\mu \equiv &\left[-\frac{\delta \mathscr{L}_{G2}}{\delta g_{\rho\sigma}}\, g_{\rho\sigma}\Big|_\nu^\mu - \frac{\delta \mathscr{L}_{G2}}{\delta \bar{g}_{\rho\sigma}}\, \bar{g}_{\rho\sigma}\Big|_\nu^\mu + \frac{\partial \mathscr{L}_{G2}}{\partial\left(\bar{\nabla}_\mu g_{\rho\sigma}\right)}\bar{\nabla}_\nu g_{\rho\sigma} - \mathscr{L}_{G2}\delta_\nu^\mu\right]\xi^\nu \\ &- {}^2\mathscr{S}_\lambda{}^{\mu\rho}\bar{\nabla}_\rho \xi^\lambda ,\end{aligned} \tag{6.3.9}$$

and below we examine its structure. Using the form of the Lagrangian (6.3.6), one easily obtains for the first term in the current (6.3.9):

$$-\frac{\delta \mathscr{L}_{G2}}{\delta g_{\rho\sigma}}\, g_{\rho\sigma}\Big|_\nu^\mu \equiv \frac{1}{8\pi}\left(\mathscr{G}_\nu{}^\mu - \bar{\mathscr{G}}_\nu{}^\mu\right) - \frac{1}{8\pi}\left(\mathfrak{h}^{\mu\rho}\delta_\nu^\sigma - \tfrac{1}{2}\mathfrak{h}^{\rho\sigma}\delta_\nu^\mu\right)\bar{R}_{\rho\sigma} . \tag{6.3.10}$$

One recognizes that the other part of the current (6.3.9) is expressed through a linear operator, $G^L_{\mu\nu}$, in (2.2.27). It is because $\bar{\nabla}_\rho \mathfrak{g}^{\mu\nu} = \bar{\nabla}_\rho \mathfrak{h}^{\mu\nu}$, thus

$$-\frac{\delta \mathscr{L}_{G2}}{\delta \bar{g}_{\rho\sigma}}\, \bar{g}_{\rho\sigma}\Big|_\nu^\mu \equiv -\frac{1}{8\pi} G^L_{\rho\nu}(\mathfrak{h})\bar{\mathfrak{g}}^{\rho\mu} . \tag{6.3.11}$$

The last part in the square brackets is exactly represented by the canonical gravitational energy-momentum (6.1.11):

$$\frac{\partial \mathscr{L}_{G2}}{\partial\left(\bar{\nabla}_\mu g_{\rho\sigma}\right)}\bar{\nabla}_\nu g_{\rho\sigma} - \mathscr{L}_{G2}\delta_\nu^\mu \equiv \boldsymbol{t}_\nu{}^\mu . \tag{6.3.12}$$

The last term in (6.3.9) is defined by

$${}^2\mathscr{S}_\lambda{}^{\mu\nu} \equiv \frac{\partial \mathscr{L}_{G2}}{\partial\left(\bar{\nabla}_\mu g_{\rho\sigma}\right)}\, g_{\rho\sigma}\Big|_\lambda^\nu + \frac{\partial \mathscr{L}_{G2}}{\partial\left(\partial_\mu \bar{g}_{\rho\sigma}\right)}\, \bar{g}_{\rho\sigma}\Big|_\lambda^\nu . \tag{6.3.13}$$

It should be noticed that the quantity (6.3.13) is antisymmetric in μ and ν and can be rewritten with the use of the spin tensor components (6.1.12) in the form:

$$^{2}\mathscr{S}_{\lambda}{}^{\mu\nu}\bar{g}^{\lambda\rho} = \boldsymbol{\sigma}^{\rho[\mu\nu]} + \boldsymbol{\sigma}^{\mu[\rho\nu]} - \boldsymbol{\sigma}^{\nu[\rho\mu]} \,. \tag{6.3.14}$$

This quantity is exactly the Belinfante correction introduced in (6.2.2), $^{2}\mathscr{S}_{\lambda}{}^{\mu\nu} = \boldsymbol{b}^{\mu\nu}{}_{\lambda}$. Summarizing (6.3.9–6.3.14), one can rewrite the current (6.3.9) in the form:

$$\boldsymbol{j}^{\mu}_{2}(\xi) \equiv {}_{2}\boldsymbol{\tau}_{\nu}{}^{\mu}\xi^{\nu} - \boldsymbol{b}^{\mu\rho\sigma}\bar{\nabla}_{\rho}\xi_{\sigma}\,, \tag{6.3.15}$$

where the energy-momentum is

$${}_{2}\boldsymbol{\tau}_{\nu}{}^{\mu} \equiv {}_{C}\boldsymbol{\tau}_{\nu}{}^{\mu} - \frac{1}{8\pi}\left(\mathfrak{h}^{\mu\rho}\bar{R}_{\rho\nu} + \bar{\mathfrak{g}}^{\mu\rho}G^{L}_{\rho\nu}(\mathfrak{h})\right) \tag{6.3.16}$$

with ${}_{C}\boldsymbol{\tau}_{\nu}{}^{\mu}$ defined in (6.1.16).

Recall the standard conceptual points. Because (6.3.8) is the identity one can rewrite the current (6.3.9) in terms of a superpotential in the form (6.1.17):

$$\boldsymbol{j}^{\mu}_{2}(\xi) \equiv \partial_{\nu}\boldsymbol{j}^{\mu\nu}_{2}(\xi) \equiv \bar{\nabla}_{\nu}\boldsymbol{j}^{\mu\nu}_{2}(\xi)\,. \tag{6.3.17}$$

To calculate the superpotential one carries out the calculations analogous to the ones from (6.1.18) to (6.1.24). Based on the identity (6.3.8), one derives a system of the Klein identities analogous to the system (6.1.19–6.1.22), which are used for transformations of (6.3.15). At the end of this procedure, one finds the superpotential,

$$\boldsymbol{j}^{\mu\nu}_{2}(\xi) = -^{2}\mathscr{S}_{\lambda}{}^{\mu\nu}\xi^{\lambda} = -\boldsymbol{b}^{\mu\nu\rho}\xi_{\rho}\,, \tag{6.3.18}$$

that is expressed solely through the quantity (6.3.11).

Looking more carefully to the identity (6.3.17), the reader can notice that it is a bi-metric identity, not a field-theoretical one. To return to the field-theoretical formulation we go back to the KBL identity (6.1.17). We see that the current (6.1.15) replicates the current (6.3.15) in many terms. Therefore, let us subtract the identity (6.3.17) from the identity (6.1.17). As a result one obtains a modified new identity:

$$\boldsymbol{j}^{\mu}_{S} \equiv \partial_{\nu}\boldsymbol{j}^{\mu\nu}_{S} \equiv \bar{\nabla}_{\nu}\boldsymbol{j}^{\mu\nu}_{S}\,. \tag{6.3.19}$$

Finalizing the application of the Noether procedure, we return to the variables $\mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu} \to \mathfrak{h}^{\mu\nu}$ of the field-theoretical formulation.

Currents and superpotentials

The current in (6.3.19) is

$$\boldsymbol{j}^{\mu}_{S} = {}_{S}\boldsymbol{\tau}_{\nu}{}^{\mu}\xi^{\nu} + \boldsymbol{z}^{\mu}_{S}(\xi)\,, \tag{6.3.20}$$

where the energy-momentum is

$$ {}_S\boldsymbol{\tau}_\nu{}^\mu = \frac{1}{8\pi}\left(\bar{g}^{\mu\rho}G^L_{\rho\nu}(\mathfrak{h}) + \mathfrak{h}^{\mu\rho}\bar{R}_{\rho\nu}\right) . \tag{6.3.21} $$

The last term in the current (6.3.20), z-term, is

$$ \begin{aligned} 16\pi \boldsymbol{z}^\mu_S(\xi) &= 16\pi\left[\boldsymbol{z}^\mu_C + \left(\boldsymbol{b}^{\mu\rho\sigma} + \boldsymbol{\sigma}^{\mu\rho\sigma}\right)\bar{\nabla}_\rho\xi_\sigma\right] \\ &= 2\left(\zeta^{\rho\sigma}\bar{\nabla}_\rho\mathfrak{h}_\sigma{}^\mu - \mathfrak{h}^{\rho\sigma}\bar{\nabla}_\rho\zeta_\sigma{}^\mu\right) - \left(\zeta_{\rho\sigma}\bar{\nabla}^\mu\mathfrak{h}^{\rho\sigma} - \mathfrak{h}^{\rho\sigma}\bar{\nabla}^\mu\zeta_{\rho\sigma}\right) \\ &\quad + \left(\mathfrak{h}^{\mu\nu}\bar{\nabla}_\nu\zeta_\rho{}^\rho - \zeta_\rho{}^\rho\bar{\nabla}_\nu\mathfrak{h}^{\mu\nu}\right) . \end{aligned} \tag{6.3.22} $$

The new superpotential in the identity (6.3.19) is

$$ \begin{aligned} \boldsymbol{j}^{\mu\nu}_S &= \boldsymbol{j}^{\mu\nu}_C + \boldsymbol{b}^{\mu\nu\rho}\xi_\rho \\ &= \frac{1}{8\pi}\left(\xi^{[\mu}\bar{\nabla}_\sigma\mathfrak{h}^{\nu]\sigma} - \xi^\sigma\bar{\nabla}^{[\mu}\mathfrak{h}^{\nu]}_\sigma + \mathfrak{h}^{\rho[\mu}\bar{\nabla}_\rho\xi^{\nu]}\right) . \end{aligned} \tag{6.3.23} $$

To convert the identity (6.3.19) into a physically meaningful conservation law, one has to use the gravitational equations in the field-theoretical form. Thus, after substituting $G^L_{\mu\nu}(\mathfrak{h})$ from the equations (2.2.26) into the identity (6.3.19) one obtains the conservation law in the form:

$$ \mathscr{I}^\mu_S = \partial_\nu\mathscr{I}^{\mu\nu}_S \equiv \bar{\nabla}_\nu\mathscr{I}^{\mu\nu}_S , \tag{6.3.24} $$

where

$$ \mathscr{I}^\mu_S = {}_S\boldsymbol{\theta}_\nu{}^\mu\xi^\nu + \boldsymbol{z}^\mu_S(\xi) \tag{6.3.25} $$

with the energy-momentum

$$ {}_S\boldsymbol{\theta}_{\mu\nu} = \boldsymbol{t}^{\text{tot}}_{\mu\nu} + \frac{1}{8\pi}\left(\mathfrak{h}_\mu{}^\rho\bar{R}_{\nu\rho} - \Phi^L_{\mu\nu}\right) . \tag{6.3.26} $$

The role of the interaction with a curved background is played by the expression in the brackets. One can use also the field-theoretical equations in the form (2.2.33) with (2.2.34), then (6.3.26) is rewritten as

$$ {}_S\boldsymbol{\theta}_{\mu\nu} = \boldsymbol{t}^{\text{eff}}_{\mu\nu} + \frac{1}{8\pi}\mathfrak{h}_\mu{}^\rho\bar{R}_{\nu\rho} . \tag{6.3.27} $$

The explicit form of (6.3.26) or (6.3.27) is

$$ {}_S\boldsymbol{\theta}_{\mu\nu} = \left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\right)\left(\mathscr{T}_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\mathscr{T}_{\pi\rho}g^{\pi\rho}\right) - \bar{\mathscr{T}}_{\mu\nu} + \boldsymbol{t}^{\text{g}}_{\mu\nu} + \frac{1}{8\pi}\mathfrak{h}_\mu{}^\rho\bar{R}_{\nu\rho} . \tag{6.3.28} $$

Superpotential $\mathscr{I}^{\mu\nu}_S$ in (6.3.24) exactly coincides with $\boldsymbol{j}^{\mu\nu}_S$ in (6.3.23) if $\mathfrak{h}^{\mu\nu}$ satisfy the field equations. Being equivalent to (6.2.18), it generalizes also the superpotential (4.2.26) for arbitrary displacement vectors ξ^α and arbitrary curved backgrounds.

Equivalence of the Belinfante corrected and the field-theoretical quantities

One sees that the superpotential (6.3.23) coincides exactly with the Belinfante corrected superpotential (6.2.18):

$$\boldsymbol{j}_S^{\mu\nu} = \boldsymbol{j}_B^{\mu\nu} . \tag{6.3.29}$$

Then, one concludes that the current (6.3.20) in (6.3.19) has to coincide exactly with the Belinfante corrected current (6.2.5). After that, because z-term (6.3.22) coincides exactly with the Belinfante corrected z-term (6.2.7),

$$\boldsymbol{z}_S^{\mu} = \boldsymbol{z}_B^{\mu} , \tag{6.3.30}$$

the energy-momentum (6.3.21) has to be equal to the Belinfante corrected energy-momentum (6.2.6),

$${}_S\boldsymbol{\tau}_\nu{}^\mu = {}_B\boldsymbol{\tau}_\nu{}^\mu , \tag{6.3.31}$$

or more explicitly,

$$\frac{1}{8\pi}\left(G^L_{\rho\nu}(\mathfrak{h})\bar{\mathfrak{g}}^{\rho\mu} + \mathfrak{h}^{\mu\rho}\bar{R}_{\rho\nu}\right) = \frac{1}{8\pi}\left(\mathscr{G}_\nu{}^\mu - \bar{\mathscr{G}}_\nu{}^\mu + \tfrac{1}{2}\mathfrak{h}^{\rho\sigma}\bar{R}_{\rho\sigma}\delta^\mu_\nu\right) + \boldsymbol{t}_\nu{}^\mu + \bar{\nabla}_\rho \boldsymbol{b}^{\mu\rho}{}_\nu . \tag{6.3.32}$$

Indeed, after bulky calculations one can stand up for this claim. Consequently the energy-momentum complexes (6.3.28) and (6.2.9) are equal as well,

$${}_S\boldsymbol{\theta}^{\mu\nu} = {}_B\boldsymbol{\theta}^{\mu\nu} . \tag{6.3.33}$$

Comparing the Einstein equations written down in the geometrical and field-theoretical forms, one can confirm this equality.

6.3.2 A family of conserved quantities and the Boulware-Deser ambiguity

The above construction of conservations laws (6.3.4–6.3.33) and conserved quantities rather straightforward but fairly tedious. To obtain the identity (6.3.19) we have subtracted the identity (6.3.7) from that (6.1.6), and relied upon the main Noether's identity

$$\pounds_\xi \mathscr{L}_{G1} + \partial_\mu(\xi^\mu \mathscr{L}_{G1}) \equiv 0 \tag{6.3.34}$$

for the scalar density

$$\mathscr{L}_{G1} = \mathscr{L}_{KBL} - \mathscr{L}_{G2} = -\frac{1}{16\pi}\mathfrak{h}^{\mu\nu}\bar{R}_{\mu\nu} , \tag{6.3.35}$$

see (6.3.5). Then, it is clear that doing certain transformations of the identity (6.3.34) from the start we, of course, will arrive the identity (6.3.19) in the field-theoretical formulation as well.

Now, let us return to Section 2.2.6, where a field-theoretical formulation related to different definitions of the metric perturbations was studied. This brought about the ambiguity (2.2.132) in the definition of a total energy-momentum. The questions arise. First, how the conservation laws, like (6.3.19), can be constructed for different definitions of variables? Second, how the Boulware-Deser ambiguity appears in definitions of superpotentials in the field-theoretical formulation?

To answer the first question one has to use the recipe suggested by the identity (6.3.34). Considering the gravitational part of the generalized dynamical Lagrangian (2.2.123) derived for arbitrary metric perturbations h^a (2.2.121), we examine the Lagrangian

$$\mathscr{L}_{a1} = h^a \frac{\delta \bar{\mathscr{L}}_H}{\delta \bar{g}^a} = -\frac{1}{16\pi} \mathfrak{h}^{\mu\nu}_a \bar{R}_{\mu\nu} . \tag{6.3.36}$$

This generalizes (6.3.35), in the sense that it uses the generic form of the independent dynamical variables $\mathfrak{h}^{\mu\nu}_a$ defined in (2.2.129). Applying the Noether procedure, one has to study now

$$\pounds_\xi \mathscr{L}_{a1} + \partial_\mu (\xi^\mu \mathscr{L}_{a1}) \equiv 0 \tag{6.3.37}$$

instead of (6.3.34). It is a family of identities corresponding to different definitions of metric perturbations in (2.2.121). This leads to a family of conserved quantities: currents and superpotentials. Step by step, one obtains

$$\boldsymbol{j}^\mu_a \equiv \partial_\nu \boldsymbol{j}^{\mu\nu}_a \equiv \bar{\nabla}_\nu \boldsymbol{j}^{\mu\nu}_a \tag{6.3.38}$$

with the current

$$\boldsymbol{j}^\mu_a(\xi) = {}_a\boldsymbol{\tau}_\nu{}^\mu \xi^\nu + \boldsymbol{z}^\mu_a(\xi) , \tag{6.3.39}$$

where the energy-momentum is

$${}_a\boldsymbol{\tau}^{\mu\nu} = \frac{1}{8\pi}\left(G^{\mu\nu}_L(\mathfrak{h}_a) + \mathfrak{h}^{\rho\mu}_a \bar{R}^\nu_\rho \right) . \tag{6.3.40}$$

The last term in the current, z-term, is

$$\begin{aligned} 16\pi \boldsymbol{z}^\mu_a(\xi) = {} & 2\left(\zeta^{\rho\sigma} \bar{\nabla}_\rho \mathfrak{h}^\mu_{(a)\sigma} - \mathfrak{h}^{\rho\sigma}_a \bar{\nabla}_\rho \zeta^\mu{}_\sigma \right) - \left(\zeta_{\rho\sigma} \bar{\nabla}^\mu \mathfrak{h}^{\rho\sigma}_a - \mathfrak{h}^{\rho\sigma}_a \bar{\nabla}^\mu \zeta_{\rho\sigma} \right) \\ & + \left(\mathfrak{h}^{\mu\nu}_a \bar{\nabla}_\nu \zeta^\rho{}_\rho - \zeta^\rho{}_\rho \bar{\nabla}_\nu \mathfrak{h}^{\mu\nu}_a \right) . \end{aligned} \tag{6.3.41}$$

A family of superpotentials, $\boldsymbol{j}^{\mu\nu}_a$ formally coincides with the one in (6.3.38), and explicitly is expressed as

$$\begin{aligned} \boldsymbol{j}^{\mu\nu}_a & = \frac{1}{8\pi} \mathfrak{h}^{\rho[\mu}_a \bar{\nabla}_\rho \xi^{\nu]} + \frac{1}{16\pi} \bar{\nabla}_\sigma \left(\bar{g}^{\rho\mu} \mathfrak{h}^{\nu\sigma}_a - \bar{g}^{\rho\nu} \mathfrak{h}^{\mu\sigma}_a - \bar{g}^{\sigma\mu} \mathfrak{h}^{\nu\rho}_a + \bar{g}^{\sigma\nu} \mathfrak{h}^{\mu\rho}_a \right) \xi_\rho \\ & = \frac{1}{8\pi} \left(\xi^{[\mu} \bar{\nabla}_\sigma \mathfrak{h}^{\nu]\sigma}_a - \xi^\sigma \bar{\nabla}^{[\mu} \mathfrak{h}^{\nu]}_{a\ \sigma} + \mathfrak{h}^{\rho[\mu}_a \bar{\nabla}_\rho \xi^{\nu]} \right) . \end{aligned} \tag{6.3.42}$$

One easily recognizes that it is the generalization to arbitrary displacement vectors and to arbitrary curved backgrounds of the family (4.2.19).

Now let us answer the question on the origin of the Boulware-Deser ambiguity. After using the field equations the current (6.3.38) transforms to

$$\mathscr{I}^{\mu}_{a}(\xi) = {}_{a}\boldsymbol{\theta}_{\nu}{}^{\mu}\xi^{\nu} + \boldsymbol{z}^{\mu}_{a}(\xi)\,, \tag{6.3.43}$$

where the energy-momentum is

$${}_{a}\boldsymbol{\theta}^{\mu\nu} = \boldsymbol{t}^{\text{tot}}_{\mu\nu}(\mathfrak{h}_a) + \frac{1}{8\pi}\left(\mathfrak{h}^{(a)\rho}_{\mu}\bar{R}_{\nu\rho} - \Phi^{L}_{\mu\nu}(\mathfrak{h}_a)\right). \tag{6.3.44}$$

Also for $\mathfrak{h}^{\mu\nu}_{a}$ which are solutions of the field equations, we have $\boldsymbol{j}^{\mu\nu}_{a} = \mathscr{I}^{\mu\nu}_{a}$ with the same form (6.3.42). Finally, with the use of the field equations the identity (6.3.38) transforms to the conservation laws:

$$\mathscr{I}^{\mu}_{a} = \partial_{\nu}\mathscr{I}^{\mu\nu}_{a} \equiv \bar{\nabla}_{\nu}\mathscr{I}^{\mu\nu}_{a}\,. \tag{6.3.45}$$

At last, taking into account the difference in perturbations (4.2.21) and linearity in variables $\mathfrak{h}^{\mu\nu}_{a}$, one finds the difference between the terms of the family of superpotentials (6.3.42):

$$\Delta\mathscr{I}^{\mu\nu}_{a} = \frac{1}{\kappa}\left(\xi^{[\mu}\bar{\nabla}_{\sigma}({}_{12}\mathfrak{h}^{\nu]\sigma}_{a}) - \xi^{\sigma}\bar{\nabla}^{[\mu}({}_{12}\mathfrak{h}^{\nu]}_{a\ \sigma}) + {}_{12}\mathfrak{h}^{\rho[\mu}_{a}\bar{\nabla}_{\rho}\xi^{\nu]}\right). \tag{6.3.46}$$

It represents the Boulware-Deser ambiguity in the definition of the superpotentials.

6.3.3 Comments on conserved quantities of three types

Boulware-Deser ambiguity

Note that the KBL approach does not depend on the choice of the variables, like $g^{\mu\nu}$, $g_{\mu\nu}$, $\mathfrak{g}^{\mu\nu}$, …, etc. As a result the Belinfante symmetrization does not depend on such a choice either and *uniquely* leads to the conservation law (6.2.4) with the superpotential (6.2.18). On the other hand, in the field-theoretical formulation we have constructed the family of the superpotentials (6.3.42). What form is more preferable? It turns out that *only* the superpotential (6.3.23) from the family (6.3.42) corresponds to the Belinfante corrected superpotential (6.2.18). This is a theoretical argument in favor of the choice of the variable $\mathfrak{h}^{\mu\nu}_{(a)} = \mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}$ in the field-theoretical formulation, resolving the Boulware-Deser ambiguity [71].

The KBL approach is not connected with the field-theoretical one directly. Indeed, the variation of the KBL Lagrangian (6.1.2) with respect to the background metric leads to the background quantities only. However, the Belinfante quantities, being equivalent to the field-theoretical ones and being obtained from the KBL quantities, can be classified as a "bridge" between the two approaches.

At last, because the field-theoretical quantities do not depend on divergences in the Lagrangian (by definition) the same property is valid for the Belinfante corrected conserved quantities owing the equivalence between these methods. This supports the claim of the Section 6.2 that the Belinfante corrected conserved quantities in general relativity do not depend on divergences in the Lagrangian. We repeat that on a more general level, in the framework of multidimensional metric theories, this property of the Belinfante procedure is proved below, see Section 7.1.4 from (7.1.85) to (7.1.87).

Integral constraints

Perhaps the most important property of conservation laws in general relativity is that the conserved quantities in a finite or infinite volume can always be expressed in terms of the surface integrals taken over the boundary of the volume or at spatial infinity. Such a representation is possible because the local value of a conserved current can be always expressed in terms of a divergence from a corresponding superpotential

$$\mathcal{J}^{\mu}(\xi) = \partial_{\nu}\mathcal{J}^{\mu\nu}(\xi) \equiv \bar{\nabla}_{\nu}\mathcal{J}^{\mu\nu}(\xi), \tag{6.3.47}$$

as shown in (6.1.30), (6.2.19) and (6.3.24). Currents, $\mathcal{J}^{\mu}(\xi)$, depend both on the perturbed metric and its derivatives and on the perturbation of the matter energy-momentum, $\delta\mathcal{T}_{\nu}{}^{\mu}$. At the same time, superpotentials, $\mathcal{J}^{\mu\nu}(\xi)$, in all cases (6.1.25), (6.2.18) and (6.3.42), contain the perturbed metric and its first (not higher) order derivatives only, and, what is the most important, they do not depend on the matter perturbations.

Let us consider a 3-dimensional spacelike hypersurface, $\Sigma := x^0 = \mathrm{const}$, which has a 2-dimansional boundary, $\partial\Sigma$. Then, by integrating (6.3.47) one obtains a conserved quantities related to a displacement vector ξ^{α} in the form:

$$\mathcal{P}(\xi) = \int_{\Sigma} dx^3 \mathcal{J}^{0}(\xi) = \oint_{\partial\Sigma} ds_i \mathcal{J}^{0i}(\xi)\,. \tag{6.3.48}$$

In the literature, as a rule, superpotentials and formule, like (6.3.48), are more often used as compared with current $\mathcal{J}^{\mu}(\xi)$. The usefulness of superpotentials was strongly advocated by Penrose [354] who introduced the notion of “quasi-local” quantities which, in the weak field limit, reduce to the ordinary conserved linear momentum and angular momentum of gravitational field in a finite volume. Many papers over the past decades have been published on the subject of the quasi-local energy, see the comprehensive review by Szabados [430]. The role and importance of superpotentials in a field theory has been emphasized by Julia and Silva [247, 414] who provided their elegant treatment and put their generic theory on a rigorous mathematical basis.

Let us come back to (6.3.48), and suppose that the boundary values of the metric, thus $\mathcal{J}^{\mu\nu}(\xi)$, on $\partial\Sigma$ are given. Then, (6.3.48) can be interpreted as a set of *integral constraints* on the perturbations of the energy-momentum $\delta\mathcal{T}_{\nu}{}^{\mu}$ for the given initial perturbations of the metric on Σ. Reciprocally, if $\delta\mathcal{T}_{\nu}{}^{\mu}$ is given, the relation (6.3.48) represents the integral constraints for the initial metric data on Σ. Among all of the integral constraints a special role is played by the integral constraints connecting the boundary values of the metric to the matter sources on Σ, see the last section of the present chapter.

The simplest connections between currents and superpotentials

We have already demonstrated in previous sections how the conserved quantities defined on a curved background are related to the classical pseudotensors and superpotentials related to a flat Minkowski space. For clarity of this comparison we consistently used the Lorentzian coordinates on the flat background. Now, we extend this analysis to the case of curvilinear coordinates on the background Minkowski space. Rosen [390, 391] was the first who found that the quadratic Lagrangian, see (1.3.9), used by Einstein to derive a conserved pseudotensor (1.4.30) can be written in a covariant form by introducing a second (background) metric. The mathematical basis of Rosen's approach was strengthen by Lichnerowicz [291]. Below we explain how the Rosen procedure works in more detail.

Let us consider a flat background, $\bar{R}^{\mu}{}_{\nu\rho\sigma} = 0$, in arbitrary coordinates. At the moment, let us go back to the "divergence dependent" energy-momentum (6.1.27) that takes on the flat background the following form

$$ {}_{C}\boldsymbol{\theta}_{\nu}{}^{\mu} = \mathscr{T}_{\nu}{}^{\mu} + {}_{R}\boldsymbol{t}_{\nu}{}^{\mu} \,, \tag{6.3.49} $$

where ${}_{R}\boldsymbol{t}_{\nu}{}^{\mu}$ defined in (6.1.11) is the energy-momentum *tensor* density given by Rosen in arbitrary coordinates as a covariantized generalization of Einstein's pseudotensor (1.4.30), ${}_{E}\boldsymbol{t}_{\nu}{}^{\mu}$. Now, in the case of a flat background, let us turn to the Klein identities (6.1.19–6.1.22), which takes on the form

$$ {}_{C}\boldsymbol{\theta}_{\nu}{}^{\mu}{}_{;\mu} = 0 \,, \tag{6.3.50} $$

$$ {}_{C}\boldsymbol{\theta}_{\nu}{}^{\mu} = -\boldsymbol{\sigma}^{\rho\mu}{}_{\nu;\rho} \,. \tag{6.3.51} $$

Thus, on a flat background ${}_{C}\boldsymbol{\theta}_{\nu}{}^{\mu}$ is a divergence of a tensor density which is not antisymmetric with respect to the upper indices but still acting like a "superpotential" for volume integrals in the Minkowski space. We find that $\boldsymbol{\sigma}^{\lambda\mu\nu} = -\mathscr{T}^{\nu\mu\lambda}$ that is Tolman's covariantized superpotential, see (1.4.32), which seems to be the first one known in the literature. This superpotential is also closely related to Freud's superpotential (1.4.39), $\mathscr{F}_{\lambda}{}^{\mu\nu}$. The relationship is defined in arbitrary coordinates by (1.4.38):

$$ \boldsymbol{\sigma}^{\lambda\mu\nu} = -\mathscr{F}^{\nu\mu\lambda} + \frac{1}{8\pi}\left(\mathfrak{h}^{\mu[\rho}\bar{\mathfrak{g}}^{\lambda]\nu}\right)_{;\rho} \,. \tag{6.3.52} $$

Since the covariant derivatives in a flat spacetime are commutative, by taking the divergence of $\boldsymbol{\sigma}^{\lambda\mu\nu}$ and using its relation with ${}_{C}\boldsymbol{\theta}_{\nu}{}^{\mu}$ we obtain

$$ {}_{C}\boldsymbol{\theta}^{\mu\nu} = \mathscr{F}^{\mu\nu\lambda}{}_{;\lambda} \tag{6.3.53} $$

that is a covariantized relation (1.4.40) where the covariantized Freud superpotential, $\mathscr{F}_{\nu}{}^{\mu\lambda}$, is given in (1.5.39).

Considering the Belinfante corrected quantities in the case of the flat background, one finds for the energy-momentum (6.2.9):

$$ {}_{B}\boldsymbol{\theta}^{\mu\nu} = \mathscr{T}_{\rho}^{(\mu}\gamma^{\nu)\rho} + \boldsymbol{t}_{B}^{\mu\nu}. \tag{6.3.54}$$

Conservation law (6.2.19) for the current (6.2.11) leads to

$$ {}_{B}\boldsymbol{\theta}_{\nu}{}^{\mu}{}_{;\mu} = 0\,, \tag{6.3.55}$$

$$ {}_{B}\boldsymbol{\theta}_{\nu}{}^{\mu} = \mathscr{P}_{\nu}{}^{\mu\rho}{}_{;\rho}\,, \tag{6.3.56}$$

where $\mathscr{P}_{\nu}{}^{\mu\rho}$ is the covariantized Papapetrou superpotential given in (6.2.17) or in another form in (1.5.42).

6.4 Criteria for the choice of conserved quantities

6.4.1 Tests of consistency

When applying various methods for constructing conservation laws one has to check the consistency of corresponding conserved quantities. There is a number of selection rules for choosing the most preferable formulation of conservation laws in various physical situations but they are not universally accepted and must be applied with care in each particular case. Nonetheless, there are certain physical principles which must be satisfied to make the conserved quantities physically meaningful and consistent. Only after satisfying such tests formulae under consideration can be thought as useful for applications. What are these principles and the consistency check points? Usually, they are based on testing the properties of exact solutions and models in general relativity which have a well-known dependence on the physical parameters, like mass, energy, angular momentum, etc. In most cases the physical consistency of the conserved quantities is tested in the weak-field approximation of general relativity by assuming the existence of the flat Minkowskian background. Such approach is rather straightforward and fairly simple.

We are interested in checking the formulae of the three approaches to building the conserved quantities presented in this chapter. Because in general relativity the currents and superpotentials in the field-theoretical approach and the Belinfante corrected method are equivalent, it is enough to check one of these two only.

A number of the consistency tests is listed below. Of course, more tests can be added in the course of development of the theory. Thus,

(i) One of the most important exact solutions in general relativity is the Schwarzschild solution. Because the constant of integration m (mass parameter) is chosen to be equal to the total mass of the system it is evident that the total energy of a system described by the Schwarzschild solution has to be equal to the mass: $E = m$ (in the present chapter we set $G = c = 1$).

Calculation of the total energy is performed the most conveniently by making use of the above mentioned approach based on the surface integration. The result of the integration provides the test for consistency of the KBL superpotential (6.1.25) and the Belinfante corrected superpotential (6.2.18). Differential laws to performing the surface integration tests are given in (6.3.53) for covariantized Freud's superpotential and in (6.3.56) for covariantized Papapetou's superpotential. Their asymptotic behavior for the Schwarzschild solution is given in (1.5.50), that leads to the acceptable result (1.5.52): $E = m$.

(ii) The consistency test of the conserved quantities based on the Schwarzschild solution is incomplete and can be extended by taking into consideration the exact solutions which admit the asymptotically flat Minkowski space. It includes all rotating and charged black hole solutions in general relativity with the mass parameter m, the rotational parameter a and the charge parameter Q. As a result of calculation of the corresponding conserved quantities with the help of surface integration of a superpotential, one has to obtain the total energy E and total angular momentum M of the central black hole expressed in terms of a physically reasonable combination of the black holes parameters, like $E = m$ and $M = ma$ for the Kerr black hole, etc.

Below in Section 6.4.2 we consider the Reissner-Nordström solution [315, 336, 386], calculate its total energy and discuss the result. Another important solution in general relativity is the Kerr solution [255, 315] that is examined in Section 6.4.3 and its total energy and total angular momentum are studied. The Kerr-Newman solution [315, 334] generalizes all of these, but we do not consider it here. As a matter of exercise, the reader is invited to test the KBL superpotential and the Belinfante corrected superpotential applying them to the Kerr-Newman black hole.

(iii) The quadrupole formula in the gravitational radiation formalism [285] states that the energy emitted in the form of gravitational waves is positive. Thus, the density of the flux of weak gravitational waves propagating on a flat background has to be positive as well. This offers another possibility for checking the self-consistency of various types of energy-momentum tensors that appear in different versions of conservation laws.

To avoid long calculations it is enough to consider simple expressions for the energy-momentum presented in (6.3.49) and in (6.3.54). Their self-consistency have been checked in Chapter 1 in the quadratic approximation by making use of equation (1.5.32). All tensors yield positively-defined energy as shown in equation (1.5.33).

(iv) For radiating isolated systems described by Bondi and others [59–61, 395] there is another important formula for the rate of energy loss corresponding to a so-called BMS radiation. This rate of the total energy loss is another test for checking the different variants of superpotentials.

We do not consider the BMS radiation test in the book, however, recommend to read the original works [252, 369, 370].

The above four possible tests of consistency of the superpotentials are available in the asymptotically flat spacetimes and in the weak-field approximation with respect to a flat background. Of course, analogous tests could be extended to curved backgrounds, however the authenticity of such tests generally may cause doubts. Nevertheless, there are solutions on curved backgrounds, which have been thoroughly studied mathematically and have well-established physical properties. These are asymptotically anti-de Sitter (AdS) solutions for black holes: generally they belong to the Reissner-Nordström-AdS family of solutions, and to the Kerr-Newman-AdS family. These general solutions are too complicated for calculations, therefore here in Section 6.4.4

(v) we will calculate, as an example, the total energy of the Schwarzschild-AdS solution only.

We would suggest to the reader to test the KBL superpotential and the Belinfante corrected superpotential by applying them for calculating the global conserved quantities for the other black holes possessing the AdS asymptotics.

6.4.2 The Reissner-Nordström solution

Because below we shall consider the asymptotically flat solutions with spherical symmetry, the spherical coordinates introduced on the flat background are the most appropriate for handling the calculations that follows. In spherical coordinates the background metric, $\gamma_{\mu\nu}$, has the form (4.1.1), for which the non-zero components of the Christoffel symbols, $C^{\alpha}{}_{\mu\nu}$, are given in (4.1.2).

The linear element of the Reissner-Nordström solution [315] reads:

$$ds^2 = -\left(1 - \frac{r_g}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{1}{1-\left(r_g/r - Q^2/r^2\right)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (6.4.1)$$

At first, we check the KBL superpotential (6.1.25) for calculating the total energy of the charged black hole represented by this solution. For the case of a flat background with translation Killing vectors for which $\xi^{\alpha}_{K;\beta} = 0$, see (1.2.7), the KBL superpotential transforms to the covariantized Freud superpotential (1.5.39),

$$\begin{aligned}\mathscr{I}_C^{\alpha\beta}(\xi_K) &= \mathscr{F}_\sigma{}^{\alpha\beta}\xi^\sigma_K = \sqrt{-\gamma}F_\sigma{}^{\alpha\beta}\xi^\sigma_K\\ &= \frac{1}{8\pi}\left(\mathfrak{g}^{\rho[\alpha}\Delta^{\beta]}{}_{\rho\sigma} + \delta^{[\alpha}_\sigma \mathfrak{g}^{\beta]\pi}\Delta^\rho{}_{\rho\pi} - \delta^{[\alpha}_\sigma\Delta^{\beta]}{}_{\mu\nu}\mathfrak{g}^{\mu\nu}\right)\xi^\sigma_K . \qquad (6.4.2)\end{aligned}$$

The surface integration (1.5.45):

$$\mathscr{P}(\xi_K) = \oint_\infty ds_1 \mathscr{I}_C^{01}(\xi_K) = \lim_{r\to\infty}\int_0^\pi d\theta\int_0^{2\pi} d\phi\, r^2\sin\theta F_\sigma{}^{01}\xi^\sigma_K \qquad (6.4.3)$$

allows us to calculate the total energy with timelike Killing vector of the Minkowski space

$$\xi_K^\alpha = (-1, \mathbf{0})\,. \tag{6.4.4}$$

Let us write the Christoffel symbols for the metric (6.4.1):

$$\begin{aligned}
\Gamma^0{}_{10} &= -\Gamma^1{}_{11} = \frac{1}{2}\,\frac{r_g/r^2 - 2Q^2/r^3}{1 - r_g/r + Q^2/r^2}\,,\\
\Gamma^1{}_{00} &= \frac{1}{2}\left(\frac{r_g}{r^2} - \frac{2Q^2}{r^3}\right)\left(1 - \frac{r_g}{r} + \frac{Q^2}{r^2}\right),\\
\Gamma^1{}_{22} &= -r\left(1 - \frac{r_g}{r} + \frac{Q^2}{r^2}\right),\ \ \Gamma^1{}_{33} = -r\sin^2\theta\left(1 - \frac{r_g}{r} + \frac{Q^2}{r^2}\right),\\
\Gamma^2{}_{21} &= \Gamma^3{}_{31} = \frac{1}{r},\ \ \Gamma^3{}_{32} = \cot\theta,\ \ \Gamma^2{}_{33} = -\sin\theta\cos\theta
\end{aligned} \tag{6.4.5}$$

and calculate the tensor $\Delta^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} - C^\mu{}_{\alpha\beta}$ in (6.4.2) with the use of (4.1.2) and (6.4.5). As a result, one arrives to the conclusion that the asymptotic behavior of the corresponding component of the superpotential entering the surface integral in (6.4.3), is

$$F_0{}^{01} \sim -\frac{r_g}{8\pi}\frac{1}{r^2}\,. \tag{6.4.6}$$

Then, immediately the integration (6.4.3) gives:

$$E = m\,. \tag{6.4.7}$$

After examining the KBL approach let us check the Belinfante corrected superpotential (6.2.18) for calculating the total energy of the charged black hole. For the case of a flat background with the translation Killing vectors it transforms to

$$\mathscr{J}_B^{\mu\nu} = \frac{1}{8\pi}\left(\delta_\sigma^{[\mu}\mathfrak{h}^{\nu]\rho}{}_{;\rho} + \mathfrak{h}^{[\mu}{}_\sigma{}^{;\nu]}\right)\xi_K^\sigma\,. \tag{6.4.8}$$

With the use of (4.1.5) we derive the field configuration corresponding to the solution (6.4.1):

$$h^{00} = -\frac{r_g/r - Q^2/r^2}{1 - r_g/r + Q^2/r^2}\,, \qquad h^{11} = -\left(\frac{r_g}{r} - \frac{Q^2}{r^2}\right). \tag{6.4.9}$$

We use these components in (6.4.8) for calculating the total energy with the timelike Killing vector (6.4.4) employed for calculation of the surface integral:

$$\mathscr{P}(\xi_K) = \oint_\infty ds_1 \mathscr{J}_B^{01}(\xi_K) = \lim_{r\to\infty}\frac{1}{16\pi}\int_0^\pi d\theta\int_0^{2\pi} d\phi\, r^2\sin\theta\left(h^{00}\gamma^{j1} + \gamma^{00}h^{j1}\right)_{;j} \tag{6.4.10}$$

that yields the physically-meaningful result (6.4.7). The same result is obtained with the use of the superpotential (6.3.23) in the field-theoretical derivation that also agrees with formula (4.1.11) for calculating the total mass of an isolated system.

At last, one has to check all the superpotentials in the family (6.3.42). We have checked out that for each of them the integration analogous to (6.4.10) gives again the same result (6.4.7) for the total energy. Thus, considering the total energy of the Reissner-Nordstróm solution, all superpotentials constructed in the present chapter are equivalent from the computational point of view.

One can see that the electric charge does not explicitly contribute to the total energy. It is not surprising because the parameter m has been chosen as a unique characteristic for the total energy of the system to correspond to the Newtonian mass in the weak-field approximation. Formally, the charge does not contribute to the energy because the part of the superpotential corresponding to the charge of the black hole falls off as $\sim 1/r^2$, that decays significantly faster than $\sim 1/r$.

6.4.3 The Kerr solution

The Kerr solution, describing the rotating black holes, is more complicated but, at the same time, more interesting for constructing conserved quantities. The line element of the Kerr solution has the form [315]:

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{r_g r}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\
&\quad + \left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2\theta\right) \sin^2\theta d\phi^2 - \frac{2 r_g r a}{\rho^2} \sin^2\theta d\phi\, c dt ; \\
\Delta &\equiv r^2 - r_g r + a^2, \\
\rho^2 &\equiv r^2 + a^2 \cos^2\theta ,
\end{aligned}
\tag{6.4.11}
$$

where the parameter $a = \text{const}$ characterizes the speed of rotation; when $a \to 0$ the Kerr solution (6.4.11) transforms to the Schwarzschild solution (4.1.3). Thus, it is interesting to construct for the solution (6.4.11) both the total energy and the total angular momentum. To carry out the calculations we choose a flat background with the flat metric (4.1.1) in spherical coordinates and the correspondent Christoffel symbols (4.1.2).

First, we check the KBL superpotential (6.1.25). For calculating the total energy with timelike Killing vector (6.4.4) we use the covariantized Freud superpotential (1.5.39), see also (6.4.2), and integration (6.4.3). We do not derive here cumbersome expressions for quantities $\Delta^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} - C^\mu{}_{\alpha\beta}$ and merely notice that a necessary component of the Freud superpotential has again the asymptotic behavior (6.4.6). Then, integration (6.4.3) gives physically-adequate result (6.4.7): $E = m$.

To construct the total angular momentum one chooses the spacelike Killing vector

$$
\xi^\alpha_K = (0, 0, 0, 1) , \qquad \xi^K_\alpha = \left(0, 0, 0, r^2 \sin^2\theta\right) . \tag{6.4.12}
$$

In this case, it is more convenient to use the form (6.1.24) for the KBL superpotential. We notice that for calculations it is sufficient to use the component $\mathcal{I}_C^{01}$ where, with the Killing vector (6.4.12), only one term survives,

$$\mathscr{I}_C^{01}(\xi_K) = \frac{1}{8\pi}\sqrt{-g}\nabla^{[0}\xi_K^{1]}\,, \tag{6.4.13}$$

which is purely a Komar part. Then, the standard integration gives the angular momentum of the Kerr black hole

$$M = \mathscr{P}(\xi_K) = \oint_\infty ds_1 \mathscr{I}_C^{01}(\xi_K) = \lim_{r\to\infty}\frac{1}{8\pi}\int_0^\pi d\theta\int_0^{2\pi} d\phi\sqrt{-g}\nabla^{[0}\xi_K^{1]} = ma\,, \tag{6.4.14}$$

which is the expected result.

Now, let us turn to the Belinfante corrected quantities (6.2.18), or the same (6.3.23) in the field-theoretical formulation. Again, to calculate the total energy we use the timelike Killing vector that leads to integration (6.4.10). Necessary components of the field configuration corresponding to (6.4.11) have a complicated form, but for us the knowing the asymptotic behavior

$$h^{00} \sim -\frac{r_g}{r}\,, \quad h^{11} \sim -\frac{r_g}{r} \tag{6.4.15}$$

is sufficient. Then, the integral formula (6.4.10) yields for the Kerr solution (6.4.11) the expected result (6.4.7).

To calculate the total angular momentum in the Belinfante corrected derivation we use the Killing vector (6.4.12) and substitute it into (6.2.18), or the same (6.3.23) in the field-theoretical formulation. Next, integration leads to formula (3.2.27), which for the space components ξ_K^k acquires the form:

$$\begin{aligned}\mathscr{P}(\xi_K) &= \oint_\infty ds_1 \mathscr{I}_S^{01}(\xi_K)\\ &= \lim_{r\to\infty}\frac{1}{16\pi}\int_0^\pi d\theta\int_0^{2\pi} d\phi\, r^2\sin\theta\Big[\left(h^{0k}\gamma^{j1} - h^{11}\gamma^{jk}\right)_{;j}\xi_k^K\\ &\quad - \left(h^{0k}\gamma^{1j} - h^{0j}\gamma^{1k}\right)\xi^K_{k;j}\Big]\,. \end{aligned}\tag{6.4.16}$$

For the solution (6.4.11) only the component h^{03}, from all of the components h^{0j}, is not zero with the asymptotic behavior

$$h^{03} \sim -\frac{r_g a}{r^3}\,. \tag{6.4.17}$$

Then, for the field-theoretical configuration (6.4.15) and (6.4.17), the calculation in (6.4.16) finally gives the acceptable result (6.4.14).

At last, one has to note that each superpotential from the family (6.3.42) in the field-theoretical derivation leads to the same results (6.4.7) and (6.4.14) for the Kerr solution (6.4.11). Thus, considering the total energy and the total angular momentum of the Kerr solution, all of the formulae for conserved quantities suggested in Sections 6.1–6.3 pass through the consistency tests.

6.4.4 The total KBL energy for the S-AdS solution

We have already calculated in Section 4.2.3 a total mass for the Schwarzschild-AdS black hole defined by the solution (4.2.30) with the use of the field-theoretical formalism. Moreover, we have checked all of the members of the family (4.2.19). Notice that the Belinfante corrected superpotential is included into this family. Thus, all of the such quantities lead to acceptable result (4.2.38): $E = m$. In what follows, we are checking the KBL superpotential (6.1.25) by calculating the total mass for the S-AdS black hole with all the assumptions of Section 4.2.3.

The KBL formulae (6.4.2–6.4.4) are also valid for the AdS background, and we use them. For the S-AdS solution the metric components are defined in (4.2.30); to find $\Delta^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \bar{\Gamma}^\rho{}_{\mu\nu}$ we use the components $\bar{\Gamma}^\rho{}_{\mu\nu}$ derived in (4.2.30) and the non-zero Christoffel symbols corresponding to the metric (4.2.30):

$$\begin{aligned}
&\Gamma^1{}_{22} = -r\left(1 + \frac{r^2}{l^2} - \frac{r_g}{r}\right), \quad \Gamma^1{}_{33} = -r\sin^2\theta\left(1 + \frac{r^2}{l^2} - \frac{r_g}{r}\right),\\
&\Gamma^0{}_{01} = -\Gamma^1{}_{11} = \frac{1}{2}\,\frac{2r/l^2 + r_g/r^2}{1 + r^2/l^2 - r_g/r},\\
&\Gamma^1{}_{00} = \frac{1}{2}\left(\frac{2r}{l^2} + \frac{r_g}{r^2}\right)\left(1 + \frac{r^2}{l^2} - \frac{r_g}{r}\right),\\
&\Gamma^2{}_{12} = \Gamma^3{}_{13} = \frac{1}{r}, \quad \Gamma^2{}_{33} = -\sin\theta\cos\theta, \quad \Gamma^3{}_{23} = \cot\theta.
\end{aligned} \tag{6.4.18}$$

Formula (6.4.3), rewritten for the Killing vector (4.2.8) is

$$\mathscr{P}(\bar{\xi}) = \oint_\infty ds_1 \mathscr{I}_C^{01}(\bar{\xi}) = \lim_{r\to\infty} \int_0^\pi d\theta \int_0^{2\pi} d\phi\, r^2 \sin\theta F_\sigma{}^{01}\bar{\xi}^\sigma \tag{6.4.19}$$

and gives

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi\, r^2 \sin\theta \left(g^{1\pi}\Delta^\rho{}_{\rho\pi} - \Delta^1{}_{\mu\nu} g^{\mu\nu}\right) = m \tag{6.4.20}$$

that is the acceptable result (4.2.38).

We conclude that all three methods of calculation of the S-AdS mass are equally acceptable as they yield one and the same result for the total energy.

6.5 The FLRW solution as a perturbation on the de Sitter background

Strongly or weakly perturbed FLRW spacetimes are naturally related to FLRW background universe which is an exact, time-dependent solution of Einstein's equations.

However, the FLRW spacetimes admit *only* six Killing vectors, each of them generates a corresponding conservation law with an associated conserved quantity. Among them, unfortunately, one cannot find energy and the Lorentz momentum. We may, however, map the FLRW spacetime (perturbed or not) on a de Sitter space, which has ten Killing vectors making up the full group of motions. One can construct ten conserved quantities corresponding to the Killing vectors. The KBL approach, which is applicable both to finite and infinitesimally small perturbations, can be used to construct such quantities. As a result, the four currents and superpotentials corresponding to the Killing vectors of the de Sitter space that are absent in FLRW spacetime describe the energy and Lorentz momentum. In the present section, we do not consider a perturbed FLRW model. We map hypersurfaces of constant cosmic time of an exact FLRW spacetime on the corresponding hypersurfaces of the de Sitter space having the same topology. The difference between FLRW and de Sitter metrics will be considered as a relative perturbation defined on the de Sitter background, and for them the conserved KBL currents and superpotentials will be constructed. At the end, we discuss how such a construction may help to describe perturbations in the FLRW universe along with all of the ten conserved quantities.

6.5.1 Spatially conformal mappings of FLRW spacetime onto de Sitter space

FLRW and de Sitter metrics

Let FLRW spacetime be described by equation (5.3.1) in the Cartesian coordinates x^k. We choose the "cosmic time" T defined by $dT = \omega(t)dt$ in the FLRW spacetime. Then the FLRW metric reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\omega^2 dt^2 + g_{kl}dx^k dx^l = -\omega^2 dt^2 + a^2 e_{kl}dx^k dx^l\,, \tag{6.5.1}$$

where $\omega = \omega(t)$ and $a = R\,(T(t)) = a(t)$. The conformal metric $e_{kl}(x^m)$ has a particular form for closed, flat or open $x^0 = t = \text{const}$ hypersurfaces given by

$$e_{kl} = \delta_{kl} + k\frac{x_k x_l}{1 - kr^2}\,, \quad e^{kl} = \delta^{kl} - kx^k x^l, \quad e = \det e_{kl} = \frac{1}{1 - kr^2}\,, \tag{6.5.2}$$

where $x_k = \delta_{km}x^m$ and $r^2 = \delta_{kl}x^k x^l$. The Hubble parameter (5.3.2) for the metric (6.5.1) has the form

$$H = \frac{1}{\omega}\frac{\dot a}{a}\,, \tag{6.5.3}$$

where a "dot" denotes differentiation with respect to t.

The metric of the de Sitter background in the same coordinates as in (6.5.1) has a similar form:

$$ds^2 = \bar g_{\mu\nu}dx^\mu dx^\nu = -\bar\omega^2 dt^2 + \bar g_{kl}dx^k dx^l = -\bar\omega^2 dt^2 + \bar a^2 e_{kl}dx^k dx^l\,, \tag{6.5.4}$$

where, $\bar{\omega}$ and $\bar{a}$ are functions of the "cosmic time" T in de Sitter space given by $dT = \bar{\omega}(t)dt$. The Hubble parameter of de Sitter space has the form

$$\bar{H} = \frac{1}{\bar{\omega}}\frac{\dot{\bar{a}}}{a}\,. \tag{6.5.5}$$

Hypersurfaces for (6.5.1) and (6.5.4) with the same t will be mapped on one another. By picking up a specific form of functions ω and $\bar{\omega}$, one fixes the correspondence between the cosmic times up to a constant. For the moment we shall fix none of them.

De Sitter Killing vectors

The ten Killing vectors of the de Sitter background, $\bar{\xi}^{\mu}$, satisfy the Killing equations

$$\bar{\nabla}_{(\rho}\bar{\xi}_{\sigma)} = 0\,, \tag{6.5.6}$$

where, as usual, the covariant derivative, $\bar{\nabla}_{\rho}$, is constructed with the use of $\bar{g}_{\mu\nu}$. The 00- and 0k-components of the Killing equations imply

$$\bar{\xi}^0 = \frac{1}{\bar{\omega}}\xi(x^k)\,, \tag{6.5.7}$$

$$\dot{\bar{\xi}}^k = -\bar{\omega}^2\bar{g}^{kl}\overset{3}{\nabla}_l\bar{\xi}^0\,, \tag{6.5.8}$$

where the component $\bar{\xi}^0$ is a function whose equation is given below, see (6.5.11), $\overset{3}{\nabla}_l$ is a e_{kl} (or $\bar{g}_{kl}$, or g_{kl}) 3-dimensional covariant derivative. It may be useful to remind the reader that indices are raised and lowered by $\bar{g}_{\mu\nu}$, as usual. Finally, the spatial part of the Killing equation (6.5.6) gives

$$e_{mk}\overset{3}{\nabla}_l\bar{\xi}^m + e_{ml}\overset{3}{\nabla}_k\bar{\xi}^m + 2\bar{\omega}\bar{H}e_{kl}\bar{\xi}^0 = 0\,. \tag{6.5.9}$$

The Hubble "constant" (6.5.5) of de Sitter space satisfies the relation

$$\frac{\dot{\bar{H}}}{\bar{\omega}} = \frac{k}{\bar{a}^2}\,, \tag{6.5.10}$$

which follows from Einstein's equations or, as the integrability condition of the equation (6.5.6). Then, if we take a partial t-derivative of (6.5.9) and make use of (6.5.8), we obtain

$$\overset{3}{\nabla}_{kl}\bar{\xi}^0 + ke_{kl}\bar{\xi}^0 = 0\,. \tag{6.5.11}$$

This equation has a solution. Then, having $\bar{\xi}^0$, we can obtain $\bar{\xi}^k$ from (6.5.8) and (6.5.9).

Explicit expressions for $\bar{\xi}^{\mu}$ and the corresponding finite group of transformations are given in Weinberg's book [456]. Any $\bar{\xi}^{\mu}$ could be a linear combination with constant coefficients of the following ten vectors:

(i) Quasi-translations in $t = \text{const}$:

$$\bar{\xi}^0_{(l)} = 0, \quad \bar{\xi}^k_{(l)} = \delta^k_l\sqrt{1-kr^2}\,. \tag{6.5.12}$$

(ii) Quasi-rotations in t = const:

$$\bar{\xi}^0_{[ls]} = 0, \quad \bar{\xi}^k_{[ls]} = \delta^{kl}x^s - \delta^{ks}x^l \,. \tag{6.5.13}$$

(iii) Time quasi-translations:

$$\bar{\xi}^0_{(0)} = -\frac{1}{\bar{\omega}}\sqrt{1-kr^2}, \quad \bar{\xi}^k_{(0)} = \bar{H}x^k\sqrt{1-kr^2} \,. \tag{6.5.14}$$

(iv) Lorentz quasi-rotations:

$$\bar{\xi}^0_{[0l]} = -\frac{1}{\bar{\omega}}x^l, \tag{6.5.15}$$

$$k = 0 \rightarrow \bar{\xi}^k_{[0l]} = \bar{H}\left[x^k x^l - \frac{1}{2}\delta^{kl}\left((r^2 - \frac{\bar{\omega}^2}{\dot{\bar{a}}^2}\right)\right], \tag{6.5.16}$$

$$k = \pm 1 \rightarrow \bar{\xi}^k_{[0l]} = \bar{H}\left[x^k x^l - k\delta^{kl}\right] \,. \tag{6.5.17}$$

It is important to notice that the Killing vectors (6.5.12) and (6.5.13) are also the Killing vectors of the FLRW spacetimes. The vectors (6.5.14) and (6.5.15–6.5.17) are conformal Killing vectors of the FLRW spacetimes. The conformal Killing vectors are discussed in the next section.

6.5.2 Superpotentials and conserved currents

To obtain the KBL superpotentials for the above model, we use the formulae presented in Section 6.1, see (6.1.25). With the metric components in (6.5.1) and (6.5.4), we calculate the difference between the two metrics $h^{\mu\nu} = (\mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu})/\sqrt{-\bar{g}}$ which we interpret as a perturbation of the de Sitter space. The non-vanishing components of the perturbation are:

$$h^{00} = \frac{1}{\bar{\omega}^2}\left(1 - \frac{\bar{\omega}}{\omega}\frac{a^3}{\bar{a}^3}\right), \quad h^{ik} = \bar{g}^{ik}\left(\frac{\omega}{\bar{\omega}}\frac{a}{\bar{a}} - 1\right) \,. \tag{6.5.18}$$

The Christoffel symbols corresponding to the metrics (6.5.1) and (6.5.4), $\Gamma^\lambda{}_{\mu\nu}$ and $\bar{\Gamma}^\lambda{}_{\mu\nu}$, and their difference $\Delta^\lambda{}_{\mu\nu}$ are given in (6.5.19–6.5.21) below. More specifically, we have:

$$\Gamma^0{}_{00} = \frac{\dot{\omega}}{\omega}, \quad \bar{\Gamma}^0{}_{00} = \frac{\dot{\bar{\omega}}}{\bar{\omega}} \rightarrow \Delta^0{}_{00} = \frac{\dot{\omega}}{\omega} - \frac{\dot{\bar{\omega}}}{\bar{\omega}} \equiv \frac{d}{dt}\ln\frac{\omega}{\bar{\omega}} \equiv \omega\mathsf{T}\,, \tag{6.5.19}$$

where the function T describes the relative shift between the two cosmic times measured in FLRW cosmic time units. Next,

$$\Gamma^k{}_{0l} = \omega H\delta^k_l, \quad \bar{\Gamma}^k{}_{0l} = \bar{\omega}\bar{H}\delta^k_l \rightarrow \Delta^k{}_{0l} = \omega\left(H - \frac{\bar{\omega}}{\omega}\bar{H}\right)\delta^k_l = \omega\,\mathsf{H}\delta^k_l\,, \tag{6.5.20}$$

where H is the relative Hubble parameter measured in units of FLRW cosmic time. Finally,

$$\Gamma^0{}_{kl} = -\frac{H}{\omega} g_{kl}, \; \bar{\Gamma}^0{}_{kl} = -\frac{\bar{H}}{\bar{\omega}} \bar{g}_{kl} \;\rightarrow\; \Delta^0{}_{kl} = -\frac{1}{\omega}\left(H - \frac{\bar{a}^2 \omega}{a^2 \bar{\omega}} \bar{H}\right) g_{kl} \,. \tag{6.5.21}$$

With $\bar{\xi}^\mu$ given by (6.5.12–6.5.17), $h^{\mu\nu}$ by (6.5.18), $\Delta^\lambda{}_{\mu\nu}$ by (6.5.19–6.5.21), the components of superpotential, defined in (6.1.25) are:

$$16\pi \mathscr{J}_C^{0k} = \sqrt{-\bar{g}}\left(\mathscr{A} \bar{g}^{kl} \bar{\nabla}_l \bar{\xi}^0 + \mathscr{B} \bar{\xi}^k\right), \tag{6.5.22}$$

$$16\pi \mathscr{J}_C^{kl} = \sqrt{-\bar{g}}\left(\mathscr{C} \bar{g}^{m[k} \bar{\nabla}_m \bar{\xi}^{l]}\right), \tag{6.5.23}$$

$$\sqrt{-\bar{g}} = \bar{\omega} \bar{a}^3 \frac{1}{\sqrt{1-kr^2}}, \tag{6.5.24}$$

where $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are functions of t:

$$\begin{aligned} \mathscr{A}(t) &= 2 - \frac{\omega a}{\bar{\omega} \bar{a}} - \frac{\bar{\omega} a^3}{\omega \bar{a}^3} \\ \mathscr{B}(t) &= \left(-2 + 3\frac{\omega a}{\bar{\omega} \bar{a}} - \frac{\bar{\omega} a^3}{\omega \bar{a}^3}\right) \frac{\bar{H}}{\bar{\omega}} - \frac{4}{\bar{\omega}} \frac{a^3}{\bar{a}^3} H \,, \\ \mathscr{C}(t) &= 2\left(\frac{\omega a}{\bar{\omega} \bar{a}} - 1\right) . \end{aligned} \tag{6.5.25}$$

The components of the conserved current $\mathscr{J}_C^\mu$ can be calculated from (6.1.26) and (6.1.27) :

$$\mathscr{J}_C^\mu \equiv {}_C\boldsymbol{\theta}^\mu{}_\nu \bar{\xi}^\nu + \boldsymbol{\sigma}^{\mu\rho\sigma} \bar{\nabla}_{[\rho} \bar{\xi}_{\sigma]} \,. \tag{6.5.26}$$

The components for the matter energy-momentum $T_\nu{}^\mu$ of an ideal fluid in the FLRW model are

$$T_0{}^0 = -\rho, \quad T_k{}^l = p\delta_k^l \tag{6.5.27}$$

and the background "matter energy-momentum" $\bar{T}_\nu{}^\mu$ in de Sitter space incorporated to ${}_C\boldsymbol{\theta}^\mu{}_\nu$ is

$$\bar{T}_\mu{}^\nu = -\frac{\Lambda}{8\pi} \delta_\mu^\nu \,. \tag{6.5.28}$$

The time component of the current (6.5.26) then, reads

$$\mathscr{J}_C^0 = -\sqrt{-\bar{g}}\left[\left(\frac{\omega a^3}{\bar{\omega} \bar{a}^3}\rho - \frac{\Lambda}{8\pi}\right) - \frac{\Lambda}{16\pi} h_\rho{}^\rho + \frac{3}{8\pi} \frac{\omega a^3}{\bar{\omega} \bar{a}^3} \mathrm{H}^2\right] \bar{\xi}^0 \equiv -\mathscr{J}(t) \bar{\xi}^0 \,. \tag{6.5.29}$$

Here,

$$h_\rho{}^\rho = -4 + 3\frac{\omega a}{\bar{\omega} \bar{a}} + \frac{\bar{\omega} a^3}{\omega \bar{a}^3} = \mathscr{C} - \mathscr{A} \,. \tag{6.5.30}$$

The spatial components of the current (6.5.26) are given by

$$\mathscr{I}_C^k = \sqrt{-\bar{g}}\left[\left(\frac{\omega a^3}{\bar{\omega}\bar{a}^3}p + \frac{\Lambda}{8\pi}\right) + \frac{\Lambda}{16\pi}h_\rho{}^\rho + \frac{3}{8\pi}\frac{\omega a^3}{\bar{\omega}\bar{a}^3}H^2 - \frac{3}{16\pi}\left(\frac{\omega^2 a}{\bar{\omega}^2\bar{a}} - \frac{a^3}{\bar{a}^3}\right)(\mathsf{T} + \mathsf{H})\bar{H}\right]\bar{\xi}^k + \frac{\sqrt{-\bar{g}}}{16\pi}\left(\frac{\omega^2 a}{\bar{\omega}^2\bar{a}} - \frac{a^3}{\bar{a}^3}\right)(\mathsf{T} + \mathsf{H})\bar{g}^{kl}\overset{3}{\nabla}_l(\bar{\omega}\bar{\xi}^0). \qquad (6.5.31)$$

The terms in the first round brackets in (6.5.29) and (6.5.31) represent the “relative energy density” and “relative pressure”, respectively. The second terms which are the product of Λ and the trace of the metric perturbation, represent the coupling to the background. The other terms are associated with the field energy and the helicity and they depend on the choice of a particular mapping of the time axes.

As a consequence of (6.5.29) and $\bar{\xi}^0$’s as given in (6.5.12–6.5.20), the conserved quantities in a volume V enclosed by a sphere of radius $r = r_0$ are all equal to zero except of the “energy”, $E(r_0)$, associated with the time quasi-translations $\bar{\xi}^0_{(0)}$ given by (6.5.14). Lorentzian quasi-rotations does not bring us about any conserved quantity because the time component of (6.5.15) has odd parity. Thus, the “energy” reads

$$E(r_0) = \frac{4\pi}{3}\bar{a}^3 r_0^3 \mathscr{I}(t)\,, \qquad (6.5.32)$$

where $\mathscr{I}(t)$ is given by (6.5.29).

We would like to emphasize that up to now the mapping has not yet been fixed. To do this one has to specify the choice of ω and $\bar{\omega}$. The most appealing mapping is one that gives $\mathscr{I}(t) = 0$ so that $E(r_0) = 0$. The advantage of this mapping is that it allows us to extend the number of conserved quantities in the FLRW universe from 6 to 10, thus, including the integrals of energy and Lorentzian momentum for physical perturbations of the FLRW universe. The ten conserved quantities are referred to the AdS background and vanish in case of the absence of the physical perturbations of gravitational field of the FLRW universe.

6.6 Integral constraints for linear perturbations on FLRW backgrounds

In the present section, we consider perturbations of the FLRW universe directly with respect to the FLRW background itself without appealing to the AdS model as a background. It allows us to demonstrate the advantages of the Belinfante method, see Section 6.3. We consider the conserved currents and superpotentials associated with the 15 conformal Killing vectors, $\underset{\Omega}{\lambda}^\mu$, of FLRW spacetimes; their detailed classification is given in next subsection. The reason for the appearance of the 15 conformal Killing vectors is that FLRW spacetimes are conformal to the Minkowski space and, hence, there are similarities between their symmetries. They correspond to 4 translations, 3 spatial rotations, 3 Lorentz boosts, 1 dilatation and 4 “accelerations”. Such similarities are helpful in geometrical interpretations. Excellent mathematical description of the conformal Killing vectors of the Minkowski space along with their physical interpretation and applications are given by Fulton, Rohrlich and Witten [181], and we refer the interested readers to their paper for more detail.

In what follows, we shall employ the 15 conformal Killing vectors to impose the integral constraints on various physical quantities inside a sphere (parameterized by r) having at a given instant of conformal cosmological time η, volume V and surface S. In other words, we are looking for integrals from the perturbations of the form (6.3.48) based on the Belinfante corrected quantities:

$$\int_V d^3x \mathscr{J}_B^{0}(\underset{\Omega}{\lambda}) = \oint_S ds_i \mathscr{J}_B^{0i}(\underset{\Omega}{\lambda}) . \tag{6.6.1}$$

Any linear combination of such integrals with the coefficients which are solely functions of time, can be reduced to the same form.

We next turn our attention to those linear combinations in which the volume integral depends only on the matter energy-momentum perturbations $\delta\mathscr{T}_\mu{}^0$ that is of the form

$$\int_V d^3x \delta\mathscr{T}_\mu{}^0 V^\mu = \oint_S ds_i \mathscr{T}^i(V^\mu) \tag{6.6.2}$$

with yet unspecified vectors V^μ. Before the application of the Belinfante corrected technique, there were known 10 integral constraints of this form: 6 of them are associated with the 6 ordinary Killing vectors of the FLRW spacetimes and the remaining 4 constraints have been introduced by Traschen [441] who found 4 additional “integral constraint vectors” V^μ. We show that they are, in fact, not independent but consist of linear combinations of the conformal Killing vectors with time dependent coefficients.

The main result of the present section is that if we apply the uniform Hubble expansion gauge studied by Bardeen [27], then all, except of one, of the 15 conformal Killing vectors (their linear combinations) are associated with the integral conservation laws of the form (6.6.2). The exception includes either the conformal time translations if $k = \pm 1$ or the conformal time “acceleration” if $k = 0$. Thus, here, we show that the Belinfante correcting method allows us to construct 4 Traschen’s integral constraint vectors and the corresponding integral constraints. A look at (6.6.2) shows that these integrals might be constructed directly from Einstein’s constraint equations. However, it is not so simple to see this, as contrasted to the Belinfante method. The integral constraints often have simple geometrical interpretations stemmed from classical mechanics. Thus, volume integrands in constraints of the type (6.6.2) can be interpreted as multipole momenta of order 0, 1 or 2. Besides, with making use of the uniform Hubble expansion gauge and a special gauge for gravitational waves, we show that the integral constraints of the type (6.6.2) are independent on the gravitational radiation.

6.6.1 A FLRW background and its conformal Killing vectors

We write the FLRW background metric in dimensionless conformal coordinates $x^\mu = \{\eta, x^k\}$, for which the metric is conformally-flat:

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(-d\eta^2 + e_{kl} dx^k dx^l) = a^2 \bar{f}_{\mu\nu} dx^\mu dx^\nu \,, \tag{6.6.3}$$

where $a(\eta)$ is the scale factor. The difference with the representations of the conformal metric here and in (5.3.15) is that here

$$\bar{f}_{\mu\nu} \equiv \mathrm{diag}\,(-1,\ e_{kl})\,, \tag{6.6.4}$$

with e_{kl} specified by the choice of the conformal coordinates in (6.5.2), whereas $\bar{\mathfrak{f}}_{\mu\nu}(x^\alpha)$ is specified by the choice of the conformal coordinates in (5.3.15) and is defined in arbitrary coordinates. The non-zero Christoffel symbols of the metric (6.6.3) are

$$\bar{\Gamma}^0{}_{00} = \mathscr{H}, \quad \bar{\Gamma}^0{}_{kl} = \mathscr{H} e_{kl}, \quad \bar{\Gamma}^m{}_{0l} = \mathscr{H} \delta^m_l, \quad \bar{\Gamma}^m{}_{kl} = kx^m e_{kl}\,, \tag{6.6.5}$$

where $\mathscr{H}$ is the dimensionless conformal Hubble parameter

$$\mathscr{H} = \frac{a_{,\eta}}{a}, \tag{6.6.6}$$

yielding $a_{,\eta} = a\mathscr{H}$, see also (5.3.4). In these notations the non-vanishing components of the Einstein tensor are

$$\bar{G}_0{}^0 = \frac{3}{a^2}(k + \mathscr{H}^2) = 8\pi \bar{T}_0{}^0\,, \tag{6.6.7}$$

$$\bar{G}_m{}^l = \frac{1}{a^2}(k + \mathscr{H}^2 + 2\mathscr{H}_{,\eta})\delta^l_m = 8\pi \bar{T}_m{}^l\,. \tag{6.6.8}$$

Now, we give a short introduction to the theory of the conformal Killing vectors on FLRW background. Let us recall that solutions (if they exist) to the standard Killing equation:

$$\pounds_\xi \bar{g}_{\mu\nu} = 0 \tag{6.6.9}$$

are the ordinary Killing vectors: $\xi^\mu = \bar{\xi}^\mu$. This means that displacements along vectors $\bar{\xi}^\mu$ do not change the metric $\bar{g}_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x)$. Conformal Killing vectors satisfy a different equation

$$\pounds_\xi \bar{g}_{\mu\nu} = \frac{1}{4} \bar{g}_{\mu\nu} \bar{g}^{\rho\sigma} \pounds_\xi \bar{g}_{\rho\sigma}, \tag{6.6.10}$$

which solutions (if they exist) are denoted $\xi^\mu = \underset{\Omega}{\lambda}{}^\mu$, and are called the conformal Killing vectors, $\underset{\Omega}{\lambda}{}^\mu$. Displacements along the conformal Killing vectors induce conformal transformations of the metric:

$$\bar{g}_{\mu\nu}(x) \to \Omega(x) \bar{g}_{\mu\nu}(x). \tag{6.6.11}$$

And, inversely, transformations (6.6.11) do not change the equation (6.6.10) and their solutions.

Solutions of equations (6.6.10) are particularly interesting in case of the FLRW background metric (6.6.3). Among all of the conformal Killing vectors, $\underset{\Omega}{\lambda}^{\mu}$, there are 15 linearly independent ones. Equation (6.6.10) is *independent* of the conformal factor and can be written in 3-dimensional notations as follows:

$$\underset{\Omega}{\lambda}^{0}{}_{,\eta} = \frac{1}{3}\overset{3}{\nabla}_k \underset{\Omega}{\lambda}^{k}, \quad \underset{\Omega}{\lambda}^{k}{}_{,\eta} = \overset{3}{\nabla}{}^{k}\underset{\Omega}{\lambda}^{0}, \quad \overset{3}{\nabla}{}^{(k}\underset{\Omega}{\lambda}^{l)} = e^{kl}\underset{\Omega}{\lambda}^{0}{}_{,\eta}\,, \tag{6.6.12}$$

where $\overset{3}{\nabla}_k$ is a 3-covariant derivative for the e_{kl} metric, $\overset{3}{\nabla}{}^{k} = e^{kl}\overset{3}{\nabla}_l$, and the first equation is equal to the trace of the third one.

In spite of the seeming simplicity of the system (6.6.12) it is not simple to solve it. Fortunately, one can use the conformal properties. Fulton with coauthors [181] constructed and studied the conformal Killing vectors of the Minkowski space in the Lorentzian coordinates X^{μ}. As is well known, the metric $\bar{f}_{\mu\nu}$ is conformal to $\eta_{\mu\nu}$, that is after making an appropriate coordinate transformation one gets: $\bar{f}_{\mu\nu} \to \Omega\eta_{\mu\nu}$ globally. Notice that such a transformation depends on the sign of the spatial curvature k, see, for example [285]. The original background metric (6.6.3) is also conformal to $\bar{f}_{\mu\nu}$. Hence, the metric (6.6.3) is conformal to the Minkowski metric,

$$ds^2 = \bar{g}_{\mu\nu}dx^{\mu}dx^{\nu} = a^2\bar{f}_{\mu\nu}dx^{\mu}dx^{\nu} \to a^2\Omega\eta_{\mu\nu}dx^{\mu}dx^{\nu}\,. \tag{6.6.13}$$

By definition, the components of the conformal Killing vectors are chosen to be the same for all conformal metrics (spacetimes) - they neither depend on Ω nor on $a^2\Omega$. Thus, to solve the conformal Killing equation for the FLRW metric, one needs the components of the conformal Killing vectors in the Lorentzian coordinates X^{μ} of the Minkowski space, which then have to be transformed into coordinates x^{μ} of the metric (6.6.3). The coordinate transformations from X^{μ} to x^{μ} has been already found in [355].

The form of the conformal Killing vectors in the Minkowski space and that in the Lorentzian coordinates have been found in [181] and we follow this work.

The Minkowski metric in the Lorentzian coordinates is

$$d\bar{s}^2 = \eta_{\mu\nu}dX^{\mu}dX^{\nu} = -dT^2 + \delta_{kl}dX^k dX^l. \tag{6.6.14}$$

There is a 15-parameter group of conformal transformations $X^{\alpha} \to \tilde{X}^{\alpha}$ such that (6.6.14) transforms to

$$ds^2 = \Phi(\tilde{X})\eta_{\mu\nu}d\tilde{X}^{\mu}d\tilde{X}^{\nu}\,. \tag{6.6.15}$$

These transformations have the form:

$$\tilde{X}^{\mu} = a^{\mu} + A_{\rho}{}^{\mu}X^{\rho} + bX^{\mu} + \frac{X^{\mu} - B^{\mu}X^2}{1 - 2B_{\mu}X^{\mu} + B^2X^2}\,, \tag{6.6.16}$$

where $\eta_{\mu\nu}A_\rho{}^\mu A_\sigma{}^\nu = \eta_{\rho\sigma}$, $B^2 = \eta_{\mu\nu}B^\mu B^\nu$, $X^2 = \eta_{\mu\nu}X^\mu X^\nu$. Variation of the equation (6.6.16) leads to the expression, coefficients of which are the components of the conformal Killing vectors:

$$\delta\tilde{X}^\mu = \mathscr{K}^\mu_{(\alpha)}\delta a^\alpha + \mathscr{K}^\mu_{([\alpha\beta])}\delta A^{[\alpha\beta]} + \mathscr{K}^\mu_{[(0)]}\delta b + \mathscr{K}^\mu_{[\alpha]}\delta B^\alpha \,. \tag{6.6.17}$$

The first two terms represent a conventional 10-parametric group of motion of the Minkowski space, which is defined by the ordinary Killing vectors, namely, by 4 vectors of translations and 6 vectors of spatial rotations and boosts [1]:

$$\mathscr{K}^\mu_{(\alpha)} = \delta^\mu_\alpha \,, \tag{6.6.18}$$

$$\mathscr{K}^\mu_{([\alpha\beta])} = \tfrac{1}{2}\left(\delta^\mu_\alpha\eta_{\beta\gamma} - \delta^\mu_\beta\eta_{\alpha\gamma}\right)X^\gamma. \tag{6.6.19}$$

The third term in (6.6.17) corresponds to the so-called dilatation, or scale, transformations:

$$\mathscr{K}^\mu_{[(0)]} = X^\mu \,. \tag{6.6.20}$$

The last term in (6.6.17) corresponds to a, so-called, “4-acceleration”

$$\mathscr{K}^\mu_{[\alpha]} = \left(\delta^\mu_\gamma\eta_{\alpha\beta} + \delta^\mu_\beta\eta_{\alpha\gamma} - \delta^\mu_\alpha\eta_{\beta\gamma}\right)X^\beta X^\gamma. \tag{6.6.21}$$

Let us denote the set of vectors (6.6.18–6.6.21) as $\mathscr{K}^\mu$. By definition, all of these 15 vectors are conformal Killing vectors both for the metric (6.6.14) and for the metric (6.6.13) in the coordinates x^α. To derive their components for the metric (6.6.3), one has to transform to the coordinates used in equation (6.6.3), $X^\mu \to x^\mu$. It transforms the components of 15 vectors (6.6.18–6.6.21), $\mathscr{K}^\mu$, to the components, $\underset{\Omega}{\lambda}{}^\mu$, which satisfy (6.6.12), and we outline them below. These transformations involve tedious algebra and we don’t provide them over here referring the reader to the original works [285] and [355].

There are 7 conformal Killing vectors, which can be written in compact form for every sign of the spatial curvature k; these are the conformal Killing vectors of time translation, $\boldsymbol{t}$, space translations, $\boldsymbol{s}_a$, and space rotations, $\boldsymbol{r}_a$; $a = 1, 2, 3$:

$$t^\mu = \delta^\mu_0, \quad s^\mu_a = \delta^\mu_a\sqrt{1-kr^2}, \quad r^\mu_a = \delta^{\mu k}\epsilon_{kal}x^l. \tag{6.6.22}$$

Other conformal Killing vectors are different for $k = 0$ and for $k = \pm1$.

The case $k = 0$. The 8 other conformal Killing vectors are the Lorentz boosts: $\boldsymbol{l}_a$, dilatation: $\boldsymbol{d}$, time acceleration: $\boldsymbol{a}_0$ and space accelerations: $\boldsymbol{a}_a$. They are respectively given by

1 Here, the signs of the Killing vectors follow the original works [181, 370].

$$
\begin{aligned}
\boldsymbol{l}_a &= \{l_a^0 = x^a,\ l_a^k = \eta\delta_a^k\}\,,\\
\boldsymbol{d} &= \{d^0 = \eta,\ d^k = x^k\}\,,\\
\boldsymbol{a}_0 &= \{a_0^0 = \eta^2 + r^2,\ a_0^k = 2\eta x^k\}\,,\\
\boldsymbol{a}_a &= \{a_a^0 = 2\eta x^a,\ a_a^k = 2x^k x^a + [\eta^2 - r^2]\delta_a^k\}\,.
\end{aligned}
\tag{6.6.23}
$$

The cases $k = \pm 1$. The 8 other conformal Killing vectors can be written in a matrix form. For this we introduce a definition of a column matrix

$$
\boldsymbol{\beta} = \begin{pmatrix} \beta^\star \\ \beta_\star \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha_{,\eta} \end{pmatrix},
\tag{6.6.24}
$$

where, $\alpha = \sin\eta$, if $k = 1$, and $\alpha = \sinh\eta$, if $k = -1$. We introduce a similar column-matrix notation for dilatation $\boldsymbol{d} \to \boldsymbol{\delta}^\star$ and time acceleration $\boldsymbol{a}_0 \to \boldsymbol{\delta}_\star$ which combination will be denoted as $\boldsymbol{\delta}$; the same rule will be applied to combine the 3 Lorentz boosts $\boldsymbol{l}_a \to \boldsymbol{\lambda}^\star$ and 3 space accelerations $\boldsymbol{s}_a \to \boldsymbol{\lambda}_\star$, to a single column matrix $\boldsymbol{\lambda}_a$,

$$
\begin{aligned}
\boldsymbol{\delta} &= \begin{pmatrix} \boldsymbol{\delta}^\star \\ \boldsymbol{\delta}_\star \end{pmatrix} = \left\{\boldsymbol{\delta}^0 = \boldsymbol{\beta}\sqrt{1 - kr^2},\ \boldsymbol{\delta}^k = \boldsymbol{\beta}_{,\eta}\sqrt{1 - kr^2}x^k\right\}\,,\\
\boldsymbol{\lambda}_a &= \begin{pmatrix} \boldsymbol{\lambda}^\star \\ \boldsymbol{\lambda}_\star \end{pmatrix} = \left\{\boldsymbol{\lambda}_a^0 = \boldsymbol{\beta}_{,\eta}x^a,\ \boldsymbol{\lambda}_a^k = \boldsymbol{\beta}e^{ka}\right\}\,.
\end{aligned}
\tag{6.6.25}
$$

Notice that for $k = 0$ we can take $\alpha = \eta$ and apply (6.6.25) to that case as well with

$$
\boldsymbol{\beta} = \begin{pmatrix} \beta^\star \\ \beta_\star \end{pmatrix} = \begin{pmatrix} \eta \\ 1 \end{pmatrix},
\tag{6.6.26}
$$

and, consequently,

$$
\begin{aligned}
\boldsymbol{\delta} &= \begin{pmatrix} \boldsymbol{d} \\ \boldsymbol{t} \end{pmatrix},\\
\boldsymbol{\lambda}_a &= \begin{pmatrix} \boldsymbol{l}_a \\ \boldsymbol{s}_a \end{pmatrix}.
\end{aligned}
\tag{6.6.27}
$$

These 4 vectors are defined in equations (6.6.22) and (6.6.23). Notice also that any linear combination of the conformal Killing vectors (6.6.22–6.6.25), respectively to $k = 0$ and $k = \pm 1$, with constant coefficients is again equivalent to a conformal Killing vector.

6.6.2 Integral relations for linear perturbations

Following (5.6.20), we define perturbations in the FLRW model in the form of conformal perturbations:

$$
ds^2 = a^2(\bar{f}_{\mu\nu} + \kappa_{\mu\nu})dx^\mu\, dx^\nu.
\tag{6.6.28}
$$

Below, we use intensively the 3-dimensional components: $\kappa_{00},\ \kappa_{0l},\ \kappa_{kl}$, indices of which are raised and lowered with the use of e_{kl}:

$$
\kappa_l{}^m = e^{mk}\kappa_{lk},\quad \kappa^{mn} = e^{mk}e^{nl}\kappa_{kl},\quad \kappa_0{}^m = e^{ml}\kappa_{0l}\,.
\tag{6.6.29}
$$

To apply the results of Sections 6.2 and 6.3 we need to express these perturbations in terms of $\mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu}$. Restricting ourself to the linear approximation, we find easily

$$\mathfrak{h}^{\mu\nu} = a^2\sqrt{-\bar{g}}(-\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} + \tfrac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma})\kappa_{\rho\sigma} = a^2\sqrt{e}(-\bar{f}^{\mu\rho}\bar{f}^{\nu\sigma} + \tfrac{1}{2}\bar{f}^{\mu\nu}\bar{f}^{\rho\sigma})\kappa_{\rho\sigma}\,. \tag{6.6.30}$$

Now, let us define 3-tensors:

$$q_l^m \equiv \kappa_l{}^m - \delta_l^m\kappa_n{}^n\,, \tag{6.6.31}$$

$$Q_l{}^m \equiv (2\mathscr{H}\kappa_{00} - \overset{3}{\nabla}_n\kappa_0{}^n)\delta_l^m + \overset{3}{\nabla}{}^m\kappa_{0l} - q_l^m{}_{,\eta}\,. \tag{6.6.32}$$

We also define by a special symbol $\mathscr{Q}$ the perturbed trace of the external curvature of the hypersurface η = const, which appears in the time components of the conserved currents. Thus, if n^μ is the unit normal vector to that hypersurface,

$$\mathscr{Q} \equiv -\left(\nabla_\mu n^\mu - \bar{\nabla}_\mu\bar{n}^\mu\right) = \tfrac{3}{2}\mathscr{H}\kappa_{00} + \tfrac{1}{2}\kappa_n{}^n{}_{,\eta} - \overset{3}{\nabla}_n\kappa_0{}^n. \tag{6.6.33}$$

The condition,

$$\mathscr{Q} = 0\,, \tag{6.6.34}$$

is the uniform Hubble expansion gauge, which was introduced by Bardeen [27].

We have now all of the elements needed to calculate the conserved currents (6.2.11) and superpotentials (6.2.18) in (6.6.1) for small perturbations by making use of the 15 conformal Killing vectors, and to write them down in a compact form. Let us recall that we are interested in the integral constraints formulated for the perturbations confined in a spatial volume considered at a constant time with spherical boundaries. Let us write, first, the current component $\mathscr{J}_B^0$ for the perturbations (6.6.28) with the notations (6.6.29–6.6.33) in the linear approximation:

$$8\pi\mathscr{J}_B^0(\underset{\Omega}{\lambda}) = 8\pi a^4\sqrt{e}\delta T_\mu{}^0\underset{\Omega}{\lambda}{}^\mu + a^2\sqrt{e}\left[\tfrac{1}{2}f_1(\underset{\Omega}{\lambda})\mathscr{Q} - \tfrac{1}{3}f_2(\underset{\Omega}{\lambda})\kappa_n{}^n + \tfrac{1}{4}\overset{3}{\nabla}_n\left(f_1(\underset{\Omega}{\lambda})\kappa_0{}^n\right)\right], \tag{6.6.35}$$

where

$$f_1(\underset{\Omega}{\lambda}) \equiv 4\mathscr{H}\underset{\Omega}{\lambda}{}^0 + \tfrac{4}{3}\overset{3}{\nabla}_k\underset{\Omega}{\lambda}{}^k\,,$$
$$f_2(\underset{\Omega}{\lambda}) \equiv \overset{3}{\nabla}{}^2\underset{\Omega}{\lambda}{}^0 + 3k\underset{\Omega}{\lambda}{}^0\,, \tag{6.6.36}$$

and $\overset{3}{\nabla}{}^2 \equiv e^{kl}\overset{3}{\nabla}_k\overset{3}{\nabla}_l$. The superpotential component $\mathscr{J}_B^{0l}$ for the perturbations (6.6.28) with the notations (6.6.29–6.6.33) in the linear approximation is

$$8\pi\mathscr{J}_B^{0l}(\underset{\Omega}{\lambda}) = \tfrac{1}{2}a^2\sqrt{e}\left[(2\mathscr{H}\kappa_0{}^l - \overset{3}{\nabla}{}^k q_k^l)\underset{\Omega}{\lambda}{}^0 + q_k^l\overset{3}{\nabla}{}^k\underset{\Omega}{\lambda}{}^0 + Q_k{}^l\underset{\Omega}{\lambda}{}^k + \kappa_{0k}\overset{3}{\nabla}{}^{[k}\underset{\Omega}{\lambda}{}^{l]}\right]. \tag{6.6.37}$$

Now, we are in a position to elaborate on the integral constraints of the type (6.6.1), where we integrate over the volume V in space Σ defined at a fixed instant of the conformal time η and restricted by a spherical boundary $\partial\Sigma$ that we denote here as S. We rewrite (6.6.1) in the form:

$$C(\underset{\Omega}{\lambda}) \equiv \int_V d^3x \mathscr{J}_B^0(\underset{\Omega}{\lambda}) = \oint_S ds_l \mathscr{J}_B^{0l}(\underset{\Omega}{\lambda}) = \oint_S \tfrac{1}{2}\epsilon_{lmn}dx^m dx^n \mathscr{J}_B^{0l}(\underset{\Omega}{\lambda}). \tag{6.6.38}$$

If this quantity does not depend on η, then $C(\underset{\Omega}{\lambda})$ can be interpreted as the motion integrals. In fact, (6.6.38) represent 15 quantities for each of the conformal Killing vectors $\underset{\Omega A}{\lambda^\mu}$; here, $A = 1, \dots, 15$. By (6.6.23–6.6.25), one can see that at least 8 quantities from $C(\underset{\Omega A}{\lambda})$ depend on η, however, one can find the linear combinations of $C(\underset{\Omega A}{\lambda})$ with time dependent coefficients, which are time independent, $c^A(\eta)C(\underset{\Omega A}{\lambda})$. Because (6.6.35) and (6.6.37) depend only linearly on $\underset{\Omega A}{\lambda^\mu}$ and their spatial derivatives, one is allowed to use the relation $c^A(\eta)C(\underset{\Omega A}{\lambda}) = C(c^A(\eta)\underset{\Omega A}{\lambda})$. Notice that $c^A(\eta)\underset{\Omega A}{\lambda^\mu}$ are not any longer the conformal Killing vectors, but we will show now that they have simple expressions and admit clear physical interpretations in an appropriate gauge.

To study the above-mentioned combinations, it is more convenient to associate the coordinates x^k in (6.6.3) with polar coordinates, where the radial coordinate $r = \sin\chi, \chi$ or $\sinh\chi$ respectively to $k = 1, k = 0$ or $k = -1$, and use notation r' as a derivative with respect to χ. Because the 7 conformal Killing vectors (6.6.22) are time independent we keep on them without changing

$$\boldsymbol{t} = \{1, 0\}, \quad \boldsymbol{s}_a = \left\{0, \delta_a^k r'\right\}, \quad \boldsymbol{r}_a = \left\{0, \epsilon_{kal}x^l\right\}. \tag{6.6.39}$$

The next 8 linear combinations are composed with time dependent coefficients. Recalling our notations (6.6.24), we use the notations $\alpha = \sin\eta$ or $\alpha = \sinh\eta$, respectively, for $k = 1$ or $k = -1$. The first 4 out of 8 time independent vectors constructed by using (6.6.23–6.6.25), have vanishing spatial components:

$$\begin{aligned}
\boldsymbol{l}_a^\dagger &= \left(\alpha_{,\eta}\boldsymbol{l}_a - \alpha\boldsymbol{b}_a\right)\big|_{k=\pm1} = \left(\boldsymbol{l}_a - \eta\boldsymbol{s}_a\right)\big|_{k=0} = \{x^a, 0\},\\
\boldsymbol{a}^\dagger\big|_{k=\pm1} &= \left(\alpha_{,\eta}\boldsymbol{a} + k\alpha\boldsymbol{d}\right)\big|_{k=\pm1} = \left\{r', 0\right\},\\
\boldsymbol{a}^\dagger\big|_{k=0} &= \left(\tfrac{1}{2}\boldsymbol{a} - \eta\boldsymbol{d} + \tfrac{1}{2}\eta^2\boldsymbol{t}\right)\big|_{k=0} = \left\{\tfrac{1}{2}r^2, 0\right\}.
\end{aligned} \tag{6.6.40}$$

The other 4 out of 8 time independent combinations (for each k) have vanishing time components, which are the 3-space conformal Killing vectors on sections $\eta = \text{const}$:

$$\begin{aligned}
\boldsymbol{d}^\dagger &= \left(\alpha_{,\eta}\boldsymbol{d} - \alpha\boldsymbol{a}\right)\big|_{k=\pm1} = \left(\boldsymbol{d} - \eta\boldsymbol{t}\right)\big|_{k=0} = \left\{0, x^k r'\right\},\\
\boldsymbol{b}_a^\dagger\big|_{k=\pm1} &= \left(\alpha_{,\eta}\boldsymbol{b}_a + k\alpha\boldsymbol{l}_a\right)\big|_{k=\pm1} = \left\{0, e^{ak}\right\},\\
\boldsymbol{b}_a^\dagger\big|_{k=0} &= \left(-\tfrac{1}{2}\boldsymbol{b}_a + \eta\boldsymbol{l}_a - \tfrac{1}{2}\eta^2\boldsymbol{s}_a\right)\big|_{k=0} = \left\{0, \delta^{ak}\tfrac{1}{2}r^2 - x^a x^k\right\}.
\end{aligned} \tag{6.6.41}$$

Substituting the components (6.6.39–6.6.41) into (6.6.35) and (6.6.37), one obtains the relations (6.6.38) corresponding to each of the vectors (6.6.39–6.6.41). We use physical (not coordinate) elements of integration in the form: $dV \equiv a^3\sqrt{e}d^3x$ and $dS_l \equiv a^2\sqrt{e}\frac{1}{2}\epsilon_{lmn}dx^m dx^n$. We also use below a conventional definition of the Hubble constant: $H = \mathscr{H}/a$. Thus, for vectors (6.6.39) the relation (6.6.38) unfolds, correspondingly, to three expressions

$$8\pi C(\boldsymbol{t}) - aH\oint_S \kappa_0{}^l dS_l = \int_V [(8\pi a\delta T_0{}^0 + 2H\mathscr{Q}) - \frac{k}{a}\kappa_m{}^m]dV$$
$$= -\tfrac{1}{2}\oint_S \overset{3}{\nabla}_k q^{kl} dS_l\,, \tag{6.6.42}$$

$$8\pi C(\boldsymbol{s}_a) = 8\pi\int_V a\delta T_a{}^0 r' dV$$
$$= \oint_S \left(\tfrac{1}{2}Q_k{}^l s_a^k - k\kappa_{0k}x^{[k}s_a^{l]}\right) dS_l\,, \tag{6.6.43}$$

$$8\pi C(\boldsymbol{r}_a) = 8\pi\int_V a\delta T_k{}^0 \epsilon_{kal}x^l dV$$
$$= \oint_S \left(\tfrac{1}{2}Q_k{}^l r_a^k - k\kappa_{0k}x^{[k}r_a^{l]} + \tfrac{1}{2}\kappa_{0k}\epsilon_{kal}\right) dS_l. \tag{6.6.44}$$

For vectors (6.6.40) the relation (6.6.38) unfolds to

$$8\pi C(\boldsymbol{l}_a^\dagger) - aH\oint_S \kappa_0{}^l x^a dS_l = \int_V \left(8\pi a\delta T_0{}^0 + 2H\mathscr{Q}\right) x^a dV$$
$$= -\tfrac{1}{2}\oint_S \left(x^a\nabla_k q^{kl} - q^{al}\right) dS_l\,, \tag{6.6.45}$$

$$8\pi C\left(\boldsymbol{a}^\dagger\right)\Big|_{k=\pm 1} - aHr'\oint_S \kappa_0{}^l dS_l = \int_V \left(8\pi a\delta T_0{}^0 + 2H\mathscr{Q}\right) r' dV$$
$$= -\tfrac{1}{2}\oint_S \left(\nabla_k q^{kl} + kq_k^l x^k\right) r' dS_l\,, \tag{6.6.46}$$

$$8\pi C\left(\boldsymbol{a}^\dagger\right)\Big|_{k=0} - \tfrac{1}{2}aH\oint_S \kappa_0{}^l r^2 dS_l = \int_V \left[\left(\tfrac{1}{2}8\pi a\delta T_0^0 + H\mathscr{Q}\right) r^2 - \frac{1}{a}\kappa_m{}^m\right] dV$$
$$= -\tfrac{1}{2}\oint_S \left(\tfrac{1}{2}\nabla_k q^{kl} r^2 - q_k^l x^k\right) dS_l\,. \tag{6.6.47}$$

For vectors (6.6.41) the relation (6.6.38) unfolds to

$$8\pi C(\boldsymbol{d}^\dagger) - r'\oint_S \kappa_0{}^l dS_l = \int_V \left(8\pi a\delta T_k{}^0 x^k + \frac{2}{a}\mathscr{Q}\right) r' dV$$
$$= \oint_S \left(\tfrac{1}{2}Q_k{}^l x^k - \kappa_0{}^l\right) r' dS_l\,, \tag{6.6.48}$$

$$8\pi C\left(\boldsymbol{b}_a^{\dagger}\right)\Big|_{k=\pm 1} + k\oint_S \kappa_0{}^l x^a dS_l = \int_V \left(8\pi a\delta T_k{}^0 e^{ak} - \frac{2k}{a}\mathscr{Q}x^a\right) dV$$
$$= \oint_S \left(\tfrac{1}{2}Q^{la} + k\kappa_0{}^l x^a\right) dS_l\,, \tag{6.6.49}$$

$$8\pi C(\boldsymbol{b}_a^{\dagger})\Big|_{k=0} + \oint_S \kappa_0{}^l x^a dS_l = \int_V \left[8\pi a\delta T_k{}^0\left(\delta^{ak}\tfrac{1}{2}r^2 - x^a x^k\right) - \frac{2}{a}\mathscr{Q}x^a\right] dV$$
$$= \oint_S \left[\tfrac{1}{2}kQ_k{}^l\left(\delta^{ak}\tfrac{1}{2}r^2 - x^a x^k\right)\right.$$
$$\left. + \kappa_0{}^l x^a + \kappa_{0k}x^{[k}\delta_a^{l]}v\right] dS_l\,. \tag{6.6.50}$$

6.6.3 Possible applications

Let us recall how *localized perturbations* are defined in a flat spacetime. Let at the initial moment $t = 0$ the density ρ be homogeneous. At the next moment $t > 0$, in the domain V, perturbations $\delta\rho$ appear, but we assume that they are absent on the boundary S of the domain. Such perturbations of density are called the localized density perturbations if they satisfy the integral relations:

$$\int_V \delta\rho dV = 0, \quad \int_V \mathbf{x}\delta\rho dV = 0\,, \tag{6.6.51}$$

which are called the *integral constraints* as well.

In general relativity, Traschen [441], considered the linear perturbations of generic form, and studied special vectors, V^μ, so-called *integral constraint vectors* satisfying a certain system of equations on spacelike sections Σ. Combining V^μ and the Einstein's equations, she constructed the relations:

$$\int_V \delta T_\mu{}^\alpha V^\mu n_\alpha dV = \oint_S T^l(\delta g_{\alpha\beta}, V^\mu) dS_l\,. \tag{6.6.52}$$

They are integral constraints in general relativity, where the right hand side depends on the metric perturbations $g_{\alpha\beta}$, their derivatives and V^μ only. The meaning of (6.6.52) is that, by imposing the boundary conditions on S, one restricts the matter perturbations inside V. It is known that there are 10 such vectors V^μ for the FLRW model. From them 6 are the aforementioned Killing vectors on the FLRW background. The other 4 vectors V^μ are exactly Traschen's integral constraint vectors [441].

Analogous to (6.6.51) the localized perturbations of the matter energy momentum in general relativity are restrained by the condition

$$\int_V \delta T_\mu{}^\alpha V^\mu n_\alpha dV = 0\,. \tag{6.6.53}$$

Recall the Sachs-Wolfe effect [396]. Any perturbation of $\delta\rho$ induces anisotropy of temperature of cosmic microwave background radiation. However, if this perturbation is *restricted* by its localization (6.6.53), the Sachs-Wolf effect is weakened with respect to the standard one [442].

Now let us turn to the integral relations (6.6.42–6.6.50). Notice that besides $\delta T_\mu{}^0$ the volume integrals also depend on $\mathscr{Q}$ and $\kappa_m{}^m$. Among them there are 4 linear combinations, which do not include $\kappa_m{}^m$ at all. They correspond to the following vectors

$$\boldsymbol{V}_0 = (\dot{a}^{-1}\boldsymbol{a}^\dagger - \boldsymbol{d}^\dagger)\big|_{k=\pm 1} \text{ or } \boldsymbol{V}_0 = (\dot{a}^{-1}\boldsymbol{t} - \boldsymbol{d}^\dagger)\big|_{k=0},$$
$$\boldsymbol{V}_a = \dot{a}^{-1}\boldsymbol{l}_a^\dagger + k\boldsymbol{b}_a^\dagger, \tag{6.6.54}$$

which are linear combinations of vectors in (6.6.39–6.6.41), and consequently are linear combinations of the conformal Killing vectors. The vectors $\boldsymbol{V}_0$ and $\boldsymbol{V}_a$ are simply the Traschen's vectors. In the Bardeen uniform Hubble expansion gauge (6.6.34) the corresponding integral relations $C(\boldsymbol{V}_0)$ and $C(\boldsymbol{V}_a)$ become Traschen's integral constraints.

In relations (6.6.42–6.6.50) we have not used any gauge restricting condition. The preferred gauge condition that significantly simplifies the integrands in (6.6.42–6.6.50) is the Bardeen gauge given by equation (6.6.34). Then, 14 volume integrands are reduced to combinations of $\delta T_\mu{}^0$ only. Thus, the correspondent 14 relations (or their combinations) are classified as the integral constraints of the type (6.6.52). Among them, 4 are newly obtained by us by using the Belinfante correction method. The "defective" relation for $k = 0$ is (6.6.47) and those for $k = \pm 1$ are (6.6.42), where along with $\delta T_\mu{}^0$ one has $\kappa_m{}^m$. The integrands in all of the integral constraints, in analogy with classical mechanics, are multipole momenta of the perturbed matter energy-momentum of the order 0, 1, 2 in components x^a, irrespectively of the sign of the spatial curvature k.

Besides the Bardeen gauge condition, the gauge condition, $\overset{3}{\nabla}_l \underset{T}{\kappa}_k{}^l = 0$, is also often used in cosmology where $\underset{T}{\kappa}_k{}^l$ is the traceless part of $\kappa_k{}^l$, see, for example, [41]. Combination of the condition, $\overset{3}{\nabla}_l \underset{T}{\kappa}_k{}^l = 0$, with the condition (6.6.34) (it was used by Bičák in an unpublished work) permits us to find out that there are 4 relations, which do not depend on the gravitational radiation irrespectively of its presence in the universe,

$$\int_V (8\pi a \delta T_0{}^0 - \frac{k}{a}\kappa_m{}^m) dV = \frac{1}{3}\oint_S \overset{3}{\nabla}{}^l \kappa_m{}^m dS_l,$$
$$\int_V 8\pi a \delta T_0{}^0 x^a dV = \frac{1}{3}\oint_S (x^a \overset{3}{\nabla}{}^l \kappa_m{}^m - e^{al}\kappa_m{}^m) dS_l. \tag{6.6.55}$$

Then the energy integral, E, and the center of mass integral, $\boldsymbol{R}$, for perturbations in the domain V are presented as

$$E \sim \int \delta T_0{}^0, \quad \boldsymbol{R} \sim \frac{\int \delta T_0{}^0 x^a}{\int \delta T_0{}^0} \tag{6.6.56}$$

and is defined by the trace $\kappa_m{}^m$ on the boundary of the domain S without dependence on the gravitational radiation. We emphasize that all of the above conclusions are valid only in the linearized approximation of the perturbation theory.

7 Conservation laws in an arbitrary multi-dimensional metric theory

Chapters from 1 to 6 are devoted fully to constructing conservation laws for perturbations and their applications in general relativity. However, in the last decades, numerous metric theories in n dimensions, which are various modifications of general relativity, have received the great interest for various reasons. They are quadratic in curvature theories, see, e. g., [126]; or theories of the Lovelock type [299]; or $f(R)$ theories [420], see also Chapter 9, etc. In these theories, there arises the necessity to study perturbations and to construct conservation laws as non-trivial differences from general relativity arise. Many results in this direction have been obtained also (see, for example, [119, 165, 253, 365, 367, 368], and reviews [364, 430] and references therein). However, a generalized formalism of constructing conservation laws for perturbations in arbitrary metric theories has not been presented. Although concrete forms of metric Lagrangians in such theories are various and certainly more complicated than in general relativity, it is natural, nevertheless, to develop the methods elaborated in general relativity. On such a basis, it is very desirable to define *unifying* rules. The present chapter is devoted to this problem.

7.1 Covariant Noether's procedure in an arbitrary field theory

In this section, for the sake of generality, we consider an arbitrary covariant field theory in n dimensions. A *direct* application of the Noether procedure leads to *non-covariant* identities and conserved quantities, see Section 1.4.1 for 4 dimensions. To construct covariant identities and conserved quantities an external (background) metric has been included in the formulation; recall the field-theoretical approach in Sections 2.2 and 6.3, the KBL construction in Section 6.1 and the Belinfante procedure in Section 6.2. Therefore, following the methods in general relativity in the covariant procedure, we operate with a background that is a given/known solution of the theory which is chosen in accordance with the problem under consideration. One of the earlier attempts to suggest covariant Noether's identities in general relativity is Ray's work [384]. In this section, following [371], we develop Ray's ideas in the framework of an arbitrary field theory.

Including a chosen background (extrinsic) metric into the Lagrangian, one upgrades partial derivatives to covariant derivatives related to this metric. As a result, the original Lagrangian becomes evidently covariant. One has to remark that by introducing the background (extrinsic) metric (and the covariant derivatives associated with it) to the Lagrangian, we don't change the theory because the Lagrangian remains the same. This trick permits us to present in a covariant form both the identities and the identically conserved quantities. We present a *new family* of the covariant

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Noether's identities and identically conserved quantities. After that we expand the *new family* using the Belinfante modification.

7.1.1 Covariant identities and identically conserved quantities

Let us turn to the Section 1.4.1, where *non-covariant* identities and conserved quantities in an arbitrary field theory were derived in four dimensions. Let us stress that all of the identities and quantities derived in Section 1.4.1 without changes, hold in the n-dimensional spacetime. We rewrite the Lagrangian in (1.2.36) examined in the Section 1.4.1,

$$\mathscr{L} = \mathscr{L}(\psi^B, \psi^B_{,\alpha}, \psi^B_{,\alpha\beta}), \tag{7.1.1}$$

that is not explicitly covariant, although the theory is covariant. Thus, the Lagrangian, $\mathscr{L}$ is to be a scalar density of weight +1. If one assumes $\psi^B = \{g_{\mu\nu}, \Phi^A\}$, all the metric theories, including general relativity, Lagrangians of which depend on the Riemann tensor algebraically, are related to this class. To provide *covariant* identities and conserved quantities, we incorporate an external (auxiliary, background) metric, $\bar{g}_{\mu\nu}$, as follows. Let us derive covariant derivatives of the fields ψ^B constructed with the use of the background metric $\bar{g}_{\mu\nu}$, see Appendixes A.2.1 and A.3,

$$\bar{\nabla}_\alpha \psi^B \equiv \partial_\alpha \psi^B + \bar{\Gamma}^\tau{}_{\alpha\rho}\, \psi^B\Big|^\rho_\tau, \tag{7.1.2}$$

$$\begin{aligned}\bar{\nabla}_{\beta\alpha} \psi^B &\equiv \partial_{\beta\alpha} \psi^B + \bar{\Gamma}^\tau{}_{\alpha\rho,\beta}\, \psi^B\Big|^\rho_\tau + \bar{\Gamma}^\tau{}_{\alpha\rho} \left(\psi^B\Big|^\rho_\tau \right)_{,\beta} + \bar{\Gamma}^\tau{}_{\beta\rho} \left(\bar{\nabla}_\alpha\, \psi^B \right)\Big|^\rho_\tau \\ &\equiv \partial_{\beta\alpha} \psi^B + \bar{\Gamma}^\tau{}_{\alpha\rho,\beta}\, \psi^B\Big|^\rho_\tau + \bar{\Gamma}^\tau{}_{\alpha\rho} \bar{\nabla}_\beta \left(\psi^B\Big|^\rho_\tau \right) \\ &\quad + \bar{\Gamma}^\tau{}_{\beta\rho} \left(\bar{\nabla}_\alpha\, \psi^B \right)\Big|^\rho_\tau - \bar{\Gamma}^\tau{}_{\alpha\rho} \bar{\Gamma}^\mu{}_{\beta\nu}\, \psi^B\, \Big|^\rho_\tau\Big|^\nu_\mu .\end{aligned} \tag{7.1.3}$$

Because the Lagrangian (7.1.1) is a scalar density, after similar substitutions

$$\partial_\alpha \psi^B \equiv \bar{\nabla}_\alpha \psi^B - \bar{\Gamma}^\tau{}_{\alpha\rho}\, \psi^B\Big|^\rho_\tau, \tag{7.1.4}$$

$$\begin{aligned}\partial_{\beta\alpha} \psi^B \equiv \bar{\nabla}_{\beta\alpha} \psi^B &- \bar{\Gamma}^\tau{}_{\alpha\rho,\beta}\, \psi^B\Big|^\rho_\tau - \bar{\Gamma}^\tau{}_{\alpha\rho} \bar{\nabla}_\beta \left(\psi^B\Big|^\rho_\tau \right) \\ &- \bar{\Gamma}^\tau{}_{\beta\rho} \left(\bar{\nabla}_\alpha\, \psi^B \right)\Big|^\rho_\tau + \bar{\Gamma}^\tau{}_{\alpha\rho} \bar{\Gamma}^\mu{}_{\beta\nu}\, \psi^B\, \Big|^\rho_\tau\Big|^\nu_\mu .\end{aligned} \tag{7.1.5}$$

it is transformed into an explicitly covariant form,

$$\mathscr{L} = \mathscr{L}(\psi^B, \psi^B_{,\alpha}, \psi^B_{,\alpha\beta}) \equiv \mathscr{L}_c(\psi^B, \bar{\nabla}_\alpha \psi^B, \bar{\nabla}_{\beta\alpha} \psi^B, \bar{g}_{\mu\nu}, \bar{R}^\alpha{}_{\mu\beta\nu}). \tag{7.1.6}$$

Here, as usual, $\bar{g}_{\mu\nu}$, $\bar{\Gamma}^\mu_{\alpha\nu}$, $\bar{\nabla}_\alpha$ and $\bar{R}^\alpha{}_{\mu\beta\nu}$ are the metric, the Christoffel symbols, the covariant derivatives and the curvature tensor of the auxiliary (background) spacetime.

Note that the left hand side of (7.1.5) is explicitly symmetric in α and β. To show this for the right hand side, one has to present the double covariant derivative as $\bar{\nabla}_{\beta\alpha}\psi^B = \bar{\nabla}_{(\beta\alpha)}\psi^B + \bar{\nabla}_{[\beta\alpha]}\psi^B$, turn to the formula (A.3.9) and make necessary algebraic transformations using the formulae from Appendix A.3.2.

To preserve the explicit covariance under variation of $\mathcal{L}_c$ the direct way is to vary the external metric $\bar{g}_{\mu\nu}$ together with the fields ψ^B. However, this way is very cumbersome, and we will choose a more economical one. It is evident that substitution of (7.1.4) and (7.1.5) does not incorporate the external metric at all, therefore the new representation $\mathcal{L}_c$ in (7.1.6) does not contain $\bar{g}_{\mu\nu}$ and its derivatives. This means that finally variation of $\mathcal{L}_c$, the same as variation of $\mathcal{L}$, has to be transformed into the *same* identity (1.4.4). We rewrite it keeping in mind that now it belongs to n-dimensional case:

$$-\left[\frac{\delta\mathcal{L}}{\delta\psi^B}\psi^B{}_{,\alpha} + \partial_\beta\left(\frac{\delta\mathcal{L}}{\delta\psi^B}\,\psi^B\big|^\beta_\alpha\right)\right]\xi^\alpha$$

$$+\,\partial_\alpha\left[\mathcal{U}_\sigma{}^\alpha\xi^\sigma + \mathcal{M}_\sigma{}^{\alpha\tau}\partial_\tau\xi^\sigma + \mathcal{N}_\sigma{}^{\alpha\tau\beta}\partial_{\beta\tau}\xi^\sigma\right] \equiv 0, \tag{7.1.7}$$

where $\mathcal{U}$, $\mathcal{M}$ and $\mathcal{N}$ are defined in (1.4.5), (1.4.6) and (1.4.7), respectively. The identity (7.1.7) is covariant in whole since it has been obtained from the covariant identity: $£_\xi\mathcal{L} + \partial_\alpha(\xi^\alpha\mathcal{L}) \equiv 0$ with preserving all the terms. The economical way of representing (7.1.7) into an explicitly covariant form uses (7.1.2–7.1.6) in (7.1.7) directly.

From the start, let us turn to the identity (1.4.12):

$$\frac{\delta\mathcal{L}}{\delta\psi^B}\psi^B{}_{,\alpha} + \partial_\beta\left(\frac{\delta\mathcal{L}}{\delta\psi^B}\,\psi^B\big|^\beta_\alpha\right) \equiv 0 \tag{7.1.8}$$

where the Lagrangian derivative,

$$\frac{\delta\mathcal{L}}{\delta\psi^A} \equiv \frac{\partial\mathcal{L}}{\partial\psi^A} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\psi^A{}_{,\alpha}}\right) + \partial_{\alpha\beta}\left(\frac{\partial\mathcal{L}}{\partial\psi^A{}_{,\alpha\beta}}\right), \tag{7.1.9}$$

in fact, is a covariant expression and can be represented in an explicitly covariant form (A.3.37), see Appendix A.3.3. Keeping this in mind and using the properties discussed in Appendix A.3, it is not difficult to show that (7.1.8) is covariant because it can be rewritten in an explicitly covariant form:

$$\frac{\delta\mathcal{L}}{\delta\psi^A}\bar{\nabla}_\alpha\psi^A + \bar{\nabla}_\beta\left(\frac{\delta\mathcal{L}}{\delta\psi^A}\,\psi^A\big|^\beta_\alpha\right) \equiv 0. \tag{7.1.10}$$

Then the identity

$$\partial_\alpha\left[\mathcal{U}_\sigma{}^\alpha\xi^\sigma + \mathcal{M}_\sigma{}^{\alpha\tau}\partial_\tau\xi^\sigma + \mathcal{N}_\sigma{}^{\alpha\tau\beta}\partial_{\beta\tau}\xi^\sigma\right] \equiv 0, \tag{7.1.11}$$

following from (7.1.7), has to be covariant also.

Now, let us turn to the relations (7.1.4) and (7.1.5) and set there ξ^σ instead of ψ^A. We use the obtained equations to change the partial derivatives of ξ^σ in (7.1.11) with the covariant ones and rewrite (7.1.11) as

$$\partial_\alpha \left[\boldsymbol{u}_\sigma{}^\alpha \xi^\sigma + \boldsymbol{m}_\sigma{}^{\alpha\tau} \bar{\nabla}_\tau \xi^\sigma + \boldsymbol{n}_\sigma{}^{\alpha\tau\beta} \bar{\nabla}_{\beta\tau} \xi^\sigma \right] \equiv 0 \tag{7.1.12}$$

where

$$\boldsymbol{u}_\sigma{}^\alpha = \mathscr{U}_\sigma{}^\alpha - \mathscr{M}_\lambda{}^{\alpha\tau} \bar{\Gamma}^\lambda_{\sigma\tau} + \mathscr{N}_\lambda{}^{\alpha\tau\rho} (\bar{\Gamma}^\lambda_{\tau\pi} \bar{\Gamma}^\pi_{\sigma\rho} - \partial_\rho \bar{\Gamma}^\lambda_{\sigma\tau}), \tag{7.1.13}$$

$$\boldsymbol{m}_\sigma{}^{\alpha\tau} = \mathscr{M}_\sigma{}^{\alpha\tau} + \mathscr{N}_\sigma{}^{\alpha\lambda\rho} \bar{\Gamma}^\tau_{\lambda\rho} - 2\mathscr{N}_\lambda{}^{\alpha\tau\rho} \bar{\Gamma}^\lambda_{\sigma\rho}, \tag{7.1.14}$$

$$\boldsymbol{n}_\sigma{}^{\alpha\tau\beta} = \mathscr{N}_\sigma{}^{\alpha\tau\beta}. \tag{7.1.15}$$

By using the new form of the Lagrangian (7.1.6) and the connections between the partial and covariant derivatives (7.1.4) and (7.1.5) we show that the new coefficients (7.1.13–7.1.15) are represented in an explicitly covariant form.

Let us begin from the last coefficient (7.1.15). Due to the evident relation (A.3.35) in Appendix A.3.3 the coefficient $\mathscr{N}$ (1.4.7) is automatically covariant, and the coefficient $\boldsymbol{n}$ is represented in the obviously covariant form

$$\boldsymbol{n}_\sigma{}^{\alpha\tau\beta} \equiv \frac{1}{2} \left[\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\beta}\psi^B)} \psi^B \Big|^\tau_\sigma + \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\tau}\psi^B)} \psi^B \Big|^\beta_\sigma \right]. \tag{7.1.16}$$

To represent $\boldsymbol{m}$ in (7.1.14) we need to use the expression $\mathscr{M}$ in (1.4.6). The first term in $\mathscr{M}$ is represented by the derivative defined in (1.2.51), let us reproduce it. For this we use (A.3.33), (A.3.35), and the rules in Appendix A.3. One gets

$$\begin{aligned} \frac{\delta \mathscr{L}}{\delta \psi^B{}_{,\alpha}} = \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\alpha}} - \partial_\beta \left(\frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\alpha\beta}} \right) &= \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_\alpha \psi^B)} \\ &+ \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\nu\mu}\psi^B)} \frac{\partial}{\partial(\bar{\nabla}_\alpha\psi^B)} \left[\bar{\Gamma}^\tau{}_{\mu\rho} (\bar{\nabla}_\nu \left(\psi^B \Big|^\rho_\tau \right) + \bar{\Gamma}^\tau{}_{\nu\rho} \left(\bar{\nabla}_\mu\, \psi^B \right) \Big|^\rho_\tau \right] \\ &- \bar{\nabla}_\beta \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha}\psi^B)} \right) + \bar{\Gamma}^\nu{}_{\beta\mu} \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha}\psi^B)} \right) \Bigg|^\mu_\nu . \end{aligned} \tag{7.1.17}$$

The other part of the expression $\mathscr{M}$ in (1.4.6) is transformed as

$$\begin{aligned} &- \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\tau\alpha}} \partial_\sigma \psi^B + \frac{\partial \mathscr{L}}{\partial \psi^B{}_{,\beta\alpha}} \partial_\beta \left(\psi^B \Big|^\tau_\sigma \right) = \\ &- \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\tau}\psi^B)} \left[\bar{\nabla}_\sigma \psi^B - \bar{\Gamma}^\nu{}_{\sigma\mu}\, \psi^B \Big|^\mu_\nu \right] \\ &+ \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\beta}\psi^B)} \left[\bar{\nabla}_\beta (\psi^B \Big|^\tau_\sigma) - \bar{\Gamma}^\nu{}_{\beta\mu}\, \psi^B \Big|^\tau_\sigma \Big|^\mu_\nu \right]. \end{aligned} \tag{7.1.18}$$

Now, combining (7.1.17) and (7.1.18), we restore $\mathscr{M}$ in (1.4.6). Then, substituting $\mathscr{M}$ in such a form, (7.1.15) and (7.1.16) into (7.1.14) and using the rules in Appendix A.3, one gets the explicitly covariant form for $\boldsymbol{m}$:

$$\begin{aligned}\boldsymbol{m}_\sigma{}^{\alpha\tau} \equiv & \left[\frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_\alpha\psi^B)} - \bar\nabla_\beta\left(\frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_{\beta\alpha}\psi^B)}\right)\right]\psi^B\big|^\tau_\sigma \\ & - \frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_{\alpha\tau}\psi^B)}\bar\nabla_\sigma\psi^B + \frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_{\alpha\beta}\psi^B)}\bar\nabla_\beta(\psi^B\big|^\tau_\sigma).\end{aligned} \tag{7.1.19}$$

To derive $\boldsymbol{u}$ in (7.1.13), we use the same strategy and obtain finally the explicitly covariant form:

$$\begin{aligned}\boldsymbol{u}_\sigma{}^\alpha \equiv & \mathscr{L}_c\delta^\alpha_\sigma + \frac{\delta\mathscr{L}_c}{\delta\psi^B}\psi^B\big|^\alpha_\sigma - \left[\frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_\alpha\psi^B)} - \bar\nabla_\beta\left(\frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_{\beta\alpha}\psi^B)}\right)\right]\bar\nabla_\sigma\psi^B \\ & - \frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_{\alpha\tau}\psi^B)}\bar\nabla_{\tau\sigma}\psi^B + \frac{1}{2}\frac{\partial\mathscr{L}_c}{\partial(\bar\nabla_{\alpha\tau}\psi^B)}\psi^B\big|^\beta_\lambda\bar R^\lambda{}_{\sigma\tau\beta}.\end{aligned} \tag{7.1.20}$$

Showing that the coefficients (7.1.13–7.1.15) are covariant, we demonstrate that the expression under divergence in (7.1.12) in whole is a vector density. Thus, keeping in mind (A.2.12), we can rewrite the identity (7.1.12) as

$$\bar\nabla_\alpha\left[\boldsymbol{u}_\sigma{}^\alpha\xi^\sigma + \boldsymbol{m}_\sigma{}^{\alpha\tau}\bar\nabla_\tau\xi^\sigma + \boldsymbol{n}_\sigma{}^{\alpha\tau\beta}\bar\nabla_{\beta\tau}\xi^\sigma\right] \equiv 0. \tag{7.1.21}$$

Opening it and equating the coefficients independently to zero at ξ^σ, $\bar\nabla_\alpha\xi^\sigma$, $\bar\nabla_{(\beta\alpha)}\xi^\sigma$ and $\bar\nabla_{(\gamma\beta\alpha)}\xi^\sigma$, we get a set of identities:

$$\bar\nabla_\alpha\boldsymbol{u}_\sigma{}^\alpha + \frac{1}{2}\boldsymbol{m}_\lambda{}^{\alpha\rho}\bar R^{\lambda}_{\sigma\,\rho\alpha} + \frac{1}{3}\boldsymbol{n}_\lambda{}^{\alpha\rho\gamma}\bar\nabla_\gamma\bar R^{\lambda}_{\sigma\,\rho\alpha} \equiv 0, \tag{7.1.22}$$

$$\boldsymbol{u}_\sigma{}^\alpha + \bar\nabla_\lambda\boldsymbol{m}_\sigma{}^{\lambda\alpha} + \boldsymbol{n}_\lambda{}^{\tau\alpha\rho}\bar R^{\lambda}_{\sigma\,\rho\tau} + \frac{2}{3}\boldsymbol{n}_\sigma{}^{\lambda\tau\rho}\bar R^{\alpha}_{\tau\rho\lambda} \equiv 0, \tag{7.1.23}$$

$$\boldsymbol{m}_\sigma{}^{(\alpha\beta)} + \bar\nabla_\lambda\boldsymbol{n}_\sigma{}^{\lambda(\alpha\beta)} \equiv 0, \tag{7.1.24}$$

$$\boldsymbol{n}_\sigma^{(\alpha\beta\gamma)} \equiv 0. \tag{7.1.25}$$

Let us discuss this system. Substituting here the expression (7.1.13–7.1.15), one can find out that the system (7.1.22–7.1.25) consists of linear combinations of the Klein identities (1.4.8–1.4.11). The identity (7.1.22) corresponds to $\partial_\alpha\mathscr{U}_\sigma{}^\alpha \equiv 0$. The latter is a consequence of (1.4.9–1.4.11). Analogously, (7.1.22) is not independent – it is a consequence of (7.1.23–7.1.25).

Since the equality (7.1.21) is identically satisfied, the current

$$\boldsymbol{i}^\alpha = -\left[\boldsymbol{u}_\sigma{}^\alpha\xi^\sigma + \boldsymbol{m}_\sigma{}^{\alpha\tau}\bar\nabla_\tau\xi^\sigma + \boldsymbol{n}_\sigma{}^{\alpha\tau\beta}\bar\nabla_{\beta\tau}\xi^\sigma\right], \tag{7.1.26}$$

must to be expressed through a divergence of a superpotential, $\boldsymbol{i}^{\alpha\beta}$, — antisymmetric tensor density, by the way:

$$\boldsymbol{i}^{\alpha} \equiv \partial_{\beta}\boldsymbol{i}^{\alpha\beta} \equiv \bar{\nabla}_{\beta}\boldsymbol{i}^{\alpha\beta}. \tag{7.1.27}$$

A double divergence of $\boldsymbol{i}^{\alpha\beta}$ is equal to zero identically, $\partial_{\beta\alpha}\boldsymbol{i}^{\alpha\beta} \equiv \bar{\nabla}_{\beta\alpha}\boldsymbol{i}^{\alpha\beta} \equiv 0$. Indeed, substituting $\boldsymbol{u}_{\sigma}{}^{\alpha}$ from (7.1.23) into the current (7.1.26) , using (7.1.24) and algebraic properties of $\boldsymbol{n}_{\sigma}{}^{\alpha\beta\gamma}$ and $\bar{R}^{\alpha}_{\ \beta\rho\sigma}$, and preserving the covariance, we reconstruct (7.1.26) into the form (7.1.27), where the superpotential is given by

$$\boldsymbol{i}^{\alpha\beta} = \left(\frac{2}{3}\bar{\nabla}_{\lambda}\boldsymbol{n}_{\sigma}{}^{[\alpha\beta]\lambda} - \boldsymbol{m}_{\sigma}{}^{[\alpha\beta]}\right)\xi^{\sigma} - \frac{4}{3}\boldsymbol{n}_{\sigma}{}^{[\alpha\beta]\lambda}\bar{\nabla}_{\lambda}\xi^{\sigma} \tag{7.1.28}$$

that is explicitly antisymmetric in α and β.

At last, the current in (7.1.26) can be rewritten as

$$\boldsymbol{i}^{\alpha} = -\left[(\boldsymbol{u}_{\sigma}{}^{\alpha} + \boldsymbol{n}_{\lambda}{}^{\alpha\beta\gamma}\bar{R}^{\lambda}_{\ \beta\gamma\sigma})\xi^{\sigma} + \boldsymbol{m}^{\rho\alpha\beta}\partial_{[\beta}\xi_{\rho]} + \boldsymbol{z}^{\alpha}\right] \tag{7.1.29}$$

where $\boldsymbol{z}$-term is defined as

$$\boldsymbol{z}^{\alpha}(\xi) = \boldsymbol{m}^{\sigma\alpha\beta}\zeta_{\sigma\beta} + \boldsymbol{n}^{\rho\alpha\beta\gamma}\left(2\bar{\nabla}_{\gamma}\zeta_{\beta\rho} - \bar{\nabla}_{\rho}\zeta_{\beta\gamma}\right), \tag{7.1.30}$$

where we used the notation,

$$\zeta_{\rho\sigma} = -\frac{1}{2}£_{\xi}\bar{g}_{\rho\sigma} = \bar{\nabla}_{(\rho}\xi_{\sigma)}, \tag{7.1.31}$$

already introduced in (6.1.14) in four dimensions. The representation in the form (7.1.29) is quite useful because the $\boldsymbol{z}$-term disappears, if ξ^{μ} is a Killing vector of the background spacetime. Then only the current (7.1.26) is determined by the energy-momentum $(\boldsymbol{u} + \boldsymbol{n}\bar{R})$-term and the spin $\boldsymbol{m}$-term.

Now, let us summarize the results. Instead of the non-covariant coefficients (1.4.5), (1.4.6) and (1.4.7), the corresponding covariant coefficients (7.1.20), (7.1.19) and (7.1.16) have been constructed. Instead of the non-covariant Klein–Noether identities (1.4.9–1.4.12), the corresponding covariant identities (7.1.23–7.1.25), (7.1.10) are introduced. By construction, the explicitly covariant current (7.1.26) is equal to the current (1.4.14) in the original form exactly: $\boldsymbol{i}^{\alpha} \equiv \mathscr{I}^{\alpha}$. One can show that for the covariant superpotentials (7.1.28) and the non-covariant superpotential (1.4.20)

$$\boldsymbol{i}^{\alpha\beta} = \mathscr{I}^{\alpha\beta} + \frac{4}{3}\partial_{\lambda}\left(\mathscr{N}_{\sigma}{}^{[\lambda\beta]\alpha}\xi^{\sigma}\right). \tag{7.1.32}$$

This means that for the antisymmetric superpotential (1.4.21), $\mathscr{I}^{\alpha\beta} = \boldsymbol{i}^{\alpha\beta}$, and, thus, it can be transformed into an explicitly covariant form.

7.1.2 Another variant of covariantization

In Section 1.4.1, all the identities and conserved quantities are derived through the partial derivatives. The order of the partial derivatives is not important because they are symmetric with respect to exchange of the indices. For example, expressions, like $\partial\mathscr{L}/\partial\psi^B{}_{,\alpha\beta}$, are symmetric in α and β. In the previous subsection, we *did not use* the symmetry of partial derivatives, *preserving the original order* of derivatives in the identities after variations. This permitted us to make a direct transition to the covariant versions of identities and conserved quantities. However, the covariant derivatives do not commute as contrasted to the partial derivatives, and this gives additional possibilities for constructing conserved quantities for perturbations.

To explain the situation better, let us consider an auxiliary Lagrangian as an example: $\mathscr{L}^{\text{test}} = \mathscr{P}_B{}^{\alpha\beta}\psi^B{}_{,\alpha\beta} + \ldots$. After direct covariantization, as in the previous subsection, it acquires the form $\mathscr{L}_c^{test} = \mathscr{P}_B{}^{\alpha\beta}\bar{\nabla}_{\beta\alpha}\psi^B + \ldots$. The variation with respect to $\psi^B{}_{,\alpha\beta}$ in the fist case gives $\mathscr{P}_B{}^{(\alpha\beta)}$. However, originally $\mathscr{P}_B{}^{\alpha\beta}$ is not necessarily symmetric in α and β, therefore in the second case, the variation with respect to $\bar{\nabla}_{\beta\alpha}\psi^B$ gives simply $\mathscr{P}_B{}^{\alpha\beta}$. Thus, unlike the first case, changing the order of the second covariant derivatives can lead to a different result. If we symmetrize α and β in the second case: $\mathscr{L}_c^{test} = \mathscr{P}_B{}^{\alpha\beta}\bar{\nabla}_{(\beta\alpha)}\psi^B + \ldots$ then we need to make sure that it is consistent with other terms in the Lagrangian.

To study the problem of a different order of the second covariant derivatives, let us change the order of indices in (7.1.6) as follows,

$$\begin{aligned}\mathscr{L}(\psi^B, \psi^B_{,\alpha}, \psi^B_{,\alpha\beta}) &\equiv \mathscr{L}_c(\psi^B, \bar{\nabla}_\alpha\psi^B, \bar{\nabla}_{\beta\alpha}\psi^B, \bar{g}_{\mu\nu}, \bar{R}^\alpha{}_{\mu\beta\nu})\\ &\equiv \mathscr{L}_c\left(\psi^B, \bar{\nabla}_\alpha\psi^B, \bar{\nabla}_{\alpha\beta}\psi^B + \psi^B\big|^\rho_\sigma\, \bar{R}_\rho{}^\sigma{}_{\alpha\beta}, \bar{g}_{\mu\nu}, \bar{R}^\alpha{}_{\mu\beta\nu}\right)\\ &\equiv \mathscr{L}^*(\psi^B, \bar{\nabla}_\alpha\psi^B, \bar{\nabla}_{\beta\alpha}\psi^B, \bar{g}_{\mu\nu}, \bar{R}^\alpha{}_{\mu\beta\nu}).\end{aligned} \tag{7.1.33}$$

Expression (7.1.33) reveals that if one changes the order of the second covariant derivatives of ψ^B one gets an additional derivative with respect to ψ^B, being proportional to the Riemann tensor.

At first, one has to check that the order of indices at second covariant derivatives does not affect the equations of motion defined by the Lagrangian derivative (7.1.9). For its explicitly covariant expression see (A.3.37) in Appendix A.3.3. Thus, for the first line in (7.1.33) one has

$$\frac{\delta\mathscr{L}_c}{\delta\psi^B} = \frac{\partial\mathscr{L}_c}{\partial\psi^B} - \bar{\nabla}_\alpha\left(\frac{\partial\mathscr{L}_c}{\partial(\bar{\nabla}_\alpha\psi^B)}\right) + \bar{\nabla}_{\alpha\beta}\left(\frac{\partial\mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha}\psi^B)}\right). \tag{7.1.34}$$

Of course, for the starred Lagrangian, the form of the Lagrangian derivative has the same standard form (7.1.34). Then, substituting the second line of (7.1.33), instead of $\mathscr{L}^*$, into the standard expression of the type (7.1.34), one obtains

$$\frac{\delta \mathscr{L}^*}{\delta \psi^B} = \frac{\partial \mathscr{L}_c}{\partial \psi^B} - \bar{\nabla}_\alpha \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_\alpha \psi^B)} \right) + \bar{\nabla}_{\alpha\beta} \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\beta} \psi^B)} \right)$$
$$+ \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha} \psi^C)} \frac{\partial}{\partial \psi^B} \left(\psi^C \Big|^\rho_\lambda \bar{R}_\rho{}^\lambda{}_{\alpha\beta} \right). \tag{7.1.35}$$

After changing the order of derivatives in the third term at the right hand side, using (A.3.9) and other formulae in Appendix A.3, one can see that, indeed the Lagrangian derivative (7.1.35) goes over to (7.1.34). However, different definitions of the covariant Lagrangian in (7.1.33) lead to different definitions of conserved quantities as we show below.

To derive the coefficients of the Klein identities related to $\mathscr{L}^*$ one has to use the same form (7.1.16), (7.1.19) and (7.1.20), of course. Then one can see that the use of the second line of (7.1.33) changes the order of the second derivatives in all the terms, like $\partial \mathscr{L}/\partial(\bar{\nabla}_{\beta\alpha}\psi^B) \to \partial \mathscr{L}/\partial(\bar{\nabla}_{\alpha\beta}\psi^B)$, *simultaneously*. This gives

$$\boldsymbol{n}^{*\,\alpha\tau\beta}_\sigma \equiv \frac{1}{2} \left[\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha} \psi^B)} \psi^B \Big|^\tau_\sigma + \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\tau\alpha} \psi^B)} \psi^B \Big|^\beta_\sigma \right], \tag{7.1.36}$$

$$\boldsymbol{m}^{*\,\alpha\tau}_\sigma \equiv \left[\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_\alpha \psi^B)} - \bar{\nabla}_\beta \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\beta} \psi^B)} \right) \right] \psi^B \Big|^\tau_\sigma$$
$$- \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\tau\alpha} \psi^B)} \bar{\nabla}_\sigma \psi^B + \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha} \psi^B)} \bar{\nabla}_\beta \left(\psi^B \Big|^\tau_\sigma \right), \tag{7.1.37}$$

$$\boldsymbol{u}^{*\,\alpha}_\sigma \equiv \mathscr{L}_c \delta^\alpha_\sigma + \frac{\delta \mathscr{L}_c}{\delta \psi^B} \psi^B \Big|^\alpha_\sigma - \left[\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_\alpha \psi^B)} - \bar{\nabla}_\beta \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\beta} \psi^B)} \right) \right] \bar{\nabla}_\sigma \psi^B$$
$$- \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha} \psi^B)} \bar{\nabla}_{\beta\sigma} \psi^B + \frac{1}{2} \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\tau\alpha} \psi^B)} \psi^B \Big|^\beta_\lambda \bar{R}^\lambda{}_{\sigma\tau\beta}. \tag{7.1.38}$$

Because all of the expressions (7.1.36–7.1.38) have been obtained from the same Lagrangian (7.1.33), they all have to satisfy the same identities (7.1.22–7.1.25) as well. Let us show this. It is not difficult to find a connection of the expressions (7.1.36–7.1.38) with the coefficients (7.1.16), (7.1.19) and (7.1.20):

$$\boldsymbol{n}^{*\,\alpha\tau\beta}_\sigma \equiv \boldsymbol{n}_\sigma{}^{\alpha\tau\beta} + \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]} \psi^B)} \psi^B \Big|^\tau_\sigma + \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\tau\alpha]} \psi^B)} \psi^B \Big|^\beta_\sigma, \tag{7.1.39}$$

$$\boldsymbol{m}^{*\,\alpha\tau}_\sigma \equiv \boldsymbol{m}_\sigma{}^{\alpha\tau} - 2 \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\tau\alpha]} \psi^B)} \bar{\nabla}_\sigma \psi^B + 2 \bar{\nabla}_\beta \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]} \psi^B)} \psi^B \Big|^\tau_\sigma \right), \tag{7.1.40}$$

$$\boldsymbol{u}^{*\,\alpha}_\sigma \equiv \boldsymbol{u}_\sigma{}^\alpha - 2 \bar{\nabla}_\beta \left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]} \psi^B)} \bar{\nabla}_\sigma \psi^B \right) + \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\tau\alpha]} \psi^B)} \psi^B \Big|^\beta_\lambda \bar{R}^\lambda{}_{\sigma\tau\beta}, \tag{7.1.41}$$

where we defined

$$\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)} \equiv \frac{1}{2}\left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\beta\alpha}\psi^B)} - \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\alpha\beta}\psi^B)}\right). \tag{7.1.42}$$

Then, a direct use of (7.1.39–7.1.41) in the identities (7.1.22–7.1.25) shows that $\boldsymbol{n}^*$, $\boldsymbol{m}^*$ and $\boldsymbol{u}^*$ satisfy them as well, like the coefficients $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{u}$. This means that if we construct a starred-current with using the rule (7.1.26):

$$\boldsymbol{i}^{*\alpha} = -\left[\boldsymbol{u}^{*\alpha}_{\sigma}\xi^\sigma + \boldsymbol{m}^{*\alpha\tau}_{\sigma}\bar{\nabla}_\tau\xi^\sigma + \boldsymbol{n}^{*\alpha\tau\beta}_{\sigma}\bar{\nabla}_{\beta\tau}\xi^\sigma\right], \tag{7.1.43}$$

then it is identically conserved. Indeed, it is easy to find that

$$\boldsymbol{i}^{*\alpha} = \boldsymbol{i}^\alpha - 2\bar{\nabla}_\beta\left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)}\pounds_\xi\psi^B\right). \tag{7.1.44}$$

Then $\partial_\alpha\boldsymbol{i}^{*\alpha} \equiv \partial_\alpha\boldsymbol{i}^\alpha$, and consequently $\partial_\alpha\boldsymbol{i}^{*\alpha} \equiv 0$. Analogous to (7.1.27), the identity

$$\boldsymbol{i}^{*\alpha} \equiv \partial_\beta\boldsymbol{i}^{*\alpha\beta} \equiv \bar{\nabla}_\beta\boldsymbol{i}^{*\alpha\beta} \tag{7.1.45}$$

exists where

$$\boldsymbol{i}^{*\alpha\beta} = \left(\frac{2}{3}\bar{\nabla}_\lambda\boldsymbol{n}^{*[\alpha\beta]\lambda}_{\sigma} - \boldsymbol{m}^{*[\alpha\beta]}_{\sigma}\right)\xi^\sigma - \frac{4}{3}\boldsymbol{n}^{*[\alpha\beta]\lambda}_{\sigma}\bar{\nabla}_\lambda\xi^\sigma. \tag{7.1.46}$$

The direct substitution of (7.1.39) and (7.1.40) into (7.1.46) gives

$$\begin{aligned}\boldsymbol{i}^{*\alpha\beta} = \boldsymbol{i}^{\alpha\beta} - 2\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)}\pounds_\xi\psi^B + \frac{2}{3}\bar{\nabla}_\rho\Bigg[\xi^\sigma\Bigg(&\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)}\,\psi^B\Big|^\rho_\sigma\\ &+ \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\alpha\rho]}\psi^B)}\,\psi^B\Big|^\beta_\sigma + \frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\rho\beta]}\psi^B)}\,\psi^B\Big|^\alpha_\sigma\Bigg)\Bigg].\end{aligned} \tag{7.1.47}$$

This looks very cumbersome, however let us discuss it. The expression in the square brackets is antisymmetric in α, β and ρ. Thus $\bar{\nabla}_\rho$ can be changed by ∂_ρ and one can see that the term in the square brackets does not contribute into the current in (7.1.45). Also, due to the Stokes's theorem this term does not contribute to the surface integrals calculated with the use of the superpotentials. Therefore, without the loss of information, we can use

$$\boldsymbol{i}^{*\alpha\beta} = \boldsymbol{i}^{\alpha\beta} - 2\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)}\pounds_\xi\psi^B, \tag{7.1.48}$$

which is in accordance with (7.1.44) and (7.1.45).

7.1.3 A new family of the identically conserved covariant Nother quantities

Let us rewrite the identities, which follow from the Lagragians $\mathscr{L}$, $\mathscr{L}_c$ and $\mathscr{L}^*$, respectively,

$$\partial_\alpha \mathscr{I}^\alpha \equiv \partial_\alpha \boldsymbol{i}^\alpha \equiv \partial_\alpha \boldsymbol{i}^{*\alpha} \equiv 0. \tag{7.1.49}$$

Besides, recall that each of the Lagrangains $\mathscr{L}$, $\mathscr{L}_c$ and $\mathscr{L}^*$ lead to the same equations of motion for ψ^B. Then adding (7.1.8) in a related form to each of the identities in (7.1.49) one obtains

$$-\left[\frac{\delta \mathscr{L}}{\delta \psi^B}\psi^B{}_{,\alpha} + \partial_\beta\left(\frac{\delta \mathscr{L}}{\delta \psi^B}\,\psi^B\Big|^\beta_\alpha\right)\right]\xi^\alpha - \partial_\alpha \mathscr{I}^\alpha \equiv 0, \tag{7.1.50}$$

$$-\left[\frac{\delta \mathscr{L}_c}{\delta \psi^B}\bar\nabla_\alpha\psi^B + \bar\nabla_\beta\left(\frac{\delta \mathscr{L}_c}{\delta \psi^B}\,\psi^B\Big|^\beta_\alpha\right)\right]\xi^\alpha - \partial_\alpha \boldsymbol{i}^\alpha \equiv 0, \tag{7.1.51}$$

$$-\left[\frac{\delta \mathscr{L}^*}{\delta \psi^B}\bar\nabla_\alpha\psi^B + \bar\nabla_\beta\left(\frac{\delta \mathscr{L}^*}{\delta \psi^B}\,\psi^B\Big|^\beta_\alpha\right)\right]\xi^\alpha - \partial_\alpha \boldsymbol{i}^{*\alpha} \equiv 0. \tag{7.1.52}$$

Each of these identities is its own form of the *unique* identity (7.1.7). Therefore, each of the currents, $\boldsymbol{i}^\alpha$ and $\boldsymbol{i}^{*\alpha}$, has equal rights. Nevertheless, which could be the preferable one from them? At a first glance it seems that it is $\boldsymbol{i}^\alpha$ because by construction, $\boldsymbol{i}^\alpha \equiv \mathscr{I}^\alpha$. On the other hand, a conservation of the symmetry of partial derivatives looks like a good idea. Then one can choose

$$\mathscr{L}(\psi^B, \psi^B{}_{,\alpha}, \psi^B{}_{,\alpha\beta}) \equiv \mathscr{L}_c\left(\psi^B, \bar\nabla_\alpha\psi^B, \bar\nabla_{(\beta\alpha)}\psi^B + \frac{1}{2}\,\psi^B\Big|^\rho_\sigma \bar R_\rho{}^\sigma{}_{\alpha\beta}, \bar g_{\mu\nu}, \bar R^\alpha{}_{\mu\beta\nu}\right) \tag{7.1.53}$$

instead of (7.1.33). However, in reality, we do not see any theoretical foundation for such a choice. Possibly, in future, applications of the constructed expressions to complicated solutions of the numerous modern modifications of general relativity will permit to do the choice. To unite the aforementioned possibilities for constructing covariant conserved quantities we suggest a covariant Lagrangian of the form:

$$\begin{aligned}\mathscr{L}(\psi^B, \psi^B{}_{,\alpha}, \psi^B{}_{,\alpha\beta}) &\equiv \mathscr{L}^\dagger\left(\psi^B, \bar\nabla_\alpha\psi^B, \bar\nabla_{\beta\alpha}\psi^B, \bar g_{\mu\nu}, \bar R^\alpha{}_{\mu\beta\nu}\right)\\ &\equiv p\,\mathscr{L}_c\left(\psi^B, \bar\nabla_\alpha\psi^B, \bar\nabla_{\beta\alpha}\psi^B, \bar g_{\mu\nu}, \bar R^\alpha{}_{\mu\beta\nu}\right)\\ &\quad + q\,\mathscr{L}^*\left(\psi^B, \bar\nabla_\alpha\psi^B, \bar\nabla_{\beta\alpha}\psi^B, \bar g_{\mu\nu}, \bar R^\alpha{}_{\mu\beta\nu}\right),\end{aligned} \tag{7.1.54}$$

where $p + q = 1$ with real p and q. The Lagrangian (7.1.54) leads to the same field equations, whereas the conservation law for (7.1.54) are defined now as

$$\boldsymbol{i}^{\dagger\alpha} \equiv \partial_\beta \boldsymbol{i}^{\dagger\alpha\beta} \tag{7.1.55}$$

with the conserved quantities $\boldsymbol{i}^{\dagger\alpha} \equiv p\,\boldsymbol{i}^{\alpha} + q\,\boldsymbol{i}^{*\alpha}$ and $\boldsymbol{i}^{\dagger\alpha\beta} \equiv p\,\boldsymbol{i}^{\alpha\beta} + q\,\boldsymbol{i}^{*\alpha\beta}$, representing in reality a *new* family of identically conserved quantities.

Turning to (7.1.44) and (7.1.48) one can rewrite the current and superpotential in (7.1.55), as

$$\boldsymbol{i}^{\dagger\alpha} = \boldsymbol{i}^{\alpha} + q\bar{\nabla}_\beta \boldsymbol{r}^{\alpha\beta}, \tag{7.1.56}$$

$$\boldsymbol{i}^{\dagger\alpha\beta} = \boldsymbol{i}^{\alpha\beta} + q\boldsymbol{r}^{\alpha\beta}; \tag{7.1.57}$$

$$\boldsymbol{r}^{\alpha\beta} \equiv -2\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)}\pounds_\xi \psi^B \tag{7.1.58}$$

with $0 \le q \le 1$, or

$$\boldsymbol{i}^{\dagger\alpha} = \boldsymbol{i}^{*\alpha} + p\bar{\nabla}_\beta \boldsymbol{r}^{*\alpha\beta}, \tag{7.1.59}$$

$$\boldsymbol{i}^{\dagger\alpha\beta} = \boldsymbol{i}^{*\alpha\beta} + p\boldsymbol{r}^{*\alpha\beta}; \tag{7.1.60}$$

$$\boldsymbol{r}^{*\alpha\beta} \equiv -2\frac{\partial \mathscr{L}_c^{\;*}}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)}\pounds_\xi \psi^B \tag{7.1.61}$$

with $0 \le p \le 1$.

A contribution from the divergence in the Lagrangian

To finalize this subsection, one has to note the following. Recall that a divergence in the Lagrangian, being irrelevant for deriving the field equations, is important (frequently even crucial) in the definition of Nœther's canonical conserved quantities. We give some necessary formulae. Let the Lagrangian $\mathscr{L}$ be added by the scalar density,

$$\mathscr{L}' = \mathscr{D}^\nu{}_{,\nu}, \tag{7.1.62}$$

where $\mathscr{D}^\nu$ is a vector density. Then the Noether identity (7.1.7) can be rewritten for (7.1.62) in the form:

$$\partial_\alpha\left(\pounds_\xi \mathscr{D}^\alpha + \xi^\alpha \mathscr{D}^\nu{}_{,\nu}\right) \equiv \bar{\nabla}_\alpha\left(\pounds_\xi \mathscr{D}^\alpha + \xi^\alpha \bar{\nabla}_\nu \mathscr{D}^\nu\right) \equiv 0. \tag{7.1.63}$$

The corresponding, additional, coefficients are

$$\boldsymbol{u}'^{\;\alpha}_{\sigma} = 2\bar{\nabla}_\beta(\delta^{[\alpha}_\sigma \mathscr{D}^{\beta]}), \tag{7.1.64}$$

$$\boldsymbol{m}'^{\;\alpha\beta}_{\sigma} = 2\delta^{[\alpha}_\sigma \mathscr{D}^{\beta]}, \tag{7.1.65}$$

$$\boldsymbol{n}'^{\;\alpha\beta\gamma}_{\sigma} = 0. \tag{7.1.66}$$

The corresponding additional current and superpotential are

$$\boldsymbol{i}'^{\alpha} = -\boldsymbol{u}'^{\;\alpha}_{\sigma}\xi^\sigma - \boldsymbol{m}'^{\;\alpha\tau}_{\sigma}\bar{\nabla}_\tau \xi^\sigma, \tag{7.1.67}$$

$$\boldsymbol{i}'^{\alpha\beta} = -\boldsymbol{m}'^{\;[\alpha\beta]}_{\sigma}\xi^\sigma. \tag{7.1.68}$$

Note that a *construction* of these quantities does not depend on the intrinsic structure of $\mathscr{D}^\nu$. Thus, a contribution from the divergence (7.1.62) to each of Lagrangians, starred and others, is the same.

7.1.4 A Belinfante corrected family of identically conserved quantities

Here, we modify the results of previous subsections with the use of the Belinfante procedure [34]. By the general Belinfante rule, see (6.2.2) for general relativity, we define a tensor density

$$\boldsymbol{s}^{\alpha\beta\sigma} \equiv -\boldsymbol{s}^{\beta\alpha\sigma} \equiv -\boldsymbol{m}_\lambda{}^{\sigma[\alpha}\bar{g}^{\beta]\lambda} - \boldsymbol{m}_\lambda{}^{\alpha[\sigma}\bar{g}^{\beta]\lambda} + \boldsymbol{m}_\lambda{}^{\beta[\sigma}\bar{g}^{\alpha]\lambda}, \tag{7.1.69}$$

which is called a Belinfante correction. Now, let us add $\bar{\nabla}_\beta(\boldsymbol{s}^{\alpha\beta\sigma}\xi_\sigma)$ to both sides of (7.1.27) and obtain a new identity:

$$\boldsymbol{i}^\alpha_B \equiv \partial_\beta \boldsymbol{i}^{\alpha\beta}_B \equiv \bar{\nabla}_\beta \boldsymbol{i}^{\alpha\beta}_B. \tag{7.1.70}$$

This modification cancels the spin term from the current (7.1.29):

$$\boldsymbol{i}^\alpha_B \equiv \left(-\boldsymbol{u}_\sigma{}^\alpha - \boldsymbol{n}_\lambda{}^{\alpha\beta\gamma}\bar{R}^\lambda{}_{\beta\gamma\sigma} + \bar{\nabla}_\beta \boldsymbol{s}^{\alpha\beta}{}_\sigma\right)\xi^\sigma + \boldsymbol{z}^\alpha_B(\xi) \equiv {}_B\boldsymbol{u}_\sigma{}^\alpha\xi^\sigma + \boldsymbol{z}^\alpha_B(\xi), \tag{7.1.71}$$

where the new $\boldsymbol{z}$-term disappears also on Killing vectors of the background:

$$\begin{aligned}\boldsymbol{z}^\alpha_B(\xi) = {} & \left(\boldsymbol{m}_\lambda{}^{\beta\alpha}\bar{g}^{\tau\lambda} + \boldsymbol{m}_\lambda{}^{\alpha\tau}\bar{g}^{\beta\lambda} - \boldsymbol{m}_\lambda{}^{\tau\beta}\bar{g}^{\alpha\lambda}\right)\zeta_{\tau\beta} \\ & + \boldsymbol{n}_\lambda{}^{\alpha\tau\beta}\left(2\bar{\nabla}_{(\beta}\zeta^\lambda_{\tau)} - \bar{\nabla}_\sigma\zeta_{\beta\tau}\bar{g}^{\lambda\sigma}\right),\end{aligned} \tag{7.1.72}$$

see the definition (7.1.31) for $\zeta_{\alpha\beta}$. Thus, the current $\boldsymbol{i}^\alpha_B$ is defined, in fact, by the modified energy-momentum tensor density ${}_B\boldsymbol{u}_\sigma{}^\alpha$ only. The new superpotential depends on the $\boldsymbol{n}$-coefficients only:

$$\boldsymbol{i}^{\alpha\beta}_B \equiv 2\left(\frac{1}{3}\bar{\nabla}_\rho \boldsymbol{n}_\sigma{}^{[\alpha\beta]\rho} + \bar{\nabla}_\tau \boldsymbol{n}_\lambda{}^{\tau\rho[\alpha}\bar{g}^{\beta]\lambda}\bar{g}_{\rho\sigma}\right)\xi^\sigma - \frac{4}{3}\boldsymbol{n}_\sigma{}^{[\alpha\beta]\lambda}\overline{D}_\lambda\xi^\sigma. \tag{7.1.73}$$

Then, due to the definition (7.1.16), the superpotential (7.1.73) vanishes for Lagrangians with only the first order derivatives. On the other hand, the superpotential (7.1.73) is well adapted to the theories with second derivatives in Lagrangians, say, algebraically depending on Riemann tensor, like in many modern gravitational theories.

All the above can be applied exactly to the starred quantities in Section 7.1.3. As a result, one obtains

$$\boldsymbol{i}^{*\alpha}_B \equiv \partial_\beta \boldsymbol{i}^{*\alpha\beta}_B \equiv \bar{\nabla}_\beta \boldsymbol{i}^{*\alpha\beta}_B, \tag{7.1.74}$$

where

$$\boldsymbol{i}_B^{*\alpha} \equiv \left(-\boldsymbol{u}_\sigma^{*\alpha} - \boldsymbol{n}_\lambda^{*\alpha\beta\gamma}\bar{R}^\lambda{}_{\beta\gamma\sigma} + \bar{\nabla}_\beta \boldsymbol{s}^{*\alpha\beta}{}_\sigma\right)\xi^\sigma + \boldsymbol{z}_B^{*\alpha}(\xi), \tag{7.1.75}$$

$$\boldsymbol{i}_B^{*\alpha\beta} \equiv 2\left(\frac{1}{3}\bar{\nabla}_\rho \boldsymbol{n}_\sigma^{*[\alpha\beta]\rho} + \bar{\nabla}_\tau \boldsymbol{n}_\lambda^{*\tau\rho[\alpha}\bar{g}^{\beta]\lambda}\bar{g}_{\rho\sigma}\right)\xi^\sigma - \frac{4}{3}\boldsymbol{n}_\sigma^{*[\alpha\beta]\lambda}\bar{\nabla}_\lambda\xi^\sigma. \tag{7.1.76}$$

The connection of the starred Belinfante corrected quantities with the current (7.1.71) and the superpotential (7.1.73) is stated with the use of (7.1.39–7.1.41),

$$\boldsymbol{i}_B^{*\alpha} \equiv \boldsymbol{i}_B^{\alpha} + \bar{\nabla}_\beta \boldsymbol{r}_B^{\alpha\beta}, \tag{7.1.77}$$

$$\boldsymbol{i}_B^{*\alpha\beta} \equiv \boldsymbol{i}_B^{\alpha\beta} + \boldsymbol{r}_B^{\alpha\beta}, \tag{7.1.78}$$

where

$$\begin{aligned}\boldsymbol{r}_B^{\alpha\beta} \equiv -2\Bigg[&\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{[\beta\alpha]}\psi^B)}\,\psi^B\Big|_\sigma^\rho\,\bar{\nabla}_\rho\xi^\sigma + \bar{\nabla}_\rho\left(\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\rho\sigma}\psi^B)}\,\psi^B\Big|_\lambda^{[\alpha}\,\bar{g}^{\beta]\lambda}\right.\\ &\left.+\frac{\partial \mathscr{L}_c}{\partial(\bar{\nabla}_{\rho\alpha}\psi^B)}\,\psi^B\Big|_\lambda^{[\sigma}\,\bar{g}^{\beta]\lambda} + \frac{\partial \mathscr{L}}{\partial(\bar{\nabla}_{\beta\rho}\psi^B)}\,\psi^B\Big|_\lambda^{[\sigma}\,\bar{g}^{\alpha]\lambda}\right)\xi_\sigma\Bigg].\end{aligned} \tag{7.1.79}$$

Analogously, one defines $\boldsymbol{r}_B^{*\alpha\beta}$ based on the Lagrangian $\mathscr{L}_c{}^*$.

The Belinfante corrected covariant conservation laws and conserved quantities for the family of the Lagrangians (7.1.54) are defined as

$$\boldsymbol{i}_B^{\dagger\alpha} \equiv \partial_\beta \boldsymbol{i}_B^{\dagger\alpha\beta} \equiv \bar{\nabla}_\beta \boldsymbol{i}_B^{\dagger\alpha\beta} \tag{7.1.80}$$

with $\boldsymbol{i}_B^{\dagger\alpha} = p\,\boldsymbol{i}_B^{\alpha} + q\,\boldsymbol{i}_B^{*\alpha}$ and $\boldsymbol{i}_B^{\dagger\alpha\beta} = p\,\boldsymbol{i}_B^{\alpha\beta} + q\,\boldsymbol{i}_B^{*\alpha\beta}$, representing a Belinfante corrected family of identically conserved quantities.

Analogously to (7.1.56) and (7.1.61), one can rewrite the current and the superpotential in (7.1.80) as

$$\boldsymbol{i}_B^{\dagger\alpha} = \boldsymbol{i}_B^{\alpha} + q\bar{\nabla}_\beta \boldsymbol{r}_B^{\alpha\beta}, \tag{7.1.81}$$

$$\boldsymbol{i}_B^{\dagger\alpha\beta} = \boldsymbol{i}_B^{\alpha\beta} + q\boldsymbol{r}_B^{\alpha\beta}; \qquad 0 \le q \le 1, \qquad \text{or} \tag{7.1.82}$$

$$\boldsymbol{i}_B^{\dagger\alpha} = \boldsymbol{i}_B^{*\alpha} + p\bar{\nabla}_\beta \boldsymbol{r}_B^{*\alpha\beta}, \tag{7.1.83}$$

$$\boldsymbol{i}_B^{\dagger\alpha\beta} = \boldsymbol{i}_B^{*\alpha\beta} + p\boldsymbol{r}_B^{*\alpha\beta}; \qquad 0 \le p \le 1. \tag{7.1.84}$$

It was remarked earlier that the conserved quantities constructed with the use of the coefficients (1.4.5–1.4.7) and the coefficients (7.1.13–7.1.15) are *unique* for the corresponding Lagrangian and in the framework of the Nœther or of the Nœther-Belinfante procedure. There is no contradiction with the results of previous and the present subsections where we suggest a *new* family of conserved quantities. It is because we have found here various possibilities to construct covariant Lagrangians, in fact we have suggested a *family* of such Lagrangians (7.1.54). Thus, for each of the Lagrangians of the family, the conserved quantities are defined by a unique way.

Zero contribution from a divergence in the Lagrangian

It is important to note that the Belinfante procedure cancels the contributions of the divergence (7.1.62) into currents and superpotentials, see (7.1.67) and (7.1.68). Let us show this. The Belinfante correction (7.1.69) constructed for $\boldsymbol{m}'$ in (7.1.65) is

$$\boldsymbol{s}'^{\alpha\beta\sigma}\xi_\sigma = 2\xi^{[\alpha}\mathscr{D}^{\beta]}. \tag{7.1.85}$$

Then, adding $\bar\nabla_\beta(\boldsymbol{s}'^{\alpha\beta\sigma}\xi_\sigma)$ and $\boldsymbol{s}'^{\alpha\beta\sigma}\xi_\sigma$ to (7.1.67) and (7.1.68), respectively, one obtains easily

$$\boldsymbol{i}'^{\alpha}_B = \boldsymbol{i}'^{\alpha} + \bar\nabla_\beta(\boldsymbol{s}'^{\alpha\beta\sigma}\xi_\sigma) = 0\,; \tag{7.1.86}$$

$$\boldsymbol{i}'^{\alpha\beta}_B = \boldsymbol{i}'^{\alpha\beta} + \boldsymbol{s}'^{\alpha\beta\sigma}\xi_\sigma = 0. \tag{7.1.87}$$

We emphasize that this result does not depend on the nature of $\mathscr{D}^\nu$ and is related to all the kinds of Lagrangians.

7.2 Conservation laws for perturbations: Three approaches

7.2.1 An arbitrary metric theory in *n* dimensions

To present the n-dimensional metric theory we consider the Lagrangian:

$$\mathscr{L}_n = -\frac{1}{2\kappa}\mathscr{L}^G(g_{\mu\nu}) + \mathscr{L}^M(g_{\mu\nu}, \Phi^A), \tag{7.2.1}$$

which depends on the metric, $g_{\mu\nu}$, and the matter, Φ^A, variables and their derivatives up to the second order. Thus the Lagrangian of the free gravitational field (metric Lagrangian), $\mathscr{L}^G$, can be thought as an algebraic function of the metric and the Riemann tensor, $\mathscr{L}^G(g_{\mu\nu}) = \mathscr{L}^G(g_{\mu\nu}, R^\alpha{}_{\rho\beta\sigma})$, that can be arbitrary with the necessary requirements for a differentiation. Here, the Einstein n-dimensional gravitational constant is

$$\kappa = 2\Omega_{n-2}G_n, \tag{7.2.2}$$

where Ω_{n-2} is the area of the unit $(n-2)$-dimensional sphere, G_n is the n-dimensional Newton's gravitational constant and $c = 1$.

Variation of (7.2.1) with respect to $g^{\mu\nu}$ leads to the gravitational equations:

$$\mathscr{E}_{\mu\nu} = \kappa \mathscr{T}_{\mu\nu}, \tag{7.2.3}$$

which generalize the Einstein's equations in the form (1.3.22). Thus the generalized Einstein's tensor density, see (1.3.15), is

$$\mathscr{E}_{\mu\nu} = \frac{\delta \mathscr{L}^G}{\delta g^{\mu\nu}}, \tag{7.2.4}$$

whereas the matter energy-momentum, $\mathscr{T}_{\mu\nu}$, is defined exactly as in (1.3.21). Variation of (7.2.1) with respect to Φ^A gives corresponding matter equations.

Below we will use also the background Lagrangian defined as

$$\bar{\mathscr{L}}_n = -\frac{1}{2\kappa}\bar{\mathscr{L}}^G + \bar{\mathscr{L}}^M, \tag{7.2.5}$$

where $\bar{\mathscr{L}}^G = \mathscr{L}^G(\bar{g}_{\mu\nu})$ and $\bar{\mathscr{L}}^M = \mathscr{L}^M(\bar{g}_{\mu\nu}, \bar{\Phi}^A)$. The corresponding background gravitational equations are

$$\bar{\mathscr{E}}_{\mu\nu} = \kappa \bar{\mathscr{T}}_{\mu\nu}, \tag{7.2.6}$$

analogously, one obtains the background matter equations. We set that the background fields $\bar{g}_{\mu\nu}$ and $\bar{\Phi}^A$ satisfy the background equations and, thus, are known (fixed).

In the present section, the subject of our attention is to be the gravitational part of the Lagrangian (7.2.1). Basing on the results of previous section, we set $\psi^A = \{g_{\mu\nu}\}$ and incorporate an external metric $\bar{g}_{\mu\nu}$ into $\mathscr{L}^G$ in (7.2.1). A representation of the Lagrangian in an "explicitly" covariant form with the use of $\bar{g}_{\mu\nu}$ is carried out following the recipe of the previous section, like in (7.1.6). We change partial derivatives with the covariant derivatives defined with the use of $\bar{g}_{\mu\nu}$. Thus,

$$\mathscr{L}^G = \mathscr{L}_c{}^G = \mathscr{L}_c{}^G(g_{\mu\nu}, \bar{\nabla}_\alpha g_{\mu\nu}, \bar{\nabla}_{\beta\alpha} g_{\mu\nu}, \bar{g}_{\mu\nu}, \bar{R}^\lambda{}_{\tau\rho\sigma}). \tag{7.2.7}$$

Now we derive the coefficients (7.1.16) and (7.1.19) for the Lagrangian

$$\mathscr{L}_c = -\frac{1}{2\kappa}\mathscr{L}_c{}^G. \tag{7.2.8}$$

Thus, the coefficients are as follow.

$$\boldsymbol{n}_\sigma{}^{\alpha\tau\beta} = -\frac{1}{4\kappa}\left[\frac{\partial \mathscr{L}_c{}^G}{\partial(\bar\nabla_{\alpha\beta}g_{\mu\nu})}\, g_{\mu\nu}\Big|^\tau_\sigma + \frac{\partial \mathscr{L}_c{}^G}{\partial(\bar\nabla_{\alpha\tau}g_{\mu\nu})}\, g_{\mu\nu}\Big|^\beta_\sigma\right], \tag{7.2.9}$$

$$\boldsymbol{m}_\sigma{}^{\alpha\tau} = -\frac{1}{2\kappa}\left[\frac{\delta \mathscr{L}_c{}^G}{\delta(\bar\nabla_{\alpha}g_{\mu\nu})}\, g_{\mu\nu}\Big|^\tau_\sigma - \frac{\partial \mathscr{L}_c{}^G}{\partial(\bar\nabla_{\alpha\tau}g_{\mu\nu})}\bar\nabla_\sigma g_{\mu\nu}\right.$$

$$\left. + \frac{\partial \mathscr{L}_c{}^G}{\partial(\bar\nabla_{\alpha\beta}g_{\mu\nu})}\bar\nabla_\beta\left(g_{\mu\nu}\Big|^\tau_\sigma\right)\right], \tag{7.2.10}$$

where $g_{\mu\nu}\Big|^\alpha_\beta = -g_{\beta\mu}\delta^\alpha_\nu - g_{\beta\nu}\delta^\alpha_\mu$ and

$$\frac{\delta \mathscr{L}_c{}^G}{\delta(\bar\nabla_{\alpha}g_{\mu\nu})} \equiv \frac{\partial \mathscr{L}_c{}^G}{\partial(\bar\nabla_{\alpha}g_{\mu\nu})} - \bar\nabla_\beta\left(\frac{\partial \mathscr{L}_c{}^G}{\partial(\bar\nabla_{\beta\alpha}g_{\mu\nu})}\right). \tag{7.2.11}$$

It is constructive to represent the coefficient (7.1.20) in a structured form:

$$\boldsymbol{u}_\sigma{}^\alpha = -\left[\frac{1}{\kappa}\mathscr{E}_\sigma{}^\alpha + \mathscr{C}_\sigma{}^\alpha + \boldsymbol{n}_\lambda{}^{\alpha\tau\beta}\bar R^\lambda{}_{\tau\beta\sigma}\right] \tag{7.2.12}$$

with the notations

$$\mathscr{E}_\sigma{}^\alpha \equiv \frac{1}{2}\frac{\delta \mathscr{L}_c{}^G}{\delta g_{\mu\nu}}\, g_{\mu\nu}\Big|^\alpha_\sigma \equiv -\frac{\delta \mathscr{L}_c{}^G}{\delta g_{\mu\alpha}} g_{\mu\sigma} \equiv \frac{\delta \mathscr{L}_c{}^G}{\delta g^{\mu\sigma}} g^{\mu\alpha}, \tag{7.2.13}$$

see (7.2.4) for the generalized Einstein's tensor density, and

$$\mathscr{C}_\sigma{}^\alpha \equiv -\frac{1}{2\kappa}\left(\frac{\partial \mathscr{L}_c{}^G}{\partial(\bar\nabla_{\alpha\beta}g_{\mu\nu})}\bar\nabla_{\beta\sigma}g_{\mu\nu} + \frac{\delta \mathscr{L}_c{}^G}{\delta(\bar\nabla_{\alpha}g_{\mu\nu})}\bar\nabla_\sigma g_{\mu\nu} - \delta^\alpha_\sigma \mathscr{L}_c{}^G\right) \tag{7.2.14}$$

for the generalized canonical energy-momentum related to the gravitational Lagrangian $\mathscr{L}_c{}^G$.

7.2.2 Canonical conserved quantities for perturbations

Incorporation of the background metric is a key point, basing on that one has the possibility to describe perturbations. As usual, perturbations are determined when a solution (dynamical) of the theory is considered as a perturbed system with respect to another solution (background) of the same theory. Then the background spacetime acquires a *physical* sense instead of being an auxiliary notion. Perturbations in such a formulation are exact (not infinitesimal or approximate), and then linear or of higher order approximations follow easily.

In this subsection, we generalize the KBL method in general relativity described in the Section 6.1. In the framework of an arbitrary metric theory, following to the KBL ideology (6.1.1), we construct the Lagrangian for pure metric perturbations:

$$\mathscr{L}_{KBL} = -\frac{1}{2\kappa}\left(\mathscr{L}_c{}^G - \bar{\mathscr{L}}_c^G + \partial_\alpha \mathscr{D}^\alpha\right). \tag{7.2.15}$$

By definition, the Lagrangian (7.2.15) has to vanish for vanishing perturbations, therefore one has to set $\mathscr{D}^\alpha$ as disappearing for vanishing perturbations. The same as in the KBL method in general relativity, there is no a necessity to concretize dynamical metric variables from the set (2.2.114).

Now, let us apply the Noether procedure developed in the Section 7.2.1 to the Lagrangian (7.2.15). As a result we obtain the analog of the identity (7.1.27):

$$\boldsymbol{j}_C^\alpha \equiv \partial_\beta \boldsymbol{j}_C^{\alpha\beta} \equiv \bar{\nabla}_\beta \boldsymbol{j}_C^{\alpha\beta} \tag{7.2.16}$$

where

$$\boldsymbol{j}_C^\alpha(\xi) \equiv \boldsymbol{i}^\alpha(\xi) - \bar{\boldsymbol{i}}^\alpha(\xi) + \boldsymbol{i}'^\alpha(\xi), \tag{7.2.17}$$

$$\boldsymbol{j}_C^{\alpha\beta}(\xi) \equiv \boldsymbol{i}^{\alpha\beta}(\xi) - \bar{\boldsymbol{i}}^{\alpha\beta}(\xi) + \boldsymbol{i}'^{\alpha\beta}(\xi). \tag{7.2.18}$$

Here, $\boldsymbol{i}^\alpha$ and $\boldsymbol{i}^{\alpha\beta}$ are defined in (7.1.29) and (7.1.28), respectively, with $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{u}$ defined in (7.2.9), (7.2.10) and (7.2.12), respectively; $\boldsymbol{i}'^\alpha$ and $\boldsymbol{i}'^{\alpha\beta}$ are defined in (7.1.67) and (7.1.68). Thus, for the identically conserved current in (7.2.16) one has

$$\boldsymbol{j}_C^\alpha = \delta\boldsymbol{u}_\sigma{}^\alpha \xi^\sigma + \delta\boldsymbol{m}^{\rho\alpha\beta}\partial_{[\beta}\xi_{\rho]} + \delta\boldsymbol{z}_C^\alpha, \tag{7.2.19}$$

where

$$\delta\boldsymbol{n}_\sigma{}^{\alpha\beta\gamma} \equiv -\left[\boldsymbol{n}_\sigma{}^{\alpha\beta\gamma} - \bar{\boldsymbol{n}}_\sigma{}^{\alpha\beta\gamma}\right], \tag{7.2.20}$$

$$\delta\boldsymbol{m}_\sigma{}^{\alpha\beta} \equiv -\left[\boldsymbol{m}_\sigma{}^{\alpha\beta} - \bar{\boldsymbol{m}}_\sigma{}^{\alpha\beta} - \frac{1}{\kappa_D}\delta_\sigma^{[\alpha}\mathscr{D}^{\beta]}\right], \tag{7.2.21}$$

$$\delta\boldsymbol{u}_\sigma{}^\alpha \equiv -\left[\boldsymbol{u}_\sigma{}^\alpha - \bar{\boldsymbol{u}}_\sigma{}^\alpha + \delta\boldsymbol{n}_\lambda{}^{\alpha\beta\gamma}\bar{R}^\lambda{}_{\beta\gamma\sigma} - \frac{1}{\kappa_D}\bar{\nabla}_\beta(\delta_\sigma^{[\alpha}\mathscr{D}^{\beta]})\right], \tag{7.2.22}$$

$$\delta\boldsymbol{z}_C^\alpha(\xi) \equiv -\left[\delta\boldsymbol{m}^{\sigma\alpha\beta}\zeta_{\sigma\beta} + \delta\boldsymbol{n}^{\rho\alpha\beta\gamma}\left(2\bar{\nabla}_\gamma\zeta_{\beta\rho} - \bar{\nabla}_\rho\zeta_{\beta\gamma}\right)\right]. \tag{7.2.23}$$

It is constructive to represent (7.2.22) in an explicit form with the use of (7.2.12)

$$\delta\boldsymbol{u}_\sigma{}^\alpha \equiv \frac{1}{\kappa}\left[\delta\mathscr{E}_\sigma{}^\alpha + \bar{\nabla}_\beta(\delta_\sigma^{[\alpha}\mathscr{D}^{\beta]})\right] + \delta\mathscr{C}_\sigma{}^\alpha, \tag{7.2.24}$$

where $\delta\mathscr{E}_\sigma{}^\alpha \equiv \mathscr{E}_\sigma{}^\alpha - \bar{\mathscr{E}}_\sigma{}^\alpha$ and $\delta\mathscr{C}_\sigma{}^\alpha \equiv \mathscr{C}_\sigma{}^\alpha - \bar{\mathscr{C}}_\sigma{}^\alpha$. Analogously, we obtain the expression for the superpotential in (7.2.16):

$$\boldsymbol{j}_C^{\alpha\beta} = \left(\frac{2}{3}\bar{\nabla}_\lambda \delta\boldsymbol{n}_\sigma{}^{[\alpha\beta]\lambda} - \delta\boldsymbol{m}_\sigma{}^{[\alpha\beta]}\right)\xi^\sigma - \frac{4}{3}\delta\boldsymbol{n}_\sigma{}^{[\alpha\beta]\lambda}\bar{\nabla}_\lambda\xi^\sigma. \tag{7.2.25}$$

In the case when the gravitational Lagrangian is chosen as the Hilbert Lagrangian, $\mathscr{L}^G = \mathscr{L}^H = \mathscr{R}$, and the divergence is chosen as in (6.1.4), $\mathscr{D}^\alpha = \boldsymbol{k}^\alpha$, the identity (7.2.16) reduces to the KBL identity in general relativity (6.1.17). Thus, the current (7.2.19) generalizes the KBL current (6.1.15), and the superpotential (7.2.25) generalizes the KBL superpotential (6.1.25).

Now, use the dynamical gravitational equations (7.2.3) and the background gravitational equations (7.2.6) in the identity (7.2.16). It turns to a physiacally sensible conservation law,

$$\mathscr{I}_C^\alpha = \partial_\beta \mathscr{I}_C^{\alpha\beta} \equiv \bar{\nabla}_\beta \mathscr{I}_C^{\alpha\beta}, \tag{7.2.26}$$

where the current is

$$\mathscr{I}_C^\alpha = {}_C\boldsymbol{\theta}_\sigma{}^\alpha\xi^\sigma + \delta\boldsymbol{m}^{\rho\alpha\beta}\partial_{[\beta}\xi_{\rho]} + \delta\boldsymbol{z}_C^\alpha \tag{7.2.27}$$

with the generalized energy-momentum

$${}_C\boldsymbol{\theta}_\sigma{}^\alpha \equiv \delta\mathscr{T}_\sigma{}^\alpha + \delta\mathscr{C}_\sigma{}^\alpha + \frac{1}{\kappa}\bar{\nabla}_\beta(\delta_\sigma^{[\alpha}\mathscr{D}^{\beta]}). \tag{7.2.28}$$

It is a generalization of the KBL energy-momentum (6.1.27). Formally the generalized superpotential $\mathscr{I}_C^{\alpha\beta}$ coincides with $\boldsymbol{j}_C^{\alpha\beta}$ in (7.2.25). The difference is that in $\mathscr{I}_C^{\alpha\beta}$ one uses the metric coefficients satisfying the field equations, whereas in $\boldsymbol{j}_C^{\alpha\beta}$ they are arbitrary.

Analogously, the starred conservation law can be constructed,

$$\mathscr{I}_C^{*\alpha} = \partial_\beta \mathscr{I}_C^{*\alpha\beta} \equiv \bar{\nabla}_\beta \mathscr{I}_C^{*\alpha\beta}, \tag{7.2.29}$$

where the current is

$$\mathscr{I}_C^{*\alpha} = {}_C\boldsymbol{\theta}_\sigma^{*\alpha}\xi^\sigma + \delta\boldsymbol{m}^{*\rho\alpha\beta}\partial_{[\beta}\xi_{\rho]} + \delta\boldsymbol{z}_C^{*\alpha} \tag{7.2.30}$$

with the generalized energy-momentum

$${}_C\boldsymbol{\theta}_\sigma^{*\alpha} \equiv \delta\mathscr{T}_\sigma{}^\alpha + \delta\mathscr{C}_\sigma^{*\alpha} + \frac{1}{\kappa}\bar{\nabla}_\beta(\delta_\sigma^{[\alpha}\mathscr{D}^{\beta]}). \tag{7.2.31}$$

The superpotential $\mathscr{I}_C^{*\alpha\beta}$ is directly defined by (7.1.46) and (7.1.68) analogous to $\mathscr{I}_C^{\alpha\beta}$.

More generally, the family of the Noether canonical conservation laws for perturbations corresponding to the representation (7.1.55) has the form:

$$\mathscr{I}_c^{\dagger\alpha}(\xi) = \partial_\alpha \mathscr{I}_c^{\dagger\alpha\beta}(\xi) \tag{7.2.32}$$

with the conserved quantities

$$\mathscr{I}_c^{\dagger\alpha} \equiv p\, \mathscr{I}_c^{\alpha} + q\, \mathscr{I}_c^{*\alpha}, \tag{7.2.33}$$

$$\mathscr{I}_c^{\dagger\alpha\beta} \equiv p\, \mathscr{I}_c^{\alpha\beta} + q\, \mathscr{I}_c^{*\alpha\beta}, \tag{7.2.34}$$

representing a family of conserved quantities if the field equations hold.

Turning to the general expressions (7.1.56–7.1.61) one can rewrite the current (7.2.33) and superpotential (7.2.34) as

$$\mathscr{I}_c^{\dagger\alpha} = \mathscr{I}_c^{\alpha} + q\bar{\nabla}_\beta \delta \boldsymbol{r}^{\alpha\beta}, \tag{7.2.35}$$

$$\mathscr{I}_c^{\dagger\alpha\beta} = \mathscr{I}_c^{\alpha\beta} + q\delta \boldsymbol{r}^{\alpha\beta}; \tag{7.2.36}$$

$$\boldsymbol{r}^{\alpha\beta} \equiv \frac{1}{\kappa}\frac{\partial \mathscr{L}_c{}^G}{\partial(\bar{\nabla}_{[\beta\alpha]} g_{\mu\nu})} £_\xi g_{\mu\nu} \tag{7.2.37}$$

with $\delta\boldsymbol{r}^{\alpha\beta} = \boldsymbol{r}^{\alpha\beta} - \bar{\boldsymbol{r}}^{\alpha\beta}$ and $0 \le q \le 1$, or

$$\mathscr{I}_c^{\dagger\alpha} = \mathscr{I}_c^{*\alpha} + p\bar{\nabla}_\beta \delta \boldsymbol{r}^{*\alpha\beta}, \tag{7.2.38}$$

$$\mathscr{I}_c^{\dagger\alpha\beta} = \mathscr{I}_c^{*\alpha\beta} + p\delta \boldsymbol{r}^{*\alpha\beta}; \tag{7.2.39}$$

$$\boldsymbol{r}^{*\alpha\beta} \equiv \frac{1}{\kappa}\frac{\partial \mathscr{L}_c{}^{*G}}{\partial(\bar{\nabla}_{[\beta\alpha]} g_{\mu\nu})} £_\xi g_{\mu\nu} \tag{7.2.40}$$

with $\delta\boldsymbol{r}^{*\alpha\beta} = \boldsymbol{r}^{*\alpha\beta} - \bar{\boldsymbol{r}}^{*\alpha\beta}$ and $0 \le p \le 1$. It is important for calculations to note that following (7.1.33),

$$\boldsymbol{r}^{*\alpha\beta} \equiv -\frac{1}{\kappa}\frac{\partial \mathscr{L}_c{}^G}{\partial(\bar{\nabla}_{[\beta\alpha]} g_{\mu\nu})} £_\xi g_{\mu\nu}, \tag{7.2.41}$$

thus, $\boldsymbol{r}^{*\alpha\beta} = -\boldsymbol{r}^{\alpha\beta}$.

Note, that even in n-dimensional general relativity there is no a difference between the starred and un-starred quantities. It is because the differences (7.2.37) and (7.2.40) vanish for covariant perturbed Hilbert Lagrangian $\mathscr{L}_c{}^G = \mathscr{L}_c{}^H$ in (6.1.3) adapted to n dimensions. Indeed, the quantity

$$\frac{\partial \mathscr{L}_c{}^H}{\partial(\bar{\nabla}_{\beta\alpha} g_{\mu\nu})} = 2\left(g^{\alpha(\mu}g^{\beta\nu)} - g^{\alpha\beta}g^{\mu\nu}\right) \tag{7.2.42}$$

after antisymmetrization in α and β, disappears. Therefore for general relativity, even in n dimensions, the conservation law (7.2.32) is a single one, not a family. Thus, for general relativity in n dimensions it is enough to consider the KBL conservation laws only.

7.2.3 The Belinfante corrected conserved quantities

Now, let us apply the Noether-Belinfante procedure developed in the Section 7.1.4 to the Lagrangian (7.2.15). As a result we obtain the analog of the identity (7.1.70):

$$\boldsymbol{j}^{\alpha}_{B} \equiv \partial_{\beta}\boldsymbol{j}^{\alpha\beta}_{B} \equiv \bar{\nabla}_{\beta}\boldsymbol{j}^{\alpha\beta}_{B} \tag{7.2.43}$$

where

$$\boldsymbol{j}^{\alpha}_{B}(\xi) \equiv \boldsymbol{i}^{\alpha}_{B}(\xi) - \bar{\boldsymbol{i}}^{\alpha}_{B}(\xi), \tag{7.2.44}$$

$$\boldsymbol{j}^{\alpha\beta}_{B}(\xi) \equiv \boldsymbol{i}^{\alpha\beta}_{B}(\xi) - \bar{\boldsymbol{i}}^{\alpha\beta}_{B}(\xi). \tag{7.2.45}$$

Let us stress that here the contribution from a divergence in Lagrangian is absent, unlike (7.2.17) and (7.2.18). Here, $\boldsymbol{i}^{\alpha}_{B}$ and $\boldsymbol{i}^{\alpha\beta}_{B}$ are defined in (7.1.71) and (7.1.73), respectively, with $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{u}$ defined in (7.2.9), (7.2.10) and (7.2.12), respectively. Thus, for the identically conserved current in (7.2.43) one has

$$\boldsymbol{j}^{\alpha}_{B} = \delta({}_{B}\boldsymbol{u}_{\sigma}{}^{\alpha})\xi^{\sigma} + \delta\boldsymbol{z}^{\alpha}_{B}(\xi), \tag{7.2.46}$$

where

$$\delta({}_{B}\boldsymbol{u}_{\sigma}{}^{\alpha}) \equiv -\left[\delta\boldsymbol{u}_{\sigma}{}^{\alpha} + \delta\boldsymbol{n}_{\lambda}{}^{\alpha\beta\gamma}\bar{R}^{\lambda}{}_{\beta\gamma\sigma} - \bar{\nabla}_{\beta}\delta\boldsymbol{s}^{\alpha\beta}{}_{\sigma}\right], \tag{7.2.47}$$

$$\begin{aligned}\delta\boldsymbol{z}^{\alpha}_{B}(\xi) = {} & \left(\delta\boldsymbol{m}_{\lambda}{}^{\beta\alpha}\bar{g}^{\tau\lambda} + \delta\boldsymbol{m}_{\lambda}{}^{\alpha\tau}\bar{g}^{\beta\lambda} - \delta\boldsymbol{m}_{\lambda}{}^{\tau\beta}\bar{g}^{\alpha\lambda}\right)\zeta_{\tau\beta} \\ & + \delta\boldsymbol{n}_{\lambda}{}^{\alpha\tau\beta}\left(2\bar{\nabla}_{(\beta}\zeta^{\lambda}_{\tau)} - \bar{\nabla}^{\lambda}\zeta_{\beta\tau}\right),\end{aligned} \tag{7.2.48}$$

with the perturbed coefficients here,

$$\delta\boldsymbol{n}_{\sigma}{}^{\alpha\beta\gamma} \equiv \boldsymbol{n}_{\sigma}{}^{\alpha\beta\gamma} - \bar{\boldsymbol{n}}_{\sigma}{}^{\alpha\beta\gamma}, \tag{7.2.49}$$

$$\delta\boldsymbol{m}_{\sigma}{}^{\alpha\beta} \equiv \boldsymbol{m}_{\sigma}{}^{\alpha\beta} - \bar{\boldsymbol{m}}_{\sigma}{}^{\alpha\beta}, \tag{7.2.50}$$

$$\delta\boldsymbol{s}^{\alpha\beta}{}_{\sigma} \equiv \boldsymbol{s}^{\alpha\beta}{}_{\sigma} - \bar{\boldsymbol{s}}^{\alpha\beta}{}_{\sigma}, \tag{7.2.51}$$

$$\delta\boldsymbol{u}_{\sigma}{}^{\alpha} \equiv \boldsymbol{u}_{\sigma}{}^{\alpha} - \bar{\boldsymbol{u}}_{\sigma}{}^{\alpha}. \tag{7.2.52}$$

It is constructive to represent (7.2.47) in an explicit form with the use of (7.2.12 – 7.2.14) and (7.2.52):

$$\delta({}_{B}\boldsymbol{u}_{\sigma}{}^{\alpha}) \equiv \frac{1}{\kappa_D}\delta\mathscr{E}_{\sigma}{}^{\alpha} + \delta\mathscr{C}_{\sigma}{}^{\alpha} + \bar{\nabla}_{\beta}\delta\boldsymbol{s}^{\alpha\beta}{}_{\sigma}, \tag{7.2.53}$$

where again $\delta\mathscr{E}_{\sigma}{}^{\alpha} \equiv \mathscr{E}_{\sigma}{}^{\alpha} - \bar{\mathscr{E}}_{\sigma}{}^{\alpha}$ and $\delta\mathscr{C}_{\sigma}{}^{\alpha} \equiv \mathscr{C}_{\sigma}{}^{\alpha} - \bar{\mathscr{C}}_{\sigma}{}^{\alpha}$.

Analogously, keeping in mind (7.1.73), we obtain the expression for the superpotential in (7.2.43):

$$\boldsymbol{j}_B^{\alpha\beta} \equiv 2\left(\frac{1}{3}\bar{\nabla}_\rho \delta \boldsymbol{n}_\sigma{}^{[\alpha\beta]\rho} + \bar{\nabla}_\tau \delta \boldsymbol{n}_\lambda{}^{\tau\rho[\alpha}\bar{g}^{\beta]\lambda}\bar{g}_{\rho\sigma}\right)\xi^\sigma - \frac{4}{3}\delta \boldsymbol{n}_\sigma{}^{[\alpha\beta]\lambda}\overline{D}_\lambda \xi^\sigma. \tag{7.2.54}$$

In the case, when the gravitational Lagrangian is chosen as the Hilbert Lagrangian, $\mathscr{L}^G = \mathscr{L}^H = \mathscr{R}$, the identity (7.2.43) reduces to the identity (6.2.4) in general relativity.

Now, let us use the dynamical gravitational equation (7.2.3) and the background gravitational equations (7.2.6) in the identity (7.2.43). It turns to a physiacally sensible conservation law,

$$\mathscr{I}_B^\alpha = \partial_\beta \mathscr{I}_B^{\alpha\beta} \equiv \bar{\nabla}_\beta \mathscr{I}_B^{\alpha\beta}, \tag{7.2.55}$$

where the current is

$$\mathscr{I}_B^\alpha = {}_B\boldsymbol{\theta}_\sigma{}^\alpha \xi^\sigma + \delta \boldsymbol{z}_B^\alpha \tag{7.2.56}$$

with the generalized energy-momentum

$${}_B\boldsymbol{\theta}_\sigma{}^\alpha \equiv \delta \mathscr{T}_\sigma{}^\alpha + \delta \mathscr{C}_\sigma{}^\alpha + \bar{\nabla}_\beta \delta \boldsymbol{s}^{\alpha\beta}{}_\sigma. \tag{7.2.57}$$

Formally the generalized superpotential $\mathscr{I}_B^{\alpha\beta}$ coincides with $\boldsymbol{j}_B^{\alpha\beta}$ in (7.2.43). The difference is that in $\mathscr{I}_B^{\alpha\beta}$ one uses the metric coefficients satisfying the field equations, whereas in $\boldsymbol{j}_B^{\alpha\beta}$ they are arbitrary. In the case of general relativity, the generalized energy-momentum (7.2.57) coincides with (6.2.9).

The starred conservation law can be constructed,

$$\mathscr{I}_B^{*\alpha} = \partial_\beta \mathscr{I}_B^{*\alpha\beta} \equiv \bar{\nabla}_\beta \mathscr{I}_B^{*\alpha\beta}. \tag{7.2.58}$$

The family of the Belinfante corrected conservation laws for perturbations has the form

$$\mathscr{I}_B^{\dagger\alpha} = \partial_\beta \mathscr{I}_B^{\dagger\alpha\beta} \equiv \bar{\nabla}_\beta \mathscr{I}_B^{\dagger\alpha\beta}. \tag{7.2.59}$$

The structure of (7.2.58) and (7.2.59) can be easily reproduced with the use of (7.1.74 – 7.1.84).

7.2.4 The field-theoretical formulation for perturbations

Here, the field equations in the framework of the field-theoretical approach (see Section 2.2 for general relativity in four dimensions) are derived for n-dimensional metric theory. For this we derive the Lagrangian (7.2.1) in a more general form:

$$\mathscr{L}_n(Q^A) = -\frac{1}{2\kappa}\mathscr{L}^G(g^a) + \mathscr{L}^M(g^a, \Phi^B), \tag{7.2.60}$$

where $Q^A = \{g^a, \Phi^A\}$ with a generalized metric variable g^a from the set (2.2.114). The field equations for the system (7.2.60) are derived in a compact form as

$$\delta \mathscr{L}_n / \delta Q^A = 0. \tag{7.2.61}$$

Let us decompose Q^A onto the background part $\bar{Q}^A$ and the dynamical part q^A, perturbations,

$$Q^A = \bar{Q}^A + q^A. \tag{7.2.62}$$

Here $q^A = \{h^a, \phi^A\}$ with a generalized metric perturbation variable h^a from the set (2.2.121). The background fields are fixed in the sense that they satisfy the background equations

$$\delta \bar{\mathscr{L}}_n / \delta \bar{Q}^A = 0\,, \tag{7.2.63}$$

where $\bar{\mathscr{L}}_n = \mathscr{L}_n(\bar{Q})$.

Following the recipe (2.2.15), we construct the dynamical Lagrangian for perturbations,

$$\begin{aligned} \mathscr{L}_n^{\text{dyn}}(\bar{Q}, q) &= \mathscr{L}_n(\bar{Q} + q) - q^A \frac{\delta \bar{\mathscr{L}}_n}{\delta \bar{Q}^A} - \bar{\mathscr{L}}_n \\ &= -\frac{1}{2\kappa} \mathscr{L}^g + \mathscr{L}^m \end{aligned} \tag{7.2.64}$$

with the Lagrangian $\mathscr{L}^g$ for free gravitational field and with the Lagrangian $\mathscr{L}^m$ for the matter interacting with the gravitational field. As in (2.2.15), the background equations should not be taken into account before the variation of $\mathscr{L}_n^{\text{dyn}}$ with respect to $\bar{Q}^A$. To obtain the field equations for the dynamical variables (perturbations) q^A one has to vary (7.2.64) with respect to q^A,

$$\frac{\delta \mathscr{L}_n^{\text{dyn}}}{\delta q^A} = 0. \tag{7.2.65}$$

Using the evident property

$$\delta \mathscr{L}_n(\bar{Q} + q) / \delta \bar{Q}^A = \delta \mathscr{L}_n(\bar{Q} + q) / \delta q^A, \tag{7.2.66}$$

the field equation (7.2.65) can be represented in the form:

$$\frac{\delta \mathscr{L}_n^{\text{dyn}}}{\delta q^A} = \frac{\delta}{\delta \bar{Q}^A} \left[\mathscr{L}_n(\bar{Q} + q) - \bar{\mathscr{L}}_n \right] = 0. \tag{7.2.67}$$

This form shows that the equations for perturbations are equivalent to the equations of the theory (7.2.61) if the background equations (7.2.63) hold. Next, defining the “background current”,

$$t_A^q \equiv \frac{\delta \mathscr{L}_n^{\text{dyn}}}{\delta \bar{Q}^A} \equiv \frac{\delta \mathscr{L}_n^{\text{dyn}}}{\delta q^A} - \frac{\delta}{\delta \bar{Q}^A} q^B \frac{\delta \bar{\mathscr{L}}_n}{\delta \bar{Q}^B}, \tag{7.2.68}$$

using the definition (7.2.64) and the property (7.2.66), one obtains another form for the equations (7.2.65):

$$-\frac{\delta}{\delta \bar{Q}^A} q^B \frac{\delta \bar{\mathscr{L}}_n}{\delta \bar{Q}^B} = t_A^q. \tag{7.2.69}$$

The left hand side here is linear in perturbations, whereas the right hand side is at least quadratic.

Now, let us separate the equations (7.2.69) into the gravitational and the matter parts as follows,

$$\mathscr{E}_a^L + \Phi_a^L = 2\kappa t_a^{\text{tot}}, \tag{7.2.70}$$

$$\Phi_A^L = t_A^m, \tag{7.2.71}$$

where

$$\mathscr{E}_a^L \equiv \frac{\delta}{\delta \bar{g}^a} g^b \frac{\delta \bar{\mathscr{L}}^G}{\delta \bar{g}^b}, \tag{7.2.72}$$

$$\Phi_a^L \equiv -2\kappa \frac{\delta}{\delta \bar{g}^a} q^B \frac{\delta \bar{\mathscr{L}}_n}{\delta \bar{Q}^B}, \tag{7.2.73}$$

$$\Phi_A^L \equiv -\frac{\delta}{\delta \bar{\Phi}^A} q^B \frac{\delta \bar{\mathscr{L}}^M}{\delta \bar{Q}^B}. \tag{7.2.74}$$

The matter equations (7.2.71) have the same form as the ones defined in general relativity (2.2.37).

To obtain the gravitational equations in the usual form, one has to contract (7.2.70) with $\partial \bar{g}^a / \partial \bar{g}^{\mu\nu}$,

$$\mathscr{E}_{\mu\nu}^L + \Phi_{\mu\nu}^L = \kappa \boldsymbol{t}_{\mu\nu}^{\text{tot}}. \tag{7.2.75}$$

Then dividing it by $\sqrt{-\bar{g}}$, one obtains

$$E_{\mu\nu}^L + \Phi_{\mu\nu}^L = \kappa t_{\mu\nu}^{\text{tot}}. \tag{7.2.76}$$

It is the generalization of the field-theoretical gravitational equations in general relativity (2.2.26). The linear operators are defined as

$$E^L_{\mu\nu} \equiv \frac{\mathscr{E}^L_{\mu\nu}}{\sqrt{-\bar{g}}} \equiv \frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\mathfrak{h}^{\alpha\beta}_a\frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}, \tag{7.2.77}$$

$$\Phi^L_{\mu\nu} \equiv \frac{\Phi^L_{\mu\nu}}{\sqrt{-\bar{g}}} \equiv -2\kappa\frac{1}{\sqrt{-\bar{g}}}\frac{\delta}{\delta\bar{g}^{\mu\nu}}\left(\mathfrak{h}^{\alpha\beta}_a\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\mathfrak{g}}^{\alpha\beta}} + \phi^B\frac{\delta\bar{\mathscr{L}}^M}{\delta\bar{\Phi}^B}\right) \tag{7.2.78}$$

with the independent gravitational variables $\mathfrak{h}^{\alpha\beta}_a$ defined as in (2.2.129).

The right hand side in (7.2.76) is defined as

$$t^{\ \text{tot}}_{\mu\nu} \equiv \frac{\boldsymbol{t}^{\ \text{tot}}_{\mu\nu}}{\sqrt{-\bar{g}}} \equiv \frac{2}{\sqrt{-\bar{g}}}\frac{\delta\mathscr{L}_n^{\ \text{dyn}}}{\delta\bar{g}^{\mu\nu}}. \tag{7.2.79}$$

It is the total symmetric energy-momentum tensor for perturbations. Following the structure of (7.2.64), it can be rewritten as a sum

$$\boldsymbol{t}^{\ \text{tot}}_{\mu\nu} \equiv \boldsymbol{t}^g_{\mu\nu} + \boldsymbol{t}^m_{\mu\nu} \equiv -\frac{1}{\kappa}\frac{\delta\mathscr{L}^g}{\delta\bar{g}^{\mu\nu}} + 2\frac{\delta\mathscr{L}^m}{\delta\bar{g}^{\mu\nu}}. \tag{7.2.80}$$

Thus, following the formulae (7.2.64) - (7.2.68),

$$\boldsymbol{t}^g_{\mu\nu} = -\frac{1}{\kappa}\left(\frac{\delta\mathscr{L}^G(\bar{g}^a + h^a)}{\delta\bar{g}^{\mu\nu}} - \frac{\delta}{\delta\bar{g}^{\mu\nu}}h^a\frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{g}^a} - \frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{g}^{\mu\nu}}\right). \tag{7.2.81}$$

An explicit expression for, $\boldsymbol{t}^g_{\mu\nu}$ in (7.2.81) can be presented, if the expression $\mathscr{L}^G$ is known. Thus with the use of (7.2.4), barred (7.2.4) and (7.2.72) $\boldsymbol{t}^g_{\mu\nu}$ in (7.2.81) can be rewritten as

$$\boldsymbol{t}^g_{\mu\nu} = -\frac{1}{\kappa}\left(\frac{\partial\bar{g}^a}{\partial\bar{g}^{\mu\nu}}\frac{\partial g^{\rho\sigma}}{\partial g^a}\mathscr{E}_{\rho\sigma}(\bar{g}^a + h^a) - \mathscr{E}^L_{\mu\nu}(\bar{g}^a,\ h^a) - \bar{\mathscr{E}}_{\mu\nu}\right). \tag{7.2.82}$$

Analogously, one derives the matter energy momentum,

$$\boldsymbol{t}^m_{\mu\nu} = \frac{\partial\bar{g}^a}{\partial\bar{g}^{\mu\nu}}\frac{\partial g^{\rho\sigma}}{\partial g^a}\mathscr{T}_{\rho\sigma}(\bar{g}^a + h^a) + \frac{1}{\kappa}\Phi^L_{\mu\nu}(\bar{g}^a,\ h^a) - \bar{\mathscr{T}}_{\mu\nu} \equiv \Delta\mathscr{T}_{\mu\nu} + \frac{1}{\kappa}\Phi^L_{\mu\nu}. \tag{7.2.83}$$

Formulae (7.2.82) and (7.2.83) show how to obtain explicit expressions for $\boldsymbol{t}^g_{\mu\nu}$ and $\boldsymbol{t}^m_{\mu\nu}$. However, summing them, as in (7.2.80), and assuming that the field equations hold, one finds that $\boldsymbol{t}^{\ \text{tot}}_{\mu\nu}$ can be simply changed by the left hand side of (7.2.75).

At last, let us consider the case of vacuum background, $\bar{\mathscr{L}}^M = 0$. Then the gravitational equations (7.2.76) acquire the form:

$$E^L_{\mu\nu} = \kappa t^{\ \text{tot}}_{\mu\nu}. \tag{7.2.84}$$

The equations (7.2.76) and (7.2.84) generalize the equations of general relativity (2.2.26) and (2.2.105), respectively; they generalize also the Deser–Tekin equations [126] constructed for the quadratic theories by direct calculations.

7.2.5 Currents and superpotentials in the field-theoretical formulation

In a series of works [122, 124–126] Deser and Tekin develop a construction of conserved charges for perturbations about vacua in metric quadratic (in curvature) gravity theories in n dimensions. They apply the Abbott and Deser procedure [1] and extended the procedure. The aim of the present subsection is to suggest an approach, which generalizes the Deser-Tekin constructions. We construct conserved currents and superpotentials corresponding to the equation (7.2.76) derived for an arbitrary metric gravitational theory, and not only in a vacuum background.

Taking into account the definition of the gravitational part of a linear operator (7.2.77), we demonstrate that conserved quantities of the system and their properties can be obtained and described analyzing *only* the scalar density

$$\mathscr{L}_1 \equiv -\frac{1}{2\kappa}\mathfrak{h}_a^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}. \tag{7.2.85}$$

It is the gravitational part of the second term in the Lagrangian (7.2.64) and generalizes the linear Lagrangian (6.3.36) in general relativity. Thus, here, we generalize and develop the field-theoretical method of constructing conserved quantities in general relativity, see Section 6.3.2. As an important case, we consider only such theories where the Lagrangian derivative $\delta\bar{\mathscr{L}}^G/\delta\bar{\mathfrak{g}}^{\alpha\beta}$ (representing the background field equations) has derivatives not higher than of second order, like the theories of the Lovelock type [299]. Although, our results can be easily generalized for theories, where the field equations have derivatives of higher orders than two, like in metric quadratic gravity theories [126].

Keeping in mind that $\mathscr{L}_1$ is the scalar density, we follow the standard technique of the Noether identities, which are universal. Dynamical variables are in (7.2.85) and we symbolically unify them into a general one $\psi^A = \{\mathfrak{h}_a^{\alpha\beta}, \bar{\mathfrak{g}}_{\mu\nu}\}$. We rewrite the initial identity (1.2.46) for the diffeomorphisms in the form:

$$£_\xi\mathscr{L}_1 + \partial_\alpha(\xi^\alpha\mathscr{L}_1) \equiv 0. \tag{7.2.86}$$

Already, we know that this identity has to lead to an identically conserved current:

$$\partial_\mu\boldsymbol{i}_1^\mu \equiv \bar{\nabla}_\mu\boldsymbol{i}_1^\mu \equiv 0. \tag{7.2.87}$$

However, we do not derive this current in the final form (1.4.14), or (7.1.26). Because $\mathscr{L}_1$ depends only on the $\mathfrak{h}_a^{\alpha\beta}$ without derivatives and on $\bar{\mathfrak{g}}_{\mu\nu}$ and their derivatives one recognizes that the current in (7.2.87) essentially depends on the $\boldsymbol{z}$-term determined by $\zeta_{\rho\sigma} = -\frac{1}{2}\pounds_\xi \bar{g}_{\rho\sigma}$ defined in (7.1.31). Therefore, we present the intermediate identity (7.2.86) in the form (1.4.3), from where we obtain:

$$\begin{aligned} \boldsymbol{i}_1^\mu &\equiv -\left(\mathscr{L}_1\xi^\mu + \xi^\nu\, \psi^A\Big|_\nu^\mu \frac{\delta\mathscr{L}_1}{\delta\psi^A}\right) + \boldsymbol{z}_S^\mu(\zeta_{\rho\sigma}) \\ &\equiv -\left(\mathscr{L}_1\xi^\mu - \frac{\xi^\sigma}{\kappa}\mathfrak{h}_a^{\rho\mu}\frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} + 2\xi^\sigma\bar{\mathfrak{g}}^{\rho\mu}\frac{\delta\mathscr{L}_1}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}\right) + \boldsymbol{z}_S^\mu, \end{aligned} \tag{7.2.88}$$

where

$$\boldsymbol{z}_S^\mu \equiv 2\zeta_{\rho\sigma}\bar{\nabla}_\nu\frac{\partial\mathscr{L}_1}{\partial\bar{g}_{\rho\sigma,\mu\nu}} - 2\frac{\partial\mathscr{L}_1}{\partial\bar{g}_{\rho\sigma,\mu\nu}}\bar{\nabla}_\nu\zeta_{\rho\sigma} \tag{7.2.89}$$

generalizes the $\boldsymbol{z}$-term (6.3.41) in general relativity. For theories, where the field equations have derivatives of higher orders than two, the structure of the current will be the same as in (7.2.88), only $\boldsymbol{z}$-term will be more complicated than in (7.2.89).

The case of a vacuum background is quite important. In this case, the background equations (7.2.6), or (7.2.63), acquire the form:

$$\frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{\mathfrak{g}}^{\rho\sigma}} = 0. \tag{7.2.90}$$

Then, in (7.2.88) one has to set $\mathscr{L}_1 = 0$. After that, assuming the existence of an arbitrary Killing vector $\bar{\xi}^\alpha$, for which $\bar{\nabla}_{(\beta}\bar{\xi}_{\alpha)} = 0$, one transforms the identity (7.2.88) into

$$-2\kappa\bar{\nabla}_\mu\left(\bar{\mathfrak{g}}^{\rho\mu}\frac{\delta\mathscr{L}_1}{\delta\bar{\mathfrak{g}}^{\rho\sigma}}\right) \equiv \bar{\nabla}^\rho\left(\frac{\delta}{\delta\bar{g}^{\rho\sigma}}\mathfrak{h}_a^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\right) \equiv \bar{\nabla}^\rho\left(\mathscr{E}^L_{\rho\sigma}\right) \equiv 0, \tag{7.2.91}$$

or $\bar{\nabla}^\rho\left(E^L_{\rho\sigma}\right) \equiv 0$, see the definition (7.2.77). Then, for the field equations on the vacuum background (7.2.84) one has the conservation law for the total energy-momentum:

$$\bar{\nabla}^\mu \boldsymbol{t}^{\text{tot}}_{\mu\nu} = 0 \tag{7.2.92}$$

that generalizes the conservation of energy-momentum (2.2.106) in general relativity on vacuum backgrounds.

Because equation (7.2.87) is the identity, the current $\boldsymbol{i}_1^\mu$ can be represented in the form of a divergence from a superpotential. To construct the latter, we use the results of Section 7.1.1 adopted for the Lagrangian $\mathscr{L}_1$. From the beginning we reset $\bar{g}_{\mu\nu} \to g_{\mu\nu}$ in $\mathscr{L}_1$ and construct the coefficients (7.1.16) and (7.1.19) with

$$\mathscr{L}_1 = \mathscr{L}_1(\psi^A, \psi^A{}_{,\alpha}, \psi^A{}_{,\alpha\beta}) \equiv \mathscr{L}_{1c}(\psi^A, \bar{\nabla}_\alpha\psi^A, \bar{\nabla}_{\beta\alpha}\psi^A), \tag{7.2.93}$$

where $\psi^A = \{\mathfrak{h}^{\mu\nu}_a, g_{\mu\nu}\}$. Then we go back, $g_{\mu\nu} \to \bar{g}_{\mu\nu}$, and obtain the simple expressions,

$$\boldsymbol{n}_{1\sigma}{}^{\lambda\mu\nu} = -2\frac{\partial \mathscr{L}_1}{\partial \bar{g}_{\rho(\mu,\nu)\lambda}}\bar{g}_{\rho\sigma}, \qquad \boldsymbol{m}_{1\sigma}{}^{\mu\nu} = 2\bar{\nabla}_\lambda\left(\frac{\partial \mathscr{L}_1}{\partial \bar{g}_{\rho\nu,\mu\lambda}}\right)\bar{g}_{\rho\sigma}. \tag{7.2.94}$$

Then, substituting these into the expression (7.1.28) for the superpotential, we can define it as the superpotential $\boldsymbol{i}_1^{\mu\nu}$, which is evidently linear in $\mathfrak{h}^{\mu\nu}_a$. Thus, we have the identity

$$\boldsymbol{i}_1^{\mu} \equiv \partial_\nu \boldsymbol{i}_1^{\mu\nu} \equiv \bar{\nabla}_\nu \boldsymbol{i}_1^{\mu\nu}, \tag{7.2.95}$$

which generalizes the identity (6.3.38) in general relativity. The superpotential has the form:

$$\boldsymbol{i}_1^{\alpha\beta} \equiv \left(\frac{2}{3}\bar{\nabla}_\lambda \boldsymbol{n}_{1\sigma}{}^{[\alpha\beta]\lambda} - \boldsymbol{m}_{1\sigma}{}^{[\alpha\beta]}\right)\xi^\sigma - \frac{4}{3}\boldsymbol{n}_{1\sigma}{}^{[\alpha\beta]\lambda}\bar{\nabla}_\lambda \xi^\sigma. \tag{7.2.96}$$

In the case of the Einstein's gravity, it reduces to the superpotential (6.3.42).

The simple form of the coefficients (7.2.94) permits us to use the convenient quantity

$$\boldsymbol{w}^{\rho\lambda|\mu\nu} = \frac{\partial \mathscr{L}_1}{\partial \bar{g}_{\rho\lambda,\mu\nu}}. \tag{7.2.97}$$

Evidently that it is symmetric in ρ and λ, and in μ and ν. Because $\boldsymbol{n}_1$ satisfies the identity of the type (7.1.25) the other not-so evident symmetry exists $\boldsymbol{w}^{\rho\lambda|\mu\nu} = \boldsymbol{w}^{\mu\nu|\rho\lambda}$. In the terms of (7.2.97) the coefficients (7.2.94) are rewritten as

$$\boldsymbol{n}_{1\sigma}{}^{\lambda\mu\nu} = -2\boldsymbol{w}_\sigma{}^{(\mu|\nu)\lambda}, \qquad \boldsymbol{m}_{1\sigma}{}^{\mu\nu} = 2\bar{\nabla}_\lambda \boldsymbol{w}_\sigma{}^{\mu|\nu\lambda}. \tag{7.2.98}$$

It is evident that these coefficients satisfy the identity (7.1.24) as well. At last, the superpotential (7.2.96) acquires the form:

$$\begin{aligned}\boldsymbol{i}_1^{\alpha\beta} &\equiv \frac{4}{3}\left(2\xi^\sigma\bar{\nabla}_\lambda \boldsymbol{w}_\sigma{}^{[\alpha|\beta]\lambda} - \boldsymbol{w}_\sigma{}^{[\alpha|\beta]\lambda}\bar{\nabla}_\lambda\xi^\sigma\right)\\ &\equiv \frac{8}{3}\bar{\nabla}_\lambda\left(\boldsymbol{w}_\sigma{}^{[\alpha|\beta]\lambda}\xi^\sigma\right) - 4\boldsymbol{w}_\sigma{}^{[\alpha|\beta]\lambda}\bar{\nabla}_\lambda\xi^\sigma .\end{aligned} \tag{7.2.99}$$

The identity (7.2.95) transforms into the physically sensible conservation law after using the gravitational equations (7.2.75)

$$\mathscr{I}_S^{\mu} = \partial_\nu \mathscr{I}_S^{\mu\nu} \equiv \bar{\nabla}_\nu \mathscr{I}_S^{\mu\nu}. \tag{7.2.100}$$

The superpotential in (7.2.100), $\mathscr{I}_S^{\mu\nu}$, formally coincides with the superpotential in (7.2.96), $\boldsymbol{i}_1^{\mu\nu}$. However, in $\boldsymbol{i}_1^{\mu\nu}$ field variables are arbitrary, whereas in $\mathscr{I}_S^{\mu\nu}$ the functions $\mathfrak{h}^{\rho\sigma}_a$ are thought as solutions to the equations (7.2.76), or (7.2.75).

The current, $\mathscr{J}_S^\mu$, in (7.2.100) is obtained from the current, $\boldsymbol{i}_1^\mu$, given in (7.2.88) after using the field-theoretical equations (7.2.76), or (7.2.75),

$$\mathscr{J}_S^\mu = {}_S\boldsymbol{\theta}_\nu{}^\mu \xi^\nu + \boldsymbol{z}_S^\mu\,, \tag{7.2.101}$$

where the energy-momentum tensor density is

$${}_S\boldsymbol{\theta}_{\mu\nu} \equiv \boldsymbol{t}^{\text{tot}}_{\mu\nu} - \frac{1}{\kappa}\Phi^L_{\mu\nu} - \bar{g}_{\mu\nu}\mathscr{L}_1 + \frac{1}{\kappa}\bar{g}_{\mu\rho}\mathfrak{h}^{\rho\sigma}_a \frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{g}^{\sigma\nu}}. \tag{7.2.102}$$

It is the generalization of the energy-momentum (6.3.26) in general relativity. Now let us turn to the total energy-momentum in the form of the sum (7.2.80) and its parts (7.2.82) and (7.2.83). Thus the energy-momentum (7.2.102) acquires the form:

$${}_S\boldsymbol{\theta}_{\mu\nu} \equiv \boldsymbol{t}^g_{\mu\nu} + \Delta\mathscr{T}_{\mu\nu} + \frac{1}{2\kappa}\bar{g}_{\mu\nu}\mathfrak{h}^{\rho\sigma}_a \frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{g}^{\rho\sigma}} + \frac{1}{\kappa}\bar{g}_{\mu\rho}\mathfrak{h}^{\rho\sigma}_a \frac{\delta\bar{\mathscr{L}}^G}{\delta\bar{g}^{\sigma\nu}}. \tag{7.2.103}$$

Here, $\boldsymbol{t}^g_{\mu\nu}$ is defined in (7.2.82), and $\Delta\mathscr{T}_{\mu\nu}$ is defined in (7.2.83). It is the generalization of the energy-momentum (6.3.27) in general relativity. In the case of vacuum background $\bar{\mathscr{E}}_{\mu\nu} = \bar{\mathscr{T}}_{\mu\nu} = 0$ the energy-momentum (7.2.103) is simplified and acquires the form:

$${}_S\boldsymbol{\theta}_{\mu\nu} \equiv \boldsymbol{t}^g_{\mu\nu} + \Delta\mathscr{T}_{\mu\nu}. \tag{7.2.104}$$

It is important to stress the following. In the framework of general relativity, the field-theoretical superpotential (6.3.23) coincides with the Belinfante corrected superpotential (6.2.16). In the case of an arbitrary metric theory this is not so. Indeed, the field-theoretical superpotential (7.2.96) is linear in metric perturbations by definition (7.2.94). At the same time, the Belinfante corrected superpotential (7.2.54) (with the others in the family (7.2.59)) is not linear in general. In the framework of general relativity, the Abbott–Deser ambiguity has been resolved by applying the correspondence between the field-theoretical and Belinfante approaches, see Sections 6.3 and 6.4. As we see now, it is impossible to use analogous arguments in the case of an arbitrary metric theory. However, using the correspondence principle between general relativity and other metric theories, one could choose again, $\mathfrak{h}^{\rho\sigma}_a = \mathfrak{h}^{\rho\sigma}$. On the other case, of course, one needs more serious arguments, like tests for various acceptable solutions.

8 Conserved quantities in the Einstein-Gauss-Bonnet gravity

8.1 Superpotentials and currents in the EGB gravity

8.1.1 Action and field equations in the EGB gravity

In the present chapter we develop and apply the results of the previous chapter to Einstein-Gauss-Bonnet (EGB) gravity. This modification of general relativity is the one of the most well-known of all modified gravitational theories. From the one side, it is the second order of the Lovelock type theories, Lagrangians of which are polynomials of the curvature tensors leaving the field equations second order [299]. From the other side, the EGB theory is the low energy limit of some string theories.

The action of the n-dimensional EGB theory with a bare cosmological term Λ_0 is

$$\begin{aligned} S &= -\frac{1}{2\kappa}\int d^n x \mathscr{L}_{EGB} + \int d^n x \mathscr{L}^M \\ &= -\frac{1}{2\kappa}\int d^n x \sqrt{-g}\,[R - 2\Lambda_0 + \alpha L_{GB}] + \int d^n x \mathscr{L}^M , \end{aligned} \tag{8.1.1}$$

where $\alpha > 0$, we restrict ourselves to $\Lambda_0 \le 0$, and the Gauss-Bonnet combination is

$$L_{GB} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 , \tag{8.1.2}$$

see, for example, [126]. Thus $\mathscr{L}_{EGB}$ plays the role of $\mathscr{L}^G$ in (7.2.1), or in (7.2.60). Below, the subscripts "$_E$" is related to the pure Einstein part of the action (8.1.1), and the subscript "$_{GB}$" is related to the Gauss-Bonnet part connected with the α-correction.

The equations of motion that follow from (8.1.1) are

$$\mathscr{E}_{\mu\nu} = \kappa \mathscr{T}_{\mu\nu} , \tag{8.1.3}$$

where

$$\mathscr{E}_{\mu\nu} \equiv \frac{\delta}{\delta g^{\mu\nu}}\mathscr{L}_{EGB} = \sqrt{-g}\left[\left(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda_0\right) + \alpha H_{\mu\nu}\right] ; \tag{8.1.4}$$

$$\begin{aligned} H_{\mu\nu} \equiv 2\Big[&RR_{\mu\nu} - 2R_{\mu\sigma\nu\rho}R^{\sigma\rho} - R_{\mu\sigma\rho\tau}R_\nu{}^{\sigma\rho\tau} - 2R_{\mu\sigma}R_\nu{}^\sigma \\ &- \tfrac{1}{4}g_{\mu\nu}\left(R^2_{\tau\lambda\rho\sigma} - 4R^2_{\rho\sigma} + R^2\right)\Big] , \end{aligned} \tag{8.1.5}$$

and $\mathscr{T}_{\mu\nu}$ is the standard matter energy-momentum defined as in (1.3.21). The expression $H_{\mu\nu}$ is a result of varying L_{GB}, and $H_{\mu\nu} \equiv 0$ if $n \le 4$.

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The equations (8.1.3) are the variant of the equations (7.2.3) for the case of the EGB gravity. The equations (8.1.3) are also

$$\mathscr{E}_{\mu\nu} = 0 \tag{8.1.6}$$

in the vacuum case.

8.1.2 Three types of superpotentials

In deriving concrete expressions for conserved quantities in the EGB gravity, we change the order of the presentation used in Sections 7.1 and 7.2, where currents have been described firstly. Indeed, considering an abstract theory, it is more natural to begin the construction from currents, then to continue the construction of superpotentials. Here, inversely, we begin from superpotentials, and finish with the currents. The reasons are as follow. First, anyway we keep in mind the principal scheme of construction in Chapter 7 and already have at hand the basic general formulae for such a construction. Second, historically superpotentials have been more popular and more desirable in applications because they give a direct way for constructing conserved charges. Third, superpotential expressions are significantly simpler than the expressions for currents.

The canonical prescription

At first, one has to define the KBL type Lagrangian for the EGB gravity. Following to (7.2.15), we derive it as

$$\mathscr{L}_{KBL} = -\frac{1}{2\kappa}\left(\mathscr{L}^{c}_{EGB} - \bar{\mathscr{L}}^{c}_{EGB} + \partial_\alpha \mathscr{D}^\alpha\right). \tag{8.1.7}$$

Following the recommendation (7.2.18), we construct the canonical superpotential in the EGB gravity,

$$\boldsymbol{j}^{*\alpha\beta}_{C}(\xi) = \boldsymbol{i}^{*\alpha\beta}(\xi) - \bar{\boldsymbol{i}}^{*\alpha\beta}(\xi) + \boldsymbol{i}'^{\alpha\beta}(\xi). \tag{8.1.8}$$

From the beginning we construct the *starred* expressions, coefficients for which are derived in Appendix C. Possibilities of the developed approaches in Section 7.2 permit us to construct families of superpotentials easily. Here we use these possibilities by applying concrete expressions in the EGB case. Thus, the pure Noether's canonical starred superpotential of the general form (7.1.46) in EGB gravity acquires the concrete form:

$$\begin{aligned}\boldsymbol{i}^{*\alpha\beta} = {} & \frac{1}{\kappa}\left(\mathrm{g}^{\rho[\alpha}\bar\nabla_\rho \xi^{\beta]} + \mathrm{g}^{\rho[\alpha}\Delta^{\beta]}_{\rho\sigma}\xi^\sigma\right)\\ & - 2\alpha\frac{1}{\kappa}\left\{\Delta^\rho_{\lambda\sigma}\mathscr{R}_\rho{}^{\lambda\alpha\beta} + 4\Delta^\rho_{\lambda\sigma}\mathrm{g}^{\lambda[\alpha}\mathscr{R}^{\beta]}_\rho + \Delta^{[\alpha}_{\rho\sigma}\mathrm{g}^{\beta]\rho}\mathscr{R}\right\}\xi^\sigma\\ & - 2\alpha\frac{1}{\kappa}\left\{\mathscr{R}_\sigma{}^{\lambda\alpha\beta} + 4\mathrm{g}^{\lambda[\alpha}\mathscr{R}^{\beta]}_\sigma + \delta^{[\alpha}_\sigma \mathrm{g}^{\beta]\lambda}\mathscr{R}\right\}\bar\nabla_\lambda\xi^\sigma,\end{aligned} \tag{8.1.9}$$

where as usual $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$ and $\mathscr{R}^{\sigma}{}_{\lambda\alpha\beta} = \sqrt{-g}R^{\sigma}{}_{\lambda\alpha\beta}$. The barred expression is simpler because $\bar{\Delta}^{\gamma}{}_{\alpha\beta} \equiv 0$:

$$\bar{\boldsymbol{\imath}}^{*\alpha\beta} = \frac{1}{\kappa}\left[\sqrt{-\bar{g}}\bar{\nabla}^{[\alpha}\xi^{\beta]} - 2\alpha\left(\bar{\mathscr{R}}_{\sigma}{}^{\lambda\alpha\beta} + 4\bar{g}^{\lambda[\alpha}\bar{\mathscr{R}}^{\beta]}_{\sigma} + \delta^{[\alpha}_{\sigma}\bar{g}^{\beta]\lambda}\bar{\mathscr{R}}\right)\bar{\nabla}_{\lambda}\xi^{\sigma}\right]. \tag{8.1.10}$$

To finalize constructing the superpotential (8.1.8) one needs to fix a divergence in the Lagrangian (8.1.7). We consider two more acceptable possibilities. At first, we follow the recommendation in the Deruelle-Katz-Ogushi paper [119], where the main requirement is that the variation of the Lagrangian has to be more economical:

$$\mathscr{D}^{\lambda}_{DKO} = -2\kappa\boldsymbol{n}^{*\lambda\alpha\beta}_{\sigma}\Delta^{\sigma}{}_{\alpha\beta}. \tag{8.1.11}$$

Thus,

$$\begin{aligned}\mathscr{D}^{\lambda}_{DKO} &= {}_{E}\mathscr{D}^{\lambda} + {}_{GB}\mathscr{D}^{\lambda}\\ &= 2\Delta^{[\alpha}{}_{\alpha\beta}\mathfrak{g}^{\lambda]\beta}\\ &\quad + 4\alpha\left(\mathscr{R}_{\sigma}{}^{\alpha\beta\lambda} - 4\mathscr{R}^{[\alpha}_{\sigma}g^{\lambda]\beta} + \delta^{[\alpha}_{\sigma}g^{\lambda]\beta}\mathscr{R}\right)\Delta^{\sigma}{}_{\alpha\beta}.\end{aligned} \tag{8.1.12}$$

Katz and Livshits [253] develop the KBL approach in n dimensions in the first order formalism that leads to

$$\begin{aligned}\mathscr{D}^{\lambda}_{KL} &= {}_{E}\mathscr{D}^{\lambda} + {}_{GB}\mathscr{D}^{\lambda}\\ &= 2\Delta^{[\alpha}{}_{\alpha\beta}\mathfrak{g}^{\lambda]\beta}\\ &\quad + 4\alpha\left(\mathscr{R}_{\sigma}{}^{\alpha\beta\lambda} - 2\mathscr{R}^{[\alpha}_{\sigma}g^{\lambda]\beta} - 2\delta^{[\alpha}_{\sigma}\mathscr{R}^{\lambda]\beta} + \delta^{[\alpha}_{\sigma}g^{\lambda]\beta}\mathscr{R}\right)\Delta^{\sigma}{}_{\alpha\beta}.\end{aligned} \tag{8.1.13}$$

As is seen, the Einstein part of both these divergences is the KBL divergence (6.1.4) in four dimensional general relativity; the Gauss-Bonnet parts are different and more complicated.

However, the Katz and Livshits approach for constructing superpotentials [253] has its own advantages. In multi-dimensional general relativity their method *uniquely* leads to the KBL superpotential (6.1.24), or (6.1.25). In EGB gravity, their superpotential, essentially connected with (8.1.13) and the GB term (8.1.2), naturally transforms to the KBL superpotential for $n = 4$. Thus, although the GB term does not affect the derivation of the field equations for $n = 4$, it plays an important role (as a criterion) in the definition of superpotentials of canonical type.

The use of the term (8.1.2) in the Lagrangian even in 4 *dimensions* turns out to be important when the other ideas are elaborated. For example, in [345] Olea includes the GB term to regularize conserved quantities, in [317] Mišković and Olea show that the standard holographic regularization procedure of AdS gravity with counter-terms is topological and, thus, can be presented by the addition of the GB term. Thus, summing these, we conclude that the choice (8.1.13) is more preferable, although considering canonical superpotentials and related charges, we use both of the possibilities (8.1.12) and (8.1.13).

At last, combining (8.1.9–8.1.13) with (7.1.65) and (7.1.68), we derive a superpotential in the canonical prescription (8.1.8) for the EGB gravity:

$$\begin{aligned} \boldsymbol{j}_C^{*\alpha\beta} &= {}_E\boldsymbol{j}_C^{*\alpha\beta} + {}_{GB}\boldsymbol{j}_C^{*\alpha\beta} \\ &= \frac{1}{\kappa}\left(\mathfrak{g}^{\rho[\alpha}\bar{\nabla}_\rho\xi^{\beta]} + \mathfrak{g}^{\rho[\alpha}\Delta^{\beta]}{}_{\rho\sigma}\xi^\sigma - \bar{\mathfrak{g}}^{\rho[\alpha}\bar{\nabla}_\rho\xi^{\beta]} + \xi^{[\alpha}{}_E\mathscr{D}^{\beta]}\right) \\ &\quad + {}_{GB}\boldsymbol{i}^{*\alpha\beta} - {}_{GB}\bar{\boldsymbol{i}}^{*\alpha\beta} + \kappa^{-1}\xi^{[\alpha}{}_{GB}\mathscr{D}^{\beta]}, \end{aligned} \tag{8.1.14}$$

where ${}_E\mathscr{D}^\beta$ and ${}_{GB}\mathscr{D}^\beta$ are derived from (8.1.12), or (8.1.13). The full Einstein part ${}_E\boldsymbol{j}_C^{\alpha\beta}$ is exactly the KBL superpotential both in four dimensional [251] and in multidimensional general relativity [119, 253].

To construct a *family* of superpotentials for perturbations in the EGB theory, we turn to the general formulae (7.1.60) and (7.2.39). To derive this family, based on (8.1.14), one has to define the difference (7.2.40) in the EGB gravity. Thus, with the use of (C.12) one calculates

$$-\frac{1}{2\kappa}\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_{[\beta\alpha]}g_{\mu\nu})} = 2\alpha\frac{1}{\kappa}\left(\mathscr{R}^{\alpha(\mu}g^{\nu)\beta} - \mathscr{R}^{\beta(\mu}g^{\nu)\alpha}\right). \tag{8.1.15}$$

As a result, one has for (7.2.40):

$$\begin{aligned} \boldsymbol{r}^{*\alpha\beta} &= -\frac{1}{\kappa}\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_{[\beta\alpha]}g_{\mu\nu})}\pounds_\xi g_{\mu\nu} \\ &= -8\alpha\frac{1}{\kappa}\left(\mathscr{R}^{[\alpha}_\rho g^{\beta]\mu} + \mathscr{R}^{\mu[\alpha}\delta^{\beta]}_\rho\right)\left(\bar{\nabla}_\mu\xi^\rho + \xi^\tau\Delta^\rho{}_{\tau\mu}\right). \end{aligned} \tag{8.1.16}$$

Finally, following (7.2.39), one obtains for the family

$$\boldsymbol{j}_C^{\dagger\alpha\beta} = \boldsymbol{j}_C^{*\alpha\beta} + p\delta\boldsymbol{r}^{*\alpha\beta}, \tag{8.1.17}$$

where $\delta\boldsymbol{r}^{*\alpha\beta} = \boldsymbol{r}^{*\alpha\beta} - \bar{\boldsymbol{r}}^{*\alpha\beta}$, and $0 \le p \le 1$.

The Belinfante correcting prescription

Now, following the recommendation (7.2.45), we construct the Belinfante corrected superpotential in the EGB gravity,

$$\boldsymbol{j}_B^{*\alpha\beta}(\xi) = \boldsymbol{i}_B^{*\alpha\beta}(\xi) - \bar{\boldsymbol{i}}_B^{*\alpha\beta}(\xi), \tag{8.1.18}$$

where we again have used the starred expressions, coefficients of which (C.17) and (C.18) are derived in Appendix C. As it has to be for the Belinfante corrected quantities, there is no dependence on a divergence in the Lagrangian. Thus, the expression (7.1.76) in EGB gravity acquires the form:

$$
\begin{aligned}
\boldsymbol{i}_B^{*\alpha\beta} &= {}_E\boldsymbol{i}_B^{*\alpha\beta} + {}_{GB}\boldsymbol{i}_B^{*\alpha\beta}\\
&= \frac{1}{\kappa}\left[\left(\delta_\sigma^{[\alpha}\bar\nabla_\lambda \mathfrak{g}^{\beta]\lambda} - \bar\nabla^{[\alpha}\mathfrak{g}^{\beta]\rho}\bar g_{\rho\sigma}\right)\xi^\sigma + \mathfrak{g}^{\lambda[\alpha}\bar\nabla_\lambda\xi^{\beta]}\right]\\
&+\alpha\frac{1}{\kappa}\bar\nabla_\lambda\left\{\mathscr{R}_\sigma{}^{\lambda\alpha\beta} + 4g^{\lambda[\alpha}\mathscr{R}_\sigma^{\beta]} + \delta_\sigma^{[\alpha}g^{\beta]\lambda}\mathscr{R} + \left[2\mathscr{R}_\tau{}^{\rho\lambda[\alpha} - 2\mathscr{R}^{\rho\lambda}{}_\tau{}^{[\alpha}\right.\right.\\
&-\left.\left. 8\mathscr{R}_\tau^\lambda g^{\rho[\alpha} + 4\mathscr{R}_\tau^\rho g^{\lambda[\alpha} + 4g^{\rho\lambda}\mathscr{R}_\tau^{[\alpha} + \mathscr{R}\left(2\delta_\tau^\lambda g^{\rho[\alpha} - \delta_\tau^\rho g^{\lambda[\alpha}\right)\right]\bar g^{\beta]\tau}\bar g_{\rho\sigma}\right\}\xi^\sigma\\
&-2\alpha\frac{1}{\kappa}\left\{\mathscr{R}_\sigma{}^{\lambda\alpha\beta} + 4g^{\lambda[\alpha}\mathscr{R}_\sigma^{\beta]} + \delta_\sigma^{[\alpha}g^{\beta]\lambda}\mathscr{R}\right\}\bar\nabla_\lambda\xi^\sigma\,.
\end{aligned}
\tag{8.1.19}
$$

The barred expression is significantly simpler

$$
\begin{aligned}
\bar{\boldsymbol{i}}_B^{\alpha\beta} &= {}_E\bar{\boldsymbol{i}}_B^{\alpha\beta} + {}_{GB}\bar{\boldsymbol{i}}_B^{\alpha\beta}\\
&= \frac{1}{\kappa}\bar{\mathfrak{g}}^{\lambda[\alpha}\bar\nabla_\lambda\xi^{\beta]} - 2\alpha\frac{1}{\kappa}\left\{\bar{\mathscr{R}}_\sigma{}^{\lambda\alpha\beta} + 4\bar g^{\lambda[\alpha}\bar{\mathscr{R}}_\sigma^{\beta]} + \delta_\sigma^{[\alpha}\bar g^{\beta]\lambda}\bar{\mathscr{R}}\right\}\bar\nabla_\lambda\xi^\sigma.
\end{aligned}
\tag{8.1.20}
$$

Thus, combining (8.1.19) and (8.1.20), we derive the superpotential in the Belinfante corrected prescription (8.1.18) for the EGB gravity:

$$
\begin{aligned}
\boldsymbol{j}_B^{\alpha\beta} &= {}_E\boldsymbol{j}_B^{\alpha\beta} + {}_{GB}\boldsymbol{j}_B^{\alpha\beta}\\
&= \frac{1}{\kappa}\left(\xi^{[\alpha}\bar\nabla_\lambda\mathfrak{h}^{\beta]\lambda} - \bar\nabla^{[\alpha}\mathfrak{h}_\sigma^{\beta]}\xi^\sigma + \mathfrak{h}^{\lambda[\alpha}\bar\nabla_\lambda\xi^{\beta]}\right)\\
&\quad + {}_{GB}\boldsymbol{i}_B^{\alpha\beta} - {}_{GB}\bar{\boldsymbol{i}}_B^{\alpha\beta}
\end{aligned}
\tag{8.1.21}
$$

In order to construct a *family* of the superpotentials for perturbations in EGB theory, one has to use (7.1.84) and (7.2.39). To derive this family based on (8.1.21), one has to define the difference $\boldsymbol{r}_B^{*\alpha\beta}$ in (7.1.84) in the EGB gravity, see (7.1.79) for $\boldsymbol{r}_B^{\alpha\beta}$. Thus, with the use of (8.1.15) one has

$$
\begin{aligned}
\boldsymbol{r}_B^{*\alpha\beta} &= 2\left[\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar\nabla_{[\beta\alpha]}g_{\mu\nu})}\, g_{\mu\nu}\Big|^\rho_\sigma\,\bar\nabla_\rho\xi^\sigma + \bar\nabla_\rho\left(\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar\nabla_{[\rho\sigma]}g_{\mu\nu})}\, g_{\mu\nu}\Big|^{[\alpha}_\lambda\,\bar g^{\beta]\lambda}\right.\right.\\
&+\left.\left.\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar\nabla_{[\rho\alpha]}g_{\mu\nu})}\, g_{\mu\nu}\Big|^{[\sigma}_\lambda\,\bar g^{\beta]\lambda} - \frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar\nabla_{[\rho\beta]}g_{\mu\nu})}\, g_{\mu\nu}\Big|^{[\sigma}_\lambda\,\bar g^{\alpha]\lambda}\right)\xi_\sigma\right]\\
&= 4\alpha\frac{1}{\kappa}\left\{-2\left(\mathscr{R}_\sigma^{[\alpha}g^{\beta]\rho} + \mathscr{R}^{\rho[\alpha}\delta_\sigma^{\beta]}\right)\bar\nabla_\rho\xi^\sigma\right.\\
&+\xi_\sigma\bar\nabla_\rho\left[\mathscr{R}_\lambda^\sigma g^{\rho[\alpha}\bar g^{\beta]\lambda} + g^{\sigma\rho}\mathscr{R}_\lambda^{[\alpha}\bar g^{\beta]\lambda} - \mathscr{R}_\lambda^{[\alpha}g^{\beta]\rho}\bar g^{\sigma\lambda}\right.\\
&+\left.\left. 2\left(\mathscr{R}^{\sigma[\alpha}\bar g^{\beta]\rho} - \mathscr{R}^{\rho[\alpha}\bar g^{\beta]\sigma} - \mathscr{R}_\lambda^\rho g^{\sigma[\alpha}\bar g^{\beta]\lambda}\right)\right]\right\}.
\end{aligned}
\tag{8.1.22}
$$

Finally, following (7.1.84), one has for the family

$$\boldsymbol{j}_B^{\dagger\alpha\beta} = \boldsymbol{j}_B^{*\alpha\beta} + p\delta\boldsymbol{r}_B^{*\alpha\beta}\,, \tag{8.1.23}$$

where $\delta\boldsymbol{r}_B^{*\alpha\beta} = \boldsymbol{r}_B^{*\alpha\beta} - \bar{\boldsymbol{r}}_B^{*\alpha\beta}$, and $0 \le p \le 1$.

The field-theoretical prescription

At last, let us construct the symmetric superpotential in the framework of the EGB theory. Let us turn to the general formula (7.2.96), or in a more convenient form (7.2.99). To concretize the calculation and to have the possibility to compare, for example, with [126], we define the perturbations in the form:

$$h^a = \varkappa_{\alpha\beta} = g_{\alpha\beta} - \bar{g}_{\alpha\beta}\,. \tag{8.1.24}$$

The Lagrangian (7.2.85) in this case is defined as

$$\mathscr{L}_1 = -\frac{1}{2\kappa}\varkappa_{\alpha\beta}\frac{\delta\bar{\mathscr{L}}_{EGB}}{\delta\bar{g}_{\alpha\beta}} = \frac{1}{2\kappa}\varkappa_{\alpha\beta}\bar{\mathscr{E}}^{\alpha\beta}\,, \tag{8.1.25}$$

where $\bar{\mathscr{E}}_{\alpha\beta}$ is the barred expression (8.1.6). Substituting the expressions (C.21) and (C.22) into (7.2.99), one obtains

$$\begin{aligned}
\boldsymbol{i}_1^{\alpha\beta} &= {}_E\boldsymbol{i}_1^{\alpha\beta} + {}_{GB}\boldsymbol{i}_1^{\alpha\beta} \\
&= \frac{\sqrt{-\bar{g}}}{\kappa}\Big(\xi_\nu\bar{\nabla}^{[\alpha}\varkappa^{\beta]\nu} - \xi^{[\alpha}\bar{\nabla}_\nu\varkappa^{\beta]\nu} + \xi^{[\alpha}\bar{\nabla}^{\beta]}\varkappa_\sigma{}^\sigma \\
&\quad - \varkappa^{\nu[\alpha}\bar{\nabla}_\nu\xi^{\beta]} + \tfrac{1}{2}\varkappa_\sigma{}^\sigma\bar{\nabla}^{[\alpha}\xi^{\beta]}\Big) \\
&\quad + \frac{4}{3}\Big(2\xi_\sigma\bar{\nabla}_\lambda\boldsymbol{w}_{GB}^{\sigma[\alpha|\beta]\lambda} - \boldsymbol{w}_{GB}^{\sigma[\alpha|\beta]\lambda}\bar{\nabla}_\lambda\xi_\sigma\Big)\,.
\end{aligned} \tag{8.1.26}$$

This expression transforms to the Deser-Tekin superpotential [126] if one chooses the AdS background and its Killing vectors $\xi^\alpha = \bar{\xi}^\alpha$.

Returning to the form of the Lagrangian (7.2.85),

$$\mathscr{L}_1 = -\frac{1}{2\kappa}\mathfrak{h}_a^{\alpha\beta}\frac{\delta\bar{\mathscr{L}}_{EGB}}{\delta\bar{\mathfrak{g}}^{\alpha\beta}}\,, \tag{8.1.27}$$

one easily obtains the generalized form of the superpotential (8.1.26), if one uses the exchange:

$$\varkappa_{\mu\nu} \to \frac{\partial\bar{g}_{\mu\nu}}{\partial\bar{\mathfrak{g}}^{\rho\sigma}}\mathfrak{h}_a^{\rho\sigma}\,. \tag{8.1.28}$$

Thus,

$$
\begin{aligned}
\boldsymbol{i}_1^{\alpha\beta} &= {}_E\boldsymbol{i}_1^{\alpha\beta} + {}_{GB}\boldsymbol{i}_1^{\alpha\beta} \\
&= \frac{1}{\kappa}\left(\mathfrak{h}_a^{\sigma[\mu}\bar{\nabla}_\sigma\xi^{\rho]} + \xi^{[\mu}\bar{\nabla}_\sigma\mathfrak{h}_a^{\rho]\sigma} - \bar{\nabla}^{[\mu}\mathfrak{h}_{a\sigma}^{\rho]}\xi^\sigma\right) \\
&\quad + \frac{4}{3}\left(2\xi_\sigma\bar{\nabla}_\lambda\boldsymbol{w}_{GB}^{\sigma[\alpha|\beta]\lambda}(\mathfrak{h}_a) - \boldsymbol{w}_{GB}^{\sigma[\alpha|\beta]\lambda}(\mathfrak{h}_a)\bar{\nabla}_\lambda\xi_\sigma\right) .
\end{aligned}
\tag{8.1.29}
$$

Of course, this expression transforms into (8.1.26), if

$$
\mathfrak{h}_a^{\rho\sigma} = \frac{\partial\bar{\mathfrak{g}}^{\rho\sigma}}{\partial\bar{g}_{\mu\nu}}\varkappa_{\mu\nu} . \tag{8.1.30}
$$

8.1.3 Three types of currents

To the best of our knowledge, unlike superpotentials, no authors have paid attention to constructing currents in modified theories. In this subsection, at least in part, we fill this gap. We construct currents in EGB gravity, based on the general expressions in an arbitrary metric theory constructed in Section 7.2. Thus, in fact, we supplement superpotentials presented in the previous subsection by the related currents.

For the sake of simplicity, we restrict ourself to only the starred currents, $\mathscr{I}_C^{*\alpha}$ and $\mathscr{I}_B^{*\alpha}$, in the conservation laws (7.2.29) and (7.2.58), respectively, using the starred expressions in Appendix C. Concerning the field-theoretical method, we consider the current, $\mathscr{I}_S^{\alpha}(\varkappa_{\alpha\beta})$, in (7.2.101) only for the perturbations $\varkappa_{\alpha\beta} = g_{\alpha\beta} - \bar{g}_{\alpha\beta}$ defined in (8.1.24). The reasons for such a simple representation are twofold. First, using the results of the Section 7.2, one can easily restore the families of the currents, $\mathscr{I}_C^{\dagger\alpha}$ and $\mathscr{I}_B^{\dagger\alpha}$, in (7.2.32) and (7.2.59), respectively, and for arbitrary types of perturbations $h^a = g^a - \bar{g}^a$ in the current $\mathscr{I}_S^{\alpha}(h^a)$ in (7.2.101). Second, in next section, the application of all the three families of superpotentials to calculate the mass of the Schwarzschild-AdS black holes (that is represented by the standard solution) does not clarify the preferable superpotential.

The canonical prescription

Let us turn to the current (7.2.30). Its structure, as one can see from (7.2.31), essentially depends on the divergence in the Lagrangian. Let us consider two types of the divergence defined by equations (8.1.12) and (8.1.13). We suppress also the $\boldsymbol{z}^*$-term in (7.2.30) as vanishing for Killing vectors $\bar{\xi}^\alpha$ in the future calculations. Thus, we rewrite the current (7.2.30) as

$$
\mathscr{I}_C^{*\alpha} = {}_C\boldsymbol{\theta}_\sigma^{*\alpha}\bar{\xi}^\sigma + \delta\boldsymbol{m}^{*\rho\alpha\beta}\partial_{[\beta}\bar{\xi}_{\rho]} . \tag{8.1.31}
$$

First, we calculate ${}_C\boldsymbol{\theta}_\sigma^{*\alpha}$ following (7.2.31). The matter part in (7.2.31) is defined by the matter part of the Lagrangian (8.1.1). It is more interesting to focus on the gravitational part in (7.2.31). Thus, we calculate (7.2.31) with the use of (C.9) and (8.1.13):

$$
\begin{aligned}
{}_{C}\boldsymbol{\theta}^{*\alpha}_{\sigma} &= \delta \mathscr{T}_{\sigma}{}^{\alpha} + \delta \mathscr{C}^{*\alpha}_{\sigma} + \frac{1}{\kappa}\bar{\nabla}_{\beta}(\delta^{[\alpha}_{\sigma}\mathscr{D}^{\beta]}) \\
&= \delta \mathscr{T}_{\sigma}{}^{\alpha} + \frac{1}{2\kappa}\delta^{\alpha}_{\sigma}\left[\bar{R}_{\rho\tau}\mathfrak{h}^{\rho\tau} - 2\Lambda_0 \delta\sqrt{-g}\right] \\
&+ \frac{\sqrt{-g}}{\kappa}\left[\left(\Delta^{\alpha}_{\rho[\tau}\Delta^{\pi}_{\pi]\sigma} + \Delta^{\alpha}_{\rho[\sigma}\Delta^{\pi}_{\pi]\tau} + \delta^{\alpha}_{\sigma}\Delta^{\pi}_{\beta[\tau}\Delta^{\beta}_{\pi]\rho} - \delta^{\alpha}_{\rho}\Delta^{\beta}_{\beta[\sigma}\Delta^{\pi}_{\pi]\tau}\right) g^{\tau\rho}\right] \\
&+ \frac{1}{\kappa}\bar{\nabla}_{\beta}(\delta^{[\alpha}_{\sigma}\,{}_{GB}\mathscr{D}^{\beta]}) + \frac{\alpha}{2\kappa}\delta^{\alpha}_{\sigma}\,\delta(\sqrt{-g}L_{GB}) \\
&+ \frac{2\alpha\sqrt{-g}}{\kappa}\left[\left(R^{\alpha\beta\rho}{}_{\tau} - 4g^{\rho[\alpha}R^{\beta]}_{\tau} + Rg^{\rho[\alpha}\delta^{\beta]}_{\tau}\right)\bar{\nabla}_{\sigma}\Delta^{\tau}_{\beta\rho}\right. \\
&+ \left. 2g^{\beta\mu}\left(\bar{\nabla}_{\beta}R^{\alpha\nu} + 2\Delta^{(\alpha}_{\beta\rho}R^{\nu)\rho}\right)\Delta^{\tau}_{\sigma(\mu}g_{\nu)\tau} - g^{\rho(\alpha}\Delta^{\nu)}_{\sigma\rho}\partial_{\nu}R\right].
\end{aligned} \tag{8.1.32}
$$

The symbol δ without subscripts, once again, means a perturbation of a quantity with respect to a background quantity: $\delta Q = Q - \bar{Q}$.

To calculate the spin term $\delta\boldsymbol{m}^{*\alpha\beta}_{\sigma}$ in (8.1.31) we return to the definition (7.2.21), use the expression (C.15), then subtract the barred (C.15), and take into account (8.1.13). Thus,

$$
\begin{aligned}
\delta\boldsymbol{m}^{*\alpha\beta}_{\sigma} &= -\left[\boldsymbol{m}^{*\alpha\beta}_{\sigma} - \bar{\boldsymbol{m}}^{*\alpha\beta}_{\sigma} - \frac{1}{\kappa}\delta^{[\alpha}_{\sigma}\mathscr{D}^{\beta]}\right] \\
&= \frac{\sqrt{-g}}{2\kappa}\left[\Delta^{\tau}_{\rho\tau}\left(2\bar{g}^{\sigma[\alpha}g^{\beta]\rho} + \bar{g}^{\sigma\rho}g^{\alpha\beta}\right) - \Delta^{\alpha}_{\rho\tau}\left(2\bar{g}^{\sigma[\tau}g^{\beta]\rho} + \bar{g}^{\sigma\rho}g^{\tau\beta}\right)\right] \\
&+ \frac{1}{\kappa}\bar{g}^{\sigma[\alpha}\,{}_{GB}\mathscr{D}^{\beta]} - \frac{2\alpha\sqrt{-g}}{\kappa}\left[R^{\alpha\tau\rho}{}_{\lambda}\Delta^{\beta}_{\tau\rho} - 2R^{\alpha(\tau\beta)}{}_{\rho}\Delta^{\rho}_{\tau\lambda}\right]\bar{g}^{\lambda\sigma} \\
&- \frac{4\alpha\sqrt{-g}}{\kappa}\left[4g^{\rho[\alpha}R^{\beta]}_{\tau}\Delta^{\tau}_{\rho\lambda} + 2R^{[\alpha}_{\lambda}g^{\tau]\rho}\Delta^{\beta}_{\tau\rho} + 2g^{\alpha[\beta}R^{\tau]}_{\rho}\Delta^{\rho}_{\tau\lambda}\right. \\
&+ g^{\tau\beta}\left(R^{\alpha}_{\rho}\Delta^{\rho}_{\tau\lambda} - R^{\rho}_{(\tau}\Delta^{\alpha}_{\lambda)\rho}\right) \\
&- \left.\left(g^{\rho\beta}\Delta^{\tau}_{\rho(\tau} + g^{\rho\tau}\Delta^{\beta}_{\rho(\tau} - g^{\tau\beta}\Delta^{\rho}_{\rho(\tau}\right)R^{\alpha}_{\lambda)}\right]\bar{g}^{\lambda\sigma} \\
&- \frac{2\alpha\sqrt{-g}}{\kappa}R\left[\Delta^{[\alpha}_{\rho\lambda}g^{\rho]\beta} + \Delta^{(\alpha}_{\rho\lambda}g^{\beta)\rho)}\right]\bar{g}^{\lambda\sigma} \\
&+ \frac{2\alpha}{\kappa}\left[2\bar{g}^{\lambda\sigma}\bar{\nabla}_{(\tau}\delta\left(\mathfrak{g}^{\tau\beta}R^{\alpha}_{\lambda)}\right) - \bar{g}^{\sigma(\alpha}\bar{\nabla}_{\rho}\,\delta\left(\mathfrak{g}^{\rho)\beta}R\right)\right].
\end{aligned} \tag{8.1.33}
$$

As expected, these expressions disappear for vanishing perturbations. The Einstein parts in (8.1.32) and (8.1.33) exactly coincide with the energy-momentum (6.1.27) and the spin tensor (6.1.12) presented in the related KBL expressions in the four dimensional general relativity. We do not present explicitly the terms with ${}_{GB}\mathscr{D}^{\alpha}$ from (8.1.12) and (8.1.13) because this does not simplify the expression as a whole.

The Belinfante correcting prescription

Here, we turn to the current in (7.2.58). Its structure, unlike the canonical case, does not depend on the spin term and the divergence in the Lagrangian. As before, we do not consider the $\boldsymbol{z}^*$-term, assuming the use of the Killing vectors. Thus, the starred expression for the current (7.2.56) acquires the form:

$$\mathscr{I}_B^{*\alpha} = {}_B\boldsymbol{\theta}_\sigma^{*\alpha}\bar{\xi}^\sigma\,, \tag{8.1.34}$$

where

$$_B\boldsymbol{\theta}_\sigma^{*\alpha} \equiv \delta\mathscr{T}_\sigma{}^\alpha + \delta\mathscr{C}_\sigma^{*\alpha} + \bar{\nabla}_\beta\delta\boldsymbol{s}^{*\alpha\beta}{}_\sigma\,. \tag{8.1.35}$$

With the use of the equations (8.1.3) one has

$$_B\boldsymbol{\theta}^{*\sigma\alpha} = \delta\mathscr{T}_\rho{}^{(\alpha}\bar{g}^{\sigma)\rho} + \kappa^{-1}\delta\mathscr{E}_\rho{}^{[\alpha}\bar{g}^{\sigma]\rho} + \delta\mathscr{C}^{*\sigma\alpha} + \bar{\nabla}_\beta\delta\boldsymbol{s}^{*\alpha\beta\sigma}\,, \tag{8.1.36}$$

where $\mathscr{E}_{\mu\nu}$ is defined in (8.1.4). Finally, substituting the explicit expressions (C.9) and (C.20) into (8.1.36) one obtains

$$\begin{aligned}
{}_B\boldsymbol{\theta}^{*\sigma\alpha} = {}& \delta\mathscr{T}_\rho{}^{(\alpha}\bar{g}^{\sigma)\rho} + \frac{1}{2\kappa}\left[\mathfrak{h}^{\rho\tau}\bar{R}_{\rho\tau}\bar{g}^{\alpha\sigma} + 2\mathfrak{h}^{\lambda[\alpha}\bar{R}_\lambda^{\sigma]} - 2\bar{g}^{\alpha\sigma}\Lambda_0\delta\sqrt{-g}\right] \\
& + \frac{1}{2\kappa}\left[\left(\mathfrak{h}^{\alpha\sigma}\bar{g}^{\rho\tau} - \bar{g}^{\alpha\sigma}\mathfrak{h}^{\rho\tau}\right)\bar{\nabla}_\tau\Delta^\lambda_{\rho\lambda} + 2\left(\mathfrak{h}^{\rho\tau}\bar{g}^{\lambda(\alpha} - \bar{g}^{\rho\tau}\mathfrak{h}^{\lambda(\alpha}\right)\bar{\nabla}_\tau\Delta^{\sigma)}_{\lambda\rho}\right] \\
& + \frac{1}{2\kappa}\left[\bar{g}^{\rho\tau}\left(\mathfrak{g}^{\alpha\sigma}\Delta^\lambda{}_{\rho\lambda}\Delta^\eta{}_{\tau\eta} + 2\mathfrak{g}^{\lambda\eta}\Delta^{(\alpha}_{\lambda\rho}\Delta^{\sigma)}_{\eta\tau}\right) + \mathfrak{g}^{\lambda\eta}\bar{g}^{\alpha\sigma}\Delta^\tau{}_{\rho\lambda}\Delta^\rho{}_{\tau\eta}\right] \\
& + \frac{1}{\kappa}\left[\bar{g}^{\rho\tau}\left(\Delta^\lambda{}_{\tau\eta}\Delta^{(\alpha}_{\lambda\rho}\mathfrak{g}^{\sigma)\eta} - 2\Delta^\lambda{}_{\tau\lambda}\Delta^{(\alpha}_{\eta\rho}\mathfrak{g}^{\sigma)\eta}\right)\right. \\
& \left. + \;\mathfrak{g}^{\lambda\eta}\left(\Delta^\tau{}_{\rho\tau}\Delta^{(\alpha}_{\lambda\eta} - \Delta^\tau{}_{\lambda\eta}\Delta^{(\alpha}_{\rho\tau} - \Delta^\tau{}_{\lambda\rho}\Delta^{(\alpha}_{\eta\tau}\right)\bar{g}^{\sigma)\rho}\right] \\
& - \frac{\alpha\sqrt{-g}}{\kappa}\bar{g}^{\lambda[\alpha}\left[R^{\sigma]}{}_{\pi\rho\tau}R_\lambda{}^{\pi\rho\tau} - 2R^{\sigma]}{}_{\pi\lambda\rho}R^{\pi\rho} - 2R^{\sigma]}_\rho R^\rho_\lambda + R^{\sigma]}_\lambda R\right] \\
& + \frac{2\alpha\sqrt{-g}}{\kappa}\bar{g}^{\lambda\sigma}\left[\left(R^{\alpha\beta\rho}{}_\tau - 4g^{\rho[\alpha}R^{\beta]}_\tau + Rg^{\rho[\alpha}\delta^{\beta]}_\tau\right)\bar{\nabla}_\lambda\Delta^\tau{}_{\beta\rho}\right. \\
& \left. - \;g^{\rho(\alpha}\Delta^{\beta)}_{\lambda\rho}\bar{\nabla}_\beta R + 2g^{\beta\mu}\left(\bar{\nabla}_\beta R^{\alpha\nu} + 2R^{\rho(\alpha}\Delta^{\nu)}_{\beta\rho}\right)\Delta^\tau{}_{\lambda(\mu}g_{\nu)\tau}\right] \\
& + \frac{\alpha}{2\kappa}\bar{g}^{\alpha\sigma}\delta\left(\sqrt{-g}L_{GB}\right) + \bar{\nabla}_\beta\delta_{GB}\hat{s}^{*\alpha\beta\sigma}\,.
\end{aligned} \tag{8.1.37}$$

The Einstein part exactly coincides with the related expression (6.2.9) in Chapter 6. Recall that, even the Einstein part in (8.1.37) (symmetrized) is not symmetric in general. Here, we do not open the divergence of the GB-part $\delta_{GB}\hat{s}^{*\alpha\beta\sigma}$ of the Belinfante correction, see (C.20), because this does not simplify the expression. At last, note that the energy-momentum (8.1.37) disappears for vanishing perturbations.

The field-theoretical prescription

At last, we turn to the current in (7.2.100) in the framework of the field-theoretical approach. Its structure, again, does not depend on the spin term and the divergence in the Lagrangian. As before, we do not consider $\boldsymbol{z}$-term, assuming the use of the Killing vectors. Thus, the expression (7.2.101) acquires the form:

$$\mathscr{I}_S^{\alpha} = {}_S\boldsymbol{\theta}_{\sigma}{}^{\alpha}\bar{\xi}^{\sigma}\,, \tag{8.1.38}$$

where the energy-momentum is defined in (7.2.103),

$$\begin{aligned} {}_S\boldsymbol{\theta}_{\mu\nu} &\equiv \boldsymbol{t}^{\text{tot}}_{\mu\nu} - \frac{1}{\kappa}\Phi^L_{\mu\nu} + \frac{1}{2\kappa}\bar{g}_{\mu\nu}\mathfrak{h}^{\rho\sigma}_a\bar{\mathscr{E}}_{\rho\sigma} + \frac{1}{\kappa}\bar{g}_{\mu\rho}\mathfrak{h}^{\rho\sigma}_a\bar{\mathscr{E}}_{\sigma\nu} \\ &= \boldsymbol{t}^{g}_{\mu\nu} + \Delta\mathscr{T}_{\mu\nu} + \frac{1}{2\kappa}\bar{g}_{\mu\nu}\mathfrak{h}^{\rho\sigma}_a\bar{\mathscr{E}}_{\rho\sigma} + \frac{1}{\kappa}\bar{g}_{\mu\rho}\mathfrak{h}^{\rho\sigma}_a\bar{\mathscr{E}}_{\sigma\nu}\,. \end{aligned} \tag{8.1.39}$$

Here, in general, $\boldsymbol{t}^g_{\mu\nu}$ and $\Delta\mathscr{T}_{\mu\nu}$ are defined in (7.2.82) and (7.2.83), respectively. However, we specificity (8.1.39) to EGB gravity (8.1.1–8.1.6) and to the decomposition (8.1.24). Then $\mathfrak{h}^{\rho\sigma}_a$ is defined in (8.1.30), and (7.2.82) and (7.2.83) are simplified to

$$\boldsymbol{t}^g_{\mu\nu} = -\frac{1}{\kappa}\left[\mathscr{E}_{\mu\nu}(\bar{g}_{\rho\sigma} + \varkappa_{\rho\sigma}) - \mathscr{E}^L_{\mu\nu}(\varkappa_{\rho\sigma}) - \bar{\mathscr{E}}_{\mu\nu}\right], \tag{8.1.40}$$

$$\Delta\mathscr{T}_{\mu\nu} = \mathscr{T}_{\mu\nu}(\bar{g}_{\rho\sigma} + \varkappa_{\rho\sigma}) - \bar{\mathscr{T}}_{\mu\nu}\,. \tag{8.1.41}$$

The expressions $\mathscr{E}_{\mu\nu}$ and $\bar{\mathscr{E}}_{\mu\nu}$ are defined in (8.1.4) and barred (8.1.4). For the vacuum background, $\bar{\mathscr{E}}_{\mu\nu} = \bar{\mathscr{T}}_{\mu\nu} = 0$, the current (8.1.39) is

$${}_S\boldsymbol{\theta}_{\mu\nu} \equiv \boldsymbol{t}^g_{\mu\nu} + \Delta\mathscr{T}_{\mu\nu} = \boldsymbol{t}^{\text{tot}}_{\mu\nu} \tag{8.1.42}$$

with

$$\boldsymbol{t}^g_{\mu\nu} = -\frac{1}{\kappa}\left[\mathscr{E}_{\mu\nu}(\bar{g}_{\rho\sigma} + \varkappa_{\rho\sigma}) - \mathscr{E}^L_{\mu\nu}(\varkappa_{\rho\sigma})\right], \tag{8.1.43}$$

$$\Delta\mathscr{T}_{\mu\nu} = \mathscr{T}_{\mu\nu}(\bar{g}_{\rho\sigma} + \varkappa_{\rho\sigma})\,. \tag{8.1.44}$$

Thus, the same as in the four dimensional general relativity, on *arbitrary* backgrounds, the current (8.1.38) is conserved with the terms supplementing $\boldsymbol{t}^{\text{tot}}_{\mu\nu}$ in (8.1.39). Thus, the current

$$\mathscr{I}_S^{\alpha} = \boldsymbol{t}^{\alpha\sigma}_{\text{tot}}\bar{\xi}_{\sigma} \tag{8.1.45}$$

is not conserved in general. However, on *vacuum* backgrounds, currents (8.1.38) and (8.1.45) coincide, and, thus, (8.1.45) is conserved also. See the discussion on the conservation of $\boldsymbol{t}^{\text{tot}}_{\mu\nu}$ on arbitrary backgrounds in general relativity.

8.2 Conserved charges in the EGB gravity

8.2.1 Charges for isolated systems

The families of superpotentials (8.1.17), (8.1.23) and (8.1.29) in Section 8.1.2 are formally constructed for arbitrary metric functions. In the case, when we consider a concrete solution of the EGB theory, these quantities acquire a physical sense, preserving the same form:

$$\boldsymbol{j}_C^{\dagger\alpha\beta} \to \mathscr{I}_C^{\dagger\alpha\beta} = {}_E\mathscr{I}_C^{\alpha\beta} + {}_{GB}\mathscr{I}_C^{\dagger\alpha\beta}\,; \tag{8.2.1}$$

$$\boldsymbol{j}_B^{\dagger\alpha\beta} \to \mathscr{I}_B^{\dagger\alpha\beta} = {}_E\mathscr{I}_B^{\alpha\beta} + {}_{GB}\mathscr{I}_B^{\dagger\alpha\beta}\,; \tag{8.2.2}$$

$$\boldsymbol{i}_1^{\alpha\beta}(\mathfrak{h}_a) \to \mathscr{I}_S^{\alpha\beta}(\mathfrak{h}_a) = {}_E\mathscr{I}_S^{\alpha\beta}(\mathfrak{h}_a) + {}_{GB}\mathscr{I}_S^{\alpha\beta}(\mathfrak{h}_a)\,. \tag{8.2.3}$$

Each of these lines represent a *family* of superpotentials. Now, we unite them into a more general one:

$$\mathscr{I}_D^{\alpha\beta} = \left\{\mathscr{I}_C^{\dagger\alpha\beta};\ \mathscr{I}_B^{\dagger\alpha\beta};\ \mathscr{I}_S^{\alpha\beta}(\mathfrak{h}_a)\right\}\,. \tag{8.2.4}$$

Each of these superpotentials are connected with its own current,

$$\mathscr{I}_D^{\alpha}(\bar{\xi}) = \partial_\beta \mathscr{I}_D^{\alpha\beta}(\bar{\xi}) = \bar{\nabla}_\beta \mathscr{I}_D^{\alpha\beta}(\bar{\xi}) \tag{8.2.5}$$

defined in Section 8.1.3 in (8.1.31), (8.1.34) and (8.1.38), if Killing vectors, $\bar{\xi}^\alpha$, are used.

Following the recommendation for constructing the conserved quantities in general relativity (1.4.24) and based on the differential conservation law (8.2.5), we construct the conserved charges in generalized form in n dimensions:

$$\mathscr{P}(\bar{\xi}) = \int_\Sigma d^{n-1}x\ \mathscr{I}_D^{0}(\bar{\xi}) = \oint_{\partial\Sigma} ds_i\ \mathscr{I}_D^{0i}(\bar{\xi})\,. \tag{8.2.6}$$

The notations for a spacelike section is as usual $\Sigma := t =$ const, and $\partial\Sigma$ is a boundary of Σ. Each of the charges (8.2.6) is related to its own Killing vector in a background spacetime.

Frequently, the definition (8.2.6) is used for calculating the integral charges of isolated systems in the spherically symmetric form :

$$\mathscr{P}(\bar{\xi}) = \oint_{\partial\Sigma} d^{n-2}x\ \mathscr{I}_D^{01}(\bar{\xi}) \tag{8.2.7}$$

with the numeration of the coordinates $x^0 = t$ and $x^1 = r$, which are more convenient at far away surfaces $\partial\Sigma$. The charges (8.2.7) become global charges, if $\partial\Sigma$ are considered at infinity $r \to \infty$.

Each of the families of superpotentials (8.2.1–8.2.3) have the Einstein part and the Gauss-Bonnet correction. In the case of four dimensional general relativity, the GB correction is absent; besides, ${}_E\mathscr{I}_C^{\alpha\beta}$, ${}_E\mathscr{I}_B^{\alpha\beta}$ are not families in this case. Recall that, testing stationary isolated systems in four dimensional general relativity with the use of ${}_E\mathscr{I}_C^{\alpha\beta}$, ${}_E\mathscr{I}_B^{\alpha\beta}$ and ${}_E\mathscr{I}_S^{\alpha\beta}(\mathfrak{h}_a)$, one does not find a preferable one because each of them gives an acceptable result. It was, for example, for asymptotically flat and asymptotically AdS spacetimes at *spatial infinity*.

A more complicated case is an isolated system at *null infinity*. In four dimensional general relativity, both the canonical and Belinfante corrected approaches give the same result coinciding with the standard Bondi-Sachs momentum [61]. Another situation is with the symmetric conserved quantities. A different choice of decompositions (2.2.120) lead to different h^a. Then the variables $\mathfrak{h}_a^{\mu\nu}$ in (8.1.29) differ one from other at the second order in perturbations, see the connection (2.2.122). Of course, this difference is explicitly incorporated in the superpotential ${}_E\mathscr{I}_S^{\alpha\beta}(\mathfrak{h}_a)$. This difference is important in a real calculation for a radiating isolated system [252, 370]. It turns out that only the choice of the metric perturbations

$$h^a = \mathfrak{h}^{\mu\nu} = \mathfrak{g}^{\mu\nu} - \bar{\mathfrak{g}}^{\mu\nu} \tag{8.2.8}$$

gives the standard Bondi-Sachs momentum [61] (not another choice gives this, including (8.1.24)). Considering different variables $\mathfrak{h}_a^{\mu\nu}$ in (8.2.3), we note also that only for (8.2.8) the Einstein parts of the Belinfante corrected and symmetric superpotentials coincide

$${}_E\mathscr{I}_B^{\mu\nu} = {}_E\mathscr{I}_S^{\mu\nu}(\mathfrak{h}), \tag{8.2.9}$$

see (6.2.18)) and (6.3.23)). This fact can be interpreted as a theoretical advantage of the choice (8.2.8).

However, in an arbitrary gravitation theory, including the EGB gravity,

$$\mathscr{I}_B^{\mu\nu} \neq \mathscr{I}_S^{\mu\nu} \tag{8.2.10}$$

even for (8.2.8). The formal reason is that the first one, $\mathscr{I}_B^{\mu\nu}$, is not linear in perturbations in general, whereas the second one, $\mathscr{I}_S^{\mu\nu}$, is linear by definition (8.1.29) in any case. In the next subsections, we consider static solutions of the EGB gravity, to which we use the families of superpotentials constructed in this subsection. Next, we show that all the three approaches give the standard mass of the S-AdS black hole, no pointing to a preferable one. The S-AdS solution is the most well-known solution in the EGB gravity with all the known properties. Only, such a solution can be used as a test. The task for the future is to find any standard solution in the EGB gravity, at least, on the basis of which one could determine a preferable superpotential among the suggested ones.

8.2.2 Superpotentials for static spherically symmetric solutions

Many interesting and important solutions to vacuum equations (8.1.6) in the EGB gravity are of the Schwarzschild form:

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 \sum_{a,b}^{n-2} q_{ab}dx^a dx^b \tag{8.2.11}$$

where, as usual, $x^0 = t$ and $x^1 = r$; $f = f(r)$, the last term describes $(n-2)$-dimensional sphere of the radius r, and q_{ab} depends on the coordinates of the sphere only. The metric-related Christoffel symbols are

$$\Gamma^1{}_{00} = \frac{ff'}{2}, \quad \Gamma^0{}_{10} = \frac{f'}{2f}, \quad \Gamma^1{}_{11} = -\frac{f'}{2f}, \quad \Gamma^a{}_{1b} = \frac{\delta^a_b}{r}, \quad \Gamma^1{}_{ab} = -rfq_{ab}, \tag{8.2.12}$$

where $\partial f/\partial r = f'$. For (8.2.11) the Riemann, Ricci tensors and curvature scalar are

$$\begin{aligned}
R_{0101} &= \frac{1}{2}f'',\\
R_{0a0b} &= \frac{1}{2}rff'q_{ab},\\
R_{1a1b} &= -\frac{rf'}{2f}q_{ab},\\
R_{abcd} &= -r^2(f-1)(q_{ac}q_{bd} - q_{ad}q_{bc});\\
R_{00} &= \frac{f}{2}\left(f'' + f'\frac{n-2}{r}\right),\\
R_{11} &= -\frac{1}{2f}\left(f'' + f'\frac{n-2}{r}\right),\\
R_{ab} &= -\left[(f-1)(n-3) + rf'\right]q_{ab};\\
R &= -\left(f'' + 2f'\frac{n-2}{r} + (f-1)\frac{(n-2)(n-3)}{r^2}\right).
\end{aligned} \tag{8.2.13}$$

As a background we consider again the solution to the equation (8.1.6) of the *same* form:

$$d\bar{s}^2 = -\bar{f}dt^2 + (\bar{f})^{-1}dr^2 + r^2 \sum_{a,b}^{n-2} q_{ab}dx^a dx^b. \tag{8.2.14}$$

We consider the solution (8.2.11) as a perturbed one with respect to (8.2.14). Then evidently all the perturbed quantities can be expressed only with the use of the deflection

$$\Delta f = f - \bar{f}. \tag{8.2.15}$$

For the metrics (8.2.11) and (8.2.14) the relation

$$-g = -\bar{g} = r^{2(n-2)} \det q_{ab} \tag{8.2.16}$$

is valid , and it is important for future calculations.

Here, we present the general formulae, which can be used for calculating the mass of an arbitrary perturbed system (8.2.11) with respect to an arbitrary background

(8.2.14). Due to the symmetry, for calculating conserved quantities one needs only the 01-component of the superpotentials in the general formula (8.2.7). We also note that an arbitrary background of the type (8.2.14) has the Killing vector

$$\bar{\xi}^{\alpha} = \{-1,\ \mathbf{0}\} \tag{8.2.17}$$

that is exactly the displacement vector which is necessary for calculating the total energy (mass) of the system. Because we will keep in mind (8.2.17) below only, we concretize it to the 0-component of the Killing vector, $\bar{\xi}^0$, that is essential only.

At first, we calculate the component $\mathscr{J}_C^{01}$ of the canonical superpotential (8.2.1) with (8.1.17) related to (8.1.8). To calculate it one has to use the expressions (8.2.11–8.2.13) and the corresponding barred expressions. With the definitions (8.1.9) and (8.1.12) one has

$$\begin{aligned}\boldsymbol{i}^{*01}(\bar{\xi}^0) + \kappa^{-1}\bar{\xi}^{[0}\mathscr{D}^{1]}_{DKO} = {} & \frac{\sqrt{-\bar{g}}}{2\kappa r}\left[\frac{r\bar{f}'}{2}\left(\frac{f}{\bar{f}} + \frac{\bar{f}}{f}\right) - (f - \bar{f})(n-2)\right] \\ & + \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^2}(n-2)\left(f - \bar{f}\right)\left(f' - rf''\right) \\ & - \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^3}(n-2)(n-3)(f-1) \times \\ & \times \left[\frac{r\bar{f}'}{2}\left(\frac{f}{\bar{f}} + \frac{\bar{f}}{f}\right) - (f - \bar{f})(n-2)\right].\end{aligned} \tag{8.2.18}$$

We have also used $\sqrt{-g} = \sqrt{-\bar{g}}$, see (8.2.16). The 01-component with the divergence (8.1.13) is calculated in the same way:

$$\begin{aligned}\boldsymbol{i}^{*01}(\bar{\xi}^0) + \kappa^{-1}\bar{\xi}^{[0}\mathscr{D}^{1]}_{KL} = {} & \frac{\sqrt{-\bar{g}}}{2\kappa r}\left[\frac{r\bar{f}'}{2}\left(\frac{f}{\bar{f}} + \frac{\bar{f}}{f}\right) - (f - \bar{f})(n-2)\right] \\ & + \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^2}(n-2)(n-3)(f - \bar{f})f' \\ & - \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^3}(n-2)(n-3)(f-1) \times \\ & \times \left[\frac{r\bar{f}'}{2}\left(\frac{f}{\bar{f}} + \frac{\bar{f}}{f}\right) - (f - \bar{f})(n-4)\right].\end{aligned} \tag{8.2.19}$$

The background expression for both of the cases (8.2.18) and (8.2.19) is the unique one

$$\bar{\boldsymbol{i}}^{*01}(\bar{\xi}^0) = \frac{\sqrt{-\bar{g}}}{2\kappa}\bar{f}' - \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^2}(n-2)(n-3)\bar{f}'(\bar{f}-1)\,. \tag{8.2.20}$$

At last, using (8.2.11–8.2.16) in (8.1.16) one finds

$$\boldsymbol{r}^{*01}(\bar{\xi}^0) = 0\,. \tag{8.2.21}$$

This means that there is no difference in the family (8.1.17) in the 01-component. Thus, combining (8.2.18–8.2.21), one obtains the 01-component of the canonical superpotential (8.2.1), of the DKO type,

$$\mathscr{I}_C^{\dagger 01}(\bar{\xi}^0) = \mathscr{I}_C^{*01}(\bar{\xi}^0) = \mathscr{I}_C^{01}(\bar{\xi}^0) = \boldsymbol{i}^{*01}(\bar{\xi}^0) - \bar{\boldsymbol{i}}^{*01}(\bar{\xi}^0) + \kappa^{-1}\bar{\xi}^{[0}\mathscr{D}^{1]}_{DKO}\,, \tag{8.2.22}$$

and of the KL type,

$$\mathscr{I}_C^{\dagger 01}(\bar{\xi}^0) = \mathscr{I}_C^{*01}(\bar{\xi}^0) = \mathscr{I}_C^{01}(\bar{\xi}^0) = \boldsymbol{i}^{*01}(\bar{\xi}^0) - \bar{\boldsymbol{i}}^{*01}(\bar{\xi}^0) + \kappa^{-1}\bar{\xi}^{[0}\mathscr{D}^{1]}_{KL}\,. \tag{8.2.23}$$

Now we turn to the Belinfante corrected derivation. Our goal is to calculate the component $\mathscr{I}_B^{\dagger 01}$ of the Belinfante corrected family of superpotentials (8.2.2) for the perturbed system (8.2.11) with respect to the background one (8.2.14) and with the displacement vector (8.2.17). Thus, exactly for (8.1.19) one has

$$\begin{aligned}\boldsymbol{i}_B^{*01}(\bar{\xi}^0) = {} & \frac{\sqrt{-\bar{g}}}{2\kappa r}\left[\frac{r\bar{f}'}{2}\left(3\frac{f}{\bar{f}} - \frac{\bar{f}}{f}\right) - rf'\left(1 - \frac{\bar{f}^2}{f^2}\right) - (f - \bar{f})(n-2)\right]\\
& - \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^2}(D-2)\left[\bar{f}\left(f' - rf''\right)\left(1 - \frac{\bar{f}}{f}\right) - ff'\left(1 - \frac{\bar{f}^2}{f^2}\right)(n-3)\right]\\
& + \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^3}(f-1)(n-2)(n-3)\left[\frac{r\bar{f}'}{2}\left(\frac{\bar{f}}{f} - 3\frac{f}{\bar{f}}\right)\right.\\
& \left. + \ (rf' - 2f)\left(1 - \frac{\bar{f}^2}{f^2}\right) + (f - \bar{f})(n-2)\right].\end{aligned} \tag{8.2.24}$$

The barred expression (8.2.24) is

$$\bar{\boldsymbol{i}}_B^{*01}(\bar{\xi}^0) = \frac{\sqrt{-\bar{g}}}{2\kappa}\bar{f}' - \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^2}(n-2)(n-3)\bar{f}'(\bar{f} - 1)\,. \tag{8.2.25}$$

Of course, it is also derived directly from (8.1.20). Using (8.2.11–8.2.16) in (8.1.22) one finds

$$\boldsymbol{r}_B^{*01}(\bar{\xi}^0) = -\frac{4\alpha\sqrt{-\bar{g}}}{\kappa r}(n-2)(f-\bar{f})\left[\frac{1}{2}f'' + \frac{1}{2r}f'(n-4) - \frac{1}{r^2}(f-1)(n-3)\right]\,. \tag{8.2.26}$$

It is evident that

$$\bar{\boldsymbol{r}}_B^{*01}(\bar{\xi}^0) = 0\,. \tag{8.2.27}$$

Now, combining (8.2.24–8.2.27), one obtains the 01-component of the family of the Belinfante corrected superpotentials (8.2.2),

$$\mathscr{I}_B^{\dagger 01}(\bar{\xi}^0) = \boldsymbol{i}^{*01}(\bar{\xi}^0) - \bar{\boldsymbol{i}}_B^{*01}(\bar{\xi}^0) + p\boldsymbol{r}_B^{*01}(\bar{\xi}^0)\,. \tag{8.2.28}$$

At last, we calculate the 01-component of the symmetric superpotential (8.2.3) in the concrete form (8.1.26). Then, for perturbations (8.1.24) of the solution (8.2.11) with respect to the background (8.2.14) has only the non-zero components:

$$\varkappa_{00} = -(f - \bar{f}), \quad \varkappa_{11} = -\frac{(f - \bar{f})}{f\bar{f}}. \tag{8.2.29}$$

With the Killing vector (8.2.17) the 01-component of the superpotential (8.1.26) acquires the form:

$$\begin{aligned}\mathscr{I}_S^{01}(\bar{\xi}^0) = \frac{8}{3}\Big[\partial_1\left(\boldsymbol{w}^{0[0|1]1}\bar{\xi}_0\right) + \bar{\xi}_0\Big(\bar{\Gamma}^0{}_{01}\boldsymbol{w}^{0[0|1]1} + \bar{\Gamma}^1{}_{11}\boldsymbol{w}^{0[0|1]1} \\ + \bar{\Gamma}^1{}_{cd}\boldsymbol{w}^{0[0|c]d}\Big)\Big] - 4\boldsymbol{w}^{0[0|1]1}\bar{\nabla}_1\bar{\xi}_0 - 4\boldsymbol{w}^{1[0|1]0}\bar{\nabla}_0\bar{\xi}_1 .\end{aligned} \tag{8.2.30}$$

It is easy to find that with (8.2.29) for both (Einstein and Gauss-Bonnet) parts in (C.22) one has $w_E^{0[0|1]1} = w_{GB}^{0[0|1]1} = 0$. Due to the symmetry $w^{1[0|1]0} = -w^{0[0|1]1}$ one has $w_E^{1[0|1]0} = w_{GB}^{1[0|1]0} = 0$ as well. Therefore, in (8.2.30) the only non-zero components are

$$\boldsymbol{w}^{0[0|c]d} = -\frac{3\sqrt{-\bar{g}}}{16\kappa}\bar{g}^{cd}\varkappa_{11}\left(1 - 2\alpha(n-3)(n-4)\frac{\bar{f}-1}{r^2}\right). \tag{8.2.31}$$

Finally, with the use of (8.2.11–8.2.17) the component (8.2.30) acquires the form:

$$\mathscr{I}_S^{01} = \frac{\sqrt{-\bar{g}}}{2\kappa r}(n-2)(f-\bar{f})\frac{\bar{f}}{f}\left[-1 + 2\alpha(\bar{f}-1)\frac{(n-3)(n-4)}{r^2}\right]. \tag{8.2.32}$$

Recall that the field-theoretical approach assumes a family of superpotentials (8.1.29) with a different definition of perturbations, as in (2.2.120), or (2.2.121). In reality, for the solutions (8.2.11) and (8.2.14), and the relation (8.2.16), one rewrites the perturbations (2.2.121) as

$$\begin{aligned} g^a &= \bar{g}^a + h^a := \\ \mathfrak{h}_{(m)}^{\mu\nu} &= \left(\sqrt{-\bar{g}}\right)^m (g^{\mu\nu} - \bar{g}^{\mu\nu}), \\ \mathfrak{h}_{(n)\mu\nu} &= \left(\sqrt{-\bar{g}}\right)^n (g_{\mu\nu} - \bar{g}_{\mu\nu}). \end{aligned} \tag{8.2.33}$$

Non-zero components of these are

$$\mathfrak{h}_{(m)}^{00} = \left(\sqrt{-\bar{g}}\right)^m \frac{f-\bar{f}}{f\bar{f}}, \qquad \mathfrak{h}_{(m)}^{11} = \left(\sqrt{-\bar{g}}\right)^m (f-\bar{f}), \tag{8.2.34}$$

$$\mathfrak{h}_{(n)00} = -\left(\sqrt{-\bar{g}}\right)^n (f-\bar{f}), \qquad \mathfrak{h}_{(n)11} = -\left(\sqrt{-\bar{g}}\right)^n \frac{f-\bar{f}}{f\bar{f}}, \tag{8.2.35}$$

Here, m and n are arbitrary real numbers, n is not dimensions. To show the difference between various definitions (8.2.33) we use, of course, (2.2.122),

$$\varkappa_{\mu\nu} = h^a \frac{\partial \bar{g}_{\mu\nu}}{\partial \bar{g}^a} + \frac{1}{2!} h^a h^b \frac{\partial^2 \bar{g}_{\mu\nu}}{\partial \bar{g}^a \partial \bar{g}^b} + \dots . \tag{8.2.36}$$

Thus, defining

$$h^{(a)}_{\mu\nu} \equiv h^a \frac{\partial \bar{g}_{\mu\nu}}{\partial \bar{g}^a} \equiv \varkappa_{\mu\nu} - \frac{1}{2!} h^a h^b \frac{\partial^2 \bar{g}_{\mu\nu}}{\partial \bar{g}^a \partial \bar{g}^b} - \dots \equiv \varkappa_{\mu\nu} + \Delta\varkappa_{\mu\nu} , \tag{8.2.37}$$

one can exchange the 01-component in (8.2.32) of the superpotential $\mathscr{J}_S^{\alpha\beta}(\varkappa)$ with the 01-component of the family of the superpotentials $\mathscr{J}_S^{\alpha\beta}(\varkappa + \Delta\varkappa)$.

8.2.3 Mass of the Schwarzschild-AdS black hole

The S-AdS solution related to the AdS solution

The Schwarzschild-AdS (S-AdS) solution of the vacuum equation (8.1.6) in the EGB gravity (8.1.2) and (8.1.1) has the form (8.2.11) with the metric coefficients

$$g_{00} = -f(r) ; \qquad g_{11} = f^{-1}(r) , \tag{8.2.38}$$

where

$$f(r) = 1 - \frac{r^2 \Lambda'}{(n-2)(n-1)} \left\{ 1 \pm \sqrt{1 - \frac{4\Lambda_0}{\Lambda'} - \frac{2(n-2)(n-1)}{\Lambda'} \frac{\mu}{r^{n-1}}} \right\} , \tag{8.2.39}$$

μ is a constant of integration, and

$$\Lambda' = -\frac{(n-2)(n-1)}{2\alpha(n-4)(n-3)} \tag{8.2.40}$$

is defined *only* by the Gauss-Bonnet term. For the sake of simplicity (to exclude numerous nuances) we restrict ourselves to $n \geq 5$.

One of the important backgrounds is the AdS one, which is defined by (8.2.39) at $\mu = 0$:

$$\bar{f}(r) = 1 - r^2 \frac{2\Lambda_{\text{eff}}}{(n-1)(n-2)} . \tag{8.2.41}$$

The effective cosmological constant:

$$\Lambda_{\text{eff}} = \frac{\Lambda'}{2} \left(1 \pm \sqrt{1 - \frac{4\Lambda_0}{\Lambda'}} \right) \tag{8.2.42}$$

due to (8.2.40) is not positive. The background Riemann, Ricci tensors and the scalar curvature are obtained with the use of (8.2.13) after substituting the barred quantity (8.2.41):

$$\bar{R}_{\mu\alpha\nu\beta} = 2\Lambda_{\text{eff}}\frac{(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\bar{g}_{\nu\alpha})}{(n-2)(n-1)},$$

$$\bar{R}_{\mu\nu} = 2\Lambda_{\text{eff}}\frac{\bar{g}_{\mu\nu}}{n-2},$$

$$\bar{R} = 2\Lambda_{\text{eff}}\frac{n}{n-2}. \tag{8.2.43}$$

The perturbation (8.2.15) acquires the concrete expression:

$$\Delta f(r) = \mp\frac{r^2\Lambda'}{(n-2)(n-1)}\left\{\sqrt{1 - \frac{4\Lambda_0}{\Lambda'} - \frac{2(n-2)(n-1)}{\Lambda'}\frac{\mu}{r^{n-1}}} - \sqrt{1 - \frac{4\Lambda_0}{\Lambda'}}\right\}. \tag{8.2.44}$$

In the linear approximation, asymptotically at $r \to \infty$ it is

$$\Delta f = \pm\left(\sqrt{1 - \frac{4\Lambda_0}{\Lambda'}}\right)^{-1}\frac{\mu}{r^{n-3}}. \tag{8.2.45}$$

We assume $1 - 4\Lambda_0/\Lambda' \neq 0$, the situation $1 - 4\Lambda_0/\Lambda' = 0$ is discussed below.

With a desire to consider the solution (8.2.39) as a black hole solution one has to choose the "−" sign (lower sign) because only then one can define a horizon radius r_+ of the black hole setting $f = 0$. Besides, there are other arguments for this choice. Boulware and Deser [71] have shown that the AdS background with the "+" sign is unstable for the graviton propagation, and, thus, is of less physical interest [87].

There is also a qualitative difference between the cases $n \geq 6$ and $n = 5$. In the first case, $n \geq 6$, vanishing of the constant of integration $\mu \to 0$ corresponds to the vanishing of the horizon $r_+ \to 0$. Thus, in this case μ really can be interpreted as a mass parameter $M = \mu$, and the AdS solution (8.2.41–8.2.43) can be interpreted as a natural background for such black holes. Then the asymptotic perturbations (8.2.44) look quite natural.

In the $n = 5$ case, the situation is different, the horizon $r_+ \to 0$ vanishes if $\mu \to \mu_0$ where

$$\mu_0 = \alpha(n-3)(n-4) = 2\alpha = -6/\Lambda'. \tag{8.2.46}$$

Then one can define a mass parameter as $M = \mu - \mu_0$, for which again $r_+ \to 0$ at $M \to 0$ follows. Thus for such a black hole in 5 dimensions it is natural to choose a vacuum background at $M = 0$ in (8.2.39):

$$\bar{f}(r) = 1 - \frac{r^2\Lambda'}{12}\left\{1 - \sqrt{1 - \frac{4\Lambda_0}{\Lambda'} + \left(\frac{12}{r^2\Lambda'}\right)^2}\right\}. \tag{8.2.47}$$

Only for negative $M = -\mu_0$, one approaches the AdS background with $\bar{f}$ in (8.2.41), μ_0 is called a gap between the AdS spacetime and a real black hole vacuum [87]. Then, for $n = 5$, the perturbation with respect to (8.2.47) is

$$\Delta f(r) = \frac{r^2\Lambda'}{12}\left\{\sqrt{1 - \frac{4\Lambda_0}{\Lambda'} + \left(\frac{12}{r^2\Lambda'}\right)^2 - \frac{24M}{r^4\Lambda'}} - \sqrt{1 - \frac{4\Lambda_0}{\Lambda'} + \left(\frac{12}{r^2\Lambda'}\right)^2}\right\}. \tag{8.2.48}$$

Asymptotically, it is

$$\Delta f(r) = -\frac{M}{r^2}\left(\sqrt{1 - \frac{4\Lambda_0}{\Lambda'}}\right)^{-1}\left[1 + \frac{6}{r^4\Lambda'}\left(M - \frac{12}{\Lambda'}\right)\right]. \tag{8.2.49}$$

Here, the main order coincides with the order of the difference between the different background quantities (8.2.47) and (8.2.41), therefore, here, we preserve the next order.

The mass of the S-AdS black hole calculated by three methods

Now we turn to calculating the mass (total energy) of the S-AdS black hole represented by the solution (8.2.39). At first we consider the canonical prescription and use the component $\mathscr{I}_C^{\dagger 01}$ in the general formula (8.2.7) under the requirement $r \to \infty$. Then it is enough to calculate linear approximation of $\mathscr{I}_C^{\dagger 01}$ (for both of the cases (8.2.22) and (8.2.23)) in the perturbation Δf in (8.2.45) with respect to (8.2.41), that is the AdS background, or in the perturbation Δf in (8.2.49) with respect to (8.2.47), that is on the "mass gap" background. A *unique* expression can be derived because in both of the cases, *only*, the main orders in (8.2.41) and (8.2.45), or in (8.2.47) and (8.2.49), contribute into the integral (8.2.7). Due to this, in the calculations the simple asymptotic relations

$$(\Delta f)' = -(n-3)\frac{\Delta f}{r}, \qquad \bar{f}' = 2\frac{\bar{f}}{r} \tag{8.2.50}$$

are used. Besides, by (8.2.21), one can use $\mathscr{I}_C^{01}$ instead of the general one $\mathscr{I}_C^{\dagger 01}$. Then one has the linear expression

$$\mathscr{I}_C^{01} = -\frac{\sqrt{-\bar{g}}}{2\kappa r}\Delta f(n-2) + \frac{\alpha\sqrt{-\bar{g}}}{\kappa r^3}\Delta f\bar{f}(n-2)(n-3)(n-4), \tag{8.2.51}$$

which is the same for both the cases (8.2.22) and (8.2.23). Substituting the main order from (8.2.41) or (8.2.47) we obtain

$$\mathscr{I}_C^{01} = -\frac{\sqrt{-\bar{g}}}{2\kappa r}\Delta f(n-2)\sqrt{1 - \frac{4\Lambda_0}{\Lambda_{EGB}}}. \tag{8.2.52}$$

Thus, both for (8.2.45), AdS background for $n \geq 6$, and for (8.2.49), "mass gap" background for $n = 5$, we have finally:

$$\mathscr{J}_C^{01} = \frac{\sqrt{-\bar{g}}}{2\kappa r}\frac{M}{r^{n-3}}(n-2). \tag{8.2.53}$$

Substituting it into (8.2.7), taking into account (7.2.2) and

$$\sqrt{-\bar{g}} = r^{n-2}\sqrt{\det q_{ij}} = r^{n-2}\Omega_{n-2}\,, \tag{8.2.54}$$

one obtains for the total energy:

$$\mathscr{P}_C(\bar{\xi}^0) = E_C = \oint_\infty d^{n-2}x\,\mathscr{J}_C^{01} = (n-2)\frac{M}{4G_n} \tag{8.2.55}$$

that is the standard accepted result obtained by using various approaches (see, for example, [344, 348] and references therein). Again recall that both the cases (8.2.22) and (8.2.23) lead to (8.2.55).

Let us turn to the case $n = 5$, but with the AdS background (8.2.41). Really the expression (8.2.52) holds for $n \geq 5$. Thus, for $n = 5$ on AdS background it could be rewritten as

$$\mathscr{J}_C^{01} = \frac{3\sqrt{-\bar{g}}}{2\kappa}\frac{M+\mu_0}{r^3} \tag{8.2.56}$$

instead of (8.2.53). After integration it could be interpreted as the total energy of the system represented by the black hole "summed" with the "mass gap" μ_0 on the AdS background:

$$E_C' = 3\frac{M+\mu_0}{4G_5}\,. \tag{8.2.57}$$

Therefore, one faces a necessity to calculate the energy of the 5D BH in the "mass gap" vacuum, as in (8.2.55).

To calculate the mass of the S-AdS black hole (8.2.39) with the use of the Belinfante corrected expressions (8.2.28). We again consider the linear approximation (8.2.45) and (8.2.49) and follow all the same steps of the previous canonical prescription from (8.2.50) to (8.2.57). Turning to the Belinfante corrected expressions we note that, unlike the canonical case (8.2.21), there exists a difference (8.2.26) in determining a family itself. However, for the solution (8.2.39) with the background (8.2.41), including variations for $n = 5$, one finds, with the use of (8.2.50), that for (8.2.26) $\boldsymbol{r}_B^{*01} \sim 1/r^{n-1}$. Thus $\boldsymbol{r}_B^{*01}$ does not contribute to integral of the type (8.2.55). Finally it turns out that all the formulae and conclusions of the canonical method are repeated exactly. Thus, one has

$$E_B = (n-2)\frac{M}{4G_n}\,, \tag{8.2.58}$$

$$E'_B = 3\frac{M+\mu_0}{4G_5}\,, \tag{8.2.59}$$

repeating (8.2.55) and (8.2.57).

The field-theoretical method requires more attention. Here, it is useful to consider the AdS background with (8.2.41) and arbitrary perturbations h^a in (8.2.33). To construct the symmetric superpotential (8.2.3) one has to use the background expressions (8.2.43) with (8.2.40–8.2.42) and with the coefficients (C.21). Then, after the exchanging (8.1.28), one obtains

$$w_{GB}^{\rho\lambda|\mu\nu}(\mathfrak{h}_a^{\rho\sigma}) = -\left(1 \pm \sqrt{1-\frac{4\Lambda_0}{\Lambda'}}\right) w_E^{\rho\lambda|\mu\nu}(\mathfrak{h}_a^{\rho\sigma}) \tag{8.2.60}$$

where $w_E^{\rho\lambda|\mu\nu}$ is the Einstein part presented in (C.21). Of course, the analogous relation holds for the antisymmetric coefficients in (C.22) by a direct anti-symmetrization of (8.2.60). The substitution of anti-symmetrized (8.2.60) into the expression (8.2.3) with (8.1.29) gives the superpotential related to an arbitrary type of EGB perturbations on the AdS background:

$$\begin{aligned}\mathscr{I}_S^{\mu\rho} &\equiv \frac{1}{\kappa}\left[1_E - \left(1 \pm \sqrt{1-\frac{4\Lambda_0}{\Lambda'}}\right)_{GB}\right] \times \\ &\times \left(\mathfrak{h}_a^{\sigma[\mu}\bar{\nabla}_\sigma\xi^{\rho]} + \xi^{[\mu}\bar{\nabla}_\sigma\mathfrak{h}_a^{\rho]\sigma} - \bar{\nabla}^{[\mu}\mathfrak{h}_a^{\rho]\sigma}\xi_\sigma\right) \\ &\equiv \mp\frac{1}{\kappa}\sqrt{1-\frac{4\Lambda_0}{\Lambda'}}\left(\mathfrak{h}_a^{\sigma[\mu}\bar{\nabla}_\sigma\xi^{\rho]} + \xi^{[\mu}\bar{\nabla}_\sigma\mathfrak{h}_a^{\rho]\sigma} - \bar{\nabla}^{[\mu}\mathfrak{h}_a^{\rho]\sigma}\xi_\sigma\right)\end{aligned} \tag{8.2.61}$$

It is the most general expression for the family of the symmetric superpotentials on AdS backgrounds.

A more concrete expression is obtained with a specification of h^a. We choose, of course, (8.1.24), substitute (8.1.30) into (8.2.61):

$$\begin{aligned}\mathscr{I}_S^{\mu\rho} &\equiv \mp\frac{\sqrt{-\bar{g}}}{\kappa}\sqrt{1-\frac{4\Lambda_0}{\Lambda'}} \times \\ &\times \left(\xi^{[\mu}\bar{\nabla}_\nu\varkappa^{\rho]\nu} - \xi_\nu\bar{\nabla}^{[\mu}\varkappa^{\rho]\nu} - \xi^{[\mu}\bar{\nabla}^{\rho]}\varkappa_\nu{}^\nu - \varkappa^{\nu[\mu}\bar{\nabla}^{\rho]}\xi_\nu - \tfrac{1}{2}\varkappa_\nu{}^\nu\bar{\nabla}^{[\mu}\xi^{\rho]}\right)\,.\end{aligned} \tag{8.2.62}$$

It is expressed through the Abbott-Deser superpotential in the Einstein theory [1], $\mathscr{I}_{AD}^{\mu\rho}$,

$$\mathscr{I}_S^{\mu\rho} = \mp\sqrt{1-4\Lambda_0/\Lambda'}\;\mathscr{I}_{AD}^{\mu\rho}\,, \tag{8.2.63}$$

see [122, 126, 348, 362].

Now, return to a more simple case, perturbed solution (8.2.39) with respect to (8.2.41), or with respect to (8.2.47) in 5 dimensions. Then the perturbations (8.2.29) in the linear approximation are

$$\varkappa_{00} = \varkappa^{11} = -\Delta f \tag{8.2.64}$$

where Δf can be defined both in (8.2.45) and in (8.2.49). To calculate the total energy (mass) of the S-AdS black hole we again follow all the same steps of the canonical and Belinfante corrected approaches with the use of (8.2.32). Again, it turns out that all the related formulae are

$$E_S = (n-2)\frac{M}{4G_n}\,, \tag{8.2.65}$$

$$E'_S = 3\frac{M+\mu_0}{4G_5}\,, \tag{8.2.66}$$

repeating (8.2.55) and (8.2.57), (8.2.58) and (8.2.59). The ambiguity presented in (8.2.33–8.2.37) does not influence (8.2.65) and (8.2.66) because it arises in the second order in perturbations with respect to (8.2.64) that does not contribute into the surface integrals.

Concluding remarks and discussions

The results of the present subsection have been obtained under the assumption that

$$\sqrt{1-\frac{4\Lambda_0}{\Lambda'}} \neq 0\,. \tag{8.2.67}$$

From the beginning we turn to the canonical and the Belinfante correcting formalisms. This factor appears in front of the 01-component of the canonical superpotential (8.2.52), the same is true for the Belinfante corrected superpotential. Thus, for

$$\Lambda' = 4\Lambda_0 \tag{8.2.68}$$

the total mass, probably, has to be treated as vanishing!? In the case of S-AdS solution this problem is resolved automatically because perturbations (8.2.45) on AdS background (8.2.41), or perturbations (8.2.49) even on non-AdS background (8.2.47) in n = 5 dimensions, have this factor in the degree "−1" that compensates (8.2.67). However, what is it in general?

This fact is remarked in the works [87, 118, 122, 348], however without detailed discussions. Deruelle and Morisava [118] (canonical prescription), and Deser, Kanik and Tekin [122] (field-theoretical formulation) have found that not only mass, but angular momentum expressions for the Kerr-AdS solution in EGB gravity have also the same problem if the condition (8.2.68) is valid. Considering this problem, some authors, refer the readers to the papers [94, 95], where the situation is explained that gravitons do not propagate on AdS backgrounds for (8.2.68). But it is not so appropriate. Indeed, in the framework of both the canonical and the Belinfante corrected derivations, this fact does not arise in general: the superpotential expressions in EGB gravity for *arbitrary* (not only static spherically symmetric or rotating symmetric) perturbations linearized around the AdS background *is not* proportional to the factor (8.2.67).

However, the idea of the papers [94, 95] by Chamseddine has a more suitable basis in the framework of the field-theoretical formulation. Of course, the field-theoretical expressions in the cases of static spherically symmetric or rotating symmetric perturbations have the same problems, like in the canonical and the Belinfante corrected derivations. But, in the case of the field-theoretical formulation, this problem is wider: indeed, in the expressions (8.2.62) and (8.2.63) the factor (8.2.67) appears for *all* the components, not only for the 01-component of the superpotentials, and for arbitrary perturbations.

Keeping the above in mind, let us derive the linear expression (7.2.77) that is the left hand side of the gravitational equations (7.2.84) on the vacuum background, like the AdS background. In the case of the EGB gravity (8.1.2) and (8.1.1), and on AdS background (7.2.76), it acquires the form:

$$E^L_{\mu\nu} \equiv \left[1_E - \left(1 \pm \sqrt{1 - \frac{4\Lambda_0}{\Lambda'}} \right)_{GB} \right] G^L_{\mu\nu} \equiv \mp\sqrt{1 - \frac{4\Lambda_0}{\Lambda'}}\, G^L_{\mu\nu}\,, \tag{8.2.69}$$

where the expression $G^L_{\mu\nu}$ is related to the Einstein part and has the form:

$$\begin{aligned} G^L_{\mu\nu} \equiv \frac{1}{2}\Big[& -\bar g^{\rho\sigma}\bar\nabla_{\rho\sigma}\varkappa_{\mu\nu} - \bar\nabla_{\mu\nu}\varkappa_\sigma{}^\sigma + \bar\nabla_{\sigma\nu}\varkappa_\mu{}^\sigma + \bar\nabla_{\sigma\mu}\varkappa_\nu{}^\sigma - \frac{4\Lambda_{\rm eff}}{n-2}\varkappa_{\mu\nu} \\ & + \bar g_{\mu\nu}\left(\bar g^{\rho\sigma}\bar\nabla_{\rho\sigma}\varkappa_\pi{}^\pi - \bar\nabla_{\rho\sigma}\varkappa^{\rho\sigma} + \frac{2\Lambda_{\rm eff}}{n-2}\varkappa_\sigma{}^\sigma \right)\Big]\,. \end{aligned} \tag{8.2.70}$$

Comparing (8.2.69) with (8.2.61) and (8.2.62), we find the same coefficient in front.

This means that the condition (8.2.68) leads to zero coefficient at the linear approximation of the EGB gravity equations around the AdS background. How can this situation be interpreted? The vanishing of the linear left hand side in (7.2.84) leads to the vanishing of the total energy-momentum. Then, if the matter is present the situation can be explained that the matter energy-momentum is compensated by the energy-momentum of the metric perturbations. If the matter is absent then the whole energy-momentum of the pure metric perturbations is to be equal to zero which means the absence of gravitons as has been explained in [94, 95].

Of course, questions connected with the condition (8.2.68) exist. It seems important to connect it to the presented approaches here with that of Regge and Teitelboim [385] in the multi-dimensional application (see, for example, [108]), and with Paddila's prescription [348] for a particular case, where the mass of the degenerate case (8.2.68) is acceptably defined.

Next, all the superpotentials, families of which are presented in (8.2.1–8.2.3), give the unique standard accepted mass (total energy) for the S-AdS black hole in EGB gravity. Thus, in this case we cannot choose a preferable conserved quantity from the families. However, numerous popular gravitational theories have various solutions. It could be useful to find among them the standard solution, like the S-AdS in EGB gravity, on the basis of which one could test the families of superpotentials constructed here.

Besides, numerous solutions in popular gravitational theories frequently have very exotic properties. In the next section, we demonstrate that the suggested families of superpotentials present an appropriate instrument for the interpretation of such solutions. We do not exclude the situation, when an unusual solution to any modified theory of gravity could be a crucial test for a choice among superpotentials.

Already we have remarked that there are infinitely many possibilities to construct conserved quantities in metric theories. In the work [373] Pitts suggested the interesting idea that instead of using many complexes, offers to use an infinite-component object that is conserved which makes sense in every coordinate system/gauge.

8.3 Interpretation of the Maeda–Dadhich exotic solutions

To understand how interesting, and consistent, etc. one or other solution from numerous solutions (frequently exotic) in multidimensional theories of gravity, one needs to examine them in detail. Some of the important characteristics of objects presented by such solutions are energy (mass), its flux, etc. In the present section, we examine exact exotic solutions in the EGB gravity in n dimensions presented in the series of works [114, 304, 305, 320]. They are the d-dimensional Kaluza–Klein type black holes (or isolated objects without horizons) with $(n - d)$-dimensional sub-manifolds. Extra dimensions have the warp-factor proportional to the GB parameter α, which is supposed to be very small. The gravitational equations, which in d dimensions describe these solutions, have a matter source in the right hand side in spite of that the starting equations in n dimensions are vacuum (without sources). This situation is described by the authors of [114, 304, 305] as purely classical example of creating matter by curvature: “matter without matter.”

The idea to create matter/energy by using the curvature effects is not new, it is supported by different approaches and it solves different problems. For example, to make inflation to be possible a pioneering proposal was advanced by Starobinsky [422], in which the high-energy density state was achieved by curved space corrections. Many other problems of modern cosmology may be solved in the framework of the multidimensional gravity by using high-order curvature invariants of Kaluza–Klein type spacetimes, see, e. g., [75] and references there in.

The claim: “matter without matter” requires a more solid foundation than it is given in the original papers [114, 304, 305]. To develop this claim we are sure that it is necessary to calculate the mass and the mass flux by classical methods. It is the main goal of the present section. For the sake of simplicity of presentation we are concentrating on examining the 3 dimensional black hole like objects in six dimensional EGB gravity described in the Molina and Dadhich paper [320]. In spite of an apparent simplicity, these toy objects are sufficiently rich in physical properties, for example, they can have a radiative regime. For calculations we use three types of superpotentials and currents constructed in Section 8.1.

8.3.1 Kaluza–Klein type 3D black holes

Maeda–Dadhich scalar equation

Maeda and Dadhich consider n-dimensional Kaluza–Klein vacuum spacetime as a solution to the equation (8.1.6) in the EGB gravity. Their main assumption is that a spacetime is to be locally homeomorphic to $\mathscr{M}^d \times \mathscr{K}^{n-d}$ with the metric

$$g_{\mu\nu} = g_{AB} \times r_0^2 q_{ab}\,; \quad A, B, \ldots = 0, \cdots, d-1, \quad a, b, \ldots = d, \cdots, n-1\,. \tag{8.3.1}$$

Thus, g_{AB} is an arbitrary Lorentzian metric on $\mathscr{M}^d$, q_{ab} is the unit metric on the $(n-d)$-dimensional space of constant curvature $\mathscr{K}^{n-d}$ with $k = 0, \pm 1$; r_0 plays the role of a small scale of extra dimensions. Thus, unlike the more common Kaluza–Klein approaches, where the extra dimensional sub-manifold is flat with $k = 0$, Maeda and Dadhich explore the cases with $k = \pm 1$. After all the above assumptions the generalized Einstein tensor density $\mathscr{E}_\mu{}^\nu$ derived in (8.1.4) is decomposed as follows

$$\begin{aligned}\mathscr{E}_A{}^B = \Bigg\{ & \left[1 + \frac{2k\alpha}{r_0^2}(n-d)(n-d-1)\right] \overset{d}{G}_A{}^B + \alpha \overset{d}{H}_A{}^B \\ & + \delta_A^B \left[\Lambda_0 - \frac{k}{2r_0^2}(n-d)(n-d-1) \right.\\ & \left. \times \left(1 + \frac{k\alpha}{r_0^2}(n-d-2)(n-d-3)\right)\right]\Bigg\} \sqrt{-\bar{g}_n}\,;\end{aligned} \tag{8.3.2}$$

$$\begin{aligned}\mathscr{E}_a{}^b = \delta_a^b \Bigg\{ & -\frac{1}{2}\left[\overset{d}{R} - 2\Lambda_0 + \frac{k}{r_0^2}(n-d-1)(n-d-2)\right] \\ & - \frac{\alpha k}{r_0^2}(n-d-1)(n-d-2) \\ & \times \left(\overset{d}{R} + \frac{k}{2r_0^2}(n-d-3)(n-d-4)\right) + \alpha \overset{d}{L}_{GB}/2\Bigg\} \sqrt{-\bar{g}_n}\,.\end{aligned} \tag{8.3.3}$$

Here, $\bar{g}_n = \det \bar{g}_{\mu\nu}$; $H_{\mu\nu}$ is the Gauss-Bonnet part (8.1.5) of the expression (8.1.4); index "d" means that a quantity is constructed with the use of g_{AB} only. Thus, keeping in mind (8.3.2) and (8.3.3), the equations (8.1.6) are decomposed into the tensorial equation

$$\mathscr{E}_A{}^B = 0 \tag{8.3.4}$$

on $\mathscr{M}^d$ and a scalar equation

$$\mathscr{E}_a{}^b = 0 \tag{8.3.5}$$

on $\mathscr{M}^d$, which is a constraint for (8.3.4). To obtain more interesting solutions one has to consider a special case studied in [114, 304, 305], when the expression (8.3.2) disappears *identically*, and the equations (8.3.4) are satisfied automatically. This situation is achieved for $d \leq 4$ only, when $\overset{d}{H}_{\mu\nu} \equiv 0$. Then constants are chosen by the way to suppress the coefficients in (8.3.4). This is possible only when $n \geq d + 2$ with $\alpha > 0$, $k = -1$ and $\Lambda_0 < 0$. Then one finds that the governing equation is a single scalar equation (8.3.5) on $\mathscr{M}^d$ only.

To significantly simplify the presentation, we consider the solutions for $n = 6$ and $d = 3$ presented in [320]. The suitable set of constraints for the constants which lead to vanishing (8.3.2) is

$$\frac{1}{r_0^2} = \frac{1}{12\alpha} = -\frac{\Lambda_0}{3}\,. \tag{8.3.6}$$

Then the unique scalar equation (8.1.9) acquires the form:

$$\overset{3}{R} = 2\Lambda_0\,. \tag{8.3.7}$$

The BTZ solution

Before studying the Maeda–Dadhich solutions it is instructive to reconsider the BTZ [20] solution by Bañados, Teitelboim and Zanelli. Their static metric is

$$\begin{aligned} ds^2 &= -fdt^2 + f^{-1}dr^2 + r^2 d\phi\,; \\ f &\equiv -r^2\Lambda_0 - \mu\,, \end{aligned} \tag{8.3.8}$$

which is a solution to the *vacuum Einstein* equations in 3 dimensions. The metric (8.3.8) represents a black hole, horizon of which is defined as usual by the condition $f = 0$ is

$$r_+^2 = -\mu/\Lambda_0\,. \tag{8.3.9}$$

The horizon for such a black hole vanishes when μ vanishes. Therefore the constant of integration μ can be called a mass parameter. Then one can see that the choice of the "black hole vacuum,"

$$\bar{f} = -r^2\Lambda_0, \tag{8.3.10}$$

$$\Delta f = -\mu\,, \tag{8.3.11}$$

as a background is natural. However, such a background is not maximally symmetric, like AdS. The AdS background for the solution (8.3.8) is

$$\bar{f} = 1 - r^2\Lambda_0\,, \tag{8.3.12}$$

$$\Delta f = -\mu - 1\,, \tag{8.3.13}$$

when $\mu = -1$. It is the analog of the massive gap $M = -\mu_0$ for the AdS black hole in 5 dimensions between the AdS spacetime (8.2.41) and a "black hole vacuum" (8.2.49).

Maeda–Dadhich static solution

One of interesting solutions to the equation (8.3.7) is the static solution, the metric of which is

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\phi\,; \qquad f \equiv \frac{r^2}{l^2} + \frac{q}{r} - \mu\,, \tag{8.3.14}$$

where μ and q are the constants of integration, and $l^2 \equiv -3/\Lambda_0$ plays the role of the AdS radius. One easily recognizes that (8.3.14) is the simplest variant of (8.2.11). The nonzero components of the Einstein tensor for the solution (8.3.14) are

$$\overset{3}{G}{}_0{}^0 = \overset{3}{G}{}_1{}^1 = \frac{1}{l^2} - \frac{q}{2r^3}\,, \qquad \overset{3}{G}{}_2{}^2 = \frac{1}{l^2} + \frac{q}{r^3} \tag{8.3.15}$$

and are calculated with the use of general expressions (8.2.13) for the curvature tensor components.

The solution (8.3.14) is more complicated than (8.3.8). Now, the equation for the event horizon, $f = 0$, leads to

$$\frac{r_+^2}{l^2} + \frac{q}{r_+} - \mu = 0 \tag{8.3.16}$$

instead of (8.3.9). For different q, the horizon positions are defined as follows.

First, for $q > 0$,

(a) $q > 2l\,(\mu/3)^{3/2}$ – no horizons,

(b) $q = 2l\,(\mu/3)^{3/2}$ – one horizon $r_+ = l\,(\mu/3)^{1/2}$,

(c) $0 < q < 2l\,(\mu/3)^{3/2}$ – two horizons, for the external horizon one has $l\,(\mu/3)^{1/2} < r_+ < l\,(\mu)^{1/2}$;

Second, for $q = 0$, – one horizon $r_+ = l\,(\mu)^{1/2}$;

Third, $q < 0$, – one horizon $r_+ > l\,(\mu)^{1/2}$.

Objects without horizons are, in fact, naked singularities.

Considering the black hole solutions, it is again natural to choose a mass parameter μ' by the way when the horizon of a black hole disappears under vanishing μ'. Then one finds that $\mu' = \mu - q/r_+$ only. The corresponding "black hole vacuum",

$$\begin{aligned} \bar{f} &\equiv \frac{r^2}{l^2} + \frac{q}{r} - \frac{q}{r_+}; \\ \Delta f &\equiv \frac{q}{r_+} - \mu = -\mu' \end{aligned} \tag{8.3.17}$$

can be chosen as a background, and that is not maximally symmetric either. Moreover, one can see that, unlike the BTZ case, such a background *depends* on the horizon radius r_+. This means that for a concrete value of r_+ its own separate background exists. Although it is a more complicate situation, but it is permissible. Besides, for the solution (8.3.14), like in the BTZ case, one can choose the maximally symmetric AdS background with

$$\begin{aligned} \bar{f} &\equiv 1 + \frac{r^2}{l^2}; \\ \Delta f &\equiv \frac{q}{r} - \mu - 1, \end{aligned} \tag{8.3.18}$$

for which parameter q is also considered as perturbation together with $\mu + 1$. The other advantage (together with maximal symmetries) of the background (8.3.18) is that, it is not related to the horizon. Therefore, it can be successfully used to study naked singularities, including the Maeda–Dadhich's naked singularities.

The radiative Vaidya-type solution

It turns out that the scalar equation (8.3.7) is satisfied not only by the static solutions, but by the radiative Vaidya-type metric with the retarded/advanced time v as well,

$$\begin{aligned} ds^2 &= -fdv^2 + 2dvdr + r^2 d\phi; \\ f &\equiv \frac{r^2}{l^2} + \frac{q(v)}{r} - \mu(v), \end{aligned} \tag{8.3.19}$$

where $\mu(v)$ and $q(v)$ are not constants now, they depend on v. The non-zero Christoffel symbols corresponding (8.3.19) are

$$\begin{aligned} &\overset{3}{\Gamma}{}^1{}_{00} = \frac{ff' - \dot{f}}{2}, \quad \overset{3}{\Gamma}{}^0{}_{00} = \frac{f'}{2}, \quad \overset{3}{\Gamma}{}^1{}_{01} = -\frac{f'}{2}, \\ &\overset{3}{\Gamma}{}^2{}_{12} = \frac{1}{r}, \quad \overset{3}{\Gamma}{}^1{}_{22} = -rf, \quad \overset{3}{\Gamma}{}^0{}_{22} = -r, \end{aligned} \tag{8.3.20}$$

where the "dot" means d/dv. Then, components of Riemann and the Ricci tensors and curvature scalar can be calculated and the non-zero components are

$$
\begin{aligned}
\overset{3}{R}{}^{0101} &= \tfrac{1}{2}f'' , \\
\overset{3}{R}{}^{0212} &= -\frac{f'}{2r^3} , \\
\overset{3}{R}{}^{1212} &= -\frac{1}{2r^3}(ff' + \dot{f}) , \\
\overset{3}{R}{}^{11} &= -\frac{1}{2r}\left[f\left(rf'' + f'\right) + \dot{f}\right] , \\
\overset{3}{R}{}^{01} &= -\frac{1}{2r}\left(rf'' + f'\right) , \\
\overset{3}{R}{}^{22} &= -\frac{f'}{r^3} , \\
\overset{3}{R} &= -\frac{1}{r}\left(rf'' + 2f'\right) .
\end{aligned}
\tag{8.3.21}
$$

The form with upper indices is more simple. The above permits us to calculate the components of Einstein tensor,

$$
\begin{aligned}
\overset{3}{G}{}_0{}^0 = \overset{3}{G}{}_1{}^1 &= \frac{1}{l^2} - \frac{q}{2r^3} , \\
\overset{3}{G}{}_0{}^1 &= \frac{\dot{\mu}r - \dot{q}}{2r^2} , \\
\overset{3}{G}{}_2{}^2 &= \frac{1}{l^2} + \frac{q}{r^3} .
\end{aligned}
\tag{8.3.22}
$$

Note that the usual 3D Einstein gravity does not have a radiating regime, therefore (8.3.22) looks as very interesting. The solution (8.3.19) is connected with the solution of the *form* (8.3.14),

$$
\begin{aligned}
ds^2 &= -f(v,r)dt^2 + f^{-1}(v,r)dr^2 + r^2 d\phi ; \qquad (8.3.23)\\
f(v,r) &\equiv \frac{r^2}{l^2} + \frac{q(v)}{r} - \mu(v) ,
\end{aligned}
$$

e. g., by the transformation

$$
dt = dv - \frac{dr}{f(v,r)} . \tag{8.3.24}
$$

Concentrating on the possibility to form Kaluza–Klein black holes discussed in [114, 304, 305], we consider only the advanced time v in (8.3.23), keeping in mind (8.3.24). If a horizon exists, one can define it for each of constant value v_0 with the corresponding quantities $\mu(v_0)$ and $q(v_0)$ in (8.3.23) analogously to the static case.

"Matter without matter"

Assuming that (8.3.14) and (8.3.19) are solutions to the Einstein's equations on $\mathscr{M}^3$, one concludes that the latter are not vacuum equations. Indeed, both (8.3.15) and (8.3.21) show that a mass/matter source $\overset{3}{T}_{AB}$ with zero trace $\overset{3}{T}{}_A{}^A = 0$ has to exist, and the Einstein's equations corresponding to (8.3.7) could be rewritten as

$$\overset{3}{R}_{AB} - \tfrac{1}{2} g_{AB} \overset{3}{R} + g_{AB}\Lambda = \kappa_3 \overset{3}{T}_{AB} \tag{8.3.25}$$

with redefined cosmological constant $\Lambda = \Lambda_0/3$ and the 3-dimensional Einstein's constant κ_3. It supports the Maeda and Dadhich interpretation [114, 304, 305] that the source in (8.3.25) is created by the compactification of $(n-d)$ extra dimensions. Below we put this claim on a more constructive and physically sensible basis by using the conservation law formalism.

8.3.2 Mass for the static Maeda–Dadhich objects

In the present section, to calculate the mass (total energy) for solutions under consideration we use the general formula (8.2.7) with each of the superpotentials (8.2.4), see (8.2.1–8.2.3), where the Killing vector is defined in (8.2.17). Thus, for calculating the total energy we use

$$M = \lim_{r\to\infty} \oint_{\partial\Sigma} d^{n-2}x \, \mathscr{I}_D^{01}(\bar{\xi}^0) \,, \tag{8.3.26}$$

that is the asymptotic value at spatial infinity (8.2.7).

The BTZ solution

As a reference example, we calculate the mass of the BTZ solution with the metric (8.3.8). Because the BTZ solution is the solution to 3D general relativity we use the *Einstein parts* of each of the superpotentials (8.2.1–8.2.3) and rewrite (8.3.26) as

$$M = \oint_{\infty} d\phi \left({}_E\mathscr{I}_D^{01}(\bar{\xi}^0) \right) . \tag{8.3.27}$$

Considering the Schwarzschild-like dynamical metric (8.3.8) with respect to the background metric of the same form (8.3.8), we can use the general formulae (8.2.22), (8.2.28) and (8.2.32). The linear approximation of the superpotentials in $\Delta f = f - \bar{f}$, that gives a contribution into (8.3.26), is

$${}_E\mathscr{I}_C^{01} = {}_E\mathscr{I}_B^{01} = {}_E\mathscr{I}_S^{01} = -\frac{\sqrt{-\bar{g}_3}}{2\kappa_3 r}\Delta f \,. \tag{8.3.28}$$

We choose a background metric both in the form (8.3.10) and in the form (8.3.12). For each of the cases the formula (8.3.27) gives, respectively,

$$M = \frac{\pi\mu}{\kappa_3} \,, \tag{8.3.29}$$

$$M = \frac{\pi(\mu+1)}{\kappa_3} \,. \tag{8.3.30}$$

The result (8.3.29) is quite acceptable (see, e. g., [119, 159]). Thus, (8.3.29) could be considered as a nice test for all the superpotentials under consideration. The result (8.3.30) represents the total energy including the mass gap related to the AdS background (8.3.12).

The Maeda–Dadhich solution

Now let us turn to the static solution (8.3.14). Because initially it is the solution of the EGB theory with the equation (8.1.6) one has to try to calculate the mass by using the *full formula* (8.2.7) with (8.2.1–8.2.4) for 6D EGB theory. Then the total background metric is to be chosen as

$$\begin{aligned} \bar{g}_{\mu\nu} &= \bar{g}_{AB} \times r_0^2 q_{ab}\,; \\ &A, B, \ldots = 0,\ 1,\ 2, \\ &a, b, \ldots = 3,\ 4,\ 5\,. \end{aligned} \tag{8.3.31}$$

The background metric $\bar{g}_{AB}$ in (8.3.31) can be chosen both in the form (8.3.17) and in the form (8.3.18).

Keeping in mind the 6D background in a non-trivial form (8.3.31), one has to outline the $(n-2)$-dimensional surface integral (8.3.26) in more detail. First, it is the far away surface considered in the $(d = 3)$-dimensional physical spacetime; second, it is integration over the $(n - d = 3)$-dimensional extra space. Then, for all the superpotentials (8.2.1–8.2.4) calculated for the solution (8.3.14) with the backgrounds (8.3.17) or (8.3.18) the mass in 6 dimensions acquires the form:

$$M = \oint_\infty d^{n-2}x\, \mathscr{I}_D^{01} = V_{r_0} \oint_\infty d\phi\, \sqrt{-\bar{g}_d} J_D^{01}\,. \tag{8.3.32}$$

As usual,

$$J_D^{01} = \frac{\mathscr{I}_D^{01}}{\sqrt{-\bar{g}_D}}\,, \tag{8.3.33}$$

and for the factor in front in (8.3.32) one has

$$V_{r_0} = \oint_{r_0} dx^{n-d} \sqrt{g_{n-d}}\,, \tag{8.3.34}$$

where

$$\begin{aligned} \bar{g}_n &= \det \bar{g}_{\mu\nu} = \bar{g}_d \cdot g_{n-d}\,, \\ \bar{g}_d &= \det \bar{g}_{AB} = -r^2\,, \\ g_{n-d} &= \det (r_0^2 q_{ab}). \end{aligned} \tag{8.3.35}$$

Factorization in (8.3.32) has been achieved from the formula (8.3.26) applied to the Maeda–Dadhich model (8.3.1–8.3.3) with the solution (8.3.14) and with a background

(8.3.31) with (8.3.17) or (8.3.18). With the use of (8.3.35) one finds that J_D^{01} depends on variables of the d-sector only. The factor V_{r_0} represents the "volume" of the extra space.

Although J_D^{01} depends on d-sector variables only, it can be rewritten as a sum of the Einstein and Gauss-Bonnet parts. Therefore, we represent (8.3.32) as

$$M = M_E + M_{GB} = V_{r_0} \oint_\infty d\phi\, r \left({}_E J_D^{01}\right) + V_{r_0} \oint_\infty d\phi\, r \left({}_{GB} J_D^{01}\right) \tag{8.3.36}$$

and analyze it. Considering the Einsteinian part of the 01-component of the superpotentials (8.2.1–8.2.4) defined in EGB theory for the solution (8.3.1) with (8.3.14) one finds that they are described only by the d-sector. Therefore for calculating the Einstein parts, the formulae of Section (8.2.2) is enough. Then, asymptotically, for both of the cases, (8.3.17) and (8.3.18), one has

$${}_E J_C^{01} = {}_E J_B^{01} = {}_E J_S^{01} = -\frac{\Delta f}{2\kappa_6 r} \tag{8.3.37}$$

that can be compared with (8.3.28) for the BTZ case. Thus, for all the superpotentials under consideration, the Einstein part in (8.3.36) gives for (8.3.17) and (8.3.18), respectively,

$$M_E = V_{r_0} \frac{\pi \mu'}{\kappa_6}, \tag{8.3.38}$$

$$M_E = V_{r_0} \frac{\pi(\mu + 1)}{\kappa_6} \tag{8.3.39}$$

that can be compared with (8.3.29) and (8.3.30) for the BTZ case; κ_6 is $n = 6$ dimensional Einstein constant.

To derive quantities analogous to (8.3.38) and (8.3.39) for the Gauss-Bonnet part in (8.3.36) one has to find the behaviour for superpotentials, like in (8.3.37). We use the general expressions (8.2.1–8.2.4) in the Gauss-Bonnet part to construct the 01-component for the solution (8.3.14) with the background (8.3.31) with (8.3.17) or (8.3.18). Such a component consists of the two parts, from which the one is pure (d = 3)-dimensional, the other one is defined by the intersecting terms of the (d = 3)-sector and the scalar curvature of the ($n - d$ = 3)-sector:

$$\overset{n-d}{R} = -\frac{6}{r_0^2} = -\frac{1}{2\alpha}. \tag{8.3.40}$$

Thus, variables of the ($n - d$ = 3)-sector are included in a constant form in (8.3.40) only. For all the three types of the GB superpotentials the pure (d = 3)-dimensional part gives a zero contribution to the integral in (8.3.36), whereas the intersecting terms have the main asymptotic approximation

$${}_{GB} J_C^{01} = {}_{GB} J_B^{01} = {}_{GB} J_S^{01} = \frac{\Delta f}{2\kappa_6 r}. \tag{8.3.41}$$

Then, the Gauss-Bonnet part in (8.3.36) gives for (8.3.17) and (8.3.18), respectively,

$$M_{GB} = -V_{r_0} \frac{\pi\mu'}{\kappa_6} , \tag{8.3.42}$$

$$M_{GB} = -V_{r_0} \frac{\pi(\mu + 1)}{\kappa_6} . \tag{8.3.43}$$

Now, summarizing (8.3.38) and (8.3.42), summarizing also (8.3.39) and (8.3.43), one finds that the total mass of the Maeda–Dadhich static solution in 6 dimensions is equal to zero. This result quite coincides with the conclusion in [88] where analogous solutions in the Lovelock gravity are studied.

However, this zero result cannot be considered as physically interesting one. Then it is more suitable to consider the solution (8.3.14) in the framework of 3 dimensional Einstein theory with the equation (8.3.25) where the sources are created by the extra dimensions. One can find out that this system from the point view of the Newtonian limit in 3 dimensions (see, e. g., [159]) has to have a total mass.

Keeping this proposal in mind, recall that the construction of superpotentials does not depend on matter sources. Thus, we do not need a concrete structure of the source in (8.3.25). All of this means that, like in the BTZ case, we have to consider the *Einstein parts* of each of the superpotentials (8.2.22), (8.2.23), (8.2.28) and (8.2.32). Although, the equations (8.3.25) are not vacuum equations, the result is analogous to the BTZ case for "black hole vacuum" background and for the AdS background, respectively,

$$M = \frac{\pi\mu'}{\kappa_3} , \tag{8.3.44}$$

$$M = \frac{\pi(\mu + 1)}{\kappa_3} . \tag{8.3.45}$$

The question arises: what is the role of the 6D EGB derivation in the interpretation of (8.3.44) and (8.3.45)? Suppressing the extra sector in EGB case, one has to consider (8.3.38) and (8.3.39) only. Then, the role of the factor V_{r_0} in (8.3.34) becomes essential. First, if the extra dimensions are not compact, then V_{r_0} diverges and the expressions (8.3.44) and (8.3.45) become meaningless. Second, if the extra dimensions are compact, then V_{r_0} is finite and a comparison (8.3.44) and (8.3.45) with (8.3.38) and (8.3.39) states

$$\kappa_3 = \frac{\kappa_6}{V_{r_0}} . \tag{8.3.46}$$

One easily recognizes in the relation (8.3.46) the main properties of the Kaluza–Klein paradigm in the creation of matter by compact extra dimensions. It is our main argument supporting Maeda and Dadhich developing the Kaluza–Klein description in EGB gravity because κ_3 in (8.3.25) is connected with the EGB spacetime by (8.3.46).

It is important to compare (8.3.44) and (8.3.45) with (8.3.29) and (8.3.30) in the BTZ case, respectively. First, for the natural black hole vacuum background case, M in

(8.3.29) for the BTZ solution is defined by the mass parameter μ only. At the same time, M in (8.3.44) for the Maeda–Dadhich solution is defined by $\mu' = \mu - q/r_+$. It is because the definition of the horizon (8.3.16) includes the parameter q. Second, for the AdS background, both (8.3.30) and (8.3.45) are identical. It seems that in the case (8.3.45), due to (8.3.18), M has to contain q. However, the fall-off of the term q/r in (8.3.18) does not contribute to (8.3.45), like in the Reisner-Nordström solution the total mass does not depend on the electric or magnetic charge, see (6.4.7).

8.3.3 Mass and mass flux for the radiative Maeda–Dadhich objects

Mass

To calculate the total mass for the radiative solution (8.3.19) we carry out calculations analogous to the ones in the previous subsection. Formally, we use the same formula (8.3.26):

$$M = \lim_{r\to\infty} \oint_{\partial\Sigma} d^{n-2}x \, \mathscr{J}_D^{01}(\bar{\xi}^0) . \tag{8.3.47}$$

Only, here, $\partial\Sigma$ is a boundary of the lightlike section Σ defined as $x^0 = v =$ const. Thus, unlike the formula (8.3.26) for static solutions, calculations with the use of (8.3.47) are related to the null infinity. Calculations both with (8.3.26) and with (8.3.47) are illustrated in the Figure 8.1.

The background for the solution (8.3.19) can be chosen in the two important cases. For the first one, the horizon radius r_+ can be calculated for each fixed v_0 as a solution to $f(v_0, r_+) = 0$ in metric (8.3.23). Then, analogously to the static case, we define background and perturbations as

$$\bar{f} \equiv \frac{r^2}{l^2} + \frac{q(v)}{r} - \frac{q(v)}{r_+} ; \tag{8.3.48}$$
$$\Delta f \equiv \frac{q(v)}{r_+} - \mu(v) = -\mu'(v) .$$

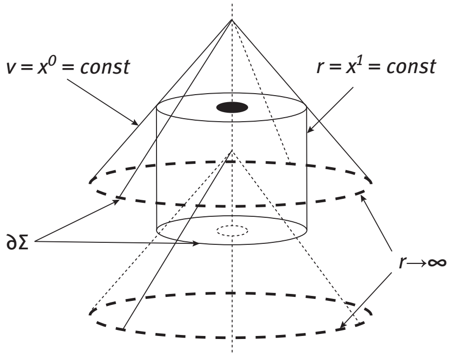

Figure 8.1: A scheme for calculating mass of the Maeda–Dadhich objects with the advanced time v.

It is a non-symmetric and non-static background, compare with (8.3.17). Our calculations for the mass are carried out on the Σ-hypersurfaces, which are just defined for each of *fixed* v_0. Thus, the corresponding background (8.3.48) is related to *this* fixed v_0. The second choice is the AdS background, like in (8.3.18):

$$\bar{f} \equiv 1 + \frac{r^2}{l^2}\,; \qquad \Delta f \equiv \frac{q(v)}{r} - \mu(v) - 1\,. \tag{8.3.49}$$

To calculate the mass for both the cases one has the possibility to use the *unique* timelike Killing vector

$$\bar{\xi}^\alpha = \{-1,\ \mathbf{0}\} \tag{8.3.50}$$

once again. However, unlike (8.2.17), here, the 0-component is related to the v-coordinate.

Again we use the general expressions for the three types of superpotentials (8.2.1–8.2.4), the necessary expressions for the radiative solution (8.3.19–8.3.22), and the Killing vector (8.3.50) with the backgrounds in (8.3.48) and (8.3.49). From the beginning, we derive the Einstein parts of all the superpotentials:

$${}_E\mathscr{I}_C^{01} = {}_E\mathscr{I}_B^{01} = {}_E\mathscr{I}_S^{01} = -\frac{\sqrt{-\bar{g}}}{2\kappa_6 r}\left(f(v,r) - \bar{f}(v,r)\right)\,. \tag{8.3.51}$$

It is surprising, but the simple expressions in (8.3.51) are exact. We do not derive the explicit Gauss-Bonnet parts, but after simple, although lengthy calculations, we derive their asymptotic (now not exact) behaviour

$${}_{GB}\mathscr{I}_C^{01} = {}_{GB}\mathscr{I}_B^{01} = {}_{GB}\mathscr{I}_S^{01} = \frac{\sqrt{-\bar{g}}}{2\kappa_6 r}\Delta f(v,r)\,. \tag{8.3.52}$$

Again, this result is valid due to intersecting (with the scalar curvature (8.3.40) in the extra dimensions) terms in the superpotentials.

Summing (8.3.51) and (8.3.52) one finds a vanishing result for (8.3.47) in 6 dimensions in the framework of the EGB gravity. Thus, one needs to repeat the interpretation of the static case and to reject the pure 6D geometrical derivation as unacceptable. Then, again we consider the equation (8.3.25) as a governing one. Repeating the steps of the previous subsection, for the non-symmetric background (8.3.48) and for the AdS background (8.3.49) one gets

$$M = \frac{\pi\mu'(v)}{\kappa_3}\,; \tag{8.3.53}$$

$$M = \frac{\pi\left(\mu(v) + 1\right)}{\kappa_3}\,. \tag{8.3.54}$$

Both of them are in a correspondence with the static case, (8.3.44) and (8.3.45). Only now M is not a constant: it changes from one Σ_{v_1} to another Σ_{v_2}. Also, because $\mu'(v)$ in (8.3.48) is connected directly with the definition of r_+ for a concrete v = const, r_+ changes with changing $\mu'(v)$. Concluding the part related to calculating mass of the Maeda–Dadhich 3D objects in 6D EGB gravity, we remark, first, that all the three types of superpoterntials, and, second, different divergences, (8.1.12) and (8.1.13), in the canonical formulation, lead to the same results. Thus, at this level we cannot to select a preferable approach.

Mass flux

The mass flux for the radiative Maeda–Dadhich 3D objects defined by the metric (8.3.19) can be obtained simply by differentiating with respect to v:

$$\dot{M} = \frac{\pi\dot{\mu}'(v)}{\kappa_3}\,; \tag{8.3.55}$$

$$\dot{M} = \frac{\pi\dot{\mu}(v)}{\kappa_3}\,. \tag{8.3.56}$$

In the first case (8.3.55), because the mass is related to the horizon, its flux describes the mass change together with changing the horizon from one slice to another,

$$\dot{\mu}'(v) = \dot{\mu}(v) - \frac{d}{dv}\left(\frac{q(v)}{r_+}\right). \tag{8.3.57}$$

In the second case (8.3.56), the mass change is related to a stable AdS space and can be calculated both for black holes and for naked singularities.

To calculate the mass flux for the radiative solution (8.3.19) in the sense of the equation (8.3.25) one can use the direct formula,

$$\dot{M} = \oint_\infty d^{n-2}x\left({}_E\mathscr{I}^1_D\right), \tag{8.3.58}$$

where, instead of the superpotentials, currents are used. Therefore, we turn to the current expressions (8.1.31), (8.1.34) and (8.1.38) in all the three approaches. One of the necessary items is energy-momentum tensor. In the terms of the “matter without matter” Kaluza–Klein approach, it is the right hand side of (8.3.25). The concrete expressions for the energy-momentum, in fact, are represented by the Einstein tensor in (8.3.22). To derive the component ${}_E\mathscr{I}^1_D(\bar{\xi}^0)$ in (8.3.58) one needs the component

$$\overset{3}{T}{}_0{}^1 = \frac{\dot{\mu}r - \dot{q}}{2\kappa_3 r^2} \tag{8.3.59}$$

only. Again returning to the expressions (8.1.31), (8.1.34) and (8.1.38), one finds asymptotically for each of the backgrounds (8.3.48) and (8.3.49), respectively,

$$
\begin{aligned}
{}_E\mathscr{I}^1_C(\bar{\xi}^0) &= {}_E\mathscr{I}^1_B(\bar{\xi}^0) = {}_E\mathscr{I}^1_S(\bar{\xi}^0) \\
&= \sqrt{-\bar{g}_3}\left(\overset{3}{T}{}_0{}^1 - \overset{3}{\bar{T}}{}_0{}^1\right)\bar{\xi}^0 = -\frac{\dot{\mu}'}{2\kappa_3}\,,
\end{aligned} \tag{8.3.60}
$$

$$
\begin{aligned}
{}_E\mathscr{I}^1_C(\bar{\xi}^0) &= {}_E\mathscr{I}^1_B(\bar{\xi}^0) = {}_E\mathscr{I}^1_S(\bar{\xi}^0) \\
&= \sqrt{-\bar{g}_3}\left(\overset{3}{T}{}_0{}^1 - \overset{3}{\bar{T}}{}_0{}^1\right)\bar{\xi}^0 = -\frac{\dot{\mu}}{2\kappa_3}\,.
\end{aligned} \tag{8.3.61}
$$

Substituting these into (8.3.58), one finds

$$\dot{M} = -\frac{\pi\dot{\mu}'}{\kappa_3}\,; \tag{8.3.62}$$

$$\dot{M} = -\frac{\pi\dot{\mu}}{\kappa_3}\,. \tag{8.3.63}$$

One can see that the difference with (8.3.55) and (8.3.56) is in another sign. There is no contradiction. First, a simple differentiation of M with respect to v gives in (8.3.55) and (8.3.56) an absolute value of the flux only, not more. Second, with using the standard expression connecting currents with superpotentials,

$$\mathscr{I}^1_D = \partial_0 \mathscr{I}^{10}_D \tag{8.3.64}$$

and taking into account the antisymmetry for the superpotentials, $\mathscr{I}^{10}_D = -\mathscr{I}^{01}_D$ used in (8.3.47), one recognizes also the correspondence in signs.

The direct calculation of the mass flux with using the three types of the current expressions is an important independent result. Maeda and Dadhich treat the matter represented by the energy-momentum at the right hand side of (8.3.25) as created by all the extra dimensions together. Of course, such a derivation differs from the standard Kaluza–Klein picture, where each of compact extra dimensions determines its own charge. Nevertheless, as we showed in the present section, in the case of *compact* dimensions the Maeda–Dadhich model is reduced to the standard Kaluza–Klein prescription. Also, we demonstrated that the created matter in (8.3.25) determines the *classically* defined mass and mass flux of the objects. Thus, in fact, we support the claim of the authors of [114, 304, 305] that their solutions represent the objects of the Kaluza–Klein type.

9 Generic gravity: Particle content, weak field limits, conserved charges

9.1 Introduction: Raisons d’être of modified gravity theory

In this chapter, we give a detailed account of conserved quantities, such as the total mass and angular momenta of asymptotically constant curvature spacetimes; and work out the particle spectrum etc. of the cosmological Einstein’s theory, quadratic gravity and the more general $f(R_{\mu\nu\sigma\rho})$ theories in their constant curvature backgrounds, namely about their (Anti)-de Sitter, (A)dS, as well flat vacua. The term *vacuum* will be used here to denote any viable maximally symmetric solution of the theory in the absence of sources. We study the Lovelock gravity as an example to our general construction. We give also the weak field limits of massive (with a Fierz-Pauli mass) and higher derivative gravity theories and compute the spin-spin, spin-orbit couplings and discuss various discontinuities arising in the massless limits that constitute a relativistic partner of the well-known van Dam-Veltman-Zakharov (vDVZ) discontinuity inflicting the interaction between static sources.

In the construction of the conserved charges, the procedure that we will adopt is that of Abbott-Deser (AD) [1] who introduced Killing charges in their work on the stability of de Sitter space (dS) in general relativity. Their method, which works for asymptotically constant curvature and flat spacetimes in arbitrary coordinates, is essentially an extension of the ADM method [12] that we discussed previously, which was valid for asymptotically flat spacetimes. In what follows we shall generalize the AD construction to generic gravity theories based on the metric following closely the construction of [124, 126]. For constant curvature backgrounds, higher order terms in the curvature contribute non-trivially to the conserved charges. Therefore, the same metric solving two different theories can have different conserved charges. Hence, unlike the ADM mass case for asymptotically flat spacetimes, the mass for asymptotically constant non-zero curvature spacetimes is not a geometric invariant of the manifold, but a conserved quantity depending on the parameters of the underlying theory.

As the bulk of this chapter will be devoted to the theories beyond general relativity, we have to explain why such modifications are needed at all. Clearly we are living in the era of “effective field theories” where we have learned that all theories we have are valid up to some energy scales where new physics, new degrees of freedom and new symmetries enter into the picture. So from this vantage point, there is a lot of motivation to study the generalizations of Einstein’s gravity with an action symbolically written of the form

$$S = \int d^n x \sqrt{-g} \left(\frac{1}{\kappa} (R - 2\Lambda_0) + \sum_{p=2}^{\infty} a_p \Big(\text{Riem}, \text{Ric}, \text{R}, \nabla\text{Riem}, \dots \Big)^p \right), \tag{9.1.1}$$

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where higher order terms can arise as a result of integrating out massive degrees of freedom in a microscopic theory, such as string theory or one might simply take general terms and build a phenomenology of the resulting theory. This type of higher derivative theories, with propagators that decay faster than $1/p^2$ at large momenta, have much better behavior in the ultraviolet and reduce to general relativity at large distances (in the infrared) bringing in only weak constraints on the couplings a_p from the solar system experiments.

Two of the problems of general relativity arise at large distances: these are the problems of accelerated expansion of the universe (what derives the current accelerated expansion of the universe?) and the rotation curves of stars in spiral galaxies (why stars far away from the bulge of the galaxy rotate around the center with constant speeds?). These large-scale problems require the introduction of dark energy and dark matter to general relativity in amounts which are far more than the visible matter and photons. This might really turn out to be the solution to these problems: namely the theory is not modified but augmented with dark energy and dark matter yet to be detected by other means than gravity. Of course it is somewhat a mystery why most of the energy/matter budget of universe is stuff that do not seem to exist in the standard model.

There is another possibility of solving these large scale problems within pure gravity (albeit modifying general relativity) without introducing dark matter and dark energy: one such route is the recent, in fact resurrected, trend of giving a tiny mass to the graviton. Logically the third possibility could be a combination of both: it is quite possible that there is some amount of dark matter and dark energy (whose nature and amount we still have to find) and also, the graviton has a tiny mass. Even though, we shall not be interested in the phenomenological aspects of large scale gravity *per se*, we shall give a discussion of massive gravity which also will play a role in the discussion of the higher derivative theories to follow in this chapter. This is because generically, once higher curvature terms are added to the Einstein-Hilbert action, besides the massless spin-2 particle of general relativity, new degrees of freedom, which are generically massive, arise. Among these theories, quadratic gravity plays a particularly important role and hence we shall spend a lot of time on figuring out its particle content exactly and construct its conserved quantities. Once quadratic gravity is understood, a large class of generic theories constitutes a rather straightforward generalization of it.

In the other extreme, at high energies or extremely small (microscopic) scales, there is no pressing experimental result that forces us to strictly modify general relativity. Of course, this does not deter us from such attempts because there are some compelling theoretical problems about general relativity in the UV scales: pure general relativity (without matter) is non-renormalizable at the two loop level in perturbation theory, while with matter, the state of affairs is even worse: it is non-renormalizable even at the one-loop level. So as a "quantum theory", general relativity is only reliable at the tree-level, namely at the level of a graviton exchange between sources. The statement that general relativity is non-renormalizable is sometimes misunderstood: it does not mean that the theory cannot be made finite, in fact, it can be made finite, but the theory loses its predictive power at high energies, since one requires infinitely many couplings to be made finite or renormalized. Hence it requires an infinite number of different measurements, or data so general relativity is at best a valid effective theory at low energies. Therefore, one must certainly find another, predictive theory of gravity at high energies.

Admittedly, there is currently no viable higher derivative theory based on the metric and additional fields (supersymmetric or not), even though there is always a hope about maximal supergravity theory being perhaps divergence-free at the several loop level, which in any case is not good enough, the theory should be divergence-free at all loops. Needless to say, when one deviates from the idea that the metric is the microscopic field and accepts that the spacetime, the metric etc. are *emergent* low energy quantities, one can build a renormalizable gravity theory. String theory is the unique example with a perturbatively valid theory of quantum gravity where the metric is not the fundamental field but appears a posteriori. Namely, gravity, as we know it becomes a low energy phenomenon. We will not have much to say on this fine endeavor, as it is out of the scope of the current work. What we are interested here in this chapter is possible deformations of Einstein's gravity in the form (9.1.1) and sometimes in the form $f(R_{\mu\nu\sigma\rho})$ that are better effective theories at high energies compared to general relativity. So, we assume that at high energies gravity is still defined within the context of Riemannian geometry, with perhaps additional scalar fields. As mentioned, among these theories, the quadratic gravity plays a special role as it is actually a renormalizable theory in four dimensions [423], but unfortunately, there is a massive ghost in the spectrum which says that, neither the flat nor the constant curvature vacuum are stable. The theory does not seem to posses a vacuum, which is of course unacceptable. Bartering unitarity with renormalizability is not a good deal for a physical theory, because the theory is a predictive one which is good but it predicts nonsense in the form of negative probabilities or higher than unity probabilities which is bad. Thus, in this sense generic quadratic gravity is not acceptable. One exception could be the 3 dimensional toy model with a special tuning of the quadratic terms which we shall briefly discuss. Less generic gravity theories such as the one with the Lagrangian density $\mathscr{L} = R + \alpha R^2$ can be unitary but not renormalizable, but of course they are not completely useless. For example this particular theory leads to a successful inflationary phase in the early universe that is consistent with the cosmological observations at this stage. The theory is called Starobinsky model and is equivalent to Einstein's gravity coupled to a self-interacting scalar field with a specific interaction potential.

First, we start the tour with the particle content of quadratic gravity in constant curvature backgrounds. The discussion is given in [437] in its full generality, here we shall expound upon some of the details skipped in the paper.

9.1.1 Conventions

In concluding these introductory remarks, let us note that the notation in the current chapter is mostly consistent with the rest of the book. The slight notational differences are introduced as the discussion is developed. One notable difference is that of the definition of a metric perturbation: here we use $h_{\mu\nu}$ defined in (9.2.3) and used in the series of related papers, whereas in the other parts of the book we use $\varkappa_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$.

In Chapters 1–8 and 10, we use the Landau-Lifshitz prescription [285] for the choice of the signs in the action functional

$$S = -\frac{1}{2\kappa}\int d^n x \mathscr{L}^G + \int d^n x \mathscr{L}^M . \tag{9.1.2}$$

For the goals of Chapters 9 (recall, of Section 3.1 also) this choice is not convenient and so we use the prescription of the textbook [315]:

$$S = +\frac{1}{2\kappa}\int d^n x \mathscr{L}^G + \int d^n x \mathscr{L}^M . \tag{9.1.3}$$

Since, below we are more interested in the particle content, non-ghost; non-tachyon nature of the spectrum and, the choice (9.1.3) is better suited. Namely at the non-relativistic limit, it corresponds to the choice

$$S = +\frac{1}{2\kappa}\int dt(K - U) \tag{9.1.4}$$

with K being the kinetic and U being the potential energy.

9.2 Particle spectrum and stability of vacuum in quadratic gravity

As the quadratic gravity theory will serve us as a “template” for more general theories to be discussed, we start with it:

$$I = \int d^n x \sqrt{-g}\left(\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma\chi_{\text{GB}}\right), \tag{9.2.1}$$

where the curvature scalar

$$\chi_{\text{GB}} \equiv R_{\mu\nu\sigma\rho}^2 - 4R_{\mu\nu}^2 + R^2 \tag{9.2.2}$$

is the Gauss–Bonnet combination which vanishes identically in three dimensions and whose variation vanishes identically in four dimensions as it can be written as a total divergence, albeit in a non-covariant way. Therefore, it is really a topological invariant of the four dimensional manifold with a boundary as long as it is a non-diverging quantity. Together with the cosmological constant and the Einstein-Hilbert piece, the Gauss-Bonnet term constitute the first three Lovelock Lagrangians which form topological invariants in their respective dimensions: 0, 2 and 4. Of course, beyond these dimensions, they give non-trivial contributions to the field equations. In four dimensions $\kappa = 16\pi G_N$ with G_N being the bare Newton's constant, but we shall keep the discussion in generic $n \geq 3$ dimensions and sometimes single out the four dimensional case.

In the theory (9.2.1), without the quadratic terms, the story is clear: the theory has a unique vacuum, that is a unique maximally symmetric solution with the cosmological constant Λ_0 which is de Sitter[1] for positive and AdS space for negative values and the flat Minkowski space when it vanishes. Given Λ_0, it is easy to show that there is a massless spin-2 excitation about this vacuum, this is simply the graviton in general relativity. The inclusion of the quadratic terms changes the vacuum structure, the particle content and the conserved charges dramatically as we shall explain below. The computations are straightforward but sometimes rather lengthy, in what follows we will give some details of the computations but the reader is advised to fill out the gaps on his/her own.

9.2.1 Curvature tensors at second order in perturbation theory

As we will need to expand various expressions up to a first order and sometimes up to a second order in perturbation theory, we collect the relevant formulas here. Let us introduce a small parameter ϵ that counts the order, then as an exact expression let us define the metric perturbation $h_{\mu\nu}$ as

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}. \tag{9.2.3}$$

Here $\bar{g}_{\mu\nu}$ is a generic background which need not be maximally symmetric for this section. From this expression, the inverse metric $g^{\mu\nu}$ can be found as

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \epsilon h^{\mu\nu} + \epsilon^2 h^{\mu\rho} h^{\nu}_{\rho} + O\left(\epsilon^3\right). \tag{9.2.4}$$

The trace of the metric perturbation is defined as $h \equiv \bar{g}^{\mu\nu} h_{\mu\nu}$. Up to second order, the Christoffel connection can be found as

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \epsilon \left(\Gamma^{\rho}_{\mu\nu}\right)_L - \epsilon^2 h^{\rho}_{\beta} \left(\Gamma^{\beta}_{\mu\nu}\right)_L + O\left(\epsilon^3\right), \tag{9.2.5}$$

where $\bar{\Gamma}^{\rho}_{\mu\nu}$ is the background metric compatible connection $\bar{\nabla}_{\rho} \bar{g}_{\mu\nu} = 0$ and the linearized connection $\left(\Gamma^{\rho}_{\mu\nu}\right)_L$ is defined to be the first order term as

1 This identification is for the mostly positive signature and dS and AdS are replaced for the mostly negative signature for our sign choice of the Riemann tensor.

$$\left(\Gamma^{\rho}_{\mu\nu}\right)_L \equiv \frac{1}{2}\bar{g}^{\rho\lambda}\left(\bar{\nabla}_\mu h_{\nu\lambda} + \bar{\nabla}_\nu h_{\mu\lambda} - \bar{\nabla}_\lambda h_{\mu\nu}\right). \tag{9.2.6}$$

Here the letter L, be it subscript or superscript, refers to the linearized forms of the corresponding tensors. Now we have to find the Riemann tensor at the second order, for this purpose let us employ the substitution

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \delta\Gamma^{\rho}_{\mu\nu}, \tag{9.2.7}$$

in the definition of the Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^{\mu}_{\sigma\nu} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\sigma\nu} - \rho \leftrightarrow \sigma \tag{9.2.8}$$

which yields

$$R^{\mu}{}_{\nu\rho\sigma} = \bar{R}^{\mu}{}_{\nu\rho\sigma} + \bar{\nabla}_\rho\left(\delta\Gamma^{\mu}_{\sigma\nu}\right) - \bar{\nabla}_\sigma\left(\delta\Gamma^{\mu}_{\rho\nu}\right) + \delta\Gamma^{\mu}_{\rho\lambda}\delta\Gamma^{\lambda}_{\sigma\nu} - \delta\Gamma^{\mu}_{\sigma\lambda}\delta\Gamma^{\lambda}_{\rho\nu}, \tag{9.2.9}$$

where the second order terms are

$$\delta\Gamma^{\rho}_{\mu\nu} = \epsilon\left(\Gamma^{\rho}_{\mu\nu}\right)_L - \epsilon^2 h^{\rho}_{\beta}\left(\Gamma^{\beta}_{\mu\nu}\right)_L. \tag{9.2.10}$$

Therefore, the Riemann tensor up to second order becomes

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} =& \bar{R}^{\mu}{}_{\nu\rho\sigma} + \epsilon\left(R^{\mu}{}_{\nu\rho\sigma}\right)_L - \epsilon^2 h^{\mu}_{\beta}\left(R^{\beta}{}_{\nu\rho\sigma}\right)_L \\ & - \epsilon^2 \bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left[\left(\Gamma^{\gamma}_{\rho\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\rho\nu}\right)_L\right] + O\left(\epsilon^3\right). \end{aligned} \tag{9.2.11}$$

We have at linear order the linearized Riemann tensor:

$$\begin{aligned} \left(R^{\mu}{}_{\nu\rho\sigma}\right)_L \equiv \frac{1}{2}\Big(& \bar{\nabla}_\rho\bar{\nabla}_\sigma h^{\mu}_{\nu} + \bar{\nabla}_\rho\bar{\nabla}_\nu h^{\mu}_{\sigma} - \bar{\nabla}_\rho\bar{\nabla}^{\mu} h_{\sigma\nu} - \bar{\nabla}_\sigma\bar{\nabla}_\rho h^{\mu}_{\nu} \\ & - \bar{\nabla}_\sigma\bar{\nabla}_\nu h^{\mu}_{\rho} + \bar{\nabla}_\sigma\bar{\nabla}^{\mu} h_{\rho\nu}\Big). \end{aligned} \tag{9.2.12}$$

Then the Ricci tensor at second order is

$$\begin{aligned} R_{\nu\sigma} =& \bar{R}_{\nu\sigma} + \epsilon\left(R_{\nu\sigma}\right)_L - \epsilon^2 h^{\mu}_{\beta}\left(R^{\beta}{}_{\nu\mu\sigma}\right)_L \\ & - \epsilon^2 \bar{g}^{\mu\alpha}\bar{g}_{\beta\gamma}\left(\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L\left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L\left(\Gamma^{\beta}_{\mu\nu}\right)_L\right) + O\left(\epsilon^3\right), \end{aligned} \tag{9.2.13}$$

and the scalar curvature at second order is

$$R = \bar{R} + \epsilon R_L + \epsilon^2 \left\{ \bar{R}^{\rho\lambda} h_{\alpha\rho} h^{\alpha}_{\lambda} - h^{\nu\sigma} \left(R_{\nu\sigma}\right)_L - \bar{g}^{\nu\sigma} h^{\mu}_{\beta} \left(R^{\beta}{}_{\nu\mu\sigma}\right)_L \right. \\ \left. - \bar{g}^{\nu\sigma} \bar{g}^{\mu\alpha} \bar{g}_{\beta\gamma} \left[\left(\Gamma^{\gamma}_{\mu\alpha}\right)_L \left(\Gamma^{\beta}_{\sigma\nu}\right)_L - \left(\Gamma^{\gamma}_{\sigma\alpha}\right)_L \left(\Gamma^{\beta}_{\mu\nu}\right)_L \right] \right\}, \tag{9.2.14}$$

where the linearized Ricci tensor and the linearized scalar curvature are defined, respectively, as

$$R^L_{\nu\sigma} \equiv \frac{1}{2} \left(\bar{\nabla}_\mu \bar{\nabla}_\sigma h^\mu_\nu + \bar{\nabla}_\mu \bar{\nabla}_\nu h^\mu_\sigma - \bar{\Box} h_{\sigma\nu} - \bar{\nabla}_\sigma \bar{\nabla}_\nu h \right), \tag{9.2.15}$$

$$R_L = \bar{g}^{\alpha\beta} R^L_{\alpha\beta} - \bar{R}^{\alpha\beta} h_{\alpha\beta}. \tag{9.2.16}$$

The above formulas [211] are valid in a generic background, such as a black hole spacetime, but for the rest of the discussion in this chapter, we shall consider the background to be a maximally symmetric one for which the above expressions are greatly simplified.

9.2.2 Field equations and the vacuum structure

Our first task is to find the field equations coming from the quadratic curvature action, but this is a long exercise which will take us astray from our path. In any case, at the end, one obtains the full source-free equations of quadratic gravity as [126]

$$\frac{1}{\kappa} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} \right) + 2\alpha R \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) \\ + (2\alpha + \beta) \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) R \\ + 2\gamma \left[R R_{\mu\nu} - 2 R_{\mu\sigma\nu\rho} R^{\sigma\rho} + R_{\mu\sigma\rho\tau} R_\nu{}^{\sigma\rho\tau} - 2 R_{\mu\sigma} R_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} \chi_{\rm GB} \right] \\ + \beta \Box \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + 2\beta \left(R_{\mu\sigma\nu\rho} - \frac{1}{4} g_{\mu\nu} R_{\sigma\rho} \right) R^{\sigma\rho} = 0. \tag{9.2.17}$$

It is not directly apparent from this expression, but one can check that in dimensions less than or equal to four, the line coming from the variation of the Gauss-Bonnet term, identically vanishes. For example, this can be shown once the Riemann tensor is decomposed in terms of the Weyl tensor, Ricci tensor and the scalar curvature (in four dimensions)

$$C_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} - g_{\mu[\nu} R_{\beta]\alpha} + g_{\alpha[\nu} R_{\beta]\mu} + \frac{R}{3} g_{\mu[\nu} g_{\beta]\alpha}, \tag{9.2.18}$$

and the four dimensional identity for the Weyl tensor

$$C_{\mu\alpha\beta\sigma} C_\nu{}^{\alpha\beta\sigma} = \frac{1}{4} g_{\mu\nu} C_{\rho\alpha\beta\sigma} C^{\rho\alpha\beta\sigma} \tag{9.2.19}$$

is employed. Note that beyond four dimensions this identity is not valid for generic spacetimes.

Let us now, first, carry out the easiest task and find the maximally symmetric solutions: let $\bar{g}_{\mu\nu}$ denote such a solution with the Riemann, Ricci tensors and the scalar curvature defined, respectively, as

$$\bar{R}_{\mu\rho\nu\sigma} = \frac{2\Lambda}{(n-1)(n-2)}\left(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\rho\nu}\right), \qquad \bar{R}_{\mu\nu} = \frac{2\Lambda}{n-2}\bar{g}_{\mu\nu}, \qquad \bar{R} = \frac{2n\Lambda}{n-2}. \tag{9.2.20}$$

The computations are going to be valid for all $n \geq 3$ dimensions. Plugging these definitions to the field equations (9.2.17), one arrives at a quadratic equation

$$\frac{\Lambda - \Lambda_0}{2\kappa} + k\Lambda^2 = 0, \qquad k \equiv (n\alpha + \beta)\,\frac{(n-4)}{(n-2)^2} + \gamma\frac{(n-3)\,(n-4)}{(n-1)\,(n-2)}. \tag{9.2.21}$$

Clearly $n = 4$ is special: in four dimensions $k = 0$ and there is a unique solution. But generically in n dimensions there are two solutions with effective cosmological constants given as

$$\Lambda_\pm = \frac{-1 \pm \sqrt{1 + 8k\kappa\Lambda_0}}{2k\kappa}. \tag{9.2.22}$$

Several remarks are apt: first, of all this non-uniqueness of vacuum is common to almost all modified gravity theories with higher powers of the curvature tensors, yet it is somewhat troubling in gravity. This is because, there is no natural choice as to which one is the true vacuum. One cannot compare them energetically, as both (for quadratic gravity) of them will be assigned a zero total energy by definition and by construction. Also, unlike the case of quantum mechanics, one cannot talk about a transition/tunneling between these vacua as the transition must change the structure at infinity (the asymptotic structure) which seems unlikely in a finite amount of time. Hence as a general remark: if these multiple vacua theories in the context of gravity are to make sense, one must find a way of selecting the true vacuum of the theory. At this stage there is no such procedure.

Leaving this complicated problem aside, let us note several observations: even for $\Lambda_0 = 0$, there is a non-flat vacuum with $\Lambda_- = -1/k\kappa$ in addition to the flat one. Secondly, one must have $8k\kappa\Lambda_0 \geq -1$ to have a maximally symmetric solution. When the bound is saturated, the two roots coalesce and for $k = 0$, one has a unique vacuum. An interesting question is what would happen if the condition $8k\kappa\Lambda_0 \geq -1$ is not satisfied? This is not clear, it is quite possible that a less symmetric solution (that does not carry as many Killing vectors as the maximal number $n(n+1)/2$), becomes the vacuum (a type of symmetry breaking takes place) or the theory may not have a vacuum and so it would be physically irrelevant. At this stage, we do not know what would be the less symmetric vacuum: it is plausible that some kind of spontaneous comptactification takes place and a vacuum of a product type in the $AdS_p \times S_{n-p}$ appears as suggested in [249]. For our current purposes, we shall not worry about this and we shall assume that there is always at least a single viable vacuum, namely a maximally symmetric solution. In what follows this maximally symmetric solution of interest will be denoted simply as Λ and we shall carry out "perturbation" theory about this vacuum which is our next task.

9.2.3 Linearization of quadratic gravity

Let us consider a generic perturbation about one of the maximally symmetric vacua, after setting $\epsilon = 1$, as

$$h_{\mu\nu} \equiv g_{\mu\nu} - \bar{g}_{\mu\nu}. \tag{9.2.23}$$

We assume that the full metric $g_{\mu\nu}$ asymptotically approaches to the background metric $\bar{g}_{\mu\nu}$, which of course puts a constraint on the perturbation. At this stage, all we need to assume is that the perturbation satisfies the required fall off conditions. For example, these fall off conditions can be dictated by the finiteness of the conserved charges for black hole type spacetimes. There is also the interesting issue of how to guarantee the smallness of the components of a tensor quantity as it changes under coordinate transformations. Of course the tacit assumption here is that there is a set of coordinates where $|h_{\mu\nu}|$ is small in the asymptotic region. Note that in the computations of the charges to follow, the perturbation need not be small in the bulk of the spacetime, as in the case for black holes.

We can now move to the linearized form of the field equations (9.2.17), which, after dropping all the terms but the $\mathcal{O}(h)$ ones, reads [126]

$$\begin{aligned} c\,\mathcal{G}^L_{\mu\nu} &+ (2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{n-2}\bar{g}_{\mu\nu}\right)R^L \\ &+ \beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}_{\mu\nu}R^L\right) = 0, \end{aligned} \tag{9.2.24}$$

where the constant c in-front of the linearized Einstein tensor is found to be

$$c \equiv \frac{1}{\kappa} + \frac{4\Lambda n}{n-2}\alpha + \frac{4\Lambda}{n-1}\beta + \frac{4\Lambda\,(n-3)\,(n-4)}{(n-1)\,(n-2)}\gamma. \tag{9.2.25}$$

All the raising, lowering and covariant derivatives are defined with respect to the background metric and the connection defined by it. For example the full non-linear cosmological Einstein tensor is

$$\mathcal{G}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}, \tag{9.2.26}$$

whose linearized form is the linearized Einstein tensor :

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^L - \frac{2\Lambda}{n-2}h_{\mu\nu}, \tag{9.2.27}$$

which is made up of the linearized Ricci tensor $R^L_{\mu\nu}$ and the linearized scalar curvature $R^L = \left(g^{\mu\nu}R_{\mu\nu}\right)^L$ given respectively as

$$R^L_{\mu\nu} = \frac{1}{2}\left(\bar{\nabla}^\sigma\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}^\sigma\bar{\nabla}_\nu h_{\mu\sigma} - \bar{\Box}h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h\right), \tag{9.2.28}$$

$$R^L = -\bar{\Box}h + \bar{\nabla}^\sigma\bar{\nabla}^\mu h_{\sigma\mu} - \frac{2\Lambda}{n-2}h. \tag{9.2.29}$$

The full Bianchi identity $\nabla_\mu\mathscr{G}^{\mu\nu} = 0$ gives rise to the linearized (or background) Bianchi identity $\bar{\nabla}_\mu\mathscr{G}^{\mu\nu}_L = 0$, which also reflects the fact that the left-over symmetry from the full-diffeomorphism group of the theory is merely the background differomorphisms of the form

$$\delta_\xi h_{\mu\nu} = \bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu, \tag{9.2.30}$$

with ξ^μ, an arbitrary function of spacetime, but it must keep the perturbation small at infinity. This restricts the allowed background diffeomorphisms.

There is an important remark about the constant c that appears in-front of the linearized Einstein term in the linearized form of the field equations (9.2.24): by the looks of it, it appears as it is the effective Newton's constant of the theory which can be defined in the weak field limit. But, this is a red-herring: there will be a further term that will contribute to c and the combination will become the effective Newton's constant, a fact which might appear to be purely a nomenclature issue at this stage of the discussion. But it is not, correct identification of the degrees of freedom and the unitarity regions as well as finding specific theories, require a proper identification of the effective Newton's constant: namely, the one measured in the experiments

Finally, we should note that there is some computation that one must carry out to arrive at (9.2.24) from the full non-linear equations. For this purpose the following linearizations given in appendix of [126] are needed

$$\left(R_{\mu\rho\nu\sigma}R^{\rho\sigma}\right)^L = \frac{2\Lambda}{n-1}\left(R^L_{\mu\nu} + \frac{1}{n-2}\bar{g}_{\mu\nu}R_L + \frac{2\Lambda}{(n-2)^2}h_{\mu\nu}\right), \tag{9.2.31}$$

$$\left(R_{\mu\rho\sigma\alpha}R_\nu{}^{\rho\sigma\alpha}\right)^L = \frac{8\Lambda}{(n-1)(n-2)}\left(R^L_{\mu\nu} - \frac{\Lambda}{n-2}h_{\mu\nu}\right), \tag{9.2.32}$$

$$\left(R_{\mu\rho\sigma\alpha}R^{\mu\rho\sigma\alpha}\right)^L = \frac{8\Lambda}{(n-1)(n-2)}R_L, \tag{9.2.33}$$

$$\left(R^\sigma_\mu R_{\nu\sigma}\right)^L = \frac{4\Lambda}{n-2}\left(R^L_{\mu\nu} - \frac{\Lambda}{n-2}h_{\mu\nu}\right), \tag{9.2.34}$$

$$\left(R_{\mu\nu}R^{\mu\nu}\right)^L = \frac{4\Lambda}{n-2}R_L, \tag{9.2.35}$$

$$\left(R^2_{\tau\lambda\rho\sigma} - 4R^2_{\sigma\rho} + R^2\right)^L = \frac{4\Lambda(n-3)}{n-1}R_L, \tag{9.2.36}$$

$$R^L_{\mu\sigma\nu\rho}\bar{g}^{\sigma\rho} = R^L_{\mu\nu} - \frac{2\Lambda}{(n-1)(n-2)}\left(h_{\mu\nu} - \bar{g}_{\mu\nu}h\right), \tag{9.2.37}$$

which follow easily for the (A)dS backgrounds. Let us also note that the GB density for the background spacetime reads

$$\bar{R}^2_{\tau\lambda\rho\sigma} - 4\bar{R}^2_{\sigma\rho} + \bar{R}^2 = \frac{4n\Lambda^2(n-3)}{(n-2)(n-1)}. \tag{9.2.38}$$

Before we move on, let us make a small digression and as a useful exercise, directly prove the background covariant conservation of the linearized Einstein tensor.

9.2.4 Explicit check of linearized Bianchi identity

Let us show that the background Bianchi identity is satisfied for the linearized Einstein tensor : $\bar{\nabla}^\mu \mathscr{G}^L_{\mu\nu} = 0$. So we need to compute the following explicitly

$$\bar{\nabla}^\mu \mathscr{G}^L_{\mu\nu} = \bar{\nabla}^\mu \left(R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^L - \frac{2}{n-2}\Lambda h_{\mu\nu} \right) \tag{9.2.39}$$

$$\begin{aligned} =& \frac{1}{2}\bar{\nabla}^\mu \left(-\bar{\Box} h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + \bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\sigma\mu} + \bar{\nabla}^\sigma \bar{\nabla}_\mu h_{\sigma\nu} \right) \\ & - \frac{1}{2}\bar{\nabla}_\nu \left(-\bar{\Box} h + \bar{\nabla}_\rho \bar{\nabla}_\sigma h^{\rho\sigma} - \frac{2\Lambda}{n-2} h \right) - \frac{2\Lambda}{n-2}\bar{\nabla}^\mu h_{\mu\nu}. \end{aligned} \tag{9.2.40}$$

One of the terms we have to handle reads

$$\bar{\nabla}^\mu \bar{\nabla}^\sigma \bar{\nabla}_\mu h_{\sigma\nu} = \left[\bar{\nabla}^\mu, \bar{\nabla}^\sigma \right] \bar{\nabla}_\mu h_{\sigma\nu} + \bar{\nabla}^\sigma \bar{\nabla}^\mu \bar{\nabla}_\mu h_{\sigma\nu}, \tag{9.2.41}$$

which after making use of the definition of the Riemann tensor and its background form becomes

$$\begin{aligned} \bar{\nabla}^\mu \bar{\nabla}^\sigma \bar{\nabla}_\mu h_{\sigma\nu} =& \bar{R}^{\mu\sigma}{}_\mu{}^\lambda \bar{\nabla}_\lambda h_{\sigma\nu} + \bar{R}^{\mu\sigma}{}_\sigma{}^\lambda \bar{\nabla}_\mu h_{\lambda\nu} + \bar{R}^{\mu\sigma}{}_\nu{}^\lambda \bar{\nabla}_\mu h_{\sigma\lambda} \\ & + \bar{\nabla}^\sigma \bar{\Box} h_{\sigma\nu} \\ =& \bar{R}^{\sigma\lambda} \bar{\nabla}_\lambda h_{\sigma\nu} - \bar{R}^{\mu\lambda} \bar{\nabla}_\mu h_{\lambda\nu} \\ & + \frac{2\Lambda}{(n-1)(n-2)} \left(\delta^\mu_\nu \bar{g}^{\sigma\lambda} - \bar{g}^{\mu\lambda}\delta^\sigma_\nu \right) \bar{\nabla}_\mu h_{\sigma\lambda} + \bar{\nabla}^\sigma \bar{\Box} h_{\sigma\nu} \\ =& \frac{2\Lambda}{(n-1)(n-2)} \left(\bar{\nabla}_\nu h - \bar{\nabla}^\lambda h_{\nu\lambda} \right) + \bar{\nabla}^\sigma \bar{\Box} h_{\sigma\nu}. \end{aligned} \tag{9.2.42}$$

Making use of these one can write $\bar{\nabla}^\mu R^L_{\mu\nu}$ as

$$\bar{\nabla}^\mu R^L_{\mu\nu} = \frac{1}{2}\left[-\bar{\nabla}^\mu \bar{\nabla}_\nu \bar{\nabla}_\mu h + \bar{\nabla}^\mu \bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\sigma\mu} + \frac{2\Lambda}{(n-1)(n-2)} \left(\bar{\nabla}_\nu h - \bar{\nabla}^\lambda h_{\nu\lambda} \right) \right], \tag{9.2.43}$$

where

$$\bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\mu\sigma} = \bar{\nabla}_\nu \bar{\nabla}^\sigma h_{\mu\sigma} + \left[\bar{\nabla}_\sigma, \bar{\nabla}_\nu\right] h^\sigma_\mu$$

$$= \bar{\nabla}_\nu \bar{\nabla}^\sigma h_{\mu\sigma} + \bar{R}_{\sigma\nu}{}^{\sigma}{}_{\lambda} h^\lambda_\mu + \bar{R}_{\sigma\nu\mu}{}^{\lambda} h^\sigma_\lambda \tag{9.2.44}$$

$$= \bar{\nabla}_\nu \bar{\nabla}^\sigma h_{\mu\sigma} + \frac{2\Lambda}{(n-1)(n-2)} \left(n h_{\mu\nu} - \bar{g}_{\mu\nu} h\right), \tag{9.2.45}$$

and similarly we also have

$$\bar{\nabla}^\mu \bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\mu\sigma} = \bar{\nabla}^\mu \bar{\nabla}_\nu \bar{\nabla}^\sigma h_{\mu\sigma} + \frac{2\Lambda}{(n-1)(n-2)} \bar{\nabla}^\mu \left(n h_{\mu\nu} - \bar{g}_{\mu\nu} h\right)$$

$$= \bar{\nabla}_\nu \bar{\nabla}^\mu \bar{\nabla}^\sigma h_{\mu\sigma} + \frac{2\Lambda}{(n-1)(n-2)} \left[(2n-1)\bar{\nabla}^\mu h_{\mu\nu} - \bar{\nabla}_\nu h\right], \tag{9.2.46}$$

and finally

$$\bar{\nabla}^\mu \bar{\nabla}_\nu \bar{\nabla}_\mu h = \left[\bar{\nabla}^\mu, \bar{\nabla}_\nu\right] \bar{\nabla}_\mu h + \bar{\nabla}_\nu \bar{\nabla}^\mu \bar{\nabla}_\mu h$$

$$= \bar{R}^\mu{}_{\nu\mu}{}^{\lambda} \bar{\nabla}_\lambda h + \bar{\nabla}_\nu \bar{\Box} h = \frac{2\Lambda}{n-2} \bar{\nabla}_\nu h + \bar{\nabla}_\nu \bar{\Box} h. \tag{9.2.47}$$

Employing these results in $\bar{\nabla}^\mu R^L_{\mu\nu}$, one arrives at

$$\bar{\nabla}^\mu R^L_{\mu\nu} = \frac{1}{2} \bar{\nabla}_\nu \left(-\bar{\Box} h + \bar{\nabla}^\mu \bar{\nabla}^\sigma h_{\mu\sigma} - \frac{2\Lambda}{n-2} h\right) + \frac{2\Lambda}{n-2} \bar{\nabla}^\mu h_{\mu\nu}, \tag{9.2.48}$$

and putting this result in (9.2.40), leads to the linearized Bianchi identity $\bar{\nabla}^\mu \mathcal{G}^L_{\mu\nu} = 0$. Of course, as the alert reader might have realized, this has been a long detour but it was a useful exercise that also verified our formulas. The easiest way of getting the linearized Bianchi identity (of any divergence-free rank two tensor, not necessarily the Einstein tensor) would simply be to linearize the full Bianchi identity as

$$\nabla^\mu \mathcal{G}_{\mu\nu} = \bar{\nabla}^\mu \bar{\mathcal{G}}_{\mu\nu} + \bar{\nabla}^\mu \mathcal{G}^L_{\mu\nu} + (\nabla^\mu)_L \bar{\mathcal{G}}_{\mu\nu} + \mathcal{O}(h^2) = 0. \tag{9.2.49}$$

The first and the third terms are zero and hence one has the linearized Bianchi identity at this order.

9.2.5 Degrees of freedom of quadratic gravity in AdS

After this intermission, let us get back to the linearized equations: it is clear that generically (9.2.24) is a fourth-order linear equation which can be factored into a product of two wave equations and all the information regarding the particle-content of the theory is contained in that equation. But to properly identify the modes (the relativistic oscillators or the normal modes, in some sense) we need to decouple the equation by choosing a better phase-space "coordinate" than $h_{\mu\nu}$. Clearly there are also spurious (gauge) modes which should be eliminated. For this purpose the above mentioned background diffeomorphisms can be used to fix the gauge, but we shall work without a choice of gauge and employ the so called auxiliary field method to identify the true degrees of freedom and their masses. As we noted above, this computation was carried out in [437] which we closely follow.

Before we indulge into the detailed discussion, a brief note on the previous literature about the particle-content of quadratic gravity is apt here. The theory itself is only a few years younger than general relativity. But the "older" literature was mostly in the context of classical physics. We take the liberty of loosely starting the modern work on this with [423] in which for flat spacetime, in four dimensions, 8 degrees of freedom were identified [424]. When one deviates from the flat background, the cosmological constant of AdS or dS also plays a role in the masses of the particles and shifts their numerical values. Namely, the scattering of the graviton (or any other particle) with the background appears as part of the mass of the graviton (or the particle at hand). In cosmological Einstein's theory, the massless graviton survives this fate and remains massless, but for quadratic gravity this is not the case. So strictly speaking, it is not only the cosmological constant that shifts the mass here, the cosmological constant conspires with the quadratic curvature terms and they together shift the mass of gravitons from their bare values in the Lagrangian.

More recently, in (A)dS, [216], the scattering amplitude between two covariantly-conserved sources was computed at the tree-level (one graviton exchange) in a slightly more general theory of quadratic gravity, that is quadratic gravity deformed with the gauge-non invariant Fierz-Pauli mass term. We shall give this computation in the discussion below. From the poles of the scattering amplitude, in principle one can read the masses, but not all poles correspond to physical particles. In [39], for $n = 3$, a specific combination of the quadratic terms ($8\alpha + 3\beta = 0$) was considered in the auxiliary formalism which has a massive spin-2 excitation and the resulting theory is called the "New Massive Gravity" (NMG). This is an interesting theory that describes a nonlinear massive gravity with 2 degrees of freedom in the perhaps simplest way possible albeit only in 3 dimensions. In [121], a truncated version of NMG which does not have the Einstein-Hilbert part was considered and the resulting theory has a massless spin-2 excitation. This theory, called K-gravity, has rather remarkable properties such as asymptotically flat black holes (black flowers with non-spherical horizons), a hitherto unnoticed sector in 3 dimensions [4, 30], since all the previously found black holes were asymptotically AdS. In 3 dimensions, for generic values of the parameters, the masses of the excitations were found in [210]. In n dimensional AdS backgrounds, when the parameters of the quadratic theory are tuned in such a way that all massive modes decouple, one ends up with the "Critical Gravity" [123, 300] with only an apparent massless spin-2 particle. All these sub-cases follow from the generic expressions reproduced below.

We can write the linearized Einstein tensor as

$$\mathcal{G}^L_{\mu\nu} \equiv \mathcal{D}\,(\bar{g})_{\mu\nu\sigma\rho}\, h^{\sigma\rho}, \tag{9.2.50}$$

where the Hermitian operator $\mathcal{D}\,(\bar{g})_{\mu\nu\sigma\rho}$ reads

$$\begin{aligned}\mathcal{D}\,(\bar{g})_{\mu\nu\alpha\beta} =& \frac{1}{2}\left(\bar{\Box} + \frac{2\Lambda}{n-2}\right)\left(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\right) - \frac{1}{2}\left(\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\bar{\nabla}_\beta + \bar{g}_{\alpha\beta}\bar{\nabla}_\mu\bar{\nabla}_\nu\right)\\ &+ \frac{1}{2}\left(\bar{g}_{\mu\alpha}\bar{\nabla}_\beta\bar{\nabla}_\nu + \bar{g}_{\nu\beta}\bar{\nabla}_\alpha\bar{\nabla}_\mu\right).\end{aligned} \tag{9.2.51}$$

Directly extending the quadratic gravity action up to $\mathcal{O}(h^2)$ is a very cumbersome task and moreover it is a rather long exercise to put the final result in an explicitly gauge invariant form (up to a boundary term). Therefore, the best way to proceed is to use the "inverse"calculus of variations and construct the action that yields the linearized field equations (9.2.24). Let that action be $I(h^2) = \int d^n x \sqrt{-\bar{g}}\mathcal{L}_2$. Then the second order Lagrangian is obtained by multiplying the linearized field equations by $-\frac{1}{2}h^{\mu\nu}$ and integrating the result over the background spacetime to arrive at the second order Lagrangian density after dropping the boundary terms

$$\mathcal{L}_2 = -\frac{1}{2}\left(c + \frac{4\Lambda\beta}{(n-1)(n-2)}\right)h^{\mu\nu}\mathcal{G}^L_{\mu\nu} + \beta\mathcal{G}^L_{\mu\nu}\mathcal{G}^{\mu\nu}_L + \left(\alpha + \frac{\beta(4-n)}{4}\right)R_L^2. \tag{9.2.52}$$

This is what one would call the free (non-interacting) version of the quadratic gravity around a maximally symmetric background. Namely, this free theory has all the perturbative degrees of freedom albeit in a coupled form plus the gauge modes. The minus sign in front of the Einsteinian piece is important as it is chosen to give the correct (positive) kinetic energy for the massless spin-2 graviton. Or equivalently, if we couple the theory to matter, that is the correct sign, from which we can also identify the effective Newton's constant as

$$\frac{1}{\kappa_{\text{eff}}} \equiv \frac{1}{\kappa} + \frac{4\Lambda(n\alpha+\beta)}{n-2} + \frac{4\Lambda\,(n-3)\,(n-4)}{(n-1)\,(n-2)}\gamma, \tag{9.2.53}$$

which incorporates the earlier-noted shift from the constant c. The numerical factor 1/2 is clear: a factor of 2 will arise in the Euler-Lagrange equation due to the Hermitian property of the operator (9.2.51). In what follows we shall make frequent use of integration by parts and the "Hermitian" property of the operator: so, not to clutter the notation, we work with Lagrangian (density) but drop the boundary terms knowing that we are really working with the actions. To be able to identify the physical modes, let us introduce two auxiliary fields $f_{\mu\nu}$ and φ to recast the Lagrangian as

$$\mathcal{L}_2 = -\frac{1}{\kappa_{\text{eff}}}\left(\frac{1}{2}h^{\mu\nu} + f^{\mu\nu}\right)\mathcal{G}^L_{\mu\nu}(h) - \frac{1}{4\beta\kappa_{\text{eff}}^2}\left(f_{\mu\nu}f^{\mu\nu} - f^2\right) + \varphi R_L - \frac{b}{2}\varphi^2, \tag{9.2.54}$$

where $f \equiv \bar{g}^{\mu\nu} f_{\mu\nu}$. To reproduce the original Lagrangian the constant b is found to be

$$b = \frac{2(n-1)}{4\alpha(n-1) + \beta n}. \tag{9.2.55}$$

Namely, integrating out the auxiliary fields in (9.2.54) gives us back our original action (9.2.52). Let us briefly show how this works. Variation with respect to φ yields

$$R_L - b\varphi = 0, \tag{9.2.56}$$

while variation with respect to $f^{\mu\nu}$ yields

$$-\frac{1}{\kappa_{\text{eff}}} \mathcal{G}^L_{\mu\nu}(h) - \frac{1}{2\beta\kappa^2_{\text{eff}}} \left(f_{\mu\nu} - \bar{g}_{\mu\nu} f \right) = 0. \tag{9.2.57}$$

Noting that the trace of the linearized Einstein tensor is

$$\bar{g}^{\mu\nu} \mathcal{G}^L_{\mu\nu}(h) = \frac{2-n}{n} R_L, \tag{9.2.58}$$

from the trace of (9.2.57), we find

$$f = -\frac{(n-2)}{n-1} \beta\kappa_{\text{eff}} R_L. \tag{9.2.59}$$

Plugging this back to the same equation we arrive at

$$f_{\mu\nu} = -2\beta\kappa_{\text{eff}} \left(\mathcal{G}^L_{\mu\nu}(h) + \bar{g}_{\mu\nu} \frac{n-2}{2(n-1)} R_L \right). \tag{9.2.60}$$

It is easy to show that plugging back φ, f, and $f_{\mu\nu}$ to the mother Lagrangian (9.2.54), one reproduces the original theory (9.2.52). Now let us take the other route: to get rid of the troublesome φR_L term, define a new field $\tilde{f}^{\mu\nu}$ as

$$f^{\mu\nu} = \tilde{f}^{\mu\nu} - \frac{2\kappa_{\text{eff}}}{n-2} \bar{g}^{\mu\nu} \varphi, \qquad f = \tilde{f} - \frac{2n\kappa_{\text{eff}}}{n-2} \varphi, \tag{9.2.61}$$

which then reduces (9.2.54) to

$$\begin{aligned} \mathcal{L}_2 = &-\frac{1}{\kappa_{\text{eff}}} \left(\frac{1}{2} h^{\mu\nu} + \tilde{f}^{\mu\nu} \right) \mathcal{G}^L_{\mu\nu}(h) - \frac{1}{4\beta\kappa^2_{\text{eff}}} \left(\tilde{f}_{\mu\nu} \tilde{f}^{\mu\nu} - \tilde{f}^2 \right) \\ &- \frac{n-1}{(n-2)\beta\kappa_{\text{eff}}} \varphi \tilde{f} + \left(\frac{n(n-1)}{\beta(n-2)^2} - \frac{b}{2} \right) \varphi^2. \end{aligned} \tag{9.2.62}$$

As φ appears without derivatives, we can integrate it out to arrive at

$$\mathscr{L}_2 = -\frac{1}{\kappa_{\text{eff}}}\left(\frac{1}{2}h^{\mu\nu} + \tilde{f}^{\mu\nu}\right)\mathscr{G}^L_{\mu\nu}(h) - \frac{1}{4\beta\kappa^2_{\text{eff}}}\left(\tilde{f}_{\mu\nu}\tilde{f}^{\mu\nu} - \tilde{f}^2\right) - \frac{1}{4\beta\kappa^2_{\text{eff}}}\xi\tilde{f}^2, \tag{9.2.63}$$

where the constant ξ is found to be

$$\xi = \frac{4\alpha(n-1) + \beta n}{4(\alpha n + \beta)}. \tag{9.2.64}$$

A further field redefinition is needed to decouple $h_{\mu\nu}$ and $\tilde{f}_{\mu\nu}$. By inspection one observes that the following shift does the job

$$h_{\mu\nu} \equiv \tilde{h}_{\mu\nu} - \tilde{f}_{\mu\nu}. \tag{9.2.65}$$

With this, our second order Lagrangian reduces to the (almost) decoupled form

$$\mathscr{L}_2 = -\frac{1}{2\kappa_{\text{eff}}}h^{\mu\nu}\mathscr{G}^L_{\mu\nu}(h) + \frac{1}{2\kappa_{\text{eff}}}f^{\mu\nu}\mathscr{G}^L_{\mu\nu}(f) - \frac{1}{4\beta\kappa^2_{\text{eff}}}\left(f_{\mu\nu}f^{\mu\nu} - (1-\xi)f^2\right), \tag{9.2.66}$$

where we removed all the tildes for notational simplicity. The first term is merely the Lagrangian of the linearized Einstein theory with an effective Newton's constant, and therefore as long as $\kappa_{\text{eff}} > 0$, it describes a massless unitary spin-2 excitation, which is the Einsteinian mode, the massless graviton that we all love to deal with. We can also observe that the second term (the kinetic part) has a wrong (positive) sign, so there will be a massive ghost. More properly, either the massless or the massive spin-2 mode is ghostlike and the conventional choice is to make the massless spin-2 mode unitary as it is the correct low energy theory. We are done with the massless part but not yet with the massive mode, for when $\xi = 0$, the $f^{\mu\nu}$ part yelds the Fierz-Pauli massive gravity (with the noted wrong-sign for the kinetic energy). For $\xi \neq 0$ as in our generic case, there is an additional massive spin-0 mode (or scalar graviton) which we still have to decouple. In order to do so and to find the masses, let us vary the action with respect to $f^{\mu\nu}$ to get

$$\mathscr{G}^L_{\mu\nu}(f) - \frac{1}{2\beta\kappa_{\text{eff}}}\left(f_{\mu\nu} - \bar{g}_{\mu\nu}f\right) - \frac{\xi}{2\beta\kappa_{\text{eff}}}\bar{g}_{\mu\nu}f = 0, \tag{9.2.67}$$

whose trace yields

$$R_L(f) + \frac{1}{(n-2)\beta\kappa_{\text{eff}}}\left(1 - n + n\xi\right)f = 0. \tag{9.2.68}$$

$\mathscr{G}^L_{\mu\nu}(f)$ satisfies the background Bianchi identity, hence double-divergence of (9.2.67) yields

$$(\xi - 1)\bar{\Box}f + \bar{\nabla}^\mu\bar{\nabla}^\nu f_{\mu\nu} = 0. \tag{9.2.69}$$

Making use of this in the trace equation and using the definition of R_L, one arrives at a massive scalar wave equation satisfied by the f-field:

$$\left(\xi\bar{\Box} + \frac{2\Lambda}{n-2} - \frac{1-n+n\xi}{(n-2)\beta\kappa_{\text{eff}}}\right)f = 0, \tag{9.2.70}$$

from which we can identify the mass of the scalar mode as

$$m_s^2 = -\frac{1}{\xi}\left(\frac{2\Lambda}{n-2} - \frac{1-n+n\xi}{(n-2)\beta\kappa_{\text{eff}}}\right), \tag{9.2.71}$$

as long as $\xi \neq 0$. Clearly, this mode decouples from the spectrum for the Fierz-Pauli tuning of $\xi = 0$. Finally, we should look at the trace-free part of (9.2.67) which yields the usual Fierz-Pauli massive graviton with the mass-square

$$m_g^2 = -\frac{1}{\beta\kappa_{\text{eff}}}. \tag{9.2.72}$$

This concludes the decoupling of the physical modes and identification of the masses for the n-dimensional quadratic gravity in (A)dS. Let us summarize our results and study some particular sub-cases. Altogether in its perturbative spectrum, quadratic gravity has a unitary massless spin-2 mode, which is the usual graviton, a massive spin-zero mode whose mass-square is given as (9.2.71) which should satisfy the Breitenlohner-Freedman bound in AdS, namely $m_s^2 \geq \frac{n-1}{2(n-2)}\Lambda$ to be a non-tachyonic mode, and a massive spin-2 ghost with the mass-square given as (9.2.72). The total number of propagating degrees of freedom in quadratic gravity is

$$\#\text{DOF} = \frac{n(n-3)}{2} + \frac{(n+1)(n-2)}{2} + 1 = n(n-2). \tag{9.2.73}$$

In terms of the parameters in the Lagrangian, more explicitly, the masses of the massive modes read

$$m_g^2 = -\frac{1}{\beta\kappa} - 4\Lambda\frac{(n-1)(\beta+\alpha n) + \gamma(n-4)(n-3)}{\beta(n-2)(n-1)}, \tag{9.2.74}$$

for the spin-2 mode, while the massive spin-0 has

$$m_s^2 = \frac{n-2}{\kappa(4\alpha(n-1)+\beta n)} + \frac{4\Lambda(n-4)\Big((n-1)(\beta+\alpha n) + \gamma(n-3)(n-2)\Big)}{(n-1)(n-2)(4\alpha(n-1)+\beta n)}, \tag{9.2.75}$$

whose flat space limits are clear. Of course, here Λ is not the bare cosmological constant in general, it is a solution to the quadratic equation we discussed before. Clearly, for a non-linear theory, as the one we have studied, by simply looking at the non-linear Lagrangian one cannot see the DOF and read their masses as the above formulas are a testament. All the terms, parameters, contribute, in a highly non-trivial way, to the masses which are only defined at a given constant curvature background. Let us consider the three and the four dimensional cases as two explicit examples as they offer interesting limits and some other special theories.

The case of $n = 3$ dimensions

The masses of the spin-2 and spin-0 modes respectively read

$$m_g^2 = -\frac{1}{\kappa\beta} - 4\left(1 + 3\frac{\alpha}{\beta}\right)\Lambda, \qquad m_s^2 = \frac{1}{(8\alpha + 3\beta)\kappa} - \frac{4(3\alpha + \beta)}{(8\alpha + 3\beta)}\Lambda, \tag{9.2.76}$$

which were also found with the usual canonical analysis in [210] for the de Sitter case. It is also important to note that, since there is no massless graviton these are the 3 degrees of freedom in 3 dimensions. This fact has remarkable consequences, for the choice $8\alpha + 3\beta = 0$, the scalar mode decouples and one can make the massive spin-2 mode unitary. This theory is called the “New Massive Gravity” (NMG) theory noted above [39] which is a theory of massive spin-2 graviton at the non-linear level. [2]

The case of $n = 4$ dimensions

The masses of the spin-2 and spin-0 modes respectively read

$$m_g^2 = -\frac{1}{\kappa\beta} - 2\left(1 + 4\frac{\alpha}{\beta}\right)\Lambda, \qquad m_s^2 = \frac{1}{2(3\alpha + \beta)\kappa}. \tag{9.2.77}$$

It is interesting to note that only in four dimensions, for (A)dS backgrounds, the massless mode does not receive a correction from the background. In addition to these two modes, there is the Einsteinian massless spin-2 graviton which altogether make up the 8 degrees of freedom whose flat space versions were given by Stelle [423]. An often studied sub-case of quadratic gravity is the Weyl-square corrected Einstein's gravity for which $3\alpha + \beta = 0$ in four dimensions. In this case the scalar mode decouples but the theory has a massive spin-2 ghost. In addition, if one drops out the Einstein term and keeps only the quadratic part, one has the so called conformal gravity [309, 389] since the combination $R_{\mu\nu}^2 - \frac{1}{3}R^2$ is equal to the Weyl-square up to a boundary term.

Einstein-Gauss-Bonnet theory

For the Einstein-Gauss-Bonnet (Einstein-GB) theory in n dimensions, since $\alpha = \beta = 0$, both massive modes decouple and only the massless mode survives as expected

2 As a side remark, let us note that NMG suffers from bulk-boundary unitarity clash, namely the theory does not have a unitary boundary conformal field theory in its 2 dimensional boundary of AdS_3 as one of its central charges of the double copy of the Virasoro algebra turns out to be negative. This unitarity clash seems to persist in all extensions of NMG with massive spin-2 gravitons [212].

since the field equations are second order. But there is a constraint on the theory for unitarity since the effective Newton's constant now reads

$$\frac{1}{\kappa_{\text{eff}}} \equiv \frac{1}{\kappa} + \frac{4\Lambda (n-3)(n-4)}{(n-1)(n-2)}\gamma = \frac{1}{\kappa} + 4\Lambda k, \tag{9.2.78}$$

which must be positive to get a positive kinetic energy for the massless graviton or equivalently, to get attractive gravity and this condition must be compatible with the existence of a maximally symmetric solution that is $1 + 8k\kappa\Lambda_0 \geq 0$. This theory arises in low energy string theory [71, 477].

Critical gravity

Critical gravity corresponds to the decoupling of the massive spin-0 mode with the choice $4\alpha(n-1)+\beta n = 0$ while tuning the cosmological constant in such a way that the massive spin-2 mode also decouples. The resulting theory is a higher derivative theory with an action of the form [123, 300]

$$S = \int d^n x\,\sqrt{-g}\left(\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 - \frac{4(n-1)}{n}\alpha R^{\mu\nu}R_{\mu\nu} + \gamma\chi_{\text{GB}})\right), \tag{9.2.79}$$

where the bare cosmological constant must be tuned as

$$\Lambda_0 = -\frac{n^2(n-1)(n-2)\left[(n-1)(n-2)\alpha + (n-3)(n-4)\gamma\right]}{8\kappa\left[(n-1)(n-2)^2\alpha + n(n-3)(n-4)\gamma\right]^2}. \tag{9.2.80}$$

Even though this theory seems promising as a possible ghost-free, perhaps renormalizable gravity theory, it turns out this is not the case. The theory has asymptotically non-AdS logarithmic modes [3, 215] which are of the wave-type and arise as both exact and as perturbative solutions. It was shown in [381] that they are ghosts and truncation of these modes renders the Hilbert space trivial, devoid of anything but the vacuum. Hence the above discussion teaches us something important: for $n \geq 4$ one must have $\beta = 0$ to avoid the massive spin-2 ghost, $n = 3$ is the exceptional case.

9.3 Particle spectrum of $f(R_{\mu\nu\sigma\rho})$ gravity in (A)dS

Now we turn our attention to the following more general theory with f an arbitrary analytic function of the Riemann tensor

$$S = \int d^n x\,\sqrt{-g} f\left(R^{\mu\nu}_{\alpha\beta}\right). \tag{9.3.1}$$

In this formulation, with two up and two down indices on the Riemann tensor, the (inverse) metric is not needed for contractions and this simplifies the ensuing discussion a little bit. For example, in this notation, the Einstein-Hilbert action (up to a multiplicative constant) simply reads

$$S_{\text{EH}} = \int d^n x \, \sqrt{-g} \delta^\alpha_\mu \delta^\beta_\nu R^{\mu\nu}_{\alpha\beta}. \tag{9.3.2}$$

Since the variations of the Kronecker delta vanish, the field equations are a little easier to get for the generic theory. At this stage one might wonder why we are not taking the more general actions of the form

$$S = \int d^n x \, \sqrt{-g} f\left(g^{\alpha\beta}, \, R^\mu{}_{\nu\rho\sigma}, \, \nabla_\rho R^\mu{}_{\nu\rho\sigma}, \ldots, \, \nabla_{\rho_1} \nabla_{\rho_2} \ldots \nabla_{\rho_m} R^\mu{}_{\nu\rho\sigma}\right), \tag{9.3.3}$$

where the covariant derivatives of the Riemann tensor and their contractions are also included ? While this is a legitimate question, one cannot find the particle content of such a theory in *full generality*: namely, the number of derivatives, and the action must be given to deal with such a theory with derivatives of the curvature tensors. But the particle content of the given in (9.3.1) can be computed with full generality in explicit form. There are two ways to do this: the first one is the conventional way of linearizing the field equations which is a rather long method which was given in detail in [213] and we shall not repeat it here. The other method, a beautiful short-cut, boils down to finding an *equivalent quadratic theory* that has the same particle content as the general theory at hand [112, 211, 233].

9.3.1 Linearization of the field equations

Let us, first, find the full non-linear field equations. The variation of the action with respect to the metric is

$$\delta_g S = \int d^n x \left(\delta \sqrt{-g} f\left(R^{\mu\nu}_{\alpha\beta}\right) + \sqrt{-g} \frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}} \delta R^{\mu\nu}_{\rho\sigma} \right). \tag{9.3.4}$$

Here, we have already picked up the first fruit of employing the up-up and down-down Riemann tensor as promised before. We need the variation of the Riemann tensor which in the usual form reads

$$\delta R^\mu{}_{\nu\rho\sigma} = \nabla_\rho \delta \Gamma^\mu_{\nu\sigma} - \nabla_\sigma \delta \Gamma^\mu_{\nu\rho}, \tag{9.3.5}$$

from which one can get the two-index up two index-down form as

$$\begin{aligned} \delta R^{\mu\nu}_{\rho\sigma} = &\frac{1}{2}\left(g_{\alpha\rho} \nabla_\sigma \nabla^\nu - g_{\alpha\sigma} \nabla_\rho \nabla^\nu\right) \delta g^{\mu\alpha} - \frac{1}{2}\left(g_{\alpha\rho} \nabla_\sigma \nabla^\mu - g_{\alpha\sigma} \nabla_\rho \nabla^\mu\right) \delta g^{\alpha\nu} \\ &- \frac{1}{2} R_{\rho\sigma}{}^\nu{}_\alpha \delta g^{\mu\alpha} + \frac{1}{2} R_{\rho\sigma}{}^\mu{}_\alpha \delta g^{\alpha\nu}. \end{aligned} \tag{9.3.6}$$

There is a slightly subtle point here: the derivative $\frac{\partial f}{\partial R^{\mu\nu}_{\alpha\beta}}$ is a four-indexed tensor but it does not in general have the symmetries of the Riemann tensor and one might consider writing it in such a way that it splits into two parts one that has those symmetries and one that does not. This would be a bit complicated but it really is redundant since at the end it is multiplied with $\delta R^{\mu\nu}_{\rho\sigma}$ and no piece of the wrong symmetry survives. Hence one should not worry at all about the symmetry issue. Then, skipping the argument of f for notational simplicity, we have

$$\begin{aligned}\delta_g S = &-\frac{1}{2}\int d^n x\, \sqrt{-g} f g_{\mu\nu}\delta g^{\mu\nu}\\ &+\frac{1}{2}\int d^n x\,\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\left(g_{\alpha\rho}\nabla_\sigma\nabla^\nu - g_{\alpha\sigma}\nabla_\rho\nabla^\nu\right)\delta g^{\mu\alpha}\\ &-\frac{1}{2}\int d^n x\,\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\left(g_{\alpha\rho}\nabla_\sigma\nabla^\mu - g_{\alpha\sigma}\nabla_\rho\nabla^\mu\right)\delta g^{\alpha\nu}\\ &-\frac{1}{2}\int d^n x\,\sqrt{-g}\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\left(R_{\rho\sigma}{}^{\nu}{}_{\alpha}\delta g^{\mu\alpha} - R_{\rho\sigma}{}^{\mu}{}_{\alpha}\delta g^{\alpha\nu}\right),\end{aligned} \tag{9.3.7}$$

which upon integration by parts and dropping the irrelevant boundary terms, for our current purpose, yields the source-free nonlinear field equations

$$\begin{aligned}&\frac{1}{2}\left(g_{\nu\rho}\nabla^\lambda\nabla_\sigma - g_{\nu\sigma}\nabla^\lambda\nabla_\rho\right)\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}} - \frac{1}{2}\left(g_{\mu\rho}\nabla^\lambda\nabla_\sigma - g_{\mu\sigma}\nabla^\lambda\nabla_\rho\right)\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\\ &\qquad -\frac{1}{2}\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\nu} - \frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\mu}\right) - \frac{1}{2}g_{\mu\nu}f = 0,\end{aligned} \tag{9.3.8}$$

which can be simplified but this form is sufficiently simple and compact to handle. The first task is to find the equation that yields the effective cosmological constants of the maximally symmetric solutions. There could be of course many vacua: if the highest order term in the action is of the form Riemann^N, one will have a polynomial of degree N with generically N solutions. [For Born-Infeld type gravity theories or when f is not in the form of a power series in the curvature, the vacuum equation need not be a polynomial of course: as an example consider $f(R) = \sin(\mu R)$.] There could of course be constraints on the parameters of the theory to get real solutions. Clearly, there is no guarantee that the solution can be explicitly found: for example, for the polynomial case beyond $N > 4$, one cannot write the explicit solution in radicals. In what follows, this does not concern us, all we need is that *there exists* at least one maximally symmetric solution about which we can perform perturbation theory and calculate the particle spectrum etc.

Clearly for the maximally symmetric spacetimes, the first line of the field equations vanishes. Therefore, we have, from the remaining part, the following equation that gives us the potential multiple vacua of the theory

$$\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{R}_{\rho\sigma}{}^{\lambda}{}_{\nu} - \left[\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{R}_{\rho\sigma}{}^{\lambda}{}_{\mu} + \bar{g}_{\mu\nu}f\left(\bar{R}^{\alpha\beta}_{\rho\sigma}\right) = 0, \tag{9.3.9}$$

where the Riemann tensor of the vacuum in this notation reads

$$\bar{R}^{\mu\lambda}_{\rho\sigma} = \frac{2\Lambda}{(n-1)(n-2)}\left(\delta^{\mu}_{\rho}\delta^{\lambda}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\lambda}_{\rho}\right) \tag{9.3.10}$$

and the bracketed terms are to be evaluated in this background. There are two terms that one needs to calculate to find the vacua:

$$f\left(\bar{R}^{\alpha\beta}_{\rho\sigma}\right), \qquad \left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}. \tag{9.3.11}$$

Let us contemplate on the meaning of this simple observation which will help us construct a new method of studying these theories : (9.3.9) says that if these two quantities are the same for any given two gravity theories with *different* actions, then those two gravity theories have the same maximally symmetric vacua. Or more explicitly, their effective cosmological constants are equal. Then, for a theory defined by a given $f\left(R^{\mu\nu}_{\alpha\beta}\right)$, if we are interested only in the vacua of the theory then we can determine the vacua by simply performing a *first order* Taylor series expansion around a yet to be determined maximally symmetric background as

$$S_{\text{ELA}} = \int d^n x\,\sqrt{-g}\left(f\left(\bar{R}^{\mu\nu}_{\alpha\beta}\right) + \left[\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\left(R^{\lambda\nu}_{\rho\sigma} - \bar{R}^{\lambda\nu}_{\rho\sigma}\right)\right). \tag{9.3.12}$$

The index ELA stands for the "equivalent linearized action", meaning S_{ELA} and (9.3.1) have the same vacua. So clearly to find the vacua of the generic theory all one needs to do is an expansion in the power series to construct this action at a first order. We can actually do better than that and continue to reduce the ELA action to the cosmological Einstein-Hilbert form as follows: let us define a parameter, ζ, via

$$\left[\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma} \equiv \zeta R. \tag{9.3.13}$$

This needs an explanation: since $\left[\partial f/\partial R^{\mu\nu}_{\rho\sigma}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}}$ is made up of the Kronecker deltas such as $\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}$, and its surviving part satisfies the symmetries of the Riemann tensor, so antisymmetrizing $\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}$ yields $\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\nu}$. Note that one does not need explicit antisymmetrization in the down indices since one has $\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\lambda} = \delta^{[\rho}_{[\mu}\delta^{\sigma]}_{\lambda]}$. Then we have

$$\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}} = \zeta\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\nu}, \tag{9.3.14}$$

which reduces ELA (9.3.12) to the Einstein-Hilbert form

$$S_{\text{ELA}} = \frac{1}{\kappa_{\text{ELA}}}\int d^n x\,\sqrt{-g}\left(R - 2\Lambda_{0,\text{ELA}}\right), \tag{9.3.15}$$

with the effective parameters defined as

$$\frac{1}{\kappa_{\mathrm{ELA}}}=\zeta,\qquad \frac{\Lambda_{0,ELA}}{\kappa_{\mathrm{ELA}}}=-\frac{1}{2}\bar{f}+\frac{n\Lambda}{n-2}\zeta, \tag{9.3.16}$$

where we utilized the shorthand notation $\bar{f}\equiv f\left(\bar{R}^{\alpha\beta}_{\rho\sigma}\right)$ and used the background value of the scalar curvature $\bar{R}=\frac{2n\Lambda}{n-2}$. It is then clear that the effective cosmological constant is simply $\Lambda=\Lambda_{0,ELA}$ which gives

$$\Lambda=\frac{n-2}{4\zeta}\bar{f}, \tag{9.3.17}$$

as the one we get from the trace of (9.3.9). The usefulness of the short-cut method is clear: one does not have to find the full nonlinear field equations to find the constant curvature vacua of the theory. Moreover this short-cut approach can be extended to the second order to study the particle content, energy and the stability of the vacua. For this latter case, the procedure is much more handy as it spares one from a rather long computation. Let us move on to this discussion.

To identify the particle content, we need to find the linearized field equations of the full theory (9.3.8) about any one of its maximally symmetric solution. For this purpose, we have to compute the following linearized quantities for the second line of (9.3.8)

$$\left[g_{\mu\nu}f\left(R^{\mu\nu}_{\alpha\beta}\right)\right]_L=h_{\mu\nu}f\left(\bar{R}^{\mu\nu}_{\alpha\beta}\right)+\bar{g}_{\mu\nu}\left[\frac{\partial f}{\partial R^{\alpha\beta}_{\rho\sigma}}\right]_{\bar{R}^{\alpha\beta}_{\rho\sigma}}\left(R^{\alpha\beta}_{\rho\sigma}\right)_L, \tag{9.3.18}$$

$$\begin{aligned}\left(\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}R_{\rho\sigma}{}^{\lambda}{}_{\nu}\right)_L&=\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\left(R^{\eta\theta}_{\alpha\tau}\right)_L\bar{R}_{\rho\sigma}{}^{\lambda}{}_{\nu}\\&\quad+\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\left(R_{\rho\sigma}{}^{\lambda}{}_{\nu}\right)_L,\end{aligned} \tag{9.3.19}$$

and the following quantity and its antisymmetric version for the first line of (9.3.8)

$$\begin{aligned}\left(g_{\nu\rho}\nabla^{\lambda}\nabla_{\sigma}\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right)_L&=\bar{g}_{\nu\rho}\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{\nabla}^{\lambda}\bar{\nabla}_{\sigma}\left(R^{\eta\theta}_{\alpha\tau}\right)_L\\&\quad+\bar{g}_{\nu\rho}\left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\alpha}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{\nabla}^{\lambda}\left(\Gamma^{\sigma}_{\sigma\alpha}\right)_L\\&\quad-\bar{g}_{\nu\rho}\left[\frac{\partial f}{\partial R^{\alpha\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{\nabla}^{\lambda}\left(\Gamma^{\alpha}_{\sigma\mu}\right)_L\\&\quad-\bar{g}_{\nu\rho}\left[\frac{\partial f}{\partial R^{\mu\alpha}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}\bar{\nabla}^{\lambda}\left(\Gamma^{\alpha}_{\sigma\lambda}\right)_L.\end{aligned} \tag{9.3.20}$$

This is a rather long and tedious computation but the crux of the matter is that we need the following three background-evaluated tensors

$$\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}, \qquad \left[\frac{\partial f}{\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\lambda}_{\rho\sigma}}, \qquad f\left(\bar{R}^{\alpha\beta}_{\rho\sigma}\right). \tag{9.3.21}$$

Once again, if these three tensors are equal for any two different theories, their particle content and vacua will be the same. They could of course differ in general at the interacting, third order, level. For this we have nothing to say here except to note that equivalent cubic curvature theory can be constructed along similar lines. So as in the case to the ELA action, we define an equivalent quadratic action (EQA) using the Taylor series expansion in the Riemann tensor up to second order as

$$\begin{aligned} S_{\text{EQA}} = \int d^n x \sqrt{-g} \Bigg\{ & f\left(\bar{R}^{\mu\nu}_{\alpha\beta}\right) + \left[\frac{\partial f}{\partial R^{\lambda\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \left(R^{\lambda\nu}_{\rho\sigma} - \bar{R}^{\lambda\nu}_{\rho\sigma}\right) \\ & + \frac{1}{2}\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \left(R^{\eta\theta}_{\alpha\tau} - \bar{R}^{\eta\theta}_{\alpha\tau}\right)\left(R^{\mu\lambda}_{\rho\sigma} - \bar{R}^{\mu\lambda}_{\rho\sigma}\right) \Bigg\}, \end{aligned} \tag{9.3.22}$$

Next, we can show that (9.3.22) reduces to the quadratic gravity whose properties we have already studied. Define the parameters α, β, and γ via

$$\frac{1}{2}\left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} R^{\eta\theta}_{\alpha\tau} R^{\mu\lambda}_{\rho\sigma} \equiv \alpha R^2 + \beta R^{\lambda}_{\sigma} R^{\sigma}_{\lambda} + \gamma\left(R^{\eta\lambda}_{\rho\sigma} R^{\rho\sigma}_{\eta\lambda} - 4 R^{\lambda}_{\sigma} R^{\sigma}_{\lambda} + R^2\right). \tag{9.3.23}$$

Once again, the left-hand side involves merely the Kronecker deltas and obeys the symmetries of the Riemann tensors $R^{\eta\theta}_{\alpha\tau}$ and $R^{\mu\lambda}_{\rho\sigma}$, therefore

$$\begin{aligned} \left[\frac{\partial^2 f}{\partial R^{\eta\theta}_{\alpha\tau}\partial R^{\mu\lambda}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} = & 2\alpha \delta^{[\alpha}_{\eta}\delta^{\tau]}_{\theta}\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\lambda} + \beta\left(\delta^{\alpha}_{[\eta}\delta^{[\rho}_{\theta]}\delta^{|\tau|}_{[\mu}\delta^{\sigma]}_{\lambda]} - \delta^{\tau}_{[\eta}\delta^{[\rho}_{\theta]}\delta^{|\alpha|}_{[\mu}\delta^{\sigma]}_{\lambda]}\right) \\ & + 12\gamma \delta^{[\alpha}_{\eta}\delta^{\tau}_{\theta}\delta^{\rho}_{\mu}\delta^{\sigma]}_{\lambda}, \end{aligned} \tag{9.3.24}$$

where the last term, clearly, should have the totally antisymmetric form since the Gauss-Bonnet combination is the quadratic Lovelock term written in terms of the generalized Kronecker deltas (to be more throughly discussed in the next section)

$$\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} R^{\nu_1\nu_2}_{\mu_1\mu_2} R^{\nu_3\nu_4}_{\mu_3\mu_4} = 4\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right), \tag{9.3.25}$$

where

$$\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} = \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\delta^{\mu_1}_{\nu_{\alpha_1}}\delta^{\mu_2}_{\nu_{\alpha_2}}\delta^{\mu_3}_{\nu_{\alpha_3}}\delta^{\mu_4}_{\nu_{\alpha_4}} = 4!\delta^{\mu_1}_{\nu_{[\alpha_1}}\delta^{\mu_2}_{\nu_{\alpha_2}}\delta^{\mu_3}_{\nu_{\alpha_3}}\delta^{\mu_4}_{\nu_{\alpha_4]}}. \tag{9.3.26}$$

Using these together with (9.3.13), one can put (9.3.22) in a more explicit form as a quadratic gravity theory

$$S_{\text{EQA}} = \int d^n x\,\sqrt{-g}\left[\frac{1}{\tilde{\kappa}}\left(R - 2\tilde{\Lambda}_0\right) + \alpha R^2 + \beta R^\lambda_\sigma R^\sigma_\lambda + \gamma\chi_{\text{GB}}\right], \tag{9.3.27}$$

where the effective Newton's constant is given as

$$\frac{1}{\tilde{\kappa}} = \zeta - \frac{4\Lambda}{n-2}\left[n\alpha + \beta + \gamma\frac{(n-2)(n-3)}{n-1}\right], \tag{9.3.28}$$

and the effective "bare" cosmological constant reads

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} = -\frac{1}{2}f\left(\bar{R}^{\alpha\beta}_{\rho\sigma}\right) + \frac{n\Lambda}{n-2}\zeta - \frac{2\Lambda^2 n}{(n-2)^2}\left[n\alpha + \beta + \gamma\frac{(n-2)(n-3)}{n-1}\right]. \tag{9.3.29}$$

The maximally symmetric solution of (9.3.27) satisfies the quadratic equation that we have seen before:

$$\frac{\Lambda - \tilde{\Lambda}_0}{2\tilde{\kappa}} + \left[(n\alpha + \beta)\frac{(n-4)}{(n-2)^2} + \gamma\frac{(n-3)(n-4)}{(n-1)(n-2)}\right]\Lambda^2 = 0. \tag{9.3.30}$$

Clearly, the quadratic appearance of this equation is a red-herring as $\tilde{\Lambda}_0$ involves the background evaluated f and this equation yields the same background as the $f\left(R^{\mu\nu}_{\alpha\beta}\right)$ theory and (9.3.15).

So we have managed to recast the generic $f\left(R^{\mu\nu}_{\alpha\beta}\right)$ as a quadratic theory up to and including $\mathcal{O}(h^2)$ about (A)dS backgrounds. This completes the discussion of the particle content of the generic theory since we have already given a detailed analysis for the quadratic gravity. So, given an f, one simply follows the recipe above to find the effective parameters and get the masses. So, generically $f\left(R^{\mu\nu}_{\alpha\beta}\right)$ theory has $n(n-2)$ degrees of freedom. In the most general form, there will always be a massive ghost which simply renders the theory unphysical and the vacuum unstable. The natural question would be to ask if there are theories free of the massive ghost and even with the massive scalar but only have the massless graviton. Lovelock theories to be discussed are of this form. Moreover, if one requires the *uniqueness* of a single viable vacuum, together with a unitary massless graviton about this vacuum, one can find rather restricted examples. One such example is the Born-Infeld gravity [214] in n dimensions given in the form

$$S = \frac{2}{\kappa\gamma}\int d^n x\left(\sqrt{-\det\left(g_{\mu\nu} + \gamma A_{\mu\nu}\right)} - (\gamma\Lambda_0 + 1)\sqrt{-g}\right), \tag{9.3.31}$$

where γ is a dimensionful parameter and the two-tensor is found to be

$$\begin{aligned} A_{\mu\nu} =& R_{\mu\nu} + \beta S_{\mu\nu} + \gamma\left(a_1 \mathscr{W}_{\mu\nu} + a_2 C_{\mu\rho\nu\sigma}R^{\rho\sigma} + \frac{\beta+1}{4}R_{\mu\rho}R^{\rho}_{\nu} + a_4 S_{\mu\rho}S^{\rho}_{\nu}\right) \\ &+ \frac{\gamma}{n}g_{\mu\nu}\Bigg(\left(\frac{(n-1)^2}{4(n-2)(n-3)} - a_1\right)\mathscr{W} \\ &- \frac{\beta}{4}R^2_{\rho\sigma} + \left(\frac{\beta(\beta+2)}{2} + \frac{n(4-3n)}{4(n-2)^2} - a_4\right)S^2_{\rho\sigma}\Bigg), \end{aligned} \tag{9.3.32}$$

where a_i are dimensionless real numbers and we defined a rank-two tensor, $\mathscr{W}_{\mu\nu}$, and its trace from the contraction of Weyl tensors as

$$\mathscr{W}_{\mu\nu} \equiv C_{\mu\rho\alpha\beta}C_{\nu}^{\ \rho\alpha\beta}, \qquad \mathscr{W} \equiv g^{\mu\nu}\mathscr{W}_{\mu\nu}. \tag{9.3.33}$$

In the spectrum of this theory, there is a *single massless graviton* about its *unique* maximally symmetric viable vacuum which solves the equation

$$(\lambda_0 + 1)\left(\frac{1}{4}x^2 + x + 1\right)^{\frac{2-n}{2}} + \frac{1}{4}x^2 - 1 = 0, \tag{9.3.34}$$

where $x \equiv \frac{2\lambda}{n-2}$ and $\lambda_0 \neq -1$ with the definitions $\lambda = \gamma\Lambda$ and so on. A proper discussion of this theory would take too long, instead we give a somewhat easier theory as an example to our general construction. As an example [see [248] for more details] consider the following quartic gravity in four dimensions

$$S = \frac{1}{2\kappa_0}\int d^4x\sqrt{-g}f(R,\chi_{\mathrm{GB}}), \tag{9.3.35}$$

where the Lagrangian density reads

$$2\gamma\mathscr{F} \equiv \left(1 + \gamma R - \frac{1}{2}\gamma^2\left(R^2 - 9\chi_{\mathrm{GB}}\right)\right)^2 - 4\lambda_0 - 1. \tag{9.3.36}$$

This theory has a unique viable vacuum about which there is only a massless graviton and no other degrees of freedom. Hence it is a close cousin of Einstein's theory.

9.3.2 Lovelock gravity

As an explicit example to the previous discussion let us study the Lovelock gravity [299] in n spacetime dimensions defined the Lagrangian density

$$\mathscr{L}_{\mathrm{Lovelock}}\left(R^{\mu\nu}_{\rho\sigma}\right) = \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q\mathscr{L}_q. \tag{9.3.37}$$

Here a_q's are dimensionful constants with different dimensions and $\left[\frac{n}{2}\right]$ corresponds to the integer part of its argument. The indices on tensors take values from the set $(0, \ldots, n-1)$. At each order we have the Euler densities as the Lagrangian densities.

$$\mathscr{L}_q = \delta^{\mu_1 \ldots \mu_{2q}}_{\nu_1 \ldots \nu_{2q}} \prod_{p=1}^{q} R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}. \tag{9.3.38}$$

Here, the front factor is the *generalized Kronecker* delta which is defined as a determinant of the following matrix formed by Kronecker deltas as

$$\delta^{\mu_1 \ldots \mu_{2q}}_{\nu_1 \ldots \nu_{2q}} \equiv \det \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_{2q}}_{\nu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_1}_{\nu_{2q}} & \ldots & \delta^{\mu_{2q}}_{\nu_{2q}} \end{vmatrix}. \tag{9.3.39}$$

Lovelock introduced this theory as the most natural generalization of Einstein's gravity in n dimensions with the well-known property that the field equations are second order in the derivatives of the metric tensor, just like in Einstein's theory. Specifically $\mathscr{L}_0$ is the cosmological constant, $\mathscr{L}_1$ is the curvature scalar yielding the Einstein-Hilbert action, $\mathscr{L}_2$ is the Gauss-Bonnet combination. On the other hand $\mathscr{L}_3$ is somewhat complicated in its explicit form:

$$\begin{aligned} \frac{\mathscr{L}_3}{8} = & -8R^{\mu\nu\rho\sigma}R_{\mu\ \rho}^{\ \tau\ \gamma}R_{\nu\tau\sigma\gamma} + 4R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\ \ \tau\gamma}R_{\rho\sigma\tau\gamma} - 24R^{\mu\nu}R^{\rho\sigma\tau}{}_{\mu}R_{\rho\sigma\tau\nu} \\ & + 3RR^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + 24R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho\nu\sigma} + 16R^{\mu\nu}R^{\rho}_{\mu}R_{\nu\rho} - 12RR^{\mu\nu}R_{\mu\nu} + R^3. \end{aligned} \tag{9.3.40}$$

9.3.3 Propagator structure of the Lovelock theory

To study the particle content and the vacuum structure of the Lovelock gravity, we know from the previous section that we have to calculate the following three quantities [113].

$$\begin{aligned} & \mathscr{L}_{\text{Lovelock}}\left(\bar{R}^{\mu\nu}_{\rho\sigma}\right), \qquad \left[\frac{\partial \mathscr{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right), \\ & \left[\frac{\partial^2 \mathscr{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma} \partial R^{\lambda\gamma}_{\alpha\beta}}\right]_{\bar{R}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)\left(R^{\lambda\gamma}_{\alpha\beta} - \bar{R}^{\lambda\gamma}_{\alpha\beta}\right). \end{aligned} \tag{9.3.41}$$

To be able to calculate these we need to study some properties of the generalized Kronecker delta which we list here.

Properties of generalized Kronecker delta

Let L be a $q \times q$ matrix, then by definition its determinant is

$$\det L = \epsilon_{\alpha_1 \ldots \alpha_q} L_{\alpha_1 1} L_{\alpha_2 2} \ldots L_{\alpha_q q}, \tag{9.3.42}$$

with the convention $\epsilon_{12\ldots 2q} = +1$. Then for the particular matrix

$$L = \begin{pmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_{2q}}_{\nu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_1}_{\nu_{2q}} & \ldots & \delta^{\mu_{2q}}_{\nu_{2n}} \end{pmatrix}, \tag{9.3.43}$$

where the index ν counts the rows, and the index μ counts the columns; i. e. one has $L_{ij} = \delta^{\mu_j}_{\nu_i}$, and for example, $L_{\alpha_1 1} = \delta^{\mu_1}_{\nu_{\alpha_1}}$, one can write $\delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} = \det L$ as

$$\delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} = \epsilon_{\alpha_1\ldots\alpha_{2q}} \delta^{\mu_1}_{\nu_{\alpha_1}} \delta^{\mu_2}_{\nu_{\alpha_2}} \ldots \delta^{\mu_{2q}}_{\nu_{\alpha_{2q}}}. \tag{9.3.44}$$

Here, note that $2q$ should be smaller than the dimension of the spacetime n, but need not to be equal to n.

Now, let us discuss how the term $\delta^{\mu_1\ldots\mu_{2k}\nu_{2k+1}\ldots\nu_{2q}}_{\nu_1\ldots\nu_{2k}\nu_{2k+1}\ldots\nu_{2n}}$ is related to $\delta^{\mu_1\ldots\mu_{2k}}_{\nu_1\ldots\nu_{2k}}$. Using (9.3.44) we can relate the $n \to n - \frac{1}{2}$ case to the $n \to n$ case as

$$\delta^{\mu_1\ldots\mu_{2k}\nu_{2k+1}\ldots\nu_{2q}}_{\nu_1\ldots\nu_{2k}\nu_{2k+1}\ldots\nu_{2q}} = [n - (2q-1)]\, \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2n-1}} \delta^{\mu_1}_{\nu_{\alpha_1}} \delta^{\mu_2}_{\nu_{\alpha_2}} \ldots \delta^{\mu_{2k}}_{\nu_{\alpha_{2k}}} \delta^{\nu_{2k+1}}_{\nu_{\alpha_{2k+1}}} \ldots \delta^{\nu_{2q-1}}_{\nu_{\alpha_{2q-1}}}. \tag{9.3.45}$$

This recursive relation reproduces a result which will be utilized in the computation of the equivalent quadratic action

$$\delta^{\mu_1\ldots\mu_{2k}\nu_{2k+1}\ldots\nu_{2q}}_{\nu_1\ldots\nu_{2k}\nu_{2k+1}\ldots\nu_{2q}} = \frac{(n-2k)!}{(n-2q)!} \delta^{\mu_1\ldots\mu_{2k}}_{\nu_1\ldots\nu_{2k}}. \tag{9.3.46}$$

With the help of this we now can compute the needed terms order by order.

Zeroth order

Let us calculate $\mathscr{L}_{\text{Lovelock}}\left(\bar{R}^{\mu\nu}_{\rho\sigma}\right)$ which has the form

$$\mathscr{L}_{\text{Lovelock}}\left(\bar{R}^{\mu\nu}_{\rho\sigma}\right) = \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \prod_{p=1}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}, \tag{9.3.47}$$

then for the background

$$\bar{R}^{\mu\nu}_{\rho\sigma} = \frac{2\Lambda}{(n-1)(n-2)} \left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\right), \tag{9.3.48}$$

one has

$$\begin{aligned} \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \prod_{p=1}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} &= \left[\frac{4\Lambda}{(n-1)(n-2)}\right]^q \delta^{\nu_1\ldots\nu_{2q}}_{\nu_1\ldots\nu_{2n}} \\ &= \left[\frac{4\Lambda}{(n-1)(n-2)}\right]^q \frac{n!}{(n-2q)!}, \end{aligned} \tag{9.3.49}$$

where the second equality follows from (9.3.46). Note that the value of this form for $q_{max} = \left[\frac{n}{2}\right]$ is same for both even and odd dimensions. Then, $\mathscr{L}_{\text{Lovelock}}\left(\bar{R}^{\mu\nu}_{\rho\sigma}\right)$ becomes

$$\mathscr{L}_{\text{Lovelock}}\left(\bar{R}^{\mu\nu}_{\rho\sigma}\right) = n! \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q \left[\frac{4\Lambda}{(n-1)(n-2)}\right]^q \frac{1}{(n-2q)!}. \tag{9.3.50}$$

First order

The first order term in the equivalent quadratic action has the form

$$\begin{aligned}\left[\frac{\partial \mathscr{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right) &= \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \sum_{r=1}^{q} \left(\prod_{\substack{p=1\\(p\neq r)}}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} \\ &\quad - \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2n}} q \left(\prod_{p=1}^{n} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right),\end{aligned} \tag{9.3.51}$$

where the term in the second line was calculated in (9.3.49); and after use of (9.3.48), the term in the first line becomes

$$\begin{aligned}&\delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \sum_{r=1}^{q} \left(\prod_{\substack{p=1\\(p\neq r)}}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} \\ &= \left[\frac{4\Lambda}{(n-1)(n-2)}\right]^{q-1} q\delta^{\mu_1\mu_2\nu_3\ldots\nu_{2q}}_{\nu_1\nu_2\nu_3\ldots\nu_{2q}} R^{\nu_1\nu_2}_{\mu_1\mu_2}.\end{aligned} \tag{9.3.52}$$

Using (9.3.46), one can further reduce this form to

$$\delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \sum_{r=1}^{q} \left(\prod_{\substack{p=1\\(p\neq r)}}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} = \left[\frac{4\Lambda}{(n-1)(n-2)}\right]^{q-1} 2q\frac{(n-2)!}{(n-2q)!}R. \tag{9.3.53}$$

This result together with (9.3.49) yields the first order term of the equivalent quadratic action as

$$\begin{aligned}&\left[\frac{\partial \mathscr{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right) \\ &= \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q q \left[\frac{4\Lambda}{(n-1)(n-2)}\right]^{q-1} \frac{2(n-2)!}{(n-2q)!} \left(R - \frac{2n\Lambda}{n-2}\right).\end{aligned} \tag{9.3.54}$$

Second order

The second order term in the equivalent quadratic action has the form

$$\left[\frac{\partial^2 \mathcal{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\lambda\gamma}_{\alpha\beta}}\right]_{\bar{R}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)\left(R^{\lambda\gamma}_{\alpha\beta} - \bar{R}^{\lambda\gamma}_{\alpha\beta}\right)$$
$$= \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \sum_{r=1}^{q} \sum_{\substack{s=1\\(r\neq s)}}^{q} \left(\prod_{\substack{p=1\\(p\neq r,s)}}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} R^{\nu_{2s-1}\nu_{2s}}_{\mu_{2s-1}\mu_{2s}}$$
$$- \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} 2(q-1) \sum_{r=1}^{q} \left(\prod_{\substack{p=1\\(p\neq r)}}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}}$$
$$\times \sum_{q=0}^{\left[\frac{n}{2}\right]} a_q \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} q(q-1) \left(\prod_{p=1}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right), \quad (9.3.55)$$

where the second and the third terms on the right-hand side were calculated in (9.3.53) and (9.3.49), respectively. On the other hand, the first term takes the form

$$\delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \sum_{r=1}^{q} \sum_{\substack{s=1\\(r\neq s)}}^{q} \left(\prod_{\substack{p=1\\(p\neq r,s)}}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} R^{\nu_{2s-1}\nu_{2s}}_{\mu_{2s-1}\mu_{2s}}$$
$$= q(q-1)\, \delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} R^{\nu_1\nu_2}_{\mu_1\mu_2} R^{\nu_3\nu_4}_{\mu_3\mu_4} \left(\prod_{p=3}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right), \quad (9.3.56)$$

after renaming the dummy indices and using the totally antisymmetric nature of $\delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}}$. Then, employing the (A)dS background Riemann tensor and using (9.3.46), one gets

$$\delta^{\mu_1\ldots\mu_{2q}}_{\nu_1\ldots\nu_{2q}} \sum_{r=1}^{q} \sum_{\substack{s=1\\(r\neq s)}}^{q} \left(\prod_{\substack{p=1\\(p\neq r,s)}}^{q} \bar{R}^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right) R^{\nu_{2r-1}\nu_{2r}}_{\mu_{2r-1}\mu_{2r}} R^{\nu_{2s-1}\nu_{2s}}_{\mu_{2s-1}\mu_{2s}}$$
$$= q(q-1) \left[\frac{4\Lambda}{(n-1)(n-2)}\right]^{q-2} \frac{(n-4)!}{(n-2q)!} 4\chi_{\text{GB}}, \quad (9.3.57)$$

where we used the short-hand notation for the Gauss-Bonnet combination. With this result and (9.3.53), (9.3.49), the second order term of the equivalent quadratic action becomes

$$\left[\frac{\partial^2 \mathscr{L}_{\text{Lovelock}}}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\lambda\gamma}_{\alpha\beta}}\right]_{\bar{R}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)\left(R^{\lambda\gamma}_{\alpha\beta} - \bar{R}^{\lambda\gamma}_{\alpha\beta}\right)$$
$$= 4\sum_{q=0}^{\left[\frac{n}{2}\right]} a_q q\,(q-1)\left[\frac{4\Lambda}{(n-1)\,(n-2)}\right]^{q-2}\frac{(n-4)!}{(n-2q)!}\chi_{\text{GB}}$$
$$-2\sum_{q=0}^{\left[\frac{n}{2}\right]} a_q q\,(q-1)\left[\frac{4\Lambda}{(n-1)\,(n-2)}\right]^{q-1}\frac{2\,(n-2)!}{(n-2q)!}\left(R-\frac{n\Lambda}{n-2}\right). \tag{9.3.58}$$

As a result, the equivalent quadratic action that has the same $O(h)$ and $O\left(h^2\right)$ expansions with the Lovelock theory (9.3.37) can be constructed from the equivalent quadratic Lagrangian density

$$\mathscr{L}_{\text{quad-equal}}\left(R^{\mu\nu}_{\rho\sigma}\right)$$
$$= -2\,(n-2)!\sum_{q=0}^{\left[\frac{n}{2}\right]}\tilde{a}_q\left[R-\frac{(q-1)\,n\Lambda}{q\,(n-2)}-\frac{(q-1)}{4\Lambda\,(q-2)}\frac{(n-1)}{(n-3)}\chi_{\text{GB}}\right], \tag{9.3.59}$$

where $\tilde{a}_q$ is defined as

$$\tilde{a}_q \equiv a_q\frac{q\,(q-2)}{(n-2q)!}\left[\frac{4\Lambda}{(n-1)\,(n-2)}\right]^{q-1}.$$

The propagator of (9.3.37) matches that of (9.3.59) which itself has exactly the same propagator as the cosmological Einstein's theory. But the effective Newton's constant is modified. Therefore at the free level one has

$$\mathscr{L}_{\text{Lovelock}}\left(h^2\right) = -\frac{1}{2\kappa_e}h^{\mu\nu}\mathscr{D}_{\mu\nu\alpha\beta}h^{\alpha\beta}, \tag{9.3.60}$$

where $\mathscr{D}_{\mu\nu\alpha\beta}$ is the propagator of the cosmological Einstein theory (9.2.51) which propagates a unitary massless spin-2 particle as long as $\kappa_e > 0$ to have a positive kinetic energy.

Let us re-write the equivalent quadratic Lagrangian (9.3.59) of the Lovelock theory as

$$\mathscr{L}_{\text{quad-equal}} = \frac{1}{\tilde{\kappa}}\left(R-2\tilde{\Lambda}_0\right)+\tilde{\gamma}\chi_{\text{GB}}, \tag{9.3.61}$$

where the parameters are defined as

$$\frac{1}{\tilde{\kappa}} \equiv -2\,(n-2)!\sum_{q=0}^{\left[\frac{n}{2}\right]} a_q\frac{q\,(q-2)}{(n-2q)!}\left[\frac{4\Lambda}{(n-1)\,(n-2)}\right]^{q-1}, \tag{9.3.62}$$

$$\frac{\tilde{\Lambda}_0}{\tilde{\kappa}} \equiv -\frac{n!}{4}\sum_{q=0}^{[\frac{n}{2}]} a_q \frac{(q-1)(q-2)}{(n-2q)!}\left[\frac{4\Lambda}{(n-1)(n-2)}\right]^q, \tag{9.3.63}$$

$$\tilde{\gamma} \equiv 2(n-4)!\sum_{q=0}^{[\frac{n}{2}]} a_q \frac{q(q-1)}{(n-2q)!}\left[\frac{4\Lambda}{(n-1)(n-2)}\right]^{q-2}. \tag{9.3.64}$$

As the above discussion shows, the vacua and the propagator of the Einstein-GB theory can be represented with cosmological Einstein's gravity that has the modified parameters as

$$S_{\text{equal-Lovelock}} = \int d^n x\sqrt{-g}\frac{1}{\kappa_e}(R-2\Lambda), \tag{9.3.65}$$

where by using (9.3.62–9.3.64), one can find that Λ satisfies the polynomial vacuum equation

$$0 = \sum_{q=0}^{[\frac{n}{2}]} a_q \frac{(n-2q)}{(n-2q)!}\left[\frac{4}{(n-1)(n-2)}\right]^q \Lambda^q, \tag{9.3.66}$$

and the effective Newton's constant κ_e becomes

$$\frac{1}{\kappa_e} = 2(n-3)!\sum_{q=0}^{[\frac{n}{2}]} a_q \frac{q(n-2q)}{(n-2q)!}\left[\frac{4\Lambda}{(n-1)(n-2)}\right]^{q-1}. \tag{9.3.67}$$

See [89, 90] who obtained the same results using different methods. We have shown that Lovelock theory has a massless spin-2 particle like in Einstein's theory. But of course at the non-linear level, Lovelock gravity is a lot more complicated and still awaits to be explored even though there is already lot of work in the literature. For example, even in the simplest of case of the Einstein-GB theory, the exact rotating black hole solution seems to be missing.

9.4 Weak field limits: Potential energy from tree-level gravitons

Let us calculate the tree-level scattering amplitude for one-graviton exchange between two covariantly conserved sources as shown in the Feynman diagram of Figure 9.1 [216]. This computation has several interesting uses: we can infer the weak field limits in a flat background and we can discuss various discontinuities that arise in different limits. For this purpose, we generalize the quadratic gravity model and add the usual linear Fierz-Pauli mass term. As we shall find the propagator of the quadratic gravity, the introduction of the mass term helps us invert the relevant operator without making a choice of a gauge, as the mass term removes gauge invariance. So, now augmenting the linearized quadratic gravity which we found in the previous section with the Fierz-Pauli mass term in (A)dS and coupling it to a covariantly conserved source, we have

$$\begin{aligned} T_{\mu\nu}(h) &= c\mathcal{G}^L_{\mu\nu} + (2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{n-2}\bar{g}_{\mu\nu}\right)R^L \\ &\quad + \beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}_{\mu\nu}R^L\right) + \frac{m^2}{2\kappa}\left(h_{\mu\nu} - \bar{g}_{\mu\nu}h\right), \end{aligned} \tag{9.4.1}$$

where the last term is the Fierz-Pauli [175] mass whose non-linear extension with a single field does not exist beyond 3 dimensions, or stated in another way this term does not come from a diffeomorphism invariant action with a single field. The relative coefficient (of -1) in the Fierz-Pauli term is the unique one that does not yield a scalar ghost as we have seen in the previous section. But even this choice fails at the non-linear level and the massive theory has the so called Boulware-Deser ghost [70]. Recall also that the coefficient of the linearized Einstein tensor was found to be

$$c \equiv \frac{1}{\kappa} + \frac{4\Lambda n}{n-2}\alpha + \frac{4\Lambda}{n-1}\beta + \frac{4\Lambda(n-3)(n-4)}{(n-1)(n-2)}\gamma. \tag{9.4.2}$$

There is a subtle issue here: we assume that (9.4.1) comes from the linearization of the quadratic gravity plus the Fierz-Pauli mass, this fixes the constant c in front of the linearized Einstein tensor. Instead if one considers (9.4.1) as the definition of the theory, not referrring to an action, of course c is arbitrary, but we take the first view.

In order to get a sense of the constraints and the particle content of the theory, let us, first, consider the trace of (9.4.1) which yields

$$\left[\left(4\alpha(n-1) + n\beta\right)\bar{\Box} - (n-2)\left(\frac{1}{\kappa} + 4k\Lambda\right)\right]R^L - \frac{m^2}{\kappa}(n-1)h = 2T, \tag{9.4.3}$$

where we have made use of the definition we introduced in the previous section:

$$k \equiv (n\alpha+\beta)\frac{(n-4)}{(n-2)^2} + \gamma\frac{(n-3)(n-4)}{(n-1)(n-2)}. \tag{9.4.4}$$

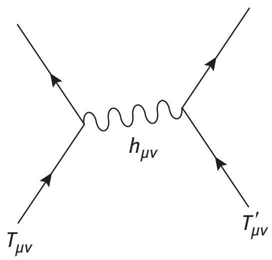

Figure 9.1: A graviton exchange between two covariantly conserved sources. This diagram gives us the amplitude in the quantum mechanical sense from which we define the low energy limit potential energy that includes mass-mass, spin-spin, spin-orbit and related terms up to desired order.

To see the rather dramatic effects of a non-vanishing Fierz-Pauli mass on the particle content of the theory, let us take the divergence and the double divergence of (9.4.1):

$$m^2\left(\bar{\nabla}^\mu h_{\mu\nu} - \bar{\nabla}_\nu h\right) = 0, \qquad m^2\left(\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\Box} h\right) = 0. \tag{9.4.5}$$

Since the linearized scalar curvature is $R^L = -\bar{\Box}h + \bar{\nabla}^\sigma \bar{\nabla}^\mu h_{\sigma\mu} - 2\Lambda h/(n-2)$, the second equation above yields $R^L = -2\Lambda h/(n-2)$. Consider, first, the *source-free* case and the flat space ($\Lambda = 0$) limit. Then $R_L = 0$ and from (9.4.3) one has $h = 0$, a traceless field and from the first equation of (9.4.5), one has $\partial^\mu h_{\mu\nu} = 0$, a transverse field, hence the field becomes transverse-traceless ($h^{TT}_{\mu\nu}$) which has $(n+1)(n-2)/2$ independent physical components that satisfy a fourth order equation as long as $\beta \neq 0$:

$$\left(\beta\partial^4 + \frac{1}{\kappa}\partial^2 - \frac{m^2}{\kappa}\right) h^{TT}_{\mu\nu} = 0, \tag{9.4.6}$$

which can be factored as

$$\beta\left(\partial^2 - m_+^2\right)\left(\partial^2 - m_+^2\right) h^{TT}_{\mu\nu} = 0, \tag{9.4.7}$$

describing two massive excitations with generically non-equal masses

$$m_\pm^2 = -\frac{1}{2\kappa\beta} \pm \frac{1}{2|\kappa\beta|}\sqrt{1 + 4\beta m^2\kappa}. \tag{9.4.8}$$

One of these excitations is a ghost, hence the theory is non-unitary, as we shall see. For $\beta = 0$ in (9.4.6), there is a single massive graviton, like in Fierz-Pauli theory, hence the α term does not play a role here.

On the other hand let us consider the $\Lambda \neq 0$ case, then the trace equation reduces to

$$\left[\left(4\alpha(n-1) + n\beta\right)\bar{\Box} - (n-2)\left(\frac{1}{\kappa} + 4\Lambda k\right) + \frac{m^2}{2\kappa\Lambda}(n-1)(n-2)\right] h = 0. \tag{9.4.9}$$

So, unless $4\alpha(n-1) + n\beta = 0$, h is a massive dynamical field. For the case of the special tuning of α and β which yields $4\alpha(n-1) + n\beta = 0$ the discussion bifurcates because the resulting equation for $n \neq 2$ is

$$\left[-\left(\frac{1}{\kappa} + 4\Lambda k\right) + \frac{m^2}{2\kappa\Lambda}(n-1)\right] h = 0, \tag{9.4.10}$$

and it has two possible solutions either $h = 0$ or the coefficient is zero. In the latter case, eliminating α in favor of β, one has the tuning of the Fierz-Pauli mass as

$$m^2 = \frac{2\Lambda\kappa}{n-1}\left(\frac{1}{\kappa} - \frac{\Lambda\beta(n-4)}{n-1} + 4\Lambda\gamma\frac{(n-3)(n-4)}{(n-1)(n-2)}\right), \tag{9.4.11}$$

at which point there arises a new higher derivative *scalar* gauge invariance of the form

$$\delta_\xi h_{\mu\nu} = \bar{\nabla}_\mu \bar{\nabla}_\nu \xi + \frac{2\Lambda}{(n-1)(n-2)} \bar{g}_{\mu\nu} \xi. \tag{9.4.12}$$

The resulting theory with one less degree of freedom, compared to the massive one, is called the "partially massless gravity", in this case, partially massless quadratic gravity. As it is clear from the above discussion, this partially massless theory exists for cosmological backgrounds even in the absence of the quadratic terms. For example, in massive Einstein's theory in four dimensions, there are 4 degrees of freedom instead of the expected 5 for a massive spin-2 field, namely, the heliciy-0 mode becomes a gauge degree of freedom. Generically in n dimensions, for flat backgrounds, in Einstein's gravity a massless spin-2 field has $n(n-3)/2$ degrees of freedom and in Einstein-Fierz-Pauli theory, a massive one has $(n+1)(n-2)/2$. Once a cosmological constant is introduced, generically these values are intact but for the special case of

$$m^2 = \frac{2\Lambda}{n-1}, \tag{9.4.13}$$

the above mentioned higher derivative scalar gauge symmetry arises and the theory has one less degree of freedom. A detailed study of this theory and its extensions to other spins were discussed by Deser-Waldron in [133, 134] following the ideas introduced in [130]. In the context of AdS/CFT applications these partially massless spin-2 fields in the bulk of the AdS lead to partially conserved sources[3] in the boundary and to non-unitary conformal field theories [141]. One possible way out of this is to consider partially massless fields in higher derivative theories as noted above. This was shown for partially massless fields in Einstein-Gauss-Bonnet-Fierz-Pauli theory in [436] where a crucial sign change removes the unitarity clash of the bulk and the boundary theory.

In the other extreme when $m^2 = 0$, the theory is invariant under background diffeomorphisms $\delta_\xi h_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$, since $\delta_\xi \mathscr{G}^L_{\mu\nu} = 0$ and $\delta_\xi R^L = 0$ which say that divergence and the double divergence do not give any constraint on $h_{\mu\nu}$ and one has the theory we have studied in the previous section.

9.4.1 Potential energy from the scattering amplitude

To relate the potential energy at the desired order to the scattering amplitude, we will use the well-known approach and compute the vacuum to vacuum transition amplitude in the path integral formalism between two sources:

$$\langle 0 | e^{-i\hat{H}t} | 0 \rangle = e^{-iUt} = W[T] = \int \mathscr{D} h_{\mu\nu} e^{iS[h,T]}, \tag{9.4.14}$$

---

3 Partially conserved source basically means the following: let L_{ij} correspond to the boundary operator corresponding to the partially massless bulk field $h_{\mu\nu}$, assuming a flat boundary, partial conservation is $\partial_i \partial_j L^{ij} = 0$, which can be generalized to non-flat backgrounds.

where t is a large time that will play no role and drop out at the end of the computation. This equation basically defines the potential energy U. In the most general form, $S[h,T]$ is the linearized action about a background $\bar{g}_{\mu\nu}$ which can be taken as

$$S[h,T] = -\int d^n x\sqrt{-\bar{g}}\Big(\frac{1}{\kappa}h^{\mu\nu}\mathscr{E}_{\mu\nu}(h) - h^{\mu\nu}T_{\mu\nu}\Big), \tag{9.4.15}$$

whose extremization with respect to $h_{\mu\nu}$ yields the linearized field equations

$$\mathscr{E}_{\mu\nu}(h) = \frac{\kappa}{2}T_{\mu\nu}. \tag{9.4.16}$$

Covariant conservation of $T_{\mu\nu}$ leads to $\bar{\nabla}_\mu\mathscr{E}^{\mu\nu}(h) = 0$. Assume now that the field $\bar{h}_{\mu\nu}$ satisfies (9.4.16) which we call the *background field* (not to be confused with the background spacetime $\bar{g}_{\mu\nu}$!). Then in the background field formalism, we make a field redefinition of the form

$$h_{\mu\nu} \to h_{\mu\nu} + \bar{h}_{\mu\nu}, \tag{9.4.17}$$

which keeps the path integral measure intact but changes the action to a decoupled form as

$$S[h,T] = -\int d^n x\sqrt{-\bar{g}}\Big(\frac{1}{\kappa}h^{\mu\nu}\mathscr{E}_{\mu\nu}(h) - \frac{1}{2}\bar{h}^{\mu\nu}T_{\mu\nu}\Big). \tag{9.4.18}$$

Since the second term does not have $h_{\mu\nu}$ one simply moves it out of the path integral and the rest yields a normalization factor resulting in

$$W[T] = \mathscr{N}\exp\Big(\frac{i}{2}\int d^n x\sqrt{-\bar{g}}\,\bar{h}^{\mu\nu}T_{\mu\nu}\Big). \tag{9.4.19}$$

To get rid of the background field and write the result in terms of the sources, let us go back to the linearized field equation and recast it in the form

$$\mathscr{O}_{\mu\nu\alpha\beta}(x)\bar{h}^{\alpha\beta}(x) = \frac{\kappa}{2}T^{\alpha\beta}(x), \tag{9.4.20}$$

where $\mathscr{O}_{\mu\nu\alpha\beta}$ is a self-adjoint operator whose explicit form depends on the theory and the background metric. Its Green's function is defined as

$$\mathscr{O}_{\mu\nu\alpha\beta}G^{\alpha\beta}{}_{\sigma\rho}(x,x') = \frac{1}{2}\Big(\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho} + \bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}\Big)\delta(x-x'). \tag{9.4.21}$$

Therefore, assuming that one can find the Green's function (in this case, a four index tensor), the particular solution of (9.4.16) can be formally written as

$$\bar{h}_{\mu\nu}(x) = \frac{\kappa}{2}\int d^n x'\,\sqrt{-\bar{g}}\,G_{\mu\nu\alpha\beta}(x,x')T^{\alpha\beta}(x'). \tag{9.4.22}$$

Homogeneous solutions are not relevant here. Plugging this back into the (9.4.19), one arrives at the usual source-source interaction

$$W = \mathcal{N} \exp\left(\frac{i\kappa}{4} \int d^n x d^n x' \sqrt{-\bar{g}(x)}\sqrt{-\bar{g}(x')} T^{\mu\nu}(x)\, G_{\mu\nu\alpha\beta}\left(x, x'\right) T^{\alpha\beta}\left(x'\right) \right). \tag{9.4.23}$$

Then we can read from this expression, the potential energy (assuming it to vanish in the absence of sources) as

$$U = -\frac{\kappa}{4t} \int d^n x \int d^n x' \sqrt{-\bar{g}(x)}\sqrt{-\bar{g}(x')} T^{\mu\nu}(x)\, G_{\mu\nu\alpha\beta}\left(x, x'\right) T^{\alpha\beta}\left(x'\right). \tag{9.4.24}$$

We could have also started with this equation as the definition of the potential energy, but the above computation is more natural as it is related to the one-graviton exchange. Note that this potential energy includes all the possible interactions between the sources, such as, mass-mass, spin-spin, spin-orbit etc. and we shall compute these in the mass-augmented quadratic gravity in what follows. For $n = 3+1$, one has $\kappa = 16\pi G$.

The reader might have realized that the non-trivial part of the computation will be finding the Green's function $G_{\mu\alpha\nu\beta}$ in a given theory. This is indeed the most challenging task in this problem: especially for the theories we are dealing since we have both a non-flat background and quadratic terms as well as non-physical degrees of freedom even in the presence of the Fierz-Pauli mass term breaking the background diffeomorphism invariance. Clearly if there are non-physical modes, then the operator $\mathcal{O}_{\mu\nu\alpha\beta}$ is not invertible, it vanishes in certain directions in the field space. So we must, first, solve this issue. In the next section we adopt a somewhat more direct technique and decompose the field into its various irreducible modes.

9.4.2 Decomposition of the graviton field and tree-level scattering amplitude

To single out the physical modes, let us employ the usual decomposition of the field in terms of its possible irreducible parts as

$$h_{\mu\nu} \equiv h^{TT}_{\mu\nu} + \bar{\nabla}_{(\mu} V_{\nu)} + \bar{\nabla}_\mu \bar{\nabla}_\nu \phi + \bar{g}_{\mu\nu}\psi, \tag{9.4.25}$$

where $h^{TT}_{\mu\nu}$ is the transverse and traceless part (namely, $\bar{\nabla}^\mu h^{TT}_{\mu\nu} = 0$ and $\bar{g}^{\mu\nu} h^{TT}_{\mu\nu} = 0$). We also have $\bar{\nabla}^\mu V_\mu = 0$ and the symmetrization (with a 1/2 factor) is implied in the vector part. Our task is to "solve" $h_{\mu\nu}$ in terms of $T_{\mu\nu}$ in the full theory (9.4.1). Taking the trace, divergence and double divergence of (9.4.25) one obtains

$$h = \bar{\Box}\phi + n\psi, \qquad \bar{\Box}h = \bar{\Box}^2\phi + \frac{2\Lambda}{n-2}\bar{\Box}\phi + \bar{\Box}\psi, \tag{9.4.26}$$

where we used $\bar{\nabla}^{\nu}\bar{\nabla}^{\mu}h_{\mu\nu} = \bar{\Box}h$, which is a condition and not a gauge choice in the massive theory as we have seen. Then, hitting the first equation of (9.4.26) with a $\bar{\Box}$, one can eliminate $\bar{\Box}\phi$ with the help of the second equation as

$$\bar{\Box}\phi = \frac{(n-1)\,(n-2)}{2\Lambda}\bar{\Box}\psi, \tag{9.4.27}$$

using this one can eliminate $\bar{\Box}\phi$.[4] Then the trace of the field becomes

$$h = \left(\frac{(n-1)\,(n-2)}{2\Lambda}\bar{\Box} + n\right)\psi. \tag{9.4.28}$$

Then using (9.4.3), we arrive at

$$\psi = \left(\frac{\Lambda}{\kappa} + 4\Lambda k - p\Lambda\bar{\Box} - \frac{m^2}{2\kappa}\,(n-1)\right)^{-1}\left(\frac{(n-1)\,(n-2)}{2\Lambda}\bar{\Box} + n\right)^{-1} T, \tag{9.4.29}$$

where we have defined

$$p \equiv \frac{4(n-1)\alpha}{n-2} + \frac{n\beta}{n-2}. \tag{9.4.30}$$

Of course inverse of an operator is the short-hand notation for the corresponding Green's function. Equation 9.4.30 expresses ψ in terms of the trace of the source, now to find the transverse traceless part of the field in terms of the source, it pays to use the Lichnerowicz operator, $\triangle_L^{(2)}$, acting on symmetric rank two tensors as:

$$\triangle_L^{(2)} h_{\mu\nu} \equiv -\bar{\Box}h_{\mu\nu} - 2\bar{R}_{\mu\rho\nu\sigma}h^{\rho\sigma} + 2\bar{R}^{\rho}{}_{(\mu}h_{\nu)\rho}. \tag{9.4.31}$$

Without giving their derivations, which would take us too far, from the above definition of this operator, one can work out the following relations compiled in [380]:

$$\begin{aligned}\triangle_L^{(2)}\nabla_{(\mu}V_{\nu)} = \nabla_{(\mu}\triangle_L^{(1)}V_{\nu)},\quad \triangle_L^{(1)}V_\mu = (-\Box + \Lambda)\,V_\mu,\quad \nabla^\mu\triangle_L^{(2)}h_{\mu\nu} = \triangle_L^{(1)}\nabla^\mu h_{\mu\nu},\\ \triangle_L^{(2)}g_{\mu\nu}\phi = g_{\mu\nu}\triangle_L^{(0)}\phi,\qquad \triangle_L^{(0)}\phi = -\Box\phi,\qquad \nabla^\mu\triangle_L^{(1)}V_\mu = \triangle_L^{(0)}\nabla^\mu V_\mu,\end{aligned} \tag{9.4.32}$$

where the notation is self-evident. With this arsenal in our hands, we can write the transverse traceless part of the linearized Einstein tensor as

$$\mathcal{G}_{\mu\nu}^{LTT} = \frac{1}{2}\left(\triangle_L^{(2)} - \frac{4\Lambda}{(n-2)}\right)h_{\mu\nu}^{TT}, \tag{9.4.33}$$

4 Assuming that the fields vanish at infinity, we actually have a stronger relation as $\phi = (n-1)\,(n-2)/2\Lambda\psi$, but this is not really needed.

using this we can find the transverse traceless part of the field in terms of the TT part of the energy-momentum tensor as

$$h_{\mu\nu}^{TT} = 2\left(\beta\bar{\Box} + c)(\triangle_L^{(2)} - \frac{4\Lambda}{n-2}) + \frac{m^2}{\kappa}\right)^{-1} T_{\mu\nu}^{TT}. \tag{9.4.34}$$

We are not done yet: the last part of the puzzle is to express the TT part of the energy-momentum tensor: since it is symmetric and covariantly conserved one can use the same decomposition as we used for the $h_{\mu\nu}$ field to arrive at

$$T_{\mu\nu}^{TT} = T_{\mu\nu} - \frac{\bar{g}_{\mu\nu}}{n-1}T + \frac{1}{n-1}\left(\bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda\bar{g}_{\mu\nu}}{(n-1)(n-2)}\right) \times \left(\bar{\Box} + \frac{2\Lambda n}{(n-1)(n-2)}\right)^{-1} T. \tag{9.4.35}$$

One might be somewhat uncomfortable as to why an apparently non-local part shows up in this equation. This is because we have demanded too much: we are expressing the irreducible transverse traceless part in terms of the full tensor, this is the reason. But this will be useful in what follows. We are now ready to compute the scattering amplitude: using (9.4.34, 9.4.35), we have

$$A = \frac{1}{4}\int d^n x\,\sqrt{-\bar{g}}T'_{\mu\nu}(x)\,h^{\mu\nu}(x) = \frac{1}{4}\int d^n x\,\sqrt{-\bar{g}}\left(T'_{\mu\nu}h^{TT\mu\nu} + T'\psi\right). \tag{9.4.36}$$

For the sake of notational simplicity, lets suppress the integral for now, altogether we have [216]

$$\begin{aligned} 4A = {} & 2T'_{\mu\nu}\left((\beta\bar{\Box} + c)(\triangle_L^{(2)} - \frac{4\Lambda}{n-2}) + \frac{m^2}{\kappa}\right)^{-1} T^{\mu\nu} \\ & + \frac{2}{n-1}T'\left((\beta\bar{\Box} + c)(\bar{\Box} + \frac{4\Lambda}{n-2}) - \frac{m^2}{\kappa}\right)^{-1} T \\ & - \frac{4\Lambda}{(n-2)(n-1)^2}T'\left((\beta\bar{\Box} + c)(\bar{\Box} + \frac{4\Lambda}{n-2}) - \frac{m^2}{\kappa}\right)^{-1}\left(\bar{\Box} + d\right)^{-1} T \\ & + \frac{2}{(n-2)(n-1)}T'\left(\frac{1}{\kappa} + 4\Lambda k - p\bar{\Box} - \frac{m^2}{2\kappa\Lambda}(n-1)\right)^{-1}\left(\bar{\Box} + d\right)^{-1} T, \end{aligned} \tag{9.4.37}$$

where defined $d \equiv 2\Lambda n/(n-2)(n-1)$. As this is the most general result, it looks cumbersome and also still a formal expression whose explicit computation has not been done yet. But for our purposes this explicit computation is not needed in its full generality: all we need is various limits from this expression. We can, first, study the pole structure of this scattering amplitude, keeping in mind that in this *curved background* the apparent poles do not directly correspond to the masses of the particles (gravitons). These apparent poles are at

$$\bar{\Box}_1 = -\frac{2\Lambda n}{(n-1)(n-2)}, \tag{9.4.38}$$

$$\bar{\Box}_{2,3} = -\frac{1}{\beta}\left(\frac{c}{2} + \frac{2\Lambda\beta}{n-2} \mp \sqrt{\left(\frac{c}{2} - \frac{2\Lambda\beta}{n-2}\right)^2 + \frac{\beta m^2}{\kappa}}\right), \tag{9.4.39}$$

$$\bar{\Box}_4 = \frac{1}{p}\left(\frac{1}{\kappa} + 4\Lambda k - \frac{m^2}{2\kappa\Lambda}(n-1)\right). \tag{9.4.40}$$

Let us now turn our attention to various limiting cases. It is well known that for this spin-2 theory, depending on the order of the limits $m^2 \to 0$, and $\Lambda \to 0$, the results differ: namely one has a discontinuity due to the non-decoupling of the scalar mode which is a pure gauge in massless theory but a dynamical degree of freedom in the massive one.

9.4.3 van Dam-Veltman-Zakharov discontinuity

In (9.4.37) the $m^2 \to 0$ and $\Lambda \to 0$ limits do not commute. To see this more explicitly let $\Lambda \to 0$ first, then one arrives at

$$4A = -2T'_{\mu\nu}\left(\beta\partial^4 + \frac{1}{\kappa}\partial^2 - \frac{m^2}{\kappa}\right)^{-1} T^{\mu\nu} + \frac{2}{n-1}T'\left(\beta\partial^4 + \frac{1}{\kappa}\partial^2 - \frac{m^2}{\kappa}\right)^{-1} T, \tag{9.4.41}$$

whose spectrum has two massive excitations with masses given in (9.4.8). There is a range of parameters for which these masses are non-tachyonic. To see whether these modes are ghosts or not, let us recast the above expression as

$$4A = -\frac{2}{\beta(m_-^2 - m_+^2)}\left(T'_{\mu\nu}\left(\frac{1}{\partial^2 - m_+^2} - \frac{1}{\partial^2 - m_-^2}\right)T^{\mu\nu} - \frac{1}{(n-1)}T'\left(\frac{1}{\partial^2 - m_+^2} - \frac{1}{\partial^2 - m_-^2}\right)T\right), \tag{9.4.42}$$

which says that unless $\beta = 0$, there is a massive ghost dues to the wrong sign of the propagator. When $\beta = 0$, one has the usual Fierz-Pauli massive gravity, with no contribution from the αR^2 term in the action. This is also a source of another discontinuity: had one started with $m^2 = 0$ from the beginning, there would be the usual massive spin-0 mode coming from the αR^2 term as we have seen in the previous sections. So the existence of m^2 through out the computation renders the α term redundant in the flat space.

As explicit examples, let us calculate the Newtonian potential energy, U, between two static point like sources given as

$$T'_{00} \equiv m_1\delta(\vec{x} - \vec{x}_1), \qquad T_{00} \equiv m_2\delta(\vec{x} - \vec{x}_2). \tag{9.4.43}$$

In three and four dimensions, respectively, one finds (whose more details will be given below)

$$\begin{aligned} U &= \frac{1}{2\beta(m_+^2 - m_-^2)} \frac{m_1 m_2}{4\pi}\left(K_0(m_- r) - K_0(m_+ r)\right) && n = 3, \\ U &= \frac{m_1 m_2}{3\beta(m_+^2 - m_-^2)} \frac{1}{4\pi r}\left(e^{-m_- r} - e^{-m_+ r}\right) && n = 4, \end{aligned} \tag{9.4.44}$$

where the distance between the sources is $r \equiv |\vec{x}_1 - \vec{x}_2|$ and $K_0(x)$ is the modified Bessel function of the second kind which whose asymptotic limits are

$$K_0(x \to 0) \to -\log x + C, \qquad K_0(x \to +\infty) \to \sqrt{\frac{\pi}{2x}} e^{-x}. \tag{9.4.45}$$

In the limit $\beta \to 0$, the potential energies become

$$U = -\frac{\kappa}{8\pi} m_1 m_2 K_0(mr) \qquad n = 3, \tag{9.4.46}$$

$$U = -\frac{4}{3}\frac{G m_1 m_2}{r} e^{-mr} \qquad n = 4, \tag{9.4.47}$$

where we used $\kappa = 16\pi G$ in four dimensions. The vDVZ discontinuity [449, 468] is apparent in four dimensions as $m \to 0$, one obtains a discretely different result from that of Newton's or Einstein's gravity in the low energy limit. This cannot be remedied by redefining the Newton's constant since then it spoils the light-bending result by 25 %. The overall conclusion from this linearized analysis is that general relativity with a massless graviton is an isolated theory from all of its $m \neq 0$. The second option, as was noted by Vainshtein long time ago [448] is that one must consider the non-linear effects: namely, the point-particle concept is not appropriate, one must use the Schwarzschild radius of one of the sources. In that case, various scales arise which can be compared to each other. More specifically, the graviton mass can then be compared to be small with respect to other scales. The crux of the matter is that somehow the theory is sensitive to the issue of the smallness of the graviton mass, obviously a dimensionful parameter. One can find the root of the problem as the non-decoupling of the helicity zero mode in the vanishing graviton mass limit to the trace of the energy-momentum tensor. These issues are well treated in the recent reviews [117, 235] as well as the non-linear bi-metric extensions of Fierz-Pauli mass term. Another possible solution to the discontinuity problem is to consider the limit $\frac{m^2}{\Lambda} \to 0$ which yields a smooth limit [232, 261, 380]. Of course this solution barters discontinuity with the non-commutativity of the limits symbolically as $[m^2 \to 0, \Lambda \to 0] \neq 0$. But it has a physical basis: graviton mass is expected to be much smaller than the cosmological constant.

9.4.4 New massive gravity redux

We discussed before that in three dimensions, a specific form of quadratic gravity, called new massive gravity (NMG), turns out to give a non-linear extension of Fierz-Pauli theory. This is unique to three dimensions since pure Einstein theory does not have a dynamical degree of freedom so the existence of the β term does not yield a conflict between the unitarity of the massless mode and the massive mode as is the situation in all dimensions greater than three. This theory was found in a different route in [39] and reproduced from the scattering amplitude computation of this section in [216] which we now repeat. The theory is valid both in (A)dS and flat backgrounds, and for the sake of simplicity, let us focus on the flat space version. So, in the (9.4.37), let us, first, set $m^2 = 0$ then $\Lambda \to 0$ to avoid the above mentioned discontinuity, Then, for generic Einstein-quadratic gravity, still in n dimensions, the scattering amplitude is

$$4A = -2T'_{\mu\nu}\left(\beta\partial^4 + \frac{1}{\kappa}\partial^2\right)^{-1}T^{\mu\nu} + \frac{2}{(n-1)}T'\left(\beta\partial^4 + \frac{1}{\kappa}\partial^2\right)^{-1}T - \frac{2}{(n-1)(n-2)}T'\left(p\partial^4 - \frac{1}{\kappa}\partial^2\right)^{-1}T, \tag{9.4.48}$$

which has 3 poles

$$\partial_1^2 = 0, \qquad \partial_2^2 = -\frac{1}{\kappa\beta} \equiv m_g^2, \qquad \partial_3^2 = \frac{1}{\kappa p} \equiv -m_s^2. \tag{9.4.49}$$

From the second pole, not to have a tachyon, we must have $\kappa\beta < 0$. Let us recast (9.4.48) as

$$4A = -2\kappa T'_{\mu\nu}\left(\frac{1}{\partial^2} - \frac{1}{\partial^2 - m_g^2}\right)T^{\mu\nu} + \frac{2\kappa}{n-1}T'\left(\frac{1}{\partial^2} - \frac{1}{\partial^2 - m_g^2}\right)T - \frac{2\kappa}{(n-1)(n-2)}T'\left(\frac{1}{\partial^2} - \frac{1}{\partial^2 - m_s^2}\right)T. \tag{9.4.50}$$

For nonzero β, the requirement of unitarity necessarily gives two conditions : $n = 3$ and $8\alpha + 3\beta = 0$. This is because the residue of the second pole requires $\kappa < 0$ for unitarity. On the other hand, this negative κ, yields a ghost sign for the residue of the massless pole unless $n = 3$. The third pole is non-tachyonic for negative $8\alpha + 3\beta$, but its residue requires positive $8\alpha + 3\beta$ for it to be a non-ghost. Hence the compromise is to set $8\alpha + 3\beta = 0$ to decouple this mode ending with the promised theory of a massive graviton with 2 degrees of freedom in 3 dimensions. The resulting theory has the action

$$I_{\text{NMG}} = \frac{1}{\kappa}\int d^3x\sqrt{-g}\Big(R - \frac{1}{m_g^2}\big(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\big)\Big), \tag{9.4.51}$$

with $\kappa < 0$. Clearly, in this expression "Einstein" part provides the mass term and the quadratic parts provide the kinetic energy in the linearized limit. This result can also be seen from the Newtonian potential for generic α and β in 3 dimensions which reads

$$U = \frac{\kappa}{8\pi}m_1m_2\Big(K_0(m_gr) - K_0(m_0r)\Big), \tag{9.4.52}$$

where in 3 dimensions $m_0^2 \equiv 1/\kappa(8\alpha + 3\beta)$. So m_0 is a massive ghost that gives a repulsive component. But, for NMG it decouples and one is left with an attractive force, since $\kappa < 0$ and the $K_0(x)$ is positive for all x and blows up at $x = 0$.

The above computation can be supported by linearized static solutions (static black hole solution in the weak field limit) [210]. Inserting the metric ansatz

$$ds^2 = -g_{00}(r)dt^2 + g_{rr}(r)dr^2 + r^2d\theta^2, \tag{9.4.53}$$

to the field equations, one finds up to a first order the following solution

$$g_{00} \approx -1 + c_1K_0\left(m_gr\right), \qquad g_{rr} \approx 1 + c_2K_1\left(m_gr\right), \tag{9.4.54}$$

where c_i are related to the mass of the source.

Next we carry out the canonical description of generic quadratic gravity in 3 dimensions as was one in [210] which provides useful insights and direct identification of degrees of freedom. Curved background case is more involved, we shall only do the flat background computation and refer the reader to [210] where the dS case was considered, from which the AdS case also follows.

The action

$$S = \int d^3x\,\sqrt{-g}\left(\frac{1}{\kappa}R + \alpha R^2 + \beta R_{\mu\nu}^2\right), \tag{9.4.55}$$

when expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, reduces at the quadratic level, up to boundary terms, to

$$S_{\mathcal{O}(h^2)} = -\frac{1}{2}\int d^3x\,h_{\mu\nu}\left(\frac{1}{\kappa}\mathcal{G}_L^{\mu\nu} + (2\alpha+\beta)\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)R_L + \beta\Box\mathcal{G}_L^{\mu\nu}\right), \tag{9.4.56}$$

with the linearized curvature tensors given as

$$\begin{aligned} \mathcal{G}_L^{\mu\nu} &= R_L^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}R_L, \qquad R_L = \partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h, \\ R_L^{\mu\nu} &= \frac{1}{2}\left(\partial_\sigma\partial^\mu h^{\nu\sigma} + \partial_\sigma\partial^\nu h^{\mu\sigma} - \Box h^{\mu\nu} - \partial^\mu\partial^\nu h\right), \qquad h = \eta^{\mu\nu}h_{\mu\nu}, \end{aligned} \tag{9.4.57}$$

with $\Box = \partial_\mu \partial^\mu = -\partial_0^2 + \nabla^2$. As we are after the canonical structure, we can decompose the $h_{\mu\nu}$ as:

$$h_{ij} \equiv \left(\delta_{ij} + \hat{\partial}_i \hat{\partial}_j\right)\phi - \hat{\partial}_i \hat{\partial}_j \chi + \left(\epsilon_{ik}\hat{\partial}_k \hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k \hat{\partial}_i\right)\xi,$$
$$h_{0i} \equiv -\epsilon_{ij}\partial_j \eta + \partial_i N_L, \qquad h_{00} \equiv N, \tag{9.4.58}$$

where the 6 arbitrary functions of that appear in the above decomposition depend on $(t, \vec{x})$. This decomposition is non-local and one has

$$\hat{\partial}_i \equiv \frac{\partial_i}{\sqrt{-\nabla^2}}. \tag{9.4.59}$$

Now we can compute the components of the linearized Einstein tensor as

$$\mathcal{G}^L_{00} = -\frac{1}{2}\nabla^2 \phi, \qquad \mathcal{G}^L_{0i} = -\frac{1}{2}\left(\epsilon_{ik}\partial_k \sigma + \partial_i \dot{\phi}\right), \tag{9.4.60}$$

$$\mathcal{G}^L_{ij} = -\frac{1}{2}\left[\left(\delta_{ij} + \hat{\partial}_i \hat{\partial}_j\right) q - \hat{\partial}_i \hat{\partial}_j \ddot{\phi} - \left(\epsilon_{ik}\hat{\partial}_k \hat{\partial}_j + \epsilon_{jk}\hat{\partial}_k \hat{\partial}_i\right)\dot{\sigma}\right], \tag{9.4.61}$$

where $\dot{\phi} = \partial\phi/\partial t$, etc. 3 functions (ϕ, σ, q) appear in these expressions and since the linearized Einstein tensor is gauge invariant, all these are gauge invariant under the transformations

$$\delta_\zeta h_{\mu\nu} = \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu. \tag{9.4.62}$$

Here, q, σ, are defined as

$$q(t, \vec{x}) \equiv \nabla^2 N - 2\nabla^2 \dot{N}_L + \ddot{\chi}, \qquad \sigma(t, \vec{x}) \equiv \dot{\xi} - \nabla^2 \eta. \tag{9.4.63}$$

With these definitions, it follows that the linearized scalar curvature is simply

$$R_L = q - \Box\phi. \tag{9.4.64}$$

So, the initial 6 arbitrary functions are reduced to 3 arbitrary functions at this stage due to the linearized Bianchi identity, $\partial_\mu \mathcal{G}_L^{\mu\nu} = 0$. Since R_L is also gauge invariant one can choose 3 which are either ϕ, σ, q; or ϕ, σ, R_L combinations. Finally with these fields we arrive at

$$S_{\mathcal{O}(h^2)} = \frac{1}{2}\int d^3x \left[\frac{1}{\kappa}\phi q + (2\alpha + \beta)\left(q - \Box\phi\right)^2 + \beta q \Box \phi\right]$$
$$+ \frac{\beta}{2}\int d^3x \left(\sigma\Box\sigma + \frac{1}{\kappa\beta}\sigma^2\right), \tag{9.4.65}$$

where the σ-field is already decoupled and it is a scalar field with mass-square $m_g^2 \equiv -1/\kappa\beta$ which is nontachyonic for $\kappa\beta < 0$ and a nonghost for $\beta > 0$, therefore $\kappa < 0$.

To be able to discuss the rest of the action, we have to know whether $2\alpha + \beta = 0$, or not. We assume that this combination is not zero (see the original paper [210] for the other route). Then q is non-dynamical and can be integrated out to get a generically higher derivative action for the ϕ field.

$$S_\phi = \frac{1}{2}\int d^3x \left[\frac{\beta(8\alpha+3\beta)}{4(2\alpha+\beta)}(\Box\phi)^2 + \frac{(4\alpha+\beta)}{2\kappa(2\alpha+\beta)}\phi\Box\phi - \frac{1}{4\kappa^2(2\alpha+\beta)}\phi^2\right]. \tag{9.4.66}$$

For generic α and β, but $2\alpha+\beta \neq 0$, (9.4.66) describes a ghost-inflected higher-derivative Pais-Uhlenbeck oscillator which can be decoupled with the definition of two new fields as

$$\varphi_1 \equiv \phi - \frac{\Box\phi}{m_g^2}, \qquad \varphi_2 \equiv \phi - \frac{\Box\phi}{m_s^2}, \tag{9.4.67}$$

which nicely turns (9.4.66) to

$$S_\phi = \frac{1}{64\kappa(2\alpha+\beta)^2}\int d^3x \left[(8\alpha+3\beta)^2\varphi_1\left(\Box - m_s^2\right)\varphi_1 - \beta^2\varphi_2\left(\Box - m_g^2\right)\varphi_2\right], \tag{9.4.68}$$

with m_g being the same as the mass of the σ-field and m_s as

$$m_s^2 = \frac{1}{\kappa(8\alpha+3\beta)}.$$

For $8\alpha+3\beta < 0$, φ_1 is nontachyonic similar to φ_2, but unlike φ_2, it describes a ghostlike excitation since its kinetic energy is negative. For the NMG case, this mode becomes infinitely massive and decouples and one ends up with

$$S_\phi = -\frac{1}{16\kappa}\int d^3x\, \varphi_2\left(\Box - m_g^2\right)\varphi_2. \tag{9.4.69}$$

In this formalism we got the masses of the unitary excitations but one thing is still missing: these two fields σ and φ_2 form helicity ± 2 modes of the massive spin-2 graviton. To be able to see this, we should, first, scale these fields to their canonical dimensions and work out the transformations under the full Lorentz-group and find their helicities which we don't do here. Instead we go back to the generic massive, quadratic gravity theory and compute the potential energy beyond the static Newtonian limit as there are unexpected results.

9.4.5 Spin-spin, spin-orbit, orbit-orbit interactions

Let us now move on beyond the Newtonian potential in these theories and compute various higher order terms coming due to the spin and orbital motion of the sources [217, 432]. We will again work in the point-particle context with two conserved sources, $\partial_\mu T^{\mu\nu} = 0$. We have to be careful about the explicit form of the energy-momentum tensor up to the desired order. Let us follow Weinberg on this [456] and define

$$T_{00} = T_{00}^{(0)} + T_{00}^{(2)}, \qquad T_{i0} = T_{i0}^{(1)}, \qquad T_{ij} = T_{ij}^{(2)}, \tag{9.4.70}$$

where one has $T^{(0)}_{00} = m\delta(\vec{x} - \vec{x}_a)$ as in the previous section and the rest reads as

$$\begin{aligned}
T^{(2)}_{00} &= \frac{1}{2}m\vec{v}^2\delta(\vec{x} - \vec{x}_a) - \frac{1}{2}J^k v^i \epsilon^{ikj}\partial_j\delta(\vec{x} - \vec{x}_a)\,,\\
T^{(1)}_{i0} &= -mv^i\delta(\vec{x} - \vec{x}_a) + \frac{1}{2}J^k\,\epsilon^{ikj}\partial_j\delta(\vec{x} - \vec{x}_a)\,,\\
T^{(2)}_{ij} &= mv^i v^j\delta(\vec{x} - \vec{x}_a) + J^l v^{(i}\epsilon^{j)kl}\partial_k\delta(\vec{x} - \vec{x}_a)\,,\\
T &= -T_{00} + \delta^{ij}T_{ij} = -m_a\delta(\vec{x} - \vec{x}_a) + \frac{1}{2}m_a\vec{v}_a^2\delta(\vec{x} - \vec{x}_a)\\
&\quad - \frac{1}{2}J^k_a v^i_a\epsilon^{ikj}\partial_j\delta(\vec{x} - \vec{x}_a)\,,
\end{aligned} \tag{9.4.71}$$

with $\vec{x}_a = \vec{x}_a(t)$ being the position of the particle with $a = 1, 2$. The above forms is dictated by the conservation requirement of the energy-momentum tensor up to $\mathscr{O}(v^2)$ and $\mathscr{O}(vJ)$. This fixes the relative coefficients and signs of the various terms. Let us show one example along these lines and prove $\partial_\mu T^{\mu 0} = 0$:

$$\begin{aligned}
\partial_0 T^{00} + \partial_i T^{i0} &= m_a\partial_0\delta(\vec{x} - \vec{x}_a) + m_a v^i_a\partial_i\delta(\vec{x} - \vec{x}_a)\\
&\quad - \frac{1}{2}J^k_a\,\epsilon^{ikj}\partial_i\partial_j\delta(\vec{x}, -\vec{x}_a)
\end{aligned} \tag{9.4.72}$$

with the last term vanishing due to symmetry obviously, and the first two terms cancel since $\partial_0\delta(\vec{x} - \vec{x}_a) = -v^l_a\partial_l\delta(\vec{x} - \vec{x}_a)$ and $\vec{v} = \frac{d\vec{x}}{dt}$, one has the desired result. Other components work similarly. Now we are ready to compute expressions of the form

$$2Ut = -\kappa T'_{\mu\nu}\frac{1}{\partial^2}T^{\mu\nu} + \frac{\kappa}{n-2}T'\frac{1}{\partial^2}T, \tag{9.4.73}$$

which in the more explicit form is actually the following double integral

$$\begin{aligned}
2Ut &= -\kappa\int d^n x d^n x' T_{\mu\nu}(x')\,G(x, x')\,T^{\mu\nu}(x)\\
&\quad + \frac{\kappa}{n-2}\int d^n x d^n x' T(x')\,G(x, x')\,T(x)\,.
\end{aligned} \tag{9.4.74}$$

Notice that instead of the tensorial Green's function we have used the scalar one which is much easier to handle. To have the consistent conventions we have

$$\partial^2_x G(x, x') = -\delta(x, x')\,, \tag{9.4.75}$$

where $\partial^2_x = -\partial^2_t + \vec{\nabla}^2$. As we shall use the retarded Green's functions, their explicit forms are easy to find for the massless case

$$\frac{1}{\partial^2} \equiv G_R(x,\, x') = \frac{\Gamma\left(\frac{n-3}{2}\right)}{4\pi^{\frac{n-1}{2}}}\frac{\delta\left[r - \left(t - t'\right)\right]}{r^{n-3}}, \tag{9.4.76}$$

and for the massive case

$$\frac{1}{\partial^2 - m_g^2} \equiv G_R\left(x, x'\right) = \frac{\left(\frac{m_g}{r}\right)^{\frac{n-3}{2}}}{(2\pi)^{\frac{n-1}{2}}} K_{\frac{n-3}{2}}\left(r\, m_g\right) \delta\left[r - \left(t - t'\right)\right], \tag{9.4.77}$$

with r being the radial distance and K_ν the modified Bessel function of the second kind. Computation in generic n dimensions is quite cumbersome [217] and does not give us much insight as there are many angular momenta, so from now on we shall stick to the four dimensions. The following relations will be used throughout the computation

$$\begin{aligned}
\partial_k r &= \frac{(x^k - x'^k)}{r} = \hat{r}^k, & \partial_k \frac{1}{r} &= -\frac{(x^k - x'^k)}{r^3} = -\frac{\hat{r}^k}{r^2}, \\
\partial_{k'} r &= -\frac{(x^k - x'^k)}{r} = -\hat{r}^k, & \partial_{k'} \frac{1}{r} &= \frac{(x^k - x'^k)}{r^3} = \frac{\hat{r}^k}{r^2}, \\
\partial_k \partial_{n'} r &= \frac{1}{r}\left(-\delta^{kn} + \hat{r}^k \hat{r}^n\right), & \partial_k \partial_{n'} \frac{1}{r} &= \frac{1}{r^3}\left(\delta^{kn} - 3\hat{r}^k \hat{r}^n\right),
\end{aligned} \tag{9.4.78}$$

which are valid as long as $r \neq 0$ which is the case at hand. (At $r = 0$, one picks up the Fermi-contact term which is relevant in quantum mechanics but not here.)

9.4.6 Gravitomagnetic effects in general relativity

Let us, first, do the computation in general relativity before we move on to massive and quadratic gravities. In the short-hand notation of suppressing the integrals, we have to compute the following

$$4Ut = -2\kappa T'_{00} \frac{1}{\partial^2} T^{00} - 4\kappa T'_{0i} \frac{1}{\partial^2} T^{0i} - 2\kappa T'_{ij} \frac{1}{\partial^2} T^{ij} + \kappa T' \frac{1}{\partial^2} T, \tag{9.4.79}$$

A detailed separate computation of each term [432] finally yields

$$\begin{aligned}
U_{GR} = &-\frac{G m_1 m_2}{r}\left(1 + \frac{3}{2}\vec{v}_1^2 + \frac{3}{2}\vec{v}_2^2 - 4\vec{v}_1 \cdot \vec{v}_2\right) - \frac{G}{r^3}\left(\vec{J}_1 \cdot \vec{J}_2 - 3\vec{J}_1 \cdot \hat{r}\, \vec{J}_2 \cdot \hat{r}\right) \\
&- \frac{G}{2r^3}\left(\frac{3m_1}{m_2}\vec{L}_2 \cdot \vec{J}_2 - \frac{3m_2}{m_1}\vec{L}_1 \cdot \vec{J}_1 - 4\vec{L}_1 \cdot \vec{J}_2 + 4\vec{L}_2 \cdot \vec{J}_1\right),
\end{aligned} \tag{9.4.80}$$

where we defined $\vec{L}_i = m_i \vec{r} \times \vec{p}_i$. This expression includes the Newtonian potential energy, plus relativistic corrections such as spin-spin and spin-orbit effects. Note that $\vec{v}_1$ and $\vec{v}_2$ are defined with respect to a frame at rest. The second bracketed term in the first line is the spin-spin interaction. It is not difficult to show that term is minimized in general relativity when two spins are minimized along the line joining the sources as shown in Figure 9.2.

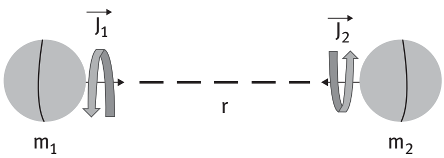

Figure 9.2: Minimum energy configuration in general relativity: In the weak field that we are working, spins are anti-parallel to each other, so the total spin of the two-body system is minimized.

9.4.7 Gravitomagnetic effects in massive gravity

For the case of Fierz-Pauli massive gravity (in short mGR) the computation is somewhat more complicated whose details are all given in [432]. Again one needs to compute

$$\begin{aligned}4Ut = &-2\kappa T'_{00}\frac{1}{\partial^2 - m_g^2}T^{00} + \frac{2\kappa}{3}T'\frac{1}{\partial^2 - m_g^2}T\\ &-4\kappa T'_{0i}\frac{1}{\partial^2 - m_g^2}T^{0i} - 2\kappa T'_{ij}\frac{1}{\partial^2 - m_g^2}T^{ij},\end{aligned} \tag{9.4.81}$$

which eventually yields

$$\begin{aligned}U_{mGR} = &-\frac{4Gm_1m_2}{3r}e^{-m_gr}\left(1+\vec{v}_1^2+\vec{v}_2^2-3\vec{v}_1\cdot\vec{v}_2\right)\\ &-\frac{Ge^{-m_gr}\left(1+m_gr+m_g^2r^2\right)}{r^3}\left(\vec{J}_1\cdot\vec{J}_2-3\vec{J}_1\cdot\hat{r}\vec{J}_2\cdot\hat{r}\frac{1+m_gr+\frac{1}{3}m_g^2r^2}{1+m_gr+m_g^2r^2}\right)\\ &-\frac{4Ge^{-m_gr}}{3r^3}\left(1+m_gr\right)\left(\frac{m_1}{m_2}\vec{L}_2\cdot\vec{J}_2-\frac{m_2}{m_1}\vec{L}_1\cdot\vec{J}_1-\frac{3}{2}\vec{L}_1\cdot\vec{J}_2+\frac{3}{2}\vec{L}_2\cdot\vec{J}_1\right).\end{aligned} \tag{9.4.82}$$

The vDVZ discontinuity in the Newtonian sector was already discussed. But in some sense "gravitomagnetic partners" of this discontinuity arises since in the $m_g \longrightarrow 0$ limit for not too large distances from (9.4.82), one obtains

$$\begin{aligned}U_{mGR} \longrightarrow &-\frac{4Gm_1m_2}{3r}\left(1+\vec{v}_1^2+\vec{v}_2^2-3\vec{v}_1\cdot\vec{v}_2\right)\\ &-\frac{G}{r^3}\left(\vec{J}_1\cdot\vec{J}_2-3\vec{J}_1\cdot\hat{r}\vec{J}_2\cdot\hat{r}\right)\\ &-\frac{4G}{3r^3}\left(\frac{m_1}{m_2}\vec{L}_2\cdot\vec{J}_2-\frac{m_2}{m_1}\vec{L}_1\cdot\vec{J}_1-\frac{3}{2}\vec{L}_1\cdot\vec{J}_2+\frac{3}{2}\vec{L}_2\cdot\vec{J}_1\right).\end{aligned} \tag{9.4.83}$$

So the spin-spin part, the second line, smoothly reduces to the general relativity expression in this limit, while a new discontinuity arises (a discrete 8/9 difference between general relativity and massive general relativity (mGR)) in the $O(v^2)$ and $O(vJ)$ terms. But the real surprise is that for large distances as observed in [217], the spin-spin

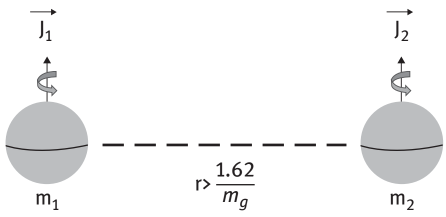

Figure 9.3: Minimum energy configuration in massive gravity, unlike the case of general relativity, at large distances the total spin is maximized and their orientation is perpendicular to the line joining the sources. For small separations, massive gravity's prediction coincide with that of general relativity.

part also gives a distinctly different answer from general relativity. To see this let us look at the large r limit of the spin-spin interaction in mGR only:

$$U_{SS} = -\frac{Ge^{-m_g r} m_g^2}{r}\left(\vec{J}_1 \cdot \vec{J}_2 - \vec{J}_1 \cdot \hat{r} \vec{J}_2 \cdot \hat{r}\right). \tag{9.4.84}$$

The relative coefficient between the two terms becomes −1 instead of −3 which makes all the difference: minimization of this energy is achieved when the total spin is maximized as shown in Figure 9.3. More precisely, for $m_g\, r \geq \frac{1+\sqrt{5}}{2} \approx 1.62$, the total spin is maximized. Getting this number (which happens to be the Golden number) requires a rather long computation given in the appendix of [217].[5] So, at large separations massive gravity predicts a rotating two-body system for equal spins while general relativity has a non-rotating system.

9.4.8 Gravitomagnetic effects in quadratic gravity

Let us consider the case of quadratic gravity without the Fierz-Pauli mass term. Then we must compute

$$U_{quad}t = -\frac{\kappa}{2}T'_{\mu\nu}\frac{1}{\partial^2}T^{\mu\nu} + \frac{\kappa}{4}T'\frac{1}{\partial^2}T + \frac{\kappa}{2}T'_{\mu\nu}\frac{1}{\partial^2 - m_\beta^2}T^{\mu\nu}$$
$$-\frac{\kappa}{6}T'\frac{1}{\partial^2 - m_\beta^2}T - \frac{\kappa}{12}T'\frac{1}{\partial^2 - m_0^2}T, \tag{9.4.85}$$

where there are two additional massive modes: a massive spin-2 graviton with $m_\beta^2 = -\frac{1}{\kappa\beta}$ and a massive spin-0 mode with $m_0^2 = \frac{1}{4\kappa(3\alpha+\beta)}$. Massless spin-2 mode of general relativity is intact and so there will be terms added to (9.4.80). Here we define $U_{quad} \equiv U_{GR} + U_2$, where

5 This unexpected result of spin flip at large separations in massive gravity was first found in a numerical simulation of Aykutlu Dane who was collaborating with one of the authors of this book for N-body simulations of weak field massive gravity. Later the analytical computation was published in [217]. What is also quite interesting is that the Golden number also shows up in non-relativistic Yukawa type force of massive gravity: one does not have stable circular orbits for the pote ntial $V(r) = -\frac{GM}{r}e^{-m_g r}$ for distances that satisfy $m_g\, r > \frac{1+\sqrt{5}}{2}$. This amusing fact, which was mentioned to us by Ferit Oktem, first, is easy to work out and left to the reader.

$$\begin{aligned}U_2t = &\frac{\kappa}{2}T'_{00}\left(\partial^2 - m_\beta^2\right)^{-1}T^{00} + \kappa T'_{0i}\left(\partial^2 - m_\beta^2\right)^{-1}T^{0i}\\&+\frac{\kappa}{2}T'_{ij}\left(\partial^2 - m_\beta^2\right)^{-1}T^{ij} - \frac{\kappa}{6}T'\left(\partial^2 - m_\beta^2\right)^{-1}T\\&-\frac{\kappa}{12}T'\left(\partial^2 - m_c^2\right)^{-1}T.\end{aligned} \tag{9.4.86}$$

The required computation long but is similar so we can simply write the final result

$$\begin{aligned}U_2 = &\frac{Gm_1m_2}{r}\left[\left(\frac{4}{3} + \frac{7}{3}\vec{v}_1^{\,2} + \frac{7}{3}\vec{v}_2^{\,2}\right)e^{-rm_\beta} - \left(\frac{1}{3} - \frac{1}{6}\vec{v}_1^{\,2} - \frac{1}{6}\vec{v}_2^{\,2}\right)e^{-rm_0}\right]\\&+\frac{4G\left(1 + rm_\beta\right)e^{-rm_\beta}}{3r^3}\left(\frac{m_1}{m_2}\vec{L}_2\cdot\vec{J}_2 - \frac{m_2}{m_1}\vec{L}_1\cdot\vec{J}_1 - \frac{3}{2}\vec{L}_1\cdot\vec{J}_2 + \frac{3}{2}\vec{L}_2\cdot\vec{J}_1\right)\\&+\frac{Ge^{-rm_0}\left(1 + rm_0\right)}{6r^3}\left(\frac{m_1}{m_2}\vec{L}_2\cdot\vec{J}_2 - \frac{m_2}{m_1}\vec{L}_1\cdot\vec{J}_1\right)\\&+\frac{Ge^{-rm_\beta}}{r^3}\left(1 + rm_\beta + r^2m_\beta^2\right)\\&\times\left(\vec{J}_1\cdot\vec{J}_2 - 3\vec{J}_1\cdot\hat{r}\vec{J}_2\cdot\hat{r}\frac{1 + rm_\beta + \frac{1}{3}r^2m_\beta^2}{1 + rm_\beta + r^2m_\beta^2}\right).\end{aligned} \tag{9.4.87}$$

Let us forget for a moment that we are working in the weak field limit valid well outside the compact sources and consider the $r \longrightarrow 0$ limit, where we expect higher curvature terms to play a role, in the potential U_{quad} which reads

$$U_{quad} \xrightarrow{r\longrightarrow 0} \frac{Gm_1m_2}{r}(v_1^2 + v_2^2) + \text{constant}. \tag{9.4.88}$$

Save this repulsive $O(v^2)$ term, in this limit all the other terms drop out as there are cancellations between Einstein and quadratic parts. Repulsive nature of quadratic gravity at small separations is expected and desired since it leads to a less divergent in the UV regime compared to Einstein's theory. But as we have shown, non-zero β gives a spin-2 ghost so the theory cannot be viable as a quantum theory.

9.4.9 Photon-photon scattering in massless and massive gravity

Before we close this section, we cannot resist to reproduce the old but very amusing result [440] about the gravitationally interacting photons (or massless particles) that move parallel or anti-parallel to each other. Zee [470] also has a nice discussion on this issue. Instead of considering the energy-momentum tensor of the photons (or more properly the electromagnetic waves) that come from Maxwell's theory, let us consider the spinless point-like particles with the four velocities given as

$$u_1^\mu = (1, 0, 0, 1), \qquad u_2^\mu = (1, 0, 0, \sigma), \tag{9.4.89}$$

where $\sigma = 1$ refers to parallel motion and $\sigma = -1$ anti-parallel motion, say in the z-direction. For the sake of simplicity, we shall work in a flat background. The energy-momentum tensor for each photon is

$$T^{\mu\nu} = Eu^\mu u^\nu, \tag{9.4.90}$$

with a vanishing trace $T = 0$. This is important since it implies that the result will be the same in general relativity and massive gravity. Then one has,

$$\begin{aligned} 4Ut &= -2\kappa T'_{\mu\nu}\frac{1}{\partial^2}T^{\mu\nu}, \\ &= -2\kappa T'_{00}\frac{1}{\partial^2}T^{00} - 4\kappa T'_{0i}\frac{1}{\partial^2}T^{0i} - 2\kappa T'_{ij}\frac{1}{\partial^2}T^{ij}. \end{aligned} \tag{9.4.91}$$

Each term, after the Green's function is inserted and the integrals are carried out yields

$$\begin{aligned} T'_{00}\frac{1}{\partial^2}T^{00} &= \frac{E_1E_2}{4\pi r}t, \qquad T'_{0i}\frac{1}{\partial^2}T^{0i} = -\frac{E_1E_2\sigma}{4\pi r}t, \\ T'_{ij}\frac{1}{\partial^2}T^{ij} &= \frac{E_1E_2\sigma^2}{4\pi r}t. \end{aligned} \tag{9.4.92}$$

Finally summing them up, one arrives at the gravitational potential energy between two massless point objects (or photons) as

$$U = \frac{-2GE_1E_2}{r}(1-\sigma)^2\,. \tag{9.4.93}$$

So, at this level of approximation, two photons that move parallel in the same direction ($\sigma = 1$) do not interact with each other while anti-parallel moving photons ($\sigma = -1$) interact with four times the expected strength. For the parallel moving photons, could this result yield something "physical" for two photons that originate from two nearby sources and travel several billion years together ? This requires more scrutiny.

9.5 Conserved charges in generic gravity

Following the notation of [126] let us review the construction of conserved charges (mass and angular momenta) for asymptotically constant curvature spacetimes in generic metric-based gravity theories. Let the full field equations coupled to a bounded, covariantly conserved matter source read

$$\Phi_{\mu\nu}(g, R, \nabla R, R^2, \ldots) = \kappa\tau_{\mu\nu}, \tag{9.5.1}$$

where in this notation R represents any possible curvature tensor. What we are given is that the "Bianchi identity" $\nabla^\mu\Phi_{\mu\nu} = 0$ holds or that the field equations come from a diffeomorphism invariant action (at least up to a boundary term as). Note that κ in this equation is the bare Newton's constant. Now let us split the full metric describing a given solution to the full equations as the sum of a background $\bar{g}_{\mu\nu}$, which solves (9.5.1) in the absence of a source term, and a deviation $h_{\mu\nu}$, that vanishes sufficiently rapidly at infinity,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{9.5.2}$$

Under this decomposition, the field equation splits to a linear part that we keep on the left and all the non-linear terms that we carry to the right eventually arriving at

$$\Phi^L_{\mu\nu} \equiv \mathscr{O}(\bar{g})_{\mu\nu\alpha\beta} h^{\alpha\beta} = \kappa T_{\mu\nu}, \tag{9.5.3}$$

where now $T_{\mu\nu}$ represents the localized matter source plus all the $\mathscr{O}(h^n)$ corrections with $n \geq 2$. So we keep the linear terms on the left and the four indexed operator $\mathscr{O}(\bar{g})_{\mu\nu\alpha\beta}$ can be found once the theory is given. [Of course what is tacitly assumed here is that the linear order terms do not vanish. As we shall see, sometimes this actually happens in certain theories for curved backgrounds and this has interesting consequences.] The background metric is assumed to satisfy the full-nonlinear source-free equations, namely

$$\Phi_{\mu\nu}(\bar{g}, \bar{R}, \bar{\nabla}\bar{R}, \bar{R}^2 \ldots) = 0. \tag{9.5.4}$$

The full Bianchi identity of the theory

$$\nabla^\mu \Phi_{\mu\nu}(g, R, \nabla R, R^2, \ldots) = \kappa \nabla^\mu \tau_{\mu\nu} = 0, \tag{9.5.5}$$

is inherited as a linearized, background Bianchi identity of the form

$$\bar{\nabla}^\mu \Big(\mathscr{O}(\bar{g})_{\mu\nu\alpha\beta} h^{\alpha\beta} \Big) = \kappa \bar{\nabla}^\mu T_{\mu\nu} = 0, \tag{9.5.6}$$

which will play an important role in what follows. Clearly we have a covariant conservation law as $\bar{\nabla}^\mu T_{\mu\nu} = 0$, which is locally equal to a partial, ordinary conservation law but it cannot be integrated to give a total charge. So we must have a true, namely $\partial_\mu j^\mu = 0$ type, conservation law. For this purpose, we have to invoke the symmetries of the background spacetime. Assume that the background metric admits a set of Killing vectors $\bar{\xi}^a_\mu$,

$$\bar{\nabla}_\mu \bar{\xi}^a_\nu + \bar{\nabla}_\nu \bar{\xi}^a_\mu = 0, \tag{9.5.7}$$

where the latin index refers to different possible Killing vectors.[6] Then it is easy to see that one has an ordinarily conserved vector density current:

$$\begin{aligned}
\bar{\nabla}_\mu \left(\sqrt{-\bar{g}} T^{\mu\nu} \bar{\xi}^a_\nu \right) &= \sqrt{-\bar{g}} \bar{\nabla}_\mu T^{\mu\nu} \bar{\xi}^a_\nu = \sqrt{-\bar{g}} \left(\partial_\mu T^{\mu\nu} \bar{\xi}^a_\nu + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} \bar{\xi}^a_\nu \right) \\
&= \sqrt{-\bar{g}} \partial_\mu T^{\mu\nu} \bar{\xi}^a_\nu + \partial_\mu \sqrt{-\bar{g}} T^{\lambda\nu} \bar{\xi}^a_\nu \\
&= \partial_\mu \left(\sqrt{-\bar{g}} T^{\mu\nu} \bar{\xi}^a_\nu \right) = 0,
\end{aligned} \tag{9.5.8}$$

6 Note that if there are no Killing vectors, then we cannot define conserved quantities and the whole procedure that will follow fails.

where we used $\partial_\mu \sqrt{-\bar{g}} = \Gamma^\rho_{\rho\nu}\sqrt{-\bar{g}}$. So the needed conserved current is $j^\mu_a = \sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}^a_\nu$. Therefore, the conserved Killing charges are expressed as

$$Q(\bar{\xi}^a) = \int_\Sigma d^{n-1}x\sqrt{-\bar{g}}T^{0\nu}\bar{\xi}^a_\nu = \oint_{\partial\Sigma} dS_i\sqrt{-\bar{g}}\mathscr{F}^{0i}\,. \tag{9.5.9}$$

Here Σ is a spatial $(n-1)$ hypersurface which is a foliation of the full spacetime $\mathscr{M}$ and $\partial\Sigma$ is its $(n-2)$ dimensional boundary; $\mathscr{F}^{0i}$ is an antisymmetric tensor obtained from $\mathscr{O}(\bar{g})$. Given the theory, one can find this antisymmetric tensor, albeit sometimes in a lengthy procedure. In passing to the surface integral, we have made use of the usual Stokes' theorem which reads

$$\int_\Sigma d^{n-1}x\,\sqrt{-g}\nabla_\mu V^\mu = \oint_{\partial\Sigma} d^{n-2}y\,\sqrt{-g}n_\mu V^\mu. \tag{9.5.10}$$

The appearance of the antisymmetric tensor (9.5.9) is not a mystery: locally one always has, for a conserved tensor, $T^{\mu\nu}\bar{\xi}^a_\nu = \bar{\nabla}_\nu\mathscr{F}^{\mu\nu}$. Owing to the antisymmetry of $\mathscr{F}^{\mu\nu}$, one has

$$\begin{aligned}\bar{\nabla}_\mu\bar{\nabla}_\nu\mathscr{F}^{\mu\nu} &= \frac{1}{2}\left[\bar{\nabla}_\mu,\bar{\nabla}_\nu\right]\mathscr{F}^{\mu\nu}\\ &= \frac{1}{2}\left(\bar{R}_{\mu\nu}{}^{\mu}{}_{\lambda}\mathscr{F}^{\lambda\nu} + \bar{R}_{\mu\nu}{}^{\nu}{}_{\lambda}\mathscr{F}^{\lambda\nu}\right)\\ &= \frac{1}{2}\left(\bar{R}_{\nu\lambda}\mathscr{F}^{\lambda\nu} - \bar{R}_{\mu\lambda}\mathscr{F}^{\mu\lambda}\right) = 0.\end{aligned} \tag{9.5.11}$$

For cosmological Einstein's theory, this procedure led in [1] to the following conserved charge expression in four dimensions:

$$Q(\bar{\xi}) = \frac{1}{8\pi G}\oint_{\partial\Sigma} dS_i\sqrt{-\bar{g}}\left(\bar{\xi}_\nu\bar{\nabla}_\beta K^{0i\nu\beta} - K^{0j\nu i}\bar{\nabla}_j\bar{\xi}_\nu\right), \tag{9.5.12}$$

where the so called superpotential $K^{\mu\alpha\nu\beta}$ is defined as

$$\begin{aligned}K^{\mu\nu\alpha\beta} &\equiv \frac{1}{2}\left(\bar{g}^{\mu\beta}H^{\nu\alpha} + \bar{g}^{\nu\alpha}H^{\mu\beta} - \bar{g}^{\mu\nu}H^{\alpha\beta} - \bar{g}^{\alpha\beta}H^{\mu\nu}\right),\\ H^{\mu\nu} &\equiv h^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}h,\end{aligned} \tag{9.5.13}$$

which has the symmetries of the Riemann tensor as

$$K^{\mu\alpha\nu\beta} = K^{\nu\beta\mu\alpha} = -K^{\alpha\mu\nu\beta} = -K^{\mu\alpha\beta\nu}. \tag{9.5.14}$$

This is the famous AD charge expression for cosmological Einstein's theory whose detailed derivation will be give below. Here one should notice that when the background Killing vector is time-like, this expression gives the total energy of the spacetime, for space-like Killing vectors it gives the total angular momentum as will be demonstrated with explicit examples later on. For globally maximally symmetric spacetimes, flat, de Sitter and AdS spaces the charges vanish by definition and by construction.

To arrive at the AD charge, as seen above the crucial step is to find the antisymmetric tensor $\mathscr{F}^{\mu\nu}$ defined as as $T^{\mu\nu}\bar{\xi}^a_\nu = \bar{\nabla}_\nu \mathscr{F}^{\mu\nu}$. For this purpose it is better to write $T^{\mu\nu} \sim \bar{\nabla}_\alpha\bar{\nabla}_\beta K^{\alpha\beta\mu\nu} + X^{\mu\nu}$ as was done in [1], since it allows us to interchange the order of the derivatives only one time in term $T^{\mu\nu}\bar{\xi}^a_\nu \sim \bar{\xi}^a_\nu\bar{\nabla}_\alpha\bar{\nabla}_\beta K^{\alpha\beta\mu\nu} + \bar{\xi}^a_\nu X^{\mu\nu}$ as follows,

$$\begin{aligned} T^{\mu\nu}\bar{\xi}^a_\nu &\sim \bar{\xi}^a_\nu\bar{\nabla}_\alpha\bar{\nabla}_\beta K^{\alpha\beta\mu\nu} + \bar{\xi}^a_\nu X^{\mu\nu} \\ &= \bar{\nabla}_\alpha\left(\bar{\xi}^a_\nu\bar{\nabla}_\beta K^{\alpha\beta\mu\nu}\right) - \left(\bar{\nabla}_\alpha\bar{\xi}^a_\nu\right)\bar{\nabla}_\beta K^{\alpha\beta\mu\nu} + \bar{\xi}^a_\nu X^{\mu\nu}. \end{aligned} \tag{9.5.15}$$

Let us find out what from $K^{\alpha\beta\mu\nu}$ and $X^{\mu\nu}$ have in the cosmological Einstein's theory whose $T^{\mu\nu}$ is

$$\begin{aligned} \kappa T^{\mu\nu} = \mathscr{G}^{\mu\nu}_L =& \frac{1}{2}\left(-\bar{\Box}h^{\mu\nu} - \bar{\nabla}^\mu\bar{\nabla}^\nu h + \bar{\nabla}^\sigma\bar{\nabla}^\nu h^\mu_\sigma + \bar{\nabla}^\sigma\bar{\nabla}^\mu h^\nu_\sigma\right) \\ &- \frac{1}{2}\bar{g}^{\mu\nu}\left(-\bar{\Box}h + \bar{\nabla}_\rho\bar{\nabla}_\sigma h^{\rho\sigma} - \frac{2\Lambda}{n-2}h\right) - \frac{2}{n-2}\Lambda h^{\mu\nu}. \end{aligned} \tag{9.5.16}$$

Reshuffling the covariant derivatives we have

$$\begin{aligned} \mathscr{G}^{\mu\nu}_L =& \bar{\nabla}_\alpha\bar{\nabla}_\beta\Big[\frac{1}{2}\big(\bar{g}^{\nu\beta}h^{\mu\alpha} + \bar{g}^{\mu\beta}h^{\alpha\nu} - \bar{g}^{\alpha\beta}h^{\mu\nu} \\ &-\bar{g}^{\mu\nu}h^{\alpha\beta} + \bar{g}^{\mu\nu}\bar{g}^{\alpha\beta}h - \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}h\big)\Big] \\ &+ \frac{\Lambda}{n-2}\left(\bar{g}^{\mu\nu}h - 2h^{\mu\nu}\right). \end{aligned} \tag{9.5.17}$$

After rearranging the terms one arrives at the superpotential $K^{\alpha\beta\mu\nu}$ given in (9.5.13). With the definition of $H^{\mu\nu}$, the remaining piece of $X^{\mu\nu}$ takes the form

$$\begin{aligned} X^{\mu\nu} &= \frac{1}{2}\bar{g}^{\nu\beta}\left[\bar{\nabla}_\alpha, \bar{\nabla}_\beta\right]h^{\mu\alpha} + \frac{\Lambda}{n-2}\left(\bar{g}^{\mu\nu}h - 2h^{\mu\nu}\right) \\ &= \frac{1}{2}\left[\bar{\nabla}_\alpha, \bar{\nabla}^\nu\right]H^{\mu\alpha} - \frac{2\Lambda}{n-2}H^{\mu\nu}. \end{aligned} \tag{9.5.18}$$

Here, $X^{\mu\nu}$ should be symmetric, and it can be explicitly shown that it reduces to a simpler form

$$X^{\mu\nu} = \frac{1}{2}K^{\mu\alpha\rho\beta}\bar{R}^\nu{}_{\alpha\rho\beta}. \tag{9.5.19}$$

Therefore, for the linearized cosmological Einstein's theory in n dimensions, we have

$$\kappa T^{\mu\nu} = \mathscr{G}^{\mu\nu}_L = \bar{\nabla}_\alpha\bar{\nabla}_\beta K^{\mu\alpha\nu\beta} + X^{\mu\nu}. \tag{9.5.20}$$

Then, moving on to the vector current $T^{\mu\nu}\bar{\xi}_\nu$, we have

$$\begin{aligned}\bar{\xi}_\nu \mathscr{G}_L^{\mu\nu} &= \bar{\xi}_\nu \bar{\nabla}_\alpha \bar{\nabla}_\beta K^{\mu\alpha\nu\beta} + X^{\mu\nu}\bar{\xi}_\nu \\ &= \bar{\nabla}_\alpha \left(\bar{\xi}_\nu \bar{\nabla}_\beta K^{\mu\alpha\nu\beta}\right) - \left(\bar{\nabla}_\alpha \bar{\xi}_\nu\right) \bar{\nabla}_\beta K^{\mu\alpha\nu\beta} + X^{\mu\nu}\bar{\xi}_\nu \\ &= \bar{\nabla}_\alpha \left(\bar{\xi}_\nu \bar{\nabla}_\beta K^{\mu\alpha\nu\beta} - K^{\mu\beta\nu\alpha}\bar{\nabla}_\beta \bar{\xi}_\nu\right).\end{aligned} \tag{9.5.21}$$

Note that not to clutter the notation, we have dropped the extra index on the Killing vector which we shall do from now on. With this expression, one gets the AD conserved charge (9.5.12).

In [126], without using the superpotential, a more straightforward expression for the conserved charges was given. Since that construction is somewhat better suited to the higher derivative theories, let us recap the reformulation of the AD charges along those lines. One has

$$\begin{aligned}2\bar{\xi}_\nu \mathscr{G}_L^{\mu\nu} &= 2\bar{\xi}_\nu R_L^{\mu\nu} - \bar{\xi}_\nu \bar{g}^{\mu\nu} R^L - \frac{4\Lambda}{n-2}\bar{\xi}_\nu h^{\mu\nu} \\ &= \bar{\xi}_\nu \Big(-\bar{\Box} h^{\mu\nu} - \bar{\nabla}^\mu \bar{\nabla}^\nu h + \bar{\nabla}_\sigma \bar{\nabla}^\nu h^{\sigma\mu} + \bar{\nabla}_\sigma \bar{\nabla}^\mu h^{\sigma\nu}\Big) \\ &\quad - \bar{\xi}^\mu \Big(-\bar{\Box} h + \bar{\nabla}_\sigma \bar{\nabla}_\nu h^{\sigma\nu} - \frac{2\Lambda}{n-2} h\Big) - \frac{4\Lambda}{n-2}\bar{\xi}_\nu h^{\mu\nu} \\ &= \bar{\nabla}_\rho \Big(\bar{\xi}_\nu \bar{\nabla}^\mu h^{\rho\nu} - \bar{\xi}_\nu \bar{\nabla}^\rho h^{\mu\nu} + \bar{\xi}^\mu \bar{\nabla}^\rho h - \bar{\xi}^\rho \bar{\nabla}^\mu h \\ &\quad + h^{\mu\nu}\bar{\nabla}^\rho \bar{\xi}_\nu - h^{\rho\nu}\bar{\nabla}^\mu \bar{\xi}_\nu + \bar{\xi}^\rho \bar{\nabla}_\nu h^{\mu\nu} - \bar{\xi}^\mu \bar{\nabla}_\nu h^{\rho\nu} + h\bar{\nabla}^\mu \bar{\xi}^\rho \Big).\end{aligned} \tag{9.5.22}$$

Of course to arrive at the right-hand side, we had to carry out several computations. But the logic is simple: try to write current in the form of a divergence of an antisymmetric two tensor. This simple observation is the crux of the matter. Then, the result can be integrated to yield

$$\begin{aligned}Q(\bar{\xi}) = \frac{1}{4\Omega_{n-2}G_n} \oint_{\partial\Sigma} \sqrt{-\bar{g}}\, dS_i \Big(&\bar{\xi}_\nu \bar{\nabla}^0 h^{i\nu} - \bar{\xi}_\nu \bar{\nabla}^i h^{0\nu} + \bar{\xi}^0 \bar{\nabla}^i h - \bar{\xi}^i \bar{\nabla}^0 h \\ &+ h^{0\nu}\bar{\nabla}^i \bar{\xi}_\nu - h^{i\nu}\bar{\nabla}^0 \bar{\xi}_\nu + \bar{\xi}^i \bar{\nabla}_\nu h^{0\nu} - \bar{\xi}^0 \bar{\nabla}_\nu h^{i\nu} + h\bar{\nabla}^0 \bar{\xi}^i \Big).\end{aligned} \tag{9.5.23}$$

The equivalence of (9.5.12) and (9.5.23) can be demonstrated as follows. Let us start the result developed with superpotential $K^{\mu\alpha\nu\beta}$ which is

$$\bar{\xi}_\nu \mathscr{G}_L^{\mu\nu} = \bar{\nabla}_\alpha \left(\bar{\xi}_\nu \bar{\nabla}_\beta K^{\mu\alpha\nu\beta} - K^{\mu\beta\nu\alpha}\bar{\nabla}_\beta \bar{\xi}_\nu\right). \tag{9.5.24}$$

Then, the first term has the form

$$\bar{\xi}_\nu \bar{\nabla}_\beta K^{\mu\alpha\nu\beta} = \frac{1}{2}\bar{\xi}_\nu \bar{\nabla}_\beta \left(\bar{g}^{\alpha\nu} H^{\mu\beta} + \bar{g}^{\mu\beta} H^{\alpha\nu} - \bar{g}^{\alpha\beta} H^{\mu\nu} - \bar{g}^{\mu\nu} H^{\alpha\beta}\right)$$

$$= \frac{1}{2}\left(\bar{\xi}^\alpha \bar{\nabla}_\beta h^{\mu\beta} - \bar{\xi}^\mu \bar{\nabla}_\beta h^{\alpha\beta} + \bar{\xi}_\nu \bar{\nabla}^\mu h^{\alpha\nu} - \bar{\xi}_\nu \bar{\nabla}^\alpha h^{\mu\nu} + \bar{\xi}^\mu \bar{\nabla}^\alpha h - \bar{\xi}^\alpha \bar{\nabla}^\mu h\right), \tag{9.5.25}$$

and the second term becomes

$$\begin{aligned} K^{\mu\beta\nu\alpha}\bar{\nabla}_\beta \bar{\xi}_\nu &= \frac{1}{2}\left(\bar{g}^{\beta\nu} H^{\mu\alpha} + \bar{g}^{\mu\alpha} H^{\beta\nu} - \bar{g}^{\alpha\beta} H^{\mu\nu} - \bar{g}^{\mu\nu} H^{\alpha\beta}\right)\bar{\nabla}_\beta \bar{\xi}_\nu \\ &= \frac{1}{2}\left(\bar{g}^{\beta\nu} h^{\mu\alpha} + \bar{g}^{\mu\alpha} h^{\beta\nu} - \bar{g}^{\alpha\beta} h^{\mu\nu} - \bar{g}^{\mu\nu} h^{\alpha\beta} + \bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} h - \bar{g}^{\beta\nu}\bar{g}^{\mu\alpha} h\right)\bar{\nabla}_\beta \bar{\xi}_\nu \\ &= \frac{1}{2}\left(-\bar{g}^{\alpha\beta} h^{\mu\nu} - \bar{g}^{\mu\nu} h^{\alpha\beta} + \bar{g}^{\mu\nu}\bar{g}^{\alpha\beta} h\right)\bar{\nabla}_\beta \bar{\xi}_\nu, \end{aligned} \tag{9.5.26}$$

since $\bar{\nabla}_{(\beta}\bar{\xi}_{\nu)} = 0$,

$$\begin{aligned} K^{\mu\beta\nu\alpha}\bar{\nabla}_\beta \bar{\xi}_\nu &= \frac{1}{2}\left(-h^{\mu\nu}\bar{\nabla}^\alpha \bar{\xi}_\nu - h^{\alpha\beta}\bar{\nabla}_\beta \bar{\xi}^\mu + h\bar{\nabla}^\alpha \bar{\xi}^\mu\right) \\ &= \frac{1}{2}\left(-h^{\mu\nu}\bar{\nabla}^\alpha \bar{\xi}_\nu + h^{\alpha\nu}\bar{\nabla}^\mu \bar{\xi}_\nu + h\bar{\nabla}^\alpha \bar{\xi}^\mu\right). \end{aligned} \tag{9.5.27}$$

Thus, one has

$$\begin{aligned} 2\bar{\xi}_\nu \mathscr{G}_L^{\mu\nu} = \bar{\nabla}_\alpha \Big(& \bar{\xi}^\alpha \bar{\nabla}_\beta h^{\mu\beta} - \bar{\xi}^\mu \bar{\nabla}_\beta h^{\alpha\beta} + \bar{\xi}_\nu \bar{\nabla}^\mu h^{\alpha\nu} - \bar{\xi}_\nu \bar{\nabla}^\alpha h^{\mu\nu} + \bar{\xi}^\mu \bar{\nabla}^\alpha h \\ & - \bar{\xi}^\alpha \bar{\nabla}^\mu h + h^{\mu\nu}\bar{\nabla}^\alpha \bar{\xi}_\nu - h^{\alpha\nu}\bar{\nabla}^\mu \bar{\xi}_\nu - h\bar{\nabla}^\alpha \bar{\xi}^\mu \Big), \end{aligned} \tag{9.5.28}$$

showing the equivalence of the two expressions.

An important issue is the symmetry of the charges: namely, they should be invariant under infinitesimal diffeomorphisms of the background which act on the perturbation as

$$\delta_\zeta h_{\mu\nu} = \bar{\nabla}_\mu \zeta_\nu + \bar{\nabla}_\nu \zeta_\mu. \tag{9.5.29}$$

And the linearized Ricci tensor transforms as

$$\delta_\zeta R^L_{\mu\nu} = \frac{2}{n-2}\Lambda \delta_\zeta h_{\mu\nu}, \tag{9.5.30}$$

yielding

$$\delta_\zeta R_L = \bar{g}^{\mu\nu}\delta_\zeta R^L_{\mu\nu} - \frac{2}{n-2}\Lambda \bar{g}^{\mu\nu}\delta_\zeta h_{\mu\nu} = 0, \tag{9.5.31}$$

and $\delta_\zeta \mathscr{G}^L_{\mu\nu} = 0$, hence $T^{\mu\nu}$ is gauge invariant so is the total charge: $\delta_\zeta Q = 0$. This invariance under small diffeomorphisms does not extend to the so called large gauge transformations which we shall discuss at the end of this chapter.

Flat space limit

The expressions we have found are valid for asymptotically constant curvature spacetimes, including the asymptotically flat ones. For the latter, in Cartesian coordinates, it is easy to see that one arrives at

$$Q(\bar{\xi}) = \frac{1}{4\Omega_{n-2}G_n}\int_{S^{n-2}} dS_i\Big(\bar{\xi}_0\big(\partial_j h^{ij} - \partial^i h_{jj}\big) + \bar{\xi}^i\partial_j h^{0j} - \bar{\xi}_j\partial^i h^{0j}\Big), \tag{9.5.32}$$

where the first bracketed terms yield the ADM energy and the remaining ones yield the total angular momentum. S^{n-2} refers to the round sphere of $n-2$ dimensions. For globally AdS or flat spaces one has vanishing charges. de Sitter case is a little tricky because of the cosmological horizon. In our construction above, we required that the background metric has proper asymptotics, and a time-like Killing vector (at least outside a compact domain) to be able to define "time" and energy. For the de Sitter space this is not the case. Let us study the following simple examples

Schwarzschild (A)dS solution

In the static coordinates the Schwarzschild-de Sitter or Schwarzschild anti-de Sitter metric, as a solution to the cosmological Einstein's theory, reads

$$ds^2 = -\Big(1-(\frac{r_0}{r})^{n-3} - \frac{r^2}{l^2}\Big)dt^2 + \Big(1-(\frac{r_0}{r})^{n-3} - \frac{r^2}{l^2}\Big)^{-1}dr^2 + r^2 d\Omega^2_{n-2}, \tag{9.5.33}$$

where $l^2 \equiv (n-2)(n-1)/2\Lambda$. For $\Lambda < 0$ we have the AdS case and for $\Lambda > 0$ we have the dS case. The background is obtained when we set $r_0 = 0$. The Killing vector $\bar{\xi}^\mu = (-1, \mathbf{0})$, is time-like for AdS everywhere but only so inside the cosmological horizon for dS since one has $\bar{g}_{\mu\nu}\bar{\xi}^\mu\bar{\xi}^\nu = -(1-r^2/l^2)$. For this Killing vector, the energy expression reduces to

$$\begin{aligned} E = &\frac{1}{4\Omega_{n-2}G_N}\int_{S^{n-2}} r^{n-2}d\Omega_{n-2} \\ &\times\Big(g_{00}\bar{\nabla}^0 h^{r0} + g_{00}\bar{\nabla}^r h^{00} + h^{0\nu}\bar{\nabla}^r\bar{\xi}_\nu - h^{r\nu}\bar{\nabla}^0\bar{\xi}_\nu + \bar{\nabla}_\nu h^{r\nu}\Big). \end{aligned} \tag{9.5.34}$$

In four dimensions, before we let $r\to\infty$, we get the result

$$E(r) = \frac{r_0}{2G}\frac{1-\frac{r^2}{l^2}}{1-\frac{r_0}{r}-\frac{r^2}{l^2}}, \tag{9.5.35}$$

from which we can see the difference between the dS and AdS cases. For the latter, we simply go to the infinity and get the expected result $E = r_0/2G \equiv M$. Namely, the integration constant r_0 that appears in the solution turns out to be the total mass of the Schwarzschild black hole spacetime in AdS. As for the dS case, the computation is only valid within the horizon, so we cannot go to infinity. All we can say is that for small black holes that do not change the location of the horizon $1-r_0/r-r^2/l^2 \approx 1-r^2/l^2$, one has the same result as in AdS. This is the best one can do for de Sitter since there does not exist strictly conserved quantities in the absence of a global time. Generically in n dimensions, the result is simply

$$E = \frac{n-2}{4G_N} r_0^{n-3}. \tag{9.5.36}$$

Clearly, one must consider the $n = 3$ case separately: For that case the corresponding solution is

$$ds^2 = -\left(1 - r_0 - \frac{r^2}{l^2}\right) dt^2 + \left(1 - r_0 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2 \tag{9.5.37}$$

for which the energy is $E = r_0/2G_3$ again but, now, r_0 is a dimensionless constant and $[G_3] = [M^{-1}]$ which was also obtained in [127]. Now let us turn into a more complicated example, where rotation is involved.

9.5.1 Mass and angular momenta of Kerr-AdS black holes in *n* dimensions

Gibbons et al. [191, 192] found the generalizations of the four dimensional Kerr-AdS metrics in n dimensions. Following closely the discussion in [122], let us calculate the conserved charges of these rather complicated metrics using the above construction for $n > 3$. (The distinct $n = 3$ case will be separately studied below in the section devoted to 3 dimensional gravity.) First, let us describe the solution in the notation of [191]: the Kerr-(Anti) de Sitter metrics are Einstein metrics that satisfy $R_{\mu\nu} = (n-1)\lambda g_{\mu\nu}$, and are given in the Kerr-Schild form[7] [222] as

$$ds^2 = d\bar{s}^2 + \frac{2m}{U}(k_\mu dx^\mu)^2, \tag{9.5.38}$$

where the background metric is the AdS metric in the following form

$$d\bar{s}^2 = -W(1-\lambda r^2)\, dt^2 + F\, dr^2 + \sum_{i=1}^{N+\epsilon} \frac{r^2 + a_i^2}{1+\lambda a_i^2}\, d\mu_i^2 + \sum_{i=1}^{N} \frac{r^2 + a_i^2}{1+\lambda a_i^2}\, \mu_i^2\, d\phi_i^2$$
$$+ \frac{\lambda}{W(1-\lambda r^2)} \left(\sum_{i=1}^{N+\epsilon} \frac{(r^2 + a_i^2)\mu_i\, d\mu_i}{1+\lambda a_i^2} \right)^2. \tag{9.5.39}$$

Here $n \equiv 2N+1+\epsilon$, with $\epsilon = 0$ for odd dimensions and $\epsilon = 1$ for even dimensions. There are N rotation parameters a_i and ϕ_i azimuthal angles and $N + \epsilon$ direction cosines that

7 Note that in this discussion we are changing our normalizations and notations a little bit to conform with [191], but it is easy to go back and forth.

satisfy

$$\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1. \tag{9.5.40}$$

The null 1-form is given as

$$k \equiv k_\mu \, dx^\mu = F\, dr + W\, dt - \sum_{i=1}^{N} \frac{a_i \mu_i^2}{1 + \lambda a_i^2} \, d\phi_i. \tag{9.5.41}$$

The three functions in the metric are found to be

$$\begin{aligned} U &= r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2 + a_i^2} \prod_{j=1}^{N} (r^2 + a_j^2), \qquad\qquad W \equiv \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{1 + \lambda a_i^2}, \\ F &\equiv \frac{1}{1 - \lambda r^2} \sum_{i=1}^{N+\epsilon} \frac{r^2 \mu_i^2}{r^2 + a_i^2}. \end{aligned} \tag{9.5.42}$$

For the time-like Killing vector $\bar{\xi}^\mu = (-1, 0, \ldots, 0)$, the total energy reduces to the expression

$$\begin{aligned} E = &\frac{1}{4\Omega_{n-2} G_N} \oint_{\partial\Sigma} dS_r \sqrt{-\bar{g}} \bigg(\bar{g}_{00} \bar{g}^{rr} \partial_r h^{00} + \frac{1}{2} h^{00} \bar{g}^{rr} \partial_r \bar{g}_{00} - \frac{m}{U} \bar{g}^{00} \partial_r \bar{g}_{00} \\ &+ 2m \partial_r U^{-1} + \frac{2m}{U} \bar{g}^{rr} \partial_r \bar{g}_{rr} - \frac{m}{U} \bar{g}^{rr} k^i k^j \partial_r \bar{g}_{ij} + \frac{m}{U} \bar{g}^{ij} \partial_r \bar{g}_{ij} \bigg), \end{aligned} \tag{9.5.43}$$

where the perturbation about the background is simply given as

$$h_{\mu\nu} = \frac{2m}{U} k_\mu k_\nu \tag{9.5.44}$$

and the above integral is to be evaluated on a sphere S^{n-2} at $r \to \infty$ for AdS. On the other hand, the same result will be valid for dS only for small black holes as discussed above. Large distance behavior of the metric components can be found to be

$$g_{00} \to \lambda W r^2, \qquad F \to -\frac{1}{\lambda r^2}, \qquad U \to r^{n-3}, \qquad k^\phi \to \frac{a_\phi}{r^2}. \tag{9.5.45}$$

There is of course no integration along the r direction, the somewhat tricky part (of the integral) comes from the determinant term since the integrand simplifies to

$$S = \sqrt{-\bar{g}} \, \frac{2m}{r^{n-2}} \Big((n-1) W - 1 \Big). \tag{9.5.46}$$

Let us now give some details of the calculation of the determinant part which can be recast as

$$\det \bar{g} = -W(1-\lambda r^2)\,F\,\det\mathscr{M}\prod_{i=1}^{N}\frac{(r^2+a_i^2)\mu_i^2}{1+\lambda a_i^2}, \tag{9.5.47}$$

where $(\mathscr{M}_{ij})$ is the matrix representing the coefficients of the form $d\mu_i d\mu_j$ in the metric whose components can be written in the following form:

$$\mathscr{M}_{ij} = A_i\delta_{ij} + B_iB_j + C_iC_j, \qquad \text{no summation}, \tag{9.5.48}$$

where the individual parts read

$$\begin{aligned} A_i &= \frac{r^2+a_i^2}{1+\lambda a_i^2}, \qquad\qquad B_i = \sqrt{\frac{r^2+a_{N+\epsilon}^2}{1+\lambda a_{N+\epsilon}^2}}\,\frac{\mu_i}{\mu_n}, \\ C_i &= \sqrt{\frac{\lambda}{W(1-\lambda r^2)}}\left(\frac{r^2+a_i^2}{1+\lambda a_i^2} - \frac{r^2+a_{N+\epsilon}^2}{1+\lambda a_{N+\epsilon}^2}\right)\mu_i. \end{aligned} \tag{9.5.49}$$

Then the determinant of this part becomes

$$\det\mathscr{M} = \prod_{i=1}^{N+\epsilon-1} A_i \sum_{i=1}^{N+\epsilon-1}\left(\frac{B_i^2}{A_i} + \frac{C_i^2}{A_i} + \sum_{j\neq i}^{N+\epsilon-1}\frac{B_i^2C_i^2}{A_iA_j} - \sum_{j\neq i}^{N+\epsilon-1}\frac{B_iB_jC_jC_i}{A_iA_j}\right), \tag{9.5.50}$$

which can be simplified to

$$\det\mathscr{M} = \frac{1}{W\mu_{N+\epsilon}^2}\prod_{i=1}^{N}\frac{1}{1+\lambda a_i^2}. \tag{9.5.51}$$

Now we are ready to do the angular integrals: for this purpose we need the following integrations:

$$\begin{aligned} &\int_\sigma \prod_{i=1}^{N+\epsilon-1}\frac{\mu_i d\mu_i}{\sqrt{1-\sum_{k=1}^{N+\epsilon-1}\mu_k^2}} = \frac{1}{(2(N+\epsilon)-3)(2(N+\epsilon)-5)\ldots.1}, \\ &\int_\sigma \prod_{i=1}^{N+\epsilon-1}\frac{\mu_i d\mu_i}{\sqrt{1-\sum_{k=1}^{N+\epsilon-1}\mu_k^2}}\mu_j^2 \\ &= \frac{2}{(2(N+\epsilon)-1)(2(N+\epsilon)-3)(2(N+\epsilon)-5)\ldots.1}, \\ &\int_\sigma \prod_{i=1}^{N+\epsilon-1}\mu_i d\mu_i = \frac{2^{1-N-\epsilon}}{(N+\epsilon)!}, \\ &\int_\sigma \prod_{i=1}^{N+\epsilon-1}\mu_i d\mu_i \mu_j^2 = \frac{2^{1-N-\epsilon}}{(N+\epsilon+1)!}. \end{aligned} \tag{9.5.52}$$

Here σ is the region where $0 < \sum_{k=1}^{N+\epsilon-1} \mu_k^2 < 1$. After all this rather long procedure one finds the energy of the Kerr-AdS black hole in n dimensions to be

$$E = \frac{m}{\Xi} \sum_{i=1}^{\frac{n-1-\epsilon}{2}} \left(\frac{1}{\Xi_i} - \left(\frac{1}{2}\right)^{1-\epsilon} \right), \tag{9.5.53}$$

where we defined

$$\Xi \equiv \prod_{i=1}^{\frac{n-1-\epsilon}{2}} (1 + \lambda a_i^2), \qquad \Xi_i \equiv 1 + \lambda a_i^2. \tag{9.5.54}$$

From this expression, one covers the known limits for $a \to 0$ and $\lambda \to 0$.

Let us consider the four dimensional case as an example, then we have $\epsilon = 1 = N$, hence a single rotation parameter a, then the energy of the Kerr-AdS black hole reads

$$E = \frac{m}{(1 + \lambda a^2)^2}, \qquad n = 4. \tag{9.5.55}$$

Similarly, we can compute the angular momenta of the Kerr-AdS metric by considering generically a Killing vector of the form

$$\bar{\xi}^\mu = (0, \ldots, 0, 1, 0, ..), \tag{9.5.56}$$

which then leads to the corresponding conserved total angular momentum

$$\begin{aligned} J &= \frac{1}{4\Omega_{n-2} G_N} \oint_{\partial\Sigma} \sqrt{-\bar{g}} dS_r \Big(\bar{g}_{\phi\phi} \bar{\nabla}^0 h^{r\phi} - \bar{g}_{\phi\phi} \bar{\nabla}^r h^{0\phi} + h^{0\nu} \bar{\nabla}^r \bar{\xi}_\nu - h^{r\nu} \bar{\nabla}^0 \bar{\xi}_\nu \Big) \\ &= -\frac{1}{4\Omega_{n-2} G_N} \oint_{\partial\Sigma} \sqrt{-\bar{g}} dS_r \bar{g}_{\phi\phi} \bar{g}^{rr} \bar{g}^{00} \partial_r h_0^\phi. \end{aligned} \tag{9.5.57}$$

Once again the integrand can be calculated to be

$$I = \sqrt{-\bar{g}} \, \frac{(n-1) 2 m a_i \mu_i^2}{r^{n-2}(1 + \lambda a_i^2)}. \tag{9.5.58}$$

Carrying out the integral, one arrives at the angular momentum in the chosen direction to be

$$J_\phi = \frac{m a_\phi}{\Xi \Xi_\phi}. \tag{9.5.59}$$

Since we have chosen an arbitrary direction, generically we can conclude that the conserved angular momenta are

$$J_i = \frac{m a_i}{\Xi \Xi_i}, \qquad i = \{1, \ldots, N\}. \tag{9.5.60}$$

Unlike the case of the energy, clearly ϵ does not appear in the angular momenta expressions, since even dimensional (say with dimension $2n$) spaces have as many independent angular momenta $(n-1)$ as the odd dimensional spaces with one lower dimension $(2n-1)$. Specifically in four dimensions we have a single conserved angular momentum given as

$$J = \frac{ma}{(1+\lambda a^2)^2}. \tag{9.5.61}$$

Of course the direction of this angular momentum is also conserved, in the coordinates we have chosen, $J = J_\phi$. Observe that we have the relation $J = aE$ for four dimensions and generically we have the following relation for *even* dimensions

$$E = \sum_i \frac{J_i}{a_i}. \tag{9.5.62}$$

The above discussion was a successful application of the AD conserved charge formulation to rotating black holes in $n > 3$ dimensional AdS spaces. We shall work out the singled out case of $n = 3$ in the section where we study the conserved charges in topologically massive gravity, a non-trivial extensions of 3 dimensional Einstein's theory. For now we turn our attention to the conserved charges in quadratic gravity

9.5.2 Conserved charges in quadratic gravity in AdS

To make this section self-contained let us briefly recap the properties of the theory that we are interested in. Recall that the generic action of the quadratic theory is

$$I = \int d^n x \sqrt{-g}\left[\frac{1}{\kappa}(R-2\Lambda_0) + \alpha R^2 + \beta R^2_{\mu\nu} + \gamma\left(R^2_{\mu\nu\sigma\rho} - 4R^2_{\mu\nu} + R^2\right)\right]. \tag{9.5.63}$$

Coupling it to a source and linearizing the field equations about a maximally symmetric background $\bar{g}_{\mu\nu}$ with an effective cosmological constant Λ yields

$$\begin{aligned}
T_{\mu\nu}(h) \equiv T_{\mu\nu}(\bar{g}) &+ \mathcal{G}^L_{\mu\nu}\left(\frac{1}{\kappa} + \frac{4\Lambda n\alpha}{n-2} + \frac{4\Lambda\beta}{n-1} + \frac{4\Lambda\gamma(n-4)(n-3)}{(n-2)(n-1)}\right)\\
&+ (2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{n-2}\bar{g}_{\mu\nu}\right)R_L\\
&+ \beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}_{\mu\nu}R_L\right)\\
&- 2h_{\mu\nu}\left(\frac{\Lambda-\Lambda_0}{2\kappa} + \Lambda^2\frac{(n-4)}{(n-2)^2}(n\alpha+\beta) + \Lambda^2\frac{\gamma(n-4)(n-3)}{(n-2)(n-1)}\right).
\end{aligned} \tag{9.5.64}$$

The first term vanishes by definition and the last term vanishes due to the vacuum equation (that determines the effective cosmological constant): namely we have

$$\frac{\Lambda-\Lambda_0}{2\kappa} + \Lambda^2\left(\frac{(n-4)}{(n-2)^2}(n\alpha+\beta) + \frac{\gamma(n-4)(n-3)}{(n-2)(n-1)}\right) = 0, \tag{9.5.65}$$

which generically yields two maximally symmetric vacua as we discussed before. Assuming that we have at least one viable solution allowed by the parameters of the theory, the linearized equations become

$$
\begin{aligned}
T_{\mu\nu}(h) = \mathcal{G}^L_{\mu\nu}\left(\frac{1}{\kappa} + \frac{4\Lambda n\alpha}{n-2} + \frac{4\Lambda\beta}{n-1} + \frac{4\Lambda\gamma(n-4)(n-3)}{(n-2)(n-1)}\right) \\
+ (2\alpha+\beta)\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{n-2}g_{\mu\nu}\right)R_L \\
+ \beta\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}_{\mu\nu}R_L\right).
\end{aligned} \tag{9.5.66}
$$

As before, we know that this is a background conserved tensor ($\bar{\nabla}^\mu T_{\mu\nu} = 0$) which can be checked explicitly with help of the expressions

$$
\begin{aligned}
\bar{\nabla}^\mu\left(\bar{g}_{\mu\nu}\bar{\Box} - \bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{2\Lambda}{n-2}g_{\mu\nu}\right)R_L = 0, \\
\bar{\nabla}^\mu\left(\bar{\Box}\mathcal{G}^L_{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}_{\mu\nu}\right)R_L = 0.
\end{aligned} \tag{9.5.67}
$$

Now, to find the total conserved charges associated with the background Killing vectors (or asymptotic Killing vectors of the full spacetime), we need to write $T_{\mu\nu}(h)\bar{\xi}^\mu$ as a surface integral. In this process, the only cumbersome term is the following

$$
\begin{aligned}
\bar{\xi}_\nu\bar{\Box}\mathcal{G}^L_{\mu\nu} = \bar{\nabla}_\alpha\left(\bar{\xi}_\nu\bar{\nabla}^\alpha\mathcal{G}^{\mu\nu}_L - \bar{\xi}_\nu\bar{\nabla}^\mu\mathcal{G}^{\alpha\nu}_L - \mathcal{G}^{\mu\nu}_L\bar{\nabla}^\alpha\bar{\xi}_\nu + \mathcal{G}^{\alpha\nu}_L\bar{\nabla}^\mu\bar{\xi}_\nu\right) \\
+ \mathcal{G}^{\mu\nu}_L\Box\bar{\xi}_\nu + \bar{\xi}_\nu\bar{\nabla}_\alpha\bar{\nabla}^\mu\mathcal{G}^{\alpha\nu} - \mathcal{G}^{\alpha\nu}_L\bar{\nabla}_\alpha\bar{\nabla}^\mu\bar{\xi}_\nu.
\end{aligned} \tag{9.5.68}
$$

Using the definition of the Killing vector, and its trace property, we have

$$
\begin{aligned}
\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{\xi}_\nu = \bar{R}^\mu_{\nu\beta\alpha}\bar{\xi}_\mu = \frac{2\Lambda}{(n-2)(n-1)}(\bar{g}_{\nu\alpha}\bar{\xi}_\beta - \bar{g}_{\alpha\beta}\bar{\xi}_\nu), \\
\Box\bar{\xi}_\mu = -\frac{2\Lambda}{n-2}\bar{\xi}_\mu,
\end{aligned} \tag{9.5.69}
$$

as well as

$$
\bar{\xi}_\nu\bar{\nabla}_\alpha\bar{\nabla}^\mu\mathcal{G}^{\alpha\nu}_L = \frac{2\Lambda n}{(n-2)(n-1)}\bar{\xi}_\nu\mathcal{G}^{\mu\nu}_L + \frac{\Lambda}{n-1}\xi^\mu R_L. \tag{9.5.70}
$$

One can show that $\bar{\xi}_\nu\bar{\Box}\mathcal{G}^L_{\mu\nu}$ can indeed be written as a surface term. More explicitly we need the following intermediate steps in this computation: let us define

$$
\bar{\xi}_\mu\left(\bar{g}^{\mu\nu}\bar{\Box} - \bar{\nabla}^\mu\bar{\nabla}^\nu + \frac{2\Lambda}{n-2}\bar{g}^{\mu\nu}\right)R_L \equiv \nabla_\mu\mathcal{F}^{\mu\nu}_1, \tag{9.5.71}
$$

and

$$
\bar{\xi}_\mu\left(\bar{\Box}\mathcal{G}^{\mu\nu}_L - \frac{2\Lambda}{n-1}\bar{g}^{\mu\nu}R_L\right) \equiv \nabla_\mu\mathcal{F}^{\mu\nu}_2. \tag{9.5.72}
$$

Let us start with the first expression and manipulate is to the desired form by pulling out a covariant derivative

$$
\begin{aligned}
&\bar{\xi}_\mu \left(\bar{g}^{\mu\nu}\bar{\Box} - \bar{\nabla}^\mu\bar{\nabla}^\nu + \frac{2\Lambda}{n-2}\bar{g}^{\mu\nu}\right) R_L \\
&= \bar{\xi}^\nu\bar{\nabla}^\mu\bar{\nabla}_\mu R_L - \bar{\xi}_\mu\bar{\nabla}^\mu\bar{\nabla}^\nu R_L + \frac{2\Lambda}{n-2}\bar{\xi}^\nu R_L \\
&= \bar{\nabla}_\mu\left(\bar{\xi}^\nu\bar{\nabla}^\mu R_L\right) - \left(\bar{\nabla}_\mu\bar{\xi}^\nu\right)\left(\bar{\nabla}^\mu R_L\right) - \bar{\nabla}^\mu\left(\bar{\xi}_\mu\bar{\nabla}^\nu R_L\right) \\
&\quad + \left(\bar{\nabla}^\mu\bar{\xi}_\mu\right)\left(\bar{\nabla}^\nu R_L\right) + \frac{2\Lambda}{n-2}\bar{\xi}^\nu R_L \\
&= \bar{\nabla}_\mu\left(\bar{\xi}^\nu\bar{\nabla}^\mu R_L - \bar{\xi}^\mu\bar{\nabla}^\nu R_L\right) - \bar{\nabla}^\mu\left(R_L\bar{\nabla}_\mu\bar{\xi}^\nu\right) \\
&\quad + R_L\bar{\nabla}^\mu\bar{\nabla}_\mu\bar{\xi}^\nu + \frac{2\Lambda}{n-2}\bar{\xi}^\nu R_L.
\end{aligned} \tag{9.5.73}
$$

Therefore one arrives at the desired result :

$$
\bar{\xi}_\mu \left(\bar{g}^{\mu\nu}\bar{\Box} - \bar{\nabla}^\mu\bar{\nabla}^\nu + \frac{2\Lambda}{n-2}\bar{g}^{\mu\nu}\right) R_L = \bar{\nabla}_\mu\left(\bar{\xi}^\nu\bar{\nabla}^\mu R_L - \bar{\xi}^\mu\bar{\nabla}^\nu R_L - R_L\bar{\nabla}^\mu\bar{\xi}^\nu\right). \tag{9.5.74}
$$

Now, considering the second term (9.5.72), we make similar manipulations and arrive at

$$
\begin{aligned}
&\bar{\xi}_\mu\left(\bar{\Box}\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}^{\mu\nu}R_L\right) \\
&= \bar{\xi}_\mu\bar{\nabla}_\rho\bar{\nabla}^\rho\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{n-1}\bar{\xi}^\nu R_L \\
&= \bar{\nabla}_\rho\left(\bar{\xi}_\mu\bar{\nabla}^\rho\mathcal{G}_L^{\mu\nu}\right) - \left(\bar{\nabla}_\rho\bar{\xi}_\mu\right)\bar{\nabla}^\rho\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{n-1}\bar{\xi}^\nu R_L \\
&= \bar{\nabla}_\rho\left(\bar{\xi}_\mu\bar{\nabla}^\rho\mathcal{G}_L^{\mu\nu} - \mathcal{G}_L^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\mu\right) - \frac{2\Lambda}{n-2}\bar{\xi}_\mu\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{n-1}\bar{\xi}^\nu R_L,
\end{aligned} \tag{9.5.75}
$$

where $\left(\bar{\nabla}_\rho\bar{\xi}_\mu\right)\bar{\nabla}^\nu\mathcal{G}_L^{\mu\rho} = 0$ due to symmetry in $\mathcal{G}_L^{\mu\rho}$ and antisymmetry in $\bar{\nabla}_\rho\bar{\xi}_\mu$. So then we have

$$
\begin{aligned}
&\bar{\xi}_\mu\left(\bar{\Box}\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{n-1}\bar{g}^{\mu\nu}R_L\right) \\
&= \bar{\nabla}_\rho\left(\bar{\xi}_\mu\bar{\nabla}^\rho\mathcal{G}_L^{\mu\nu} - \bar{\xi}_\mu\bar{\nabla}^\nu\mathcal{G}_L^{\mu\rho} - \mathcal{G}_L^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\mu + \mathcal{G}_L^{\mu\rho}\bar{\nabla}^\nu\bar{\xi}_\mu\right) \\
&\quad + \bar{\xi}_\mu\left[\bar{\nabla}_\rho, \bar{\nabla}^\nu\right]\mathcal{G}_L^{\mu\rho} - \mathcal{G}_L^{\mu\rho}\bar{R}^\sigma{}_\rho{}^\nu{}_\mu\bar{\xi}_\sigma \\
&\quad - \frac{2\Lambda}{n-2}\bar{\xi}_\mu\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{n-1}\bar{\xi}^\nu R_L \\
&= \bar{\nabla}_\rho\left(\bar{\xi}_\mu\bar{\nabla}^\rho\mathcal{G}_L^{\mu\nu} - \bar{\xi}_\mu\bar{\nabla}^\nu\mathcal{G}_L^{\mu\rho} - \mathcal{G}_L^{\mu\nu}\bar{\nabla}^\rho\bar{\xi}_\mu + \mathcal{G}_L^{\mu\rho}\bar{\nabla}^\nu\bar{\xi}_\mu\right) \\
&\quad + \bar{\xi}_\mu\left[\bar{\nabla}_\rho, \bar{\nabla}^\nu\right]\mathcal{G}_L^{\mu\rho} - \mathcal{G}_L^{\mu\rho}\bar{R}^\sigma{}_\rho{}^\nu{}_\mu\bar{\xi}_\sigma \\
&\quad - \frac{2\Lambda}{n-2}\bar{\xi}_\mu\mathcal{G}_L^{\mu\nu} - \frac{2\Lambda}{n-1}\bar{\xi}^\nu R_L.
\end{aligned} \tag{9.5.76}
$$

Finally collecting all the pieces together, conserved Killing charges of the quadratic gravity for asymptotically (A)dS spacetimes becomes

$$Q(\bar{\xi}) = \left(\frac{1}{\kappa} + \frac{4\Lambda n\alpha}{n-2} + \frac{4\Lambda\beta}{n-1} + \frac{4\Lambda\gamma(n-4)(n-3)}{(n-2)(n-1)}\right)\int d^{n-1}x\,\sqrt{-\bar{g}}\bar{\xi}_\nu \mathscr{G}_L^{0\nu}$$
$$+ (2\alpha+\beta)\int dS_i\sqrt{-\bar{g}}\left(\bar{\xi}^0\bar{\nabla}^i R_L + R_L\bar{\nabla}^0\bar{\xi}^i - \bar{\xi}^i\bar{\nabla}^0 R_L\right)$$
$$+ \beta\int dS_i\sqrt{-\bar{g}}\left(\bar{\xi}_\nu\bar{\nabla}^i\mathscr{G}_L^{0\nu} - \bar{\xi}_\nu\bar{\nabla}^0\mathscr{G}_L^{i\nu} - \mathscr{G}_L^{0\nu}\bar{\nabla}^i\bar{\xi}_\nu + \mathscr{G}_L^{i\nu}\bar{\nabla}^0\bar{\xi}_\nu\right). \tag{9.5.77}$$

As before, this expression unifies all the Killing charges. The first line is left as a volume integral which we have already dealt with as the AD expression corresponding to the cosmological Einstein's theory which receives a non-trivial multiplicative factor in front coming from the quadratic terms in the curvature. As expected, the same solution or geometry can have different conserved charges in different theories. Unlike the ADM mass of the asymptotically flat geometries, the mass or energy defined as above is not a geometric invariant of the underlying manifold. This of course does not say that asymptotically AdS manifolds do not have an ADM-type geometric invariant, they might indeed have such geometric invariants; but, the crucial point is that the conserved charges here are defined via the field equations of the theory and hence dynamics is involved in addition to the asymptotic geometry. The geometry alo ne does not define the conserved charges is the physical theory. To reiterate: these are conserved charges (time independent) exactly under the condition that field equations are satisfied.

By now, the above formalism has been applied to various solutions of quadratic gravity in generic dimensions in the literature. Let us consider two simple cases: first, a putative solution which is asymptotically an AdS black hole of the quadratic theory. The second one will be the Boulware-Deser solution of the Einstein-GB theory. For the first case, whatever the exact solution is, we require its asymptotic form to be

$$h_{00} \approx +\left(\frac{r_0}{r}\right)^{n-3}, \qquad h^{rr} \approx +\left(\frac{r_0}{r}\right)^{n-3} + O(r_0^2), \tag{9.5.78}$$

and the rest of the components are identically zero or they vanish faster in these coordinates. Therefore, asymptotically we have an Einstein space with the linearized Ricci tensor given as

$$R^L_{\mu\nu} = \frac{2\Lambda}{n-2}h_{\mu\nu}, \tag{9.5.79}$$

which yields $R_L = \bar{g}^{\mu\nu}R^L_{\mu\nu} - \frac{2\Lambda}{n-2}h = 0$ and thus $\mathscr{G}^L_{\mu\nu} = 0$ in the asymptotic region hence there is no contribution to the total charges from the second and third lines of (9.5.77). So then the energy of this solution to general quadratic gravity reads

$$E = \left(\frac{1}{\kappa} + \frac{4\Lambda n\alpha}{n-2} + \frac{4\Lambda\beta}{n-1} + \frac{4\Lambda\gamma(n-4)(n-3)}{(n-2)(n-1)}\right)\frac{n-2}{4G}r_0^{n-3} \qquad n \geq 4, \tag{9.5.80}$$

subject to the condition (9.5.65). Of course generically, this energy could be negative or positive: one does not have a positive energy theorem here. The front factor can also vanish yielding vanishing charges for all asymptotically constant curvature or flat solutions of the theory. This is not yet a well understood issue (or more modestly, we do not yet understand the vanishing charge issue for non-vacuum solutions), therefore, we do not want to discuss it here in detail but note that in the previous literature "zero-energy" issue in the context of gravity has been tackled in several places. For example in [72] zero energy of the purely quadratic model in the case of asymptotically flat spacetimes was considered as "confinement" of gravity or energy. This conclusion was questioned in [126] as it fails to be true for asymptotically AdS spacetimes for generic quadratic gravity, but there always arises a particular theory with zero conserved charges for generic gravity, whose meaning is still somewhat unclear. In an attempt to remedy this issue, another definition of conserved charges , built on the decay of curvatures, and not the metric as was done above for quadratic gravity, was given in [132] for which the only theory that has zero energy is the Gauss-Bonnet theory.

As a second example, let us now consider the much studied Einstein-GB theory, assuming $\Lambda_0 = 0$, the exact non-rotating solution is [71],

$$ds^2 = -g_{00}dt^2 + g_{rr}dr^2 + r^2 d\Omega_{n-2}, \tag{9.5.81}$$

where

$$-g_{00} = g_{rr}^{-1} = 1 + \frac{r^2}{4\kappa\gamma(n-3)(n-4)}\left\{1 \pm \left\{1 + 8\gamma(n-3)(n-4)\frac{r_0^{n-3}}{r^{n-1}}\right\}^{\frac{1}{2}}\right\}, \tag{9.5.82}$$

whose asymptotic behavior seems to come with the wrong signs (namely, opposite to those of the putative solution discussed above)

$$h_{00} \approx -\left(\frac{r_0}{r}\right)^{n-3}, \qquad h^{rr} \approx -\left(\frac{r_0}{r}\right)^{n-3} + O(r_0^2), \tag{9.5.83}$$

which is actually a blessing since the front factor in the energy expression also becomes negative for the Einstein-GB theory and so one has overall a positive energy given as $E = (n-2)r_0^{n-3}/4G$. This theory has a supersymmetric extension, therefore we expect that it has a positive energy theorem.

9.6 Miscellaneous issues about conserved charges

The above construction has been general, there are some specific issues and further generalizations and reformulations which are often useful. Without going into to much detail, in what follows we will list these cases.

9.6.1 Conserved charges of *f*(Riemann) theories

With the tools in our hands we can find the conserved charges for asymptotically (A)dS spacetimes of the generic gravity whose action is of the form [111]

$$S = \int d^n x \sqrt{-g} f\left(R^{\mu\nu}_{\rho\sigma}\right). \tag{9.6.1}$$

Here f is an arbitrary but at least twice differentiable scalar function of the Riemann tensor. To by-pass the explicit procedure of linearizing the field equations, earlier in the discussion of the particle content of this theory, we have shown that we can construct an equivalent quadratic action of the form

$$S_{\text{EQA}} = \int d^n x \sqrt{-g} f_{\text{quad-equal}}\left(R^{\mu\nu}_{\rho\sigma}\right) \tag{9.6.2}$$

$$= \int d^n x \sqrt{-g}\left[\frac{1}{\kappa}(R - 2\Lambda_0) + \alpha R^2 + \beta R^\mu_\nu R^\nu_\mu + \gamma\left(R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} - 4R^\mu_\nu R^\nu_\mu + R^2\right)\right], \tag{9.6.3}$$

where the equivalent quadratic Lagrangian is defined as the Taylor series expansion of the full Lagrangian about the maximally symmetric background up to second order in the curvature tensor:

$$f_{\text{quad-equal}}\left(R^{\mu\nu}_{\rho\sigma}\right) = \sum_{i=0}^{2} \frac{1}{i!}\left[\frac{\partial^i f}{\partial\left(R^{\mu\nu}_{\rho\sigma}\right)^i}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} \left(R^{\mu\nu}_{\rho\sigma} - \bar{R}^{\mu\nu}_{\rho\sigma}\right)^i. \tag{9.6.4}$$

As we have shown earlier S and S_{EQA} have the same vacua and free particle properties. Or more specifically, up to and including $\mathcal{O}(h^2)$ expansions about AdS, these two actions yield the same results. For the construction of the conserved charges, we need up to $\mathcal{O}(h^2)$ in the action or up to $\mathcal{O}(h)$ in the field equations. So given a theory, one needs to calculate

$$\left[\frac{\partial f}{\partial R^{\mu\nu}_{\rho\sigma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma} \equiv \zeta R, \tag{9.6.5}$$

$$\frac{1}{2}\left[\frac{\partial^2 f}{\partial R^{\mu\nu}_{\rho\sigma}\partial R^{\alpha\beta}_{\lambda\gamma}}\right]_{\bar{R}^{\mu\nu}_{\rho\sigma}} R^{\mu\nu}_{\rho\sigma} R^{\alpha\beta}_{\lambda\gamma} \equiv \alpha R^2 + \beta R^\lambda_\sigma R^\sigma_\lambda + \gamma\left(R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\mu\nu} - 4R^\mu_\nu R^\nu_\mu + R^2\right), \tag{9.6.6}$$

where $\zeta, \alpha, \beta, \gamma$ are to be determined from these equations. Three parameters, α, β and γ will appear exactly in the equivalent quadratic action (9.6.3). The other remaining two parameters of (9.6.3) are to be determined from

$$\frac{1}{\kappa} = \zeta - \left(\frac{4\Lambda}{n-2} (n\alpha + \beta) + \frac{4\Lambda (n-3)}{n-1} \gamma \right), \tag{9.6.7}$$

$$\frac{\Lambda_0}{\kappa} = -\frac{1}{2} f\left(\bar{R}^{\mu\nu}_{\rho\sigma}\right) + \frac{\Lambda n}{n-2}\zeta - \frac{2\Lambda^2 n}{(n-2)^2}(n\alpha + \beta) - \frac{2\Lambda^2 n (n-3)}{(n-1)(n-2)}\gamma. \tag{9.6.8}$$

Then, the gravitational charges of the $f\left(R^{\mu\nu}_{\rho\sigma}\right)$ theory is given as

$$Q_f(\bar{\xi}) = \left(\frac{1}{\kappa} + \frac{4\Lambda n}{n-2}\alpha + \frac{4\Lambda}{n-2}\beta + \frac{4\Lambda (n-3)(n-4)}{(n-1)(n-2)}\gamma \right) Q_{\text{Einstein}}(\bar{\xi}), \tag{9.6.9}$$

where the $Q_{\text{Einstein}}(\bar{\xi})$ is computed with $\kappa = 1$ and we have discarded the vanishing terms for asymptotically AdS spacetimes. Here, note that α, β, γ, κ are to be found from (9.6.5–9.6.7), and the effective cosmological constant Λ satisfies (9.2.21). See [8] for a similar formulation of conserved charges in generic f(Riemann) theories.

An example: Charges of Born-Infeld gravity (BINMG)

As a non-trivial application of the above formalism, let us calculate the mass and angular momentum of the BTZ black hole [20] in the Born-Infeld extension of new massive gravity (BINMG) [208] defined with the action

$$S_{\text{BINMG}} = -4m^2 \int d^3x \left(\sqrt{-\det\left(g_{\mu\nu} - \frac{1}{m^2} G_{\mu\nu}\right)} - \left(1 - \frac{\lambda_0}{2}\right)\sqrt{-g} \right), \tag{9.6.10}$$

where $G_{\mu\nu}$ is the Einstein tensor (without a cosmological constant). Remarkably, the theory has a *unique* AdS vacuum about which there is a massive spin-2 excitation and no other degrees of freedom. Since it is a 3 dimensional theory, there are two helicity modes ± 2, this is an infinite order extension of the NMG theory which was a non-linear extension of the Fierz-Pauli massive gravity. Hence this theory is a rather unique extension of the linear massive gravity in 3 dimensions. All other finite or infinite order extensions in curvature suffer from the non-uniqueness of the vacuum [352, 415]. The effective cosmological parameter ($\lambda \equiv \Lambda/m^2$) and the mass of the spin-2 excitation are found to be

$$\lambda = -\lambda_0 \left(1 - \frac{\lambda_0}{4}\right), \quad \lambda_0 < 2, \qquad m_g^2 = m^2 \left(1 + \frac{\lambda_0}{2}\right)^2, \tag{9.6.11}$$

which we shall derive below from the equivalent quadratic action formalism. In 3 dimensions, since the Ricci and Riemann tensors contain the same amount of information is suffices to work with the Ricci tensor and compute the following 3 quantities to describe the vacuum, particle content and the conserved quantities of this theory:

$$f_{\text{quad-equal}}\left(R^\mu_\nu\right) \equiv \sum_{i=0}^{2} \frac{1}{i!} \left[\frac{\partial^i f}{\partial \left(R^\mu_\nu\right)^i} \right]_{\bar{R}^\mu_\nu} \left(R^\mu_\nu - \bar{R}^\mu_\nu\right)^i, \tag{9.6.12}$$

where $\bar{R}^{\mu}_{\nu} \equiv 2\lambda m^2 \delta^{\mu}_{\nu}$. For the BINMG action we get

$$f\left(\bar{R}^{\mu}_{\nu}\right) = 4m^2\left[\left(1 - \frac{\lambda_0}{2}\right) - (1+\lambda)^{3/2}\right],$$
$$\left[\frac{\partial f}{\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} R^{\alpha}_{\beta} = -(1+\lambda)^{1/2} R, \tag{9.6.13}$$
$$\frac{1}{2}\left[\frac{\partial^2 f}{\partial R^{\rho}_{\sigma}\partial R^{\alpha}_{\beta}}\right]_{\bar{R}^{\mu}_{\nu}} R^{\rho}_{\sigma} R^{\alpha}_{\beta} = \frac{1}{m^2}(1+\lambda)^{-1/2}\left(R^{\mu}_{\nu}R^{\nu}_{\mu} - \frac{3}{8}R^2\right),$$

which require $\lambda > -1$. Observe that the NMG tuning appears at the quadratic order. Therefore, one can simply read the effective parameters of the equivalent quadratic action from (9.6.5) and (9.6.6) as

$$\zeta = -\sqrt{1+\lambda}, \qquad \beta = -\frac{8}{3}\alpha = \frac{1}{m^2\sqrt{1+\lambda}}. \tag{9.6.14}$$

Using this result in (9.6.7) and (9.6.8), one gets[8]

$$\frac{1}{\kappa} = -\frac{1+\frac{\lambda}{2}}{\sqrt{1+\lambda}}, \tag{9.6.15}$$
$$\frac{\Lambda_0}{\kappa} = m^2\left[\lambda_0 - 2 + \frac{1}{\sqrt{1+\lambda}}\left(2+\lambda-\frac{\lambda^2}{4}\right)\right]. \tag{9.6.16}$$

As we found before, the mass of the spin-2 excitation in quadratic gravity is simply

$$m_g^2 = -\frac{1}{\beta\kappa_{\text{eff}}}, \qquad \frac{1}{\kappa_{\text{eff}}} = -\sqrt{1+\lambda}, \tag{9.6.17}$$

which yields the mass quoted above (9.6.11). To find λ, recall the quadratic equation that determines the potential vacua of quadratic gravity:

$$\frac{\Lambda - \Lambda_0}{2\kappa} + k\Lambda^2 = 0, \tag{9.6.18}$$

where for this case $k = -(3\alpha + \beta)$. Inserting these one obtains the unique solution in (9.6.11). Now let us calculate the mass and angular momentum of an explicit metric.

In [209, 330], it was shown that the rotating BTZ black hole

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2\left(N^{\phi} dt + d\phi\right)^2, \tag{9.6.19}$$

8 There could be a potentially confusing point here: Λ_0 is the bare cosmological constant of the equivalent quadratic action and it is *not* $m^2\lambda_0$.

where

$$N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{a^2}{4r^2}, \qquad N^\phi(r) = -\frac{a}{2r^2}, \tag{9.6.20}$$

is a solution to BINMG theory under the condition

$$\lambda = -\lambda_0\left(1 - \frac{\lambda_0}{4}\right), \quad \lambda_0 < 2. \tag{9.6.21}$$

This condition is equivalent to the earlier condition for the existence of a maximally symmetric vacuum (9.6.11) which is quite natural since the BTZ black hole is locally equivalent to the AdS vacuum and differs only in its global identifications. Then by using (9.6.9) the mass and the angular momentum of the BTZ black hole in BINMG can be found as

$$E = -M\sqrt{1+\lambda}, \qquad J = -a\sqrt{1+\lambda}. \tag{9.6.22}$$

Both of these quantities are quite reasonable as they are simply in the form $E = M/\kappa_{\text{eff}}$ and $J = a/\kappa_{\text{eff}}$. Observe that as expected from (9.6.9) the charges are scaled yet their ratio is intact. This result matches with [330] where the charges were calculated using thermodynamics arguments for black holes.

9.6.2 Conserved charges of topologically massive gravity

Let us now study a theory which is only diffeomorphism invariant up to a boundary term. This is the topologically massive gravity (TMG) introduced in [128, 129]

$$S = \frac{1}{\kappa}\int d^3x\left[\sqrt{-g}(R - 2\Lambda) + \frac{1}{2\mu}\epsilon^{\alpha\beta\gamma}\Gamma^\mu_{\alpha\nu}\left(\partial_\beta\Gamma^\nu_{\ \gamma\mu} + \frac{2}{3}\Gamma^\nu_{\ \beta\rho}\Gamma^\rho_{\ \gamma\mu}\right)\right], \tag{9.6.23}$$

where $\epsilon^{\alpha\beta\gamma}$ is the totally antisymmetric Levi-Civita symbol, which, as a tensor density, has the same weight as $\sqrt{-g}$. As opposed to pure 3 dimensional Einstein's gravity, this is a dynamical theory of gravity, albeit a parity-non-invariant one with a single massive helicity mode with helicity +2 or −2 but not both. The mass of this lonely mode is $m^2 = \mu^2 + \Lambda$, see [210] for a detailed canonical analysis of this theory. The source-free field equations are given as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{9.6.24}$$

where the added part is the Cotton tensor defined as

$$C^{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\epsilon^{\mu\alpha\beta}\nabla_\alpha\left(R^\nu_{\ \beta} - \frac{1}{4}\delta^\nu_{\ \beta}R\right). \tag{9.6.25}$$

So the full theory is third order in the derivative of the metric tensor and hence the parity non-invariance. The Cotton tensor which symmetric, covariantly conserved and traceless, is defined only in 3 dimensions in this form as the ϵ-tensor appears explicitly.[9] We need the linearized form of the Einstein and Cotton tensors in 3 dimensions which are simply found to be

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^L - 2\Lambda h_{\mu\nu}, \tag{9.6.26}$$

$$C^{\mu\nu}_L = \frac{1}{\sqrt{-\bar{g}}}\epsilon^{\mu\alpha\beta}\bar{g}_{\beta\sigma}\bar{\nabla}_\alpha \left(R^{\sigma\nu}_L - 2\Lambda h^{\sigma\nu} - \frac{1}{4}\bar{g}^{\sigma\nu} R_L\right). \tag{9.6.27}$$

Actually it is much better to recast the linearized Cotton tensor into an explicitly symmetric form as

$$C_L{}^{\mu\nu} = \frac{1}{2\sqrt{-\bar{g}}}\left(\epsilon^{\mu\alpha}{}_\beta\bar{\nabla}_\alpha\mathcal{G}_L{}^{\nu\beta} + \epsilon^{\nu\alpha}{}_\beta\bar{\nabla}_\alpha\mathcal{G}_L{}^{\mu\beta}\right). \tag{9.6.28}$$

Using this form, and following the procedure introduced above to get the vector current, one can show that [125]

$$\begin{aligned} 2\bar{\xi}_\nu C_L{}^{\mu\nu}\sqrt{-\bar{g}} = \bar{\nabla}_\alpha\left(\epsilon^{\mu\alpha\beta}\mathcal{G}^L{}_{\nu\beta}\bar{\xi}^\nu + \epsilon^{\nu\alpha}{}_\beta\mathcal{G}_L{}^{\mu\beta}\bar{\xi}_\nu + \epsilon^{\mu\nu\beta}\mathcal{G}^{L\,\alpha}{}_\beta\bar{\xi}_\nu\right) \\ + \epsilon^{\alpha\nu}{}_\beta\mathcal{G}_L{}^{\mu\beta}\bar{\nabla}_\alpha\bar{\xi}_\nu. \end{aligned} \tag{9.6.29}$$

Then given the background Killing vector we have the conserved quantity

$$\begin{aligned} Q(\bar{\xi}) &= \int_\Sigma d^2x\sqrt{-\bar{g}}T^{0\nu}\bar{\xi}_\nu = \int_\Sigma d^2x\sqrt{-\bar{g}}\left\{\mathcal{G}_L{}^{0\nu}\bar{\xi}_\nu + \frac{1}{\mu}C_L{}^{0\nu}\bar{\xi}_\nu\right\} \\ &\equiv Q_{\text{E}} + Q_{\text{C}}. \end{aligned} \tag{9.6.30}$$

We already know how the Einsteinian part is written as a surface integral. The Cotton part also can be written as a surface integral when one realizes that

$$\bar{X}^\beta \equiv \epsilon^{\alpha\nu\beta}\bar{\nabla}_\alpha\bar{\xi}_\nu, \tag{9.6.31}$$

is also a background Killing vector. Therefore, we have the rather nice charge expression for TMG as

$$Q(\bar{\xi}) = Q_{\text{E}}\left(\bar{\xi} + \frac{1}{\mu}\bar{X}\right) + \frac{1}{\mu}\oint_{\partial\Sigma} dS_i\left(\epsilon^{0i\beta}\mathcal{G}^L{}_{\nu\beta}\bar{\xi}^\nu + \epsilon^{\nu i}{}_\beta\mathcal{G}_L{}^{0\beta}\bar{\xi}_\nu + \epsilon^{0\nu\beta}\mathcal{G}^{L\,i}{}_\beta\bar{\xi}_\nu\right). \tag{9.6.32}$$

9 There is a rank-3 tensor, also called the Cotton tensor, defined in all dimensions.

So, to compute the conserved charge in TMG corresponding to the Killing vector $\bar{\xi}^{\mu}$, one must compute the corresponding expression in the Einstein's theory with the shifted Killing vector $\bar{\xi}^{\mu} + \frac{1}{\mu}\epsilon^{\alpha\nu\mu}\bar{\nabla}_{\alpha}\bar{\xi}_{\nu}$. As an example consider the rotating BTZ solution to TMG with the metric (9.6.19)[10] for which the vacuum is defined as $M = 0$, $a = 0$ in (9.6.19)

$$ds^2 = -\frac{r^2}{\ell^2}\,dt^2 + \frac{\ell^2}{r^2}\,dr^2 + r^2\,d\phi^2\,.$$

Then energy and angular momentum are found as

$$E = M - \frac{a}{\mu\ell^2} \qquad \text{and} \qquad J = a - \frac{M}{\mu}\,.$$

So clearly the same solution, or the spacetime, has different conserved charges in different theories. As we have noted several times, this is to be expected since these conserved charges are dynamical, not geometric invariants. In the specific case of $M = \mu a$ the angular momentum vanishes and furthermore setting $\mu^2\ell^2 = 1$, the energy also vanishes [346]. This interesting limit gives rise to the so called Chiral Gravity [290, 308] which only has a left single copy of the Virasoro algebra in the boundary of AdS, instead of the usual double copy [76]. Chiral gravity initially raised hope that one could define a quantum theory of gravity via the AdS/CFT correspondence. Namely, the boundary chiral CFT can be used to calculate high energy properties of bulk gravity. But this issue is not clear yet as there arise solutions in the theory which are not asymptotically AdS exactly in the chiral limit. These solutions, in the usual coordinates have logarithmic behavior and induce a logarithmic, non-unitary conformal field theory on the boundary [207].

9.6.3 Conserved charges from the symplectic structure for generic backgrounds

There is an interesting connection between symplectic structure (an antisymmetric two form on the phase space) and the conserved charges of a theory which we shall show here. It is well-known since the work [109] that canonical quantization, which apparently requires a choice of time to define *canonical* momenta, can actually be done in a *covariant* way. The combination "covariant canonical" might seem like an oxymoron but the construction follows by defining a symplectic 2-form ω in the phase space, let is call the phase space of the theory to be Z. The symplectic 2-form on the phase space is all that one needs to do a "covariant canonical" quantization granted that it is closed ($\delta\omega = 0$) and non-degenerate except for the gauge directions. Therefore ω, as a matrix, has no zero eigenvalues and its inverse exists. The crux of the matter is this: in local coordinates, q^I, of the phase space, the fundamental Poisson

10 All Einstein spacetimes, such as the BTZ solution solve TMG trivially since their Cotton tensor vanishes identically. But the charges are modified.

bracket is simply $\{q^I, q^J\} = \omega^{IJ}$. Here we shall not be interested in the quantization of the gravity theory, but we shall give another derivation of the conserved charges in the more general setting of background admitting Killing vectors but they are not necessarily maximally symmetric. We will consider the topologically massive gravity (TMG) in 3 dimensions but the Einsteinian part will be valid for all generic n dimensions. The construction was given in [332] which we follow. TMG action is[11]

$$S = \int d^3x \left[\sqrt{-g}R + \frac{1}{2\mu}\epsilon^{\alpha\beta\gamma}\Gamma^{\mu}_{\alpha\nu}\left(\partial_\beta \Gamma^{\nu}{}_{\gamma\mu} + \frac{2}{3}\Gamma^{\nu}{}_{\beta\rho}\Gamma^{\rho}{}_{\gamma\mu}\right)\right], \tag{9.6.33}$$

We need to carefully vary the action with respect to the metric and collect the boundary terms that we shall employ, hence we have

$$\delta S = \delta S_{EH} + \delta S_{CS}, \tag{9.6.34}$$

with the Einstein-Hilbert term in the explicit form as

$$\begin{aligned}\delta S_{EH} &= \delta \int d^3x \sqrt{-g}R \\ &= \int d^3x \sqrt{-g}\delta g^{\mu\nu} G_{\mu\nu} + \int d^3x \partial_\alpha \Big(\sqrt{-g}g^{\mu\nu}\delta\Gamma^{\alpha}{}_{\mu\nu} - \sqrt{-g}g^{\alpha\mu}\delta\Gamma^{\nu}{}_{\mu\nu}\Big).\end{aligned} \tag{9.6.35}$$

Variation of the Chern-Simons term is a rather long exercise which we hope the reader carries out to find at the end

$$\begin{aligned}\delta S_{CS} &= \delta \int d^3x \frac{1}{2\mu}\epsilon^{\alpha\beta\gamma}\Gamma^{\mu}_{\alpha\nu}\left(\partial_\beta \Gamma^{\nu}{}_{\gamma\mu} + \frac{2}{3}\Gamma^{\nu}{}_{\beta\rho}\Gamma^{\rho}{}_{\gamma\mu}\right) \\ &= \frac{1}{2\mu}\int d^3x \epsilon^{\alpha\beta\gamma}\delta\Gamma^{\mu}_{\alpha\nu}R^{\nu}{}_{\mu\beta\gamma} + \int d^3x \partial_\alpha \left(-\frac{1}{2\mu}\epsilon^{\alpha\nu\sigma}\Gamma^{\rho}{}_{\nu\beta}\delta\Gamma^{\beta}{}_{\sigma\rho}\right) \\ &= \frac{1}{\mu}\int d^3x \sqrt{-g}\delta g^{\mu\nu} C_{\mu\nu} \\ &\quad + \int d^3x \partial_\alpha \left[-\frac{1}{\mu}\epsilon^{\alpha\nu\sigma}\left(\tilde{R}^{\rho}{}_{\sigma}\delta g_{\nu\rho} + \frac{1}{2}\Gamma^{\rho}{}_{\nu\beta}\delta\Gamma^{\beta}{}_{\sigma\rho}\right)\right],\end{aligned} \tag{9.6.36}$$

where in the last line we have used the Cotton tensor defined as

$$C^{\mu\nu} = \frac{\epsilon^{\mu\beta\gamma}}{\sqrt{-g}}\nabla_\beta \tilde{R}^{\nu}{}_{\gamma}, \tag{9.6.37}$$

given in terms of the so called Schouten tensor which itself reads in 3 dimensions as $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$. Therefore we have the source-free TMG equations

$$G_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{9.6.38}$$

11 The addition of a cosmological constant does not change the ensuing discussion.

which require the vanishing of the scalar curvature $R = 0$. The boundary terms in the variation of the action add up to

$$\Lambda^\alpha \equiv \Lambda^\alpha_{EH} + \Lambda^\alpha_{CS}, \tag{9.6.39}$$

where the individual parts read

$$\Lambda^\alpha_{EH} = \sqrt{-g}g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - \sqrt{-g}g^{\alpha\mu}\delta\Gamma^\nu{}_{\mu\nu}, \tag{9.6.40}$$

$$\Lambda^\alpha_{CS} = -\frac{1}{\mu}\epsilon^{\alpha\nu\sigma}\left(\tilde{R}^\rho{}_\sigma\delta g_{\nu\rho} + \frac{1}{2}\Gamma^\rho{}_{\nu\beta}\delta\Gamma^\beta{}_{\sigma\rho}\right). \tag{9.6.41}$$

These boundary terms, which are 1-forms on the phase space of the theory, can be recycled to construct a symplectic 2-form as follows: first, we construct a 2-form *current* J^α

$$J^\alpha = J^\alpha_{EH} + J^\alpha_{CS}, \tag{9.6.42}$$

where the Einstein-Hilbert piece is given as the variation of the corresponding boundary term as

$$\begin{aligned} J^\alpha_{EH} &\equiv -\frac{\delta\Lambda^\alpha_{EH}}{\sqrt{-g}} \\ &= \delta\Gamma^\alpha{}_{\mu\nu}\wedge\left(\delta g^{\mu\nu} + \frac{1}{2}g^{\mu\nu}\delta\log g\right) - \delta\Gamma^\nu{}_{\mu\nu}\wedge\left(\delta g^{\alpha\mu} + \frac{1}{2}g^{\alpha\mu}\delta\log g\right), \end{aligned} \tag{9.6.43}$$

and the Chern-Simons part similarly reads as

$$J^\alpha_{CS} \equiv -\frac{\delta\Lambda^\alpha_{CS}}{\sqrt{-g}} = \frac{1}{\mu}\frac{\epsilon^{\alpha\nu\sigma}}{\sqrt{-g}}\left(\delta\tilde{R}^\rho{}_\sigma\wedge\delta g_{\nu\rho} + \frac{1}{2}\delta\Gamma^\rho{}_{\nu\beta}\wedge\delta\Gamma^\beta{}_{\sigma\rho}\right). \tag{9.6.44}$$

Then, the object of our interest, the symplectic 2-form on the phase space of TMG, is defined as an integral over a 2 surface as

$$\omega = \int_\Sigma d\Sigma_\alpha\sqrt{-g}J^\alpha, \tag{9.6.45}$$

which more explicitly reads

$$\begin{aligned} \omega = \int_\Sigma d\Sigma_\alpha\sqrt{-g} &\times\Bigg(\delta\Gamma^\alpha{}_{\mu\nu}\wedge\left(\delta g^{\mu\nu} + \frac{1}{2}g^{\mu\nu}\delta\log g\right) - \delta\Gamma^\nu{}_{\mu\nu}\wedge\left(\delta g^{\alpha\mu} + \frac{1}{2}g^{\alpha\mu}\delta\log g\right) \\ &+ \frac{1}{\mu}\frac{\epsilon^{\alpha\nu\sigma}}{\sqrt{-g}}\left(\delta\tilde{R}^\rho{}_\sigma\wedge\delta g_{\nu\rho} + \frac{1}{2}\delta\Gamma^\rho{}_{\nu\beta}\wedge\delta\Gamma^\beta{}_{\sigma\rho}\right)\Bigg). \end{aligned} \tag{9.6.46}$$

Of course, up to this point we have not done much except give names to these objects. We have to show that the desired properties nicely addressed in [109] are indeed satisfied. The easiest part is to show that the two-form is closed, $\delta\omega = 0$, it is clear from above that it is indeed closed without the use of field equations as $\delta^2 = 0$. The hardest part of the computation is to show that the current is covariantly conserved, $\nabla_\alpha J^\alpha = 0$ modulo the field equations and their variations,

$$\delta G_{\mu\nu} + \frac{1}{\mu}\delta C_{\mu\nu} = 0. \tag{9.6.47}$$

This computation takes at least several pages of algebra, here we give some of the intermediate steps. Let us split the covariant divergence of the current as

$$\nabla_\alpha J^\alpha \equiv I_1 + I_2 + \frac{1}{\mu}I_3, \tag{9.6.48}$$

where the individual parts read

$$I_1 \equiv \frac{1}{2}\nabla_\alpha\Big(g^{\mu\nu}\delta\Gamma^\alpha_{\ \mu\nu} \wedge \delta\log g - g^{\alpha\mu}\delta\Gamma^\nu_{\ \mu\nu} \wedge \delta\log g\Big), \tag{9.6.49}$$

$$I_2 \equiv \nabla_\alpha\Big(\delta\Gamma^\alpha_{\ \mu\nu} \wedge \delta g^{\mu\nu} - \delta\Gamma^\nu_{\ \mu\nu} \wedge \delta g^{\alpha\mu}\Big), \tag{9.6.50}$$

and

$$I_3 \equiv \nabla_\alpha\left[\frac{\epsilon^{\alpha\nu\sigma}}{\sqrt{-g}}\left(\delta\tilde{R}^\rho_{\ \sigma} \wedge \delta g_{\nu\rho} + \frac{1}{2}\delta\Gamma^\rho_{\ \nu\beta} \wedge \delta\Gamma^\beta_{\ \sigma\rho}\right)\right]. \tag{9.6.51}$$

With the help of the Palatini identity,

$$\delta R_{\mu\nu} = \nabla_\alpha\delta\Gamma^\alpha_{\ \mu\nu} - \nabla_\mu\delta\Gamma^\alpha_{\ \nu\alpha}, \tag{9.6.52}$$

and the explicit form of $\delta\Gamma$ in terms of the metric and the symmetries of the involved tensors one can reduce I_1 and I_2 to the following forms:

$$I_1 = \frac{1}{2}g^{\mu\nu}\delta R_{\mu\nu} \wedge \delta\log g + g^{\mu\nu}\delta\Gamma^\alpha_{\ \mu\nu} \wedge \delta\Gamma^\lambda_{\ \alpha\lambda}, \tag{9.6.53}$$

$$I_2 = \delta R_{\mu\nu} \wedge \delta g^{\mu\nu} - g^{\mu\nu}\delta\Gamma^\alpha_{\ \mu\nu} \wedge \delta\Gamma^\lambda_{\ \alpha\lambda}. \tag{9.6.54}$$

Using the variation of the field equations, the sum of these two parts becomes

$$I_1 + I_2 \equiv \frac{1}{\mu}I_4, \tag{9.6.55}$$

where I_4 explicitly reads

$$I_4 = \delta C^{\mu\nu} \wedge \left(\delta g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\delta\log g\right) - C^{\mu\nu}\delta g_{\mu\nu} \wedge \delta\log g. \tag{9.6.56}$$

Now we must find the variation of the Cotton tensor under arbitrary changes of the metric: the result is

$$\delta C^{\mu\nu} = \frac{\epsilon^{\mu\beta\gamma}}{\sqrt{-g}} \left(-\frac{1}{2} \nabla_\beta \tilde{R}^\nu{}_\gamma \delta \log g + \nabla_\beta \delta \tilde{R}^\nu{}_\gamma + \tilde{R}^\sigma{}_\gamma \delta \Gamma^\nu{}_{\beta\sigma} \right). \tag{9.6.57}$$

Making use of this in I_4 one arrives at

$$I_4 = \frac{\epsilon^{\mu\beta\gamma}}{\sqrt{-g}} \left(\tilde{R}^\sigma{}_\gamma \delta \Gamma^\nu{}_{\beta\sigma} + \nabla_\beta \delta \tilde{R}^\nu{}_\gamma \right) \wedge \delta g_{\mu\nu}. \tag{9.6.58}$$

Recasting I_3 in the form

$$I_3 = -\frac{\epsilon^{\mu\beta\gamma}}{\sqrt{-g}} \left(\nabla_\beta \delta \tilde{R}^\nu{}_\gamma \wedge \delta g_{\mu\nu} + g_{\lambda\mu} \delta \tilde{R}^\nu{}_\gamma \wedge \delta \Gamma^\lambda{}_{\beta\nu} + \delta \Gamma^\nu{}_{\mu\sigma} \wedge \nabla_\beta \delta \Gamma^\sigma{}_{\gamma\nu} \right), \tag{9.6.59}$$

and making use of this result one obtains

$$\nabla_\alpha J^\alpha = \frac{\epsilon^{\mu\beta\gamma}}{\mu\sqrt{-g}} \delta \Gamma^\nu{}_{\beta\sigma} \wedge \left(\delta \left(g_{\mu\nu} \tilde{R}^\sigma{}_\gamma \right) + \nabla_\mu \delta \Gamma^\sigma{}_{\gamma\nu} \right). \tag{9.6.60}$$

One more step is needed to show that this expression vanishes: for this purpose we need the following 3 dimensional identities that come from the definition of the ϵ-symbol

$$\epsilon^{\mu\beta\gamma} R^\sigma{}_{\mu\gamma\nu} = \epsilon^{\mu\beta\gamma} \left(\delta^\sigma{}_\gamma \tilde{R}_{\mu\nu} + \tilde{R}^\sigma{}_\gamma g_{\mu\nu} \right), \tag{9.6.61}$$

$$\epsilon^{\mu\beta\gamma} \delta R^\sigma{}_{\mu\gamma\nu} = \epsilon^{\mu\beta\gamma} \delta^\sigma{}_\gamma \delta \tilde{R}_{\mu\nu} + \epsilon^{\mu\beta\gamma} \delta \left(\tilde{R}^\sigma{}_\gamma g_{\mu\nu} \right). \tag{9.6.62}$$

So finally we have shown that our symplectic current is covariantly conserved $\nabla_\alpha J^\alpha = 0$. But, we are not done yet: we now have to show that that ω is diffeomorphism invariant both in the full solution space and in the more relevant quotient space of solutions modulo the diffeomorphism group.

Diffeomorphism invariance on the space of solutions

We must show that our symplectic 2-form ω has vanishing components in the pure gauge directions. For this purpose let us decompose the variation of the metric into non-gauge and pure gauge parts as

$$\delta g'_{\mu\nu} = \delta g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{9.6.63}$$

where ξ is a 1-form on the cotangent space of the phase space. Under this decomposition the relevant tensors split as the Lie derivative of the associated tensors, $\mathcal{L}_\xi T$, with respect to the vector ξ, hence the connection and the Ricci tensor decompose as

$$\delta \Gamma'^\lambda{}_{\mu\nu} = \delta \Gamma^\lambda{}_{\mu\nu} + \nabla_\mu \nabla_\nu \xi^\lambda + R_\nu{}^\lambda{}_{\mu\beta} \xi^\beta, \tag{9.6.64}$$

$$\delta \tilde{R}'^\mu_\nu = \delta \tilde{R}^\mu_\nu + \xi^\beta \nabla_\beta \tilde{R}^\mu_\nu + \tilde{R}^\mu{}_\beta \nabla_\nu \xi^\beta - \tilde{R}_{\nu\beta} \nabla^\beta \xi^\mu, \tag{9.6.65}$$

which then lead to a change in the symplectic current of the Einstein-Hilbert part as computed in [109]

$$\begin{aligned}\Delta J^{\alpha}_{EH} =& \nabla_\mu X^{\mu\alpha}_{EH} + R^{\mu\alpha}\left(\xi_\mu \wedge \delta \log g + 2\xi^\nu \wedge \delta g_{\mu\nu}\right) + R^{\mu\nu}\delta g_{\mu\nu} \wedge \xi^\alpha \\ &+ \delta R \wedge \xi^\alpha + 2\xi_\mu \wedge \delta R^{\alpha\mu},\end{aligned} \tag{9.6.66}$$

where $X^{\mu\alpha}_{EH}$ is an antisymmetric tensor which turns out to be

$$\begin{aligned}X^{\mu\alpha}_{EH} =& \nabla^\mu \delta g^{\nu\alpha} \wedge \xi_\nu + \delta g^{\nu\alpha} \wedge \nabla_\nu \xi^\mu + \frac{1}{2}\delta \log g \wedge \nabla^\alpha \xi^\mu + \nabla_\nu \delta g^{\mu\nu} \wedge \xi^\alpha \\ &+ \nabla^\mu \delta \ln g \wedge \xi^\alpha - (\alpha \leftrightarrow \mu)\,.\end{aligned} \tag{9.6.67}$$

For the moment let us consider the pure Einstein-Hilbert theory, then the first term in (9.6.66) is a boundary term which vanishes for sufficiently fast decaying metric variations and the last four terms vanish yielding a diffeomorphism invariant two form ω_{EH} on the quotient space of classical solutions. This result is valid in generic n dimensions.

Let us turn to the full TMG case: we must also compute the change in the Chern-Simons part which reads

$$\begin{aligned}\mu\Delta J^{\alpha}_{CS} = \frac{\epsilon^{\alpha\nu\sigma}}{\sqrt{-g}}\Bigg(& \left(-\tilde{R}_{\beta\sigma}\nabla^\beta \xi^\rho + \tilde{R}^\rho{}_\beta \nabla_\sigma \xi^\beta + \nabla_\beta \tilde{R}^\rho{}_\sigma \xi^\beta\right) \wedge \delta g_{\nu\rho} \\ &+ \delta \tilde{R}^\rho{}_\sigma \wedge \left(\nabla_\rho \xi_\nu + \nabla_\nu \xi_\rho\right) + \left(\nabla_\nu \nabla_\beta \xi^\rho + R_\beta{}^\rho{}_{\nu\gamma} \xi^\gamma\right) \wedge \delta \Gamma^\beta{}_{\sigma\rho}\Bigg).\end{aligned} \tag{9.6.68}$$

Now a highly nontrivial computation is needed to bring this into the desired form: the guiding principle is the following : collect terms in the form $\nabla_\mu X^{\mu\alpha}_{CS}$ with an antisymmetric tensor $X^{\mu\alpha}_{CS}$, plus terms that will cancel the remaining non-boundary terms in the Einstein-Hilbert part (9.6.66). For this purpose we need the following identities

$$\nabla_\beta \delta \tilde{R}^\beta{}_\sigma = \frac{1}{4}\nabla_\sigma \delta R + \delta\Gamma^\lambda{}_{\beta\sigma}\tilde{R}^\beta{}_\lambda - \delta\Gamma^\lambda{}_{\beta\lambda}\tilde{R}^\beta{}_\sigma, \tag{9.6.69}$$

$$\epsilon^{\mu\alpha\beta}\xi^\nu = g^{\mu\nu}\epsilon^{\rho\alpha\beta}\xi_\rho + g^{\alpha\nu}\epsilon^{\mu\rho\beta}\xi_\rho + g^{\beta\nu}\epsilon^{\mu\alpha\rho}\xi_\rho, \tag{9.6.70}$$

which eventually yield the desired expression

$$\begin{aligned}\mu\Delta J^{\alpha}_{CS} =& \nabla_\mu X^{\mu\alpha}_{CS} + C^{\mu\alpha}\Big(\xi_\mu \wedge \delta \log g + 2\xi^\nu \wedge \delta g_{\mu\nu}\Big) \\ &+ C^{\mu\nu}\delta g_{\mu\nu} \wedge \xi^\alpha + 2\xi_\mu \wedge \delta C^{\alpha\mu},\end{aligned} \tag{9.6.71}$$

where $X^{\mu\alpha}_{CS}$ is an antisymmetric tensor found to be

$$X^{\mu\alpha}_{CS} = \frac{\epsilon^{\alpha\mu\sigma}}{\sqrt{-g}}\left(-\delta\Gamma^\beta{}_{\sigma\rho} \wedge \nabla_\beta \xi^\rho + 2\delta\tilde{R}^\nu{}_\sigma \wedge \xi_\nu + \tilde{R}^\rho{}_\gamma \delta g_{\sigma\rho} \wedge \xi^\gamma + \tilde{R}^\beta{}_\sigma \delta g_{\beta\rho} \wedge \xi^\rho\right). \tag{9.6.72}$$

We must sum (9.6.66) and (9.6.71) and use the field equations and their variations. This leads to the conclusions that ω has no components in the pure gauge directions for sufficiently fast decaying metric variations (note that from the current to symplectic form, we do an integration hence dropping the boundary term for fast decaying fields makes sense.)

Conserved charge construction from the symplectic structure

We are now ready to pick up our main fruit: the conserved charges for a generic background admitting a Killing symmetry. For this purpose let us consider the diffeomorphisms which are isometries of the background spacetime: Namely we have the Killing equation[12] $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ which leads to an identical vanishing of the variation of the current as

$$\Delta J^\alpha = \nabla_\mu \left(X^{\mu\alpha}_{EH} + \frac{1}{\mu} X^{\mu\alpha}_{CS} \right) = 0. \tag{9.6.73}$$

From this point on, what we have done in our earlier discussion of charge construction follows verbatim leading to

$$\begin{aligned} Q = \frac{1}{2\pi} \oint_{\partial\Sigma} dS_\alpha \sqrt{-g} \Bigg[\Bigg(& \nabla^0 h^{\nu\alpha} \xi_\nu + h^{\nu\alpha} \nabla_\nu \xi^0 - \frac{1}{2} h \nabla^\alpha \xi^0 \\ & + \nabla_\nu h^{0\nu} \xi^\alpha - \nabla^0 h \xi^\alpha - (\alpha \leftrightarrow 0) \Bigg) \\ & + \frac{1}{\mu} \frac{\epsilon^{0\alpha\sigma}}{\sqrt{-g}} \left(-\delta\Gamma^\beta_{\sigma\rho} \nabla_\beta \xi^\rho + 2\delta\tilde{R}^\nu{}_\sigma \xi_\nu + \tilde{R}^\rho{}_\gamma h_{\sigma\rho} \xi^\gamma + \tilde{R}^\beta{}_\sigma h_{\beta\rho} \xi^\rho \right) \Bigg], \end{aligned} \tag{9.6.74}$$

where $2\delta\Gamma^\beta_{\sigma\rho} = g^{\beta\lambda} \left(\nabla_\sigma h_{\rho\lambda} + \nabla_\rho h_{\sigma\lambda} - \nabla_\lambda h_{\sigma\rho} \right)$ and $\delta\tilde{R}^\nu{}_\sigma = \delta \left(g^{\nu\lambda} \tilde{R}_{\lambda\sigma} \right)$. This is the expression for conserved charges in TMG valid for not only asymptotically flat or AdS spacetimes, but also for asymptotically non-AdS spacetimes. All that is required from the background is the existence of a Killing vector. For the BTZ black hole, the formula gives the same result as before. But let us consider somewhat more complicated examples of non-Einstein type where the Cotton part also contributes to the explicit solution.

Examples: Conserved charges for non-Einstein solutions of TMG

1: Logarithmic solution of TMG at the chiral point

At the so called "chiral point" that we discussed which corresponds to the chiral gravity limit of TMG, for which $\mu\ell = 1$, where $\ell^2 = -\frac{1}{\Lambda}$, the following metric solves TMG [184]:

12 Please note that we do not use an over-bar notation for the background values of the vectors and tensors in this section, but it should not cause a confusion.

$$ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2\Big(N_\theta(r)dt - d\theta\Big)^2 + N_k(r)\Big(dt - \ell d\theta\Big)^2, \tag{9.6.75}$$

where

$$N(r) = \frac{r^2}{\ell^2} - m + \frac{m^2\ell^2}{4r^2}, \;\; N_\theta(r) = \frac{m\ell}{2r^2}, \;\; N_k(r) = k\log\Big(\frac{2r^2 - m\ell^2}{2r_0^2}\Big). \tag{9.6.76}$$

Defining the background as $m = k = 0$, our formula (9.6.74) yields the energy (using the Killing vector $\xi^\mu = (-1, 0, 0)$) and the angular momentum (using the Killing vector $\xi^\mu = (0, 0, 1)$) as

$$E = 4k, \;\; J = 4k\ell. \tag{9.6.77}$$

Here we took $8G = 1$. These are the same charges as the ones found in [184], employing the counter-term approach, and in [318], using the first order formalism, and in [54].

2: Spacelike stretched black holes

The following metric solves TMG for any value of μ

$$ds^2 = -N(r)dt^2 + \ell^2 R(r)(d\theta + N^\theta(r)dt)^2 + \frac{\ell^4 dr^2}{4R(r)N(r)}, \tag{9.6.78}$$

where the metric functions are given as

$$R(r) \equiv \frac{r}{4}\Big(3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+r_-(\nu^2+3)}\Big), \tag{9.6.79}$$

$$N(r) \equiv \frac{\ell^2(\nu^2+3)(r-r_+)(r-r_-)}{4R(r)}, \tag{9.6.80}$$

$$N^\theta(r) \equiv \frac{2\nu r - \sqrt{r_+r_-(\nu^2+3)}}{2R(r)}, \tag{9.6.81}$$

where[13] $\nu = -\frac{\mu\ell}{3}$. This complicated solution describes a spacelike stretched black hole for $\nu^2 > 1$ with $r_\pm$ as inner and outer horizons. This type of solutions to TMG was found by Nutku [337] and Gurses [221] and studied in [54, 69, 308, 318]. The conserved charges of this metric was discussed in the latter works. Using (9.6.74) and defining the background to be $r_\pm = 0$ and using the Killing vectors[14] $\xi^\mu = (-1/\ell, 0, 0)$ and $\xi^\mu = (0, 0, 1)$ we get the energy as

$$E = \frac{\big(3+\nu^2\big)}{3\nu}\Big(\nu(r_+ + r_-) - \sqrt{(3+\nu^2)\,r_+r_-}\Big), \tag{9.6.82}$$

13 Note that with this choice of sign and with the convention $\epsilon^{tr\theta} = 1$, the metric solves the TMG equations.

14 To keep the energy dimensionless we rescale the Killing charge.

and the angular momentum as

$$J = \frac{\ell}{24\nu}\Big(2(10\nu^4 - 15\nu^2 + 9)(r_+^2 + r_-^2) + 18(\nu^2 - 1)(\nu^2 - 2)r_+ r_- \\ + \nu(5\nu^2 - 9)(r_+ + r_-)\sqrt{(3+\nu^2)\, r_+ r_-}\Big). \tag{9.6.83}$$

Both E and J turn out to be finite in a highly nontrivial way: Einstein-Hilbert and Chern-Simons parts give divergent results separately, but they yield a finite result when added. Energy computed here is exactly the same as the one given in [54, 69, 308]. However, the angular momentum, J, differs from the one, $\mathscr{J}$, given in those papers. $\mathscr{J}$ is a linear combination of E and J given above. The relation is as follows:

$$\mathscr{J} = c_1 J + c_2 \ell E, \tag{9.6.84}$$

where c_1 and c_2 are complicated constants.

9.6.4 Generic scalar-tensor theory in *n* dimensions

In the above discussion we have assumed minimal coupling with the matter fields and also assumed that we have localized matter in a compact region of spacetime. Here we relax the minimal coupling assumption and consider a generic scalar field coupled non-minimally to the cosmological Einstein's theory defined by the action [131]

$$S = \frac{1}{2\kappa}\int d^n x \sqrt{-g}\, U(\phi)\Big(R(g) + 2\Lambda_0 - W(\phi)\partial_\mu \phi \partial^\mu \phi - V(\phi) + H(\phi)\mathscr{L}_{\mathrm{m}}\Big), \tag{9.6.85}$$

where $g_{\mu\nu}$ is sometimes called the Jordan frame metric, the "sigma model" metric or the "string frame" metric.The term $\mathscr{L}_{\mathrm{m}}$ includes all the matter fields except the scalars. Instead of working out the conserved charges in this theory from scratch, we can go to the Einstein frame with the field redefinition

$$g^E_{\mu\nu} \equiv U(\phi)^{\frac{2}{n}} g_{\mu\nu}, \tag{9.6.86}$$

which transforms the action to

$$S = \frac{1}{2\kappa}\int d^n x \sqrt{-g^E}\Big(R(g^E) + 2\Lambda_0\Big) + S_M, \tag{9.6.87}$$

where the matter sector reads

$$S_M = \frac{1}{2\kappa}\int d^n x \sqrt{-g^E}\Big(\frac{n-1}{n-2}(\partial_\mu \log U(\phi))^2 - W(\phi)\partial_\mu \phi \partial^\mu \phi \\ - U(\phi)^{\frac{2}{2-n}}\Big(V(\phi) - H(\phi)\mathscr{L}_{\mathrm{m}}\Big)\Big),$$

in which we have dropped the boundary terms since they play no role in our charge definition and we have assumed that the bare cosmological constant does not change, if it does, we can easily take care of that. What is more alarming is the fact that the scalar field could decay more slowly than the required fall of conditions to have finite charges. Here we assume that is not the case. So we already know that we have the AD expression for asymptotically AdS spacetimes which we write now as

$$Q(\bar{g}^E, \bar{\xi}^E) = \frac{1}{4\,\Omega_{n-2}\,G_N} \oint_{\partial\Sigma} dS_i \sqrt{-\bar{g}^E} q^{i0}(\bar{\xi}^E). \tag{9.6.88}$$

Now consider the inverse of the transformation (9.6.86) on the full metric $g_{\mu\nu} = U(\phi)^{-\frac{2}{n}} g^E_{\mu\nu}$ which yields

$$\bar{g}_{\mu\nu} = U(\phi)^{-\frac{2}{n}} \bar{g}^E_{\mu\nu}, \qquad h_{\mu\nu} = U(\phi)^{-\frac{2}{n}} h^E_{\mu\nu}. \tag{9.6.89}$$

We have

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{9.6.90}$$

given that

$$\nabla^E_\mu \xi^E_\nu + \nabla^E_\nu \xi^E_\mu = 0, \tag{9.6.91}$$

which is correct as long as $\xi^\mu = U(\phi)^{-\frac{2}{n}} \xi^\mu_E$ and $\xi^\mu_E \partial_\mu U(\phi) = 0$ is satisfied. (Note that this condition is to be expected since, for example if ξ^μ_E is a time-like Killing vector such as $(-1, \mathbf{0})$, $U(\phi)$ cannot depend on time if $\xi^\mu = (-U(\phi)^{-\frac{2}{n}}, \mathbf{0})$ is to be a Killing vector. Clearly similar reasoning works for the other Killings that are related to angular momenta. With all these we can find how the integrand of the conserved charge (9.5.9) transforms as

$$\begin{aligned}\sqrt{-\bar{g}} q^{i0}(\xi) = U^{-\frac{2}{n}} \sqrt{\bar{g}^E} \Bigg(& q^{i0}(\xi^E) - \frac{3}{n} \xi^E_\nu h^{i\nu}_E \partial^0 \log U + \frac{3}{n} \xi^E_\nu h^{0\nu}_E \partial^i \log U \\ & - \frac{n-1}{n} \xi^i_E h^{0\nu}_E \partial_\nu \log U + \frac{n-1}{n} \xi^0_E h^{i\nu}_E \partial_\nu \log U \Bigg).\end{aligned} \tag{9.6.92}$$

Therefore if $U(\infty) = 1$ then $g_{\mu\nu}$ and $g^E_{\mu\nu}$ have the "same" charges, otherwise they have different charges for the same solution. So this construction also sets conditions on the conformal scaling of the charges. This procedure can also be generalized to higher derivative models.

10 Conservation laws in covariant field theories with gauge symmetries

It was shown in previous chapters that the standard Noether's procedure leads to well-defined and unique expressions for the conserved *canonical* currents of the field perturbations associated with the diffeomorphism invariance. The present chapter explores the application of the Noether formalism to generally-covariant and, simultaneously, gauge-invariant field theories. This class of theories is rather broad in physics and includes even pure gravitational theories like general relativity if gravitational field is described by a set of dynamic variables such as vector fields of tetrads instead of the metric tensor field. We show that in the gauge field theories the Noether procedure breaks down and there is no unique method of constructing conserved canonical currents following from the diffeomorphism invariance, which are simultaneously both generally-covariant and gauge-invariant.

The thorough study of the problem requires application of a pretty complicated mathematical apparatus of modern differential geometry including fiber bundles, jet bundles, gauge-natural bundles, a principal (Ehresmann) connection, modified Lie derivatives, etc. The goal of this book is not to introduce the reader to all subtlties of this branch of mathematics but to outline the principal difficulties in the construction of the conserved currents in the gauge theories. A close inspection of the problem reveals that in reality the problem has a kinematic character: it has no relation to dynamics that follows from the choice of the Lagrangian. Therefore, we illustrate the origin of the difficulties by making use of simple but physically important examples.

The chapter is a brief review of the main results obtained by the researchers who have been studying the problem under discussion with corresponding references to literature. In Section 10.1 we consider a generally-covariant version of the Yang–Mills theory with the gauge fields interacting with a multiplet of massless, charged scalar fields having the intrinsic symmetry described by the group $\mathrm{SU}(N)$. Section 10.2 discusses the tetrad formulation of general relativity from the point of view of a gauge field theory where the gauge freedom is associated with the rotations of the tetrad in the tangent space at points of the spacetime manifold. The basic features of the generic differential-geometric approach to the problem are outlined in Section 10.3.

10.1 Conserved quantities in generally-covariant Yang–Mills theories

10.1.1 The Yang–Mills theories

Let $\Phi = \{\Phi^a(x)\}$ be a multiplet of n charged, massless, complex scalar fields interacting with the Yang–Mills fields of the matrix type $\boldsymbol{A}_\mu = \{(A_\mu)^a{}_b(x)\}$; the small Roman

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indexes $a, b, \ldots$ take values in $\{1, \ldots, n\}$ and the Greek indexes belong to the spacetime. As usual, by the sign † we denote a Hermitian conjugation. It is assumed that the Yang–Mills fields, $\boldsymbol{A}_\mu$, are anti-Hermitian matrices: $\boldsymbol{A}^\dagger_\mu = -\boldsymbol{A}_\mu$.

In matrix notation, the generally-covariant Lagrangian of such a physical system has the form [266]:

$$L = L_A + L_{\Phi A} = \frac{1}{8} g^{\kappa\mu} g^{\lambda\nu} \mathrm{Tr}\,(\boldsymbol{F}^\dagger_{\kappa\lambda} \boldsymbol{F}_{\mu\nu}) + g^{\mu\nu} (\mathscr{D}_\mu \Phi)^\dagger (\mathscr{D}_\nu \Phi). \tag{10.1.1}$$

Here, $g^{\mu\nu} = g^{\mu\nu}(x)$ is a fixed inverse metric tensor of the Riemannian spacetime; Tr means an operation of calculating the trace of the matrix with respect to the Roman indices; and $\mathscr{D}_\nu$ is a *generally-covariant gauge* derivative

$$\mathscr{D}_\nu \Phi \equiv \nabla_\nu \Phi + \boldsymbol{A}_\nu \Phi\ , \tag{10.1.2}$$

such that its Hermitian conjugation is

$$\mathscr{D}_\nu \Phi^\dagger \equiv \nabla_\nu \Phi^\dagger - \Phi^\dagger \boldsymbol{A}_\nu \tag{10.1.3}$$

where ∇_ν denotes a generally-covariant derivative on spacetime manifold constructed with the use of the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$, defined in Appendix (A.2.2). Usually, the second term in the right hand side of (10.1.2) (and, correspondingly, in (10.1.3)) includes a coupling constant that characterizes the strength of physical interaction between the scalar and Yang–Mills fields. We set this constant equal to 1 to simplify equations. It is achieved by choosing a special system of units, and a non-zero coupling constant can be easily restored by a redefinition of $\boldsymbol{A}_\nu$.

The anti-symmetric tensor,

$$\boldsymbol{F}_{\mu\nu} = \nabla_\mu \boldsymbol{A}_\nu - \nabla_\nu \boldsymbol{A}_\mu + [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + [\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]\,, \tag{10.1.4}$$

is the anti-Hermitian matrix of the Yang–Mills field strength tensor, $\boldsymbol{F}^\dagger_{\mu\nu} = -\boldsymbol{F}_{\mu\nu}$, and the square brackets denote a commutator of fields, $[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \equiv \boldsymbol{A}_\mu \boldsymbol{A}_\nu - \boldsymbol{A}_\nu \boldsymbol{A}_\mu$ which is not vanishing in the most general case of non-commuting matrices $\boldsymbol{A}_\mu$.

The Lagrangian (10.1.1) is invariant under the intrinsic gauge transformations:

$$\Phi \quad \to \quad \widetilde{\Phi} = \boldsymbol{u}\Phi, \tag{10.1.5}$$

$$\Phi^\dagger \quad \to \quad \widetilde{\Phi}^\dagger = \Phi^\dagger \boldsymbol{u}^\dagger, \tag{10.1.6}$$

$$\boldsymbol{A}_\nu \quad \to \quad \widetilde{\boldsymbol{A}}_\nu = \boldsymbol{u}\boldsymbol{A}_\nu \boldsymbol{u}^\dagger + \boldsymbol{u}\partial_\nu \boldsymbol{u}^\dagger \tag{10.1.7}$$

with an arbitrary unitary matrix, $\boldsymbol{u} = \{u^a{}_b(x)\}$, such that

$$\boldsymbol{u}^\dagger = \boldsymbol{u}^{-1}\ . \tag{10.1.8}$$

Indeed, substitution of (10.1.5–10.1.7) leaves the tensor $\boldsymbol{F}_{\mu\nu}$ and derivatives of scalar fields gauge-covariant,

$$\widetilde{\boldsymbol{F}}_{\mu\nu} = \boldsymbol{u}\boldsymbol{F}_{\mu\nu}\boldsymbol{u}^\dagger\ , \qquad \mathscr{D}_\nu \widetilde{\Phi} = \boldsymbol{u}\left(\mathscr{D}_\nu \Phi\right)\ , \qquad \mathscr{D}_\nu \widetilde{\Phi}^\dagger = \left(\mathscr{D}_\nu \Phi^\dagger\right) \boldsymbol{u}^\dagger\ . \tag{10.1.9}$$

The set of all the transformations (10.1.5–10.1.8) forms an infinite-dimensional Lie group which we denote as Gau. The corresponding infinite-dimensional Lie algebra is defined by the infinitesimal transformation of the group Gau, and is denoted as $\mathfrak{gau}$. The group of diffeomorphisms of spacetime and its corresponding algebra will be denoted as Diff and $\mathfrak{diff}$, respectively.

The generally-covariant gauge derivative $\mathscr{D}_\nu$ is a generalization of a covariant gauge derivative D_ν from flat to curved spacetime. The covariant gauge derivative in a spacetime is defined by the rule:

$$D_\nu \Phi \equiv \partial_\nu \Phi + \boldsymbol{A}_\nu \Phi\,, \qquad D_\nu \Phi^\dagger \equiv \partial_\nu \Phi^\dagger - \Phi^\dagger \boldsymbol{A}_\nu\,, \tag{10.1.10}$$

which should be compared with (10.1.2), (10.1.3). The reader should pay attention to the fact that the generally-covariant gauge derivative of scalar fields yields the same result as the covariant gauge derivative, $\mathscr{D}_\nu \Phi = D_\nu \Phi$, but the two derivatives are conceptually different operators. Indeed, in the general case of tensor fields the action of the derivative $\mathscr{D}_\nu$ is both Gau-covariant and Diff-covariant while the action of the derivative D_ν is only Gau-covariant because the operator of the partial derivative ∂_ν entering definition of D_ν is not generally-covariant. A necessity to distinguish D_ν and $\mathscr{D}_\nu$ becomes evident in Section 10.2.

We call the transformations (10.1.5–10.1.8) intrinsic because they do not influence the coordinates in spacetime. The group of these unitary transformations in each of points x of the spacetime is denoted as $\mathsf{U}(N)$. Below, we will be interested in the, so called, connected component of this group denoted $\mathsf{SU}(N)$ and consisting of all matrices with determinant $\det\ \mathsf{U}(\mathrm{N}) = +1$. Therefore, from now on we assume that the Gau-transformation matrices, $\boldsymbol{u} \in \mathsf{SU}(N) \subset \mathsf{U}(N)$. Any element of $\mathsf{SU}(N)$ can be represented in the exponential form:

$$\boldsymbol{u} = e^{-i\varepsilon^a \boldsymbol{t}_a}\,, \tag{10.1.11}$$

where $\varepsilon^a = \varepsilon^a(x)$ are real-valued parameters of the gauge transformations depending on spacetime coordinates $x = \{x^\alpha\}$, and $\boldsymbol{t}_a = \{(t_a)^b{}_c\}$ are called generators of the gauge transformations. Parameters ε^a has no spacetime indices and behave as scalar functions with respect to spacetime transformations. The generators are *constant* (independent of spacetime coordinates x^α) Hermitian matrices, $\boldsymbol{t}_a^\dagger = \boldsymbol{t}_a$, forming a basis of the Lie algebra $\mathfrak{su}(N)$ satisfying the commutation relationship

$$[\boldsymbol{t}_a, \boldsymbol{t}_b] = i\mathrm{c}^c{}_{ab}\boldsymbol{t}_c\,. \tag{10.1.12}$$

Here, $\mathrm{c}^c{}_{ab} = \mathrm{c}^c{}_{[ab]}$ are the structure constants of the algebra $\mathfrak{su}(N)$. For the sake of simplicity, we assume that the generators $\boldsymbol{t}_a$ are selected in the adjoint representation, that is they are simply identified with the structure constants as

$$(t_a)^c{}_b = i\mathrm{c}^c{}_{ab}\,. \tag{10.1.13}$$

Recall that $a, b, \ldots = 1, 2, \ldots, n$, where $n = N^2 - 1$.

In what follows, it will be more convenient to work with the real-valued components of the Yang–Mills fields with respect to the basis of generators, ${A^a}_\mu = {A^a}_\mu(x)$, and those of their strength tensor, ${F^a}_{\mu\nu} = {F^a}_{\mu\nu}(x)$. They are defined by relations,

$$\boldsymbol{A}_\nu = -i{A^a}_\nu \boldsymbol{t}_a \,, \qquad \boldsymbol{F}_{\mu\nu} = -i{F^a}_{\mu\nu}\boldsymbol{t}_a \,. \tag{10.1.14}$$

Notice that the real-valued components of the Yang–Mills field ${A^a}_\mu$ do not compose a contravariant vector with respect to the transformations of the group Gau because $\boldsymbol{A}_\mu$ transforms in accordance with (10.1.7). The real-valued Yang–Mills field strength tensor is expressed in terms of the real-valued Yang–Mills fields as follows:

$$\begin{aligned} {F^a}_{\mu\nu} &= \nabla_\mu {A^a}_\nu - \nabla_\nu {A^a}_\mu + {\mathrm{c}^a}_{bc}{A^b}_\mu {A^c}_\nu \\ &= \partial_\mu {A^a}_\nu - \partial_\nu {A^a}_\mu + {\mathrm{c}^a}_{bc}{A^b}_\mu {A^c}_\nu, \;. \end{aligned} \tag{10.1.15}$$

As for the derivatives, we have

$$\mathscr{D}_\nu \Phi^a = \nabla_\nu \Phi^a + {A^c}_\nu {\mathrm{c}^a}_{cb}\Phi^b = \partial_\nu \Phi^a + {A^c}_\nu {\mathrm{c}^a}_{cb}\Phi^b \,, \tag{10.1.16}$$

$$\mathscr{D}_\nu \Phi^\star_a = \nabla_\nu \Phi^\star_a - {A^c}_\nu {\mathrm{c}^b}_{ca}\Phi^\star_b = \partial_\nu \Phi^\star_a - {A^c}_\nu {\mathrm{c}^b}_{ca}\Phi^\star_b \,. \tag{10.1.17}$$

Here, $\Phi^\star_a = (\Phi^a)^*$ where "*" means the complex conjugation. Then, in the component notation, the Lagrangian (10.1.1) can be recast in the form:

$$L = L_A + L_{\Phi A} = \frac{1}{4}\delta_{ab}\, g^{\kappa\mu} g^{\lambda\nu} {F^a}_{\kappa\lambda}{F^b}_{\mu\nu} + g^{\mu\nu}(\mathscr{D}_\mu \Phi^\star_a)(\mathscr{D}_\nu \Phi^a), \tag{10.1.18}$$

where, $\delta_{ab} = -\frac{1}{2}\mathrm{Tr}\,(\boldsymbol{t}_a \boldsymbol{t}_b) = \frac{1}{2}{\mathrm{c}^c}_{ad}{\mathrm{c}^d}_{bc}$, is the *unit* Killing-Cartan metric. In the case under consideration the Killing-Cartan metric coincides with the Kroneker symbol, $\delta_{ab} = \{+1, +1, \ldots, +1\}$ and is used to raise and to lower the group indexes $a, b, c, \ldots$.

10.1.2 Field equations and the Noether current

To obtain the differential equations which govern the evolution and interaction of the fields Φ^a, $\Phi^\star_a$ and ${\mathbf{A}^a}_\mu$, and to construct the Noether current, $\mathscr{J}^\mu$ we apply the general variational procedure described in Chapter 1 of this book for the Lagrangian of the Yang–Mills theory (10.1.1). To this end we calculate the total variation, $\delta' S$, of the action functional

$$S = \int_{\Sigma_1}^{\Sigma_2} d^4x\, \mathscr{L} \,, \tag{10.1.19}$$

where, $\mathscr{L} = \sqrt{-g}L$, is the Lagrangian density, L is given in (10.1.1), $g = \det g_{\mu\nu}$, $d^4x = dx^0 dx^1 dx^2 dx^3$, and the domain of integration is restricted by two spacelike hypersurfaces, Σ_1 and Σ_2. The total variation, can be split in sum of two terms

$$\delta' S = \delta S + \delta_\Sigma S, \tag{10.1.20}$$

where the variation δS is caused by variations of the fields only without variations of the boundaries,

$$\delta S = \int_{\Sigma_1}^{\Sigma_2} d^4x\, \delta\mathscr{L} = \int_{\Sigma_1}^{\Sigma_2} d^4x\, \sqrt{-g}\,[\delta(\ln\sqrt{-g})L + \delta L]\,, \tag{10.1.21}$$

and the variation $\delta_\Sigma S$ is caused by variations of the boundary hupersurfaces of the domain of integration without changing the values of the fields,

$$\delta_\Sigma S = \int_{\Sigma_1}^{\Sigma_2} d^4x\, \partial_\mu(\mathscr{L}\delta x^\mu)\,. \tag{10.1.22}$$

Calculating variations on the right hand side of (10.1.21) is reduced to taking the Lagrangian derivatives with respect to all the fields which are the metric tensor $g^{\mu\nu}$, the Yang–Mills fields $A^a{}_\mu$, and the scalar fields Φ^a along with their conjugates $\Phi^\star_a$. Variation with respect to the metric tensor is performed as usual in accordance with the rules described, for example, in Appendix A.2.4, recall also the relation,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}. \tag{10.1.23}$$

Recall that the Yang–Mills field, $A^a{}_\nu$, is not gauge-covariant because it is transformed non-homogenously under gauge transformations. On the other hand, its variation, $\delta A^a{}_\nu$, represents a contravariant vector with respect to the intrinsic index a. Generally-covariant derivative applied to the intrinsic vector reads,

$$\mathscr{D}_\mu \delta A^a{}_\nu = \nabla_\mu \delta A^a{}_\nu + A^b{}_\mu \mathrm{c}^a{}_{bc}\delta A^c{}_\nu\,. \tag{10.1.24}$$

Then, from the relation (10.1.15–10.1.17) one finds

$$\delta F^a{}_{\mu\nu} = 2\mathscr{D}_{[\mu}\delta A^a{}_{\nu]}\,, \tag{10.1.25}$$

$$\delta\mathscr{D}_\nu\Phi^a = \mathscr{D}_\nu\delta\Phi^a + \delta A^c{}_\nu \mathrm{c}^a{}_{cb}\Phi^b\,, \tag{10.1.26}$$

$$\delta\mathscr{D}_\nu\Phi^\star_a = \mathscr{D}_\nu\delta\Phi^\star_a - \delta A^c{}_\nu \mathrm{c}^b{}_{ca}\Phi^\star_b\,. \tag{10.1.27}$$

Making use of the above-derived formulae in the Yang–Mills part, L_A, of the Lagrangian, we get its variation in the following form

$$\begin{aligned}\delta L_A &= \frac{1}{2}\delta_{ab}\,g^{\kappa\lambda}F^a{}_{\kappa\alpha}F^b{}_{\lambda\beta}\delta g^{\alpha\beta} + \frac{1}{2}\delta_{ab}\,g^{\kappa\mu}g^{\lambda\nu}F^a{}_{\kappa\lambda}\delta F^b{}_{\mu\nu}\\ &= \frac{1}{2}F^{a\,\alpha}{}_\mu F_{a\,\alpha\nu}\delta g^{\mu\nu} - \mathscr{D}_\mu F_a{}^{\mu\nu}\,\delta A^a{}_\nu + \nabla_\mu(F_a{}^{\mu\nu}\delta A^a{}_\nu).\end{aligned} \tag{10.1.28}$$

Variation of the scalar field part, $L_{\Phi A}$, of the Lagrangian reads

$$\begin{aligned}\delta L_{\Phi A} &= \mathscr{D}_\mu\Phi^\star_a\mathscr{D}_\nu\Phi^a\delta g^{\mu\nu} + g^{\mu\nu}\left(\delta\mathscr{D}_\nu\Phi^\star_a\;\mathscr{D}_\mu\Phi^a + \mathscr{D}_\mu\Phi^\star_a\;\delta\mathscr{D}_\nu\Phi^a\right)\\ &= \mathscr{D}_\mu\Phi^\star_a\mathscr{D}_\nu\Phi^a\delta g^{\mu\nu} - \delta\Phi^\star_a\;\mathscr{D}_\mu\mathscr{D}^\mu\Phi^a - \mathscr{D}_\mu\mathscr{D}^\mu\Phi^\star_a\;\delta\Phi^a\\ &\quad -\mathrm{c}^b{}_{ac}\left(\Phi^\star_b\mathscr{D}^\nu\Phi^c - \mathscr{D}^\nu\Phi^\star_b\Phi^c\right)\delta A^a{}_\nu\\ &\quad +\nabla_\mu\left(\delta\Phi^\star_a\;\mathscr{D}^\mu\Phi^a + \mathscr{D}^\mu\Phi^\star_a\;\delta\Phi^a\right).\end{aligned} \tag{10.1.29}$$

Combining all the obtained results, one finds the total variation of the action to be

$$\begin{aligned}\delta' S = \int_{\Sigma_1}^{\Sigma_2} d^4x\,\sqrt{-g}\,\Bigg\{&\Bigg[\frac{1}{2}\left(F^{a\,\alpha}{}_\mu F_{a\,\alpha\nu} - \frac{1}{4}g_{\mu\nu}F^{a\,\alpha\beta}F_{a\,\alpha\beta}\right)\\ &+ \mathscr{D}_\mu\Phi^\star_a\mathscr{D}_\nu\Phi^a - \frac{1}{2}g_{\mu\nu}\mathscr{D}_\alpha\Phi^\star_a\mathscr{D}^\alpha\Phi^a\Bigg]\delta g^{\mu\nu}\\ &- \left[\mathscr{D}_\mu F_a{}^{\mu\nu} + \mathrm{c}^b{}_{ac}\left(\Phi^\star_b\mathscr{D}^\nu\Phi^c - \mathscr{D}^\nu\Phi^\star_b\Phi^c\right)\right]\delta A^a{}_\nu\\ &- \delta\Phi^\star_a\left(\mathscr{D}_\mu\mathscr{D}^\mu\Phi^a\right) - \left(\mathscr{D}_\mu\mathscr{D}^\mu\Phi^\star_a\right)\delta\Phi^a\\ &+ \nabla_\mu\left(F_a{}^{\mu\nu}\delta A^a{}_\nu + \delta\Phi^\star_a\;\mathscr{D}^\mu\Phi^a + \mathscr{D}^\mu\Phi^\star_a\;\delta\Phi^a + L\delta x^\mu\right)\Bigg\}.\end{aligned} \tag{10.1.30}$$

The Noether theorem demands that if the action is invariant with respect to variations of its arguments induced by transformations of a group, then, related identities and conserved quantities exist. Hence, we require that the variation of the action (10.1.30) must vanish which means that the integrand of (10.1.30) is equal to zero. It brings about the main Noether's identity (1.2.52),

$$\frac{\delta\mathscr{L}}{\delta g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\delta\mathscr{L}}{\delta A^a{}_\nu}\delta A^a{}_\nu + \delta\Phi^\star_a\frac{\delta\mathscr{L}}{\delta\Phi^\star_a} + \frac{\delta\mathscr{L}}{\delta\Phi^a}\delta\Phi^a \equiv \partial_\mu\mathscr{J}^\mu\,, \tag{10.1.31}$$

where we have used notations of the Lagrangian derivatives to denote separate terms in (10.1.30) associated with the variations of the fields, and the vector density $\mathscr{J}^\mu$ standing under the operator of divergence forms the Noether current of the type (1.2.58).

The first term in (10.1.31) defines the metric energy-momentum tensor, as usual, see (1.3.21) and (1.3.25),

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}}{\delta g^{\mu\nu}} = \frac{1}{\sqrt{-g}}\mathscr{T}_{\mu\nu}\,, \tag{10.1.32}$$

which consists of two terms

$$T^{\mu\nu} = T^{\mu\nu}_A + T^{\mu\nu}_{\Phi A}\,, \tag{10.1.33}$$

where

$$T^{\mu\nu}_{A} = F^{a\,\alpha\mu}F_{a\,\alpha}{}^{\nu} - \frac{1}{4}g^{\mu\nu}F^{a\,\alpha\beta}F_{a\,\alpha\beta}\,, \tag{10.1.34}$$

is the tensor of energy-momentum of the Yang–Mills field, and

$$T^{\mu\nu}_{\Phi A} = \mathscr{D}^{\mu}\Phi^{\star}_{a}\,\mathscr{D}^{\nu}\Phi^{a} + \mathscr{D}^{\nu}\Phi^{\star}_{a}\,\mathscr{D}^{\mu}\Phi^{a} - g^{\mu\nu}\mathscr{D}_{\alpha}\Phi^{\star}_{a}\,\mathscr{D}^{\alpha}\Phi^{a}\,, \tag{10.1.35}$$

is the energy-momentum of the scalar fields. The second term in (10.1.31) is determined by

$$\frac{\delta\mathscr{L}}{\delta A^{a}{}_{\nu}} = -\left(\mathscr{D}_{\mu}\mathscr{F}_{a}{}^{\mu\nu} + \boldsymbol{j}_{a}{}^{\nu}\right)\,. \tag{10.1.36}$$

It is the operator of the Yang–Mills field equations

$$\mathscr{D}_{\mu}\mathscr{F}_{a}{}^{\mu\nu} + \boldsymbol{j}_{a}{}^{\nu} = 0 \tag{10.1.37}$$

with $\mathscr{F}_{a}{}^{\mu\nu} \equiv \sqrt{-g}F_{a}{}^{\mu\nu}$, and

$$\boldsymbol{j}_{a}{}^{\nu} \equiv \sqrt{-g}j_{a}{}^{\nu} \equiv \sqrt{-g}\,\mathsf{c}^{b}{}_{ac}\left(\Phi^{\star}_{b}\,\mathscr{D}^{\nu}\Phi^{c} - \mathscr{D}^{\nu}\Phi^{\star}_{b}\,\Phi^{c}\right)\,, \tag{10.1.38}$$

being a current of the scalar fields which is a source of the Yang–Mills field. The third and fourth terms in (10.1.31) are determined by

$$\frac{\delta\mathscr{L}}{\delta\Phi^{\star}_{a}} = -\sqrt{-g}\mathscr{D}_{\mu}\mathscr{D}^{\mu}\Phi^{a}, \qquad \frac{\delta\mathscr{L}}{\delta\Phi^{a}} = -\sqrt{-g}\mathscr{D}_{\mu}\mathscr{D}^{\mu}\Phi^{\star}_{a}\,. \tag{10.1.39}$$

They are the operators of the scalar field equations

$$\mathscr{D}_{\mu}\mathscr{D}^{\mu}\Phi^{a} = 0, \qquad \mathscr{D}_{\mu}\mathscr{D}^{\mu}\Phi^{\star}_{a} = 0\,. \tag{10.1.40}$$

The right hand side of (10.1.31) is a generally-covariant divergence of the Noether current

$$\mathscr{J}^{\mu} = -\left[\mathscr{F}_{a}{}^{\mu\nu}\delta A^{a}{}_{\nu} + \sqrt{-g}\left(\delta\Phi^{\star}_{a}\,\mathscr{D}^{\mu}\Phi^{a} + \mathscr{D}^{\mu}\Phi^{\star}_{a}\,\delta\Phi^{a}\right) + \mathscr{L}\delta x^{\mu}\right]\,. \tag{10.1.41}$$

This current is determined by the variations induced by both diffeomorphisms on a spacetime manifold (extrinsic transformations) and gauge (intrinsic) transformations, and corresponds to invariance of the physical system under these transformations.

10.1.3 Conserved quantities corresponding to the gauge invariance

First, let us construct the current and the superpotential corresponding to the gauge invariance of the action defined by the Lagrangian (10.1.1). In this case $\delta x^{\mu} = 0$. Then, from the formulae (10.1.5–10.1.7), (10.1.11), (10.1.14) and (10.1.13) one can conclude that in the case of the gauge transformations induced by the infinitesimal parameters $\varepsilon^a = \varepsilon^a(x)$, the fields are transformed to the new values

$$\begin{aligned} \Phi^a &\to \widetilde{\Phi}^a = \Phi^a + \delta_\varepsilon \Phi^a , & (10.1.42)\\ \Phi^\star_a &\to \widetilde{\Phi}^\star_a = \Phi^\star_a + \delta_\varepsilon \Phi^\star_a , & (10.1.43)\\ A^a{}_\nu(x) &\to \widetilde{A}^a{}_\nu = A^a{}_\nu + \delta_\varepsilon A^a{}_\nu , & (10.1.44)\end{aligned}$$

where the gauge variations of the fields read

$$\begin{aligned} \delta_\varepsilon \Phi^a &= \varepsilon^c \mathrm{c}^a{}_{cb} \Phi^b , & (10.1.45)\\ \delta_\varepsilon \Phi^\star_a &= -\varepsilon^c \mathrm{c}^b{}_{ca} \Phi^\star_b , & (10.1.46)\\ \delta_\varepsilon A^a{}_\nu &= -A^c{}_\nu \mathrm{c}^a{}_{cb} \varepsilon^b - \partial_\nu \varepsilon^a . & (10.1.47)\end{aligned}$$

One can see that the last variation is represented by the generally covariant gauge derivative of the parameters ε^a:

$$\mathscr{D}_\nu \varepsilon^a = \nabla_\nu \varepsilon^a + A^c{}_\nu \mathrm{c}^a{}_{cb} \varepsilon^b . \tag{10.1.48}$$

After substituting expressions (10.1.45–10.1.47) into formula (10.1.41), one finds the Noether current,

$$\mathscr{J}^\mu[\varepsilon] = \mathscr{F}_a{}^{\mu\nu} \mathscr{D}_\nu \varepsilon^a + \boldsymbol{j}_a{}^\mu \varepsilon^a . \tag{10.1.49}$$

To use this formula for constructing the superpotential, we carry out the chain rule in the first term. Taking into account the relation (10.1.36), one obtains

$$\mathscr{J}^\mu[\varepsilon] = \left(\mathscr{D}_\lambda \mathscr{F}_a{}^{\lambda\mu} + \boldsymbol{j}_a{}^\mu\right) \varepsilon^a + \mathscr{D}_\nu(\mathscr{F}_a{}^{\mu\nu}\varepsilon^a) = -\frac{\delta\mathscr{L}}{\delta A^a{}_\mu}\varepsilon^a + \mathscr{D}_\nu(\mathscr{F}_a{}^{\mu\nu}\varepsilon^a). \tag{10.1.50}$$

Here, the last term, being a divergence of an antisymmetric tensor density, is reduced to

$$\mathscr{D}_\nu \left(\mathscr{F}_a{}^{\mu\nu}\varepsilon^a\right) = \nabla_\nu \left(\mathscr{F}_a{}^{\mu\nu}\varepsilon^a\right) = \partial_\nu \left(\mathscr{F}_a{}^{\mu\nu}\varepsilon^a\right) . \tag{10.1.51}$$

The current (10.1.50) is conserved $\partial_\mu \mathscr{J}^\mu = 0$ if the Yang–Mills field equation (10.1.37) hold that is “on-shell” according to the field-theoretical glossary. In this case, the equality (10.1.50) can be formulated in the form of a conservation law:

$$\mathscr{J}^\mu[\varepsilon] = \partial_\nu \mathscr{J}^{\mu\nu}[\varepsilon] , \tag{10.1.52}$$

where

$$\mathscr{J}^{\mu\nu}[\varepsilon] = \mathscr{F}_a{}^{\mu\nu}\varepsilon^a \tag{10.1.53}$$

is a superpotential $\mathscr{J}^{\mu\nu}$ satisfying $\partial_{\mu\nu} \mathscr{J}^{\mu\nu} \equiv 0$.

Integrating (10.1.52) and implementing the Gauss integration theorem yield an integral expression for the charges of the scalar fields which are the sources of the Yang–Mills field:

$$Q[\varepsilon] = \int_{\Sigma} ds_\mu \mathscr{J}^\mu[\varepsilon] = \frac{1}{2}\oint_{\partial\Sigma} ds_{\mu\nu} \mathscr{J}^{\mu\nu}[\varepsilon] \,, \tag{10.1.54}$$

where ds_μ and $ds_{\mu\nu}$ are *coordinate* elements of integration on Σ and $\partial\Sigma$, respectively.

Let us make some remarks related to the obtained results (10.1.49–10.1.54).

(i) To the best of our knowledge, Uzes [447] was the first person who has constructed the gauge invariant currents of the type (10.1.49) depending on an arbitrary gauge parameter $\boldsymbol{\varepsilon}(x)$ in the Yang–Mills theories.

(ii) Only the Yang–Mills field gives a contribution to the gauge-invariant superpotential $\mathscr{J}^{\mu\nu}[\varepsilon]$, the scalar fields Φ^a or $\Phi^\star_a$, do not contribute. Recall that in the case of currents corresponding to diffeomorphisms, the contribution to superpotentials is given only by the fields having a non-zero spin. One can see that this assertion remains valid for the currents corresponding to the gauge transformations.

(iii) Because the superpotential (10.1.53) does not contain the scalar fields Φ^a (or $\Phi^\star_a$) one is tempted to conclude that the value of the related charge (10.1.54) is independent of the behavior of the scalar fields at all. However, this conclusion is erroneous. Indeed, the consistency of the system of the Yang–Mills field equation (10.1.37) requires the fulfillment of the conditions $\mathscr{D}_\mu j_a{}^\mu = 0$. However, this is possible only if the equations of motion for the scalar fields (10.1.40) are taken into account. We repeat that the integral representation of the charge (10.1.54) is valid on-shell only which means that the field equations for all fields entering the problem are satisfied.

(iv) At last, the structure of the (10.1.49), (10.1.53) and (10.1.54) itself shows that the current $\mathscr{J}^\mu[\varepsilon]$, the superpotential $\mathscr{J}^{\mu\nu}[\varepsilon]$ and the charge $Q[\varepsilon]$, corresponding to the gauge invariance of the system, *are simultaneously both generally-covariant and gauge invariant*. This observation is essential for the discussion which follows in the next section. As we will see, in case of conserved quantities following from the diffeomorphism invariance, the situation is different. Application of the standard Noether's procedure does not bring about the gauge invariant currents and the formalism must be extended to deal with the new realm of the Yang–Mills theories!

10.1.4 Conserved quantities corresponding to the diffeomorphism invariance

Let us now turn to constructing currents and superpotentials related to diffeomorphisms. Recall that the fields Φ^a and $\Phi^\star_a$ are transformed as scalars under diffeomorphisms, and spacetime components of the field $A^a{}_\mu$ are transformed as a covariant vector. The variations of the field variables induced by infinitesimal diffeomorphisms

$$x^\mu \quad \to \quad \widetilde{x}^\mu = x^\mu + \xi^\mu \ , \tag{10.1.55}$$

are defined by their Lie derivatives $£_\xi$ along the infinitesimal vector field $\boldsymbol{\xi} = \{\xi^\mu(x)\}$:

$$\Phi^a \quad \to \quad \widetilde{\Phi}^a = \Phi^a + \delta_\xi \Phi^a, \tag{10.1.56}$$

$$\Phi^\star_a \quad \to \quad \widetilde{\Phi}^\star_a = \Phi^\star_a + \delta_\xi \Phi^\star_a, \tag{10.1.57}$$

$$A^a{}_\nu \quad \to \quad \widetilde{A}^a{}_\nu = A^a{}_\nu + \delta_\xi A^a{}_\nu, \tag{10.1.58}$$

where

$$\delta_\xi \Phi^a = £_\xi \Phi^a = -\xi^\alpha \partial_\alpha \Phi^a \ , \tag{10.1.59}$$

$$\delta_\xi \Phi^\star_a = £_\xi \Phi^\star_a = -\xi^\alpha \partial_\alpha \Phi^\star_a \ , \tag{10.1.60}$$

$$\delta_\xi A^a{}_\nu = £_\xi A^a{}_\nu = -\xi^\alpha \partial_\alpha A^a{}_\nu - A^a{}_\alpha \partial_\nu \xi^\alpha \ . \tag{10.1.61}$$

Variations (10.1.59–10.1.61) contain partial derivatives from the fields which are not generally-covariant gauge derivatives. Because our goal is to construct conserved quantities that are simultaneously generally-covariant and gauge-invariant, we attempt to reach this goal by reformulating the Lie derivatives in (10.1.59–10.1.61) in a manifestly covariant form. To this end we introduce parameters

$$\sigma^a = \sigma^a[\xi] \equiv A^a{}_\alpha \xi^\alpha \ , \tag{10.1.62}$$

which generally-covariant gauge derivative is, by definition, equal to

$$\mathscr{D}_\nu \sigma^a = \nabla_\nu \sigma^a + A^c{}_\nu \mathrm{c}^a{}_{cb} \sigma^b = \partial_\nu \sigma^a + A^c{}_\nu \mathrm{c}^a{}_{cb} \sigma^b \ . \tag{10.1.63}$$

It should be emphasized that σ^a is not an intrinsic vector because it is proportional to the gauge field $A^a{}_\alpha$ which has the non-tensorial transformation law (10.1.7).

Keeping in mind the definitions (10.1.15–10.1.17) of the covariant gauge derivatives for scalar fields along with the definition of the generators (10.1.13), and accounting for (10.1.62), one finds that equations (10.1.59–10.1.61) can be recast to the generally-covariant form,

$$£_\xi \Phi^a = -\xi^\alpha \mathscr{D}_\alpha \Phi^a + \sigma^c \mathrm{c}^a{}_{cb} \Phi^b \ , \tag{10.1.64}$$

$$£_\xi \Phi^\star_a = -\xi^\alpha \mathscr{D}_\alpha \Phi^\star_a - \sigma^c \mathrm{c}^b{}_{ca} \Phi^\star_b \ , \tag{10.1.65}$$

$$£_\xi A^a{}_\nu = -F^a{}_{\alpha\nu} \xi^\alpha - \mathscr{D}_\nu \sigma^a \ , \tag{10.1.66}$$

where the tensor $F^a{}_{\alpha\nu}$ has been defined in (10.1.15).

Substituting the diffeomorphism-induced variations of the fields (10.1.64–10.1.66) into general expression for the Noether current (10.1.41) yields for the diffeomorphism-associated Noether's current, $\mathscr{J}^\mu[\xi]$, the following expression:

$$\mathscr{J}^{\mu}[\xi] = \sqrt{-g}\left[\left(F^{a\,\alpha\mu}F_{a\,\alpha\nu} - \frac{1}{4}\delta^{\mu}_{\nu}F^{a\,\alpha\beta}F_{a\,\alpha\beta}\right)\xi^{\nu}\right.$$
$$+\left(\mathscr{D}^{\mu}\Phi^{\star}_{a}\,\mathscr{D}^{\nu}\Phi^{a} + \mathscr{D}_{\nu}\Phi^{\star}_{a}\,\mathscr{D}^{\mu}\Phi^{a} - \delta^{\mu}_{\nu}\mathscr{D}_{\alpha}\Phi^{\star}_{a}\,\mathscr{D}^{\alpha}\Phi^{a}\right)\xi^{\nu}$$
$$\left.+F_{a}{}^{\mu\nu}\,\mathscr{D}_{\nu}\sigma^{a} + c^{b}{}_{ac}\left(\Phi^{\star}_{b}\,\mathscr{D}^{\mu}\Phi^{c} - \mathscr{D}^{\mu}\Phi^{\star}_{b}\,\Phi^{c}\right)\sigma^{a}\right]\,. \tag{10.1.67}$$

By making use of the definition of the energy-momentum tensor (10.1.33) and the scalar field current (10.1.38), we rewrite the current in a more concise form,

$$\mathscr{J}^{\mu}[\xi] = \mathscr{T}^{\mu}{}_{\nu}\xi^{\nu} + \mathscr{F}_{a}{}^{\mu\nu}\mathscr{D}_{\nu}\sigma^{a} + \boldsymbol{j}_{a}{}^{\mu}\sigma^{a}\,. \tag{10.1.68}$$

To obtain a superpotential corresponding to the current (10.1.68), we carry out the chain rule in the second term of the last formula. Taking into account (10.1.32) and (10.1.36) one finds

$$\mathscr{J}^{\mu}[\xi] = 2g^{\mu\rho}\frac{\delta\mathscr{L}}{\delta g^{\rho\nu}}\xi^{\nu} - \frac{\delta\mathscr{L}}{\delta A^{a}{}_{\mu}}\sigma^{a}[\xi] + \nabla_{\nu}\left(\mathscr{F}_{a}{}^{\mu\nu}\sigma^{a}[\xi]\right)\,. \tag{10.1.69}$$

The second term in the right side of (10.1.69) disappears on-shell that is when the Yang–Mills field equations are satisfied, see (10.1.37). As for the first term in the right hand side of (10.1.69), it does not vanish because we consider the theory of the Yang–Mills and scalar fields interacting on a fixed background spacetime. We conclude that in case of the Yang–Mills field equations hold, the canonical current (10.1.69) consists of two terms,

$$\mathscr{J}^{\mu}_{C} = \mathscr{T}^{\mu}{}_{\nu}\xi^{\nu} + \nabla_{\nu}\left(\mathscr{F}_{a}{}^{\mu\nu}A^{a}{}_{\rho}\xi^{\rho}\right)\,, \tag{10.1.70}$$

where the first term depends on the metrical energy-momentum tensor density introduced in (10.1.32) and is classified as a symmetrical current:

$$\mathscr{J}^{\mu}_{S} = \mathscr{T}^{\mu}{}_{\nu}\xi^{\nu}\,. \tag{10.1.71}$$

We notice that the structure of the current (10.1.70) is identical to that given in (1.2.135),

$$\mathscr{J}^{\mu}_{C} = \mathscr{J}^{\mu}_{S} - \nabla_{\nu}(\boldsymbol{b}^{\mu\nu}{}_{\rho}\xi^{\rho}_{K})\,, \tag{10.1.72}$$

which has been already discussed earlier in this book in the framework of an arbitrary field theory in the Minkowski space admitting Killing vectors ξ^{α}_{K}. Assuming that the fixed background of the Yang–Mills theory is the Minkowski space and comparing (10.1.70) with (10.1.72) we conclude that $\boldsymbol{b}^{\mu\nu}{}_{\rho} = -\mathscr{F}_{a}{}^{\mu\nu}A^{a}{}_{\rho}$, and the expression under divergence in (10.1.70) can not be included to the superpotential which is used for constructing conserved charges of the matter fields. This divergence only connects canonical $\mathscr{J}^{\mu}_{C}$ and symmetrical currents $\mathscr{J}^{\mu}_{S}$. The reader is referred to the discussion following (1.2.135) for the physical meaning of the divergence $\nabla_{\nu}(\boldsymbol{b}^{\mu\nu}{}_{\rho}\xi^{\rho}_{K})$.

Let us now analyze the obtained results.

(i) Notice that the very last terms in each expression for the Lie derivative (10.1.64–10.1.66) are equivalent to the variations of the fields generated by the infinitesimal gauge transformation with the infinitesimal parameter $\sigma^a[\xi]$ defined in (10.1.62). This statement can be easily proven by replacing $\varepsilon^a \rightarrow \sigma^a$ in (10.1.45–10.1.47) and comparing the result with the last terms in right sides of (10.1.64–10.1.66). Thus, the attempt to reformulate Lie derivatives in terms of the generally-covariant gauge derivatives *entangles the diffeomorphism variations of the fields with their gauge variations* .

(ii) Formula (10.1.62) points out that the parameters $\sigma^a[\xi]$ of the diffeomorphism-dependent gauge transformation is not gauge covariant because of the coupling with the Yang–Mills field $A^a{}_\alpha$. The Yang–Mills field gauge transformation is not homogeneous, see (10.1.7). Therefore, by picking up a specific value of the transformation matrix $\boldsymbol{u}$, one can always find a gauge in which the parameters $\sigma^a[\xi]$ disappear along an arbitrary given curve, but not in full spacetime. This observation tells us that apparently the *Lie derivative is not gauge-covariant* due to its dependence on the gauge-dependent parameter $\sigma^a[\xi]$.

(iii) Due to the gauge non-covariance of the parameters $\sigma^a[\xi]$, the Noether current $\mathscr{J}^\mu[\xi]$ introduced in (10.1.68) is not a gauge invariant object but it remains generally-covariant. Of course, such a result is not satisfactory from a physical point of view and requires more studying. We focus on analysis of the origin of this problem in the rest of the present section.

10.1.5 Modified Lie derivative

How can one improve this situation? The main idea is to attempt to construct the gauge invariant current by modifying the concept of the Lie derivative in order to make it both generally and gauge covariant. It seems plausible to fulfill this goal by adding to the variation of the fields, δ_ξ, induced by diffeomorphism, $\boldsymbol{\xi} = \{\xi^\alpha(x)\}$, yet another variation, δ_ε, induced by the gauge transformation of the fields, (10.1.45–10.1.47), and depending on parameter, $\boldsymbol{\varepsilon} = \{\varepsilon^a(x)\}$ which we do not consider as an intrinsic contravariant vector from now on. The purpose of introducing this supplementary parameter is to compensate the gauge-noninvariance of the parameter, $\sigma^a[x, \xi(x)] = A^a{}_\alpha(x)\xi^\alpha(x)$, that appears in expressions (10.1.64–10.1.66) of the Lie derivative after replacing the partial derivative, ∂_α, with its gauge-invariant counterpart, $\mathscr{D}_\alpha$, as explained in Section 10.1.4. It is evident that the parameter ε^a cannot be merely a function of the spacetime coordinates, $x = \{x^\alpha\}$, as it was in case of the pure gauge transformations (10.1.42–10.1.44) but must depend on the diffeomorphism vector ξ^α as well, because a modified Lie derivative is supposed to be uniquely associated with the standard Lie derivative, $£_\xi$, on spacetime manifold. If we shall be able to find out such a parameter, $\varepsilon^a = \varepsilon^a[x, \xi(x)]$, the modified Lie derivative will be both gauge and generally-covariant.

The additional parameters, ε^a of the supplementary gauge transformation combines linearly with the gauge-transformation parameters σ^a induced by diffeomorphism on spacetime manifold, so that the overall variations are

$$\zeta^a = \sigma^a + \varepsilon^a \,, \tag{10.1.73}$$

where we have used the notations: $\zeta^a \equiv \zeta^a[x, \xi(x)]$, $\sigma^a \equiv \sigma^a[x, \xi(x)]$, and $\varepsilon^a \equiv \varepsilon^a[x, \xi(x)]$. The parameters ζ^a are supposed to be both generally- and gauge-covariant. It is important to stress the following:

(i) from now and up to the end of present section, applying the gauge transformations (10.1.45–10.1.47), we consider $\varepsilon^a \equiv \varepsilon^a[x, \xi(x)]$ *only*, not $\varepsilon^a = \varepsilon^a(x)$;

(ii) because $\varepsilon^a \equiv \varepsilon^a[x, \xi(x)]$ this means that diffeomorphism variations in a spacetime induced by the displacement vectors ξ^α are "lifted" into space of gauge transformations.

The total infinitesimal change of the fields induced simultaneously by the diffeomorphism and the gauge transformation is directly obtained by adding up two sets of variations, (10.1.42–10.1.44), –now with parameters $\varepsilon^a = \varepsilon^a[x, \xi(x)]$–, and (10.1.64–10.1.66),resulting in

$$\Delta_\xi \Phi^a = -\xi^\alpha \partial_\alpha \Phi^a + \varepsilon^c \mathrm{c}^a{}_{cb} \Phi^b, \tag{10.1.74}$$

$$\Delta_\xi \Phi^\star_a = -\xi^\alpha \partial_\alpha \Phi^\star_a - \varepsilon^c \mathrm{c}^b{}_{ca} \Phi^\star_b, \tag{10.1.75}$$

$$\Delta_\xi A^a{}_\nu = -\xi^\alpha \partial_\alpha A^a{}_\nu - A^a{}_\alpha \partial_\nu \xi^\alpha - \partial_\nu \varepsilon^a - A^c{}_\nu \mathrm{c}^a{}_{cb} \varepsilon^b \,. \tag{10.1.76}$$

These variations are written down with the use of partial derivatives. For the sake of convenience, we call this presentation as the one given in a, so-called, $\boldsymbol{\partial}$-$\boldsymbol{t}$ basis with the generators $(\xi^\alpha \partial_\alpha,\ \varepsilon^a[x, \xi(x)]\boldsymbol{t}_a)$. In (10.1.74–10.1.76), we have introduced a new notion – a *modified variation*:

$$\Delta_\xi \equiv £_\xi + \delta_\varepsilon \,. \tag{10.1.77}$$

We stress that the index, "ξ", at the symbol Δ_ξ is induced not only by the displacement vector ξ^α itself, but ξ^α being included in $\varepsilon^a = \varepsilon^a[x, \xi(x)]$ also. The variations (10.1.74–10.1.76) can be written down with the use of generally-covariant gauge derivatives:

$$\Delta_\xi \Phi^a = -\xi^\alpha \mathscr{D}_\alpha \Phi^a + \zeta^c \mathrm{c}^a{}_{cb} \Phi^b, \tag{10.1.78}$$

$$\Delta_\xi \Phi^\star_a = -\xi^\alpha \mathscr{D}_\alpha \Phi^\star_a - \zeta^c \mathrm{c}^b{}_{ca} \Phi^\star_b, \tag{10.1.79}$$

$$\Delta_\xi A^a{}_\nu = -F^a{}_{\alpha\nu} \xi^\alpha - \mathscr{D}_\nu \zeta^a, \tag{10.1.80}$$

where the gauge-covariant, as assumed, parameters ζ^a are defined in (10.1.73). We call this presentation as the one given in a $\mathscr{D}$-$\boldsymbol{t}$ basis with the generators $(\xi^\alpha \mathscr{D}_\alpha,\ \zeta^a[x, \xi(x)]\boldsymbol{t}_a)$.

We will define a modified Lie derivative by the rule

$$\mathfrak{L}_\xi = \Delta_\xi , \tag{10.1.81}$$

where Δ_ξ is given in (10.1.77). Because the theory is supposed to be covariant both with respect to diffeomorphisms and to the gauge transformations, the modified Lie derivative, $\mathfrak{L}_\xi$, must be subject to these symmeties as well. However, the identification (10.1.81) itself does not satisfy this requirement. Therefore, in addition to (10.1.81) we must to require *the algebra of the modified Lie derivative be consistent with (isomorphic to) the algebra of the standard Lie derivatives $\pounds_\xi$.*

We recall that the Lie derivative is induced by spacetime diffeomorphisms and is defined by the rule, $\pounds_\xi = \delta_\xi$. Algebra of the Lie derivatives can be understood by calculating their commutator along vector fields $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. It states that the commutator obeys the rule $[\pounds_{\xi_1}, \pounds_{\xi_2}] = \pounds_{\xi_3}$ where $\boldsymbol{\xi}_3 = [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]$ with the standard definition:

$$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2] \equiv \xi_1^\beta \partial_\beta \xi_2^\alpha - \xi_2^\beta \partial_\beta \xi_1^\alpha . \tag{10.1.82}$$

The "lifting" rule (10.1.77) must be compatible with the algebra of the Lie derivatives $\pounds_\xi$, which means that the commutator of two modified Lie derivatives, namely, $\mathfrak{L}_{\xi_1}$ defined by parameters ξ_1^α, $\varepsilon_1^a \equiv \varepsilon^a[x, \xi_1(x)]$, and $\mathfrak{L}_{\xi_2}$ defined by parameters ξ_2^α, $\varepsilon_2^a \equiv \varepsilon^a[x, \xi_2(x)]$ must be constructed formally by the same rule,

$$\left[\mathfrak{L}_{\xi_1}, \mathfrak{L}_{\xi_2}\right] = \mathfrak{L}_{\xi_3} , \tag{10.1.83}$$

where the status of the index, "ξ", at the symbol $\mathfrak{L}_\xi$ is the same as of the one in (10.1.77), thus, $\mathfrak{L}_{\xi_3}$ is defined by a set of two parameters ξ_3^α, $\varepsilon_3^a \equiv \varepsilon^a[x, \xi_3(x)]$.

The requirement (10.1.83) is not sufficient for making up the self-consistent algebra of the operators of the modified Lie derivatives. The parameters $\xi_3^\alpha, \varepsilon_3^a$ defining the modified Lie derivative $\mathfrak{L}_{\xi_3}$ must be uniquely expressed in terms of the commutator $[\xi_1, \xi_2]$ that is $\xi_3^\alpha = [\xi_1, \xi_2]^\alpha$ and $\varepsilon_3^a = \varepsilon^a\left(x, [\xi_1, \xi_2]\right)$. This last requirement is called the condition of *functoriality* and it imposes a certain limitation on the functional structure of the parameter, $\boldsymbol{\varepsilon} = \{\varepsilon^a[x, \xi(x)]\}$, introduced in (10.1.73). *One of the main goals of the end of the present section is to derive out the functoriality condition and to discuss it.*

Keeping the above in mind, before formulating the functoriality condition when the parameters $\varepsilon^a = \varepsilon^a[x, \xi(x)]$ are left undetermined, we will operate with the *modified variations*, Δ_ξ, defined in (10.1.77) only. To formulate the functoriality condition/equation more explicitly and to find out the functional structure of the parameter $\boldsymbol{\varepsilon}$, we begin from studying the commutator of the modified variations Δ_ξ. It is more mathematically convenient to calculate the commutator, first, in the $\boldsymbol{\partial}$-$\boldsymbol{t}$ basis of generators which is based on the partial derivatives. It eliminates from the calculation the covariant derivatives which are more difficult to operate with. Besides, it excludes the gauge parameter $\sigma^a = A^a{}_\alpha \xi^\alpha$ induced by the spacetime diffeomorphism and allows us to focus on the appearance of the gauge-parameter $\varepsilon^a = \varepsilon^a[x, \xi(x)]$ only.

The parameter σ^a will be introduced to the algebra at the next step of transformation from the basis ∂-$\boldsymbol{t}$ to $\mathscr{D}$-$\boldsymbol{t}$ basis.

A short review

Let us make a short excursion to history. It looks like the idea of the modified Lie derivatives (10.1.81) has been invented in the fundamental work on quantum electrodynamics by Heisenberg and Pauli [228] who discovered that transformations of electromagnetic and spinor fields to a new inertial frame must be accompanied by additional gauge transformations to retain the initial gauge condition imposed on the system. The modern view onto the approach see in the book [53, chap. 14.3, pp. 73].

Jackiw [244] studied the gauge non-invariance of the canonical energy-momentum tensor in the non-Abelian gauge theories in the Minkowski space. He proposed to restore the gauge-invariance of the tensor by making use of a gauge transformation that makes the parameter $\zeta^a = 0$ in (10.1.73) and suppresses the gauge non-covariant terms in the expression for the energy-momentum tensor. The corrected energy-momentum tensor turned out to be exactly the Belinfante symmetrized energy-momentum tensor of the Maxwell type, see (10.1.33) and (1.2.142–1.2.145) for electrodynamics in the Minkowski space. Unfortunately, the lift with $\zeta^a = 0$ is not functorial as it was noticed in [244, 245].

A modified Lie derivative was studied in 1950-th by Yano [466] by the methods of differential geometry. The interested reader can find other useful theoretical approaches to build the modified Lie derivative in a review by Godina and Matteuccii [194] and in the monographs [172, 264].

Concerning a construction of diffeomorphic currents, Giachetta and Sardanashvily [187, 188] and Sardanashvily [401, 402] developed a general method in arbitrary gauge theories. A particular attention was paid to special cases of topological field theories with the intrinsic gauge symmetries of the Chern-Simons type. Their gauge invariance is not so manifest due to the fact that the Lagrangians of such theories belong to the secondary characteristic classes and, hence, are invariant only up to a total divergence under gauge transformations. As a result, construction of the gauge-invariant canonical diffeomorphic currents in the Chern-Simons theories meets additional difficulties as compared to the gauge theories of the Yang–Mills type.

The related problem was studied comprehensively in the series of the works by Francaviglia with coauthors [5, 6, 64–68, 169], by Giachetta et al. [189], and Obukhov et al. [341].

10.1.6 Commutator of the modified variations Δ_ξ in ∂-t basis

In order to calculate the commutator of two variations Δ_ξ we execute consecutively two infinitesimal transformations defined in (10.1.77) with undetermined parameters $\varepsilon^a \equiv \varepsilon^a[x, \xi(x)]$. The first one, $\Delta_1 \equiv \Delta_{\xi_1}$, is with parameters $\{\xi_1^\alpha, \varepsilon_1^a\}$, and the second one,

$\Delta_2 \equiv \Delta_{\xi_2}$, is with parameters $\{\xi_2^\alpha, \varepsilon_2^a\}$. Each of the fields from the set $\phi = \{\Phi^a, \Phi^\star_a, A^a{}_\alpha\}$ will change accordingly as

$$\phi \underset{\xi_1,\varepsilon_1}{\longrightarrow} \phi_1 = \phi + \Delta_1\phi \underset{\xi_2,\varepsilon_2}{\longrightarrow} \phi_{21} = \phi_1 + \Delta_2\phi_1 \ ,$$

resulting in

$$\phi_{21} = \phi + \Delta_1\phi + \Delta_2(\phi + \Delta_1\phi) = \phi + \Delta_1\phi + \Delta_2\phi + \Delta_2\Delta_1\phi \ . \tag{10.1.84}$$

Analogously, we make two infinitesimal transformations with the same parameters but in a reversed order,

$$\phi \underset{\xi_2,\varepsilon_2}{\longrightarrow} \phi_2 = \phi + \Delta_2\phi \underset{\xi_1,\varepsilon_1}{\longrightarrow} \phi_{12} = \phi_2 + \Delta_1\phi_2 \ ,$$

resulting in

$$\phi_{12} = \phi + \Delta_2\phi + \Delta_1(\phi + \Delta_2\phi) = \phi + \Delta_2\phi + \Delta_1\phi + \Delta_1\Delta_2\phi \ . \tag{10.1.85}$$

Now, we compare the results of the transformations (10.1.84) and (10.1.85) by calculating their commutator

$$[\Delta_2, \Delta_1]\phi \equiv \phi_{21} - \phi_{12} = \Delta_2\Delta_1\phi - \Delta_1\Delta_2\phi \ . \tag{10.1.86}$$

The calculations are straightforward but tedious.

To overview the results of the calculation let us introduce a new notation, $\boldsymbol{V} = \{V^a{}_b\}$, for the matrix of the differential operator of the first order entering the total variation of the scalar fields. For the two variations we have,

$$V_1{}^b{}_c \equiv -\delta^a_b \xi_1^\alpha \partial_\alpha + \varepsilon_1^c \mathrm{c}^a{}_{cb} \ , \qquad V_2{}^a{}_b \equiv -\delta^a_b \xi_2^\alpha \partial_\alpha + \varepsilon_2^c \mathrm{c}^a{}_{cb} \ . \tag{10.1.87}$$

In terms of this operator the commutator of the two infinitesimal variations of the field Φ^a is

$$[\Delta_2, \Delta_1]\Phi^a = \left(V_2{}^a{}_b V_1{}^b{}_c - V_1{}^a{}_b V_2{}^b{}_c\right)\Phi^c = [\boldsymbol{V}_2, \boldsymbol{V}_1]^a{}_c \Phi^c \ , \tag{10.1.88}$$

and the task of carrying out the algebra of transformations for the scalar field Φ^c goes over to calculating the commutator of the matrix differential operators $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. Calculation reveals that the commutator (10.1.88) is a differential operator of the first order

$$[\boldsymbol{V}_2, \boldsymbol{V}_1] = \boldsymbol{V}_3 \ , \tag{10.1.89}$$

where $\boldsymbol{V}_3 = \{V_3{}^a{}_b\}$, and

$$V_3{}^a{}_b \equiv -\delta^a_b \xi^\alpha_3 \partial_\alpha + \varepsilon^c_3 \mathrm{c}^a{}_{cb} \,, \tag{10.1.90}$$

has exactly the same structure as shown in (10.1.87) with a set of parameters

$$\xi^\alpha_3 \equiv \xi^\beta_1 \partial_\beta \xi^\alpha_2 - \xi^\beta_2 \partial_\beta \xi^\alpha_1 = [\xi_1, \xi_2]^\alpha \,, \tag{10.1.91}$$

$$\varepsilon^c_3 \equiv \xi^\alpha_1 \partial_\alpha \varepsilon^c_2 - \xi^\alpha_2 \partial_\alpha \varepsilon^c_1 - \mathrm{c}^c{}_{ab} \varepsilon^a_1 \varepsilon^b_2 \,, \tag{10.1.92}$$

describing a new infinitesimal variation, Δ_3. Recalling the dependence $\varepsilon^a \equiv \varepsilon^a[x, \xi(x)]$ and taking into account the result (10.1.91) one concludes that $\varepsilon^a_3 \equiv \varepsilon^a[x, [\xi_1, \xi_2]]$ that already signals that the variations Δ_ξ are to be functorial. However, the functional structure of $\varepsilon^a[x, \xi(x)]$ is left unknown and the relation (10.1.92) restricts it. At last, finally (10.1.91) and (10.1.92) state

$$[\Delta_2, \Delta_1]\Phi^a = \Delta_3 \Phi^a \,. \tag{10.1.93}$$

Calculation of the commutator for the field $\Phi^\star_a$ ends up with the same result.

Total variation of the Yang–Mills field, $A^a{}_\alpha$, given by (10.1.76), is more involved than the total variation of the scalar fields. It can be written down in terms of a differential operator of the first order $\boldsymbol{W} = \{W^{a\beta}{}_{b\alpha}\}$ as

$$\Delta_\xi A^a{}_\alpha = W^{a\beta}{}_{b\alpha} A^b{}_\beta - \partial_\alpha \varepsilon^a \,, \tag{10.1.94}$$

where the components of the operator read

$$W^{a\beta}{}_{b\alpha} \equiv -\delta^a_b \delta^\beta_\alpha \xi^\rho \partial_\rho - \delta^a_b \partial_\alpha \xi^\beta - \varepsilon^d \mathrm{c}^a{}_{bd} \delta^\beta_\alpha \,. \tag{10.1.95}$$

It is important to recall that the quantity $\Delta_\xi A^a{}_\alpha$ is transformed like an intrinsic contravariant vector under gauge transformations unlike $A^a{}_\alpha$ itself whose law of transformation (10.1.7) is not homogeneous. Therefore, the total infinitesimal variation of $A^a{}_\alpha$ given in (10.1.94) contains the term $\partial_\alpha \varepsilon^a$. It is not the case for $\Delta_\xi A^a{}_\alpha$ because it is transformed as a contravariant intrinsic vector under the gauge transformations so that its transformation depends only on the transformation matrix $\boldsymbol{W}$. In other words, we shall have

$$\Delta_2 \Delta_1 A^a{}_\alpha = W^{a\beta}_2{}_{b\alpha} (\Delta_1 A^a{}_\alpha) = W^{a\beta}_2{}_{b\alpha} \left(W^{b\gamma}_1{}_{c\beta} A^c{}_\gamma - \partial_\beta \varepsilon^b_1 \right) \,, \tag{10.1.96}$$

and a similar expression will be valid for $\Delta_1 \Delta_2 A^a{}_\alpha$ with the corresponding change of indexes 1 and 2. The commutator of the two total variations of the Yang–Mills field $A^a{}_\alpha$ results in

$$[\Delta_2, \Delta_1] A^a{}_\alpha = [\boldsymbol{W}_2, \boldsymbol{W}_1]^{a\gamma}{}_{c\alpha} A^c{}_\gamma - (W^{a\beta}_2{}_{b\alpha} \partial_\beta \varepsilon^b_1 - W^{a\beta}_1{}_{b\alpha} \partial_\beta \varepsilon^b_2). \tag{10.1.97}$$

After not so lengthy calculations, we obtain

$$[\boldsymbol{W}_2, \boldsymbol{W}_1]^{a\gamma}{}_{c\alpha} = W_3^{a\gamma}{}_{c\alpha}\,, \tag{10.1.98}$$

where

$$W_3^{a\gamma}{}_{c\alpha} \equiv -\delta^a_c \delta^\gamma_\alpha \xi^\rho_3 \partial_\rho - \delta^a_c \partial_\alpha \xi^\beta_3 - \varepsilon^b_3 \mathrm{c}^a{}_{cb} \delta^\gamma_\alpha, \tag{10.1.99}$$

where the parameters ξ^α_3 and ε^a_3 are given above in (10.1.91), (10.1.92). We also find out that the last term in the right side of (10.1.97) is reduced to

$$W_2^{a\beta}{}_{b\alpha} \partial_\beta \varepsilon^b_1 - W_1^{a\beta}{}_{b\alpha} \partial_\beta \varepsilon^b_2 = \partial_\alpha \varepsilon^a_3\,, \tag{10.1.100}$$

where again the parameter $\boldsymbol{\varepsilon}_3$ is the same as in (10.1.92). Substitution of (10.1.98), (10.1.100) into (10.1.97) yields

$$[\Delta_1, \Delta_2] A^a{}_\alpha = \Delta_3 A^a{}_\alpha\,, \tag{10.1.101}$$

where $\Delta_3 A^a{}_\alpha$ is given by formula (10.1.94) with the parameters ξ^α_3 and ε^a_3 given in (10.1.91), (10.1.92).

Thus, for all the fields under consideration one has

$$[\Delta_1, \Delta_2] = \Delta_3, \tag{10.1.102}$$

where the correspondence between the parameters of the total variations is provided by (10.1.91), (10.1.92) correspondingly. Commutation relations (10.1.102) confirm that the commutator of modified variations is indeed a differential operator of the first order and definition (10.1.81) with the requirement of functoriality can be applied to convert modified variations, Δ_ξ, into modified Lie derivatives, $\mathfrak{L}_\xi$.

10.1.7 Commutator of the modified variations Δ_ξ in $\mathscr{D}$-$\boldsymbol{t}$ basis

To obtain the commutator of the modified variations Δ_ξ in a covariant form we use the $\mathscr{D}$-$\boldsymbol{t}$ basis with generators $\xi^\alpha \mathscr{D}_\alpha, \zeta^a \boldsymbol{t}_a$. It means that we now take the same variation Δ_1 expressed in terms of the parameters $\{\xi^\alpha_1, \zeta^a_1 = \sigma^a_1 + \varepsilon_1\}$, and commute it with the variation Δ_2 expressed in terms of the parameters $\{\xi^\alpha_2, \zeta^a_2 = \sigma^a_2 + \varepsilon^a_2\}$. Subsequent transition from the $\boldsymbol{\partial}$-$\boldsymbol{t}$ basis of generators to the $\mathscr{D}$-$\boldsymbol{t}$ basis is performed in all formulae of the previous section by expressing the partial derivatives, ∂_μ, through the gauge-covariant derivatives, $\mathscr{D}_\mu$. This replacement does not change the commutation relation (10.1.102) which remains valid in $\mathscr{D}$-$\boldsymbol{t}$ basis as well. Nonetheless, the *form* of the presentation of parameters $\{\xi^\alpha_3, \zeta^a_3 = \sigma^a_3 + \varepsilon^a_3\}$ of the variation Δ_3 in the right side of (10.1.102) will be modified. This form is what we are looking for.

Parameter ξ_3^α is equal to the commutator of two vector field, $[\xi_1, \xi_2]^\alpha$, as shown in (10.1.91). To convert this expression to a covariant form we replace the partial derivatives with generally-covariant ones, and use the symmetry of the Christoffel symbols, $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$. We obtain,

$$\begin{aligned}\xi_3^\alpha &= \xi_1^\beta(\nabla_\beta \xi_2^\alpha - \Gamma^\alpha_{\gamma\beta}\xi_2^\gamma) - \xi_2^\beta(\nabla_\beta \xi_1^\alpha - \Gamma^\alpha_{\gamma\beta}\xi_1^\gamma) \\ &= \xi_1^\beta \nabla_\beta \xi_2^\alpha - \xi_2^\beta \nabla_\beta \xi_1^\alpha \ .\end{aligned} \tag{10.1.103}$$

Formula (10.1.103) elucidates that the parameter of diffeomorphism, $\boldsymbol{\xi}_3 = \{\xi_3^\alpha\}$, is both generally-covariant and gauge-invariant.

Resulting parameter of the variation Δ_3 of the commutator (10.1.102) is a sum of two parameters, $\zeta_3^a = \sigma_3^a + \varepsilon_3^a$. Parameter σ_3^a is defined in (10.1.62), and must have the following form

$$\sigma_3^c \equiv A^c{}_\alpha \xi_3^\alpha = A^c{}_\alpha \left(\xi_1^\beta \nabla_\beta \xi_2^\alpha - \xi_2^\beta \nabla_\beta \xi_1^\alpha\right) \ , \tag{10.1.104}$$

where we have used the result (10.1.103). By making use of the chain rule, definition of the covariant gauge derivative $\mathscr{D}_\alpha$, given in terms of the generally-covariant derivative ∇_α and the connection $A^a{}_\alpha$, we derive

$$A^c{}_\alpha \xi_1^\beta \nabla_\beta \xi_2^\alpha = \xi_1^\alpha \mathscr{D}_\alpha \sigma_2^c - \xi_1^\alpha \xi_2^\beta \left(\nabla_\alpha A^c{}_\beta + \mathrm{c}^c{}_{ab} A^a{}_\alpha A^b{}_\beta\right) \ , \tag{10.1.105}$$

$$A^c{}_\beta \xi_2^\alpha \nabla_\alpha \xi_1^\beta = \xi_2^\alpha \mathscr{D}_\alpha \sigma_1^c - \xi_1^\alpha \xi_2^\beta \left(\nabla_\beta A^c{}_\alpha + \mathrm{c}^c{}_{ab} A^a{}_\beta A^b{}_\alpha\right) \ . \tag{10.1.106}$$

After making use of these equations in (10.1.104) along with the definition (10.1.15) of the strength tensor of the Yang–Mills field, $F^a{}_{\alpha\beta}$, we obtain

$$\sigma_3^c \equiv \xi_1^\alpha \mathscr{D}_\alpha \sigma_2^c - \xi_2^\alpha \mathscr{D}_\alpha \sigma_1^c - \mathrm{c}^c{}_{ab} \sigma_1^a \sigma_2^b - F^c{}_{\alpha\beta} \xi_1^\alpha \xi_2^\beta \ . \tag{10.1.107}$$

It allows us to express the gauge parameter σ_3^a in manifestly generally-covariant form in terms of the parameters of the first and second diffeomorphisms, $\xi_1^\alpha = \xi_1^\alpha(x)$, $\xi_2^\alpha = \xi_2^\alpha(x)$, and associated with them gauge parameters $\sigma_1^a = \sigma^a[x, \xi_1(x)]$ and $\sigma_2^a = \sigma^a[x, \xi_2(x)]$.

Parameter ε_3^a defined in (10.1.92), is reformulated in terms of the covariant gauge derivatives with the help of equation (10.1.48). We get,

$$\varepsilon_3^a = \xi_1^\alpha \mathscr{D}_\alpha \varepsilon_2^a - \xi_2^\alpha \mathscr{D}_\alpha \varepsilon_1^a - \mathrm{c}^a{}_{bc}(\varepsilon_1^b \varepsilon_2^c + \sigma_1^b \varepsilon_2^c + \varepsilon_1^b \sigma_2^c) \ . \tag{10.1.108}$$

Notice that this equation points out that the *gauge parameter ε_3^a is not gauge-covariant* although it is generally-covariant. Adding up (10.1.107) and (10.1.108) we obtain the covariant form for the parameter ζ_3^a which we were looking for. Summarizing, we conclude that the parameters, ξ_3^a and ζ_3^a, defining the infinitesimal variation Δ_3 in the commutation relation (10.1.102), are given in the $\mathscr{D}$-$\boldsymbol{t}$ basis by the following formulae,

$$\xi_3^\alpha = [\xi_1, \xi_2]^\alpha \,, \tag{10.1.109}$$

$$\zeta_3^c = \xi_1^\alpha \mathscr{D}_\alpha \zeta_2^c - \xi_2^\alpha \mathscr{D}_\alpha \zeta_1^c - \mathrm{c}^c{}_{ab} \zeta_1^a \zeta_2^b - F^c{}_{\alpha\beta} \xi_1^\alpha \xi_2^\beta \,, \tag{10.1.110}$$

where each vector ζ^c is a sum, $\zeta^c = \sigma^c + \varepsilon^c$, in accordance with (10.1.73). Equations (10.1.109) and (10.1.110) represent a covariant form of the relationships (10.1.91) and (10.1.92).

10.1.8 The functoriality condition

In order to convert the modified variations, Δ_ξ, to the modified Lie derivatives, $\mathfrak{L}_{\xi_3}$, one must use the functoriality concept as it has been stressed in Section 10.1.5. Let us repeat this requirement. *The parameters $\xi_3^\alpha, \varepsilon_3^a$ defining the modified Lie derivative $\mathfrak{L}_{\xi_3}$ in $\left[\mathfrak{L}_{\xi_1}, \mathfrak{L}_{\xi_2}\right] = \mathfrak{L}_{\xi_3}$ must be uniquely expressed in terms of the commutator $[\xi_1, \xi_2]$ that is $\xi_3^\alpha = [\xi_1, \xi_2]^\alpha$ and $\varepsilon_3^a = \varepsilon^a(x, [\xi_1, \xi_2])$, that is $\zeta_3^a = \zeta^a(x, [\xi_1, \xi_2])$.* Because $\zeta_3^a = \zeta^a(x, \xi_3)$ the relation (10.1.109) tells that ζ^a satisfies this requirement anyway, whereas the relation (10.1.110) must be considered as an equation restricting a functional structure of ζ^c. Thus, we explore the (10.1.110) as the functoriality equation, and rewrite it in the form of an inhomogeneous equation:

$$\zeta^c\,[[\xi_1, \xi_2]] + \xi_2^\alpha \mathscr{D}_\alpha \zeta^c[\xi_1] - \xi_1^\alpha \mathscr{D}_\alpha \zeta^c[\xi_2] + \mathrm{c}^c{}_{ab} \zeta^a[\xi_1] \zeta^b[\xi_2] = -F^c{}_{\alpha\beta} \xi_1^\alpha \xi_2^\beta, \tag{10.1.111}$$

where the right hand side is defined by the known strength tensor $F^c{}_{\alpha\beta} = F^c{}_{\alpha\beta}(x)$ defined in (10.1.17).

It can be easily understood that $\zeta^c[\xi] = \sigma^c[\xi]$ is a particular solution of the inhomogeneous equation (10.1.111). Indeed, substituting $\zeta^a = \sigma^a$ to this equation makes it identity,

$$\sigma_3^c + \xi_2^\alpha \mathscr{D}_\alpha \sigma_1^c - \xi_1^\alpha \mathscr{D}_\alpha \sigma_2^c + \mathrm{c}^c{}_{ab} \sigma_1^a \sigma_2^b \equiv -F^c{}_{\alpha\beta} \xi_1^\alpha \xi_2^\beta \,, \tag{10.1.112}$$

which is fulfilled after implementing the definition of the parameter $\sigma^a = A^a{}_\alpha(x)\xi^\alpha(x)$, and the commutation relation $\xi_3^\alpha = [\xi_1, \xi_2]^\alpha$.

Because a general solution ζ^a of the inhomogeneous equation (10.1.111) is a sum, $\sigma^a + \varepsilon^a$, and $\sigma^a = A^a{}_\alpha(x)\xi^\alpha(x)$ satisfies the inhomogeneous equation (10.1.112), a *general* solution ε^a has to satisfy the corresponding homogeneous equation. Thus, the task of exploration of the functionality equation (10.1.111) is reduced to finding the general solution of a homogeneous equation for functional $\varepsilon^a = \varepsilon^a[\xi]$ which is more convenient to write down in terms of partial derivatives with the

$$\varepsilon^c\,[[\xi_1, \xi_2]] + \xi_2^\alpha \partial_\alpha \varepsilon^c[\xi_1] - \xi_1^\alpha \partial_\alpha \varepsilon^c[\xi_2] + \mathrm{c}^c{}_{ab} \varepsilon^a[\xi_1] \varepsilon^b[\xi_2] = 0 \,. \tag{10.1.113}$$

We are looking for the solution of this equation in the class of functionals which are supposed to make the parameter ζ^c both gauge-covariant that makes the modified Lie derivative both gauge- and generally-covariant. The simplest solution from this class of functions is

$$\bar{\varepsilon}^a = -\bar{A}^a{}_\alpha \xi^\alpha , \tag{10.1.114}$$

where $\bar{A}^a{}_\alpha$ is a pure gauge connection of the group $\mathsf{SU}(N)$ for which the tensor of strength (10.1.17) of the filed $\bar{A}^a{}_\alpha$ vanishes identically, $\bar{F}^c{}_{\alpha\beta} \equiv 0$. For example, we can pick up

$$\bar{A}^a{}_\alpha = -v^a{}_b \, \partial_\alpha \eta^b, \tag{10.1.115}$$

where $v^a{}_b$ is determined by the relation $u^a{}_c \; \partial_\alpha (u^{-1})^c{}_b \; = \; \mathrm{c}^c{}_{ab} v^c{}_d \; \partial_\alpha \eta^d$, where $\boldsymbol{u}$ is a matrix of a finite gauge transformation with arbitrary fixed finite parameters, $\eta^a(x)$, of gauge transformations. The fact that (10.1.114) is indeed a solution, is checked by a direct inspection after substituting it to the homogeneous equation (10.1.113). Combining the particular solution σ^c with the solution $\bar{\varepsilon}^c$ of the homogeneous equation we get the parameter

$$\zeta^a = \left(A^a{}_\alpha - \bar{A}^a{}_\alpha\right) \xi^\alpha , \tag{10.1.116}$$

which is both gauge- and generally-covariant because it is built as a difference of two connections that is transformed as a tensor under the gauge transformations. This proves functoriality of the modified Lie derivative for the class of the solutions of the functorial equation (10.1.113).

Disadvantage of the proposed solution (10.1.116) is that it can be amended with an arbitrary gauge-*covariant*, $\varepsilon^a_{\mathrm{cov}}$, solution that can be obtained from the functoriality condition equation (10.1.113) as well. For example, the solution to the equation (10.1.113) can be searched under the assumption that $\varepsilon^a_{\mathrm{cov}}$ is modeled by linear combinations of generally-covariant derivatives of ξ^α of various orders by using $\eta^a(x)$ which participate in (10.1.115). From the physical point of view it is not satisfactory. Theoretical physicists always try to avoid such a situation as it makes the results poorly defined. Whether or not *well defined generic solutions* of the functorial equation (10.1.111) exist we don't know yet.

The problem has been thoroughly studied in the series of works by Francaviglia et al. [5, 6, 64–68, 169], by Giachetta et al. [189], and Obukhov et al. [341]. Active research in this direction is still in progress.

Generally-covariant and gauge-invariant canonical currents

We remark that the action functional (10.1.1) is invariant under the infinitesimal variations for the generic variable ϕ:

$$\phi \to \tilde{\phi} = \phi + \mathfrak{L}_\xi \phi, \tag{10.1.117}$$

where $\zeta^a[\xi]$ included in the modified Lie derivative, $\mathfrak{L}_\xi$, satisfies the functoriality equation (10.1.111). It means that a current corresponding to this "modified diffeomorphism" can be constructed.

The simplest way to derive it is as follows. Using the gauge-covariant parameters $\zeta^a[\xi]$ in the formula (10.1.68), instead of non-covariant one $\sigma^a[\xi]$, we can construct expressions for the current in the form:

$$\mathscr{J}^\mu[\xi] = \mathscr{T}^\mu{}_\nu \xi^\nu + \mathscr{F}_a{}^{\mu\nu} \mathscr{D}_\nu \zeta^a[\xi] + \boldsymbol{j}_a{}^\mu \zeta^a[\xi] . \tag{10.1.118}$$

It is both manifestly generally-covariant and manifestly gauge-invariant. Nevertheless, it is not fully satisfactory because this current contains the functional $\zeta^a[\xi]$ which is not defined uniquely. It is evident that even for the same physical system at the same state, a different choice of functionals $\zeta^a[\xi]$ leads to different values of physically measurable quantity defined by the current.

10.2 Conservation laws in the tetrad formalism of general relativity

10.2.1 Tetrads and gravitational field

Interaction of tensor (bosonic) fields of matter with gravity is described in a Lagrangian theory of gravity directly through the coupling with the metric tensor which is a primary dynamical variable associated with the gravitational field. This description fails if one needs to study the interaction of spinor (fermionic) fields of matter with gravity which is essentially more complicated and requires an extension of the Lagrangian formalism [343] .

The tetrad is a set of four orthonormal contravariant vector fields $\mathrm{e}_i{}^\mu = \mathrm{e}_i{}^\mu(x)$. Here and below, the Greek indexes take in values 0,1,2,3, and numerate the spacetime tensor components – spacetime indexes. Small Latin indexes i, j, k, ... = 0,1,2,3 numerate the tetrad vectors and tetrad components of geometric objects. By definition of orthonormality the tetrad vectors satisfy the relation,

$$g_{\alpha\beta} \mathrm{e}_i{}^\alpha \mathrm{e}_j{}^\beta = \eta_{ij}, \tag{10.2.1}$$

where η_{ij} is a constant matrix with components η_{ij} = diag $(-1, 1, 1, 1)$, and $g_{\alpha\beta} = g_{\alpha\beta}(x)$ is the spacetime metric. The basis of the covariant vectors $\mathrm{e}^i{}_\mu = \mathrm{e}^i{}_\mu(x)$ is dual to the tetrad basis $\mathrm{e}_i{}^\mu$ and is called a cotetrad. The duality assumes that the cotetrad vectors are subject to the condition,

$$\mathrm{e}^i{}_\alpha \mathrm{e}_j{}^\alpha = \delta^i_j. \tag{10.2.2}$$

With the use of (10.2.1) and (10.2.2) one can easily obtain some other useful algebraic relations, for example,

$$e_i{}^\alpha e^i{}_\beta = \delta^\alpha_\beta \,, \tag{10.2.3}$$

$$g^{\alpha\beta} e^i{}_\alpha e^j{}_\beta = \eta^{ij} \,, \tag{10.2.4}$$

where η^{ij} is a constant matrix with the components $\eta^{ij} = \text{diag}\,(-1,1,1,1)$, matrices η_{ij} and η^{ij} are inverses of each other. In what follows, we shall often use formulae (10.2.1–10.2.4) without explicit referencing. It is worth mentioning that the spacetime (Greek) indexes are raised and lowered with the use of the metric $g_{\alpha\beta}$, whereas the tetrad indices are moved up and down with the use of the Lorentzian metric η_{ij}. It should be also noticed that the tetrad basis vectors $e_i{}^\alpha$ are connected by the real-valued commutation coefficients, $c^i{}_{jk}$, in the commutator of two tetrads

$$[e_j{}^\alpha, e_k{}^\alpha] = c^i{}_{jk} e_i{}^\alpha \,, \tag{10.2.5}$$

with $c^i{}_{jk} = c^i{}_{[jk]}$. Notice that the commutation coefficients are not constants but depend on spacetime coordinates, $c^i{}_{jk} = c^i{}_{jk}(x)$.

All spacetime tensors of the theory can be expressed as a linear combination of their tetrad components with the vector and covector basis. For example, for a contravariant spacetime vector, $V^\alpha = V^\alpha(x)$, we have

$$V^\alpha = V^i e_i{}^\alpha \,, \tag{10.2.6}$$

while for a covariant spacetime vector

$$V_\alpha = V_i e^i{}_\alpha \,, \tag{10.2.7}$$

where $V^i = V^i(x)$ and $V_i = V_i(x)$ are the tetrad and cotetrad components of the vector field. In particular, decomposing a tetrad, $e^i{}_\alpha$ as a vector with respect to the tetrad basis, we have

$$e_i{}^\alpha = \delta^j_i e_j{}^\alpha \,, \tag{10.2.8}$$

which means that the tetrad components in its own basis are constant numbers, $\delta^j_i = \text{diag}(1,1,1,1)$. The same conclusion is valid with regard to the components of the cotetrad.

Relations (10.2.1–10.2.4) allow us to represent the components of the spacetime metric in terms of the tetrads, namely,

$$g_{\mu\nu} = \eta_{ij} e^i{}_\mu e^j{}_\nu \,, \qquad g^{\mu\nu} = \eta^{ij} e_i{}^\mu e_j{}^\nu \,. \tag{10.2.9}$$

From this equation we can immediately derive a relationship between the determinant, $g = \det g_{\mu\nu}$ of the metric tensor and the determinant, e, of the tetrad

$$\sqrt{-g} = \mathrm{e}\,, \qquad \mathrm{e} \equiv \det(\mathrm{e}^i{}_\mu)\,. \tag{10.2.10}$$

Relation (10.2.9) allows us to interpret the metric tensor field as having been *constructed from the tetrad vectors* which can be considered as dynamical field variables of gravity in the Lagrangian formalism instead of the metric tensor. This convention admits a specific gauge invariance stemming from the demand that neither measured values of physical fields nor physical content of the field equations depend on the orientation of the tetards. It means that the formulation of the theory admits arbitrary local rotations (the Lorentzian rotations) of the tetrad fields which is an intrinsic gauge symmetry. For example, the metric tensor $g_{\mu\nu}$ does not change under the local Lorentzian rotations of the tetrad/cotetrads:

$$\mathrm{e}^i{}_\mu \to \tilde{\mathrm{e}}^i{}_\mu = \Lambda^i{}_j \mathrm{e}^j{}_\mu\,, \qquad \mathrm{e}_i{}^\mu \to \tilde{\mathrm{e}}_i{}^\mu = {}^{-1}\Lambda^j{}_i \mathrm{e}_j{}^\mu\,, \tag{10.2.11}$$

in the sense that

$$g_{\mu\nu} = \eta_{ij}\mathrm{e}^i{}_\mu \mathrm{e}^j{}_\nu = \eta_{ij}\tilde{\mathrm{e}}^i{}_\mu \tilde{\mathrm{e}}^j{}_\nu = g_{\mu\nu}\,. \tag{10.2.12}$$

Here $\Lambda^i{}_j = \Lambda^i{}_j(x)$ is a matrix of the Lorentz transformations determined by the relation,

$$\eta_{ij}\Lambda^i{}_k \Lambda^j{}_l = \eta_{kl}\,, \qquad \eta^{ij} = \eta^{kl}\,{}^{-1}\Lambda^i{}_k\,{}^{-1}\Lambda^j{}_l\,, \tag{10.2.13}$$

and ${}^{-1}\Lambda^i{}_j = {}^{-1}\Lambda^i{}_j(x)$ is the inverse matrix, ${}^{-1}\Lambda^i{}_j \Lambda^j{}_k = \delta^i_k$. Thus, each point of spacetime manifold with a fixed value of the coordinates $x = \{x^\alpha\}$, is equipped with a local tetrad basis which is defined only up to a set of the local Lorentzian rotations forming the group $\mathrm{O}(1,3)$. We will consider only the transformations related to connected component of this group denoted as $\mathrm{SO}(1,3)$.

10.2.2 Connections and derivatives

To use a technique of the previous section we need to define a gauge-covariant derivative in the tetrad formalism to represent a complete analogue of the gauge derivative in the Yang–Mills theory that has been discussed in Section 10.1. It is based on the local Lorentzian connection associated with the gauge freedom of the local Lorentzian rotations. This connection is also called $\mathrm{SO}(1,3)$-connection represented in the form of a square matrix, $\boldsymbol{A}_\mu = \{(A_\mu)^i{}_j(x)\}$, of order $n = 4$ for each of spacetime index "$_\mu$", and that is an antisymmetric real matrix: $A_{ij\mu} = A_{[ij]\mu}$. More conventional notation of the connection components is $A^i{}_{j\mu} \equiv (A_\mu)^i{}_j$.

Under the gauge transformations (10.2.11) the connection is transformed by the rule (cf. (10.1.7)):

$$A^i{}_{j\mu} \quad \to \quad \tilde{A}^i{}_{j\mu} = \Lambda^i{}_k A^k{}_{l\mu}{}^{-1}\Lambda^l{}_j + \Lambda^i{}_k \partial_\mu{}^{-1}\Lambda^k{}_j \, . \tag{10.2.14}$$

Let us denote $\boldsymbol{F}_{\mu\nu} = \{F^i{}_{j\mu\nu}(x)\}$ the curvature tensor of the Lorentzian connection $\boldsymbol{A}_\alpha$:

$$F^i{}_{j\mu\nu} \equiv \partial_\mu A^i{}_{j\nu} - \partial_\nu A^i{}_{j\mu} + A^i{}_{k\mu} A^k{}_{j\nu} - A^i{}_{k\nu} A^k{}_{j\mu} \, . \tag{10.2.15}$$

Notice that tensor $\boldsymbol{F}_{\mu\nu}$ is analogous to the Faraday tensor of the field strength of the Yang–Mills field $\boldsymbol{A}_\alpha$. At the moment, it should be distinguished from the Riemann tensor of curvature of spacetime. In what follows, we shall use the following convenient notations for the projections of the SO(1, 3) connection curvature on the tetrad basis:

$$F^j{}_\nu = \mathrm{e}_i{}^\mu F^{ij}{}_{\mu\nu}, \qquad F = \mathrm{e}_j{}^\nu F^j{}_\nu \, . \tag{10.2.16}$$

The SO(1, 3)-covariant derivative D_ν is defined in a full correspondence with the rule introduced in previous section. For example, for a contravariant (intinsic) vector V^i and for a covector $V_i = \eta_{ij} V^j$ we introduce gauge-covariant derivatives:

$$D_\nu V^i = \partial_\nu V^i + A^i{}_{j\nu} V^j \, , \qquad D_\nu V_i = \partial_\nu V_i - A^j{}_{i\nu} V_j \, . \tag{10.2.17}$$

Although the derivative D_ν is SO(1, 3)-covariant, it is not generally-covariant. The generally-covariant derivative is denoted by ∇_ν, and is defined, as usual, for a contravariant vector V^μ and for a covector $V_\mu = g_{\mu\nu} V^\nu$ as

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda \, , \qquad \nabla_\nu V_\mu = \partial_\nu V_\mu - \Gamma^\lambda_{\mu\nu} V_\lambda \, , \tag{10.2.18}$$

with $\Gamma^\mu_{\lambda\nu}$ – the Christoffel symbols on the spacetime manifold. The result of action of ∇_μ is generally-covariant, but is not SO(1, 3)-covariant.

Combining the formulae (10.2.17) and (10.2.18), we introduce *generally-covariant gauge* derivatives:

$$\mathscr{D}_\nu V^i{}_\mu = \partial_\nu V^i{}_\mu - \Gamma^\lambda_{\mu\nu} V^i{}_\lambda + A^i{}_{j\nu} V^j{}_\mu \, . \tag{10.2.19}$$

$$\mathscr{D}_\nu V_i^\mu = \partial_\nu V_i^\mu + \Gamma^\mu_{\lambda\nu} V_i^\lambda - A^j{}_{i\nu} V_j^\mu \, , \tag{10.2.20}$$

which are both generally-covariant and gauge-covariant. Tetrads/cotetrads are exactly this type of the geometric objects which have the indices belonging to the two spaces. Therefore, their generalized covariant differentiation obeys the same rules. More specifically,

$$\mathscr{D}_\nu e^i{}_\mu = \nabla_\nu e^i{}_\mu + A^i{}_{j\nu} e^j{}_\mu = D_\nu e^i{}_\mu - \Gamma^\lambda_{\mu\nu} e^i{}_\lambda, \tag{10.2.21}$$

$$\mathscr{D}_\nu e_i^\mu = \nabla_\nu e_i^\mu - A^j{}_{i\nu} e_j^\mu = D_\nu e_i^\mu + \Gamma^\mu_{\lambda\nu} e_i^\lambda. \tag{10.2.22}$$

In general case of an arbitrary theory of gravity, SO(1, 3)-connection is considered as an independent physical field which dynamics is governed by the field Lagrangian but not the kinematic behavior of the tetrads introduced on spacetime manifold. In the present section, for the sake of simplicity, we consider merely a tetrad formulation of general relativity where the gravitational field can be *uniquely* represented by tetrads. To construct a related Lagrangian we will use the contraction of the curvature tensor, $F^i{}_{j\mu\nu}$, defined in (10.2.15), in the same way as in the metric representation of general relativity one uses the scalar of the Riemannian tensor. Therefore, from now on, we require that SO(1, 3)-connection $A^i{}_{j\nu}$ is fully consistent with the definition of the Christoffel symbols. The consistentcy is achieved by imposing the law of the differentiation of tetrad/cotetrad:

$$\mathscr{D}_\nu e^i{}_\mu \equiv 0\,, \qquad \mathscr{D}_\nu e_j^\lambda \equiv 0\,. \tag{10.2.23}$$

This law employed in (10.2.21) and (10.2.22) immediately allows us to express the SO(1, 3)-connection in terms of the spacetime covariant derivatives of tetrad/cotetrad

$$A^i{}_{j\nu} = e^i{}_\mu \nabla_\nu e_j^\mu = -e_j^\mu \nabla_\nu e^i{}_\mu\,. \tag{10.2.24}$$

Going further on, we make use of equations (10.2.15) and (10.2.24), and establish a one-to-one link between the curvature, $F^i{}_{j\mu\nu}$, of the SO(1, 3)-connection and the Riemannian tensor $R^\kappa{}_{\lambda\mu\nu}$ (1.3.2), the Ricci tensor $R_{\mu\nu}$ (1.3.3) and the Ricci scalar R (1.3.4) of spacetime manifold.[1] After tedious calculations, we arrive to simple equations,

$$F^{ij}{}_{\mu\nu} = e^i{}_\kappa e^j{}_\lambda R^{\kappa\lambda}{}_{\mu\nu}, \qquad R^{\kappa\lambda}{}_{\mu\nu} = e_i^\kappa e_j^\lambda F^{ij}{}_{\mu\nu}; \tag{10.2.25}$$

$$F^i{}_\nu = e^i{}_\mu R^\mu{}_\nu, \qquad R^\mu{}_\nu = e_i^\mu F^i{}_\nu; \tag{10.2.26}$$

$$F = R;\,, \tag{10.2.27}$$

where $F^i{}_\nu$ and F have been defined in (10.2.16).

10.2.3 Variation of the Hilbert action

After the short introduction to the tetrads, cotetrads and the connections let us now discuss the dynamic aspects of this theory and the method of construction of conserved quantities in the tetrad representation of general relativity. Using formulae (10.2.25–10.2.27) and relation (10.2.10) the Lagrangian density of general relativity can be rewritten in the tetrad form:

1 The tetrad representation of $R^\kappa{}_{\lambda\mu\nu}$, $R_{\mu\nu}$ and R can be found, for example, in the textbook [285].

$$\mathscr{L} = \sqrt{-g}\,R = \mathrm{e}\,L\ , \tag{10.2.28}$$

where

$$L = F = \mathrm{e}_i{}^{\mu}\mathrm{e}_j{}^{\nu}F^{ij}{}_{\mu\nu}\ . \tag{10.2.29}$$

Therefore, we consider a theory with the action functional:

$$S = -\frac{1}{2\kappa}\int_{\Sigma_1}^{\Sigma_2} d^4x\,\mathscr{L}\ , \tag{10.2.30}$$

where $\mathscr{L}$ is fully expressed in terms of the tetrad/cotetrad vectors, and we choose the cotetrad vectors $\mathrm{e}^i{}_{\mu}$ as independent dynamic variables.

We calculate the total variation $\delta' S$ of the action functional (10.2.30) as usual, and obtain

$$\delta' S = \delta_{\Sigma} S + \delta S\ , \tag{10.2.31}$$

where

$$\delta_{\Sigma} S = -\frac{1}{2\kappa}\int_{\Sigma_1}^{\Sigma_2} d^4x\,\mathrm{e}\,\nabla_{\mu}\left(L\delta x^{\mu}\right)\ , \tag{10.2.32}$$

$$\delta S = -\frac{1}{2\kappa}\int_{\Sigma_1}^{\Sigma_2} d^4x\,\delta(\mathrm{e}\,L)\ . \tag{10.2.33}$$

Calculation of variation $\delta(\mathrm{e}\,L)$ in (10.2.33) consists of a few steps. First of all, we use the Leibnitz rule to write

$$\delta(\mathrm{e}\,L) = \delta\mathrm{e}\,L + \mathrm{e}\,\delta L\ , \tag{10.2.34}$$

and apply the known formula for variation of the determinant,

$$\delta\mathrm{e} = \mathrm{e}\,\mathrm{e}_i{}^{\mu}\delta\mathrm{e}^i{}_{\mu}\ . \tag{10.2.35}$$

Taking into account the specific form of the Lagrangian L in (10.2.29) and definition (10.2.16), we also find,

$$\delta L = 2F^j{}_{\nu}\delta\mathrm{e}_j{}^{\nu} + \mathrm{e}_i{}^{\mu}\mathrm{e}_j{}^{\nu}\delta F^{ij}{}_{\mu\nu}\ . \tag{10.2.36}$$

Varying relation (10.2.2) we express the variation of the tetrad $\delta\mathrm{e}_j{}^{\nu}$ through the variation of the cotetrad $\delta\mathrm{e}^i{}_{\mu}$,

$$\delta\mathrm{e}_j{}^{\nu} = -\mathrm{e}_j{}^{\mu}\mathrm{e}_i{}^{\nu}\delta\mathrm{e}^i{}_{\mu}\ . \tag{10.2.37}$$

Varying relation (10.2.15) we express the variation $\delta F^{ij}{}_{\mu\nu}$ of the SO(1, 3)-curvature through the variation $\delta A^{i}{}_{j\nu}$ of the SO(1, 3)-connection,

$$\delta F^{ij}{}_{\mu\nu} = \eta^{jk}\delta F^{i}{}_{k\mu\nu} = \eta^{jk}\left(\mathscr{D}_{\mu}\delta A^{i}{}_{k\nu} - \mathscr{D}_{\nu}\delta A^{i}{}_{k\mu}\right) . \tag{10.2.38}$$

Recall that the variation of the connection $\delta A^{i}{}_{k\nu}$ is a tensor of the second rank in the intrinsic space and a covector in spacetime. Hence, its derivatives which appear in the right hand side of (10.2.38) can be calculated by applying the rules of differentiation (10.2.19) and (10.2.20),

$$\mathscr{D}_{\mu}\delta A^{i}{}_{k\nu} = \nabla_{\mu}\delta A^{i}{}_{k\nu} + A^{i}{}_{l\mu}\delta A^{l}{}_{k\nu} - A^{l}{}_{k\mu}\delta A^{i}{}_{l\nu} . \tag{10.2.39}$$

Eventually, combining the results of the previous steps and taking into account relations (10.2.26), (10.2.27), and the rule of the differentiation (10.2.23) of the tetrad vectors, we obtain,

$$\delta(\mathrm{e}L) = -2\mathrm{e}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R\right)\mathrm{e}_{i}{}^{\nu}\delta\mathrm{e}^{i}{}_{\mu} + 2\mathrm{e}\,\mathscr{D}_{\mu}\left(\mathrm{e}_{[i}{}^{\mu}\mathrm{e}_{j]}{}^{\nu}\eta^{jk}\delta A^{i}{}_{k\nu}\right) . \tag{10.2.40}$$

Because the expression in the last parentheses does not have free tetrad indices, the derivative $\mathscr{D}_{\mu}$ is reduced to taking merely a covariant derivative ∇_{μ}.

Substituting (10.2.40) into the total variation of the action (10.2.31) and applying the requirement of the Noether theorem, $\delta' S \equiv 0$, one obtains the main Noether's identity in the form:

$$-\frac{1}{2\kappa}\frac{\delta\mathscr{L}}{\delta\mathrm{e}^{i}{}_{\mu}}\delta\mathrm{e}^{i}{}_{\mu} \equiv \partial_{\mu}\mathscr{J}^{\mu} , \tag{10.2.41}$$

where we have introduced shorthand notations

$$\frac{\delta\mathscr{L}}{\delta\mathrm{e}^{i}{}_{\mu}} = -2\mathrm{e}\,G^{\mu}{}_{\nu}\mathrm{e}_{i}{}^{\nu}, \qquad G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R , \tag{10.2.42}$$

for the Lagrangian derivative and the Einstein tensor $G^{\mu}{}_{\nu}$, as well as

$$\mathscr{J}^{\mu} = \frac{\mathrm{e}}{2\kappa}\left(2\mathrm{e}_{[i}{}^{\mu}\mathrm{e}_{j]}{}^{\nu}\eta^{jk}\delta A^{i}{}_{k\nu} + R\delta x^{\mu}\right) , \tag{10.2.43}$$

for the Noether current. From formula (10.2.42) it follows that the gravitational field equations defined in the tetrad formalism of general relativity with the action functional (10.2.30), $\delta\mathscr{L}/\delta\mathrm{e}^{i}{}_{\mu} = 0$, coincide with the usual vacuum Einstein equations, $G^{\mu}{}_{\nu} = 0$.

10.2.4 The Noether current

Let us transform the Noether current (10.2.43) to a new form which is more convenient for calculations of conserved quantities in the tetrad formalism. Our goal is to express the variation of the connection, $\delta A^i{}_{k\nu}$, in terms of the variations of the metric tensor and those of the tetrads. To this end, we notice that a covariant derivative from the variation of the tetrad is

$$\nabla_\nu \left(\delta \mathrm{e}^i{}_\mu\right) = \partial_\nu \left(\delta \mathrm{e}^i{}_\mu\right) - \Gamma^\lambda_{\mu\nu} \delta \mathrm{e}^i{}_\lambda \, . \tag{10.2.44}$$

However, the operations of taking the partial derivative, ∂_α, and the variation, δ, commute,

$$\partial_\alpha \left(\delta \mathrm{e}^i{}_\mu\right) = \delta \left(\partial_\alpha \mathrm{e}^i{}_\mu\right) \, . \tag{10.2.45}$$

Hence, we conclude that

$$\delta(\nabla_\nu \mathrm{e}^i{}_\mu) = \nabla_\nu(\delta \mathrm{e}^i{}_\mu) - \delta\Gamma^\lambda{}_{\mu\nu} \mathrm{e}^i{}_\lambda, \tag{10.2.46}$$

The connection coefficients, $A^i{}_{k\nu}$, are expressed in terms of the covariant derivative, ∇_α, as shown in (10.2.24),

$$A^i{}_{k\nu} = -\mathrm{e}_k{}^\alpha \nabla_\nu \mathrm{e}^i{}_\alpha \, . \tag{10.2.47}$$

Taking variation from both sides of (10.2.47) and accounting for formulae (10.2.37) and (10.2.46), we get

$$\begin{aligned} \delta A^i{}_{k\nu} &= -(\delta \mathrm{e}_k{}^\alpha)\nabla_\nu \mathrm{e}^i{}_\alpha - \mathrm{e}_k{}^\alpha \delta(\nabla_\nu \mathrm{e}^i{}_\alpha) \\ &= \mathrm{e}_k{}^\beta \mathrm{e}_l{}^\alpha (\nabla_\nu \mathrm{e}^i{}_\alpha) \delta \mathrm{e}^l{}_\beta + \mathrm{e}_k{}^\alpha \left[\mathrm{e}^i{}_\lambda \delta\Gamma^\lambda{}_{\alpha\nu} - \nabla_\nu(\delta \mathrm{e}^i{}_\alpha)\right] . \end{aligned} \tag{10.2.48}$$

Now, converting (10.2.47),

$$\nabla_\nu \mathrm{e}^i{}_\alpha = -A^i{}_{j\nu} \mathrm{e}^j{}_\alpha, \tag{10.2.49}$$

and making use of this and the relations (10.2.21–10.2.23), we transform the expression for $\delta A^i{}_{k\nu}$ to the form:

$$\begin{aligned} \delta A^i{}_{k\nu} &= -\mathrm{e}_k{}^\alpha (\nabla_\nu \delta \mathrm{e}^i{}_\alpha + A^i{}_{j\nu} \delta \mathrm{e}^j{}_\alpha) + \mathrm{e}_k{}^\mu \mathrm{e}^i{}_\lambda \delta\Gamma^\lambda{}_{\mu\nu} \\ &= -\mathrm{e}_k{}^\alpha \mathscr{D}_\nu (\delta \mathrm{e}^i{}_\alpha) + \mathrm{e}_k{}^\mu \mathrm{e}^i{}_\lambda \, \delta\Gamma^\lambda{}_{\mu\nu} \\ &= -\mathscr{D}_\nu (\mathrm{e}_k{}^\alpha \delta \mathrm{e}^i{}_\alpha) + \mathrm{e}_k{}^\mu \mathrm{e}^i{}_\lambda \, \delta\Gamma^\lambda{}_{\mu\nu}. \end{aligned} \tag{10.2.50}$$

The product $(\mathrm{e}_k{}^{\alpha}\delta\mathrm{e}^i{}_{\alpha})$ in the first term on the right hand side of formula (10.2.50) is rearranged as follows,

$$\mathrm{e}_k{}^{\alpha}\delta\mathrm{e}^i{}_{\alpha} = \eta_{kl}g^{\alpha\beta}\mathrm{e}^l{}_{\alpha}\delta\mathrm{e}^i{}_{\beta} = \eta_{kl}g^{\alpha\beta}\left(\mathrm{e}^{(l}{}_{\alpha}\delta\mathrm{e}^{i)}{}_{\beta} + \mathrm{e}^{[l}{}_{\alpha}\delta\mathrm{e}^{i]}{}_{\beta}\right) . \tag{10.2.51}$$

Noticing that

$$\begin{aligned} g^{\alpha\beta}\mathrm{e}^{(l}{}_{\alpha}\delta\mathrm{e}^{i)}{}_{\beta} &= \frac{1}{2}g^{\alpha\beta}\delta(\mathrm{e}^l{}_{\alpha}\mathrm{e}^i{}_{\beta}) = -\frac{1}{2}\delta g^{\alpha\beta}\mathrm{e}^l{}_{\alpha}\mathrm{e}^i{}_{\beta} \\ &= \frac{1}{2}g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}\mathrm{e}^l{}_{\alpha}\mathrm{e}^i{}_{\beta} , \end{aligned} \tag{10.2.52}$$

we eventually get,

$$\mathrm{e}_k{}^{\alpha}\delta\mathrm{e}^i{}_{\alpha} = \eta_{kl}\left(\frac{1}{2}\mathrm{e}^l{}_{\alpha}\mathrm{e}^i{}_{\beta}g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu} + g^{\alpha\beta}\mathrm{e}^{[l}{}_{\alpha}\delta\mathrm{e}^{i]}{}_{\beta}\right) . \tag{10.2.53}$$

Expression for the variation $\delta\Gamma^{\lambda}{}_{\mu\nu}$ making up the very last term in formula (10.2.50) can be expressed in terms of the variations of the metric tensor with the help of the well-known formula [285, 315]

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}\left(\nabla_{\mu}\delta g_{\alpha\nu} + \nabla_{\nu}\delta g_{\alpha\mu} - \nabla_{\alpha}\delta g_{\mu\nu}\right) . \tag{10.2.54}$$

Substitution of (10.2.53), (10.2.54) in (10.2.50) leads to the desired expression for the variation of the connection coefficients:

$$\delta A^i{}_{k\nu} = \mathrm{e}_k{}^{\mu}\mathrm{e}^{i\lambda}\nabla_{[\mu}\delta g_{\lambda]\nu} - \eta_{kl}\mathscr{D}_{\nu}\left(g^{\alpha\beta}\mathrm{e}^{[l}{}_{\alpha}\delta\mathrm{e}^{i]}{}_{\beta}\right) . \tag{10.2.55}$$

Making use of (10.2.55) in (10.2.43) brings about the Noether current as a function of the variations of the coordinates, the metric tensor and the tetrad basis:

$$\mathscr{J}^{\mu} = \frac{\mathrm{e}}{2\kappa}\left\{R\delta x^{\mu} - 2g^{\mu\alpha}g^{\nu\beta}\nabla_{[\alpha}\delta g_{\beta]\nu} + 2\mathrm{e}_i{}^{\mu}\mathrm{e}_j{}^{\nu}\mathscr{D}_{\nu}\left(g^{\alpha\beta}\mathrm{e}^{[i}{}_{\alpha}\delta\mathrm{e}^{j]}{}_{\beta}\right)\right\} . \tag{10.2.56}$$

This expression can be further simplified by taking into account the consistency condition (10.2.23) of the tetrad vectors. This equation allows us to reduce the derivative $\mathscr{D}_{\nu}$ to the covarant derivative on spacetime manifold by taking the tetrad vectors under the sign of the derivative $\mathscr{D}_{\nu}$ and noticing that the object to which the derivative $\mathscr{D}_{\nu}$ is now applied is a scalar with regard to the tetrad indices. More specifically, the last term in the right hand side of (10.2.56) is simplified to

$$\begin{aligned} \mathrm{e}_i{}^{\mu}\mathrm{e}_j{}^{\nu}\mathscr{D}_{\nu}\left(g^{\alpha\beta}\mathrm{e}^{[i}{}_{\alpha}\delta\mathrm{e}^{j]}{}_{\beta}\right) &= \mathscr{D}_{\nu}\left(g^{\alpha\beta}\mathrm{e}_i{}^{\mu}\mathrm{e}_j{}^{\nu}\mathrm{e}^{[i}{}_{\alpha}\delta\mathrm{e}^{j]}{}_{\beta}\right) \\ &= \nabla_{\nu}\left(g^{\alpha\beta}\mathrm{e}_i{}^{\mu}\mathrm{e}_j{}^{\nu}\mathrm{e}^{[i}{}_{\alpha}\delta\mathrm{e}^{j]}{}_{\beta}\right) \\ &= \nabla_{\nu}\left(g^{\alpha[\mu}\mathbf{e}_i{}^{\nu]}\delta\mathbf{e}^i{}_{\alpha}\right) . \end{aligned} \tag{10.2.57}$$

This reduces (10.2.56) to

$$\mathscr{J}^{\mu} = \frac{\mathrm{e}}{2\kappa}\left(R\delta x^{\mu} - 2g^{\mu\alpha}g^{\nu\beta}\nabla_{[\alpha}\delta g_{\beta]\nu}\right) + \frac{1}{\kappa}\partial_{\nu}\left(\mathrm{e}\, g^{\alpha[\mu}\mathrm{e}_{i}{}^{\nu]}\delta\mathrm{e}^{i}{}_{\alpha}\right) . \tag{10.2.58}$$

Formula (10.2.58) for the Noether current is presented in the most elegant and convenient way to offer calculation of the conserved quantities and superpotentials corresponding to both the intrinsic gauge invariance (Lorentzian rotations) and diffeomorphic invariance of the theory that are discussed in next sections.

10.2.5 Conserved quantities corresponding to the local Lorentz invariance

The intrinsic gauge invariance of general relativity in the tetrad formalism is represented by the infinitesimal rotations of tetrads generated by the matrix of the Lorentzian rotations introduced earlier in (10.2.11) with the parameters $\varepsilon^{ij} = \varepsilon^{[ij]}$ such that

$$\Lambda^{i}{}_{j} \approx \delta^{i}_{j} + \varepsilon^{i}{}_{j} , \qquad {}^{-1}\Lambda^{i}{}_{j} \approx \delta^{i}_{j} - \varepsilon^{i}{}_{j} . \tag{10.2.59}$$

For this type of symmetry the variations of the coordinates and the metric tensor are nil,

$$\delta_{\varepsilon}x^{\mu} = 0 , \qquad \delta_{\varepsilon}g_{\mu\nu} = 0 , \tag{10.2.60}$$

while the variations of the tetrads/cotetrads are given by

$$\delta_{\varepsilon}\mathrm{e}^{i}{}_{\mu} = \varepsilon^{i}{}_{j}\mathrm{e}^{j}{}_{\mu} , \qquad \delta_{\varepsilon}\mathrm{e}_{j}{}^{\mu} = -\varepsilon^{i}{}_{j}\mathrm{e}_{i}{}^{\mu}. \tag{10.2.61}$$

The conserved current corresponding to the gauge symmetry with respect to the Lorentz rotations is given by (10.2.58) where the variations of the corresponding quantities are taken from (10.2.60) and (10.2.61). It shows that the first term in the round brackets in the right hand side of (10.2.58) vanishes while the second term yields a conserved current, $\mathscr{J}^{\mu}[\varepsilon]$, that can be expressed in terms of a superpotential, $\mathscr{J}^{\mu\nu}[\varepsilon]$, as follows,

$$\mathscr{J}^{\mu}[\varepsilon] = \partial_{\nu}\mathscr{J}^{\mu\nu}[\varepsilon] , \qquad \mathscr{J}^{\mu\nu}[\varepsilon] = -\frac{\mathrm{e}}{\kappa}\varepsilon^{ij}\mathrm{e}_{i}{}^{\mu}\mathrm{e}_{j}{}^{\nu} . \tag{10.2.62}$$

It is interesting to notice that the current $\mathscr{J}^{\mu}[\varepsilon]$ transforms to a divergence of the superpotential $\mathscr{J}^{\mu\nu}[\varepsilon]$ off-shell, that is without making use of the gravitational field equations, $G^{\mu}{}_{\nu} = 0$.

10.2.6 Conserved quantities corresponding to the diffeomorphism invariance

In order to construct the conserved quantities corresponding to the diffeomorphism invariance of general relativity, it is instructive to split the Noether current (10.2.58) in two parts,

$$\mathscr{J}^{\mu} = \mathscr{J}^{\mu}_{\mathrm{h}} + \mathscr{J}^{\mu}_{\mathrm{e}}, \tag{10.2.63}$$

where

$$\mathscr{J}^{\mu}_{\mathrm{h}} = \frac{\mathrm{e}}{2\kappa}\left(R\delta x^{\mu} - 2g^{\mu\alpha}g^{\nu\beta}\nabla_{[\alpha}\delta g_{\beta]\nu}\right)\,, \tag{10.2.64}$$

$$\mathscr{J}^{\mu}_{\mathrm{e}} = \frac{\mathrm{e}}{\kappa}\nabla_{\nu}\left(g^{\alpha[\mu}\mathrm{e}_i{}^{\nu]}\delta\mathrm{e}^i{}_{\alpha}\right)\,. \tag{10.2.65}$$

Let us consider, first, the contribution $\mathscr{J}^{\mu}_{\mathrm{h}}$ in (10.2.64). For infinitesimal diffeomorphisms with parameter ξ^{μ}, one has

$$\delta_{\xi}x^{\mu} = \xi^{\mu}\,, \tag{10.2.66}$$

$$\delta_{\xi}g_{\mu\nu} = \pounds_{\xi}g_{\mu\nu}\,, \tag{10.2.67}$$

where $\pounds_{\xi}g_{\mu\nu}$ is the Lie derivative of the metric tensor that is well-known [315],

$$\pounds_{\xi}g_{\mu\nu} = -\nabla_{\mu}\xi_{\nu} - \nabla_{\nu}\xi_{\mu}\,. \tag{10.2.68}$$

From here,

$$\nabla_{[\alpha}\delta_{\xi}g_{\beta]\nu} = -\nabla_{[\alpha}\nabla_{\beta]}\xi_{\nu} - \tfrac{1}{2}(\nabla_{\alpha}\nabla_{\nu}\xi_{\beta} - \nabla_{\beta}\nabla_{\nu}\xi_{\alpha})\,. \tag{10.2.69}$$

Changing the order of the covariant derivatives in (10.2.69), recalling that a commutator of two covariant derivatives is expressed in terms of the Riemann curvature tensor and using the Ricci identity for the Riemann tensor, we get

$$\nabla_{[\alpha}\delta_{\xi}g_{\beta]\nu} = R^{\gamma}{}_{\nu\alpha\beta}\xi_{\gamma} - \nabla_{\nu}\nabla_{[\alpha}\xi_{\beta]}\,. \tag{10.2.70}$$

Replacing (10.2.66) and (10.2.70) in (10.2.64), one finds

$$\mathscr{J}^{\mu}_{\mathrm{h}}[\xi] = -\frac{1}{\kappa}\sqrt{-g}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R\right)\xi^{\nu} + \nabla_{\nu}\left(\frac{1}{\kappa}\sqrt{-g}\,\nabla^{[\mu}\xi^{\nu]}\right) \tag{10.2.71}$$

or

$$\mathscr{J}^{\mu}_{\mathrm{h}}[\xi] = -\frac{1}{\kappa}\sqrt{-g}\,G^{\mu}{}_{\nu}\xi^{\nu} + \partial_{\nu}\mathscr{K}^{\mu\nu}[\xi], \tag{10.2.72}$$

$$\mathscr{K}^{\mu\nu}[\xi] = \frac{1}{\kappa}\sqrt{-g}\,\nabla^{[\mu}\xi^{\nu]}, \tag{10.2.73}$$

where we have used the relation $\mathrm{e} = \sqrt{-g}$. The first term in the right hand side of (10.2.72) vanishes on-shell for $G^{\mu}{}_{\nu} = 0$, whereas the second term is a divergence of a superpotential $\mathscr{K}^{\mu\nu}[\xi]$ which turns out to be exactly the Komar superpotential derived earlier in (1.4.80). This result is not surprising because the part $\mathscr{J}^{\mu}_{\mathrm{h}}$ of the Noether current (10.2.63) exactly coincides with the current employed in the standard metric formulation of general relativity with the Hilbert action (1.3.1).

Let us examine the contribution induced by the term $\mathscr{J}_{\mathrm{e}}^{\mu}[\xi]$ in (10.2.65). Under the diffeomorphism with the infinitesimal parameter ξ^{α}, the variation of the cotetrad $\{\mathrm{e}^{i}{}_{\alpha}\}$, reads

$$\delta_{\xi}\mathrm{e}^{i}{}_{\mu} = \pounds_{\xi}\mathrm{e}^{i}{}_{\mu} \,, \tag{10.2.74}$$

$$\pounds_{\xi}\mathrm{e}^{i}{}_{\mu} = -\xi^{\alpha}\nabla_{\alpha}\mathrm{e}^{i}{}_{\mu} - \mathrm{e}^{i}{}_{\alpha}\nabla_{\mu}\xi^{\alpha} = \sigma^{i}{}_{j}\mathrm{e}^{j}{}_{\mu} - \mathrm{e}^{i}{}_{\alpha}\nabla_{\mu}\xi^{\alpha} \,, \tag{10.2.75}$$

where we have used (10.2.21) and (10.2.23), and the matrix parameter is defined as

$$\sigma^{i}{}_{j} \equiv A^{i}{}_{j\alpha}\xi^{\alpha}. \tag{10.2.76}$$

Substituting (10.2.74) to the right side of (10.2.65) and making necessary simplifications, we arrive to the conclusion that the current $\mathscr{J}_{\mathrm{e}}^{\mu}[\xi]$ is expressed in terms of a superpotential $\mathscr{J}_{\mathrm{e}}^{\mu\nu}[\xi]$ such that

$$\mathscr{J}_{\mathrm{e}}^{\mu}[\xi] = \partial_{\nu}\mathscr{J}_{\mathrm{e}}^{\mu\nu}[\xi] \,, \qquad \mathscr{J}_{\mathrm{e}}^{\mu\nu}[\xi] = -\mathscr{K}^{\mu\nu}[\xi] - \frac{\mathrm{e}}{\kappa}\sigma^{ij}\mathrm{e}_{i}{}^{[\mu}\mathrm{e}_{j}{}^{\nu]} \,, \tag{10.2.77}$$

where the first term in the expression for the superpotencial $\mathscr{J}_{\mathrm{e}}^{\mu\nu}[\xi]$ coincides exactly with the Komar superpotential (10.2.73) but taken with the opposite sign.

Adding up the two contributions, $\mathscr{J}_{\mathrm{h}}^{\mu}[\xi]$ and $\mathscr{J}_{\mathrm{e}}^{\mu}[\xi]$, and assuming that the gravitational field equations are fulfilled, we derive the total conserved current, $\mathscr{J}^{\mu}[\xi]$, corresponding to the invariance of the theory with respect to diffeomorphisms. The Komar superpotential drops out of the final expression for the current $\mathscr{J}^{\mu}[\xi]$, which takes on the form of a covariant derivative from a total superpotential, $\mathscr{J}^{\mu\nu}[\xi]$,

$$\mathscr{J}^{\mu}[\xi] = \partial_{\nu}\mathscr{J}^{\mu\nu} \,, \qquad \mathscr{J}^{\mu\nu}[\xi] = -\frac{\mathrm{e}}{\kappa}\sigma^{ij}\mathrm{e}_{i}{}^{[\mu}\mathrm{e}_{j}{}^{\nu]} \,. \tag{10.2.78}$$

Expression (10.2.78) points out to the same kind of conceptual difficulties like those we have already discussed in the Section 10.1.4 with regard to the Yang–Mills theory. The parameter of the "comoving" gauge transformation $\boldsymbol{\sigma} = \{\sigma^{ij}\}$ defined in (10.2.76), is not gauge-covariant as it depends on the $\mathrm{SO}(1,3)$-connection coefficients $A^{i}{}_{j\alpha}$ which is not an intrinsic tensor.

10.2.7 The Kosmann lift and the Komar superpotential

The diffeomorphism-related superpotential (10.2.78) depending on the non-covariant set of parameters σ^{ij} has exactly the same functional structure as the superpotential (10.2.62) associated with the gauge transformations (Lorentzian rotations) with arbitrary parameters ε^{ij}. Therefore, they can be naturally added up to get a resulting superpotential $\mathscr{J}^{\mu\nu}$. The net Noether's current $\mathscr{J}^{\mu}$ is a divergence from the (on-shell) superpotential,

$$\mathscr{J}^{\mu} = \partial_{\nu}\mathscr{J}^{\mu\nu} \,, \qquad \mathscr{J}^{\mu\nu} = -\frac{\mathrm{e}}{\kappa}\zeta^{ij}\mathrm{e}_{i}{}^{[\mu}\mathrm{e}_{j}{}^{\nu]} \,, \tag{10.2.79}$$

where the parameters

$$\zeta^{ij} = \varepsilon^{ij} + \sigma^{ij} \ . \tag{10.2.80}$$

The superpotential $\mathscr{J}^{\mu\nu}$ is still not gauge-invariant but the fact that the parameters entering the superpotential form a linear combination, reveals a possible remedy of this deficiency. The idea is to replace the arbitrary parameters ε^{ij} of the rotations with their counterparts depending on displacement vectors, ξ^α, of diffeomorphisms which will compensate the non-covariance of the parameters σ^{ij} making all together the generic parameter ζ^{ij} and the superpotential $\mathscr{J}^{\mu\nu}$, and respectively the current $\mathscr{J}^\mu$, fully covariant.

The first method is to pick some a pure-gauge or, let say, background connection $\bar{A}^i{}_{j\alpha}$ of the group SO(1, 3) as a reference, and contract it with the vector of diffeomorphism ξ^α. The pure-gauge connection is given by the very last term in (10.2.14) and we are to pick up the reference connection, $\bar{A}^i{}_{j\alpha}$, in the following form

$$\bar{A}^i{}_{j\alpha} = -\bar{\Lambda}^i{}_k \partial_\mu {}^{-1}\bar{\Lambda}^k{}_j \ , \tag{10.2.81}$$

where, $\bar{\Lambda}^i{}_k = \bar{\Lambda}^i{}_k(x)$, is an arbitrary matrix from SO(1, 3) and, ${}^{-1}\bar{\Lambda}^k{}_j = {}^{-1}\bar{\Lambda}^k{}_j(x)$, is its inverse, cf. (10.2.13). The SO(1, 3)-curvature defined in (10.2.15) is nil ("flat") for the pure gauge connection (10.2.81), that is $\bar{F}^i{}_{j\mu\nu} = 0$. In this case, the generic parameter

$$\zeta^i{}_j = \left(A^i{}_{j\alpha} - \bar{A}^i{}_{j\alpha}\right)\xi^\alpha \ , \tag{10.2.82}$$

is a difference between the two connections which is a tensor under arbitrary transformations of the group SO(1, 3). Picking up the parameter, $\zeta^{ij} = \zeta^i{}_k \eta^{kj}$, in the form (10.2.82) makes the Noether current (10.2.79) both gauge and diffeomorphism invariant. Nonetheless, this solution suffers from the indeterminacy of the reference connection $\bar{A}^i{}_{j\alpha}$ which choice is not directly determined by physics.

Perhaps, a better physical solution resolving the problem of the non-covariance of the Noether current in the tetrad formulation of general relativity, is based on the idea of making use of a modified Lie derivative which determines the variation of the tetrad field entering the term $\mathscr{J}^\mu_{\mathrm{e}}$ in the net Noether's current (10.2.63). The overall ideology behind this idea is similar to that we have discussed in Section 10.1.5 of the present chapter but now it should be applied to the case of SO(1, 3) group which substitutes SU(N) group in case of the Yang–Mills theory.

We define the modified variation Δ_ξ of the tetrad as a linear combination of two variations: $\delta_\xi = \pounds_\xi$, induced by diffeomorphism ξ^α and defined in (10.2.74), and δ_ε induced by the Lorentz boost and defined in (10.2.61) with the parameters ε^{ij} which is yet undetermined but supposed to be a functional of $\boldsymbol{\xi}$: $\varepsilon^{ij} = \varepsilon^{ij}[\xi]$, cf. (10.1.77),

$$\Delta_\xi \mathrm{e}^i{}_\alpha = \delta_\xi \mathrm{e}^i{}_\alpha + \delta_\varepsilon \mathrm{e}^i{}_\alpha = \left(\varepsilon^i{}_j + \sigma^i{}_j\right)\mathrm{e}^j{}_\alpha - \mathrm{e}^i{}_\mu \nabla_\alpha \xi^\mu \ . \tag{10.2.83}$$

The functional dependence of $\varepsilon^{ij}[\xi]$ can be found from the physical demand that the net Noether's current and its corresponding superpotential in the tetrad formulation of general relativity must be equal to those of the metric-based formulation of the same theory. We already know that the covariant superpotential emerging in general relativity is the Komar superpotential (10.2.73) which also originates in the tetrad formulation from the first part of the Noether current $\mathscr{J}^{\mu}_{\mathrm{h}}$ as shown in (10.2.72). It means that we have to look for the total variation of the tetrad field, $\Delta_\xi \mathrm{e}^i{}_\mu$, annihilating the second part of the Noether current, $\mathscr{J}^{\mu}_{\mathrm{e}} = 0$, that is

$$g^{\alpha[\mu} \mathrm{e}_i{}^{\nu]} \Delta_\xi \mathrm{e}^i{}_\alpha = 0 \,. \tag{10.2.84}$$

Substituting the right hand side of (10.2.83) for $\Delta_\xi \mathrm{e}^i{}_\alpha$ in (10.2.84), and solving it for ε^{ij}, we get,

$$\varepsilon^{ij} = -\sigma^{ij} - \mathrm{e}^i{}_\alpha \mathrm{e}^j{}_\beta \nabla^{[\alpha} \xi^{\beta]} \,. \tag{10.2.85}$$

Consequently, the generic matrix parameter (10.2.80)

$$\zeta^{ij} = -\mathrm{e}^i{}_\alpha \mathrm{e}^j{}_\beta \nabla^{[\alpha} \xi^{\beta]} \,. \tag{10.2.86}$$

This relation is called *the Kosmann lift*. Making use of the Kosmann lift in (10.2.79), immediately gives us the Komar superpotential

$$\mathscr{J}^{\mu\nu} = \mathscr{K}^{\mu\nu}[\xi] \,, \tag{10.2.87}$$

making a solid justification for the application of the tetrad formalism for finding the covariant conserved currents and superpotenials in general relativity.

Functoriality of the Kosmann lift

The variation Δ_ξ defines the modified Lie derivative by the rule $\mathfrak{L}_\xi = \Delta_\xi$, generalizing the standard Lie derivative rule, $\pounds_\xi = \delta_\xi$. Similarly to (10.1.81), we have

$$\mathfrak{L}_\xi = \pounds_\xi + \delta_\varepsilon \,, \tag{10.2.88}$$

where the variation δ_ε is due to the gauge transformation from the group $\mathsf{SO}(1,3)$. Since the gauge transformations don't affect the metric, we have

$$\mathfrak{L}_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)} \,. \tag{10.2.89}$$

Now, we substitute the Kosmann lift parameter, ε^{ij}, defined in (10.2.85) to calculate $\delta \mathrm{e}^i{}_\alpha$, in accordance with (10.2.61), and make use of this result in the definition of the modified Lie derivative (10.2.88) where the standard Lie derivative of the tetrad is given in (10.2.74). The outcome of the calculation yields the modified Lie derivative of the tetrad in the explicit form,

$$\mathfrak{L}_\xi \mathrm{e}^i{}_\mu = -\frac{1}{2} \mathrm{e}^i{}_\alpha g^{\alpha\beta} \pounds_\xi g_{\beta\mu} \,. \tag{10.2.90}$$

Let us make two diffeomorphism transformations, ξ_1^α and ξ_2^α, one after another and denote the modified Lie derivatives taken along these vector fields as $\mathfrak{L}_1 = \mathfrak{L}_{\xi_1}$ and $\mathfrak{L}_2 = \mathfrak{L}_{\xi_2}$ respectively. Then, we calculate the commutator of the two derivatives and find out that

$$[\mathfrak{L}_2, \mathfrak{L}_1]\mathrm{e}^i{}_\alpha = \mathfrak{L}_3\mathrm{e}^i{}_\alpha - 2\mathrm{e}_j{}^\beta \left(\mathfrak{L}_{[1}\mathrm{e}^i{}_\beta\right)\left(\mathfrak{L}_{2]}\mathrm{e}^j{}_\alpha\right) , \qquad (10.2.91)$$

where we have denoted $\mathfrak{L}_3 = \mathfrak{L}_{\xi_3}$ and $\xi_3 = \{\xi_3^\alpha\}$, with $\xi_3^\alpha = [\xi_1, \xi_2]^\alpha$ defined in (10.1.82). Equation (10.2.91) reveals that the commutator of the two modified Lie derivatives is not reduced in general case to a modified Lie derivative taken along the commutator of the two diffeomorphisms – there appears a non-linear term being proportional to the square of the modified Lie derivatives from the diffeomorphisms under consideration. It is inconsistent with the functoriality of the algebra of the Lie derivatives pointing out that the Kosmann lift is not functorial in the most general case. On the other hand, we notice that in case when, at least one of the diffeomorphisms in the commutator (10.2.91), is a Killing vector ξ_K^α the Kosmann lift becomes functorial because in this case the result of the action of the modified Lie derivative on the tetrad is trivial: $\mathfrak{L}_{\xi_K}\mathrm{e}^i{}_\alpha = 0$.

A short review

The non-invariance of canonical diffeomorphic currents in the tetrad formulation of general relativity with respect to the local Lorentz rotations of tetrads was pointed out and clarified by Møller in well-known works [322–324]. In the following years, the conserved quantities constructed by Møller were rediscovered many times by different researchers and analyzed by various methods (see, for example, [338] and references therein).

In the works by Aros et al. [14, 15] new conserved quantities for asymptotically anti-de Sitter space-times have been constructed in general relativity by making use of the tetrad formalism. The authors have applied manifestly covariant methods but could not obtain gauge-covariant expressions for the diffeomorphic currents. These results attracted attention and became a basis for more deeper studies by Obukhov and Rubilar [339–341] who showed that a modification of the Lie derivative by the rule (10.2.88) permits to reconstruct the expressions obtained in the above-mentioned works of Aros with coauthors, to gauge covariant counterparts. In the works [341] and [340] this technique has been generalized to the case of teleparallel models of gravity and arbitrary generally-covariant gauge theories, respectively.

The classical Kosmann lift has been discovered and studied in papers [274–277] which are devoted to constructing Lie derivatives for spinor fields. Penrose and Rindler [355, chap. 6.6, p. 102] studied a conformal structure of spacetime and introduced their own definition of Lie derivatives for spinor fields that differs from the Kosmann's one. Both the Kosmann and Penrose lifts are physically justified for good reasons but they are functorial only on Killing and conformal Killing vector fields, respectively. Other lifts have been suggested for tetrads and spinors by Levitskii and Yappa [286, 287], Bilyalov [44–49], Francaviglia et al. [166, 168, 171], Sardanashvily [397, 399, 403], and in a recent work by Helfer [231]. In interesting work by Godina et al. [195] relationships between the dual Kosmann lift, the Witten lift [463], the complex Nester-Witten form [333] and the Penrose quasi-local 4-momentum [354, 355, chap. 9.10, p. 432] has been found and analyzed to some extent.

In the present section, the derivation of the Kosmann lift from the correspondence principle of the Noether current in the metric-based and tetrad formulations of general relativity is a new result.

10.3 Fiber bundles and the Noether theorem

We have demonstrated in previous two sections that the Noether theorem applied in gauge field theories to construct the canonical currents corresponding to diffeomorphisms, suffers a setback – these currents are either not transformed in gauge-invariant manner or depend on the choice of correcting lift which is not unique. This problem has deep geometrical reasons which can be better understood by studying fiber bundles and their sections. In what follows, we introduce the reader to the abstract mathematical theory of the fiber bundles and explain the break down of the Noether theorem in the gauge theories from the geometric point of view.

10.3.1 Diffeomorphisms, automorphisms and functorial lift

Modern differential geometry associates the state of a physical field with a section of a fiber bundle having spacetime as a base. This idea is used for analysis of theories with the intrinsic gauge symmetries as well as for those without these symmetries. However, there are essential differences between these theories from the differential-geometric point of view. To explain this in more detail, let us introduce the necessary machinery of the fiber bundles.

Let $\mathscr{E}(E, M, \pi, F)$ denote a fiber bundle with the total space E, the base M, the typical fiber F, and the canonical projection π from the total space E to the base manifold M. The fiber bundle is locally a product space $U \times F$ called a local trivialization, where U is a small neigborhood of a point x on the base space M. Globally the fiber bundle may have a non-trivial topological structure. The spaces E, M, and F are required to be smooth manifolds admitting local diffeomorphisms forming the groups $\mathsf{Diff}(E)$, $\mathsf{Diff}(M)$ and $\mathsf{Diff}(F)$, respectively. Among all the diffeomorphisms of the manifold E, a special role is played by the diffeomorphisms which preserve the fiber structure of $\mathscr{E}$, that is they transform fibers of $\mathscr{E}$ to fibers of $\mathscr{E}$. One calls such transformations as the *automorphisms* of $\mathscr{E}$. Their collection forms a group that is denoted $\mathsf{Aut}(\mathscr{E})$.

Let $\{x^\mu, y^a\}$ be a local trivialization chart, that is a local system of coordinates on E satisfying the projection condition,

$$\pi: \quad \{x^\mu, y^a\} \quad \rightarrow \quad \{x^\mu\}. \tag{10.3.1}$$

Here, $x = \{x^\mu\}$ are coordinates on the base; $y = \{y^a\}$ are coordinates on the fibers. In the local trivialization (10.3.1) any automorphism of $\mathscr{E}$ is described by the formulae of the form:

$$\begin{cases} x^\mu \quad \rightarrow \quad \tilde{x}^\mu = f^\mu(x); & (10.3.2) \\ y^a \quad \rightarrow \quad \tilde{y}^a = \phi^a(x,y) & (10.3.3) \end{cases}$$

with some differentiable functions $f^\mu(x)$ and $\phi^a(x,y)$. The infinitesimal generators of automorphisms are projectable vector fields, $\boldsymbol{\eta}(x,y) = \boldsymbol{\xi}(x) + \boldsymbol{\zeta}(x,y)$ over E, with some infinitesimal functions $\boldsymbol{\xi}(x) = \{\xi^\mu(x)\}$ and $\boldsymbol{\zeta}(x,y) = \{\zeta^a(x,y)\}$ such that

$$\boldsymbol{\eta}(x,y) = \xi^\mu(x)\partial_\mu + \zeta^a(x,y)\partial_a\,. \tag{10.3.4}$$

Under the canonical projection π (10.3.1) any automorphism of $\mathscr{E}$ (10.3.2), (10.3.3) is projected into diffeomorphism of the base M:

$$\pi: \quad \{f^\mu(x), \phi^a(x,y)\} \quad \rightarrow \quad \{f^\mu(x)\}. \tag{10.3.5}$$

Then, a vector field $\boldsymbol{\eta}(x,y)$ over E (10.3.4) is projected to vector field $\boldsymbol{\xi}(x)$ over M:

$$\pi^*: \quad \boldsymbol{\eta}(x,y) \quad \rightarrow \quad \boldsymbol{\xi}(x) = \{\xi^\mu(x)\}, \tag{10.3.6}$$

that is an infinitesimal generator of diffeomorphism of the base M. Therefore, *an arbitrary automorphism of the fiber bundle $\mathscr{E}(E,M,\pi,F)$ induces a well defined (finite or infinitesimal) diffeomorphism of its base M.*

Let us now assume that

$$x^\mu \quad \rightarrow \quad \tilde{x}^\mu = f^\mu(x) \tag{10.3.7}$$

is a diffeomorphism of the base M. Can we, using the standard canonical procedure, associate (10.3.7) with a *unique* automorphism of $\mathscr{E}$ (10.3.2), (10.3.3) satisfying the condition (10.3.5)? In other words, can we associate with an infinitesimal vector field $\boldsymbol{\xi}(x)$ over M a *unique* infinitesimal vector field $\boldsymbol{\eta}(x,y)$ over E (10.3.4) satisfying the condition (10.3.6)? In the most general case the answer is negative – the knowledge of functions $f^\mu(x)$ (or the base generators $\xi^\mu(x)$) do not allow us to fix functions $\phi^a(x,y)$ (or the fiber generators $\zeta^a(x,y)$) which can be chosen arbitrary. Generally speaking, a whole *family* of automorphisms of $\mathscr{E}$ corresponds to any of diffeomorphisms of the base M. To pick up a specific member of this family we need a procedure called a lift.

The lift ℓ is a rule defined on the fiber bundle $\mathscr{E}(E,M,\pi,F)$ and allowing us to associate a *unique* automorphism of $\mathscr{E}$ with each diffeomorphisms of the base M, that is

$$\ell: \quad f^\mu(x) \quad \rightarrow \quad \{f^\mu(x), \phi^a(x,y)\}. \tag{10.3.8}$$

It is evident that the differential ℓ^* of the lift ℓ can be used to lift the infinitesimal generators of diffeomorphisms $\boldsymbol{\xi}(x)$ of the base M to the infinitesimal generators of automorphisms $\boldsymbol{\eta}(x,y)$ of the total space E,

$$\ell^* : \quad \boldsymbol{\xi}(x) \quad \to \quad \boldsymbol{\eta} = \{\xi^\mu(x), \zeta^a(x,y)\}. \tag{10.3.9}$$

The lift ℓ is called *canonical*, if its definition does not require any additional geometric structures. In opposite case the lift is called non-canonical. The lift l is called *functorial*, if it preserves the composition law

$$\ell(f_2 \circ f_1) = \ell(f_2) \circ \ell(f_1)\,, \qquad \forall f_1, f_2 \in \mathsf{Diff}(M)\,. \tag{10.3.10}$$

Here, the symbol $\circ$ denotes an operation of composition of two maps: at the left – two diffeomorphisms $\{f_1^\mu(x)\}$ and $\{f_2^\mu(x)\}$ of the base M, and at the right – the corresponding to them automorphisms $\{f_1^\mu(x), \phi_1^a(x,y)\}$ and $\{f_2^\mu(x), \phi_2^a(x,y)\}$ of the fiber bundle $\mathscr{E}$. In terms of the infinitesimal generators, the functoriality condition for the lift ℓ reads,

$$\ell^*\,([\xi_1, \xi_2]) = [\ell^*(\xi_1), \ell^*(\xi_2)]\ , \qquad \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathfrak{diff}(M)\,. \tag{10.3.11}$$

Here, the square brackets in the left side of the equation denote the commutator (also called the Lie brackets) of the vector fields in the tangent space TM of the base M, and those in the right side do that of the vector fields in the tangent space TE of the total space E. From the point of view of the group theory, the functorial lift is a homomorphism from the group $\mathsf{Diff}(M)$ into the group $\mathsf{Aut}(\mathscr{E})$ (respectively, from the algebra $\mathfrak{diff}(M)$ into the algebra $\mathfrak{aut}(\mathscr{E})$). The canonical functorial lift is called *the natural lift*. Fiber bundles, possessing a natural lift, are called *the natural bundles*. The field theories which configuration bundles are natural bundles are called *natural field theories*.

10.3.2 Field theories without intrinsic gauge symmetry as natural field theories

In the field theories not possessing an intrinsic gauge symmetry, an initial geometrical structure is a spacetime manifold M. Physical fields are typically geometrical objects (vectors, tensors, tensor densities, linear affine connections, etc.) on M. Using the language of the fiber bundles, one says that the states of physical fields are given by sections of the bundles of the geometrical objects. *All such bundles are natural bundles*. For the sake of simplicity, let us illustrate the last claim on the example of the tangent bundle $\mathscr{T}\mathscr{M}(TM, M, \pi, V)$ – a tensor bundle of rank (1, 0). Let $\{x^\mu, v^\nu\}$ be a local trivialization of $\mathscr{T}\mathscr{M}$. Then, for each diffeomorphism

$$x^\mu \quad \to \quad \tilde{x}^\mu = f^\mu(x) \tag{10.3.12}$$

of the base M, one can construct the automorphism

$$\begin{cases} x^\mu \quad \to \quad \tilde{x}^\mu = f^\mu(x); & (10.3.13)\\ v^\nu \quad \to \quad \tilde{v}^\nu = \dfrac{\partial f^\nu}{\partial x^\alpha} v^\alpha & (10.3.14)\end{cases}$$

of the tangent bundle $\mathscr{T}\mathscr{M}$. The lift ℓ,

$$\ell: \quad f^\mu \quad \to \quad \{f^\mu, (\partial_\alpha f^\nu)v^\alpha\} \tag{10.3.15}$$

is a canonical one, because for its definition, one does not require additional structures. It is also a functorial lift because, by the well known chain rule, if $\tilde{x}^\mu = f_1^\mu(x)$ and $\hat{x}^\mu = f_2^\mu(\tilde{x})$, then, the composition law (10.3.10) is equivalent to

$$\hat{x}^\mu = f_2^\mu(f_1(x)) \equiv f_3^\mu(x) \tag{10.3.16}$$

and

$$\frac{\partial f_3^\mu}{\partial x^\alpha} = \frac{\partial f_2^\mu}{\partial \tilde{x}^\beta}\frac{\partial f_1^\beta}{\partial x^\alpha}. \tag{10.3.17}$$

Thus, the lift (10.3.15) is natural. It is evident that these arguments are also valid for matter fields represented by tensor densities of an arbitrary rank and weight, or the Christoffel symbols.

In terms of the infinitesimal generators, functoriality of the lift (10.3.15) is equivalent to the well known property of the standard Lie derivatives $£_\xi$ of the geometrical objects along the generators $\boldsymbol{\xi}(x)$, namely

$$[£_{\xi_2}, £_{\xi_1}] = £_{\xi_3}\ , \tag{10.3.18}$$

where

$$\xi_3^\mu = [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]^\mu = \xi_1^\alpha \partial_\alpha \xi_2^\mu - \xi_2^\alpha \partial_\alpha \xi_1^\mu. \tag{10.3.19}$$

Summing up the above discussion, we conclude that, *in natural theories, the presence of the well-defined canonical conserved currents following from the diffeomorphism invariance is based fully on the existence of the natural lifts in natural bundles*.

10.3.3 Field theories with intrinsic gauge symmetry as gauge-natural field theories

The other situation is in generally-covariant field theories possessing an intrinsic gauge symmetry with a structure group G. Here, the initial geometrical object is not a spacetime manifold M, but a principal fiber bundle $\mathscr{P}(P, M, \pi, G)$ with the total space P, a base M, a canonical projection π, and a typical fiber G. A state of physical fields possessing the internal (gauge) freedom is typically given by the sections of the fiber bundles associated with the principal fiber bundle $\mathscr{P}$.

Among all the diffeomorphisms of the total space P, the *principal automorphisms* play a special role. These transformations not only conserve the fiber structure of $\mathscr{P}$, but they are also consistent with the action of the structure group on fibers. In a local trivialization $\{x^\mu, z^a\}$,

$$\pi: \quad \{x^\mu, z^a\} \quad \to \quad x^\mu, \tag{10.3.20}$$

an arbitrary principal automorphism is described by formulae

$$\begin{cases} x^\mu \quad \rightarrow \quad \tilde{x}^\mu = f^\mu(x); & (10.3.21) \\ z^a \quad \rightarrow \quad \tilde{z}^a = (\Phi(x) \circ \boldsymbol{z})^a. & (10.3.22) \end{cases}$$

where, $x = \{x^\mu\}$ are coordinates on the base, $\boldsymbol{z} = \{z^a\}$ are coordinates on fibers; $\forall x \in M$, $\Phi(x) = \{\phi^a(x)\} \in G$, and the symbol $\circ$ means a group multiplication of two elements of the group G. Infinitesimal generators of principal automorphisms are projectable equivariant vector fields $\boldsymbol{\tau}$ over E,

$$\boldsymbol{\tau}(x, z) = \xi^\mu(x)\partial_\mu + \varepsilon^a(x)\boldsymbol{t}_a(z). \qquad (10.3.23)$$

Here, $\boldsymbol{t}_a(z)$ represents a basis for the vertical right-invariant vector field over G; $\boldsymbol{\xi}(x) = \{\xi^\mu(x)\}$, and $\boldsymbol{\varepsilon}(x) = \{\varepsilon^a(x)\}$ are some infinitesimal functions. A collection of all principal automorphisms (10.3.21), (10.3.22) forms the group P-Aut($\mathscr{P}$) and the infinitesimal generators of principal automorphisms (10.3.23) form the algebra $\mathfrak{p}$-$\mathfrak{aut}(\mathscr{P})$.

The group P-Aut($\mathscr{P}$) contains canonically defined subgroup Gau($\mathscr{P}$) consisting of vertical transformations,

$$\begin{cases} x^\mu \quad \rightarrow \quad \tilde{x}^\mu = x^\mu; & (10.3.24) \\ z^a \quad \rightarrow \quad \tilde{z}^a = (\Phi(x) \circ z)^a. & (10.3.25) \end{cases}$$

Infinitesimal generators of such transformations are vertical equivariant vector fields over E

$$\boldsymbol{\varkappa}(x, z) = \varepsilon^a(x)\boldsymbol{t}_a(z). \qquad (10.3.26)$$

They represent the sub-algebra $\mathfrak{gau}(\mathscr{P}) \subset \mathfrak{p}$-$\mathfrak{aut}(\mathscr{P})$.

The same as in the general fiber bundle (see Section 10.3.1), each principal automorphism of $\mathscr{P}(P, M, \pi, G)$ induces the well defined diffeomorphism of the base M:

$$\pi: \quad \{f^\mu(x), \phi^a(x)\} \quad \rightarrow \quad f^\mu(x)\,. \qquad (10.3.27)$$

Transformations related to the subgroup Gau($\mathscr{P}$) induce identical transformations, id, of the base M,

$$\pi: \quad \{\text{id}, \Phi(x)\} \quad \rightarrow \quad \text{id}. \qquad (10.3.28)$$

Therefore, from the physical point of view, the transformations (10.3.28) are purely intrinsic gauge transformations. The inverse procedure is ambiguous. In general, there is no a natural lift on the principal fiber bundles.

Fiber bundles $\mathscr{E}(E, M, \pi, F)$ connected with $\mathscr{P}(P, M, \pi, G)$ - a principal fiber bundle such that each principal automorphism of $\mathscr{P}$ (10.3.21), (10.3.22) creates a principal automorphism of $\mathscr{E}$ by the canonical procedure, are called the *gauge-natural bundles*. Field theories, whose configuration bundles are the gauge-natural bundles, are called the *gauge-natural field theories*. The Yang–Mills theories and the tetrad general relativity from Sections 10.1 and 10.2 are typical examples of the gauge-natural field theories.

In the gauge-natural field theories, the total group of symmetry is the group $\mathsf{P\text{-}Aut}(\mathscr{P})$. Because the group $\mathsf{P\text{-}Aut}(\mathscr{P})$ contains canonically defined subgroup (10.3.24), (10.3.25) of vertical transformations $\mathsf{Gau}(\mathscr{P})$ one can construct well-defined and physically-meaningful canonical conserved currents in such theories associated with their gauge invariance.

On the other hand, in the gauge-natural field theories, construction of unambiguous conserved quantities corresponding to the diffeomorphism-invariance (energy, momentum, angular momentum) faces difficulties because there is no canonical way to lift the diffeomorphism of the base M to a *unique* principal automorphism of the principal fiber bundle $\mathscr{P}(P, M, \pi, G)$. There are infinitely many principal automorphisms of $\mathscr{P}$ projecting to a given diffeomorphism of M. We do not know which concrete principal automorphism from this collection has to be chosen to obtain the "correct" expressions for energy, momentum and angular momentum of the system. Because principal automorphisms projecting onto the same diffeomorphism differ only by an element of $\mathsf{Gau}(\mathscr{P})$, the conserved canonical currents corresponding to diffeomorphism-invariance of the theory are defined only up to an arbitrary gauge transformation.

In an equivalent way, one can describe the situation as follows. In a principal fiber bundle $\mathscr{P}(P, M, \pi, G)$, there is a canonical equivariant method to select a vertical vector space $VP \subset TP$ consisting of vectors, which under the action of π^* are projected to nil vectors of TM. Using the canonical method, this permits to define a subgroup $\mathsf{Gau}(P)$ of the group $\mathsf{P\text{-}Aut}(P)$ that is the group of purely gauge transformations. At the same time, there is no canonical method to select a horizontal subspace $HP \subset TP$, such that

$$TP = HP \oplus VP. \tag{10.3.29}$$

Therefore, there is an ambiguity in selecting from the total symmetry group $\mathsf{P\text{-}Aut}(P)$ a subgroup describing purely diffeomorphism-invariance of the theory. The failure to construct a functorial lift on $\mathscr{P}$ is related to the ambiguity in building the horizontal subspace $HP \subset TP$.

If a principal connection, $A^a{}_\mu$, (in physics terms – the Yang–Mills field) is given on $\mathscr{P}$, then there is a canonical horizontal lift

$$\ell^* : \quad \boldsymbol{\xi}(x) = \xi^\alpha(x)\partial_\alpha \quad \to \quad \boldsymbol{\tau}(x,z) = \xi^\alpha(x)D_\alpha\,, \tag{10.3.30}$$

defined with the help of the gauge-covariant derivative (cf. (10.1.10) and (10.2.17))

$$D_\alpha = \partial_\alpha + A^a{}_\alpha(x)\boldsymbol{t}_a(z)\,. \tag{10.3.31}$$

Unfortunately, this lift is functorial only if the principal connection, $A^a{}_\mu$, has zero curvature, $F^a{}_{\mu\nu}$, but it is not so interesting from the physical point of view. The naive lift arising in Sections 10.1.4 and 10.2.6,

$$\ell^* : \quad \boldsymbol{\xi}(x) = \xi^\alpha(x)\partial_\alpha \quad \to \quad \boldsymbol{\xi}(x,z) = \xi^\alpha(x)\partial_\alpha \,, \tag{10.3.32}$$

with the replacement

$$\xi^\alpha\partial_\alpha = \xi^\alpha(x)D_\alpha - \sigma^a[\boldsymbol{\xi}(x)]\, \boldsymbol{t}_a(z)\,, \qquad \left(\sigma^a \equiv A^a{}_\mu\xi^\mu\right) \tag{10.3.33}$$

does not help to solve the problem because the vector field (10.3.32) is not equivariant and cannot represent the infinitesimal generator of a principal automorphism of $\mathscr{P}$.

Thus, from the differential-geometric point of view, *the absence in gauge-natural field theories of unambiguous canonical conserved quantities following from diffeomorphism-invariance of the theory is explained by the absence of natural horizontal lift in the gauge-natural bundles*.

A short review

The gauge-natural bundles, as a geometrical basis for studying the gauge field theories, have been introduced in the work by Eck [146]. The modern understanding that the differential geometrical principles constitute the basis for constructing generally-covariant and gauge invariant conserved quantities in the natural and gauge-natural field theories has been laid down in the works by Dubois-Violette and Madore [143], by Francaviglia et al. [161–163, 167, 173], by Cianci, Vignolo and Bruno [104], by Palese and Winterroth [174, 349, 350]. A general presentation of the fiber bundle approach to the gauge theories can be found in reviews [164, 170, 312]. Differential-geometric properties of natural and gauge-natural field theories can be found in a few recent monographs [172, 190, 264, 400].

10.3.4 Fixing the horizontal lift

The above discussion elucidates that in gauge-natural bundles, the construction of a horizontal lift ℓ,

$$\ell^* : \ \boldsymbol{\xi}(x) = \xi^\alpha(x)\partial_\alpha \quad \to \quad \boldsymbol{\eta}(x,z) = \xi^\alpha(x)\partial_\alpha + \zeta^a[\xi(x)]\, \boldsymbol{t}_a(z), \tag{10.3.34}$$

necessarily requires an introduction of additional geometrical structures to define the vector field $\boldsymbol{\zeta} = \{\zeta^a[\xi(x)]\}$. It is evident that the introduction of such structures is to be based on some physically-motivated criteria which should be mathematically consistent as well. Below, we discuss three criteria imposed on the field ζ^a – gauge-covariance, locality and functoriality.

The gauge-covariance is equivalent to the requirement that the field $\boldsymbol{\zeta}[\boldsymbol{\xi}] = \zeta^a[\xi(x)]\boldsymbol{t}_a(z)$ represents the right-invariant vertical vector field in TP. Concerning the locality, the quantities $\zeta^a[\xi(x)]$ have to be local linear functionals of $\xi^\alpha(x)$. The functoriality condition (10.3.11) demands the integrability of the system of partial differential equations:

$$\begin{cases} \delta x^\mu = \xi^\mu(x); & (10.3.35) \\ \delta z^a = \zeta^c[\xi(x)](\boldsymbol{t}_c(z))^a. & (10.3.36) \end{cases}$$

In another word, the condition (10.3.11) requires the field $\boldsymbol{\eta}$ be an infinitesimal generator of a *finite* principal automorphism of $\mathscr{P}$, see (10.3.34). Mathematical analysis of the equation (10.3.11) provided in Section 10.1.8 for the case of the Yang–Mills theory, shows that, unfortunately, the three criteria mentioned above are insufficient to determine a unique horizontal functorial lift, so that additional criteria must be found (if they exist).

At present, no other *purely mathematical* criteria to build the equivariant horizontal lift in the gauge-natural bundles have yet been formulated. This may be an indication to existence of a more profound reason to our incapability. Resolving the problem may require to look more carefully to the *physical* foundations of the gauge field theories on fiber bundles like a correspondence principle used in Section 10.2.

It is worth emphasizing that the ambiguity in the choice of the horizontal lift leads to the ambiguity in the definition of the canonical Noether's current in the form of a gauge non-invariant term that appears as a divergence of superpotential as shown in (10.1.70) and (10.2.77). The appearance of this term changes the algebra of the corresponding conserved charges leading to the appearance of so-called, *central charges*. *The presence of the central charges in the theory can lead to physically observable effects* . For example, the papers [91, 92, 306] prove that a known algebra of the central charges permits us to define the entropy of black holes. The presence of the central charges can reflect the existence of quantum anomalies and/or the Schwinger terms. At present, a comprehensive study of these questions is vigorously pursued by theoretical physicists around the globe but this research program is still far from being completed.

Appendix A: Tensor quantities and tensor operations

The goal of this appendix is to define various mathematical objects residing in spacetime continuum, to describe their geometric properties and the rules of their differentiation that are used in calculations throughout the book.

A.1 Tensors and tensor densities

From the mathematical point of view a spacetime is a smooth manifold covered by mutually overlapping coordinate charts and endowed with a metric **g** having the Lorentzian (pseudo-Euclidean) signature. We choose the signature of the metric to be $\mathrm{sign}(\mathbf{g}) = \{-, +, +, +\}$ in 4 dimensions, or $\mathrm{sign}(\mathbf{g}) = \{-, +, +, \dots, +\}$ with $n - 1$ positive signs in the more general case of n-dimensional pseudo-Riemannian manifold. By definition the manifold can be covered by a set of local coordinate charts x^α with $\alpha = 0, 1, 2, 3$ in general relativity or $\alpha = 0, 1, 2, \dots, n-1$ in a more general theory, where $n \geq 4$ is a finite whole number defining a dimension of the manifold; sometimes we consider also $n = 3$. Coordinates are convenient for mathematical manipulations with geometric objects. At the same time, tensor algebra allows us to conduct calculations in arbitrary coordinates.

In the present appendix, the discussion is focused on spacetime manifolds without torsion with the metric **g** with components $g_{\alpha\beta} = g_{\alpha\beta}(x)$ that obey the Einstein equations or some generic gravity equations. The metric uniquely defines the Christoffel symbols (A.2.2) and the curvature tensor (A.2.15) which are primary geometric objects in general relativity and in any other metric-based gravitational theory. It is *important to point out* that in some cases the manifold of the metric-based theory of gravity can be endowed with a second metric and the additional geometrical objects corresponding to it. The second metric in these theories is used to consider perturbations of the gravitational field which are described by the primary metric. In such a case, the second metric describes geometry of an unperturbed (background) spacetime manifold and must obey the Einstein equations. In case when one considers the perturbations of a strong gravitational field, the background metric is usually chosen as an exact solution of Einstein's equation — the Schwarzschild metric, the Kerr metric, the Friedmann-Lemître-Robertson-Walker metric, etc. In case of weak gravitational fields (like that in the solar system) the background metric is chosen as the metric $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(x)$ of the Minkowski space. In the most general case the background metric $\bar{g}_{\alpha\beta} = \bar{g}_{\alpha\beta}(x)$ is chosen as a metric on a generic curved manifold with a known (fixed) geometry. We *emphasize* that the technique of Appendix A is applicable to any metric-based theory of gravity irrespectively of the choice of the background metric.

Mathematical objects in spacetime manifold can be classified according to their behavior under transformation of the local coordinate charts

$$x'^{\alpha} = x'^{\alpha}(x^{\beta}) . \tag{A.1.1}$$

One assumes that the transformation equation (A.1.1) are not degenerated and can be resolved with respect to x^{α}: $x^{\alpha} = x^{\alpha}(x'^{\beta})$ that is the inverse coordinate transformation exists and is well-defined. This means that the Jacobian, J, for these transformations does not vanish at any point in the domain of overlapping of the two coordinate charts,

$$J = \det \left\| \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \right\| \neq 0 . \tag{A.1.2}$$

Consider a smooth scalar function of n coordinates, $f = f(x)$ where we have omitted the coordinate index, $x^{\alpha} \equiv x$ for simplicity as it does not cause misunderstanding. The function $f(x)$ can be interpreted as a scalar field defined in the spacetime. After making the coordinate transformation (A.1.1), one has

$$f(x) = f[x(x')] = f'(x') , \tag{A.1.3}$$

where f' is different from f. Although the form of $f'(x')$ differs from that of $f(x)$, the numerical values of these functions are identically the *same* at each of point of the manifold covered by the two coordinate charts, x and x'. Such scalar functions are called *scalar* fields or simply *scalars*. Each of coordinates x^{α} is, for example, a scalar function on the manifold and should not be considered as components of a vector.

The metric is a tensor field of a second rank,

$$g_{\alpha\beta} = g_{\alpha\beta}(x) , \tag{A.1.4}$$

defined on the entire manifold. The metric is a symmetric tensor, thus, having $n(n+1)/2$ independent components. The metric defines the metric properties of spacetime that is it determines how to measure distances and angles on the manifold. For example, the distance (also called the interval) between two nearby points with coordinates x^{α} and $x^{\alpha} + dx^{\alpha}$ is calculated with the help of the metric tensor as

$$ds^2 = \sum_{\alpha,\beta=0}^{n-1} g_{\alpha\beta} dx^{\alpha} dx^{\beta} . \tag{A.1.5}$$

Using Einstein's summation convention over repeated indices one can rewrite the interval (A.1.5) as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta . \tag{A.1.6}$$

The value of this quantity does not depend on the choice of coordinates as the interval is a scalar function.[1]

The infinitesimally-small increment dx^α is a differential of coordinates which form a vector. It is easy to find how the components of this vector are transformed under coordinate transformation (A.1.1),

$$dx'^\alpha = dx^\beta \frac{\partial x'^\alpha}{\partial x^\beta} . \tag{A.1.7}$$

Generalization of the infinitesimal vector dx^α is a contravariant vector field **A** represented by a set of its n components, A^α, residing in a tangent space of spacetime manifold. By definition A^α changes under coordinate transformation (A.1.1) exactly as dx^α in (A.1.7):

$$A'^\alpha(x') = A^\beta(x) \frac{\partial x'^\alpha(x)}{\partial x^\beta} . \tag{A.1.8}$$

Now, let us define a new object $dx_\alpha \equiv g_{\alpha\beta} dx^\beta$ obtained from dx^β by contracting it with the metric tensor with respect to index β. In terms of dx_α the spacetime interval (A.1.6) takes on the following form, $ds^2 = dx_\alpha dx^\alpha$. Applying the coordinate transformation (A.1.1) and taking into account that the interval is a scalar, $ds'^2 = dx'_\alpha dx'^\alpha = dx_\alpha dx^\alpha = ds^2$, one easily finds the law of transformation for dx_α which reads

$$dx'_\alpha = dx_\beta \frac{\partial x^\beta}{\partial x'^\alpha} . \tag{A.1.9}$$

Similarly to the definition of dx_α we can introduce a new object $\tilde{\mathbf{A}}$ with components

$$A_\alpha = g_{\alpha\beta} A^\beta , \tag{A.1.10}$$

which is called a *covariant* vector $\tilde{\mathbf{A}}$ or simply a covector. Covectors reside in a cotangent space – the space which is dual to the tangent space. The metric tensor establishes isomorphism between vectors and covectors in the sense of operation (A.1.10) which is called the operation of lowering indices of vector A^β.

Many mathematical operations with tensors require introduction of the inverse matrix of the metric tensor, $g^{\alpha\beta}$, which is defined by making use of the identity,

1 Contemporary theoretical physics prefers to operate with equations and geometric objects by making use of a symbolic language being independent on coordinates. This formulation is elegant but in practice it is often preferable to work in some coordinates in order to carry out mathematical computations and numerical simulations, hence we prefer to operate with components of tensors having coordinate indices explicitly.

$g^{\alpha\gamma}g_{\gamma\beta} \equiv \delta^{\alpha}_{\beta}$. Then, with the use of $g^{\alpha\beta}$ one can raise the indices of covectors, $A^{\alpha} = g^{\alpha\beta}A_{\beta}$. Keeping in mind (A.1.9), one states that components of covectors are transformed as

$$A'_{\alpha}(x') = A_{\beta}(x)\frac{\partial x^{\beta}(x')}{\partial x'^{\alpha}} . \tag{A.1.11}$$

Recalling that the line-element (A.1.6) is a scalar, $ds'^2 = ds^2$, combining this with (A.1.7) and (A.1.9), one derives the coordinate transformation for the metric,

$$g'_{\alpha\beta}(x') = g_{\gamma\delta}(x)\frac{\partial x^{\gamma}}{\partial x'^{\alpha}}\frac{\partial x^{\delta}}{\partial x'^{\beta}} , \tag{A.1.12}$$

and that for the inverse metric,

$$g'^{\alpha\beta}(x') = g^{\gamma\delta}(x)\frac{\partial x'^{\alpha}}{\partial x^{\gamma}}\frac{\partial x'^{\beta}}{\partial x^{\delta}} . \tag{A.1.13}$$

An important quantity is the determinant of the metric $g = g(x) \equiv \det g_{\alpha\beta}$, which can be shown to transform under coordinate transformations as

$$g'(x') = J^{-2}(x)g(x) . \tag{A.1.14}$$

So, it is not a scalar because of the presence of the square of the Jacobian, J, of the coordinate transformation in (A.1.14). The general name for such geometric objects is tensor density. The object $\mathscr{F} \equiv \sqrt{-g}f$, which is a product of $\sqrt{-g}$ with a scalar function, $f = f(x)$, is called a scalar density. Combining (A.1.3) and (A.1.14), one finds the coordinate transformations for the scalar density,

$$\mathscr{F}'(x') = J^{-1}(x)\mathscr{F}(x) . \tag{A.1.15}$$

Any object ($\mathscr{P}$) with the transformation law (A.1.15), although without relation to $\sqrt{-g}$, is a scalar density also.

The geometric objects described above are the simplest examples of *tensors* – a scalar f is a tensor of rank 0, a vector A^{α} is a contravariant tensor of rank 1, a covector A_{α} is a covariant tensor of rank 1, a scalar density $\mathscr{F}$ is a tensor density of weight +1 and rank 0. Generalizing this nomenclature, one calls a geometric object $\boldsymbol{Q}$ k-contravariant, l-covariant tensor density of rank $k + l$ and weight $+n$, if its components are transformed under coordinate transformation (A.1.1) as follows:

$$\mathscr{Q}'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x') = J^{-n}(x)\mathscr{Q}^{\kappa\lambda\ldots\tau}_{\mu\nu\ldots\omega}(x)\overbrace{\frac{\partial x'^{\alpha}}{\partial x^{\kappa}}\frac{\partial x'^{\beta}}{\partial x^{\lambda}}\cdots\frac{\partial x'^{\gamma}}{\partial x^{\tau}}}^{k}\underbrace{\frac{\partial x^{\mu}}{\partial x'^{\pi}}\frac{\partial x^{\nu}}{\partial x'^{\rho}}\cdots\frac{\partial x^{\omega}}{\partial x'^{\sigma}}}_{l} . \tag{A.1.16}$$

Notice that the transformation (A.1.16) is carried out at a single point of the manifold. However, the value of $\mathscr{Q}^{\kappa\lambda\ldots\tau}_{\mu\nu\ldots\omega}(x)$ in the right side of this equation is a function of the "old" coordinates x of this point while the transformed value $\mathscr{Q}'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x')$ in the left side of (A.1.16) depends on the "new" coordinates x' of the same point of the manifold.

Let us list some important properties of tensor densities. Tensor densities, $\boldsymbol{P}$ and $\boldsymbol{Q}$, of the same type (that is the same rank, weight and the same positions of indices) can be added forming a new tensor density of the same type, $\boldsymbol{R} = \boldsymbol{P} + \boldsymbol{Q}$, or in components

$$\mathscr{R}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} = \mathscr{P}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \mathscr{Q}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\,. \tag{A.1.17}$$

Tensor densities of different types cannot be added.

Indices of tensor densities can be lowered or raised with the help of the metric tensor or its inverse, for example,

$$\mathscr{Q}^{\kappa\alpha\beta\ldots\gamma}{}_{\rho\ldots\sigma} = g^{\kappa\pi}\mathscr{Q}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\,, \qquad \mathscr{Q}_{\kappa\pi\rho\ldots\sigma}{}^{\beta\ldots\gamma} = g_{\kappa\alpha}\mathscr{Q}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\,, \tag{A.1.18}$$

and so on. The contravariant index of tensor density of type $k + l$ can be directly contracted with the covariant one, thus, producing a tensor density of type $(k - 1) + (l - 1)$ and the same weight, for example

$$\mathscr{P}^{\beta\ldots\gamma}_{\rho\ldots\sigma} \equiv \mathscr{Q}^{\alpha\beta\ldots\gamma}_{\alpha\rho\ldots\sigma}\,. \tag{A.1.19}$$

Equation (A.1.18) points out that one can also contract either contravariant or covariant indices of a tensor density of type $k + l$ with the help of the metric tensor

$$\mathscr{R}_{\pi\rho\ldots\sigma}{}^{\lambda\ldots\gamma} \equiv g_{\alpha\beta}\mathscr{Q}^{\alpha\beta\lambda\ldots\gamma}_{\pi\rho\ldots\sigma}\,. \tag{A.1.20}$$

As a result, one obtains a tensor density of the same weight, but of the reduced rank $(k - 2) + l$.

Next, one defines an outer tensor product, $\boldsymbol{R} = \boldsymbol{P} \otimes \boldsymbol{Q}$, of two different tensor densities: $\boldsymbol{P}$ of weight n and rank $k+l$, and $\boldsymbol{Q}$ of weight m and rank $p+q$, as an algebraic multiplication of their components,

$$\mathscr{R}^{\alpha\beta\ldots\gamma\kappa\lambda\ldots\tau}_{\pi\rho\ldots\sigma\mu\nu\ldots\omega} = \mathscr{P}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\,\mathscr{Q}^{\kappa\lambda\ldots\tau}_{\mu\nu\ldots\omega}\,. \tag{A.1.21}$$

The resulting tensor density $\boldsymbol{R}$ is of weight $n + m$ and rank $(k = p) + (l + q)$.

One has also to define the nil tensor: it is the tensor whose components vanish identically in a certain coordinate chart. Then, by the law of tensor transformation (A.1.16), one concludes that it has nil components in all coordinates. Also, by (A.1.21), one concludes that a tensor product of the nil tensor density with an arbitrary tensor density yields a nil tensor density once again.

A.2 Derivatives

Mathematical operation of differentiation is one of the most important (and difficult) subjects of theoretical physics and mathematics of manifolds. It is not the goal of this book to go into a rigorous theory of the derivatives and the associated connections. We merely provide the reader with the constructive definition and properties of the derivatives being relevant to the calculations described in the main part of the book. More detailed description of this important mathematical subject can be found in excellent textbooks on the theory of gravity [178, 285, 315] or in any modern textbook on differential geometry.

A.2.1 Covariant derivatives and the Christoffel symbols

The covariant differentiation of tensors and tensor densities originated from the fact that a partial derivative of a tensor density is not a tensorial quantity. For example, the partial derivative $\partial_\beta A^\alpha$ of a contravariant vector A^α is not transformed as a tensor. To preserve the tensorial character of $\partial_\beta A^\alpha$, one has to modify the partial differentiation by adding an additional term proportional to an affine connection,

$$\nabla_\beta A^\alpha \equiv \partial_\beta A^\alpha + \Gamma^\alpha{}_{\beta\gamma} A^\gamma \,. \tag{A.2.1}$$

The tensorial operation (A.2.1) is called the covariant derivative of a vector field A^α. The components of $\Gamma^\alpha{}_{\beta\gamma}$ are called the Christoffel symbols which are expressed solely in terms of the metric tensor and its partial derivatives,

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} \left(\partial_\beta g_{\gamma\rho} + \partial_\gamma g_{\beta\rho} - \partial_\rho g_{\beta\gamma} \right) \,. \tag{A.2.2}$$

By making coordinate transformation (A.1.1) one can see that the Christoffel symbols are transformed as follows,

$$\Gamma'^\alpha{}_{\beta\gamma} = \Gamma^\pi{}_{\rho\sigma} \frac{\partial x'^\alpha}{\partial x^\pi} \frac{\partial x^\rho}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\gamma} + \frac{\partial^2 x^\rho}{\partial x'^\beta \partial x'^\gamma} \frac{\partial x'^\alpha}{\partial x^\rho} \,. \tag{A.2.3}$$

Clearly, the Christoffel symbols are not components of a tensor quantity because they do not satisfy the law of transformation of tensors (A.1.16). For example, the Christoffel symbols may vanish in one coordinate chart but be non-zero in another one. Non-tensorial character of the Christoffel symbols is exactly the desired property which compensates the non-tensorial behavior of $\partial_\beta A^\alpha$, and makes it tensor. Analogously to (A.2.1) one can introduce a covariant derivative of a covector

$$\nabla_\beta A_\alpha \equiv \partial_\beta A_\alpha - \Gamma^\gamma{}_{\alpha\beta} A_\gamma \,, \tag{A.2.4}$$

where the reader should notice the important minus sign in the front of the second term. Equations (A.2.1) and (A.2.4) are reciprocal. At last, generalizing (A.2.1–A.2.4) one has for a covariant derivative of an arbitrary tensor density of weight $+n$ the following, rather long definition,

$$\nabla_\lambda Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} \equiv \partial_\lambda Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} - n\Gamma^{\mu}{}_{\lambda\mu} Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}$$
$$+ \Gamma^{\alpha}{}_{\lambda\mu} Q^{\mu\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \Gamma^{\beta}{}_{\lambda\mu} Q^{\alpha\mu\ldots\gamma}_{\pi\rho\ldots\sigma} + \ldots + \Gamma^{\gamma}{}_{\lambda\mu} Q^{\alpha\beta\ldots\mu}_{\pi\rho\ldots\sigma}$$
$$- \Gamma^{\nu}{}_{\lambda\pi} Q^{\alpha\beta\ldots\gamma}_{\nu\rho\ldots\sigma} - \Gamma^{\nu}{}_{\lambda\rho} Q^{\alpha\beta\ldots\gamma}_{\pi\nu\ldots\sigma} - \ldots - \Gamma^{\nu}{}_{\lambda\sigma} Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\nu} . \tag{A.2.5}$$

We emphasize that the operation of the covariant derivative of a tensor density or rank $k + l$ and weight $+n$ does not change its weight but increases its rank to $k + (l + 1)$. Let us now recall the important properties of covariant derivatives.

First, using (A.2.2) and (A.2.5) one finds that the covariant derivative of the metric tensor and its inverse vanish,

$$\nabla_\gamma g_{\alpha\beta} = 0 , \qquad \nabla_\gamma g^{\alpha\beta} = 0 . \tag{A.2.6}$$

The same is true for the determinant g of the metric tensor or any differentiable scalar function $f(g)$ of it, $\nabla_\gamma f(g) = (\partial f/\partial g)\nabla_\gamma g = 0$. Thus, in calculations of the covariant derivatives, the metric or functions depending on the metric, can be pulled out of the operator of the covariant derivative. Equation (A.2.6) tells us that the affine connection and the associated covariant derivative are metric compatible. It is this case when the affine connection is reduced to the Christoffel symbols. In more general cases of the manifolds with torsion the covariant derivative from the metric tensor may not vanish but we shall not consider this complication over here.

Second, by making use of definition (A.2.5), we can derive the rules of taking a covariant derivative from the sum and product of two arbitrary tensor densities, $\boldsymbol{P}$ and $\boldsymbol{Q}$,

$$\nabla_\lambda \left(\mathscr{P}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \mathscr{Q}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} \right) = \nabla_\lambda \mathscr{P}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \nabla_\lambda \mathscr{Q}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} \tag{A.2.7}$$

$$\nabla_\lambda \left(\mathscr{P}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} \mathscr{Q}^{\kappa\lambda\ldots\tau}_{\mu\nu\ldots\omega} \right) = \mathscr{Q}^{\kappa\lambda\ldots\tau}_{\mu\nu\ldots\omega} \nabla_\lambda \mathscr{P}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \mathscr{P}^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} \nabla_\lambda \mathscr{Q}^{\kappa\lambda\ldots\tau}_{\mu\nu\ldots\omega} . \tag{A.2.8}$$

Third, we list some differential relations, which follow from the definition of the Christoffel symbols (A.2.2) which are useful in practical calculations:

$$\Gamma^{\beta}{}_{\lambda\beta} = \tfrac{1}{2} g^{\mu\nu} \partial_\lambda g_{\mu\nu} = \partial_\lambda \sqrt{-g} , \tag{A.2.9}$$

$$\partial_\lambda g_{\alpha\beta} = g_{\alpha\rho} \Gamma^{\rho}{}_{\beta\lambda} + g_{\beta\rho} \Gamma^{\rho}{}_{\alpha\lambda} , \tag{A.2.10}$$

$$\partial_\lambda g^{\alpha\beta} = -g^{\alpha\rho} \Gamma^{\beta}{}_{\rho\lambda} - g^{\beta\rho} \Gamma^{\alpha}{}_{\rho\lambda} . \tag{A.2.11}$$

Fourth, the transformation (A.2.5) for a vector density $\mathscr{A}^\alpha$ of weight +1 and for an antisymmetric tensor density, $\mathscr{B}^{\alpha\beta} = -\mathscr{B}^{\beta\alpha}$, of weight +1 permits one to reduce the covariant derivatives from the mentioned tensor densities to partial derivatives,

$$\nabla_\alpha \mathscr{A}^\alpha = \partial_\alpha \mathscr{A}^\alpha , \tag{A.2.12}$$
$$\nabla_\beta \mathscr{B}^{\alpha\beta} = \partial_\beta \mathscr{B}^{\alpha\beta} . \tag{A.2.13}$$

These relations are often employed in variational calculus on manifolds.

A.2.2 The curvature tensor

Commutator of two covariant derivatives is a linear differential operator which defines the curvature (Riemann) tensor, $R^{\alpha}{}_{\beta\mu\nu}$, see [178, 285, 315],

$$[\nabla_\mu, \nabla_\nu]A^\alpha \equiv \nabla_{\mu\nu}A^\alpha - \nabla_{\nu\mu}A^\alpha = A^\beta R^\alpha{}_{\beta\mu\nu}\,, \tag{A.2.14}$$

where we denoted $\nabla_{\mu\nu} \equiv \nabla_\mu\nabla_\nu$, and

$$R^\mu{}_{\alpha\nu\beta} \equiv \partial_\nu\Gamma^\mu{}_{\alpha\beta} - \partial_\beta\Gamma^\mu{}_{\alpha\mu} + \Gamma^\rho{}_{\alpha\beta}\Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\alpha\rho}\Gamma^\rho{}_{\beta\nu}\,. \tag{A.2.15}$$

With the use of (A.2.6) one can derive the value of the commutator of two covariant derivatives applied to a covector A_α. Generalizing (A.2.14) for an arbitrary tensor density results in

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} &= Q^{\lambda\beta\ldots\gamma}_{\pi\rho\ldots\sigma}R^\alpha{}_{\lambda\mu\nu} + Q^{\alpha\lambda\ldots\gamma}_{\pi\rho\ldots\sigma}R^\beta{}_{\lambda\mu\nu} + \ldots + Q^{\alpha\beta\ldots\lambda}_{\pi\rho\ldots\sigma}R^\gamma{}_{\lambda\mu\nu}\\ &\quad - Q^{\alpha\beta\ldots\gamma}_{\lambda\rho\ldots\sigma}R^\lambda{}_{\pi\mu\nu} - Q^{\alpha\beta\ldots\gamma}_{\pi\lambda\ldots\sigma}R^\lambda{}_{\rho\mu\nu} - \ldots - Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\lambda}R^\lambda{}_{\sigma\mu\nu}\,.\end{aligned} \tag{A.2.16}$$

Notice that in relations, like (A.2.14–A.2.16) the weight of a tensor density does not play any role because the terms depending on weight are mutually canceled out.

A.2.3 Lie derivative

The important notion of a Lie derivative was introduced in Section 1.2.3, see (1.2.72–1.2.82). Here, we derive some formal mathematical properties of this derivative which are employed in calculations. Thus, considering an infinitesimal coordinate transformation:

$$x'^\alpha = x^\alpha + \xi^\alpha(x)\,, \tag{A.2.17}$$

one defines the Lie derivative of a tensor density $\boldsymbol{Q} = \{Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\}$ as a variation between $\boldsymbol{Q}$ and its transformed value $\boldsymbol{Q}'$ obtained by applying transformation (A.1.16) generated by (A.2.17) and shifting it on the manifold along ξ^α to the point with the same coordinates $x = \{x^\alpha\}$,

$$£_\xi Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x) \equiv Q'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x) - Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x)\,. \tag{A.2.18}$$

The operational formula for the Lie derivative is derived from the variation

$$\delta' Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} \equiv Q'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x') - Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x)\,, \tag{A.2.19}$$

where $Q'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x')$ is related to $Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x)$ by the tensor transformation (A.1.16) induced by (A.2.17). Making Taylor series expansion of the matrix of the partial derivatives of the coordinate transformation (A.2.17) with respect to the small vector ξ^α and keeping only the linear terms, result in

$$\begin{aligned}\delta' Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} = &-n\partial_\mu\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\\ &+\partial_\mu\xi^\alpha Q^{\mu\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \partial_\mu\xi^\alpha Q^{\alpha\mu\ldots\gamma}_{\pi\rho\ldots\sigma} + \ldots + \partial_\mu\xi^\gamma Q^{\alpha\beta\ldots\mu}_{\pi\rho\ldots\sigma}\\ &-\partial_\pi\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\mu\rho\ldots\sigma} - \partial_\rho\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\pi\mu\ldots\sigma} - \ldots - \partial_\sigma\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\mu}\,.\end{aligned} \tag{A.2.20}$$

In addition to this, one has to make a Taylor series expansion of $Q'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x')$ around the point x which yields

$$Q'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x') = Q'^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x) + \xi^\mu\partial_\mu Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x)\,, \tag{A.2.21}$$

or

$$\delta' Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} = £_\xi Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x) + \xi^\mu\partial_\mu Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x)\,. \tag{A.2.22}$$

Combining (A.2.18–A.2.22) one finds

$$\begin{aligned}£_\xi Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x) = &-\xi^\mu\partial_\mu Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}(x) - n\partial_\mu\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\\ &+\partial_\mu\xi^\alpha Q^{\mu\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \partial_\mu\xi^\alpha Q^{\alpha\mu\ldots\gamma}_{\pi\rho\ldots\sigma} + \ldots + \partial_\mu\xi^\gamma Q^{\alpha\beta\ldots\mu}_{\pi\rho\ldots\sigma}\\ &-\partial_\pi\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\mu\rho\ldots\sigma} - \partial_\rho\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\pi\mu\ldots\sigma} - \ldots - \partial_\sigma\xi^\mu Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\mu}\,.\end{aligned} \tag{A.2.23}$$

We notice that *all* partial derivatives in the right side of equation (A.2.23) can be simultaneously replaced with the covariant derivatives because the terms containing the Christoffel symbols cancel each other, see (1.2.82). Thus, $£_\xi \boldsymbol{Q}$ is a tensor density of the same type as $\boldsymbol{Q}$. Now, we list some important properties of the Lie derivative.

The Lie derivative commutes with a partial (but not a covariant) derivative

$$\partial_\alpha\left(£_\xi Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\right) = £_\xi\left(\partial_\alpha Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\right)\,. \tag{A.2.24}$$

This property allows us to show that the Lie derivative of a tensor density $\boldsymbol{Q} = \boldsymbol{Q}\left(q, q_\alpha, q_{\alpha\beta}\right)$ which depends on some variable $q \equiv q(x)$ and its partial derivatives, $q_\alpha \equiv \partial_\alpha q(x)$, $q_{\alpha\beta} \equiv \partial_{\alpha\beta} q(x)$, can be represented as follows,

$$£_\xi Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} = \frac{\partial Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}}{\partial q}£_\xi q + \frac{\partial Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}}{\partial q_\alpha}£_\xi q_\alpha + \frac{\partial Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}}{\partial q_{\alpha\beta}}£_\xi q_{\alpha\beta}\,. \tag{A.2.25}$$

This relationship can be rewritten as an algebraic sum of a Lagrange derivative (which is explained in next subsection) and a collection of terms which is a total divergence,

$$£_\xi Q^{\alpha\beta...\gamma}_{\pi\rho...\sigma} = \frac{\delta Q^{\alpha\beta...\gamma}_{\pi\rho...\sigma}}{\delta q} £_\xi q + \frac{\partial}{\partial x^\alpha}\left(\frac{\delta Q^{\alpha\beta...\gamma}_{\pi\rho...\sigma}}{\delta q_\alpha} £_\xi q + \frac{\partial Q^{\alpha\beta...\gamma}_{\pi\rho...\sigma}}{\partial q_{\alpha\beta}} £_\xi q_\beta\right) . \tag{A.2.26}$$

This property of the Lie derivative is used for the derivation of Noether's theorems.

Among a number of other important properties of the Lie derivative we recall the following:

$$£_\xi \delta^\rho_\tau = 0 , \tag{A.2.27}$$

$$£_\xi\left(\mathscr{Q}^{\alpha\beta...\gamma}_{\pi\rho...\sigma} + \mathscr{P}^{\alpha\beta...\gamma}_{\pi\rho...\sigma}\right) = £_\xi \mathscr{Q}^{\alpha\beta...\gamma}_{\pi\rho...\sigma} + £_\xi \mathscr{P}^{\alpha\beta...\gamma}_{\pi\rho...\sigma} , \tag{A.2.28}$$

$$£_\xi\left(\mathscr{P}^{\alpha\beta...\gamma}_{\pi\rho...\sigma}\mathscr{Q}^{\kappa\lambda...\tau}_{\mu\nu...\omega}\right) = \mathscr{P}^{\alpha\beta...\gamma}_{\pi\rho...\sigma} £_\xi \mathscr{Q}^{\kappa\lambda...\tau}_{\mu\nu...\omega} + \mathscr{Q}^{\kappa\lambda...\tau}_{\mu\nu...\omega} £_\xi \mathscr{P}^{\alpha\beta...\gamma}_{\pi\rho...\sigma} , \tag{A.2.29}$$

$$£_{[\zeta\xi]} \mathscr{Q}^{\kappa\lambda...\tau}_{\mu\nu...\omega} = £_\zeta £_\xi \mathscr{Q}^{\kappa\lambda...\tau}_{\mu\nu...\omega} - £_\xi £_\zeta \mathscr{Q}^{\kappa\lambda...\tau}_{\mu\nu...\omega} , \tag{A.2.30}$$

where in the last equation (A.2.30) the Lie derivative is calculated along the commutator of two vector fields, $[\zeta\xi] = \xi^\rho \zeta^\alpha{}_{,\rho} - \zeta^\rho \xi^\alpha{}_{,\rho} = \xi^\rho \nabla_\rho \zeta^\alpha - \zeta^\rho \nabla_\rho \xi^\alpha$.

At last, we would like to draw attention of the reader to the fact that the operation of taking the Lie derivative can be applied not only to tensor densities but also to some non-tensorial geometrical objects like the Christoffel symbols (A.2.2). Applying definition (A.2.18) to the Christoffel symbols

$$£_\xi \Gamma^\alpha{}_{\beta\gamma}(x) = \Gamma'^\alpha{}_{\beta\gamma}(x) - \Gamma^\alpha{}_{\beta\gamma}(x) , \tag{A.2.31}$$

and accounting for the transformation law (A.2.3) one recognizes that the Lie derivative (A.2.31) is a tensor transforming accordingly under coordinate transformations.

A.2.4 Variational and Lagrangian derivatives

Mathematical theory of perturbations of physical fields on manifolds including the gravitational field itself, relies on the least action principle applied to a functional

$$S = \int_\Omega d^4x \mathscr{L} , \tag{A.2.32}$$

called the action. Variational derivative is a mathematical operation that appears when the least action principle is used to derive the field equations which are the extremal of the action. The scalar density, $\mathscr{L}$, that appears in the integrand of the functional (A.2.32) is called the Lagrangian. It has weight +1 and can be rewritten, for example, in the form $\mathscr{L} = \sqrt{-g}L$, where L is a scalar function depending on the physical fields under consideration. We assume that the Lagrangian depends on matter and metric variables as well as on their first and second derivatives,

$$\mathscr{L} = \mathscr{L}\left(Q^A, Q^A_{,\alpha}, Q^A_{,\alpha\beta}; g_{\mu\nu}, g_{\mu\nu,\alpha}, g_{\mu\nu,\alpha\beta}\right) . \tag{A.2.33}$$

Here the terms depending on the first and second derivatives are analogs of the velocity and the acceleration in the non-relativistic mechanics of point-like particles. The variable Q^A can be a tensor density of an arbitrary type and weight with the covariant and/or contravariant indices which are combined in a single (economic) index A which is more convenient to handle long and cumbersome tensorial operations. The mathematical rules of operation with the economic index notations are explained in the next section of the appendix.

A certain care should be taken in choosing the set of independent dynamical variables and their derivatives for doing variational calculus. The reason is that in the most cases the dynamical variables entering the Lagrangian are tensor fields whose contravariant and covariant components differ. Therefore, it is more preferable to operate with a set of dynamical variables of the same tensor type which shorten calculations and simplify equations. The reason behind this recommendation is that covariant and contravariant components of a tensor field are dual mathematical objects which are interrelated via the metric tensor. Therefore, derivatives of the contravariant components differ from the covariant ones by a number of additional terms involving derivatives of the metric tensor which may lead to unnecessary complication of calculations and/or erroneous interpretation of physics.

Variational derivative from the action S taken with respect to the field variable Q^A relates the variation, δS, of the action S to the variation, δQ^A of the field variable. In order to get an operational formula for the variational derivative we take the variation of the action

$$\delta S = \int_\Omega d^4x \delta \mathscr{L} \,, \tag{A.2.34}$$

where

$$\delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial Q^A} \delta Q^A + \frac{\partial \mathscr{L}}{\partial Q^A{}_{,\alpha}} \delta Q^A{}_{,\alpha} + \frac{\partial \mathscr{L}}{\partial Q^A{}_{,\alpha\beta}} \delta Q^A{}_{,\alpha\beta} \,, \tag{A.2.35}$$

is a functional increment of $\mathscr{L}$ expressed in terms of the variations of its arguments. The variational derivative is obtained after we make use of the commutation relations, $\delta Q^A{}_{,\alpha} = (\delta Q^A)_{,\alpha}$ and $\delta Q^A{}_{,\alpha\beta} = (\delta Q^A)_{,\alpha\beta}$, integrate by parts the terms with the partial derivatives, and single out a total divergence in the right side of (A.2.35). The total divergence is reduced to a surface term in the integral (A.2.34) which vanishes on the boundary $\partial\Omega$ of the volume Ω of integration, if the boundary conditions

$$\delta\, Q^A\big|_{\partial\Omega} = 0, \qquad \left(\delta Q^A{}_{,\alpha}\right)\big|_{\partial\Omega} = 0 \,, \tag{A.2.36}$$

are imposed on the variation of the field variable Q^A and its first derivative. Thus, the variation of S is reduced to

$$\delta S = \int_\Omega d^4x \frac{\delta \mathscr{L}}{\delta Q^A} \delta Q^A \,, \tag{A.2.37}$$

where

$$\frac{\delta \mathscr{L}}{\delta Q^A} \equiv \frac{\partial \mathscr{L}}{\partial Q^A} - \frac{\partial}{\partial x^\alpha}\frac{\partial \mathscr{L}}{\partial Q^A{}_{,\alpha}} + \frac{\partial^2}{\partial x^\alpha \partial x^\beta}\frac{\partial \mathscr{L}}{\partial Q^A{}_{,\alpha\beta}} \ . \tag{A.2.38}$$

We call the operator (A.2.38) the *Lagrangian derivative* following the nomenclature adopted in the book [266]. More common practice is to call (A.2.38) as a variational derivative and to denote it as $\delta S/\delta Q^A$. This notation seems to be somewhat misleading as the derivative (A.2.38) is directly applied to the Lagrangian $\mathscr{L}$ but not to the action S.

Let us assume that there is another Lagrangian, $\tilde{\mathscr{L}}$, which differs from the original one, $\mathscr{L}$, in (A.2.33) by a total divergence depending only on the field variables Q^A and their first derivatives $Q^A_{,\alpha}$

$$\tilde{\mathscr{L}} = \mathscr{L} + \partial_\beta \mathscr{D}^\beta\left(Q^A, Q^A{}_{,\alpha}\right) \ . \tag{A.2.39}$$

It is well known [236, 316] and can be easily checked by inspection that the operator of the Lagrangian derivative (A.2.38) applied to the total divergence yields a zero:

$$\frac{\delta}{\delta Q^A}\left(\frac{\partial \mathscr{D}^\alpha}{\partial x^\alpha}\right) \equiv 0 \ . \tag{A.2.40}$$

Thus, the Lagrangian derivative of $\tilde{\mathscr{L}}$ and $\mathscr{L}$ yields the same result. In fact, it is straightforward to prove that the Lagrangian derivative (A.2.38) applied to a *partial* derivative of an arbitrary tensor density, $\boldsymbol{F} = \boldsymbol{F}\left(Q^A, Q^A_{,\alpha}, Q^A_{,\alpha\beta}\right)$, vanishes

$$\frac{\delta}{\delta Q^A}\left(\frac{\partial \boldsymbol{F}}{\partial x^\alpha}\right) \equiv 0 \ . \tag{A.2.41}$$

However, this property does not hold for the covariant derivative of the tensor density in the most general case [316].

Because the Lagrangian is a scalar density of weight +1, and Q^A along with δQ^A are tensor densities as well, the reader may suggest that the Lagrangian derivative (A.2.38) derived from (A.2.37) is a covariant object which does not depend on the choice of coordinates. This is indeed true and we prove the covariance of the Lagrangian derivative (A.2.38) in the next section of this appendix, see (A.3.37).

Lagrangian derivative of the Lagrangian (A.2.33) with respect to the metric tensor $g_{\mu\nu}$ is defined by the same rule like (A.2.38) that reads

$$\frac{\delta \mathscr{L}}{\delta g_{\mu\nu}} \equiv \frac{\partial \mathscr{L}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x^\alpha}\frac{\partial \mathscr{L}}{\partial g_{\mu\nu,\alpha}} + \frac{\partial^2}{\partial x^\alpha \partial x^\beta}\frac{\partial \mathscr{L}}{\partial g_{\mu\nu,\alpha\beta}} \ . \tag{A.2.42}$$

It does not look like a covariant expression at a first glance. However, after careful inspection of each terms given in the next section of this appendix we prove that (A.2.42) does not depend on the choice of coordinates and is covariant, see (A.3.44).

The proof is not quite straightforward and requires some critical thinking because the covariant derivative of the metric tensor $\nabla_\alpha g_{\mu\nu} = 0$, see (A.2.6). Let us note some important properties of the Lagrangian derivatives.

The Lagrangian derivatives are not linear operators. For example, they do not obey the Leibniz's rule [199]. More specifically, for a product of two geometric objects, $\mathscr{F} = \mathscr{F}\left(Q^A, Q^A{}_{,\alpha}, Q^A{}_{,\alpha\beta}\right)$ and $\mathscr{T} = \mathscr{T}\left(Q^A, Q^A{}_{,\alpha}, Q^A{}_{,\alpha\beta}\right)$, the Lagrangian derivative

$$\frac{\delta(\mathscr{F}\mathscr{T})}{\delta Q^A} \neq \frac{\delta(\mathscr{F})}{\delta Q^A}\mathscr{T} + \mathscr{F}\frac{\delta(\mathscr{T})}{\delta Q^A}\,, \tag{A.2.43}$$

in the most general case.

The chain rule with regard to the Lagrangian derivative exists in two forms. First, let us consider a geometric object $\mathscr{F}$ depending on Q^A and its first and second derivatives $\mathscr{F} = \mathscr{F}\left(Q^A, Q^A{}_{,\alpha}, Q^A{}_{,\alpha\beta}\right)$ where Q^A is a function of a variable P^B without its derivatives, that is $Q^A = Q^A(P^B)$. Then, the Lagrangian derivative reads

$$\frac{\delta\mathscr{F}}{\delta P^A} = \frac{\delta\mathscr{F}}{\delta Q^B}\frac{\partial Q^B}{\partial P^A}\,, \tag{A.2.44}$$

which can be confirmed by inspection [379]. The chain rule (A.2.44) simplifies the calculation of the Lagrangian derivative of $\mathscr{F}$ with respect to variable P^A if we already know the Lagrangian derivative of $\mathscr{F}$ with respect to Q^A. For example, the Lagrangian derivative with respect to the contravariant metric tensor, $g^{\mu\nu}$, can be immediately found from (A.2.42) by applying the chain rule,

$$\frac{\delta\mathscr{L}}{\delta g^{\mu\nu}} = \frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}}\frac{\delta\mathscr{L}}{\delta g_{\alpha\beta}} = -g_{\alpha\mu}g_{\beta\nu}\frac{\delta\mathscr{L}}{\delta g_{\alpha\beta}}\,. \tag{A.2.45}$$

By the same token we can get the Lagrangian derivative with respect to the contravariant metric tensor expressed in terms of the Lagrangian derivative with respect to the Gothic metric,

$$\frac{\delta\mathscr{L}}{\delta g^{\mu\nu}} = \frac{\partial \mathfrak{g}^{\alpha\beta}}{\partial g^{\mu\nu}}\frac{\delta\mathscr{L}}{\delta \mathfrak{g}^{\alpha\beta}} = \sqrt{-g}\left(\delta^\alpha_\mu\delta^\beta_\nu - \tfrac{1}{2}g^{\alpha\beta}g^{\mu\nu}\right)\frac{\delta\mathscr{L}}{\delta \mathfrak{g}^{\alpha\beta}}\,, \tag{A.2.46}$$

and so on.

The second chain rule is applied in case when one has a geometric object $\mathscr{F} = \mathscr{F}(Q^A)$, depending merely on a variable Q^A (without derivatives) which depends on the variable P^A along with its first and second derivatives, $Q^A = Q^A\left(P^B, P^B_{,\alpha}, P^B_{,\alpha\beta}\right)$. In this situation the chain rule reads [379]

$$\frac{\delta\mathscr{F}}{\delta P^A} = \frac{\partial\mathscr{F}}{\partial Q^B}\frac{\delta Q^B}{\delta P^A}\,. \tag{A.2.47}$$

The chain rules (A.2.46) and (A.2.47) are frequently used in calculations present in the book.

A.3 Introduction to economic tensor operations

A.3.1 Economic index notations

The above equations show that the operations with tensor densities of high rank and weight involving all tensor indices lead in most cases to long and very cumbersome expressions. In order to simplify the tensor equations with a large number of indices we introduce condensed, economic notations for them. To this end, an arbitrary tensor density $Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}$ which appears, for example in (A.1.16), will be denoted as

$$Q^A \equiv Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} , \tag{A.3.1}$$

where a collective index A combines both contravariant and covariant tensor indices in a single piggyback notation. This economic notation of tensor densities is used consistently throughout the book to shorten equations. With the use of notation (A.3.1) an outer product is defined:

$$R^C \equiv P^A Q^B , \tag{A.3.2}$$

where the set of indices C unites the set A and B; it can be a short writing of the formula (A.1.21). The lower collective index is useful also. Thus, for example, for a concrete set of indices in (A.3.1) we can define

$$P_A \equiv P^{\pi\rho\ldots\sigma}_{\alpha\beta\ldots\gamma} . \tag{A.3.3}$$

Thus, the operation of a contraction of collective indices with the use of the Einstein rule is useful for calculations also:

$$R \equiv P_A Q^A . \tag{A.3.4}$$

Another useful device to manipulate with long tensor expressions is a linear operator $|^{\mu}_{\nu}$ of permutations of indices of the tensor density of weight $+n$ [371]

$$\begin{aligned} Q^A\Big|^{\mu}_{\nu} \equiv Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma}\Big|^{\mu}_{\nu} = &-n\delta^{\mu}_{\nu} Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \delta^{\alpha}_{\nu} Q^{\mu\beta\ldots\gamma}_{\pi\rho\ldots\sigma} + \delta^{\beta}_{\nu} Q^{\alpha\mu\ldots\gamma}_{\pi\rho\ldots\sigma} + \ldots + \delta^{\gamma}_{\nu} Q^{\alpha\beta\ldots\mu}_{\pi\rho\ldots\sigma} \\ &-\delta^{\mu}_{\pi} Q^{\alpha\beta\ldots\gamma}_{\nu\rho\ldots\sigma} - \delta^{\mu}_{\rho} Q^{\alpha\beta\ldots\gamma}_{\pi\nu\ldots\sigma} - \ldots - \delta^{\mu}_{\sigma} Q^{\alpha\beta\ldots\gamma}_{\pi\rho\ldots\nu} . \end{aligned} \tag{A.3.5}$$

A double permutation operator is again a permutation operator being naturally defined as

$$Q^A\Big|^{\alpha}_{\beta}\Big|^{\mu}_{\nu} \equiv \left(Q^A\Big|^{\alpha}_{\beta}\right)\Big|^{\mu}_{\nu} \equiv Q^B\Big|^{\mu}_{\nu} \tag{A.3.6}$$

where $Q^B \equiv Q^A\Big|^{\alpha}_{\beta}$.

The permutation operator seems to appear for the first time in textbook by Mitskevich [316], however our definition of $|^{\mu}_{\nu}$ is simpler and more efficient for conducting tensor operations. Equation (A.3.5) applied to the metric tensor, $Q^A \equiv g_{\mu\nu}$, reads $g_{\mu\nu}\Big|^{\alpha}_{\beta} = -\delta^{\alpha}_{\mu} g_{\beta\nu} - \delta^{\alpha}_{\nu} g_{\mu\beta}$; the same equation applied to the Kronecker delta, $Q^A = \delta^{\nu}_{\mu}$, yields $\delta^{\nu}_{\mu}\Big|^{\alpha}_{\beta} = 0$; for a scalar density $Q^A = \mathscr{L}$ one has $\mathscr{L}|^{\alpha}_{\beta} = -\delta^{\alpha}_{\beta}\mathscr{L}$, and so on. The abstract form of the symbolic permutation operation, $Q^A\Big|^{\alpha}_{\beta}$, is very helpful as it drastically shortens long tensor calculations including covariant, Lie and Lagrangian derivatives. The ploy here is that we don't need to write down all tensor indices at the intermediate steps of calculation - they can be easily restored at the end of the calculation by expanding the collective indices of tensor densities with the help of (A.3.5) and substituting them to the final result.

To demonstrate the advantages of the economic notations (A.3.1) and (A.3.5) we rewrite the long formula (A.2.5) for the covariant derivative in the economic form:

$$\nabla_{\lambda} Q^A = \partial_{\lambda} Q^A + Q^A\Big|^{\mu}_{\nu} \Gamma^{\nu}{}_{\lambda\mu} \,. \tag{A.3.7}$$

Economic expression of formula (A.2.23) for the Lie derivative takes on the following form:

$$£_{\xi} Q^A = -\xi^{\mu}\partial_{\mu} Q^A + Q^A\Big|^{\mu}_{\nu}\partial_{\mu}\xi^{\nu} \,. \tag{A.3.8}$$

The reader can check by inspection that the economic form of the permutation operator, $Q^A\Big|^{\alpha}_{\beta}$, also emerges in the (antisymmetric) commutator of two covariant derivatives (A.2.16):

$$\nabla_{\mu\nu} Q^A - \nabla_{\nu\mu} Q^A = Q^A\Big|^{\alpha}_{\beta} R^{\beta}{}_{\alpha\mu\nu} \,. \tag{A.3.9}$$

Economic notations (A.3.1) and (A.3.5) can be effectively extended to a *set* of tensor densities as well. For example, let us consider a set of k tensor densities,

$$\phi^A = \{Q^{A_1}, Q^{A_2}, \dots, Q^{A_k}\} \,, \tag{A.3.10}$$

where each member Q^{A_i} ($i = 1, 2, \dots, k$) of the set is a tensor density of a certain rank and weight like that shown in (A.3.1). Thus, the components of the set, ϕ^A, consist of all admissible components of the members of the set. We can easily define the partial derivative of the set ϕ^A by the rule

$$\partial_{\mu}\phi^A \equiv \{\partial_{\mu} Q^{A_1}, \partial_{\mu} Q^{A_2}, \dots, \partial_{\mu} Q^{A_k}\} \,. \tag{A.3.11}$$

A similar definition is applied to the covariant derivative of the set ϕ^A,

$$\nabla_{\mu}\phi^A \equiv \{\nabla_{\mu} Q^{A_1}, \nabla_{\mu} Q^{A_2}, \dots, \nabla_{\mu} Q^{A_k}\} \,, \tag{A.3.12}$$

and for its Lie derivative

$$£_{\xi}\phi^A \equiv \{£_{\xi} Q^{A_1}, £_{\xi} Q^{A_2}, \dots, £_{\xi} Q^{A_k}\} \,. \tag{A.3.13}$$

Partial derivative with respect to the set ϕ^A considered as an independent variable, is defined as follows

$$\frac{\partial}{\partial\phi^A} \equiv \left\{\frac{\partial}{\partial Q^{A_1}}, \frac{\partial}{\partial Q^{A_2}}, \dots, \frac{\partial}{\partial Q^{A_k}}\right\}. \qquad \text{(A.3.14)}$$

A similar definition is introduced for the Lagrangian derivatives with respect to the set,

$$\frac{\delta}{\delta\phi^A} \equiv \left\{\frac{\delta}{\delta Q^{A_1}}, \frac{\delta}{\delta Q^{A_2}}, \dots, \frac{\delta}{\delta Q^{A_k}}\right\}. \qquad \text{(A.3.15)}$$

The economic index notations introduced in this appendix are very helpful in variational and differential calculus of tensor and/or gauge fields propagating on geometric manifolds.

A.3.2 Algebra of economic index notations

Now, we derive some useful rules of algebraic operations involving economic index notations which are to help the reader to understand the calculations conducted in the main part of the book.

First of all, we notice that the permutation operator (A.3.5) obeys the chain rule,

$$(Q^A P^B)\Big|^{\alpha}_{\beta} = Q^A\Big|^{\alpha}_{\beta} P^B + Q^A P^B\Big|^{\alpha}_{\beta}. \qquad \text{(A.3.16)}$$

and the commutator of two permutation operators is a difference of two permutation operators,

$$Q^A\Big|^{\beta}_{\rho}\Big|^{\tau}_{\alpha} - Q^A\Big|^{\tau}_{\alpha}\Big|^{\beta}_{\rho} = \delta^{\beta}_{\alpha}\, Q^A\Big|^{\tau}_{\rho} - \delta^{\tau}_{\rho}\, Q^A\Big|^{\beta}_{\alpha}. \qquad \text{(A.3.17)}$$

The property (A.3.16) can be applied to derive equation for the permutation operator applied to a covariant derivative of tensor density:

$$\left(\nabla_\alpha Q^A\right)\Big|^{\tau}_{\rho} = \nabla_\alpha\left(Q^A\Big|^{\tau}_{\rho}\right) - \delta^{\tau}_{\alpha}\nabla_\rho Q^A. \qquad \text{(A.3.18)}$$

It can be easily understood if we remember that the covariant derivative ∇_α is a rank-1 tensor. Then, the left side of (A.3.18) can be viewed as a tensor product of two tensors to which the permutation operator is applied. Accounting for equation (A.3.16) and $\nabla_\alpha\big|^{\tau}_{\rho} = -\delta^{\tau}_{\alpha}\nabla_\rho$ (which is a direct consequence of (A.3.5) applied to ∇_α) we get (A.3.18).

Taking a Lagrangian derivative of the Lagrangian depending on tensor fields Q^B is facilitated by applying an equation (notice contraction with respect to the corresponding economic indices B)

$$\frac{\partial\mathscr{L}}{\partial(\nabla_\alpha Q^B)}\, Q^B\Big|^{\rho}_{\tau} = \left(\frac{\partial\mathscr{L}}{\partial(\nabla_\alpha Q^B)} Q^B\right)\Bigg|^{\rho}_{\tau} - \left(\frac{\partial\mathscr{L}}{\partial(\nabla_\alpha Q^B)}\right)\Bigg|^{\rho}_{\tau} Q^B, \qquad \text{(A.3.19)}$$

which follows from (A.3.16). It is also instructive to notice that the first term in the right side of (A.3.19), to which the permutation operator is applied, is a vector density of weight +1. Hence, it can be written down explicitly by making use of (A.3.5) as follows:

$$\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\alpha Q^B)}Q^B\right)\bigg|^{\rho}_{\tau} = -\delta^{\rho}_{\tau}\frac{\partial\mathscr{L}}{\partial(\nabla_\alpha Q^B)}Q^B + \delta^{\alpha}_{\tau}\frac{\partial\mathscr{L}}{\partial(\nabla_\rho Q^B)}Q^B\,. \tag{A.3.20}$$

The covariant derivative (A.3.7) applied to calculate divergence of a vector density, $\mathscr{A}^\alpha$, in (A.2.12) and that of antisymmetric tensor density, $\mathscr{B}^{\alpha\beta}$, in (A.2.13), evidently yields

$$\mathscr{A}^\alpha\big|^{\rho}_{\tau}\,\Gamma^{\tau}_{\alpha\rho} = 0\,, \tag{A.3.21}$$

$$\mathscr{B}^{\alpha\beta}\big|^{\rho}_{\tau}\,\Gamma^{\tau}_{\beta\rho} = 0\,. \tag{A.3.22}$$

An example of application of equation (A.3.21) is equation (A.3.20) contracted with the Christoffel symbols, which gives:

$$\left(\frac{\partial\mathscr{L}}{\partial(\nabla_\alpha Q^B)}Q^B\right)\bigg|^{\rho}_{\tau}\,\Gamma^{\tau}_{\alpha\rho} = 0\,. \tag{A.3.23}$$

One more useful relation connecting the double permutation operator with the partial derivatives of tensor densities is:

$$\frac{\partial\left(Q^A\big|^{\alpha}_{\beta}\right)}{\partial Q^B}\,Q^B\big|^{\rho}_{\tau} = Q^A\big|^{\alpha}_{\beta}\big|^{\rho}_{\tau}\,. \tag{A.3.24}$$

We remark that a major part of calculations used to derive basic results present in the book, are long and tedious, and it is virtually impossible to list all particular relations involving the economic index notations. Nonetheless, we have explained the main idea of how to conduct calculations with the economic index notations and hope that the information provided in Appendices A.3.1 and A.3.2 will be sufficient for the reader to repeat calculations without obstacles.

A.3.3 Covariant expressions

The techniques of the economic index notations and permutation operator allow us to represent both the Lie derivatives (A.3.8) and the Lagrangian derivatives (A.2.38) and (A.2.42) in explicitly covariant form which is important in the field theories on geometric manifolds. The procedure of doing this is rather straightforward and is given below.

Lie derivatives

We rewrite the partial derivatives from the field Q^A and a vector field ξ^ν entering the right side of (A.3.8), in terms of the covariant derivatives,

$$Q^A{}_{,\mu} = \nabla_\mu Q^A - \Gamma^\nu{}_{\mu\rho}\, Q^A\big|^\rho_\nu\,; \tag{A.3.25}$$
$$\xi^\nu{}_{,\mu} = \nabla_\mu \xi^\nu - \Gamma^\nu{}_{\mu\rho}\xi^\rho\,. \tag{A.3.26}$$

Substituting these expressions into (A.3.8), one obtains an explicitly covariant expression for the Lie derivative:

$$£_\xi Q^A = -\xi^\mu \nabla_\mu Q^A + Q^A\big|^\nu_\mu \nabla_\nu \xi^\mu\,. \tag{A.3.27}$$

Lagrangian derivatives

In order to derive the covariant expression for the Lagrangian derivative we represented the Lagrangian (A.2.33) in an explicitly covariant form:

$$\mathscr{L} \equiv \mathscr{L}_c = \mathscr{L}_c\left(Q^A, \nabla_\alpha Q^A, \nabla_{\alpha\beta} Q^A; g_{\mu\nu}, R^\alpha{}_{\mu\beta\nu}\right)\,. \tag{A.3.28}$$

We notice that this general form includes, as a particular case, a Lagrangian of arbitrary covariant metric gravitational theory with a minimal coupling:

$$\mathscr{L}_c = -\frac{1}{2\kappa}\mathscr{L}^G\left(g_{\mu\nu}, R^\alpha{}_{\mu\beta\nu}\right) + \mathscr{L}^M\left(Q^A, \nabla_\alpha Q^A, \nabla_{\alpha\beta} Q^A; g_{\mu\nu}\right)\,. \tag{A.3.29}$$

Let us use (A.3.7) and a covariant derivative from both sides of (A.3.7) to express the covariant derivatives in (A.3.28) in terms of the partial ones:

$$\nabla_\alpha Q^A = Q^A{}_{,\alpha} + \Gamma^\tau{}_{\alpha\rho}\, Q^A\big|^\rho_\tau\,, \tag{A.3.30}$$
$$\begin{aligned}\nabla_{\beta\alpha} Q^A &= \nabla_\beta\left(\nabla_\alpha Q^A\right) = \left(Q^A{}_{,\alpha} + \Gamma^\tau{}_{\alpha\rho}\, Q^A\big|^\rho_\tau\right)_{,\beta} + \left(\nabla_\alpha Q^A\right)\big|^\nu_\mu \Gamma^\nu{}_{\beta\mu}\\ &= Q^A{}_{,\alpha\beta} + \Gamma^\tau{}_{\alpha\rho,\beta}\, Q^A\big|^\rho_\tau + \Gamma^\tau{}_{\alpha\rho}\nabla_\beta\left(Q^A\big|^\rho_\tau\right) + \Gamma^\tau{}_{\beta\rho}\left(\nabla_\alpha Q^A\right)\big|^\rho_\tau\\ &\quad -\Gamma^\tau{}_{\alpha\rho}\Gamma^\mu{}_{\beta\nu}\, Q^A\big|^\rho_\tau\big|^\nu_\mu\,.\end{aligned} \tag{A.3.31}$$

Now, with the use of (A.3.30) and (A.3.31) let us represent separate terms in (A.2.38) in terms of the partial derivatives with respect to the field Q^A and its covariant derivative. The first term is rewritten as

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial Q^A} &= \frac{\partial\mathscr{L}_c}{\partial Q^A} + \frac{\partial\mathscr{L}_c}{\partial(\nabla_\alpha Q^B)}\frac{\partial(\nabla_\alpha Q^B)}{\partial Q^A} + \frac{\partial\mathscr{L}_c}{\partial(\nabla_{\beta\alpha} Q^B)}\frac{\partial(\nabla_{\beta\alpha} Q^B)}{\partial Q^A}\\ &= \frac{\partial\mathscr{L}_c}{\partial Q^A} + \frac{\partial\mathscr{L}_c}{\partial(\nabla_\alpha Q^B)}\frac{\partial}{\partial Q^A}\left(\Gamma^\rho{}_{\alpha\tau}\, Q^B\big|^\tau_\rho\right)\\ &\quad + \frac{\partial\mathscr{L}_c}{\partial(\nabla_{\beta\alpha} Q^B)}\frac{\partial}{\partial Q^A}\Big[\Gamma^\rho{}_{\alpha\tau,\beta}\, Q^B\big|^\tau_\rho\\ &\quad + \left(\delta^\sigma_\alpha\, Q^B\big|^\tau_\rho\big|^\mu_\nu - \delta^\sigma_\rho\delta^\tau_\alpha\, Q^B\big|^\mu_\nu\right)\Gamma^\rho{}_{\beta\tau}\Gamma^\nu{}_{\mu\sigma}\Big]\,.\end{aligned} \tag{A.3.32}$$

To rewrite the second term in (A.2.38) we need equation,

$$\frac{\partial \mathcal{L}}{\partial Q^A{}_{,\alpha}} = \frac{\partial \mathcal{L}_c}{\partial(\nabla_\alpha Q^A)} + \frac{\partial \mathcal{L}_c}{\partial(\nabla_{\nu\mu} Q^B)} \frac{\partial(\nabla_{\nu\mu} Q^B)}{\partial(\nabla_\alpha Q^A)} = \frac{\partial \mathcal{L}_c}{\partial(\nabla_\alpha Q^A)} + \frac{\partial \mathcal{L}_c}{\partial(\nabla_{\nu\mu} Q^B)} \frac{\partial}{\partial(\nabla_\alpha Q^A)} \left[\Gamma^\tau{}_{\mu\rho} \nabla_\nu \left(Q^B\Big|_\tau^\rho \right) + \Gamma^\tau{}_{\nu\rho} \left(\nabla_\mu Q^B \right)\Big|_\tau^\rho \right] , \tag{A.3.33}$$

which follows from (A.3.31). Making use of this equation the second term in (A.2.38) takes on the following form:

$$-\frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial Q^A{}_{,\alpha}} \right) = -\nabla_\alpha \left(\frac{\partial \mathcal{L}_c}{\partial(\nabla_\alpha Q^A)} \right) + \Gamma^\tau{}_{\alpha\rho} \left(\frac{\partial \mathcal{L}_c}{\partial(\nabla_\alpha Q^A)} \right)\Bigg|_\tau^\rho - \partial_\alpha \left(\frac{\partial \mathcal{L}_c}{\partial(\nabla_{\nu\mu} Q^B)} \frac{\partial}{\partial(\nabla_\alpha Q^A)} \times \left[\Gamma^\tau{}_{\mu\rho} \nabla_\nu \left(Q^B\Big|_\tau^\rho \right) + \Gamma^\tau{}_{\nu\rho} \left(\nabla_\mu Q^B \right)\Big|_\tau^\rho \right] \right) . \tag{A.3.34}$$

To rewrite the third term in (A.2.38) we note that

$$\frac{\partial \mathcal{L}}{\partial Q^A{}_{,\alpha\beta}} = \frac{\partial \mathcal{L}_c}{\partial(\nabla_{\beta\alpha} Q^A)} , \tag{A.3.35}$$

and one has

$$\frac{\partial^2}{\partial x^\alpha \partial x^\beta} \left(\frac{\partial \mathcal{L}}{\partial Q^A{}_{,\alpha\beta}} \right) = \nabla_{\alpha\beta} \left[\frac{\partial \mathcal{L}_c}{\partial(\nabla_{\beta\alpha} Q^A)} \right] - \Gamma^\tau{}_{\alpha\rho} \left[\nabla_\beta \frac{\partial \mathcal{L}_c}{\partial(\nabla_{\beta\alpha} Q^A)} \right]\Bigg|_\tau^\rho - \partial_\alpha \left[\Gamma^\tau{}_{\beta\rho} \left(\frac{\partial \mathcal{L}_c}{\partial(\nabla_{\beta\alpha} Q^A)} \right)\Bigg|_\tau^\rho \right] . \tag{A.3.36}$$

Summing up equalities (A.3.32), (A.3.34) and (A.3.36) one finds that the first terms in the right hand sides survive while all other terms are mutually canceled out. Thus, the Lagrangian derivative (A.2.38) is represented in an explicitly covariant form:

$$\frac{\delta \mathcal{L}}{\delta Q^A} = \frac{\delta \mathcal{L}_c}{\delta Q^A} = \frac{\partial \mathcal{L}_c}{\partial Q^A} - \nabla_\alpha \frac{\partial \mathcal{L}_c}{\partial(\nabla_\alpha Q^A)} + \nabla_{\alpha\beta} \frac{\partial \mathcal{L}_c}{\partial(\nabla_{\beta\alpha} Q^A)} . \tag{A.3.37}$$

We emphasize that the order of indices in the covariant derivatives entering the last term in right side of (A.3.37) is important because the covariant derivatives do not commute.

Now, let us rewrite the Lagrangian derivative (A.2.42) with respect to the metric in an explicitly covariant form. In this case, we vary the set of the metric tensor, $g_{\mu\nu}$, the Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$, and the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ as a set of independent variables with the correspondening variations:

$$\delta \mathcal{L}_c = \frac{\partial \mathcal{L}_c}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\partial \mathcal{L}_c}{\partial \Gamma^\alpha{}_{\mu\nu}} \delta \Gamma^\alpha{}_{\mu\nu} + \frac{\partial \mathcal{L}_c}{\partial \Gamma^\alpha{}_{\mu\nu,\beta}} \delta \Gamma^\alpha{}_{\mu\nu,\beta} + \frac{\partial \mathcal{L}_c}{\partial R^\alpha{}_{\beta\mu\nu}} \delta R^\alpha{}_{\beta\mu\nu} , \tag{A.3.38}$$

where the variations of the Christoffel symbols and the Riemann tensor are tensors that can be expressed in terms of the variation $\delta g_{\mu\nu}$ of the metric tensor [456]

$$\begin{aligned} \delta\Gamma^{\alpha}{}_{\mu\nu} &= \tfrac{1}{2}g^{\alpha\sigma}\left[\nabla_\nu(\delta g_{\sigma\mu}) + \nabla_\mu(\delta g_{\sigma\nu}) - \nabla_\sigma(\delta g_{\mu\nu})\right] \,, && \text{(A.3.39)} \\ \delta R^{\alpha}{}_{\beta\mu\nu} &= \nabla_\mu(\delta\Gamma^{\alpha}{}_{\beta\nu}) - \nabla_\nu(\delta\Gamma^{\alpha}{}_{\beta\mu}) \,. && \text{(A.3.40)} \end{aligned}$$

It is evident that the first and fourth terms in (A.3.38) are covariant. Thus, one has to be convinced in the covariance of the second and the third terms. Let us substitute there an explicit expression for the Lagrangian (A.3.28), then up to a divergence one has

$$\begin{aligned} \frac{\delta\mathscr{L}_c}{\delta\Gamma^{\alpha}{}_{\mu\nu}}\delta\Gamma^{\alpha}{}_{\mu\nu} &\equiv \left[\frac{\partial\mathscr{L}_c}{\partial\Gamma^{\alpha}{}_{\mu\nu}} - \partial_\beta\left(\frac{\partial\mathscr{L}_c}{\partial\Gamma^{\alpha}{}_{\mu\nu,\beta}}\right)\right]\delta\Gamma^{\alpha}{}_{\mu\nu} \\ &= \left[\frac{\partial\mathscr{L}_c}{\partial(\nabla_\sigma Q^A)}\frac{\partial(\nabla_\sigma Q^A)}{\partial\Gamma^{\alpha}{}_{\mu\nu}} + \frac{\partial\mathscr{L}_c}{\partial(\nabla_{\pi\sigma} Q^A)}\frac{\partial(\nabla_{\pi\sigma} Q^A)}{\partial\Gamma^{\alpha}{}_{\mu\nu}}\right. \\ &\quad \left. - \partial_\beta\left(\frac{\partial\mathscr{L}_c}{\partial(\nabla_{\pi\sigma} Q^A)}\frac{\partial(\nabla_{\pi\sigma} Q^A)}{\partial\Gamma^{\alpha}{}_{\mu\nu,\beta}}\right)\right]\delta\Gamma^{\alpha}{}_{\mu\nu} \\ &= \left[\frac{\partial\mathscr{L}_c}{\partial(\nabla_\sigma Q^A)}\, Q^A\Big|^{\rho}_{\tau} + \frac{\partial\mathscr{L}_c}{\partial(\nabla_{\sigma\pi} Q^A)}\,\left(\nabla_\pi Q^A\right)\Big|^{\rho}_{\tau}\right. \\ &\quad \left. - \nabla_\pi\left(\frac{\partial\mathscr{L}_c}{\partial(\nabla_{\pi\sigma} Q^A)}\right) Q^A\Big|^{\rho}_{\tau}\right]\delta\Gamma^{\tau}{}_{\sigma\rho} \,. \end{aligned} \quad \text{(A.3.41)}$$

This evidently is covariant.

Thus, (A.3.38) can be rewritten up to a divergence in the form

$$\delta\mathscr{L}_c = \frac{\partial\mathscr{L}_c}{\partial g_{\mu\nu}}\delta g_{\mu\nu} + \frac{\delta\mathscr{L}_c}{\delta\Gamma^{\alpha}{}_{\mu\nu}}\delta\Gamma^{\alpha}{}_{\mu\nu} + \frac{\partial\mathscr{L}_c}{\partial R^{\alpha}{}_{\beta\mu\nu}}\delta R^{\alpha}{}_{\beta\mu\nu} \,. \quad \text{(A.3.42)}$$

Now, we replace (A.3.39), (A.3.40) in (A.3.42) and single out a total divergence again. It yields

$$\delta\mathscr{L}_c = \frac{\delta\mathscr{L}_c}{\delta g_{\mu\nu}}\delta g_{\mu\nu} \,, \quad \text{(A.3.43)}$$

and the covariant Lagrangian derivative of the Lagrangian with respect to the metric tensor reads

$$\begin{aligned} \frac{\delta\mathscr{L}_c}{\delta g_{\mu\nu}} &= \frac{\partial\mathscr{L}_c}{\partial g_{\mu\nu}} - \frac{1}{2}\nabla_\alpha\left(g^{\sigma\mu}\frac{\delta\mathscr{L}_c}{\delta\Gamma^{\sigma}{}_{\nu\alpha}} + g^{\sigma\nu}\frac{\delta\mathscr{L}_c}{\delta\Gamma^{\sigma}{}_{\mu\alpha}} - g^{\sigma\alpha}\frac{\delta\mathscr{L}_c}{\delta\Gamma^{\sigma}{}_{\mu\nu}}\right) \\ &\quad + \nabla_{\alpha\beta}\left(g^{\sigma\mu}\frac{\partial\mathscr{L}_c}{\partial R^{\sigma}{}_{\alpha\beta\nu}} + g^{\sigma\nu}\frac{\partial\mathscr{L}_c}{\partial R^{\sigma}{}_{\mu\beta\alpha}} - g^{\sigma\alpha}\frac{\partial\mathscr{L}_c}{\partial R^{\sigma}{}_{\mu\beta\nu}}\right) . \end{aligned} \quad \text{(A.3.44)}$$

Appendix B: Retarded functions

B.1 Lorentz invariance of retarded potentials

We use a prime in the appendices exclusively as a label for time and spatial coordinates which are used as variables of integration in volume integrals (see, for example, equations (B.1.2), (B.1.3), and so on). It should not be confused with the time derivative with respect to the conformal time used in the main text of the Chapter 5.

Let us consider an inhomogeneous wave equation for a scalar field, $V = V(\eta, \boldsymbol{X})$, written down in a coordinate chart $X^\alpha = (X^0, X^i) = (\eta, \boldsymbol{X})$,

$$\Box V = -4\pi\sigma_X \ , \tag{B.1.1}$$

where $\Box \equiv \eta^{\alpha\beta}\partial_{\alpha\beta}$, $\partial_\alpha = \partial/\partial X^\alpha$, and $\sigma_X = \sigma_X(\eta, \boldsymbol{X})$ is the source (a scalar function) of the field V with a compact support (bounded by a finite volume in space). Equation (B.1.1) has a solution given as a linear combination of advanced and retarded potentials. Let us focus only on the retarded potential which is more common in physical applications. Advanced potential can be treated similarly.

We assume the field, V, and its first derivatives vanish at past null infinity. Then, the retarded solution (retarded potential) of (B.1.1) is given by an integral,

$$V(\eta, \boldsymbol{X}) = \int_{\mathcal{V}} \frac{\sigma_X(\zeta, \boldsymbol{X}')\, d^3X'}{|\boldsymbol{X} - \boldsymbol{X}'|} \ , \tag{B.1.2}$$

where

$$\zeta = \eta - |\boldsymbol{X} - \boldsymbol{X}'| \ , \tag{B.1.3}$$

is the retarded time, and we assume the fundamental speed $c = 1$. Physical meaning of the retardation is that the field V propagates in spacetime with the fundamental speed c from the source σ_X, to the point with coordinates $X^\alpha = (\eta, \boldsymbol{X})$ where the field V is measured in correspondence with equation (B.1.2). Left side of equation (B.1.1) is Lorentz-invariant. Hence, we expect that solution (B.1.3) must be Lorentz-invariant as well. As a rule, textbooks prove this statement for a particular case of the retarded (Liénard–Wiechert) potential of a moving point-like source but not for the retarded potential given in the form of the integral (B.1.2). This appendix fulfils this gap.

Lorentz transformation to coordinates, $x^\alpha = (t, \boldsymbol{x})$ linearly transforms the isotropic coordinates $X^\alpha = (\eta, \boldsymbol{X})$ of the FLRW metric as follows

$$x^\alpha = \Lambda^\alpha{}_\beta X^\beta \ , \tag{B.1.4}$$

where the matrix of the Lorentz boost [315]

$$\Lambda^0{}_0 = \gamma\ , \qquad \Lambda^i{}_0 = \Lambda^0{}_i = -\gamma\beta^i\ , \qquad \Lambda^i{}_j = \delta^{ij} + \frac{\gamma - 1}{\beta^2}\beta^i\beta^j\ , \tag{B.1.5}$$

the boost four-velocity $u^\alpha = \{u^0, u^i\} = u^0\{1, \beta^i\}$ is constant, and

$$\gamma = u^0 = \frac{1}{\sqrt{1-\beta^2}}\ , \tag{B.1.6}$$

is the constant Lorentz-factor.

The inverse Lorentz transformation is given explicitly as follows

$$\eta = \gamma(t + \boldsymbol{\beta}\cdot\boldsymbol{x})\ , \tag{B.1.7}$$

$$\boldsymbol{X} = \boldsymbol{r} + \frac{\gamma^2}{1+\gamma}(\boldsymbol{\beta}\cdot\boldsymbol{r})\boldsymbol{b}\ , \tag{B.1.8}$$

where

$$\boldsymbol{r} = \boldsymbol{x} + \boldsymbol{\beta}t\ , \tag{B.1.9}$$

and the boost three-velocity, $\boldsymbol{\beta} = \{\beta^i\} = \{u^i/u^0\}$.

Let us reiterate (B.1.2) by introducing a one-dimensional Dirac's delta function and integration with respect to time η,

$$V(\eta, \boldsymbol{X}) = \int_{-\infty}^{\infty}\int_{\mathcal{V}} \frac{\sigma_X(\eta', \boldsymbol{X}')\delta(\eta' - \zeta)\, d\eta' d^3X'}{|\boldsymbol{X} - \boldsymbol{X}'|}\ , \tag{B.1.10}$$

where ζ is the retarded time given by (B.1.3). Then, we transform coordinates $X'^\alpha = (\eta', \boldsymbol{X}')$ to $x'^\alpha = (t', \boldsymbol{x}')$ with the Lorentz boost (B.1.4). The Lorentz transformation changes functions entering the integrand of (B.1.10) as follows,

$$\sigma(\eta', \boldsymbol{X}') = \sigma_X(t', \boldsymbol{x}')\ , \tag{B.1.11}$$

$$|\boldsymbol{X} - \boldsymbol{X}'| = \sqrt{|\boldsymbol{r} - \boldsymbol{r}'|^2 + \gamma^2[\beta\cdot(\boldsymbol{r} - \boldsymbol{r}')]^2}\ , \tag{B.1.12}$$

where the coordinate difference

$$\boldsymbol{r} - \boldsymbol{r}' = \boldsymbol{x} - \boldsymbol{x}' + \boldsymbol{\beta}(t - t')\ . \tag{B.1.13}$$

The coordinate volume of integration remains Lorentz-invariant

$$d\eta' d^3X' = dt' d^3x'\ . \tag{B.1.14}$$

Let us denote $F_\eta(\eta') \equiv \eta' - \zeta$ where ζ is given by (B.1.3). After making the Lorentz transformation this function changes to

$$F_\eta(\eta') = F_t(t') = \gamma\left[t' - t - \boldsymbol{\beta}\cdot\left(\boldsymbol{x}-\boldsymbol{x}'\right)\right] + \sqrt{|\boldsymbol{x}-\boldsymbol{x}'|^2 - (t'-t)^2 + \gamma^2[\boldsymbol{\beta}\cdot(\boldsymbol{x}-\boldsymbol{x}') - (t'-t)]^2}\,, \tag{B.1.15}$$

where we have used equations (B.1.7), (B.1.8) and (B.1.12) and relationship $\gamma^2\beta^2 = \gamma^2 - 1$, to perform the transformation. Integral (B.1.10) in coordinates x^α becomes

$$V(t,\boldsymbol{x}) = \int_{-\infty}^{\infty}\int_{\mathcal{V}} \frac{\sigma_X(t',\boldsymbol{x}')\delta(F_t(t'))\,dt'd^3x'}{\sqrt{|\boldsymbol{r}-\boldsymbol{r}'|^2 + \gamma^2[\boldsymbol{\beta}\cdot(\boldsymbol{r}-\boldsymbol{r}')]^2}}\,, \tag{B.1.16}$$

The delta function has a complicated argument $F_t(t')$ in coordinates x^α. It can be simplified with a well-known formula

$$\delta\left[F_t(t')\right] = \frac{\delta(t'-s)}{\dot{F}_t(s)}\,, \tag{B.1.17}$$

where $\dot{F}_t(s) \equiv \left[dF_t(t')/dt'\right]_{t'=s}$, and s is one of the roots of equation $F_t(t') = 0$ that is associated with the retarded interaction. It is straightforward to confirm by inspection that the root is given by formula,

$$s = t - |\boldsymbol{x}-\boldsymbol{x}'|\,. \tag{B.1.18}$$

The time derivative of function $F_t(t')$ is

$$\dot{F}_t(t') = \gamma + \gamma^2\frac{\beta^2(t'-t) - \boldsymbol{\beta}\cdot(\boldsymbol{x}-\boldsymbol{x}')}{\sqrt{|\boldsymbol{x}-\boldsymbol{x}'|^2 - (t'-t)^2 + \gamma^2[\boldsymbol{\beta}\cdot(\boldsymbol{x}-\boldsymbol{x}') - (t'-t)]^2}}\,. \tag{B.1.19}$$

After substituting $t' = s$, with s taken from equation (B.1.18), we obtain,

$$\dot{F}_t(s) = \frac{1}{\gamma}\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{|\boldsymbol{x}-\boldsymbol{x}'| + \boldsymbol{\beta}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\,. \tag{B.1.20}$$

Performing now integration with respect to t' in equation (B.1.16) with the help of the delta-function, we arrive to

$$V(t,\boldsymbol{x}) = \int_{\mathcal{V}} \frac{\sigma_X(s,\boldsymbol{x}')\,d^3x'}{\dot{F}_t(s)|\boldsymbol{X}-\boldsymbol{X}'|_{t'=s}}\,, \tag{B.1.21}$$

where $|\boldsymbol{X}-\boldsymbol{X}'|_{t'=s}$ must be calculated from (B.1.12) with $t' = s$ where s is taken from (B.1.18). It yields

$$\dot{F}_t(s)|\boldsymbol{X}-\boldsymbol{X}'|_{t'=s} = |\boldsymbol{x}-\boldsymbol{x}'| \,, \tag{B.1.22}$$

and proves that the retarded potential (B.1.2) is Lorentz-invariant

$$\int_{\mathscr{V}} \frac{\sigma_X(\zeta,\boldsymbol{X}')\,d^3X'}{|\boldsymbol{X}-\boldsymbol{X}'|} = \int_{\mathscr{V}} \frac{\sigma_X(s,\boldsymbol{x}')\,d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|} \,. \tag{B.1.23}$$

We have verified the Lorentz invariance for the *scalar* retarded potential. However, it is not difficult to check that it is valid in case of a source $\sigma_{\alpha_1\alpha_2\dots\alpha_l}$ that is a tensor field of rank l. Indeed, the Lorentz transformation of the source leads to $\Lambda^{\beta_1}{}_{\alpha_1}\Lambda^{\beta_2}{}_{\alpha_2}\dots\Lambda^{\beta_l}{}_{\alpha_l}\sigma_{\beta_1\beta_2\dots\beta_l}$ but the matrix $\Lambda^{\alpha}{}_{\beta}$ is constant, and can be taken out of the sign of the retarded integral. Because of this property, all mathematical operations given in this appendix for a scalar retarded potential, remain the same for the tensor of any rank. Hence, the Lorentz invariance of the retarded integral is a general property of the wave operator in the Minkowski space.

B.2 Retarded solution of the sound-wave equation

Let us consider an inhomogeneous sound-wave equation for a scalar function $\mathscr{U} = \mathscr{U}(\eta,\boldsymbol{X})$ describing a perturbation propagation in a medium. This equation written down in the isotropic coordinates $X^\alpha = (\eta,\boldsymbol{X})$, reads

$$\Box_s \mathscr{U} = -4\pi\tau_X \,, \tag{B.2.1}$$

where $\tau_X = \tau_X(\eta,\boldsymbol{X})$ is the source of $\mathscr{U}$ having a compact support, and the sound-wave differential operator $\Box_s$ was defined in (5.7.17). It is Lorentz-invariant and reads

$$\Box_s = \Box + \left(1-\frac{c^2}{c_s^2}\right)\bar{\mathrm{v}}^\alpha\bar{\mathrm{v}}^\beta\partial_{\alpha\beta} \,, \tag{B.2.2}$$

where $\bar{\mathrm{v}}^\alpha$ is four-velocity of motion of the medium with respect to the coordinate chart, c_s is the constant speed of sound in the medium, and we keep the fundamental speed c in the definition of the operator for dimensional purposes. We assume that $c_s < c$. The case of $c_s = c$ is treated in Section B.1, and the case of $c_s \geq c$ makes a formal mathematical sense in discussion of the speed of propagation of gravity in alternative theories of gravity since the equation describing propagation of gravitational potential $\mathscr{U}$ has the same structure as (B.2.1) after formal replacement of c_s with the speed of gravity c_g [270, 462]. In particular, in the Newtonian theory the speed of gravity $c_g = \infty$, and the operator (B.2.2) is reduced to the Laplace operator

$$\Delta = \Box + \mathrm{v}^\alpha\mathrm{v}^\beta\partial_\alpha\partial_\beta = \bar{\pi}^{\alpha\beta}\partial_{\alpha\beta} \,, \tag{B.2.3}$$

where the constant projection operator, $\bar{\pi}^{\alpha\beta}$, has been defined in (5.6.23).

We are looking for the solution of (B.2.1) in the Cartesian coordinates $x^\alpha = (t, \boldsymbol{x})$ moving with respect to the isotropic coordinates X^α with constant velocity β^i. Transformation from X^α to x^α is given by the Lorentz transformation (B.1.4). In coordinates X^α the four-velocity $\bar{v}^\alpha = (1, 0, 0, 0)$. Therefore, in these coordinates, equation (B.2.1) is just a wave equation for the field $\mathscr{U}$ propagating with speed c_s. It has a well-known retarded solution,

$$\mathscr{U}(\eta, \boldsymbol{X}) = \int_{\mathscr{V}} \frac{\tau_X(\eta_s, \boldsymbol{X}')\, d^3X'}{|\boldsymbol{X} - \boldsymbol{X}'|} , \tag{B.2.4}$$

where

$$\eta_s = \eta - \frac{c}{c_s}|\boldsymbol{X} - \boldsymbol{X}'| , \tag{B.2.5}$$

is the retarded time.

Equation (B.2.1) is Lorentz-invariant. Hence, its solution must be Lorentz-invariant as well. Our goal is to prove this statement. To this end, we take solution (B.2.4) and perform the Lorentz transformation (B.1.7), (B.1.8). We recast the retarded integral (B.2.4) to another form with the help of one-dimensional delta-function

$$\mathscr{U}(\eta, \boldsymbol{X}) = \int_{\infty}^{\infty} \int_{\mathscr{V}} \frac{\tau_X(\eta', \boldsymbol{X}')\delta(\eta' - \eta_s)\, d\eta' d^3X'}{|\boldsymbol{X} - \boldsymbol{X}'|} . \tag{B.2.6}$$

It looks similar to (B.1.2) but one has to remember that the retarded time η_s differs from ζ that was defined in (B.1.3) on the characteristics of the null cone defined by the fundamental speed c. Transformation of functions entering integrand in (B.2.6) is similar to what we did in Section B.1 but, because $c_s \neq c$, calculations become more involved. It turns out more preferable to handle the calculations in tensor notations, making transition to the coordinate language only at the end of the transformation procedure.

Let us consider two events with the isotropic coordinates $X^\alpha = (\eta, \boldsymbol{X})$ and $X'^\alpha = (\eta', \boldsymbol{X}')$. We postulate that in the coordinate chart, x^α, these two events have coordinates, $x^\alpha = (t, \boldsymbol{x})$, and, $x'^\alpha = (t', \boldsymbol{x}')$, respectively. We define the components of a four-vector, $r^\alpha = (t' - t, \boldsymbol{x} - \boldsymbol{x}')$ which is convenient for doing mathematical manipulations with the Lorentz transformations. For instance, the Lorentz transformation of the Euclidean distance between the spatial coordinates of the two events, is given by a

$$|\boldsymbol{X} - \boldsymbol{X}'| = \sqrt{\bar{\pi}_{\alpha\beta} r^\alpha r^\beta} , \tag{B.2.7}$$

where $\bar{\pi}^{\alpha\beta}$ is the operator of projection on the hyperplane being orthogonal to $\bar{v}^\alpha$ (the same operator as in (B.2.3)). Equation (B.2.7) is a Lorentz-invariant analogue of expression (B.1.12) and matches it exactly. Transformation of the source, $\tau_X(X^\alpha) = \tau_x(x^\alpha)$ is fully equivalent to that of σ_X as given by equation (B.1.11). Coordinate volume of integration transforms in accordance with (B.1.14). We need to transform the argument, $\eta' - \eta_s$, of delta-function which we shall denote in coordinates X^α as $f_\eta(\eta') \equiv \eta' - \eta_s$. The argument is a scalar function which is transformed as $f_\eta(\eta') = f_t(t')$ where,

$$f_t(t') = -\bar{v}_\alpha r^\alpha + \frac{c}{c_s}\sqrt{\bar{\pi}_{\alpha\beta} r^\alpha r^\beta} \,. \tag{B.2.8}$$

Transformation of the delta-function in the integrand of integral (B.2.6) is

$$\delta\left[f_t(t')\right] = \frac{\delta(t' - \varsigma)}{\dot{f}_t(\varsigma)} \,, \tag{B.2.9}$$

where $\dot{f}_t(\varsigma) \equiv \left[df_t(t')/dt'\right]_{t'=\varsigma}$, and ς is one of the roots of equation $f_t(t') = 0$ that is associated with the retarded interaction. Eventually, after accounting for transformation of all functions and performing integration with respect to time, integral (B.2.6) assumes the following form

$$\mathscr{U}(t, \boldsymbol{x}) = \int_{\mathscr{V}} \frac{\tau_x(\varsigma, \boldsymbol{x}')d^3x'}{\dot{f}_t(\varsigma)|\boldsymbol{X} - \boldsymbol{X}'|_{t'=\varsigma}} \,, \tag{B.2.10}$$

where $|\boldsymbol{X} - \boldsymbol{X}'|_{t'=\varsigma}$ denotes the expression (B.2.7) taken at the value of $t' = \varsigma$. What remains is to calculate the instant of time, ς, and the value of functions entering denominator of the integrand in (B.2.10).

Calculation of ς is performed by solving equation $f_t(\varsigma) = 0$, that defines the characteristic cone of the sound waves, and has the following explicit form,

$$\left[\eta_{\alpha\beta} + \left(1 - \frac{c_s^2}{c^2}\right)\bar{v}_\alpha \bar{v}_\beta\right] r^\alpha r^\beta = 0 \,, \tag{B.2.11}$$

which is derived from (B.2.8). This is a quadratic algebraic equation with respect to the time variable $r^0 = \varsigma - t$. It reads

$$A(\varsigma - t)^2 + 2B(\varsigma - t) + C = 0 \,, \tag{B.2.12}$$

where the coefficients A, B, C of the quadratic form are,

$$A = -1 + \left(1 - \frac{c_s^2}{c^2}\right)\gamma^2 \,, \tag{B.2.13}$$

$$B = -\left(1 - \frac{c_s^2}{c^2}\right)\gamma^2 \beta \cdot (\boldsymbol{x} - \boldsymbol{x}') \,, \tag{B.2.14}$$

$$C = |\boldsymbol{x} - \boldsymbol{x}'|^2 + \left(1 - \frac{c_s^2}{c^2}\right)\gamma^2 \left[\beta \cdot (\boldsymbol{x} - \boldsymbol{x}')\right]^2 \,, \tag{B.2.15}$$

and $\gamma = 1/\sqrt{1 - \beta^2}$ is the Lorentz factor. Equation (B.2.12) has two roots corresponding to the advanced and retarded times. The root corresponding to the retarded-time solution of (B.2.12) is

$$\varsigma = t - \frac{1}{A}\left(B - \sqrt{B^2 - AC}\right) , \tag{B.2.16}$$

or, more explicitly,

$$\varsigma = t - |\boldsymbol{x} - \boldsymbol{x}'| \frac{\left(1 - \frac{c_s^2}{c^2}\right)\gamma^2\,(\beta\cdot\boldsymbol{n}) + \sqrt{1 - \left(1 - \frac{c_s^2}{c^2}\right)\gamma^2\left[1 - (\beta\cdot\boldsymbol{n})^2\right]}}{1 - \left(1 - \frac{c_s^2}{c^2}\right)\gamma^2} , \tag{B.2.17}$$

where the unit vector $\boldsymbol{n} = (\boldsymbol{x} - \boldsymbol{x}')/|\boldsymbol{x} - \boldsymbol{x}'|$. After some algebra equation (B.2.17) can be simplified to

$$\varsigma = t - \frac{\alpha_s}{c_s}|\boldsymbol{x} - \boldsymbol{x}'| , \tag{B.2.18}$$

where

$$\alpha_s = \frac{1 - \beta^2}{1 - \frac{\beta^2}{c_s^2}}\left[\sqrt{1 + \left(1 - \frac{c^2}{c_s^2}\right)\gamma^2(\beta\times\boldsymbol{n})^2} - \left(1 - \frac{c^2}{c_s^2}\right)\gamma^2(\beta\cdot\boldsymbol{n})\right] . \tag{B.2.19}$$

Coefficient α_s defines the speed of propagation of the sound waves, $v_s \equiv c_s/\alpha_s$, as measured by observer moving with speed β^i with respect to the Hubble flow. Thus, the value of the speed of sound, v_s, depends crucially on the motion of observer.

Derivative of the function, $\dot{f}_t(\varsigma)$, is given by

$$\dot{f}_t(\varsigma) = \frac{\partial f_t}{\partial r^\alpha}\frac{\partial r^\alpha}{\partial \varsigma} , \tag{B.2.20}$$

where the partial derivative $\partial r^\alpha/\partial\varsigma = \delta^\alpha_0 = (1, 0, 0, 0)$. Making use of (B.2.8), the partial derivative

$$\frac{\partial f_x}{\partial r^\alpha} = -\bar{\nabla}_\alpha + \frac{c}{c_s}\frac{\bar{\pi}_{\alpha\beta}r^\beta}{\sqrt{\bar{\pi}_{\alpha\beta}r^\alpha r^\beta}} , \tag{B.2.21}$$

which has to be calculated at the instant of time, $t' = \varsigma$, where ς is given by (B.2.18).

In order to calculate the denominator in the integrand in (B.2.10), we account for (B.2.7), (B.2.11) and combine (B.2.20), (B.2.21) together. We get

$$|\boldsymbol{X} - \boldsymbol{X}'|\dot{f}_x(\varsigma) = \frac{c}{c_s}\left[r_\alpha + \left(1 - \frac{c_s^2}{c^2}\right)\bar{\nabla}_\alpha\bar{\nabla}_\beta r^\beta\right]\delta^\alpha_0 . \tag{B.2.22}$$

It is straightforward to check that after using (B.2.16) the above equation is reduced to $|\boldsymbol{X} - \boldsymbol{X}'|\dot{f}_x(\varsigma) = (c/c_s)\sqrt{B^2 - AC}$, or more explicitly,

$$|\boldsymbol{X} - \boldsymbol{X}'|\dot{f}_x(\varsigma) = |\boldsymbol{x} - \boldsymbol{x}'|\sqrt{1 + \left(1 - \frac{c^2}{v_s^2}\right)\gamma^2(\beta \times \boldsymbol{n})^2}\,, \tag{B.2.23}$$

Finally, the retarded Lorentz-invariant solution of (B.2.1) is

$$\mathscr{U}(t, \boldsymbol{x}) = \int_{\mathscr{V}} \frac{\tau_x(\varsigma, \boldsymbol{x}')}{\sqrt{1 + \gamma^2\left(1 - \frac{c^2}{c_s^2}\right)(\beta \times \boldsymbol{n})^2}} \frac{d^3x'}{|\boldsymbol{x} - \boldsymbol{x}'|}\,, \tag{B.2.24}$$

with the retarded time ς calculated in accordance with (B.2.18). This solution reduces to the retarded potential (B.1.23) in the limit of $c_s \to c$.

Appendix C: Auxiliary expressions in EGB gravity

Here, we derive necessary coefficients to construct currents and superpotentials in the framework of the EGB gravity in Chapter 8. Let us derive the Lagrangian defined in (8.1.2) and (8.1.1):

$$\mathscr{L}^G = \mathscr{L}_{EGB} = \sqrt{-g}\left[R - 2\Lambda_0 + \alpha\left(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2\right)\right] . \tag{C.1}$$

To construct the necessary coefficients $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{u}$ in (7.2.9), (7.2.10) and (7.2.12), respectively, or related starred coefficients, one has to covariantize $\mathscr{L}_{EGB}$, like in (7.2.7), (7.2.8),

$$\mathscr{L}_c = -\frac{1}{2\kappa}\mathscr{L}^c_{EGB} . \tag{C.2}$$

We repeat necessary for covariantization expressions, which are hold in all dimensions, not only in 4 dimensions:

$$\Delta^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \bar{\Gamma}^\alpha_{\mu\nu} = \tfrac{1}{2}g^{\alpha\rho}\left(\bar{\nabla}_\mu g_{\rho\nu} + \bar{\nabla}_\nu g_{\rho\mu} - \bar{\nabla}_\rho g_{\mu\nu}\right) , \tag{C.3}$$

$$\bar{\nabla}_\rho g_{\mu\nu} = g_{\tau\mu}\Delta^\tau_{\rho\nu} + g_{\tau\nu}\Delta^\tau_{\rho\mu} , \tag{C.4}$$

$$R^\lambda{}_{\tau\rho\sigma} = \bar{\nabla}_\rho\Delta^\lambda_{\tau\sigma} - \bar{\nabla}_\sigma\Delta^\lambda_{\tau\rho} + \Delta^\lambda_{\rho\eta}\Delta^\eta_{\tau\sigma} - \Delta^\lambda_{\eta\sigma}\Delta^\eta_{\tau\rho} + \bar{R}^\lambda{}_{\tau\rho\sigma} . \tag{C.5}$$

To demonstrate possibilities of the approach we suggest to calculate the starred coefficients $\boldsymbol{n}^*$ and $\boldsymbol{m}^*$ instead of (7.2.9) and (7.2.10),

$$\boldsymbol{n}^{*\,\alpha\tau\beta}_\sigma = -\frac{1}{4\kappa}\left[\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_{\beta\alpha}g_{\mu\nu})}\, g_{\mu\nu}\Big|^\tau_\sigma + \frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_{\tau\alpha}g_{\mu\nu})}\, g_{\mu\nu}\Big|^\beta_\sigma\right] , \tag{C.6}$$

$$\begin{aligned}\boldsymbol{m}^{*\,\alpha\tau}_\sigma = -\frac{1}{2\kappa}\Bigg[&\left(\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_\alpha g_{\mu\nu})} - \bar{\nabla}_\beta\frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_{\alpha\beta}g_{\mu\nu})}\right) g_{\mu\nu}\Big|^\tau_\sigma \\ &- \frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_{\tau\alpha}g_{\mu\nu})}\bar{\nabla}_\sigma g_{\mu\nu} + \frac{\partial\mathscr{L}^c_{EGB}}{\partial(\bar{\nabla}_{\beta\alpha}g_{\mu\nu})}\bar{\nabla}_\beta\big(g_{\mu\nu}\big|^\tau_\sigma\big)\Bigg] ,\end{aligned} \tag{C.7}$$

see (7.1.36) and (7.1.37). The starred coefficient $\boldsymbol{u}^*$ instead of (7.2.12) is

$$\boldsymbol{u}^{*\,\alpha}_\sigma = -\left[\frac{1}{\kappa}\mathscr{E}_\sigma{}^\alpha + \mathscr{C}^{*\,\alpha}_\sigma + \boldsymbol{n}^{*\,\alpha\tau\beta}_\lambda\bar{R}^\lambda{}_{\tau\beta\sigma}\right] , \tag{C.8}$$

see (7.1.38), where

$$\mathscr{C}^{*\alpha}_{\sigma} \equiv -\frac{1}{2\kappa}\left[\frac{\partial \mathscr{L}^{c}_{EGB}}{\partial(\bar{\nabla}_{\beta\alpha}g_{\mu\nu})}\bar{\nabla}_{\beta\sigma}g_{\mu\nu} + \left(\frac{\partial \mathscr{L}^{c}_{EGB}}{\partial(\bar{\nabla}_{\alpha}g_{\mu\nu})} - \bar{\nabla}_{\beta}\frac{\partial \mathscr{L}^{c}_{EGB}}{\partial(\bar{\nabla}_{\alpha\beta}g_{\mu\nu})}\right)\bar{\nabla}_{\sigma}g_{\mu\nu} - \delta^{\alpha}_{\sigma}\mathscr{L}^{c}_{EGB}\right]. \tag{C.9}$$

To derive the above coefficients it is useful to present the next derivatives:

$$\begin{aligned}
\frac{\partial \mathscr{L}^{c}_{EGB}}{\partial g_{\mu\nu}} &= \frac{\partial \mathscr{L}^{c}_{E}}{\partial g_{\mu\nu}} + \frac{\partial \mathscr{L}^{c}_{GB}}{\partial g_{\mu\nu}} \\
&= -\sqrt{-g}\left[R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}(R - 2\Lambda_0)\right] \\
&\quad +2\sqrt{-g}\left[g^{\rho(\mu}\Delta^{\nu)}_{\alpha[\sigma}\Delta^{\alpha}_{\rho]\tau} + g^{\alpha\rho}\Delta^{(\mu}_{\tau[\sigma}\Delta^{\nu)}_{\rho]\alpha} + g^{\rho(\mu}\bar{\nabla}_{[\sigma}\Delta^{\nu)}_{\rho]\tau}\right]g^{\tau\sigma} \\
&\quad -2\alpha\sqrt{-g}\Big[R^{\mu\tau\rho\sigma}R^{\nu}{}_{\tau\rho\sigma} - 4R^{\mu\rho}R^{\nu}_{\rho} + RR^{\mu\nu} \\
&\quad -\frac{1}{4}g^{\mu\nu}\left(R_{\lambda\tau\rho\sigma}R^{\lambda\tau\rho\sigma} - 4R_{\rho\sigma}R^{\rho\sigma} + R^2\right)\Big] \\
&\quad +4\alpha\sqrt{-g}\left[g^{\lambda(\mu}\Delta^{\nu)}_{\alpha\sigma}\Delta^{\alpha}_{\rho\tau} + g^{\alpha\lambda}\Delta^{(\mu}_{\tau\sigma}\Delta^{\nu)}_{\rho\alpha} + g^{\lambda(\mu}\bar{\nabla}_{\sigma}\Delta^{\nu)}_{\rho\tau}\right]R_{\lambda}{}^{\tau\rho\sigma} \\
&\quad -16\alpha\sqrt{-g}\left[g^{\rho(\mu}\Delta^{\nu)}_{\alpha[\sigma}\Delta^{\alpha}_{\rho]\tau} + g^{\alpha\rho}\Delta^{(\mu}_{\tau[\sigma}\Delta^{\nu)}_{\rho]\alpha} + g^{\rho(\mu}\bar{\nabla}_{[\sigma}\Delta^{\nu)}_{\rho]\tau}\right]R^{\tau\sigma} \\
&\quad +4\alpha\sqrt{-g}\left[g^{\rho(\mu}\Delta^{\nu)}_{\alpha[\sigma}\Delta^{\alpha}_{\rho]\tau} + g^{\alpha\rho}\Delta^{(\mu}_{\tau[\sigma}\Delta^{\nu)}_{\rho]\alpha} + g^{\rho(\mu}\bar{\nabla}_{[\sigma}\Delta^{\nu)}_{\rho]\tau}\right]Rg^{\tau\sigma}\,;
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
\frac{\partial \mathscr{L}^{c}_{EGB}}{\partial(\bar{\nabla}_{\alpha}g_{\mu\nu})} &= \frac{\partial \mathscr{L}^{c}_{E}}{\partial(\bar{\nabla}_{\alpha}g_{\mu\nu})} + \frac{\partial \mathscr{L}^{c}_{GB}}{\partial(\bar{\nabla}_{\alpha}g_{\mu\nu})} \\
&= 2\sqrt{-g}\left[\Delta^{\alpha}_{\sigma\rho}g^{\sigma[\rho}g^{\mu]\nu} + g^{\alpha\sigma}\Delta^{(\mu}_{\sigma\rho}g^{\nu)\rho} - g^{\alpha(\mu}\Delta^{\nu)}_{\sigma\rho}g^{\sigma\rho}\right] \\
&\quad +4\alpha\sqrt{-g}\left[2R^{\alpha\sigma\rho(\mu}\Delta^{\nu)}_{\sigma\rho} - \Delta^{\alpha}_{\sigma\rho}R^{\sigma\mu\nu\rho}\right] \\
&\quad -4\alpha\sqrt{-g}\Big[2R^{\alpha\sigma}\Delta^{(\mu}_{\sigma\rho}g^{\nu)\rho} - 2g^{\alpha(\mu}\Delta^{\nu)}_{\sigma\rho}R^{\sigma\rho} + 2g^{\alpha\sigma}\Delta^{(\mu}_{\sigma\rho}R^{\nu)\rho} \\
&\quad - 2R^{\alpha(\mu}\Delta^{\nu)}_{\sigma\rho}g^{\sigma\rho} + \Delta^{\alpha}_{\sigma\rho}R^{\sigma\rho}g^{\mu\nu} + \Delta^{\alpha}_{\sigma\rho}g^{\sigma\rho}R^{\mu\nu} - 2\Delta^{\alpha}_{\sigma\rho}R^{\sigma(\mu}g^{\nu)\rho}\Big] \\
&\quad +4\alpha\sqrt{-g}R\left[\Delta^{\alpha}_{\sigma\rho}g^{\sigma[\rho}g^{\mu]\nu} + g^{\alpha\sigma}\Delta^{(\mu}_{\sigma\rho}g^{\nu)\rho} - g^{\alpha(\mu}\Delta^{\nu)}_{\sigma\rho}g^{\sigma\rho}\right]\,;
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
\frac{\partial \mathscr{L}^{c}_{EGB}}{\partial(\bar{\nabla}_{\beta\alpha}g_{\mu\nu})} &= \frac{\partial \mathscr{L}^{c}_{E}}{\partial(\bar{\nabla}_{\beta\alpha}g_{\mu\nu})} + \frac{\partial \mathscr{L}^{c}_{GB}}{\partial(\bar{\nabla}_{\beta\alpha}g_{\mu\nu})} \\
&= \sqrt{-g}\left[g^{\alpha(\mu}g^{\nu)\beta} - g^{\alpha\beta}g^{\mu\nu}\right] \\
&\quad +2\alpha\sqrt{-g}\Big[2R^{\alpha(\mu\nu)\beta} - 4R^{\alpha(\mu}g^{\nu)\beta} + 2g^{\mu\nu}R^{\alpha\beta} + 2g^{\alpha\beta}R^{\mu\nu} \\
&\quad + R\left(g^{\alpha(\mu}g^{\nu)\beta} - g^{\alpha\beta}g^{\mu\nu}\right)\Big]\,.
\end{aligned} \tag{C.12}$$

Substituting (C.11) and (C.12) into (C.6) and (C.7) one obtains

$$\begin{aligned}
\boldsymbol{n}^{*\lambda\alpha\beta}_{\sigma} &= {}_{E}\boldsymbol{n}^{*\lambda\alpha\beta}_{\sigma} + {}_{GB}\boldsymbol{n}^{*\lambda\alpha\beta}_{\sigma} \\
&= \frac{\sqrt{-g}}{2\kappa}\left[g^{\alpha\beta}\delta^{\lambda}_{\sigma} - g^{\lambda(\alpha}\delta^{\beta)}_{\sigma}\right] \\
&+\frac{\alpha\sqrt{-g}}{\kappa}\left[-2R_{\sigma}{}^{(\alpha\beta)\lambda} - 4R^{\lambda}_{\sigma}g^{\alpha\beta} + 4R^{(\alpha}_{\sigma}g^{\beta)\lambda} + R\left(g^{\alpha\beta}\delta^{\lambda}_{\sigma} - g^{\lambda(\alpha}\delta^{\beta)}_{\sigma}\right)\right] ;
\end{aligned} \tag{C.13}$$

$$\begin{aligned}
\boldsymbol{m}^{*\alpha\beta}_{\sigma} &= {}_{E}\boldsymbol{m}^{*\alpha\beta}_{\sigma} + {}_{GB}\boldsymbol{m}^{*\alpha\beta}_{\sigma} \\
&= -\frac{\sqrt{-g}}{2\kappa}\left[\delta^{\alpha}_{\sigma}\Delta^{\beta}_{\rho\tau}g^{\rho\tau} - 2\Delta^{\alpha}_{\sigma\rho}g^{\beta\rho} + \Delta^{\rho}_{\rho\sigma}g^{\alpha\beta}\right] \\
&+\frac{2\alpha\sqrt{-g}}{\kappa}\left[R^{\alpha\tau\rho}{}_{\sigma}\Delta^{\beta}_{\tau\rho} - 2R^{\alpha(\tau\beta)}{}_{\rho}\Delta^{\rho}_{\tau\sigma}\right] \\
&+\frac{4\alpha\sqrt{-g}}{\kappa}\Big[4g^{\rho[\alpha}R^{\beta]}_{\tau}\Delta^{\tau}_{\rho\sigma} + 2R^{[\alpha}_{\sigma}g^{\tau]\rho}\Delta^{\beta}_{\tau\rho} + 2g^{\alpha[\beta}R^{\tau]}_{\rho}\Delta^{\rho}_{\tau\sigma} \\
&- g^{\tau\beta}\left(\bar{\nabla}_{(\tau}R^{\alpha}_{\sigma)} + R^{\rho}_{(\tau}\Delta^{\alpha}_{\sigma)\rho} - R^{\alpha}_{\rho}\Delta^{\rho}_{\tau\sigma}\right)\Big] \\
&-\frac{\alpha\sqrt{-g}}{\kappa}\left[\left(\delta^{\alpha}_{\sigma}\Delta^{\beta}_{\rho\tau}g^{\rho\tau} - 2\Delta^{\alpha}_{\sigma\rho}g^{\beta\rho} + \Delta^{\rho}_{\rho\sigma}g^{\alpha\beta}\right)R - 2\delta^{(\alpha}_{\sigma}g^{\tau)\beta}\partial_{\tau}R\right] .
\end{aligned} \tag{C.14}$$

To calculate (C.9) we use (C.10–C.12),

$$\begin{aligned}
\mathscr{C}^{*\alpha}_{\sigma} &= {}_{E}\mathscr{C}^{*\alpha}_{\sigma} + {}_{GB}\mathscr{C}^{*\alpha}_{\sigma} \\
&= -\frac{\sqrt{-g}}{2\kappa}\left[2g^{\rho[\tau}\bar{\nabla}_{\sigma}\Delta^{\alpha]}_{\rho\tau} - \delta^{\alpha}_{\sigma}(R - 2\Lambda_0)\right] \\
&+\frac{2\alpha\sqrt{-g}}{\kappa}\Big[\left(R^{\alpha\beta\rho}{}_{\tau} - 4g^{\rho[\alpha}R^{\beta]}_{\tau} + Rg^{\rho[\alpha}\delta^{\beta]}_{\tau}\right)\bar{\nabla}_{\sigma}\Delta^{\tau}_{\beta\rho} \\
&+ 2g^{\beta\mu}\left(\bar{\nabla}_{\beta}R^{\alpha\nu} + 2R^{\rho(\alpha}\Delta^{\nu)}_{\beta\rho}\right)\Delta^{\tau}_{\sigma(\mu}g_{\nu)\tau} - g^{\rho(\alpha}\Delta^{\beta)}_{\sigma\rho}\bar{\nabla}_{\beta}R\Big] \\
&+\frac{\alpha\sqrt{-g}}{2\kappa}\delta^{\alpha}_{\sigma}\left(R^{2}_{\mu\nu\rho\sigma} - 4R^{2}_{\mu\nu} + R^{2}\right) .
\end{aligned} \tag{C.15}$$

With the use of (8.1.6), (C.15) and (C.13) we construct the coefficient (C.8), $\boldsymbol{u}^*$, in EGB gravity.

On the basis of (C.13) and (C.14) we construct the following combination, which are necessary for constructing canonical superpotential of the form (7.1.46) and Belinfante corrected superpotential (7.1.76). Thus

$$\begin{aligned}
\boldsymbol{m}^{*[\alpha\beta]}_{\sigma} &= {}_{E}\boldsymbol{m}^{*[\alpha\beta]}_{\sigma} + {}_{GB}\boldsymbol{m}^{*[\alpha\beta]}_{\sigma} \\
&= \frac{\sqrt{-g}}{2\kappa}\left[2\Delta^{[\alpha}_{\sigma\rho}g^{\beta]\rho} - \delta^{[\alpha}_{\sigma}\Delta^{\beta]}_{\rho\tau}g^{\rho\tau}\right]
\end{aligned} \tag{C.16}$$

$$+ \frac{\alpha\sqrt{-g}}{\kappa}\left[2R_{\sigma}{}^{\rho\tau[\alpha}\Delta^{\beta]}_{\rho\tau} - 2\left(2R^{\alpha\beta\rho}{}_{\tau} + R^{\rho[\alpha\beta]}{}_{\tau}\right)\Delta^{\tau}_{\rho\sigma}\right.$$
$$+ 2R^{\rho}_{\sigma}\Delta^{[\alpha}_{\rho\tau}g^{\beta]\tau} - 10R^{[\alpha}_{\rho}g^{\beta]\tau}\Delta^{\rho}_{\sigma\tau} + 4R^{[\alpha}_{\sigma}\Delta^{\beta]}_{\tau\rho}g^{\tau\rho} + 2g^{\rho[\alpha}\bar{\nabla}_{\rho}R^{\beta]}_{\sigma}$$
$$\left. + R\left(2\Delta^{[\alpha}_{\sigma\rho}g^{\beta]\rho} + \Delta^{[\alpha}_{\rho\tau}\delta^{\beta]}_{\sigma}g^{\rho\tau}\right) + \delta^{[\alpha}_{\sigma}g^{\beta]\rho}\bar{\nabla}_{\rho}R\right] ;$$

$$\boldsymbol{n}^{*[\alpha\beta]\lambda}_{\sigma} = {}_{E}\boldsymbol{n}^{*[\alpha\beta]\lambda}_{\sigma} + {}_{GB}\boldsymbol{n}^{*[\alpha\beta]\lambda}_{\sigma} \tag{C.17}$$
$$= \frac{3\sqrt{-g}}{4\kappa}\delta^{[\alpha}_{\sigma}g^{\beta]\lambda} + \frac{3\alpha\sqrt{-g}}{2\kappa}\left[R_{\sigma}{}^{\lambda\alpha\beta} + 4g^{\lambda[\alpha}R^{\beta]}_{\sigma} + \delta^{[\alpha}_{\sigma}g^{\beta]\lambda}R\right] ;$$

$$\boldsymbol{n}^{*\tau\rho[\alpha}_{\lambda}\bar{g}^{\beta]\lambda}\bar{g}_{\rho\sigma} = {}_{E}\boldsymbol{n}^{*\tau\rho[\alpha}_{\lambda}\bar{g}^{\beta]\lambda}\bar{g}_{\rho\sigma} + {}_{G)}\boldsymbol{n}^{*\tau\rho[\alpha}_{\lambda}\bar{g}^{\beta]\lambda}\bar{g}_{\rho\sigma} \tag{C.18}$$
$$= \frac{\sqrt{-g}}{4\kappa}\left[2g^{\rho[\alpha}\bar{g}^{\beta]\tau}\bar{g}_{\rho\sigma} - g^{\tau[\alpha}\delta^{\beta]}_{\sigma}\right]$$
$$+\frac{\alpha\sqrt{-g}}{2\kappa}\left[2R_{\lambda}{}^{\rho\tau[\alpha} - 2R^{\rho\tau}{}_{\lambda}{}^{[\alpha} - 8R^{\tau}_{\lambda}g^{\rho[\alpha}\right.$$
$$\left. + 4R^{\rho}_{\lambda}g^{\tau[\alpha} + 4g^{\rho\tau}R^{[\alpha}_{\lambda} + R\left(2\delta^{\tau}_{\lambda}g^{\rho[\alpha} - \delta^{\rho}_{\lambda}g^{\tau[\alpha}\right)\right]\bar{g}^{\beta]\lambda}\bar{g}_{\rho\sigma} .$$

Now, with the use of (C.14) we derive the starred Belinfante correction (7.1.69),

$$\boldsymbol{s}^{*\alpha\beta\sigma} \equiv -\boldsymbol{s}^{*\beta\alpha\sigma} \equiv -\boldsymbol{m}^{*\sigma[\alpha}_{\lambda}\bar{g}^{\beta]\lambda} - \boldsymbol{m}^{*\alpha[\sigma}_{\lambda}\bar{g}^{\beta]\lambda} + \boldsymbol{m}^{*\beta[\sigma}_{\lambda}\bar{g}^{\alpha]\lambda} , \tag{C.19}$$

in the EGB gravity

$$\boldsymbol{s}^{*\alpha\beta\sigma} = {}_{E}\boldsymbol{s}^{*\alpha\beta\sigma} + {}_{GB}\boldsymbol{s}^{*\alpha\beta\sigma} \tag{C.20}$$
$$= \frac{\sqrt{-g}}{\kappa}\left[\Delta^{[\alpha}_{\tau\rho}\bar{g}^{\beta]\sigma}g^{\tau\rho} + \Delta^{\rho}_{\lambda\rho}g^{\sigma[\alpha}\bar{g}^{\beta]\lambda} - \Delta^{\sigma}_{\lambda\rho}g^{\rho[\alpha}\bar{g}^{\beta]\lambda}\right.$$
$$\left. - 2\Delta^{[\alpha}_{\lambda\rho}\bar{g}^{\beta]\lambda}g^{\rho\sigma} + \Delta^{[\alpha}_{\lambda\rho}g^{\beta]\rho}\bar{g}^{\sigma\lambda}\right]$$
$$+\frac{2\alpha\sqrt{-g}}{\kappa}\left[\bar{g}^{\lambda[\alpha}\left(R^{\beta]\sigma\tau}{}_{\rho} - 2R^{\beta]\tau\sigma}{}_{\rho}\right)\Delta^{\rho}_{\lambda\tau} + \bar{g}^{\lambda[\alpha}\Delta^{\beta]}_{\tau\rho}R^{\sigma\tau\rho}{}_{\lambda}\right.$$
$$\left. + \bar{g}^{\lambda[\alpha}R^{\beta]\tau\rho}{}_{\lambda}\Delta^{\sigma}_{\tau\rho} + \bar{g}^{\sigma\lambda}\left(R_{\lambda}{}^{\tau\rho[\alpha}\Delta^{\beta]}_{\tau\rho} - \tfrac{3}{2}R^{\alpha\beta\tau}{}_{\rho}\Delta^{\rho}_{\lambda\tau}\right)\right]$$
$$+\frac{4\alpha\sqrt{-g}}{\kappa}\left[\left(R^{\tau}_{\lambda}g^{\rho(\sigma}\Delta^{\alpha)}_{\tau\rho} - g^{\tau\rho}R^{(\sigma}_{\lambda}\Delta^{\alpha)}_{\tau\rho} - g^{\sigma\alpha}R^{\tau}_{\rho}\Delta^{\rho}_{\tau\lambda}\right.\right.$$
$$\left. + \bar{\nabla}_{(\tau}R^{(\sigma}_{\lambda)}g^{\alpha)\tau} + R^{\rho}_{(\tau}\Delta^{(\sigma}_{\lambda)\rho}g^{\alpha)\tau}\right)\bar{g}^{\beta\lambda}$$
$$-\left(R^{\tau}_{\lambda}g^{\rho(\sigma}\Delta^{\beta)}_{\tau\rho} - g^{\tau\rho}R^{(\sigma}_{\lambda}\Delta^{\beta)}_{\tau\rho} - g^{\sigma\beta}R^{\tau}_{\rho}\Delta^{\rho}_{\tau\lambda}\right.$$
$$\left. + \bar{\nabla}_{(\tau}R^{(\sigma}_{\lambda)}g^{\beta)\tau} + R^{\rho}_{(\tau}\Delta^{(\sigma}_{\lambda)\rho}g^{\beta)\tau}\right)\bar{g}^{\alpha\lambda}$$
$$+\left(2g^{\tau[\alpha}R^{\beta]}_{\rho}\Delta^{\rho}_{\tau\lambda} - R^{\tau}_{\lambda}g^{\rho[\alpha}\Delta^{\beta]}_{\tau\rho} + g^{\tau\rho}R^{[\alpha}_{\lambda}\Delta^{\beta]}_{\tau\rho}\right.$$
$$\left.\left. -\bar{\nabla}_{(\tau}R^{[\alpha}_{\lambda)}g^{\beta]\tau} - R^{\rho}_{(\tau}\Delta^{[\alpha}_{\lambda)\rho}g^{\beta]\tau}\right)\bar{g}^{\sigma\lambda}\right]$$

$$
\begin{aligned}
&+\frac{2\alpha\sqrt{-g}}{\kappa}\left[\left(g^{\sigma[\alpha}\bar{g}^{\beta]\lambda}\Delta^{\rho}_{\lambda\rho} - \Delta^{(\sigma}_{\lambda\rho}g^{\alpha)\rho}\bar{g}^{\beta\lambda} + \Delta^{(\sigma}_{\lambda\rho}g^{\beta)\rho}\bar{g}^{\alpha\lambda}\right.\right.\\
&\left.\left.-\bar{g}^{\sigma[\alpha}\Delta^{\beta]}_{\lambda\rho}g^{\lambda\rho} - \bar{g}^{\sigma\lambda}g^{\rho[\alpha}\Delta^{\beta]}_{\lambda\rho}\right)R + \left(\bar{g}^{\lambda[\alpha}g^{\beta]\sigma} + \bar{g}^{\sigma[\alpha}g^{\beta]\lambda}\right)\partial_\lambda R\right] .
\end{aligned}
$$

Now we present the quantity (7.2.97) calculated for $\mathscr{L}_1$ in (8.1.25) the EGB gravity. For simplicity we derive a tensor quantity, w, connected with a density as $\sqrt{-\bar{g}}w^{\rho\lambda|\mu\nu} = \boldsymbol{w}^{\rho\lambda|\mu\nu}$. Thus,

$$
\begin{aligned}
&w^{\rho\lambda|\mu\nu} = w_E^{\rho\lambda|\mu\nu} + w_{GB}^{\rho\lambda|\mu\nu} = \\
&-\frac{1}{4\kappa}\left[\bar{g}^{\rho\lambda}\varkappa^{\mu\nu} + \bar{g}^{\mu\nu}\varkappa^{\rho\lambda} - \bar{g}^{\rho(\mu}\varkappa^{\nu)\lambda} - \bar{g}^{\lambda(\mu}\varkappa^{\nu)\rho} + \varkappa^{\sigma}_{\sigma}\left(\bar{g}^{\rho(\mu}\bar{g}^{\nu)\lambda} - \bar{g}^{\rho\lambda}\bar{g}^{\mu\nu}\right)\right]\\
&-\frac{\alpha}{2\kappa}\left\{\varkappa^{\sigma}_{\sigma}\left[\left(\bar{g}^{\rho(\mu}\bar{g}^{\nu)\lambda} - \bar{g}^{\rho\lambda}\bar{g}^{\mu\nu}\right)\bar{R} - 2\left(\bar{g}^{\rho(\mu}\bar{R}^{\nu)\lambda} + \bar{R}^{\rho(\mu}\bar{g}^{\nu)\lambda}\right)\right.\right.\\
&+\left. 2\left(\bar{g}^{\rho\lambda}\bar{R}^{\mu\nu} + \bar{R}^{\rho\lambda}\bar{g}^{\mu\nu}\right) + 2\bar{R}^{\rho(\mu\nu)\lambda}\right] - 2\left(\varkappa^{\rho\lambda}\bar{R}^{\mu\nu} + \bar{R}^{\rho\lambda}\varkappa^{\mu\nu}\right)\\
&+\left(\bar{g}^{\rho\lambda}\varkappa^{\mu\nu} + \bar{g}^{\mu\nu}\varkappa^{\rho\lambda} - \bar{g}^{\rho(\mu}\varkappa^{\nu)\lambda} - \bar{g}^{\lambda(\mu}\varkappa^{\nu)\rho}\right)\bar{R} + 2\left(\varkappa^{\rho(\mu}\bar{R}^{\nu)\lambda} + \bar{R}^{\rho(\mu}\varkappa^{\nu)\lambda}\right)\\
&+4\left(\bar{R}^{(\rho}_{\sigma}\bar{g}^{\lambda)(\mu}\varkappa^{\nu)\sigma} + \bar{R}^{(\mu}_{\sigma}\bar{g}^{\nu)(\rho}\varkappa^{\lambda)\sigma}\right) - 4\left(\bar{g}^{\rho\lambda}\bar{R}^{(\mu}_{\sigma}\varkappa^{\nu)\sigma} + \bar{g}^{\mu\nu}\bar{R}^{(\rho}_{\sigma}\varkappa^{\lambda)\sigma}\right)\\
&-2\left(\bar{g}^{\rho(\mu}\bar{g}^{\nu)\lambda} - \bar{g}^{\rho\lambda}\bar{g}^{\mu\nu}\right)\bar{R}^{\tau}_{\sigma}\varkappa^{\sigma}_{\tau} - 4\left(\bar{R}_{\sigma}{}^{(\rho\lambda)(\mu}\varkappa^{\nu)\sigma} + \bar{R}_{\sigma}{}^{(\mu\nu)(\rho}\varkappa^{\lambda)\sigma}\right)\\
&+\left. 2\varkappa_{\sigma\tau}\left(\bar{R}^{\sigma\mu\tau(\rho}\bar{g}^{\lambda)\nu} + \bar{R}^{\sigma\nu\tau(\rho}\bar{g}^{\lambda)\mu}\right) + 2\varkappa_{\sigma\tau}\left(\bar{g}^{\rho\lambda}\bar{R}^{\sigma\mu\nu\tau} + \bar{g}^{\mu\nu}\bar{R}^{\sigma\rho\lambda\tau}\right)\right\} .
\end{aligned}
\tag{C.21}
$$

This expression has to be antisymmetrized to construct the field-theoretical superpotential (7.2.99):

$$
\begin{aligned}
&w^{\rho[\lambda|\mu]\nu} = w_E^{\rho[\lambda|\mu]\nu} + w_{GB}^{\rho[\lambda|\mu]\nu} = \\
&-\frac{3}{8\kappa}\left(\bar{g}^{\rho[\lambda}\varkappa^{\mu]\nu} - \bar{g}^{\nu[\lambda}\varkappa^{\mu]\rho} + \varkappa^{\sigma}_{\sigma}\bar{g}^{\nu[\lambda}\bar{g}^{\mu]\rho}\right)\\
&-\frac{3\alpha}{4\kappa}\left\{\varkappa^{\sigma}_{\sigma}\left[\bar{g}^{\nu[\lambda}\bar{g}^{\mu]\rho}\bar{R} + 2\bar{g}^{\rho[\lambda}\bar{R}^{\mu]\nu} - 2\bar{g}^{\nu[\lambda}\bar{R}^{\mu]\rho} - \bar{R}^{\rho\nu\lambda\mu}\right]\right.\\
&+\left(\bar{g}^{\rho[\lambda}\varkappa^{\mu]\nu} - \bar{g}^{\nu[\lambda}\varkappa^{\mu]\rho}\right)\bar{R} + 2\left(\varkappa^{\nu[\lambda}\bar{R}^{\mu]\rho} - \varkappa^{\rho[\lambda}\bar{R}^{\mu]\nu}\right)\\
&+2\left(\varkappa^{\sigma[\lambda}\bar{g}^{\mu]\rho}\bar{R}^{\nu}_{\sigma} - \varkappa^{\sigma[\lambda}\bar{g}^{\mu]\nu}\bar{R}^{\rho}_{\sigma}\right) + 2\left(\varkappa^{\sigma\rho}\bar{g}^{\nu[\lambda}\bar{R}^{\mu]}_{\sigma} - \varkappa^{\sigma\nu}\bar{g}^{\rho[\lambda}\bar{R}^{\mu]}_{\sigma}\right)\\
&-2\bar{g}^{\nu[\lambda}\bar{g}^{\mu]\rho}\varkappa^{\sigma}_{\tau}\bar{R}^{\tau}_{\sigma} + 4\left(\bar{R}_{\sigma}{}^{[\lambda\mu][\rho}\varkappa^{\nu]\sigma} + \bar{R}_{\sigma}{}^{[\rho\nu][\lambda}\varkappa^{\mu]\sigma}\right)\\
&+\left. 2\varkappa_{\sigma\tau}\left(\bar{R}^{\sigma\nu\tau[\lambda}\bar{g}^{\mu]\rho} - \bar{R}^{\sigma\rho\tau[\lambda}\bar{g}^{\mu]\nu}\right)\right\} .
\end{aligned}
\tag{C.22}
$$

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