Hydrodynamic Instability and Transition to Turbulence

FLUID MECHANICS AND ITS APPLICATIONS
Volume 100

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Hydrodynamic Instability and Transition to Turbulence

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Foreword

Akiva Moiseevich Yaglom (1921–2007) was a major figure of turbulence research, closely associated with the giant scientist Andrei Nikolaevich Kolmogorov. For more biographical information on A.M. Yaglom, see the obituary by Peter Bradshaw (in Flow, Turbulence and Combustion, vol. 80 (3), 287–289, April 2008), a copy of which follows this Foreword.

Together with Andrei Monin, A.M. Yaglom was the author of "Statistical Fluid Mechanics: Mechanics of Turbulence" (MIT Press vol. 1 1971, vol. 2 1975, republished by Dover Publications, 2007). With editing help from John L. Lumley, this was a much augmented and revised English edition of the Russian originals published in 1965 (vol. 1) and 1967 (vol. 2) by Nauka Press, Moscow. These books, usually known as MY1 and MY2, by far the most detailed account of the subject (close to 2000 pages), had become the standard references in turbulence research when Yaglom emigrated to the USA in 1992. Still very active, he set out to revise Monin and Yaglom's book, starting with those aspects of vol. 1 dealing with "Hydrodynamic Instabilities and Transition to Turbulence," a subject which had grown very much since 1971 and which he was following closely. After about ten years of work, four monographs totalling more than 800 pages were produced in preliminary form by the Center for Turbulence Research (CTR, Stanford University) with considerable editing help from Peter Bradshaw. This material (CTR Chap. 2–5) covered only about one quarter of the topics of MY1 but in a coherent and self-consistent way suitable for a standalone publication rather than a new edition of MY1. In 2004, at the 10th European Turbulence Conference (Trondheim) Yaglom discussed the matter with the publisher Springer and this is how the project of the present book was born. During the next three years, A.M. Yaglom prepared an introductory chapter (Chap. 1 of the present book) and worked on two additional chapters which unfortunately could not be completed before his death. The present book thus contains an introductory chapter and the CTR Chap. (2–5) entitled, respectively, "Basic experimental facts and introduction to linear stability theory", "More about linear stability theory; studies of the initial-value problem", "Stability to finite disturbances: energy method and Landau's equation" and "Further weakly-nonlinear approaches to laminar-flow stability: Blasius boundary layer flow as a paradigm".

Publication of the present book has been further delayed because the original files of the CTR monographs and of the figures could not be located. Fortunately many people helped in restoring the material. Thanks are thus due to Peter Bradshaw, Julia Yaglom, Yuni Rodman-Yaglom, Naomi Sherman, Victor Privalsky and many others. Thanks are also due to Springer Verlag and particularly to Nathalie Jacobs, Anneke Pot and Crest Premedia Solutions, Pune. Each chapter ends with the author's own acknowledgments in which he expresses his gratitude to those who helped him scientifically or otherwise.

Uriel Frisch, Nice

In memory of Akiva M. Yaglom

Akiva Moiseevich Yaglom died in Boston, MA on 13 December 2007, after a short illness. Akiva and his brother Isaak were born on 6 March 1921 (correct date: there are other versions) in Kharkov, Ukraina (then part of the Soviet Union). They were said to be as alike as two drops of water. Isaak was also a distinguished mathematician, and the brothers wrote several books together. The family moved to Moscow in 1925. Isaak died in 1986, and Akiva said that only a twin could understand what it was like to lose a twin. Perhaps the first of Akiva's many honors was a prize in the Moscow Mathematical Olympiads, a competition for high school students, in 1938. Akiva shared this prize with his brother Isaak. The prize was presented to him by Andrei N. Kolmogorov, one of the organizers of the competition. Kolmogorov remembered him when they met in 1941, and in spring 1943 invited Akiva to do graduate work with him in Moscow. Kolmogorov's interest in encouraging young mathematicians led, in rather similar circumstances, to his acquisition of Aleksandr M. Obukhov as a graduate student: the third of these famous meteorologists, Andrei S. Monin entered Kolmogorovs group by a more conventional route.

The present writer does not know if these applied statisticians ever discussed the probability that their given names should all begin with the same letter. In 1994, Akiva contributed a detailed and affectionate review A.N. Kolmogorov as a fluid mechanician and founder of a school in turbulence research to vol. 26 of Annual Reviews of Fluid Mechanics. Some of the material in the present obituary comes from that review. Its frontispiece shows Yaglom and Kolmogorov, flanked by Kraichnan and Millionshchikov, at a 1961 conference in Marseille. Yaglom's undergraduate studies at Moscow University were interrupted by the Great Patriotic War (World War II) and in the Autumn of 1941, when the invasion was nearing Moscow and many of its citizens were moved to safer locations, he transferred to Sverdlovsk, about 1300 km East of Moscow, where Moscow University was evacuated. He had previously volunteered for military service but was rejected because of poor eyesight: he said that most of his friends who joined up were killed and that his rejection was probably great good luck for him. Like his fortuitous discovery by Kolmogorov, it was certainly great good luck for Fluid Mechanics.

Yaglom remained interested in fundamental physics while working on turbulence. After he finished graduate study, I.E. Tamm and V.L. Ginsburg offered him a post at the Lebedev Physics Institute of the Russian Academy of Sciences, He was informed that part of his work would be connected to the atomic bomb project. That offer provided an exceptional opportunity to pursue his interest in fundamental physics, but he declined it since he disliked the idea of developing a bomb for Stalin. He therefore took a post at the Institute of Theoretical Geophysics of the Russian Academy of Sciences (now the A.M. Obukhov Institute of Atmospheric Physics). He intended to return to theoretical physics as soon as he could do so without being involved in the fission bomb project. However, working in theoretical physics while remaining uninvolved in military projects never became possible in Soviet Russia, and Yaglom remained at the Institute (a third piece of great good luck for Fluid Mechanics). He also rose to Full Professor at Moscow University. He and his wife June emigrated to the United States in 1992, when he was 70 years old but still working enthusiastically on scientific problems. He had kept up contacts in the West, both personal and postal, and the present writer knew him as a valuable pen friend for many years before meeting him.

At the suggestion of his friend the late Marten Landahl of the Department of Aeronautics and Astronautics at M.I.T., Akiva settled there with the title of Research Fellow, and continued scientific work. Until the last, he worked 10–12 hours a day, going to his MIT office almost every day and spending long hours in his home office. His work was supported by the Stanford/NASA Center for Turbulence Research (CTR), to which he made several working visits, and later by the Poduska Family Foundation. He also did some consulting work in the Boston area. He travelled extensively to conferences and other events, including several visits to Russia.

Yaglom was the author or co-author of six books and 120 scientific papers, by no means all on turbulence. One of the books written in collaboration with his brother was translated into English as Information Theory, and was very recently described as the principal Russian text [on this subject]. Another, multi-volume, book by the twins was Challenging Mathematical Problems with Elementary Solutions (latest edition 2007) based partly on problems prepared for the Moscow Mathematical Olympiads which continue to this day.

Right up to the time of his death, he was working on the revision of Statistical Hydromechanics, universally known as Monin & Yaglom and covering instability and transition as well as turbulence. His work on the Instability volume is being assembled for publication. Monin died on 22 Sept. 2007, less than three months before his junior author, but he had been inactive for some years. Akiva's honors, as well as the above-mentioned high-school prize which really founded his career, included the degree of Doctor of Science (awarded in Russia for a corpus of work rather than a single thesis), the Otto Laporte Award of the American Physical Society (1988) and the Lewis Fry Richardson Medal of the European Geosciences Union (2008). The award was announced before Akiva died: his widow June was invited to accept the medal for him in Vienna in April.

Richardson's classic book "Weather Prediction by Numerical Process" was published just one year after Yaglom was born. Now, at the time of his death, there are plans to build a computer dedicated to realizing Richardson's dream numerical solutions for the whole of Earth's atmosphere, but with rather more computing power than Richardson's humorous concept of a myriad (nominally 10,000) human operators of mechanical calculating machines. Like Kolmogorov, Akiva Yaglom took a great interest in school-level education, and in his later years he supported Shalom House in Brighton, MA, which is a school dedicated to combining orthodox Jewish education with high-grade courses in mathematics and science. Books were his life-long passion, and he collected a unique library of several thousand volumes.

Let Debra Spinks, Administrative Manager of CTR at the time of Akiva's visits to Stanford, have the last word: I remember best that he was a fascinating, gentle soul.

Peter Bradshaw, Stanford University

Contents

Chapter 1
The Equations of Fluid Dynamics and Some of Their Consequences

1.1 Principal Equations of Fluid Mechanics

It is well known that the overwhelming majority of both natural and man-made flows of fluids[1] do not vary smoothly in space and time but fluctuate in a quite disordered manner, exhibiting sudden and irregular (but still continuous) space- and time-variations. Such irregular flows are called "turbulent". A very large amount of information is required to describe the whole of a field of turbulence[2] in space and time, but as a rule only statistical properties, such as time averages are useful to scientists and engineers. Various semi-empirical, approximate methods have been devised to calculate the simpler statistical averages directly.

In contrast to turbulent flows, smooth and regular motions of fluids are called "laminar", the name coming from the erroneous concept of layers of fluid sliding over one another. Laminar flows are produced in fluid initially at rest by regular forces, generating ordered motions described by smoothly changing solutions of the fluid dynamics equations. Turbulent flows usually arise from small disturbances produced in laminar flows by extraneous disturbances. Except at low Reynolds numbers, even small influences usually grow, increasing in disorder until a fully-turbulent state is reached. The process of growth of small random disturbances leading to transformation of a regular laminar flow into a fully disordered turbulent motion is the subject of this book. However we must begin with a brief review of the classical equations of fluid mechanics.

Most attention will be paid to the simplest case of incompressible fluid of constant density ρ. In this case the flow velocity $\mathbf{u}(\mathbf{x}, t) = \{u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t)\}$ will

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[1] As usual the word 'fluid' denotes here any liquid or gaseous medium.

[2] The word turbulence (Latin *turbulentia*) originally refers to the disorderly motion of a crowd (*turba*). It was first used (in a sense close to that accepted today) around the year 1500 by the famous painter (and, as it was discovered much later, also a remarkable inventor and scientist) Leonardo da Vinci (who used the Italian spelling *la turbolenza*; see Frisch (1995, p. 112). However Leonardo did not write his scientific notes for publication and, being left-handed, his writings must be read with the help of a mirror. Leonardo's notes remained little known up to the modern times, and by this reason the word 'turbulence' in its scientific meaning is often attributed to Kelvin; see, e.g., Lamb (1932), Sect. 366.

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satisfy the equation of mass conservation ("continuity") in the simple form:

$$\frac{\partial u_\alpha}{\partial x_\alpha} = 0. \tag{1.1}$$

(Here and henceforth, we will always adopt Einstein's summation convention, according to which whenever an index occurs twice in a single-term expression, the summation is carried out over all possible values of this index; hence $\partial u_\alpha / \partial x_\alpha$ has here the same meaning as $\sum_{\alpha=1}^{3} (\partial u_\alpha / \partial x_\alpha)$). In the case of fluid of variable density where $\rho = \rho(\mathbf{x}, t)$, the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_\alpha)}{\partial x_\alpha} = 0. \tag{1.2}$$

However, as mentioned above, in this book the fluid will usually be assumed to be incompressible with constant density; therefore Eq. (1.2) will be used only rarely.

The fundamental dynamic equations of fluid motion express Newton's second law of conservation of momentum applied to a small volume of fluid. In the case of incompressible fluid of constant density ρ, they have the form

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} = X_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \Delta u_i, \quad i = 1, 2, 3. \tag{1.3}$$

Here $X_i = X_i(\mathbf{x}, t)$ is the value at the point $\mathbf{x}$ and the time t of the ith component of the externally-applied "body" force per unit mass of fluid, if any, $p = p(\mathbf{x}, t)$ is the pressure, and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid (while μ is the ordinary "molecular" viscosity). As usual, $\Delta = \partial^2 / \partial x_\alpha \partial x_\alpha$ denotes the Laplace operator (Laplacian): in a compressible flow the viscous term is more complicated (see below). The three Eq. (1.3) are the classical Navier-Stokes (briefly, N-S) equations which describe the motions of incompressible viscous fluids of constant density.

Fluids with a kinematic viscosity ν so small that its influence on the fluid motions may be disregarded are called either inviscid, or ideal, or perfect. Then the last term in the Eq. (1.3) can be discarded, giving the simpler Euler dynamic equations for the flow of an ideal fluid, incompressible or compressible:

$$\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha} = X_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i}, \quad i = 1, 2, 3. \tag{1.4}$$

In the case of a compressible fluid the fluid density $\rho = \rho(\mathbf{x}, t)$ may vary in time and from point to point; therefore the equations of motion of a viscous fluid are more complicated than Eq. (1.3). For the general case of flow of a viscous compressible fluid the fundamental dynamic equations have the form

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_i u_\alpha)}{\partial x_\alpha} = \rho X_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_\alpha} \left[\mu \left(\frac{\partial u_i}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_i} - \frac{2}{3} \frac{\partial u_\beta}{\partial x_\beta} \delta_{i\alpha} \right) \right] + \frac{\partial}{\partial x_i} \left(\zeta \frac{\partial u_\beta}{\partial x_\beta} \right), \quad i = 1, 2, 3, \tag{1.5}$$

where $\delta_{i\alpha} = 1$ if $i = \alpha$ and $= 0$ if $i \neq \alpha$ (see e.g., Landau and Lifshitz (1987), Sect. 10, or any of the numerous monographs devoted to laminar flows, such as those of Goldstein (1938) and of Lagerstrom (1964)[3]). We see that these equations include not one coefficient of viscosity μ but two such coefficients: the ordinary viscosity μ and the so-called second viscosity ζ. Note also that in many compressible-fluid flows the dependence of the coefficients μ and ζ on spatial coordinates (due usually to their dependence on the variable temperature T of the fluid) is so small that these coefficients may be assumed constant. Then Eq. (1.5) may be simplified to

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \Delta u_i + \left(\zeta + \frac{\mu}{3}\right)\frac{\partial^2 u_\alpha}{\partial x_i \partial x_\alpha}, \quad i = 1, 2, 3. \tag{1.6}$$

The four Eqs. (1.2) and (1.5) (or (1.6)) describing the motions of compressible fluids, contain five unknown functions: ρ, p, u_1, u_2 and u_3. Therefore these equations do not form a closed system (thermodynamic equations are needed as well: see below). In practice, however, compressible fluid flows often have quite small variations of the density ρ so that it may be adequate to assume that ρ is constant (for more detailed discussion of this assumption see, e.g., Landau and Lifshitz (1987), Sect. 10). In such cases the four Eqs. (1.2) and (1.5) (or (1.6)), supplemented by the appropriate initial and boundary conditions, give acceptable values of the four functions $u_i(\mathbf{x}, t)$, $i = 1, 2, 3$, and $p(\mathbf{x}, t)$.

Now let us return to flows of incompressible (constant-density) fluids. Here the full system of fluid-dynamic equations comprises only the simple form (1.1) of the equation of continuity and the three N-S Eq. (1.3). These four equations contain four unknown functions p, u_1, u_2 and u_3 of the variables $\mathbf{x}$ and t, but in fact pressure p may be easily eliminated from Eq. (1.3). We need only take the curl of both sides of Eq. (1.3), which in tensor notation is written as the operator $\varepsilon_{k\beta i}\frac{\partial}{\partial x_\beta}$. (Here $\varepsilon_{k\beta_i}$ is a completely skew-symmetric tensor of the third rank, sometimes called the unit alternating tensor, and the indices k, β, i can take any of the values 1, 2 and 3. Thus the operator is equal to zero if two, or all three, of the indices are the same: $\varepsilon_{k\beta_i} = +1$ if (k, β, i) are in cyclic order (1, 2, 3; 2, 3, 1 or 3, 1, 2) and $\varepsilon_{k\beta_i} = -1$ if they are in anticyclic order). Assuming, for simplicity, that there are no external forces X_i we arrive at the system of three equations

$$\frac{\partial \eta_k}{\partial t} + u_\alpha \frac{\partial \eta_k}{\partial x_\alpha} - \eta_\alpha \frac{\partial u_k}{\partial x_\alpha} = \nu \Delta \eta_k, \quad k = 1, 2, 3, \tag{1.7}$$

where

$$\eta_k = \varepsilon_{k\beta\alpha}\frac{\partial u_\alpha}{\partial x_\beta} \tag{1.8}$$

[3] All material presented in this chapter can be found in a great number of textbooks and monographs on fluid mechanics. Therefore the few references presented here must be considered as only some examples of numerous books containing the stated results.

are the three components of the vorticity vector. In principle, the three velocity components $u_i(\mathbf{x}, t)$ may be determined from Eqs. (1.7) and (1.8). Then the pressure field $p(\mathbf{x}, t)$ is the solution of the Poisson equation,

$$\Delta p = -\rho \frac{\partial^2 (u_\alpha u_\beta)}{\partial x_\alpha \partial x_\beta}, \tag{1.9}$$

that is obtained by applying the operation $\partial/\partial x_i$ to the N-S Eq. (1.3) with $X_i = 0$. From Eq. (1.9) it follows that, to within a harmonic function of $\mathbf{x}$,

$$p(\mathbf{x}) = \frac{\rho}{4\pi} \int \frac{\partial^2 [u_\alpha(\mathbf{x}') u_\beta(\mathbf{x}')]}{\partial x'_\alpha \partial x'_\beta} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}, \tag{1.10}$$

where the integration is taken over the whole volume of the fluid. In the case of a flow occupying the whole unbounded space the supplementary harmonic function must take a constant value. Since only derivatives of the pressure appear in the equations of motion, the constant term in the expression for pressure plays no role at all; thus here Eq. (1.10) may be considered as being fully accurate. However, for flows in finite regions the harmonic addition to Eq. (1.10) must be determined from the boundary conditions for the pressure; here in a number of cases it may be also proved that this addition must take a constant value which may be ignored.

The main part of this book will be devoted to flows of fluids of constant density, satisfying the N-S Eq. (1.3) and the continuity Eq. (1.1). However in Chap. 2 we will also consider "thermally-inhomogeneous" flows, in which the fluid temperature T (and hence also the fluid density ρ) depends on the spatial point $\mathbf{x}$ and possibly the time t. In a gravitational field, this can lead to so-called "buoyant convection", in which the flow is determined mainly by buoyancy rather than (say) horizontal pressure gradients. Therefore we must consider equations which are valid in all conditions between buoyant convection and "forced convection": the latter is the name used for ordinary flows with negligible buoyancy effects.

In the flow of a thermally-inhomogeneous compressible fluid the continuity and dynamical Eqs. (1.2) and (1.5) (or (1.6)) contain five unknowns and hence do not form a closed system. To close this system one must add a fifth equation, obviously for some temperature-dependent quantity—e.g. the equation of the budget of the total energy per unit volume of moving fluid. This energy is equal to $\rho(u^2/2 + e)$, where $u = u(\mathbf{x}, t)$ is the magnitude $|\mathbf{u}|$ of the flow velocity $\mathbf{u}(\mathbf{x}, t)$ at point $\mathbf{x}$ and time t, and $e = e(\mathbf{x}, t)$ is the thermal internal energy of unit mass of fluid, a function of temperature. An alternative, and indeed more popular, choice is the total *enthalpy*, but the total-energy equation is easier to understand. The general equation for the total-energy budget can be found, e.g., in the books of Landau and Lifshitz (1987), Sect. 49, Monin and Yaglom (1971), Sect. 1.5, and Lagerstrom (1964), Sect. B.2. It has the form

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho e\right) = -\frac{\partial}{\partial x_\alpha}[\rho u_\alpha (u^2/2 + e + p/\rho) - u_\beta \sigma_{\beta\alpha} - \kappa \partial T/\partial x_\alpha] + u_\alpha \rho X_\alpha, \tag{1.11a}$$

where $\rho(u^2/2+e)$ is the total energy per unit volume of moving fluid, $\sigma_{\beta\alpha}=\sigma_{\alpha\beta}$ is the viscous stress tensor,

$$\sigma_{\alpha\beta} = \mu\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \frac{2}{3}\frac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta}\right) + \zeta\frac{\partial u_\gamma}{\partial x_\gamma}\delta_{\alpha\beta} \tag{1.11b}$$

(compare with Eq. (1.5)) and κ is the coefficient of thermal conductivity of the fluid (see any of the above-mentioned books). Using Eqs. (1.2), (1.5) and some known thermodynamic equations[4], we can convert Eq. (1.11a) into the following balance equation for the entropy s of unit mass of a perfect gas:

$$\rho T\left(\frac{\partial s}{\partial t} + u_\alpha\frac{\partial s}{\partial x_\alpha}\right) = \sigma_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial}{\partial x_\alpha}\left(\kappa\frac{\partial T}{\partial x_\alpha}\right). \tag{1.12}$$

Equations (1.11a) and (1.12) are equivalent, and either may be used in a closed system of equations for flows of a compressible (and/or thermally inhomogeneous) fluid.

To obtain such a closed system of equations for the five unknown functions, we must express the entropy s as a function of the pressure p and the density ρ (or the temperature T) with the aid of the general equations of thermodynamics and the equation of state of the fluid (gas or liquid) which connects the quantities p, ρ, and T. Let us begin with a *perfect gas*, which has the well-known equation of state

$$p = R\rho T \tag{1.13}$$

where T is the absolute temperature (measured in degrees Kelvin) and the constant R is equal to the difference $c_{\mathrm{p}} - c_{\mathrm{v}}$ (here c_{p} and c_{v} are specific heats of the gas at constant pressure and at constant volume, respectively). It is easy to show that in the present case $s = -R\ln\rho + c_v\ln T + \text{const.} = -R\ln p + c_p\ln T + \text{const.}$ (see, e.g., Landau and Lifshitz (1980), Sect. 43). Combining these equations with the entropy Eq. (1.12) and using the equation of state (1.13) it is possible to transform the energy-balance Eq. (1.11a) into the general equation for the "heat" budget in the flow of a perfect gas

$$c_v\rho\left(\frac{\partial T}{\partial t} + u_\alpha\frac{\partial T}{\partial x_\alpha}\right) = -p\frac{\partial u_\alpha}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}\left(\kappa\frac{\partial T}{\partial x_\alpha}\right) + \rho\varepsilon, \tag{1.14}$$

or the equivalent form

$$c_v\rho\left(\frac{\partial T}{\partial t} + u_\alpha\frac{\partial T}{\partial x_\alpha}\right) = \frac{\partial p}{\partial t} + u_\alpha\frac{\partial p}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha}\left(\kappa\frac{\partial T}{\partial x_\alpha}\right) + \rho\varepsilon, \tag{1.14a}$$

[4] This use of the thermodynamic quantities and equations may raise some doubts, since a fluid flow with nonzero gradients of velocity and temperature does not constitute a system in thermodynamic equilibrium. However in all the books cited here it is explained that in the case of the moderate gradients encountered in real fluid flows, the fundamental thermodynamic quantities may be defined in such a way that all their ordinary properties, and the corresponding equations will be valid (for more details see, e.g., Sect. 49 of the book by Landau and Lifshitz (1987) or one of the more special publications, such, as, e.g., the paper by Tolman and Fine (1948) and the books by Chapman and Cowling (1952) and Hirschfelder et al. (1954)).

where in both cases $\rho\varepsilon = \sigma_{\alpha\beta}\partial u_\alpha/\partial x_\beta$ (see, e.g., Monin and Yaglom (1971), Sect. 1.5). Monin and Yaglom (1971) will henceforth be referred to as MY1. It is easy to show that ε is the rate at which energy per unit mass of fluid is transferred from kinetic energy to thermal internal energy via the work done by the flow (the velocity gradients) against the viscous stresses. In cases where there are additional heat sources in the flow produced by external radiation, chemical reactions, phase transitions, etc., a further source tern ρQ must be added to the right-hand sides of Eqs. (1.11a), (1.12), (1.14) and (1.14a). However such cases will not be considered in this book.

Above, we considered only flows of perfect gases having the very simple equation of state (1.13). However a heat-balance equation of the form (1.14a) may also be obtained for thermally-inhomogeneous flows of purely-viscous liquids. Here, instead of the equation of state (1.13), we must use the equation for thermal expansion of liquids

$$\rho - \rho_0 = -\beta\rho_0(T - T_0) \tag{1.15}$$

where ρ and ρ_0 are the densities of the liquid at temperatures T and T_0, and β is the coefficient of thermal expansion of this liquid (if β is constant, T_0 is arbitrary). It can be shown that in this case the heat-balance Eq. (1.14a) becomes

$$c_p\rho\left(\frac{\partial T}{\partial t} + u_\alpha\frac{\partial T}{\partial x_\alpha}\right) = \beta T\left(\frac{\partial p}{\partial t} + u_\alpha\frac{\partial p}{\partial x_\alpha}\right) + \frac{\partial}{\partial x_\alpha}\left(\kappa\frac{\partial T}{\partial x_\alpha}\right) + \rho\varepsilon \tag{1.16}$$

while in any incompressible fluid $\dfrac{\partial u_\beta}{\partial x_\beta} = 0$ and the expression for ε can be simplified to

$$\varepsilon = \frac{1}{2}\nu\sum_{\alpha,\beta}\left(\frac{\partial u_\alpha}{\partial u_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)^2 \tag{1.17}$$

(see, for example, Squire (1953)). The coefficient β for ordinary liquids is usually very small (for example, $\beta \approx 1.5 \times 10^{-4}$ per degree for water at 15°C). Thus in Eq. (1.16) we may ignore the term containing this coefficient, and obtain

$$\frac{\partial T}{\partial t} + u_\alpha\frac{\partial T}{\partial x_\alpha} = \chi\Delta T + \frac{\varepsilon}{c_p} \tag{1.18}$$

where $\chi = \kappa/c_p\rho$ is the thermal diffusivity of any fluid, which we will assume to be constant in liquids. The term ε/c_p on the right-hand side of Eq. (1.18) describes the general heating of this medium caused by the internal friction of the liquid; this heating under real-life conditions usually plays an insignificant role and in the majority of cases it may be neglected. Then Eq. (1.18) simplifies still further and transforms into the ordinary equation of heat conduction in any moving medium with constant properties:

$$\frac{\partial T}{\partial t} + u_\alpha\frac{\partial T}{\partial x_\alpha} = \chi\Delta T. \tag{1.19}$$

Let us now emphasize that the heat-conduction Eq. (1.19) is applicable not only to liquids, but also to gases, provided that the velocity of the gas is much less than the corresponding sound velocity $a = (\gamma p/\rho)^{1/2}$, where $\gamma = c_p/c_v$, i.e. provided that the Mach number u/a is low. It may be also shown that at such flow velocities, variations in pressure p are small compared to the absolute pressure and play a considerable smaller role in Eq. (1.14a) than variations in the temperature T. Therefore terms containing p on the right sides of Eq. (1.14a) may be ignored. Then we may easily obtain Eq. (1.18) for gases and then, if we also ignore heating of the gas due to viscous dissipation of kinetic energy we get Eq. (1.19).

Note, however, that comparison of Eqs. (1.18) and (1.14) shows that in Eq. (1.14), in contrast to Eq. (1.14a), it is impossible to ignore the additional "pressure work" term $-p\partial u_\alpha/\partial x_\alpha$, since the local compressions and expansions caused by variations of density during heating and cooling must be taken into account in the heat equation, even in the case of small velocities (low Mach number). However, at low Mach number the velocity field may be assumed completely "incompressible" (at least in the cases when the temperature differences in the flow are small in comparison with its absolute temperature T_0). Under these conditions, it may also be assumed in the expression for ε that $\partial u_\alpha/\partial x_\alpha = 0$, i.e. ε may once again be simplified to the form Eq. (1.17). In the case of an incompressible medium $Tds = de$ (where, as usual, ds and de are increments of the entropy and of the internal energy of unit mass of fluid). Thus, in an incompressible fluid ε will be exactly equal to the increase of internal energy per unit mass per unit time, i.e., to the amount of kinetic energy dissipated (transformed into heat) per unit time per unit mass of fluid (cf. Landau and Lifshitz (1987), Sect. 16). Although this quantity may generally be ignored in the calculation of the temperature field, it is nevertheless a very important physical characteristic of turbulent flow, where energy is extracted from the fluctuating part of the velocity field as well as the mean.

The exceptionally important role played by the rate of the energy dissipation in the mechanics of turbulent flows became obvious after the development by Kolmogorov (1941a, b) of the general theory of "locally isotropic turbulence" from which it follows that the small-scale structure of any turbulent flow with high enough Reynolds number $\mathrm{Re} = UL/\nu$ (where U and L are the velocity and length scales characterizing the large-scale structure of the flow considered) is isotropic, universal and depends only on two physical parameters ε and ν. The description of Kolmogorov's famous theory, its numerous applications and further developments comprise the main part of the book by Monin and Yaglom (1971) and of many other books and survey papers (those by Hunt et al. (1991); Frisch (1995); Nelkin (1994); Yaglom (1981, 2004) and Lundgren (2004) are only a few examples). However in this book Kolmogorov's theory will not be considered (but the energy-dissipation rate ε will be met with several times).

Let us now stress that Eq. (1.19) is identical in form to the well-known diffusion equation

$$\frac{\partial \vartheta}{\partial t} + u_\alpha \frac{\partial \vartheta}{\partial x_\alpha} = \chi \Delta \vartheta \tag{1.20}$$

that describes the variation in concentration $\vartheta(\mathbf{x}, t)$ of an arbitrary "passive admixture" (a contaminant which has no effect on the dynamics of the flow, otherwise known as a "passive scalar"). The only difference is that in the latter case the coefficient χ must be interpreted not as the coefficient of thermal conductivity, but as the molecular coefficient of diffusion of the contaminant in the fluid of the base flow.

The "passive admixture" assumption implies that the velocity field $u_i(\mathbf{x}, t)$ may be determined, independently of the distribution of ϑ, from the ordinary system of equations for the flow of a incompressible fluid, and then substituted into Eq. (1.20) to calculate ϑ. If we take ϑ to be temperature then the temperature differences between different points of the fluid must be sufficiently small that the changes they produce in the physical properties of the fluid have no effect on the flow (this is quite a serious consideration in liquids because the viscosity changes rapidly with temperature). At the same time the temperature differences must be large enough for heating due to viscous dissipation or any unwanted sources of heat to be neglected, so that Eq. (1.19) can be used instead of Eq. (1.18). In this case the temperature inhomogeneities will simply move with the fluid, at the same time being smoothed out by molecular conduction.

We have already mentioned the effect of buoyancy on flows in a gravitational field, and this is another important case of flows in which the temperature cannot be considered as a passive admixture is that of non-uniformly heated fluids in the presence of the gravitational field. "Free convection" in which the motion is driven entirely by buoyancy forces, is sufficiently different from "forced convection" with negligible buoyancy effects that it is helpful to derive equations of motion for this special case free-convection equations. We will consider here only a somewhat simplified version of the most general free-convection equations, which will be adequate for the flows to be discussed in this book.

Let us assume first of all that the velocity in the flows to be considered here is small enough for us to ignore variations in the density produced by variations of pressure (but not of temperature). Therefore we can use the ordinary equation of continuity (1.1) and N-S Eq. (1.3) but we must include in the latter the body force $\mathbf{X} = -g\mathbf{e}_z$ (where g is the gravitational acceleration and $\mathbf{e}_z$ is the unit vector of the vertical axis $Ox_3 = Oz$) and remember that the density ρ depends on temperature T (when discussing flows with variable fluid properties it is convenient to define T as the absolute temperature). Let us also assume that $T = T_0 + T_1$ where T_0 is a constant reference temperature, say the temperature in the unheated part of the flow (if any) and T_1 is a small deviation from T_0. It is clear that then $\rho = \rho_0 + \rho_1$, where ρ_0 is the constant density corresponding to temperature T_0 and $\rho_1 = \rho - \rho_0$ is given by the equation

$$\rho_1 = -\beta\rho_0 T_1 \tag{1.21}$$

(the coefficient of thermal expansion $\beta = -1/\rho_0(\partial\rho_0/\partial T)_p$ for an ideal gas satisfying Eq. (1.13) is equal to $1/T_0$). Note that if $T = T_0 =$ constant and $\rho = \rho_0 =$ constant, the pressure p_0 will not be constant but will decrease with increasing height $z = x_3$

as the weight of the column of fluid above the point considered decreases:

$$p_0(x_3) = -\rho_0 g x_3 + \text{const.} \tag{1.22}$$

Putting $p = p_0 + p_1$, $\rho = \rho_0 + \rho_1$, we have, to first-order accuracy,

$$\frac{1}{\rho}\frac{\partial p}{\partial x_3} = \frac{1}{\rho_0}\frac{\partial p_0}{\partial x_3} + \frac{1}{\rho_0}\frac{\partial p_1}{\partial x_3} - \frac{\rho_1}{\rho_0^2}\frac{\partial p_0}{\partial x_3} = -g + \frac{1}{\rho_0}\frac{\partial p_1}{\partial x_3} + g\beta T_1.$$

From this it follows that the third Navier–Stokes equation (for the velocity component in the z or x_3 direction) becomes

$$\frac{\partial u_3}{\partial t} + u_\alpha \frac{\partial u_3}{\partial x_\alpha} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x_3} + \nu \Delta u_3 - g\beta T_1 \tag{1.23}$$

(the last term on the right side of this equation, which includes T_1, quantitatively confirms the qualitatively obvious fact that in buoyant flows the temperature cannot be considered as a passive admixture). The first and second N-S equations may be written in the usual form (1.3) with ρ replaced by the constant reference density ρ_0, while p is taken to mean the height-dependent deviation p_1 of the true pressure from its reference value p_0. Finally, the equation for the temperature, as always when the medium may be assumed incompressible, is the equation for heat conduction in a moving fluid

$$\frac{\partial T_1}{\partial t} + u_\alpha \frac{\partial T_1}{\partial x_\alpha} = \chi \Delta T_1 \tag{1.24}$$

(as usual, we ignore the term including the rate of energy dissipation ε).

The set of five approximate equations presented here, (1.1), (1.3) with $i = 1$ and 2, (1.23), and (1.24), which contains five unknown functions $u_i(\mathbf{x}, t)$, $i = 1$, 2 and 3, $p_1(\mathbf{x}, t)$ and $T_1(\mathbf{x}, t)$, describe the free (buoyant) convection of the fluid. These equations are often called the *Boussinesq equations* (or the *Boussinesq approximation* for fluid-flow equations) because they were used in the old book of Boussinesq (1903). However Joseph (1976), Sect. 64, was apparently the first to point out that practically the same equations (and also the same modifications) were used still earlier by Oberbeck (1879, 1888). Therefore in Joseph's book of 1976 these equations (and also some related systems) of free-convection equations were called not the Boussinesq equations but the Oberbeck–Boussinesq equations: justice was done after nearly a century. According to Joseph, the prevalence of the term 'Boussinesq equations' is due to Rayleigh, who did not know Oberbeck's papers and therefore used just this term in his paper of 1916 which became extremely popular. Some authors of subsequent books and papers have adopted Joseph's re-naming of the equations were in fact first obtained by Oberbeck (see, e.g., the books by Drazin and Reid (1981) and Drazin (2002)). However, for simplicity we will follow custom and refer to the 'Boussinesq equations'. Note that the closed system of five differential equations with five unknowns presented above will be applicable to all the convective flows discussed in this book, while detailed analysis of the conditions of applicability of this and allied systems of convective-flow equations may be found in the books by Chandrasekhar (1961); Joseph (1976) and Koschmieder (1993) and in the papers by Spiegel and Veronis (1960) and Michaljan (1962).

1.2 Some Examples of Exact and Approximate Solutions of the Equations

In Sect. (1.1) we discussed the general form of the fluid-dynamical equations, but in fact the full equations will be used only rarely in this book. Here, we will consider a few exact solutions of these equations that will be used repeatedly below (some further solutions may be found in the survey of Wang (1991)). All these solutions correspond to simple laminar flows whose stability to small disturbances will be examined later in this book. We will begin with the idealized steady and streamwise-homogeneous two-dimensional plane-parallel flow in the *Ox* direction between two infinite parallel walls in the planes $z = z_1$ and $z = z_2$ at a finite distance H from each other. (Below we will use the ordinary Cartesian coordinates $(x, y, z) = (x_1, x_2, x_3)$ and for definiteness will assume that the two walls are at $z = 0$ and $z = H$ where $H > 0$). The external force X_i will usually be assumed to be zero.

Let us note immediately that in the case of steady plane-parallel flows of viscous fluid with the velocity field $\mathbf{u}(\mathbf{x}) = \{u(z), 0, 0\}$, writing the general vorticity Eq. (1.7) for the vorticity component $\eta^2 = \partial u/\partial z - \partial w/\partial x$ (spanwise) yields the very simple result $d^3u(z)/dz^3 = 0$. This means that the function $u(z)$ must be a polynomial of z with no term of higher order than z^2. Similarly, when the x-component velocity varies with the "spanwise" coordinate y as well as with z, but all other velocity components remain zero so that $\mathbf{u}(\mathbf{x}) = \{u(y, z), 0, 0\}$, two Eq. (1.7) for the vorticity components η_2 and η_3 imply that here $\Delta_2 u(y, z) = \text{const.}$, where Δ_2 denotes the two-dimensional Laplacian.

1. *Plane Couette flow* (briefly, PCF). This is the flow between two parallel walls moving in the x-direction with two different constant velocities U_1 and U_2 driving the fluid in the same direction. We will set $U_2 - U_1 = 2U_0 > 0$ (the inequality $2U_0 > 0$ specifies that the upper wall has the greater velocity (U_2), and the factor 2 is added because in PCF studies the half-difference of wall velocities is traditionally used as the velocity scale). Since the equations of fluid mechanics are invariant in the change to a new inertial ('Galilean') system of spatial coordinates, only the difference $2U_0$ affects the flow while the mean velocity $U_3 = (U_1 + U_2)/2$ characterizes the unimportant transport of the whole mass of fluid with constant velocity U_3. The velocity of the fluid in the case considered here is everywhere parallel to the *Ox* axis while all fluid-dynamical quantities depend only on the coordinate $z = x_3$. Hence the continuity Eq. (1.1) and the second N-S Eq. (1.3) with $i = 2$ are satisfied identically here, while the first and the third N-S equations take the forms

$$\frac{d^2u}{dz^2} = 0, \quad \frac{dp}{dz} = 0, \tag{1.25}$$

where $u(z) = u_1(z)$ is the streamwise, and only, velocity component. Hence here p is constant over any cross-section of the flow and $u(z) = az + b$. In addition the boundary conditions $u(0) = U_1$, $u(H) = U_2$ imply that $u(z) = U_1 + (U_0/H_1)z$, where $H_1 = H/2$. We see that this laminar flow has a linear velocity profile which, of course, satisfies the condition $d^3u(z)/dz^3 = 0$ for viscous flow. In addition to the form of the velocity profile, it is often desirable to evaluate some simpler characteristics which help us to compare different flows. One characteristic (dimensionless parameter) that proved to be very useful was introduced for special purposes (which will be explained in Sect. 2.1) by Reynolds (1883), but later it proved very useful in general: it was named the *Reynolds number* and denoted by Re. Re is the dimensionless combination UL/ν, where U is some typical velocity of the large-scale features of the flow, L is the corresponding length scale, and ν, as usual, denotes the kinematic fluid viscosity. Taking the half-difference of wall velocities U_0 and the half-distance between the walls H_1 as the most convenient velocity and length scales of the large-scale features of this flow (called the *plane Couette flow* for reasons which will be explained below), the Reynolds number in its most popular form becomes $\mathrm{Re} = U_0 H_1/\nu$.

Laminar plane Couette flow is described by the simplest exact solution of the N-S equations, and because of its simplicity, it soon attracted the attention of some of the founders of the modern theoretical fluid mechanics. See, for example, the references in Chap. 2 of this book to the early studies of PCF stability by Kelvin and Rayleigh, and by Lorentz (1907), who regarded the study of Couette flow as a valuable introduction to more difficult studies of pipe-flow stability. However setting up a good approximation to plane-parallel flow between infinite parallel planes in the laboratory, where the planes are necessarily of finite size, is not a simple task. Therefore experiments on this flow (which will be briefly considered in Chap. 2) were first carried out only in the second half of the 20th century.

2. *Plane Poiseuille flow* (briefly, PPF). Now let us consider a steady two-dimensional flow between stationary walls at $z = 0$ and $z = H$ due to the action of an external force applied at infinity and producing a negative streamwise pressure gradient $\partial p/\partial x < 0$, independent of y and z. It is clear that the velocity is everywhere along the Ox axis and depends only on z, so that $u(z) = u_1(z)$ is again the only nonzero component of the velocity field $\mathbf{u}(\mathbf{x}, t)$. Exactly as in plane Couette flow, the continuity Eq. (1.1) and the second N-S Eq. (1.3) with $i = 2$ are satisfied identically. However the first and the third N-S Eq. (1.3) now take the forms

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 u}{dz^2}, \quad \frac{\partial p}{\partial z} = 0 \tag{1.26}$$

(where instead of the kinematic viscosity ν the ordinary viscosity $\mu = \rho\nu$ now appears). The second Eq. (1.26) shows that, again as in plane Couette flow, the pressure p is independent of z, i.e. constant over any cross-section of the flow, depending only on x. Moreover, using the first Eq. (1.26) it is easy to show that the streamwise pressure gradient dp/dx is independent of x (and therefore $dp/dx = -\Delta_i p/l$ where $-\Delta_i p/l$ is the normalized drop of the pressure between the planes $x = x_0$ and $x = x_0 + l$ which is the same for all values of x_0 and l). The same equation shows that the velocity profile $u(z)$ has the form: $u(z) = [(dp/dx)/2\mu]z^2 + az + b$ where a and b are constants. (This velocity profile clearly also has zero third derivative). The constants a and b can be easily determined from the standard no-slip boundary conditions on immovable solid walls: $u(z) = 0$ for $z = 0$ and $z = H$. These boundary conditions together with the first Eq. (1.26) lead to the following final expression for the velocity profile in *plane*

Poiseuille flow (the reason for the name will be explained below):

$$u(z) = \frac{1}{2\mu}\frac{dp}{dz}\left[\left(z - \frac{H}{2}\right)^2 - \frac{H^2}{4}\right]. \tag{1.27}$$

Equation (1.27) shows that velocity profiles $u(x, z)$ depend here only on z and hence have the same parabolic form at all values of x with a maximum value $-(dp/dx)H^2/8\mu$ in the middle of the fluid layer, $z = H/2$. Using this maximum velocity as the velocity scale U_0 of this flow and its half-width $H_1 = H/2$ as the flow length-scale L we obtain the representative Reynolds number for this flow as $\text{Re} = U_0 H_1/\nu = -(dp/dx)H^3/16\rho\nu^2$.

As in the case of plane Couette flow, plane Poiseuille flow in a layer between two infinite parallel walls is clearly a mathematical idealization, but an adequate approximation is much more easily realized in the laboratory. Therefore experimental studies of PPFs began much earlier, and have been much more numerous, than similar studies of plane Couette flow.

1. *Plane Couette-Poiseuille flow* (briefly, PCPF). We have already shown that the velocity profile $u(z)$ in any steady plane-parallel flow of viscous fluid must be a polynomial in z with no term of higher order than z^2. Therefore the general steady parallel flow in the Ox direction between solid, but not necessarily stationary, walls at $z = 0$ and $z = H$, is the so-called *plane Couette-Poiseuille flow* (sometimes called also *the plane Poiseuille-Couette flow*; see., e.g., Cowley and Smith (1985) or Balakumar (1997)) where the velocity $u(z)$ is represented by a quadratic polynomial of z including both the linear component (corresponding to some plane Couette flow) and the quadratic (parabolic) component (corresponding to some plane Poiseuille flow). This means that such a velocity profile corresponds to the fluid motion in the Ox-direction which is produced by a constant negative longitudinal pressure gradient $-dp/dx$ combined with x-wise motion of one wall (or of both walls with different velocities). Since only the difference of wall velocities is of importance, we will assume for simplicity that only the upper wall is moving in the Ox direction with constant positive velocity $2U_w$ while the lower wall is stationary.

The velocities $\mathbf{u}(\mathbf{x})$ of all the PCPF family have only one non-zero component, the streamwise velocity, $u(z)$, which depends on only one coordinate, z. The linear-plus-parabolic profile that satisfies the boundary conditions is

$$u(z) = \frac{1}{2\mu}\frac{dp}{dz}\left[\left(z - \frac{H}{2}\right)^2 - \frac{H^2}{4}\right] + \frac{2U_w}{H}z \tag{1.28}$$

where dp/dz is the constant negative pressure gradient generating the Poiseuille component of the flow, and $2U_w$ is the positive streamwise velocity of the upper wall producing the Couette component. Note that $u(z)$ is just the sum of the Poiseuille component given by Eq. (1.27) and the Couette component discussed above. Equation (1.28) is an exact solution of the N-S equation for the general steady plane-parallel fluid flow between two walls. It is a very trivial solution: the N-S equations are nonlinear (strictly, quasi-linear) which means that in general we cannot simply add two solutions to get a third—but this is what we have just done. The explanation is that the nonlinear terms in the N-S equations are those that represent the acceleration of the fluid, which is zero in all the PCPF family so only the linear terms appear. Another possible representation of the velocity profile of Eq. (1.28) is:

$$U_1(z_1) = (4 - A)z_1 - (4 - 2A)z_1^2 \tag{1.28a}$$

where $z_1 = z/H$ (hence $0 \le z_1 \le 1$), $U_1(z_1) = u(z_1 H)/u(H/2)$ and $\mathrm{A} = u(H)/u(H/2)$. Here A is a dimensionless parameter which characterizes the relative contribution of the Couette component to the combined flow and grows with the increase of this contribution. As defined here, A varies from zero in pure plane Poiseuille flow to 2 in pure Couette flow. Some additional results related to stability properties of various Couette-Poiseuille flows will be presented later in this book.

Now we will pass from two-dimensional plane-parallel flows in a layer between two parallel planes to some other laminar flows which will be also considered at greater length later in this book. Let us begin with a flow very often met with in real life; steady laminar flow in a straight circular pipe of constant diameter D, generated by a constant negative pressure gradient.

4. *Circular Poiseuille* (or *Hagen-Poiseuille*) *flow in a pipe* (briefly, CPF). Let us assume that the Ox axis coincides with the axis of an infinitely long pipe. Then the velocity $\mathbf{u}$ of a steady flow in this pipe will have only one, streamwise, nonzero component which will be independent of the time t and of the streamwise coordinate x and thus will have the form $u_1(y, z)\mathbf{e}_x = u(y, z)\mathbf{e}_x$. The continuity Eq. (1.1) is satisfied identically, as in the other flows discussed in this section, and as before the second and third N-S Eq. (1.3) imply that $\partial p/\partial y = \partial p/\partial z = 0$, i.e. that the pressure $p = p(x)$ is constant over any cross-section of the pipe. For the circular pipe it is convenient to use polar coordinates (r, ϕ) in the (y, z) plane with origin on the Ox axis. Then by the circular symmetry of the pipe, $u(r, \phi) = u(r)$. From this it follows that in cylindrical coordinates the first N-S Eq. (1.3) has the form

$$\frac{dp}{dx} = \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right). \tag{1.29}$$

Since the left-hand side of this equation depends only on x and the right-hand side only on r, both sides must have the same constant value. In particular, $dp/dx = \text{const} = -\Delta_l p/l$, where $\Delta_l p$ is the pressure drop along a pipe section of length l. Taking into account the boundary condition $u(D/2) = 0$ and the fact that $u(r)$ is a bounded function of r, we obtain from Eq. (1.29) the following result:

$$u(r) = \left[\left(-\frac{dp}{dx}\right)/4\mu\right](R^2 - r^2), \tag{1.30}$$

where $R = D/2$ is the pipe radius. We see that as in plane Poiseuille flow the velocity profile (along any diameter) is parabolic with the maximum velocity, on the pipe axis, equal to $U_0 = -\frac{dp}{dx}\frac{R^2}{4\mu}$. The Reynolds number is most often defined as $\mathrm{Re} = U_0 R/\nu$. (In engineering work, in pipes of any shape the maximum velocity U_0 is often replaced by the cross-sectional-mean velocity: for a circular pipe,

$U_m = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R u(r)r\,dr\,d\varphi = U_0/2$. The reason is that the volume flow rate, the quantity of greatest interest, is just U_m times the cross-sectional area. Then the length scale is usually taken as the pipe diameter $D = 2R$ is used. The Reynolds number has the same value whichever pair of scales is used; $U_{\mathrm{m}}D/\nu = U_0R/\nu$.

As before, the solution (1.30) of the N-S equations clearly refers to an idealized flow, since the pipe is taken to be infinitely long, and the flow to be strictly independent of x. However measurements showed that the flow becomes independent of x ("fully developed") at acceptably small distances from the pipe inlet, of the order of R^2U_0/ν. (According to numerical calculations by Schiller (1934), cited also in the popular survey of Goldstein (1938), Vol. **1**, Sect. 139, the pipe flow can be considered as being fully-developed at $x \approx 0.1R^2U_0/\nu$). Experimental on laminar flows in circular pipes began quite early in the development of modern fluid dynamics. Hagen (1839) and Poiseuille (1840–1841) independently, and almost simultaneously, studied liquid flows in round pipes and measured, in particular, the dependence of the volume flow rate on the pressure gradient dp/dx). Their results agree excellently with Eq. (1.30); for a more detailed description of the history and the results of these early studies of pipe flows see, e.g., the surveys by Rouse and Ince (1957) and Sutera and Skalak (1993). These early experiments motivated the later naming of flow in a citcular pipe "Poiseuille flow" (sometimes, more justly, "Hagen-Poiseuille flow"). (The name of "plane Poiseuille flow" was given to the flow produced by a constant pressure gradient in the gap between two parallel planes because of its resemblance to circular Poiseuille flow). The results given here for flows in pipes were first obtained by Stokes (1845) (who also derived Eq. (1.32), below).

5. *Circular Couette* (or *Taylor-Couette*) *flow between two concentric rotating cylinders* (briefly, CTCF). Our next example is also a flow within a cylindrical boundary, the steady motion of a fluid in the annulus between two infinite coaxial concentric cylinders of radii R_1 and $R_2 > R_1$, rotating about their axis at different angular velocities Ω_1 and Ω_2. The pressure drop along the axis will be assumed to be zero, so all hydrodynamic quantities are independent of the x coordinate, measured along the axis of the cylinders. Let us again introduce polar coordinates (r, ϕ) in the plane Oyz perpendicular to the axis of the cylinders. Then it follows from symmetry that only the velocity component $u_\phi = u$ is nonzero, and that the velocity u and the pressure p depend only on the coordinate r. From this it follows that Eq. (1.1) and the first of the N-S Eq. (1.3) are satisfied identically, while the second and third Eq. (1.3), converted to polar coordinates, have the forms

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{u^2}{r}, \quad \frac{\mathrm{d}^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0. \tag{1.31}$$

The general solution of the second Eq. (1.31) has the form: $u(r) = ar + b/r$; then the boundary conditions $u(r) = \Omega_1R_1$ for $r = R_1$ and $u(r) = \Omega_2R_2$ for $r = R_2$ determine the function $u(r)$ uniquely as:

$$u(r) = \frac{\Omega_2R_2^2 - \Omega_1R_1^2}{R_2^2 - R_1^2}r + \frac{(\Omega_1 - \Omega_2)R_1^2R_2^2}{R_2^2 - R_1^2}\frac{1}{r}. \tag{1.32}$$

Using the first Eq. (1.31) it is now easy to find the function $p(r)$ but we shall not need this result here. Note only that the flow given by Eq. (1.32) is similar in many respects to plane Couette flow, considered above as Example 1—there we had to do with the flow between two different planes parallel to the same Oxy-plane and moving in the same Ox-direction with different velocities, while now we are dealing with the flow between two concentric cylinders rotating with different angular velocities. This latter flow is sometimes called the *circular Couette flow*; in fact it was first experimentally investigated by Couette (1888, 1890) in his studies of viscous friction in liquids. After circular Couette flow was so named, his name was also applied to the flow between two infinite parallel planes in relative motion, to emphasize its similarity to the cylindrical case actually investigated by Couette. Note that after the publications of the results of brilliant experimental and theoretical studies of the stability of flow between concentric cylinders rotating with different angular velocities which were performed by G. I. Taylor (1923) (for more details of these studies see Chap. 2 below), this flow has often been called *Taylor–Couette flow*.

6. *Boundary layers on solid surfaces*. Up to now we have discussed exact solutions of the Navier–Stokes equations describing some simple laminar flows in spatial regions finite in the direction of at least one coordinate axis. In all these cases the nonlinear terms in the N-S equations representing the acceleration of the fluid were zero: the remaining linear equations are much simpler to solve. Now we consider the idealized plane-parallel flow in a space which is unbounded for $x \leq 0$, and in which the fluid far upstream has constant velocity U in the Ox direction. The "spanwise" line $x = 0$, $z = 0$, $-\infty < y < \infty$ is the leading edge of an infinitely wide solid body in the region $0 \leq x \leq L$ (where L may be equal to infinity). A typical body might be one with the $x{-}z$ plane section of a lifting airfoil, but a simpler case for discussion is a thin flat plate. We seek a solution for the flow above the plate, i.e. in the half-space $z > 0$. In an inviscid flow the velocity would still be U everywhere. Since there is no upper boundary to constrain it, the real, viscous flow never becomes "fully developed" with zero acceleration like the internal flows considered above, but continues to depend on x. Exact (algebraic) solution of the full N-S equations, including the nonlinear acceleration terms, is not possible; and numerical solutions of these equations are complicated and expensive. Therefore we will discuss only approximations to the N-S equations and their—simpler—numerical solutions, which are adequately accurate in many flows at high Reynolds number Re (crudely speaking, "low viscosity" flows) over flat plates or other bodies which are important in real life. It is natural to think that at high Re the viscous terms in the N-S equations will be much smaller than the nonlinear terms and may be ignored, i.e., the fluid may be assumed inviscid. However this assumption is invalid near solid surfaces because on such surfaces the flow of a viscous fluid with any value of viscosity ν must satisfy the "no-slip" and "no-permeability" boundary conditions, according to which all components of flow velocity on these surfaces must be equal to zero. Therefore when a fluid, even with arbitrarily small viscosity, flows over a solid surface, there must be a layer near the surface in which viscosity significantly affects the motion, because there is a region of significant velocity gradient (rate of shear strain) as the velocity increases from zero at the surface to a finite value somewhere within the fluid. It is natural to think

that at very small values of ν the viscosity-dependent layer will be very thin but it will be seen that its existence can lead to large differences between the ideal inviscid flow and the real-life one. The concept of a viscosity-dependent layer near a solid surface was introduced by Prandtl (1904), who proposed the name *Grenzschicht*, translated as *boundary layer*. Prandtl noted that the thickness of a boundary layer must decrease with increase of the Reynolds number Re and he also presented a simplified set of equations of motion for such layers. Since the appearance of Prandtl's classical paper, a great deal of effort has been devoted to boundary-layer studies. The results have been discussed in practically all fluid-mechanics textbooks, and at greater length in numerous specialized monographs and survey papers. (As typical examples we may mention the books we have repeatedly cited above, by Monin and Yaglom (1971, Sect. 1.4) and by Landau and Lifshitz (1987, Sect. 39). Examples of special literature devoted to boundary layers include the survey by Nickel (1973), monographs by Goldstein (1938); Schlichting (1951); Loitsianskii (1962); Lagerstrom (1964); Cebeci and Cousteux (1999), and the fundamental treatise by Schlichting and Gersten (2000) representing the 8th English edition of Schlichting's famous book, first published in 1951 and republished many times in various countries (in particular in 9 German and 8 English editions, successively revised and enlarged). Note also that in his survey of the early development of boundary-layer studies in Russia, Loitsianskii (1970) mentioned that flows near solid bodies in moving fluids were briefly considered in the early 1880s and 1890s by two famous Russian scientists D. I. Mendeleyev and N. E. Zhukovskii (the latter name in non-Russian languages is sometimes spelled in the French phonetic form, Joukowski) who both stressed the great practical importance of such flows but did not propose quantitative models.

Now we consider the above-mentioned simple case of the boundary layer in a two-dimensional parallel flow of fluid of small kinematic viscosity $\nu = \mu/\rho$ having constant velocity U in the Ox-direction, with a flat plate in the plane $z = 0$ with its leading edge at $x = 0$ and its trailing edge at $x = \infty$. By definition of a two-dimensional flow, the lateral velocity $u_2 = v$ is zero everywhere. A *boundary layer* will be formed, in which the streamwise flow velocity $u_1 = u$ will change from $u = 0$ at $z = 0$ up to some value practically indistinguishable from U at the upper edge of the layer. At a sufficiently large value of x, L say, the Reynolds number $\mathrm{Re} = UL/\nu$ will be large; therefore it is natural to expect that the thickness δ of the boundary layer at $x = L$, compared with L, will be small.[5] Within the boundary layer, δ is a typical length scale in the Oz-direction, while L is a typical length scale in the Ox-direction. U is the typical scale of velocity anywhere in the flow. Therefore in this layer $\partial^2 u/\partial z^2$ will have the same order as U/δ^2 and thus the main viscous term of the first N-S

[5] It seems natural to expect that the thickness δ will decrease with the increase of flow viscosity (a proof of this statement will be presented somewhat later). Note also that the value of δ cannot be determined uniquely since the boundary layer has no strictly defined upper edge and the velocity $u = u(x, z)$ of flow in this layer tends with increase of z to the free-stream velocity U only asymptotically as $z \to \infty$. Hence δ must be defined somewhat artificially. In practice it is often taken to be equal to the distance from the plate at $z = 0$ up to the level z at which u attains a given, sufficiently great fraction of U, e.g., 0.99 U (in this case δ is μ sometimes denoted by δ_{99}). Some other possible definitions of the thickness δ will be indicated later.

equation $\nu(\partial^2 u/\partial z^2)$ will have the order of $\nu U/\delta^2$. However for validity of the first N-S equation, the main viscous term must have there the same order of magnitude as the main its nonlinear term $u\partial u/\partial x$, evidently having the order of U^2/L (while all other terms will have considerably smaller orders of magnitude). Hence here $U^2/L \propto \nu U/\delta^2$, i.e.,

$$\delta \propto \sqrt{\frac{\nu L}{U}} = \frac{L}{\sqrt{\mathrm{Re}}}, \quad \mathrm{Re} = \frac{UL}{\nu}. \tag{1.33}$$

(Alternatively, it is clear that if the value of δ at $x = L$ depends only on U, L and ν, the relation in Eq. (1.33) is the only dimensionally-correct one). We see that the relative thickness of the boundary layer is proportional to $(\mathrm{Re})^{-1/2}$, i.e. δ/L decreases rapidly with increasing Re. In general,

$$\delta \propto \sqrt{\frac{\nu x}{U}} \quad \text{at the point } x. \tag{1.34}$$

Note that in the boundary layer on a flat plate, $\partial u/\partial z \infty U/\delta$. Therefore the viscous shear stress at the solid surface, $\tau(x) = \mu(\partial u/\partial z)_{z=0}$ is of order $\mu U/\delta$ so, using Eq. (1.33), we find that the local skin friction coefficient $C_f = \tau/\rho U^2$ varies as $(\mathrm{Re})^{-1/2}$.

Let us now briefly discuss the calculation of the two-dimensional (briefly, 2D) velocity field $\{u(x, z), w(x, z)\}$ of a boundary layer in steady flow over a long (and infinitely wide) plate. The N-S and continuity equations have the forms

$$\begin{aligned} u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \\ u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \end{aligned} \tag{1.35}$$

where $w = u_3$ (and, as above, $u = u_1$). Recall now that within the boundary layer the velocity u is equal to a finite fraction of the free-stream velocity U and thus it has the same order as U. Hence here $\partial u/\partial x$ and $\partial u/\partial z$ have the orders of U/L and U/δ respectively. Thus integrating the third Eq. (1.35) with respect to z would show that within the boundary layer the velocity w has the order of $\delta(U/L) = (\delta/L)U$, i.e. w is approximately $(\mathrm{Re})^{1/2}$ times less than u. Further, analyzing the second Eq. (1.35) it is easy to show that all its terms, with the exception of the term containing $\partial p/\partial z$, have orders not greater than $U^2\delta/L^2 = (U^2/L)(\delta/L)$; hence for the equation to balance, the term containing $\partial p/\partial z$ must also have an order not greater than this. Ignoring terms of such low orders we can disregard the second Eq. (1.35) as a whole, using it only to show that within the boundary layer $\partial p/\partial z = 0$ to a good approximation, and thus at any point of the layer p is equal the pressure in the free stream above this point. Therefore in the first Eq. (1.35) the partial derivative $\partial p/\partial x$ can be replaced by dp/dx and may be determined from the first Euler Eq. (1.4) of ideal-fluid dynamics.

Taking into account that in a thin boundary layer $\nu\partial^2 u/\partial z^2 \gg \nu\partial^2 u/\partial x^2$, we obtain, for the case where $U = \text{const.}$, the following system of two equations determining the velocity components $u(x, z)$ and $w(x, z)$ in the two-dimensional boundary layer on a flat plate (by definition $v = 0$):

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2},$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{1.36}$$

The pressure-gradient term should be a total derivative, *dp/dx*

These are the equations derived by Prandtl in the year 1904. They are to be solved with the boundary conditions: $u = w = 0$ for $z = 0$, $0 \le x \le L$; $u(x, z) \to U(x)$ as $z \to \infty$. (In the case of unsteady flow the first Eq. (1.36) will include a contribution $\partial u/\partial t$ to the left-hand side (the acceleration following the motion of the fluid) and it will be necessary to supplement the spatial boundary conditions at $z = 0$ and $z \to \infty$ by initial conditions at $t = 0$. However this case will not be considered here.

After the appearance of Prandtl's paper of 1904 both Prandtl himself and some other authors proposed several new derivations of Eq. (1.36) showing, in particular, how these equations must be modified to make them applicable to a number of boundary layers more general than those considered in the original paper. In particular, it was then proved that, slightly modified, Prandtl's equations are applicable also to boundary layers in flows over surfaces of moderate curvature, such as airfoils. Related results may be found in cited above monographs by Goldstein (1938); Lagerstrom (1964); Loitsianskii (1962); Cebeci and Cousteux (1999), and Schlichting and Gersten (2000). Here we will discuss the numerical calculations of Blasius (1908) for the flow which we have already discussed qualitatively: the boundary layer on a very long and wide flat plate in the half-plane $z = 0$, $x > 0$, with steady flow in the half-space $z > 0$ having everywhere constant velocity U in the Ox direction. In such a flow the pressure p is the same everywhere; therefore the pressure-gradient term may be omitted from the first Eq. (1.36). For simplicity Blasius assumed that both the width D and the length L of the plate are infinite. Thus we will discuss here the solutions of two simple equations

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1.37}$$

which satisfy the conditions

$$u = w = 0 \quad \text{for } z = 0, x \ge 0; \quad u \to U \text{ as } z \to \infty \tag{1.38}$$

By virtue of the second Eq. (1.37), the "continuity equation", both velocity components u and w may be expressed in terms of the unique stream function $\psi(x, z)$

$$u = \partial\psi/\partial z, \quad w = -\partial\psi/\partial x. \tag{1.39}$$

So that the calculations could be applied to boundary layers at any Reynolds number and on any scale, Blasius chose to work in dimensionless variables: The above-mentioned statement that u/U and $w\sqrt{x/U\nu}$ must depend only on the combination in (1.33) corresponds to the following form of the stream function ψ:

$$\psi = \sqrt{\nu x U}\, f(\zeta), \quad \text{where } \zeta = z\sqrt{U/\nu x}. \tag{1.40}$$

The dimensionless z coordinate is proportional to z/δ, and the stream function has the only possible dimensionally-correct form using the present independent variables. We then have

$$u = Uf'(\zeta), \quad w = 0.5\sqrt{\nu U/x}(\zeta f' - f). \tag{1.41}$$

where w was obtained by integrating the continuity equation. Substituting the two Eqs. (1.41) into the first Eq. (1.37), we obtain the following differential equation of the third order for the function $f\ (\zeta)$:

$$ff'' + 2f''' = 0 \tag{1.42a}$$

while (1.38) leads to following boundary conditions for this equation

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{1.42b}$$

Three boundary conditions are necessary and sufficient for determination of the unique solution of the third-order differential Eq. (1.42a). Note that Eqs. (1.41) and (1.40) show that the vertical profiles of velocities $u(x, z)$ and $w(x, z)$ in laminar boundary layers on a large horizontal flat plate at different values of parameters U, ν and x will always be (geometrically) similar to each other. Also, if in Eq. (1.40) the combination νx is replaced by $2\nu x$, then all the subsequent arguments will undergo only some trivial changes but the final equation replacing (1.42a) will have the slightly simpler form:

$$ff'' + f''' = 0 \tag{1.42c}$$

(with boundary conditions. The form (1.42c) is now used more frequently than Blasius' Eq. (1.42a) in both current research and textbooks, for example the comparatively new popular books of Drazin and Reid (1981); Schlichting and Gersten (2000) and Schmid and Henningson (2001)).

In his numerical solution of Eq. (1.42a) with boundary conditions (1.42b) Blasius (1908) represented $f(\zeta)$ by a power series near $\zeta = 0$ and by an asymptotic series for $\zeta \to \infty$ and then combined these two representations. Some later authors used more accurate numerical methods, but all led to results close to those found by Blasius; see, e.g., again the above-mentioned books by Goldstein (1938); Lagerstrom (1964); Cebeci and Cousteux (1999), and, first of all, the most complete book of Schlichting and Gersten (2000) and references therein.[6] Given the solution for $f(\zeta)$, profiles

[6] Note, in particular, that an exact solution of Eq. (1.42a) under conditions (1.42b) was obtained long ago by the famous mathematician Weyl (1942).

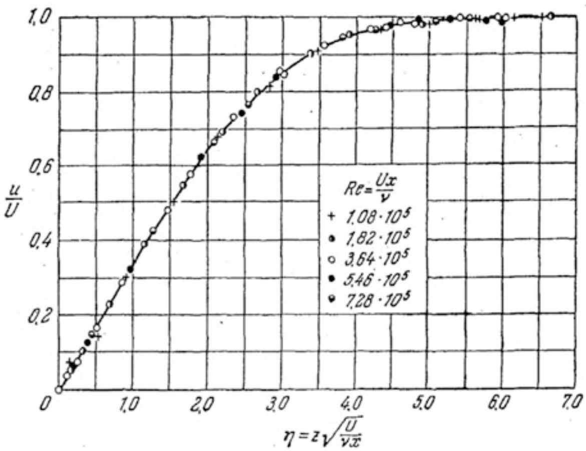

Fig. 1.1 Comparison of the computed by Blasius (1908) profiles of the longitudinal flow velocity $u(x, z)$ in the boundary layer on a flat plate with given by experimental data of Nikuradze (1942) describing the results of his measurements made in a laboratory boundary layer on a horizontal plate placed in a big wind tunnel

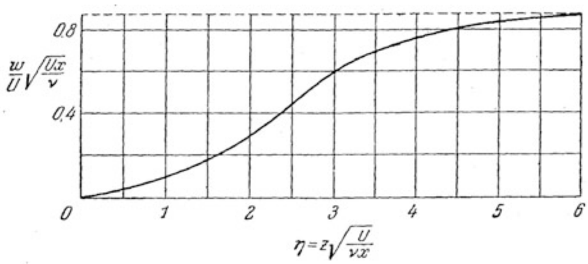

Fig. 1.2 Blasius' computed z-profile of the vertical flow velocity $w(x, z)$ in the boundary layer over a flat plate

of velocities $u(z)$ and $w(z)$ can be computed and compared with experimental data: This was a useful check on the approximate numerical method used by Blasius. Some early measurements of velocities in the boundary layers on a flat plate at $z = 0$ will be briefly discussed at the end of Sect. 2.1 (see, in particular, Fig. 2.4 there). In addition to this, an expressive early example of the comparison of wind-tunnel measurements of $u(x, z)$ in a flat-plate boundary layer with the results of numerical computations is shown in Fig. 1.1. Here Blasius' computed values of the function $u(\zeta)/U$ are compared with rather accurate experimental data of Nikuradze (1942). The agreement is excellent.[7] Blasius' computations of the normalized vertical velocity $w\sqrt{x/\nu U}$ shown in Fig. 1.2, which also agrees rather well with the much less numerous available measurements.

For the purposes of this book, we do not need to discuss Blasius' pioneering results, or later and more extensive computations of laminar boundary layers, in more detail. Somewhat more attention will be paid here to the determination of the

[7] This remarkable figure was included in the first German edition of the well-known book by Schlichting (1951) and then it was reproduced in all revised re-publications of this book and in many other books and survey papers dealing with flat-plate boundary layers.

boundary-layer thickness δ representing the main vertical (i.e., normal to the plate) length scale of the boundary-layer on a flat plate.

The question about the evaluation of the thickness δ, which was already briefly broached above, where formula (1.34), following from dimensional analysis, was given for the x-dependence of this thickness (equated there to the height δ_{99} at which $u(x, z) = 0.99U$). However this formula was incomplete since there was omitted the unknown numerical coefficient (which could not be found by dimensional analysis). The results of Blasius' calculations give the opportunity to compute a more exact value of $\delta = \delta_{99}$. In fact Blasius did this computation and; it showed that the thickness of a flat-plate boundary layer is

$$\delta \approx 5\sqrt{\frac{\nu x}{U}}. \tag{1.43}$$

The numerical factor is not an exact integer.

Let us remember that Eq. (1.43) refers to a rather arbitrary definition of boundary-layer thickness $\delta = \delta_{99}$. This definition of δ was widely used in boundary-layer studies during a number of years but later some scientists began to stress that it has no physical justification. It seemed more realistic for them to use some definition of the boundary-layer thickness which is directly related to the vertical distribution of the velocity $u(x, z)$ within the flat-plate boundary layer and thus has some definite physical meaning. As a result at least three new definitions of the flat-plate boundary-layer thickness were proposed. They are the so-called *displacement thickness* δ_1, *momentum thickness* δ_2 and *energy thickness* δ_3 defined by the equations:

$$\delta_1 = \int_0^\infty \left[1 - \frac{u(z)}{U}\right] dz, \quad \delta_2 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dz, \quad \delta_3 = \int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dz. \tag{1.44}$$

Note that symbols δ^* and θ are often used for the displacement and momentum thicknesses, respectively.

Using the results of Blasius' computations of the function $u(x)/U$, it is possible to compute the values of all three boundary-layer thicknesses (1.44); these computations led to the following results:

$$\delta_1 \approx 1.72\sqrt{\frac{\nu x}{U}}, \quad \delta_2 \approx 0.66\sqrt{\frac{\nu x}{U}}, \quad \delta_3 \approx 1.04\sqrt{\frac{\nu x}{U}}. \tag{1.45}$$

Note that all these new boundary-layer thicknesses are considerably smaller than the thickness $\delta = \delta_{99}$ of Eq. (1.43). The physical meaning of the thicknesses δ_1 and δ_2, which explains the names used here, is discussed, e.g., in the book by Monin and Yaglom (1971), pp. 50–52; for more information about all three thicknesses, see the book by Schlichting and Gersten (2000).

Above, the Reynolds number of the boundary-layer flow on a flat plate of length L was defined as $\mathrm{Re} = UL/\nu$, while near a point with given value of x the combination

$\mathrm{Re} = Ux/\nu$ was used as the local Reynolds number. However, for discussion of local properties the length scale chosen is usually the local boundary-layer thickness, so the most relevant Reynolds number is based on that thickness is most suitable, but in this case it will be necessary for the researcher to decide which of the four above definitions of the length δ is the most relevant.

It was noted above that in the early investigations of the boundary-layer flows only the thickness $\delta = \delta_{99}$ was commonly used. One of the earliest measurements of the dependence of this thickness on x in a good laboratory model of the Blasius boundary layer was performed by Hansen (1928) whose results will be shown in Fig. 2.4 of Sect. 2.1. These results at small and moderate values of x agree quite well with Eq. (1.43) confirming that δ grows with x as $\sqrt{x}$ (only the numerical coefficient in Eq. (1.43) found by Hansen was closer to 5.5 than to 5.0 but such small difference was, of course, not striking at all when results of so early and difficult laboratory measurements were considered). Much more important was Hansen's discovery that the validity of Eq. (1.43) was confirmed only at not too large values of x, while at larger x the growth of δ with x became much faster than was predicted by Eq. (1.34). This circumstance will be explained in Sect. 2.1.

Up to now we have considered only boundary layers on flat plates, the velocity outside the boundary layer (and therefore the pressure) being the same everywhere. A more complicated problem arises when the speed of flow over a flat plate at $z = 0$, $x \geq 0$, (or a body of finite thickness near the $(\mathbf{x}, \mathbf{z})$ plane) varies as $U = U(x)$. Here the first Eq. (1.37) will include the additional term $U\partial U/\partial x$ on the right-hand side indicating that the pressure $p(x)$ in the flow above the plate depends on x. On the other hand the second Eq. (1.37) and the boundary conditions (1.38) in such cases have the same forms as in the case of Blasius' boundary layer. The second Eq. (1.37) implies the possibility of expressing the velocities u and w in terms of the stream function $\psi(x, z)$ by Eq. (1.39) Moreover, the boundary conditions (1.38) and the arguments leading to Eqs. (1.40), (1.41) and to the boundary conditions (1.42b) for the function $f(\zeta)$ undergo now almost no modifications, but the basic Eq. (1.42a) determining this function will now include an additional term (or terms) produced by the term $U\partial U/\partial x$ in the first Eq. (1.37) and depending on the function $U(x)$. Theoretical studies of boundary layers in streamwise-varying flows yielded many papers, books and monographs, including, in particular, the papers by Falkner and Skan (1931); Hartree (1937); Goldstein (1939) and Mangler (1943), and special review sections in the books of Goldstein (1938), Chap. IV, Drazin and Reid (1981), pp. 231–233, Lagerstrom (1964), Sect. B.14, Panton (1996), Sect. 20.5, Cebeci and Cousteux (1999), Sect. 4.3, and Schlichting and Gersten (2000), Sect. 7.2. If the external-flow velocity is a function of x, Eq. (1.42a) or (1.42c) for the stream function are replaced by a more general equation of the form:

$$f''' + \alpha_1 ff'' + \alpha_2 - \alpha_3 (f')^2 = 0, \tag{1.46}$$

where α_1, α_2 and α_3 are three constants but the function $f(\zeta)$ satisfies the same boundary conditions, (1.42b). There was much interest in a special case, the so-called *Falkner–Skan boundary layers* to which correspond equations of the form

(1.46) where either $\alpha_1 = 1$ or $\alpha_1 = 2$, but in both cases $\alpha_2 = \alpha_3 = \beta$ (the letter β is traditionally used here). If $\alpha_1 = 1$, we obtain

$$f''' + ff'' + \beta[1 - (f')^2] = 0, \tag{1.46a}$$

While if $\alpha_1 = 2$ coefficient 2 must be added to the second term of the left side, as required by Eq. (1.46). (The two versions of this equation naturally correspond to the two versions (1.42a) and (1.42c) of the main equation of Blasius' theory of the "flat plate" boundary layer and again are due to two very similar possible normalizations of the stream function ψ and the z-wise coordinate ζ). In the extensive literature on laminar boundary layers both forms of Eqs. (1.42a) and (1.46a) may be found; in fact these forms lead to very close results differing only by some normalizing factors. Let us also note that the third form of the Falkner–Skan equation was used in the book by Cebeci and Cousteux (1999), p. 65, where coefficient β is replaced by coefficient m directly connected with the pressure gradient in the internal flow (and in the case of constant pressure gradient coinciding with coefficient m in Eq. (1.47), below).

The Falkner–Skan Eq. (1.46a) with $\beta = 0$ clearly describes Blasius' boundary layer on a flat plate in a flow of constant velocity parallel to the plate. For $0 < \beta < 1$ Eq. (1.46a) corresponds to the boundary layer in a flow past a two-dimensional wedge where $dp/dx < 0$—i.e. the velocity $U(x)$ increases with x – while for $\beta < 0$ this equation describes the flow past a corner where $dp/dx > 0$ and the velocity decreases when x increases (for more details see Schlichting and Gersten's book; some information about this matter may be also found in other monographs mentioned above Eq. (1.46)). Note that if $\beta \neq 2$, then Eq. (1.46a) corresponds to the boundary layer below an external stream whose velocity $U = U(x)$ obeys a power law:

$$U(x) \sim x^m, \quad \text{where } m = \beta/(2 - \beta) \text{ and hence } \beta = 2m/(m + 1) \tag{1.47}$$

(the case where $\beta = 2$ is here excluded; then $m = \infty$ and velocity $U(x)$ grows exponentially when x is increasing). Thus the Falkner–Skan equation is applicable to the boundary layer above a flat plate if the external flow over a considerable region of the x-axis can be approximated well enough by a power law (1.47), and in this case the value of β is connected with the exponent m of the velocity law by Eq. (1.47). We see that $0 < \beta < 2$ corresponds to favorable pressure gradients accelerating the flow, while the Falkner–Skan profiles with $\beta < 0$ correspond to adverse pressure gradients retarding the flow. In Fig. 1.3, taken from the book of Panton (1996), dimensionless velocity profiles in Falkner–Skan boundary layers are plotted for a number of positive and negative values of the parameter m. These profiles show the dependence of the normalized streamwise velocity $u(z)/U$ on the normalized distance from the surface $\eta = z\sqrt{(U/\nu x)}$). Here $U(x)$ is the external flow velocity and $u(z)$ is the velocity at the point (x, z) within the boundary layer. The profile for $m = 0$ is of course Blasius' velocity profile from Fig. 1.2, repeated for comparison, while graphs for positive and negative values of m correspond to boundary layers in flows with favorable and adverse pressure gradients, respectively.

Some solutions of Falkner–Skan Eq. (1.46a) and of more general Eq. (1.46) were considered in Sect. 7 of Schlichting and Gersten's book, Sect. 4.2 of the book of

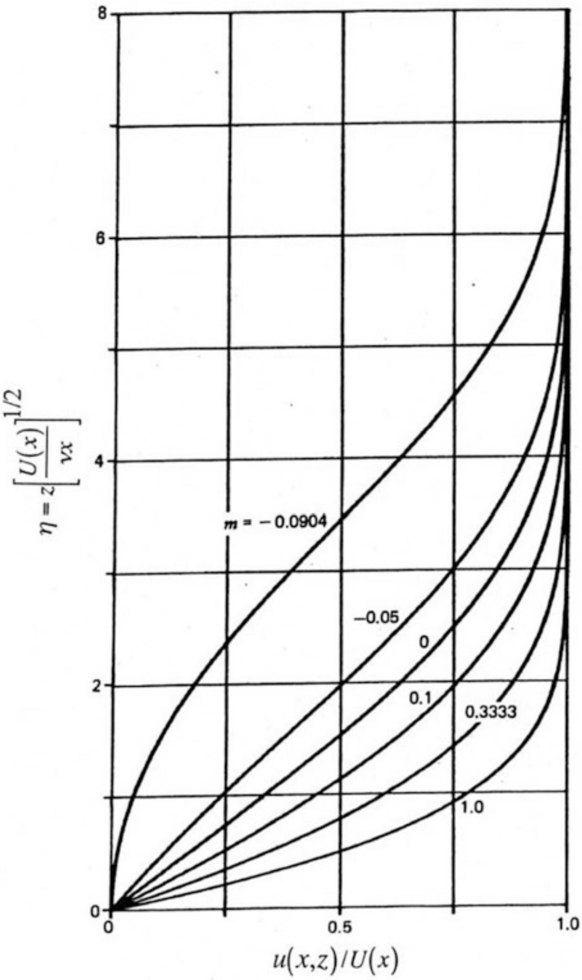

Fig. 1.3 Longitudinal velocity profiles $u(x,z)/U(x)$ in Falkner–Skan laminar boundary layers with mean velocity $U(x) \sim x^n$ at different values of the exponent m. (After Panton (1996))

Cebeci and Cousteux, and in the other books and papers cited above Eq. (1.46). Note also that mathematical properties of Eq. (1.46a) with $\beta \geq 0$ were studied by Weyl (1942) who found their exact solutions in cases where $\beta = 0$ (as mentioned above) and $\beta = 1/2$, while for other $\beta > 0$ he developed very effective methods for numerical solution of these equations. Moreover, the form of the Falkner–Skan velocity profile $u(z)/U$ corresponding to very large values of β was additionally computed by Lagerstrom (1964), pp. 125–129 (see also Drazin and Reid (1981), pp. 231–232) who also explained the physical conditions in which such a flow may arise.

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Chapter 2
Basic Experimental Facts and Introduction to Linear Stability Theory

2.1 Concept of Turbulence; Transition to Turbulence in Tubes, Channels, and Boundary Layers

The difference between laminar and turbulent regimes of flow is revealed in a number of phenomena which are of great significance for many engineering problems. For example, the action of a flow on rigid walls (i.e., friction on walls) in the case of a turbulent regime is considerably greater than in the case of a laminar regime (since the transfer of momentum in a turbulent medium is much more intense). The presence of irregular fluctuations of velocity leads also to a sharp increase in mixing (transfer) of heat or mass as well as momentum: extremely intense mixing is often considered as the most characteristic feature of turbulent motion. The increase in mixing implies an increase in the apparent diffusivities of the fluid (viscosity, thermal conductivity, etc.). For all these reasons, the determination of the conditions of transition from a laminar to a turbulent regime is a very pressing problem. Moreover, finding the mechanism of the initiation of turbulence must surely aid our understanding of its general nature and facilitate the study of the laws of turbulent flow, which are of great importance in practical work.

Unfortunately, many features of the transition from laminar to turbulent flow are not completely clear at present. Different opinions have been expressed even about the basic reasons why almost all flows become turbulent. For example, in 1934 the famous French mathematician Leray suggested that transition to turbulence is related to a breakdown of validity of solutions of the three-dimensional Navier-Stokes equations, which describe laminar fluid flow. However, this suggestion was not supported by the majority of experts in fluid mechanics. Nowadays nearly all scientists adhere to the old idea of Reynolds (1883), according to which a turbulent flow is also described by the Navier-Stokes equations, but the smooth solution of these equations, which corresponds to a laminar flow, becomes unstable to small perturbations of the flow parameters, which are always present in real life.

At present, there is an enormous body of literature devoted to the study of hydrodynamic instability and transition to turbulence. This literature includes a number of general expositions of the subject (e.g., by Lin (1961a); Chandrasekhar (1961); Betchov and Criminale (1967); Joseph (1976); Goldshtik and Shtern (1977); Drazin

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and Reid (1981); Georgescu (1985), and Zhigulev and Tumin (1987)); several dozens of monographs on various special topics related to this subject (the books by Eckhaus (1965); Gershuni and Zhukhovitskii (1972); Kachanov et al. (1982); Yudovich (1984); Straughan (1992); Koschmieder (1993), and Chossat and looss (1994) are typical examples); many specialized collections of papers (see, e.g., Eppler and Fasel (1980); Meyer (1981); AGARD Reps. (1984, 1994); Swinney and Gollub (1985); Kozlov (1985); Arnal and Michel (1990); Hussaini and Voigt (1990); Reda et al. (1991); Hussaini et al. (1992), Ashpis et al. (1993), Kobayashi (1995), and Corke et al. (1996)); many dozens (if not hundreds) of review papers (those by Schlichting (1959); Dryden (1959); Shen (1964); Drazin and Howard (1966); Stuart (1971a); Reshotko (1976); Monin (1978, 1986); Morkovin (1988, 1991); Bayly et al. (1988); Herbert (1988); Reed and Saric (1989); Kachanov (1994), and Reed et al. (1996) are only a few examples); and several thousand journal papers and reports. Also, many books and papers consider the general theory of stochastization and transition to chaotic behavior for various classes of dynamic systems, which is apparently related to transition to turbulence in fluid flows (this relation will be discussed later in this book). Of course, it is impossible to consider in one book, even if only briefly, all topics related to instability and transition. Below we shall confine ourselves to a short description of the most important (and firmly established) results for a few quite simple flows which are often encountered in practice, and devote most of our attention to the principles involved. For the details of the mathematical calculations and a more complete description of the experiments, and also for matters relating to more complex flows, the reader is referred to the aforementioned sources and/or special articles.

The first results on the conditions for the transition to turbulence were obtained by Hagen (1839). Hagen studied flows of water in straight circular tubes of fairly small diameter and established that, with gradual decrease of the viscosity of water (brought about by increasing its temperature), the velocity of flow for the same pressure drop first increases to some limit and then begins to decrease again. The jet of water issuing from the tube before this limit is reached has a smooth form, but after passing through this limit it shows sharp fluctuations. Hagen also showed that a variation in the nature of the flow may be effected by changing the pressure drop (i.e., the mean velocity) or the radius of the tube; he could not, however, obtain any general criterion for the transition from laminar to turbulent flow.

In the second half of the 19th century, important contributions to the understanding of the nature of turbulent flows were made in france by B. de Saint Venant and J. Boussinesq (see, e.g., Boussinesq (1877, 1897) and the discussion of the early history of turbulence studies by Frisch (1995), Sect. 9.6.1). It was shown that turbulence is produced by oscillatory motions of fluid elements which increase the fluid viscosity. So, the important notion of the *turbulent* (or *eddy*) *viscosity,* which will be widely used later in this book, was introduced very early on. However, the general criterion for transition to turbulence was established only by Reynolds (1883) who used the concept of mechanical similarity of fluid flows.

The Reynolds criterion is that the flow will be laminar so long as the Reynolds number $\mathrm{Re} = UL/\nu$ (where U and L are typical velocity and length scales of the flow and ν is the kinematic viscosity of a fluid) does not exceed some critical value Re_{cr}, while for $Re > Re_{cr}$ it will be turbulent. It is well known (and also is explained in Sect. 1.3 of MY1) that Reynolds number has the significance of a ratio of characteristic values of the forces of inertia and viscosity. The inertia forces lead to the approach of initially remote fluid volumes and thus contribute to the formation of small-scale inhomogeneities in the flow. The viscous forces, on the other hand, lead to the equalizing of velocities at neighboring points, i.e., to the smoothing of small-scale inhomogeneities. Thus for small Re, when viscous forces predominate over the inertia forces, there can be no sharp inhomogeneities in the flow, i.e., the fluid-mechanical quantities will vary smoothly and the flow will be laminar. For large Re, on the other hand, the smoothing action of the viscous forces will be weak, and irregular fluctuations—sharp, small-scale inhomogeneities—will arise in the flow, i.e., the flow will become turbulent.

To verify this criterion experimentally and to measure the value of Re_{cr} for a *circular Poiseuille flow,* Reynolds carried out a series of experiments with water flows in glass tubes connected to a reservoir. In these experiments a source of colored liquid was placed on the axis of the tube at the intake. For small Re, the colored water took the form of a thin, clearly defined filament, indicating a laminar regime of flow. As Re increased, at the instant of passing through the critical value, the form of the colored jet sharply changed; at quite a small distance from the intake into the tube, waves appeared in the filament; further on, separate eddies were formed, and towards the end of the tube the whole of the liquid was colored. If in such an experiment the flow is illuminated by an electric spark, it may be seen that the colored mass consists of more or less distinct swirls which indicate the presence of vorticity. For subcritical Re numbers close to Re_{cr}, transient phenomena are observed in a laminar flow. According to Reynolds these phenomena consist of the intermittent appearance of short-term bursts ("flashes") of high-frequency fluctuations which fill the entire cross section of the tube but only for fairly short sections of its length. In the initial part of the tube, with $Re > Re_{cr,}$ the flow has a similar character. However, as Re increases, the length of the initial part where the flow is not entirely turbulent decreases rapidly; at large enough values of Re, the flow usually becomes turbulent at a short distance from the beginning of the tube.

Subsequent experimental studies of tube flows are very numerous; the work by Schiller (1932, 1934); Binnie and Fowler (1947); Rotta (1956); Lindgren (1957, 1959–1963, 1965, 1969); Leite (1959); Wygnanski and Champagne (1973); Wygnanski et al. (1975); Rubin et al. (1980); Teitgen (1980); Bandyopadhyay (1986); Huang and Huang (1989); Breuer and Haritonidis (1991); Darbyshire and Mullin (1995), and Draad et al. (1995) are just typical examples. Interesting data were also obtained by O'Sullivan and Breuer (1994) from direct numerical simulations of tube flows; an excellent presentation of some of the experimental results for such flows can be found in the textbook by Tritton (1988), Sects. 2.6 ad 18.3. It was found in the above-mentioned works that for $Re > Re_{cr}$, but in the intermediate ("transitional") regime described above, the mean fraction of the time in which a turbulent regime is observed at a given point (the "coefficient of intermittency") increases monotonically

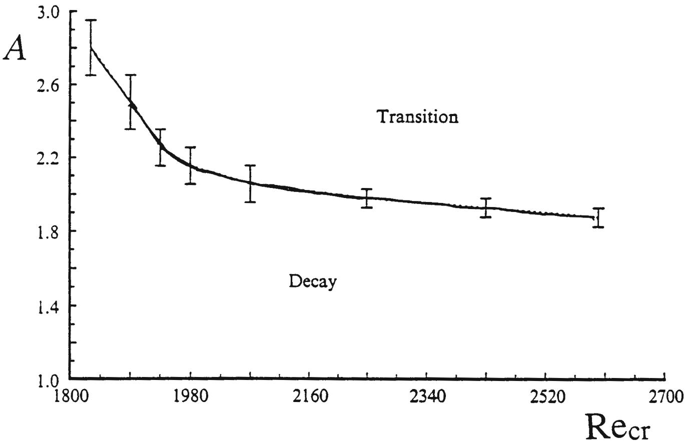


Fig. 2.1 Approximate dependence of the critical Reynolds number Re_{cr} for a tube flow on the amplitude A (measured in some special units) of a disturbance produced by a short jet ejected in the tube through a hole. The error bars indicate the range of amplitudes within which decay of the disturbance or transition to turbulence are both possible. (After Darbyshire and Mullin (1995))

with increase of the distance x from the intake into the tube (or from another cross-section where an initial disturbance was generated). This increase is explained by the fact that the local velocity of the leading edge of a "turbulent flash" filling the whole cross section exceeds the local velocity of its trailing edge. (Reynolds' "flash" is now usually called a "turbulent slug"; this name has superseded the earlier name of "turbulent plug".) This shows that laminar fluid near each end of the slug is brought into turbulent motion and makes the slug longer. As a result, some slugs occasionally overtake others and then the two coalesce to form a single, longer slug. It was also found by Wygnanski and his collaborators that two different types of intermittent turbulent formations can occur in a circular tube. The first type is represented by the above-mentioned slugs and is usually caused by small disturbances in the boundary layer at the initial portion of the tube wall. The slugs usually appear at relatively large values of Re (most often above 3000, if $Re = U_m D/\nu$ is formed from the flow velocity averaged over the tube cross-section, U_m, and the tube diameter D). The formations of the second type are called "turbulent puffs"; they are generated by larger disturbances at the entrance of the tube and can be observed at lower values of Re. Both the puffs and the slugs can be also produced by artificial disturbances within the fully developed Poiseuille flow (see, e.g., the papers by Darbyshire and Mullin, by Draad et al. and Fig. 2.1). At small enough values of Re and small amplitude of the initial disturbance, puffs generated through an artificial disturbance eventually decay and the flow reverts to laminar behavior, while at higher Reynolds numbers and/or disturbance amplitudes, puff splitting occurs and two puffs are formed from one. At even higher values of Re, the puffs coalesce and form a slug and then transition to developed turbulence can occur (Rubin et al. (1980); Breuer and Haritonidis (1991); Darbyshire and Mullin (1995)). The streamwise velocity variations are usually quite different for slugs and puffs (see, e.g., the above-cited papers by Wygnanski and Champagne, Wygnanski et al., Teitgen, Bandyopadhyay, Huang and Huang, and Darbyshire and Mullin; examples of instantaneous velocity fields in planes interacting a slug are presented by Draad et al.). Note also that the slugs usually appear randomly in time at a small distance from the tube intake, but under some conditions they can also originate periodically; see Tritton (1988), and Stassinopoulos et al. (1994) for possible explanations.

Reynolds' experiments were carried out in tubes of various diameters with a smooth intake from the reservoir. Variation of $\mathrm{Re} = U_m D/\nu$ was accomplished by varying the rate of flow or the viscosity of the water (by altering the temperature) or by changing to a tube of different diameter D. The value of $\mathrm{Re}_{\mathrm{cr}}$ in these experiments was on the average close to 12,830. However, such results were obtained only by exercising the greatest care to eliminate disturbances in the water entering the tube. Reynolds' investigations showed that the value of $\mathrm{Re}_{\mathrm{cr}}$, corresponding to the transition from laminar to turbulent flow, depends considerably on these disturbances (or, as we say, the "initial turbulence," which is determined principally by whether the entry to the tube is sharp or streamlined). Therefore, $\mathrm{Re}_{\mathrm{cr}}$ may prove to differ very much between different experiments. If the tube entry is sharp, disturbances are produced by vortices separating from the leading edge. These disturbances are usually characterized by relatively short-time, high-frequency fluctuations of velocity. Their intensity U'/U_m (where U' is the typical magnitude of streamwise velocity fluctuation) may be fairly large, but, if Re is low enough, the disturbances do not alter the laminar nature of the flow and have no significant effect on the mean velocity profile and the pressure drop. For such Reynolds numbers, the disturbances which arise are attenuated as they travel downstream. If the Reynolds number is increased, then at the instant that it attains its critical value (which depends on the intensity of the disturbances and possibly on their scale and frequency), the disturbances suddenly generate turbulence. It is found that when the value of $\mathrm{Re}_{\mathrm{cr}}$ corresponding to transition to turbulence is the smaller, the greater the intensity of the disturbances. References to a number of successive experiments in which the transition to turbulence was delayed to higher and higher values of $\mathrm{Re}_{\mathrm{cr}}$ by reducing the intake disturbances by various means can be found in MY1 on pp. 75–76 (this delay is called *persistence of the laminar regime*). Here we only note that in the recent experiments by Draad et al. (1995) the tube flow was laminar at $\mathrm{Re} = 53{,}000$ while much earlier Pfenninger (1961) showed that a tube flow can be fully laminar even at $\mathrm{Re} = 100{,}000$.

We see that the Reynolds number in itself is not a unique criterion for transition to turbulence; for flow in a tube it is apparently impossible to find a universal value $\mathrm{Re}_{\mathrm{cr}}$ such that for $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ the flow is bound to be turbulent. The maximum Reynolds number at which the initial disturbances decay and hence laminar flow is maintained is a function of U'/U_m, $\mathrm{Re}_{\mathrm{cr}}(U'/U_m)$ say, which decreases with increasing U'/U_m. To show this, Darbyshire and Mullin (1995) studied a laminar water flow in a long tube where disturbances of several types having various amplitudes A were injected into the flow through a hole (or several holes) in the tube placed $70D$ downstream of the tube entry. They found that in this case the value $\mathrm{Re}_{\mathrm{cr}}$, such that at $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ the disturbances decay while at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ transition to turbulence occurs, decreases with the increase of A, but the function $\mathrm{Re}_{\mathrm{cr}}(A)$ is in this case "fuzzy" and not single-valued since at any Re there is a short range of amplitudes within which either outcome is possible (see Fig. 2.1; similar graphs were also given in this paper for some other types of disturbance).

The value of $\mathrm{Re}_{\mathrm{cr}}$ depends, in general, not only on disturbance amplitude but also on other characteristics of the initial disturbances and on the flow conditions producing the disturbances. For example, since disturbances can be generated by roughness of the inner surface of the tube, the value of $\mathrm{Re}_{\mathrm{cr}}$ must depend on the tube roughness and decrease with increasing roughness height. However, in this book we shall not consider this effect (which is of importance not only for tube flows but also for other flows along solid walls), and here we restrict ourselves to references only to the old survey by Goldstein (1938) and the experimental investigations of roughness influence on tube-flow transition to turbulence by Lindgren (1959–1963), Parts VII and VIII. Some more recent studies of roughness effect will be mentioned later when the stability of boundary-layer flows is considered.

Without knowing the degree of disturbance of the laminar flow, we can apply only a rather weak criterion to indicate the conditions under which only a laminar regime of flow is possible. To find this criterion we must determine the Reynolds number $\mathrm{Re}_{\mathrm{cr\ min}}$ which Corresponds to the transition from laminar to turbulent flow for the largest possible level of disturbance of the laminar flow at the intake into the tube. For $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr\ min}}$ the flow will always remain laminar, i.e., any disturbance, regardless of its intensity, will be damped.

Experiments to measure $\mathrm{Re}_{\mathrm{cr\ min}}$ were performed by Reynolds himself. Since in these experiments it is necessary to introduce into the tube fluid that is as disturbed as possible, the method of dye injection is clearly unsuitable here. As a result we must determine the transition to turbulent regime in some other way(for example, by the variation of the skin friction law which determines the dependence of the mean velocity on the pressure drop). In Reynolds' experiments the minimum critical value of $\mathrm{Re} = U_m D/\nu$ was found to be $\mathrm{Re}_{\mathrm{cr\ min}} \approx 2030$. Values close to this (usually lying between 1750 and 2300) were also obtained in subsequent investigations by Binnie and Fowler (1947); Leite (1959); Lindgren (1957, 1959–1963); Sibulkin (1962); Wygnanski et al. (1975); Breuer and Haritonidis (1991); and Darbyshire and Mullin (1995). In particular, Wygnanski et al. and Breuer and Haritonidis found that $\mathrm{Re}_{\mathrm{cr\ min}} \approx 2200$, while Darbyshire and Mullin, who considered several types of disturbances and take into account the fuzziness of the dependence of $\mathrm{Re}_{\mathrm{cr}}$ on disturbance amplitude, obtained a slightly smaller estimate, $\mathrm{Re}_{\mathrm{cr\ min}} \approx 1760$.

Similar results were also found for flows in channels of rectangular cross-section. Transition to turbulence in such flows was studied, for instance, by Davies and White (1928); Sherlin (1960); Narayanan and Narayana (1967); Patel and Head (1969); Karitz et al. (1974); Nishioka et al. (1975); Carlson et al. (1982); Kozlov and Ramazanov (1980, 1984); Herbert (1983, 1984); Nishioka and Asai (1985); Alavyoon et al. (1986); Klingmann (1989, 1992), and Klingmann and Alfredsson (1990). In most of this work the aspect ratio (width-to-depth ratio of channel) was high enough to justify regarding the channel flow as a *plane Poiseuille flow* (i.e., the flow between infinite parallel planes at $z = 0$ and $z = H$ produced by a constant pressure gradient dp/dx and having a parabolic velocity profile; see, e.g., pp. 34–35

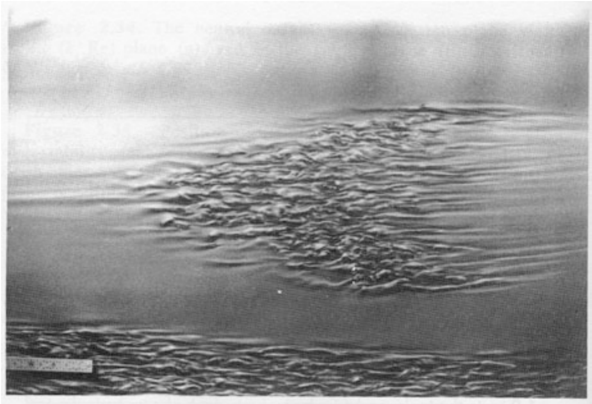


Fig. 2.2 "Turbulent spot" in the transitional flow in a water channel made visible by addition of small aluminium flakes to water. (After Cantwell et al. (1978))

in MY1). In fact, the strict conditions for this are not clear and, as was stated in a review paper by Herbert (1983), this may explain some part of the discrepancies in the data from different laboratories. However, we will not dwell on this circumstance, and for simplicity will call all the flows discussed below "plane Poiseuille flows".

According to thorough measurements by Carlson et al. and Alavyoon et al. transition to turbulence in a plane Poiseuille flow occurs at a Reynolds number $\mathrm{Re} = UH_1/\nu$ (where H_1 is the half-depth of the channel and U is the maximum velocity at channel axis) slightly exceeding 1000; more precisely, according to those authors, $\mathrm{Re}_{cr} \approx 1100$ for such flow. Nearly the same estimate $\mathrm{Re}_{cr} = 1000$ was suggested by Orszag and Kells (1980) on the basis of direct numerical simulations of the time evolution of a plane Poiseuillc flow with some combinations of two- and three-dimensional wave-like disturbances superposed on it. The available data also show that the value of Re_{cr} can be considerably increased by reducing the disturbances in the flow. Karnitz et al., by reducing U'/U to 0.3 %, maintained the flow laminar up to $\mathrm{Re} = 5000$ and Nishioka et al. found that at a level of 0.05 % plane Poiseuille flow remains laminar up to $\mathrm{Re} \approx 8000$. Patel and Head (1969) were apparently the first to note that the transition of a plane Poiseuille flow to turbulence begins with the sudden appearance of bursts of intense velocity fluctuations in the form of isolated "turbulent spots". (Such sopts were in fact detected as far back as the early Fifties in boundary layer transition, which will be considered below.) Later the flow-visualization studies by Carlson et al. (1982) and Alavyoon et al. (1986), hot-wire measurements by Henningson and Alfredsson (1987), Klingmann (1989, 1992), and Klingmann and Alfredsson (1990), and numerical simulations of disturbance development by Henningson et al. (1987); Henningson (1989), and Henningson and Kim (1991) revealed many additional features of these spots and of the transition process initiated by them in a plane Poiseuille flow (see, e.g., and excellent picture of a turbulent spot in a water flow resembling plane Poiseuille flow in Fig. 2.2 taken from the paper by Cantwell et al. (1978)), It was shown in papers mentioned above that after the first spots appear and begin to expand, new spots continue to appear (i.e., the turbulent region is spreading) and then the spots begin merging with each other; as a result the whole flow becomes turbulent. Many other interesting details of the structure of turbulent spots have been deduced from experimental and numerical data; some of them will be mentioned below.

Let us now consider data on transition to turbulence of a *plane Couette flow* between two infinite parallel planes, one of which is stationary and the other moving with constant velocity U. This flow has a linear velocity profile (see, e.g., pp. 33–34 in MY1), and is the simplest exact solution of the Navier-Stokes equations, but it is more difficult to make a satisfactory laboratory test rig for plane Couette flow than for flows in a circular tube or rectangular channel. Therefore it is not surprising that there have not been very many successful laboratory studies of a plane Couette flow. The first of them were apparently due to Reichardt (1956, 1959), and Robertson (1959); the more recent experiments that include attempts to determine the value of Re_{cr} for plane Couette flow are those by Leutheusser and Chu (1971); Aydin and Leutheusser (1979, 1987, 1991); Tillmark and Alfredsson (1991, 1992); Daviaud et al. (1992); Dauchot and Daviaud (1994, 1995), and Malerud et al. (1995), see also Bech et al. (1995) and the summary report by Tillmark (1995) where the description of some other experiments can be found. There have also been attempts to estimate the critical Reynolds number for a plane Couette flow from studies of circular Couette flow between rotating cylinders of radii R_1 and $R_2 > R_1$ at different dimensionless widths $\gamma = (R_2 - R_1)/R_1$ by examination of the limit of Re_{cr} as $\gamma \to 0$ (Wendt (1933); Taylor (1936b); Tillmark and Alfredsson (1992)). Moreover, plane Couette flows can now be rather accurately simulated numerically (see the description of several such simulations by Bech et al. (1995)). Such simulations were used for the determination of Re_{cr} and for detailed study of the transition process by Orszag and Kells (1980), and by Lundbladh and Johansson (1991).

The difficulties in setting up a plane Couette flow experimentally may explain part of the spread in measured values of Re_{cr}. Another part is apparently due to the absence of a rigorous and unique definition of Re_{cr}. Usually this number is determined in the experiments by introducing some particular disturbance in the flow and examining its further development. It is, as a rule, assumed that at small enough Re the disturbance must decay; beginning from some value of Re it produces a spot similar to that shown in Fig. 2.2; and only if $\text{Re} \geq \text{Re}_{cr}$ will the spot persist after the disturbance is removed. However, the value of Re_{cr} determined in this way can naturally depend on the nature and intensity of the disturbance.

Values of Re_{cr} for a plane Couette flow obtained before 1990 were in the wide range of 260–750, where Re is defined in terms of half the channel depth H_1 and half the velocity difference U_m. (For example, according to Leutheusser and Chu (1971); and Aydyn and Leutheusser (1979, 1987, 1991), $\text{Re}_{\text{cr}} \approx 260$–300, while according to Reichardt (1956) $\text{Re}_{\text{cr}} \approx 750$). The early numerical-simulation study by Orszag and Kells (1980) showed only that $\text{Re}_{\text{cr}} < 1000$. The results of more recent experiments are considerably closer to each other. So, the experimental results by Tillmark and Alfredsson (1992), and Tillmark (1995) agree excellently with the results obtained by Lundbladh and Johansson (1991) from direct numerical simulation: all these studies lead to the conclusion that apparently $\text{Re}_{cr} \approx 360$ (the same estimate was given by Bech et al. (1995)). The close estimate $\text{Re}_{cr} \approx 370$ was found in two quite different experiments by Daviaud et al. (1992), and Malerud et al. (1995). However Dauchot and Daviaud (1994, 1995), who used the same apparatus as Daviaud et al., pay attention to the difference between two critical Reynolds numbers: the number $\mathrm{Re}_{cr\,\min}$ such that at $\mathrm{Re} < \mathrm{Re}_{cr\,\min}$ any disturbance must decay, and the number Re_{cr} guaranteeing the possibility of formation of a persistent spot. Dauchot and Daviaud stressed that at $\mathrm{Re} > \mathrm{Re}_{cr\,\min}$ a persistent spot is formed only If the disturbance amplitude A exceeds some critical value A_{cr}, and they studied the dependence of $A = A_{cr}(\mathrm{Re})$ on Re. It was-found that this dependence is well represented by a power law of the form $A_{cr}(\mathrm{Re}) \propto (\mathrm{Re}-\mathrm{Re}_{\mathrm{c}})^{-\alpha}$ where $\mathrm{Re}_{\mathrm{c}} \approx 325$ and $\alpha > 0$. Hence $\mathrm{A}_{cr}(\mathrm{Re}) \to \infty$ as $\mathrm{Re} \to \mathrm{Re}_{\mathrm{c}}$ and this gives grounds for concluding that $\mathrm{Re}_{cr\,\min} = \mathrm{Re}_c \approx 325$. The difference between $\mathrm{Re}_{cr\,\min}$ and Re_{cr}, and the dependence of the effect of the disturbance on its amplitude, can explain most of the scatter in experimental values of Re_{cr} for a plane Couette flow. The flow-visualization studies by Tillmark and Alfredsson, Deviaud et al., and Dauchot and Daviaud, the precise optical measurements by Malerud et al., and the careful treatment by Lundbladh and Johansson of data from direct numerical simulations in the presence of superposed disturbances, have revealed many details of the transition to turbulence. All these studies showed that the general features of transition in a plane Couette flow are qualitatively similar to those for transition in a plane Poiseuille flow, though quantitative characteristics in the two flows are somewhat different.

Results for transition to turbulence similar to those given above have been obtained in studies of boundary layer flows around solid bodies. Transition to turbulence in the boundary layers on the wings and bodies of aircraft, and the bodies of ground and marine vehicles, very strongly affects the drag and is therefore.of paramount importance, especially for aviation. Moreover, transition plays an important role in many other engineering processes. Therefore it is not surprising that an enormous body of literature (which includes a great part of the material in review paper and specialized collections referred to at the beginning of this section) is devoted to boundary—layer stability and transition studies. Here we shall consider only the simplest case of the boundary layer formed on a flat plate by a flow of constant velocity U, flowing parallel to the plate; moreover, at present we shall restrict ourselves to consideration of a few classical facts discovered rather early. Some results of more recent and more refined experimental studies of transitions in boundary-layer flows will be considered later, together with results of related theoretical investigations.

The Reynolds number of the boundary layer may be defined, e.g., by $\mathrm{Re}_\delta = U\delta/\nu$, where δ is the thickness of the boundary layer. Alternatively we may use the more easily measured quantity $\mathrm{Re}_{\mathrm{x}} = Ux/\nu$, where x is the distance from the leading edge of the plate measured along the flow. The numbers Re_δ and Re_{x} are connected by a functional dependence; for example, for a laminar flow without longitudinal pressure gradient it is known that $\mathrm{Re}_\delta \approx 5(\mathrm{Re}_{\mathrm{x}})^{1/2}$ (see, e.g., Eq. (1.49) in MY1). Proceeding downstream, both Re_δ and Re_x increase, and at some point x_{cr}, they attain the "critical value" when the flow sharply changes its nature and becomes turbulent. Thus, for $x < x_{\mathrm{cr}}$ (more precisely, for $\mathrm{Re}_\delta < \mathrm{Re}_{\delta\mathrm{cr}}$ and $\mathrm{Re}_{\mathrm{x}} < \mathrm{Re}_{\mathrm{xcr}}$) the flow in the boundary layer will be laminar while for $x > x_{\mathrm{cr}}$ (i.e., $\mathrm{Re}_\delta > \mathrm{Re}_{\delta\mathrm{cr}}$ and $\mathrm{Re}_{\mathrm{x}} > \mathrm{Re}_{\mathrm{xcr}}$) it will be turbulent. In the neighborhood of x_{cr} a "mixed regime" is formed in which only discrete bursts of turbulence are observed; these arise at some points in the form of turbulent spots which grow in size and coalesce with each other as they move downstream. The appearance of turbulent spots leads to the realization, at points near x_{cr}, of alternating laminar and turbulent flow conditions. It is clear that at the beginning of the transition region both the frequency of occurrence and the duration of the turbulent state are quite small, while at the end of it the flow becomes laminar only very rarely and briefly. It should be noted that the transition region can be quite long, a significant fraction of x_{cr}, especially if disturbances in the external flow are weak.

We see that transition to turbulence in boundary-layer flows proceeds similarly to transition in plane Poiseuille and Couette flows. Note that transition via turbulent spots was in fact first discovered in boundary layers by Emmons (1951) and Emmons and Bryson (1951); see, e.g., Schlichting (1959); Sect. 14, Elder (1960); and Coles (1962). (Only a few rather old sources are indicated here since we restrict ourselves to a rough, simplified description of the transition process. There is extensive experimental and theoretical literature devoted to the study of turbulent spots, their origin, structure, and evolution; some of the results obtained and a number of additional references can be found, e.g., in the surveys by Tritton (1988); Sect. 18.2, and Riley and Gad-el-Hak (1985); and in the recent theoretical paper by Conrado and Bohr (1995).) The study of the appearance and subsequent evolution of "turbulent spots" in plane Poiseuille and Couette flow was of course stimulated by the boundary layer results.

The first measurements of the critical Reynolds number of a boundary-layer flow were made quite early. In 1924 Burgers and Van der Hegge Zijnen, and in 1928 Hansen, made such measurements in wind tunnels (again see Schlichting (1959) and also Fig. 2.4 in MY1, borrowed from Hansen's paper). According to the data of these authors (confirmed also by data from some subsequent investigations)

$$\mathrm{Re}_{\mathrm{xcr}} = (Ux/\nu)_{\mathrm{cr}} \sim 3 \times 10^5 \div 5 \times 10^5,$$

which corresponds to

$$\mathrm{Re}_{\delta\mathrm{cr}} = (U\delta/\nu)_{\mathrm{cr}} \sim 2750 \div 3500.$$

Later it was shown that, as in the case of flows in tubes and plane channels, the critical Reynolds number of a boundary layer depends considerably on the disturbance level of the ambient flow. Depending on this level $\mathrm{Re}_{\mathrm{xcr}}$ may vary from 1×10^5 to almost 3×10^6 (see Fig. 2.3, which illustrates the dependence of $\mathrm{Re}_{\mathrm{x\,cr}}$ on U'/U, where $3U'^2$ is the mean square velocity fluctuation of the ambient flow). According to this Figure (which is taken from a relatively old survey but on the whole agrees satisfactorily enough with modern data) $\mathrm{Re}_{\mathrm{x\,cr\,min}}$ is of the order 10^5 for boundary-layer flow and $\mathrm{Re}_{\mathrm{xcr}}$ tends to a definite limit (of the order 3×10^6) as the intensity of existing disturbances tends to zero. Note however that in Fig. 2.3 the disturbances are characterized only by the corresponding mean square velocity fluctuation, measured with a hot wire, though in fact $\mathrm{Re}_{\mathrm{xcr}}$ can also depend on some other disturbance characteristics (e.g., on typical length scale; see in this respect Eq. (2.1) below). Furthermore

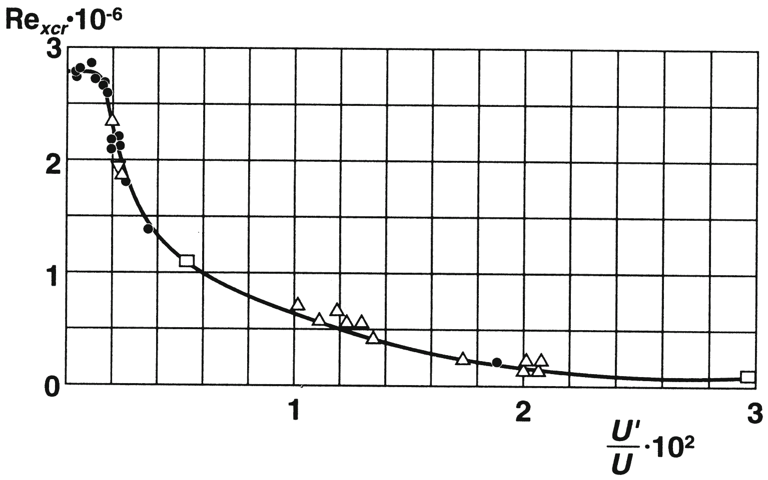


Fig. 2.3 Dependence of the critical Reynolds number for the boundary layer on a flat plate on the free-stream disturbance level (After Dryden (1959)). (The different symbols on the figure denote the data of different investigators)

sound waves in the wind tunnel can affect transition: a hot wire can record velocity fluctuations due to sound waves as well as turbulence, but a global mean square velocity fluctuation may not be an adequate standard of comparison for results from different tunnels with different mixes of sound waves and true turbulence.

Stimulation of transition to turbulence by disturbances in the flow also explains the fact that the critical Reynolds number is considerably smaller for flow along a rough plate with natural or artificial irregularities. According to numerous measurements, even one isolated wall irregularity can bring about the transition of a laminar houndary layer into a turbulent one, provided that this irregularity is large enough. Even more important is the effect of a number of irregularities scattered over the whole plate (see also a similar remark about the roughness effect on tube-flow transition above in this section, the references there, and the reviews and collections of papers cited at the beginning of the section, which often discuss the wall-roughness effect; Morkovin (1990b) is only one example of such discussion). The value of Re_{xcr} is also changed considerably by the wall curvature and/or even quite small longitudinal pressure gradients in the incident flow (positive (adverse) pressure gradients usually destabilize the flow while negative (favorable) gradients as a rule increase stability). In accord with the suggestion by Morkovin (1969), the strong effect of the above-mentioned factors on boundary-layer transition is often called "receptivity" (to external disturbances, roughness, wall curvature, longitudinal pressure gradients, etc.); in this respect see also Sect. 2.92 of this. book. However, here we shall only refer again to the literature indicated at the beginning of this section, and briefly consider one special effect produced by a longitudinal pressure gradient which is often of great practical importance and can also shed some light upon the possible Influence of disturbance scale on Re_{xcr}.

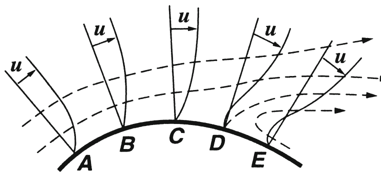


Fig. 2.4 Schematic form of the streamlines (*dotted*) and the velocity profiles (*solid*) at different points in the flow past a right cylinder

2.2 Flow Past Solid Bodies; Boundary-Layer Separation the Drag Crisis and the Dependence of $\mathrm{Re}_{\mathrm{cr}}$ on Intensity and Length Scale of Available Disturbances

Transition to turbulence in a flow of viscous fluid past a solid body may occur not only in the boundary layer but, alternatively, in the wake behind the body, whether the wake begins at the trailing edge (streamlined body) or at separation points part way along the body (bluff body).

The formation of the turbulent wake of a bluff body is generally connected with the retarding action of a positive longitudinal pressure gradient in the flow. Let us consider, for example, a right circular cylinder with an irrotational flow perpendicular to its axis (see Fig. 2.4 which shows the flow past the upper part of the cylinder). Outside the boundary layer the fluid may be assumed ideal and its motion irrotational. The streamlines of this potential motion are closest together on the upper part of the cylinder (point *C*) where the tangential velocity attains a maximum. By the well-known Bernoulli equation

$$u^2/2 + p/\rho = \text{const}$$

the pressure in the outer flow will attain a minimum at *C*, so that it will decrease along *AC* and increase along *CE*. Such changes in the pressure along the surface of the body will also take place in the boundary layer (since across the boundary layer the pressure hardly varies). Consequently, on *CE* the fluid in the boundary layer must move in the direction of increasing pressure, which leads to retardation. This retardation will have the strongest effect, of course, on the fluid particles moving very close to the surface of the cylinder, i.e., possessing the least velocity. At some point *D* downstream from *C*, these particles will come to a standstill, and beyond *D* they will be moving backwards with respect to the fluid particles further from the cylinder surface which have not been so strongly retarded. The formation of reversed flow on the surface of the body beyond *D* forces the outer flow away from the surface of the cylinder; and "separation of the boundary layer" from the surface occurs, with formation of a dividing stream-surface *DF* in the fluid. It is clear that the velocity *U* decreases sufficiently rapidly beyond *C,* separation of the boundary layer is bound to occur. Even if the boundary layer is laminar before separation, after separation it will behave as a free mixing layer and will quickly become turbulent (for considerably smaller Re than for an unseparated boundary layer since ther presence of the wall has a stabilizing effect on the flow.) The dividing stream-surface *DF,* which at infinite

Reynolds number is a surface of tangential discontinuity of the velocity, is very unstable (see below) and is quickly transformed into one or more vortices. In the region *FDE* beyond the dividing stream-surface a large-scale vortex is formed close to the cylinder, with a second such vortex being formed on the lower part of the cylinder. These vortices separate in turn from the cylinder, are carried downstream, and gradually are dispersed; in their place new vortices are formed (see, e.g., Photos 42–48 in the album by Van Dyke (1982)).

As a result, a turbulent wake is formed behind the body in which the motion is rotational, while outside the layer the motion is irrotational (i.e. potential). In fact, the fluid outside the boundary layer may be assumed to be ideal; it follows, therefore, that during its motion, the circulation of the velocity along any closed contour is conserved, and hence in the case of steady motion the curl of the velocity is constant along streamlines. Therefore, it is evident that a region of turbulent rotational flow at a distance from a body can only arise when streamlines leave the boundary layer (in which the motion will be rotational due to the viscosity), i.e., when the fluid from the boundary layer is mixed with that of the region outside it.

It is also clear that the streamlines cannot leave the region of flow in which the curl of the velocity is nonzero, i.e., the region of the turbulent wake (although they can enter the wake from the region of potential flow). In other words, fluid can flow into the turbulent wake from the potential (irrotational) region but cannot flow out of the turbulent wake. At the same time, fluctuations of velocity can penetrate from the truly-turbulent wake into the region of potential flow, although with considerable attenuation. In fact, for potential motion of an incompressible fluid, the dynamical vorticity equations (obtained by applying the operation of taking the curl to the Navier-Stokes equations of motion) will be satisfied identically. Therefore, in this case the flow will be described by the single condition of incompressibility $\partial u_i/\partial x_i = 0$. This condition is equivalent to Laplace's condition $\nabla^2\phi = 0$ for the velocity potential ϕ, which defines the velocity: $u_i = \partial\phi/\partial x_i$. Let z be the coordinate across the wake. Then the field $\phi(x, y, z)$, which describes the velocity fluctuations, may conveniently be decomposed into periodic components of the form $\phi = \phi_0(z)\ \exp\ [i(k_1x + k_2y)]$. From $\nabla^2\phi = 0$, it follows that $d^2\phi_0/dz^2 = k^2\phi_0$, where $k = (k_1^2 + k_2^2)^{1/2}$ is the wave number, inversely proportional to the horizontal (Oxy-plane) length scale of the periodic fluctuations under consideration. Discarding the physically meaningless solution for ϕ_0 which increases with increasing z, we find that the attenuation of the fluctuation amplitude in the region $z > 0$ is given by the factor e^{-kz}. Therefore, the quicker the fluctuations are attenuated, the smaller the scale. Consequently, at a sufficient distance inside the potential motion, only comparatively smooth large-scale fluctuations arise. For such fluctuations the energy dissipation does not play a large role; thus almost all the dissipation in the flow will take place in the rotational turbulent wake.[1]

[1] The analysis described is due to Landau; see, e.g., Landau and Lifshitz (1987), Sect. 35, or any of the numerous previous editions of this book. This analysis plays the central role in the detailed investigations of the velocity field outside a turbulent region of the flow that were begun by Phillips (1955) and Stewart (1956).

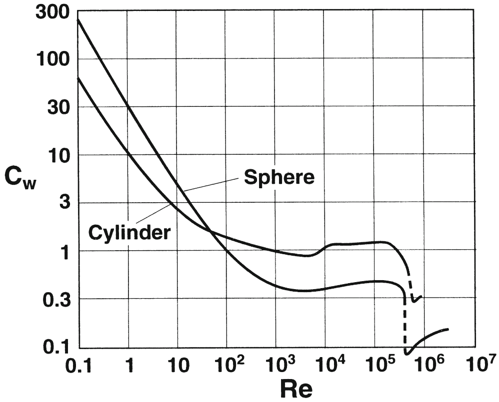


Fig. 2.5 Dependence of the drag coefficients of a sphere and a circular cylinder on the Reynolds number

The considerable energy dissipation in the whole region of the turbulent wake leads to a considerable increase in drag of bodies having separated boundary layers. As a rule, this drag will be the smaller the narrower the turbulent wake, i.e., the further downstream separation occurs. For sufficiently large Reynolds numbers (but assuming that the boundary layer remains laminar up to the separation point) the drag coefficient $C_w = \frac{W}{0.5\rho U^2 S}$ (where W is the total drag and S is the area of the body or its cross section) is independent of Re because the position of the separation point is independent of Re (as is found from the equation $(\partial u/\partial z)_{z=0} = 0$, in which it may be shown that the Reynolds number does not occur). However, when we reach the Reynolds number at which the boundary layer becomes turbulent just before the separation point for a laminar boundary layer, separation will move downstream and may be suppressed altogether: in this case the turbulent wake becomes considerably narrower and the drag of the body decreases sharply (perhaps to a small fraction of its previous value). This phenomenon is called the *drag crisis*. It is explained by the fact that the momentum transfer within a boundary layer increases sharply when it becomes turbulent. Therefore, the entrainment of fluid from the high-speed outer flow by the boundary layer is considerably increased, so that the fluid particles in the boundary layer can penetrate much further in the direction of increasing pressure than in the case of a laminar boundary layer.

The drag crisis in the case of flow past a sphere was first observed by Eiffel (1912). The transition from large to small drag occurs in this case for Reynolds number $\text{Re} = UD/\nu$ (where D is the diameter of the sphere) close to 5.0×10^5; according to Eiffel the drag coefficient C_w decreases approximately from 0.5 at $\text{Re} = 10^5$–0.15 at $\text{Re} = 10^6$. Later, it was also found that the minimum value of C_w in very careful experiments is less than 0.1. The coefficient C_w for a circular cylinder behaves in the same way. The dependence of the drag coefficient for a sphere and a cylinder on Re is shown in Fig. 2.5. It is clear from the discussion in Sect. 2.1 that the drag crisis will arise the earlier the greater the disturbance level of the ambient flow, i.e., the smaller the critical Reynolds number for transition to a turbulent regime in the boundary layer. This is confirmed clearly by the experiments by Prandtl (1914) who achieved passage through the drag crisis in the flow past a sphere by fitting a wire ring around the sphere, i.e., by introducing additional disturbances into the flow to cause transition to turbulence in the boundary layer.

The necessary condition for separation of the boundary layer is an increase of pressure in the direction of flow along some part of the surface of the body. This condition is satisfied not only for flow past convex surfaces, but also in other cases, e.g., for flow along a flat plate if the ambient velocity decreases with increasing distance x from the leading edge, or flow in an expanding conical tube (diffuser) or sharply bent tube.

The formation of the turbulent wake behind a solid body is just one case of the transition to turbulence of free (unbounded) flows such as jets, wakes, and mixing layers separating two unbounded flows of different velocities. Some remarks related to the other cases will be made later in this book.

The separation of the boundary layer under the action of a negative pressure gradient may also explain to a certain extent the effect of disturbances in the ambient flow on the value of Re_{xcr}. We may assume that this effect is connected with the generation by disturbances of fluctuating pressure gradients, which leads to the formation at some random points of unstable S-shaped velocity profiles (as at point E, Fig. 2.4) stimulating the transition to turbulence. On the basis of this hypothesis, G.I. Taylor (1936a) tried to estimate the form of the dependence of Re_{xcr} on the disturbance characteristics (later, Wieghardt (1940) simplified Taylor's arguments).

Taylor proceeded from the approximate Karman-Pohlhausen theory of a laminar boundary layer in the presence of a longitudinal pressure gradient dp/dx. According to the theory the shape of the velocity profile at a given x depends only on a single dimensionless parameter $\Lambda = -\frac{\delta^2}{\nu U\rho}\frac{dp}{dx}$ where δ is the boundary layer thickness. In a constant-pressure turbulent boundary layer (on a flat plate) $dp/dx = 0$, where now p designates the mean pressure, but fluctuations of pressure may also exist. Taylor proposed that in this case not the mean pressure gardient (which is equal to zero) but the root-mean-square (RMS) value of the fluctuating pressure gradient is an important physical parameter. Therefore the character of the fluid motion at a station $x = \text{constant}$ is supposed to be determined in this case by the parameter $\Lambda = -\frac{\delta^2}{uU\rho}\frac{\delta p'}{\delta x}$ (where p' is the pressure fluctuation and $\delta/\delta x$ signifies the RMS value of the derivative ∂/∂x). In other word, according to Taylor, the point of transition from laminar to turbulent flow is determined by the parameter Λ attaining some critical value Λ_{cr}. But the equations of motion show that $-\frac{1}{p}\frac{\delta p'}{\delta x}$ must be of the same order of magnitude as $u'\frac{\delta u'}{\delta x} = \frac{1}{2}\frac{\delta u'^2}{\delta x}$ where u$'$ is the flucuation of the longitudinal velocity in the ambient flow. Furthermore, we may put $\frac{\delta u'^2}{\delta x} \sim \frac{U'^2}{\lambda}$, where U' is a RMS value of the velocity fluctuations, and λ is the so-called Taylor miscroscale of turbulence which is determined from the condition $\left(\frac{\delta u'}{\delta x}\right)^2 = \frac{U'^2}{\lambda^2}$ (this scale will be used several times in later sections of the book). The scale λ may be expressed in terms of the external (integral) scale of turbulence L (which defines the typical length scale of the largest eddies) as follows: the mean rate of energy dissipation $\varepsilon \sim \nu(\delta u'/\delta x)^2$ is proportional to $\nu U'^2/\lambda^2$, but it is known that, for large values of $U'L/\nu$, it is also

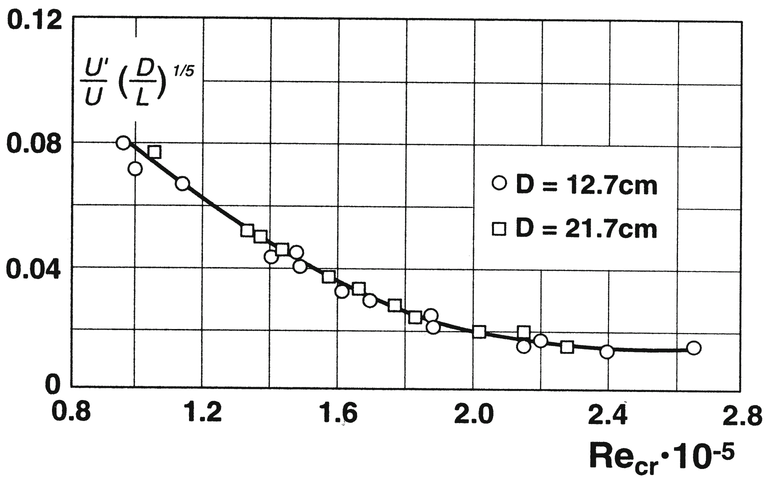


Fig. 2.6 Dependence of Re_{cr} on $\frac{U'}{U}\left(\frac{D}{L}\right)^{1/5}$ for flow past a sphere

proportional to U'^3/L (see, e.g., Sreenivasan (1995)). Consequently,

$$\lambda \sim L\left(\frac{U'L}{\nu}\right)^{-1/2}, \quad \frac{1}{\rho}\frac{\delta p'}{\partial x} \sim \frac{U'^2}{\lambda} \sim \left(\frac{U'^5}{\nu L}\right)^{1/2}.$$

Taking into account that for a laminar boundary layer on a flat plate $\delta \sim (\nu x/U)^{1/2}$, we obtain

$$\Lambda = -\frac{\delta^2}{\nu U\rho}\frac{\delta p'}{\delta x} \sim \left(\frac{U'}{U}\right)^{5/2}\left(\frac{x}{L}\right)^{1/2}\left(\frac{Ux}{\nu}\right)^{1/2}.$$

Thus, taking $\mathrm{Re}_{xcr} = \varphi(\Lambda_{cr})$, we obtain

$$\mathrm{Re}_{xcr} = \left(\frac{Ux}{\nu}\right)_{cr} = F\left[\frac{U'}{U}\left(\frac{x}{L}\right)^{1/5}\right]. \tag{2.1}$$

We may take the length L in this equation to be some characteristic dimension of the device generating the turbulence (for example, if the turbulence is set up by a grid in a wind-tunnel, then, not too far downstream, L will be roughly equal to the distance between the rods of the grid). We see that according to this theory Re_{xcr} depends not on U'/U alone but on the product $(U'/U)(x/L)^{1/5}$

The result (2.1) has been deduced here for the case of flow past a flat, plate. However, it may be generalized to flows past other bodies by replacing the coordinate x with s, the distance from the point where the flow impinges on the body to the point of transition of a boundary layer from a laminar to a turbulent regime, reckoned along the contour of the body, or by some characteristic dimension of the body D, and by using, instead of $\mathrm{Re}_x = Ux/\nu$, $\mathrm{Re}_s = Us/\nu$ or $\mathrm{Re} = UD/\nu$. In spite of the lack of rigor in its deduction, this result is in very good agreement with experimental data (see, e.g., Fig. 2.6 which gives the data of Dryden et al. (1937), who measured the value of Re_{cr} for boundary layers on spheres of different radii with different values of the intensity and scale of the turbulence in the free stream).

Local separations of a boundary layer due to fluctuations of the Iongitudinal pressure gradient, producing short-time unstable S-shaped profiles of velocity at some random points, apparently play an important part in the generation of turbulence in the near-wall region of a boundary layer with $Re > Re_{cr}$ (or $> Re_{cr\ min}$ in the case of a high level of external disturbances). The solid wall always exerts a stabilizing influence on the flow in its vicinity, but if $Re > Re_{cr}$, significant velocity fluctuations can be usually observed even at relatively small distances from the wall. These fluctuations affect the instantaneous values of longitudinal pressure gradients and lead to the appearance of local flow instabilities. As a result of these instabilities a complex system of specialized vortical structures appears in the near-wall region of the flow; this system has been thoroughly investigated during the last three decades. Recent theoretical and experimental work suggests that the physical processes In the near-wall region may play a crucial role in transition to turbulence in flows bounded by solid walls (e.g., in the formation of turbulent spots and slugs). We will consider these topics in more detail later.

2.3 Hydrodynamic Instability

It has been mentioned already that at present it is universally accepted that the velocity and pressure fields in any fluid flow, whether laminar or turbulent, are solutions of the equations of fluid mechanics (Navier-Stokes equations) satisfying given initial and boundary conditions. Steady laminar flow, in particular, is described by steady solution of these equations; however, in the case of turbulent flow, each individual example of the flow corresponds to a very complex nonsteady solution. The non-existence of laminar flow for sufficiently high Reynolds numbers (in spite of the fact that the equations of fluid mechanics have a steady solution for any Re) clearly shows that not every solution of Navier-Stokes equations corresponds to a real fluid motion. It is natural to associate this with the familiar assertion that the solution of dynamic equations describing the real motion must surely be stable to small disturbances. In other words, small disturbances of the motion, which are always present, must be damped in time so as not to change the general nature of this motion. In the other case, when small disturbances increase with time, the motion deviates considerably from the original solution, which continues to exist but no longer describes the real motion.

Therefore, we may expect that the value of Re_{cr} corresponds to the point on the Re-axis at which stability is lost; for $Re < Re_{cr}$ the laminar flow is stable, and with $Re > Re_{cr}$ it is unstable and apparently becomes turbulent under the influence of the existing small disturbances. (It will be shown later that this assertion is in fact not always true. In some cases the loss of stability leads to the transformation of the given laminar flow into another, more complex, laminar flow, and the transition to turbulence occurs only after several losses of stability of the resulting laminar flows. However, we shall not discuss such a possibility here.) If our expectation is met, then, by the mathematical study of stability applied to laminar solutions of fluid dynamics equations, we may (at least in principle) theoretically determine the corresponding critical Reynolds number. This explains the exceptional attention given to the study of hydrodynamic stability by many great scientists beginning with O. Reynolds, Lord Kelvin, and Lord Rayleigh.

Turbulence is characterized by very complicated irregular fluctuations of the velocity and the other fluid-mechanical fields. In general, the flow of fluid at a given instant of time is determined by the values of the collection of all independent fluid-dynamic and thermodynamic fields characterizing the instantaneous state of the moving fluid. (In the simplest case of incompressible fluid, this collection consists of the pressure and the three components of the velocity field $\boldsymbol{u}(\boldsymbol{x}) = \{u_1(\boldsymbol{x}), u_2(\boldsymbol{x}), u_3(\boldsymbol{x})\}$, which is solenoidal (having vanishing divergence), in the whole three-dimensional space or its region filled by the fluid. If the fluid is thermally inhomogeneous, the temperature field $T(\boldsymbol{x})$ must be also included in the collection; in the study of admixture dispersion by fluid turbulence, the field of concentration $c(\boldsymbol{x})$ must be taken into account; for compressible fluid both the fields of density and temperature (or entropy) must be added to the list.) For the fluid flow the *phase space* at a given instant t consists of all admissible values of these fields; a set of fixed values determines a *phase point* ω. Temporal evolution of the flow corresponds to a *phase trajectory* $\omega(t)$, which is a curve in the phase space. (For steady flow this trajectory degenerates into a point; for periodic flow it is a closed curve, i.e., a loop or cycle.) The set of phase trajectories $\omega(t) = T_t\omega(0)$ where $-\infty < t < \infty$, corresponding to all phase points $\omega(0)$, determine a group of mappings of the phase space into itself, which is called the *phase flow* and describes the temporal evolution of the fluid flows corresponding to all initial conditions.

The fluid flow can be considered as continuous movement of a phase point along the corresponding phase trajectory, i.e., as a *dynamic system* in the infinite dimensional (functional) phase space Ω. An infinte dimensional dynamic system is a very complicated mathematical entity, but in fact it can always be approximated by a simpler dynamic system in a finite dimensional space. For this purpose the values of all the fluid mechanical fields must be replaced by values of a finite number of "generalized coordinates" which determine the fields with sufficiently high accuracy. The number of these coordinates will then give the effective number of "degrees of freedom" of the flow considered.

Let us consider the simplest case of incompressible fluid having constant temperature. To define the generalized coordinates we may begin from a decomposition of the fluid motion into elementary components. These components must be such that the sum of all their energies is equal to the total energy of the flow while the state of each of them is characterized by a fairly small number of parameters. The parameters of all the elementary components of motion will be generalized coordinates of the flow, and the number of these coordinates which can vary under given external conditions will be the total number of degrees of freedom of the flow. From a mathematical veiwpoint, decomposition of the motion into elementary components is equivalent to the expansion of the velocity field in terms of an orthogonal system of functions. In such an expansion each of the functions will describe the velocity field of the corres ponding elementary component of the motion, while the coefficients of the expansion will be generalized coordinates of the flow. The choice of an appropriate orthogonal system of functions is dictated by the form of the flow boundaries and the conditions imposed at these boundaries. For flows in a finite volume, the complete system of orthogonal functions will always be countable (i.e., the functions of the system can be numbered by consecutive integers 1, 2, 3, ...); thus, such flows will have no more than a countable set of generalized coordinates. The concept of a flow in an infinite space is always a mathematical idealization; using this idealization we must often apply families of functions determined by values of continuous parameters and to allow the existence of a continuous spectrum (such examples will be encountered below in this chapter). However, this is of no importance, since in bounded space-time regions any fluctuation may be approximated as accurately as it is desirable by a countable (and even a finite) collection of harmonic oscillations.

For steady laminar flow the values of the generalized coordinates will be defined uniquely by the given external and boundary conditions so that the number of degrees of freedom of a laminar flow is zero. The number of degrees of freedom of a turbulent flow in a finite volume will be very great, but it is also finite. In fact when the velocity field is expanded in a series in terms of orthogonal functions, the various components will describe elementary motions of different scales. As the order of the component increases indefinitely, the corresponding scale tends to zero. However, due to the viscosity, fluctuations of too small a scale cannot exist. With steady external conditions the coefficients of the expansion of the velocity field in terms of orthogonal functions of sufficiently high order will be independent of time. This means that the number of degrees of freedom of the flow will be finite. Also, the number of degrees of freedom must increase with decrease of the coefficient of viscosity, in other words, with the increase of Re. According to the estimate of Landau and Lifshitz (see, e.g., Monin and Yaglom (1975), p. 349), the number of degrees of freedom of a turbulent flow in a finite volume will be proportional to $\mathrm{Re}^{9/4}$ for large enough Re, where Re is the Reynolds number of the overall flow. Consequently, the number of degrees of freedom will increase rapidly with the increase of Re, and for developed turbulence with large Reynolds numbers it will reach enormous values.

When the flow is approximated by a finite-dimensional dynamic system, its development in time may depend not only on the values of the generalized coordinates, but also on the values of the corresponding generalized velocities. Then the phase space of a system must be determined as the space of all the simultaneous values of generalized coordinates and velocities, and the phase trajectory is a line in this phase space.

Let us consider the evolution of a fluid flow under fixed, steady conditions (which include the presence of a constant influx of energy necessary for the existence of a steady flow of viscous fluid) but with variable initial conditions. Decomposition of the flow into elementary components permits us to consider it as a set of interacting elementary nonlinear oscillators. Each oscillator may obtain some energy directly from the external influx or from the other oscillators and also may lose energy to viscous dissipation or by transport to other oscillators. Under such conditions self-excited oscillations may arise. The possibility of such oscillations arising is determined by the relationship between the energy E^+ acquired by the oscillator and

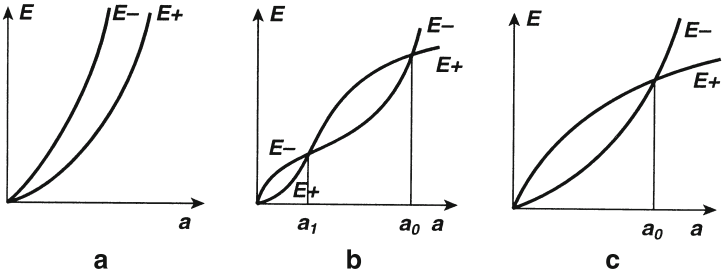


Fig. 2.7 Different variants of the dependence of the energy gained and lost by an oscillator on the amplitude of the oscillations

the energy E^- lost by it, for various amplitudes a of the oscillations (see Fig. 2.7). If $E^- > E^+$ for all amplitudes (Fig. 2.7a), then the oscillations will evidently be damped for any initial amplitude, and the system will be stable to any disturbances. if $E^- < E^+$ for $a_1 < a < a_0$, but $E^- > E^+$ for $a < a_1$ or $a > a_0$ (Fig. 2.7b), then oscillations with initial amplitude $a < a_1$ will be damped, but those with initial amplitude $a > a_1$ will increase until their amplitude attains the equilibrium value a_0. In this case, the system will be stable to small disturbances but unstable to disturbances of sufficiently large amplitude (such a system is called a system with *hard self-excitation*). Finally, if $E^+ > E^-$ however small the amplitude (Fig. 2.7c), the system will be unstable to infinitely small disturbances (i.e. unconditionally unstable) and will practically always be in a regime of self-excited oscillations with amplitude a_0 (system with *soft self-excitation*).

It will be shown below that in fluid flows all three situations shown in Fig. 2.7 may arise. However, it is a very difficult task to find the conditions determining which situation actually exists in a given flow, and the solution is known only for some special cases.

2.4 Simple Examples of Unconditionally Unstable Fluid Flows

The existence of $\mathrm{Re}_{\mathrm{cr\,min}}$, and the dependence of $\mathrm{Re}_{\mathrm{cr}}$ for flows in tubes, channels, and boundary layers on the initial disturbance level, show that for a range of Re values exceeding $\mathrm{Re}_{\mathrm{cr\,min}}$ these flows represent self-excited systems with hard excitation. Now we shall show some simple examples of fluid motions that are unstable even with respect to infinitely small disturbances, i.e., are systems with soft excitation.

One of the simplest examples of an unconditionally unstable flow is the above-mentioned flow near a surface of tangential velocity discontinuity. In this case, the unconditional instability may be explained qualitatively with the aid of simple physical considerations. Let us consider an ideal fluid of zero viscosity, two layers of which slide over each other with equal and opposite velocities U and $-U$, forming a surface of discontinuity of velocity. (The case of arbitrary unequal velocities U_1 and U_2 of two layers can be reduced to this one by means of transformation to a new system

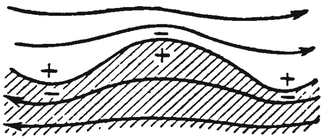


Fig. 2.8 Schematic form of the streamlines and the pressure distribution close to a disturbed surface of a tangential velocity discontinuity

of coordinates moving with the velocity $(U_1 + U_2)/2$ in the streamwise direction.) Let us assume that as a result of some disturbance on the suface of discountinuity, a small-amplitude wave is formed (see Fig. 2.8). For simplicity, we assume that this wave is nonprogressive. Under these circumstances, the streamlines above the wave-crest will draw closer together, i.e., the velocity will increase, while in the troughs the streamlines will become farther apart and the velocity will decrease. According to Bernoulli's equation, $u^2/2 + p/\rho =$ constant, the pressure will fall above the crest and rise in the troughs (in Fig. 2.8 this is denoted by the plus and minus signs). Thus a transverse pressure gradient arises in the fluid, tending to increase the amplitude of the wave. Later, this increase in amplitude leads to the wave disintegrating into individual vortices, forming the beginning of the turbulent zone.

In a real fluid, of course, the waves which arise can be progressive, but the processes of their evolution are similar. In a viscous fluid the sliding of two layers over one another is impossible, and instead of the surface of discontinuity there will be a narrow transition layer between the two flows, in which the velocity profile will be S-shaped. The investigation of the stability of such a layer will be more complicated; however, here also both theory and experiment show that it is very unstable (see below, Sect. 2.93).

Two fluid layers moving with different velocities in the same direction parallel to the dividing surface of "velocity discontinuity" can be produced relatively easily in the laboratory; the evolution of surfaces of discontinuity of this type can also be observed in experiments with a jet issuing from an orifice and then expanding in a space filled with the same (but motionless) fluid. The instability of a surface of tangential velocity (and density) discontinuity was first noted by Helmholtz (1868) and then was rigorously studied by W. Thomson, the future Lord Kelvin (see Kelvin (1871)). Later Rayleigh (1883) considered the instability of a surface of density discontinuity in the presence of a gravitational force perpendicular to this surface, while Taylor (1950) investigated the instability of the arbitrary surface of discontinuity between two fluids accelerated in the direction perpendircular to it. At present the instability of surfaces of discontinuity, separating two regions of flow, filled with the same or different fluids, moving with different velocities, is usually called the *Kelvin* (or *Kelvin-Helmholtz*) *instability*, if it is not affected by external force, and the *Rayleigh-Taylor* instability, if there is an accelerating force normal to the surface. The instability of a surface of tangential velocity discontinuity and some other simple cases of the Kelvin-Helmholtz instabilities are considered in many textbooks (e.g., in those by Lamb (1932); Sect. 2.32; Landau and Lifshitz (1987); Sect. 29; Tritton (1988); Sect. 17.6; and panton (1996) Sect. 22.2); see also Drazin and Reid (1981) and the survey paper by G. Birkhoff (1962). A survey of work on the Rayleigh-Taylor instability was given by Sharp (1984).

Another simple example of unconditional instability is the equilibrium, in a gravitational field, of stationary stratified fluid with variable density $\rho = \rho(z)$ increasing with height. It is clear that with any function $\rho(z)$, the equations of motion of incompressible fluid will allow a solution $\boldsymbol{u}(x, y, z, t) = 0$, corresponding to a state of rest; the gravitational field will only produce a vertical pressure variation according to the law

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g, i.e., p(z) = g\int_z^\infty \rho(z')dz' + \text{const.} \tag{2.2}$$

Now suppose that as a result of some disturbance, some element of the fluid is displace from level z to a new level $z' = z + h$. If the density ρ decreases with height, then for $h > 0$ the element will tend to move downwords under the force of gravity, and for $h < 0$ it will tend to rise under the action of buoyancy, so that the equilibrium will be stable. However, if the density increases with height, then for any value of h the displaced element will tend to move even further from its orginal position, and the state of equilibrium will be unconditionally unstable. Moreover, for an ideal (inviscid) fluid the equations of motion will also have a steady solution for any density profile $\rho = \rho(z)$ and any profile of the x-component of velocity $u = u\,(z)$ (with zero components of velocity along the other axes). Using the same argument, this flow will be unconditionally unstable for $d\rho/dz > 0$. For $d\rho/dz < 0$, the question of stability of the flow is considerably more complex; at this early stage in the discussion we can only say, by similarity, that the criterion of stability here must be expressed in terms of the so-called *Richardson number*, i.e., the dimensionless parameter

$$Ri = -\frac{\frac{g}{\rho}\frac{d\rho}{dz}}{\left(\frac{\partial u}{\partial z}\right)^2}. \tag{2.3}$$

The case of a fluid that is stratified with respect to the z axis is of great importance for *geophysical fluid mechanics*. In fact, flows in the atmosphere and ocean are almost always stratified; the dependence of density $\rho(\text{z})$ on z arises here from the temperature profile $T(z)$ and (in the case of an ocean) the salinity profile $c(z)$. (For simplicity, only the influence of temperature will be considered below.) However, in this case the incompressibility assumption, used above, is often not a satisfactory approximation since it ignores the following effect: when fluid moves vertically its temperature changes and fluid either expands and therefore cools or is compressed and therefore warms up. To take this effect into account, we must use the equation of state and the elementary thermodynamic identities (for more detailed presentation see, e.g., Landau and Lifshitz (1987), Sect. 4).

Assuming that the displacement of fluid elements occurs at constant entropy, we find that an element displaced from level z to level $z + h$ will be lighter for $h > 0$ than the surrounding fluid, but heavier for $h < 0$ if, and only if

$$\frac{dT}{dz} < -\frac{gT}{c_p V}\left(\frac{\partial V}{\partial T}\right)_p, \tag{2.4}$$

where T is now the absolute temperature and V is the specific volume. Condition (2.4) will be a condition of unconditional instability of the state of rest in the presence of a temperature profile $T = T(z)$. When the medium may be considered as an ideal gas, $\left(\frac{\partial V}{\partial T}\right)_P = \frac{R}{p} = \frac{V}{T}$, so that the criterion of instability will take the form

$$\frac{dT}{dz} < -\frac{g}{c_p} = -\frac{\gamma - 1}{\gamma}\frac{g}{R}, \; \gamma = \frac{c_p}{c_v}, \tag{2.5}$$

(the criterion will be found in this form in all textbooks on dynamical meteorology). In meteorology, $G_a = \frac{\gamma-1}{\gamma}\frac{g}{R}$ is called the *adiabatic temperature gradient* or *adiabatic lapse rate* (for air, this gradient is approximately 1°C/100 m). Cases for which—(dT/dz) is greater than, equal to, or less than G_a are called, respectively, stable, neutral, or unstable stratification.

Another representation of the conditions of instability (2.4) or (2.5) is often used in meteorology; this is connected with the introduction of the so-called *potential temperature*, defined by

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}, \tag{2.6}$$

where p_0 is some standard pressure (usually taken as the normal sea-level pressure), instead of the ordinary temperature T. By the entropy equation for an ideal gas, $c_p \ln \theta = s + \text{const.}$, where s is the entropy of unit mass of fluid. Therefore, the potential temperature does not vary in an adiabatic process, so that θ is equal to the temperature which the air would attain if brought adiabatically to standard pressure p_0. It is easy to see that $\frac{d\theta}{dz} \sim \frac{dT}{dz} - G_a$. Thus, using the concept of the potential temperature, the instability criterion (2.4) may be formulated as follows: the state of rest will be unstable if $d\theta/dz < 0$ (i.e., if the potential temperature decreases with height) and stable if $d\theta/dz > 0$ (i.e., if the potential temperature increases with height).

If an arbitrary wind velocity profile exists, the motion in the case of unstable stratification will likewise be unstable; for stable strafification, however, the stability or instability of the motion must be determinded in some way by the value of the Richardson number

$$Ri = \frac{\frac{g}{T}\left(\frac{dT}{dz} - G_a\right)}{\left(\frac{du}{dz}\right)^2} = \frac{\frac{g}{\theta}\frac{d\theta}{dz}}{\left(\frac{du}{dz}\right)^2}. \tag{2.7}$$

2.5 Linear Stability Theory and Method of Normal Modes

Of course, the conditions for instability of a fluid flow are by no means as easy to find in every case as in the above examples. Quite often determination of such conditions is a complicated problem, and to solve it a number of sophisticated methods has been developed. These methods and their numerous applications make up the contents of a special branch of science called *hydrodynamic stability theory*. Some of the results from this theory will be briefly presented in the following sections of this chapter.

The simplest means of investigating hydrodynamic stability is the general method of small disturbances. For simplicity, we will consider a flow of incompressible fluid of constant density ρ (and hence also of constant temperature T). The method of small disturbances is based on writing the velocity field $u_i(\boldsymbol{x}, t)$ and the pressure $p(\boldsymbol{x}, t)$ which satisfy the dynamic equations in the form $u_i = U_i + u_i{}'$, $p = P + p'$, where $U_i(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}, t)$ are particular solutions of the equations under investigations, and $u_i{}'$, p' are small disturbances. Substituting the given expressions for u_i and p into dynamic equations, and ignoring all the products of two disturbances, we obtain linear equations for u_i' and p' whose solution represents the first approximation of the method of small disturbances. Later, if desirable, the next approximation can be considered that takes into account also the terms quadratic with respect to first-order solutions, and so on. However, here we shall limit ourselves to the first approximation only; it is usually called the *method of small disturbances*.

Thus, the method of small disturbances uses only approximate linear equations for the disturbances and neglects all the higher-order corrections which are important in the case of finite disturbances. Because of this the application of this method to the study of hydrodynamic instability is often called the *linear theory of hydrodynamic stability* or the *theory of stability with respect to infinitesimal disturbances*. Note that instability of a given laminar flow with respect to infinitesimal disturbances does not mean that the flow necessarily becomes turbulent; it only shows that the original laminar flow can no longer exist and must be transformed into some other flow. Nevertheless, the determination of conditions for such an instability is clearly an important first step in the investigations of laminar-turbulent transition. Therefore it is not surprising that for more than 100 years (beginning in the 70 s and 80 s of the 19th century) much work has been devoted to the development of the linear stability theory. Let us now consider some general approaches and specific results obtained in the course of this work.

In the case of a flow of incompressible fluid, the dynamic equations are the Navier-Stokes equations. Substituting in them $U_i + u'_i$ and $P + p'$ instead of u_i and p, ignoring quadratic terms in disturbances, assuming that external forces X_i are absent, and taking into account that U_i and P themselves satisfy the equations of motion, we obtain linear equations for u_i' and p' in the form

$$\frac{\partial u_i'}{\partial t} + U_\alpha \frac{\partial u_i'}{\partial x_\alpha} + u_\alpha' \frac{\partial U_i}{\partial x_\alpha} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \nabla^2 u_i',$$

$$\frac{\partial u_\alpha'}{\partial x_\alpha} = 0. \tag{2.7a, b}$$

Differentiating Eq. (2.7a) with respect to x_i, summing with respect to i, and using Eq. (2.7b), we may obtain a Poisson equation for p' (similar to Eq. (1.9′) in MY1), which expresses p' in terms of u'_i. Hence the solution of Eq. (2.7) will be determined by fixing only the initial values $u_i'(\boldsymbol{x}, 0)$ of the functions $u_i'(\boldsymbol{x}, t)$. We may thus (at least in principle) attempt to establish conditions guaranteeing the existence of such initial values $u_i(x, 0)$ that the corresponding solution of the initial value problem will not be damped as time tends to infinity. These conditions will then be conditions for the instability, with respect to infinitesimal disturbances, of the solution U_i, P of the Navier-Stokes equations. If there exists a solution of Eq. (2.7) that increases with t without bound, then the solution U_i, p of the Navier-Stokes equations is called strictly unstable. (According to this terminology, the *strict instability* of a flow differs from the ordinary *instability* because neutrally stable flows—i.e. those for which the corresponding Eqs. (2.7) have no solutions growing in time, but have some solutions which are neither damped nor increasing with t- are unstable but not strictly unstable. However, below we shall usually not distinguish between ordinary and strict instabilities.) Of course, if the solution U_i, P is stable with respect to infinitesimal disturbances, it may nevertheless be unstable with respect to finite disturbances u'_i, p' (described by essentially nonlinear equations). To verify whether this is so or not quite different methods of investigation are needed; some of them will be considered later in this book.

When the solution $U_i = U_i(\boldsymbol{x})$, $P = P(\boldsymbol{x})$ describes a steady laminar flow of fluid, the coefficients of the system of equations (2.7) will evidently be time-independent. In this case, the system will have particular solutions of the form

$$\boldsymbol{u}'(\boldsymbol{x}, t) = e^{-i\omega t}\boldsymbol{f}_\omega(\boldsymbol{x}),\ p'(\boldsymbol{x}, t) = e^{-i\omega t}g_\omega(\boldsymbol{x}), \tag{2.8}$$

the time-dependence of which is given by the exponential factor $e^{-i\omega t}$ with, generally speaking, a complex "frequency" ω. (Here and below, when complex functions are used to describe real flow characteristics, it is always assumed that these characteristics are equal to the real parts of the functions considered.) The permissible values of the characteristic frequency ω and the corresponding amplitudes $\boldsymbol{f}_\omega(\boldsymbol{x})$, $g_\omega(\boldsymbol{x})$ will then be determined from the eigenvalue problem for a linear system of partial differential equations. When the coefficients of this system are independent of some space coordinates, the number of independent variables in the system may be reduced by assuming that the dependence of the amplitudes $\boldsymbol{f}_\omega$ and g_ω on the corresponding coordinates will also be exponential, with a given "wave number" (i.e., the spatial scale of the disturbance is prescribed in the directions of the coordinate axes along which the undisturbed flow is homogeneous.) Thus, for example, if the undisturbed flow depends only on the coordinate x_3 then we may put

$$\boldsymbol{f}_\omega(\boldsymbol{x}) = e^{i(k_1x_1+k_2x_2)}\boldsymbol{f}_{\omega;k_1,k_2}(\boldsymbol{x}_3), \quad g_\omega(\boldsymbol{x}) = e^{i(k_1x_1+k_2x_2)}g_{\omega;k_1,k_2}(\boldsymbol{x}_3); \tag{2.9}$$

where the characteristic frequency $\omega = \omega(k_1, k_2)$ and the amplitudes $\boldsymbol{f}_{\omega;k_1,k_2}$ and $g_{\omega;k_1,k_2}$ will be determined from the eigenvalue problem for a system of ordinary differential equations containing the parameters k_1 and k_2. Similar equations, with $\exp[i(k_1x_1 + k_2x_2)]$, $\boldsymbol{f}_{\omega;k_1,k_2}(x_3)$ and $g_{\omega;k_1,k_2}(x_3)$ replaced by $\exp(ik_1x_1)$, $\boldsymbol{f}_{\omega;k_1}(x_2, x_3)$ and $g_{\omega;k_1}(x_2, x_3)$, will be obtained for flows which depend essentially on two coordinates x_2 and x_3.

The stability analysis based on the study of elementary ("normal") disturbances whose time dependence is described by simple exponential factors $e^{-i\omega t}$, where ω

is a complex constant, was well known in the first half of the nineteenth century in applications to mechanics of systems of with a finite number of degrees of freedom. In the second half of that century it was adapted by Stokes, Rayleigh, and Kelvin to problems in fluid mechanics. Disturbances that depend exponentially on time (i.e., are proportional to $e^{-i\omega t}$) are usually called the *normal components, normal modes,* or *eigenmodes* of the problem. Stability analysis by the *method of normal modes* equates the instability of the dynamical system considered to the existence of at least one normal mode with a real value of the constant ω (and equates strict instability to the existence of at least one increasing mode with positive imaginary part of ω). In the framework of this method, it is convenient to define the critical Reynolds number $Re_{\rm cr}$ as the smallest value of Re at which a real eigenfrequency ω first appears; then at $\mathrm{Re} = \mathrm{Re}_{\rm cr}$ the flow will be unstable while at $\mathrm{Re} > \mathrm{Re}_{\rm cr}$ it will be strictly unstable. Hence, determination of the conditions for hydrodynamic instability by the mormal-mode method can be reduced to the investigation of a definite eigenvalue problem for a system of differential equations. The applications of this method to a number of specific problems will be considered in the next four sections.

It is reasonable, however, to begin with some general remarks related to the normal-mode method. The method is, in fact, based on the assumption that any disturbance of the laminar flow considered can be represented by a (finite or infinite) sum of normal modes depending exponentially on the time $\boldsymbol{t}$. Hence, it is always assumed here (but usually not stated explicitly) that an infinite number of discrete eigenvalues ω exists for this kind of stability problem and that the system of the corresponding eigenfunctions $\boldsymbol{f}_{\omega}(\boldsymbol{x})$ is *complete* in the space of all vector functions $\boldsymbol{f} = (f_1, f_2, f_3)$ corresponding to possible fluid flows (i.e., those satisfying the continuity equation $\partial f_i/\partial x_i = 0$ and all the necessary regularity and boundary conditions). (The pressure eigenfuncions $g_{\omega}(\boldsymbol{x})$ are not mentioned here, since it can be assumed that p' has been already expressed in terms u_i'.) If there is a spatial homogeneity of the flow with respect to one or two coordinates, then it must be assumed that the functions $\boldsymbol{f}_{\omega;k_1}(x_2, x_3)$ or $\boldsymbol{f}_{\omega;k_1,k_2}(x_3)$, where the wave numbers k_1 or k_1 and k_2 are fixed and $\omega = \omega(k_1)$ or $\omega(k_1, k_2)$ passes through all the eigenfrequencies corresponding to given values of wave numbers, form a complete system in space of admissible vector functions of two or one variables.

The problem of completeness in the theory of hydrodynamic stability is not a simple one. The concept of completeness (i.e., of the coincidence of the linear span for a given infinte system of functions with the entire space of all the functions considered) requires the introduction of some norm $\|\boldsymbol{f}\|$ in function space permitting the distance $\|\boldsymbol{f} - \boldsymbol{g}\|$ between two functions to be determined (this is necessary for the limit of an infinite series of functions to be defined). For most of the usual quadratic norms (given, e.g., by kinetic energy per unit volume), the function space is a Hilbert space, with scalar product $(\boldsymbol{f}, \boldsymbol{g})$ determined by the given norm (so that $\|\boldsymbol{f}\| = (\boldsymbol{f}, \boldsymbol{f})^{1/2}$). (For information about Hilbert spaces and linear operators in them see, e.g., Halmos (1951); Riesz and Sz.-Nagy (1955); Dunford and Schwartz (1958, 1963, 1971)), Reed and Simon (1972); Akhiezer and Glazman (1980); or Griffel (1981). For an elementary introduction to applications of Hilbert space methods to fluid mechanics see Doering and Gibbon (1995).) Then a linear differential equation can be represented as an operator equation $\Lambda \boldsymbol{f} = 0$ where Λ is a linear operator in Hilbert space. If the operator Λ is *self-adjoint,* i.e., $(\Lambda \boldsymbol{f}, \boldsymbol{g}) = (\boldsymbol{f}, \Lambda \boldsymbol{g})$ for any two functions $\boldsymbol{f}$ and $\boldsymbol{g}$, all the eigenvalues of Λ are real, and conditions for the completeness of the system of corresponding eigenfuctions can be determined from the well-known spectral theory of self-adjoint operators. Note also that in this case the two eigenfunctions corresponding to different eigenvalues are necessarily orthogonal in the Hilbert space; this facilitates the expansion of any given function into eigenfunctions and makes such expansions much more useful.

However, the eigenvalue problems which appear in the theory of hydrodynamic stability are those of a linear non-self-adjoint operators in a function space. Non-self-adjoint linear operators can have complex eigenvalues, and here the spectral theory exists only for some special classes of operators and is more complicated and less complete then the spectral theory of self-adjoint linear operators; see, e.g., Gokhberg and Krein (1965); and Dunford and Schwartz (1971). Nevertheless, some general criteria for the completeness of the system of eigenfunctions of non-self-adjoint operators (and the related associated functions appearing when there are degenerate eigenvalues) were given by Keldysh (1951); (see also Keldysh and Lidskii (1963)) and Naimark (1954). The first mathematical proof of the eigenfunction expansion theorem (i.e., of the existence of an infinite number of eigenvalues to which corresponds a complete system of eigenfuncions) for a problem of hydrodynamic stability theory was given by Haupt (1912), who studied stability of plane Couette flow with respect to two-dimensional disturbances. However, this early paper did not attract much attention, and the next study of the completeness problems of hydrodynamic stability theory appeared only in 1960, when Schensted investigated stability of plane Poiseuille (or Couette-Poiseuille) flow with respect to two-dimensional disturbances and of circular Poiseuille flow with respect to axisymmetric disturbances. She proved the eigenfunction expansion theorem for both these problems. More general results (related to arbitrary flows within a finite volume) were found by Yudovich (1965, 1984) who referred to the general theorem by Keldysh, used sophisticated mathematical techniques, and formulated his results in a form intended for mathematicians. (Note that the cases of flows, which are either homogeneous along the direction of x_1-axis and have finite area of (x_2, x_3) –cross-section or are homogeneous in the planes $x_3 = \text{const.}$ and have finite extent in the direction of the axis x_3 can be reduced to cases of flows in finite three-dimensional domains if the wave numbers are fixed for directions of homogeneity.) More direct formulations of eigenfunction expansion theorems for eigenvalue problems from the linear theory of hydrodynamic stability were given by Di Prima and Habetler (1969); Sattinger (1970) (see also Sect. 2.3 of the book by Georgescu (1985)); and Herron (1980, 1982, 1983). Di Prima and Habetler used Naimark's result to prove the eigenfunction expansion theorem for a wide class of non-self-adjoint eigenvalue problems, including the problems arising in the linear stability analysis of a plane-parallel flow between two parallel plates, a flow between two concentric rotating cylinders, and an immovable horizontal layer of fluid heated from below. Sattinger studied a wide class of fluid flows in a bounded three-dimensional domain and used, for the proof of the completeness theorem, some general results related to the Hilbert-Schmidt class of linear operators which are presented in the book by Dunford and Schwartz (1963). The paper by Herron (1980) is devoted to plane-parallel flows of a stratified fluid of variable density and will therefore be referred later (in Sect. 2.83). In his next paper (Herron (1982)) the case of "nearly parallel" flows between solid walls at $z=0$ and $z=H$ was considered. Such flows have velocity $\boldsymbol{U}(\boldsymbol{x}) = \{U(x,z), 0, W(x,z)\}$ where $|W| \ll |U|$, $|\partial U/\partial x| \ll |\partial U/\partial z|$, and the x derivative of W is also fairly small. Then it is possible, to a first approximation, to neglect the dependence of the velocity on x and to assume that $\boldsymbol{U}(\boldsymbol{x}) = \{U(z), 0, W(z)\}$. The study of the stability of such flows with respect to infinitesimal two-dimensional wave-like disturbances leads to a non-self-adjoint eigenvalue problem different from the problems considered by Di Prima and Habetler and by Sattinger. Herron used some general results by Gohberg and Krein (1965) and with their aid proved the eigenfunction expansion theorem for the nearly-parallel case too; in addition he showed also that the number of unstable eigenvalues (having a positive imaginary part) is here always finite (maybe equal to zero). Finally, Herron (1983) investigated a more complicated case of a nearly-parallel flow in a domain unbounded in the z-direction; this paper will be discussed later.

When the system of eigenfunctions is complete, any initial value $u(\boldsymbol{x}, 0)$ can be expanded in a series in terms of these eigenfunctions. Thus the general solution of the initial value problem for Eqs. (2.7) may be expressed as a superposition of elementary ("normal") modes. Therefore it may seem that the general stability problem is reducible here to the corresponding eigenvalue problem. In other words, it seems that to answer the question, whether the given flow is stable or not, it is sufficient to determine whether all or not all characteristic frequencies ω have a negative imaginary part $\Im m\omega < 0$.[2] Where there is a spatial homogeneity with respect to one or more coordinates, the various characteristic frequencies ω will, generally speaking, depend on the spatial "scales" of the disturbances (i.e., on the wave numbers k_1 or k_1 and k_2) and on Re. As $\mathrm{Re} \to 0$, the imaginary parts of all frequencies ω will tend to negative values (because for $\rho = \text{const.}$, the state of rest is always stable). However, as Re increases, the imaginary parts of certain frequencies may increase, and finally become at first equal to zero and then positive. The value of Re at which the real frequency first appears is just the critical Reynolds number Re_{cr} given by the eigenvalue analysis. This value of Re_{cr} can be determined from the equation $\max_j \Im m\omega_j = 0$ where $\omega_j = \omega_j(\mathrm{Re})$ (or, if there is a spatial homogeneity, $\omega_j = \omega_j(\mathrm{Re}, k_1)$ or $\omega_j = \omega_j(\mathrm{Re}, k_1, k_2)$ is the jth eigenvalue and the maximum is taken over all values of j (and, in a homogeneous case, all values of k_1 or k_1 and k_2). Of course, this critical value of Re corresponds to instability with respect to arbitrarily small disturbances and therefore it might be denoted by $\mathrm{Re}_{cr\ max}$ in the notation of Sect. 2.1.

A number of applications of this method for determination of the critical value Re_{cr} (or, more precisely, $\mathrm{Re}_{cr\ max}$) to specific fluid flows will be considered in Sects.

[2] In the case of parallel (or nearly-parallel) flows homogeneous in the Ox_l direction the spatial formulation of the stability problem is also possible (and often is even the more natural). In this formulation the frequency ω is assumed to be real and fixed but the longitudinal wave number k_1 is now the unknown (and in general complex) eigenvalue. However, we shall postpone the consideration of the spatial stability problems until Sect. 2.9.

2.6–2.9. It will be seen that, according to laboratory measurements, in some cases the basic laminar flow ceases to exist at $\mathrm{Re} = \mathrm{Re}_{cr}$ exactly, and for $\mathrm{Re} > \mathrm{Re}_{cr}$ it is transformed into a new laminar flow which is stable at Re exceeding (but not too much) the observed value of Re_{cr}. (The transition to a turbulent regime occurs here only after loss of stability of this new laminar flow (or even after several consecutive such losses of stability), at a Reynolds number considerably greater than $\mathrm{Re}_{cr} = \mathrm{Re}_{cr\,max}$.) However, in some other flows the transition to turbulence observed in laboratory experiments always occurs at values of Re much smaller than the value $\mathrm{Re}_{cr\,max}$ predicted by eigenvalue analysis. One possible explanation of this discrepancy is that the eigenvalue analysis is valid for infinitesimal disturbances only; therefore, it is possible that, for the finite disturbances which exist in all real experiments, nonlinear effects are important and lead to loss of stability at $\mathrm{Re} < \mathrm{Re}_{cr\,max}$. This explanation seems to be natural, and for many year it was generally accepted and repeated in numerous publications (including MY1 and many more recent sources, e.g., the excellent books by Drazin and Reid (1981); and Tritton (1988)). However, recently it was shown that failure of eigenvalue analysis may often have also other reasons related to the internal inadequacy of this method. Hence some supplementary general remarks about the eigenvalue approach must be made here.

Note first of all that the possibility of expanding an arbitrary solution of the system (2.7) in a series in terms of particular solutions of the form of Eq. (2.8) occurs often but not always. In particular, the situation is more complicated if the system (2.7) is singular (i.e., for example, if a coefficient of a leading derivative in this system becomes zero at some point) or if the flow region is unbounded in the direction of the flow non-homogeneity. In this case, the completeness of the system of eigenfunctions cannot be proved simply, and even the very concept of the eigenfunction (and of the eigenvalue) must be defined with care. Often (both for non-homogeneous flows and for flows homogeneous in some directins, with fixed wave numbers k_1 or k_1 and k_2) a continuous part of the spectrum of eigenvalues arises, with corresponding eigenfunctions satisfying unusual boundary conditions or possessing a more complicated structure (e.g., not vanishing at infinity or having discontinuities of the derivatives at a singular point). In applications, these "improper" eigenfunctions sometimes simply go unnoticed; then the system of simple "elementary solutions" (eigenodes of the form (2.8)) is obviously incomplete (see, e.g., Lin (1961b); Lin and Benny (1962); Case (1962); Drazin and Howard (1966); Drazin and Reid (1981)). The correct form of the eigenfunction expansion theorem states here that any initial value can be represented as a sum over all the eigenfunctions corresponding to a discrete spectrum, supplemented by an integral over the improper eigenfunctions corresponding to a continuous spectrum (which needs special rigorous definition). This clearly complicates both the formulation and the proof of the theorem. Nevertheless, even in cases where a continuous spectrum exists, the corrected completeness theorem is often (though not always) valid.

Consider, for example, the important case of a plane-parallel flow in the half-space $0 \leq z < \infty$ with the Blasius velocity profile (described, e.g., in Sect. 1.4 of MY1), which represents a useful model of the boundary-layer flow over a flat plate. Here the eigenfrequency spectrum includes both the discrete and the continuous parts, but the eigenfunction expansion theorem (which implies the completeness of the system of all eigenfunctions) is valid, as was shown by Salwen and Grosch (1981) (for more details see Sect. 2.92). A much more general case was considered by Herron (1983): he studied the linear stability problem for a wide class of 'nearly parallel' flows with the velocity profile $\{U(z), 0, W(z)\}$ in the half-space $0 \leq z < \infty$ or in the whole space $-\infty < z < \infty$. (The flows considered include convenient models for boundary-layer flows in the presence of streamwise pressure gradient and for plane jets, wakes, and mixing layer.) For the resulting eigenvalue problems, the strict definitions of the spectrum, eigenfunction, and expansion theorem were given, and then the eigenfunction expansion theorem was proved under sufficiently general regularity conditions.

In cases where the eigenfunction expansion theorem is proved, it is possible, in principle, to determine the "eigenfrequencies" ω corresponding to all (proper and improper) eigenfunctions, and then to apply the eigenvalue approach to the solution of the stability problem. However, other circumstances often make direct applications of the eigenvalue analysis difficult. Up to this point we have in fact assumed (without indicating this explicitly) that all the eigenvalues ω_j are distinct; in other words, we have neglected possible degeneracies of the frequency spectra. Such neglect does not take into account the fact that, as was discovered some time ago (in particular, by Schensted (1960) and Betchov and Criminale (1966)), the frequency spectrum in some flows necessarily contains degenerate (i.e., multiple or coalescing) eigenfrequencies. What is more, it was shown later that in many cases the degeneracies in the frequency spectrum are due to existing symmetries of the stability problem and therefore they *must* occur (see, e.g., Langford et al. (1988); Shanthini (1989) or Chossat and Iooss (1994) and the references therein). It is known that in the case of a multiple eigenfrequency ω_j of a non-self-adjoint operator, the corresponding eigenfunction of the form (2.8) must be supplemented by associated functions containing polynomial factors, in addition to the exponential function of time. Hence, if ω_j is real, these functions will correspond to unstable solutions which grow algebraically in time, and if $\Im m\omega_j < 0$, then the corresponding solutions will grow algebraically at small values of t and begin to decay exponentially only at times of the order of $T_j = 2\pi/\omega_j$. In this case, however, it is possible that at shorter times the algebraic growth will amplify the initial small disturbances so much that linear stability theory will cease to be applicable to them. Moreover, there are purely physical reasons showing that algebraic growth of small three-dimensional disturbances is quite common in fluid mechanics and may play a very important role in transition to turbulenc; see, e.g., Landahl (1980, 1993) and Chap. 3 of this book.

Besides the algebraic growth produced by degeneracies of the frequency spectrum, there are other possible causes for strong initial growth of infinitesimal disturbances, even in cases where all the eigenfrequencies of the linear system are distinct and lie in the lower half-plane of the complex-variable plane. It was noted above that self-adjoint linear equations (i.e., equations corresponding to self-adjoint linear operators in a Hilbert space of functions) have the following valuable property: the eigenfunctions corresponding to two different eigenvalues are necessarily orthogonal to each other (i.e., represent two orthogonal vectors of Hilbert space). This property is valid not only for self-adjoint equations but also for a wider class of the so-called *normal* linear operators L in a Hilbert space, having the property that $LL^* = L^*L$ where L^* is the operator adjoint to L, i.e., such that $(L\boldsymbol{f}, \boldsymbol{g}) = (\boldsymbol{f}, \boldsymbol{L}^*\mathrm{g})$ for any $\boldsymbol{f}$ and $\mathbf{g}$ (see, e.g., Dunford and Schwartz (1958, 1971); Kato (1976, 1982) or Pazy (1983) for more information about such operators). The linearized fluid dynamics equations of linear stability theory include non-self-adjoint operators in all really interesting cases, but in some flows the corresponding operators prove to be normal while in others they are nonnormal (and sometimes even "very far from normal"). The strict meaning of the words "very far from normal" will be explained later; here it is enough to say that for "very strongly nonnormal" operators the eigenfunctions corresponding to different eigenvalues are not only non-orthogonal, but even nearly linearly dependent. It is clear that even in the case where such eigenfunctions form a complete system of functions, it is not reasonable to base the analysis on the expansions of arbitrary functions in terms of this system. It can be shown that in this case, even if all the eigenfrequencies are stable (i.e. lie in the lower half-plane), a small initial disturbance may sometimes be amplified by an enormously large factor before the asymptotic exponential decay becomes apparent. Moreover, this exponential decay may become fictitious since the asymptotic behavior is very sensitive here to the inevitable pre-existing small perturbations (see, e.g., Schmid et al. (1993)). This shows that results from the linear stability theory obtained by eigenvalue analysis of equations corresponding to nonnormal operators must always be considered with caution.

Nevertheless it will be shown below that in some cases eigenvalue analysis leads to brilliant scientific results which agree excellently with the experimental data and represent important steps in the development of fluid mechanics. Also, even the results of this analysis whose physical relevance is now in doubt are often quite important for proper understanding of subsequent works. Therefore the next subsections will be devoted to results of the linear stability theory obtained with the aid of eigenvalue analysis.

2.6 Linear Stability Analysis of Flow Between Two Rotating Cylinders

One important linear stability problem, which is amenable to complete mathematical analysis, is related to studies of stability for a steady circular Couette flow between two rotating coaxial cylinders. This flow can be modeled relatively easily in a laboratory[3] and in fact the first apparatus producing a fluid flow between one stationary and one rotating cylinder were built quite early (independently by Couette (1888); and Mallock (1888) for measuring fluid viscosity; for further details see Donnelly

[3] Of course, the coaxial cylinders have a finite length in any laboratory model, while the theory usually deals with the idealized case of a flow between infinite cylinders. However, both the experimental data and results of theoretical computations show that if the cylinders are long enough, the effects of finite length are significant only near the ends of cylinders and can be neglected when the middle part of the flow field is considered.

(1991, 1992)). These early experiments attracted attention to the study of stability of circular Couette flow; two great scientists, Lord Rayleigh and Sir Geoffrey (G.I.) Taylor, were central figures in the early developments of these studies, and the results obtained by them are now a classical part of fluid mechanics.

Let R_1 and Ω_1 be the radius and angular velocity of the inner cylinder, and $R_2 > R_1$ and Ω_2 those of the outer cylinder. In cylindrical coordinates r,φ,z with Oz axis along the axis of the cylinders, the velocity field of a circular Couette flow will be defined by the familiar equations (see, e.g., Eq. (1.28) in MY1).

$$\begin{aligned} &U_r = U_z = 0, \quad U_\varphi = U(r) = Ar + \frac{B}{r}, \\ &A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad B = -\frac{R_1^2 R_2^2(\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}, \end{aligned} \tag{2.10}$$

First, let us ignore the effect of viscosity. Then we may define the criterion of instability from the following elementary physical considerations, which are not completely rigorous but seem quite convincing. In a steady laminar flow, the centrifugal force acting on an element of the fluid will be balanced by the radial pressure gradient. Now, let an element of mass m move under the action of the velocity disturbance from a position with coordinate r_0 to a position with coordinate $r > r_0$. Then by the law of conservation of angular momentum $mrU(r)$, its velocity in the new position will equal $r_0U(r_0)/r$. Therefore, a centrifugal force $m\frac{r_0^2}{r^3}U^2(r_0)$ will now act on this fluid element. If this force is greater than the radial pressure gradient at a distance r from the axis, then this force will move the considered fluid element still further outwards, and hence the initial position of the element will be unstable. The radial pressure gradient at a distance r is equal in magnitude to the undisturbed value of centrifugal force at this distance. Therefore, the condition of instability (which was established for inviscid Couette flow by Rayleigh (1880, 1916b)), will have the form

$[r_0U(r_0)]^2 - [rU(r)]^2 > 0$ for some r_0 and $r > r_0$, or, in other words,

$$\frac{d}{dr}[rU(r)]^2 < 0 \text{ for some } r. \tag{2.11}$$

(This derivation of Rayleigh's criterion, which differs a little from the original arguments by Rayleigh, is due to Karman (1934); see also Di Prima and Swinney (1985), pp. 142–143; and Koschmieder (1993), pp. 206–207.)

Following Coles (1965, 1967), and taking into account that $U(r)/r$ is the angular velocity of the flow and that $d(rU)/rdr$ is the axial vorticity, we may easily rephrase the criterion (2.11) as follows: *a flow is unstable if the vorticity* (*local rotation*) *is opposite in sense to the angular velocity* (*overall rotation*). In such (or slightly modified) form, this criterion is apparently valid for many inviscid circulatory flows (cf. Joseph (1976), Chap. 6, and Mutabazi et al. (1992)).

Using Eq. (2.10), Rayleigh's instability criterion may be reduced to the form $\left(\Omega_2 R_2^2 - \Omega_1 R_1^2\right) U < 0$. If the cylinders are rotating in opposite directions, then U will change sign somewhere between the cylinders, and in this case the flow will certainly be unstable. When both rotate in the same direction, we may put $\Omega_1 > 0$, $\Omega_2 > 0$, and then $U(r) > 0$ everywhere; in this case the Rayleigh criterion of instability takes the form

$$\mu = \frac{\Omega_2}{\Omega_1} < \left(\frac{R_1}{R_2}\right)^2. \tag{2.12}$$

When the inner cylinder is fixed and the outer one rotates, then $\Omega_1 = 0$ while $U(r)$ has everywhere the same sign as Ω_2; hence, in this case the flow must be stable according to Rayleigh's criterion. In the opposite case, when only the inner cylinder rotates, $\Omega_2 = 0$ while Ω_1 and $U(r)$ are of the same sign; so in this case the flow must always be unstable.

A rigorous mathematical derivation of the result (2.12), by the method of small disturbances applied to inviscid flow between rotating cylinders, was given by Synge (1933, 1938a) for the case of axisymmetric disturbances (i.e., independent of φ). (In this respect see also Chandrasekhar (1960, 1961); Shen (1964); Warren (1976); Joseph (1976); Drazin and Reid (1981); Georgescu (1985); and Lortz (1993) where many additional details of Synge's derivation were given and the eigenvalue problem was considered also for the case of arbitrary normal modes.) The general case of non-axisymmetric infinitesimal disturbances (proportional to $\exp(\mathrm{i}(kz + n\varphi - \omega t))$) was also considered by Krueger and Di Prima (1962); Bisshopp (1963) and Warren (1976) (see also the book by Drazin and Reid). These authors found that in the case of inviscid flow between rotating cylinders all the non-axisymmetric modes of disturbance have smaller growth rates than that for the most unstable axisymmetric mode (at least if the gap between cylinders is narrow, i.e., R_1/R_2 is close to one).

Rayleigh's inviscid instability criterion for circular Couette flow clearly disagrees with early experimental findings by Couette (1888, 1890), in the case where the inner cylinder is fixed and the outer one is rotated, according to which there is a change in the flow structure at Some specific value of Ω_2. This change is apparently caused by the loss of stability at this Ω_2, whereas according to Rayleigh's criterion the flow must be stable for any value of Ω_2. On the other hand, this criterion agrees well with findings of another early experimenter, Mallock (1888, 1896), who found that if $\Omega_2 = 0$ but $\Omega_1 \neq 0$ (i.e., if only the inner cylinder rotates), then instability occurs at any value of Ω_1. However, this experimental result was inconsistent with conclusions from another paper by Rayleigh, published in 1913. Later Taylor (1921) constructed an apparatus in which both cylinders could rotate independently and found that Rayleigh's criterion (2.12) often does not hold when the cylinders rotate in opposite senses (i.e., are counter-rotating). He concluded that this inconsistency was probably due to neglect of the effect of viscosity. This circumstance forced him to arrange a thorough theoretical investigation of the instability conditions for circular Couette flow of a viscous fluid, supplemented by careful experimental verification of the results. This undertaking culminated in the appearance of his brilliant theoretical and experimental paper (Taylor 1923) whose importance for all the subsequent development of the investigations of hydrodynamic instability cannot be overestimated.

Since in circular Couette flow between sufficiently long cylinders the undisturbed velocity field (4.10) depends only on the r-coordiante, the appropriate form of Eqs.

(2.8) and (2.9) is

$$u_r'(\boldsymbol{x},t) = e^{i(kz+n\varphi-\omega t)}\boldsymbol{f}^{(r)}(r), u_\varphi'(\boldsymbol{x},t) = e^{i(kz+n\varphi-\omega t)}\boldsymbol{f}^{(\varphi)}(r),$$
$$u_z'(\boldsymbol{x},t) = e^{i(kz+n\varphi-\omega t)}\boldsymbol{f}^{(z)}(r), p'(\boldsymbol{x},t) = e^{i(kz+n\varphi-\omega t)}g(r). \tag{2.13}$$

Here k is the axial wave number (and $2\pi/k$ is the wavelength of the disturbance in the Oz direction); n, the azimuthal wave number, is a nonnegative integer determining the dependence of the disturbance on the angle φ; *and* $\boldsymbol{f}(r) = \boldsymbol{f}_{\omega;k,n}(r) = [f^{(r)}(r), f^{(\varphi)}(r), f^{(z)}(r)]$ and $g(r) = g_{\omega;k,n}(r)$ are the r-dependent "amplitudes" of the disturbance, with given axial wavenumber k, azimuthal wavenumber n, and characteristic frequency ω. Substituting from Eqs. (2.10) and (2.13) into the general system of equations (2.7), and taking into account the boundary conditions $u(r,\varphi,z,t) = 0$ for $r = R_1$ and $r = R_2$, we arrive at the eigenvalue problem determining the spectrum of permissible frequencies for given k and n. It may be shown (see, for example, Di Prima (1961)), that this problem, after certain transformations (including, in particular, the elimination of $\boldsymbol{f}^{(z)}$ and g), may be reduced to the following system of two differential equations in two unknown functions $\boldsymbol{f}^{(\mathrm{r})}$ and $\boldsymbol{f}^{(\varphi)}$:

$$\begin{aligned}
&\frac{d}{dr}\left[N\left(\frac{d\boldsymbol{f}^{(r)}}{dr}+\frac{\boldsymbol{f}^{(r)}}{r}\right)\right]-k^2\left(N+\frac{\nu}{r^2}\right)\boldsymbol{f}^{(r)}\\
&= -2k^2\left(A+\frac{B}{r^2}-i\frac{n\nu}{r^2}\right)\boldsymbol{f}^{(\varphi)} - in\frac{d}{dr}N\left(\frac{f^{(\varphi)}}{r}\right),\\
&-k^2\left(N+\frac{\nu}{r^2}\right)\boldsymbol{f}^{(\varphi)}-\frac{n^2}{r}N\left(\frac{\boldsymbol{f}^{(\varphi)}}{r}\right)\\
&= 2k^2\left(A-i\frac{n\nu}{r^2}\right)\boldsymbol{f}^{(r)}-\frac{in}{r}N\left(\frac{d\boldsymbol{f}^{(r)}}{dr}+\frac{\boldsymbol{f}^{(r)}}{r}\right),
\end{aligned} \tag{2.14}$$

where A and B are determinded from Eq. (2.10) and N is the differential operator

$$N = -\nu\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}-k^2\right)-i\left(\omega-\frac{nU(r)}{r}\right). \tag{2.15}$$

The boundary condition for which the sixth-order system (2.14) must be solved take the form

$$\boldsymbol{f}^{(r)}(r) = \boldsymbol{f}^{(\varphi)}(r) = \frac{d\boldsymbol{f}^{(r)}}{dr} = 0 \text{ for } r = R_1 \text{ and r} = R_2. \tag{2.16}$$

The boundary-value problem (2.14–2.16) has no singularities. It follows from the general theorems about systems of this type that for any values of k and n the system has an infinite number of eigenvalues $\omega_j = \omega_j(k,n,\Omega_1,\Omega_2,R_1,R_2,\nu)$. Furthermore, the corresponding set of eigenfunctions $(\boldsymbol{f}^{(\mathrm{r})},\boldsymbol{f}^{(\varphi)})$ is complete in the Hilbert space of pairs of functions on the interval $R_1 \le r \le R_2$, satisfying conditions (2.16) and having finite values of the integral of $\left|f^{(r)}\right|^2+\left|f^{(\varphi)}\right|^2$ (see, e.g., the literature cited in Sect. 2.5 above and the Remarks to Chap. II in the book by Joseph (1976) which contain addtional references). Therefore, the values at $t=0$ of the corresponding functions (2.13), where k passes through all the real values and n through all the integer values, is complete in the space of all permissible initial values $[\boldsymbol{u}'(\boldsymbol{x},0), p'(\boldsymbol{x},0)]$. This fact justifies the attempt to apply the eigenvalue method of the Linear stability theory, i.e., to study the set of eigenvalues $\omega_j(k, n, \Omega_1, \Omega_2, R_1, R_2, \nu)$ in order to find the conditions guaranteeing that all the eigenvalues lie in the lower half-plane $\Im\mathrm{m}\,\omega < 0$.

However, instead of solving the complete eigenvalue problem (2.14–2.16), all early stability studies dealing with circular Couette flow (beginning with Taylor's work (1923)) assumed that $n=0$, i.e., they considered only axisymmetric velocity disturbances, which do not depend on φ. The reason for this was that data from almost all early experiments (with the exception of data by Lewis (1928) which were unnoticed at first) gave the impression that when the Reynolds number Re increases the disturbance that first becomes unstable is always axisymmetric. Under the assumption that $n=0$, the system (2.14–2.15) can be considerably simplified and reduced to the form

$$\begin{aligned}\left(L-k^2+\frac{i\omega}{\nu}\right)(L-k^2)f^{(r)}(r) &= \frac{2k^2}{\nu}\left(A+\frac{b}{r^2}\right)f^{(\varphi)}(r),\\ \left(L-k^2+\frac{i\omega}{\nu}\right)f^{(\varphi)}(r) &= \frac{2}{\nu}Af^{(r)}(r),\end{aligned} \tag{2.17}$$

where

$$L=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}=\frac{d}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right). \tag{2.17$'$}$$

The boundary-value problem (2.17–2.16) for fixed k has an infinite number of eigenvalues $\omega_j(k,\Omega_1,\Omega_2,R_1,R_2,\nu)$ but the corresponding eigenfunctions will not determine a complete system of functions in the space of all permissible initial velocity fields $\boldsymbol{u}'(\boldsymbol{x})$ (if only because all the corresponding values of $\boldsymbol{u}'(\boldsymbol{x})$ are independent of φ). Nevertheless, the region of the (Ω_1,Ω_2)-plane to which there correspond unstable disturbances of the form $\exp[i(kz-\omega t)]\,\boldsymbol{f}(r)$ (i.e., disturbances of this form with $\Im m\omega \geq 0$) was identified for a number of years with the whole region of instability [see, e.g., Landau and Lifshitz (1944, 1953, 1958); Lin (1961a); Chandrasekhar (1961); Stuart (1963)). Although it was shown later that this identification is not exactly correct, we will begin with results valid for disturbances with $n=0$ and only after this consider the general case.

According to many early measurements the disturbance that first becomes unstable is not only axisymmetric but also nonoscillatory; in other words, this disturbance corresponds to an eigenmode with $n=0$ and purely imaginary eigenfrequency ω. Relying on this observation, Taylor (1923) limited his investigation of the eigenvalue problem (2.17–2.16) by studying only the case of purely imaginary values of ω; the same assumption has been used by many subsequent investigators of the subject. Later, several authors tried to prove mathematically that $\Re\mathrm{e}\,\omega=0$ for the most unstable mode; but nobody could find such a proof under general conditions. Apparently the strongest result was obtained by Yih (1972), who showed that if both cylinders are rotating in the same sense and Rayleigh's condition (2.12) is valid (i.e., $0 \le \mu \le (R_1/R_2)^2$), then at any fixed value of Re $= \Omega_1 r_1^2/\nu \ge 0$ there exists an infinite number of axisymmetric eigenmodes, and to all these modes (both stable and unstable) there correspond purely imaginary eigenfrequencies ω_j. Since in 1972 there was no doubt that at $\mu > (R_1/R_2)^2$ unstable modes do not exist at all, Yih's result showed that in the case of cylinders rotationg in the same sense ω must be purely imaginary for the most unstable eigenmode, if such eigenmode exists. Therefore the loss of stability takes place here when an eigenvalue $\omega = 0$ appears. For flows between counter-rotating cylinders ($\mu < 0$) this result was not proved (and now it is known that it is incorrect, if $\mu = \Omega_2/\Omega_1$ is negative and large enough in absolute value; see below). However, it has long been assumed that experimental data show that the most unstable mode is *always* axisymmetic and nonoscillatory. If it is true, then for determination of the boundary of the instability region (the so-called "neutral curve") it is enough to assume that $\omega = 0$ in the system (2.17–2.16) and then to find for which values of Ω_1 and Ω_2 the obtained simplified system of equation has a solution. The fact that the transition from stability to instability proceeds through a steady state, corresponding to zero eigenvalue $\omega = 0$, is often called in fluid mechanics the "principle of exchange of stabilities" (this term was first introduced by H. Poincare more than 100 years ago in a somewhat different context, and its use has a long history; see, e.g., Joseph (1976), Remarks to Chap. II, and Drazin and Reid (1981), p. 12). Now it is known, however, that this principle has a limited domain of validity; in particular, it is not valid for circular Couette flows if μ is negative and large in absolute value (again see the discussion later in this section). The system for the case where $n = 0$, (2.17–2.16), includes five dimensionless parameters, $R_1/R_2 > 0, \Omega_2/\Omega_1, \kappa = k(R_2 - R_1) = kd, \omega^0 = \omega/\Omega_1$, and Re $= \Omega_1 R_1 d/\nu$ (if $\Omega_1 = 0$, then it must be replaced by Ω_2 in the last two equations). Hence, here $\omega^0 = \omega^0(R_1/R_2\Omega_2/\Omega_1, \kappa, \mathrm{Re})$. For fixed values of R_1/R_2, Ω_2/Ω_1 and κ this system of equations allows us to determine the smallest (and usually unique) value of Re, $\mathrm{Re}(\kappa)_{\mathrm{cr}}$ say, such that $\omega^0(R_1/R_2, \Omega_2/\Omega_1, \kappa, \mathrm{Re}(\kappa)_{cr}) = 0$ (i.e., the axisymmetric disturbance with dimensionless wave number κ is neutral at $\mathrm{Re} = \mathrm{Re}(\kappa)_{\mathrm{cr}}$). Thus, $\mathrm{Re}(\kappa)_{\mathrm{cr}}$ is the critical Reynolds number for axisymmetric disturbances with axial wave number $k = \kappa d$. The schematic form of the function $\mathrm{Re}(\kappa)_{\mathrm{cr}}$ is shown in Fig. 2.9, taken from Tritton's book (1988); the shape of the neutral curve in the (Re, κ)-plane given here is closest to that for $\Omega_2/\Omega_1 = 0$, $d/R_1 \ll 1$ (see also Fig. 2.15a, b but note that they correspond to quite different values of Ω_2/Ω_1 and d/R_1). The points (Re,κ) outside the area bounded by the curve $\mathrm{Re} = \mathrm{Re}(\kappa)_{\mathrm{cr}}$ correspond to stable disturbances, with $\Im m\omega < 0$, and the points inside this area to unstable disturbances. The minimum value of $\mathrm{Re}(\kappa)_{\mathrm{cr}}$, which corresponds to the leftmost point of the $\mathrm{Re}(\kappa)_{\mathrm{cr}}$ curve, is the value of $\mathrm{Re}_{\mathrm{cr}} = \mathrm{Re}_{\mathrm{cr\ min}}$ according to linear stability theory, while the corresponding value of $\kappa_{\mathrm{cr}} = k_{\mathrm{cr}}\ d$ determines the wave number k_{cr} of the disturbance which first becomes unstable as Re increases. Results of accurate numerical computations of the function $\mathrm{Re} = \mathrm{Re}(\kappa)_{\mathrm{cr}}$ for $\Omega_2/\Omega_1 = 0$ and a number of values of R_1/R_2 in the range $0.975 \le R_1/R_2 \le 0.1$ can be found in the papers by Di Prima and Eagles (1977); and Dominguez-Lerma et al. (1984).

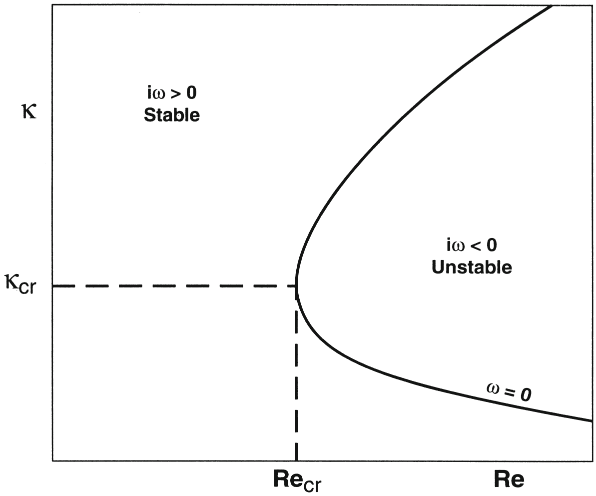


Fig. 2.9 Schematic form of the *neutral-stability curve* $Re = Re(\kappa)_{cr}$ in the (Re, κ) plane (after Tritton (1988)). (This curve depends on values of R_2/R_1 and Ω_2/Ω_1; the given shape is close to that for $\Omega_2/\Omega_1 = 0$ and $(R_2\text{-}R_1)/R_1 \ll 1$)

The numerical solution of the system (2.17–2.16) was not a simple problem in the precomputer era, even for the special case where $\omega = 0$. This is the reason why Taylor (1923), who first performed such computations, considered only the narrow-gap case where $d = R_2 – R_1 \ll R_1$. He used an expansion of the solution of Eqs. (2.17) in terms of a special set of orthonormal functions, which reduced the differential equation to an infinite system of linear algebraic equations. Later a number of other methods of numerical solution was proposed, both for Eqs. (2.17) and for the more complicated Eqs. (2.14). Many results of the computations performed, supplemented by numerous additional references, can be found, e.g., in the books by Chandrasekhar (1961); Joseph (1976); Goldshtik and Shtern (1977); Drazin and Reid (1981); Koschmieder (1993); and Chossat and Iooss (1994); and in the papers by Sparrow et al. (1964); Walowit et al. (1964); Roberts (1965); Babenko et al. (1982); Dominguez-Lerma et al. (1984); Di Prima and Swinney (1985); Afendikov and Babenko (1985); Babenko and Afendikov (1985); Langford et al. (1988); Takhar et al. (1989a, b); Gwa and Cohen (1992); Gebhardt and Grossmann (1993); and many others; see also the extensive bibliography and valuable survey paper by Tagg (1992, 1994). The appearance of general-pupose digital computers and subsequent advances in computer technology substantially affected the development of these investigations and made it possible to carry out in the 1960s and 1970s many computations which seemed impossible earlier, and in the 1980s and 1990s to carry out much more extensive and diverse computation than those done before 1980. Modern computers also allowed the eigenvalue and normal-mode computations to be supplemented by results of direct numerical simulations of disturbance development in circular Couette flows (i.e., numerical solutions of the appropriate nonlinear equations of motion); see, e.g., Jones (1981, 1982, 1985); Moser et al. (1983); Marcus (1984); and Hirschberg (1992). The experimental methods used for laboratory verifications of computational results have also improved greatly during recent years (see, e.g., Weidman (1989); and Donnelly (1992)).

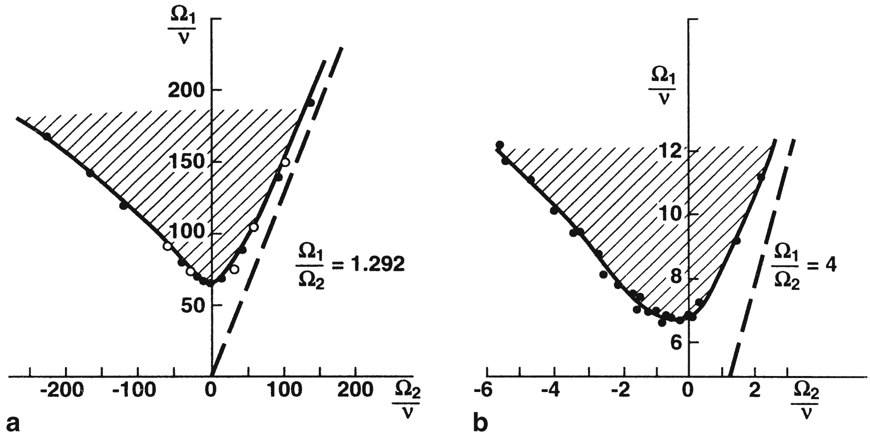


Fig. 2.10 Position of the region of instability in the plane (Ω_1, Ω_2) for Couette flow between rotating cylinders, for $R_2/R_1 = 1.13$ **a** and $R_2/R_1 = 2$ **b**. The region of instability is shaded in the figure; the *black dots* represent experimental results while the *white circles* in figure *a* and the *solid line* in figure *b* are found by computations. The *dotted lines* indicate the boundary of the region of instability for the corresponding flow of an inviscid fluid (after Rayleigh)

As the first example of results from numerical calculations, let us consider Fig. 2.10a, taken from Taylor's classic paper of 1923. This fugure shows the region of the (Ω_1, Ω_2)-plane in which unstable axisymmetric disturbances are possible for $R_1/R_2 = 0.880$. The instability region is shaded in this figure; the small white circles represent the results of Taylor's calculations of the points belonging to the neutral curve in the (Ω_1/ν, Ω_2/ν)-plane, while the black dots show the measured values of (Ω_1/ν, Ω_2/ν) at which instability first appears as the angular velocity Ω_2 is slowly increased. We see that theoretical and experimental results agree excellently with each other and determine the same neutral curve (the solid line in Fig. 2.10a). This figure and the similar figure for $R_1/R_2 = 0.743$, also given in Taylor's paper, represent the first great (even epoch-making) success of the theory of hydrodynamic stability, which was confirmed by the results of numerous subsequent investigations. The neutral curves in the (Ω_1, Ω_2)-plane (or, what is the same, in the (Re_1, Re_2) or (Re_1^*, Re_2^*)—plane, where $Re_i = \Omega_i R_i d/\nu, Re_i^* = \Omega_i R_i^2/\nu$, i $= 1, 2$), calculated for a number of other values of $R_1/R_2 = \eta$ (ranging from 0.1 to 0.964) and for various ranges of $\Omega_2/\Omega_1 = \mu$, are presented, and often compared with the available data, in papers by Chandrasekhar (1958); Chandrasekhar and Elbert (1962); Donnelly (1962); Sparrow et al. (1964); Walowit et al. (1964); Snyder (1968b); Di Prima and Swinney (1985) and others. In particular, a summary of calculated and measured neutral curves in the (Re_1^*, Re_2^*)-plane found before 1968 is presented in Snyder's paper (1968b). Here, as one more example, we show in Fig. 2.10b the instability curve for $R_1/R_2 = 0.5$ (which is a typical wide-gap case) calculated by Chandrasekhar (1958) and compared with data by Donnelly and Fultz (1960), which are indicated by black dots in the figure (see also Chandrasekhar (1961); Chandrasekhar and Elbert (1962); and Donnelly (1962)). Note that in both cases represented in Fig. 2.10

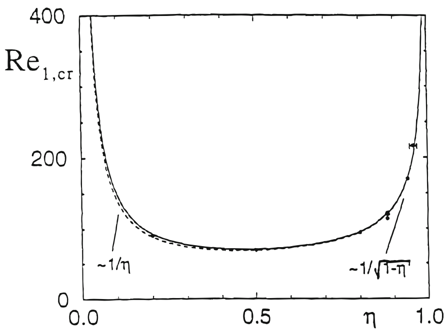


Fig. 2.11 Dependence of the critical Reynolds number $Re_{1,cr}$ for Couette flow between inner rotating and outer stationary cylinders on the radius ratio $\eta = R_1/R_2$. The *solid* and *dashed lines* represent the results of numerical computations by Gebhardt and Grossmann (1993); and the approximate equation by Esser and Grossmann (1996); respectively, while *dots* show experimental results by different investigators (after Gebhardt and Grossmann and Esser and Grossmann)

(and also for all other values of R_1/R_2), the calculations predict unstable disturbances only if $\Omega_1/\Omega_2 < (R_1/R_2)^2$ (i.e., within the region of instability for inviscid Couette flow). This fact agrees with the relatively old theoretical result by Synge (1938b); (see also Chandrasekhar (1961), Sect. 70) who showed that in the case where $\Omega_1/\Omega_2 > (R_1/R_2)^2$ instability with respect to axisymmetric disturbances will not occur for any value of Re. It is natural to assume that in this case circular Couette flow will also be stable at any Re with respect to any non-axisymmetric infinitesimal disturbance. This last assumption has not so far been proved in full generality, but it agrees with all available experimental data; if it is correct, it means that here the viscosity has only a stabilizing effect. Figure 2.10 shows also that when $\Omega_2/\Omega_1 = \text{const} < (R_1/R_2)^2$, instability to infinitesimal disturbances must arise if the value of Re is sufficiently high.

Accurate computations of the linear growth rate $\gamma = \Im m\omega$ for the most unstable disturbance at several supercritical values of $Re > Re_{cr}$ were performed by Di Prima and Eagles (1977); and Dominguez-Lerma et al. (1984), for Couette flows with $\Omega_2 = 0$ and various values of $\eta = R_1/R_2$. Di Prima and Swinney (1985) listed a number of computed values of $Re_{1,cr} = (\Omega_1 R_1 d/\nu)_{cr}$, where $d = R_2 - R_1$, for Couette flows with $\Omega_2 = 0$ and various values of η (in Fig. 2.10a, b $Re_{1,cr}$ corresponds to points of intersection of the neutral curve and ordinate axis). Later Gebhardt and Grossmann (1993) computed the values of the function $Re_{1,cr}(\eta)$ for a great number of values of η; their results are shown in Fig. 2.11 together with the available experimental results.

Esser and Grossmann (1996) tried to find an analytic expression, albeit approximate, for the function $Re_{1,cr}(\eta)$ by generalizing to the case of circular Couette flow of viscous fluid the arguments of Rayleigh (1916b); and Karman (1934) which lead, in the inviscid case, to the instability criterion (2.12). For this it was necessary to take into account the effect of viscosity on the motion of fluid particles (i.e., the loss of fluid particle energy and momentum produced by viscous force) and to determine the condition under which the hypothetical exchange of two neighboring fluid rings produced by their virtual motions will lead to increase of the kinetic energy (i.e., such exchange will be energetically disadvantageous). By introducing some reasonable hypotheses the authors obtained for the function $\mathrm{Re}_{1,\mathrm{cr}}(\mathrm{h})$ an expression of the form

$$\mathrm{Re}_{1,\mathrm{cr}}(\eta) = \frac{1}{\alpha^2} \frac{(1+\eta)^2}{2\eta[(1-\eta)(3+\eta)]^{1/2}},$$

where $\alpha \approx 0.155$ (this value of the constant α was determined from the comparison of the results for $\mathrm{Re}_{1,\mathrm{cr}}$ as $\eta \to 1$(in the "narrow-gap limit") with values obtained from Eqs. (2.17)). The resulting function $\mathrm{Re}_{1,\mathrm{cr}}(\eta)$ is shown in Fig. 2.11 by a dashed line; as one can see, it agrees excellently with the results of numerical calculations.

A similar method was applied by Esser and Grossmann to determine an analytic approximation for the function $\mathrm{Re}_1 = \mathrm{Re}_1$ (Re_2), which corresponds to the neutral curve in the (Re_1, Re_2) plane. The equation obtained depends on η; for $\eta = 0.2$, 0.5 and 0.964 it implies forms of the neutral curve which agree quite satisfactorily with the experimental data by Donnelly and Fultz (1960); and Snyder (1968b). These approximate equations for functions $\mathrm{Re}_{1,\mathrm{cr}}(\eta)$ and $\mathrm{Re}_1(\mathrm{Re}_2)$ shed additional light on the physical mechanism of considered instability.

According to the discussion above, both experimental data and the results of numerical calculations show that transition from stability to instability in circular Couette flows very often occurs when, for some value k_{cr} of the axial wave number, one of the eigenvalues ω_j becomes zero (the cases where transition occurs differently will be considered later). At slightly supercritical Reynolds numbers (when $0 < \mathrm{Re} - \mathrm{Re}_{\mathrm{cr}} \ll \mathrm{Re}_{\mathrm{cr}}$) there will be a narrow band of unstable eigenmodes. In this case the most unstable mode (which has the greatest rate of growth) is the mode that loses stability first of all, and at slightly supercritical values of Re this mode suppresses all the other unstable modes. Moreover, in circular Couette flow the amplitude growth rate of the most unstable mode usually decreases with increasing amplitude, and as $t \to \infty$ the amplitude tends to a finite limiting value. (This process depends on the nonlinear terms in the equations of motion and will be considered at greater length later). Therefore, instability in circular Couette flow usually leads to the development of a new (secondary) steady motion with an axisymmetric velocity field of the form $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$ where $\boldsymbol{u}' = \exp(ik_{cr}z)\boldsymbol{f}(r)$. This motion consists of a series of so-called *Taylor vortices* (discovered, calculated and described by Taylor in his pioneering paper of 1923)—cellular toroidal vortices spaced regularly along the axis of the cylinders at distances close to k_{cr}/π. (Since the value of k_{cr} differs not too much from π/d in many cases, the vertical scale of these vortices is often nearly the same as their horizontal scale $R_2 - R_1$). The streamline pattern of these vortices, computed from the corresponding eigenfunctions $\boldsymbol{f}(r)$, is shown in Fig. 2.12 (for more precise plots, see, e.g., Drazin and Reid (1981)); excellent flow visualization photographs of the vortices can be found, e.g., in books by Joseph (1976); Drazin and Reid (1981); Van Dyke (1982); Tritton (1988); Koschmieder (1993); and Chossat and looss (1994). With further increase of the Reynolds number in Couette flow, there will be a whole series of · transformations of the flow structure leading finally to fully developed turbulent flow. This part of the story will be discussed at length later.

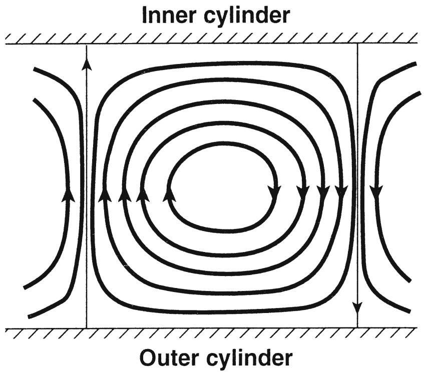


Fig. 2.12 Streamline pattern of vortex disturbance for flow between rotating cylinders. (After Shen (1964))

We have already explained that consideration only of axisymmetric disturbances, in Taylor's paper (1923) and in a long series of subsequent investigations, was based on some observational results but had no rigorous theoretical justification. However, Taylor himself noticed the appearance of non-axisymmetric vortical structures in flow between rotating cylinders, though he considered this as a phenomenon related to a secondary nonlinear instability. Non-axisymmetric disturbances in some circular Couette flows (related to cases of counter- rotating cylinders with rather large negative value of the ratio $\mu = \Omega_2/\Omega_1$) were also observed in the work by Lewis (1928), but this work did not attract much attention at the time and became widely known only in the late 1960s. Nevertheless, from the beginning of the 1960s several attempt were mde to investigate theoretically the behavior of non-axisymmetric disturbances of the form (2.13), where $n \neq 0$, in laminar Couette flows between rotating cylinders.

The early history of these investigations is described at length in MY1, pp. 104–107; it will be repeated here more briefly but will be supplemented by results from more recent work. In the first studies of the general eigenvalue problem (2.14–2.16) by Di Prima (1961); and Roberts (1965), only zero and positive values of μ were considered, and it was found that at such values of μ the disturbance which first becomes unstable is always axisymmetric. In other words, it was found that $n_{cr} = 0$ for $\mu \geq 0$, where n_{cr} is the azimuthal wave number of the disturbance which becomes unstable first of all when Re increases. However, a somewhat surprising result of these early investigations was that the critical Reynolds number for non-axisymmetric disturbances with small non-zero values of n was only slightly larger (by a few percent in all cases) than the value of Re_{cr} for the most unstable axisymmetric disturbance with $n = 0$. The first work in which an asymptotic analysis of the eigenvalue problem (2.14–2.16) was carried out for $\mu < 0$ was the dissertation of Krueger (1962). According to his results, if $\mu < -0.8$ and the gap between the cylinders is very small (i.e., $d/R_1 \ll 1$), then $n_{cr} = 1$, i.e., some non-axisymmetric disturbance with $n = 1$ becomes unstable at a smaller value of Re than any axisymmetric disturbance. A more complete investigation was made by Krueger et al. (1966). They calculated, under the assumption that either $d/R_1 \to 0$ or $d/R_1 = 1/20$, the values of $\mathrm{Re}_{1,cr} = (\Omega_1 R_1 d/\nu)_{cr}$ for $0 \leq n \leq 5$ and $0 \geq \mu \geq -1.25$ (in fact the so-called *Taylor number* Ta (a special dimensionless combination of $R_1, R_2, \Omega_1, \Omega_2$ and ν which is

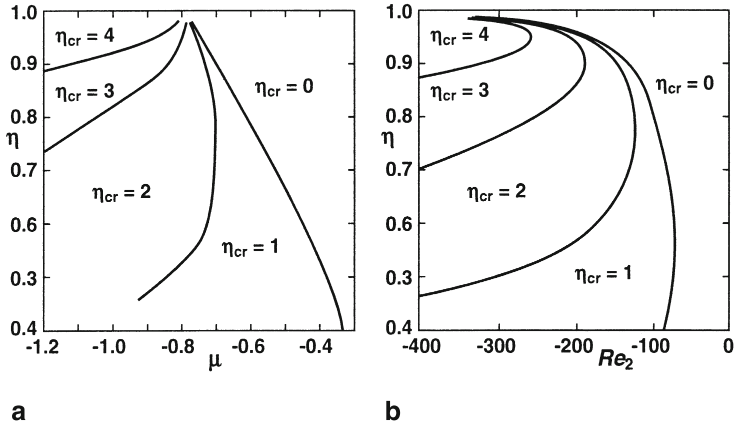


Fig. 2.13 Regions in the (μ, η) plane **a** and in the (Re_2, η) plane **b** in which normal modes with different azimuthal wave number n first becomes unstable as Re_1 is increasing. (After Langford et al. (1988))

proportional to $(Re_1)^2$ if $\Omega_2 = 0$ and d/R_1 has a fixed value) was used here instead of Re). It was found by Krueger et al. that for $\mu < -0.78$ the most unstable disturbance is non-axisymmetric, and that the value of n_{cr} is in general increasing when μ decreases (for example, if $d/R_1 = 1/20$, then $n_{cr} = 0$, 1, 3, 4, and 5 for $\mu = -0.70$, -0.80, -0.90, -1.00, and -1.25, respectively). Krueger et al. also found that the difference between the minimum value Ta_{cr} or Re_{cr} (which corresponds to $n = n_{cr}$) and the value of Ta_{cr} or Re_{cr} determined from an examination of axisymmetric disturbances with $n = 0$ only, increases with the increase of both $-\mu$ and $\delta = d/R_1$, but in all cases considered it as very small (usually only a few percent). These results were refined and augmented by the results of subsequent stability calculations for circular Couette flows between counter-rotating cylinders by Romashko (1981); Demay and Iooss (1984); and Afendikov and Babenko (1985); all these results also agree satisfactorily with the early experimental data of Lewis (1928); Nissan et al. (1963) and Snyder (1968a). However, the most complete computational and experimental results for the stability of laminar flows between counter-rotating cylinders with respect to infinitesimal disturbances were obtained by Langford et al. (1988); supplemented by Tagg et al. (1990) (see also Tagg (1994)). Langford et al. numerically solved the eigenvalue problem (2.14–2.16) (transformed by them into a more convenient form) for $n = 0$, 1, 2, 3, and 4 and various values of the dimensionless parameters $\kappa = kd, \eta = R_1/R_2, \mu = \Omega_2/\Omega_1$, and $Re_2 = \Omega_2 R_2 d/\nu = Re_1\mu/\eta$ (where $Re_1 = \Omega_1 R_1 d/\nu$) in the ranges $\kappa \geq 0, 0.4 \leq \eta < 1, -1.2 < \mu < -0.3$, and $0 > Re_2 > -400$. Since the determination of stability boundaries was the main interest, most attention was given to eigenvalues ω with vanishing imaginary part $\Im m\omega = 0$ (computed separately for five values of n). The results of the computations included the value $n = n_{cr}$, which corresponds to the disturbance that first becomes unstable, and also the regions in the (μ, η) and (Re_2, η) planes which correspond to various values of n_{cr} (see Fig. 2.13a, b; no computations were made for $n > 4$, and higher values

Table 2.1 Bicritical points computed for four particular values of $\eta = R_1/R_2$, with corresponding critical values of some important parameters (after Langford et al. (1988))

$n_{cr} \to (n+1)_{cr}$	$Re_{2,cr}$	$Re_{1,cr}$	μ	κ_{cr} for n_{cr} wave	κ_{cr} for $(n+1)_{cr}$ wave	$\omega^{(r)}{}_{cr}/n\Omega_1$ for n_{cr} wave	$\omega^{(n)}{}_{cr}/(n+1)\Omega_1$ for $(n+1)_{cr}$ wave
$\eta = 0.500$							
$0 \to 1$	−73.70	95.25	−0.3874	4.079	3.813	0	0.2813
$1 \to 2$	−300.85	175.94	−0.8550	6.354	5.280	0.3365	0.2825
$\eta = 0.736$							
$0 \to 1$	−87.71	114.82	−0.5622	3.631	3.590	0	0.3345
$1 \to 2$	−126.45	132.91	−0.7004	3.872	3.796	0.3670	0.3216
$2 \to 3$	−326.98	202.37	−1.1890	3.765	4.722	0.3702	0.3047
$\eta = 0.800$							
$0 \to 1$	−99.25	129.55	0.6129	3.571	3.551	0	0.3445
$1 \to 2$	−124.60	141.80	0.7030	3.726	3.678	0.3663	0.3355
$2 \to 3$	−243.40	186.67	1.0430	3.871	4.114	0.3731	0.3174
$\eta = 0.883$							
$0 \to 1$	−128.93	166.88	−0.6822	3.516	3.511	0	0.3546
$1 \to 2$	−142.81	173.83	−0.7254	3.592	3.574	0.3654	0.3505
$2 \to 3$	−189.68	193.78	−0.8643	3.704	3.708	0.3711	0.3418
$3 \to 4$	−345.74	248.16	−1.2300	3.779	4.070	0.3688	0.3253

of n_{cr} might correspond to some small regions in the upper left-hand corners of the figures). The dividing lines in Fig. 2.13a and b correspond to flows where at least two different neutrally stable wave disturbances can exist at $Re_1 = Re_{1,cr}$ (Langford et al. called such points the "bicritical points"). Values of the functions $\kappa_{cr}(Re_2)$ and $\omega^{(r)}_{cr}(Re_2)$ (where $\omega^{(r)} = \Re e\omega$) change discontinuously at the bicritical points while the functions $Re_{1,cr}$ (Re_2) are continuous here but have discontinuous derivatives (see Table 2.1 and Figs. 4–6 in the paper by Langford et al.).

Langford et al. supplemented their eigenvalue computations by detailed experimental investigation. The radius ratio η of their apparatus was gradually increased in steps of 0.05 and for all values of η experiments of two types were performed. In the first type, one cylinder speed was held constant (usually the outer so that Re_2 was fixed) and the other cylinder speed was adjusted until transition to non-Couette type of flow was observed. Such observations were made for many values of Re_2 varying from 0 to −400. In the experiments of the second type the speed ratio μ was held constant (and varied from measurement to measurement in the range from about −0.3 to −1.2) but the values of Ω_1 and of $-\Omega_2 = -\mu\Omega_1$ were both increased until transition was observed. At all the observed transitions the values of Re_1, Re_2, μ, η, κ, and $\omega^{(r)}/\Omega_1$ were tabulated and the results obtained were listed in a special document suitable for distribution (for more details see the original paper). Comparison of the experimental and theoretical results showed excellent agreement (see e.g. Fig. 2.14).

Let us recall that the forms of both the neutral curves shown in Fig. 2.10 were found from computational results for Couette-flow stability with respect to axisymmetric $(n = 0)$ disturbances. Also recall that when Re_1 increases, a non-axisymmetric disturbance can become unstable earlier than any axisymmetric disturbance only if $-\mu$

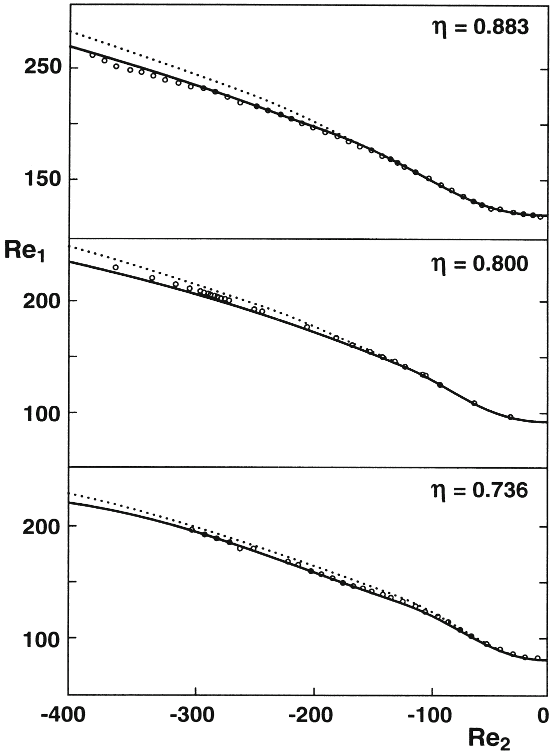


Fig. 2.14 True neutral-stability curves $\mathrm{Re}_1 = \mathrm{Re}_{1,\mathrm{cr}}(\mathrm{Re}_2)$ (*solid*) and curves $\mathrm{Re}_1 = \mathrm{Re}_{1,\mathrm{cr}}(\mathrm{Re}_2, 0)$ (*dotted*) for three values of η. *Small circles* indicate experimentally determined critical values of $\mathrm{Re}_{1,\cdot}$. (After Langford et al. (1988))

is quite high (at $\mu < -0.7$ if $\eta = 0.88$, and at $\mu < -0.4$ if $\eta = 0.5$; see Table 2.1). Therefore, results related to instability with respect to disturbances with higher values of n can affect only the far left part of the two above-mentioned curves. Moreover, these results imply that the changes in this part of the curves are so small that they apparently cannot be detected in Fig. 2.10a and b. In fact, it has already been noted that Krueger et al. (1966) found that the true critical Reynolds number $\mathrm{Re}_{1,\mathrm{cr}}$ (which corresponds to the disturbance, with any n, that first becomes unstable) always differed very little from the value of Re_1 at which some axisymmetric disturbance became unstable. These results were confirmed by Langford et al. (1988) who treated much more extensive data. They computed the values of the minimum Reynolds number $\mathrm{Re}_{1,\mathrm{cr}}(\mathrm{Re}_2, n)$ at which, for given Re_2, a neutrally stable disturbance with the azimuthal wave number n first appears. They covered the range $-400 < \mathrm{Re}_2 < 0$ and the following values of n and η: $n = 0, 1, 2, 3,$ and 4; $\eta = 0.883, 0.800, 0.736,$ and 0.500 (just these four values of η are represented in Table 2.1). At given n the function $\mathrm{Re}_{1,\mathrm{cr}}(\mathrm{Re}_2,\ n)$ determines the boundary, in the (Re_1, Re_2)-plane, of the region of Couette-flow instability with respect to disturbances of the form (2.13) where n is fixed. These curves (for the abovementioned values of n and η) were presented by Langford et al. together with the true neutral curve $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}(\mathrm{Re}_2) = \min_n[R_{1,\mathrm{cr}}(\mathrm{Re}_2, n)]$. It was found that the difference between the "conditional neutral curve" $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}(\mathrm{Re}_2, 0)$, which corresponds to axisymmetric disturbances,

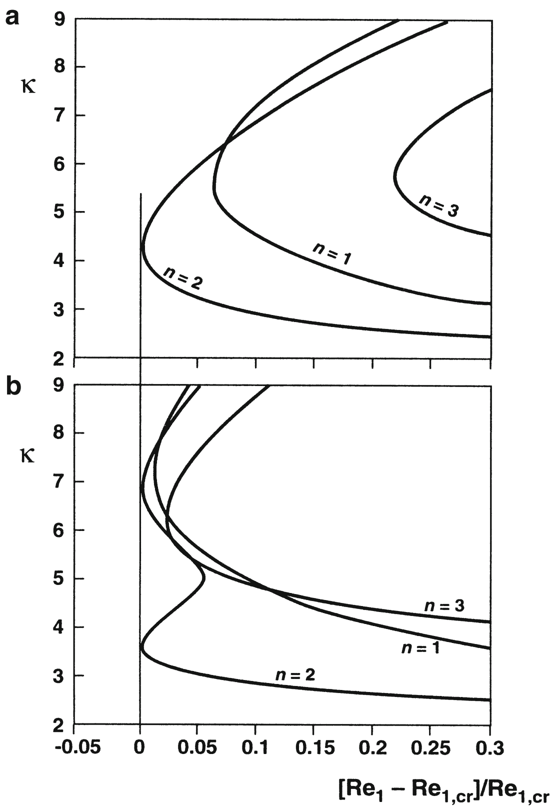


Fig. 2.15 Neutral-stability curves $\mathrm{Re}_1 = \mathrm{Re}_{1,\mathrm{cr}}(\kappa,n)$ in the (Re_1,κ) plane for $n = 1$, 2 and 3, and the cases where $(\mu, \eta) = (-1, 0.64)$, $\mathrm{Re}_{1,\mathrm{cr}} = 169.1$ **a** or $(\mu,\eta) = (-1.4, 0.64)$, $\mathrm{Re}_{1,\mathrm{cr}} = 246.3$ **b**. (After Tagg (1994))

and the unconditional (true) neutral curve $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}(\mathrm{Re}_2)$ is negligible except at large negative values of Re_2 and is rather small in all cases considered (see Fig. 2.14, which shows some of the computed results together with the corresponding experimental data). However, for larger negative values of μ, taking account of instability to non-axisymmetric disturbances leads to a considerable change of the neutral curve in the (Ω_1, Ω_2) [or $(\mathrm{Re}_1, \mathrm{Re}_2)$] plane (see, e.g., schematic Fig. II.6 in the book by Chossat and Iooss (1994)).

Results of some additional computations of stability characteristics for Couette flows between counter-rotating cylinders (i.e., for $\mu < 0$) were presented by Tagg et al. (1990); and Tagg (1994). In particular, these papers contain data allowing determination, for several combinations of μ and η values, of the neutral curves $\mathrm{Re}_1 = \mathrm{Re}_{1,\mathrm{cr}}(\kappa, n)$ in the (Re_1, κ) plane, which correspond to disturbances with given azimuthal wave number n (recall that the schematic graph of the function $\mathrm{Re}_1(\kappa)_{\mathrm{cr}}$ in Fig. 2.9 corresponds to the special case where $n = 0$, $\mu = 0$, and $1 - \eta \ll 1$). In Fig 2.15 (taken with some modification from Tagg (1994)) the curves $\mathrm{Re}_1 = \mathrm{Re}_{1,\mathrm{cr}}(\kappa, n)$, where $n = 1$, 2, and 3, are shown for $(\mu,\eta) = (-1, 0.64)$ and $(\mu,\eta) = (-1.4, 0.64)$. We see that in both cases the disturbance which first becomes unstable has azimuthal wave number $n = 2$; however the second of these two cases is somewhat peculiar since here two non-axisymmetric normal modes with the same azimuthal wave number $n = 2$ but different axial wave numbers, $\kappa_{1,\text{cr}}$ and $\kappa_{2,\text{cr}}$, become unstable at practically the same value of $\text{Re}_1 = \text{Re}_{1,\text{cr}}$.

However the main topic investigated by Tagg et al. (1990) is only indirectly related to stability studies for normal wave-like modes of the form (2.13). The modes (2.13) with $n = 0$, real k and purely imaginary ω correspond to axisymmetric standing waves, which grow or decay depending on the sign of $\Im m\omega$. If $n \neq 0$ and ω is purely imaginary, the disturbance (2.13) is a standing helical wave. However, if $\Re e\omega = \omega^{(r)} \neq 0$, then the distrubance (2.13) is a propagating wave which is moving either in the axial direction Oz (if $n = 0$) or in a spiral manner (if $n \neq 0$). Therefore, if $\omega^{(\text{r})} \neq 0$, then it is desirable to take into account that in the case of propagating disturbances there are two different types of instability. Namely, the instability of a disturbance can be distinguished according to whether it is *convective* (the disturbance grows in time only in the reference frame moving with the disturbance but it decays to zero at any fixed position) or *absolute* (the disturbance grows in time at fixed position). Tagg et al. paid most attention to the determination of the types of instability for disturbances in Couette flows between counter-rotating cylinders. The distinction between convective and absolute instabilities was first noted about the middle of this century, and later it became quite important in plasma physics; however, more recently this distinction has attracted much interest in fluid mechanics also. To distinguish between the two types of instability, it is necessary to consider the initial value problem for a disturbance localized at $t = 0$, i.e., here it is not enough to study only the behavior of separate normal modes. Since the convective/absolute alternative is especially important for fluid flows in tubes, channels, boundary layers, jets, and wakes, we shall consider this topic at greater length in Sect. 2.93, where the main results of the paper by Tagg et al. (1990) will also be described.

In the case of strongly counter-rotating cylinders (large negative μ), the normal mode which first becomes unstable when the Reynolds number is increasing is oscillatory (i.e., here $\Re e\omega \neq 0$). Therefore, in this case transition from stability to instability produces some kind of time- periodic flow and not a steady Taylor-vortex flow as sketched in Fig. 2.12. It can be shown that two different types of flow can appear in this case at $\text{Re} = \text{Re}_{\text{cr}}$—either spiral vortices traveling along the cylinder axis, or so-called "ribbons" that travel only in the azimuthal direction. The choice between these possibilities depends on nonlinear mode interactions and cannot be determined by linear stability theory*;* see. e.g., Tagg (1994) or, for more details, Chossat and Iooss (1994). Of course, if $\mu < 0$, a series of transitions to more and more complicated forms of flow also appears, in an orderly sequence, as Re increases, until finally the flow takes the form of disordered turbulent motion. In the remarkable early paper by Coles (1965), it was first shown that a great number of different flow states can coexist in the parameter space corresponding to flows between rotating cylinders. This result was later confirmed and augmented by other authors; see e.g. the expressive Fig. 2.16, taken from the paper by Andereck et al. (1986). More data for the various forms of supercritical circular Couette flow can be found in the collection of papers edited by Andereck and Hayot (1992). However, results of this type cannot

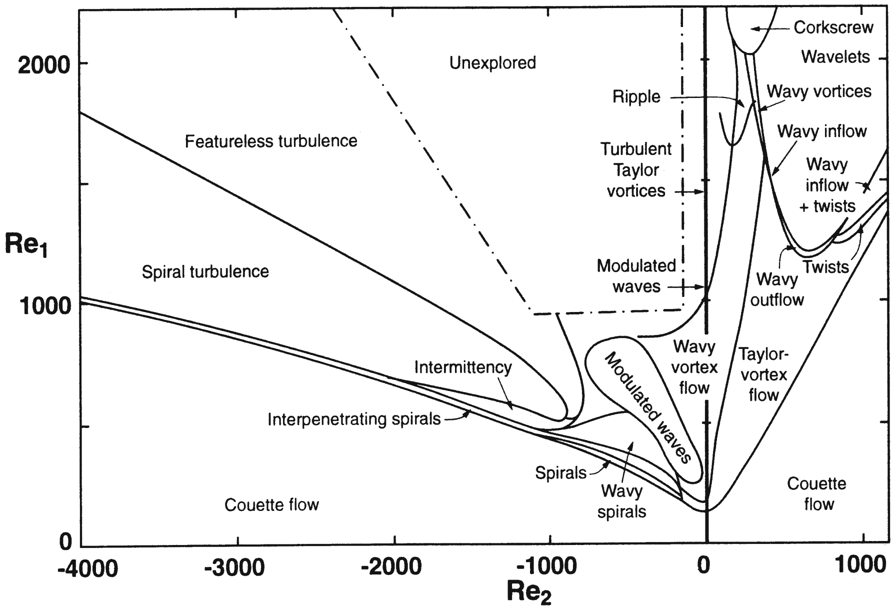


Fig. 2.16 Regions of different flow states in the (Re_1, Re_2) plane for the experiments where the outer cylinder speed Ω_2 (and hence also value of Re_2) is fixed and the inner cylinder speed Ω_1 is slowly increasing. (After Andereck et al. (1986))

be explained by the linear theory of hydrodynamic stability; therefore, they will be not considered in this chapter of the book.

It was also found by Coles (1965) and by some other authors that transitions to and from a given flow state when the Reynolds number at first increases and then decreases is often accompanied by hysteresis effects, and that a non-Couette state can sometimes exist even at $Re < Re_{cr}$. The last result can be explained by the temporal growth of some stable (i.e., asymptotically decaying) modes (see the final part of Sect. 2.5 and also the next chapter of this book) or by nonlinear interactions of stable modes. These possibilities stimulate interest in the study of damped eigenmodes below the stability threshold. It was, in particular, shown by Ko and Cohen (1987); and Gwa and Cohen (1992) that even at $\mu \geq 0$, where neutral and most unstable modes are necessarily nonoscillatory, there can exist damped axisymmetric oscillatory modes (with $n = 0$, $\Re e\omega \neq 0$). More extensive damped-mode data were presented by Gebhart and Grossmann (1993) who computed a number of characteristics (including a great number of subcritical eigenvalues, forms of several individual eigenfunctions, and flow patterns) for both axisymmetric and non-axisymmetric damped modes at positive, zero, and negative values of μ.

In the theoretical investigations mentioned in this section, the ideal circular Couette flow between infinite concentric rotating cylinders was assumed. However, in the experiments the cylinders always had a finite length L, and although the ratio R_2/L was usually rather small, the end effects could distort the agreement between the experimental and calculated characteristics. Many related studies in the theory of hydrodynamic stability have included additional factors affecting the flow between rotating cylinders. There have been studies taking into account possible small eccentricity of the two cylinders, the influence of axial or circumferential pressure (or temperature) gradients, and/or axial, circumferential, or radial magnetic fields, or the replacement of the usual fluid by a non-Newtonian fluid, etc. Some of these additional effects are considered in the books by Chandrasekhar (1961); Joseph (1976); Goldshtik and Shtern (1977); Drazin and Reid (1981); Koschmieder (1993); Andereck and Hayot (1992); and Chossat and Iooss (1994); references to many papers on these subjects can be found in the comprehensive bibliography and subsequent survey by Tagg (1992, 1994). However we will not discuss these subjects here.

2.7 Linear Stability Analysis for a Layer of Fluid Heated from Below

Now we pass to consideration of another classical stability problem where the predictions which follow from the linear stability theory agree quite well with the results of laboratory experiments. This problem concerns the conditions for stability of an immovable horizontal layer of homogeneous fluid in the presence of a vertical temperature gradient $dT/dz \neq 0$.

Assume that a layer of fluid is bounded by walls at $z = 0$ and $z = H$, having fixed temperatures T_0 and T_1. Also assume that the temperature difference $\Delta T = T_0 - T_1$ is not too high in absolute value, so that $|\Delta T|/T_m \ll 1$ where $T_m = (T_0 + T_1)/2$ and T_0 and T_1 are absolute temperatures measured in degrees Kelvin. Then the velocity and temperature fields $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{T}(\boldsymbol{x}, t)$ will satisfy with good accuracy the so-called Boussinesq (or Oberbeck-Boussinesq; see, e.g., MY1, Sect. 1.5, and Joseph (1976), Sect. 54) and heat conduction equations

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + \delta_{i3}[g\beta(T - T_0) - g], \quad i = 1, 2, 3, \tag{2.18}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.19}$$

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \chi \nabla^2 T, \tag{2.20}$$

where p is the pressure, $\rho_0 =$ constant is the density corresponding to temperature T_0 (or to temperature T_m which is supposed to differ little from T_0) and β is the coefficient of thermal expansion of fluid (which is equal to $1/T_m$ for an ideal gas). Equations (2.18–2.20), with temperature boundary conditions $T(x_1, x_2, x_3, t) = T(x, y, z, t) = T_0$ for $z = 0$ and $T = T_1$ for $z = H$, clearly have the steady solution

$$\begin{aligned} &u_i = 0, \quad i = 1, 2.3, \quad T = T(z) = T_0 - \gamma z, \\ &p = p(z) = p_0 - g\rho_0(z + \gamma\beta z^2/2) \end{aligned} \tag{2.21}$$

where $\gamma = \Delta T/H$ (i.e., $-\gamma$ is the vertical temperature gradient), and $p_0 = p(0)$. This solution corresponds to the state of rest. (Of course, if the molecular viscosity and thermal diffusivity are neglected, i.e. it is assumed that $\nu = \chi = 0$, then the vertical temperature profile $T(z)$ can take arbitrary initial values in a state of rest.) The study of the stability for the state of rest is just the problem that will be considered in this section.

It was noted in Sect. 2.4 that in the case of an ideal fluid with $\nu = \chi = 0$, the state of rest is stable when temperature $T(z)$ is increasing with height and is unstable when $T(z)$ is decreasing with height. Hence, if $\nu = \chi = 0$, then the solution (2.21) of the equations of motion is stable if $T_1 > T_0$ and unstable if $T_1 < T_0$. If was long assumed that in the case of a real viscous and heat-conducting fluid, the same condition $T_1 < T_0$ will also be necessary and sufficient for the instability of the state of rest. However, it was proved by Rayleigh (1916a) (who was himself surprised by the unexpected result obtained) that in a viscous heat conducting fluid the solution (2.21) of equations (2.18–2.20) will be unstable only if the temperature difference ΔT is greater than some positive 'critical value' ΔT_{cr} (depending on H, ν and χ).

Rayleigh's theoretical investigation was stimulated by experimental results by Bénard (1900, 1901), who studied carefully convective motions in thin layers (thickness of the order of 1 mm) or various liquids above a heated horizontal metal plate (detailed description of Bénard's experiments can be found in Koschmieder's book (1993)). For this reason, convection in a horizontal layer of fluid heated from below is often called the *Bénard* (or *Bénard-Rayleigh*) *convection.* Unfortunately, Bénard's name proved to be somewhat misleading, since it was shown in the middle of the 20th century that the fluid motions observed by Bénard were in fact mostly driven not by buoyancy forces, which produce ordinary thermal convection, but by variation of the temperature-dependent surface tension at the upper free surface of liquid (see, e.g., Block (1956); Pearson (1958); Zierep and Oertel (1982); Tritton (1988) Sect. 4.5; Koschmieder (1993), Chap. 3). Surface-tension-driven vertical motions in fluid layers are now often called *Marangoni convection;* see, e.g., Zierep and Oertel (1982). However, not all authors have found this name appropriate. Therefore Nield (1968); and Koschmieder (1993) proposed to call the surface-tension-driven convection in a fluid layer (first really observed by Bénard) the *Bénard convection,* while for the buoyancy-driven convection in such a layer, Nield and also Thompson and Sogin (1966)) used the name *Rayleigh-Jeffreys convection* (Jeffreys' work will be considered below) and Koschmieder used the name *Bénard-Rayleigh convection*. However, these the surface-tension-driven convection will not be considered.

Rayleigh applied the normal-mode approach of the linear theory of hydrodynamic stability to the study of convection in a layer heated from below. Putting $T = T(z) + T'$, $p = p(z) + p'$ and then linearizing Eqs. (2.18–2.20) with respect to the disturbances u_i, T' and p' we obtain the following system of five equestion with five unknowns:

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 u_i + \delta_{i3} g\beta T', \quad i = 1, 2, 3, \tag{2.22}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.23}$$

$$\frac{\partial T'}{\partial t} - \delta_{i3}\gamma u_i = \chi \nabla^2 T'. \tag{2.24}$$

Taking the divergence ($\partial/\partial x_i$) of vector Eq. (2.22), we find that $\nabla^2 p' = \rho_0 g\beta \partial T/\partial z, z = x_3$. Applying the Laplace operator $\Delta = \nabla^2$ to the third Eq. (2.22) we can now eliminate p' from this equation; the resulting equation and Eq. (2.24) form a system of two equations with two unknowns, $w = u_3$ and T'. It is easy to eliminate either w or T' from this system; this leads to the following equation for $T'(\boldsymbol{x}, \mathrm{t})$

$$\nabla^2\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\frac{\partial}{\partial t} - \chi\nabla^2\right)T' - g\beta\gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)T' = 0, \tag{2.25}$$

(where $x = x_1$, $y = x_2$) and to exactly the same equation for the unknown $w(\boldsymbol{x}, \mathrm{t})$. The first two of Eqs. (2.22) together with Eq. (2.23) also allow $u_1 = u$ and $u_2 = v$ to be determined; see, e.g., Drazin and Reid (1981) or Koschmieder (1993).

If we transform the derived equations to dimensionless variables $\xi = x/H, \eta = y/H, \zeta = z/H, \tau = t\chi/H^2$ and after this, following eqs. (2.8) and (2.9), seek $T'(\boldsymbol{x}, t)$ in the form of a product

$$T'(x,t) = \Delta T \exp\left[i(k_1\xi + k_2\eta - \omega\tau)\right]\theta(\zeta), \tag{2.26}$$

we obtain from Eq. (2.25) the following one-dimensional problem

$$\left(\frac{d^2}{d\varsigma^2} - k^2\right)\left(\frac{d^2}{d\varsigma^2} - k^2 + i\omega\right)\left(\frac{d^2}{d\varsigma^2} - k^2 + \frac{i\omega}{\mathrm{Pr}}\right)\theta + k^2\mathrm{Ra}\theta = 0, \tag{2.27}$$

where $k^2 = k_1^2 + k_2^2, \omega$ is the unknown eigenvalue, $\mathrm{Pr} = \nu/\chi$ is the Prandtl number, and the new dimensionless combination

$$\mathrm{Ra} = \frac{g\beta\gamma H^4}{\nu\chi} = \frac{g\beta\Delta T H^3}{\nu\chi}, \tag{2.27$'$}$$

is the so-called *Rayleigh number.* (This combination became well-known after its appearance in Rayleigh's paper (1916a) but it was in fact first introduced by L. Lorenz in 1881; see Joseph (1976), Sect. 54.) Assuming that

$$w(x,t) = (\chi/H)\exp\left[i(k_1\xi + k_2\eta - \omega t)\right]W(\zeta), \tag{2.26$'$}$$

We obtain for $W(\zeta)$ the same Eq. (2.27) with the function $\theta(\zeta)$ replaced by $W(\zeta)$.

The boundary conditions on rigid surfaces of constant temperature are

$$T' = w = \partial w/\partial z = 0, \tag{2.28}$$

(the last two of these clearly follow from the usual no-slip condition). It was also assumed by Rayleigh that at free surfaces of constant temperature the temperature disturbance T', the vertical velocity w, and the tangential stress all vanish. From this it follows that at free surfaces

$$T' = w = \partial^2 w/\partial z^2 = 0. \tag{2.28$'$}$$

Using Eqs. (2.22–2.24) it is now easy to show that boundary conditions (2.28) and (2.28′) for functions $T'(\boldsymbol{x}, t)$ and $w(\boldsymbol{x}, \mathrm{t})$ of the exponential forms (2.26) and (2.26′) can be also rewritten as follows

$$W = W' = W^{(iv)} - (2k^2 - i\omega/\,\mathrm{Pr})W'' = 0 \text{ on rigid surfaces,} \tag{2.29}$$

$$W = W'' = W^{(iv)} = 0 \text{ on free surfaces,} \tag{2.29$'$}$$

Or, if we are studying the eigenvalue problem for $\theta(\zeta)$,

$$\theta = \theta'' = \theta''' - (k^2 - i\omega)\theta' = 0 \text{ on rigid surfaces,} \tag{2.30}$$

$$\theta = \theta'' = \theta^{(iv)} = 0 \text{ on free surfaces,} \tag{2.30$'$}$$

where primes and superscript$^{(iv)}$ symbolize differentiation of the corresponding order with respect to ζ. The eigenvalue problem (2.27) with boundary conditions (2.30–2.30′) and the same problem for $W(\zeta)$ with boundary conditions (2.29–2.29′) include three dimensionless parameters: Pr, Ra, and the dimensionless wave number k. The Prandtl number Pr characterizes the physical properties of the fluid, and for any given fluid its value is fixed. (Possible dependence of v and χ on temperature is neglected here; note that in gases the Prandtl number is really nearly constant.) Consequently, for a given fluid and fixed types of walls (rigid or free) there will correspond to any given values of k and Ra an associated set of eigenvalues ω_j (k, Ra). Note that in the idealized case of two free surfaces the eigenvalue problems for the functions $W(\zeta)$ and $\theta(\zeta)$ coincide; here, therefore, it is evident that the sets of eigenvalues $\omega_j(k, \mathrm{Ra})$ for both problems are the same. In the more realistic cases of two rigid surfaces or one rigid and one free surface, the boundary conditions for the functions $W(\zeta)$ and $\theta(\zeta)$ are not the same. However it is clear from physical reasons that here also the eigenvalues $\omega_j(k, \mathrm{Ra})$, which determine the complex ‘frequencies’ of the corresponding normal modes of the convective flow, must be the same for the two eigenvalue problems considered.

Rayleigh was interested first of all in elucidating the qualitative features of the convection process. Therefore, in his paper (1916a) he analyzed only the mathematically much simpler (but physically unreal) problem of convection in a layer of fluid between two free boundaries at constant temperatures. We know that this problem is reducible to the eigenvalue problem (2.27) with boundary conditions (2.30′) for $\zeta = 0$ and $\zeta = 1$. This type of eigenvalue problem can be solved and analyzed rather easily. It is not difficult to show that all solutions of Eq. (2.27) satisfying the specified conditions are of the form $\theta(\zeta) = \sin \pi j \zeta$, $j = 1, 2, \ldots$ Substituting these solutions into Eq. (2.27) we obtain, for any integer value of j, a quadratic equation which determines two eigenvalues ω_{j1} and ω_{j2}. It easy to verify that if $\mathrm{Ra} \le 0$ (i.e., if $\Delta \mathrm{T} \le 0$

so that the temperature of the lower boundary is not higher than that of the upper boundary), then $\Im m\omega_{j1} < 0$ and $\Im m\omega_{j2} < 0$ for any j and any wave number k. Hence for $\Delta T \leq 0$ all normal modes of disturbance are stable, and hence the state of rest is stable too. In this case, all eigenvalues ω_{j1} and ω_{j2} are purely imaginary for sufficiently low values of $|\Delta T|$, but at higher values of $|\Delta T|$ some eigenvalues turn out to be complex (with negative imaginary parts). However, when $\mathrm{Ra} > 0$ (i.e, the lower boundary has higher temperature than the upper one), all the eigenvalues ω_{j1} and ω_{j2} are purely imaginary (i.e., the principle of exchange of stabilities is valid), but now the imaginary parts of all the eigenvalues are non-positive only if Ra is small enough. However, if Ra exceeds some critical value $\mathrm{Ra}_{\mathrm{cr}}$, the imaginary parts of some eigenvalues become positive, showing that the state of rest is unstable.

It is also easy to determine the exact value of $\mathrm{Ra}_{\mathrm{cr}}$ and the value, k_{cr}, of the wave number corresponding to the normal mode which first becomes unstable. In fact, to find the critical values of Ra and k, it is sufficient to consider Eq. (2.27) with $\omega = 0$, i.e.,

$$\left(\frac{d^2}{d\varsigma^2} - k^2\right)^3 \theta + k^2 \mathrm{Ra}\theta = 0, \tag{2.31}$$

If we substitute the solution $\theta = \sin \pi j \zeta$ in Eq. (2.31), we obtain the critical Rayleigh number $\mathrm{Ra}(k, j)_{\mathrm{cr}}$ which corresponds to the jth eigenmode of neutral disturbances (with $\omega = 0$) having given dimensionless wave number k:

$$\mathrm{Ra}(k, j)_{\mathrm{cr}} = \left(j^2\pi^2 + k^2\right)^3 / k^2. \tag{2.32}$$

It is clear from this equation that for any given k the minimum critical Rayleigh number corresponds to the first eigenmode, i.e., to a disturbance with $j = 1$. Hence the value of $\mathrm{Ra}_{\mathrm{cr}}$ can be determined as the minimal value of $\mathrm{Ra}(k, 1)_{\mathrm{cr}} = \mathrm{Ra}(k)_{\mathrm{cr}}$ over all values of k: $\mathrm{Ra}_{\mathrm{cr}} = \min_k[(\pi^2 + k^2)^3/k^2]$. Now elementary computation leads to the results

$$\mathrm{Ra}_{\mathrm{cr}} = 27\pi^4/4 \approx 657.5, \quad k_{\mathrm{cr}} = \pi\sqrt{2}/2 \approx 2.2. \tag{2.33}$$

Results (2.33) for the idealized model of a fluid layer bounded by two free surfaces were found by Rayleigh (1916a), who tried to use them to explain Bénard's experimental findings, if only qualitatively. First of all he was interested in Bénard's observation that when the state of rest loses its stability a specific "cellular regime" is formed in the layer of fluid: the flow decomposes into a number of regular vertical cells (now called *Bénard's cells*), often having the form of identical regular hexagonal prisms where fluid is ascending in the middle of cells and descending near their boundaries.[4] This observation makes one think that a cellular solution of Boussinesq's equations replaces the state of rest at the onset of instability. At

[4] Later it was discovered that such order of updrafts and downdrafts is typical only for liquids, while in the case of cellular convection in gases, gas is usually descending in the middle and ascending near the boundary of a cell [see, e.g., Koschmieder (1993)].

slightly supercritical Rayleigh numbers $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{cr}}$, the amplitude of the most unstable cellular solution will at first increase exponentially with time, but then its growth rate will gradually decrease and the amplitude will tend to a finite limiting value corresponding to a steady finite-amplitude cellular convection.

It was stressed by Rayleigh that in his theory the value of $k^2 = k_1^2 + k_2^2$ determines only the typical horizontal size of possible cells but not their form. In fact, the exponential function $\exp \Im[i(k_1\xi + k_2\eta)]$ can be replaced in Eqs. (2.26) and (2.26′) by an arbitrary linear combination of such functions with different values of k_1 and k_2 but a fixed value of $k_1^2 + k_2^2$. And what is more, instead of such combinations it is also possible to use, in Eqs. (2.26) and (2.26′), an even more general arbitrary function $\phi(\xi, \eta)$ satisfying the so-called *membrane* (or *Helmholtz's*) *equation*

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} + k^2\phi = 0. \tag{2.34}$$

It is clear that solutions of this equation can describe quite different forms of cells. Rayleigh did not find the solution corresponding to hexagonal cells and considered only the simplest "cellular solution" with square cells, but he also estimated the value of k for the case of hexagonal cells of given size and showed that the value in Eq. (2.33) of k_{cr} does not differ too much from the value of k corresponding to the hexagonal cells observed by Bénard. Later, Christopherson (1940) determined the exact solution of Eq. (2.34) describing regular hexagonal cells (this solution can now be found in the books by Gershuni and Zhukhovitskii (1972); Drazin and Reid (1981); and Koschmieder (1993)) while Bisshopp (1960) discussed some more general cellular solutions.

Bénard's experimental results stimulated attempts by a number of scientists to explain the appearance of hexagonal cells after the onset of convection with the aid of the non-linear Boussinesq equations. However, these works lost much of their interest when it became known that Bénard's experiments have little relation to natural buoyancy-driven convection. For the same reason the closeness of Rayleigh's theoretical value for k_{cr} to the value of k corresponding to Bénard's observations seems now to be accidental. However, Rayleigh's paper (1916a) has retained all its importance as a classical contribution to to linear theory of hydrodynamic stability, containing the first derivation of a non-trivial condition for convective instability.

In the case of the more realistic problems of convection in a layer of fluid between two rigid boundaries of fixed temperatures, or between a rigid boundary below and a free surface above, some complicated analytic expressions can be also found for the functions $\theta(\zeta)$ and $W(\zeta)$. However, here these expressions are of little help since the eigenvalues ω_j must now be determined from complicated transcendental equations which can be solved only numerically. The approximate calculation of values for $\mathrm{Ra}_{\mathrm{cr}}$ and k_{cr} in the case of two rigid or one rigid and one free boundaries was first made by Jeffreys (1926, 1928). Jeffreys' results for the case of two rigid boundaries proved to be sufficiently precise, but in the case of one rigid and one free surface, the boundary conditions used on the upper free surface were later found to be unsatisfactory. More accurate results for this case were obtained by Low (1929); and Pellew and Southwell

(1940). Still later, modern computers allowed all critical values to be determined with very high accuracy. The values accepted today are: $\mathrm{Ra}_{\mathrm{cr}} = 1707.8, k_{cr} = 3.117$ for the case of two rigid boundaries, $\mathrm{Ra}_{\mathrm{cr}} = 1100.7$, $k_{\mathrm{cr}} = 2.682$ for one rigid and one free boundary (see, e.g., Drazin and Reid (1981); and Koschmieder (1993); for the rigid-rigid case more precise values having many supplementary significant digits can be found in the paper by Dominiguez-Lerma et al. (1984)). We see that the presence of rigid boundaries increases the value of $\mathrm{Ra}_{\mathrm{cr}}$ considerably and Increases also the values of k_{cr} but to a lesser degree. Pellew and Southwell (1940) gave a simple proof of the general "principle of exchange of stabilities", introduced by Jeffreys (1926) as an assumption. According to this principle all the eigenvalues ω_j are real in the case of a horizontal layer of fluid heated from below. The proof of this statement can now be found, e.g., in the books by Gershuni and Zhukhovitskii (1972); Joseph (1976); Drazin and Reid (1981); and Koschmieder (1993) (the first two of these also contain some generalizations of the stated result). The values of $\mathrm{Ra}_{\mathrm{cr}}$ and k_{cr} (and also the values of the critical Rayleigh numbers $\mathrm{Ra}(k)_{\mathrm{cr}}$ for disturbances with fixed wave number k, which give the 'neutral curve' in the (Ra, k)-plane) are determined by Eq. (2.31), and it is clear that they do not depend on the Prandtl number Pr. However, the growth rates $\sigma = \Im m\omega$ of supercritical unstable disturbances at $\mathrm{Ra} > \mathrm{Ra}_{\mathrm{cr}}$, (and also the decay rates $-\sigma = -\Im m\omega$ of subcritical stable disturbances at $\mathrm{Ra} < \mathrm{Ra}_{\mathrm{cr}}$) must be determined from Eq. (2.27); therefore, they depend on Ra, k, and Pr. It was shown above that in Rayleigh's case of two free boundaries these rates are the roots of a quadratic equation with coefficients depending on the above-mentioned parameters; in cases of rigid-rigid or rigid-free boundaries their evaluation is more complicated and includes some numerical computations. (Note also that the case of one rigid and one free boundary is computationally more complex than the cases of two boundaries of the same type, where the problem exhibits an additional mirror symmetry with respect to the plane $z = H/2$. Therefore, in the "rigid-free' case, in contrast to 'free-free' and 'rigid-rigid' cases, the set of all eigenfunctions $W(\zeta)$ and $\theta(\zeta)$ does not decompose into two subsets of functions, even and odd with respect to the mid-point $\zeta = 1/2$. For more details of problems with asymmetric boundary conditions see, e.g., Clever and Busse (1993, 1995)). The function $\mathrm{Ra}(k)_{\mathrm{cr}}$ for Rayleigh's free-free case was indicated above; some results for the rigid-rigid and rigid-free cases were given by Pellew and Southwell (1940) [see also Fig. 2.17, taken from the review paper by Busse (1989)). Values of growth rates σ at small positive values of $\mathrm{Ra} - \mathrm{Ra}_{\mathrm{cr}}$ and k close to k_{cr} are given for the free-free case by analytical equations, but for the rigid-rigid case some numerical computations are also needed; see, e.g., the papers by Newell and Whitehead (1969); Cross (1980); and Dominigues-Lerma et al. (1995). Some results for asymmetric boundary conditions were computed by Kvernvold (1979); and Clever and Busse (1993, 1995) who, however, paid most attention to nonlinear stability analysis.

Most experimental verifications of the results from the linear stability theory of fluid layers heated from below concern comparisons of the theoretical values for $\mathrm{Ra}_{\mathrm{cr}}$ with the laboratory data. The loss of stability for the state of rest of a fluid layer (i.e., the onset of convection) can be detected visually as the appearance of a cellular structure in the fluid if some visualization technique is used and the value of

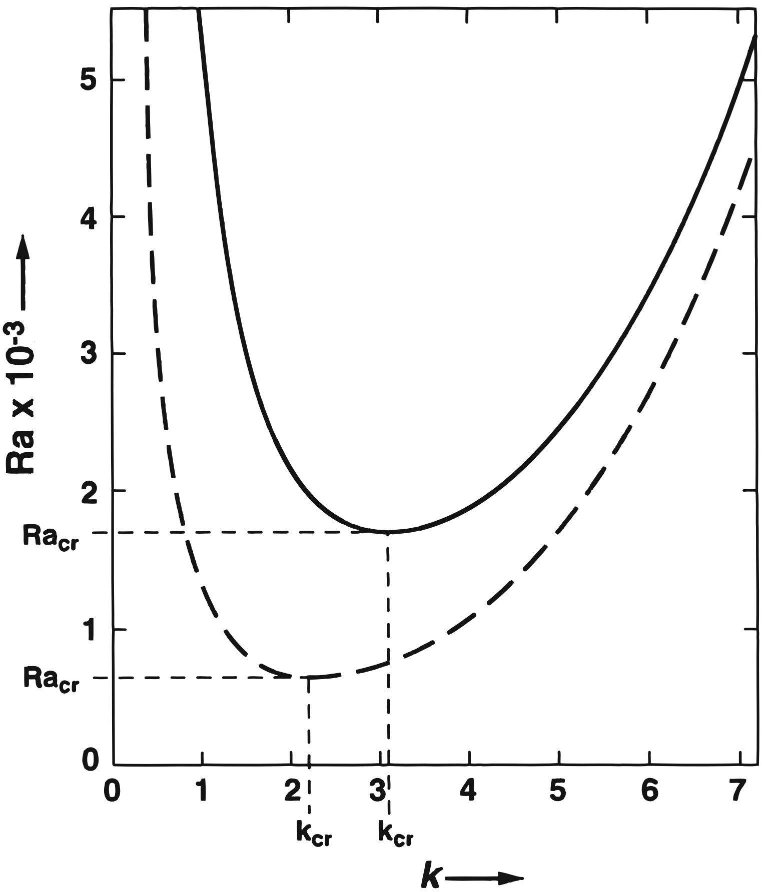


Fig. 2.17 The neutral-stability curves of the onset of convection in the (k, Ra) plane for a fluid layer bounded by two rigid walls (*solid line*) and by two free surfaces (*dashed line*). (After Busse (1989)).

$\Delta T = T_0 - T_1$ is increased gradually (or ΔT is held constant but the gas pressure is increased and hence ν and χ are decreased; see Thompson and Sogin (1966)). Instead of visual observation, which are always subjective to a certain degree, it is posible to use measurements of the heat transfer through the fluid layer at different values of Ra; such measurements are easily made and they provide the most accurate results. In the state of rest the rate of heat transfer across unit area of the layer, Q, is clearly given by the equation $Q = \kappa \nabla T = \kappa \Delta T / H$ where $\kappa = c_p \rho \chi$ is the molecular coefficient of thermal conductivity; hence Q is proportional here to ΔT and Ra. After the onset of convection the molecular conduction of heat through the fluid begins to be supplemented by heat transfer by vertical fluid motions. Therefore, Q begins to exceed $\kappa\ \Delta T/H$, and this produces the "break" in the heat transfer curve at $\text{Ra} = \text{Ra}_{\text{cr}}$, showing the dependence of Q (or the Nusselt number $Nu = QH/\kappa \Delta T$)

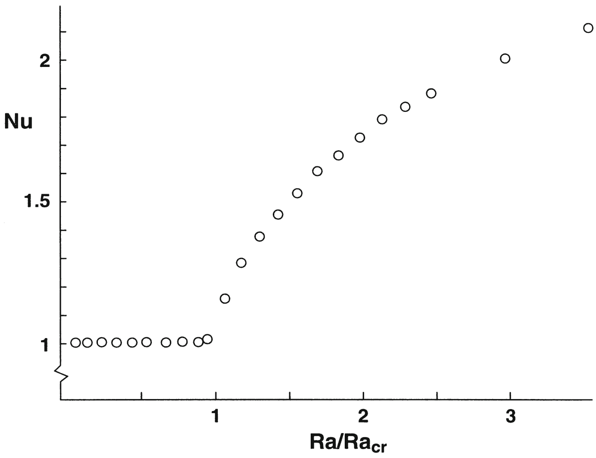


Fig. 2.18 Dependence of the Nusselt number $Nu = QH/\kappa\Delta T$ on Ra/Ra$_{cr}$, where Ra$_{cr}$ is the value of Ra at which Nu begins to grow above the value $Nu = 1$, for silicone oil (according to Koschmieder and Pallas (1974)). The value of Ra$_{cr}$ was found to be 1675, i.e., only 2 % smaller than the theoretical value 1708

on ΔT or Ra; Fig. 2.18, taken from the paper by Koschmieder and Pallas (1974), is typical. (Similar graphs were constructed by these authors for two other types of oil having different viscosities and Prandtl numbers; the corresponding values of Ra$_{cr}$ in both cases differ from the theoretical value 1708 by no more than 4 %). Surveys of the results obtained by this method for fluid layers filling large aspect ratio containers with $L/H \gg 1$ (where L is the width, or diameter, of the container), bounded at the bottom and top by rigid boundaries, are presented in the books by Drazin and Reid (1981, Sect. 12); and Koschmieder (1993, Sect. 5.4); one more impressive example is given by Tritton (1988, Fig. 22.1). According to all these sources, results of the numerous available measurements of Ra$_{cr}$ agree with the theoretical value 1708 to within about 5 %.

Another method for verification of the linear theory of fluid-layer instability is based on comparison of the theoretical value for k_{cr} with measured sizes of the cells that appear at the onset of convection. It is natural to assume that if (Ra −Ra$_{cr}$)/Ra$_{cr} \ll 1$, then the resulting "cellular pattern" (often called also the "planform of convection") must be described by the solution of Eq. (2.34) with the dimensionless wave number k very close to k_{cr}. (The shape of the cells is also a very important characteristic of the cellular pattern, but it cannot be used for verification of the theory since the linear instability theory does not determine the shape. Moreover, the experimental data collected by Koschmieder (1993) in Sect. 4.2 of his book show that the cell shape is strongly affected by the size and form of the horizontal cross-section of the container and by its lateral boundaries, whereas the theory assumes that the cross-section is infinite and has no lateral boundaries at all.) In practice the exact solution of Eq. (2.34) describing the cellular pattern is usually not employed to determine k, but it is assumed that the typical horizontal size ("width") of cells arising at the onset of convection is equal to half the wavelength $\lambda = 2\pi H/k$ corresponding to the dimensionless wave number k. The "measured value" of k deduced from this can then be compared with the theoretical value of k_{cr}. Note that the ever-present lateral boundaries restrict the set of possible "measured values" of k since the cross-section of the container must be filled by an integer number of nominally identical cells of width $\lambda/2 = \pi H/k$. Nevertheless, it seems natural to think that in the case of containers of large aspect ratio, the lateral boundaries and the finite cross-section will be of little significance for the observed process of convection. If so, then the width of the cells appearing at the onset of convection must be close to $\lambda_{cr}/2 = \pi H/k_{cr}$.

Two examples of such experimental determinations of k_{cr} for the fluid layer between two rigid plates are presented in the book by Drazin and Reid (1981), p. 60; one more such example is analyzed at greater length by Koschmieder (1993), p. 83. The general conclusion from these examples is that this method for experimental determination of k_{cr} leads to values which agree within a couple of percent with the theoretical value of k_{cr} for a fluid layer between two rigid surfaces. Thus, the comparisons of the measured values for Ra_{cr} and k_{cr} with results of theoretical computations show that there is no reason to doubt that the normal-mode approach to the linear theory of hydrodynamic stability quite accurately describes the onset of convection in a layer of fluid between two rigid walls.

In a number of relatively old publications (including also MY1, p. 111), it was stated that for layers of fluid between one rigid and one free boundary at constant temperatures the theoretical value $\mathrm{Ra}_{cr} \approx 1100$ is confirmed by experimental data with the same accuracy as for layers between two rigid boundaries. However, these statements were based on rather old data whose accuracy is unsatisfactory according to modern standards. In fact it is very difficult to provide an accurate fulfillment of boundary conditions (2.28′) (and hence also of (2.29′) and (2.30′)) on real free surfaces of liquid layers, and to guarantee the absence there of supplementary effects affecting the motions of liquid elements. In particular, it is very difficult to preserve strictly constant temperature at the upper free boundary of a layer of liquid, and the existence of temperature variations implies variations of surface tension, which 0 produce supplementary motions. It has been already noted above that surface-tension variations apparently played the main role in Bénard's experiments; it is also possible to show that in these expriments the boundary conditions (2.28′) used by Rayleigh in his model computations are not all valid at the upper boundaries of the liquid layers. An attempt to construct a convection theory based on real conditions at the free boundary of a liquid layer under air was made by Nield (1964), who took into account both the buoyancy and the surface tension effects and obtained estimates for Ra_{cr} and k_{cr} which differ considerably from the values that follow from Rayleigh's conditions (2.28′) applied to the upper boundary. Nield's estimates do not contradict Koschmieder's (1967) measurements of convection in a liquid layer under air. Note also that sufficiently strong convective motions of a liquid can deform its free surface, and this deformation can substantially affect the instability characteristics and sometimes lead to violation of the principle of exchange of stabilities, making the convection pattern not stationary but oscillatory (see, e.g., Gershuni and Zhukhovitskii (1972); and Benguria and Depassier (1987)).

Let us also note that not all Rayleigh's boundary conditions (2.28) are strictly valid on all rigid surfaces bounding fluid layers. Of course, conditions related to the vertical velocity w are evident consequences of the no-slip and no-permeability conditions, which are always valid in a viscous fluid. However the assumption of complete absence of temperature fluctuations at the rigid boundary (i.e., the exact constancy of boundary temperature) is often only a poor approximation. This approximation is adequate for a massive boundary with infinite thermal conductivity, i.e., it is appropriate in the case where a layer of liquid is bounded by a thick plate of copper or other metal; however, in other cases special measures must be taken to guarantee its validity. On the other hand, in the case of an insulating solid boundary the boundary condition $T' = 0$ must be replaced by the condition $\partial T'/\partial z = 0$, introduced by Jeffreys (1928); in some other cases, constant values of the heat flux, or some linear combination of temperature and heat flux, must be assumed instead of constant values of temperature (see Sparrow et al. (1964)). Values of $\mathrm{Ra}_{\mathrm{cr}}$ and k_{cr} corresponding to some other boundary conditions which are appropriate, e.g., in the cases of a fluid layer bounded by rigid plates of finite thermal conductivities having infinite or given finite thicknesses, or by a deformable free upper boundary, or by a free boundary at non-constant temperature, can be found in the papers by Hurle et al. (1967); Nield (1968); Jakeman (1968); Benguria and Depassier (1987); Clever and Busse (1995); and the book by Gershuni and Zhukhovitskii (1972).

Rayleigh's problem on the stability conditions for an infinite horizontal fluid layer is a simple example from an extensive class of problems related to convective instabilities of fluids. This class includes the generalizations of Rayleigh's problem to horizontal layers of non-Boussinesq or non-Newtonian fluids, to layers with internal sources of heat, layers fixed in a rotating coordinate frame, layers of conducting fluids in the presence of electric and/or magnetic fields, layers of density-stratified fluids; it also includes stability studies for fluids at rest in bounded containers of various forms (in particular, studies of the influence of lateral walls on convection in bounded fluid layers; see, e.g., Sect. 5.2 in Koschmieder (1993)) and for non-horizontal fluid layers; and a great number of problems about stability conditions for various flows of thermally inhomogeneous fluids. Some of these problems are considered in the books by Chandrasekhar (1961); Gershuni and Zhukhovitskii (1972); and Gershuni et al. (1989); there is also a great deal of literature in journals and collections of papers. However it is not possible to consider even a small part of this material here.

2.8 Introduction to the Linear Stability Theory of Parallel Fluid Flows

Above, we considered two examples of the application of the linear theory of hydrodynamic stability to particular fluid flows. These examples were selected because the presence of a special supplementary force (a centrifugal force in the example studied in Sect. 2.6 and an Archimedean force of buoyancy in Sect. 2.7) leads to a comparatively simple form of the resulting eigenvalue problem, which permits one to obtain quite definite results allowing experimental verification. However, from both the purely theoretical and the applied points of view, the study of stability for flows in tubes and boundary layers on solid bodies seems to be much more important and challenging. Unfortunately, the application of the linear stability analysis to these flows encounters serious mathematical difficulties, which even today cannot be considered to have been completely solved. Therefore we postponed the discussion of these stability problems until now, and below we will confine ourselves to consideration of the simplest cases of tube and boundary-layer flows, and of some other sufficiently simple flows where the application of mathematical methods is not too complicated.

The laminar flow in a circular tube is parallel and axially symmetric, and the boundary layer along a flat plate is an example of approximately (but not exactly) plane-parallel flow in the half-space $z \geq$ o. However, we shall begin with the analysis of a simpler case, the strictly plane-parallel laminar two-dimensional flow in a plane channel bounded by planes $z=0$ and $z=H$. Let us assume that the velocity of this flow is everywhere parallel to the axis $Ox_1 = Ox$ and is given by a function depending only on the coordinate z. Note that according to the Navier-Stokes equations, in the case of a steady plane-parallel flow of viscous fluid with the velocity $U(x, y, z) = U_1(z)e_1$ (where e_1 is the unit vector parallel to Ox), the function $U(z)$ must be polynomial and at most quadratic in z. Hence strictly plane-parallel flow between walls at $z=0$ and $z=H$ can always be represented as a linear combination of the plane Couette flow with linear velocity profile and the plane Poiseuille flow with parabolic velocity profile. However, keeping in mind that we are interested in strictly plane-parallel flows as possible models of more complicated real flows, we shall also consider arbitrary profiles of $U_1(z)$ in the hope that the results obtained may be applicable to flows that are only approximately plane-parallel.

2.8.1 The Orr-Sommerfeld Equation

Let us apply the method of small disturbances to a plane-parallel flow in a channel between walls at $z=0$ and $z=H$ having a given velocity profile $U_1(z)$. Here Eqs. (2.7) clearly have the form

$$\frac{\partial u_i'}{\partial t} + U_1 \frac{\partial u_i'}{\partial x} + w \frac{dU_1}{dz} \delta_{i1} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \nabla^2 u_i', \quad i = 1, 2, 3, \tag{2.35}$$

$$\frac{\partial u_i'}{\partial x_i} = 0, \tag{2.36}$$

where $x = x_1$, $z = x_3$, $w = u_3$ and primes denote disturbances of the primary fields. Taking the divergence $\partial/\partial x_i$ of (2.35) and using (2.36) we obtain

$$\nabla^2 p' = -2\rho \frac{dU_1}{dz} \frac{\partial w}{\partial x}. \tag{2.37}$$

Then, taking $\partial/\partial x_3 = \partial/\partial z$ of this equation and $\nabla^2 = \partial^2/\partial x_i \partial x_i$ of the third Eq. (2.35), we can easily eliminate p' from the latter equation and transform it to the form

$$\left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x} \right) \nabla^2 w - \frac{d^2 U_1}{dz^2} \frac{\partial w}{\partial x} - \nu \nabla^4 w = 0. \tag{2.38}$$

In the case of solid walls at $z=0$ and $z=H$ the boundary conditions for w will be: $w=\partial w/\partial z=0$ at $z=0$ and H.

Applying the normal-mode method of linear stability theory, we can replace the general Eqs. (2.8) and (2.9) by a single equation

$$w(\boldsymbol{x},t)=U_0\exp\left[i(k_1\xi+k_2\eta-\omega\tau)\right]W(\zeta), \tag{2.39}$$

where dimensionless variables $\xi=x/H, \eta=y/H, \zeta=z/H,$ and $\tau=tU_0/H$ are used instead of dimensional variables x, y, z, t, and w is normalized by a velocity scale U_0. (As a suitable velocity scale U_0, a typical value of the non-perturbed velocity U(z), e.g., the maximum of $U(z)$, can be taken.) Of course, equations of the form (2.39) must be assumed to be valid for $u'=u_1'$, $v=u_2'$, and p' too, but they can be omitted here. It is convenient and usual to represent the dimensionless 'frequency' ω as

$$\omega=k_1c, \tag{2.40}$$

where c is the dimensionless 'streamwise phase velocity'. Substitution of (2.39) and (2.40) into (2.38) gives the equation

$$(U-c)\left(\frac{d^2W}{d\zeta^2}-k^2W\right)-\frac{d^2U}{d\zeta^2}W+\frac{i}{k_1\mathrm{Re}}\left(\frac{d^4W}{d\zeta^4}-2k^2\frac{d^2W}{d\zeta^2}+k^4W\right)=0, \tag{2.41}$$

(where $U(z)=U_1(z)/U_0$ is the dimensionless velocity profile, $k^2={k_1}^2+{k_2}^2$ and $\mathrm{Re}=U_0H/\nu$) with boundary conditions

$$W(0)=W'(0)=W(1)=W'(1)=0, \tag{2.42}$$

(here primes symbolize differentiation on ζ). This is the famous *Orr-Sommerfeld* (or (*O-S*) *equation* which for many years was the main equation of hydrodynamic stability theory studied in many hundreds of papers. For special cases of plane Poiseuille and Couette flows, Eq. (2.41) was first derived by Kelvin (1887); however, at present it is usually called by the names of Orr (1907); and Sommerfeld (1908) who both considered only the simplest case of plane Couette flow but independently reduced the investigation of stability to solution of the eigenvalue problem relating to the two-dimensional form of this equation.

Equation (2.41) is in fact a generalization of the equation most often called the O-S equation. Equation (2.41) includes three given parameters: k_1, $k=(k^2)^{1/2}$, and Re, and the unknown, and in general complex, eigenvalue $c=c^{(\mathrm{r})}+ic^{(\mathrm{i})}$. The eigenvalue problem (2.41–2.42) will usually have nontrivial (i.e., non-zero) solutions only for some discrete (finite or infinite) set of eigenvalues $c_j=c_j^{(r)}+ic_j^{(i)}$ depending on the parameters k_1, k (or k_2) and Re. It is possible to show that $c_j^{(i)}<0$ for any j, k_1 and k_2 at small enough values of Re; thus, here all the normal modes of disturbance with any wave numbers k_1, k_2 are decreasing exponentially with time, i.e., they are stable. Now let Re increase; then at some critical value $\mathrm{Re}(k_1, k_2)_{\mathrm{cr}}$ (depending on k_1 and k_2)

of Re, an eigenvalue c_j having vanishing imaginary part $\Im m c_j = c_j^{(i)} = 0$ appears for the first time, while at a slightly greater Reynolds number $\mathrm{Re} > \mathrm{Re}(k_1,\ k_2)_{\mathrm{cr}}$ the imaginary part $c_j^{(i)}$ becomes positive. The value $\mathrm{Re}(k_1,\ k_2)_{\mathrm{cr}}$ is then the critical Reynolds number for normal modes of disturbance with dimensionless wave numbers k_1 and k_2.

The critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ of the plane-parallel flow with velocity profile $U_1(z)$, which determines the threshold for the loss of stability with respect to infinitesimal wave-like disturbances, is equal to $\min(\mathrm{Re}(k_1, k_2)_{\mathrm{cr}})$ where the minimum is taken over all values of k_1 and k_2. The determination of $\mathrm{Re}_{\mathrm{cr}}$ was considered for many years to be the main problem of the general (or, at least, of the linear) theory of hydrodynamic stability.

Let us now show that when seeking the value of $\mathrm{Re}_{\mathrm{cr}}$ it is sufficient to only consider particular two-dimensional velocity disturbances of the form

$$w(x,t) = w(x,y,t) = U_0 \exp\left[i(k\xi - \omega\tau)\right] W(\zeta). \tag{2.43}$$

In this case k_1 must clearly be replaced by k in Eqs. (2.41) and (2.41) hence we obtain the equation

$$(U - c)(W'' - k^2 W) - U''W + \frac{i}{k\,\mathrm{Re}}(W^{(iv)} - 2k^2 W'' + k^4 W) = 0 \tag{2.44}$$

(with the same boundary conditions (2.42)), where the superscripts again denote differentiation with respect to ζ. Equation (2.44) contains only two parameters, k and Re. Now, denoting $\mathrm{Re}k_1/k = \mathrm{Re}k_1/k(k_1^2 + k_2^2)^{1/2}$ by Re^*, we can rewrite Eq. (2.41) in a form identicle to Eq. (2.44) but with

Re replaced by $\mathrm{Re}^* < \mathrm{Re}$. Therefore, if at some value of Re Eq. (2.41) has a non-trivial solution for some values k_1 and $k_2 \neq 0$ corresponding to a (complex) eigenvalue c_j, then Eq. (2.44), in which $k = \left(k_1^2 + k_2^2\right)^{1/2}$, necessarily has a non-trivial solution corresponding to the same eigenvalue c_j at a *smaller* value $\mathrm{Re}^* = \mathrm{Re}k_1/(k_1^2 + k_2^2)^{1/2}$ of the Reynolds number. In particular, if a normal mode of disturbance of the form (2.39) with horizontal wave numbers k_1 and $k_2 \neq 0$ is unstable at some value of Re, then a two-dimensional normal mode of the form (2.43) with $k = (k_1^2 + k_2^2)^{1/2}$ will be unstable at an even smaller value of Re, equal to Re k_1/k. We see that the disturbance that becomes unstable at the smallest value of Re is always two-dimensional and has the form of a wave propagating in the x direction. Hence only such waves need be considered when the determination of $\mathrm{Re}_{\mathrm{cr}}$ is the main aim, i.e., here it is possible to limit oneself to consideration of Eq. (2.44) only. Just this equation (and not the more general Eq. (2.41)) is usually called the *Orr-Sommerfeld* (or *O-S*) *equation.*

The stated result regarding the possibility of reducing Eq. (2.41) to the form (2.44) is due to Squire (1933); it is often called *Squire's theorem,* while the transformation $k_1 \to k$, $Re \to Rek_1/k$ (and the related transformation of fluid dynamic quantities converting the three-dimensional normal mode of disturbance at Reynolds number Re into a two-dimensional normal mode at smaller value of Re corresponding to the same eigenvalue c) is called *Squire's transformation.* It is usually assumed that Squire's theorem means that only two-dimensional disturbances of the form (2.43) need be considered in the linear stability theory of plane-parallel flows. For such disturbances $v(\boldsymbol{x},\ t) = 0$ and $\partial u/\partial x + \partial w/\partial z = 0$, where $u,\ v,\ w$ are velocity disturbances directed along axes Ox, Oy and Oz. Therefore, instead of velocity field $\boldsymbol{u}'(\boldsymbol{x}, t) = \{u(x, z, t), 0, w(x, z, t)\}$ we can use a stream function $\psi(x,\ z,\ t)$ {of the form $\psi(x, z, t) = \exp[ik(\xi - c\tau)]\psi(\zeta)\}$, determining the velocity components by the equations

$$u = \partial\psi/\partial\zeta, \quad w = -\partial\psi/\partial\xi. \tag{2.45}$$

It is easy to see that the stream function amplitude $\psi(\zeta)$ will also satisfy the O-S equation (2.44)

$$(U - c)\left(\psi'' - k^2\psi\right) - U''\psi + \frac{i}{kRe}\left(\psi^{(iv)} - 2k^2\psi'' + k^4\psi\right) = 0, \tag{2.44$'$}$$

and boundary conditions corresponding to (2.42)

$$\psi(0) = \psi'(0) = \psi(1) = \psi'(1) = 0. \tag{2.42$'$}$$

It was noted by Squire in 1933 (but was seldom indicated in subsequent works), that in transforming Eqs. (2.35–2.36) into the single Eq. (2.38) for $w(\boldsymbol{x},\ t)$, one discards possible solutions of the initial equations satisfying the condition $w(\boldsymbol{x},\ t) = 0$. In this case also $p'(\boldsymbol{x},\ t) = 0$ and, therefore, both horizontal components of velocity fluctuation u_1 and u_2 (primes at the symbols denoting velocity fluctuations are now omitted) satisfy the same equation $(\partial/\partial t + U_1\partial/\partial x)u_i = \nu\Delta u_i, i = 1, 2$. For normal modes which are proportional to $\exp[\mathrm{i}(k_1\xi + k_2\eta - k_1 c\tau))$ it follows from (2.36) that $k_1u_1 + k_2u_2 = 0$ and the dimensionless amplitudes $V_1(\zeta)$ and $V_2(\zeta) = -(k_1/k_2)V_1(\zeta)$ of two non-zero velocity components satisfy the equation

$$V_j'' - k^2V_j - ik_1\,\mathrm{Re}\,(U - c)V_j = 0, \quad j = 1, 2, \tag{2.46}$$

with boundary conditions $V_j(0) = V_j(1) = 0$. For given values of $k_1 \neq 0$ and $k_2 \neq 0$, Eq. (2.46) has a spectrum of eigenvalues $c_j(k_1,\ k_2,\ \mathrm{Re})$ that supplements the eigenvalue spectrum of the O-S equation (2.44) and corresponds to a new family of three-dimensional normal modes of disturbance. However, it is easy to show that all the new modes are stable and hence unimportant for the determination of instability conditions. In fact, multiplying Eq. (2.46) by the complex conjugate function $V^*{}_j$, integrating with respect to ζ from 0 to 1, and considering the real part of the resulting equation, we get

$$-\int_0^1 \left(\left|dV_j/d\zeta\right|^2 + k^2\left|V_j\right|^2\right) d\zeta - \Im mck_1\mathrm{Re}\int_0^1 \left|V_j\right|^2 d\zeta = 0, \tag{2.47}$$

This shows that $\Im mc < 0$. Therefore the normal mode that first loses its stability when Re is increasing is always two-dimensional and independent of η (i.e., of y). Hence, in determining the condition for instability by the method of normal modes, we need only study the eigenvalue problem for the O-S equation (2.44).

The eigenvalues of the O-S equation depend on the parameters k and Re, i.e., they have the form $c_j(k, \mathrm{Re})$ where j is the number of the eigenvalue. The critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ is then the smallest value of Re for which $\min_{\mathrm{k,j}}[\Im m c_j(k, \mathrm{Re})] = 0$, where the minimum is taken over all real values of k and integer values of j. Similarly the critical wave number k_{cr} is such that $\Im m c_j(k_{\mathrm{cr}}, \mathrm{Re}_{\mathrm{cr}}) = 0$ for the value of j corresponding to the neutrally stable normal mode (say, for $j = 1$). Note that here, generally speaking, the real part $c_{1(\mathrm{r})}$ of the eigenvalue $c_1(k_{\mathrm{cr},}\ \mathrm{Re}_{\mathrm{cr}})$ with vanishing imaginary part is different from zero. This means that the "principle of exchange of stabilities" is not valid here, and at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ the neutrally stable mode of disturbance represents a wave propagating along the Ox axis with phase velocity $c_1^{(r)}$. For values of Re somewhat larger than $\mathrm{Re}_{\mathrm{cr}}$, a fairly small range of k values close to k_{cr} exists for which $\Im m c_1(k, \mathrm{Re})$ is positive (and the real part $c_1^{(r)}(k, \mathrm{Re})$ of $c_1(k, \mathrm{Re})$ is nonzero). Waves with values of k from this range will form a wave packet, which will grow with time and simultaneously move downstream (with the group velocity of the packet). In this respect, the instability under discussion differs from that of the flows considered in Sects. 2.6 and 2.7, where unstable disturbances usually did not move and, at a given point, grew to form finite-amplitude Taylor vortices or Bénard cells.

The range of k values having the property that at given Re (slightly exceeding $\mathrm{Re}_{\mathrm{cr}}$) the inequality $\Im m c_1(k, \mathrm{Re}) = c_1^{(i)}(k, \mathrm{Re}) > 0$ is valid (i.e., the two-dimensional normal mode of disturbance corresponding to wave number k is unstable) corresponds to the set of unstable two-dimensional normal modes. However, three-dimensional normal modes can also be unstable if $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$. Therefore the investigation of the behavior of three-dimensional modes of disturbance in a steady plane-parallel fluid flow at supercritical Reynolds numbers $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ is also of interest. Such an investigation was carried out, in particular, by Watson (1960); and Michael (1961). They showed that within the framework of the normal-mode method of the linear stability theory for any plane-parallel flow there always exists a range of values of Re, $\mathrm{Re}_{\mathrm{cr}} < \mathrm{Re} < \mathrm{Re}_1$, within which, of all the unstable wave disturbances, the most rapidly increasing disturbance (i.e., that having the greatest rate of growth $\Im m c_j$) is necessarily two-dimensional. (This statement will not be correct for all values of Re; furthermore, in the case of disturbances with fixed wave number k, the most unstable disturbance is three-dimensional for certain values of k). Therefore, assuming that at slightly supercritical values of Re the most unstable wave disturbance will suppress all the others so that only its amplitude grows, we must expect that a finite range of supercritical Reynolds numbers will exist, within which the flow will differ from the initial plane-parallel flow only by the superposition on it of the most unstable (and hence two-dimensional) normal mode of disturbance. Some further information about three-dimensional waves unstable at a given supercritical value of Re can be found in the paper by Magen and Patera (1986).

Note now that, according to the available experimental data on transitions to turbulence of steady plane-parallel and almost plane-parallel flows, three-dimensional disturbances often begin to play a fundamental role right from the first appearance of hydrodynamic instability, i.e., at least for $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ and in some cases even for values of Re smaller than the value of $\mathrm{Re}_{\mathrm{cr}}$ calculated by the normal-mode method (see, e.g., Klebanoff et al. (1962); Kachanov et al. (1982); Herbert (1988); Kachanov

(1994) and also Sect. 2.1 above and Chap. 3 below). These unexpected results were usually explained by the authors considering this topic (including the present ones; see MY1, p. 114) by the apparent influence of the inadequately-studied non-linear effects. However, recent developments in the linear theory of hydrodynamic instability led to the conclusion that the above-mentioned experimental facts can also be explained within the framework of linear disturbance theory if we replace the normal-mode method by a more general approach which will be considered later.

It was implicitly assumed above that the Orr-Sommerfeld eigenvalue problem (2.44–2.42) (or, what is the same, (2.44′–2.42′)) has a purely discrete eigenvalue spectrum c_j, $j = 1, 2, \ldots$. This statement was, in fact, first proved by Lin (1961b); (see also Drazin and Reid (1981), p. 156) under the condition that the function $U_1(z)$ (or, what is the same, $U(\zeta) = U_1(z)/U_0$), which describes the velocity profile and hence is determined only for $0 \le z \le H$(or $0 \le \zeta \le 1$), can be continued to the whole plane of a complex variable as an analytic entire function of z (or ζ). (This condition is clearly valid for linear and quadratic velocity profiles describing Couette, Poiseuille, and Couette-Poiseuille flows.) Later the same result was proved under more general conditions by Schensted (1960); Yudovich (1965); Di Prima and Habetler (1969); and Herron (1982); (cf. also Sattinger (1970); Yudovich (1984); Georgescu (1985) and Sect. 2.5 of this book). All these authors also proved that the corresponding system of eigenfunctions $W_{\mathrm{j}}(\zeta)$ [or $\psi_{\mathrm{j}}(\zeta)$) is complete in the space of all functions of ζ, $0 \le \zeta \le 1$, square-integrable and vanishing at the end points together with its derivatives. Therefore, any function of ζ having a continuous derivative which vanishes with the function itself at $\zeta = 0$ and $\zeta = 1$ can be expanded into an uniformly convergent series in terms of the eigenfunctions of the O-S equation. Similarly, any two-dimensional disturbance of the velocity (or stream-function) field can be expanded in a convergent series in terms of corresponding normal modes. Much additional information about these eigenfunction and normal-mode expansions can be found in the book by Drazin and Reid (1981). A number of useful inequalities for the eigenvalues of the O-S problem (2.44–2.42) was found by Joseph (1968, 1969); Yih (1969, 1973); Georgescu (1970) and Warren (1976). Some of these inequalities, which give sufficient conditions for stability of plane-parallel viscous flows with respect to infinitesimal wave-like disturbances, are considered in the books by Joseph (1976); Drazin and Reid (1981); and Georgescu (1985).

It was mentioned above that the O-S equation is also often applied as a reasonable first approximation to stability studies for so-called *nearly plane-parallel flows*—steady two-dimensional flows with the velocity field $\boldsymbol{U}(\boldsymbol{x}) = \{U(x, z), 0, W(x, z)\}$, where $|W| \ll |U|$ and $|\partial U/\partial x| \ll |\partial U/\partial z|$. The class of nearly plane-parallel flows includes many important flows, for example boundary layers on flat plates, plane jets issued from thin linear apertures along the *Oy* axis, plane wakes behind long thin cylinders, and mixing zones between two plane-parallel flows, flowing one above the other with different velocities. In these cases it is often possible to neglect the z component of the velocity and the dependence of the x component on z, i.e., to consider the flow as being strictly plane-parallel in the first approximation (see some specific examples in Sects.2.92 and 2.93; another approximation was used in the papers by Herron referred in Sect. 2.5). Considering the flow as plane-parallel we can use the appropriate O-S equation; however, the O-S equation must now be valid in an unbounded region—either the halfspace $z \geq 0$ or the whole space. (In these cases the length scale H must of course be defined not as the total thickness of the flow region but as some typical vertical scale of the unbounded flow.) The boundary conditions at infinity are just the same here as at solid walls in the case of channel flow: both $W(\zeta)$ and $W'(z)$ [or $\psi(\zeta)$ and $\psi'(\zeta)$) must vanish at infinity, i.e., tend to zero as $\zeta \to \infty$ or, for flows in the whole space, $\zeta \to \pm\infty$. However, in the case of unbounded flow, the statement that the eigenvalue spectrum of the O-S problem is purely discrete is incorrect—here this eigenvalue problem has, as a rule, both discrete and continuous spectra. The continuous spectra of the O-S equations will be considered at greater length later in this book, where the stability problems for some particular unbounded nearly plane-parallel flows will be discussed.

2.8.2 *The Rayleigh Equation and the Stability Analysis for Plane-Parallel Flows of an Inviscid Fluid*

The O-S eigenvalue problem (2.44–2.42) is quite complicated and its solution requires the use of sophisticated and cumbersome mathematical procedures. Since the available data show that the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ is very large for many plane-parallel flows, it seems natural to expect that, for Reynolds numbers near or above $\mathrm{Re}_{\mathrm{cr}}$, the terms, of Eq. (2.44) which contain the factor Re^{-1} and describe the action of the viscous forces will be small compared with the other terms, which are independent of Re. If so, then we may first consider the fluid as ideal (i.e., inviscid) and ignore the terms of (2.44) which are proportional to Re^{-1}.Thus, instead of (2.44) we obtain the abridged equation

$$(U - c)(W'' - k^2 W) - U'' W = 0, \tag{2.48}$$

(primes denote differentiation), which was studied in detail by Rayleigh (1880, 1887, 1895, 1913) and is now called the *Rayleigh equation.* The same equation will of course be valid, in the case of an ideal fluid, for the stream-function amplitude ψ. Rayleigh's equation determines the amplitudes of two-dimensional wave-like disturbances (proportional to $\exp ik(x - ct)\}$). Within the framework of the normal-mode approach to hydrodynamic stability theory, the limitation to such disturbances can again be justified by the Squire transformation, showing that if there exists an unstable three-dimensional wave (i.e., a disturbance proportional to $\exp ik_1(x, - ct) + k_2 y\}$ where $\Im mc > 0$), then a two-dimensional wave also exists which is more unstable (i.e., faster growing) than the initial three-dimensional wave (see, e.g., Drazin and Howard (1966) or Drazin and Reid (1981)). Note also that, in the case of a two-dimensional unstable flow of ideal fluid, the fastest growing wave is always two-dimensional, though this statement can be incorrect for a viscous fluid; see, e.g., Gaster (1970). Moreover, no analog of the Squire theorem exists for arbitrary (not wave-like) disturbances. This fact did not attract much attention in the past, but, as we shall see, it proves to be rather important.

For better conformity with the notation used in most other work, we will not yet transform the dynamic equations to dimensionless independent and dependent variables. Therefore, U, W and c will now have the dimension of velocity (and ψ the dimension of velocity multiplied by length); U, W and ψ will depend on z (where $0 \le z \le H$; note that notation $U(z)$ is now used instead of $U_1(z)$, and primes will denote differentiation on z (and not on $\zeta = z/H$)). Rayleigh's equation is not of the fourth but of the second order, and this is naturally seen as a considerable simplification. We cannot require that four boundary conditions (2.42) (or (2.42′)) be satisfied by a second-order equation. It is known, however, that for a flow of inviscid fluid only the normal component of the velocity must be equal to zero on solid walls. It is easy to see that from this it follows that conditions (2.42) and (2.42′) must be replaced here by the condition $W = 0$ and $k\psi = 0$ on the walls (and also at infinity). Moreover, in a plane-parallel flow of ideal fluid the velocity profile $U(z)$ can be arbitrary.

A number of examples where the Rayleigh equation can be explicitly solved in terms of some elementary or known transcendental functions of mathematical physics was collected by Russell (1994). Note however that although Eq. (2.48) is mathematically simpler, the behavior of its solutions may in some respects be more complicated than those of the O-S equation (2.44). This is because in Eq. (2.44) the coefficient of the highest-order derivative of the unknown function is a non-zero constant, while in Eq. (2.48) this coefficient depends on the variable z, and if neutrally stable modes are studied it can vanish at some point z_0 where the velocity of the undisturbed flow $U(z)$ is equal to the phase velocity c of the considered normal mode. (Recall that only the neutrally stable modes with real values of c need be studied for determination of the instability criterion.) It was proved quite early by Rayleigh that *the phase velocity c of any neutral wave mode will always be between the minimum and the maximum velocity of the undisturbed flow* (i.e., $U_{\min} \le c \le U_{\max}$), so that a point z_0 where $U - c = 0$ always exists within the flow. This point z_0 is a singular point of the Rayleigh equation and the presence of such a point complicates the analysis considerably.

There are several different proofs of Rayleigh's statement, above; one of the simplest, which is due to Drazin and Howard (1966), is the following. Let us rewrite Eq. (2.48) in the form

$$\{(U-c)^2 F'\}' - k^2 (U-c)^2 F = 0, \quad F = W/(U-c). \tag{2.48′}$$

Now multiply this equation by the complex conjugate function F^* and then integrate it from $z = 0$ to $z = H$. Since $W(0) = W(H) = 0$ (and hence also $F(0) = F(H) = 0$), integrating by parts we obtain for nonsingular F that

$$\int_0^H (U-c)^2 \left\{ |F'|^2 + k^2 |F|^2 \right\} dz = 0. \tag{2.49}$$

The last equation shows that F cannot be nonsingular when c is real, and therefore for real c the difference $U - c$ must vanish at some point, z_0 say.

Equation (2.49) can be rewritten as

$$\int_0^H [U(z) - c]^2 \Phi(z) dz = 0, \quad \text{where } \Phi(z) > 0 \tag{2.49'}$$

(and $\Phi(z)$ is nonsingular for any complex c with non-zero imaginary part). If $c = c^{(r)} + ic^{(i)}$ is a complex number with $c^{(i)} \neq 0$, then the imaginary and real parts of (2.49′) imply the relations

$$\begin{aligned} \int_0^H U\Phi dz &= c^{(r)} \int_0^H \Phi dz, \\ \int_0^H U^2 \Phi dz &= [(c^{(r)})^2 + (c^{(i)})^2] \int_0^H \Phi dz. \end{aligned} \tag{2.49''}$$

The first of these two equations shows that $U(z) - c^{(r)}$ must change sign somewhere, and hence $U - c^{(r)}$ *must vanish at some point* z_0 *for any eigenvalue* c, so that $U_{\min} \leq \Re\mathrm{e}\, c \leq U_{\max}$ always (this result, which was also known to Rayleigh, generalizes the result given above for real eigenvalues c). Now let $U_{min} = a$ and $U_{max} = b$; then clearly $(U - a)(U - b) \leq 0$ everywhere. Using Eqs. (2.49″) we easily obtain that

$$0 < \int_0^H (U - a)(U - b)\Phi dz = \left\{ \left[c^{(r)} - \frac{1}{2}(a + b) \right]^2 + \left[c^{(i)} \right]^2 - \frac{1}{4}(b - a)^2 \right\} \int_0^H \Phi dz$$

and hence

$$\left[c^{(r)} - \frac{1}{2}(a + b) \right]^2 + \left[c^{(i)} \right]^2 \leq \left[\frac{1}{2}(b - a) \right]^2.$$

This means that *all the eigenvalues* $c = c^{(\mathrm{r})} + ic^{(\mathrm{i})}$ *of the Rayleigh equation with* $c^{(\mathrm{i})} > 0$ *(i.e., those corresponding to unstable modes) lie in a semicircle in the complex c-plane with center at the point* $(U_{\max} + U_{\min})/2$ *of the real axis and radius* $(U_{\max} - U_{\min})/2$ *(i.e., having the segment* $(U_{\min}, U_{\max})$ of the real axis as its diameter). (In other words, any eigenvalue c, real or complex, must lie in or on the circle having this segment as its diameter.) This is the *semicircle theorem* of Howard (1961). (Another simple proof was given by Warren (1976) who also proved an analog of this theorem for the Couette flow between concentric rotating cylinders.) A number of generalizations and sharpenings of this result is also known; see, e.g., Drazin and Howard (1966); Sattinger (1967); Kochar and Jain (1979); Jain and Kochar (1983); Makov and Stepanyants (1984) and also Sects. 2.83 and 2.84 below.

In addition to the semicircle theorem, many other bounds for the eigenvalues c (depending on the velocity profile $U(z)$ and often on the wavenumber k too) can be also found in the available literature; see, e.g., Drazin and Howard (1966); and Craik (1972). These bounds give additional necessary conditions for instability (more exactly, for the existence of exponentially growing wave-like disturbances in the flow) and restrict the values of the wavenumbers of such growing disturbances.

Since $U(z) - c \approx U'(z_0)(z - z_0)$ at points z near the singular point z_0, it follows that if $U'(z_0) \neq 0$, $W''(z)$ will tend to infinity as $[U''(z_0)W(z_0)]/[U'(z_0)\{z - z_0\}]$ when $z \to z_0$. Hence, $W'(z)$ will be proportional to $\text{In}(z - z_0)$ in the vicinity of z_0; also $\psi'(z) = \partial\psi/\partial z$, which is proportional to the streamwise velocity of disturbance $u(x, z)$, will be proportional to $\text{In}(z - z\text{o})$. We see that near the singular point the velocity of the neutral mode of disturbance tends to infinity and is described by a multivalued function, which gives rise to a difficult question: which branch of the multivalued function must be taken here? The usual method of overcoming all the resulting difficulties consists of going back from Rayleigh's abridged equation to the complete O-S equation, but then we lose the advantage of the simplicity of Eq. (2.48). In fact, the available methods that use this approach are quite cumbersome; see, e.g., Lin (1961a), Chap. 8, Drazin and Howard (1966); and Drazin and Reid (1981). Another method, which permits us to get rid of the singularity, consists of supplementing the Rayleigh equation not with viscous terms, but with terms quadratic in the velocity disturbances (see e.g. Stuart (1971b)); however, this method lies outside linear stability theory. One more method that allows the relative simplicity of the inviscid theory to be used consists in replacing the normal-mode approach by an analysis of the corresponding initial-value problem; this method will be considered at length in the next chapter.

There is also the further difficulty that if c in Eq. (2.48) is a complex eigenvalue (corresponding to the eigenfunction $W(z)$), then the complex conjugate $c*$ of c will also be an eigenvalue of this equation (corresponding to the eigenfunction $W*(z)$). Consequently, in addition to the damped normal mode, the equations for the vertical velocity and for the stream function will always have a solution in the form of a growing wave. Therefore, for an inviscid fluid it is meaningless to define the stable case as the case when only damped normal modes occur. Thus the very definition of stability based on the consideration of normal modes (i.e., elementary wave-like solutions) must be changed. Stability must now be defined as the absence of growing (and hence also of damped) wave-like disturbances. Therefore, the normal-mode criterion for stability is now that the Rayleigh equation has only real eigenvalues c for any value of k.

One more difficulty is that the singularity of Eq. (2.48), which is connected with the coefficient of the highest derivative becoming zero, leads to the presence, for a wide range of conditions, of only a finite number of discrete eigenvalues c (see, e.g., Rosencrans and Sattinger (1966); and Sattinger (1967)). Hence, it is clear that the set of all normal modes (the stability of which is determined by the value of the imaginary part of the corresponding eigenvalue c) is not complete here, and not all disturbances can be approximated by a linear combination of such modes. Therefore, it is natural to think that a continuous spectrum of eigenvalues c must exist here in addition to the usual discrete spectrum. In fact, the existence of such a continuous spectrum can be proved rather easily, as was shown by Rayleigh ((1894), Vol. 2, pp. 391–400). However, surprisingly, the importance of this fact was disregarded (and the continuous spectrum was not taken into account), not only by Rayleigh himself but also by all other workers until much later. Note that the eigenfunctions $W(z)$ [or $\psi(z)$] for eigenvalues of the continuous spectrum do not satisfy the ordinary "regularity conditions" and they are often not ordinary functions at all but the so-called "generalized" "functions" (or "distributions") systematically studied only in the second half of the twentieth century (see, e.g., Schwartz (1961); Gel'fand and Shilov (1958); Lighthill (1968), or Lumley (1970)). The very great difference between spectra (i.e., sets of all eigenvalues) and eigenfunctions of the complete O-S equation and the abridged Rayleigh equation clearly shows that there is no simple connection between the eigenvalues and eigenfunctions of these two equations, so that eigenvalues and eigenfunctions of the abridged equation cannot be obtained from those of the complete equation by simple passage to the limit as $\nu \to 0$ (cf. Case (1961); Lin (1961b); and Lin and Benney (1962)).

The continuous spectrum of the Rayleigh equation was studied by Faddeev (1972) at the suggestion of Arnol'd. The main idea of his work is quite simple (see also Dikii (1976)). Let us denote by Λ the operator $d^2/dz^2 - k^2$ in the space of functions in the segment $0 \le z \le H$, vanishing at the end-points $z = 0$ and $z = H$. Then Rayleigh's eq. (2.48) can be written in the form

$$(U - U''\Lambda^{-1})\Omega = c\Omega, \quad \Omega = \Lambda W. \tag{2.50}$$

We see that c is the eigenvalue of the sum of two linear operators—the operator of multiplication on the given function $U(z)$ and the operator $U''\Lambda^{-1}$. The first of these operators clearly has a continuous spectrum consisting of all the values of the function $U(z)$, i.e., of all the real numbers in the range from $U_{\min}$ to $U_{\max}$ (the corresponding eigenfunctions here are Dirac's δ functions, which are the best-known generalized functions). The second operator $U''\Lambda^{-1}$ can be represented as an integral operator in the space of functions on a finite interval. Such operators belong to the class of so-called completely continuous (compact) operators, which always have purely discrete spectra. It is natural to expect that such an addition will not change the continuous spectrum of the eigenvalue problem (2.50) which will always fill the interval $(U_{\min}, U_{\max})$.

The rigorous proof of this statement forms the most important part of Faddeev's paper. He begins by rewriting Eq. (2.50) in the form of an integral equation, which includes the Green function of the operator Λ. It follows from this representation that the operator on the left-hand side of (2.50) belongs to the class of operators studied by Friedrichs (1948) in his theory of the perturbations of continuous spectra (see also Faddeev (1964); and Dunford and Schwartz (1971)). From the results of Friedrichs' theory, it follows that in fact the continuous part of the spectrum of problem (2.50) consists of the above-mentioned finite interval; and the discrete spectrum, if it is infinite, can have accumulation points (limits of sequences of discrete eigenvalues) only on the same interval.

The investigation of the discrete spectrum of the problem (2.50) is more complicated, and here only partial results were obtained by Faddeev. He noted that, according to a theorem of Rayleigh (which will be proved below), if $U''(z)$ does not vanish (i.e., if the velocity profile has no inflection points) then the problem (2.50) has no discrete eigenvalues at all; hence in this case its spectrum consists only of the interval $(U_{\min}, U_{\max})$. If the velocity profile $U(z)$ is monotone and $U''(z)$ vanishes at only one point, then there exists a value $K > 0$ such that for $k^2 > K$ the spectral problem (2.50) has only a continuous spectrum while for $k^2 < K$ the continuous spectrum is supplemented by one pair of discrete complex-conjugate eigenvalues. In the case where $U(z)$ is monotone but $U''(z)$ vanishes at a finite number m of points, there can be several (not more than $1 + m/2$) pairs of complex discrete eigenvalues. However, if the velocity profile is not monotone, the discrete spectrum can have a considerably more complicated form.

Note also that much more general results were recently obtained by Vishik (1996) who described the continuous spectrum of linear equations for infinitesimal disturbances of arbitrary three-dimensional flow of ideal fluid. For such flows a continuous spectrum can have many different (and often quite complicated) forms.

Let us now consider some general results of the stability theory of plane-parallel inviscid flows. The important work of Rayleigh (mentioned above) served as a starting point of the theory. In particular he showed, as early as 1880, that if $U''(z)$ *is nonzero everywhere within the flow, then the abridged eq.* (2.48) *cannot possess complex eigenvalues* c *with* $\Im m c \neq 0$. The proof of this theorem is very simple. Rewriting Eq. (2.48) in the form

$$W'' - k^2 W - \frac{U'' W}{U - c^{(r)} - i c^{(i)}} = 0,$$

multiplying throughout by the complex conjugate function W^* and integrating from $z = 0$ to $z = H$ (i.e., over the whole thickness of the fluid layer) we obtain

$$\int_0^H \left(\left| W' \right|^2 + k^2 |W|^2 \right) dz + \int_0^H \frac{\left(U - c^{(r)} + i c^{(i)} \right) U'' |W|^2}{\left(U - c^{(r)} \right)^2 + \left(c^{(i)} \right)^2} dz = 0. \qquad (2.51)$$

With $c^{(i)} \neq 0$, the imaginary part of this equation can become zero only if $U''(z)$ changes sign somewhere between $z = 0$ and $z = H$. This proves Rayleigh's statement. (The important strengthening of this classical result by Faddeev (1972) was discussed above.)

It was widely accepted for many years that the Rayleigh theorem gave complete proof of the stability (i.e., the absence of growing disturbances) of any plane-parallel inviscid flow whose velocity profile does not possess an inflection point at which $U''(z) = 0$. However, the situation is, in fact, not so simple. The theorem states only that in the case of such flows no discrete complex eigenvalues of the Rayleigh equation can exist. But it was explained above that the Rayleigh equation has not only a discrete but also a continuous spectrum.

Therefore, in order to use normal-mode analysis to prove that no growing infinitesimal disturbance can exist in a flow with a velocity profile without inflection points, it is necessary to show that the continuous spectrum of Eq. (2.48) is also real, and that every infinitesimal velocity disturbance can be represented in terms of (ordinary or generalized) normal modes corresponding to eigenvalues belonging either to the discrete, or to the continuous, spectrum of Rayleigh's equation. As was explained above, a strict proof of the theorem about the reality of the continuous spectrum of Eq. (2.48) was given by Faddeev (1972); however, he did not consider the corresponding eigenfunction expansion theorem (which apparently has not been investigated up to now). Let us also recall the important remarks at the end of Sect. 2.5 about the possible existence of algebraically-growing solutions, and about the deficiency of expansions in terms of eigenfunctions in the case of non-normal operators. Hence it is clear that the question of the connection between the validity of Rayleigh's condition of absence of inflection points in the velocity profile and the stability of the flow needs further investigation.

Some of the recent results related to this question will be considered in the next chapter of this book. Here we shall only note that some related partial results were obtained long ago by Case (1960a, 1962); and Dikii (1960a) on the basis of study of the initial-value problem for the dynamic equation (2.38) with $\nu = 0$. However, their results (which will be discussed at greater length later) concern only two-dimensional disturbances (independent of y), with the velocity field $\boldsymbol{u}(\boldsymbol{x}, t) = \{u(x, z, t), 0, w(x, z, t)\}$. It was also shown by Dikii (1976) that, for some important classes of velocity profiles $U(z)$ guaranteeing the absence of complex eigenvalues c in Eq. (2.48) (and including all the profiles without inflection points), the absence of growing infinitesimal velocity disturbances independent of y can be proved with the aid of the conservation law (2.57) (see the text following this equation). Another approach to the study of the connection between the absence of points where $U''(z) = 0$ and stability of flow with velocity profile $U(z)$ with respect to two-dimensional velocity disturbances, which is applicable even to finite (i.e., not infinitesimal) such disturbances, was proposed by Arnol'd (1965); it will be considered in a later Chapter). However, all the above-mentioned results concern only two-dimensional velocity disturbances. Since no theorem similar to Squire's is valid for disturbances of arbitrary form, the stated results do not guarantee the absence of growing three-dimensional infinitesimal disturbances in cases when Rayleigh's equation has no complex eigenvalues. And, in fact, it will be shown below that according to recent developments in the theory of hydrodynamic stability, growing disturbances (both infinitesimal and finite) can appear much more often in fluid flows than was thought earlier.

The Rayleigh theorem gives only a necessary, but not a sufficient, condition for the existence of complex eigenvalues c. A stronger necessary condition for the existence of exponentially-growing wave-like disturbances in a plane-parallel inviscid flow was given by Fjørtoft (1950). Let $c = c^{(r)} + ic^{(i)}$ be a complex eigenvalue with $c^{(i)} \neq 0$: if we add the real part of equation (2.51) to the imaginary part of the same equation multiplied by $(c^{(r)} - K)/c^{(i)}$ where K is an arbitrary constant, we obtain

$$\int_0^H \frac{(U - K)U''|W|^2}{|U - c|^2} dz < 0. \tag{2.52}$$

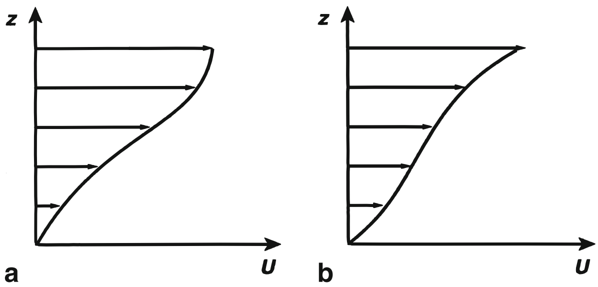


Fig. 2.19 Two types of velocity profiles $U(z)$ with one inflection point; Fjørtoft's condition is valid for profile b but not valid for profile a

This shows that *if there exists a constant K such that* $[U(z)-K]\,U''(z) \geq 0$ *for any z, then an eigenvalue c with* $c^{(i)} \neq 0$ *cannot exist.* Fjørtoft's condition is clearly a generalization of Rayleigh's, since if $U''(z)$ has the same sign everywhere then Fjørtoft's condition is clearly valid for any K such that $|K| > \max|U(z)|$ and $KU''(z)<0$. On the other hand, if $U''(z)$ vanishes at some point $z=z_0$ and $K=U(z_0)=U_s$, then we find that a complex eigenvalue c can exist only if $U''(U-U_s)$ is negative somewhere in the field of flow. In particular, if $U(z)$ is a monotonic function and $U''(z)$ vanishes at one point z_0 only, then for the existence of growing wave-like solution the inequality $U''(U-U_s)<0$ must be fulfilled at every $z \neq z_0$. Thus, for example, in a flow with the velocity profile shown in Fig. 2.19a unstable wave-like disturbances can exist, but in a flow with the velocity profile shown in Fig. 2.19b they cannot exist.

It was noted above that if Rayleigh's equation (2.48) has no complex eigenvalues, then it is possible only to prove that the corresponding inviscid flow is stable with respect to two-dimensional infinitesimal disturbances, and even this proof is in general rather difficult. However, if the absence of complex eigenvalues c is related to the non-fulfillment of Rayleigh's or Fjørtoft's necessary condition, then a much simpler proof based on a special conservation law can be given.

In the case of an inviscid fluid, the linearized dynamic equation for the stream function $\psi(x, z, t)$ of a two-dimensional velocity disturbance $\boldsymbol{u}(\boldsymbol{x}, t)=\{u(x, z, t), 0, w(x, z, t)\}$ clearly has the form of Eq. (2.38) with $\nu=0$:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Delta_2\psi - U''\frac{\partial\psi}{\partial x}=0, \tag{2.53}$$

where $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$ is the two-dimensional Laplacian. If the dependence of ψ on x and $\boldsymbol{t}$ is given by an exponential factor $\exp[ik(x-ct)]$, Eq. (2.53) becomes Rayleigh's equation (2.48) for the stream-function amplitude $\psi(z)$; but assuming only that $\psi(x, z, t)=\exp(ikx)\,\psi(z, t)$ we obtain

$$(\partial/\partial t + ikU)(\psi'' - k^2\psi) - ikU''\psi = 0, \tag{2.54}$$

where primes again denote differentiation on z. Let us now multiply Eq. (2.54) by the complex conjugate function ψ^* and integrate it on z over all the thickness of the flow (i.e., from $z=0$ to $z=H$). The real part of the resulting equation can then be written as

$$-\frac{d}{dt}\int_0^H \left(\left|\psi'\right|^2 + k^2|\psi|^2\right) dz + \Re e \left[ik\int_0^H U\left(\psi'' - k^2\psi\right)\psi * dz\right] = 0. \quad (2.55)$$

(This procedure is analogous to the derivation of Eq. (2.49) from Eq, (2.48′).) If K is now a constant such that $[U(z) - K]/U''(z)$ is a continuous function of z, then multiplying Eq. (2.54) by $\left(\psi'' * - k^2\psi*\right)(U - K)/U''$, integrating the product from $z = 0$ to $z = H$, and taking the real part of the resulting equation, we obtain

$$-\frac{d}{dt}\int_0^H \frac{U - K}{U''}\left|\psi'' - k^2\psi\right|^2 dz + \Re e\left[ik\int_0^H U(\psi'' * - k^2\psi*)\psi dz\right] = 0. \quad (2.56)$$

The sum of Eqs. (2.55) and (2.56) has the form

$$\frac{d}{dt}\int_0^H \left[\left|\psi'\right|^2 + k^2|\psi|^2 + \frac{U - K}{U''}\left|\psi'' - k^2\psi\right|^2\right] dz = 0. \quad (2.57)$$

Equation (2.57) describes a new integral invariant of fluid motion which is quadratic with respect to the stream function $\psi(z, t)$. If now $(U - K)/U'' > 0$ for all values of z, and $\psi(z)$, $\psi'(z)$ and $\psi''(z)$ (i.e., the stream function, velocity of flow, and vorticity) are very small at $t = 0$ (i.e., the initial disturbance is very small), then according to (2.57) the integrals of $|\psi|^2$, $\left|\psi'\right|^2$ and $\left|\psi''\right|^2$ will be small at any $t > 0$. Hence, if the Fjørtoft condition is valid, then the smallness of disturbance at the initial moment guarantees the smallness of the mean square values of stream function, velocity and vorticity of the disturbance at all values of $\boldsymbol{t}$. This shows that under the specified conditions the flow is stable with respect to two-dimensional infinitesimal disturbances.

The conservation law (2.57) is due to Dikii (1976). It is related to Arnol'd's method (1965) of stability investigation, which used the non-linear fluid dynamic equations and is therefore also applicable to finite disturbances.

Fjørtoft's necessary condition for the existence of wave-like unstable modes of disturbance is more restrictive than the older condition due to Rayleigh. However, in general this new condition is also insufficient. In particular, it was shown by Tollmien (1935) that in the case where $U(z) = \sin z$ and the solid walls are placed at $z_1 < 0$ and $z_2 > 0$, Rayleigh's equation has no complex eigenvalues if $z_2 - z_1 < \pi$, and it is easy to verify that the Fjørtoft condition is valid here (cf. also Drazin and Howard (1966), p. 35, or Drazin and Reid (1981), p. 136). In the same paper, Tollmien showed that the presence of an inflection point of the velocity profile is sufficient for the existence of at least one complex eigenvalue c in the very important cases of (i) channel flows with velocity profiles symmetric with respect to the midplane $z = H/2$ and monotonically increasing in the lower half of the channel (as shown in Fig. 2.20), and (ii) flows of boundary-layer type with velocity profiles of the form shown in Fig. 2.19a (see also Lin (1961a), Sect. 8.2; Drazin and Howard (1966); and Dikii (1976), Sect. 8).

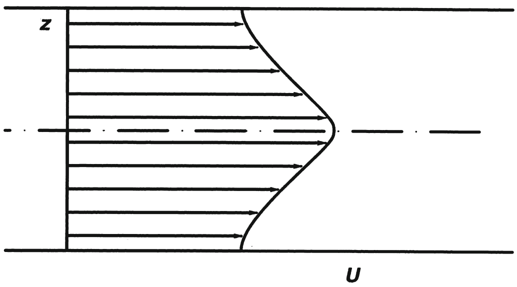


Fig. 2.20 Velocity profile of a channel flow where, according to Tollmien's result, there exists an unstable wave-like disturbance

Another necessary condition for the existence of complex eigenvalues c of Rayleigh's equation, which is also sufficient in the case of a monotonic profile $U(z)$ with only one inflection point, was given by Rosenbluth and Simon (1964); (see also Dikii (1976); and Craik (1972)).

Conditions that are sufficient for the existence of an eigenvalue c with $\Im\text{m}\ c > 0$ clearly imply that the corresponding flows are unstable with respect to infinitesimal disturbances. However, conditions for the non-existence of eigenvalues c such that $\Im\text{m}\ c > 0$, which attracted so much attention in the past when they were considered to be genuine conditions for hydrodynamic stability, have now partly lost their importance. In fact, it is now clear that these conditions show only that the corresponding flows are stable with respect to a special (relatively restricted) class of infinitesimal disturbances. This circumstance will be discussed at greater length in the next chapter.

2.8.3 Stability Analysis for Plane-Parallel Flows of a Stratified Inviscid Fluid

In the earlier part of this section, only flows of a homogeneous fluid having constant density ρ were considered. However, flows of inhomogeneous fluids with variable density $\rho(\boldsymbol{x})$ appear quite often in various applications. In particular, the study of such flows has great importance to geophysics. The density of atmospheric air depends on the air temperature T (and also on the pressure p, but this last dependence can be often neglected as is done when the Boussinesq approximation is applied to equations of motion) while the density of oceanic water depends on both the temperature T and the salinity s (dependence on pressure is here negligible for most purposes). Since the mean temperature, salinity (and also pressure), and, therefore, the density almost always vary with height (or depth) z, flows in the atmosphere and ocean are usually stratified. Having these examples in mind, we shall now briefly consider the problem of hydrodynamic instability for plane-parallel flows of incompressible stratified fluid with given velocity and density profiles $U(z)$ and $\rho(z)$. Note that by convention z is chosen vertically upwards in the atmosphere, vertically downwards in the ocean. Since geophysical flows are usually characterized by very high values of the Reynolds number, we shall neglect the action of viscosity, i.e. assume that the fluid is inviscid.

Studies of instability for stratified fluid flows have a rather long history. Apparently, the first problem considered, the stability of a plane-parallel flow in an unbounded space, having constant velocity and density U_1 and ρ_1 for $z<0$ and different but constant velocity and density U_2 and ρ_2 for $z>0$, was briefly outlined by Helmholtz (1868); and investigated by Kelvin (1871). (These papers were mentioned in Sect. 2.4 in relation to Kelvin-Helmholtz instability.) The stability of a stationary layer of stratified fluid (i.e., the case where $U=0$) was studied by Rayleigh in 1883. For one special case of stratified plane-parallel flow with continuous profiles $U(z)$ and $\rho(z)$, important results about stability were obtained by G.I. Taylor in 1914, but were published by this author only much later, at first because of the war and then because he was waiting in vain for experimental data confirming his theoretical results (see Taylor (1931)). However, even today there are many unsolved problems in the theory of stability of stratified flows.

The fluid dynamic equations for an inviscid incompressible fluid of variable density have the form

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}-\delta_{i3}g\rho, \quad i=1,2,3. \tag{2.58}$$

$$\frac{\partial u_j}{\partial x_j}=0, \tag{2.59}$$

$$\frac{\partial \rho}{\partial t}+u_j\frac{\partial \rho}{\partial x_j}=0. \tag{2.60}$$

(If the variations of the density ρ are due only to the variations of the temperature T, we may use the variable T in these equations instead of ρ; see, e.g., Sect. 2.7. Note also that not only the molecular viscosity, but also the molecular diffusivity of mass and/or temperature (which can affect the evolution of the field $\rho\ (\boldsymbol{x}, t)$), are assumed to be negligibly small in these equations.) For any given profiles $U(z)$ and $\rho\ (z)>0$, Eqs. (2.58–2.60) have the following stationary solution

$$u_i=u_i(z)=U(z)\delta_{i1}, \quad \rho=\rho(z), \quad p=p(z)=\begin{cases} p(0)-\int\limits_0^z \rho(z')dz' & \text{for} \quad z>0, \\ p(0)+\int\limits_z^0 \rho(z')dz' & \text{for} \quad z<0. \end{cases} \tag{2.61}$$

To study stability of this solution, we write, as usual,

$$u_i=u_i(z)+u_i'(\boldsymbol{x},t), \quad \rho=\rho(z)+\rho'(\boldsymbol{x},t), \quad p=p(z)+p'(\boldsymbol{x},t), \tag{2.62}$$

and substitute (2.62) into (2.58–2.60). Then, we obtain a system of five equations with five unknowns $u'_{i,}\ i=1,2,3$, ρ' and p'.

It is easy to eliminate from this system all the variables except $u'_3 = w$. Then we arrive at the following equation for w

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \left(\Delta w + \frac{\rho'}{\rho}\frac{\partial w}{\partial z}\right) - \left(U'' + \frac{\rho'}{\rho}U'\right)\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial w}{\partial x} - g\frac{\rho'}{\rho}\Delta_2 w = 0, \quad (2.63)$$

where $x = x_1$, Δ and Δ_2 are three-dimensional and two-dimensional (horizontal) Laplacians, and primes now denote not disturbances, but differentiation of profiles $U(z)$ and $\rho(z)$ on z. In the case of a homogeneous fluid of constant density, $\rho' = 0$ and Eq. (2.63) reduces to Eq. (2.38) with $\nu = 0$ (cf. also Eq. (2.53)). Seeking for normal modes and therefore using Eqs. (2.39) and (2.40) in which the variables are made dimensionless with length and velocity scales H and U_0 characterizing the vertical thickness of the flow and its typical velocity, we obtain the equation

$$(U - c)\left(W'' - k^2 W\right) - U''W - \frac{\rho'}{\rho}\frac{gH}{U_0^2}\frac{k^2 W}{k_1^2(U - c)} + \frac{\rho'}{\rho}[(U - c)W' - U'W] = 0, \quad (2.64)$$

where now $U = U(\zeta) = U(z)/U_0$ is dimensionless and primes denote derivatives with respect to dimensionless length $\zeta = z/H$. Equation (2.64) generalizes Rayleigh's equation (2.48), to which it reduces when $\rho' = 0$. Here c is the unknown eigenvalue, and the boundary conditions require W to vanish at solid walls and at infinity. In the case of discontinuous profiles $U(z)$ and $\rho(z)$ (which attracted considerable attention in the past but will not be considered here), special boundary conditions must be satisfied at the discontinuity points; see, e.g., Drazin and Howard (1966); Dikii (1976); and Drazin and Reid (1981).

Equation (2.64) has many similarities to Rayleigh's equation: here too if c is a complex eigenvalue then the complex conjugate value c^* is also an eigenvalue (corresponding to the eigenfunction W^*) and the coefficient of the highest-order derivative necessarily becomes zero at some point in the flow, making this point singular. Note also that in the case of stratified fluid, the singularity at the point where $U(z) = c$ survives even if $\nu \neq 0$; see, e.g., Dikii (1960b). If both the viscosity and the heat or mass diffusivity differ from zero, we obtain a non-singular eigenvalue problem, but this time it is of the sixth order; see, e.g., Koppel (1964); and Herron (1980).[5]

Equations (2.63) and (2.64) contain three terms proportional to ρ'/ρ. The term which is proportional to g describes the influence of density variations on the buoyancy (or gravitational lowering) of fluid particles while the other two terms describe

[5] It was also shown by Herron that if the thickness H of the fluid layer is finite, the general results by Di Prima and Habetler (1969) can be applied to this non-singular eigenvalue problem of the sixth order to show that here an infinite sequence of discrete eigenvalues c_j always exists, and the set of corresponding eigenfunctions $W_j(\zeta)$ is complete in the functional space of all admissible functions $W(\zeta)$. In the case of a singular eigenvalue problem, such a statement is of course incorrect.

the influence of these variations on particle inertia. In many applications $\rho(z)$ varies with height much more slowly than $U(z)$, so that $|\rho'/\rho| \ll 1$ (since the length scale H determines the distance over which the velocity U changes significantly), but the Froude number $\mathrm{Fr} = U_0/(gH)^{1/2}$ is also much smaller than 1 and therefore

$$Ri^* = -\frac{gH}{U_0^2}\frac{\rho'}{\rho} = -\frac{gH^2}{U_0^2}\frac{1}{\rho}\frac{d\rho}{dz}, \tag{2.65}$$

is not negligibly small. (Since U_0/H is the typical scale for the vertical shear dU/dz, Ri^* is, in fact, the *overall Richardsom number* of the flow.) In such cases, we can neglect the terms of equations (2.63) and (2.64) which contain ρ'/ρ but do not contain g. Then, we obtain the simpler equations

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 \Delta w - U''\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial w}{\partial x} = \frac{\rho'}{\rho} g\Delta_2 w, \tag{2.63$'$}$$

$$(U-c)(W'' - k^2W) - U''W = \frac{gH}{U_0^2}\frac{\rho'}{\rho}\frac{k^2}{k_1^2(U-c)}W, \tag{2.64$'$}$$

(recall that in Eq. (2.63′) all the variables are dimensional while in Eq. (2.64′) they are dimensionless). The approximation leading to Eqs. (2.63′–2.64′) is similar to the Boussinesq approximation (and will simply be called the Boussinesq approximation below); it reduces to the ordinary Boussinesq approximation in cases where the density variations are due only to variations of temperature. Equation (2.64′) represents the simplest generalization of Rayleigh's equation to the case of nonhomogeneous fluid; it is often called the *Taylor-Goldstein equation* since it was used in early works by Taylor (1931); and Goldstein (1931) (but in fact was simultaneously introduced in a more general form also by Haurwitz (1931)).

Yih (1955) (see also Drazin and Reid (1981), Sect. 44) applied Squire's transformation $k_1 \to k$, $k_2 \to 0$ to Eq. (2.64′) (which does not differ from Eq. (2.64) in this respect). This transformation shows that to each three-dimensional wave-like disturbance with the horizontal wave numbers (k_1, k_2) there corresponds a two-dimensional normal mode with the same value of complex streamwise velocity c but with wave numbers $((k_1^2 + k_2^2)^{1/2}, 0)$ and smaller Froude number $\mathrm{Fr}_2 = k_1\mathrm{Fr}/(k_1^2 + k_2^2)^{1/2}$ (i.e., smaller velocity scale U_0). Thus, if the three-dimensional wave is unstable, i.e., $\Im m c = c^{(i)} > 0$, then the corresponding two-dimensional wave is also unstable and has magnified growth rate $\left(k_1^2 + k_2^2\right)^{1/2} c^{(i)}$ (instead of $k_1 c^{(i)}$). Therefore, the most rapidly growing (i.e., most unstable) normal mode must be two-dimensional; for this reason only such modes are usually considered. (Restriction to two-dimensional disturbances does not seem to be justified at present; this circumstance will be explained in the next chapter but at this place we shall pay no attention to it.) For two-dimensional modes the eigenvalue problem arising in the study of hydrodynamic stability for plane-parallel flows of inviscid stratified fluid has the form

$$(U-c)\left(W'' - k^2W\right) - U''W + [Ri^*/(U-c)]W$$

$$+ (\rho'/\rho)\left[(U-c)W' - U'W\right] = 0, \tag{2.66}$$

or in cases where the Boussinesq approximation is used,

$$(U-c)\left(W'' - k^2W\right) - U''W + [Ri^*/(U-c)]W = 0, \tag{2.66$'$}$$

(this simplified form of equation (2.64′) is usually called the Taylor-Goldstein equation). In Eqs. (2.63) and (2.63′), the Laplace operators Δ and Δ_2 must be replaced by the operators $(\partial^2/\partial x^2 + \partial^2/\partial z^2)$ and $\partial^2/\partial x^2$ in cases where only two-dimensional disturbances are considered. Note also that in the case of a two-dimensional disturbance its velocity field $\{u(x,z,t), 0, w(x,z,t)\}$ may be expressed in terms of the stream function $\psi(x,\ z,\ t)$ (of the form $U_0H\ \exp[ik(\zeta - c\tau)]\psi(\zeta)$, where $\zeta = x/H$, $\zeta = z/H$, and $\tau = U_0t/H$, if a normal mode is considered). The functions $\psi(x,\ z,\ t)$ and $\psi(\zeta)$ clearly satisfy the same equations as $w(x,\ z,\ t)$ and $W(\zeta)$, with boundary conditions $\partial\psi/\partial x = 0$ (i.e., $\psi =$ const.) and $k\psi = 0$ on the rigid boundaries and at infinity.

Rayleigh (1883) considered the simple but rather important case of the stability problem for a stationary stratified fluid layer (i.e., $U(z) = 0$). Here the Taylor-Goldstein equation (2.65′) written in ordinary dimensional variables (note that no velocity scale U_0 exists in this case) takes the form

$$c^2\left(W'' - k^2W\right) + N^2W = 0, \tag{2.67}$$

where $N = (-g\rho'/\rho)^{1/2}$ is the so-called *Brunt- Väisälä frequency* (which is real in the case of stable stratification where $\rho' = d\rho/dz < 0$). Rayleigh proved that a stationary layer is stable if and only if $N^2 > 0$ everywhere (i.e., the density is decreasing with height and stratification is everywhere stable). In this case Eq. (2.67) determines the velocity spectrum of *internal gravity waves.* Rayleigh determined this spectrum for the special case where $\rho(z) = \rho_0\ \exp\ (-z/H)$ and hence $N^2 = g/H =$ const. More general results for internal waves can be found, for example, in the books by Krauss (1966); and Yih (1980) see also Sect. 44.2: in Drazin and Reid (1981) and the collection edited by Mobbs and King (1993).

In the case of an arbitrary velocity profile $U(z)$, it is also natural to think that the flow will be definitely unstable if $d\rho/dz$ is somewhere positive, showing that light fluid is located below heavier fluid. Therefore, the most interesting case is where $d\rho/dz < 0$ (and hence $N^2(z) > 0$ and $\mathrm{Ri}^* > 0$) everywhere, and the stabilizing effect of the decrease of density with elevation is competing with the destabilizing effect of velocity shear. To study this case we shall follow Howard (1961) and transform the equation (2.66) for $W(\zeta)$ into an equation for the function $F_n(\zeta) = (U-c)^{n-1}W$. (In fact Howard used Boussinesq's approximation and therefore replaced Eq. (2.66) by the simpler Eq. (2.66′), but this replacement produced only minor changes in the derivations and did not change the main corollaries.) It is easy to verify that Eq. (2.66) implies the following equation for F_n:

$$\begin{aligned}&[\rho(U-c)^{2(1-n)}F'_n]' - \{n[\rho(U-c)^{1-2n}U']' + \rho k^2(U-c)^{2(1-n)}\\ &\quad + \rho(U-c)^{-2n}[n^2(U')^2 - Ri^*]\}F_n = 0.\end{aligned} \tag{2.68}$$

Multiplying this equation by the complex conjugate function F^*_{n} and integrating it from $\zeta = 0$ to $\zeta = 1$ (i.e., through the full thickness of the fluid layer) we obtain the identity

$$\begin{aligned}\int_0^1 \{\rho(U-c)^{2(1-n)}(\left|F_n'\right|^2 + k^2|F_n|^2) + n[\rho(U-c)^{1-2n}U']'|F_n|^2 \\ + \rho(U-c)^{-2n}[n^2(U')^2 - Ri^*]|F_n|^2\}d\zeta = 0.\end{aligned} \tag{2.69}$$

(If the Boussinesq approximation is used, then ρ can be taken outside the square brackets in the first two terms of Eq. (2.68) and the second term of the integrand in Eq. (2.69).) Considering Eq. (2.69) with different values of n, Howard derived a number of stability properties (see also Drazin and Howard (1966), Sect. V).

Putting $n = 1$, Howard obtained a curious (though not very useful) estimate for $c^{(i)} = \Im mc$, first found by Synge (1933). For $n = 0$, Eq. (2.69) can be represented in the form (2.49′) with the equality sign replaced by the inequality sign $>$. It is clear that in this case the proof of Howard's semicircle theorem given in Sect. 2.82 can be repeated without any change; the additional term containing Ri* (which leads to the necessity of replacing $=$ with $>$) only strengthens the inequalities used, and was in fact omitted by Howard. Note also that for $n = -1$, Eq. (2.69) takes especially simple form if the Boussinesq approximation is used and $U'(z) = \text{constant}$ (i.e., for a stratified Couette flow). It is easy to see that it then follows from this equation with $n = -1$ that if $\mathrm{Ri} \leq -2$ everywhere within such a flow, real eigenvalues c cannot exist (this remark is due not to Howard but to Kuo (1963); see Sect. 3.23 in Chap. 3 for more details). Finally, putting $n = 1/2$ in (2.69) and taking the imaginary part of the identity obtained, Howard found that

$$-c^{(i)}\int_0^1 \rho\left\{\left|F_{1/2}'\right|^2 + k^2\left|F_{1/2}\right|^2 - \left[\frac{1}{4}(U')^2 - Ri^*\right]\frac{\left|F_{1/2}\right|^2}{|U-c|^2}\right\}d\zeta = 0 \tag{2.70}$$

and hence, if $c^{(i)} \neq 0$,

$$\int_0^1 \rho\left[\frac{1}{4}(U')^2 - Ri^*\right]\frac{\left|F_{1/2}\right|^2}{|U-c|^2}d\zeta > 0. \tag{2.71}$$

Therefore, *no complex eigenvalues c can exist, if*

$$\frac{Ri^*}{(U')^2} = -\frac{g d\rho/dz}{\rho[dU/dz]^2} = Ri \tag{2.72}$$

is greater than 1/4 everywhere in the flow. This is the sufficient condition for the absence of unstable two-dimensional normal modes of disturbance in a plane-parallel flow of stratified fluid, which was first proved in a more complicated way by Miles (1961) under the additional condition that the functions $U(z)$ and $\rho(z)$ are both analytic (this condition was later found to be unnecessary).

Let us now make some remarks about these results. It was noted above that in his derivation of the semicircle theorem, Howard neglected the term characterizing stratification. However it is natural to expect that stable stratification must contract the domain of possible complex values of c. This expectation led to the appearance of a series of papers devoted to the refinement of the semicircle theorem for flows of stably stratified fluids, with the aid of estimates for the term containing Ri*. Apparently the first results of this type were given by Banerjee and Jain (1972); Banerjee et al. (1974); and Banerjee et al. (1978), who found reduced domains for c—values dependent on both stability characteristic and velocity distribution $U(z)$. Then Kochar and Jain (1979) used two equations (2.69) with $n = 0$ and $n = 1/2$ to prove the following *semiellipse theorem,* which sharpened the theorem by Howard: *In the case of a stably stratified plane-parallel flow of inviscid incompressible fluid the complex phase velocities* c *of unstable normal modes of disturbance must lie inside a semiellipse in the upper half-plane of the complex plane, with a major axis coinciding with the segment* $U_{\min} \leq c^{(r)} \leq U_{\max}$ *of the real axis and a minor axis whose length is smaller than* $U_{\max} - U_{\min}$ *by the factor* $\left\{[1 + (1 - 4Ri_m)^{1/2}]/2\right\}^{1/2}$, *where* Ri_m *is the minimum value of the Richardson number* (*2.3*). (It is assumed here that $\mathrm{Ri}_m \leq 1/4$ since it is known that otherwise the unstable normal modes cannot exist at all.) However, the stated semiellipse theorem is in some respect weaker than Miles' result, since the theorem gives a semielliptic domain even for $\mathrm{Ri}_m = 1/4$ and Miles proved that no unstable wave-like disturbances can exist if $\mathrm{Ri}_m = 1/4$. Later, therefore, Jain and Kochar (1983) used more precise estimates for the terms of Eqs. (2.69), and obtained an improved form of the semiellipse theorem, specifying a domain for values of $c^{(\mathrm{i})}$ which disappears at $\mathrm{Ri}_m = 1/4$. A further refinement of the Howard-Kochar-Jain theorem was made by Makov and Stepanyants (1984); they took into account that the available computations of eigenvalues c for specific velocity and density profiles $U(z)$ and $\rho(z)$ (e.g. those by Drazin and Howard (1966); Turner (1973); and Gossard and Hooke (1975)) showed that the location of these eigenvalues in the complex plane depends strongly on the wave number k of the disturbance considered. Therefore Makov and Stepanyants repeated, more carefully, all the manipulations with identities (2.68) made by Howard and improved by Kochar and Jain, preserving and estimating accurately the terms proportional to k^2, which were omitted by previous authors. As a result they obtained new boundaries for the domains of eigenvalues c with $\Im\mathrm{m}c > 0$, corresponding to different values of k. The new domain for $k = 0$ coincides with the semiellipse found by Kochar and Jain (1979) but for $k \neq 0$ it proves to be located inside this semiellipse, being considerably smaller in area and agreeing much better with numerical and laboratory determinations of the eigenvalues c. Some other k -dependent bounds for the eigenvalues c of the Taylor-Goldstein equation were given by Craik (1972); while Russell (1994) presented some examples where this equation can be explicitly solved.

Note now that Miles' condition (namely, $\mathrm{Ri} > 1/4$ everywhere) is only sufficient, but not necessary, for the absence of complex eigenvalues c of the Taylor-Goldstein eigenvalue problem (both in its original form (2.66′) or the more precise form (2.66) that does not use the Boussinesq approximation). In fact the real spectrum of eigenvalues, which includes not only discrete values c but also the continuous spectrum which is always present, is not determined by the minimum value of Ri alone but depends strongly on the exact forms of the profiles of $\rho(z)$ and $U(z)$ (and also varies with the wave number k). For example, the case where $U(z) = az, \rho(z) = \rho_0 \exp(-bz), 0 \leq z < \infty$, was studied by Taylor (1931), who found the arguments showing that apparently if $\mathrm{Ri} = gb/a^2 > 1/4$, then an infinite sequence of real eigenvalues c_j exists, but there are no complex eigenvalues, while if $0 < \mathrm{Ri} < 1/4$ then there are no eigenvalues (either real or complex) at all. (Taylor did not prove these statements rigorously; they were proved only much later independently by Dyson (1960); and Dikii (1960c).) Since it was known from the classical paper by Richardson (1920) that increasing Ri increases the stability of the flow, the sharp change of the eigenvalue spectrum at $\mathrm{Ri} = 1/4$ was interpreted at first by Prandtl and Schlichting as an indication that the flow is stable if $\mathrm{Ri} > 1/4$ but unstable if $\mathrm{Ri} < 1/4$ (see, e.g., Prandtl (1949); and Schlichting (1959, 1979)). However, later Case (1960b); and Dikii (1960b) independently analyzed the general solution of the appropriate initial-value problem and showed that the flow considered by Taylor is in fact stable with respect to any infinitesimal two-dimensional disturbance (and hence also to any normal mode of disturbance, whether two- or three-dimensional) at any non-negative value of Ri (this result will be discussed at greater length in the next chapter). Hence we see that violation of Miles' condition does not imply that unstable normal modes necessarily exist.

Several additional sufficient conditions for non-existence of unstable two-dimensional wave-like disturbances were given by Yih (1970, 1974a, 1980). This author showed that *if both the functions* $U(z)$ *and* $\rho(z)$ *are analytic,* $\rho(z)$ *is monotone decreasing and* $U(z)$ *is monotone increasing with z, and either* $(\rho U')' > 0$ *and* $(\log \rho)'' > 0$ *everywhere or* $U'' < 0$ *and* $(\log \rho)'' < 0$ *everywhere, then no complex eigenvalues c can exist.* In the case of homogeneous fluid $\rho\,(z) = \text{constant}$; hence in this case both the indicated conditions reduce to Rayleigh's condition of the absence of inflection points on the velocity profile, which is valid for any (analytic or nonanalytic) profile $U(z)$. Therefore it is natural to hope that the analyticity assumptions used by Yih are in fact also not necessary for validity of the stated results.

The search for sufficient conditions for the existence of unstable normal nodes of disturbance in density-stratified inviscid shear flows has not so far been very successful. Nevertheless, some interesting (though relatively restricted) conditions were given by Yih in the papers and book indicated above, and by Baines and Mitsudera (1994) who followed Yih (1974b) in studies of the physical mechanism of stratified-flow instability. Note again in this respect (cf. Sect. 2.82 above) that existence of an eigenvalue c with $\Im\mathrm{m}\, c > 0$ definitely shows that the flow considered is unstable with respect to infinitesimal disturbances, while non-existence of such eigenvalues only signifies the non-existence of exponentially growing disturbances of a special form.

Many additional results and calculated specific examples from linear stability theory for plane-parallel stratified flows (mostly, but not always, considered to be inviscid) can be found in particular in the papers by Drazin and Howard (1966); Howard and Maslowe (1973); Strehle (1979); Engevik et al. (1985); Caulfield (1994); and Baines and Mitsudera (1994) and books by Turner (1973); Gossard and Hooke (1975); Dikii (1976); Yih (1980) and Drazin and Reid (1981). These

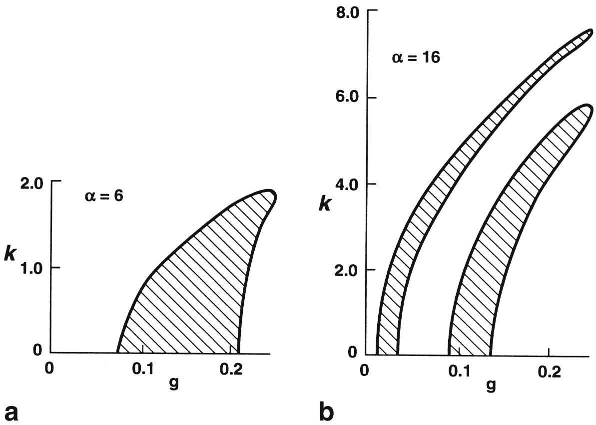


Fig. 2.21 Neutral-stability curves in the plane (g, k) (where all the parameters and variables made dimensionless so that $(-\rho'/\rho) = \text{Ri}^*$) for stratified Couette flow with $U(z) = z$ and $-\rho'/\rho = 1 + \alpha^2 z^2$, $-1 \leq z \leq 1$, in cases where $\alpha = 6$ **a** and $\alpha = 16$ **b**. (After Howard and Maslowe (1973); the regions of instability are shaded in the figure.)

papers and books contain great numbers of important supplementary references and many graphs of 'neutral curves' in the (Fr, k)-plane (or planes with coordinates $(g, k) \infty (\text{Fr}^{-2}, k)$, (N^*, k), or (Ri^*, k), where N^* and Ri^* denote some overall values, independent of z, of the Brunt-Väisälä frequency and Richardson number) separating the region where complex eigenvalues c with $\Im\text{m}\, c > 0$ exist from the region where there are no complex eigenvalues c at all. It was found, in particular, that the 'neutral curves' for plane-parallel flows of stratified fluids are often multiply-connected, i.e., consist of several isolated curves (see, e.g., Fig. 2.21, taken from the paper by Howard and Maslowe (1973)).

2.8.4 Remarks Concerning Linear Stability Theory of Axisymmetric and Some Other Non-Plane-Parallel Flows

Up to this point, only the stability of plane-parallel flows has been considered in this section though arbitrary parallel flows were mentioned in its title. A general parallel steady flow (i.e., one having parallel streamlines) has a velocity field of the form $\boldsymbol{U}(\boldsymbol{x}) = \{U(y, z), 0, 0\}$ where the points (y, z) pass through all the plane Oyz or a given part of it. (The function $U(y, z)$ must satisfy the two-dimensional Laplace equation $\Delta_2 U = 0$ if the fluid has non-zero viscosity but can be arbitrary in the case of an inviscid fluid.) For plane-parallel flows the function $U(y, z)$ depends on only one coordinate (denoted by z above). Now we shall consider another important case, that of axisymmetric flows where $U(y, z) = U(r)$ depends only on $r = (y^2 + z^2)^{1/2}$.

As in the stability problem of Sect. 2.6 (where the primary flow was also axisymmetric but not parallel), it is natural to use cylindrical coordinates r, ϕ, x (in Sect. 2.6 the coordinate x was replaced by z).

Transforming Eqs. (2.7) to these coordinates, we obtain the following system of four linear partial differential equations with four unknowns, u'_r, u'_ϕ, u'_x and u', which are functions of the variables r, ϕ, x and t:

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u'_r=-\frac{1}{\rho}\frac{\partial p'}{\partial r}+\nu\left(Lu'_r-\frac{2}{r^2}\frac{\partial u'_\phi}{\partial \phi}\right),$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u'_\phi=-\frac{1}{\rho r}\frac{\partial p'}{\partial \phi}+\nu\left(Lu'_\phi+\frac{2}{r^2}\frac{\partial u'_r}{\partial \phi}\right),$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u'_x+\frac{dU}{dr}u'_r=-\frac{1}{\rho}\frac{\partial p'}{\partial x}+\nu\left(Lu'_x+\frac{u'_x}{r^2}\right),$$

$$\frac{\partial r u'_r}{\partial r}+\frac{\partial u'_\phi}{\partial \phi}+\frac{\partial r u'_x}{\partial x}=0, \tag{2.73}$$

where

$$L=\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial x^2}. \tag{2.73$'$}$$

Seeking for normal modes of disturbance, we must represent the unknown function in a form similar to (2.13) with z replaced by x; moreover, as in Sects. 2.81–2.83, we shall introduce the phase velocity c satisfying the relation $\omega = kc$. Then the partial differential equations with the unknowns u'_r, u'_ϕ, u'_x and p'/ρ will be transformed into ordinary differential equations in the unknown functions $f^{(r)}$, $f^{(\phi)}$, $f^{(x)}$ and g of one variable r. It is easy to see that in the case of an inviscid fluid with $\nu = 0$ these equations take the form

$$ik(U-c)f^{(r)}+g'=0,$$

$$ik(U-c)f^{(\phi)}+(in/r)g=0,$$

$$ik(U-c)f^{(x)}+U'f^{(r)}+ikg=0,$$

$$ikf^{(x)}+[f^{(r)}]'+f^{(r)}/r+inf^{(\phi)}/r=0, \tag{2.74}$$

where now (and also below) primes denote differentiation on r. Eliminating all the unknown functions except $f^{(\mathrm{r})}(r) = F(r)$ (here $F(r)$ is only a new shorter notation) from this system, we obtain a single second-order differential equation of the form

$$(U-c)\frac{d}{dr}\left\{\frac{r}{n^2+k^2r^2}\frac{d(rF)}{dr}\right\}-(U-c)F-\frac{d}{dr}\left(\frac{rU'}{n^2+k^2r^2}\right)rF=0. \tag{2.75}$$

The corresponding boundary conditions state that $F(r) = 0$ on the solid boundaries of the flow (which clearly must have cylindrical form) and $F(r) \to 0$ as $r \to \infty$ in the case of unbounded flow; while $F(0) = 0$ if $n \neq 1$, and $F(0)$ is bounded if $n = 1$, in the case of a flow enveloping the axis $r = 0$ (see, e.g., Batchelor and Gill (1962)). Equation (2.75) and the indicated boundary conditions form the eigenvalue problem, which determines the possible values of c.

Equation (2.75) is due to Rayleigh (1892), who founded the theory of hydrodynamic stability of ideal (inviscid) fluid flows in the axisymmetric case too. Note that, as in the case of plane-parallel flows, when transforming Eqs. (2.74) (or their generalizations to the case where $\nu \neq 0$) into the single equation for the unknown $f^{(r)} = F(r)$, we lost solutions of the initial equations satisfying the conditions $f^{(r)} = 0$. However, it is easy to show that all the normal modes satisfying this condition are stable, i.e., are such that $\Im mc \leq 0$ (independently of the value of the viscosity ν; see Lew (1955); and Schade (1962)). Therefore, these supplementary solutions need not be considered when the method of normal modes is used for determination of the instability conditions.

Equation (2.75) is an analog of Rayleigh's equation (2.48) relating to the stability theory for plane-parallel flows of inviscid fluid. There is, however, an important difference between these two equations: Eq. (2.48) includes, except for c, only one supplementary parameter k, while two such parameters, k^2 and n^2, enter Eq. (2.75). This is because no analog of Squire's theorem is valid for axisymmetric parallel flows; therefore, it is impossible here to specialize to disturbances whose geometrical form depends on a single parameter. (The original papers by Rayleigh give the impression that he foresaw this difference since, studying the stability of plane-parallel flows, he considered only two-dimensional velocity disturbances, but passing to the axisymmetric case he considered general wave-like disturbances proportional to $\exp\{i(kx + n\phi - \omega t)\}$.)

Let us now divide Eq. (2.75) by $U - c$, and then multiply the resulting equation by the complex conjugate $(rF)^*$ of rF and integrate the imaginary part of the product over the whole range of r-values within the flow. Then we get the following result

$$c^{(i)} \int_{r_1}^{r_2} |q(r)|^2 Q'(r) dr = 0 \tag{2.76}$$

where $c^{(\mathrm{i})} = \Im mc$, $q(r) = rF(r)/(U - c)$, $Q(r) = rU'(r)/(n^2 + k^2 r^2)$, and r_1 and r_2 are the boundaries of the flow (the cases $r_1 = 0$ and/or $r_2 = \infty$ are not excluded). Therefore, the necessary condition for the existence of complex eigenvalues $c = c^{(r)} + ic^{(i)}$ where $c^{(i)} > 0$ (i.e. of exponentially growing wave-like disturbances) is that $Q'(r)$ should change sign at some point of the flow. This condition, which can be considered as the translation to the case of axisymmetric flows of Rayleigh's condition of the necessity of an inflection point in the velocity profile of a plane-parallel flow, was also demonstrated by Rayleigh.

Batchelor and Gill (1962) showed that the stronger necessary condition by Fjørtoft for the existence of exponentially growing wave-like disturbances in a plane-parallel inviscid flow can also be easily transferred to the case of inviscid axisymmetric parallel flows. For this we must only add to the real part of Eq. (2.75) multiplied by $rF^*/(U - c)$ the imaginary part of the same product multiplied by $(c^{(r)} - K)/c^{(i)}$, where K is some constant and it is assumed that $c^{(i)} \neq 0$. Integrating the sum over r we easily get the relation

$$\int_{r_1}^{r_2} |q(r)|^2 [U(r) - K] Q'(r) dr < 0 \tag{2.77}$$

which is similar to relation Eq. (2.52). It follows from Eq. (2.77) that a complex eigenvalue c cannot exist if there exists a constant K such that $[U(r)-K]Q'(r) \geq 0$ for any r (cf. Fjørtoft's related condition given in Sect. 2.82).

Multiplying Eq. (2.75) by $q^*(r) = [rF(r)/(U-c)]^*$ and then integrating the product with respect to r, Batchelor and Gill found the relation

$$\int_{r_1}^{r_2} (U-c)^2 \left\{ \frac{r}{n^2+k^2r^2} \left| q'(r) \right|^2 + \frac{1}{r} |q(r)|^2 \right\} dr = 0. \tag{2.78}$$

This equation is of the same form as (2.49′) though the function Φ now has a different meaning. Therefore Eq. (2.78) leads to the same consequences as Eq. (2.49). First of all, it is clear that Eq. (2.78) cannot be satisfied if c is real and outside the range of U. (If c is real and inside the range of U, the derivation of (2.78) fails, since $q(r)$ has a singularity at the point where $U(r) = c$.) Moreover, as in the case of Eq. (2.49) it follows from Eq. (2.78) that $c^{(r)} = \Re e c$ cannot lie beyond the range of possible values of $U(r)$ (this result was known to Rayleigh), and that Howard's semicircle theorem, which restricts the range of possible values of c in the complex c -plane in the case of a plane-pareallel primary flow, is also valid for an axisymmetric parallel primary flow (and has exactly the same formulation).

For the special case of axisymmetric disturbances (where $n = 0$), Schade (1962) derived some supplementary results concerning their stability. In particular he showed that the results of Tollmien (1935), concerning the simple sufficient conditions for inviscid instability of some important classes of plane-parallel flows, can be transferred to axiymmetric flows. (Note that in the case of sufficient conditions for instability it is enough to show that under the conditions considered there exists at least one exponentially growing disturbance of any form).

Stability studies for parallel axisymmetric flows of a viscous fluid ($\nu \neq 0$) are much more complicated. Here also it is possible to obtain, from Eqs. (2.73–2.73′), a system of four linear equations for the amplitudes $f^{(r)}, f^{(\phi)}, f^{(x)}$, and g of a normal mode, but now they will differ from (2.74) by a number of additional terms proportional to ν. These terms very much complicate the elimination of all the unknowns except one from the system, and render such an elimination not very useful since it always leads to a very cumbersome equation which does not yield to analysis. Therefore the elimination has usually been carried out on the assumption that only axisymmetric disturbances with $n = 0$ are relevant. In this case, clearly, Eqs. (2.73–2.73′) decompose into the system of three equations for the unknowns u'_r, u'_x and p' and one equation (the second Eq. (2.73)) for the unknown $u_\phi{}'$. This circumstance considerably simplifies the subsequent analysis (see, e.g., Synge (1938a), Pretsch (1941), and other papers cited in Sect. 2.94 below and in the book MY1 in connection with the examination of results of stability studies for a circular Poiseuille flow).

Let us now say a few words about the normal mode approach to the stability theory of general parallel steady flows with velocity field $\boldsymbol{U}(\boldsymbol{x}) = \{U(y, z), 0, 0\}$. For simplicity we shall restrict ourselves to consideration of flows of an ideal (inviscid) fluid with $\nu = 0$. The stability theory of general parallel flows of ideal fluid was begun by Hocking (1968); its subsequent development was stimulated by related geophysical problems (see, e.g., the papers by Blumen (1971, 1975) and references therein). Seeking for normal modes we must assume that the disturbances u', v', w' and p'/ρ are proportional to $\exp(ik(x-ct))$ with coefficients of proportionality (the wave amplitudes) dependent on y and z. Linearized dynamic equations for the disturbances and the incompressibility condition then give a system of four linear partial differential equations for the four amplitudes of velocity and pressure waves. Eliminating from this system all the unknowns except the amplitude $g = P(x, y)$ of the pressure wave, Blumen obtained the following equation for P

$$[U-c)^{-2}P_y]_y + [(U-c)^{-2}P_z]_z - k^2(U-c)^{-2}P = 0 \tag{2.79}$$

where the subcripts y and z denote differentiation on the indicated variables. The fluid dynamic equations imply that $v' \propto P_y, w' \propto P_z$. The rigid boundaries of the flow can consist of planes parallel to the Ox-axis or, more generally, have the form of a cylindrical surface of arbitrary cross-section with the axis parallel to Ox. The boundary condition on any smooth rigid wall has the form $\partial P/\partial n = 0$ where n denotes the direction normal to the wall. In particular, if the flow is bounded by solid walls at $y = y1$, $y = y2$, $z = z1$ and $z = z2$, the boundary conditions are:

$$Py = 0 \quad \text{for} \quad y = y1, y2, \quad Pz = 0 \quad \text{for} \quad z = z1, z2. \tag{2.80}$$

In the case of unbounded flow the boundary conditions at infinity will be

$$P_y \to 0, P_z \to 0 \quad as \quad (y^2 + z^2)^{1/2} \to \infty.$$

Consider the simplest case where the flow is bounded by the planes $y = y1, y2$; and $z = z1$, $z2$. Multiplying Eq. (2.79) by P^*, the complex conjugate of P, and then integrating over the flow field and using boundary conditions (2.80), we obtain the equation

$$\int_{y_2}^{y_1}\int_{z_1}^{z_2} (U-c^*)^2 \frac{|P_y|^2 + |P_z|^2 + k^2|P|^2}{|(U-c)^2|^2} dydz = 0. \tag{2.81}$$

(The same equation will be valid in the case of flow in a tube domain of arbitrary bounded or unbounded cross-section, with a smooth boundary; but in this case the integration will be taken over the corresponding cross-section.) Eq. (2.81) is again of the form (2.49′) and hence it implies the same corollaries as the latter equation. If c is real, then (2.81) clearly cannot be satisfied for c outside the range of $\mathbf{U} = U(y, z)$ (note that the derivation of (2.81) fails if $U - c$ vanishes somewhere). If c is complex, then it follows from (2.81) that Howard's semicircle theorem is valid here and has exactly the same formulation as in particular cases of plane-parallel or axisymmetric primary flows.

Having in mind possible geophysical applications, Blumen (1971, 1975) considered from the beginning the case of a general parallel flow of stratified fluid, having a density$\rho = \rho(z)$ which depends on the vertical coordinate z. As in the problem studied in Sect. 2.83 the dynamic equations will have the form (2.58–2.60), though now the primary laminar flow will have a velocity field of the form $u_i = u_i(y,z) = U(y,z)\delta_i 1$ (and the density and pressure fields will again be determined by Eqs. (2.61)). In geophysical applications it is often convenient to use different vertical and horizontal length scales H and L; therefore, Blumen normalized x and y by L, z by H, horizontal velocities U, c, u' and v' by the horizontal velocity scale U_0, and w' by U_0H/L; for $t, p'/\rho$ *and* ρ' the ordinary scales $L/U_0, U_0^2$ and ρ_0 were used. Assuming that the disturbances $u'v', w', p'/\rho$ and ρ' are proportional to $\exp[ik(x-ct)]$ (where here and later the symbols x and t, and also y and z, denote normalized dimensionless coordinates and time) and eliminating from the system of five equations for the five wave amplitudes all the unknowns except $g = P(y, z)$, Blumen obtained the following differential equation for P, generalizing equation (2.79):

$$\left\{\frac{P_y}{(U-c)^2}\right\}_y - \left\{\frac{k^2 Pz}{Ri* - (\gamma k)^2 (U-c)^2}\right\}_z - \frac{k^2 P}{(U-c)^2} = 0 \tag{2.82}$$

where Ri* denotes the overall Richardson number (2.65) and $\gamma = H/L$. Multiplying this equation by P^* and then integrating the product over the flow field (aasumed to be bounded by the planes indicated in (2.80)), we obtain

$$\int_{y_1}^{y_2}\int_{z_1}^{z_2} (U-c^*)^2 \left\{\frac{\left|P_y\right|^2 + k^2|P|^2}{\left|(U-c)^2\right|^2} + \frac{(\gamma k)^2 |P_z|^2}{\left|Ri* - (\gamma k)^2 (U-c)^2\right|^2}\right\} dydz$$

$$= \int_{y_1}^{y_2}\int_{z_1}^{z_2} \frac{Ri^* k^2 |P_z|^2}{\left|Ri* - (\gamma k)^2 (U-c)^2\right|^2} dydz. \tag{2.83}$$

If Ri* > 0, then Eq. (2.83) can be represented in the form (2.49′) with the equality sign = replaced by the inequality sign >. Exactly as in Sect. 2.83, it follows from this that Howard's semicircle theorem is again valid and has the same form as in the case of a plane-parallel flow of constant-density fluid.

Examples of more detailed investigations of stability properties for non-stratified and stratified parallel flows with velocity $U(y, z)$ depending on two variables can be found in the above-mentioned papers by Hocking and Blumen.

Simple general results also follow from the application of the normal-mode method to the study of stability for steady flows with straight streamlines, everywhere parallel to fixed plane (say the z-plane)but directed differently at different distances from this plane. In this case the velocity vector has the form $\mathbf{U}(\boldsymbol{x}) = \{U(z), V(z), 0\}$. Studies of such flows can have important geophysical applications since flows in atmospheric and oceanic boundary layers have velocity fields of this type (where z is the vertical coordinate); in aeronautical engineering such flows can be used as models of the boundary layers of swept wings.

Seeking for normal modes, we must assume that the disturbances u', v', w' and p'/ρ (for the time being the density is assumed to be constant) are proportional to exp

$\{i(k_1x+k_2y-\omega t)\}$ with coefficients (wave amplitudes) depending on z. It is easy to verify that under the assumption that the viscosity is equal to zero, the resulting equation for the coefficient $W(z)$ of the vertical velocity disturbance w' will have the form:

$$(kQ-\omega)(W''-k^2W)-kQ''W=0 \tag{2.84}$$

where $k=(k_1^2+k_2^2)^{1/2}, Q=Q(z)=[k_1U(z)+k_2V(z)]/k$ and primes denote differentiation on z, while in the case of a viscous fluid the zero on the right hand side of (2.84) must be replaced by the term $-i\nu(d^2/dz^2-k^2)^2W$ (see, e.g., Landahl and Mollo-Christensen (1992)). Writing $\omega=kc$ as usual, we can easily see that Eq. (2.84) differs from the classical Rayleigh equation (2.48) only by replacement of the function $U(z)$ by the function $Q(z)$ which depends on both components U and V of the velocity $U(x)$. Similarly, the equation for a viscous fluid differs from the classical O-S equation (written in dimensional variables) only by replacement of the velocity profile $U(z)$ by $Q(z)$. Therefore all the results from the linear theory of plane-parallel flows presented above in Sects. 2.81 and 2.82 can be reformulated for flows with velocity field $\{U(z), V(z), 0\}$ with the aid of simple replacement of the function $U(z)$ by $Q(z)$.

Let us now assume that the fluid is stratified, i.e., its density $\rho(z)$ depends on the vertical coordinate z, while the velocity $\boldsymbol{U}(\boldsymbol{x})$ is the same as above. In this case, as was shown by Russell (1994), the equation for the amplitude $W(z)$ of the normal mode of vertical velocity disturbance will have the form of the Taylor-Goldstein equation (2.66) (or (2.66′) if the Boussinesq approximation is used) but the function $U(z)$ will again be replaced by the function $Q(z)$. It follows from this that all the results derived by the normal-mode method for stratified plane-parallel flows of ideal fluid can be reformulated as results related to flows of a stratified ideal fluid having a velocity field $\boldsymbol{U}(\boldsymbol{x})$ of the form $\{U(z), V(z), 0\}$.

2.9 Applications of Normal-Mode Stability Analysis to Specific Parallel and Nearly Parallel Flows

In Sect. 2.8 several general results of the normal-mode stability analysis of parallel fluid flows were presented and discussed. In this section we shall consider applications of this analysis to particular classes of parallel and nearly parallel laminar flows such as flows in wide rectangular channels (“two-dimensional” in the first approximation), circular tubes, boundary layers, and free flows in jets, wakes and mixing layers.

2.9.1 Plane-parallel Flows Bounded by Solid Walls: Couette, Poiseuille and Couette-Poiseuille Flows

Steady plane-parallel flows in a domain bounded by two parallel solid walls at $z=0$ and z $=H$ are apparently the most simple laminar flows. Let the axis Ox be directed along the flow velocity $U(z)$. It has already been noted above that the Navier-Stokes equations show that the function $U(z)$ must be a polynomial of degree not greater than two. Hence the class of flows considered consists of *plane Couette flows* between two walls moving with respect to each other, *plane Poiseuille flows* where the walls are immovable and the flow is produced by a constant longitudinal pressure gradient *dP/dx,* and general *Couette-Poiseuille flows* where one of the walls is moving with respect to the other and there is also a constant streamwise pressure gradient of either sign.

A plane Couette flow with a linear velocity profile of the form $U(z) = U_0 z/H$ represents the simplest solution of the Navier-Stokes equations; in particular, the vanishing of the second derivative $U''(z)$ considerably simplifies both Rayleigh's and Orr-Sommerfeld's equations. Therefore it is not surprising that the study of the stability of this flow has attracted considerable attention for many years. In MY1 and in the book by Drazin and Reid (1981) a number of references can be found to the rather old stability investigations of Couette flow by Kelvin, Rayleigh, Orr, L. Hopf, von Mises, and Southwell and Chitty made at the end of the 19th or the beginning of 20th centuries. The methods used in these papers are often not sufficiently rigorous by modern standards and all of them study only some special, and not general, modes of disturbance. It is however important to note that in none of these works was any evidence found of instability (i.e., of existence of disturbances that do not tend to zero as $t \to \infty$). Note also that Orr (1907) showed that some disturbances can increase considerably when t is increasing from the initial value $t = 0$ before they begin to decrease tending to zero (this work will be discussed at greater length in the next chapter).

A number of subsequent applications of the normal-mode method of linear stability anlysis to a plane Couette flow is also considered in the above-mentioned two books. Therefore, here we shall mention only a few works not indicated in MY1: Shtern (1969, 1970); Davey (1973); Gallagher (1974); Reid (1979); and Davis and Morris (1983). In these papers, and older papers mentioned in MY1, various asymptotic and numerical methods were applied to the Couette-flow Orr-Sommerfeld equation; moreover, Reid (1979) found some exact solutions of this equation. Some of the papers also cantain interesting mathematical results; for instance, Grohne (1954) proved the existence of a sequence of higher modes of wave-live disturbances in a plane Couette flow at given k and Re. Later, the complex eigenvalues c (k, Re) corresponding to these higher modes were precisely computed by a number of investigators (see, e.g., Gallagher's paper (1974); and the books by Betchov and Criminale (1967); Goldshtik and Shtern (1977); and Drazin and Reid (1981)). However, in all the published work only stable wave-like disturbances were found (i.e., the imaginary parts of all the eigenvalues c_j were negative). Therefore, most investigators were convinced that no unstable infinitesimal wave-like disturbances exist in plane Couette flow, even before the rigorous proof of this statement was finally given by Romanov (1971, 1973) (quite different proof was later indicated by Herron (1991)). This discovery seems to contradict the experimental data showing that any steady laminar flow becomes unstable at large enough values of Re (recall in particular the experimental studies of plane Couette flows described in Sect. 2.1). To explain this contradiction it was usually assumed that a plane Couette flow is unstable for finite disturbances (this last assumption was first formulated by Rayleigh in 1914). It will, however, be shown in the next chapter that the apparent contradiction can also have quite another explanation, related to some general defects of the normal-mode method of the linear stability analysis.

Let us now consider stability studies of *plane Poiseuille flow* with a parabolic velocity profile of the form $U(z) = U_{max}[1 - (2z/H - 1)^2]$ where $U_{\max}$ is the maximum flow velocity in the center of the channel. (Recall that a plane Poiseuille flow is usually considered as a natural model of flow in a wide enough channel of rectangular cross-section.) Since $U''(z) = -8U_{max}/H^2$ does not vanish here, it was generally assumed for many years that a flow of ideal fluid with such a velocity profile is stable with respect to infinitesimal disturbances, by virtue of Rayleigh's theorem. (In the next chapter we shall discuss questions about the validity of this assumption at greater length.) It seemed also natural to think, in the early years of stability studies, that viscosity can only increase the stability of the flow.[6] At the same time, experiments definitely showed that laminar flows in wide rectangular channels are always unstable at large values of Re (again see Sect. 2.1). Therefore the mathematical study of stability of a plane Poiseuille flow attracted much attention from quite early times (an incorrect proof of stability of such flow was published by Kelvin in 1887). The most important early paper devoted to this subject was due to Heisenberg (1924), who investigated in detail the asymptotic behavior of the solution of the corresponding Orr-Sommerfeld equation with large value of Re (i.e., small ν) and proved that for sufficiently large Reynolds numbers a plane Poiseuille flow will become unstable. This result, which confirmed the statement that viscosity can cause instability, seemed paradoxical to many people; since it was obtained by sophisticated mathematical procedures, and on a physical (rather than *n* mathematical) level of rigor, it aroused serious doubt for a long time. Another kind of approximate analysis was applied to the Orr-Sommerfeld equation by Tollmien (1929, 1930), but this author did not consider Poiseuille flows. Only in 1945 did Lin's careful calculations, based on asymptotic expansions of a special type, confirm (and make more precise) Heisenberg's conclusion on the instability of Poiseuille flow for large (but finite) Re (see also Lin (1961a)). Later it was also shown that Heisenberg's method of asymptotic analysis can rather easily be made rigorous and quite useful (see, e.g., Drazin and Reid (1981) and Tsugé and Sakai (1985)).

The main improvement of Heisenberg's results by Lin consists of sufficiently accurate computation of the shape of the neutral stability curve $\max_j [\Im m c_j(k, \mathrm{Re})] = 0$ in the (k, Re) plane. This curve corresponds to neutrally stable disturbances and separates the region of points (k, Re) such that only stable two-dimensional wave-like disturbances with a wave number k can exist for given Re, from the region of points

[6] Note that this "natural idea" did not seem to be completely evident to Reynolds (1883) [see also Drazin and Reid (1981), p. 124] and its groundlessness was convincingly shown rather early by Prandtl (1921, 1922) in his studies of parallel flows with velocity profiles consisting of segments of straight lines. However, the physical mechanism of the very interesting phenomenon of instability caused by viscosity remains to be obscure and leading to conflicting opinions up to now [see, e.g., the old arguments by Prandtl (1921) and Lin (1961a, Chap. 4) and the recent discussion by Baines et al. (1996)]. Therefore it is not surprising that earlier the suggestions of such a phenomenon permanently aroused strong disrust.

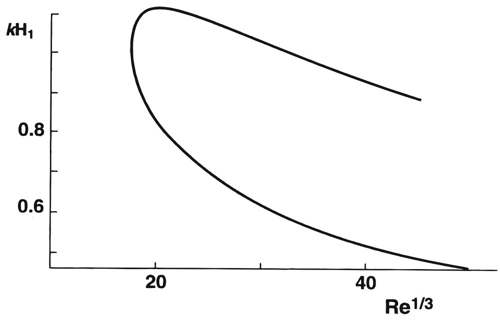


Fig. 2.22 The form of the neutral curve in the (*k,* Re) plane for a plane Poiseuille flow according to calculations of Lin (1945, 1961a)

(*k,* Re) corresponding to possible unstable two-dimensional waves. Later this neutral curve was recalculated by a number of authors using the newly-available electronic computers and different numerical techniques. The results obtained were, as a rule, sufficiently close to those of Lin shown in Fig. 2.22 (here, as in Sect. 2.1, $\mathrm{Re} = UH_1/\nu$ is formed from the maximal velocity $U = U_{max}$ and the channel half-width $H_1 = H/2$). The point of the neutral curve which is farthest to the left determines the values of the critical Reynolds number $\mathrm{Re_{cr}}$ and critical wave number k_{cr}; according to Lin $\mathrm{Re_{cr}} \approx 5300$, $k_{\mathrm{cr}} \approx 1/H_1$. Several examples of close (but not identical) estimates for $\mathrm{Re_{cr}}$ and k_{cr} obtained by other authors during the Fifties and Sixties are presented in MY1; however, the most precise results were found a little later by Orszag (1971) (and were confirmed also by Chock and Schechter (1973) who used another numerical method): $\mathrm{Re_{cr}} = 5772.22$, $k_{\mathrm{cr}} = 1.02/H_1$ (the corresponding form of the neutral curve differs very little from that shown in Fig. 2.22). As $\mathrm{Re} \to \infty$, both branches of the neutral curve (upper and lower) tend to zero (the upper branch as $\mathrm{Re}^{-1/11}$, and the lower as $\mathrm{Re}^{-1/7}$; see, e.g., Lin (1961a)). Thus, as Re increases, wave disturbances with fixed but not too large k lie first in the stability region (i.e., are damped), then fall in the region of instability, and, finally, again prove to be stable. Therefore, when $\mathrm{Re} \to \infty$ (i.e., $\nu \to 0$) all wave-like disturbances become stable.

Shen (1954) and Grosch and Salven (1968) calculated the form of the curves $\max_j[\Im m c_j(k, \mathrm{Re})] = \mathrm{const.}$, which determine the rates of growth for unstable wave-like disturbances of plane Poiseuille flow at different values of k and Re. Grohne (1954) proved that the Orr-Sommerfeld equation also has an infinite number of higher normal modes in the case of the basic Poiseuille velocity profile. He carried out the first eigenvalue computations for the four different modes and found that all the eigenvalues c corresponding to higher modes have negative imaginary parts (i.e., correspond to damped wave disturbances) at any values of k and Re. Hence, only the first eigenvalue $c1$, corresponding to the first mode, has an imaginary part that becomes positive for some values of k if Re is high enough. This result was confirmed by a number of other authors who studied the higher eigenvalues c_j in more detail (and discovered the existence ofseveral families of higher modes); see e.g. Grosch and Salven (1968); Orszag (1971); Mack (1976); Antar (1976); Stocker and Duck (1995). In the books by Goldshtik and Shtern (1977); and Drazin and Reid (1981) some information about the higher modes of eigenvalues is presented, together with several examples of eigenfunctions corresponding to eigenvalues c (*k,* Re).

It was indicated in Sect. 2.1 that in some experiments, where special measures were taken to reduce the level of pre-existing disturbances, the flow in a wide rectangular channel was maintained laminar at values of Re of the order of (or even slightly greater than) the value $\mathrm{Re}_{\mathrm{cr}} \approx 6000$ determined by normal-mode stability analysis of plane Poiseuille flow. However, these experiments are quite exotic; under normal conditions transition to turbulence is usually observed in wide rectangular channels at Reynolds numbers as low as $\mathrm{Re} \approx 1000$. Such early manifestations of instability were for many years attributed to the effects of nonlinear terms in the equations of motion. However, recently a quite different explanation of the observed failure of the linear normal-mode analysis was proposed; it will be considered in the following chapter of this book.

The most general steady, plane-parallel flow of viscous fluid in a layer $0 \le z \le H$ bounded by solid walls is the combined *plane Couette-Poiseuille flow* with a (dimensionless) velocity profile of the form

$$U_1(z_1) = (4 - A)z_1 - (4 - 2A)z_1^2, \quad 0 \le z_1 \le 1, \quad z_1 = z/H,$$
$$U_1\,(z_1) = U\,(z_1 H)\,/\,U\left(\frac{1}{2}H\right).$$

Here $A = U_1(1) = U(H)/U\left(\frac{1}{2}H\right)$ is the nondimensional parameter which is equal to zero for pure Poiseuille flow and to two for pure Couette flow. The stability of Couette-Poiseuille flow was studied by the normal-mode method by Potter (1966); Hains (1967); and Reynolds and Potter (1967), and their results are in substantial agreement. Some additional information about eigenvalues and neutral curves for this flow can be found in papers by Cowley and Smith (1985); and Stocker and Duck (1995).

According to these studies, superimposing a Couette flow on a Poiseuille flow always has a stabilizing effect, and increases $\mathrm{Re}_{\mathrm{cr}}$ considerably. (For example, changing the upper wall velocity from zero (for Poiseuille flow) to 10 % of the maximum Poiseuille velocity leads to an increase in $\mathrm{Re}_{\mathrm{cr}}$ of 236 %.) Moreover, when A increases from the value $A = 0$ corresponding to pure Poiseuille flow, the plane Couette-Poiseuille flow becomes stable to all infinitesimal wave-like disturbances (i.e., $\mathrm{Re}_{\mathrm{cr}}$ tends to infinity) when A reaches a value of about 0.55—long before the linear (Couette) velocity profile is reached at $A = 2$.

2.9.2 Nearly Plane-parallel Boundary-layer Flows

It has been already said earlier (in Sect. 2.1, p. 17) that the most important applications of hydrodynamic stability theory are related to studies of various boundary-layer flows. Transition to turbulence in such flows is accompanied by large changes in frictional drag and heat-transfer rate at the wall, which often have great engineering importance. It is therefore only natural that the literature on boundary-layer instability is very extensive, and only a very small part of it can be discussed here.

Let us begin with the simplest idealized case of a laminar boundary layer on a flat plate of infinite width and length (occupying the half-plane $0 \le x < \infty$, $-\infty < y < \infty$, $z = 0$) in a fluid moving with given constant velocity U in the Ox direction. Here the thickness of the boundary layer δ is increasing with x and the two components of fluid velocity, $u = u(x, z)$ and $w = w(x, z)$, differ from zero and depend on coordinates x and z. Hence this boundary-layer flow is not plane-parallel. However, it is known that at large values of $\mathrm{Re} \equiv \mathrm{Re}_\delta = U\delta/\nu$ (i.e., at very high values of $\mathrm{Re}_x = Ux/\nu \propto \mathrm{Re}^2{}_\delta$) the thickness δ increases very slowly, the vertical velocity w is much smaller than u, and the dependence of U on x is quite weak: specifically,

$$\frac{d\delta}{dx} \propto \frac{1}{\mathrm{Re}}, \frac{|w|}{|u|} \propto \frac{1}{\mathrm{Re}} \quad \text{and} \quad \frac{|\partial u/\partial x|}{|\partial u/\partial z|} \propto \frac{z}{x} < \frac{\delta}{x} \propto \frac{1}{\mathrm{Re}}.$$

Therefore, in studying wave disturbances of a boundary layer with wavelengths much smaller than $\mathrm{Re} \times \delta$, it is possible to use the plane-parallel approximation, i.e., to consider the flow as being plane-parallel with only one non-zero velocity component $u = U(z)$, where $U(z)$ is identical to the Blasius velocity profile of a laminar constant-pressure boundary layer. This profile has the form

$$U(z) = Uf(\zeta), \text{ where } \zeta = \sqrt{\frac{U}{\nu}}\frac{z}{\sqrt{x}}, \quad f(\zeta) = \phi'(\zeta) \tag{2.85a}$$

[$(\nu x/U)^{1/2} = \alpha\delta$, where $\alpha \approx 0.2$, is considered here as a constant length) and $\phi(\zeta)$ is the solution of the following boundary-value problem for a nonlinear ordinary differential equation:

$$\phi\phi'' + 2\phi''' = 0, \quad \phi(0) = \phi'(0) = 0, \quad \phi'(\infty) = 1. \tag{2.85b}$$

In the plane-parallel approximation, the stability problem for a boundary-layer flow is quite similar to those for Couette and Poiseuille flows, differing only by the replacement of a flow region of finite thickness H by the semi-infinite region $0 \le z < \infty$ and by a more complicated form of the velocity profile—now given not by an explicit formula but by a third-order differential equation with definite boundary conditions.

The first attempts to calculate the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ and the neutral-stability curve $\max_j[\Im\mathrm{m}c_j(k, \mathrm{Re})] = 0$ for a boundary layer flow having the Blasius velocity profile were due to Prandtl's collaborators Tollmien (1929) and Schlichting (1933a) (see also the surveys by Schlichting (1959, 1979)). Both of them used some approximations valid only at high values of Re (i.e., small ν) and also some special expansions for solutions of the inviscid Rayleigh equation (2.48), while the Blasius velocity profile was replaced in these papers by a simple polynomial approximation {see Drazin and Reid's book (1981) for more details}. According to Tollmien's calculations $\mathrm{Re}^*{}_{\mathrm{cr}} \approx 420$ (where $\mathrm{Re}^* = U\delta^*/\nu$, $\delta^* \approx 0.35\delta$ is the displacement thickness corresponding to the Blasius velocity profile) and $k_{\mathrm{cr}}\delta^* \approx 0.34$, while according to Schlichting $\mathrm{Re}^*{}_{\mathrm{cr}} \approx 575$, $k_{\mathrm{cr}}\,\delta^* \approx 0.23$; the general form of the neutral-stability curve found by both authors was approximately the same. (In terms of the more easily measured Reynolds number $\mathrm{Re}_x = Ux/\nu$, Tollmien's result implies that $\mathrm{Re}_{x\,\mathrm{cr}} \approx 6 \times 10^4$,

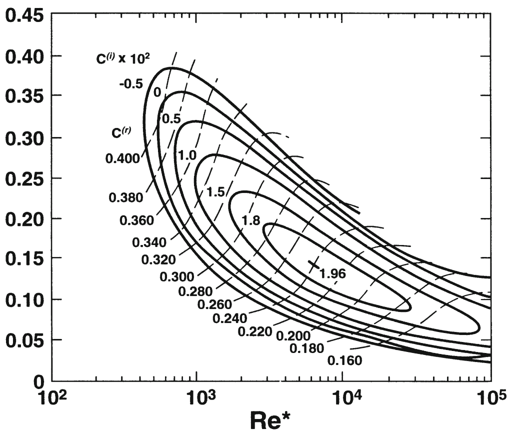


Fig. 2.23 Curves of constant amplification rates $c^{(i)}$ = const. and of constant phase velocity $c^{(r)}$ = const. for wave disturbances in a plane boundary-layer flow with Blasiu's velocity profile. (After Obremski et al. (1969))

while according to Schlichting $Re_{x\ cr} \approx 1.1 \times 10^5$.) Later Lin (1945, 1961a) and Shen (1954) repeated these calculations using slightly different asymptotic expansions and polynomial approximation for the velocity profile. Both sets of results proved to be close to Tollmien's: $Re^*_{cr} \approx 421$, $k_{cr}\delta^* \approx 0.37$. The neutral curve $\Im m\, c_1(k, Re) = 0$ (where c_1 is the eigenvalue with the greatest imaginary part) has a form similar to that in plane Poiseuille flow (but now the upper and lower branches decrease asymptotically, as Re increases, as $Re^{-1/10}$ and $Re^{-1/4}$). The curves $\Im m c_1$ (k, Re) = const., which determine the growth rates of various unstable wave disturbances, were computed, in particular, by Schlichting (1933a); Shen (1954); Wazzan et al. (1966); and Obremski et al. (1969); see e.g. Fig. 2.23. The results of some subsequent computations related to boundary-layer stability will be discussed later.

Tollmien's and Schlichting's results on boundary layer stability at first provoked the strong skepticism of many fluid mechanics experts since the values of Re_{cr} obtained seemed to be too low (cf. the empirical values of $Re_{x\ cr}$ given in Sect. 2.1) and the predicted appearance of two-dimensional waves before transition to turbulence contradicted all the available observational data, in which waves were not found. Therefore, it was often assumed in the 1930s and early 1940s that the linear stability theory is useless for the understanding of transition to turbulence (see, e.g., the paper by Taylor (1939) published in the Proceedings of the 5th International Congress of Applied Mechanics, and the discussion of this subject at the Congress). However, this skeptical attitude to the linear stability theory disappeared after the publication of the classical paper by Schubauer and Skramstad (1947), describing the results of their experiments made in the early Forties and later becoming one of the most frequently cited papers in fluid mechanics.

The experiments of Schubauer and Skramstad were carried out in the wind tunnel of the National Bureau of Standards in Washington, D.C., having extremely low initial turbulence. In this tunnel the parameter U'/U (where U' is the root-mean-square value of velocity fluctuations outside the boundary layer) can be as low as 0.0003–0.0002 when certain precautionary measures are taken. This proves to be very important, since there are some data showing that when U'/U exceeds roughly

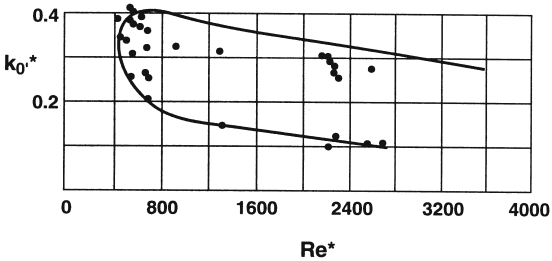


Fig. 2.24 Comparison of the wave numbers of neutral disturbances in the boundary layer on a flat plate observed by Schubauer and Skramstad (*dots*) with neutral curve calculated by Lin

0.002 (as in all of the older tests), the transition to turbulence in the boundary layer is apparently affected strongly by free-stream disturbances (either in accordance with Taylor's scheme described in Sect. 2.2, or as a result of the mechanism of interaction of the free-stream disturbances with the boundary layer flow studied by Criminale (1967), or in some other way[7]). However, when $U'/U < 0.002$ and the Reynolds number Re is increasing, at some value of Re random two-dimensional disturbances of sinusoidal form appear in the flow, and under certain conditions their amplitude increases downstream in accordance with the deductions of the linear disturbance theory. The presence of such regular oscillations was demonstrated as early as 1940 by Schubauer and Skramstad, using careful hot-wire anemometer observations. Later, for more accurate verification of the theoretical deductions, they used a thin metal ribbon placed in the boundary layer, set in oscillation by an electromagnet and producing artificial disturbances of fixed frequency ω. They found a number of neutral (neither growing nor damped) almost purely sinusoidal fluctuations of velocity, which corresponded with satisfactory precision to points on the neutral curve in the stability diagram (see Fig. 2.24).

After the work by Schubauer and Skramstad (1947), vibrating ribbon experiments became a popular method of studying the evolution of disturbances in various laminar boundary layers and other near-wall laminar flows; see, e.g., the papers by Klebanoff et al. (1962); Ross et al. (1970); Reynolds and Saric (1986); Asai and Nishioka (1989) and Klingmann et al. (1993) and the surveys by Kachanov et al. (1982); Kachanov (1994) and Saric (1990, 1996) containing many additional references. All experiments made at low enough values of U'/U confirmed that a gradual increase of Re leads to the appearance of two-dimensional sinusoidal waves in a boundary-layer flow. These waves, which agree well with the early theoretical predictions

[7] Many features of the mechanisms that cause the external (i.e., free-stream or environmental) disturbances to enter the boundary layer and generate unstable internal disturbances stimulating transition to turbulence are unknown at present. Morkovin was the first who in 1969 began to stress the importance of studying the related processes and proposed to call them the *boundary layer receptivity*. Later the studies of receptivity took a quite important part in transition investigations; see, e.g., Reshotko (1984), Goldstein and Hultgren (1989), Kozlov and Ryzhov (1990), Morkovin (1990a), Saric (1992, 1996), Voke and Yang (1995), Duck et al. (1996), and papers on receptivity in the collections edited by Hussaini and Voigt (1990) and Kobayashi (1995) where many additional references can also be found. Note, however, that the word *receptivity* is also often used to denote the influence of any factors stimulating boundary layer transition – e.g., of roughness, wall inhomogeneities or leading-edge properties.

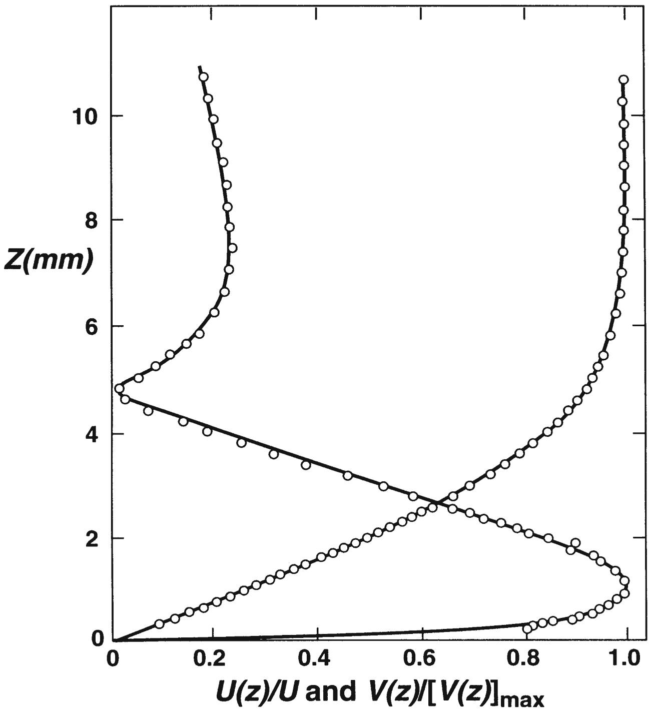


Fig. 2.25 Comparison of experimental data by Reynolds and Saric (1986) for normalized velocity profile $U(z)/U$ and amplitudes of streamwise-velocity TS-wave $V(z)/(V(z))_{\max}$ (*circles*) with theoretical predictions (*solid lines*). The data are from the vibrating-ribbon experiment where $f = w/2\Pi = 50\text{hz.}, U = 9.6\text{m/s(and Re} = (Ux/V)^{1/2} = 780, F = wv/U^2 = 56.4 \times 10^{-6}), [V(z)]_{\max}/U = 1.5\%$

by Tollmien and Schlichting, are now known as *Tollmien-Schlichting waves* (or *T-S waves*). The results of the experiments, as a rule, agree satisfactorily (and sometimes excellently) with deductions from the linear stability theory; as an example, Fig. 2.25 shows the data obtained by Reynolds and Saric (1986); (and given also in Saric's surveys (1990, 1996)). Figure 2.25 shows that the measured velocity profile $U(z)$ of a flat-plate boundary layer agrees excellently with the Blausius profile while the measured amplitudes $f_{\omega,k}^{(u)}(z) = V(z)$ of wave disturbance (TS wave) of the streamwise velocity produced by a vibrating ribbon agree excellently with values of $V(z)$ given by the solution of the corresponding Orr-Sommerfeld eigenvalue problem (in its spatial formulation, where ω is a given constant and k is the unknown eigenvalue; see below).

Note now that the disturbances produced by a vibrating ribbon differ in fact from the ordinary normal modes with fixed real wave numbers k and complex angular frequencies $\omega = kc = \omega^{(r)} + i\omega^{(i)}$, determined by the eigenvalue problem (2.44) with given boundary conditions. Gaster (1962, 1965) was one of the first who noted that here the model of a spatially growing (or damping) wave (2.43) with fixed real frequency ω (equal to the frequency of ribbon oscillations) and complex wave number $k = k^{(\mathrm{r})} + ik^{(\mathrm{i})}$ seems to be more appropriate (in fact the suitability of such model for hydrodynamic-stability studies was noted also much earlier by Orr (1907, p. 16) and then by Landau and Lifshitz (1953, 1958)). In the case of such modes the values of $k = k(\omega, \mathrm{Re})$ must be found from the eigenvalue problem (2.44) where $c = \omega/k$, ω is a given constant and k is the unknown complex eigenvalue. Then the real part $k^{(r)}$ determines the wavelength of the disturbance while the imaginary part $k^{(\mathrm{i})}$ determines the rate of spatial damping (or growth) of the wave amplitude with the increase of x. Just such damping or growth rates were in fact observed by Schubauer and Skramstad. To convert their results from spatial to temporal rates of change (the only case for which theoretical estimates were then available), Schubauer and Skramstad transformed distances x into time intervals t by dividing x by the phase velocity $c^{(r)}$ of the disturbance. It was, however, noted by Gaster (1962) that this procedure is not rigorous and, if there is any dispersion, it is more reasonable to transform lengths x into time intervals using the group rather than the phase velocity. According to Gaster (1962, 1965) the Schubauer-Skramstad transformation has satisfactory precision only in the case of disturbances with comparatively small amplification rates, for which the group and phase velocities are roughly equal. For the general case, Gaster developed a reasonable approximate procedure of conversion from spatially to temporally growing modes (see also the discussion of his work by Drazin and Reid (1981); Mack (1984) and Saric (1992, 1996)).

For modern electronic computers the solution of the eigenvalue problem (2.44) for unknown complex k (with fixed real ω) is not much more complicated than determination of the unknown complex eigenvalues c (with fixed real k); the fact that c enters the equation linearly and k nonlinearly does not now play an important part. Therefore, it is not surprising that a number of computations of spatially growing normal modes in the Blasius boundary layer were also made in the last thirty years (the results of the earliest among them are presented in Sect. 54 of the book by Betchov and Criminale (1967)).[8]

Most of the early computations of eigenvalues and eigenfunctions for both the temporally and spatially growing normal modes used some combination of numerical methods applied at small and moderate values of Re and asymptotic expansions of the desired solutions at large values of Re. The study of asymptotic expansions plays an important part in the well-known book by Drazin and Reid (1981) on hydrodynamic stability and continues to be quite important at the present time; see, e.g., the survey by Cowley and Wu (1994) and Healey's papers (1995a, b). However, rapid improvements in electronic computers made possible direct numerical

[8] In reality, the model of spatially growing normal modes is also an approximation, which is more precise than the model of temporally growing modes but nevertheless does not describe the results of the vibrating ribbon experiments exactly. A more accurate analysis of the evolution of disturbances generated by such ribbons was developed by Ashpis and Reshotko (1990), Gaster and Sengupta (1993), Sengupta et al. (1994), and Hill (1995). Modern computers also permit direct numerical simulation of the evolution of such disturbances in a boundary-layer flow; see, e.g., Spalart and Yang (1987), Fasel and Konzelmann (1990), Kleiser and Zang (1991), and also the paper by Rist and Fasel (1995) where a slightly different model of disturbance generator was used.

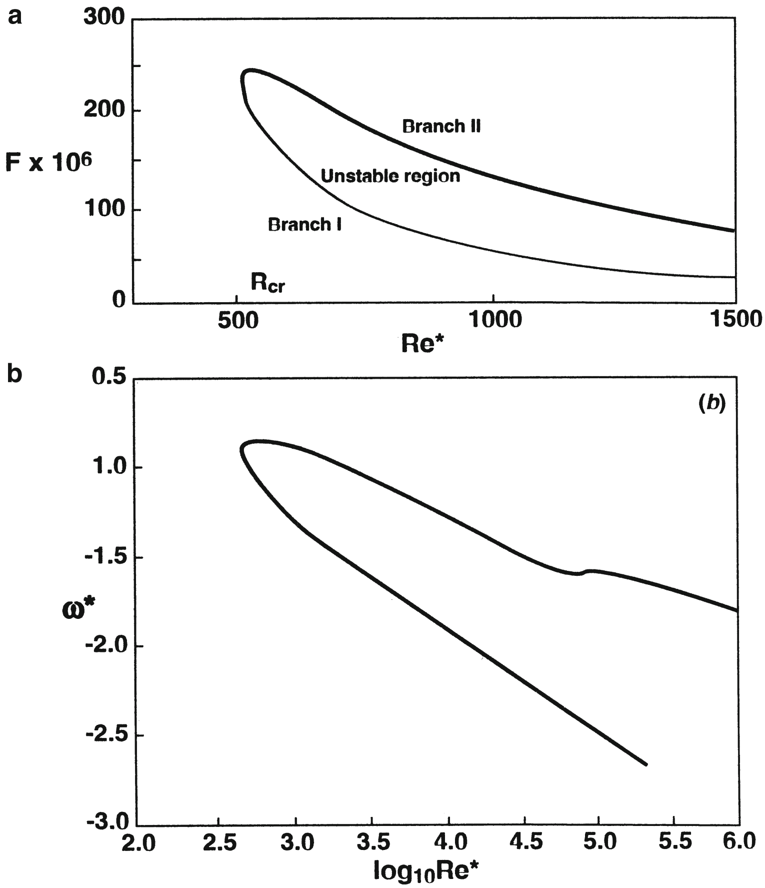


Fig. 2.26 **a** The form of the neutral curve for a plane boundary-layer flow in the (F, Re*) plane where $F = \omega\nu/U^2$, according to calculations by Klingmann et al. (1993). **b** Neutral curve for a plane boundary layer on a log-log plot (where $\omega^* = \omega\delta^*/U$) according to Healey (1995b).

solutions of the Orr-Sommerfeld equations and determination of the corresponding eigenvalues and eigenfunctions for a wide range of Re values, without any use of asymptotic approaches. Apparently the first direct computation of the spatial neutral curve (relating to spatially growing modes, i.e., represented by a curve in (ω, Re*)-plane) for the Blasius plane-parallel boundary layer was due to Jordinson (1970). His main results (namely: $\mathrm{Re}^*_{\mathrm{cr}} \approx 520$, $k_{\mathrm{cr}}\delta^* \approx 0.30$), surprisingly, proved to be closer to those of Schlichting than to those of Tollmien, Lin, and Shen. However, the form of the neutral curve in the (ω, Re*)-plane found by Jordinson was later confirmed by Kümmerer (in his 1973 Dissertation; see also Kümmerer (1976)) and by Davey (1982), and it also agrees satisfactorily with recent numerical results by Klingmann et al. (1993) (see Fig. 2.26a) and by Healey (1995a, b) who carried out extended computations of the neutral curve for a very wide range of Re* values (much beyond

10^5; see Fig. 2.26b). Therefore, this form seems to be reliable. However, the experimental data by Schubauer and Skramstad agree better with the older (and apparently less precise) findings by Tollmien and Lin than with those by Jordinson.

The British scientists collaborating with Jordinson assumed at first that the apparent contradiction was due to the non-parallel character of real boundary layers, which was neglected in all the above-mentioned theoretical calculations. Therefore, Barry and Ross (1970) revised Jordinson's calculations, including in the Orr-Sommerfeld equation some small terms approximately describing the increase of the boundary-layer thickness with x. These corrections brought the neutral curve closer to that found by Lin (in particular, they slightly reduced the value of Re^*_{cr}). It was found by Ross et al. (1970) that the new curve agrees better with the old data of Schubauer and Skramstad and with the new data of similar experiments carried out by the authors in the low-turbulence wind tunnel at the University of Edinburgh. Therefore, it was concluded by Ross et al. that nonparallelism of the boundary-layer flow cannot be neglected in computations of its stability characteristics (see also Drazin and Reid (1981), p. 227). However, subsequent development of theoretical and experimental studies of boundary-layer stability cast doubt on the validity of this statement.

A number of papers on the influence of the non-parallel character of a boundary layer on its stability was published after 1970. The works by Bouthier (1972, 1973); Ling and Reynolds (1973); Gaster (1974); Saric and Nayfeh (1975, 1977); Smith (1979); Van Stijn and Van de Vorren (1983); and Bridges and Morris (1986) are typical examples; see also similar developments in papers by Corner et al. (1976) and Lakin and Grosch (1955). All the authors mentioned supplemented the Orr-Sommerfeld equation by additional terms, describing approximately the influence of the growth of the boundary-layer thickness, and used asymptotic expansions taking into account the slowness of horizontal changes in comparison with vertical ones. Note, however, that both the additional terms and the asymptotic expansions were selected differently by different authors. Therefore, it is not surprising that the results obtained proved to be different, too (though all authors came to the conclusion that nonparallelism destabilizes the flow, i.e., increases the domain of unstable disturbances in the (ω, Re)-plane). Hence, the published papers do not permit a reliable quantitative estimate to be made for the influence of growth of the boundary-layer thickness on its stability.

Another method, which fully uses the wonderful possibilities of modern computers, was applied to the same problem by Fasel and Konzelmann (1990). These authors studied the evolution of two-dimensional wave-like disturbances in a non-parallel laminar boundary layer by numerical solution of the Navier-Stokes equations with the appropriate initial and boundary conditions. The results obtained were then compared with those following from the Orr-Sommerfeld equation corresponding to the plane-parallel Blasius model of a boundary layer. Here, in other words, direct numerical simulation (DNS) of the Navier-Stokes flow dynamics is used for determination of the effect of non-parallelism. Note that DNS began to be used in hydrodynamic-stability and transition studies as far back as the late 1970s and was immediately found to be quite fruitful; see, e.g., the paper by Wolf et al. (1978) and the surveys by Kleiser and Zang (1991); Joslin et al. (1993) and Reed (1994) devoted to this topic. As to Fasel and Konzelmann' s work, it showed that the non-parallel neutral curve obtained by the DNS method differs very little from the Jordinson curve corresponding to the plane-parallel model of a Blasius boundary layer, but differs more significantly from most of the results obtained when non-parallel corrections were taken into account (the only exception was found for some results by Gaster (1974) which led to rather small deviations of the non-parallel neutral curve from the parallel one; this is also true for results by Ling and Reynolds (1973); which were not mentioned by Fasel and Konzelmann). A similar conclusion was reached by Bertolotti et al. (1992) who also considered the DNS of plane-wave development within the Blasius boundary layer with increasing thickness, and simultaneously applied to the same problem the "Parabolized Stability Equations", simplified partial differential equations of parabolic type which proved to be a very convenient tool for solving the stability problem (for more information about this method see Joslin et al. (1993) and Herbert (1994)). The results obtained with the aid of these two rather different approaches agreed well; this agreement confirms the accuracy of both methods. The neutral curve obtained by Bertolotti et al. for a boundary layer with growing thickness did not deviate much from the neutral curve for the plane-parallel boundary layer shown in Fig. 2.26 and the observed small deviation could by no means be held responsible for discrepancies between results of some vibrating-ribbon experiments and theoretical results for parallel flow. (Recall that the results on vibrating-ribbon experiments by Reynolds and Saric (1986) agree excellently with the deductions from parallel-flow stability theory; see e.g. Fig. 2.25.) Hence, it must be concluded that observed discrepancies apparently had no relation to the growth of the boundary-layer thickness.

The discrepancy between theoretical and experimental estimates of the neutral curve may be related to incomplete fulfillment in laboratory facilities of the strict requirements used in theoretical derivations. Saric (1990, 1996) and Bertolotti et al. (1992) discussed possible violations of theoretical assumptions in real experiments. Apparently it is most difficult to guarantee the absence of small pressure gradients at the leading edges of flat plates of finite thickness. Hence, such gradients can probably explain the observed significant deviations of a number of previous experimental results from the theoretical conclusions obtained for parallel *constant-pressure* boundary layers. This circumstance was taken into account in recent vibrating-ribbon experiments by Klingmann et al. (1993) made in the new low-turbulence wind tunnel at the Royal Institute of Technology in Stockholm. In these experiments, exceptional attention was given to fulfillment of the constant-pressure requirement with the highest attainable accuracy. These precautions resulted in much better agreement with the parallel-flow theory than that found in earlier experiments of the same type (see Fig. 2.27). Relying on the results of these experiments and on calculations by Fasel and Konzelmann (1990) and Bertolotti et al. (1992); Saric (1996) and Reed et al. (1996) came to the conclusion that in the absence of leading-edge pressure gradients the parallel-flow theory is good for all Reynolds numbers used in experiments.

Let us now stress again that the critical Reynolds number Re_{cr} of a boundary-layer flow determined from the left-most point of the neutral-stability curve is always considerably lower than the values of Re at which the real transition to turbulence takes place. This may be explained by remarking that the linear stability theory

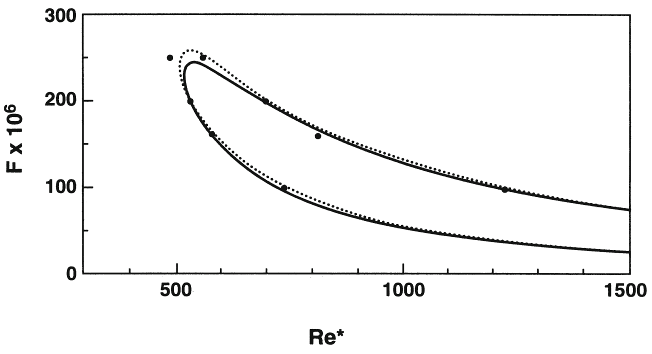


Fig. 2.27 Comparison of the neutral-stability points in (F, Re*) plane found in experiments by Klingmann et al. (1993) in a flat-plate boundary layer with theoretical neutral curves calculated for the plane-parallel model of a boundary layer (*solid line*) and according to the non-parallel theory of Gaster (1974) (*dotted line*)

determines only the value of $\mathrm{Re}_{\mathrm{cr\,min}}$, and instability to infinitesimal disturbances does not have to be accompanied by instantaneous transition to the turbulent regime. It seems reasonable to assume that at the instability point oscillations occur in the flow, are amplified as they move downstream, and only for some larger value of x lead to transition to developed turbulence. However, this explanation loses some of its plausibility because the path to transition that usually occurs, both in wind- and water-tunnels and in real life, is the so-called *by-pass transition,* where the initial disturbances are so large that the appearance of small-amplitude wave-like disturbances (the TS waves) and their subsequent growth in accordance with the linear theory are *by-passed,* i.e., cannot be observed. The phenomenon of by-pass transition has attracted much attention (see, e.g., Morkovin (1969, 1991, 1993); Breuer and Kuraishi (1993); Reshotko (1994); Saric (1992, 1994, 1996), Reed et al. (1996)) but the details are strongly dependent on the disturbance field and the topic will not be considered here. Note only that it considerably diminishes the possibility of reliable determination of the transition point x from the results of the linear stability theory. Nevertheless, since prediction (and/or control) of the boundary-layer transition point is enormously important for many engineering applications, there have been many attempts to use the linear theory of TS-waves to develop approximate methods of prediction the boundary-layer transition point. The most well-known (and widely used) such method is the so-called e^n method, proposed independently by A.M.O. Smith and J.L. Van Ingen in the mid-1950s and based on the results of linear stability theory for the growth of wave amplitudes. This very rough engineering method will not be considered here (but see, e.g., Arnal (1984, 1994); Mack (1984); Saric (1994, 1996); Arnal et al. (1995) and Reed et al. (1996) where some other applications of the linear theory are also considered); it is mentioned mainly to stress that linear theory of boundary-layer stability continues to have many applications to important practical problems.

The wide applicability of conclusions from the linear theory of boundary-layer stability naturally stimulated attempts to develop this theory further and to carry out analogous calculations of the neutral curve and the stability diagram (containing graphs of the curve $\Im\mathrm{m}\ c_1(k, \mathrm{Re}) = \mathrm{const.}$ [or $\Im\mathrm{m}\ c_1(\omega, \mathrm{Re}) = \mathrm{const.}$)) for many other types of boundary layers met in practice (e.g., for velocity profiles differing from the Blasius profile, for boundary layers in the presence of external forces or in fluids with dynamic equations different from the Navier-Stokes equations, etc.). The forms of $U(z)$ studied included boundary-layer profiles corresponding to flows past curved surfaces or the presence of a favorable or adverse pressure gradient, to flows over a porous plate when there is suction or blowing of the fluid through the plate, to boundary layers with heat transfer, compressible and/or three-dimensional boundary layers, hydromagnetic boundary layers and many others. A number of results of such calculations can be found in the books by Betchov and Criminale (1967); Levchenko et al. (1975); Drazin and Reid (1981); Kachanov et al. (1982); Zhigulev and Tumin (1987); in the survey papers by Mack (1984) and Arnal (1984, 1994); and in numerous scientific papers and reports (those by Wazzan et al. (1966, 1967, 1986); Obremski et al. (1969) and Nayfeh and Padhye (1980) are typical examples; see also the collections of papers listed at the beginning of Sect. 2.1). A specific example of such investigations is the calculations by Wazzan et al. and Obremski et al., of stability characteristics for the Falkner-Skan (FS) velocity profiles, which are self-similar boundary-layer velocity profiles corresponding to a free-stream velocity $U(x)$ varying with x as x^m. The profiles $U(x, z)$ can be described in this case by the same Eq. (2.85a) as in the case of the Blasius profile, but Eq. (2.85b) for $\phi(\zeta)$ now must be supplemented by a term $+\beta(1 - \phi'^2)$ on the left-hand side, where $\beta = 2m/(1+m)$. The FS profiles are often also used to describe non-self-similar boundary layers with arbitrary pressure gradient, choosing the local value of m as $m = (x/U)$ dU/dx the pressure gradient is favorable if $m > 0$ and adverse if $m < 0$, while $m = 0$ corresponds to the Blasius profile. Some results of computations by Obremski et al. (related to the plane-parallel model of FS boundary layers) are shown in Figs. 2.28a, b and 2.29.

Note that the form of the neutral curves in both the (k, Re) and the (ω, Re) planes depends considerably upon whether or not the velocity profile $U(z)$ has a point of inflection (i.e., whether d^2U/dz^2 becomes zero for some $z > 0$). In the latter case, the neutral curve will have the same character as in cases of plane Poiseuille flow or the Blasius boundary layer, i.e., as $\mathrm{Re} \to \infty$ both its branches asymptote to the abscissa (cf. Figs.2.22 and 2.25). Thus, when the velocity profile has no inflection point, any wave disturbances of fixed wavelength (or fixed frequency) will finally become stable if the Reynolds number increases sufficiently. However, when the velocity profile does possess an inflection point, the upper branch of the neutral curve will have a finite asymptote as $\mathrm{Re} \to \infty$ (see Fig. 2.30), and the ordinate of this asymptote depends upon the distance of the inflection point from the wall. In other words, there exists a range of wavelengths (or frequencies) such that corresponding wave disturbances are unstable however large the value of the Reynolds number. Obviously, this is connected directly with the definitive role played by the presence of an inflection point in the velocity profile, in the inviscid problem of instability with respect to wave disturbances.

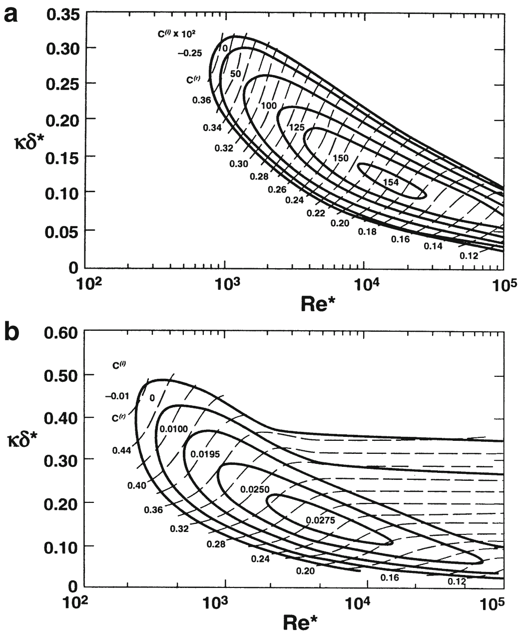


Fig. 2.28 Curves $c^{(i)} = \text{const.}$ and $c^{(r)} = \text{const.}$ in the (k, Re*) plane for the velocity profiles from the Falkner-Skan family corresponding to $\beta = 0.05$ **a** and $\beta = -0.05$ **b**. (After Obremski et al. (1969))

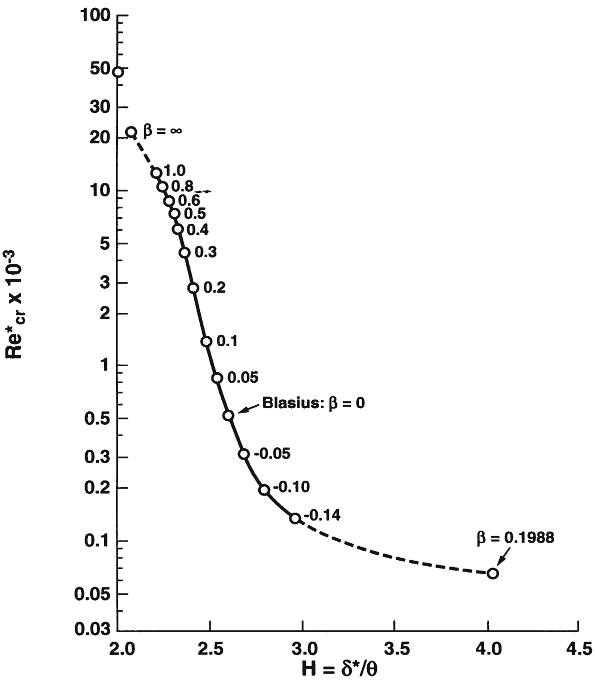


Fig. 2.29 Dependence of the critical Reynolds number Re^*_{cr} for the Falkner-Skan family of boundary-layer profiles on the parameter β and on the determined by β value of the boundary-layer form-factor $H = \delta^*/\theta$ (where θ is the momentum thickness of the boundary layer). (After Drazin and Reid (1981) who used the results of computations by Obremski et al.)

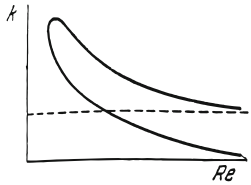


Fig. 2.30 Schematic form of the neutral curve in the (*k,* Re) plane for a flow with velocity profile possessing an inflection point (e.g., for a boundary layer in the presence of an adverse pressure gradient)

Above, most attention was paid to the curve of neutral stability $\Im m c_1$ $(k, \mathrm{Re}) = 0$ [or $\Im m c_1$ $(\omega, \mathrm{Re}) = 0$], which depends only on the first (the most unstable or, in a stable region of (k, Re) or (ω, Re) plane, the least stable) eigenvalue c_1 of the Orr-Sommerfeld temporal or spatial eigenvalue problem. However, the other eigenvalues are also interesting in some cases. It has been already mentioned in Sect. 2.5 that the completeness of the system of eigenfunctions was proved for the temporal eigenvalues of the boundary-layer stability problem by Salwen and Grosch in 1981. In this proof the functions corresponding to all discrete eigenvalues and also to all points of the continuous spectrum were included in the eigenfunction system. Therefore, to study the temporal or spatial evolution of an arbitrary disturbance with the aid of the normal-mode decomposition, the whole eigenvalue spectrum of the Orr-Sommerfeld equation must be known. The higher eigenvalues are also of interest in some approaches to the practically important problem of boundary Iayer receptivity, mentioned above. This justifies a brief discussion of the results of studies of the higher eigenvalues.

Recall that it was proved by Schensted (1960) and Di Prima and Habetler (1969); for the case of a plane-parallel flow of finite thickness (and by Herron (1982) for similar nearly plane-parallel flows) that for any velocity profile there exists an infinite number of discrete eigenvalues, and the system of corresponding eigenfunctions is complete in the appropriate functional space. In the case of parallel boundary-layer flow in the half-space $0 \le z < \infty$, the system of eigenfunctions corresponding to discrete eigenvalues cannot be complete because of the existence of the continuous spectrum, but it was natural to think that some higher eigenvalues of the Orr-Sommerfeld problem might exist here too. Therefore, the problem of finding them (in both temporal and spatial formulation) was bound to attract attention.

Apparently Jordinson (1971) was the first to carry out an approximate computation of several higher eigenvalues of the spatial and temporal Orr-Sommerfeld problem (i.e., assuming that either k or $w = ck$ is the unknown complex eigenvalue while the other parameters are fixed and real) for a plane-parallel flow with the Blasius velocity profile. (Some of the numerical values found by him were corrected in subsequent publications but all the qualitative results were found to be correct.) A simplified method of eigenvalue computation was then suggested by Gaster and Jordinson (1975) who gave some examples of its applications. Later the higher eigenvalues of the stability problem for parallel boundary-layer flow were studied, in particular, by Corner et al. (1976); (in spatial formulation), Mack (1976); Kümmerer (1976); Antar (1976) and Antar and Benek (1978) (in temporal formulation) and by Murdock and Stewartson (1977) and Lakin and Grosch (1955) (for both formulations of the problem). (Corner et al. and Lakin and Grosch also considered some small corrections to eigenvalues, approximately describing the effect of nonparallelism of the flow, but these corrections will be not discussed here.) The obtained results give the impression that the number of discrete eigenvalues of the problems is apparently finite in both formulations (and relatively small, maybe equal to six or seven), but that there also exists a continuous spectrum of eigenvalues (this was formulated most explicitly by Murdock et al. and Antar et al.).

Lakin and Grosch (1955) carried out the most detailed investigation of the higher normal modes to date. Studying the simpler case of temporal eigenvalues they apply the mathematical method of Lidskii and Sadovnichii (1968) (developed by these authors for the study of the Orr-Sommerfeld eigenvalue problem in a finite domain) to the case of a semi-infinite region. This method permitted a quite rigorous proof that, if the profile $U(z)$ has derivatives of all orders (a condition which is satisfied by the Blasius function), then for fixed Re and k there can be only a finite number of discrete eigenvalues c. This mathematical theorem agrees well with the available computational results (including those by Lakin and Grosch themselves).

The computation of the spatial eigenvalues $k = k^{(r)} + ik^{(i)}$ (at fixed real ω) is more complicated, and to date no mathematical theorems related to them have been proved. However, it has been already mentioned that the results of computations of such eigenvalues show that their number at given values of ω and Re is also finite. The fact that in the case of parallel flow with a Blasius velocity profile (or another profile of the same type) discrete and continuous spectra exist for both temporal and spatial Orr-Sommerfeld eigenvalue problems is now well known (see, e.g., Murdock and Stewartson (1977); Grosch and Salwen (1978a, b); Salwen and Grosch (1981); Lakin and Grosch (1955) and Herron (1983)). For the temporal problem Grosch and Salwen (1978a) showed that if the complex phase velocity $c = \omega/k$ is taken as the unknown eigenvalue and both c and k are made dimensionless by the length and velocity scales L and U (where L may be selected arbitrarily and U is the free-stream velocity), then the continuous spectrum at given values of k and Re fills the half-line

$$c = 1 - i\frac{ak}{\text{Re}}, 1 \leq a < \infty, \tag{2.86}$$

in the complex c-plane. (This result was later discussed by Craik (1991).) The real part 1 of all values (2.86) shows that all the normal modes corresponding to the continuous spectrum have the same phase velocity U, while the negative imaginary parts show that all these modes are damped. It was shown by Grosch and Salwen that to each eigenvalue (2.86) there correspond two different ‘eigenmodes’. The completeness of the system of all normal modes (corresponding to a discrete or to a continuous spectrum) in the space of all two-dimensional disturbances was proved for this temporal eigenvalue problem by Salwen and Grosch (1981) (who used the results by Gustavsson (1979)) and by Herron (1983).

The situation with the spatial version of the Orr-Sommerfeld eigenvalue problem for plane-parallel flow in the half-plane $0 \leq z < \infty$ having a Blasius (or related) velocity. profile proved to be more complicated. Grosch and Salwen (1978a) found only one branch of a continuous spectrum for the spatial eigenvalue problem. This branch fills a curve in the upper half-plane of the complex k-plane; hence all the corresponding eigenmodes are damped. However, slightly later Grosch and Salwen (b) discovered that the continuous spectrum of the eigenvalue problem in fact includes four different branches consisting of points in the upper half-plane. A detailed description of the four branches of the continuous spectrum and of the eigenfunctions corresponding to them was given by Salwen and Grosch (1981) who, however, could not prove the completeness of the system of eigenfunctions belonging either to discrete or to continuous spectra. Apparently, the completeness theorem for this spatial eigenvalue problem has not been proved up to now.

2.9.3 Plane-Parallel Flows in an Unbounded Space: Models for Plane Jets, Wakes, and Mixing Layers. Convective and Absolute Instabilities

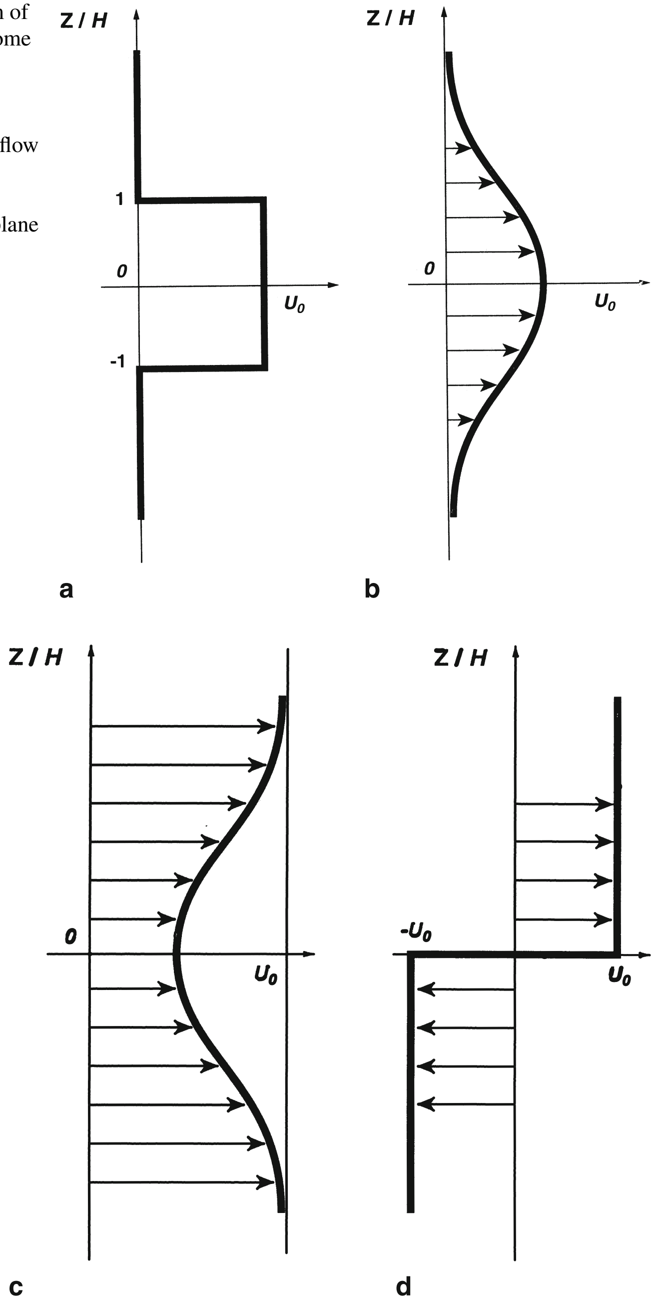
Now we shall consider the stability problem for certain plane-parallel flows in an unbounded space $-\infty < z < \infty$ which model plane laminar jets, wakes, and mixing layers. Schematic forms of the velocity profiles for these flows are shown in Fig. 2.31a–f. Here profiles 2.31a, b represent two models of a jet, while Fig. 2.31c gives a model of a plane-wake profile, and in Fig. 2.31d–f three models of a profile produced by two adjoining flows of different velocities are depicted.

Let us begin with a jet velocity profile of the type shown in Fig. 2.31b, described by the equation

$$U(z) = U_0 \sec\mathrm{h}^2 \frac{z}{H} = \frac{4U_0}{(e^{z/H} + e^{-z/H})^2}. \tag{2.87}$$

It is easy to see that this velocity profile has inflection points (i.e., $U''(z) = 0$) at $z/H = \pm[\ln(2+\sqrt{3})]/2 \approx \pm 0.6585$ (where $U(z) = 2U_0/3$). During the 1960s it was shown by several authors that the flow of an inviscid fluid with such a velocity profile is unstable with respect to infinitesimal wave-like velocity disturbances; see, e.g., Drazin and Howard (1966); Betchov and Criminale (1967), Sect. I.6; or Drazin and Reid (1981), Sect. 31.9, where the main results of the corresponding stability analysis are presented. However, in studying such a flow it seems natural to assume that $\nu \neq 0$. This is because, according to the calculations of Schlichting (1933b) and Bickley (1937) (see also Goldstein (1938), Vol. I, Sect. 57; or Schlichting (1979), Chap. IX, Sect. f), equation (2.87) represents, with good accuracy, the similarity solution of the Navier-Stokes equations for the longitudinal velocity profile $u_1 = U$ in a laminar plane jet (issuing from an infinitely thin linear aperture along the line $x = 0$, $z = 0$, into a space filled with the same fluid). Here the parameters U_0 and H

Fig. 2.31 Schematic form of the velocity profiles for some plane-parallel flows in an unbounded space: **a** and **b**—models of a plane-jet flow; **c**—plane wake; **d**—flow with a tangential velocity discontinuity; **e** and **f**—mixing layers of two plane flows of different velocity



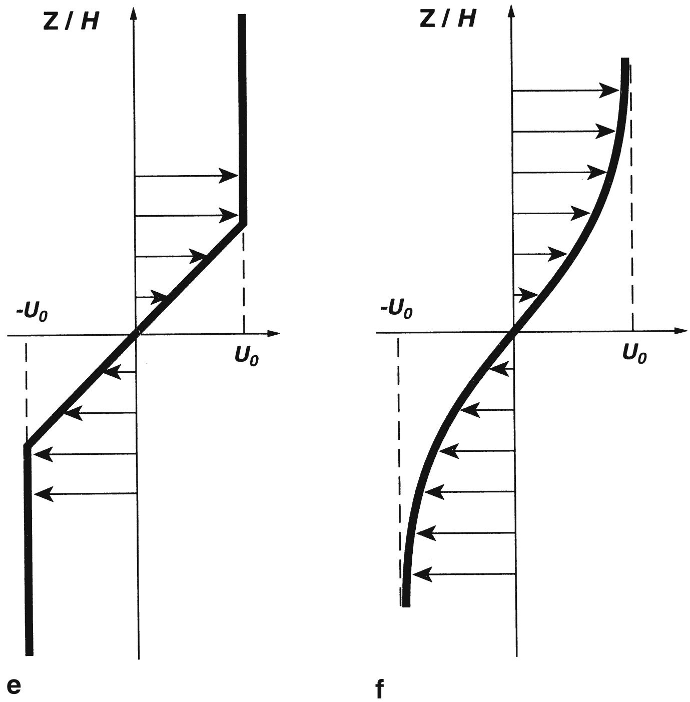


Fig. 2.31 (Continued)

will depend on x only comparatively weakly, while the transverse velocity, $u_3 = w$, will be small in comparison with the longitudinal component. Thus a plane-parallel flow of viscous fluid with profile (2.87) may be considered as a model of a *plane jet* at a great distance from the aperture. Just in this connection its stability has been analyzed by the normal-mode method by several authors. References to a number of relatively old such works dating from the late 1950s and early 1960s can be found in the book MY1 and the paper by Ko and Lessen (1969); here we shall mention in addition only the results of the digital solution of the corresponding Orr-Sommerfeld equation found by Kaplan (1964); (see also Betchov and Criminale (1967); Sect. 16) and Silcock (1975); (see also Drazin and Reid (1981), Sect. 31.9).

As has been explained above, the normal-mode method of determination of the neutral curve in the plane (k, Re) [or (ω, Re)) for a given flow is based on the investigation of an eigenvalue problem for the corresponding Orr-Sommerfeld equation (2.44) (or (2.44′)). The unboundedness of the flow restricts the discussion to those solutions of these equations which are damped at infinity. In this case, moreover, because of the symmetry of the profile (2.87), all the eigenfunctions W(z) and $\psi(z)$ may be divided into even and odd with respect to z, corresponding respectively to antisymmetric and symmetric disturbances of the horizontal velocity u_1. Experiments, the results of inviscid analysis (which show that if $\nu = 0$ the disturbances with

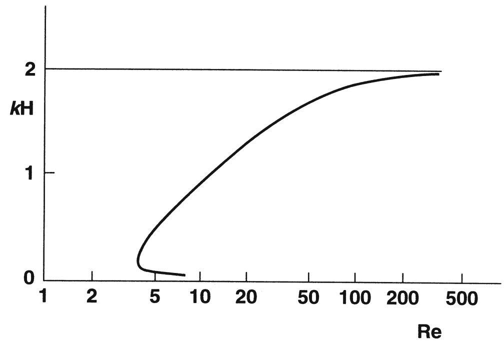


Fig. 2.32 The neutral curve in the (k, Re) plane for a plane jet with velocity profile (2.87), according to the calculations of Kaplan (1964)

even $\psi(z)$ always have larger growth rate than those with odd $\psi(z)$), and also some crude theoretical estimates give the impression that the most unstable disturbances are always those with even $\psi(z)$ (and $W(z) = -ik\psi(z)$). Therefore, most authors confine themselves, in detailed calculations, to the case of the region $0 \leq z < \infty$ and the boundary conditions $\psi(\infty) = \psi'(\infty) = 0$ and $\psi'(0) = \psi'''(0) = 0$ corresponding to even modes. The results of all calculations carried out before 1975 agreed fairly well with each other; they showed that $\mathrm{Re}_{\mathrm{cr}}$, where $\mathrm{Re} = U_0H/\nu$, is very small here (close to 4.0; the smallness of $\mathrm{Re}_{\mathrm{cr}}$ is typical of flows with velocity profiles having an inflection point) while k_{cr} is close to $0.25/H$. Moreover, according to these results the region of unstable wave numbers expands monotonically as Re increases; see, e.g., Fig. 2.32 where the results of Kaplan's computations are presented. (This author also found the form of stability curves $\Im\mathrm{m}c = c^{(i)}(k, \mathrm{Re}) = \mathrm{const.}$ and $\Re\mathrm{e}c = c^{(\mathrm{r})}(k, \mathrm{Re}) = \mathrm{const.}$; see Betchov and Criminale (1967), Fig. 16.1.) The paper by Soprunenko (1965) was apparently the first where results related to stability of both even and odd modes were given; according to this paper $\mathrm{Re}_{\mathrm{cr}} \approx 90$ for odd modes (i.e., it is much greater than $\mathrm{Re}_{\mathrm{cr}}$ for even modes, as was expected). The early estimates of $\mathrm{Re}_{\mathrm{cr}}$ for even and odd modes, and also the conclusion from Fig. 2.32 that as $\mathrm{Re} \to \infty$ all disturbances with $k < 2/H$ are unstable, were confirmed by Silcock (1975), who carried out the most accurate and comprehensive stability computations for both antisymmetric and symmetric wave-like disturbances of a flow with velocity profile (2.87). However, his neutral curves (represented in Fig. 2.33) show that in the case considered the instability region has in fact a more complicated shape than that found in previous studies: it does not expand strictly monotonically with Re, but includes also a supplementary small stability region at small values of k/H (not exceeding several hundredths) and moderate values of $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$. Drazin and Reid (1981), p. 235 (see also Drazin (1961)), state that the existence of this new stability region was also earlier conjectured by J.T. Stuart on the basis of a careful asymptotic study of stability for very long waves (i.e., small values of k).

Note now that for real jets the value $\mathrm{Re} \approx 4$ is generally attained in a region of the flow in which the jet cannot be considered plane-parallel. Apparently Ko and Lessen (1969) were the first who carried out calculations of plane-jet stability characteristics based not only on the plane-parallel model of a jet, but also on a simplified 'nearly plane-parallel' model which took into account, although only approximately, the

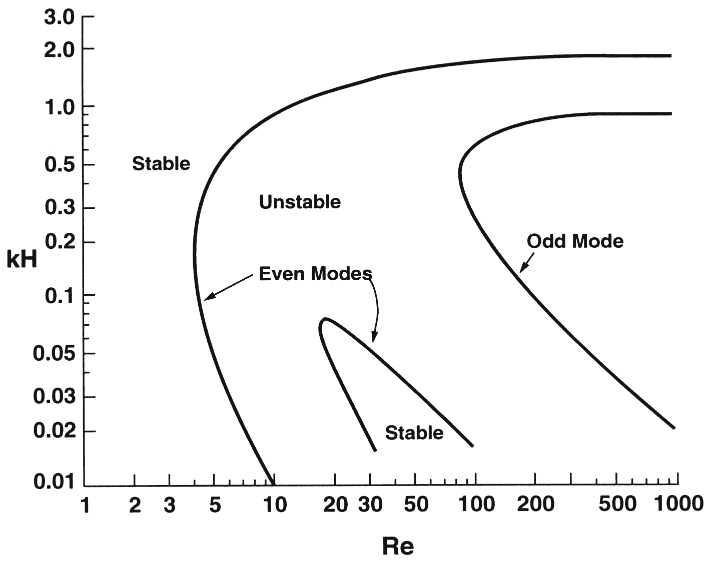


Fig. 2.33 Form of neutral curves for even and odd modes in the case of a plan jet with velocity profile (2.87), according to Silcock (1975). (After Drazin and Reid (1981))

non-parallelism of the flow. They investigated the spatially growing normal modes (with real ω and complex k) and found that their non-parallel flow corrections increased the value of Re_{cr} for a plane-jet flow from $\mathrm{Re}_{cr} \approx 4.0$ to $\mathrm{Re}_{cr} \approx 12.4$ and therefore led to results that do not contradict the rather scattered experimental data of Sato and Sakao (1964). However, Ling and Reynolds (1973) criticized the non-parallel model of Ko and Lessen and instead suggested the use of a perturbation expansion about the parallel-flow solution (in powers of a small 'parameter of non-parallelism') of solutions of the temporal stability problem for a non-parallel-flow model. Their method is based on some crude approximations having a limited region of applicability; it led to the conclusion (which contradicts the results of all other authors) that the non-parallel corrections significantly change the shape of the low-wavenumber part of the neutral stability curve in the (k. Re)-plane for both the plane-jet and plane-wake flows, but this change *reduces* (and not *increases*) the value of Re_{cr}. Other nearly-parallel-flow models of spatial stability problem, which take into account only some of the non-parallel effects, were considered by Haaland (1972) and Bajaj and Garg (1977); in both these studies it was found that $\mathrm{Re}_{cr} \approx 11.5$ for a slightly non-parallel plane jet. Later Garg (1981) investigated a more complete model of a nearly-parallel plane jet with the velocity profile (2.87), similar to the model used by Bouthier (1972, 1973) for the non-parallel boundary-layer flow with a Blasius velocity profile. Garg found that according to this model $\mathrm{Re}_{cr} \approx 20$ (and noted that this result also does not contradict the data of Sato and Sakao (1964)). However, all the non-parallel models proposed use some non-rigorous assumptions whose accuracy is unclear (cf. the corresponding discussion in Sect. 2.92). It would be interesting to investigate the influence of non-parallelism on the plane-jet stability by direct numerical simulation (see again the related discussion in Sect. 2.92), but such an investigation has apparently never been carried out.

The determination of the value for Re_{cr} and calculation of the neutral curve in the (k, Re) (or (ω, Re)) plane for the plane-jet flow requires, as in the case of other plane-parallel or nearly plane-parallel flows, the investigation of only the lower normal mode of wave-like disturbances. It is, however, natural to assume that higher-order modes of disturbance also exist in this flow. We do not know any works dealing with the higher disturbance modes of the flow with velocity profile (2.87), but for a simplified model of a plane jet with the polygonal velocity profile shown in Fig. 2.31a, several such modes were found by Yamada (1964). This author studied the corresponding Orr-Sommerfeld equation (using the natural "matching condition" at the corners of the velocity profile) and found that in this case $\mathrm{Re}_{cr} \approx 4$ and $k_{cr} \approx 0.25/H$ for antisymmetric velocity disturbances, whereas $\mathrm{Re}_{cr} \approx 20$ and $k_{cr} \approx 1/H$ for symmetric disturbances, in reasonable agreement with the results for the jet with a smooth velocity profile. Moreover, Yamada also disclosed the existence of a number of eigenmodes of both antisymmetric and symmetric types, with only the first mode leading to unstable dIsturbances and the, values of Re_{cr} and k_{cr} mentioned above. The analogy between the normal disturbance modes of the boundary-layer flow and the even (or odd) modes in plane-parallel jet flows suggests that the total number of discrete higher modes in a jet flow must be finite; however, this assumption has apparently never been proved.

It is also natural to expect that a continuous spectrum of wave-like disturbances must also exist in the case of plane-parallel jet flows in an unbounded space. In fact, the existence of such a spectrum for a wide class of model jet velocity profiles $U(z)$ was proved by Grosch and Salwen (1978a); (see also the paper by Herron (1983) who proved the existence of the continuous spectrum of frequencies ω or the phase velocities c for a very wide class of plane-parallel or nearly plane-parallel velocity profiles, in the half-space or the entire unbounded space). Grosch and Salwen considered both the temporal (k real, c and $\omega = kc$ complex) and spatial (ω real, k and $c = \omega/k$ complex) formulations of the nomal-mode problem and also investigated the form of 'generalized eigenfunctions' corresponding to the points of the continuous spectrum (detailed results were given for the velocity profile (2.87) and were also discussed by Craik (1991)). It was shown that in the temporal formulation the continuous spectrum at given values of k and Re fills the half-line

$$c = -i\frac{ak}{\mathrm{Re}}, 1 \leq a < \infty, \tag{2.88}$$

in the complex c-plane, and to each point of this half-line there correspond two 'eigenmodes' both having the form of temporally-decaying standing waves. The completeness of the system of the eigenmodes corresponding to either a discrete or a continuous spectrum, and the expansion theorem expressing any regular two-dimensional disturbance in terms of eigenmodes, were proved under rather general conditions by Herron (1983).

In the case of spatial formulation only a part of the full continuous spectrum was found in the paper (1978a) by Grosch and Salwen; this part fills a definite curve in the complex c-plane and the corresponding modes form a family of spatially decaying traveling waves. Other parts form three different branches; their general description is given by Salwen and Grosch (1981); however, they paid no special attention to the case of a jet flow, and the completeness of their system of eigenmodes was not proved.

Consider now the case of a *two-dimensional wake* behind a solid body covering the line $x = 0, z = 0$ in a steady uniform flow having constant velocity U_0 parallel to the Ox axis. Such a wake has a velocity profile of the form shown in Fig. 2.31c. In the particular case of a wake of flat plate parallel to the flow direction, it is known (see, for example, Goldstein (1933); or Goldstein (1938), Vol. II, Sect. 2.48; or Schlichting (1979), Chap. IX, Sect. e) that at not too small distances from the solid body the wake can often be considered as being two-dimensional and having a 'Gaussian' velocity profile of the form

$$U(z) = U_0[1 - \alpha \boldsymbol{e}^{-\ln 2(z/H)^2}]. \tag{2.89}$$

Here $\alpha = [U_0 - U(0)]/U_0$ is the dimensionless velocity defect on the center line and the factor ln2 is included in the exponent to make the length H equal to the half-width of the wake, determined by condition $U_0 - U(H) = \frac{1}{2}[U_0 - U(0)]$. The linear stability of even disturbance modes in such a wake flow was investigated (in both temporal and spatial formulations) by Wazzan et al. (1973). (They assumed that $\alpha = 0.692$ to correspond to the experimental data of Sato and Kuriki (1961), but in fact, if $U\text{o} - U(0)$ is used as the velocity scale, results do not depend on the value of α). According to Wazzan et al. $\text{Re}_{\text{cr}} = \{[U_0 - U(0)]H/\nu\}_{\text{cr}} \approx 3.55$ for the 'Gaussian wake'; the forms of the neutral curves they found are shown in Fig. 2.34, in (k, Re) and (ω, Re) planes. (The question about the possible existence of a supplementary stability region similar to that found by Silcock for a jet flow has not been considered in the available literature.) The curves of constant spatial amplification rates $k^{(i)}H =$ const. in the (ω, Re)-plane were also computed by Wazzan et al., together with the characteristics of the most unstable disturbances at several values of Re, which were found to be in satisfactory agreement with the data of Sato and Kuriki for the wake region where the plane-parallel approximation and linear stability theory seem to be applicable. Another computation of the neutral curve (in the (k, Re)-plane) for the wake flow with velocity profile (2.89) was made by Ling and Reynolds (1973), who also considered approximate non-parallel flow corrections which proved to be similar to those found by these authors for a plane-jet flow.

One more important type of flow in an unbounded space is that for which $U(z) \to \text{U}_0$ as $z \to \infty$ and $U(z) \to -U_0$ as $z \to -\infty$. The simplest flow of this type is the idealized flow with a broken velocity profile shown in Fig. 2.31d; this flow describes a plane surface of tangential velocity discontinuity. More real profiles of the same type are shown in Fig. 2.31e, f (the last one has two symmetric inflection points); they correspond to the *laminar mixing layer* of two plane-parallel flows, flowing one above the other with different velocities. As has been already mentioned in Sect. 2.4, the inviscid instability of flows with the velocity profile shown in Fig. 2.31d was strictly proved as early as Helmholtz (1868); and Kelvin (1871); as for the flow represented

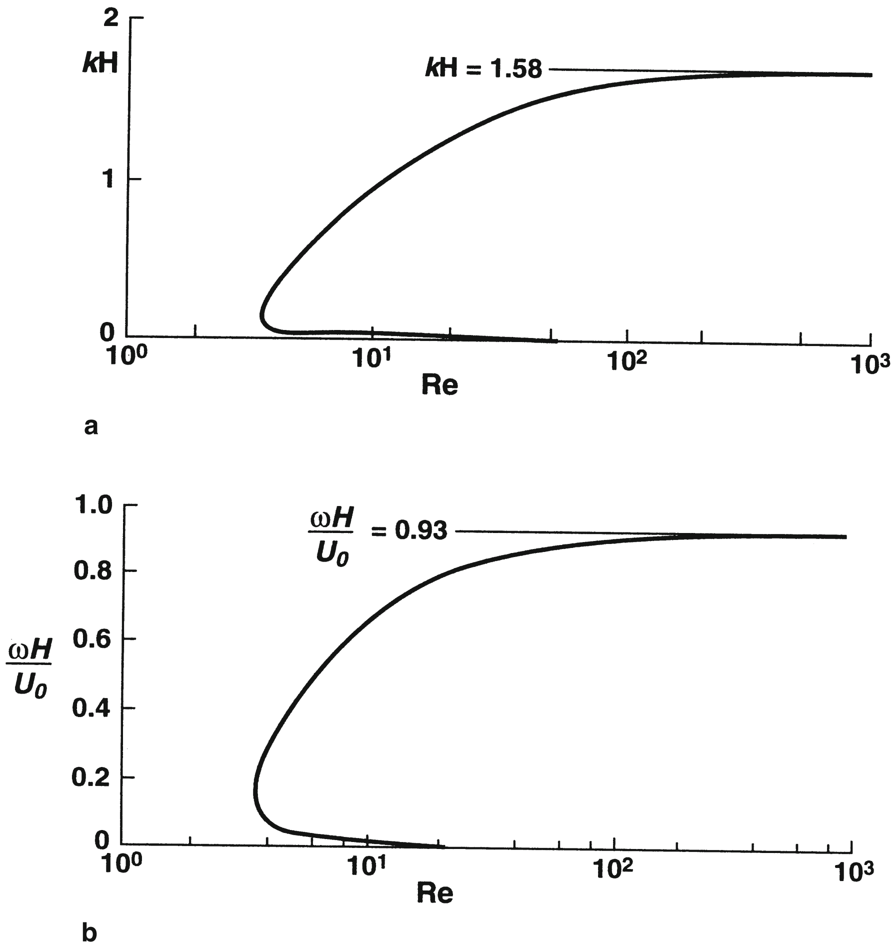


Fig. 2.34 The neutral curves for a Gaussian plane wake in the (k, Re) plane **a** and in the (ω, Re) plane **b**, according to Wazzan et al. (1973)

by Fig. 2.31e, its instability was proved by Rayleigh (1894). The temporal and streamwise evolution of unstable wave-like disturbances in the flow of an inviscid fluid with the hyperbolic-tangent velocity profile of the form shown in Fig. 2.31f was also investigated some time ago, in every detail, by Michalke (1964, 1965); and Gotoh (1965); see also the books by Betchov and Criminale (1967); and Drazin and Reid (1981). Taking the viscosity into account, the stability of a flow with the hyperbolic-tangent velocity profile was studied for the case of relatively large values of Re by Lessen (1950); who found that here $\mathrm{Re}_{cr} < 20$ (in Lin (1961a), it was deduced erroneously from this that in this case Re_{cr} is close to 20). However, later, Tatsumi and Gotoh (1960) showed that the plane-parallel flow of a viscous fluid in an unbounded space, corresponding to a plane mixing zone of two parallel flows, is,

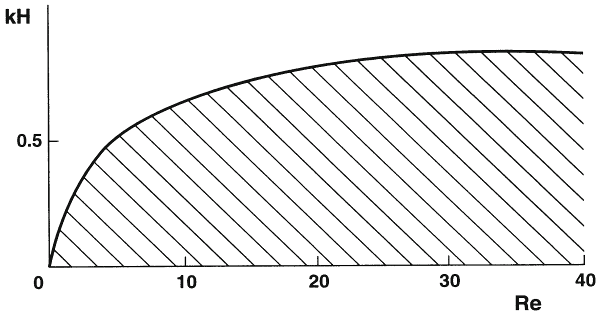


Fig. 2.35 Position of the neutral curve and the region of instability (*shaded*) in the (k, Re) plane for a mixing layer (with the hyperbolic-tangent velocity profile) between two, plane-parallel flows, according to the data of Betchov and Szewczyk (1963)

in fact, unstable at all values of Re for a very wide range of possible velocity profiles (i.e., here $\mathrm{Re}_{cr} = 0$). The form of the neutral curve in the (k, Re)-plane for a plane mixing zone with the hyperbolic-tangent velocity profile $U(z) = U_0 \tanh(z/H)$ was carefully calculated by Betchov and Szewczyk (1963) by numerical integration of the corresponding Orr-Sommerfeld equation; the results obtained are shown in Fig. 2.35 (for another presentation of these results see Drazin and Reid (1981), Fig. 4.28). The details of the integration procedure used, and many additional results, may be found in Betchov and Criminale (1967), Sect. 13.

The higher-order normal modes of wake and mixing layer flows have apparently never been studied. The continuous spectrum of normal disturbances in such flows was briefly considered by Grosch and Salwen (1978a).

Up to this point, only wave-like disturbances (with either the frequency ω or the wave number k being complex) have been considered in Sects. 2.8 and 2.9. However, a parallel fluid flow is always an *open flow* where fluid particles do not remain at any time within a fixed bounded domain but are advected downstream without limit. Here, therefore, it is not enough merely to know whether growing wave disturbances can exist in the flow or not. If disturbances growing with time exist, then it is also important to know whether such a disturbance, localized at $t = 0$ in some bounded domain, will be swept away by the mean motion (thus leading to the original undisturbed state at any fixed location) or whether this disturbance will spread out in all directions, leading eventually to exponential growth everywhere. As it has been already noted in Sect. 2.6 when the paper by Tagg et al. (1990) was discussed, in the first case the flow is called *convectively unstable* while in the second case it is *absolutely unstable.* (Note that the concepts of convective and absolute instabilities are not Galilean invariant, and a given convectively unstable flow can become absolutely unstable after transition to another inertial system of coordinates. Hence, the use of these notions in fact implicates that the system of coordinates used is constrained by some physical condition (e.g., by the requirement that a solid wall, or the aperture of a nozzle, or the solid body generating a wake, must be motionless).)

This terminology was apparently introduced in some studies of plasma instabilities (see, e.g., the surveys by Briggs (1964); and Bers (1983) and other sources related to physics of plasma mentioned by Huerre and Monkewitz (1985, 1990)). However, in fact the distinction between convective and absolute instabilities was noted much earlier, in particular by Twiss (1951) and Landau and Lifshitz (1953). The useful criterion for absolute instability, which apparently also originated in plasma physics, is closely related to the normal-mode method of linear stability theory but is based on consideration of generalized space-time waves with complex values of both ω and k.

Let us recollect that in the temporal formulation of the instability problem the wave number k is assumed to be real and fixed, while $\omega = \omega\ (k, \text{Re})$ are complex eigenvalues of the Orr-Sommerfeld eigenvalue problem. In the spatial formulation ω is real and fixed and $k = k\ (\omega, \text{Re})$ are complex eigenvalues. If, however, both ω and k are assumed to be complex, then the Orr-Sommerfeld equation and boundary conditions imply a 'dispersion relation' of the form

$$D(k, \omega, \text{Re}) = 0. \tag{2.90}$$

Here D is the function which at given Re determines the possible values of the pair (k, ω) of complex numbers for which the Orr-Sommerfeld equation has a solution satisfying the given boundary conditions. The computation of the dispersion relation (2.90) requires only the determination of a great number of eigenvalues $\omega(k, \text{Re})$ for various complex values of k and positive values of Re.

The eigenvalue problem of hydrodynamic stability theory with both k and ω complex was first considered by Betchov and Criminale (1966) for some examples of jet and wake velocity profiles and $\text{Re} = \infty$ (i.e., in the inviscid approximation). They determined numerically the dependence of complex k on complex $c = \omega/k$ and found that this dependence is apparently analytic everywhere, with the exception of a few singular points in the complex c-plane. Later, Gaster (1968) investigated this general eigenvalue problem for the full Orr-Sommerfeld equation with $v \neq 0$. He explained why the singularities discovered by Betchov and Criminale must occur, and showed that the singularities in the c -plane are at the points where $dc/dk = 0$, while the singularities in the ω-plane (if the dependence of ω on k and vice versa are studied) correspond to points where $d\omega/dk = 0$, and hence the wave packets have zero group velocity. Moreover, Gaster estimated the contribution of the singularities to the perturbations generated by a pulse (i.e., localized initially at a point), which was proved to be very significant.

The results mentioned are in fact used in the derivation of the general criterion for absolute instability, which was first proposed by Briggs (1964) and then became standard. Here the resolved dispersion relation $\omega = \omega(k, \text{Re})$ (where ω and k are complex) is used and attention is paid mainly to the corresponding branch-point singularities in the complex ω-plane, i.e., to points $\omega_s = \omega_s^{(r)} + i\omega_s^{(i)}$ where $\partial\omega_s(k, \text{Re})/\partial k = 0$. According to the criterion, the type of the instability is determined by the location of the points ω_s: if all the singularities lie in the lower half-plane (i.e., $\omega_s^{(i)} < 0$ for all points ω_s), then it is a *convective instability,* but if there is a point ω_s in the upper half-plane (so that $\omega_s^{(i)} > 0$), then the flow is *absolutely unstable.* For the proof of this criterion and further discussion related to it see, e.g., the papers by Huerre and Monkewitz (1985, 1990) and Huerre (1987); cf also the book by Lifshitz and Pitaevskii (1981), Sect. 62.

One of the first proofs of the convective nature of instability in a fluid flow was given by Iordanskii and Kulikovskii (1965). These authors considered plane Poiseuille flow and showed that at high enough Reynolds number its instability is convective. Later Deissler (1987) combined the asymptotic analysis for large values of Re with the results of numerical solution of the Orr-Sommerfeld equation for complex k and ω in a wide range of Re, and showed that plane Poiseuille flow is convectively unstable for any Re larger than Re_{cr}, where Re_{cr} is the critical Reynolds number of the linear stability theory. (Note that in the book by Landau and Lifshitz (1987), Sect. 28; it was stated that the nature of the instability in a plane Poiseuille flow is unknown for moderately large values of Re; this remark now is not true.) For plane-parallel boundary-layer flow with the Blasius velocity profile, the existing calculations (in particular by Gaster (1968, 1975) and Tam (1981)) suggest that here the instability also has a convective nature.

Investigations of the nature of instability in mixing layers, jets and wakes in an unbounded space are very numerous; the papers by Koch (1985); Huerre and Monkewitz (1985, 1990); Hultgren and Aggarwal (1987); Monkewitz (1988); Yang and Zebib (1989); Hannemann and Oertel (1989); Pavithran and Redekopp (1989); and the collection of papers edited by Eckelmann et al. (1993) containing a great number of additional references, are just typical examples. In these flows the nature of instability often depends upon the values of some specific parameters and can change when these values are changed. So, for example, in the case of a mixing layer with the velocity profile $U(z) = U_0 \tanh(z/H) + U_1$, which is unstable at any value of Re, the nature of the instability depends upon the value of the ratio $\mathrm{S} = U_0/U_1$ (the dependence on U_1 demonstrates that the nature of the instability is not Galilean invariant). According to Huerre and Monkewitz (1985) at $\mathrm{Re} = \infty$ (i.e., in an inviscid fluid) the instability of this mixing layer will be convective if $\mathrm{S} < \mathrm{S}_{cr} \approx 1.315$, but for $\mathrm{S} > \mathrm{S}_{cr}$ the flow will be absolutely unstable. Later Pavithran and Redekopp (1989) considered a more complex model of a weakly-compressible mixing layer between two streams of different velocities and temperatures, having hyperbolic-tangent profiles of $U(z)$ and $T(z)$; here the regions of convective and absolute instabilities depend on several parameters and often have complicated shapes. Results by Betchov and Criminale (1966) and Mattingly and Criminale (1972) imply that at $\mathrm{Re} = \infty$ a plane-parallel wake flow with the velocity profile $U(z) = U_1\ [1\text{-Q}\ \mathrm{sech}^2\ (z/H)]$ will be convectively unstable when $\mathrm{Q} < 0.94$ and absolutely unstable when $\mathrm{Q} > 0.94$. Later, more detailed studies of convective and absolute instabilities in wake flows with various values of Re and various forms of the velocity profile were carried out by Koch (1985); Monkewitz and Nguyen (1987); Hultgren and Aggarwal (1987); Monkewitz (1988); Hannemann and Oertel (1989); Yang and Zebib (1989); Wallace and Redekopp (1992) and some other authors; see also the survey by Oertel (1990) and the collection edited by Eckelmann et al. (1993). In these papers the distributions of the regions of convective and absolute instabilities were studied in different wakes and for different values of Re and it was shown that in all cases an absolutely unstable region begins to form only when Re exceeds a definite critical value (close to 20 in the case of a cylinder wake), and grows with increasing Re. Note also that according to Monkewitz and Nguyen (1987) the wakes behind two-dimensional bluff bodies can sometimes be convectively unstable near the body, absolutely unstable further downstream, and again convectively unstable still further. Jet flows are convectively unstable (but this is apparently not documented in the available literature); the absolute instability appears here only when a jet (which can be either plane or round) is heated or otherwise has variable density (see, e.g., Monkewitz and Sohn (1988); Sreenivasan et al. (1989); Huerre and Monkewitz (1990); and Krizhevsky et al. (1996) and references therein).

Until this point we have considered convective and absolute instabilities only for flows of a fluid moving in a space that extends to infinity. However these concepts are also meaningful in cases where undisturbed fluid particles remain in bounded flow regions but the unstable disturbances have the form of growing waves propagating to infinity. Just such a case occurs in circular Couette flow between strongly counter-rotating cylinders. It has already been mentioned in Sect. 2.6 that Tagg et al. (1990) investigated conditions under which the instability of such a flow at $\mathrm{Re} \geq \mathrm{Re}_{\mathrm{cr}}$ will be of convective or of absolute type. They computed the lines in the (η, μ) plane dividing the regions with different values of the critical azimuthal wave number n_{cr} for first appearing (as Re_1 increased) absolute instability and found that the line dividing regions where $n_{\mathrm{cr}} = 0$ and $n_{\mathrm{cr}} = 1$ is nearly identical to the line computed by Langford et al. (1988) who considered arbitrary instabilities of any type (see Fig. 2.13a in Sect. 2.6), while in other cases the new dividing lines are placed significantly to the left of the corresponding lines in Fig. 2.13a.

Another pair of conflicting instability concepts, which attracted much attention in recent years, is given by local/global concepts of instability. These concepts refer to nearly plane-parallel flows where the mean velocity is slightly nonuniform in the streamwise direction. There, *local instability* means the instability of a plane-parallel flow with the local velocity profile, while the term *global instability* is used for the presence of instability modes involving the entire flow field. These concepts will not be considered here; a good survey of the related results was given by Huerre and Monkewitz (1990) and Monkewitz et al. (1993).

2.9.4 Circular Poiseuille Flow and Certain Other Axisymmetric Flows

The problem of the instability and transition to turbulence of Poiseuille flow in a circular tube is probably the most intriguing and interesting of all stability problems, closely related to the classical experiments of O. Reynolds described in Sect. 2.1. However, this problem is also very difficult and, in spite of many attempts to solve it, until now only a few rigorous results have been obtained in this field. Therefore, our discussion of this stability problem will of necessity be rather short.

Circular Poiseuille flow corresponds to the axisymmetric steady solution of the Navier-Stokes and continuity equations satisfying the condition that the velocity field $\boldsymbol{u}(\boldsymbol{x}) = \{U(r), 0, 0\}$ vanishes at $r = R$, where R is the tube radius. According to this solution $U(r) = A(R^2 - r^2)$, where $A = [-\partial p/\partial x]/4\rho\nu = \mathrm{const}$. As was explained in

Sect. 2.84, linearized equations for the velocity component and pressure disturbances, written in cylindrical coordinates r, ϕ, x, have in this case the form (2.73–2.73′) with $U(r)$ as given above.

As in Sect. 2.84, we shall consider only the normal modes of disturbance, i.e., we shall assume that the velocity and pressure disturbances are proportional to $\exp\{i[k(x-ct)+n\phi]\}$ with coefficients ('wave amplitudes') depending only on r. In the inviscid approximation (i.e., for $\nu = 0$), equations for these amplitudes have the form (2.74), which implies equation (2.75) for the amplitude $F(r)$ of the radial velocity component. If $n = 0$ and $U(r) \sim (R^2 - r^2)$, then the last term on the left-hand side of Eq. (2.75) vanishes. It is easy to see that in this case no discrete eigenvalues c exist and the eigenvalue problem related to Eq. (2.75) has only a continuous spectrum filling the interval $[U_{min}, U_{max}] = [0, AR^2]$. (This situation is exactly the same as that for plane Couette flow of an inviscid fluid.) However, in the spatial formulation, i.e., when $\omega = kc$ is fixed and real an k is a complex eigenvalue, the inviscid Poiseuille-flow eigenvalue problem has an infinite sequence of easily-determined purely-complex eigenvalues kj with $\Im m k_j > 0$, i.e., corresponding only to damped waves; see Gill (1965). When $n \neq 0$, the explicit determination of the eigenvalues is not so simple but if the temporal eigenvalue problem is considered (i.e., k is real), then Eq. (2.76) can be applied. It is easy to show that in this case the function $Q'(r)$ in Eq. (2.76) does not change sign inside the flow; hence the circular Poiseuille flow in an inviscid fluid can have no modes of disturbance growing with time for any value of n.

In the case of viscous fluid with $\nu \neq 0$, the determination of the possible normal modes of disturbance for the circular Poiseuille flow is much more difficult, and here our knowledge is still rather poor. For a rather long time, almost all authors considered only the simplest case of axisymmetric normal modes of disturbance (i.e. independent of ϕ so that $n = 0$); see, e.g., Sexl (1927a, b); Pretsch (1941); Pekeris (1948); Schensted (1960); Gill (1965; here the spatial eigenvalue problem was studied), Betchov and Criminale (1967), Sect. 56; Davey and Drazin (1969); and Drazin and Reid (1981), Sect. 31.2. A number of additional references can be found in these sources and in the book MY1. It has already been noted in Sect. 2.84 that in this case the system of Eqs. (2.73–2.73′) can be transformed into a system of three equations for the unknowns $f^{(r)}$, $f^{(x)}$ and g, and one equation for the unknown $f^{(\phi)}$. The last equation can be solved independently, and usually the boundary conditions imply that the zero solution $f^{(\phi)} = 0$ must be used. Then the unknowns $f^{(x)}$ and g can be eliminated from the remaining three equations leading to a singular fourth-order differential equation for $f^{(r)}$ (having a singular point at $r = 0$), which, together with the appropriate boundary conditions, define an eigenvalue problem which determines possible values of c or k.

Schensted (1960) proved that this eigenvalue problem (in its temporal formulation, i.e., when c is the unknown eigenvalue) has an infinite set of discrete eigenvalues and the corresponding system of eigenfunctions is complete (i.e., any axisymmetric initial disturbance can be expanded in these functions). In the other work mentioned above, results of numerous computations of eigenvalues and eigenfunctions were presented, and it turned out that all the eigenvalues found correspond to stable modes (i.e., here $\Im m c < 0$ or, in the spatial formulation, $\Im m k > 0$). Therefore, all the experts long ago became confident that the circular Couette flow is stable with respect to any infinitesimal axisymmetric disturbance, though a rigorous proof of this fact was found only in 1991 by Herron. (Before this the above-mentioned confidence was support by the profound mathematical analogy between the eigenvalue problems related to stability of plane Couette flow and stability of circular Poiseuille flow for axisymmetric disturbances and by the available relatively old experimental data by Leite (1959); Reshotko (1958); Kuethe and Raman (1959) and Kaskel (1961), who studied streamwise evolution of some axisymmetric disturbances produced artificially in the initial part of tube flow.)

The stability problem for the general case of arbitrary nonaxisymmetric disturbances ($n \neq 0$) in circular Poiseuille flow proved to be even more complicated than that for axisymmetric disturbances. Results related to this general problem began to appear only rather recently. Betchov and Criminale (1967) even assumed that circular Poiseuille flow may be unstable to some infinitesimal nonaxisymmetric disturbances. However, the observed possibility of extending the laminar regime of a tube flow to extremely high values of Re by decreasing the disturbance level inclined most people to believe that the Poiseuille flow in a tube is stable with respect to any infinitesimal disturbance. Sharing this popular opinion, and taking into account also he results of a few early computations of eigenvalues $c = c(k, n, \mathrm{Re})$ for some nonaxisymmetric disturbances with $n \neq 0$, Monin and Yaglom (in MY1) and Drazin and Reid (1981) included the assertion of the stability of circular Poiseuille flow with respect to any infinitesimal disturbance in the texts of their books. This assertion was. supported in MY1 by references to the paper by Lessen et al. (1968), where the normal modes with $n = 1$ were computed for a range of k and Re values, and to a preliminary announcement by Salwen and Grosch (1968) about the results of computations of some normal modes with $n \leq 5$, while Drazin and Reid referred only to a subsequent more complete publication by Salwen and Grosch (1972). (An error in the computations described by Salwen and Grosch (1968, 1972); was later corrected by Salwen et al. (1980).) However, progress in computer performance led to the appearance during the 1970s and early 1980s of a number of works where the eigenvalues $c(k, n, \mathrm{Re})$ corresponding to the lower normal mode of the circular-Couette-flow stability problem were calculated for wide ranges of integers n (sometimes up to $n = 30$), and for large portions of the (k, Re)-plane extending up to very high values of Re; see, in particular, the papers by Vilgelmi et al. (1973); Kalugin et.al. (1976), Vanderborck and Platten (1978), Salwen et al. (1980), the book by Goldshtik and Shtern (1977) and also the paper by Garg and Rouleau (1972) devoted to consideration of the spatial formulation (ω real and fixed, k complex) of the Poiseuille-flow stability problem for both axisymmetric ($n = 0$) and the simplest nonantisymmetric ($n = 1$) disturbances. All the eigenvalues found in these works (and also in all other available reliable sources) correspond only to *stable normal modes.* Therefore, at present, in spite of the absence of a strict proof, there is no doubt that unstable normal modes of disturbance, either axisymmetric or nonaxisymmetric, do not exist in a circular Couette flow.

The stability problem for the fluid flow in an annular channel between two concentric cylinders, produced by a pressure gradient parallel to the axis of the cylinders, generalizes in some respects the problems for circular Poiseuille flow and for plane Poiseuille flow since both these flows can be considered as limiting cases of the flow in an annular channel. A number of results obtained in the course of solution of this general stability problem can be found in the books by Joseph (1976) and Goldshtik and Shtern (1977).

Besides the circular Poiseuille flow there are other interesting and practically important axisymmetric flows, in particular axisymmetric jet and wake flows. An axisymmetric jet originates when a fluid under the action of over-pressure flows through a circular aperture into a space which we shall assume to be filled with the same fluid at rest; an axisymmetric wake is formed when an uniform fluid flow of constant velocity $\boldsymbol{u} = \{U_0, 0, 0\}$ meets an axisymmetric (with respect to Ox axis) solid body. These flows are not strictly parallel—their diameter increases with distance from the flow origin, and therefore the radial velocity component does not vanish and the axial velocity component depends not only on r but also on x. However, if the Reynolds number is large enough, then in an axisymmetric jet (and quite often in a wake too) the radial velocity component will be small compared with the axial component and the axial velocity will change with x much more slowly than with r. In other words, under these conditions the flow will be *nearly parallel* and it can be considered locally (when only the flow in a short interval of the Ox axis is of interest), with relatively good precision, as being *parallel* with a velocity field of the form $\{U(r), 0, 0\}$.

Below, most attention will be given to axisymmetric jets; wakes will be mentioned only occasionally. Let us begin with the inviscid theory, where the normal-mode method can be reduced to the eigenvalue problem for Rayleigh's equation (2.75). Exactly as for the plane Rayleigh equation (2.48), Eq. (2.75) has only either real eigenvalues or pairs of complex conjugate eigenvalues c, and has a singularity at the point r_c (which can be complex if $U(r)$ is an analytic function and c is complex) where $U(r) = c$. This singularity considerably Complicates the correct physical interpretation of the complex conjugate eigenfunctions $F(r)$ and $F^*(r)$ corresponding to eigenvalues c and c^*; see, e.g., Lin (1961a), pp. 123–26; and Batchelor and Gill (1962), pp. 535–536. However, these complications do not concern the amplified modes, which are most interesting for determination of the instability conditions.

The first model example of an axisymmetric jet of inviscid fluid unstable with respect to axisymmetric disturbances, was also given by Rayleigh (1879). Much later, Batchelor and Gill (1962) systematically considered the normal-mode stability theory for axisymmetric jets of inviscid fluid. In their paper, the jet flow was assumed to be parallel, and detailed computations were carried out for two models of the velocity profile $U(r)$: an idealized discontinuous 'top-hat' profile (i.e., $U(r) = U_0$ for $r < R$, $U(r) = 0$ for $r > R$; cf. Fig. 2.31a), which had already been considered by Rayleigh, and a more realistic smooth profile of the form

$$U(r) = \frac{U_0}{(1 + r^2/R^2)^2}. \tag{2.91}$$

The 'top-hat' profile models a jet issuing from the open final cross section of a tube at small distances from this cross section; profile (2.90) is a good approximation (valid at a great distance from the jet origin) in the case of a strong jet produced by a point source in an unbounded space filled with motionless fluid (see, e.g., Landau and Lifshitz (1987), Sect. 23, where results of Landau's paper of 1943 are presented, or Batchelor and Gill (1962) and Lessen and Singh (1973)). In the case of the top-hat profile there exists only one mode of discrete eigenvalues $c = c(k, n)$ of the temporal stability problem. Equation (2.75) cannot be used directly in the case of a discontinuous profile $U(r)$; so another method was used by Batchelor and Gill. It permitted them to determine analytically the physically-plausible eigenvalues $c(k, n)$; their imaginary parts proved to be positive at all values of k and n, showing that amplified normal modes exist here for any k and n, i.e., the flow is very unstable. In the case of a profile $U(r)$ of the form (2.91), numerical computation showed here that only wave disturbances with $n = 1$ appear to be unstable for some values of k. Similar results were also obtained by Sato and Okada (1966) for the 'Gaussian' axisymmetric wake (i.e., a wake with velocity profile $U(r) = U_0\ [1 - \alpha \exp\{-\ln 2(r/R)^2\}]$; cf. Eq. (2.89)). In this case the inviscid stability theory showed that unstable wave-like disturbances may again exist only for $n = 1$; moreover, the computed growth rates for unstable disturbances turn out to agree well with the experimental data of Sato and Okada for a laboratory axisymmetric wake having a Gaussian velocity profile (with $\alpha = 0.3$).

The paper by Batchelor and Gill stimulated the appearance of a number of other publications devoted to the study of stability of axisymmetric jets by the normal-mode method. In the interesting survey of this topic by Michalke (1984), containing many additional references, several velocity profiles $U(r)$ used in such studies by him and some other authors were listed. Michalke then selected five of these profiles (both profiles considered by Batchelor and Gill were included) for subsequent analysis. The Batchelor-Gill profiles depend on one-dimensional parameter R characterizing the jet radius; the other three selected profiles depend on R and on one more dimensionless parameter b, characterizing the shape of the profile. By varying the value of b it is possible to approximate the observed jet velocity profiles at different distances from the jet origin.

In contrast to Batchelor and Gill, Michalke considered not the temporal but the spatial version of the stability theory (ω fixed and real, k complex), which is more suitable for comparison of the theoretical results with the available observations. This replacement does not cause many changes (but now the fact that k is a complex eigenvalue does not imply that k^* is also an eigenvalue). In particular, in a jet with a top-hat velocity profile, unstable waves with any values of ω and n can exist, while in the case of profile (2.91) unstable waves correspond only to $n = 1$ and a restricted range of ω. In the case of the other profiles considered by Michalke, waves growing with x exist for some restricted ranges of frequencies ω, at least when $n = 0$ and 1 but apparently for higher values of n also (cf. the paper by Mattingly and Chang (1974) where for one of these profiles results of calculations were presented (and compared with the experimental data) for $n = 0$, 1, and 2).

The most significant discrepancy between results of temporal and spatial stability theory consists in the existence, according to the spatial theory, of additional normal modes of wave-like disturbances in some frequency ranges, having no analogues in the temporal theory. These new modes were discovered by Michalke and were called by him "irregular" Since they have some unusual properties. The physical meaning of these modes is not clear; therefore, they will not be considered here.

It has been already explained that in the case of an axisymmetric flow of viscous fluid with $\nu \neq 0$, the equations for normal modes of disturbance become much more complicated than in the case of inviscid flow. Nevertheless, in the late 1960s the numerical solution of these more complicated equations also became practicable with newly-available computers. In consequence of this, a number of works on applications of viscous stability theory to free axisymmetric flows appeared in a short time. One of the first was the dissertation by Burridge (1968) devoted to the stability of a round jet with the velocity profile (2.91). Based on the results of Batchelor and Gill, Burridge investigated only the temporal stability of the first nonaxisymmetric disturbance with $n = 1$, and found numerically that in this case $\mathrm{Re}_{\mathrm{cr}} = (U_0R/\nu)_{\mathrm{cr}} \approx 37.5$ (and $k_{\mathrm{cr}} = 0.43/R$). Slightly later Kambe (1969) published results of the inviscid and viscous stability analysis for the axisymmetric jet with a parabolic velocity profile (having a slope discontinuity at the jet edge). He found that such a jet is stable with respect to axisymmetric ($n = 0$) disturbances but for small enough values of ν (including $\nu = 0$) it is unstable to disturbances with $n = 1$ or 2 (and apparently with any higher values of n also). Calculations of the eigenvalues $c(k, n, \mathrm{Re})$ for this case showed that here $\mathrm{Re}_{\mathrm{cr}} \approx 32.8$ for disturbances with $n = 1$, but it takes considerably greater values for disturbances with $n > 1$ (i.e., the first nonaxisymmetric disturbances are more unstable than all the others).

Contents of the papers by Lessen and Singh (1973) and by Mollendorf and Gebhart (1973), published almost simultaneously, are partially overlapping: both contain the results of spatial normal-mode stability computations of eigenvalues $k\,(\omega, n, \mathrm{Re})$ for a viscous round jet with velocity profile (2.91). (Lessen and Singh, who did not know about Burridge's dissertation, also gave the results of temporal stability analysis.) It was found in both papers that the axisymmetric modes are stable at any value of Re and that instability is possible only for nonaxisymmetric disturbances with $n = 1$. The values of $\mathrm{Re}_{\mathrm{cr}}$ obtained agreed quite well with each other (and with Burridge's earlier results): $\mathrm{Re}_{\mathrm{cr}} = 37.9$ according to Lessen and Singh, and $\mathrm{Re}_{\mathrm{cr}} = 37.6$ according to Mollendorf and Gebhart. Later Morris (1976) repeated the same spatial stability computations once more and found that $\mathrm{Re}_{\mathrm{cr}} = 37.64$, $k_{\mathrm{cr}} = 0.44/R$, $\omega_{\mathrm{cr}} = 0.1U_0/R$; apparently these figures are the most reliable.

A neutral stability curve in the (ω, Re)-plane and stability charts (the graphs of curves $k^{(i)}$ (ω, Re) = const. and $k^{(r)}$ (ω, Re) = const.) can also be found in all three of the above-mentioned papers. Lessen and Singh also gave similar curves in the (k, Re)-plane, related to temporal stability and amplification rates. Moreover, they noted that according to the figures presented, the viscous amplification rates (both temporal $c^{(i)}$ and spatial $k^{(i)}$) are greater than the corresponding inviscid rates for certain ranges of wavenumbers or frequencies. Some values of the viscous rates even exceed the largest values of the inviscid rates, thus showing that for some values of Re a jet of viscous fluid is more unstable than the inviscid jet. This discovery evidently contradicts the popular belief that viscosity always plays a stabilizing role in free shear flows with a point of inflection. It also implies the existence of some closed

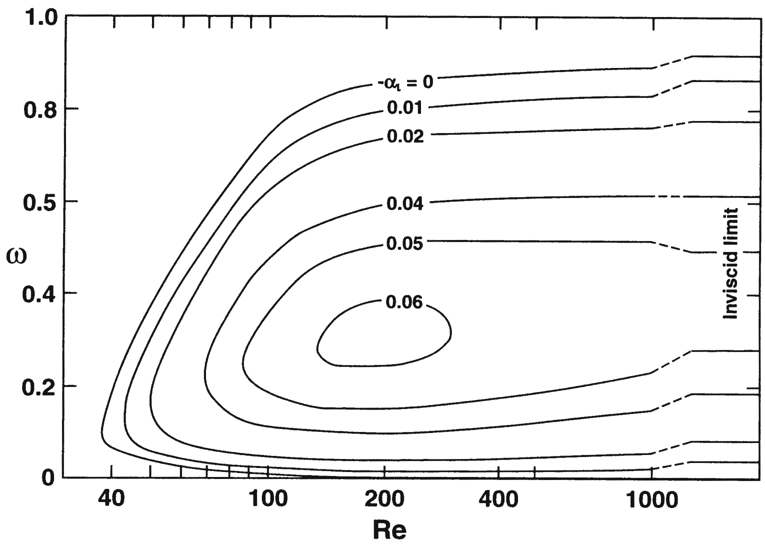


Fig. 2.36 Curves of constant amplification rate in the (ω, Re) plane for a round jet with the velocity profile (2.90), according to Morris (1976)

curves of constant amplification rate in the (k, Re) and (ω, Re) planes and even suggests the possibility of existence of an additional minor neutral stability curve, similar to that first predicted for the case of a plane jet by Stuart (and by Drazin (1961)) and later found by Silcock (see Fig. 2.33). The last remark was made by Lessen and Singh but was not confirmed by the results of their computations (nor of those by Mollendorf and Gebhart), which were insufficiently precise for this aim. Only the more thorough computations by Morris (1976) permitted the discovery of some closed curves of constant spatial amplification rate; see Fig. 2.36. As to the suggested possibility of the existence of an additional minor stability region inside the main neutral curve, it has not been proved (nor disproved) up to now.

The papers by Lessen and Singh, Mollendorf and Gebhart, and Morris also contain much additional information about the normal-mode stability properties of the jet flow considered. In particular, several example of eigenfunctions of the corresponding temporal and spatial eigenvalue problems are depicted there, together with graphs showing the dependence on Re and ω of some stability characteristics of disturbance modes with $n = 0$, 1 and 2; a few examples of eigenvalues corresponding to higher normal modes are also given (but nothing is said about the continuous spectrum of normal modes, which apparently also exists in this case). Besides this, all the three papers mentioned also include results of normal-mode viscous stability calculations for some axisymmetric free flows other than a simple jet with the velocity profile (2.91).

In the first of these papers, the temporal and spatial stability analysis of a round Gaussian wake with velocity profile $U(r) = U_0\ [1-\alpha\ \exp\{-\ln 4(r/R)^2\}]$ was also considered. (Above in this section and in Eq. (2.89) the factor -ln2 was used instead of -ln4 in the equation for the Gaussian wake profile. The factor -ln2 means that the length scales R and H are the so-called half-wake radius and thickness, i.e., the distances at which the velocity defect is one-half of its greatest value at the center of the wake. However, since in Eq. (2.91) R is the radial distance at which the velocity is one-quarter of its maximal value, it is convenient now to use the 'quarter-wake radius' as the length scale R; therefore -ln2 is replaced by -ln4.) Both the temporal (in the (k, Re) plane) and the spatial (in the (ω, Re) plane) neutral stability curves for the wake (given by Lessen and Singh together with graphs for curves of constant amplification rates) show that here $\mathrm{Re}_{cr} = \{[U_0 - U(0)]R/\nu\}_{cr} \approx 32.6$. This value does not differ much from the value of Re_{cr} for a jet flow (note that the length and velocity scales are selected similarly for the two flows). Examination of the curves where $c^{(i)} = \text{const.}$ and $k^{(i)} = \text{const.}$ shows, however, that the amplification rates increase with $\mathrm{Re} - \mathrm{Re}_{cr}$ considerably faster in a wake than in a jet. Hence it is possible to conclude that the axisymmetric wake is more unstable than the axisymmetric jet. At the same time, comparison of values for Re_{cr} given here and in Sect. 2.93 shows that plane jets and wakes are more unstable than the axisymmetric ones.

Morris supplemented his computation of stability characteristics for the fully developed jet far from the jet orifice, where its velocity profile is given by Eq. (2.91), by computations of the same characteristics for two other forms of jet velocity profile corresponding to two stages of transition from the top-hat velocity profile very close to the jet exit to the profile (2.91) far away. (Both these forms were proposed by Michalke (1971) and were also used in his survey of 1984, which included a summary of Morri's results.) For these two profiles some reasonable values of the additional parameter b were selected and then the values of Re_{cr} were computed for the axisymmetric ($n = 0$) and the first nonaxisymmetric ($n = 1$) modes of disturbance. The dependence of the spatial amplification rate $k^{(i)}$ on ω was also computed for various values of Re. It was found that, in all cases considered, Re_{cr} for axisymmetric disturbances is higher than for nonaxisymmetric disturbances with $n = 1$, so that the latter are more unstable than the former.

Mollendorf and Gebhart supplemented the stability computations for the ordinary jet by similar computations for the more complicated case of a vertical jet with given profiles of vertical velocity $U(r)$ and temperature $T(r)$ in the presence of thermal diffusivity and buoyancy (i.e., the Boussinesq form of the dynamic equations and the heat budget equation replaced the Navier-Stokes equations). According to the results obtained, positive buoyancy destabilizes vertical jet flow. However, the volume limitations for this survey do not permit us to consider the contents of this paper in greater detail or to discuss the influence of other physical factors which can significantly affect the stability of jet flows. Note only that Michalke's survey (1984) contains special sections devoted to consideration of the influence on the jet stability of the Mach number effect (i.e., of fluid compressibility at high velocities), temperature effect (for the heated jets), and external flow effect (when the jet orifice is moving with respect to the surrounding fluid; see also Michalke (1993) and references there). Since at present the most important jets are clearly those which are issued from apertures of airplane jet engines, it is clear that the last three physical factors have very great engineering importance; therefore, it is not surprising that there are many works devoted to their investigation.

The determination of corrections for nonparallelism of axisymmetric steady jet flows requires the replacement of the 'parallel' primary velocity field $\boldsymbol{u}(\boldsymbol{x}) = \{U(r), 0, 0\}$ by a more precise (and general) model of the form $\boldsymbol{u}(\boldsymbol{x}) = \{U(r, x), V(r, x), 0\}$. Here V is the radial velocity, and it is assumed that axial variations with x of velocity components are much slower than their radial variations with r, while the absolute value of V is much smaller than that of U. It is then reasonable to replace the coordinate x by the 'slow variable' $X = \varepsilon x$ (where ε is a small 'parameter of nonparallelism') and to rewrite the velocity field in a form $\boldsymbol{u}(\boldsymbol{x}) = \{U_1(r, X), \varepsilon \mathrm{V}_1(r, X), 0\}$. After that it may be assumed that the disturbances can be expanded in powers of ε; in particular, for an axial velocity disturbance u' generated by a vibrating point force of angular frequency ω, this expansion has the form

$$u'(x, r, \phi, t) = [u_o(r, X) + \varepsilon u_1(r, X) + \ldots] \times \exp\{i[k(X)x + n\phi - \omega t]\} \quad (2.92)$$

(cf. Bouthier (1972); and Gaster (1974)). The determination of the most interesting functions $u_0(r, X)$ (which includes the factor describing the slow growth or damping of the disturbance amplitude with x, produced by nonparallelism) and $k(X)$ (which determines the variation of the wave length with x) requires the introduction of some supplementary hypotheses (determining, in particular, the mean jet velocity components $U_1(r, X)$ and $V_1(r, X)$) and the completion of complicated computations using the solution of the corresponding parallel-flow problem. Therefore Crighton and Gaster (1976), who first used such an approach to determine non-parallel flow effects on jets, calculated only the evolution of axisymmetric disturbances ($n = 0$) in inviscid ($\nu = 0$) jet flow. They assumed that the axial velocity $U(r, x)$ was described by a specific profile equation proposed by Michalke (1971), with parameters slowly varying with x in accordance with experimental data; the radial velocity component V was then determined from the continuity equation. The results of the numerical calculations showed that the growth of the unstable normal modes with x was not purely exponential, but proved to be restricted to a certain peak value of the amplitude by non-parallel flow effects (earlier, such restrictions had always been attributed to non-linear effects). The calculated disturbance modes agreed satisfactorily with the organized structures observed in a round jet by Crow and Champagne (1971). Later Plaschko (1979, 1983); (see also Michalke (1984), Sect. 3) applied the same approach to nonaxisymmetric disturbances ($n = 1$ and 2) in inviscid jet flow, and found even better agreement with the available experimental data for orderly structures in round jets. As to the computations of similar non-parallel flow corrections for axisymmetric viscous jets, nothing is known to us about this matter.

The available literature contains many other examples of normal-mode stability computations for various classes of laminar flows. However, space limitations make it impossible to include more material in this chapter.

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Chapter 3
More About Linear Stability Theory: Studies of the Initial-Value Problem

3.1 Beginning of the Story: The Works of Kelvin and Orr

The normal-mode method of the linear stability theory, which was considered in Chap. 2, deals only with special "wave-like" infinitesimal disturbances of a given laminar flow. This method equates the strict instability of a steady flow to the existence of at least one wave-like disturbance (proportional to $e^{-i\omega t}$ and, in the case of homogeneity in the streamwise direction Ox, also to e^{ikx} which grows exponentially as $t \to \infty$ or, in the spatial formulation, as $x \to \infty$), and states that ordinary instability means that there exists a wave-like disturbance which is not damped at infinity. (The adjectives "strict" and "ordinary" will be omitted below in all cases where the difference between two types of instability is unimportant or it is clear from context which instability is considered.) However, is this definition of instability always appropriate? Is it not more reasonable to call a flow unstable, if there exists at least one small disturbance *of any form* which grows without bound after a long-enough time? Moreover, in practice even a bounded but large-enough initial growth of a small disturbance can violate the applicability of the linear stability theory, and make the flow unstable whatever be the asymptotic behavior of this disturbance according to linear theory. In Sect. 2.5 we have already noted in this respect that practical usefulness of the method of normal modes must not be exaggerated. In this chapter this topic will be considered at greater length.

A study of the time evolution of an arbitrary infinitesimal disturbance requires consideration of the solution of the general initial-value problem for linear equations (2.7a,b), obtained by linearization of the Navier-Stokes equations with respect to disturbances $u'_i, i = 1, 2, 3$, and p'. The first attempt to construct the general solution of such an initial-value problem, for the simplest case of a plane Couette flow with linear velocity profile $U(r) = bz, 0 \leq z \leq H$, was made quite early by Kelvin (1887a) (who was still called William Thomson at this time, but was created Baron Kelvin of Largs in 1892). He found a family of exact solutions of Eq. (2.7) for this flow, which depended on three wave numbers k_1, k_2, k_3 (where $k_3 = n\pi/H$ with an integer value of n) and two amplitude coefficients W_0 and V_0 (the second of them for the spanwise velocity component $u'_2 = v$). Kelvin's solution for the vertical velocity $u'_3 = w$ has the form

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$$w(\boldsymbol{x},t) = \frac{W(t)}{k^2 + (k_3 - k_1 bt)^2} \exp i[k_1 x + k_2 y + (k_3 - k_1 bt)z] + w^{(0)}(\boldsymbol{x},t) \qquad (3.1)$$

where $W(t) = W_0 \exp\{-\nu t[K^2 - k_1 k_3 bt + (k_1{}^{2/3})b^2t^2]\}, k^2 = k_1^2 + k_2^2, K^2 = k_1^2 + k_2^2 + k_3^2$ and $w^{(0)}$ $(\boldsymbol{x}, t)$ is the vertical velocity component corresponding to the solution of Eqs. (2.7) which satisfies the initial condition $w^{(0)}$ $(\boldsymbol{x}, 0) = 0$ for any $\mathbf{x}$ and the boundary conditions guaranteeing that $w(\boldsymbol{x},t) = \partial w(\boldsymbol{x},t)/\partial z = 0$ for any x, y and t, if $z = 0$ or $z = H$. (Similar solutions found by Kelvin for disturbances of the other two velocity components and the pressure may be omitted here.) Since $w(\boldsymbol{x},0) = (W_0/K^2)\exp[i(k_1 x + k_2 y + k_3 z)]$, the Fourier analysis allows one to represent any initial value of the vertical velocity disturbance in the form of an integral of the function $w(\boldsymbol{x}, 0)$ over all real values of k_1 and k_2 and a sum over all integer values of n. Noting now that the solution (3.1) decreases exponentially as $t \to \infty$ ($W(t)$ is an exponentially decreasing function and it seems natural to suppose that because of this $w^{(0)}$ $(\boldsymbol{x}, t)$ must also fall off exponentially with time), Kelvin came to the conclusion that any infinitesimal disturbance of a plane Couette flow must tend asymptotically to zero as t increases to ∞, i.e. that this flow is stable with respect to all such disturbances.

Kelvin's conclusion was disputed by Rayleigh (1892) and Orr (1907). Rayleigh's criticism was mainly directed at the arguments presented in Kelvin's paper (1887b), where Eq. (2.41) (now usually called the Orr-Sommerfeld equation) was first derived, but was used erroneously for proving the stability of plane Poiseuille and Couette flows since only real, and not complex, values of the frequency ω were considered by the author. Latter Orr noted (and Rayleigh agreed) that a similar objection can be applied to Kelvin's arguments in the paper (1887a), since here the function $w^{(0)}$ $(\boldsymbol{x}, t)$ was represented as an integral of a harmonic fluctuation $f(\boldsymbol{x},\omega)e^{i\omega t}$ over all real values of ω, while in fact the possible complex values of ω should also be taken into account. Thus, Kelvin considered only some special solutions of the initial value problem, and therefore their asymptotic decay did not prove the stability of Couette flow. (A more modern modification of the same criticism was given by Marcus and Press (1977), who showed that Kelvin's reasoning can be used to prove the linear stability of a flow with the linear velocity profile $\boldsymbol{U}(\boldsymbol{x}) = \{bz, 0, 0\}$ in an unbounded space $-\infty < z < \infty$, but not in a layer of finite thickness between two solid walls.) So, it became clear long ago that Kelvin's proofs (1887a, b) of the stability of plane Couette and Poiseuille flows to infinitesimal disturbances contained incorrigible flaws. Therefore, for a long time very little attention was paid to these papers in the literature on fluid mechanics.

However, Kelvin's papers of 1887 contained some valuable arguments as well. It has already been noted that in the paper (1887b) the very important equation (2.41) was derived. in the paper (1887a) an exact solution of the linearized fluid dynamics equations was found, which decays algebraically (as t^{-2}) as $t \to \infty$, if $\nu = 0$; it was quite different from exponentially decaying (or growing) normal-mode solutions, which Rayleigh began to study (for inviscid flows) a little earlier and which for almost a century completely ousted from the theory of hydrodynamic stability the study of solutions with algebraic asymptotic behavior. Moreover, it was mentioned in passing in the same paper that "solution $w^{(0)}(\boldsymbol{x}, t)$ rises gradually from zero at $t = 0$ and later comes asymptotically to zero again as t increases to infinity." Discussing this paper, Orr (1907) remarked that for some values of viscosity ν and wave numbers k_i, $i = 1, 2, 3$, the whole Kelvin solution (3.1) also at first rises rapidly with time from its initial value at $t = 0$, and only later begins to decrease, tending to zero—this important remark by Orr will be considered in detail below.

Durnig the 20th century interest in exact solutions of dynamic equations was generally growing, and this led to revitalization, in the second half of this century, of attention to "Kelvin's modes of disturbance" (3.1). These modes were then re-examined by a number of scientists, some of whom (e.g., Moffatt (1967); Rosen (1971); Marcus and Press (1977)) apparently did not know Kelvin's old results, and rediscovered them (for more details see the interesting review by Craik and Criminale (1986)). It was, in particular, pointed out in this review that a single Kelvin mode is, in fact, an exact solution not only of the linearized equations (2.7) where $\boldsymbol{U} = \{bz, 0, 0\}$, but also of the full Navier-Stokes equations for the disturbed velocity field $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U} + \boldsymbol{u}\ (\boldsymbol{x}, t)$ with $\mathbf{U}$ as above. This interesting fact (which is, however, not especially important for stability studies since superpositions of exact solutions of nonlinear equations usually do not satisfy them, while stability theory has to do with superpositions of modes) was unknown to Kelvin and Orr. It was mentioned in passing by Moffatt (1967) and, according to Craik and Criminale, was independently discovered after 1965 (i.e., about 80 years after the appearence of Kelvin's paper) by a number of people (including both the authors) and was apparently first discussed in a publication on hydrodynamic stability only by Tung (1983). The above-cited review also contains a number of references to papers where Kelvin's solution was generalized to flows with a linear velocity profile incorporating either Coriolis force (Tung (1983) is just one of them), or density stratification, or both these effects; some of the results relating to stratified flows will be presented later in this chapter. Moreover, Craik and Criminale also found exact "Kelvin-like" solutions of the Navier-Stokes equations for the velocity field $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$ (without any restriction on the sizes of summands $\boldsymbol{U}$ and $\boldsymbol{u}'$) where $U_i(\boldsymbol{x}, t) = b_{ij}(t)x_j + b_i(t), (i, j = 1, 2, 3,$ and summation over the repeated index j is, as usual, assumed), so that the "basic velocity field" $\boldsymbol{U}\ (\boldsymbol{x}, t)$ is here, in general, neither steady nor parallel.

Let us now return to Orr's paper (1907). Here it is remarked (on pp. 74–75) that a superposition of an infinite number of functions $\exp(i\omega_n t)$, where $\Im\mathrm{m}\,\omega_n \geq 0$ for all n, "may at some time have a value which is exceedingly great compared with its initial value, and may even become infinite"; this is in fact a strong criticism of the method of normal modes (which, at the same time, underwent significant development in Orr's paper). Orr pointed out that a stability investigation requires the study of the time evolution of a general solution of the disturbance initial-value problem, but he limited himself to consideration only of some special exact solutions of it. In Part I of his paper the stability of inviscid (ideal) fluid flows was investigated, so the main references here were to Rayleigh's papers (1880, 1887, 1892, 1895) devoted to ideal-fluid stability studies. For an inviscid plane Couette flow with velocity profile $U(z) = bz, 0 \leq z \leq H$, Orr found an exact solution for the vertical velocity of the disturbance $w\ (\boldsymbol{x}, t)$ which, after some simple transformations, can be represented in the form

$$w(\boldsymbol{x},t)=\frac{W_0\exp[i(k_1x+k_2y)]}{k^2+(k_3-k_1bt)^2}\left[e^{i(k_3-k_1bt)z}-e^{i(k_3-k_1bt)H}\frac{\sinh kz}{\sinh kH}\right.$$
$$\left.-\frac{\sinh k(H-z)}{\sinh kH}\right] \qquad (3.2)$$

where, as in (3.1), $k=(k_1+k_2)^{1/2}$. Solution (3.2) can be considered as the limiting form of Kelvin's solution (3.1) as $\nu\to 0$. The last two terms in the brackets here correspond to the term $w^{(0)}(\boldsymbol{x},t)$ in (3.1); they provide the fulfilment of the boundary condition that $w=0$ at $z=0$ and $z=H$, but do not vanish at $t=0$ [therefore here $w(\boldsymbol{x},0)\neq(W_0/K^2)\exp\{i\ (k_1x+k_2y+k_3z)\}$]. However, replacing these two terms by slightly more complicated combination of trigonometric and hyperbolic functions, it is not difficult to obtain an exact solution with just such an initial value of the vertical velocity; see Orr (1907), pp. 26–27, or Drazin and Howard (1966), p. 28. Expressions for the other velocity components corresponding to solution (3.2) or Orr's related solution are not so simple; they were given by Orr only for the case of a two-dimensional disturbance where $v=0$ and $k_2=0$.

Orr pointed out that the solutions obtained imply the existence of small disturbances of inviscid plane Couette flow, which can grow indefinitely before they begin to decay with time. In fact, let us assume that $k_1>0$, $k_3>0$ and $b>0$ and exclude from consideration the close vicinity of solid walls, where $z\leq\varepsilon H$ or $z>H$ $(1-\varepsilon)$ for some very small number ε. Then it is possible to choose kH large enough (i.e., the horizontal wavelengths small enough compared to the flow thickness H) to make negligibly small the contributions of the second and third terms in the brackets on the right side of Eq. (3.2). In such a case $|w(\boldsymbol{x},t)|$ will grow with time from its initial value $|w(\boldsymbol{x},0)|=|w|_o$ to a maximum $|w|_{\max}$ at time $t_{\rm opt}\approx k_3/k_1\ b$ with $|w|_{\max}/|w|_0\approx(k^2+k_3^2)/k^2=1+k_3^2/(k_1^2+k_2^2)$. This shows that $|w|$ can reach an arbitrarily large value if k_3 is chosen to be large enough (and $t_{\rm opt}$ then also becomes very large). According to Eq. (3.2), $|w(\boldsymbol{x},t)|$ diminishes with time without limit (asymptotically as $(t-t_{\rm opt})^{-2}$) after the critical time $t_{\rm opt}$; however, if the disturbance grows greatly at smaller values of t, then the validity of the above equation (which follows from the linear stability theory) at $t>t_{\rm opt}$ becomes quite questionable. This is the reason why Orr said that, according to his results, plane Couette flow of inviscid fluid is *practically unstable,* and this can explain the flow instabilities observed in other, but similar, types of flows.

In the case of a two-dimensional disturbance with $k_2=0$, similar results were obtained by Orr for the temporal evolution of the corresponding streamwise velocity component and kinetic energy density per unit mass, T^*, of the disturbance. It was found that for some values of k_1 and k_3 these quantities also increase greatly when time increases from $t=0$ up to a certain critical time $t_{\rm cr}$ and only after this time do they decrease, tending to zero; see also an account of these results by Orr given by Farrell (1982). Plane-parallel inviscid flow with an arbitrary continuous velocity profile $U(z)$, $0\leq z\leq H$, was also considered by Orr; however, no strict proofs were obtained for this case and only some qualitative reasons were presented, which gave the impression that as a rule the situation here does not differ very much from that of a plane Couette flow.

Special attention was given by Orr in Part I to the important case of Poiseuille flow in a circular tube. This flow has the parabolic velocity profile $U(r) = A(R^2 - r^2)$, $0 \le r \le R$, and only the most simple axisymmetric two-dimensional velocity disturbances of the form $\boldsymbol{u}'(x,t) = \{u(r,x,t),\ 0, w(r,x,t)\}$, where $u = u'_x$ and $w = u'_r$ are streamwise and radial velocity components, were considered in his paper. Analyzing the time evolution of such disturbances in an inviscid fluid, Orr found that if $w(r,x,0) = U_0 \exp\left[i\left\{k_1 x + (k_2 r)^2\right\}\right]$ (the initial value of the component u can be easily determined in this case from the continuity equation), then the values of the disturbance amplitude and of its kinetic energy both increase with time at first (and the values of k_1 and k_2 may be chosen to give whatever growth is wanted) and only later begin to decay, tending to zero as $t \to \infty$. Orr supposed that the existence of such strongly growing disturbances can explain instability of a tube flow as studied by Reynolds (1883). (Note however that Orr's results do not agree well with the results of recent more accurate computations which will be considered below in Sect. 3.34. These new results suggest that only nonaxisymmetric disturbances can undergo substantial transient growth in a tube flow.)

In Part II of his paper Orr turned to the stability problem for viscous flows and therefore most attention was paid to Kelvin's papers (1887a, b). Orr presented detailed analysis of errors made by Kelvin in his reasoning and then considered the special exact solution (3.1) of linearized dynamic equations for disturbed plane Couette flow of viscous fluid. He explained that the supplementary solution $w^{(0)}(\boldsymbol{x}, t)$ of these equations, which provided the fulfilment of the boundary conditions at the walls, can be made as small as is wanted everywhere except in the close vicinity of the walls, if the wave numbers $k_i, i = 1, 2, 3$, are chosen to be very large compared with $1/H$ (i.e., wavelengths in all directions are much smaller that the thickness of the flow). Hence, if $k_i H \gg 1$ for all three values of i and the point $\boldsymbol{x}$ is not too close to a wall, then the time evolution of the first term on the right side of (3.1) will play the main part. The numerator of this term decreases exponentially with time [at first as $\exp(-\nu K^2 t)$ and finally as $\exp\{-\frac{1}{3}\nu(k_1 b)^2 t^3\}$], but the denominator also decreases with time until $t = t_{cr} = k_3/k_1 b$, and therefore it is clear that, if the viscosity ν is sufficiently small, the disturbance $w(\boldsymbol{x}, t)$ will at first grow with time in spite of the exponential decrease of $W(t)$. It is clear that if ν is so small that the decrease of $W(t)$ between $t = 0$ and $t = t_{cr}$ is negligible, the growth of $|w(\boldsymbol{x}, t)|$ may be made as large as desired by appropriate choice of wave numbers k_i. According to Orr, the existence of disturbances having such properties show that the plane Couette flow of a viscous fluid is *practically unstable* for sufficiently small viscosity ν (i.e. for sufficiently high values of the Reynolds number $\mathrm{Re} = H^2 b/\nu$).

Orr also made some approximate calculations for the case of two-dimensional disturbances, where $k_2 = 0$ and $v(x,t) = 0$. He determined the size of a disturbance by specifying its kinetic energy density T^*, allowed moderate values for $k_1 H$ and $k_3 H$, and used two modifications of Kelvin's solution (3.1) which satisfied two different boundary conditions at the walls (both simplifying the standard conditions

Table 3.1 Characteristics of plane-wave disturbances optimally growing in plane couette flow at various values of Re. (After Butler and Farrell (1992))

Re	t	k_1	k_2	$E_{max}/E(0)$
4000	467	0.0088	1.60	18956
2000	234	0.0175	1.60	4739
1000	117	0.035	1.60	1184.6
500	59	0.067	1.60	296.0
250	30.2	0.12	1.61	73.9
125	16.1	0.144	1.63	18.55
62.5	8.2	0.0024	1.65	4.87
31.25	3.21	0	1.62	1.50

of vanishing velocity there). It was found that for moderate values of $k_1 H$ and $k_3 H$ the maximal growth of kinetic energy T^* not only depends on $\text{Re} = H^2 b/\nu$, but is also very sensitive to the form of boundary conditions. According to these calculations, at $\text{Re} \approx 1900$, the maximal value of $T^*(t)/T^*(0)$ can be close to 10,000, at least for one form of the boundary conditions used. This is, of course, only a crude estimate (since it was obtained for incorrect boundary conditions) but it strengthens Orr's conclusion about the practical instability of the flow considered, in spite of the asymptotic approach of $T^*(t)/T^*(0)$ to zero as $t \to \infty$. (The crude estimate by Orr of the maximum possible growth of the disturbance kinetic energy in Couette flow may be compared with the results of the first computation of this maximum by Butler and Farrell (1992), presented in Sect. 3.33, Table 3.1; note however that $\text{Re} = H^2 b/4\nu$ in this table.)

It is worth noting that for Orr himself the proof of practical *instability* to infinitesimal disturbances of a plane Couette flow and of some other simple flows of an inviscid or slightly viscous fluid was apparently the main aim of his investigation. (This explains why Orr did not study the general initial-value problem for an arbitrary disturbance; for his purposes it was enough to consider only special exact solutions of disturbance equations). Curiously enough, although his paper of 1907 became a standard reference in all the literature on hydrodynamic stability, it was usually referred only in relation to the so-called Orr-Sommerfeld equation, which was, in particular, widely used (at first unsuccessfully and then successfully) to prove the *stability* (in the sense accepted in the normal-mode method) of Couette flow with respect to infinitesimal disturbances. At the same time, all other Orr's results (except, perhaps, those on the "energy method" of nonlinear hydrodynamic stability theory, which will be considered later in this book) were almost never mentioned in the literature for many decades (Willke's paper (1972) was apparently one of the earliest exceptions to this). It was only recently that Orr's concept of practical instability and his results related to it achieved wide popularity, began to be cited frequently, and became cornerstones of modern, quite sensational, developments in the linear theory of hydrodynamic stability—described, in particular, by Trefethen et al. (1993) and Grossmann (1995, 1996). Let us now consider these developments.

3.2 Studies of the Inviscid Initial-Value Problem for Disturbances in Plane-Parallel Flows

3.2.1 Discussion of General Results and Associated Examples

Many years passed by after Kelvin's unsuccessful attempt (1887a) to find the general solution of the initial-value problem for an infinitesimal disturbance in a particular steady laminar fluid flow before the next such attempt was made. One of the first new publication on the initial-value approach to hydrodynamic-stability theory was the interesting paper by Eliassen et al. (1953) who studied evolution of two-dimensional disturbances in a plane-parallel flow of an inviscid stratified fluid with the velocity profile $U(z) = bz$ and density profile $\rho(z) = \rho_0 \exp(-az), 0 \le z \le H$. However, these authors themselves commented that their mathematical derivations were not rigorous (see also critical remarks about this work by Dikii (1960a) and Hartman (1975)). Later a more rigorous approach to the same problem was made by Case (1960b); Dikii (1960a) (in both these papers it was assumed that $H = \infty$,) and some other authors. These works will be considered at greater length in Sect. 3.23. For now, we will discuss the more simple case of a plane-parallel flow of inviscid homogeneous (i.e., constant-density) fluid, whose stability was also studied by the initial-value-problem method by Case (1960a) and Dikii (1960b).

Let us assume at first, as Case and Dikii did, that the disturbance is two-dimensional, i.e., $\boldsymbol{u}'(\boldsymbol{x}, t) = \{u(x, z, t), 0, w(x, z, t)\}$. Then, substituting this disturbance and the mean velocity $\boldsymbol{U} = \{U(z), 0, 0\}$ into the linearized dynamical equations (2.7) with $v = 0$, and then eliminating the unknowns u and p' from the system obtained, we come to the following equation (often referred to as the *Rayleigh equation in space and time*) for the unknown function $w(x, z, t)$ satisfying the boundary conditions $w(x, z, t) = 0$ at $z = 0$ and $z = H$:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) w - U''(z)\frac{\partial w}{\partial x} = 0. \tag{3.3}$$

Here $U'' = d^2U/dz^2$; as in Chap. 2, we will denote by primes both differentiations on z and fluctuations of fluid-dynamic quantities, hoping that this will not cause confusion. Equation (3.3) has the same form as Eq. (2.53) for the stream function $\psi(x, z, t)$ of a two-dimensional disturbance, and it differs from the more general Eq. (2.38) only by the absence of terms containing $\partial/\partial y$ and v.

The method of normal modes consists of finding the "wave-like" solutions of Eq. (3.3) (proportional to $e^{i(kx-\omega t)} = e^{ik(x-ct)}$). The set of "eigenvalues" ω (or $c = \omega/k$), for which wave-like solutions exist, form the discrete frequency (or phase-velocity) spectrum of flow disturbances (depending, generally speaking, on k). It is however well known that the set of all wave-like solutions of Eq. (3.3) is not complete (i.e., their linear combinations do not exhaust all the possible disturbances) since a continuous spectrum also exists here (see Sect. 2.82). It has been already mentioned in this book (cf. also Drazin and Reid (1981), Sect. 21) that usually only a finite number of wave-like solutions of Eq. (3.3) exists for a given flow at each value of the wavenumber k. In the simplest case of a plane Couette flow, where $U''(z) \equiv 0$, it is very easy to show that wave-like solutions do not exist at all; here, therefore, the discrete spectrum is empty at any k. It follows from the results of Faddeev (1972) and Dikii (1976) that such solutions also cannot exist in the case of any velocity profile $U(z)$ having no inflection points, i.e., such that $U''(z)$ does not vanish within the flow (the absence here of complex eigenvalues c was proved as far back as 1880 by Rayleigh). All this made clear the inadequacy of the method of normal modes for the linear theory of hydrodynamic stability (at least, for inviscid fluids) and was an important stimulant for renewal of studies based on the consideration of the general initial-value problem.

The complicated form of the normal-mode spectrum of Eq. (3.3) suggests that double Fourier transforms with respect to x and t are not convenient for the study of the corresponding initial-value problem. Case (1960a, b) and Dikii (1960a, b) both found that combined Fourier-Laplace transforms (which were earlier applied to the solution of some initial-value problems arising in the linear theory of hydrodynamic stability by Eliassen et al. (1953) and Miles (1958)) are much more suitable for this purpose. Let us take a Fourier transform with respect to x and a Laplace transform with respect to t of Eq. (3.3). Then the unknown function $w(x, z, t)$ is replaced by the Fourier-Laplace integral

$$\hat{w}(k, p; z) = \int_{0}^{\infty} e^{-pt} dt \int_{-\infty}^{\infty} e^{-ikx} w(x, z, t) dx. \tag{3.4}$$

(Here the Fourier integral indicates that Fourier components with given wave number k are, considered, i.e., it is assumed that $w(x, z, t) \infty e^{ikx.}$ However, the assumption about the proportionality of w to $e^{i\omega t}$, which is a cornerstone of the normal-mode method, is not used here.) Applying the Fourier-Laplace transform to all terms of Eq. (3.3) we obtain

$$\left[\{p + ikU(z)\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) - ikU''(z)\right]\hat{w}(k, p; z) = \left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{w}(k; z, 0) \tag{3.5}$$

where $\hat{w}(k; z, t)$ is the Fourier transform with respect to x of the function $w(x, z, t)$. Replacement of the variable p by $c = ip/k$ transforms (3.5) to the form

$$ik\left[\{U(z) - c\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) - U''(z)\right]\hat{w}(k, -ikc; z) = w_0(z) \tag{3.5$'$}$$

where w_0 (k, z) coincides with the right-hand side of Eq. (3.5). So, instead of the homogeneous partial differential equation (3.3) we now have to treat the inhomogeneous ordinary differential equation (3.5) or (3.5′) with its right-hand side determined by the initial value $w(x, z, 0)$. When the solution $\hat{w}(k, p; z)$ is found, the vertical velocity $w(x, z, t)$ can be determined by the inversion formula for a Fourier-Laplace integral (3.4):

$$w(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\hat{w}(k,p;z)dp \tag{3.6}$$

where $t > 0, -\infty < x < \infty, 0 < z < H$, and γ is chosen so that the integration contour in the complex p-plane is to the right of all singularities of the integrand.

The solution of the inhomogeneous equation (3.5′) can be written as

$$\hat{w}(k,-ikc;z) = \int_0^H G(z,z';k,c)w_0(z')dz' \tag{3.7}$$

where $G(z,z';k,c)$ is the appropriate Green's function (the solution of the same equation with the Dirac delta function $\delta(z-z')$ on the right). The Green's function G can be given as

$$G(z,z',k,c) = \frac{w_2(z)w_1(z')}{ik[U(z')-c]W(c)} \quad \text{for} \quad z' < z, \tag{3.8}$$

$$= \frac{w_1(z)w_2(z')}{ik[U(z')-c]W(c)} \quad \text{for} \quad z' > z,$$

where $w_1(z)$ and $w_2(z)$ are solutions to the homogeneous part of (3.5) satisfying conditions $w_1(0) = w_2(H) = 0$ and $w_1'(H) = w_1'(0) = 1$, while $W(c) = w_1w_2' - w_2w_1' = w_1(H) = -w_2(0)$ is the Wronskian of these two solutions (all primes denote here differentiation on z). For the special case of a plane Couette flow, where $U''(z) \equiv 0$, it is easy to find the explicit expression of the function G in terms of hyperbolic functions (see, e.g., Case (1960a); Drazin and Howard (1966); Dikii (1976); Drazin and Reid (1981); or Henningson et al. (1994); and also Criminale et al. (1991) where three different representations of this function are given). Equations (3.6–3.8) determine the general solution of the initial-value problem for the vertical velocity w, and the same equations with functions w's replaced by ψ's give the solution of the initial-value problem for the stream function $\psi(x,z,t)$ (which at any non-zero value of wavenumber k also satisfies the zero boundary conditions). In the case of plane Couette flow, the solution corresponding to the initial value of w or ψ represented by a single Fourier component naturally coincides with the solution found by Orr (1907) which falls off as t^{-2} when $t \to \infty$. For the much more general case of arbitrary, but sufficiently smooth, initial conditions, Case (1960a) found that in plane Couette flow $|w(x,z,t)|$ and $|\psi(x,z,t)|$ at any point (x, z) usually decrease as t^{-1} when $t \to \infty$. However, exact determination of the exponent in the decay law is a tricky problem and Case's results do not agree with the earlier deduction by Eliassen et al. (1953). who found that for rather general initial conditions $|w(x,z,t)| = |\partial\psi/\partial x|$ decays in Couette flow as t^{-2} when $t \to \infty$, and it is only $|u(x,z,t)| = |\partial\psi/\partial z|$ that decays as t^{-1}. The estimate by Eliassen et al. of the decay of $|w(x,z,t)| = |\partial\psi/\partial x|$ in Couette flow was later confirmed by Engevik (1966) and Brown and Stewartson (1980).

Dikii (1960b) used his solution of the initial-value problem for small disturbances in a plane Couette flow for the proof of its stability of another type with respect to such disturbances. Namely, he showed that for some wide enough class of smooth initial values the quantities $|w(x,z,t)|$ and $|\psi(x,z,t)|$ at any values of x and z are functions of t which are bounded by some constants, decreasing to zero when the initial values of w and ψ and of their spatial derivatives of the first two orders tend to zero. The different formulations of results by Dikii and the authors mentioned above were due to the fact that Eliassen et al., Case, Engevik, and Brown and Stewartson were looking for conditions of "asymptotic stability," i.e., of dying-out at infinity, of any small enough disturbance, while Dikii studied conditions for Lyapunov's stability, which means that any disturbance remains bounded at any t by a constant, which can be made arbitrarily small by sufficiently strong diminution of the initial disturbance.[1] Neither of these definitions of stability conflicts with Orr's "practical instability" mentioned in Sect. 3.1–in the case of "Lyapunov stability," this is because k is now assumed to be fixed, so an increase of k_3, as in Orr's arguments, increases the derivatives of the initial values.

In the case of an arbitrary smooth velocity profile $U(z)$ no explicit formula for the Green's function G can be found. Therefore, we must now investigate the asymptotic behavior of the second integral on the right side of Eq. (3.6), where $\hat{w}(k,p;z)$ is given by Eq. (3.7). For this aim it is convenient to deform the contour of integration on p to the left and thus to transform it into a new contour which is confined to the left half-plane of the complex-variable plane except for some loops surrounding the singularities of the function $G(z,z',k,c) = G(z,z',k,ip/k)$. It was shown by Dikii (1960b, 1976) and Case (1960a) (see also Drazin and Howard (1966), p. 31) that the only substantial singularities of this functions are poles at zeros of $W(c)$, i.e., at such values of c that the corresponding homogeneous version of Eq.(3.5′) has a solution $w(z)$ satisfying the conditions $w(0) = w(H) = 0$. For these and only these c's, wave-like solutions of Eq. (3.3), proportional to $e^{ik(x-ct)}$, exist and hence these c's form the discrete phase-velocity spectrum of the stability problem considered. The poles at zeros of $W(c)$ (under very broad conditions there are no more than a finite number of them at any k) make wave-like contributions to the vertical velocity w (or stream function ψ) with given longitudinal wavenumber k, and these contributions are proportional to e^{-ikct} (or have the form of these exponential functions multiplied by powers of t in the case of multiple eigenvalues c). The remaining part of the integral corresponds to a continuous spectrum of phase velocities (this spectrum is responsible for the singularities of G, which are due to the vanishing of $U(z') - c$]; asymptotic behavior of this part can be investigated as in the case of Couette flow, and for smooth enough velocity profiles $U(z)$ the results are the same as for this special case (see again the above-mentioned publications by Case and Dikii). We see that, in spite of the fact that in an inviscid plane-parallel flow there usually exist only a few possible wave-like disturbances, any sufficiently smooth, two-dimensional

[1] According to Lyapunov, a trajectory $U_0(t), 0 \le t < \infty$, of a dynamic system in a phase space with a norm $||U||$ is stable, if for any $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that for any initial value $U(0)$ satisfying the inequality $||U(0) - U_0(0)|| < \delta(\varepsilon)$ the inequality $||U(t) - U_0(t)|| < \varepsilon$ is valid for any t. For more details about such stability and discussion of its application to fluid mechanics, see Sect. 4.1 in Chap. 4 of this series. Lyapunov's stability clearly depends on the selection of the norm $||U||$ which in studies by Dikii included the absolute values of the function and its two derivatives on z.

initial disturbance (it was found later that both stated conditions are essential) can grow in such a flow, without bound as $t \to \infty$, only at the expense of unstable wave-like disturbances, i.e., only in cases where there exist complex or multiple real eigenvalues of the corresponding Rayleigh's equation.

At first sight, the results of Dikii and Case, related to the general velocity profile, give the complete solution of the initial-value problem for small enough disturbances of steady inviscid flows; and, apart from this, they rehabilitate the normal-mode approach, showing that instability of these flows can be produced only by unstable normal modes. However, this first impression is incorrect. To say nothing of the fact that both these authors were dealing only with the simplest plane-parallel flows with smooth velocity profiles $U(z)$, we must stress again that here only smooth two-dimensional disturbances were studied (in spite of the fact that there is no analog of Squire's theorem valid for initial-value problems) and the possible strong transient growth of initially small disturbances (studied long ago by Orr for both two-dimensional and three-dimensional disturbances) was not even mentioned. Thus, important restrictions of the problem were accepted in the above-mentioned papers, and many questions related to the initial-value-problem approach were left there unsolved.

Interesting results about the asymptotic behavior of three-dimensional infinitesimal velocity disturbances of inviscid steady flows were obtained by Arnold (1972). He showed that in the case of some such flows the growth of three-dimensional unstable disturbances differs considerably from the case of the more ordinary flows and disturbances usually considered in the linear theory of hydrodynamic stability. In some exceptional flows studied by this author, an infinite number of very different types of unstable disturbances can exist and the absolute value of the disturbance vorticity $\zeta = \{\zeta_1, \zeta_2, \zeta_3\}$ can grow exponentially with time, regardless of the character, and location in the complex-variable plane, of the spectrum of exponents ω corresponding to "normal modes," i.e., to velocity disturbances proportional to $e^{i\omega t}$. All these exceptional flows are strictly three-dimensional and fairly complicated; therefore, the will not be considered in this book. However, it was remarked by Bogdat'eva and Dikii (1973) (see also Dikii (1976), Sect. 9) that Arnold's arguments show also that in the case of three-dimensional disturbances of a steady inviscid flow the length $|\zeta|$ of the disturbance vorticity vector can grow without bound with time, even in simple plane-parallel flows with velocity $U = \{U(z), 0, 0\}$ having no complex eigenvalues ω of the corresponding Rayleigh's equation (and thus being stable according to normal-mode formulation of linear stability theory). The growth of vorticity is linear in time in these simple flows and it does not indicate that the flow is unstable in the ordinary sense since all velocity components (and also the vertical component of vorticity) are here bounded for all values of t and only the horizontal vorticity components are rising without bound.

Bogdat'eva and Dikii based their modification of Arnold's arguments on the study of evolution of three-dimensional disturbances in a plane-parallel flow with given velocity profile $U(z)$. Here the equation for the vertical velocity $w(x, t) = w(x, y, z, t)$

has the form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 w - U''(z)\frac{\partial w}{\partial x} = 0 \tag{3.9}$$

(cf. again Chap. 2, Eq. (2.38)), i.e., it differs from Eq. (3.3) only by replacement of the two-dimensional Laplacian by the three-dimensional Laplacian $\nabla^2 = \Delta$. Note now that in the case of two-dimensional disturbances, where $\boldsymbol{u}'(\boldsymbol{x},t) = \{u(x,z,t), 0, w(x,z,t)\}$, it is enough to have only an equation for w, since here, when w is known, $u(x,z,t)$ can be easily determined from the continuity equation $\partial u/\partial x + \partial w/\partial z = 0$ (and the pressure disturbance, if needed, can be determined form Eq. (2.37)). However, for general three-dimensional disturbances the values of $w(x,y,z,t)$ do not determine the velocity field $\boldsymbol{u}'(\boldsymbol{x},t) = \{u,v,w\}$; therefore, in this case at least one more equation is needed. (As the third and fourth equations needed for determination of all the velocity-component and pressure disturbances, the continuity Eq. (2.36) and (2.37) can then be used).

The most convenient equation to supplement Eq. (3.9) is the equation for the vertical vorticity component $\zeta_3 = \partial v/\partial x - \partial u/\partial y$. It follows easily from Eq. (2.35) that this equation has the form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta_3 - U'\frac{\partial w}{\partial y} = 0. \tag{3.10}$$

When w is determined from Eq. (3.9), (3.10) allows ζ_3 to be determined, and when w and ζ_3 are known, the continuity Eq. (2.36) allows the horizontal velocities u and v to be found (cf. Eq.(3.15) below). Note also, that Eqs. (3.9) and (3.10) imply the equation

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[U''\frac{\partial \zeta_3}{\partial x} - U'\frac{\partial \nabla^2 w}{\partial y}\right] = 0, \tag{3.11}$$

which shows that the combination in the brackets can depend only on $x-Ut$, y, and z. let us now apply the Fourier transformation with respect to horizontal coordinates x and y and consider, for the sake of simplicity, only one Fourier component. This means that the velocity and vorticity components of the disturbance considered are assumed to be proportional to $\exp[i(k_1x + k_2y)]$ with amplitudes depending on z and t (but, contrary to the normal-mode method, the form of dependence on t is now not restricted). Let $k_1 \neq 0$ (the case where $k_1 = 0$ will be considered later); then it follows easily from Eq. (3.11) that

$$\zeta_3 = \frac{k_2U'}{k_1U''}(w'' - k^2w) + e^{ik_1Ut}\left[\zeta_3 - \frac{k_2U'}{k_1U''}\left(w'' - k^2w\right)\right]_{t=0} \tag{3.12}$$

where again $k^2 = k_1^2 + k_2^2$. (If the initial-value problem has already been solved for the vertical velocity w, Eq. (3.12) gives the explicit solution of the same problem for the vertical vorticity ζ_3). Note now that if $U''(z) \neq 0$ for all values of z (i.e., if Rayleigh's condition is valid) and also in more general cases where Fjørtoft's condition is valid (and hence there exists such constant velocity K that $[U(z)-K]/U''(z)$ is a continuous nonnegative function of z) the functions $|w|, |w'|$ and $|w'')$ are bounded by some constants c_0, c_1, and c_2 which do not depend on t. (This statement follows, in particular, from Dikii's conservation law

$$\frac{d}{dt}\int_0^H [|w'|^2 + k^2|w|^2 + \frac{U-K}{U''}|w'' - k^2w^2|^2]dz = 0 \tag{3.13}$$

which can be proved in exactly the same way as its special form (2.57) was proved for the case of two-dimensional disturbances, if Eq. (2.53) is replaced by Eq. (3.9).[2]) Hence it follows from Eq. (3.12) that for flows where $U''(z) \neq 0$ everywhere, $|\zeta_3|$ is also bounded by some constant. Equation (3.12) also shows that $|\zeta_3|$ does not tend to zero as $t \to \infty$ even if the vertical velocity does, since the second term on the right-hand side of this equation represents a harmonic oscillation with fixed amplitude. However, this amplitude decreases to zero when the initial values of ζ_3, w, and w'' tend to zero; therefore the behavior of the vertical vorticity is not in conflict with Liapunov's stability (with appropriate definition of the norm) of the flow with respect to infinitesimal disturbances.

If $U''(z)$ vanishes at some point (or points), then Eq. (3.12) implies that $|\zeta_3 U''|$ is bounded by some constant. Therefore, in this case $|\zeta_3|$ can possibly grow with time without bound, at inflection points of the velocity profile.

For a given Fourier component of the disturbance the definition of the vorticity component z_3 and the equation of continuity take the forms

$$\zeta_3 = ik_1v - ik_2u, w' = -ik_1u - ik_2v, \tag{3.14}$$

where $w' = \partial w/\partial z$, and hence

$$u = i\left(k_1w' + k_2\zeta_3\right)/k^2, v = i\left(k_2w' - k_1\zeta_3\right)/k^2. \tag{3.15}$$

It follows from this that in Rayleigh's case (when $U''(z) \neq 0$ everywhere $|u|$ and $|v|$ do not tend to zero as $t \to \infty$ but are bounded by some constants (which become zero when the initial disturbance and its first and second derivatives on z tend to zero).

However, the horizontal vorticity components ζ_1 and ζ_2 in this case can increase infinitely when $t \to \infty$. To illustrate this Bogdat'eva and Dikii considered the simplest solution of Eqs. (3.9) and (3.10) where $w = 0$ (i.e., $\boldsymbol{u}'(\boldsymbol{x}, t) = \{u(x, y, z, t), v(x, y, z, t), 0\}$). Then Eq. (3.12) shows that here the Fourier components of the vertical vorticity have the form $\zeta_3 = A(z)\exp[i\{k_1(x - U(z)t) + k_2y\}]$ where $A(z)\exp[i(k_1x + k_2y)] = \zeta_3(x, y, z, 0)$. According to Eq. (3.15), components u and v are proportional to ζ_3 when $w = 0$; hence these three functions of x, y, z, and t all have the same form. Therefore, here $-\partial v/\partial z = \zeta_1$ and $\partial u/\partial z = \zeta_2$ include terms of the

[2] It is true that Eq. (3.13) implies only that if the initial values of $|w|$, $|w'|$ and $|w''|$ are small enough, then their root-mean-square values will be bounded by some small constants at any value of t. However, using results of the initial-value-problem investigations, it is possible to prove that in fact the values of these functions of t and z will be uniformly bounded by some small constants for all $t > 0$ and $0 < z < H$; see, e.g., Dikii (1976).

form $B(z)k_1tU'(z)\exp[i\{k_1(x-U(z)t)+k_2y\}]$ representing harmonic oscillations with amplitudes growing linearly with time. If $w(x,y,z,0)\neq 0$, then the arguments become somewhat more complicated but the situation here, as a rule, is the same as for disturbances with vanishing vertical velocity. In fact, according to Eq. (3.12), in this case $\zeta_3(x,y,z,t)$ also includes the summand of the same form as above (with $A(z)$ equal to the initial Fourier amplitude of the combination in the brackets on the right-hand side of Eq. (3.12)), and, according to Eq. (3.15), u and v include the summands of this form too. However, ζ_1 and ζ_2 by definition include the derivatives $-\partial v/\partial z$ and $\partial u/\partial z$, respectively, and this implies that these vorticity components include harmonic oscillations with amplitudes proportional to time t. Thus we see that the horizontal velocity components u and v of a three-dimensional small disturbance of a plane-parallel steady flow with velocity profile $U(z)$ without inflection points are bounded at any t by small constants (but do not go asymptotically to zero as $t\to\infty$) while horizontal vorticity components ζ_2 and ζ_3 can here increase indefinitely with time. Whether under such conditions a flow must be called stable or unstable depends on the precise definition of the term "stability" employed (in particular, Arnold (1972) regarded the unbounded growth of vorticity as an indication of flow instability).

The existence of the strong dependence of the evolution of a disturbance on smoothness of its initial value was demonstrated by Willke (1972) on some rather peculiar examples relating to two-dimensional disturbances in an inviscid plane Couette flow. However, he began with consideration of the old Orr's solution (3.2), where $k_2=0$, of the linearized inviscid Navier-Stokes equations for the velocity field $\boldsymbol{u}=\boldsymbol{U}+\boldsymbol{u'}$, where $\boldsymbol{U}=\{bz,0,0\}$, $\boldsymbol{u'}=\{u(x.z.t),0,w(x,z,t)\}$, $0<z<H$. He neglected in this solution the terms involving hyperbolic functions, which are of importance only in close proximity to the solid walls at $z=0$ and $z=H$, and instead of the vertical velocity $w(x,z,t)$ he used the stream function $\psi(x,z,t)$ which satisfies the same equation as w. It has in fact already been noticed in Sect. 3.1 that, if k_1,k_3 and b are positive, then solution (3.2) for ψ implies that $|\psi(x,t)|=|\psi|$ increases from the initial value $|\psi|_0$ at $t=0$ to the value $|\psi|\max\approx[1+(k_3/k_1)^2]|\psi|_0$ at $t=t_{\max}\approx k_3/k_1b$, and then decreases, tending to zero as $t\to\infty$. According to Eq. (3.2) $|\psi|$ falls off like $(t-t_{\max})^{-2}$ on either side of the time $t_{\max}$. Willke noted in this respect that the above conclusions (with appropriate change of the values for $|\psi|_{\max}$ and $t_{\max}$) are valid not only for wave-like disturbances, where $\psi(x,z,0), w(x,z,0)$ and also the initial vorticity $\zeta(x,z,0)=-(\partial^2/\partial x^2+\partial^2/\partial z^2)\psi(x,z,0)=\partial u(x,z,0)/\partial z-\partial w(x,z,0)/\partial x$ are proportional to $\exp[i(k_1x+k_3z)]$, but also for disturbances with much more general initial values of the form $\zeta(x,z,0)=\exp(ik_1x)f(z)$, where $k_1\neq 0$ and the function $f(z)$ is twice continuously differentiable. (To obtain this last result, which according to Willke, was already known to Orr, it is only necessary to expand the function $f(z), 0\leq z\leq H$, into Fourier's series and apply Orr's solution to all Fourier components. Note that proportionality to t^{-2} of the asymptotic decay rate of $|w|$ was later proved by Henningson et al. (1994) also for smooth three-dimensional disturbances in Couette flow with $k_1\neq 0$; See below about this matter.)

Then Willke considered more complicated cases where the initial disturbance is very irregular and does not satisfy the smoothness requirements used in previous investigations of Couette-flow stability. His investigation of these cases employed a nonstandard mathematical technique and some subtle analytical results; the conclusion obtained will be described only briefly below.

Willke assumed that the initial disturbance was specified not by a smooth ordinary function but by a "generalized function" (or, what is the same, a "distribution") which can have any degree of irregularity (see literature on such functions listed in Sect. 2.82, p. 84). In such a case it is natural to look for solutions of the corresponding dynamic equations which are also represented by generalized functions, i.e., to use the generalized-function (or else distribution-theoretic) approach to these differential equations (see the book by Gel'fand and Shilov (1958) devoted to discussion of this approach). This allowed Willke to analyse rather easily the laws of growth and decay for arbitrarily irregular solutions and to find estimates for the dependence of the highest possible growth rate on a numerical characteristic of the degree of irregularity of the generalized function describing the initial disturbance. To show that his estimates are strict, Willke considered a special sequence of complicated solutions represented by lacunary series of solutions of the form (3.2) (i.e., by infinite sums of functions of this form with $k_2 = 0$, fixed value of k_1, rapidly increasing values of k_3 and decreasing amplitudes W_0's; these sums do not converge at fixed points (x, z, t) but converge in some special sense to a definite generalized function). The first term of the above-mentioned sequence of solutions is an ordinary (but nondifferentiable) function; all the further terms are generalized functions related to first- or higher-order derivatives of continuous nondifferentiable functions. (These generalized functions can be accepted as flow variables in the same way as more common examples of such functions which include Dirac's δ-functions and their derivatives.) With the aid of some analytical results Willke showed that his "generalized solutions" can match all the growth-rate bounds found by him for irregular two-dimensional disturbances of a Couette flow. It turned out that these solutions can grow (for any length of time) like any positive integer power of t, and then decay arbitrarily slowly (like an arbitrarily small negative fractional power of t). The transient growth proportional to a high positive power of t can be reached only for very irregular disturbances represented by complicated generalized functions, but arbitrarily slow ultimate decay is possible for disturbances with continuous, but not differentiable, initial vorticity $\zeta(x, 0)$.

Willke's paper (1972) was devoted to investigation of rather exotic two-dimensional disturbances of an inviscid plane Couette flow, which are interesting only theoretically but not in practice (at least until now). However, in the 1980s and 1990s many more realistic examples of disturbance developments in plane-parallel inviscid flows were also studied, and some of them were again related to plane Couette flow. As typical examples we can mention the papers by Shepherd (1985) and Farrell (1987) where development of simple two-dimensional disturbances to an inviscid plane Couette flow in unbounded space $-\infty < x, y, z < \infty$ was considered as a proper model of some important meteorological phenomena. The unboundedness of space makes unnecessary the terms of Eq. (3.2) containing hyperbolic functions, which were added to satisfy the boundary conditions at the walls; hence here Orr's solution corresponding to two-dimensional disturbances with $k_2 = 0$ takes the form: $w(x,t) \propto \psi(\boldsymbol{x},t) \propto [{k_1}^2 + (k_3 - k_1 bt)^2]^{-1} \exp[i(k_1 x + (k_3 - k_1 bt)z)]$, while $\zeta(\boldsymbol{x},t) \propto \exp[i(k_1 x + (k_3 - k_1 bt)z)]$. Using these equations Shepherd studied the evolution of a standing wave composed of a pair of two-dimensional Orr's waves with the same amplitude W_0 and wave vectors $(k_1, 0, k_3)$ and $(k_1, 0, -k_3)$. If $k_1 > 0, k_3 > 0$ and $b > 0$, then the first of these two waves will gain energy from the mean motion until time $k_3/k_1 b$ and will lose it after this time, while the second wave will lose energy at any time t. As to the total energy density of a standing wave, it will decrease monotonically if $k_1^2 > 3k_3^2$, i.e., $\theta = \arctan\ (k_3/k_1) < \pi/6$, and will at first increase and then decrease if $\theta > \pi/6$. For the isotropic collection of standing waves with homogeneous circular distribution of angles θ the total energy was found to remain constant in time, while for some other simple distributions of wave-vector directions moderate transient growth of energy was discovered. Farrell (1987) tried to estimate the value of the inverse-shear time scale b^{-1} appropriate for modeling the mid-latitude free-atmosphere processes and found that it is typically of the order of 10 h. Therefore, he concluded that the asymptotic laws of wave development in a steady Couette flow at $t \gg b^{-1}$ are usually irrelevant for modeling real atmospheric processes since for such times this model is unsuitable, but the transient growth of waves can serve as a reasonable model of the initial stage of the development of a disturbance at the expense of the energy of mean atmospheric motion. The accumulated wave energy can then be transferred to some quasi-stationary large- or medium-scale atmospheric structures (e.g., in cyclogenesis) or be spent to generate modal disturbances whose subsequent development must be studied within the framework of the normal-mode theory. In this respect Farrell studied the energetics of the solitary-wave and wave-packet developments, and considered the temporal evolution of the Couette flow disturbances for a number of specific initial values of the corresponding stream function (such as the "checkerboard initial value" $\psi(x,z,0) = A\cos(k_1 x)\cos(k_2 z)$; Shepherd's isotropic wave packet where $\psi(x,z,0) = AJ_0(kr)$, J_0 is the Bessel function and $r = (x^2 + z^2)^{1/2}$; a Gaussian isotropic wave packet where $\psi(x,z,0) = A\exp[-(kr)^2)]$; and an anisotropic localized disturbance where $\psi(x,z,0) = A\exp[-(k_0 r)^2]\cos(k_1 x + k_2 z)$.) Most attention was paid to the last of these examples, where a considerable transient growth of disturbance energy (depending on and increasing with $s = k_2/k_1$) was found and where the time evolution of the disturbance shape agreed qualitatively with data of some meteorological observations.

General three-dimensional disturbances in bounded inviscid plane Couette flow between walls at $z = 0$ and $z = H$ can be analyzed by the method applied by Case (1960a) and Dikii (1960b) to the study of two-dimensional disturbances. Let us replace the one-dimensional Fourier transform on the left-hand side of Eq. (3.4) by the two-dimensional Fourier transform of the function $w(x,y,z,t)$ with respect to horizontal coordinates x and y (or, what is the same, assume that $w(x,y,z,t) = \tilde{w}(k_1,k_2,z,t)\exp\{i(k_1 x + k_2 y)\}$ and take the Laplace transform of $\tilde{w}(k_1,k_2,z,t)$ with respect to t). Using Eq. (3.9) instead of (3.3), it is easy to show that the resulting Fourier-Laplace transform of $w(x,y,z,t)$ (or the Laplace transform of $\tilde{w}((k_1,k_2,z,t))$, which we will denote by $\hat{w}(k_1, k_2, p; z)$, satisfies an equation very like Eq. (3.5). The only differences are that the two factors ik entering Eq. (3.5) must now be replaced by ik_1; $\tilde{w}(k; z, 0)$ must be replaced by $\tilde{w}(k_1, k_2, z, 0)$, the Fourier transform of $w(x, y, z, 0)$ with respect to x and y (or the value of the coefficient $\tilde{w}(k_1, k_2, z, t)$ at $t = 0$) while k^2 must be interpreted as $k_1^2 + k_2^2$. Denoting the Laplace-transform variable p by $-ik_1c$ we arrive at an equation of the form (3.5′) for $\hat{w}(k_1, k_2, -ik_1c; z)$ with ik replaced by ik_1 and $w_0(z) = (\partial^2/\partial z^2 - k^2)\tilde{w}(k_1, k_2, z, 0)$. Solution of this inhomogeneous linear equation can again be represented in the form (3.7), with Green's function G given by (3.8) with k replaced by k_1 (recall that in the case of Couette flow an explicit expression of the function G can be easily obtained). When $\hat{w}(k_1, k_2, p; t)$ is known, the vertical velocity w can be determined by the inversion formula for either a composite triple Fourier-Laplace integral generalizing (3.6) or, if w is assumed to be proportional to $\exp[i(k_1x + k_2y)]$, a one-dimensional Laplace integral.

The general expression for $w(x, y, z, t)$ obtained is close to that found by Case (1960a) for two-dimensional disturbances in Couette flow. Henningson et al. (1994) showed that according to this expression $\tilde{w}(k_1, k_2, z, t)$ decays as t^{-2} as $t \to \infty$ if $k_1 \neq 0$ and the above function $w_0(z)$ is smooth enough. Transient growth of the vertical velocity at small and moderate values of k_1t must also occur here for reasons explained by Orr as for back as 1907. However, the temporal behavior of the horizontal velocity components is different and the simplest way to show this is based on the study of the vertical vorticity ζ_3.

The simple Eq. (3.12) cannot be used in the case of Couette flow, where $U''(z) = 0$ at all values of z. However, when one Fourier component of the disturbance is studied and hence derivatives $\partial/\partial x$ and $\partial/\partial y$ can be replaced by factors ik_1 and ik_2, Eq. (3.10) for ζ_3 can be easily integrated to yield the result

$$\zeta_3(z, t) = \zeta_3(z, 0)e^{-ik_1U(z)t} + ik_2U'(z)e^{-ik_1U(z)t}\int_0^t w(z, t')e^{ik_1U(z)t'}dt' \qquad (3.16)$$

(the dependence of the vorticity ζ_3 and velocity w on horizontal coordinates, given by the factor $\exp[i(k_1x + k_2y)]$, is not indicated here). The first term represents the advection of the initial vertical vorticity by the flow velocity $U(z)$, while the second term represents the integrated effect of the vertical velocity, the so-called *lift-up* effect (see Landahl (1975)). Note that, according to Eqs. (3.15), the horizontal velocity components u and v include terms proportional to ζ_3; so this effect consists of the generation of horizontal velocity perturbations by lifting-up of the fluid elements in the presence of the mean shear. The lift-up effect increases with increasing U', as it must, and with decreasing spanwise wavelength $2\pi/k_2$ (for a physical explanation of this last dependence see, e.g., Henningson (1988) or Henningson et al. (1994)). In cases where vertical velocity disturbances decay fast enough as $t \to \infty$, the integral in (3.16) converges to a finite limit and the second term on the right side describes a *permanent scar* in the disturbance, convecting downstream with the local mean velocity, discovered by Landahl (1975) (see also Bogdat'eva and Dikii (1973) and Gustavsson (1978)). In Rayleigh's plane-parallel flows, where $U''(z) \neq 0$ for all z, the value of the scare is given by the last term in brackets on the right side of (3.12); for a plane Couette flow it was shown by Henningson et al. (1994) that $\zeta_3(z,t) \approx \zeta_3(z,0)\exp[-ik_1bzt] + ik_2w(z,0)\,bt$ at small values of t and $\zeta_3(z,t) \approx \left[\zeta_3(z,0) - i\pi\frac{k_2}{k_1}\frac{\sin hkz \sin hk(H-z)}{k\sin hkH}w_0(z)\right]e^{-ik_1bzt}$ at large values of t, where b and $w_0(z)$ have the same meaning as above. We see that $\zeta_3(t)$, aside from the convected initial value, contains a term which grows linearly for short times and for large times represents a permanent scar convected downstream, depending on the initial value of the vertical velocity w and on the wave numbers k_1, k_2, and $k = (k_1^2 + k_2^2)^{1/2}$.

Three-dimensional disturbances of an inviscid plane Couette flow between solid walls were also studied by Criminale and Drazin (1990) and Criminale et al. (1991). Their method for solution of the general initial-value problem was based on the transition to the "convected coordinate system" ($\xi = x - U(z)t, y, z$) used much earlier by Kelvin (1887a) and Orr (1907) (see also Craik and Criminale (1986)). Criminale and Drazin considered two particular solutions of the initial-value problem, while Criminal et al. found the explicit non-Fourier-transformed form of the general solution and then considered at length the case where $w(x,y,z,0) \propto \exp[i(k_1x + k_2y)]$. Most attention was paid here to the particular case where $(\partial^2/\partial z^2 - k^2)w(x,y,z,0) = \exp[i(k_1x + k_2y)]W_0(z)$ and $W_0(z) = W_0(z; z_0, L)$ has the form of a rectangular pulse of unit area with the center at $z = z_0$ and the thickness $2L$ (hence $W_0(z) \to \delta(z - z_0)$ as $L \to 0$). Since equations obtained for the velocity components (u,v,w) proved to be complicated, an integrated positive measure of disturbance size (the "energy," whilch for purely two-dimensional disturbances coincided with the ordinary kinetic energy density) was introduced. Evaluation of this measure showed that solution of the initial-value problem considered usually decays monotonically with time (or, as an exception, preserves their size); hence, the phenomenon of the transient disturbance growth is here mostly lacking. The rate of decay is practically independent of the position of the pulse $W_0(z)$, but depends strongly on its relative thickness $\mu = L/H$, dimensionless wave number $\kappa = kH$, and wave-vector orientation $\theta = \arctan(k_2/k_1)$, growing with increasing μ and κ and with a decrease of θ from $\pi/2$ to zero. In particular, if $\theta = 0$ (i.e., the disturbance is two-dimensional) and either $\mu = 0$ or $\kappa \ll 1$, then the "energy" of the disturbance remains constant with time; the same is true for cases where $\mu \neq 0$ and κ is not small but $\theta = \pi/2$. Moreover, if $\mu = 0$ (i.e., $W_0(z) = \delta(z - z_0)$) and $\theta \neq 0$ (i.e., the disturbance is really three-dimensional), then the horizontal velocities velocities u and v at $z = z_0$ grow linearly with time and at $|z - z_0|/H \ll 1$ their growth is practically linear up to very large values of bt (i.e., here there is a considerable transient algebraic growth of the disturbance). This shows again that in cases of singular initial conditions the behavior of disturbances can differ considerably from that for smooth initial values (see also the paper by Willke (1972) discussed above).

Criminale and Drazin (1990) considered, along with the case of a plane Couette flow, development of disturbances in two other steady inviscid plane-parallel flows with piecewise linear velocity profiles: in two-layered unbounded flow where $U(z) = b_1z$ for $z > 0, U(z) = b_2z$ for $z < 0$, and $b_2 \neq b_1$; and in a piecewise-linear model of a boundary layer where $0 \leq z < \infty$, and $U(z) = bz$ for $0 \leq z < H$, while

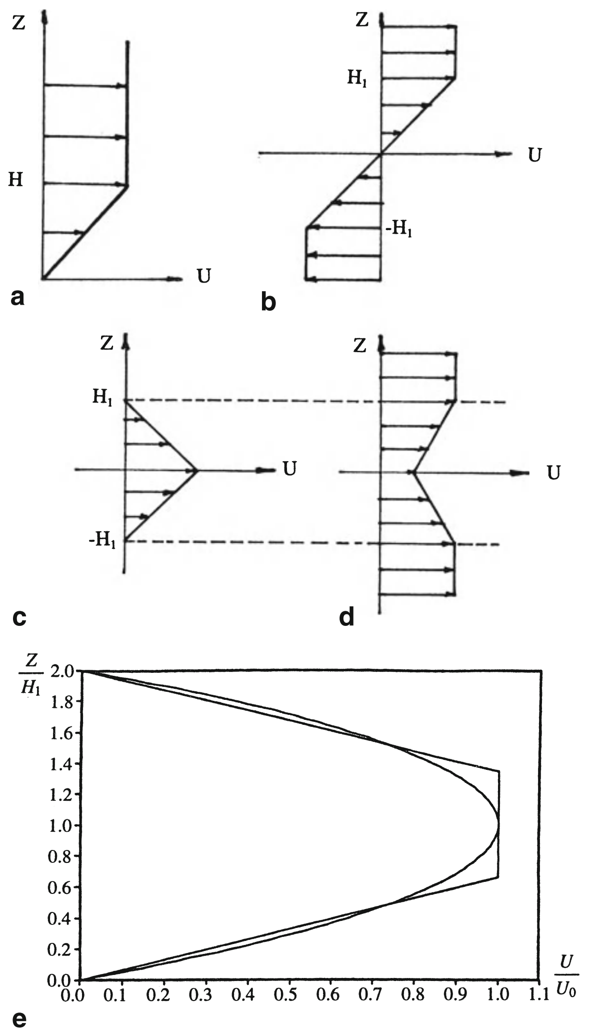


Fig. 3.1 Piecewise-linear models of velocity profiles for some plane-parallel fluid flows: **a** model of a boundary-layer profile used by Gustavsson (1978) and Criminale and Drazin (1990); **b** model of a mixing-layer profile used by Bun and Criminale (1994) and Criminale et al. (1995); **c** and **d** models of plane-jet and plane-wake profiles by Criminale et al. (1995); **e** model of a plane Poiseuille-flow profile by Henningson (1988) compared with the exact parabolic profile

$U(z) = bH = U_0$ for $z > H$ (see Fig. 3.1a). For disturbances in these flows the following schematic initial conditions at $t = 0$ were used; (*a*) unit point pulse of velocity, (*b*) unit point pulse of vorticity, (*c*) monochromatic three-dimensional plane wave of velocity, and (*d*) a similar wave of vorticity. The possibility of stimulation of nonlinear effects by transient algebraic growth of an initially small disturbance was discussed by the authors, and such growth was illustrated by results related to the case of the initial condition (*c*), first considered, for disturbances of a Couette flow, by Orr (1907).

Later Bun and Criminale (1994) and Criminale et al. (1995) applied the initial-value-problem approach to detailed study of the evolution of three-dimensional disturbances in schematic piecewiselinear models of an inviscid plane mixing layer (with profile $U(z)$ shown in Fig. 3.1b (and also in Fig. 2.31e in Chap. 2)) and (in the second of those papers) also of a plane jet (Fig. 3.1c) and a plane wake (Fig. 3.1d). It was mentioned in Sect. 2.93 that as long ago as 1894 Rayleigh proved that unstable normal modes (growing exponentially as $t \to \infty$) exist in a plane mixing layer with the velocity profile given in Fig. 3.1b. It is easy to show that the same statement is also true for inviscid piecewise-linear free shear flows in an unbounded space with the velocity profiles shown in Figs. 3.1c and 3.1d. (Recall that in Sect. 2.93 models of viscous plane jets and wakes with analytic velocity profiles, differentiable everywhere, were considered and the results showed that such flows definitely have unstable normal modes of disturbance in the inviscid case too; see also the remarks following Eq. (2.87), which contains a number of references related to this topic. In the case of piecewise-linear jet and wake models the corresponding proofs are even simpler since here the exact analytic solutions for equations of the linear stability theory may be used instead of the approximate numerical solutions used in the cases of analytic velocity profiles.) It might be concluded from this that consideration of the general initial-value problem for piecewise-linear plane free flows in an unbounded space is superfluous, since the classical normal-mode theory has already proved that these flows are unstable with respect to small disturbances, which can grow here as exp $(\omega^{(i)} t)$ as $t \to \infty$, where $\omega^{(i)}$ is the greatest imaginary part of the discrete eigenvalues of Rayleligh's equation (2.48) with $c = \omega/k$. However, results by Bun and Criminale (1994) and by Crilminale et al. (1995) show that this conclusion is incorrect in many cases. According to these results the behavior of three-dimensional disturbances in these flows is dominated by the exponential growth of unstable normal modes only for very large times, while for earlier times the transient algebraic growth, which is in fact due to the continuous spectrum of Rayleigh's equation, plays the main part. This transient growth can lead to a quite substantial rise of the velocity disturbances before the exponentially-growing normal modes become dominant. Just this rise apparently produces the early nonlinear transformation of the whole flow structure which has often been observed experimentally. Similar results were deduced by Criminale et al. from the numerical solution of the appropriate initial-value problem for the case of inviscid jets and wakes with differentiable analytic velocity profiles.

Let us now mention two other investigations of the initial-value problem for small disturbances in plane-parallel steady inviscid flows with piecewise linear velocity profiles. In the PhD thesis by Gustavsson (1978) a general solution of the problem was given for the same piecewise linear model of a boundary-layer flow that was later considered by Criminale and Drazin (and is sketched in Fig. 3.1a). Gustavsson paid special attention to the case of a localized three-dimensional initial disturbance. Then Henningson (1988) applied Gustavsson's method to study the evolution of disturbances in the piecewise linear model of a plane Poiseuille flow shown in Fig. 3.1e. Note also the paper by Breuer and Haritonidis (1990) who numerically solved the initial-value problem for a localized disturbance in a plane-parallel boundary-layer flow with the Blasius velocity profile. (As mentioned above, in the case of curved velocity profiles the initial-value problem for small velocity disturbances can only be solved numerically.) In these investigations (and also in the survey by Henningson and Alfredsson (1996)) it was stressed that the general solution of the initial–value problem includes terms of two different types, which are directly separated in analytical solutions and can also be detected in numerical results.

The first of these types is represented by terms of the Fourier–transformed solutions for velocity components which contain the factor $e^{-ik_1U(z)t}$ (see e.g. Eqs. (3.1), (3.2), (3.11) and (3.16) above, which include a number of such terms). Here the inversion of the Fourier transform leads to functions of y, z and $\xi = x - U(z)t$; this indicates that the corresponding disturbances are convected streamwise with the local flow velocity $U(z)$. Such disturbances were, in fact, first discovered by Kelvin (1887a) and Orr (1907) and were also at length studied by Criminale and his collaborators in the papers indicated above; they often undergo considerable transient growth followed by decline. These disturbances were called *convective* by Gustavsson (1978); in the case of inviscid flow they correspond to a continuous spectrum of Rayleigh's equation (which, for bounded flows, fills the interval $U_{\min} \leq c \leq U_{\max}$ of the real axis; see Sect. 2.82, p. 85). Besides convective terms, Fourier transforms of solutions of the initial-value problem also include "terms of the second type," proportional to $e^{-ik_1c(k)t}$, where $k = (k_1^2 + k_2^2)^{1/2}$ and $c(k)$ is a special function that appears in the course of the solution. In some cases there are several functions $c(k)$, appearing in different terms of the second type; these functions can be either real or complex, and they can also be determined by analysis of the corresponding Rayleigh's equation (which, according to Chap. 2, has the same form (2.48) for two- and three-dimensional disturbances, and includes only k but not k_1 and k_2). Inversion of the Fourier transform translate these terms into functions of x-$c(k)t$, which describe three-dimensional waves with the wave vector $k = (k_1, k_2)$ and streamwise phase velocity $c(k)$ (or $\Re e\ c(k)$ if $c(k)$ is complex), and are related to normal modes studied in Chap. 2.[3] Since the frequency $\omega = k_1c(k)$ (or frequencies $\omega_i = k_1c_i(k)$, if there are several functions $c(k)$) of waves corresponding to second-type terms) depend on k, these waves are *dispersive*; therefore Gustavsson called the collection of these waves the *dispersive* disturbances.

The normal-mode approach to stability theory paid most attention to individual normal modes with given values of k and c (or ω). However, in studies of the initial-values problems, all plane waves making non-zero contribution to the Fourier expansion of the initial disturbance must be simultaneously taken into account. In the important case of an initially-localized disturbance, the Fourier expansion includes a vast collection of waves with different wave vectors $k = (k_1, k_2)$. Hence here the laws of wave-packet evolution must be applied.

The convective disturbances are convected streamwise with velocity $U(z)$; hence their evolution is relatively simple. However, the evolution of dispersive disturbances with angular velocities $\omega = k_1c(k)$ is more complicated. According to kinematic wave theory (see, e.g., Landau and Lifshitz (1958, 1987), Sects. 66, 67, or Whitham

[3] They are not identical to normal modes since the functions $c(k)$ do not coincide with the discrete eigenvalues of Rayleigh's eigenvalue problem, which do not exist in many important cases. In fact, functions $c(k)$ correspond to limits, as Re $\to \infty$, of discrete eigenvalues of the Orr-Sommerfeld eigenvalue problem for given profile $U(z)$; their determination from the Rayleigh equation requires careful examination of the analytic continuation fo this equation into the complex-variable plane.

(1974)), if there is a wave packet which is concentrated in a bounded spatial region and is composed of waves with various wave numbers (k_1, k_2), then the group of waves with the "central wave" of the form $A(z)\exp\{i[k_1(x-ct)+k_2 y]\}, c = c(k) = c\left(\sqrt{k_1^2+K_2^2}\right)$, is moving, not simply streamwise with the phase velocity $c\,(k)$ but with the two-component horizontal group velocity $\boldsymbol{C} = \{C_x(k_1,k_2), C_y(k_1,k_2)\}$ where

$$C_x = \frac{\partial\omega}{\partial k_1} = c + \frac{{k_1}^2}{k}\frac{dc}{dk}, C_y = \frac{\partial\omega}{\partial k_2} = \frac{k_1 k_2}{k}\frac{dc}{dk}. \tag{3.17}$$

If the initial disturbance is concentrated in a close vicinity of the point $(0, 0, z_o)$, then at time $t > 0$ its dispersive waves with wave numbers (k_1, k_2), where $k_1^2 + k_2^2 = k^2$ is fixed, will form a packet whose horizontal projection will be concentrated near the point (x, y) where

$$\frac{x}{t} = c + \frac{k_1^2}{k}\frac{dc}{dk}, \quad \frac{y}{t} = \frac{k_1 k_2}{k}\frac{dc}{dk}. \tag{3.17$'$}$$

Gustavsson (1978) noted that Eqs. (3.17′) imply the following simple result

$$\left(\frac{x}{t} - c - \frac{k}{2}\frac{dc}{dk}\right)^2 + \left(\frac{y}{t}\right)^2 = \left(\frac{k}{2}\frac{dc}{dk}\right)^2. \tag{3.18}$$

It follows from this that in the case of disturbance initially located near the point $(0, 0, z_0)$, the dispersive wave components with wave numbers (k_1, k_2), where ${k_1}^2 + {k_2}^2 = k^2 =$ const., spread horizontally over a circle whose center at time t is at the point $(ct + ktdc/dk,\ 0)$, with radius $(kt/2)dc/dk$. The location of these circles corresponding to different values of k is shown, for piecewise-linear models of boundary-layer and plane Poiseuille flows, in figures presented by Gustavsson (1978); (see also Henningson (1988); Henningson et al. (1994); and Henningson and Alfredsson (1996)). (In Poiseuille flow there are two different functions $c_1(k)$ and $c_2(k)$ corresponding to disturbances symmetric and antisymmetric with respect to the channel midplane $z = H/2$, However, the waves with phase velocity $c_1(k)$ are characterized by much greater spreading than waves with velocity $c_2(k)$, which moreover take quite different values in the cases of piecewise-linear and of real, parabolic, Poiseuille profiles.) Note also that, according to the above-mentioned papers, the spreading of localized disturbances by wave dispersion (i.e., the dispersive effect) forms only a small part of the total disturbance spreading, which is mainly due to Landahl's lift-up effect mentioned above.

Henningson (1988) and Breuer and Haritonidis (1990), who considered quite different flows, both made careful calculations for the case where the initial disturbance had the form of two pairs of counter-rotating eddies, schematically shown in Fig. 3.2. Here the initial streamwise velocity disturbance u is equal to zero and therefore $v = -\partial\psi/\partial z, w = \partial\psi/\partial y$ where the stream function $\psi(x, y, z)$ is very close to zero everywhere outside a small spatial region surrounding the "central point" with coordinates $(0, 0, z_0)$. [This form of the initial disturbance was chosen be-

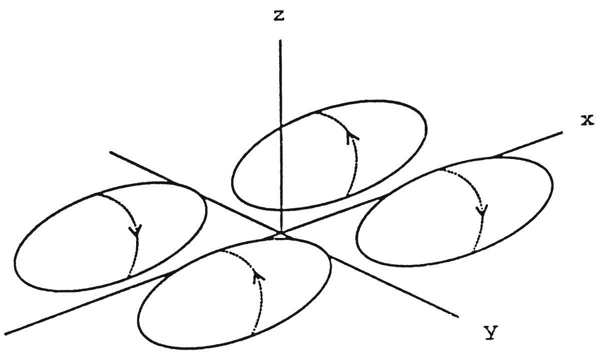


Fig. 3.2 Schematic shape of the initial velocity field for a localized disturbance considered by Russell and Landahl (1984), Henningson (1988), and Breuer and Haritonidis (1990)

cause it was used for similar purposes by Russell and Landahl (1984).) In Figs. 3.3a, b taken from Breuer and Haritonidis (1990) (and reprinted also by Henningson et al. (1994)), contours of vertical and horizontal disturbance velocities w and u are plotted in the (x, z) plane for $y = 0$ and for several values of t. The distribution of the vertical velocity does not change qualitatively with t, but typical values of w decrease, and the entire structure moves downstream with a velocity approximating the typical group velocity of waves in the boundary layer. Simultaneously, the velocity distribution expands in the streamwise direction, which also agrees well with theoretical predictions for dispersive disturbances. Contours of $w = \text{const.}$ in the (x, y) plane, also presented by Breuer and Haritonidis (1990) for one value of z, show quite definitely the development of the wave-packet-like character of the vertical velocity distribution with increasing time t. The only feature in Fig. 3.3a resembling convective disturbances is the patch of low-speed fluid moving streamwise at large heights (the edge of the boundary layer is located near $z = 3\delta^*$) ahead of the main disturbance, with a speed approaching the free-stream velocity U_0.

The distributions of the streamwise disturbance velocity u shown in Fig. 3.3b contrast strongly with the distributions of w. Since initially $u = 0$, transient growth of streamwise velocity clearly must take place. Computational results in Fig. 3.3b demonstrate that the growth of $|u|$ is dominated by the lift-up effect. This effect at first produces a region of negative values of u which travels at the local undisturbed velocity and is immediately followed by a high-speed region of fluid. The mean velocity gradient existing in the lower part of the boundary layer generates the tilting of the shear layer between low-speed and high-speed regions and the stretching of the structure in the x direction; as a result, an inclined shear layer is formed whose intensity and streamwise length increase with time. Thus, the streamwise velocity disturbances are mainly of a convective nature.

Constant-velocity contours in the (x, y) plane were also presented by Henningson (1988) (see also Henningson et al. (1994)) for the piecewise-linear model of plane Poiseuille flow, showing normal and streamwise disturbance velocities at a fixed value of z and several values of t. These contours show the same typical features as the later results of Breuer and Haritonidis. Here again vertical velocity disturbances w are mostly dispersive, and their contours show that a wave-packet with wave-crests

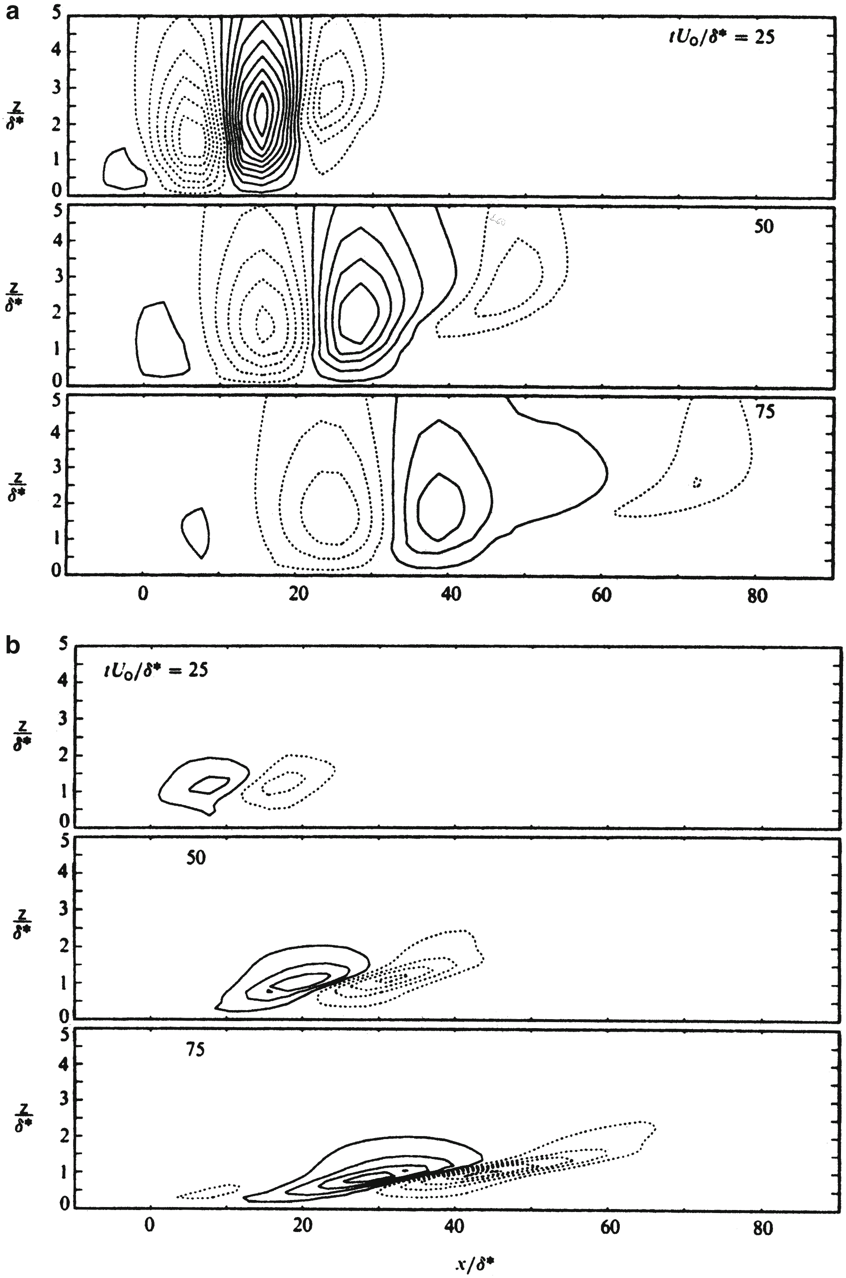


Fig. 3.3 Computations by Breuer and Haritonidis (1990) of contours in the (x, z)-plane of the vertical velocity $w(x, y, z, t)$ (**a**), and the streamwise disturbance velocity $u(x, y, z, t)$ (**b**) in an inviscid boundary-layer flow at $y = 0$ and several values of t. Velocities, lengths, and times are scaled with the free-stream velocity U_0, displacement thickness δ^*, and ratio δ^*/U_0, respectively. *Solid* and *dotted lines* represent positive and negative velocity values; contour spacing is $0.2w_0$ for w-contours, and to $2w_0$ for u-contours, where w_0 is the maximum value of $w(x, y, z, 0)$

swept back at 45° emerges rather quickly from the initial disturbance. In this case the amplitude of w disturbances also decreases with time, while the disturbance as a whole spreads in the horizontal plane. In contrast to this, the streamwise component u grows quickly, and a moderate values of t its typical value exceeds that of w more than tenfold and is dominated, not by the wavepacket, but by an intense shear layer. Comparison of these results with those of Breuer and Haritonidis gives the impression that the main features of the disturbance development are not too sensitive to the details of the undisturbed velocity profile.

Breuer and Haritonidis (1990) also performed an experimental investigation of disturbance evolution in a laboratory boundary layer on a flat plate. The initial disturbance was created by the impulsive motion, first up and then down, of a small flush-mounted membrane at the wall and thus consisted of two small-amplitude three-dimensional disturbances of opposite signs. The observed disturbance evolution during small enough initial time intervals was found to be in good qualiltative agreement with the results of inviscid calculations, showing the rapid formation of an intense inclined shear layer and a strong increase of streamwise disturbance velocity. Further downstream, viscous effects were detected; at not too small initial amplitude of disturbance, the nonlinearity clearly manifests itself.

3.2.2 Further Examples of Unstable Disturbances in Inviscid Plane-Parallel Flows

It was shown above that transient growth of initially small disturbances often takes place in plane-parallel flows of inviscid fluid, and can lead to immense increases of disturbance size and energy. It was also proved that, under rather general conditions, the horizontal components u and v of the disturbance velocity do not decay with time. Now we will consider several specific examples of unstable disturbances, some of which, apparently, can be of importance in many flows encountered in nature and engineering.

It was mentioned, in particular, by Willke (1972) and Criminale et al. (1991) that in a plane-parallel inviscid flow the vertical velocity w of a small-enough disturbance, which is independent of the streamwise coordinate x (i.e., such that $k_1 = 0$ in its Fourier representation) does not damp with time. It was, however, simply shown by Ellingsen and Palm (1975) that if in the case of such disturbance the velocity shear $U'(z)$ of the primary flow and the initial vertical disturbance velocity w do not vanish, then the streamwise velocity u of the disturbance increases indefinitely with time so that the flow is clearly unstable with respect to this disturbance. In fact, in this case Eqs. (2.35) and (2.37), with $n = 0$, take the forms

$$\frac{\partial u}{\partial t} + U'w = 0, \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\partial w}{\partial t} = -\frac{1}{p}\frac{\partial p}{\partial z}, \tag{3.19}$$

$$\nabla^2 p = 0, \tag{3.20}$$

where $\nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$ and the disturbances of flow variables are denoted by small letters without primes. Since $p(y, z, t)$ denotes a pressure disturbance vanishing at infinity, it follows from (3.20) that $p = 0$. Equation (3.20) shows that all nonconstant Fourier components of p vanish and thus $p = 0$ is the only physically acceptable solution; recall also that to prove instability it is enough to show that there exists a growing solution.) This implies that w (and also v) do not depend on time; therefore, according to the first Eq. (3.19)

$$u(y, z, t) = u(y, z, 0) - U'(z)w(y, z, 0)t, \tag{3.21}$$

and hence, if $U'(z) \neq 0$ and $w(y, z, 0) \neq 0$, then $|u|$ increases linearly with time. This result of Ellingsen and Palm was apparently one of the first examples of disturbances growing algebraically (and not exponentially) without bound as $t \to \infty$.

Of course, disturbances independent of x, which preserve their intensity from $-\infty$ to ∞ in the streamwise direction, are not those that are really encountered in turbulent flows. It was, however, shown by Landahl (1980) that similar arguments can also be applied to the much more important class of three-dimensional localized initially-infinitesimal disturbances, vanishing at time $t = 0$ outside some bounded region surrounding the coordinate origin. Let us now present his arguments.

In the general case where independence of x is not assumed, the partial derivative $\partial/\partial t$ must be replaced in Eqs. (3.19) by $D/Dt = \partial/\partial t + U(z)\partial/\partial x$, while zero right-hand sides of the first Eq. (3.19) and Eq. (3.20) must be replaced by $-\rho^{-1}\partial p/\partial x$ and $-2\rho U'(z)\partial w/\partial x = -\partial(2\rho U' w)/\partial x$, respectively (see Eqs. (2.35) and (2.37) in Chap. 2). Let us now integrate the corrected form of the first Eq. (3.19) over the whole x-axis. Then for the localized disturbance we obtain

$$\frac{\partial \overline{u}}{\partial t} = -\overline{w}U'(z), \quad \text{where} \quad \overline{u} = \int_{-\infty}^{\infty} u dx, \quad \overline{w} = \int_{-\infty}^{\infty} w dx, \tag{3.22}$$

and it is assumed that both the integrals in the given definitions of $\overline{u}$ *and* $\overline{v}$ exist. Integration over x of the corrected third of the Eqs. (3.19) yields

$$\frac{\partial \overline{w}}{\partial t} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z}, \quad \text{where} \quad \overline{p} = \int_{-\infty}^{\infty} p dx. \tag{3.23}$$

However, it follows from the solution of the corrected Eq. (3.20) (i.e., from Eq. (2.37)) that p may be represented as a derivative with respect to x of a finite integral, which vanishes for large values of x. Hence $\overline{p} = 0$, and from Eq. (3.23) it follows that $\overline{w}$ is independent of time. Integration of Eq. (3.22) then gives

$$\overline{u}(y, z, t) = \overline{u}(y, z, 0) - U'(z)\overline{w}(y, z, 0)t. \tag{3.24}$$

This result has the same form as Eq. (3.21), but now velocities of a localized flow disturbance integrated over the x-axis replace the x-independent point values. It is clear that the new equation has an immeasurably wider domain of application than Eq. (3.21) due to Ellingsen and Palm.

Equation (3.24) does not imply that the velocity u itself necessarily increases with time. In fact, the integrated velocity $\overline{u}$ could increase because the disturbance spreads streamwise without becoming more intense, and Landahl showed that this is exactly what occurs. He carefully inspected the general solution of the initial-value problem for a localized three-dimensional disturbance in a steady plane-parallel flow between solid walls, with velocity profile $U(z)$, and proved that under rather general conditions, guaranteeing the absence of exponentially-growing normal modes in the flow, the streamwise propagation speed of such disturbance lies between minimum and maximum values of $U(z), U_{\min}$ and $U_{\max}$, and that asymptotically the front of the disturbance propagates just with the velocity $U_{\max}$ and its back with the velocity $U_{\min}$. Then, using Eq. (3.24) Landahl showed that as $t \to \infty$

$$E(y,z,t) \equiv \frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2+w^2)dx > \frac{[U'(z)]^2\overline{w}^2(y,z,0)t}{2(U_{\max}-U_{\min}+\Delta)} \tag{3.25}$$

for any $\Delta > 0$. Hence, if $U'(z) \neq 0$ and $\overline{w}(y,z,0) = \overline{w}_0 \neq 0$, then the total kinetic energy $E(t)$ grows at least as fast as linearly as $t \to \infty$, but this growth is naturally explained by linear growth of the streamwise extent of the disturbed region. Landahl's analysis of the initial-value-problem solution for a three-dimensional localized disturbance shows that here, under wide conditions, $|w|$ decays as t^{-1} at large times; since the size of the disturbed region is growing linearly in time, such decay is consistent with constancy of the integrated vertical velocity $\overline{w}(t)$. However the same analysis leads to the conclusion that, in the case of a localized three-dimensional disturbance, the value of $|u|$ remains bounded as $t \to \infty$ (see also Eqs. (3.15) and (3.12) above). Hence the linear growth of the integrated streamwise velocity, which follows from Eq. (3.24), must be explained by streamwise elongation of the disturbed region. Such elongation clearly transforms any group of small localized three-dimensional disturbances of a shear flow into a streaky structure–a collection of longitudinal narrow streaks of either high-speed or low-speed fluid. At present there are numerous data, both from the flow-visualization experiments and from direct numerical simulations, which show that longitudinal streaky structures arise very often in transitional and fully turbulent shear flows. Such structures are typical, in particular, for flows behind the "turbulent spots" that appear during the initial stage of transition to turbulence (see, e.g., Sect. 2.1, Chap. 2), and in the near-wall regions of turbulent flows bounded by solid walls. The widespread occurrence of streaky turbulent structures gives ground for the suggestion that Landahl's (1980) algebraic growth of the energy of streamwise velocity disturbances can be of fundamental importance in many flows where transition to turbulence and formation of complicated eddy structures occur.

Landahl's simplified stability analysis of 1980 predicted the asymptotic behavior at $t \to \infty$ of the streamwise and vertical velocity disturbances integrated over the x-axis, $\overline{u}(y,z,t)$ and $\overline{w}(y,z,t)$. However for better understanding of the nature of eddy structures generated by algebraic instability it was necessary to study, at greater length, the behavior of disturbances in fluid flows of various types. Having this in mind and taking into account the important role played by near-wall flows in all phenomena related to interactions of fluid flows with the contiguous solid bodies (in particular, in fluid friction and heat or mass exchange), Landahl (1990, 1993a, b, 1996, 1997) gave much attention to the evolution of a weak three-dimensional disturbance arising in the near-wall layer of a turbulent boundary layer.

Landahl (1993a, b) stressed that there are three different physical processes involving (and affecting) weak disturbances: *i*) disturbance interaction with the shear of the undisturbed flow; *ii*) viscous damping of velocity disturbances; and *iii*) their nonlinear interactions with themselves. The first two processes are described by the linearized dynamic equations, and only the study of the third process requires the use of the full nonlinear Navier-Stokes equations. These three processes are characterized by the following quite different time scales: *i*) shear-interaction time scale $t_s = [dU/dz|_{z=0}]^{-1}$; *ii*) viscous-interaction scale $t_\nu = [L^2\nu^{-1}(dU/dz)^{-2}]^{1/3}$; and *iii*) nonlinear-interaction scale $t_n = L/u^{(0)}$, where L is the typical streamwise length of the initial disturbance and $u^{(0)}$ is the scale of the streamwise disturbance velocity. Taking into account observational data on weak velocity disturbances in the near-wall regions of turbulent boundary layers on a flat plate, Landahl estimated that here typically $t_\nu/t_s \approx 20$, $t_n/t_s \approx 100$. Hence usually $t_s \ll t_\nu \ll t_n$ and therefore the viscous damping beging to play a role at relatively late times (namely, at $t = \mathrm{O}(t_\nu)$); at earlier times the evolution of a disturbance can be accurately enough described by the inviscid linear stability theory. Moreover, the nonlinear terms of dynamic equations need not be taken into account until still later (at $t = \mathrm{O}(t_n)$). In this section, devoted to the inviscid linear theory, we shall consider only the initial stage of the evolution of weak disturbances in a boundary layer flow.

Let us restrict ourselves to the plane-parallel model of a boundary-layer flow, which neglects the influence of the weak nonparallelism caused by the dependence of the boundary-layer thickness on x. Then the inviscid linearized dynamic equations will have the form of Eqs. (2.35–2.36) with $\nu = 0$, implying Eqs. (2.37) and (2.38), again with $\nu = 0$ (cf. also remarks preceding Eqs. (3.22) and (3.23)). Replacing the streamwise coordinate x by the convected coordinate $\xi = x - U(z)t$, and then integrating the first two of Eqs. (2.35) and Eq. (2.38) with $\nu = 0$ (i.e., the first two Eqs. (3.19) and (3.9) corrected as above) over "Lagrangian time," following a fluid element moving with the undisturbed velocity $U(z)$, Landahl obtained the equations:

$$u(\xi, y, z, t) = u_0(\xi, y, z) - U'(z)l - \Pi_x/\rho, \tag{3.26a}$$

$$v(\xi, y, z, t) = v_0(\xi, y, z) - \Pi_y/\rho, \tag{3.26b}$$

$$\nabla^2 w(\xi, y, z, t) = \nabla^2 w_0(\xi, y, z) + U'' l_x. \tag{3.26c}$$

In these equations the subscript zero denotes the initial values, subscripts x and y denote the partial derivatives, and the *liftup* of a fluid element l and the *pressure impulse* Π are determined as follows

$$l(x, y, z, t) = \int_0^t w(x - U(z)(t - t_1), y, z, t_1)dt_1, \tag{3.27}$$

$$\Pi(x, y, z, t) = \int_0^t p(x - U(z)(t - t_1), y, z, t_1)dt_1.$$

Substituting the definition (3.27) of the liftup l into Eq. (3.26c) we obtain an integro-differential equation for the vertical velocity w. On the other hand, substituting the explicit solution of the Poisson equation (3.26c) (see Eq. (3.30) below) in the first Eq. (3.27) we arrive at an intergro-differential equation for the liftup l. Differentiating Eq. (3.26a) on x and Eq. (3.26b) on y and taking into account that $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$ for an incompressible fluid, we find the following two-dimensional Poisson equation for the pressure impulse Π:

$$\nabla_h^2 \Pi = -w_z - u_{o\xi}(\xi, y, z) - v_{0y}(\xi, y, z) + U'(z)l_x, \quad \nabla_h^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{3.28}$$

where again subscript zero denotes initial values while subscripts ξ, y and x denote differentiation. Hence Π can be expressed in terms of w, l, and the initial disturbance velocities. When Π and l are known, horizontal velocities u and v can be easily found from Eqs. (3.26a,b). (The value of l determines the most important "lift-up term" $U'(z)l$ of the streamwise-velocity equation (3.26a), which was in fact first introduced long ago by Prandtl (1925); see also Landahl (1985)). Equations (3.26–3.28) were applied by Landahl (1993a, 1996, 1997) to determination of the asymptotic ("long-time") behavior of all the velocity and vorticity components of a localized disturbance with initial velocities u_0, v_0 and w_0 vanishing outside a bounded domain having the streamwise length scale L and the "center" at a point with $x = y = 0$.

Analysis of the behavior of l, w, u and v for $t \to \infty$ (i.e., for $T \to \infty$, where $T = tU'(z)$ is the dimensionless time) proves to be quite complicated but the explicit solution (3.30) of the Poisson equation (3.26c) nevertheless allows one to obtain some interesting results. Studying the vertical velocity w, Landahl found that

$$w(x, y, z, t) \approx f_1(\xi, y, z)T^{-1} \quad \text{for} \quad \mathrm{T} \gg 1, \tag{3.29a}$$

where f_1 is a function of three variables determined by the initial values of $\nabla^2 w$. This result clearly agrees with the conclusion about integrated velocity $\overline{w}$ given in Landahl (1980). The long-time behavior of the streamwise velocity u was found to be particularly complicated but it is also admissible to analysis. For fixed bounded values of $|\xi|/L$ Landahl showed that

$$u(\xi, y, z, t) \approx -U'(z)f_2(\xi, y, z)T\gamma \quad \text{for} \quad T \gg 1, \tag{3.29b}$$

where f_2 is another function of three variables given by some special integral transform of $\nabla^2 w_o(x, y, z)$ while $\gamma = zU''(z)/U'(z)$. As to the spanwise velocity

$v(x, y, z, t)$, its values at $T \gg 1$ may be determined with satisfactory accuracy from the approximate equality

$$\partial u/\partial x + \partial v/\partial y \approx 0. \tag{3.29c}$$

We see that the horizontal velocity components approach a "frozen shape" in a coordinate system moving with the local velocity $U(z)$, and simultaneously decay or grow (depending on the sign of γ) algebraically with time. For velocity profiles without inflection points $U''(z) < 0$ for all z. Hence in flows with such profiles the disturbances will decay algebraically at the rate $\gamma = \gamma\ (z)$ which takes very small values near the wall (and tends to zero when $z \to 0$).

Different results were found by Landahl for the case of a fixed streamwise location (i.e., for fixed x where $|x| = \mathrm{O}(L)$). Here

$$u(x, y, z, t) = f_3(y, z) \quad \ln T + f_4(y, z) \quad \text{for} \quad T \gg 1, \tag{3.29d}$$

where f_3 and f_4 are some integral transforms of the function $F(y, z) = \int_{-\infty}^{\infty} \nabla^2 w_0(x, y, z)dx$. Therefore, there will be a logarithmic growth with time of the streamwise velocity of a disturbance before the algebraic decay takes over. Such logarithmic growth was observed by Lundbladh (1993) in numerical solutions of the linearized Navier-Stokes equations describing the evolution of a weak localized disturbance in plane Couette and Poiseuille flows (which differ from the boundary-layer flow studied by Landahl but apparently must be subjected to the same asymptotic laws). Streamwise velocities of disturbances independent of x lead to the appearance of streaks where fluid is flowing with velocity unequal to $U(z)$. The streaks have cross-flow structure which is approximately independent of x, while their lengths grow with time in proportion to $tU(z)$.

The formation of streaks in the near-wall region of a turbulent boundary layer, first observed by Kline et al. (1967) and later confirmed and investigated by many authors, was the main subject of Landahl's paper (1990). Here, in particular, some results of the mostly inviscid numerical calculations were given for the case of the development of a localized disturbance in a boundary layer with velocity profile U(z) close to the mean-velocity profiles observed in turbulent boundary layers. The initial shape of the disturbance was a pair of counter-rotating streamwise rolls (similar to those shown in Fig. 3.2) either fully symmetric or slightly non-symmetric with respect to the plane $y = 0$. The results obtained (partially presented in Fig. 3.4) showed that the streaks are weakly represented in the case of a symmetrical initial structure, but even small initial asymmetry in the spanwise direction y makes them much longer and more persistent.

To find the unknown function $w(x, y, z, t)$ the Poisson equation (3.26c) may be handled by standard methods. According to known results for this equation in the half-space $-\infty < x, y < \infty, 0 \le z < \infty$, with zero boundary conditions at $z = 0$ and at infinity, the formal solution of Eq. (3.26c) may be written as

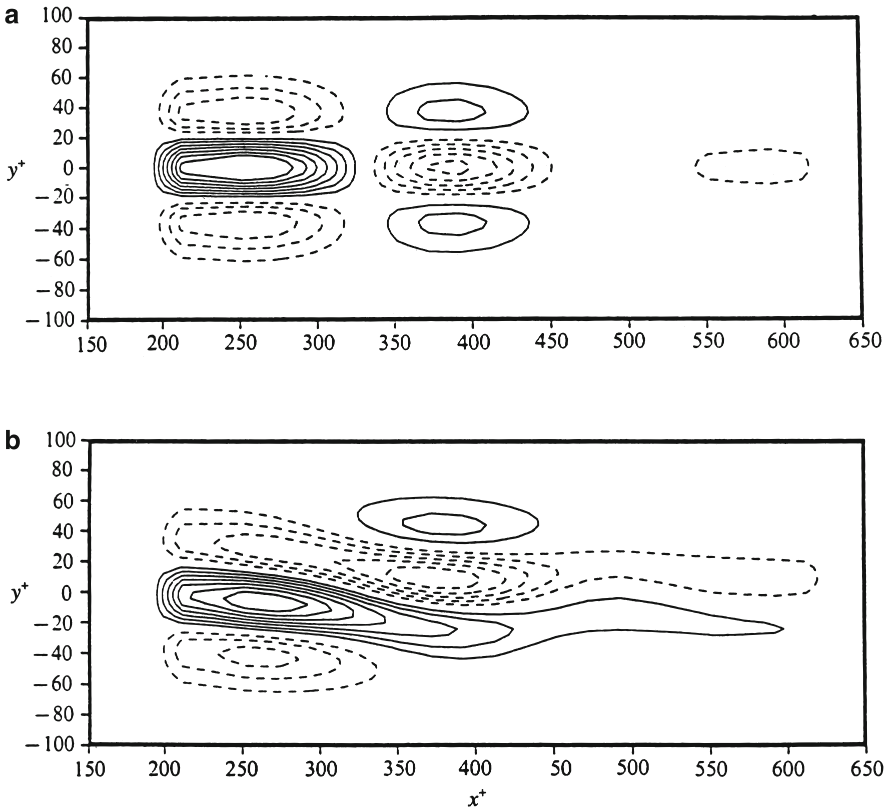


Fig. 3.4 Contours of constant streamwise velocity u in the plane $z^+ \equiv zu^*/\nu = 15$ of a boundary layer at $t^+ \equiv t(u^*)^2/\nu = 40$ [where $u^* = (\nu dU/dz^{|}z = 0)^{1/2}$], for **a** symmetrical initial structure of the disturbance, and **b** slightly asymmetrical initial structure (after Landahl (1990)). *Solid* and *dotted lines* represent positive and negative values, respectively; contours start at $u = -W_0$, with spacing equal to $0.25w_0$ where w_0 is the velocity scale characterizing the intial distribution of vertical velocity

$$w = -\frac{1}{4\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{0}^{\infty} dz' [\nabla^2 w_0(\xi', y', z') + U''(z'))l_x(x', y', z', t)]\left[\frac{1}{R} - \frac{1}{R_1}\right], \tag{3.30}$$

where $\xi' = x' - U(z')t$ and

$$R = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}},$$
$$R_1 = [(x-x')^2 + (y-y')^2 + (z+z')^2]^{\frac{1}{2}}. \tag{3.30a}$$

In Eq. (3.30), it is convenient to replace integration over x' by integration over ξ'; then dx', x' and $x-x'$ must be replaced in this equation by $d\xi', \xi' + U(z')t$ and $\xi - \xi' + [U(z) - U(z')]t$. This explicit solution was the basis of the asymptotic analysis by Landahl whose main results were briefly outlined above.

3.2.3 Initial-Value Problems for Disturbances in Inviscid Stratified Flows

Applications of the method of normal modes to stability analysis of inviscid stratified plane-parallel flows, whose horizontal velocity $U(z)$ and density $\rho(z)$ depend on the vertical coordinate z, were considered in Chap. 2 Sect. 2.8.3. Recall now that at the beginning of Sect. 3.21 it was pointed out that the papers by Eliassen et al. (1953), Dikii (1960a) and Case (1960b), devoted to applications of the initial value method of stability analysis to some stratified plane-parallel flows, were among the earliest papers using such an approach to hydrodynamic stability theory. Let us additionally remark that in the papers by Miles (1958), Hartman (1975), and Brown and Stewartson (1980), which were also mentioned in Sect. 3.21 as examples of this approach, the primary flows considered were also stratified. We see therefore that publications dealing with applications of the initial-value method of stability analysis to stratified flows are definitely not rare. Hence it seems reasonable to devote some space to consideration of such publications.

As everywhere in Sect. 3.2 we shall assume that the primary flow is plane-parallel and inviscid. Let us begin with the paper by Dikii (1960a), which differs from the other above-mentioned papers by the definition of stability used. Namely, Dikii proved the Lyapunov stability of exponentially-stratified Couette flows, where $U(z) = bz,\ \rho(z) = \rho_0 \exp(-az)$, in a half-space $0 \leq z < \infty$ (bounded by a flat solid wall at $z=0$), while the other authors interpreted stability as the absence of disturbances growing unboundedly with time. It was also assumed by Dikii that $\mathrm{Ri} = ga/b^2 > 0$, i.e., that the density is decreasing with height and hence the density stratification is stable. (It seems to be obvious that in the case of unstable stratification, where $\rho(z)$ increases with z and $\mathrm{Ri}<0$ everywhere, the flow will be unstable; a proof of this fact will be indicated below in this section.) We have already mentioned in Sect. 2.83 that this problem was first studied by Taylor (1931) by the method of normal modes; this author presented convincing arguments showing that the spectrum of eigenvalues c for the eigenvalue problem related to Eq. (2.66′) (at present usually called the Taylor-Goldstein, or T-G, equation) is here quite different in cases where $\mathrm{Ri} > 1/4$ and where $0 < \mathrm{Ri} < 1/4$. However, Taylor's results did not imply a clear answer to the question about the stability (or instability) of the flow to small disturbances. Therefore Dikii did not give much attention to the normal modes, but studied the solution of the initial-value problem for Eq. (2.63), which determines the vertical velocity w of a disturbance. The disturbance was assumed to be two-dimensional and hence w depended only on x, z and t. Note also that the Boussinesq approximation, which simplifies all the equations, was not used by

Dikii, but he noticed that the introduction of this approximation would not change his results.

To find the required solution, Dikii first replaced the streamwise coordinate x in Eq. (2.63) by the convected coordinate $\xi = x - bzt$, and then applied to the unknown function $w(\xi, z, t)$ the combined Fourier-Laplace transform (where Fourier and Laplace transforms were taken with respect to variables ξ and z, respectively). As a result, an ordinary differential equation of the second order was obtained for the Fourier-Laplace transform $\hat{w}(k, p; t)$ of $w(\xi, z, t)$, where this transform was considered as a function of t dependent on two parameters, the Fourier- and Laplace-transform variables k and p. The solution of this equation was then found in the form of a sum of two indefinite integrals of some combinations of hypergeometric functions with weight functions determined by complicated integral equations, which included initial values of w and $\partial w/\partial t$. Applying the inversion formula (similar to Eq. (3.6)) to the Fourier-Laplace transform $\hat{w}(k, p; t)$, Dikii found the general solution $w(\xi, z, t)$ of the initial value problem. A cumbersome investigation of the asymptotic behavior of this solution allowed him to prove that if initial values of w and $\partial w/\partial t$ are smooth enough, then for any $\mathrm{Ri} > 0$ the absolute value of the vertical velocity $|w(\xi, z, t)|$ remains bounded at any time $t > 0$ by a constant which can be made arbitrarily small by sufficient diminution of the absolute values of these initial values. This statement just proved the Lyapunov stability of the considered stratified flows with respect to small and smooth initial disturbances. In his paper Dikii paid most attention to a single Fourier component of the disturbance, i.e., to the case where $w(x, z, t) = e^{ikx}W(z, t)$. In this case asymptotic behaviors of $|w(x, z, t)|$ and $|w(\xi, z, t)|$, when $t \to \infty$ but the other two independent variables have fixed values, are the same; however, in some other cases, considered below, they can differ considerably.

The initial-value approach was also used for the study of time evolution (first of all as $t \to \infty$) of disturbances in exponentially stratified Couette flows (where the vertical extent $0 \leq z < \infty$ was sometimes replaced by $0 \leq z \leq H < \infty$ or $-\infty < z < \infty$ with appropriate change of the boundary conditions) by Eliassen et al. (1953), Case (1960b), Kuo (1963), Hartman (1975), Chimonas (1979), Brown and Stewartson (1980), and Farrell and Ioannou (1993a) (this list surely is not complete). Since these papers are quite typical of applications of the initial-value approach to stability of stratified flows, we shall consider below, almost exclusively, flows with linear velocity and exponential density profiles. Note that in all the above-mentioned papers the Boussinesq approximation was used, and, as a rule, only two-dimensional disturbances were studied, with the aid of the combined Fourier-Laplace transform (3.4) with respect to variables (x,t), applied to the vertical velocity $w(x, z, t)$ (or, what is practically the same, to the stream function $\psi(x, z, t)$) (rarely-met deviations from this procedure will be noted below). However, the investigation of the asymptotic behavior of these transforms as $t \to \infty$ proved to be sophisticated and requiring great skill; therefore it is not surprising that some of the results obtained were inaccurate and differed from more precise results given in other publications.

In the early investigation by Eliassen et al., where the thickness H was assumed to be finite, arguments were presented which made very plausible the assumption (which later was proved to be correct) that for $\mathrm{Ri} > 1/4$ the T-G eigenvalue problem has an infinite number of real discrete eigenvalues (note that for the case where $0 \leq z < \infty$ the existence of such real eigenvalues was proved by Taylor (1931); however, if $-\infty < z < \infty$, then, as we shall see later, no discrete eigenvalues exist at any value of Ri). Eliassen et al. assumed also that the eigenfunctions corresponding to the set of all discrete eigenvalues form a complete system in a space of admissible initial values of w or ψ; however, this assumption was proved later to be incorrect. (If it were correct, any solution $w(x, z, t)$ or $\psi(x, z, t)$ would be representable in the form of a linear combination of neutral normal modes, i.e., would be a bounded undamped function of t; hence the flow would be stable. It was in fact later proved to be stable, but the proof turned out to be not so simple.) As for the case where $0 < \text{Ri} < 1/4$, Eliassen et al., relying on some nonstrict arguments, came to the conclusion that in this case the T-G eigenvalue problem has no discrete eigenvalues and that the flow is stable, since here, as $t \to \infty, |w| \infty t^{-1}$ and $|u| \infty t^{\mu-1/2}$, where $\mu = (1/4 - \text{Ri})^{1/2}$ (here and below μ always denotes the positive value of the square root of $1/4 - \text{Ri}$, or that having a positive imaginary part). For $0 > \text{Ri} > -3/4$ (i.e., $1/2 < \mu < 1$) Eliassen et al. found that the T-G eigenvalue problem also has no discrete eigenvalues, but the flow is unstable since here again $|u| \infty t^{\mu-1/2}$ as $t \to \infty$ (however $|w| \infty t^{\mu-3/2}$, i.e., it tends to zero). Moreover, it was also found in this paper that for $\text{Ri} < -3/4$ there exists at least one pair of complex eigenvalues (the number of such pairs increases with decrease of Ri) of the corresponding T-G eigenvalue problem; therefore, here the flow is unstable and some disturbances in it grow exponentially with time. This last conclusion and the majority of the results on discrete spectra and on flow stability or instability were later rigorously proved by other authors; however certain suggested asymptotic relations were found to be incorrect.

Case (1960b) followed Taylor and studied stability of exponentially-stratified Couette flows in the half-space $0 \leq z < \infty$. Based on the analytical results by Dyson (1960) (which were independently found also by Dikii (1960c)) he proved rigorously that for such flows at any $\text{Ri} > 0$ there are no complex eigenvalues satisfying the T-G eigenvalue problem, but $\text{Ri} > 1/4$ then at each wave number k there exists an infinite number of real eigenvalues, while for $0 < \text{Ri} < 1/4$ there are either one or zero real eigenvalues at any k (see also discussion of this topic in Sect. 2.83). However, it was also showed by Case that in the case considered the T-G equation at any $\text{Ri} > 0$ has a continuous spectrum which fills the half-lines $0 < c < \infty$ and $-\infty < c < 0$. According to Case's calculations (which were later found to be inaccurate), $|w|$ includes not only undamped oscillations, corresponding to discrete real eigenvalues, but also a contribution from the continuous spectrum which is represented by a function tending to zero as $t^{-1/2}$, if $\text{Ri} > 1/4$, and as $t^{\mu-1/2}$, if $0 < \text{Ri} < 1/4$; therefore Case concluded that the flow is stable at any $\text{Ri} > 0$.

Kuo (1963) considered the general solution of the initial-value problem for a three-dimensional vertical velocity disturbance $w(x, y, z, t)$ in an exponentially-stratified Couette flow, having either stable or unstable stratification and filling either a layer of finite thickness H or a half-space $0 \leq z < \infty$. The solution found by him used a preliminary transformation from $w(x, y, z, t)$ to a Fourier-Laplace transform $\hat{w}(k_1, k_2, p; z)$, combining a two-dimensional Fourier integral with respect to the horizontal coordinates and a Laplace integral with respect to time. Following Taylor (1931), Kuo showed that the study of stability for three-dimensional waves proportional to $\exp[i(k_1x + k_2y)]$ can be reduced to the corresponding two-dimensional problem with modified Richardson number $\mathrm{Ri}_1 = (1 + k_2^2/k_1^2)\,\mathrm{Ri} > Ri$ (cf. also the corresponding discussion in Sect. 2.83). Therefore, it is enough to consider only two-dimensional disturbances below. Note also that most attention was given by Kuo to the investigation of normal-mode disturbances; since we are currently discussing the initial-value problem, the results of this investigation will be only briefly indicated here.

First, Kuo noticed that in the case of a stratified Couette flow, where $U''(z) \equiv 0$, the Boussinesq approximation implies the following simple form of the integral relation (2.69) found by Howard (1961) for $n = -1$:

$$\int_0^H \rho\{(U - c)^4(|\partial F_{-1}/\partial z|^2 + k^2|F_{-1}|^2) - (U - c)^2(2 + \mathrm{Ri}\,)(U')^2|F_{-1}|^2\}dz = 0 \tag{3.31}$$

where c is an eigenvalue satisfying the T-G eigenvalue problem and, contrary to Eq. (2.69), all the variables are now assumed to be dimensional (here the height H can take both finite and infinite values). If $\mathrm{Ri} \leq -2$ then for real c the integrand in the left-hand side of Eq. (3.31) is everywhere positive, and hence no real eigenvalues c can exist in this case. For exponentially stratified Couette flows in an infinite layer, where $H = \infty$, Kuo showed that complex eigenvalues c exist if and only if $\mathrm{Ri} < -2$ (i.e., for $\mathrm{Ri} < -2$ wave-like disturbances exponentially growing with time surely exist). In addition to this he also calculated the number of discrete eigenvalues c for any $\mathrm{Ri} < 1/4$ (for $\mathrm{Ri} > 1/4$ this number is infinite while for $\mathrm{Ri} < 1/4$ it is always finite) and repeated without criticism Case's conclusion that $|w| \propto t^{\mu - 1/2}$ as $t \to \propto$, if $0 < \mathrm{Ri} < 1/4$, supplementing it with the statement that this conclusion holds also for $0 > \mathrm{Ri} > -2$ (i.e., for $1/2 < \mu < 3/2$ when $t^{\mu-1/2}$ grows unboundedly with t).

For a Couette flow of finite height H having unstable exponential stratification (so that $\rho^{-1}\partial\rho/\partial z = b > 0$) Kuo found that here complex eigenvalues c (and hence exponentially growing normal modes of disturbance) exist only for $\mathrm{Ri} < -3/4$, while for $1/4 > \mathrm{Ri} > -3/4$ no discrete eigenvalues exist at any wavenumber k. Therefore the time evolution of the disturbance velocity w in this case is determined by the contribution from the continuous spectrum of T-G eigenvalues. According to Kuo, this contribution leads to the same asymptotic law as for $H = \infty$, so that here again $|w| \infty t^{\mu - 1/2}$ for large enough values of t. Moreover, Kuo also investigated the spectrum of discrete eigenvalues c (which depends on k) at various values of Ri for both $H = \infty$ and finite H. (Note that according to (3.31) Kuo's real eigenvalues c in cases of strong stability (large negative Ri) must be fictitious. In fact, the corresponding eigenfunctions have singularities and therefore do not represent true solutions; cf. discussion by Eliassen et al. (1953)). Finally, Kuo investigated the shapes of the most unstable disturbances in unstably-stratified Couette flows and found that they can explain the results of some meteorological observations and laboratory experiments. However, we have no space to discuss this matter in more details.

Later Chimonas (1979) analyzed anew the asymptotic behavior of disturbances to exponentially stratified Couette flow, for $-\infty < z < \infty$ and Ri < 1/4. (The unboundedness of the low domain simplifies all the computations, while it seems probable that most of the asymptotic results will be the same as for flows in a bounded layer or half-space.) According to Chimonas, if $\xi = x - bzt$, then in the case of unbounded flow $|w(\xi, z, t)| \propto t^{2\mu - 1}$ at large values of t and fixed values of ξ and z. Chimonas explained the difference between his result and that found by Case by the fact that Case, contrary to him, determined the asymptotic behavior of $|w(x, z, t)|$ at fixed values of x and z by assuming that the initial disturbance was of bounded extent in x. It is clear that then the disturbance velocity $w\ (x,\ z,\ t)$ must fall off at a fixed point $(x,\ z)$ more rapidly with t than at fixed values of (ξ, z) and that here the physically most interesting behavior is that at fixedξ and not at fixed x. In addition Chimonas also determined the asymptotic behavior as $t \to \infty$ of the horizontal velocity, density and pressure disturbances $u(x, z, t)$, $\rho'(x, z, t)$ and $p'(x, z, t)$. He found that at Ri $< 1/4$, $|p'|$ decays as $t^{2\mu-1}$, but both $|u|$ and $|\rho'|$ grow with t as $t^{2\mu}$ (i.e., without limit). Proceeding from this, Chimonas asserted that at 0 < Ri < 1/4 exponentially-stratified inviscid Couette flows are unstable. This assertion contradicted the results of the other available investigations of the same topic and therefore from the very beginning seemed to be dubious; later an error in Chimonas' analysis was indicated by Brown and Stewartson (1980).

Correct asymptotic relations replacing those suggested by Chimonas (and also results found by Eliassen et al. Case, and Kuo) were published by Hartman (1975). His results incorporate also the earlier results by Phillips (1966), Chap. 5, and Booker and Bretherton (1967) relating to development of internal waves in stably-stratified ocean or atmosphere; so it is reasonable to begin with some conclusions from the latter two publications. To investigate the influence of the velocity shear on the evolution of oceanic internal waves, Phillips used a model example of wave development in an exponentially-stratified inviscid Couette flow filling an unbounded space. In this respect he considered particular solutions of Eq. (2.63′) for small disturbances, of the form

$$w(x, y, z, t) = W(t) \exp[i(k_1\xi + k_2 y)] = W(t) \exp[i(k_1 x + k_2 y - k_1 btz)], \tag{3.32}$$

where $\xi = x - bzt$ and $b = \partial U/\partial z = U'(z)$ is a constant velocity shear. (Phillips explained that his model, in which the dependence of the velocity shear U'(z) and Brunt-Väisälä frequency $N(z) = (-g\rho'/\rho)^{1/2}$ on z was neglected, is appropriate only for wave motions of small vertical scale; therefore the dependence of these motions on z was also neglected. However the possible dependence on y was taken into account, in contrast to all the work considered above in this subsection except that of Kuo.) Phillips noted that the general solution of the second-order differential equation for $W(t)$ implied by Eq. (2.63′) can be expressed in terms of hypergeometric functions, but in his book only the case of strong stability (or weak shear), where Ri $= N^2/b^2 \gg 1$, was studied at length. He showed that in this case the solution obtained represents a three-dimensional wave motion whose amplitude, wave numbers and direction of propagation are slowly changing with time. According to this solution, the amplitude and the wavelength (which is inversely proportional to the length of the three-dimensional wavenumber vector) are continuously decreasing and the direction of propagation is approaching the vertical axis. These features of the wave motions considered agree more or less satisfactorily with some features of real oceanic internal waves, but for us here the most important discovery by Phillips is his finding that in the case of very strong stability (i.e. for $\mathrm{Ri} \gg 1) W(t) = |w(x, y, z, t)|$ is asymptotically proportional to $t^{-3/2}$ as $t \to \infty$. Wave amplitudes for the horizontal velocity components $u(x, y, z, t)$ and $v(x, y, z, t)$ were also found to decrease, but more gradually, only as $t^{-1/2}$. Hence the motion becomes more and more horizontal with time and its mean kinetic-energy density per unit mass $T^*(t)$ decreases asymptotically as t^{-1}. However, the rapid decrease of the vertical scale leads to unlimited increase of the vertical gradients of u and v and hence also of horizontal components of the vorticity field, ζ_1 and ζ_2.

Booker and Bretherton (1967) further developed Phillips′ approximate theory. They were primarily interested in atmospheric internal waves, and paid most attention to wave motions near the critical height z_{cr} where the undisturbed velocity $U(\mathrm{z_{cr}})$ coincides with the phase velocity c of the wave. For the present discussion it is important that these authors also analyzed the general solution of the initial-value problem for the wave-like vertical-velocity disturbance $w(x, z, t)$ (variable y is absent here since only two-dimensional waves were considered), for a flow model which included a layer where both $U'(z)$ and $N(z)$ took constant values. The results obtained included the discovery of some particular solutions which are valid within this layer (and in the case of exponentially-stratified Couette flow in the whole space too, a fact mentioned by the authors in passing), under the condition that $\mathrm{Ri} = N^2/U'^2 > 1/4$. Asymptotically (i.e., for large enough values of t) these solutions have the forms of damped waves

$$w(x, z, t) = W_1(z)t^{\mu-3/2}e^{ik(x-bzt)} + W_2(z)t^{-\mu-3/2}e^{ik(x-bzt)} \tag{3.33}$$

where, as usual, $\mathrm{b} = U'(z) = \mathrm{const.}$ and $\mu = (1/4-\mathrm{Ri})^{1/2}$ (hence μ has a purely imaginary value here). Corresponding solutions for the horizontal velocity component have the forms

$$u(x, z, t) = U_1(z)t^{\mu-1/2}e^{ik(x-bzt)} + U_2(z)t^{-\mu-1/2}e^{ik(x-bzt)}, \tag{3.34}$$

showing that horizontal velocity also decays, but more slowly. These equations represent a slight refinement (concerning the admissibility of the amplitude dependence on z) of the asymptotic laws found by Phillips, but Booker and Bretherton discovered that these laws are valid not only for $\mathrm{Ri} \gg 1$ but for any Ri exceeding 1/4.

Hartman (1975) considered only the idealized model of exponentially-stratified Couette flow in an unbounded space, and found a simple form of the general solution to the initial-value problem for an infinitesimal two-dimensional disturbance $\{u(x, z, t), 0, w(x, z, t)\} = \mathbf{u}(x, z, t)$. Instead of the vertical velocity w or the stream function ψ chosen as the dependent variables in many previous studies, Hartman solved the initial-value problem for the non-zero vorticity component $\zeta = \partial u/\partial z -$

$\partial w/\partial x = -\nabla^2\psi$. Using the convected "spatial coordinates" $(\xi, z) = (x - btz, z)$ he found that the dependence on the variable t of the two-dimensional Fourier transform $\hat{\zeta}(k_1, k_2; t)$ of $\zeta(\xi, z, t)$, with respect to coordinates (ξ, z), can be determined from a second-order differential equation, whose solution for the given initial values of ζ and $\partial\zeta/\partial t$ at $t = 0$ can be simply expressed through standard hypergeometric functions. This solution was then skillfully used for the investigation of wave-packet propagation in unbounded stratified Couette flow under the condition that Ri > 1/4 supplemented by comparison of the results obtained with those of Booker and Bretherton (1967). It was also mentioned that this solution can be applied to the determination of the behavior of localized disturbances in unbounded stratified Couette flow at 0 < Ri < 1/4, but this specific application is omitted from Hartman's paper. However, he described the asymptotic behavior of his solutions at large values of t (more correctly, of the dimensionless time $T = U'(z)\,t = bt$) and these results are most interesting for the present discussion.

According to Hartman the main terms of the asymptotic expressions for the general solution of the initial-value problem have the following forms:

$$\hat{\zeta}(k_1, k_2; t) \approx a_1 t^{\mu+1/2} + a_2 t^{-\mu+1/2}, \quad \text{if} \quad \mathrm{Ri} \neq 1/4, \tag{3.35a}$$

$$\hat{\zeta}(k_1, k_2; t) \approx a_3 t^{1/2} \ln t, \quad \text{if} \quad \mathrm{Ri} = 1/4, \tag{3.35b}$$

at $T = bt \gg 1$, where the coefficients a_1, a_2 and a_3 depend on k_1, k_2 and the initial conditions. We see that vorticity is growing without limit at any Ri. Now, using the simple relationship between the Fourier transforms of the vorticity ζ and the stream function ψ given by Hartman, it is easy to find the main terms of asymptotic expressions for the Fourier transforms $\hat{w}(k_1, k_2; t)$ and $\hat{u}(k_1, k_2; t)$ of the vertical and horizontal velocity components $w(x, z, t)$ and $u(x, z, t)$. Obtained in this way, asymptotic relations at Ri $\neq$ 1/4 have the form, recalling that $\mu = (1/4 - \mathrm{Ri})^{1/2}$

$$\hat{w}(k_1, k_2; t) = c_1 t^{\mu-3/2} + c_2 t^{-\mu+3/2}, \tag{3.36}$$

$$\hat{u}(k_1, k_2; t) = d_1 t^{\mu-1/2} + d_2 t^{-\mu+1/2}, \tag{3.37}$$

while for Ri = 1/4 (i.e., $\mu = 0$) an additional logarithmic factor must be included in the right-hand parts. These equations imply that T^* (k_1, k_2; t), the averaged density per unit mass of the kinetic energy for a wave-component of the disturbance with wave numbers (k_1, k_2), satisfies the relationships:

$$\begin{aligned} &T^*(k_1, k_2; t) \propto t^{2\mu-1} \quad \text{if} \quad \mathrm{Ri} < 1/4, \\ &T^*(k_1, k_2, t) \propto t^{-1} \quad \text{if} \quad \mathrm{Ri} > 1/4. \end{aligned} \tag{3.38}$$

Later Farrell and Ioannou (1993a) supplemented Hartman's results (3.35–3.38) by similar results for Fourier transforms of the density disturbance, $\hat{\rho}(k_1, k_2; t)$, and of the averaged total energy density (per unit mass), $E(k_1, k_2; t)$, for the disturbance component with wave numbers (k_1, k_2). (The total energy density can be represented as $E(k_1, k_2; t) = T^*(k_1, k_2; t) + V(k_1, k_2; t)$ where $V(k_1, k_2; t)$ is the density of the potential energy disturbance for fluid elements of variable density $\rho(\boldsymbol{x},\ t)$ in a gravitational force field.) The asymptotic relationships are:

$$\hat{\rho}(k_1, k_2; t) = e_1 t^{\mu-1/2} + e_2 t^{-\mu-1/2}, \tag{3.39}$$

$$E(k_1, k_2; t) \propto t^{2\mu-1} \quad \text{if} \quad \text{Ri} < 1/4,\ E(k_1, k_2; t) \propto t^{-1} \quad \text{if} \quad \text{Ri} > 1/4. \tag{3.40}$$

Recall that wave numbers (k_1, k_2) correspond to waves of the form exp $[i\{k_1(x - bzt) + k_2 y\}]$; therefore in the material space the decay laws (3.35–3.40) are related to asymptotic behavior at fixed $(x - bzt, z)$ and not at a fixed point (x, z).

Equations (3.36) and (3.37) clearly agree with the results found by Phillips (1966) and Booker and Bretherton (1967) for decay laws relating to waves in an unbounded stably-stratified Couette flow. However, Phillips derived these laws only for the case where Ri$\,\gg 1$, and Booker and Bretherton generalized them to the wider class of stratified Couette flows with Ri$\,>1/4$. Now we see that these results are true for these flows with any value of Ri, positive, zero, or negative, the only exception being Ri$\,=1/4$ exactly, where there is a degeneracy (the merging of two solutions) which produces the appearance of a slight correction. According to Eqs. (3.38) and (3.40), the growth or decrease of energy of wave-like disturbances in an unbounded exponentially-stratified Couette flow is always algebraic, not exponential. This proves that for this flow the Taylor-Goldstein equation has no discrete eigenvalues at any value of Ri (as explained earlier, this statement is incorrect in cases where the Couette flow considered has one or two solid boundaries).

Brown and Stewartson (1980) also considered the question of the precise form of decay laws for waves in an unbounded exponentially-stratified Couette flow. They did not mention the paper by Hartman (1975) and apparently did not know about it, but their main result is the same as that found by Hartman: it consists in the confirmation of Eq. (3.33), proved by Booker and Bretherton (1967) for the case where Ri$\,>1/4$, supplemented by the proof that this result is in fact true for any Ri (the slight correction needed at Ri$\,=1/4$ was unnoticed here). Brown and Stewartson also indicated the error in the derivations by Case (1960b) and Chimonas (1979).

We have already mentioned the paper by Farrell and Ioannou (1993a). This paper is also devoted to the study of development of small two-dimensional disturbances to an inviscid, exponentially-stratified Couette flow with Ri$\,\geq 0$, filling either an unbounded space or a layer between two parallel walls. However, here the authors pay most attention, not to asymptotic results for $t \to \infty$, but to transient development of disturbances during finite time intervals. First of all they are interested in the possibility of considerable growth of disturbances during the early stage of their development, as first discovered, for the case of a Couette flow of constant-density fluid, by Orr (1907).

Farrell and Ioannou remark that, according to results by Phillips (1966) and Hartman (1975), in the case of an unbounded exponentially-stratified Couette flow, the time-dependent amplitude $\hat{\zeta}(k_1, k_2; t) \equiv Z(t)$ of a Fourier component

$Z(t)\exp[i(k_1\xi + k_2z)]$ of the vorticity field can be explicitly expressed in terms of the standard hypergeometric functions, and for given initial conditions it can be accurately determined by numerical integration of the relevant second-order differential equation. When $Z(t)$ is known, it is easy to determine also the Fourier amplitudes $\Psi(t)$, $W(t)$, $U(t)$ and $R(t)$ of the disturbance stream function ψ, the velocity components w and u, and the density ρ'' that satisfy the asymptotic relationships (3.35–3.37) and (3.39). Farrell and Ioannou do not consider the asymptotics further, but pose a question about the initial conditions which yield greatest growth of the total disturbance energy in a specific time $T_{opt} = (bt)_{opt}$ (the corresponding disturbance is then called the *optimal disturbance* for the time T_{opt}). However, to make this question fully definite it is necessary to clarify what is meant in this case by "initial conditions."

For unique determination of the development of a flow disturbance it is necessary to give the initial values of two of its independent fluid-dynamic fields, e.g., fields of the vorticity ζ and its time derivative $\partial\zeta/\partial t$ (as Hartman did), or of the stream function ψ (uniquely determining two velocity components u and w) and the density ρ' (this second choice was made by Farrell and Ioannou). Since the finding of the optimal value for two fields $\psi(x,z,0)$ and $\rho'(x,z,0)$ is a complicated task, and plane waves form an orthogonal basis in the functional space of functions in an unbounded space, Farrell and Ioannou restricted themselves to determination of only the optimum plane-wave initial values of ψ and ρ'. In this case the initial conditions are given by the initial values Ψ_0 and R_0 of amplitudes for the stream-function and density waves, and by the corresponding wave numbers k_1 and k_2; the relative energy growth $G(t) = E(t)/E(0)$ is dependent only on the wave-number ratio $s = k_2/k_1$ and not on k_1 and k_2 individually. So, for the determination of the optimal wave disturbance, it is only necessary to find the maximum value of a function of three variables Ψ_0, R_0 and s.

At fixed value of t, the maximal value $G_{\max}(t)$ of $G(t)$, and the values of the three variables corresponding to its maximum, clearly depend on the choice of the time t. In Sect. 3.1 we referred to Orr (1907) to mention that the maximal growth $|w|_{\max}/|w|_0$ of the vertical velocity for a plane-wave disturbance in an inviscid plane Couette flow of constant-density fluid can be made as large as desired, if it is permitted to increase indefinitely the time t_{opt} at which this growth is reached. Farrell and Ioannou calculated the values $G_{\max}(t)$ for different values of t and Ri; the results obtained are shown in Fig. 3.5. It was found that for small values of t (measured in the shear units b^{-1}, i.e., given by values of $T = bt$) the function $G_{\max}(T, Ri)$ is practically independent of Ri and only slightly exceeds unity, but later on it begins to increase and becomes significantly dependent on both variables, increasing indefinitely with T and decreasing with Ri. Farrell and Ioannou remarked that, in real geophysical flows, ambient fluctuations usually impose a time scale beyond which the growth of disturbances according to the theory is definitely disrupted. Therefore the computations of the function $G_{\max}(T, Ri)$ for very large values of T are practically useless. As a reasonable representative value they selected $T_{\text{opt}} = 6$ in their study, noting that other choices for T_{opt} usually do not change the results qualitatively. In Fig. 3.6 the values of $G(T) = E(T)/E(0)$ for the optimal plane wave corresponding to $T_{\text{opt}} = 6$ are plotted against T for a number of nonnegative values of Ri. We see that for zero and small positive values of Ri the value of $G(T_{\text{opt}})$ for $T_{\text{opt}} = 6$ is in the range from

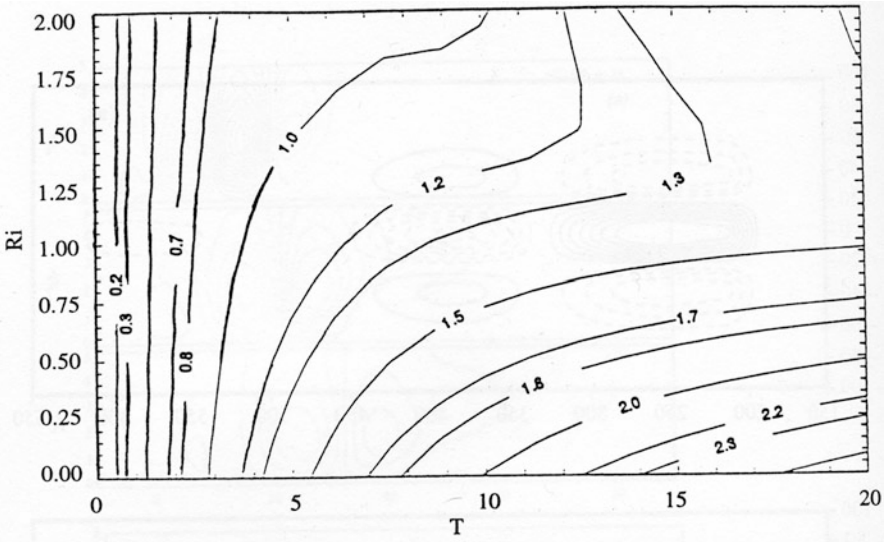


Fig. 3.5 Maximal energy growth $G_{\max}(T, \mathrm{Ri})$ for plane-wave disturbances with optimal values of Ψ_0, R_0, and s in a stratified Couette flow, as a function of the time of maximal growth $T = bt$ and Ri. The contour values are those of $\log_{10} G_{\max}$ (T, Ri). (After Farrell and Ioannou (1993a))

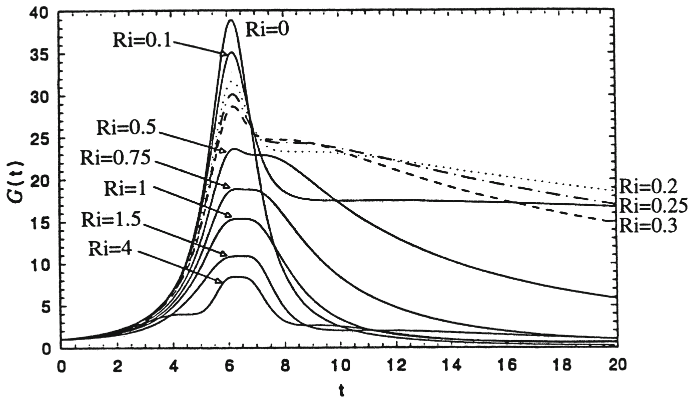


Fig. 3.6 Dependence on $T = bt$ of the normalized energy $G(T) = E(T)/E(0)$ of a plane-wave disturbance in a stratified Couette flow having maximal possible energy growth at $T = 6$, for different values of Ri. (After Farrell and Ioannou (1993a))

25 to 40, while at $\mathrm{Ri} = 4$, it is close to 7. Note the appearance of near-persistence of $G(T)$ (i.e., of the total energy $E(T)$) at $T > T_{\mathrm{opt}}$ when Ri is in the range $0.1 < \mathrm{Ri} < 0.3$; such persistence can have some practical importance. Similar graphs of the functions $T^*(t)/T^*(0)$, and $V(t)/V(0)$, showing the dependence of kinetic and potential energies of the optimal disturbance on t and Ri, are also presented by Farrell and Ioannou, together with some data characterizing the optimal initial values and discussion of the results obtained.

In the case of a stratified Couette flow in a layer between two parallel solid walls (for example, in a channel of finite depth) the problem proves to be more complicated, since here there are no closed-form solutions of the dynamic equations satisfying the required boundary conditions. The corresponding T-G equation has in this case a continuous spectrum of eigenvalues filling the real-axis interval $U_{\min} \le c \le U_{\max}$ (exactly as in the case of a constant-density flow in a channel; see Sect. 2.82) supplemented, for Ri > 1/4, by an infinite number of real discrete eigenvalues. Farrell and Ioannou approximately determined the optimal disturbances in this case, by replacing the differential equations in the interval $0 \le z \le H$ by finite-difference equations in a domain with a sufficiently large number N of mesh points. The algebraic finite-difference equations then have only a finite number $2N$ of discrete eigenvalues c_j, and the general solution can be represented by a linear combination of the corresponding eigenvectors multiplied by $\exp[ik(x - c_j t)]$. The energy $E(t)$ takes the form of a Hermitian positive-definite quadratic form of $2N$ variables, with coefficients depending on t. Finding of the maximum value for $G(t) = E(t)/E(0)$ is now a more difficult problem than in the case of an unbounded Couette flow, but this problem is also accessible to modern computers. Computations made for different values of N showed that results for $N = 30$ are as a rule sufficiently accurate in this case. Using this approximate method, the authors presented two examples of optimal initial disturbances (corresponding to $T_{\text{opt}} = 6$, Ri = 0.2 and 0.75, and fixed k) and of their forms at T = $T_{\text{opt}} = 6$ and $T = 15$, supplemented by graphs of the same type as in Fig. 3.6 showing the dependence of the total, kinetic, and potential energies of the optimal disturbance (again for $T_{\text{opt}} = 6$) on Ri and T.

In this subsection, only studies of hydrodynamic stability of exponentially-stratified Couette flows have been discussed so far. Moreover, with the exception of the works by Kuo and by Phillips, only two-dimensional disturbances were considered in these studies. Note in this respect that the first applications of the normal-mode stability investigations to various inviscid stratified flows were made very early (more than a hundred years ago by Kelvin (1871), who studied the case of an unbounded flow having very simple discontinuous velocity and density profiles, and then by Taylor (1931) and Goldstein (1931), who considered several more complicated examples of profiles $U(z)$ and $\rho(z)$). Later many other normal-mode stability studies of inviscid stratified flows, involving a great number of new examples, were carried out. However, until now only a few papers have appeared in the physics and engineering literature on applications of the initial-value method to stability studies for stratified non-Couette flows. True, Chimonas (1979), in addition to the main example of unbounded exponentially-stratified Couette flow, presented some general results relating to unbounded flows with arbitrary profiles $U(z)$ and $\rho(z)$, but the derivation of these results had the same defect which invalidated Chimonas' results for Couette flows. Moreover, Brown and Stewartson (1980) noted in passing that in the case of arbitrary smooth profiles $U(z)$ and $\rho(z)$ it is only necessary to replace the constants $U'(z) = b$ and Ri in the main terms (3.33) of the asymptotic expansions for $w(x, z, t)$ (or $\psi(x, z, t)$) by functions $U'(z)$ and Ri(z) depending on z, but apparently no proof for this assertion has been published.

However, in the above remark about the rarity of papers on the subject in the physics and engineering literature the mention of "physics and engineering" was meaningful. It was stressed at the beginning of Sect. 2.83 that atmospheric and oceanic flows represent the most important practical examples of stratified fluid flows. Therefore, it is not surprising that stability of stratified flows is discussed more often, and usually at greater length, in the geophysical literature. In fact, the above mentioned works of Taylor, Dikii, Case, Dyson, Eliassen et al., Phillips, Booker and Bretherton, Farrell and Ioannou, and Hartman all originated from geophysical problems. And in the geophysical literature many other publications can be found where the initial-value method of stability investigations is applied to some particular inviscid atmospheric and oceanic flows, for determination of either the asymptotic laws of the disturbance evolution or its transient development. As typical examples we can mention the papers by Pedlosky (1964), Burger (1962), Yamagata (1976a, b), Farrell (1982, 1988b, 1989), Tung (1983), and Farrell and Moore (1992). However, the flows considered in these papers are not so simple as the exponentially-stratified Couette flows and often involve some additional geophysical factors (e.g., the baroclinity or the Coriolis force) which require additional space for description and discussion of the corresponding examples. Unfortunately, space limitations make it impossible to include this material in the present book.

3.3 The Initial-Value Problem for Viscous Parallel Flows

In Sect. 3.2 the fluid was assumed inviscid, but any real fluid (with the sole exception of liquid helium in the state of superfluidity) has viscosity $\nu \neq 0$, and this can significantly affect the flow. Therefore, inviscid fluid mechanics is an approximation, which in cases where $\mathrm{Re} \gg 1$ often (but, of course, not always) has relatively high accuracy. It was noted in Sect. 3.22 that, according to Landahl (1993a, b), in studies of the evolution of weak localized disturbances in the near-wall region of a boundary layer, viscous effects can be neglected only during an initial time interval of duration $t \ll t_\nu$ where t_ν is the so-called viscous-interaction time scale determined by the values of the viscosity ν, the mean-velocity shear dU/dz, and the streamwise length scale of the disturbance, L. In general, the role of viscous effects can be determined only by comparison of the deductions from inviscid theory with the results of more complete theory which takes viscosity into account.

Let us recall that in the first attempt by Kelvin (1887a) to solve the initial-value problem for a weak flow disturbance (which proved to be unsuccessful but nevertheless led to discovery of some important new results) the flow considered was a viscous Couette flow, and that later Orr (1907) also considered development of disturbances with given initial values in such viscous flow. Orr showed that there exist initial values which lead to very great growth of disturbances during the beginning stage of their evolution and proposed on this basis the important concept of "practical instability." It has already been indicated, in Sect. 3.21, that the early results by Kelvin and Orr only began to attract attention many years after their appearance, when they stimulated a significant new development of the initial-value approach to flow-stability investigations. Now we will discuss the present situation concerning the initial-value problems for disturbances in viscous flows.

3.3.1 Wave-Packet Approximations for Solutions of the Initial-Value Problem

The first new attempts to solve the initial-value problem for disturbances to some viscous laminar flows, which appeared after Kelvin's (1887a) and Orr's (1907) old papers, had no relation to these early works but were made with the purpose of explaining the results of observations of flow instabilities collected during the 1940s and 1950s. The normal-mode method of linear stability theory, well known at that time, connected the instability of a plane-parallel flow with the appearance of the so-called Tollmien-Schlichting (for short, T-S) waves-two-dimensional plane waves growing exponentially with time. It was indicated in Sect. 2.92 that at first the T-S stability theory seemed to be unsuitable as an explanation of real flow instabilities, since the available boundary-layer observations did not confirm the existence of T-S waves. Later such waves were observed in the brilliant wind-tunnel experiments by Schubauer and Skramstad, and their development was found to agree excellently with the theoretical predictions by Tollmien and Schlichting. However, in these experiments a plane wave was artificially excited in the upstream part of the flow, while for other shapes of initial disturbances transition to turbulence usually began with the appearance of "turbulence spots," which grew, coalesced with each other, and gradually filled up all the flow domain (cf. Sects. 2.91 and 2.92). Thus, although the Schubauer-Skramstad experiments proved the accuracy of the T-S theory, questions nevertheless arose about the reconciliation of known theoretical results with experimental data, because the latter showed that flow instabilities are in most cases not accompanied by the appearance of T-S waves.

It seems natural to try to explain this phenomenon by supposing that at supercritical Reynolds numbers (even slightly supercritical, i.e., when $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ but $(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})/\mathrm{Re}_{\mathrm{cr}} \ll 1$) the whole group of different T-S waves is usually simultaneously excited and forms a wave packet where individual waves are masked and thus are hardly observable. Note that at any $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ there exists one most unstable mode, with the maximal value of $\Im\mathrm{m}\,\omega = \omega^{(i)}$ where $\omega = k_1 c$ is an eigenvalue of the Orr-Sommerfeld (O-S) eigenvalue problem (2.41–2.42). (According to Sect. 2.81, the eigenvalue $\omega = \omega_j\ (k_1, k_2)$ at given Re depends on two wave numbers, k_1 and k_2, and the integer j; therefore max $\omega^{(i)}$ must be taken with respect to all possible values of k_1, k_2, and j.) For any k_1 and k_2, let $j = 1$ correspond to the most unstable mode (or, if there are no unstable modes, then to the least stable), so that $\max_j \omega_j^{(i)}(k_1, k_2) = \omega_1^{(i)}(k_1, k_2)$. Variations of the function $\omega_1^{(i)}(k_1, k_2) = s(k_1, k_2)$ with k_1 and k_2 are as a rule smooth and gradual, and hence its maximum in a wave-number plane is rather broad. Therefore, at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ there exists a number of different unstable normal modes with nearly the same growth rate, and an unstable initial disturbance can include many of them. This makes convincing the above-mentioned explanation of the non-observability of individual T-S waves in the majority of experiments on boundary-layer instability.

The arguments given above are qualitative and they naturally stimulate a number of related quantitative studies of the initial-value problems for disturbances in slightly supercritical steady viscous flows. One of the first such studies was due to Benjamin (1961) (see also Drazin and Reid (1981), Sect. 47.1) who used the representation of the solution for the initial-value problem in terms of normal modes. It was explained in Sects. 2.81 and 2.91 that in a plane-parallel flow of a finite thickness there exists, for any values of horizontal wave numbers (k_1, k_2) and Reynolds number Re, an infinite set of normal modes for the vertical velocity disturbance w, having the form

$$w_j(\boldsymbol{x}, t) = W_j(z) \exp [i(k_1 x + k_2 y - \omega_j t)], \quad j = 1, 2 \ldots \tag{3.41}$$

(similar expressions are also valid for the horizontal velocity disturbances u and v and pressure disturbance p'). Here $\omega_j = \omega_j^{(r)} + i\omega_j^{(i)} = \omega_j^{(r)} + is_j$ are complex eigenvalues of the O-S problem and $W_j(z)$ are the corresponding eigenfunctions, ω_j and $W_j(z)$ both depending on k_1, k_2 and Re. If the set of eigenfunctions $W_j(z)$ is complete in the space of admissible vertical profiles of w, the general solution of the initial-value problem for the vertical-velocity disturbance can be presented in the form

$$w(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} w_j(k_1, k_2) W_j(z) \exp [i\{k_1 x + k_2 y - \omega_j^{(r)}(k_1, k_2)t\} + s_j(k_1, k_2)t] dk_1 dk_2 \tag{3.42}$$

where $w_j(k_1, k_2)$ are coefficients in the expansion of the two-dimensional Fourier transform $\hat{w}(k_1, k_2, z)$ of the initial value $w(x, 0)$ in a series of terms proportional to eigenfunctions $W_j(z)$. However, the form (3.42) of the required solution is too cumbersome to be useful.

To simplify this result, Benjamin noted that, at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ there is usually not more than one unstable mode for given k_1 and k_2, while all the other modes are stable. Let us again assume that, for any k_1 and k_2, $j = 1$ corresponds to the most unstable (or, if all modes are stable, to the least stable) normal mode, with the greatest value of $\omega_j^{(i)} = s_j$. Making the natural assumption that the main contribution to the asymptotic behavior of $w(x, t)$ as $t \to \infty$ is due to the most unstable modes, it is now possible to omit all terms with $j \neq 1$ from the right-hand side of (3.42). Such neglect simplifies Eq. (3.42), but not enough to make it easily applicable to real fluid flows.

It is however natural to suppose that the most unstable normal mode with the greatest rate of growth (i.e., the wave with the maximal value $s = \max_{k_1,k_2} s_1(k_1, k_2)$ of the imaginary part of the eigenvalue ω_1), together with a group of modes with $j = 1$ and wave numbers k_1 and k_2 close to those for the most unstable mode (and hence corresponding to values of $s_1(k_1, k_2)$ close to s), will after some initial time fully dominate all the other modes. Relying on this assumption Benjamin (1961) (and also Criminale (1960) and Criminale and Kovasznay (1962); see below) proposed to replace the functions s_1 (k_1, k_2) and $\omega_1^{(r)}(k_1, k_2)$ in equations of the form of

Eq. (3.42) by their Taylor's expansions in the neighborhood of the point $(k_1^{(0)}, k_2^{(0)})$ where $s_1(k_1, k_2)$ takes its maximal value s, and to preserve in these expansions only terms not higher than second order. Recall now that according to Watson's result (1960), mentioned in Sect. 2.81, the most unstable normal mode is necessarily two-dimensional for a substantial range of $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, so that $(k_1^{(0)}, k_2^{(0)}) = (k_0, 0)$. Using such value of $(k_1^{(0)}, k_2^{(0)})$ in the above-mentioned Taylor's expansions, and assuming that the initial disturbance $w(x,0)$ was localized near the point where $x = 0$ and $y = 0$, Benjamin derived from Eq. (3.42) the following asymptotic result:

$$w(x, t) \approx W(x, y, z)t^{-1}e^{st} \quad \text{as} \quad t \to \infty. \tag{3.43}$$

In Eq. (3.43) the dependence of the amplitude W on z is determined by the eigenfunction corresponding to the most unstable mode, while for given z this amplitude takes the maximal value at $x = U_g t$, $y = 0$ (where $U_g = (\partial\omega_1^{(r)}/\partial k_1)_{k_1=k_0,k_2=0}$ is the group velocity of the most unstable wave) and is negligibly small outside of an ellipse in the (x, y)-plane with the center at the point $(U_g t, 0)$ and semiaxes proportional to $t^{1/2}$ (i.e., with the area proportional to t). We see that a localized disturbance is convected downstream at the group velocity U_g in the form of an expanding elliptically-shaped perturbed region. Note also that in this case the wave-packet amplitude does not grow exponentially with t, as do the amplitudes of individual normal modes, but, due to interference of wave-packet components, as $t^{-1}e^{st}$.

The theory sketched above is approximate and its degree of accuracy cannot be easily determined. However Benjamin (1961) showed that the results obtained describe, satisfactorily enough, some results of his experiments on a slightly unstable film flowing down in inclined plate. He also mentioned the possibility of applying the same approach to study boundary-layer instabilities, and referred in this respect to a lecture by Criminale, which was later published as a report (see Criminale (1960)) and still later was used as the basis of the interesting paper by Criminale and Kovasznay (1962).

Criminale and Kovasznay considered the initial-value problem for localized disturbances in a plane-parallel boundary-layer flow with the Blasius velocity profile $U(z)$. In such flow there is no infinite family of discrete eigenvalues ω_j determining a set of eigenfunctions $W_j(z)$ complete in an appropriate function space—as it was explained in Sect. 2.92, only a few discrete eigenvalues exist here, but they are supplemented by a continuous spectrum. Therefore, the form (3.42) of the general solution for a vertical-velocity initial-value problem is inapplicable in this case. However, this form is not needed in arguments based on the assumption that the contribution of all higher normal modes (corresponding to either a discrete or a continuous spectrum) to a disturbance development is negligibly small in comparison to the contribution of the most unstable modes. Since at any values of Ri, k_1, and k_2 in a Blasius boundary layer, there exists "the first mode" with the greatest value of $\omega^{(i)}$, Eq. (3.42) can be applied to this flow too, if the equality symbol in this equation is replaced by symbol $\approx$, index j is replaced by 1, and the summation symbol in the integrand is omitted.

The approach to the initial-value problem used by Criminale (1960) and Criminale and Kovasznay (1962), coincides with that used by Benjamin (1961), but their investigation is much more comprehensive. The authors used the data of previous computations of the O-S eigenvalues and eigenfunctions corresponding to the unstable two-dimensional waves in a Blasius boundary layer, and supplemented them by some new numerical results relating to unstable three-dimensional (oblique) waves. As the initial condition for vertical velocity disturbance at a level z_1 in the outer portion of the boundary layer (the exact value of z_1 is of small importance according to the results obtained) Criminale and Kovasznay considered an axisymmetric Gaussian pulse with small enough standard deviation. Generating reasonably-truncated Taylor's expansions of $\omega^{(r)}(k_1,k_2)$ and $\omega^{(i)}(k_1,k_2) = s(k_1,k_2)$ around the point of maximum amplification, the authors analytically determined the behavior of $w(x,y,z_1,t)$ at small and large values of dimensionless time $\tau = Uk_0t$ (where U is the free-stream velocity and k_0 is the streamwise wave number of the most unstable wave). For intermediate values of τ, some numerical computations were presented. The main terms of the asymptotic equations found for $\tau \to \infty$ agreed with Eq. (3.43) and with the conclusion obtained by Benjamin (1961), that at large t the distribution of vertical velocity of the disturbance in planes $z = \text{const.}$ has the shape of an expanding elliptical wave packet traveling downstream at the group velocity.

Benjamin's and Criminale and Kovasznay's papers stimulated a number of subsequent investigations of developments of weak localized disturbances in slightly-supercritical plane-parallel flows of viscous fluid. In particular, Tam (1967) applied to this problem the general method used by Case (1960a) and Dikii (1960b) for study of disturbance development in plane-parallel inviscid flows. Case and Dikii looked for the general solution of the two-dimensional Eq. (3.3), while Tam instead considered the three-dimensional viscous equation (2.38) of the form

$$\left[\frac{\partial}{\partial t} + U(z)\frac{\partial}{\partial x}\right]\nabla^2 w - U''(z)\frac{\partial w}{\partial x} - \nu\nabla^4 w = 0 \tag{3.44}$$

where $0 \le z \le H$ and where $w = w(x,y,z,t)$ and $\partial w/\partial z$ vanish at $z=0$ and $z=H$. Therefore Tam replaced the simple Fourier-Laplace integral (3.4) by the triple Fourier-Laplace integral

$$\hat{w}(k_1,k_2,p;z) = \int_0^\infty e^{-pt}dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y)} w(x,y,z,t)\,dx\,dy. \tag{3.45}$$

Equations (3.44) and (3.45) imply the following equation for the Fourier-Laplace transform $\hat{w}(k_1,k_2,p;z)$:

$$\left[\{p + ik_1U(z)\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) - ik_1U''(z) - \nu\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2\right]\hat{w}(k_1,k_2,p;z) = w_0(k_1,k_2;z) \tag{3.46}$$

where

$$k^2 = k_1^2 + k_2^2,\ w_0(k_1,k_2;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y)}\nabla^2 w(x,y,z,0)\,dx\,dy \tag{3.47}$$

(and hence $w_0(k_1, k_2; z)$ depends only on the initial value $w(x, y, z, 0)$). It is easy to see that the left-hand side of Eq. (3.46) coincides with the O-S equation (2.41) where $-ik_1c = -i\omega = p$. According to (3.45–3.47), the solution of Eq. (3.44) corresponding to the above boundary and initial conditions can be written as

$$w(x, y, z, t) = \frac{1}{8\pi^3 i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{H} \int_{\gamma - i\infty}^{\gamma + i\infty} G(z, z'; k_1, k_2, p) w_0(k_1, k_2; z)$$

$$e^{i(k_1 x + k_2 y) + pt} dp dz' dk_1 dk_2. \tag{3.48}$$

Where $G(z, z'; k_1, k_2, p)$ is the Green function of Eq. (3.46) for the indicated boundary conditions (cf. Eqs. (3.6) and (3.7)). Similarly to the case of inviscid flow, the Green function G can be explicitly constructed if four linearly independent solutions of homogeneous Eq. (3.46) are known. Moreover, again similarly to the inviscid case, the asymptotic behavior of the solution (3.48) for $t \to \infty$ is fully determined by the singularities of the function G of the complex variable p laying on the right-hand half of the p-plane, and the only substantial singularities are poles at points corresponding to eigenvalues $c = ip/k_1$ of the corresponding O-S eigenvalue problem.

In the case of a slightly supercritical flow only a pair of simple poles of G lies in the right-hand half of the p-plane, and these poles exist only for $\boldsymbol{k} = (k_1, k_2)$ lying within some small vicinity of the point $\boldsymbol{k}_0 = (k_0, 0)$ corresponding to the most unstable normal mode. Therefore, only the values of $\boldsymbol{k}$ within this vicinity of $\boldsymbol{k_0}$ make significant contributions to the asymptotic behavior of $\boldsymbol{w}(\boldsymbol{x}, t)$, and for any such k only the contributions of the complex values of $p = -i\omega$ which correspond to the poles of G need to be taken into account. To estimate contributions of these poles to the integral on the right side of (3.48), Tam used the general equation determining the functional form of $(p - p_0)G$ in the vicinity of the pole p_0 of G, and the trinomial Taylor-series approximation of the functions $s(k_1, k_2) = \Re e p(k_1, k_2)$ and $\omega^{(i)}(k_1, k_2) = -\Im m p(k_1, k_2)$ for normal modes with wavenumber vectors $\boldsymbol{k} = (k_1, k_2)$ close to $\boldsymbol{k}_0$. Substituting all these equations into Eq. (3.48), Tam derived from it the same asymptotic equation for the behavior of disturbance $w(x, y, z, t)$ as $t \to \infty$ as was found by Benjamin (1961) and Criminale and Kovasznay (1962). In conclusion, he also noted that the asymptotic shape of an expanding ellipsoidal disturbance is similar to the shape of turbulent spots often observed in laminar flows at the beginning of their transition to turbulence.

A more detailed analysis of the solution of the initial-value problem for an initially localized disturbance to plane-parallel viscous flow with $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ was performed by Easthope and Criminale (1992), for the case of a model boundary layer with the piecewise linear velocity profile shown in Fig. 3.1a. The authors applied the Fourier transform with respect to horizontal coordinates to all terms of Eq. (3.44), then, using the simplicity of the profile $U(z)$, found an analytical equation describing with good accuracy the dependence of w on t and z, and finally determined the other velocity components and pressure and inverted the Fourier transform numerically. They plotted the function $w(x, y, z, t)$ for $z \approx 0.1H$ and several values of t, and also showed the spreading of the wave packet with time. The results obtained agreed well with the asymptotic predictions by Benjamin (1961) and Criminale and Kovasznay (1962) and describe in more details the initial stage of packet development. In particular, these results duplicated some of the experimental findings (corresponding to relatively small values of t) by Gaster and Grant (1975) (whose experiments will be discussed below), although they were insufficient to explain the data obtained by these authors for larger values of t. Note also that, according to Easthope and Criminale's results, the vertical velocity w rises rapidly at small values of t but then begins to decay, while the components u and v (and the vertical vorticity ζ_3) continue to increase at least linearly with t. Therefore, the computations confirmed Landahl's (1980) general predictions based on quite different arguments.

General solutions of the initial-value problem for initially localized disturbances to the boundary-layer flow were considered also at first by Gustavsson (1979) and Hultgren and Gustavsson (1981) and then by Brevdo (1995a, b). These authors replaced real boundary layers with gradually growing thickness by a model plane-parallel flow in the half-space having the Blasius velocity profile. Gustavsson, and Hultgren and Gustavsson obtained some particular results (which will be considered in Sect. 3.32 below) about the stability properties of disturbances at such flow at subcritical values of Re. Brevdo studied in detail asymptotic behavior of wave packets in the considered model flow at supercritical values of Re; he proved, in particular, that for a wide range of Re values exceeding Re_{cr} the plane waves and wave packets in this flow can be only convectively, but not absolutely, unstable (cf. Sect. 2.93 in Chap. 2 where, in particular, a similar result by Deissler (1987) relating to a plane Poiseuille flow was indicated).

Another approach to the asymptotic analysis of approximate wave-packet solutions of the initial-value problem for localized weak disturbances was proposed by Gaster (1968a) who applied it first to the Blasius boundary-layer flow and then, jointly with Davey (see Gaster and Davey (1968)), to the highly unstable two-dimensional wake in unbounded space having a Gaussian velocity profile. Here the wave-packets produced by an initial pulsed disturbance in a point in the fluid were represented by the following equation

$$w(x,y,z,t) = \int_{c_1}\int_{c_2} \hat{w}(k_1,k_2;z)\exp\left[i\left(k_1\frac{x}{t}+k_2\frac{y}{t}-\omega\right)t\right]dk_1dk_2 \tag{3.49}$$

where $\hat{w}(k_1,k_2;z)$ is the two-dimensional Fourier transform of the initial value $w(x,y,z,t,0)\omega = \omega(k_1,k_2)$ is the complex eigenvalue of the O-S problem corresponding to the most unstable wave with the wave-number vector $\boldsymbol{k}=(k_1,k_2)$, and the integration paths c_1 and c_2 in the complex k_1- and k_2-planes are obtained from the horizontal axes $-\infty < k_j < \infty$, $j = 1, 2$, by continuous deformations placing them above all the singularities of the integrand.

To find the asymptotic behavior of $w(\boldsymbol{x},t)$ for $t \to \infty$, it is necessary to determine the asymptotic of an integral whose integrand includes the exponential of a function $\Psi(k_1,k_2)$ multiplied by a large factor t. Gaster (1968a) followed his analysis for the vibrating ribbon problem (presented in Gaster (1965), cf. Sect. 2.92) by applying the method of steepest descent to evaluation of the integral on the right-hand side of

(3.48). For this it was necessary to expand the function Ψ about its stationary saddle points where

$$c_{gx} = \frac{\partial\omega}{\partial k_1} = \frac{x}{t}, c_{gy} = \frac{\partial\omega}{\partial k_2} = \frac{y}{t} \tag{3.50}$$

and c_{gx} and c_{gy} are group velocities of the wave packet in the x and y directions. However the precise location of the saddle points in the complex $(\boldsymbol{k}, \omega) = (k_1, k_2, \omega)$ space is a tricky problem requiring some complicated computations. At first both Gaster (1968a) and Gaster and Davey (1968) did not take into account all the complications involved, and therefore the asymptotic wave-packet shapes found by them proved to be distorted by some spurious contributions. Later Gaster's result for boundary-layer wave packets was corrected by the author himself (see Gaster (1979, 1981, 1982a) and also Craik (1981, 1982)), while Gaster and Davey's results for wave packets in an unbounded plane wake were corrected by Gaster's student Jiang (1991). At the same time some other methods of the wave-packet shape evaluation were also developed by Gaster (1975); Landahl (1972, 1982); and Craik (1981, 1982).

A simple theoretical model of early stages of wave-packet development in the Blasius boundary layer was proposed by Gaster (1975) as an explanation of experimental data collected at the same time by Gaster and Grant (1975). These data were from careful hot-wire-anemometer measurements of the development of a disturbance produced by a short acoustic pulse injected in the flat-plate boundary layer over a large plate in a wind tunnel through a small hole near the leading edge of the plate. According to the results obtained, the resulting wave packet is roughly elliptic in plane view at small distances from its origin, but further downstream the disturbed region spreads out (roughly in proportion to elapsed time t) and distorts, becoming distinctly bowed into a crescent shape. Gaster's model represents a wave packet as a superposition of a large number of wave-like normal modes of the ordinary form

$$u(x, t) = u(z)\exp\left[i(k_1x + k_2y - \omega t)\right] \tag{3.51}$$

(cf. Eq. (3.41); now u is used instead of w since only the streamwise velocity of disturbance was measured by Gaster and Grant). However, contrary to earlier wave-packet models, Gaster supposed that k_2 and ω are two given real constants while the streamwise wavenumber $k_1 = k_1(k_2, \omega)$ is a complex function of two variables equal to the most unstable eigenvalue (i.e., that having the numerically-greatest negative imaginary part) of the corresponding spatial O-S eigenvalue problem. Hence, a wave packet was considered as a superposition of spatially (and not temporally) growing waves. Moreover, all higher normal modes and also the continuous spectrum of eigenvalues in the O-S problem in a half-sapce were ignored by Gaster, exactly as was done in the other studies mentioned above. That is, he used the following approximation to the general solution of the initial-value problem for the streamwise velocity,

$$u(\boldsymbol{x}, t) \approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(k_2, \omega; z) \exp\left[i\{k_1, (k_2, \omega)x + k_2 y + \omega t\}\right] dk_2 d\omega \qquad (3.52)$$

where $u(k_2, \omega; z)$ is the Fourier transform, with respect to y and t, of the value $u(0, y, z, t)$ of the streamwise velocity disturbance at $x = 0$.

Note also that in fact Gaster introduced in his paper of 1975 (and studied at greater length in Gaster (1982b), for two-dimensional packets but for a much wider range of x-values) some corrections of the simple Eqs. (3.51) and (3.52), which approximately described the influence on the packet development of the slow growth of boundary-layer thickness δ or displacement thickness δ^* (and or Re) with streamwise distance x. However, consideration of the proposed corrections would take too much space here, so only a simplified version of the arguments given in the original publications will be presented.

To evaluate the right-hand side of (3.52), Gaster (1975) replaced the double integral by an appropriate integral sum. First of all, he carefully calculated a great number of eigenvalues $k_1(0, \omega)$ for two-dimensional waves with $k_2 = 0$ and different values of the wavenumber k_1, also varying the Reynolds number $\mathrm{Re}_{\delta^*} = U\delta^*/\nu$ (Where δ^* is the displacement thickness of the boundary layer) within a range corresponding to the streamwise locations of the measurements of Gaster and Grant (1975). The eigenvalues corresponding to three-dimensional (oblique) modes with $k_2 \neq 0$ were computed from the eigenvalues for two-dimensional waves with the aid of Squire's transformation, described in Sect. 2.81, which can also be applied to spatial formulation of the eigenvalue problem. Gaster and Grant's measurements include time records of the streamwise velocity $u(\boldsymbol{x}, t)$ at a fixed value of z and various values of x and y. The amplitudes $u(k_2, \omega; z)$ entering the integrand in Eq. (3.52) represent the two-dimensional Fourier transform of the function $u(0, y, z, t)$ with respect to y and t, for fixed value of z. For a disturbance produced by a narrow acoustic pulse of short duration the dependences on both y and t must be close to Dirac δ-functions, and therefore it was natural to assume that the initial (ω, k_2)-spectrum must be quite flat in both frequency ω and spanwise wavenumber k_2. Gaster and Grant found that this assumption agreed well enough with frequency-wavenumber spectral data at different distances x from the hole in the plate where the disturbance was introduced, and therefore this assumption was used by Gaster (1975) as a reasonable first approximation.

The amplification (or attenuation) of various normal modes was determined from the measured values of the (ω, k_2)-spectra for values of $u(x, y, z, t)$ at the selected value of z and various x. At the same time this amplification could also be calculated by the linear theory of hydrodynamic stability, determining the most unstable values of $k_1(k_2, \omega)$ for any given values of k_2,ω, and Re. According to Gaster (1975) the measured and calculated values of the amplification agreed well, giving additional confirmation of the satisfactory accuracy of the linear stability theory and of the approximation (3.52). The calculated shapes of the disturbed regions, i.e., of the regions where the relative amplification of the wave-packet power exceeded some appropriate threshold value, also agreed well with the experimental data by Gaster and Grant at not-too-large values of x. In other words, Gaster's theoretical model predicted with satisfactory accuracy the observed variations of the overall shape of the disturbed region and the way it expands as the wave packet traveled downstream. Note, however, that good agreement between the results of normal-mode summation and the observations was found only for the early stages of wave-packet development. For larger values of x the observed shape of the packet was found to be more distorted and Gaster (1975) attributed these definite discrepancies between theory and experiment to nonlinear effects.

Another explanation of these discrepancies is the possibility that the summation of normal modes is insufficient for the determination of long-time behavior, the main contribution to which is due to saddle points of the dispersion relation $\omega = \omega(k_1, k_2, \mathrm{Re})$ for complex wavenumbers k_1 and k_2. It has been already noted above that the asymptotic saddle-point analysis was first applied to wave-packet development by Gaster (1968a) and Gaster and Davey (1968) but since the exact form of the dispersion relation is usually unknown and cannot be easily determined, these attempts were not wholly successful. Later Gaster (1979, 1982a) thoroughly analyzed the applications of the saddle-point method to the evaluation of long-time development of two-dimensional (2D) wave packets (composed of 2D waves) in a plane-parallel Blasius boundary layer. He found that the method gives accurate results for all but very short times after the generation of the disturbance, and he also theoretically estimated the errors for various simplified asymptotic representations of 2D packets. For three-dimensional (3D) wave packets Gaster (1981) considered some approximate asymptotic representations (including that used by Benjamin and by Criminale and Kovasznay), and estimated their accuracies (which was found to be sufficiently good) by comparison of the results obtained with those given by numerical integration. Independently Craik (1981, 1982) proposed evaluating the developments of wave packets generated by short-term localized 3D disturbances in unstable plane-parallel flows by a saddle-point method, using simplified algebraic models of the 3D dispersion relations. For models containing enough free numerical parameters it is possible to achieve good agreement with the available results of computations of the O-S eigenvalues, and Craik then showed that the saddle-point method leads to conclusions which also agree well with the results for wave-packet development. In particular, Craik's models imply the initial elliptic shape of a packet, its subsequent bending to a crescent shape, and the expected behavior of packets in cases where Re considerably exceeds $\mathrm{Re}_{\mathrm{cr}}$.

One more method for evaluation of the wave packet development was first used for a special purpose by Landahl (1972) who later (in 1982 and 1985) developed it further and applied it to representation of evolution for rather general localized 3D packets of waves growing both in space and time. This new method is based on the kinematic wave theory by Whitham (1965, 1974) (see also the short presentation by Landahl and Mollo-Christensen (1992), Cap. 6). Whitham's theory in its original version dealt only with conservative waves; therefore, its application by Landahl (1972) to waves with small dissipation at first gave rise to some criticism. However, later formal extension of kinematic wave theory to the case of such waves was developed by several authors (in particular, by Jimenez and Whitham (1976) and

Chin (1980)). This extension confirmed Landahl's results of 1972 and showed (see Landahl (1982, 1985)) that the kinematic wave theory leads to results equivalent to those given by the saddle-point method and can be used also for representation of some nonlinear features of wave-packet propagation.

Advances in computer technology led, during the last decade, to a number of investigations of wave-packet developments in laminar boundary-layer flows by direct numerical simulations (DNS), i.e., by numerical solutions of Navier-Stokes equations with the initial and boundary conditions corresponding to a steady boundary-layer flow disturbed at the instant $t = 0$ by a strongly localized small disturbance. Some examples of such simulations were described, in particular, by Lenz (1986), who considered only two-dimensional disturbances, and by Fasel et al. (1987) and Konzelmann (1990). Note that the modern development of computational methods makes it unnecessary to linearize the navier-Stokes equations with respect to flow disturbances, and hence permits combined DNS study of the nonlinear stage of wave-packet development and its initial linear stage. Moreover, the boundary-layer development can also be numerically simulated independently of the computation of disturbances, and the influence of the growth of the boundary-layer thickness is automatically taken into account. We will not consider here details of the available DNS results for wave-packet development, but simply note that the results of the above-mentioned authors show excellent qualitative and fully satisfactory quantitative agreement with the experimental results obtained by Gaster and Grant (1975).

3.3.2 Resonance and Degeneracy Growths of Disturbances in Subcritical Flows

Section 3.31 was devoted mostly to consideration of disturbance development in supercritical laminar flows, with $\text{Re} > \text{Re}_{\text{cr}}$, and to discussion of some possible reasons for the so-called *by-pass transition* to turbulence (see Sect. 2.92), where no T-S waves are observed. In this and the following sections, the main case to be discussed will be that of subcritical flows, with $\text{Re} < \text{Re}_{\text{cr}}$, and a possible explanation will be given of the *subcritical transitions* that are frequently observed (e.g., in plane Couette and circular Poiseuille flows, where $\text{Re}_{\text{cr}} = \infty$, and in plane Poiseuille flows, where $\text{Re}_{\text{cr}} \approx 5770$, while transition to turbulence usually occur at $\text{Re} \approx 1000$; see Sects. 2.1 and 2.9). For this purpose it will be necessary to pay more attention to general solutions of the initial-value problems for localized disturbances in laminar flows.

The general solution (3.48) of the initial-value problem for the disturbance vertical velocity $w(x, y, z, t)$ in a laminar channel flow was considered above in the discussion of the paper by Tam (1967). The same solution was later studied by Gustavsson (1979) with application to subcritical disturbance development in a Blasius boundary layer (strictly, a plane-parallel flow in the half-space $0 \leq z < \infty$ with the Blasius velocity profile). Gustavsson used a slightly different, but equivalent, form of the integrand in Eq. (3.48); he expressed the Green's function G explicitly, in terms of four linearly independent solutions of the homogeneous Eq. (3.46). Like Tam, he was mainly interested in the asymptotic behavior of the vertical velocity $w(\boldsymbol{x}, t)$ as $t \to \infty$, which is determined by the contribution to w made by singular points of the integrand in the complex p-plane. However, in the case of a Blasius boundary layer, the simple poles of the integrand, at values of $ip = \omega$ equal to discrete eigenvalues ω_j of the corresponding O-S equation, are supplemented by a branch point. Therefore in this case the path of integration in the p-plane must be deformed around the branch cut ending at the branch point. As in the case of a channel flow, the poles at points $p_j = -i\omega_j$ contribute to the log-time behavior of w, the summands proportional to $e^{-i\omega_j \psi}$, which describe the asymptotic behavior of the T-S wave components of the vertical velocity, depending exponentially on time. Now, however, the loop enclosing the branch cut also makes a non-zero contribution to the value of w, which determines the component of w generated by the continuous spectrum of the O-S eigenvalues. Gustavsson showed that this component also eventually decays exponentially, but at small values of t its dependence on time is algebraic (with exponent depending on the shape of the initial disturbance). He also showed that the duration of the initial period of algebraic variation increases, and the rate of exponential decay as $t \to \infty$ decreases, with increase of the disturbance length scale.

More detailed analysis of Gustavsson's solution of the initial-value problem for a disturbance in a Blasius boundary layer was carried out by Hultgren and Gustavsson (1981) for the special case of disturbances having a streamwise length scale l very-large compared to the boundary layer thickness δ. (It was explained by the authors that since the parallel-flow assumption was used, l had to satisfy the double inequality $\delta \ll l \ll \delta\,\mathrm{Re}$.) In this case $k_1\delta \ll 1$, and it is easy to show that then the integrand to the inverse Laplace transform in Eq. (3.48) does not possess any poles, and hence only a branch cut in the p-plane (describing the continuous-spectrum contribution) must be taken into account. Applying some algebraic manipulations to Gustavsson's solution for $w(\boldsymbol{x}, t)$, Hultgren and Gustavsson found that if $l/\delta \gg 1$ (so that the dependence of w on x can be neglected in the first approximation), then inside the boundary layer $w(y, z, t) \approx w(y, z, 0) = w_0(y, z)$ (i.e., the vertical velocity essentially remains constant) for small times $t \ll \delta^2/\nu$, but $w \propto U_0(t\nu/\delta^2)^{-2}$ (i.e., it decays as $t-{}^2$) for large times $t \gg \delta^2/\nu$. The results for $w(y, z, t)$ were then used to find the asymptotic behavior of horizontal velocity disturbances. Using Eq. (2.35) for the streamwise velocity disturbance u, and the continuity equation (2.36), and neglecting the dependence of u and the pressure disturbance p on x, Hultgren and Gustavsson obtained for u and the spanwise velocity v the following two equations

$$\frac{\partial u}{\partial t} - \nu\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u = -U'w, \tag{3.53}$$

$$\frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}.$$

Here the vertical velocity disturbance $w(y, z, t)$ was known from the above results; therefore Eqs. (3.53) could be used for finding the horizontal components of the disturbance velocity. The second Eq. (3.53) allows the spanwise velocity component $v(y, z, t)$ to be found quite easily; it leads to obvious results which will be omitted here. The first Eq. (3.53) has the form of a heat-conduction (or diffusion) equation with a source term. Hultgren and Gustavsson showed that the solution of this equation corresponding to a given initial value $u(y, z, 0) = u_0(y, z)$ can be written as a sum of two terms T_1 and T_2, the first of which depends on $u_0(y, z)$ and the second on the solution for $w(y, z, t)$. For an arbitrary value of t, this solution must be evaluated numerically, but its asymptotic behavior at small and large times is given by simple analytic expressions. At small times $t \ll \delta^2/\nu$ the term T_1 differs from the initial value $u_0(y, z)$ only by a small viscous correction, while the main part of T_2 has the form $-U'(z)w_0(y, z)t$ (again with a small viscous correction).Thus the inviscid result of Ellingsen and Palm (1975) (see Eq. (3.21)) was recovered (with a viscous correction and a viscous limit of validity) directly from the solution of the initial-value problem. At times $t \gg \delta^2/\nu$, it was found that the streamwise velocity disturbance $u(y, z, t)$ decays as t^{-2} in both the boundary layer and the free stream.[4]

Thus, the algebraic initial growth of $u(\boldsymbol{x}, t)$ was derived by Hultgren and Gustavsson (1981) from a solution of the initial-value problem for disturbances with a large streamwise scale. More complete analysis of the considered general solution of the initial-value problem was carried out by Brevdo (1995a, b); these papers were already mentioned in Sect. 3.31. Note now that there are also many other special cases where solutions of the initial-value problem imply the algebraic growth of disturbance velocities; see, for example, the discussion of Easthope and Criminale's paper (1992) in Sect. 3.31. In Sect. 3.2, when discussing the inviscid Eq. (3.9), it was noted that for two-dimensional disturbances, of the form $\{u(x, z, t), 0, w(x, z, t)\}$, the only dynamical equation needed is that for the vertical velocity disturbance w; in this case, once w is known, the streamwise velocity disturbance u follows from continuity (and, if needed, the pressure disturbance p can also be determined easily). However, if general three-dimensional disturbances are considered, then to find the whole velocity field, at least one more dynamic equation is needed. In Sect. 3.2, the inviscid Eq. (3.10) for the vertical vorticity component $\zeta_3 = \partial v/\partial x - \partial u/\partial y$ was recommended as such a supplementary equation.

The same considerations clearly apply to viscous flows. In this case, Eq. (3.9) for the vertical velocity disturbance $w(\boldsymbol{x}, t)$ in a steady plane-parallel flow must be replaced by Eq. (3.44), which includes a viscous term. Also, Eq. (3.10) for ζ_3 must now also be supplemented by an additional viscous term turning it into

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta_3 - \nu\nabla^2\zeta_3 = U'\frac{\partial w}{\partial y}. \qquad (3.54)$$

[4] These results were found for the plane-parallel model of the Blasius boundary layer. In reality the thickness of a boundary layer increases with x and this must lead to gradual weakening of the influence of viscosity. This effect was studied by Luchini (1996) who found that in the model of a boundary layer with the thickness depending on x a three-dimensional disturbance can exist which algebraic growth produced by the lift-up effect overcomes the viscous damping. Therefore, within the limits of the linear stability theory and of the model of a boundary layer of infinite streamwise extent, this disturbance is growing at all times.

In the case of purely horizontal velocity disturbances, where $w(x,t) \equiv 0$, Eq. (3.54) for the vorticity ζ_3 has the same form as the equations for both horizontal velocity components $u(\boldsymbol{x}, t)$ and $v(\boldsymbol{x}, t)$ given for this case in Sect. 2.81 (before Eqs. (2.46)). It was explained there that in steady plane-parallel flows these equations for u and v describe some normal modes having the form of horizontal-velocity waves, in addition to the better-known T-S waves. These new modes (sometimes called the *Squire modes* in contrast to the more ordinary *Orr-Sommerfeld modes* where w satisfies the O-S equation) always decay with time and therefore may be ignored when the normal-mode approach to the linear theory of hydrodynamic stability is used. It can also be shown (see, e.g., Reddy et al. (1993); Reddy and Henningson (1993); Henningson et al. (1994); or Schmid and Henningson (2001)) that, according to linear stability theory, the energy of any horizontal-velocity disturbance $\{u(x, y, z, t),\ v(x, y, z, t),\ 0\}$ always decays monotonically with time (in contrast to the case of vertical velocity disturbances $w(\boldsymbol{x}, t)$ where the possibility of very large initial growth had been proved already by Orr (1907)). Hence, one might think that infinitesimal disturbances with zero vertical velocity can also be omitted in stability studies using the initial-value-problem approach. However, this conclusion is incorrect, since it does not follow from the above-mentioned results that horizontal velocity components and vertical vorticity of a disturbance are irrelevant in the general case where all components of the velocity vector $\boldsymbol{u}(\boldsymbol{x}, t) = \{u(\boldsymbol{x}, t), v(\boldsymbol{x}, t), w(\boldsymbol{x}, t)\}$ differ from zero.

In the general case, Eqs. (3.44) and (3.54) form a closed system of two equations, with two unknowns w and ζ_3. Equation (3.44) may be solved independently from Eq. (3.54), and the solution obtained for w then substituted into Eq. (3.54). Let us consider the normal modes of disturbance which are proportional to $\exp[i(k_1\xi + k_2\eta - \omega\tau)]$ (where, as in Sects. 2.81 and 2.92, $\xi = x/H, \eta = y/H$, and $\tau = tU_0/H$ are dimensionless horizontal coordinates and time, H and U_0 being appropriate length and velocity scales). Then the dimensionless vertical-velocity amplitude $W(\zeta)$ (where $\zeta = z/H$) will satisfy the O-S equation (2.41) (with $c = \omega/k_1$), while the dimensionless vertical-vorticity amplitude $Z(\zeta)$ will satisfy an equation having a left side of the same form as in Eq. (2.46) (again with $c = \omega/k_1$), but the non-zero term $ik_2U'W(\zeta)$ on the right side.

In Chap. 2 it was explained that in the case of the O-S eigenvalue problem (i.e., for the O-S equation with the appropriate boundary conditions) there corresponds, to any values of k_1, k_2, and Ri, an infinite (in the case of flows in channels of finite depth) or finite (for plane-parallel flows in an unbounded or semibounded space) set of eigenvalues $\omega_j(k_1, k_2, \mathrm{Re})$ determining a set of O-S waves. Another set of eigenvalues $\omega_j^0(k_1, k_2, \mathrm{Re})$ corresponds to the Squire (briefly, Sq) eigenvalue problem (i.e., to the Sq equation (2.46) with the appropriate boundary conditions), and determines a set of Sq waves. As was said above, the Sq waves always decay, since $\Im\mathrm{m}\, \omega_j^0(k_1, k_2, \mathrm{Re}) < 0$ for any values of $j,\ k_1,\ k_2$ and Re (for information about the eigenvalues ω_j^0 see, e.g., Davey and Reid (1977) where the same eigenvalue problem appeared in a different context). However, $\Im\, m\omega_j(k_1, k_2, \mathrm{Re})$ is negative for any j, k_1 and k_2 only if $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$.

Let us assume that $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$; then all O-S and Sq waves decay as τ (i.e., t) tends to infinity. Therefore the flow is stable from the standpoint of the normal-mode approach to linear stability theory. Note, however, that the Sq waves represent vorticity waves corresponding to "free oscillations" of the vorticity and horizontal velocity fields, while Eq. (3.54) contains on the right side a "force" $U'\partial w/\partial y$.[5] Here, therefore, "forced," not "free," solutions of the Sq problem must be considered.

For $\mathrm{Re} < \mathrm{Re}_{cr}$ viscous effects lead to damping of all wave-like disturbances as $t \to \infty$. It is however known that even forced oscillations that eventually die out can be strongly amplified initially in the case of a resonance, i.e., when a frequency of free oscillations of (say) a mechanical structure coincides with a frequency of the applied force. Therefore, it is natural to investigate whether a resonance can occur in forced excitation of vertical-vorticity waves, and if so what will be its consequences.

Solutions of the homogeneous Eq. (3.54) for a wide range of conditions can be expanded into Sq waves, while a force can be represented by a series of O-S waves. Therefore a resonance is possible here if values of k_1, k_2 and Re exist, such that $\omega_j(k_1, k_2, \mathrm{Re}) = \omega_i^0(k_1, k_2, \mathrm{Re})$ for some integers j and i. Apparently Gustavsson and Hultgren (1980) were the first to formulate the resonance problem of linear hydrodynamic stability theory, and to study it for the case of a plane Couette flow. They began with numerous computations of the Couette-flow O-S and Sq eigenvalues belonging to the first four eigenvalue modes. Instead of complex frequencies ω they used the complex phase velocities $c = \omega/k_1$, which depend on two variables $k = (k_1{}^2 + k_2{}^2)^{1/2}$ and k_1Re (while in case of the Sq eigenvalues, $c + ik^2/k_1\mathrm{Re} = c$' depends only on k_1Re). The computations showed that for any value of k at least two values of k_1Re exist, such that $c(k,\ k_1\mathrm{Re}) = c^0(k,\ k_1\mathrm{Re})$ (where c and c^0 are the Couette-flow O-S and Sq eigenvalues) for either the first or the second eigenalue mode. If $c = c^0$, then clearly $\omega(k_1, k_2, \mathrm{Re}) = \omega^0(k_1, k_2, \mathrm{Re})$ for $k_2 = (k^2 - k_1{}^2)^{1/2}$. Thus, for corresponding values of k_1, k_2,Re (and maybe also for some other still-unknown values of these variables), a resonant excitation of the vertical vorticity (and horizontal velocity) waves can occur in a plane Couette flow. In such cases the corresponding wave-like solutions of Eq. (3.54) (and the related horizontal-velocity waves too) will include a resonance term depending on time as $\tau e^{-i\omega\tau}$. Since $\Im\mathrm{m}\,\omega = \omega^{(i)} < 0$, this term will eventually die out, but at first, as long as $-\omega^{(i)}\tau \ll 1$, it will grow linearly with time. Gustavsson and Hultgren found that the slowest exponential decay (and hence the longest period of the resonance growth of a disturbance, and the largest value of an amplitude at the end of this period) are usually reached for $k \approx 2$ (but at $k \approx 2$ there exists a broad wave packet consisting of waves with similar growth properties). At large values of Re the structures which grow most significantly in the initial period have a streamwise elongated shape, and the duration of their period of growth increases with Re. In the limit $\mathrm{Re} \to \infty$, the linear resonant growth is real-

[5] The physical mechanism of the "force effect" is rather simple: If $U' \neq 0$, the vertical velocity w leads to vertical displacements of fluid particles transferring their original streamwise velocity to a new height with different mean velocity U, i.e., producing additional disturbances of the streamwise velocity (Landahl's lift-up effect mentioned in Sect. 3.2). If $U' \neq 0$, and $\partial w/\partial y \neq 0$, then the lift-up effect will vary with the span wise coordinate y creating regions of non-zero derivative $\partial u/\partial y$ and hence acting as a source of vertical vorticity. Note also that the first Eq. (3.53) describes forced streamwise velocity oscillations where the force on the right side represents the lift-up effect.

ized at all times (in full accordance with Ellingsen and Palm's (1975) and Landahl's (1980) results considered in Sect. 3.22).

An investigation of possible resonances in a subcritical plane Poiseuille flow was performed by Gustavsson (1981, 1986) (see also Benney and Gustavsson (1981)). Since there were no data to determine whether the coincidences $c = c^0$ are possible or impossible in this case, Gustavsson calculated anew a number of the corresponding O-S and Sq eigenvalues c and c^0. He found that, contrary to the case of a plane Couette flow, in the case of a plane Poiseuille flow resonances can occur only for certain isolated points (k, k_1Re). A number of such Poiseuille-flow "resonance points" (where $c = c^0$) was indicated in Gustavsson's papers (1981, 1986) where it was also stated that their number is apparently infinite.

It has been mentioned above that resonant excitation of disturbances is usually responsible for only a part of the total values of $u(\boldsymbol{x}, t)$, $v(\boldsymbol{x}, t)$, and $\zeta_3(x, t)$. As a rule, initial disturbances include many different Fourier components, and generate wave packets in which waves corresponding to resonant values of k and k_1Re are masked by all the other waves. However, even for the Fourier component of a disturbance with such wave numbers k_1 and k_2 that $((k_1{}^2 + k_2{}^2)^{1/2}, k_1\text{Re})$ is a resonance point in the (k, k_1Re)-plane, the wave amplitude is not exactly proportional to $\tau e^{-ik_I c\tau}$. According to Gustavsson's general solution of the corresponding initial-value problem, the resonant term (i.e., the contribution of the resonance pole of the integrand in Eq. (3.48) in the complex p-plane) has the form $[r_1 + r_2 k_1 \tau] \exp[i(k_1\xi + k_2\eta - k_1 c\tau)]$ where r_1 and r_2 are complex numbers (depending on initial conditions and the parameters ζ, k_1, k_2, and Re). Hence the time-dependent wave amplitude, $R(\tau)$, is equal to $(r_1 + r_2 k_1 \tau)\exp(k_1 \Im\text{m}\, c\tau)$ where $c = c(k,\ k_1\text{Re})$ is the joint O-S and Sq eigenvalue. We see that the amplitude $R(\tau)$ includes two terms, the first of which decreases exponentially (since $\Im\text{m}\, c\tau < 0$ in a subcritical flow) while the second at first grows linearly (this is just the resonance growth) and only later begins to decay. It is easy to see that the general character of the time evolution for the wave disturbance considered depends on the sign of the difference $\Re e(r_2/r_1) - \Im\text{m}\, c$; only if it is positive will the amplitude $R(\tau)$ grow initially and decay at later times (see, e.g., Shanthini (1989)).

Gustavsson (1986) computed time-dependent amplitudes $R(\tau)$ of the resonant vertical-vorticity waves for a number of resonance parameters (k_1, k_2,Re) and initial values $w_0(k_1, k_2;\ z)$ (defined by Eq. (3.47)). It turned out that all the computed amplitudes decay monotonically with time. This shows that here the contributions of a monotonically-decreasing term with coefficient r_1 usually dominate the disturbance development. Of course, Gustavsson's computations covered only a limited range of conditions but, nevertheless, his results cast doubt on the assumption that the resonance mechanism is the main cause of the observed transient growth of flow disturbances.

Studies of the possible resonances in the case of plane-parallel boundary-layer flows were carried out by Benney and Gustavsson (1981), who investigated three examples of a velocity profile $U(\zeta)$ but published only results for the Blasius profile. For boundary layers only a finite number of discrete eigenvalues c (or $\omega = k_1 c$) exists, so here there are not too many choices for possible direct resonances. Calculations of both O-S and Sq eigenvalues suggested that in boundary-layer flows no exact resonances (i.e., no values of k and k_1Re such that $c(k, k_1\mathrm{Re}) = c^0(k, k_1\mathrm{Re})$) occur for any of the considered velocity profiles. (Later Jang et al. (1986) discovered that an exact resonance, which may be physically important, exists in a turbulent boundary layer, where the velocity profile is quite different from that in laminar layers. However, this topic is beyond the scope of this chapter of our book.) Moreover, Benney and Gustavsson found that in the case of laminar boundary layers some near-resonances exist, i.e., here the difference $c - c^o$ can take quite small absolute values. The authors stated that such near-resonances can also produce substantial growth of disturbances, which can lead to important consequences when the nonlinear mechanisms of the disturbance development, also considered in their paper, are taken into account. However, we have no space to discuss this matter here.

Gustavsson (1989) also studied the forcing mechanisms and resonances occurring in disturbed Poiseuille flow in a circular tube. Equations for small disturbances of an axisymmtric laminar flow were given in Sect. 2.84, where cylindrical coordinates, r, ϕ, x were used instead of rectangular coordinates x, y, z. Gustavsson showed that in a tube flow, resonances can occur only in the case of non-axisymmetric disturbances (depending on ϕ). (Recall that in plane-parallel flows resonances are possible only for three-dimensional disturbances, depending on the spanwise coordinate y.) Therefore, Gustavsson considered only the normal modes with the azimuthal wave number $n \neq 0$.

Gustavsson used Eqs. (2.73) to obtain a system of four homogeneous differential equations for the r-dependent amplitudes g, $f^{(r)}$, $f^{(\phi)}$ and $f^{(x)}$ of normal modes corresponding to reduced pressure p/ρ and three components (u_r, u_ϕ, u_x) of disturbance velocity. (This system differs from Eqs. (2.74) by viscous terms which were omitted in Sect. 2.84.) Then he eliminated all unknowns except g from the system, and thus found a sixth-order homogeneous differential equation for the pressure amplitude $g(r)$. The dimensionless form of this equation utilizes the normalized radial coordinate r/R instead of r (a dimensional radial coordinate will not be used below and therefore r will later denote just the normalized radial coordinate) and includes the following dimensionless parameters: $k, n, c = \omega/k$, and $Re = U_0R/\nu$, where R and U_0 are the tube radius and the centerline velocity, k is the streamwise wave number multiplied by R, and c is the phase velocity of the modal pressure wave divided by U_0. The boundary conditions require that all three velocity components vanish at the tube wall (i.e., at $r = 1$) and are finite at the tube axis; they allowed Gustavsson to obtain six boundary conditions for the function $g(r)$. The boundary conditions, together with the equation for g, form an eigenvalue problem determining a set of eigenvalues $c_j(k, n, \mathrm{Re})$ for the given values of k, n and Re.

The third Eq. (2.73) relating to streamwise disturbance velocity u_x leads to the following dimensionless u_x-amplitude equation

$$\tilde{\nabla}^2 f^{(x)} - ik\,\mathrm{Re}\,(U - c^0) f^{(x)} = \mathrm{Re}\,(U' f^{(r)} + ikg),$$

$$\tilde{\nabla}^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - k^2 - \frac{n^2}{r^2}, \tag{3.55}$$

where now $U(r) = 1 - r^2$ is the velocity of the circular Poiseuille flow divided by U_0, a prime denotes the derivative *d/dr* and c^0 is the dimensionless streamwise velocity of the u_x-mode. The homogeneous version of Eq. (3.55), where the right-hand side is replaced by zero, together with boundary conditions requiring that $f^{(x)} = 0$ for $r = 1$ and tends to a finite value when $r \to 0$, form the u_x-mode eigenvalue problem determining the set of eigenvalues $c_i^0(k, n, \mathrm{Re})$. The right side of Eq. (3.55) contains a linear combination of functions g and $f^{(r)}$, and it is easy to deduce from Eqs. (2.73.) that $U' f^{(r)} = i\tilde{\nabla}^2 g/2k$. Therefore, the inhomogeneous Eq. (3.55) can be regarded as an equation describing forcing of the streamwise velocity component by a specific force linearly depending on $g(r)$ (i.e., in fact by the pressure force). A resonance occurs here when, for some values of $k, n \geq 1$, and Re, a pressure eigenvalue $c_j(k, n, \mathrm{Re})$ coincides with some streamwise-velocity eigenvalue $c_i^0(k, n, \mathrm{Re})$.

To find resonances, Gustavsson calculated values of discrete eigenvalues c_j and c_i^0 for a great number of combinations of values for k^2, kRe and n, where k was assumed to be complex. He found that resonances are very numerous (apparently there are infinitely many of them) and listed 36 resonances presenting the corresponding values of k, n, and Re together with the values for the phase velocity and damping rate of resonance waves. However, no attempt to investigate the possible resonance growth of velocity disturbances was made.

Above, we considered a number of studies of the initial algebraic growth of small disturbances in steady parallel viscous flows. Special attention was given to resonance effects, which play an important part in many mechanical problems. Let us now recall that in Sect. 2.5, in the introductory discussion of the normal-mode approach to linear theory of hydrodynamic stability, it was indicated that initial algebraic growth of disturbances can also be caused by the degeneracies of the frequency spectra, i.e., by coalescences of frequencies for some pairs of normal modes. Some references to papers on this topic were given in Sect. 2.5; here we shall briefly consider only the papers by Koch (1986), Jones (1988), and Shanthini (1989) devoted to investigations of degeneracies in the Orr-Sommerfeld spectra of some standard steady plane-parallel flows.

Let us, however, begin with a general remark. According to the above discussion, resonant growth of small disturbances is due to a coalescence of two eigenvalues belonging to spectra of two different fluid-dynamic fields, while degeneracy growth is due to a coalescence of two eigenvalues belonging to the spectrum of one such field. These two mechanisms clearly have some internal similarity; therefore, it is sometimes said (see, e.g., Koch's paper) that degeneracy growth is caused by a resonance between two normal modes of the same fluid-dynamic field. At the same time it is also possible to look at this matter the other way round. The two equations (3.44) and (3.54), with the appropriate boundary conditions, form a two equation system for the unknown vector field $\{w(\boldsymbol{x}, t), \zeta_3(\boldsymbol{x}, t)\} = q(\boldsymbol{x}, t)$ determining (together with the equation of continuity (2.7b)) all three components of the disturbance velocity $\boldsymbol{u}(\boldsymbol{x},t)$. The normal modes of the vector field $\boldsymbol{q}$ are the solutions of the system which are proportional to exp $\{i(k_1 x + k_2 y - \omega t)\}$. It is clear that for any given values of k_1 and k_2, the finding of such solutions is reducible to solution of a two-equation eigenvalue problem determining the spectrum of the admissible eigenfrequencies ω_j.

It was shown by Henningson and Schmid (1992) (see also Eq. (3.66) in Sect. 3.33) that this spectrum of the field $\boldsymbol{q}$ consists of all the O-S eigenvalues $\omega_j, j = 1, 2, \ldots,$ and all the Sq eigenvalues $\omega_i^0, i = 1, 2, \ldots$ (recall that in the case of a channel flow both the O-S and Sq frequency spectra are discrete and infinite). Hence the resonance condition: $\omega_j = \omega_i^o$ for some j and i, determines a part of the degeneracies in the combined spectrum of the two-equation eigenvalue problem relating to the vector field $\boldsymbol{q}(\boldsymbol{x}, t)$, while degeneracies in the O-S and Sq spectra determine two other parts of the set of all such degeneracies. This shows that the distinction between resonances and degeneracies is in fact even smaller than it seems at first. The situation with the tube-flow resonances studied by Gustavsson (1989) is completely similar to that in the case of channel flows: here also resonances form a part of the degeneracies in the spectrum of eigenfrequencies of the vector field $\{p(x, t)/\rho, u_x(\boldsymbol{x}, t)\}$ (see Schmid and Henningson (1994)).

Now we will pass to consideration of the O-S-spectrum degeneracies. Koch (1986) and Shanthini (1989) both used Gustavsson's (1979, 1986) general solution of the initial-value problem for a small disturbance in a steady plane-parallel viscous flow to determine the contribution to the disturbance development of a double eigenvalue of the corresponding O-S eigenvalue problem. To such an eigenvalue there corresponds a double pole of the integrand in Eq. (3.48) in the complex p-plane. The contribution of such a pole to the inverse Laplace transform in this equation is of the same form as the resonant contribution considered above,$(r_1 + r_2 k_1 t)\ \exp(-i\omega_o t)$ where ω_o is the complex double eigenvalue. (Now we return to the dimensional variables and assume that the O-S eigenvalue problem is formulated for the unknown eigenfrequencies ω. If phase velocities $c = \omega/k_1$ are the sought-for eigenvalues, then, of course, ω_o must be replaced by $k_1 c_o$. Moreover, in the case of spatial formulation of the O-S eigenvalue problem, the Laplace transform must be carried out with respect to x, so that here $p = ik_1$, where $k_1 = k_1(\omega, k_2, \mathrm{Re})$ are the unknown eigenvalues, and the contribution of the double pole$p_o = ik_1^o$ in the p-plane to the disturbance amplitude has the form $(r_1 + r_2 x)\ \exp(ik_1^o x)$).

Note that, in contrast to the resonant growth which is possible only for three-dimensional disturbances, the degeneracy growth can take place for either three-dimensional or two-dimensional disturbances. For two-dimensional disturbances $k_2 = 0$ and the O-S eigenvalue depend on only two parameters, $k_1 = k$ and $k\mathrm{Re}$ (in the case of temporal formulation of the problem or ω and Re) (in the case of spatial formulation). Jones (1988) studied multiple eigenvalues of the temporal O-S eigenvalue problem for a plane Poiseuille flow, considering only symmetric two-dimensional normal modes of disturbances $\{u(x, z, t),\ 0,\ w(x, z, t)\}$. (Since the velocity profile of a Poiseuille flow in a channel bounded by walls at $z = 0$ and $\mathrm{z} = H$ is symmetric with respect to the channel midplane $\mathrm{z} = H/2$, the vertical-velocity amplitude $W(\mathrm{z})$ of a normal mode is always eilther symmetric or antisymmetric with respect to this plane, i.e., is represented by either an even or an odd function of $z_1 = z - H/2$. According to this, the normal modes fall into symmetric and antisymmetric ones; the most unstable mode is always symmetric.) Jones recomputed the Poiseuille-flow O-S eigenvalues and found 16 double eigenvalues c_i for symmetric normal modes in flows with values of $\mathrm{R} = k\mathrm{Re}$ in the range $0 < \mathrm{R} < 6000$. All the double eigenvalues c_i found have negative imaginary parts (i.e., correspond to damped modes) and real parts close to each other. However, Jones made no attempt to estimate the possible transient amplitude growth for the degenerating modes he found.

Koch also paid great attention to double O-S eigenvalues in plane Poiseuille flow, although he stated that his prime object was to study spectral degeneracies in boundry-layer flows. Like Jones, he considered only symmetric normal modes but concentrated on the spatial eigenvalues k_1 (ω, k_2, Re) where the values $k_2 \neq 0$ were also permitted. He began with accurate computation of a number of eigenvalues $k_1(\omega)$ for the case where $k_2 = 0$ and $\mathrm{Re} = 10^4$; here no double eigenvalues were found. Then he studied the more general case where both variables k_1 and ω take complex values and, following Gastser (1968b) and Gaster and Jordinson (1975), he determined six branch points in the complex k_1-plane which correspond to modal degeneracies (i.e., to singularities of the "dispersion relation" $\mathrm{D}(k_1, \omega) = 0$; cf. Sect. 2.93 above). After this he began to vary the value of Re (keeping k_2 zero) in the hope that $\Im\mathrm{m}\omega$ would vanish for some of the singular points (k_1, ω) However, in the range $10^3 \leq \mathrm{Re} \leq 2.10^4$, no singular points on the realω-axis were discovered. The next step was to vary the spanwise wave number k_2; then at least one degeneracy of the spatial O-S modes was found at real values of ω, k_2 and Re, but at a high value of Re which in practice would definitely correspond not to laminar but to fully turbulent plane Poiseuille flow. Hence, Koch discovered no degeneracies of the spatial O-S spectrum which might possibly induce certain growth of disturbances.

However, application of the same procedure to a Blasius boundary-layer flow (beginning with computations at $k_2 = 0$ and $\mathrm{Re} = 580$, where $\mathrm{Re} = (U_o x/\nu)^{1/2} = {\mathrm{Re}_\mathrm{x}}^{1/2} = \mathrm{Re}\,\delta^*/1.72$, with subsequent varying of values for Re and k_2) was found to be more fruitful. Here again no coalescences of spatial eigenvalues $k_1(\omega, k_2, \mathrm{Re}\,)$ were obtained for $k_2 = 0$ and real values of ω and Re, although the complex singular points (k_1, ω) were numerous here, Then the value of k_2 was varied, still at $\mathrm{Re} = 580$, and the double complex eigenvalue $k_1 \approx 0.21 + 0.07i$ was found for $\omega = 0.1$ and $k_2 = 0.283$ (all the variables are given in dimensionless form). The spatial spectral degeneracy found was easily traced to other Reynolds numbers (degeneracy values of k_1, k_2, and ω for five values of Re up to $\mathrm{Re} = 2200$ were presented). Koch also indicated the variation of the degeneracy frequency ω and spanwise wave number k_2 with the parameter β characterizing the family of the Falkner-Skan velocity profiles (where $\beta = 0$ corresponds to the Blasius boundary layer, see Sect. 2.92). Thus, it was shown that in the case of boundary-layer flows there are some double spatial O-S eigenvalues which possibly can induce certain growth of disturbances.

Shanthini (1989) studied the degeneracies of the temporal O-S eigenvalues in plane Poiseuille flow for two-dimensional and three-dimensional O-S modes of both symmetric and antisymmetric types. He discovered several new double eigenvalues, and thoroughly investigated the first six degeneracies (four for symmetric and two for antisymmetric modes) to determine which of these degeneracies can produce growth of flow disturbances and what maximal amplitude can then be reached. It was found that growth is possible only in the cases of the first symmetric and first antisymmetric degeneracies, where amplitudes can become at most seven and two times larger that their initial values, respectively. The unexplored degeneracies of higher orders seemed to Shanthini to be unpromising as growth sources; moreover, the majority of them correspond to supercritical values of Re and hence cannot play any role in the "by-pass" transition process. Nevertheless, later Reddy and Henningson (1993) examined the maximal energy growths $G^* = \max_{t>0}[E(t)/E(0))$ (an exact definition of this quantity will be given in the next subsection) for disturbances corresponding to four other degeneracies of the Poiseuille-flow O-S eigenvalues, and found that in these cases the values of G^* are in the range from 1.00 to 5.15. Similar calculations were made by Reddy and Henningson for four degeneracies of O-S eigenvalues they found in plane Couette flow; here $1.00 \leq G^* \leq 1.30$.

Summing up the results of this subsection, we may say that they confirm that resonances and degeneracies can produce some contributions to the often-observed transient growth of flow disturbances. However, none of the investigators whose work was considered above found a resonance or degeneracy growth rate large enough to explain the numerous experimental and computational data showing very significant transient growth of flow disturbances. (See in this respect the discussion of the inadequacy of resonance and degeneracy mechanisms of disturbance energy growth in the papers by Butler and Farrell (1992), p. 145, and Reddy and Henningson (1993), Sect. 6.2, which will be considered later in this book.) Note also, that among the works discussed above the greatest growth was found by Hultgren and Gustavsson (1981) who did not refer to resonance or degeneracy mechanisms at all (they considered the case where the O-S equation has only continuous eigenvalue spectrum), but investigated some special solutions of the general initial-value problem. Therefore it seems natural to think that some other mechanism of disturbance growth must exist which is more universal and more effective than resonance and degeneracy mechanisms. To seek such a mechanism, the general solutions of the initial-value problems for flow disturbances will be considered at greater length in the next subsection.

3.3.3 *Complete Solutions of the Initial-Value Problem and Transient Growth of Disturbances in Plane-Parallel Flows*

Tam's (1967) general solution (3.48) of the initial-value problem for the vertical velocity $w(\boldsymbol{x},t)$ of a small disturbance to a steady plane-parallel flow, and its more explicit form given by Gustavsson (1979), were mentioned several times in Sects. 3.31–3.32. However, the general solution was used by these authors only to find the part of the complete vertical-velocity field which determines the asymptotic behavior of $w(\boldsymbol{x},t)$ as $t \to \infty$. Now we shall consider some studies in which the complete solutions of the initial-value problem for three-dimensional disturbance velocity $\boldsymbol{u}(\boldsymbol{x},t) = \{u(\boldsymbol{x},t), v(\boldsymbol{x},t), w(\boldsymbol{x},t)\}$ (or, what is equivalent, for $\{w(x,t),\ \zeta_3(x,t)\}$ where $\zeta_3 = \partial v/\partial x - \partial u/\partial y$) were applied to investigation of the behavior of small disturbances in the initial stage of their evolution.

Gustavsson's (1981, 1986) representation of the exact solution to the initial-value problem for the vertical velocity w of a three-dimensional disturbance in plane

Poiseuille flow was rewritten at greater length by Shanthini (1989), while an equally detailed solution for the vertical vorticity ζ_3 in plane Couette flow was given by Gustavsson and Hultgren (1980). Both these explicit solutions were given for the three-dimensional Fourier-Laplace transforms of the fields studied, and in the cited papers they were used only for evaluation of the disturbance growth caused by spectral degeneracies or resonances (see Sect. 3.32). Later Gustavsson (1991) combined his previous results to obtain, for the case of plane Poiseuille flow, a complete solution of the initial-value problem for both fields $w(\boldsymbol{x},t)$ and $\zeta_3(\boldsymbol{x},t)$, which fully determine the three-dimensional velocity $\boldsymbol{u}(\boldsymbol{x},t)$ of a disturbance. He applied this solution to the special case where a single O-S normal mode of the vertical velocity $w(\boldsymbol{x},t)$, excited the vorticity field $\zeta_3(\boldsymbol{x},t)$. In this case $w(\boldsymbol{x},t) = W(z)\exp[i(k_1x + k_2y - \omega t))$ and this allows the solution of Eq. (3.54) for ζ_3 to be simplified. The initial value $\zeta_3(\boldsymbol{x},0)$ was set equal to zero in this paper, and hence only the induced vertical vorticity was included in the solution of initial-value problem. The given values of the O-S mode and the derived values of $\zeta_3(\boldsymbol{x},t)$ were used by Gustavsson to determine the kinetic energy of a disturbance

$$T^*(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{H}(u^2 + v^2 + w^2)dxdydz. \tag{3.56}$$

Using a Fourier representation with respect to horizontal coordinates, Eq. (3.15) allows us to rewrite Eq. (3.56) in the general case as

$$T^*(t) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{H}\frac{1}{k^2}(|\hat{\zeta}_3|^2 + |\hat{w}'|^2 + k^2|\hat{w}|^2)dk_1dk_2dz \tag{3.56$'$}$$

where $k^2 = k_1{}^2 + k_2{}^2$, while the circumflex and prime respectively denote a Fourier transform and a derivative with respect to z. According to (3.56′), the energy density in the wave-number plane, $E(k_1, k_2; t)$, is given by

$$E(k_1, k_2; t) = \int_{0}^{H}\frac{1}{2k^2}\left(|\hat{\zeta}_3|^2 + |\hat{w}'|^2 + k^2|\hat{w}|^2\right)dz. \tag{3.57}$$

Gustavsson considered the case of subcritical Reynolds numbers, $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$, (where $\mathrm{Re}_{\mathrm{cr}} \approx 5772$) and carried out the energy computations for a single O-S mode of the vertical velocity, in the hope that results for the least-damped mode would indicate the possible maximal rate of transient growth of energy. Following his paper, we will non-dimensionalize the spatial coordinates, time, wave numbers, and flow variables by the use of the Poiseuille-flow maximum (centerline) velocity U_o and the channel half-depth $H_1 = H/2$ as units of velocity and length, and define $\mathrm{Re} = U_oH_1/\nu$. The dimensionless amplitude of the forcing O-S mode was chosen by Gustavsson so that $E(k_1, k_2; 0) = 1$ (where k_1 and k_2 are the wave numbers of the exciting w-mode and $\zeta_3(0) = 0$). He then showed that the dimensionless energy density of the induced normal vorticity

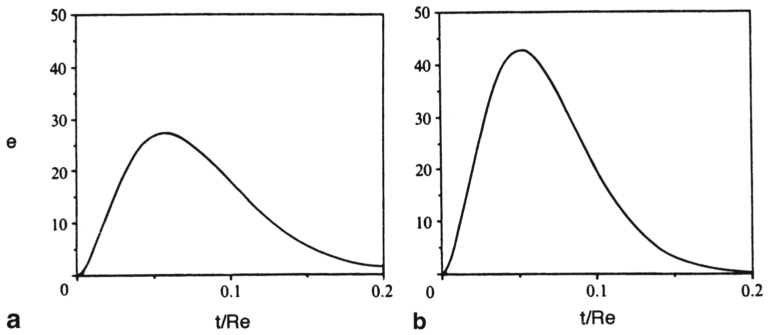


Fig. 3.7 Dependence on t/Re (where $t = t^* U_0/H_1$ is the dimensionless time) of the energy density $e(t)$ for normal vorticity induced in a plane Poiseuille flow by the least-damped symmetric **a** and antisymmetric **b** O-S normal modes of the vertical velocity with energy density $E(0) = 1$, in the case where $\mathrm{Re} = 1000$, $k_1 = 0.1$, $k = (k_1{}^2 + k_2{}^2)^{1/2} = 1$. (After Gustavsson (1991))

$$e(k_1, k_2; t) = \frac{1}{2k^2} \int_0^2 |\hat{\zeta}_3|^2 dz$$

depends only on k_1, k_2 (or k) and $t/\mathrm{Re} = t^* \nu / H_1{}^2$ (where t^* is the dimensional time)

The computations showed that when Re is not exceptionally small, the energy density $e(t)$ at first grows with time much faster than the energy of the forcing O-S mode decays due to viscosity. Therefore, the total kinetic energy density $E(t)$, which includes $e(t)$, grows rapidly with time and can greatly exceed the initial energy density $E(0) = 1$. Some of the results obtained for $\mathrm{Re} = 1000$, $k_1 = 0.1$ and $k = 1$ and the least-damped symmetric and antisymmetric (with respect to the channel midplane) O-S vertical-velocity modes are presented in Fig. 3.7. (Note that in a plane Poiseuille flow the symmetric vertical-velocity mode excites the antisymmetric vertical vorticity mode and vice versa). In Fig. 3.8, again for $\mathrm{Re} = 1000$, contours in the (k_1, k_2)-plane are shown for $e(k_1, k_2, t)$, the energy density of the normal vorticity excited by the least-damped symmetric O-S mode, at $t = 80$ (which is close to the time when the maximum value of $e(k_1, k_2, t)$ is reached for $k_2, \approx 2, k_1 = 0$). According to Fig. 3.8, at the highly-subcritical Reynolds number $\mathrm{Re} = 1000$, the kinetic energy density of the induced disturbance can take values which are almost two hundred times greater than the initial energy $E(0)$. Gustavsson also showed that at other values of Re, $e(k_1, k_{2,}\, t)$ in the region of the $(k_1, k_{2)}$-plane with substantial energy growth is approximately proportional to Re^2, if k_1 and t are rescaled in proportion to Re^{-1} and Re, respectively. Hence at subcritical values of Re higher than 1000 the ratios $e(t)/E(0)$ and $E(t)/E(0)$ can considerably exceed 1000. If the forcing O-S mode of the vertical velocity is not the least-damped symmetric or antisymmetric mode, then the growth of $e(t)$ is not so great but, nevertheless, computations show that at $\mathrm{Re} = 1000$ second and even third symmetric and antisymmetric modes can also produce substantial transient growth of disturbance energy. Figure 3.8 also shows that the main growth is achieved for small values of k_1 (corresponding to streamwise wave-lengths much greater that the full

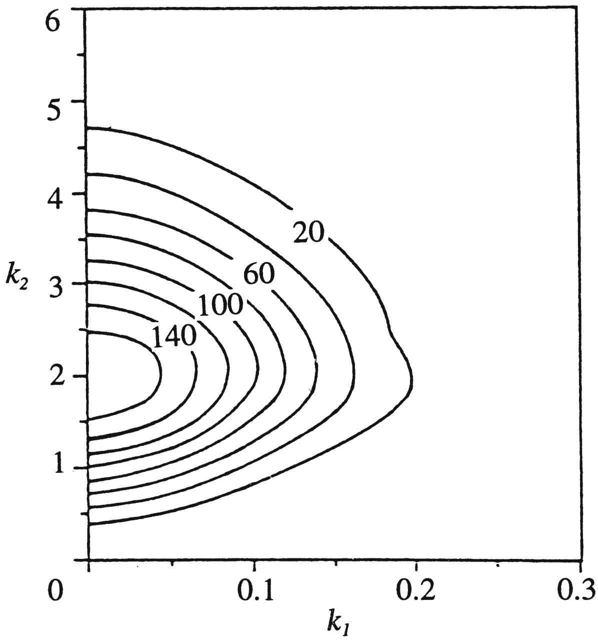


Fig. 3.8 Contours in the (k_1, k_2)-plane for the energy density $e(k_1, k_2; t)$, at $t = 80$, of the vertical vorticity ζ_3 induced in a plane Poiseuille flow with Re = 1000 by the least-damped normal O-S mode of the vertical velocity with $E(k_1, k_2; 0) = 1$. The labels are values of the energy density; the adjacent contours correspond to increments of 20 in the energy-density values. (After Gustavsson (1991))

height of the flow and the span wise wavelength of a disturbance). Hence the forcing most effectively generates streamwise-elongated flow structures, whose amplitudes can reach rather high values before viscous decay becomes appreciable. This is, in fact, true also for supercritical Poiseuille flows at not-too-high values of Re, where an unstable O-S mode exists but in many cases grows much more slowly during the initial stage of evolution than the induced vertical vorticity ζ_3 (see, e.g., Farrell (1988a), p. 2094, and Reddy and Henningson (1993), Fig. 9).

Gustavsson's student Diedrichs (1996) computed the energy growth in a plane Couette flow for a disturbance with $\zeta_3(x, 0) = 0$ and $w(\boldsymbol{x},0)$ corresponding to the least-damped O-S mode of the vertical velocity, either symmetric or antisymmetric with respect to the channel midplane. On the basis of results by Gustavsson (1991) and Butler and Farrell (1992) showing that the greatest growth is most often obtained for structures infinitely elongated in the streamwise direction, Diedrichs confined his study to x-independent disturbances with $k_1 = 0$. Hence the O-S modes considered were of the form $w(\boldsymbol{x},t) = W(z)\ \exp[i(ky - \omega t))$ (such modes are sometimes called the Stokes modes). According to Diedrichs' computations for Couette flow with Re = 1000 (where the channel half-depth H_1 and the half-difference of the wall velocities U_o are taken as the length and velocity scales), in the case of vertical vorticity forcing by the least-damped symmetric Stokes mode, the maximum growth of energy density is obtained for $k = 1.66$, where $E(t)/E(0) \approx 1157$ for $t \approx 139$. Considerably smaller growth occurs when the least-damped antisymmetric Stokes mode induces the growth of vertical vorticity; here the maximum value of E(t)/E(0) is close to 116 (and is reached at $t \approx 46$ for the optimal spanwise wave number $k \approx 2.72$).

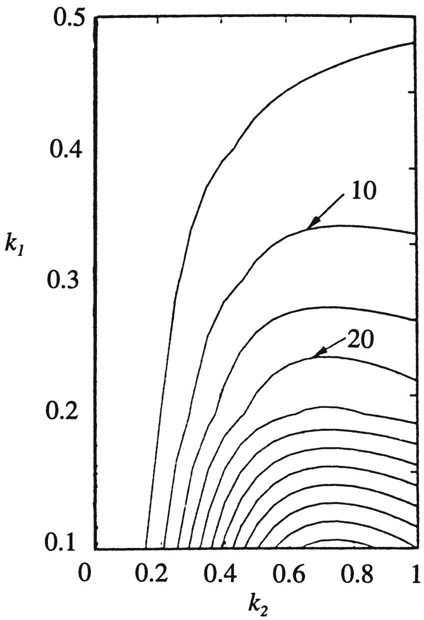


Fig. 3.9 Contours in the (k_1, k_2)-plane for the maximum energy growth $G(k_1, k_2) = \max_t[E(k_1, k_2, t)/E(k_1, k_2, 0)]$ of the plane-wave disturbance induced in a Blasius boundary layer with $\mathrm{Re} = U_0\,\delta^*/\nu = 500$ by the least-damped O-S mode of the vertical velocity, under the condition that the initial vertical vorticity is equal to zero. The adjacent contours correspond to increments of 5 in the G-values. (After Breuer and Kuraishi (1994))

These results agree quite well with the results of Butler and Farrell (1992) which will be considered later. Diedrichs also investigated the disturbance growth in some other channel flows with more complicated velocity profiles (either "Couette-like" or "Poiseuille-like"), where even greater growth of energy can be achieved than in ordinary Couette and Poiseuille flows; however, these results will not be considered here.

Transient growth of small disturbances in various boundary-layer flows was studied by Breuer and Kuraishi (1994). Following Gustavsson (1991), these authors also paid most attention to disturbances with fixed horizontal wave numbers k_1 and k_2, having initially-zero vertical vorticity ζ_3 and a vertical velocity w corresponding to the least-damped O-S mode; however, the case of a localized initial disturbance with the shape depicted in Fig. 3.2 was also briefly considered. The boundary layers investigated were mostly three-dimensional (i.e., with non-zero cross-stream velocity $V(z)$ as well as a velocity component $U(z)$ parallel to the free-stream velocity outside the boundary layer, as is typical for boundary layers over swept wings) and also often had non-zero pressure gradient. Such more complicated boundary layers will not be considered in this series; therefore, only some of Breuer and Kuraishi's results for a simple two-dimensional Blasius boundary layer are shown in Fig. 3.9. In this figure the contours in the (k_1, k_2)-plane of the maximum energy growth $G(k_1, k_2) = \max_t > 0[E(k_1, k_2; t)/E(k_1, k_2; 0))$ are presented for the case of a disturbance imposed at $t = 0$, with $\mathrm{Re}\,\delta^* = U_o\delta^*/\nu = 500$ (here U_o and δ^* are the free-stream velocity and boundary-layer displacement thickness, respectively, and k_1 and k_2 are made dimensionless with δ^*). We see that, as in the case of plane Couette flow, the maximum possible energy growth is substantial and occurs for disturbances with quite small values of k_1 (thus, strongly streamwise-elongated) and finite k_2 (in the range from 0.6 to 0.8). The greatest numerical values of energy growth in Fig. 3.9 are considerably smaller than in Fig. 3.8, but the computation procedure used by Breuer and Kuraishi did not permit reliable estimates of $G(k_1, k_2)$ for $k_1 < 0.1$, while in Fig. 3.8 the greatest growth rates do correspond to very small values of k_1.

Let us now assume that, in a given plane-parallel channel flow, $w(x,0) = W(z)$ $\exp[i(k_1 x + k_2 y)]$ where $W(z)$ is an arbitrary function. If the growth of disturbance energy has already been computed for cases where the forcing is due to a single O-S mode of vertical velocity, then to find the disturbance development in the case of arbitrary $W(z)$ we need only expand this $W(z)$ into O-S eigenfunctions and then superpose the solutions corresponding to normal-mode components of $W(z)$. More general results, related to behavior in the real time-space of the vorticity $\zeta_3(x, t)$ induced by arbitrary vertical-velocity disturbance $w(\boldsymbol{x}, t)$, can be obtained by expanding the initial value $w(\boldsymbol{x}, 0)$ in a two-dimensional Fourier integral, applying the above results to individual Fourier components, and then carrying out the inverse Fourier transformation. Some computations of this type were performed for the case of a localized initial disturbance in a plane Poiseuille flow by Henningson (1991) and Henningson et al. (1993), whose results will be considered later in this subsection. A simpler approach was used by Criminale et al. (1997) who investgated the problem of the transient growth of disturbances in plane Couette and plane Poiseuille flows, based on the direct numerical solution of Eqs. (3.44) and (3.54) for $w(\boldsymbol{x}, t)$ and $\zeta_3(x, t)$ with given initial values $w(\boldsymbol{x}, 0)$ and $\zeta_3(x, 0)$. The initial values were assumed to be represented by two-dimensional Fourier integrals, but the subsequent expansion of individual Fourier components into O-S and Sq eigenfunctions was not used in this paper. Dimensionless forms of Eqs. (3.44) and (3.55) (where all independent and dependent variables were again non-dimensionalized by using the undisturbed velocity at the channel midplane, U_o, and the channel half-depth, H_1, as units of velocity and length) imply the following equations for two-dimensional Fourier transforms, $\hat{w}(k_1, k_2; z, t)$ and $\hat{\zeta}(k_1, k_2; z, t)$, of $w(x, y, z, t)$ and $\zeta_3(x, y, z, t)$:

$$\left[\left\{\frac{\partial}{\partial t} - ik_1 U(z)\right\}\left(\frac{\partial^2}{\partial z^2} - k^2\right) + ik_1 U''(z)\right]\hat{w} - \frac{1}{\text{Re}}\left(\frac{\partial^2}{\partial z^2} - k^2\right)^2 \hat{w} = 0, \quad (3.58)$$

$$\left[\frac{\partial}{\partial t} - ik_1 U(z)\right]\hat{\zeta} - \frac{1}{\text{Re}}\left(\frac{\partial^2}{\partial z^2} - k^2\right)\hat{\zeta} = -ik_2 U'(z) w. \quad (3.59)$$

By shifting the origin of z to the channel midplane, the vertical extent of the flow can be transformed to the segment $-1 \leq z \leq 1$. Then the dimensionless undisturbed velocity $U(z)$ becomes equal to $1 - z^2$ or z for plane Poiseuille or Couette flow, respectively, while boundary conditions take the form:

$$\hat{w}(-1, t) = \hat{w}(1, t) = \hat{w}'(-1, t) = \hat{w}'(1, t) = 0, \hat{\zeta}(-1, t) = \hat{\zeta}(1, t) = 0. \quad (3.60)$$

In Eq. (3.58–3.60), as always, $k^2 = k_1^2 + k_2^2$ and primes denote derivatives with respect to z.

Criminale et al. solved the differential equations (3.58), (3.59) by a finite-difference method on a uniform grid. The results were verified by grid-independence checks and by recomputing known results for O-S eigenvalues and eigenfunctions. It was found that the computational scheme allowed the values of the energy density $E(k_1, k_2; t) = E(t)$ for both Couette and Poiseuille flows to be determined relatively swiftly for any given values of k_1, k_2 and Re, and arbitrary initial values $\hat{w}(k_1, k_2; z, 0) = W_o(z)$ and $\hat{\zeta}(k_1, k_2; z, 0) = Z_o(z)$. The authors published some results of computations for various combinations of five forms of the function $W_0(z)$ (viz. $W_0 = A_o(1-z^2)^2 \equiv W_o^{(1)}(z), (A_o/n\pi)(\cos n\pi - \cos n\pi z) \equiv W_0{}^{(2)}(z)$, $W_o^{(2)}(z)(4\pi\lambda)^{-1/2}e^{-z^2/4\lambda} = W_o^{(3)}(z)$, and two other forms which were anitsymmetric in z) with three forms of the function $Z_o(z)$ (viz. $Z_o = 0$, $A_1 \cos[(2n-1)\pi z]/2$, and $A_1 \sin n\pi z$), where A_o, A_1, and λ are positive constants and n is an integer. At first they considered the case of two-dimensional disturbances with $k_2 = 0$. Here, results were obtained for a disturbance with $k_1 = 1.48$ in plane Poiseuille flow with Re = 5000 and a disturbance with $k_1 = 1.21$ in plane Couette flow with Re = 1000. (The choice of values for k_1 was motivated by the results of Butler and Farrell (1992), considered later in this section.) If $n = \lambda = 1$ and $A_o = A_1$, then, for all combinations of initial values $W_o(z)$ and $Z_o(z)$ considered, the disturbance decays monotonically with time in Poiseuille flow, while in Couette flow some combinations of the initial values W_o and Z_o lead to minor disturbance growth, increasing the energy density $E(t)$ by less than a factor of two. These conclusions seemed somewhat strange since, according to the above-mentioned paper by Butler and Farrell, substantial growth can occur for two-dimensional disturbances in both Poiseuille and Couette flows. Therefore Criminale et al. continued their study by considering the case of the second above-mentioned form for $W_o(z)$ with $n > 1$, corresponding more closely to disturbances for which Farrell (1988b) and Butler and Farrell (1992) found maximum growth. Then it was found that in plane Poiseuille flow the second form of $W_o(z)$, together with the assumption that $\zeta_3(\boldsymbol{x}, 0) = 0$ (i.e., the first form of $Z_o(z)$), leads to maximal growth of disturbance energy for $n = 7$, when the maximum value $E(t)/E(0) = 12$ is reached at $t = 14.1$. Similarly, in the Couette flow the greatest growth was found for the same combination of functions $W_o(z)$ and $Z_o(z)$ if $n = 3$, when $E(t)/E(0)$ reaches a maximum value of 4.8 at $t = 7.8$. The calculated values of the times when the maximum growths are reached agree well with values obtained by Butler and Farrell (1992) for quite different initial conditions but the maximum growths are appreciably smaller than those found in the latter paper.

In the case of three-dimensional disturbances, with $k_2 \neq 0$, much greater growth can occur. Again on the basis of the results of Butler and Farrell (1992), Criminale et al. gave special attention to consideration of disturbances with $k_1 = 0$, $k_2 = 2.044$ in plane Poiseuille flow with Re = 5000, and disturbances with $k_1 = 0$, $k_2 = 1.66$ in plane Couette flow with Re = 1000. All possible combinations of the above-mentioned forms for vertical velocity and vorticity profiles $W_o(z)$ and $Z_o(z)$ were studied, with a number of values for n, λ, A_0 and A_1, but results were given in the paper only for $n = \lambda = 1$ (leading to maximal disturbance growth in the cases considered) and $A_1 = A_0$, if $\zeta_3(\boldsymbol{x}, 0) \neq 0$. In Fig. 3.10 the functions $G(t) = E(t)/E(0)$ are shown for three-dimensional disturbances in plane Poiseuille and Couette flows

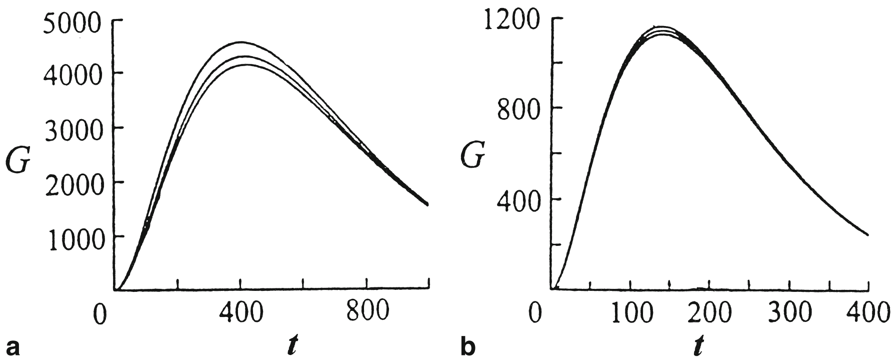


Fig. 3.10 Dependence on dimensionless time t of the energy-growth function $G(t) = E(t)/E(0)$ for a plane-wave disturbance with $k_1 = 0$ and $k_2 = 2.044$ in plane Poiseuille flow with Re = 5000 **a**, and a plane-wave disturbance with $k_1 = 0$, $k_2 = 1.66$ in plane Couette flow with Re = 1000 **b**, for various initial values of the vertical velocity and zero initial value of the vertical vorticity. The three curves correspond to the values $W_{o(1)}(z)$, $W_{o(2)}(z)$, and $W_{o(3)}(z)$ of the initial vertical-velocity amplitude $W_o(z)$. (After Criminale et al. (1997))

having the above-mentioned values of k_1 and k_2, zero initial vertical vortilcity $\zeta_3(x, 0)$ and the three forms of the initial vertical-velocity profile $W_o(z)$ indicated above. We see that the three forms of $W_o(z)$ lead to very similar forms of the function $G(t)$, with maximum values close to 4500 or 1150 in the case of Poiseuille or Couette flow, respectively. These maximum values of $G(t)$, and also the times when the maxima are reached, proved to be only slightly different from values found by Butler and Farrell (1992) for the "optimal initial conditions", which will be considered later. However Criminale et al. found that for two $W_o(z)$ profiles that were antisymmetric in z, the disturbance growth for $\zeta_3(\boldsymbol{x}, t) = 0$ was much smaller than that shown in Fig. 3.10.

Non-zero initial vertical vorticity $\zeta_3(x, 0) = Z_o(z) \exp[i(k_1 x + k_2 y)]$ also considerably reduces the values of $G(t)$. This can be explained by the fact that the initial (non-induced) vertical vorticity contributes to the value of $E(0)$ but decays rapidly with time, while only the induced vertical vorticity grows with time and produces growth of $E(t)$. It was shown by Criminale et al. that, in the case of the first $W_o(z)$ profile introduced by them, the function $G(t)$ nevertheless grows quite appreciably with time for both non-zero forms of $Z_o(z)$ with $A_1 = A_o$, while for the other four forms of $W_o(z)$ considered, the maximum of $G(t)$, in cases where $Z_o(z) \neq 0$, proves to be many times smaller than that for the first form.

Computations by Criminale et al. also confirmed the conclusion of Gustavsson (1991) that $G^* = \max_{t>0} G(t)$ is almost exactly proportional to Re^2, while the time t^* when this maximum is reached is proportional to Re. Since, according to Fig. 3.10, $G^* \approx 4540$ for Poiseuille flow with Re = 5000 and ≈ 1150 for Couette flow with Re = 1000, we deduce that, for the initial conditions considered, $G^* \approx 180$ for Poiseuille flow with Re = 1000 (in good agreement with the result by Gustavsson relating to another initial condition), while $G^* \approx 29000$ for Couette flow with Re = 5000.

Let us now consider some other approaches to investigation of transient disturbance growth. Henningson (1991) (see also Henningson et al. (1994); Henningson and Alfredsson (1996); and Schmid and Henningson (2001)) used the general Eqs. (3.44) and (3.54) somewhat differently. Studying the disturbance development in a plane Poiuseuille flow, he also assumed that the horizontal wave numbers k_1 and k_2 of disturbance vertical velocity and vorticity w and ζ_3 are fixed (and thus $w(x,t) = \hat{w}(k_1,k_2;z,t)\ \exp[i(k_1x+k_2y)], \zeta_3(\mathbf{x},t) = \hat{\zeta}(k_1,k_2;z,t)\ \exp[i(k_1x+k_2y)]$) where $\hat{w}(k_1,k_2;z,t)$ and $\hat{\zeta}_3(k_1,k_2;z,t)$ satisfy Eqs. (3.58) and (3.59)). However, instead of solving these equations, he expanded the functions $\hat{w}(k_1,k_2;z,t)$ and $\hat{\zeta}_3(k_1,k_2;z,t)$ in eigenfunctions of the O-S and Sq equations (2.41) and (2.42) corresponding to wave numbers k_1 and k_2. The possibility of expanding $\hat{w}(k_1,k_2;z,t)$ in the O-S eigenfunctions follows from the completeness of the system of these eigenfunctions, proved by Schensted (1960) and Di Prima and Habetler (1969) (see Sect. 2.5); the general solution $\hat{\zeta}(k_1,k_2;z,t)$ of Eq. (3.59) may be represented as the sum of the general solution of the corresponding homogeneous equation with zero right-hand side (this summand may be expanded in Sq modes) and some particular solution of the inhomogeneous equation. Henningson showed that such a particular solution can be constructed rather easily for the case where $\hat{w}(k_1,k_2;z,t)$ is represented by single O-S mode or by a given linear combination of such modes. Therefore, his approach avoids using the complicated Fourier-Laplace-transformation technique for determination of the general solution of the initial-value problem.

In the case where $\hat{w}(k_1,k_2;z,t)$ is given by a single O-S normal mode (say the first, i.e., the least stable one) and hence $\hat{w}(k_1,k_2;z,t) = \hat{w}(z,t) = AW_1(z)e^{-i\omega_1 t}$ (where $W_1(z)$ and ω_1 are the first O-S eigenfunction and eigenfrequency, and A is an arbitrary coefficient), Henningson's solution has the form

$$\zeta_3(\mathbf{x},t) = \left[\sum_{j=1}^{\infty} C_j e^{-i\omega_j^o t} + \sum_{j=1}^{\infty} D_j^{(1)} \frac{e^{-i\omega_1 t} - e^{-i\omega_j^o t}}{\omega_1 - \omega_j^o}\right] \zeta_{3j}(z) e^{i(k_1x+k_2y)}, \qquad (3.61)$$

where $\zeta_{3j}(z)$, ω_j and ω_j^0 denote the j^{th} Sq eigenfunction and O-S and Sq eigenfrequencies, respectively, while C_j and $D_j^{(1)}$ are coefficients in the expansions of the functions $\hat{\zeta}(k_1,k_2;z,0) = \hat{\zeta}(z,0)$ and $-iAk_2U'(z)W_1(z)$ into Sq eigenfunctions $\zeta_{3j}(z), 1 \le j < \infty$. (This solution, where all the frequencies and coefficients are complex, can be checked easily by direct substitution into Eq. (3.59).) In the more general case where

$$\hat{w}(k_1,k_2;z,t) = \sum_m A_m W_m(z) e^{-i\omega_m t} \qquad (3.62)$$

(i.e., $\hat{w}$ is given by a sum of O-S modes), the solution (3.61) clearly takes the form

$$\zeta_3(x,t) = \left[\sum_j C_j e^{-i\omega_j^o t} + \sum_{m,j} D_{mj}^{(1)} \frac{e^{-i\omega_m t} - e^{-i\omega_j^o t}}{\omega_m - \omega_j^o}\right] \zeta_{3j}(z) e^{i(k_1x+k_2y)}. \qquad (3.63)$$

A Taylor series expansion of the right-hand side of (3.63) in powers of time gives, for small values of t, the result

$$\zeta(\mathbf{x},t) = \sum_{j=1}^{\infty}[C_j(1 - i\omega_j^o t) - i\sum_m D_{mj}^{(1)}t + O(t^2)]\zeta_{3j}(z)e^{i(k_1x+k_2y)} \tag{3.64}$$

(where O symbolizes order of magnitude), This shows that the normal vorticity (and hence, by virtue of (3.15), the horizontal-component velocities u and v also) initially grow linearly with time. Moreover, Henningson noted that, according to an asymptotic expansion given by Drazin and Reid (1981), p. 159 at small k_1Re and large Re the eigenfrequencies ω_m and ω_j^0 are inversely proportional to Re. Hence, at high Reynolds numbers and small enough values of k_1, all the O-S and Sq eigenfrequencies take values close to zero and they coalesce as Re $\to \infty$. This means that at high Reynolds numbers and small streamwise wave numbers a number of near-resonances and near-degeneracies necessarily exists. This circumstance can explain the substantial growth of disturbances elongated in the streamwise direction in flows with large Re.

Summing all the terms in the right-hand side of (3.62) which do not tend to zero as Re $\to \infty$, Henningson (1991) (see also Henningson et al. (1994); Schmid and Henningson (2001)) found that at small values of k_1Re and large Re

$$\hat{\zeta}(z,t) = \hat{\zeta}(z,0) - ik_2U'(z)\hat{w}(z,0)t + O\left(\frac{t}{\text{Re}}\right). \tag{3.64$'$}$$

According to (3.15) this equation coincides, when Re $\to \infty$, $t = O(1)$, and $k_1 = 0$ with the known inviscid result (3.21) due to Ellingsen and Palm.

Henningson also calculated a number of values for the coefficients $D_j^{(1)}$ in Eq. (3.61) (note that in the case of a non-self-adjoin eigenvalue problem, the eigenfunctions of the adjoint problem are needed for the computation of the expansion coefficients; see, e.g., Schensted (1960); Eckhaus (1965); Betchov and Criminale (1967); Joseph (1976); or Schmid and Henningson (2001)). It was found that values of $D_j^{(1)}$ are sometimes many tens (or even many hundreds) times greater than the maximum amplitude of the driving O-S mode. Henningson then used Eq. (3.61) for calculation of the time development of the normal-vorticity amplitude in plane Poiseuille flow, disturbed by a superimposed vertical velocity represented by the least-stable O-S mode with the maximum of the amplitude $AW_1(z)$ equal to one. He assumed that $\zeta_3(\boldsymbol{x},0) = 0$, and either Re $= 3000$, $k_2 = 1$ with varying k_1, or $k_1 = 0$, $k_2 = 1$ and varying Re. In all these cases significant initial transient growth of vorticity amplitude was observed, increasing with decreasing k_1 (i.e., increasing streamwise wavelength) and increasing Re. For small values of k_1 and not-too-small subcritical Re (equal to or exceeding 3000), it was found that the maximum amplitude of the normal vorticity can be fifty or more times larger than the amplitude of the initial normal-velocity wave. These results clearly agree well with those obtained by Gustavsson (1991).

Solution (3.63) was applied by Henningson to the study of the normal vorticity development in plane Poiseuille flow, produced by a localized initial disturbance of the shape shown in Fig. 3.2. The results obtained were compared with direct numerical simulation of the same vorticity development (i.e., with numerical solution of the corresponding Eqs. (3.44) and (3.54)). These results of Henningson (1991) were developed further by Henningson et al. (1993), whose paper will be considered later in this section.

Strong renewed interest in transient disturbance growth, arising in the late 1980s and early 1990s, led several authors to pay special attention to the "optimal initial conditions" providing maximal growth of the initial disturbance. One of the first investigations of this type was due to Farrell (1988a) who studied the development of two-dimensional disturbances in plane Poiseuille and Couette flows. Since $\zeta_3 \equiv \partial v/\partial x - \partial u/\partial y = 0$ for a two-dimensional disturbance, where $\boldsymbol{u}(\boldsymbol{x},t) = \{u(x,z,t), 0, w(x,z,t)\}$, only Eqs.(3.44) and (3.58) (where $\partial w/\partial y$ and k_2 are equal to zero) are of importance in this case. Farrell based his analysis on consideration of a Fourier component $\psi(z,t)e^{ikx}$ of the stream function $\psi(x,z,t)$. (Then$-ik\psi(z,t)e^{ikx}$ will be the corresponding Fourier component of the vertical velocity $w(x,z,t)$; hence it makes no difference whether stream function of vertical velocity is considered.) To simplify the computations, he approximated differential equation on the interval $0 \le z \le H$ (here we use dimensional independent and dependent variables again) by finite-difference equations with a sufficiently high number N of mesh points (cf. in Sect. 3.23 the description of a similar approach used by Farrell and Ioannou (1993a)). Thus, the function $\psi(z,t)$ and the O-S eigenvalue problem (relating to the case where $\psi(z,t) = \psi(z)e^{-i\omega t}$) were replaced, respectively, by the vector-function $\psi(t)$ and by the finite-difference version of the classical O-S problem dealing with a system of N linear algebraic equations. The value of N was chosen to be 100, giving results practically indistinguishable from exact ones. The algebraic eigenvalue problem has N discrete eigenvalues ω_j (or $c_j = \omega_j/k$) and N eigenvectors ψ_j. The N-dimensional vector $\psi(0)$ corresponding to the Fourier amplitude $\psi(z,0)$ of the initial stream function $\psi(x,z,0)$ (such correspondence will be denoted below as $\psi(z,0) \Rightarrow \psi(0)$) can be expanded in eigenvectors ψ_j of the discretized O-S equation. Let this expansion be

$$\psi(0) = \sum_{j=1}^{N} a_j \psi j;$$

then the evolution in time of the initial disturbance is described by the equation

$$\psi(z,t)e^{ikx} \Rightarrow \psi(t)e^{ikx} = \sum_{j=1}^{n} a_j \psi j e^{ik(x-c_j t)}. \qquad (3.65)$$

The initial conditions are now given by the vector $\boldsymbol{a}$ with components, $a_j, j = 1, \ldots, N$. To make the concept of "optimal initial conditions" definite, we must introduce a measure for the disturbance magnitude and determine what specific meaning is given to the word "optimal". Farrell considered two different measures of disturbance magnitude: the simplest L_2 measure, corresponding to the norm

$$||\psi||_L = \left[\int_0^H |\psi(z,t)|^2 dz \right]^{1/2},$$

and the energy measure based on the norm

$$||\psi||_E = \left\{ \frac{1}{2} \int_0^H [k^2|\psi(z,t)|^2 + |\psi'(z,t)|^2] dz \right\}^{1/2} = \left\{ \frac{1}{2} \int_0^H [|u|^2 + |w|^2] dz \right\}^{1/2}$$

where a prime again denotes a derivative with respect to z. Both the measures (squares of the corresponding norms) were approximated by positive-definite quadratic forms of variables a_j, with coefficients depending on t. As to the meaning of the "optimality", it must be chosen in accordance with the optimization problem being solved. Two such problems were investigated by Farrell: i) Determination of the minimum initial disturbance exciting a chosen normal mode of unit measure (for example, the least stable or, if $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, the unstable mode), and ii) Determination of the shape of the initial disturbance that produced the maximum growth of the disturbance magnitude over a fixed reasonably chosen time interval. For the two measures of disturbance magnitude given above, both problems can be reduced to the relatively simple problem of finding a conditional maximum (or minimum) for a quadratic form of N variables. The most striking feature of the solution of the first problem (relevant to the best way to produce the most persistent wave) was the discovery that the optimal initial conditions for generation of a given mode are very different from this mode itself. Therefore, it is most advantageous here not to put the energy available at the time $t = 0$ directly into the mode which one wants to excite, but to use this energy quite differently.

Farrell's problem ii) is more interesting to us, being much closer to the topic of this subsection. It represents an attempt to estimate maximal possible transient growths of a disturbance over various finite time intervals (cf. again the discussion of the paper by Farrell and Ioannou (1993a) in Sect. 3.23). Farrell showed that the required initial disturbance is given by the eigenvector of some specific $N \times N$ matrix corresponding to its greatest eigenvalue, while the eigenvalue itself determines the relative growth of the disturbance magnitude. Then he illustrated this general result by two examples. The first of these concerned the development of the disturbance with $kH_1 = 1$ that grows maximally in the L_2 or energy norm (both were considered) over the first 20 time units (i.e., from $t = 0$ to $t = 20\,\mathrm{H_1/U}_o$ where, as usual, H_1 and U_o are the channel half depth and the maximum undisturbed velocity) in a plane Poiseuille flow with supercritical Reynolds number $\mathrm{Re} = 10^4$. The precise form of the optimum disturbance depends on the choice of the norm but many general features of this disturbance are common in both cases and differ strikingly from the features of the unstable normal mode with the same value of k. However, in course of its development the form of the optimal disturbance approaches the form of the unstable mode. The maximal kinetic energy density of the disturbance which can be reached over 20 time units is 61 times greater than the initial disturbance energy density;

and the transient growth rate found was nearly two orders of magnitude greater than the rate of the exponential growth of the unstable mode at $\mathrm{Re} = 10^4$. The second example considered dealt with a disturbance having the same wave number k as above and growing maximally in the energy norm over 12 time units in Couette flow with $\mathrm{Re} = 10^3$ (now the velocity U_o entering the definition of Re is the half-difference of wall velocities). Here the growth rate was smaller than in the first example, and the maximal energy density exceeded the initial energy density only by a factor of about 11.

Later some supplementary results concerning maximally growing two-dimensional disturbances in plane Poiseuille and Couette flows were mentioned in passing by Butler and Farrell (1992) who gave their main attention to development of three-dimensional disturbances. In particular they stated that, in the case of a Couette flow with $\mathrm{Re} = 1000$, comparison of the optimal two-dimensional disturbances found for various dimensionless wave numbers $\kappa = kH_1$ and growth periods $\tau = tU_o/H_1$ shows that the maximal transient growth of the kinetic energy density $E(t) = ||\psi(x, z, t)||_E^2$ occurs for a disturbance with $\kappa = 1.21$ where $E(t)/E(0)$ reaches the maximal value 13 at $\tau \approx 9$ but then begins to decrease. In the case of a plane Poiseuille flow a similar investigation was carried out for $\mathrm{Re} = 5000$ (i.e., for a subcritical value, half the supercritical Re used in Farrell's paper (1988a)). It was found that here $\kappa = 1.48$ for the most strongly growing two-dimensional disturbance, whose energy density $E(t)$ reaches a value close to $46E(0)$ at $\tau \approx 14$ but then also starts to decrease. These results stimulated Criminale et al. (1997) to choose the same values of κ and Re in a subsequent study of two-dimensional disturbance growth in plane Couette and Poiseuille flows. It was noted above that the maximum energy growths found in this paper for the initial conditions considered was appreciably smaller than those obtained by Butler and Farrell for optimal initial conditions.

The maximum energy growth found by Farrell and by Butler and Farrell for two-dimensional disturbances in a plane Poiseuille flow were much smaller than those found, for example, by Gustavsson (1991) and Henningson (1991), who analyzed the evolution of some particular disturbances but did not solve optimization problems. Recall that the last-mentioned two authors both considered three-dimensional disturbances and stressed that the growth mechanisms studied by them were in principle three-dimensional. In fact, two-dimensional disturbances produce no forcing of the vertical vorticity (and horizontal velocities) by vertical velocity, which plays such an important part in the dynamics of three-dimensional disturbances. The principal growth mechanism for two-dimensional disturbances is based on the extraction of the energy from the sheared mean flow by disturbances through the action of the "Reynolds stress of a disturbance" uw having a sign opposite to that of the velocity shear of the undisturbed flow dU/dz (see in this respect the energy-balance equation (3.74) in Sect. 3.4 and the papers by Farrell and Butler and Farrell cited above). However, according to all available data, this mechanism is much less efficient than the mechanism of vertical-velocity forcing of vertical vorticity.

The above discussion makes it clear that Farrell's paper (1988a) must be considered only as an introduction to the main part of the paper by Butler and Farrell (1992) devoted to application of Farrell's variational procedure to the study of the development of small three-dimensional disturbances in plane-parallel Couette, Poiseuille, and Blasius boundary-layer flows. In this case Eqs. (3.44) and (3.54) (or, if the horizontal Fourier components of disturbances are studied, Eqs. (3.58) and (3.59)) must be solved simultaneously. Butler and Farrell looked for normal-mode solutions, i.e. assumed that $w(\boldsymbol{x}, t) = W(z) \exp[i(k_1 x + k_2 y - \omega t)]$, $\zeta_3(\boldsymbol{x}, t) = Z(z) \exp[i(k_1 x + k_2 y - \omega t)]$. These expressions were substituted into Eqs. (3.44) and (3.54) leading, together with the usual wall boundary conditions, to an eigenvalue problem determining the spectrum of complex eigenvalues ω_j at given real values of k_1 and k_2. For plane Couette and Poiseuille flows the set of eigenvalues is discrete and the system of corresponding eigenfunctions is complete in the Hilbert space of vector functions {W(z), Z(z)} equipped with the energy norm. For a Blasius boundary layer the situation is more complicated but this circumstance is immaterial for the work considered here, where the differential equations are in all cases replaced by finite-difference ones. In fact, it the functions $W(z)$ and $Z(z)$, where $0 \leq z \leq H$, are replaced by N-dimensional vectors $\mathbf{W}$ and $\mathbf{Z}$, representing the values at N finite-difference locations, then the resulting algebraic eigenvalue problem, relating to a $(2N \times 2N)$-matrix, will always have $2N$ eigenvalues. Now any vector pair ($\mathbf{W}$,$\mathbf{Z}$) may be expanded in the corresponding eigenvectors. Hence, the initial disturbance may be represented as

$$\{\mathbf{W}(0), \mathbf{Z}(0)\} = \left\{ \sum_{j=1}^{2N} a_j \mathbf{W}_j, \sum_{j=1}^{2N} a_j \mathbf{Z}_j \right\},$$

and at time t this disturbance will be transformed into

$$\{\mathbf{W}(t), \mathbf{Z}(t)\} = \left\{ \sum_{j=1}^{2N} a_j \mathbf{W}_j e^{-i\omega_j t}, \sum_{j=1}^{2n} a_j \mathbf{Z}_j e^{-i\omega_j t} \right\}.$$

According to Eq. (3.56′), the square of the energy norm for three-dimensional disturbances has the form

$$||(w, \zeta_3)||_L^2 = \frac{1}{2} \int_0^H \left\{ |w|^2 + k^{-2} \left(|\partial w/\partial z|^2 + |\zeta_3|^2 \right) \right\} dz.$$

Butler and Farrell showed that at time t the expansion coefficients $(a_1, \ldots, a_{2N})$ of the "optimal initial disturbance" coincide with the components of the first eigenvector (i.e., that corresponding to the greatest eigenvalue) of some specific $2N \times 2N$ matrix, which depends on t, while its greatest eigenvalue is equal to the optimal-disturbance energy gain $E(t)/E(0)$ (where the kinetic energy density $E(t)$ is given by the square of the energy norm). Thus, finding the optimal initial disturbances and computing the corresponding energy growths is reduced to solving some tedious but quite standard problems of linear algebra. Some results obtained in this way are presented in Butler and Farrell's paper.

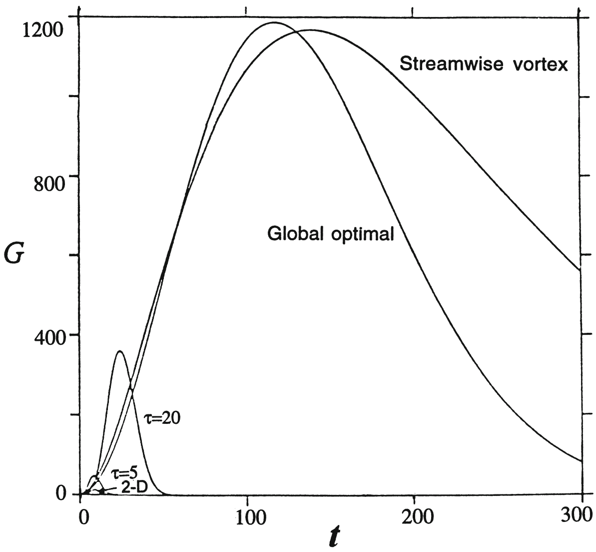


Fig. 3.11 Plot of the energy-growth function $G(t) = E(t)/E(0)$ for the globally optimal, x-independent optimal (streamwise vortex), and two-dimensional (2-D) optimal disturbances, and for disturbances which grow the most in time $t_{\text{opt}} = \tau$ equal to 5 and 20 time units, in Couette flow with Re = 1000. (After Butler and Farrell (1992))

Results related to the optimal (i.e., most strongly growing) wave-like two-dimensional disturbance to plane Couette flow with Re = 1000 were described above. They were computed simply for comparison with similar results relating to three-dimensional disturbances. Since the previous results of other authors (e.g., those by Gustavsson shown in Fig. 3.8) showed that three-dimensional disturbances which do not vary in the streamwise direction (i.e., those with $k_1 = 0$) apparently grow more strongly than all the others, Butler and Farrell first studied the development of such disturbances in the same case of Couette flow with Re = 1000. They found that maximum energy growth is achieved for a disturbance with $k_2H_1 = 1.66$. (Only non-dimensional wave numbers kH_1 and times tU_o/H_1 will be used later in this subsection, so they will be denoted below simply by symbols k and t. Also the frequencies ω and coordinates x, y, z will now always be assumed to be non-dimensionalized.) For a disturbance with $k_2 = 1.66$, the energy density $E(t)$ increases up to $1166E(0)$ at $t = 138$. Such large energy growth is utterly surprising for a flow which was so long considered to be absolutely stable. Note, however, that it is achieved for a streamwise-unbounded disturbance (having the form of an infinite streamwise vortex) and is reached after a rather long development time.

Butler and Farrell then arranged a search for a globally optimal disturbance among those with any values of k_1 and k_2 in order to check whether streamwise-vortex disturbances were in fact the most strongly growing. They found that this is not so, since the most strongly growing wave-like disturbance in a Couette flow with R = 1000 depends on all three coordinates, has the wave numbers $k_1 = 0.035$ and $k_2 = 1.60$, and reaches its maximum energy $E_{\max}$ (equal to $1185E(0)$ at $t = 117$. This disturbance is also strongly elongated in the streamwise direction (since $k_1 \ll 1$) and requires a long time to reach the maximal energy (since $t \gg 1$), and its maximum growth is only slightly larger than that for the streamwise vortex. However, as shown in Fig. 3.11, the streamwise vortex decays considerably more slowly, after reaching maximum energy, than the globally optimal disturbance. In Fig. 3.11 the dependence of the energy density on time is also shown for the optimal two-dimensional disturbance and for disturbances achieving the maximal energy admissible at $t = 20$ and at $t = 5$. In the latter two cases the rates of energy growth at small values of time are greater, but the maxima are much lower than those for the optimal streamwise vortex or the globally optimal disturbance.

Results by Butler and Farrell relating to globally optimal disturbances in Couette flows with Re varying from 31 to 4000 are presented in Table 3.1. We see that the values of the spanwise wave number k_2 are nearly the same for all these disturbances. At the same time the streamwise wavelength $2\pi/k_1$ (which at all values of Re is much greater than the flow thickness $2H_1$ or the spanwise wavelength $2\pi/k_2$) and the time t for maximum growth to occur both increase as Re, and the maximum energy growth $E_{\max}/E(0)$ increases as $(\mathrm{Re})^2$, at high Reynolds numbers.

In the case of plane Poiseuille flow it was shown that the globally optimal disturbance has the form of a streamwise vortex with $k_1 = 0$. At $\mathrm{Re} = 5000$, maximum energy growth was determined to be $E(t) = 4897E(0)$, reached at $t = 379$, for a spanwise wave with $k_2 = 2.04$ and with stream function $\psi(y, z, t)$ antisymmetric in z (where it is assumed that $-H_1 \le z \le H_1$ and the stream function is defined by equations $-\partial\psi/\partial y = w$ and $\partial\psi/\partial z = v$). Unlike Couette flow (where nothing similar occurs), Poiseuille flow also contains a second set of strongly-increasing disturbances independent of x, whose energy growths are about half of those for globally optimal disturbances but whose stream function is symmetric in z (and the disturbances themselves are optimal in the set of all disturbances with symmetric stream functions). At $\mathrm{Re} = 5000$ the symmetric optimal disturbance has the spanwise wave number $k_2 = 2.64$ and its energy grows from $E(0) = 1$ up to the maximum value $E(t) \approx 2819$ at $t \approx 270$. Note that Gustavsson (1991) and Diedrichs (1996), who studied the growth of the vertical vorticity induced by the least-damped symmetric or antisymmetric O-S modes in plane Poiseuille and Couette flows, both found maxima of $E(t)/E(0)$, and times t needed to reach these maxima, which are rather close to values obtained by Butler and Farrell for corresponding optimal disturbances. Moreover, Criminale et al. (1997), who studied some non-modal disturbances with $k_1 = 0$ and optimal values of k_2 (found by Butler and Farrell for x-independent disturbances), also obtained, for both Poiseuille and Couette flows, values for maxima of $E(t)/E(0)$ and for times t which are very close to those given by Butler and Farrell for optimal disturbances. These facts lead one to believe that the transient turbulence growth is not very sensitive to the shape of the initial disturbance.

As to the boundary-layer flow, it was modeled by a plane-parallel flow in the half-space $0 \le \zeta < \infty$ with the Blasius velocity profile $U(z)$. (See, however, footnote 4 on p. 64 relating to the influence of the gradual thickening of a boundary layer.) Recall now that results found for Poiseuille flow show that the most strongly growing disturbances are confined to the shear regions near the walls. Since the velocity shear $U'(z)$ decreases rapidly with z in a Blasius boundary layer, it seemed reasonable to assume that, to find the most strongly growing disturbances, the boundary-layer flow may be replaced by a synthetic channel flow with the Blasius velocity profile and with the upper wall at a height z where the velocity $U(z)$ is practically independent of z (i.e., indistinguishable from the free-stream velocity U_0). Just such a model was used by Butler and Farrell; its adequacy was verified by comparison of a few first O-S eigenvalues computed for this model (more precisely, for its finite-difference approximation) with those computed for a semi-infinite boundary layer with the smooth Blasius velocity profile. (Of course, changing to this model greatly changes the spectrum of O-S eigenvalues: for a real Blasius boundary layer it consists of a few discrete eigenvalues supplemented by a continuous spectrum, while in a channel flow the O-S spectrum is discrete and infinite, but in the case of a finite-difference approximation it is finite. However, the form of the O-S spectrum does not, as a rule, significantly affect the transient growth of disturbances and the forms of the most strongly growing structures; see, e.g., Farrell and Ioannou (1993c).) Butler and Farrell defined the Blasius boundary layer by the Reynolds number based on the free-stream velocity U_0 and the displacement thickness $\delta^*, \mathrm{Re}_{\delta^*} = U_o\delta^*/\nu; U_0$ and δ^* were also used to make wave numbers k and times t' dimensionless.

The globally optimal disturbance in this synthetic-channel model of a boundary layer was found to be a streamwise vortex independent of x. At $\mathrm{Re}_{\delta^*} = 1000$ this disturbance has spanwise wave number $k_2 = 0.65$ (which is close to the optimal value of k_2 in Fig. 3.9), and its energy density grows from the value $E(0)$ up to a value $E_{\max} = 1514E(0)$, reached at $t = 778$, while for the optimal two-dimensional disturbance with $k_2 = 0$ in this flow, $k_1 = 0.42$, and the energy gain $E_{\max}/E(0) = 28$ at $t = 45$. At other values of Re_{δ^*} the globally optimal disturbances have the same value of k_2, but the time t when the maximum energy is reached and the value of the maximum energy growth, $E_{\max}/E(0)$, are proportional to Re_{δ^*} and $(\mathrm{Re}_{\delta^*})^2$, respectively. Later Butler and Farrell (1993) tried to apply their method of finding optimally growing small disturbances to a turbulent boundary layer, with a mean velocity profile quite different from the Blasius one. The idea was to seek an explanation of some well-known, but until now inexplicable, features of near-wall regions in turbulent boundary layers (cf. also the work by Jang et al. (1986), referred to in Sect. 3.32, which had a similar purpose). However, discussion of the 1993 paper by Butler and Farrell must be postponed until after a general introduction to near-wall turbulent flows.

Some other methods for determination of the optimally growing wave-like disturbances with given horizontal wave numbers k_1, k_2 in plane Poiseuille and Couette flows, and computation of the corresponding growth functions $G(k_1, k_2; t) = E(k_1, k_2; t)/E(k_1, k_2; 0)$ (which also depend on Re), were proposed by Reddy and Henningson (1993) and Criminale et al. (1997). Reddy and Henningson used their method, which will be described shortly, to obtain a number of new results relating to characteristics of the optimally growing disturbances and values of their maximal growth $G^*(k_1, k_2) = \max_{t>0} G(k_1, k_2; t)$ for various values of k_1, k_2 and Re. In particular, they plotted growth contours $G^*(k_1, k_2) = \text{const.}$, in the (k_1, k_2)-plane for several values of Re, and, for two-dimensional disturbances with $k_2 = 0$, plotted growth contours of $G^* = G^*(k_1, \mathrm{Re})$ in the (k_1, Re)-plane. (Recall that a disturbance is called "optimally growing." or, more precisely, "optimally growing from $t = 0$ till time t" if it produces the greatest value of $G(k_1, k_2; t)$; this disturbance clearly depends on t. However, the "globally optimal" disturbance, for which $G(k_1, k_2; t)$ as a function of t has the highest maximum, does not depend on t.) In their computations Reddy and Henningson used the expansion of functions $\hat{w}(k_1, k_2; z, t)$ and $\hat{\zeta}(k_1, k_2, z, t)$ in the O-S and Sq eigenfunction, which follows from Eq. (3.63). In fact, according to this equation, the general solution $\{\hat{w}(k_1, k_2; z, t), \hat{\zeta}(k_1, k_2; z, t)\}$ of Eqs. (3.58), (3.59) can be represented as

$$\{\hat{w}(k_1, k_2; z, t), \hat{\zeta}(k_1, k_2; z, t)\} = \sum_{m=1}^{\infty} A_m \mathbf{q}_m(z) e^{-i\omega_m t} + \sum_{j=1}^{\infty} B_j \mathbf{p}_j(z) e^{-i\omega_j^o t} \tag{3.66}$$

where $\mathbf{q}_m(z) = \{W_m(z), V_m(z)\}$ and $\mathbf{p}_\mathrm{j}(z) = \{0, k\zeta_{3j}(z)\}$ are definite two-dimensional vector-functions of z. These vector-functions are simply the eigenfunctions of the two-dimensional eigenvalue problem arising from Eqs. (3.58–3.59) when the derivatives $\partial/\partial t$ are replaced by $-i\omega$, where ω is the unknown eigenvalue (see, e.g., Henningson and Schmid (1992)). Therefore, Eq. (3.66) represents the expansion of the solution of Eqs. (3.58–3.59) in the corresponding two-dimensional eigenfunctions. (For the sake of simplicity it is assumed here that all eigenvalues ω' s are distinct.) Reddy and Henningson proposed arranging the terms on the right-hand side of Eq. (3.66) in order of decreasing imaginary parts of the corresponding eigenvalues ω_k (either ω_m or ω_j^o), and preserving only a limited number N of terms, taking into account that $\Im\mathrm{m}\, \omega_k \ll 0$ for large values of k so that the corresponding terms of (3.66) would be negligibly small for practically any $t > 0$. The truncation of the series (3.66) was combined with a special vertical discretization (differing from the simplest uniform grid discretization) which reduces the computation of unknown eigenvalues and eigenfunctions to problems from linear algebra. All this makes the numerical procedure somewhat different from that used by Butler and Farrell (the latter is less economic in number of arithmetic operations needed).

The results of the computations were presented by Reddy and Henningson in a number of figures and tables. As examples, in Figs. 3.12a–3.14 contours of $G^*(k_1, k_2) = G^*(k_1, k_2;\, \mathrm{Re})$ are shown for Poiseuille flow, in the $(k_1,\ \mathrm{Re})$-plane for the case of two-dimensional disturbances, having $k_2 = 0$, and in the (k_1, k_2)-plane for the case where $\mathrm{Re} = 1000$; also, contours of the normalized function $k^2 G^*(k_1, k_2)/(k_2 \mathrm{Re})^2$ are presented in the $(k, k_1\mathrm{Re})$-plane, where $k = (k_1^2 + k_2^2)^{1/2}$, for both Couette and Poiseuille flows with two different values of Re. Figure 3.14 confirms the conclusion by Reddy and Henningson that, at all not too small values of Re, $k^2 G^*(k_1, k_2)/k_2\mathrm{Re})^2$ is almost a function of k and $k_1\mathrm{Re}$ alone, especially at low values of $k_1\mathrm{Re}$. Therefore the curves in Fig. 3.14 make possible the determination of the values $G^*(k_1, k_2)$ for plane Couette and Poiseuille flows at any not-too-small value of Re. In particular, Trefethen et al. (1993) used this result, together with some data of other authors, to compute the contours of $G^*(k, \mathrm{Re}) = \max_{k_1^2 + k_2^2 = k^2} G^*(k_1, k_2, \mathrm{Re})$ for Poiseuille flow. Their results presented in Fig. 3.12b supplement Fig. 3.12a and show to what extent the growth of three-dimensional disturbances can exceed the growth of two-dimensional ones. The contours of $G^*(k_1, 0,\ \mathrm{Re})$ and $G^*(k,\ \mathrm{Re})$ for plane Couette flow are similar to those in Fig. 3.12 (the first of them are presented in the paper by Reddy and Henningson) but they contain no shaded regions, since no unstable normal modes exist in a plane Couette flow at any value of Re. Some other results of Reddy and Henningson's paper will be briefly considered in Sect. 3.4.

Criminale et al. (1997) proposed to use expansion of the values $\{\hat{w}(k_1, k_2; z, 0), \hat{\zeta}(k_1, k_2; z, 0)\}$ in a Fourier series with respect to the vertical coordinate z and then to solve Eqs. (3.58), (3.59) with the initial conditions represented by a properly truncated series, and to find the optimal growth function $G(k_1, k_2; t)$ by solving the variational problem relating to the values of Fourier coefficients. According to these authors, such a procedure has an advantage over that based on the expansion (3.66) used by Reddy and Henningson, but only its application to the study of the optimal two-dimensional disturbance in a plane Poiseuille flow at Re = 5000, which was found by Butler and Farrell (1992), briefly outlined in their paper.

Schmid et al. (1994) and Lundbladh et al. (1994) (these two papers strongly overlap) also considered the problem of transient growth for small disturbances in a plane Poiseuille flow, but for the spatial evolution of disturbances (in the streamwise x-direction), rather than the temporally-evolving case discussed above. Therefore, it was assumed that $w(\boldsymbol{x},t)$ and $\zeta_3(\boldsymbol{x},t)$ are proportional to $\exp\{i(k_1 x + k_2 y - \omega t)\}$, where k_2 and ω are given real values while k_1 is the unknown complex eigenvalue. Then Eqs. (3.44) and (3.54) imply the known O-S equation (Eq. (3.58) with $\partial/\partial t$and $\hat{w}$ replaced by $i\omega$ and W, respectively) for the vertical-velocity amplitude $W(z)$, and an inhomogeneous equation for the amplitude $Z(z)$ of the vertical vorticity ζ_3 (Eq. (3.59) with the same replacements as above and $\hat{\zeta}$ replaced by Z). These two equations, supplemented by the usual boundary conditions at the walls $z = 0$ and $z = H$, form an eigenvalue problem in which the eigenvalue k_1 appears in powers up to the fourth. This complicates the solution, but the authors showed that this eigenvalue problem can be transformed, with the aid of some manipulations, to a standard linear eigenvalue problem for the system of three differential equations, to which known numerical techniques can be applied. Then the authors discretized the eigenvalue problem in the vertical z-direction, transforming it into an algebraic eigenvalue problem, expanded the velocity field in terms of the eigenvectors found, and truncated this expansion to reduce the number of arithmetic operations needed. Using the appropriate definition of the local "energy density" $E(k_2, \omega; x) = E(x)$ (introduced by Henningson and Schmid (1994)), they reduced the determination of the maximal energy amplification $G(x) = \max_{\boldsymbol{u}(0)} E(x)/E(0)$ (where the maximum is taken with respect to all "initial" velocity disturbances $\boldsymbol{u}(0, y, z, t)$ with given values of k_2 and ω) to the solution of an algebraic variational problem very similar to that studied by Reddy and Henningson (1993). In Fig. 3.15a the results given in both indicated papers for the function $G^*(k_2, \omega) = \max_{x>0} G(k_2, \omega; x)$ at Re = 2000 are presented as contours in the (ω, k_2) -plane. They show that, at this Re, the maximum value of $G^*(k_2, \omega)$, which is close to 100, is reached at $\omega = 0$ (i.e., for a steady disturbance) and $k_2 \approx 2$ (as usual for a plane-Poiseuille primary flow, the wave numbers, frequencies, coordinates, and flow variables are made dimensionless by using the half-depth H_1 and the maximum Poiseuille-flow velocity U_0 as unit length and velocity). In Fig. 3.15b graphs of the function $G(2, 0; x)$ are given for Re = 500, 1000, and 2000. These graphs show that the decrease of Re leads to a significant decrease of $G^* = \max_x G(x)$ but has less effect on the streamwise coordinate $x_{\max}$ where the

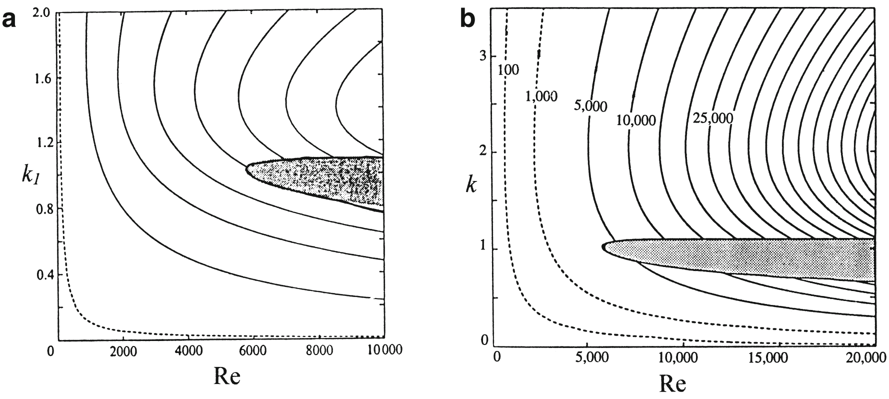


Fig. 3.12 **a** Contours of $G^*(k_1, 0, \mathrm{Re})$ in the (k_1, Re)-plane for plane Poiseuille flow (after Reddy and Henningson (1993)). Dotted line: $G^* = 1$; *solid lines*, from left to right: $G^* = 10, 20, 30, \ldots, 70$; in the shaded region exponentially growing O-S modes exist and hence unbounded energy-growth is possible. **b** Contours of $G^*(k, \mathrm{Re}) = \max_{k_1^2+k_2^2=k^2} G^*(\mathrm{k}_1, k_2, \mathrm{Re})$ in the (k, Re)-plane for plane Poiseuille flow (after S. Reddy, whose results were published in slightly different form by Trefethen et al. (1993), and in the form presented here in the book by Panton (1996)). Two *dotted lines*: $G^* = 100$ and 1000, the left-most solid curves correspond to increments of 5000 in the G^*-vlaues; the shaded region has the same meaning as in Figure **a**

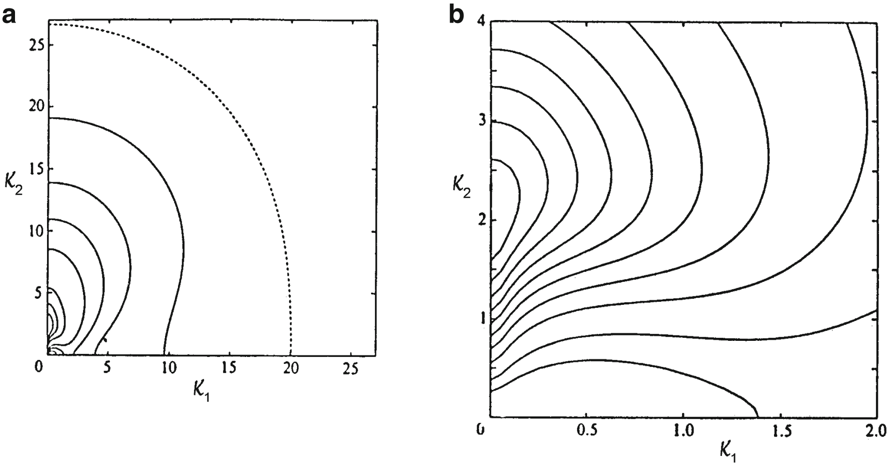


Fig. 3.13 Contours of $G^*(k_1, k_2, \mathrm{Re})$ in the (k_1, k_2)-plane for plane Poiseuille flow with $\mathrm{Re} = 1000$ (after Reddy and Henningson (1993)). **a** *Dotted line*: $G^* = 1$; *solid lines*, from outer to inner: $G^* = 2$, 5, 10, 20, 60, 100, 140. **b** Lower-left part of (**a**); lines from outer to inner correspond to $G^* = 10$, 20, 40, ..., 140, 160, 180

value G^* is reached. The cusps on the curves reflect the dependence of the optimal disturbance on x: at cusp points the shape of the optimal w-disturbance switches from symmetry to antisymmetry with respect to the channel midplane.

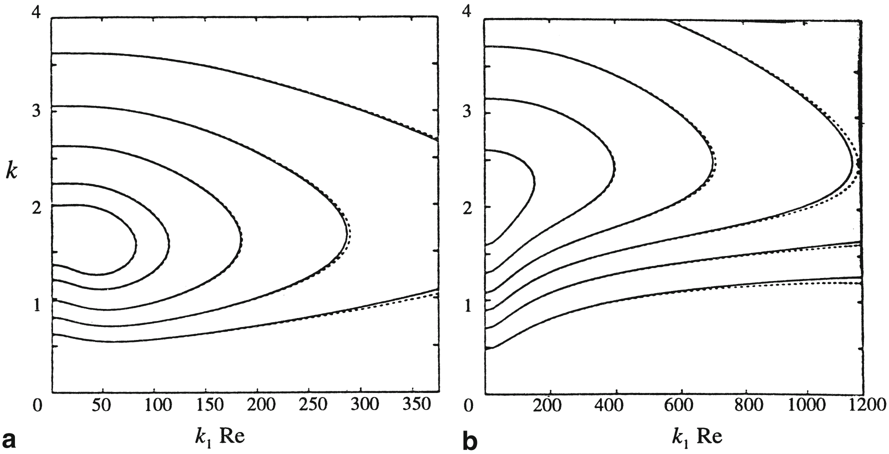


Fig. 3.14 Contours of $k^2G^*(k_1, k_2, \mathrm{Re})/k_2{}^2\mathrm{Re}^2$ in the $(k, k_1\ \mathrm{Re})$-plane for plane Couette and Poiseuille flows (after Reddy and Henningson (1993)). **a** For Couette flows with Re = 1000 (*solid lines*) and Re = 500 (*dotted lines*, sometimes merging with solid ones); the contours, from outer to inner, correspond to values: 0.4, 0.6, 0.8, 1.0, 1.1 ($\times$ 10^{-3}). **b** For Poiseuille flows with Re = 3000 (*solid lines*) and Re = 1500 (*dotted lines*); the contours from outer to inner correspond to values: 0.3, 0.6, 0.9 1.2, 1.5, 1.8 ($\times$ 10^{-4})

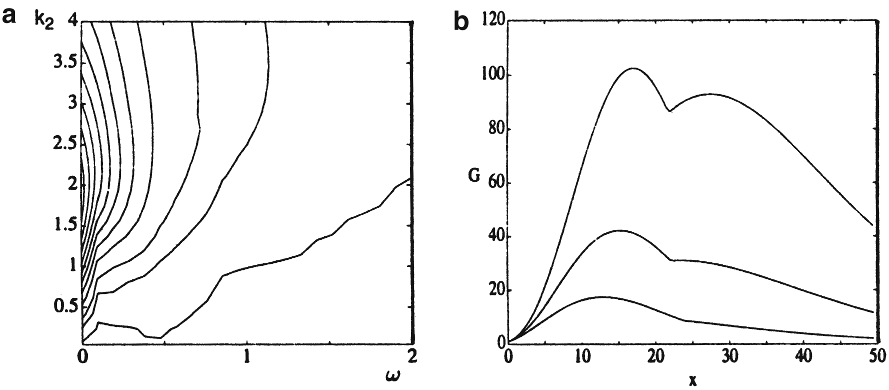


Fig. 3.15 Spatial growth of disturbance energy in plane Poiseuille flows (after Schmid et al. (1994) and Lundbladh et al. (1994)). **a** Level contours of $G^*(k_2, \omega, \mathrm{Re})$ in the (k_2, ω)-plane for Re = 2000. The contours, from outer to inner, corresponds to levels: 5, 10, 20, . . . ,100. **b** Plots of the functions $G(k_2, \omega, \mathrm{Re}; x)$ for $k_2 = 2$, $\omega = 0$, and various values of Re; the curves correspond. (From *bottom* to *top*) to Re = 500, 1000, and 2000

Spatial development of small disturbances in a plane Poiseuille flow was also studied by Criminale et al. (1997), by the method of direct numerical simulation, i.e., by numerical solution, at Re = 5000, of the complete Navier-Stokes (N-S) equations for the disturbed velocity $\boldsymbol{U}(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}, t)$ and the deduced pressure

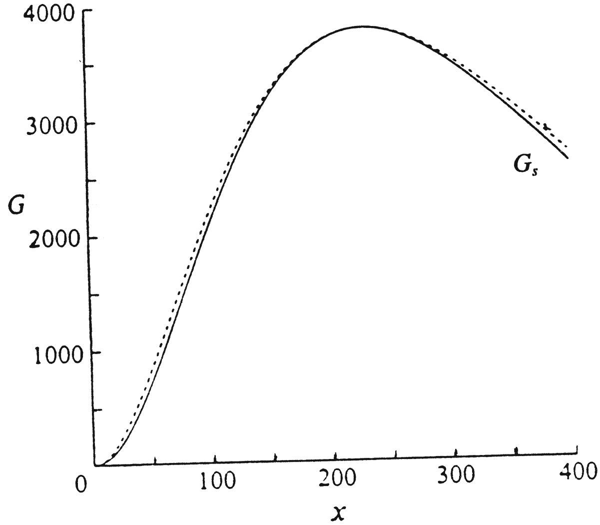


Fig. 3.16 Computed spatial energy-growth curve $G_s(x)$ for disturbance with near-optimal (with respect to temporal growth) initial conditions in plane Poiseuille flow with Re = 5000. The dashed line represents the temporal energy-growth curve $G(t)$ computed for the same Re and initial values, and rescaled in both coordinates to make the maxima of spatial and temporal growth curves equal and corresponding to the same abscissa. (After Criminale et al. (1997))

$[P(\boldsymbol{x}) + p(\boldsymbol{x}, t)]/\rho$. Here ρ is the density, $\boldsymbol{U}(\boldsymbol{x}) = \{1 - z^2, 0, 0\}, -1 \leq z \leq 1$, and $P(\boldsymbol{x})/\rho = P_o/\rho - 2\boldsymbol{x}/\mathrm{Re}$ are Poiseuille-flow velocity and deduced pressure, while $\boldsymbol{u}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ are velocity and pressure fluctuations of a small disturbance (all variables are assumed to be non-dimensionalized). The boundary conditions are the spatially-initial ("inflow") condition $\boldsymbol{u}(0, y, z, t)$ at $x = 0$ and the usual solid-surface conditions at $z = \pm 1$. The authors chose time-independent (steady) inflow conditions: $\zeta_3(0, y, z) = 0$ and $w(0, y, z) = A_o(1 - z^2)^2 \cos 2y$. This model $W_o^{(1)}(z)$ of the vertical-velocity amplitude was used in this paper when the temporal disturbance development was studied, with wavenumber values $k_1 = 0$, $k_2 = 2$. These wavenumbers correspond to temporal disturbance growth close to the maximum possible at Re = 5000 (see Table 3.1 and Fig. 3.10a above). From the continuity equation, the horizontal velocity components at $x = 0$ are $u(0, y, z) = 0$, $v(0, y, z) = 2A_o z(1 - z^2)^2 \sin 2y$. The initial amplitude A_o was taken to be 10^{-7}; this small value was chosen to guarantee that the growing disturbance remains small so that the results agree with those implied by linearized dynamic equations. The computed spatial growth curve $G_s(x) = E(x)/E(0)$ for the given inflow conditions is shown in Fig. 3.16. This growth curve has a shape similar to those shown in Fig. 3.10a for temporal growths of some fixed disturbances in Poiseuille flow at the same Reynolds number. The $G_s(x)$-curve has a much higher peak, and is considerably smoother, than the upper $G(x)$-curve in Fig. 3.15b, which corresponds to a lower Reynolds number Re = 2000 and concerns not a fixed disturbance but a family of disturbances depending on x (and is in fact composed of two separate growth curves, for symmetric and antisymmetric w-disturbances). To illustrate graphically the similarity between spatial and temporal growth curves, Criminale et al. also computed the temporal growth curve $G(t) = E(t)/E(0)$ in Poiseuille flow at Re = 5000, for the disturbance having an initial velocity $\boldsymbol{u}(x, y, z, 0)$ equal to the inflow velocity

$\boldsymbol{u}(0, y, z, t)$ defined above (this was possible since this $\boldsymbol{u}(0, y, z, t)$ was chosen to be independent of t). This new growth curve was then rescaled; abscissa t was replaced by $x/0.561$, and ordinate G by $G_T = 0.831G$ (the factors were chosen to make the maxima of the spatial and temporal growth curves $G_s(x)$ and $G_T(x)$ coincide in magnitude and location in x). The function $G_T(x)$ obtained is also shown in Fig. 3.16; as can be seen, it differs only insignificantly from $G_s(x)$.

Farrell and Ioannou (1993b) studied the optimally growing temporally-evolving three-dimensional disturbances of infinite plane. Couette flow having constant shear $U'(z)=b$ in the unbounded space—$\infty < z < \infty$. For such a flow the O-S equation has no discrete eigenvalues and hence the O-S spectrum is purely continuous. However, here the investigation of the disturbance development is simplified by the existence in this case of a complete set of analytic solutions which are orthogonal (in the inner products corresponding to both the L_2 and energy norm) and which have the form of plane waves, with constant horizontal wave numbers k_1 and k_2 and vertical wave number k_3 depending lineary on time. (Recall that such solutions were in fact first found by Kelvin (1887a) and Orr (1907) but then were forgotten for a long time; see Sect. 3.1, and particularly Eqs. (3.1) and (3.2) where on the right-hand sides the last term in (3.1) and the last two terms in (3.2) must be omitted in the case of infinite Couette flow without walls.)

The initial value problem for disturbances in an infinite couette flow was solved in full generality (with inclusion of an external force and a mass source affecting the flow) by Criminale and Smith (1994). However, here the imposed generality of the problem statement and the search for fundamental solutions and universally applicable generalized Green's function led to rather complicated equations, and restricted the authors mainly to the inviscid solution (providing the correct leading term in cases where the time t is not too long and the disturbance length scale l is not too small). In contrast to this, Farrell and Ioannou considered only some simple particular solutions where the initial values $w(\boldsymbol{x},0)$ and $\zeta_3(\boldsymbol{x}, 0)$ both have plane-wave or checker-board shape (i.e., are proportional either to $\exp\{i(k_1x + k_2y + k_3z)\}$ or to $\cos(k_1x)\cos(k_2y)\cos(k_3z)$) If the optimal growth is interpreted as the maximal energy-density growth $G(t) = E(t)/E(0)$ attainable at some time t by appropriate choice of the values for W_0, Z_0, k_1, k_2, k_3 (i.e., for amplitudes of initial plane-wave or checkerboard structures and for three wave-number components), then the optimal values of $G(t)$ will be the same for viscous and inviscid fluid and will increase infinitely with time t. (This is so since the influence of viscosity can be arbitrarily decreased by a large enough increase of disturbance length scale l determined by the wave numbers k_i.) Therefore, Farrell and Ioannou (1993b) first computed the maximal growth $G(T_{\text{opt}})$ attainable in an inviscid fluid in a specific non-dimensional time $T_{\text{opt}} = (bt)_{\text{opt}}$ (cf. the discussion of the related paper by Farrell and Ioannou (1993a) in Sect. 3.23). The values of $G(T_{\text{opt}})$ were found to be different for single-plane-wave and checkerboard initial conditions. However, for large values of T_{opt}, of the order of several tens or a hundred, they usually take high values of the order of 10^3 or even 10^4.

To take into account the effect of viscosity, it was enough to consider only the initial conditions constrained to have a fixed value of k_1. Then Re may be defined as $bl^2/\nu =$

$bk_1^2/\pi^2\nu$, the optimal values of k_2/k_1 and k_3/k_1 may be uniquely determined for any given value of T_{opt}, and the value of $G(T_{\text{opt}})$ will be a function of Re. For example, if $k_1 = 1$, then for Re = 100, 1000, 10000 and checkerboard initial conditions, it was found that maximum growth $G(T_{\text{opt}})$ is achieved at $T_{\text{opt}} = 7$, 15, 30, and is equal to 12.5, 109, 707, respectively. It was also noted that in real flows the ambient turbulent fluctuations usually interfere with the regular growth of a velocity disturbance by disrupting the corresponding flow structures; therefore, the results for large values of T_{opt} are often of no physical meaning. Bearing this in mind, Farrell and Ioannou presented a contour plot of $G(T)$ in the (k_1, k_2)-plane for Re = 1000 and checkerboard initial conditions, with a moderate value of $T_{\text{opt}} = 10$. It was found that in this case $G_{\max}(10) \approx 115$, which can be reached for practically all $k < 1$ if $\tan^{-1}(k_2/k_1) \approx 63°$. In another paper, Farrell and Ioannou (1993c) showed that the contour plot of $G_{\max}(T)$ for $T = 10$ in their paper (1993b), is similar in many respects to contours of $G_{\max}$ for plane Couette and Poiseuille flows in channels of finite height. This similarity reflects the similarity in the flow structures responsible for maximal growth of disturbance energy in the three flows considered.

Practically all specific computations of transient disturbance growth considered above in this subsection concerned horizontally unbounded disturbances (in most cases plane waves with given horizontal wave numbers, but also unbounded checkerboard structures). However, real disturbances appearing in natural, engineering, and laboratory flows of fluids are most oft en due to some factors affecting only a finite fluid volume and producing a localized initial disturbance. Note that in Sects. 3.22 and 3.23 much attention was given to papers by Henningson (1988), Breuer and Haritonidis (1990), and Landahl (1980, 1990, 1993a, 1996) devoted to study of evolution of localized disturbances in an inviscid fluid, while in this section the papers by Henningson (1991) and Henningson et al. (1993) on the same evolution in viscous fluids have been mentioned already. Now it is the time to dwell at somewhat greater length on the results of the last-mentioned paper.

The major part of this paper was devoted to numerical investigation of the development of a localized disturbance in a plane Poiseuille flow. The case of a disturbance initially consisting of two pairs of counter-rotating vortices (see Fig. 3.2) filling the height of the channel and symmetric with respect to its midplane was investigated in most detail. (Recall that this is just the case that was earlier studied by Henningson (1988) and (with a change of the z-dependence to one appropriate for the boundary-layer flow) by Russell and Landahl (1984); and Breuer and Haritonidis (1990); and was also considered by Henningson et al. (1990); and Henningson (1991)). However, in the paper by Henningson et al. (1993) an initial disturbance consisting of the same two pairs of eddies, but rotated around the vertical z-axis by some angle θ, was also considered (such rotation clearly changes the wave-number composition of the initial disturbance). Moreover, to investigate the sensitivity of the results obtained to changes in the primary flow or initial disturbance, the authors also considered a quite different initial disturbance shape (used only in the study of nonlinear effects which will not be discussed here) and added some remarks on comparison with the related boundary-layer results (corresponding to a disturbance having similar general shape but different z-dependence). Developing further the approach by Henningson

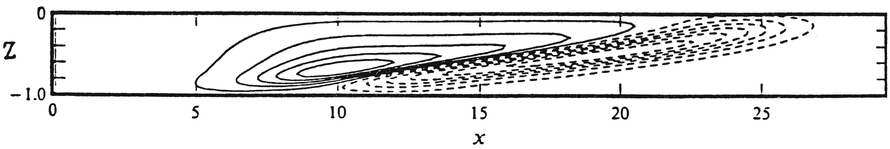


Fig. 3.17 Contours in the (*x, z*)-plane of streamwise velocity $u(x, y, z, t)$ at $y = 0$ and $t = 25$ of a localized disturbance, sketched in Fig. 3.2, in plane Poiseuille flow with Re = 3000. *Solid* and *dashed lines* correspond to positive and negative *u*-values. (After Henningson et al. (1993))

(1991), Henningson et al. (1993) computed the time evolution of the disturbances by two different methods. The first was the method of direct numerical simulation (DNS), i.e., of numerical solution of the N-S equations for the case of small enough amplitude of the initial disturbance. The second method used a truncated eigenfunction expansion of the Fourier-transformed initial data $\{\hat{w}(k_1, k_2; z, 0), \hat{\zeta}(k_1, k_2; z, 0)\}$ (cf. The description above of the paper by Reddy and Henningson (1993)) combined with the subsequent application of Eq. (3.15) and inverse Fourier transformation of the resulting Fourier components of velocity and vorticity.

The DNS results given by Henningson et al. (1993) for Re = 3000 show strong initial growth of the streamwise velocity component during a few tens of time units (as usual, length and time units are set equal to H_1 and H_1/U_0) with formation of the familiar wave-packet structure at the rear (later time) of the disturbance and the streaky structure at the front (see also Fig. 3.3 and the experimental data by Klingmann and Alfredsson (1991) and Klingmann (1992), revealing similar features). According to these results, the streamwise-velocity amplitude quickly becomes about one order of magnitude larger than the amplitude of the vertical velocity, although the streamwise velocity at $t = 0$ was zero. The contours of constant streamwise velocity in the (*x, z*) plane, shown in Fig. 3.17 for the lower half of the channel, clearly demonstrate the stretching of the disturbance structure in the streamwise direction (the initial horizontal spread of the disturbance was close to two units), with generation of alternating high-speed and low-speed regions and formation of inclined shear layers between them (cf. again Fig. 3.3b). Figure. 3.18a shows the dependence of the disturbance energy on time for a localized initial disturbance similar to that shown in Fig. 3.2, and for the same disturbance rotated by the angle $\theta = 10°$, $20°$ and $45°$. (This figure was based on the DNS data but the eigenfunction-expansion computations gave practically the same results.) We see that some initial energy growth is present at any θ, but for $\theta = 0$ it is significant only at small values of t and is quickly replaced by decay, while increasing the value of θ increases the energy growth. (This is because rotation of the disturbance in the physical space rotates the wavenumber spectrum in the (k_1, k_2)-plane and increases the contribution of the region of small values of k_1. Recall also Landahl's result of 1990, illustrated with Fig. 3.4, that breaking of the symmetry of the initial disturbance with respect to a plane $y = \text{const.}$ increases the transient growth and leads to generation of streaky structures corresponding to small values of k_1.) Computation of the percentage contribution to disturbance energy at different values of t from three components of velocity, and from the vertical vorticity as such, showed that at $\theta = 0$ the percentage contributions of both the vertical and spanwise velocities w and v decrease monotonically with time, while the contributions of u and ζ_3 increase monotonically and at $t \geq 15$ the contribution of ξ_3 fully dominates all the others; essentially the same results are valid also for the cases where $\theta \neq 0$. These results agree well with the assumption that forcing of the vertical vorticity by the vertical velocity plays the main part in the transient growth of small disturbances. Similar results were obtained by Henningson et al. for the development of disturbances of the type shown in Fig. 3.2 in a Blasius boundary layer (see Fig. 3.18b, where some results for a boundary layer with $\text{Re}\,\delta^* = 1000$ are presented; time is measured here in δ^*/U_0 units).

The study of the transient growth of disturbances in plane parallel flows will be continued in Sect. 3.4. However, now we will passs to consideration of the very important case of circular Poiseuille flow (flow in a round tube).

3.3.4 Transient Growth of Small Disturbances in the Poiseuille Flow in a Round Tube

In Chap. 2, in the very beginning of Sect. 2.9.4, it was pointed out that the problem of stability of Poiseuille flow in a round tube is apparently the most challenging and mysterious problem in the theory of hydrodynamic stability. It was also noted there that this problem has been studied repeatedly by a number of first-rate scientists but nevertheless is still far from being solved. However, in Sect. 2.94 only applications of the classical normal-mode method to this problem were considered. Now we will consider studies of transient disturbance growth in circular Poiseuille flow in a tube, which represent a natural extension of the investigations described in the previous subsection.

Just as in the case of plane-parallel flows, the first attempt to consider this topic was made by Orr (1907). He could not overcome the analytical difficulties appearing in the general case and was forced to simplify the statement of the problem. Therefore, in Chap. II of Part I of his paper, he confined himself to the study of a flow of inviscid fluid in a tube of radius R, where a disturbance which was small (*i.e.*, satisfying the linearized equations) and axisymmetric, having the velocity $\boldsymbol{u}(\boldsymbol{x}, t) = \{u_r(r,\ x,\ t),\ 0, u_x(r,\ x,\ t)\}$ (*i.e.*, no circumferential component u_ϕ) was superimposed on the primary steady axisymmetric Poiseuille flow with the velocity $\boldsymbol{U}(\boldsymbol{x}) = \{U(r),\ 0,\ 0\},\ U(r) = A(R^2 - r^2)$. As was indicated in the concluding part of Sect. 3.1, Orr considered solutions of the initial-value problem for the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ corresponding to several initial values $u_r(r,\ x,\ 0)$ of the radial velocity u_r (since $u_\phi = 0$, the axial velocity u_x can be easily determined from values of u_r with the help of the equation of continuity given in Sect. 2.84). He found a class of initial conditions (indicated in Sec 3.1) which for some values of the parameters implies a very great increase of the disturbance kinetic energy with time before the ultimate decay as $t \to \infty$. Therefore, he concluded that the circular Poiseuille flow of an inviscid fluid is *practically unstable* (cf. again Sect. 3.1). In the beginning of

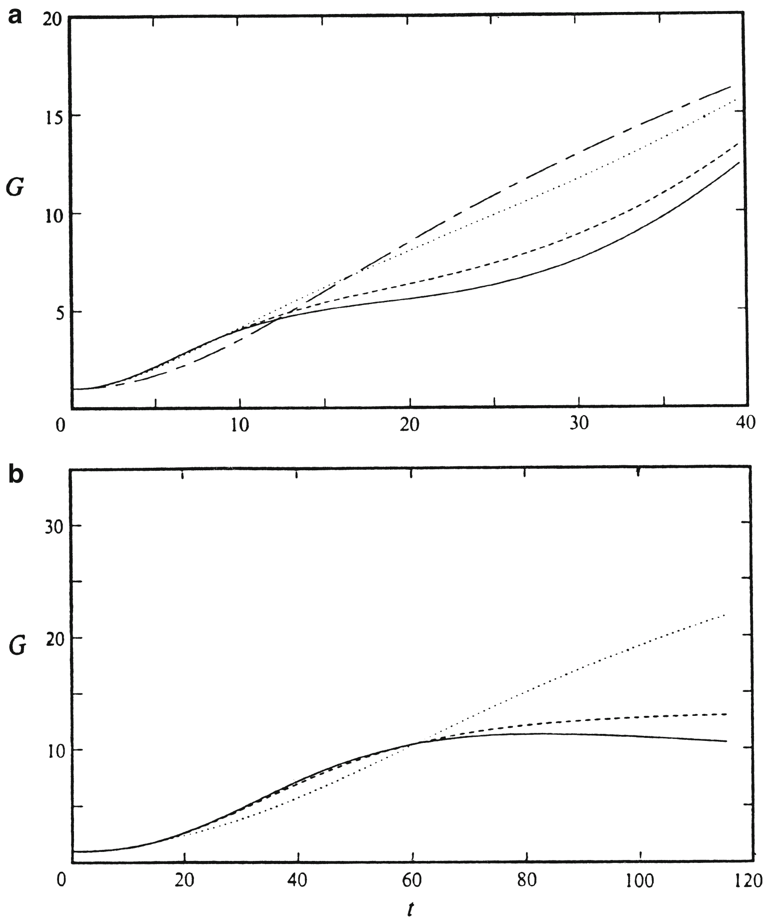


Fig. 3.18 **a** Energy-growth function $G(t) = E(t)/E(0)$ for some localized disturbances in plane Poiseuille flow with Re = 3000. Each disturbance either had the initial shape sketched in Fig. 3.2 (*solid line*) or its initial shape was obtained from that in Fig. 3.2 by rotation around the z-axis by the angle $\theta = 10°$, 20°, or 45° (*dashed*, *dotted*, and *chain-dashed lines*, respectively). **b** Energy-growth function $G(t)$ for localized disturbances in the boundary-layer flow with $\mathrm{Re} = U_0\,\delta^*/\nu = 1000$ having an initial shape either similar to that shown in Fig. 3.2 (*solid line*) or obtained from it by rotation by the angle $\theta = 10°$ or 45° (*dashed* and *dotted lines*, respectively); time is measured in δ^*/U_0 units. (After Henningson et al. (1993))

Part II of his paper Orr also noted that although he could not obtain similar results for viscous flow in a circular tube, he considered a proof of instability of the inviscid flow as also a proof of instability for the case where viscosity is not exactly zero but is small enough. However, these results by Orr had been completely forgotten when, about 80 years later, the subject was investigated anew by several scientists.

The first new study of the circular-Poiseuille-flow instability by a method going beyond the limits of the traditional normal-mode approach was due to Boberg and Brosa (1988). They tried to develop a nonlinear theory describing all stages of transition of a tube flow to turbulence. However, a very important part was played in their investigation by a study of a purely linear mechanism of extracting the kinetic energy from the mean flow, by transiently-growing small nonaxisymmetric disturbances with azimuthal wave number $n = \pm 1$. Boberg and Brosa carried out a numerical simulation of the corresponding process, based on the expansion of the disturbed velocity field into the so-called "Stokes functions" which are in fact the eigenfunctions of the N-S equations corresponding to very slow motion of fluid. Their severe truncations of the expansions used made many of their results only qualitatively relevant to the real onset of turbulence in a tube, although some important features of real disturbance development were described with satisfactory accuracy. The paper by Boberg and Brosa was also quite important as the trigger for many subsequent investigations of instability and transition by non-standard methods.

Slightly later, the paper by Gustavsson (1989) was entirely devoted to transient disturbance growth in circular Poiseuille flow. In this paper the important role of nonaxisymmetric disturbances in the disturbance-growth mechanism was revealed, and the equation which describes the forcing of the streamwise velocity of a disturbance by its pressure was derived. However, Gustavsson gave his main attention to the search for resonances between disturbance pressure and streamwise velocity, assuming that such resonances can make substantial contribution to algebraic growth of disturbances. This was the reason why his paper was considered in Sect. 3.32, devoted to resonances and degeneracies. However, later it became clear that resonance contribution does not play a substantial part in the disturbance growth (see the discussion of this topic in Sect. 3.33). Therefore, Gustavsson's paper of 1989 cannot now be considered as of great importance.

Later Gustavsson's student Bergström (1992) investigated the development of small nonaxisymmetric, x-independent, disturbances in Poiseuille tube flow. This work was stimulated by results by Ellingsen and Palm (1975) and Hultgren and Gustavsson (1981) who found that x-independent disturbances often grow algebraically with time in plane-parallel flows (see Sects. 3.22 and 3.32 above). It was also taken into account that the absence of x-dependence considerably simplified the stability analysis.

In fact, neglecting all x-derivatives in Eq. (2.73) (and hence assuming that the normal modes of a disturbance are proportional to $e^{i(n\phi-\omega t)}$) it is easy to obtain a simple fourth-order differential equation (with coefficients depending on n and $\mathrm{Re} = U_o R/\nu$, where U_o is the centerline velocity) for the amplitude $f^{(r)}(r)$ of the radial-velocity normal mode. This equation, with the physically-obvious boundary conditions for $f^{(\mathrm{r})}(r)$ at $r = 0$ and $r = R$, forms the radial eigenvalue problem, which was found to be easily solvable in terms of Bessel functions. In particular, it was shown that the dimensionless eigenvalues ω_j, $j = 1, 2, \ldots$ (below, all independent and dependent variables will be non-dimensionlized by appropriate combinations of R and U_0) are given by the equation $\omega_j = -i\hat{J}^2_{n+1,j}(\mathrm{Re})^{-1}$ where $\hat{J}_{n+1,j}$ is the j^{th} zero of the Bessel function J_{n+1}. Since all ω_j have negative imaginary and vanishing real parts, the normal modes derived are non-oscillating and damped. When u_r is known, the azimuthal velocity u_ϕ can be found from the continuity equation (the last of Eqs. (2.73)); in particular, $u_\phi = (i/n)\partial(ru_r)/\partial r$ for velocity disturbances with azimuthal wave number n. Proceeding further, one may obtain, for the amplitude $f^{(x)}(r)$ of the streamwise-velocity mode, a second-order inhomogeneous differential equation with the term $\mathrm{Re}U'f^{(r)}(r)$ on the right. This term describes forcing of x-independent stream-wise velocity disturbances by a radial velocity disturbance $u_r(r, \phi, t)$. Below, most attention will be given to development of the streamwise velocity $u_x(r, \phi, t)$ induced by a normal mode of the radial-velocity field.

Bergström fixed the value of $n \geq 1$ of the azimuthal wave number, assuming that $u_x(r, \phi, t) = u_x(r, t)\, e^{in\phi}$ and that $u_r(r, \phi, t) = f_j^{(r)}(r)e^{i(n\phi - \omega_j t)}$ is represented by a single (j^{th}) radial-velocity normal mode corresponding to this wave number. Moreover, he also accepted the initial condition $u_x(r, \phi, 0) = 0$, thus considering only the component of the streamwise velocity u_x which is induced by the radial-velocity disturbance. He showed that it is possible to find an explicit expression for the Laplace transform

$$\hat{u}_x(p, r) = \int_0^\infty e^{-pt} u_x(r, t)dt$$

of the function $u_x(r, t)$ and then to determine this function itself by inverse Laplace transformation. Normalizing the initial radial-velocity disturbance $u_r(r, \phi, 0)$ by the condition that $E_r(0) = 1$ (where the "radial kinetic energy" $E_r(t)$ is defined as the integral of $(u_r(r, \phi, t))^2/2$ over the tube cross-section), Bergström calculated the values of the "streamwise kinetic energy" $E_x(t)$ (defined similarly to $E_r(t)$) for a number of values for n, j, Re, and t (where t is measured in the R/U_0 units). Since it was shown that $E_x(t)/(\mathrm{Re})^2$ depends only on t/Re but not on t and Re separately, only the dependence of $E_x(t)/(\mathrm{Re})^2$ on n, j, and $t/\mathrm{Re} = T$ had to be determined.

According to the above results, $E_r(t) = \exp(-|\omega_j|t) = \exp(-\hat{J}^2_{n+1,j}T)$ falls off exponentially with t. Since $u_\phi \propto \partial(ru_r)/\partial r$, it follows that the "azimuthal kinetic energy" $E_\phi(t)$ (the integrated valued of $u_\phi^2/2$) also decays as $\exp(-|\omega_j|t)$. However, the streamwise kinetic energy $E_x(t)$ behaves differently. $E_x(0) = 0$ since $u_x(r, \phi, 0) = 0$, but when t increases, $E_x(t)/(\mathrm{Re})^2$ at first grows algebraically with time, reaches a maximum value $E_0 = E_0(n, j)$ at some value $T_0(n, j)$ of t/Re, and then begins to decay (asymptotically at the same rate as the energies $E_r(t)$ and $E_\phi(t)$). Results of computations (partially presented in Fig. 3.19) clearly showed that the disturbance component with $n = 1$ and $j = 1$ dominates the total growth of kinetic energy in the initial stage of disturbance development. Note that since the maximal values of $E_x(t)$ is proportional to $(\mathrm{Re})^2$, it increases rapidly if Re increases (*i.e.*, ν decreases). According to Bergström's computations, if $\mathrm{Re} = 1000$ and $E_r(0) = 1$, then $\max_{t>0} E_x(t) \approx 167$ for $n = 1, j = 1$, but this maximum value decreases quickly with n and even quicker with j (see Fig. 3.19).

Subsequent computations of disturbance development in Poiseuille tube flow, relating to the general case of disturbances depending on three coordinates x, r, and ϕ, were carried out by Bergström (1993a), Schmid and Henningson (1994) and O'Sullivan and Breuer (1994). In this case normal modes are proportional

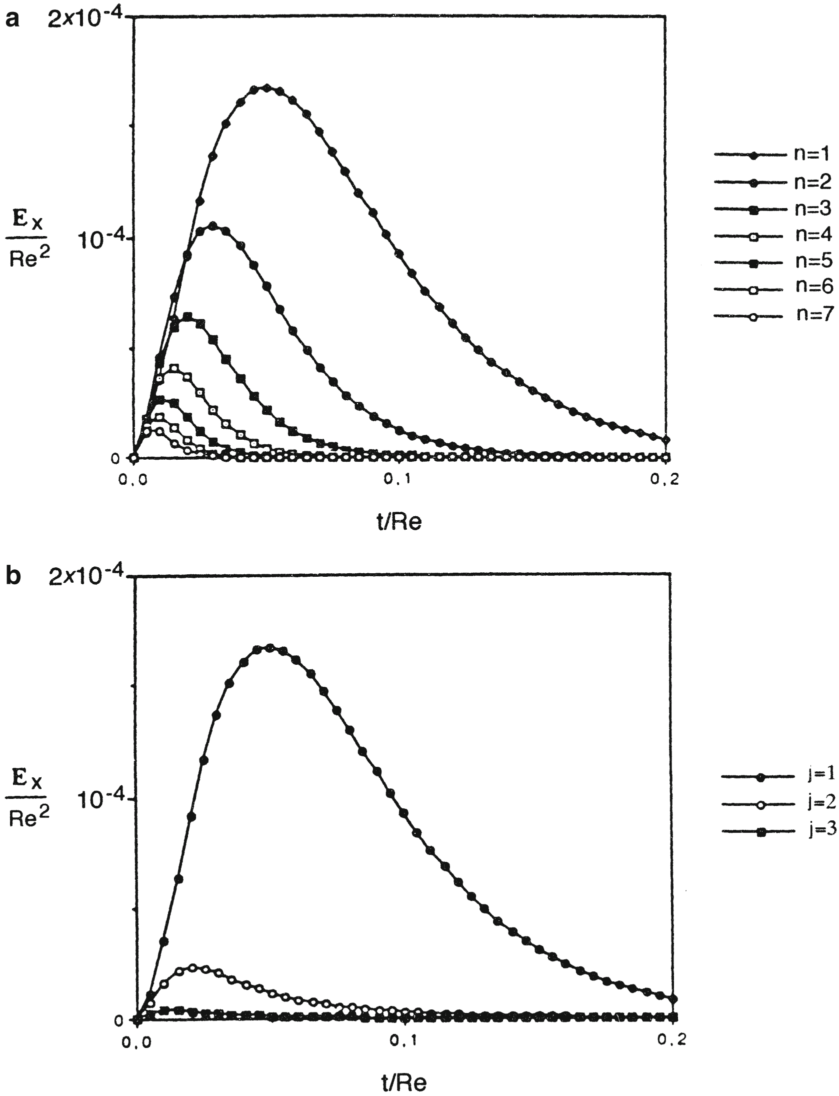


Fig. 3.19 The normalized energy-growth function $E_x(t)/(\mathrm{Re})^2$ for the streamwise kinetic energy $E_x(t)$ of an x-independent disturbance in circular Poiseuille flow induced by the radial-velocity disturbance represented by the jth normal mode with azimuthal wave number n having initial kinetic energy $E_r = 1$, for **a** $j = 1$, $n = 1, 2, \ldots, 7$, and **b** $n = 1$, $j = 1$, 2 or 3. (After Bergström (1992))

to $e^{i(kx+n\phi-\omega t)} = e^{i[k(x-ct)+n\phi]}$; therefore, the wave number k also appears, and frequency ω can be replaced by phase velocity $c = \omega/k$, Bergström (1993a), like Gustavsson (1989), began his investigation with the derivation, from dynamic equations (2.73), of a sixth-order homogeneous differential equation for the pressure-mode amplitude $g\ (r)$. This equation and the boundary conditions appropriate to it form the pressure-field eigenvalue problem, with coefficients depending on k, n and Re. This problem determines the set of eigenvalues ω_j or c_j, where $j = 1, 2, \ldots$ However, contrary to Gustavsson's approach, Bergström did not supplement the equation for $g(r)$ by the inhomogeneous differential equation (3.55) for the streamwise amplitude $f^{(x)}(r)$. Instead of this he showed that when $g(r)$ and c are known, $f^{(x)}(r)$ and the radial-velocity amplitude $f^{(r)}(r)$ can be determined with the help of equations of the form

$$f^{(r)}(r) = \mathrm{L}g(r), f^{(x)}(r) = Mf^{(r)}(r) + Ng(r), \tag{3.67}$$

where L, M, and N are three linear differential operators (with coefficients depending on k, n, Re and c), while $f^{(\phi)}(r)$ can be determined from values of $f^{(r)}(r)$ and $f^{(x)}(r)$ with the help of the equation of continuity. Therefore, according to his analysis, solution of the pressure-field eigenvalue problem allows the normal modes of all velocity components to be found as well.

Bergström solved the pressure-field eigenvalue problem numerically and found a number of complex eigenvalues c_j and eigen-function $g_j(r)$. Using Eqs. (3.67) and (2.74) he also evaluated the amplitudes $f_j^{(x)}(r)$, $f_j^{(\phi)}(r)$, and $f_j^{(r)}(r)$ of normal velocity modes. Then he considered disturbances represented by finite linear combinations of normal modes

$$\mathbf{u}(x, r, \phi, t) = \sum_{j=1}^{N} A_j \mathbf{u}_j(r) e^{\iota[k(x - c_j t) + n\phi]}, \mathbf{u}_j = \left\{ f_j^{(x)}, f_j^{(r)}, f_j^{(\phi)} \right\}. \tag{3.68}$$

The kinetic energy density $E\ (k, n; t) = E(t)$ of such a disturbance is given by a certain positive-definite quadratic form of coefficient A_j, and hence the method of Butler and Farrell (1992) can now be applied to determination of the maximum possible value of the energy growth $G(t) = \max_{\mathbf{A}} E(t)/E(0)$, where $\mathbf{A} = \{A_1, \ldots, A_N\}$ determines the shape of the initial velocity field $\mathbf{u}(\boldsymbol{x}, 0)$. The value of $G(t)$ depends on k, n and Re, and can also depend on the number N of normal modes considered, and the selection of these modes. However, if the modes are numbered in order of decreasing imaginary parts of the eigenvalues c_j (*i.e.*, in order of increasing mode-damping), the computational results show that an increase of N above some not-too-high lmiting values leaves the value of $G(t)$ practically unchanged. Therefore, the dependence on N is immaterial, if N is not too small. The dependence of $G(t)$ on t is shown in Fig. 3.20a for Re $= 1000$, $n = 1$, and several values of k. This figure shows that the maximal possible growth $G^* = \max_{t>0} G(t)$ increases with decreasing k (*i.e.*, here again the greatest growth occurs for streamwise-elongated disturbances), although at small values of time the disturbances with smaller streamwise wavelength can grow faster than strongly-elongated structures. Note that even at $k = 0.01$ (wavelength $l \approx 600R$) the largest energy amplification $G^* = 72.4$ is considerably smaller than the value $\max_{t>0} E_x(t)/E_r(0) \approx 167$ found by Bergström in 1992 for a non-optimal

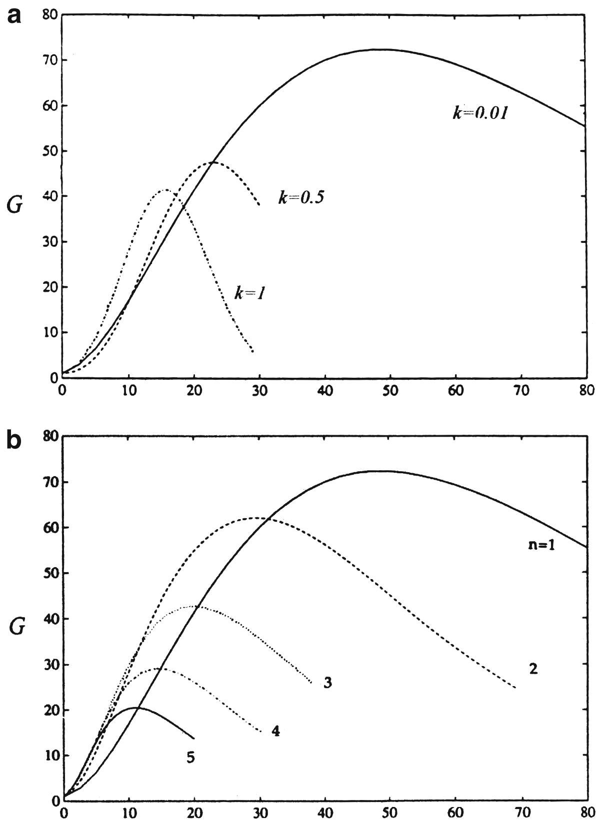


Fig. 3.20 The growth functions $G(t)$ for optimally-growing disturbances in circular Poiseuille flow with $\mathrm{Re} = U_0 R/\nu = 1000$ having wave numbers **a** $n = 1$, $k = 0.01$, 0.5, 1; and **b** $n = 1, 2, \ldots, 5$, $k = 0.01$. (After Bergström (1993a))

x-independnet disturbance with $k = 0$ (see Fig. 3.19). Dependence of $G(t)$ on n is shown in Fig. 3.20b for the case where $\mathrm{Re} = 1000$ and $k = 0.01$; according to this figure, $G^* = \max_t >_0 G(t)$, at small k, decreases with increase of n, but at small values of t the disturbances with greater azimuthal wave numbers can grow faster than that with $n = 1$. (At larger values of k the disturbances with $n \geq 1$ can sometimes grow more than that with $n = 1$; see Fig. 3.21 below.) It was also shown by Bergström that the energy of the streamwise velocity component usually dominates the growing energy, especially for streamwise-elongated disturbances with small values of k.

Schmid and Henningson (1994) and O'Sullivan and Breuer (1994) employed a form of the linearized dynamic equations which differs from that used by Gustavsson and Bergström. In both these 1994 papers, the authors followed Bergström's paper of 1993 by considering the normal modes with fixed wavenumbers k and n and unknown frequency ω or phase velocity $c = \omega/k$. However, in these papers the four equations (2.73), with the unknown functions u_x, u_r, u_ϕ and p, were replaced by two equations, for the radial velocity u_r and the radial vorticity $\zeta_r = \partial u_\phi/\partial x - \partial u_x/\partial r\phi$, as was suggested long ago by Burridge and Drazin (1969). The definition of the vertical vorticity and the equation of continuity make it easy to express the streamwise and azimuthal velcotiy components u_x and u_ϕ in terms of u_r and ζ_r; in particular, for disturbances proportional to $e^{i(kx+n\phi)}$ we obtain

$$u_x = \frac{ik}{rK^2}\frac{\partial(ru_r)}{\partial r} + \frac{in}{rK^2}\zeta_r, \quad u_\phi = \frac{in}{r^2K^2}\frac{\partial(ru_r)}{\partial r} - \frac{ik}{K^2}\zeta_r \tag{3.69}$$

where $K^2 = k^2 + n^2/r^2$. Equations (3.69) are similar to Eqs. (3.15), which express horizontal velocity components of a normal-mode disturbance in a plane-parallel flow in terms of vertical velocity and vorticity; thus, these new equations may be considered as representing the form taken by Eqs. (3.15) in cylindrical coordinates. Burridge and Drazin (1969) showed that for disturbances proporiotnal to $e^{i(kx+n\phi)}$ the linearized dynamic equations (2.73) can be reduced to the following system of two equations for functions $-iru_r = \phi$ and $i\zeta_r/K^2r = \psi$ associated with u_r and ζ_r:

$$\left[\left(\frac{\partial}{\partial t} + ikU\right)\Xi - ikK^2r\left(\frac{U'}{K^2r}\right)'\right]\phi = \frac{1}{\mathrm{Re}}[\Xi^2\phi - 2kn\Xi\psi],$$
$$\left(\frac{\partial}{\partial t} + ikU\right)\psi - i\frac{nU'}{K^2r^3}\phi = \frac{1}{\mathrm{Re}}\left[\Theta\psi + \frac{2kn}{K^4r^4}\Xi\phi\right] \tag{3.70}$$

Here K has the meaning indicated above, a prime denotes differentiation with respect to r, and Ξ and Θ are the following second-order differential operators:

$$\Xi = \frac{1}{r^2} - \frac{1}{r}\frac{d}{dr}\left(\frac{1}{K^2r}\frac{d}{dr}\right), \Theta = K^4r^2 - \frac{1}{r}\frac{d}{dr}\left(K^2r^3\frac{d}{dr}\right). \tag{3.70$'$}$$

In Eq. (3.70) it may be assumed that the unknown functions $\phi = \phi\ (r,\ t)$ and $\psi = \psi\ (r, t)$ depend only on r and t, and represent "amplitudes" preceding the factor $e^{i(kx+n\phi)}$. Assuming now that the disturbance depends exponentially on time, one may replace $\partial/\partial t$ by $-i\omega$ (or $-ikc$) and consider only the stationary amplitudes $\phi(r)$ and $\psi(r)$. We now supplement Eqs. (3.70), (3.70′) by the boundary conditions at $r = 1$ and $r = 0$. At $r = 1, \phi = \phi' = \psi = 0$, while at $r = 0$ we have $\phi = \phi' = 0$ if $n = 0$; or $\phi = \psi = 0$, with ϕ' finite, if $n = \pm 1$; or $\phi = \phi' = \psi = 0$ if $n \geq 2$. We thus obtain the general vector eigenvalue problem of the linear theory of tube-flow stability (see, e.g., Schmid and Henningson (1994)). This problem determines the sets of eigenvalues ω_j (or c_j) and eigenfunctions $\phi_j,\ \psi_j$ allowing the normal modes of all velocity components to be found.

Schmid and Henningson (1994) solved the eigenvalue problem numerically, and found a great number of eigenvalues ω_j and eigenfunctions f_j and ψ_j corresponding to various combinations of wavenumbers k and n. Following Reddy and Henningson, they numbered the eigensolutions in order of increasing mode-damping and then, fixing the values of k and n, expanded vector $\mathbf{q}(r,t) = \{\phi\ (r,\ t),\ \psi\ (r,\ t)\} = \{-iru_r(r,\ t), i\zeta_r(r,\ t)/K^2r\}$ into the eigensolutions

$$\mathbf{q}(r,t) = \sum_{j=1}^{N} A_j\mathbf{q}_j(r)e^{-i\omega_j t}, \mathbf{q}_j(r) = \{\phi_j(r), \psi_j(r)\}. \tag{3.71}$$

(the integer N determines the degree of the series truncation). Using Eqs. (3.69) it is easy to show that the energy density $E(k, n; t) = E(t)$, which also determines the energy norm $||\mathbf{q}(r, t)||_{\mathrm{E}} = [E(t)]^{1/2}$ in the space of vector-functions $\mathbf{q}(r; t)$, is given here by the equation

$$E(k, n; t) \equiv E(t) = \pi \int_0^1 \left[\frac{|\phi'|^2}{K^2 r^2} + \frac{|\phi|^2}{r^2} + K^2 r^2 |\psi|^2 \right] r dr. \qquad (3.72)$$

Equations (3.71) and (3.72) allow the energy $E(t)$ to be represented as a positive-definite quadratic form of the coefficients A_j. Hence here again the variational method of Butler and Farrell (1992) may be used for determination of the optimally-growing disturbance and the maximum energy growth $G(t) = \max_{\mathbf{A}} (E(t)/E(0))$ where $\mathbf{A} = \{A_1, \ldots, A_N\}$. As above, $G(t)$ depends here on the values of k, n, and Re, while the dependence on N was found to be immaterial if N is not too small (it was indicated by Schmid and Henningson that for $N > 7$ the results vary with N by less than 1.2 %).

In Fig. 3.21 the calculated functions $G(t)$ are presented for $k = 0, 0.1$, and 1, $n = 1$, 2, 3, and 4, and Re = 3000. For $k = 0$ it was proved by Bergström (1992) (and was confirmed by Schmid and Henningson (1994) and by O'Sullivan and Breuer (1994)) that $G(t)/(\mathrm{Re})^2$ depends only on t/Re; therefore the data in Figs. 3.21a and 3.21b refer to any values of Re. In Figs. 3.21c and 3.21d, values of time on the abscissa are also divided by Re but here this does not make the data sufficient for determination of the energy growth at any Re. For $n = 0$ (axisymmetric disturbances) some transient growth was found under the condition that $k\mathrm{Re} > 370$ but it proved to be quite small (here $\max_{t>0} G(t) \le 3$ in all cases investigated). Note also that Bergström (1992) appeared to find a larger energy growth for disturbances with $k = 0$ than that indicated in Fig. 3.21a, but he normalized $E(t)$ by the initial energy of only one radial velocity component. Therefore, data in Fig. 3.19 must be rescaled for comparison with data in Fig. 3.21a (cf. Fig. 4 by O'Sullivan and Breuer (1994) who performed the necessary rescaling and found that it brings Bergström's results much closer to those by Schmid and Henningson). As to the results of Bergström (1993a) presented in Fig. 3.20, they do not contradict those in Fig. 3.21.

Contours of $G^*(k, n, \mathrm{Re}) = \max_{t>0} G(t)$ in the (Re, kRe)-plane were also given by Schmid and Henningson separately for $n = 0, 1, 2$, and 3; they show that for $n > 0$ the maximum energy amplification increases with decreasing k and that $G^*/(\mathrm{Re})^2$ becomes practically independent of Re at large values of the Reynolds number. Some examples of the initial velocity fields leading to the maximum energy growth are presented in Schmid and Henningson's paper, together with some other results which will be discussed in Sect. 3.4.

O'Sullivan and Breuer (1994) also made computations related to those performed by Schmid and Henningson (1994). In both these papers, Eqs. (3.70) and (3.70′) with the unknown functions $\phi(r, t)$, $\psi(r, t)$ were used to find the eigenvalues ω_j

or $c_j = \omega_j/k$ and the eigenfunctions $\phi_j(r)$, $\psi_j(r)$. However, these results were applied by O'Sullivan and Breuer only to the study of the tube-flow eigenvalues and for evaluation of some specific initial conditions. They followed Gustavsson (1991) and assumed that $u_r(\boldsymbol{x}, 0)$ corresponds to some normal-mode solution of the linearized stability equations (3.70) and (3.70′) while the initial radial vorticity $\zeta_r(\boldsymbol{x}, 0)$ is equal to zero. Two other velocity components, $u_x(\boldsymbol{x}, 0)$ and $u_\phi(\boldsymbol{x}, 0)$, can then easily be determined with the help of Eq. (3.69) and the initial pressure $p(\boldsymbol{x}, 0)$ can be found from dynamic equations (2.73). The resulting initial conditions are clearly non-modal since infinitely many normal modes are needed to annihilate the non-zero radial vorticity entering the full normal-mode solution. However, such initial conditions seem to be interesting since they guarantee strong transient growth of radial vorticity $\zeta_r(\boldsymbol{x}, t)$. Moreover, comparison of Gustavsson's results of 1991 with those of Butler and Farrell (1992) suggests that, in the case where $u_r(\boldsymbol{x}, 0)$ corresponds to the least-stable normal mode, the energy growth must be close to the optimal one.

O'Sullivan and Breuer did not try to expand the chosen initial values in normal modes $\{\phi_j, \psi_j\}$, $j = 1, 2, \ldots$, but used direct numerical simulation, *i.e.*, numerical solution of the N-S equations in the cylindrical domain with given initial and boundary conditions. The numerical factor entering the normal-mode solution representing $u_r(\boldsymbol{x}, 0)$ was chosen to be small enough to make nonlinear effects unimportant even when the velocity disturbances grow by more than four orders of magnitude. The DNS results are shown in the paper for a number of initial normal modes of radial velocity, several values of n, and values of k varying from 0 to 0.5. The results showed considerable transient growth of disturbances in almost all cases considered, in some cases comparable with the optimal growth determined by Schmid and Henningson. It was also shown that growth curves $G(t)$ at $\mathrm{Re} = 1000$ and $\mathrm{Re} = 2000$ are practically the same if the doubling of Re is accompanied by the halving of k.

There were only a few experiments on tube flows which gave data that can be compared with the above theoretical conclusions. However, Bergström's (1993b, 1995) results are worth mentioning in this respect. In the first of the indicated papers, results of the measurements by the laser-doppler anemometer of the spatial developments of disturbances in a tube water flow were presented. The non-axisymmetric localized disturbances (corresponding to azimuthal wave numbers $n = 1$ and 5) were introduced in flows with several values of Re through 60 small holes made in the tube wall at a fixed value of the streamwise coordinate x. Then the disturbance amplitudes were measured at a number of downstream positions. In the experiments described in the second paper, localized initial disturbances of an air flow in a tube were produced, at several values of Re, by two jets induced radially into the tube by diametrically opposed loudspeakers and then amplitudes of the streamwise disturbance velocity were measured by a hot-wire anemometer at different radial and axial positions. Transiently-growing disturbances (growing at first and later beginning to decay) with theoretically reasonable streamwise velocity were detected in both experiments; especially conclusive results were obtained in the second set of experiments, where the evolution with x of the peak amplitude and spatial distribution of the streamwise velocity were investigated in detail. It was shown that the disturbances having most significant transient growth correspond to $n = 1$ and are

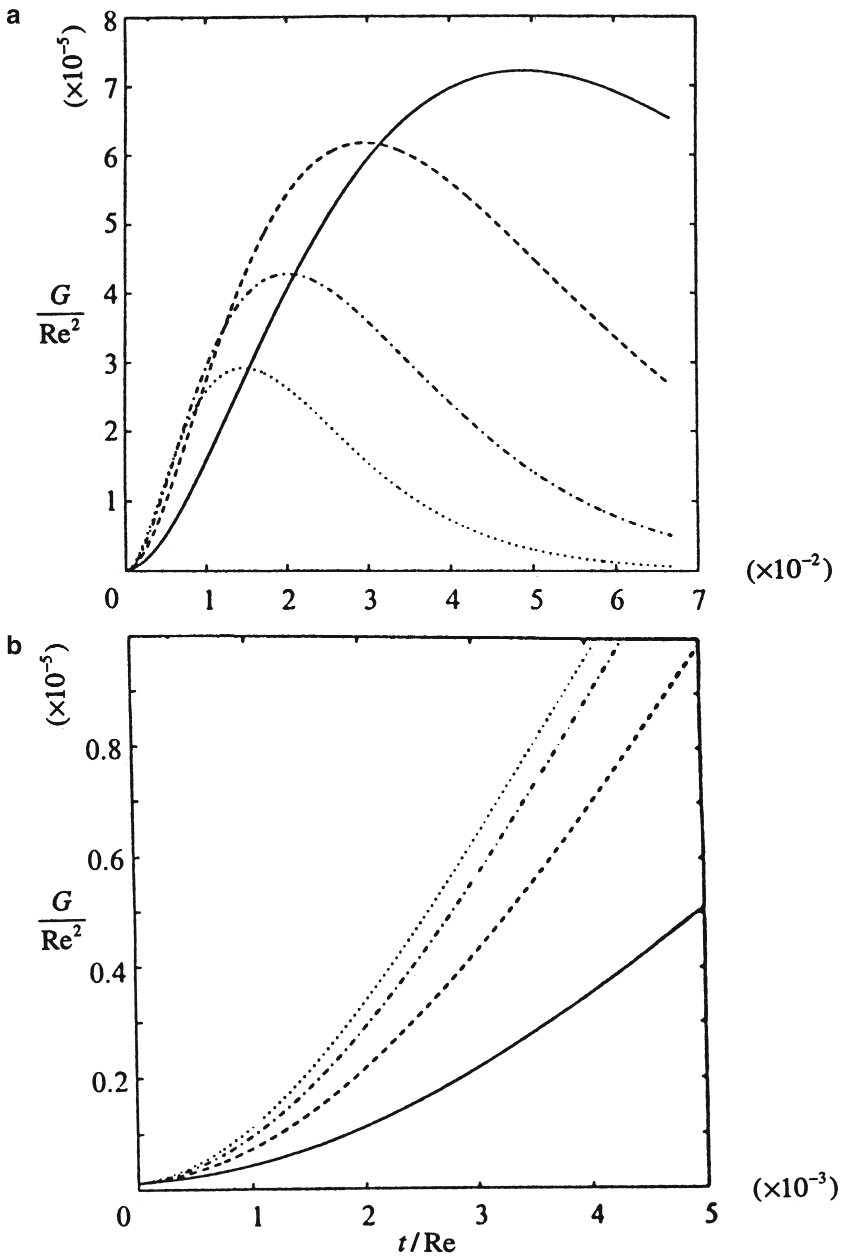
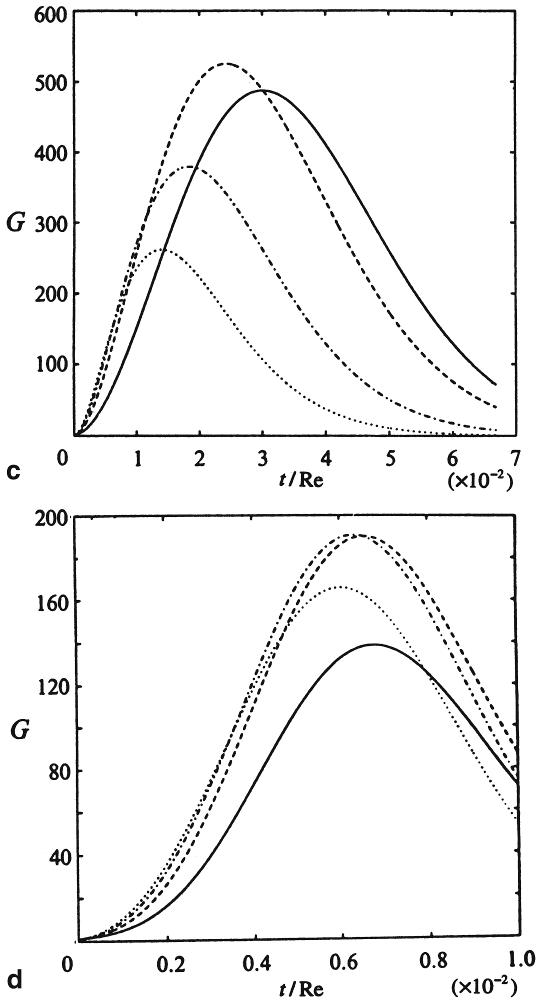


Fig. 3.21 The growth functions $G(t)$ for optimally-growing disturbances in circular Poiseuille flow with $\mathrm{Re} = 3000$ having wave numbers $n = 1$, 2, 3, or 4 and **a** $k = 0$; **b** $k = 0$, data for small times; **c** $k = 0.1$; and **d** $k = 1$. The *solid*, *dashed*, *chain-dashed*, and *dotted lines* correspond to $n = 1, 2, 3,$ and 4, respectively; in **a** and **b** the scaling of G by $(\mathrm{Re})^2$ and t by Re makes the graphs correct at any value of Re. (After Schmid and Henningson (1994))

Fig. 3.21 (Continued)



streamwise elongated; these results agree well with available theoretical predictions. The maximal growth of the disturbance energy was found to be increasing with the Reynolds number, but in all cases this growth was considerably smaller than that calculated for the case of optimally growing disturbances. However, this is only natural since the produced initial disturbances were clearly far from optimal. In any case, it was important that the reality of the transient disturbance growth was experimentally confirmed. As to the above theoretical results, indicating the possibility of the energy growth of tube-flow disturbance by three or even more orders of magnitude, they definitely indicate a mechanism that can affect the transition of tube flows to turbulence. Note again that such transition, which was first described by Hagen in 1839 and then was carefully investigated by Reynolds in 1883, remains up to now unexplained.

3.4 Some General Remarks About Transient Growth of Small Disturbances in Parallel Fluid Flows

Results of Sects. 3.33–3.34 show that in a viscous fluid with a subcritical value of Re, weak disturbances, whose evolution is described by linearized dynamic equations, often grow very significantly with time at first, and only later begin to decay. For $\mathrm{Re} > \mathrm{Re}_{cr}$ the situation is not too much different since here the rate of transient growth of small disturbances often greatly exceeds the rate of growth of the unstable normal mode (having the form of the O-S wave in the case of a plane-parallel flow). As a result the nonlinear interactions of transiently growing weak disturbances can become quite important, not only in subcritical flows with $\mathrm{Re} < \mathrm{Re}_{cr}$ but also in supercritical flows at times when the unstable normal mode is still very weak and practically unobservable. Thus, in both these cases the nonlinear interactions can lead to the so-called *bypass transition* to turbulence where the normal modes play no role at all (see Sect. 2.92).

The above arguments may produce the impression that transition studies must be based mainly on the nonlinear theory while the linear development of small disturbances is here only of secondary interest. However, in fact the solutions of the linearized initial-value problems are here also of primary importance since they provide the initial conditions for the subsequent nonlinear development. Moreover, some authors even suggested that the exact form of non-linear interactions of not-too-small disturbances is in fact of secondary importance. According to these authors, the only requirement limiting the form of nonlinear interactions producing transition to turbulence is the following one: they must lead to not-too-late breakdown of growing disturbances, which transfers the accumulated energy, prior to its appreciable viscous decay, to numerous small disturbances subjected again to transient growth in accordance with the linear stability theory (see, e.g., the remarkable early paper by Boberg and Brosa (1988) and the recent survey by Baggett and Trefethen (1997)). This suggestion (which is not taken for granted by everybody) stimulated the appearance of a number of quite different simple low-dimensional models of nonlinear interactions leading, as a rule, to relatively similar conclusions which do not contradict the available data; see, in particular, the two above-mentioned papers and the papers by Trefethen et al. (1993), Gebhardt and Grossmann (1994), Baggett et al. (1995), and Grossman (1996), which will be considered at greater length in Chap. 4.

Henningson and Reddy (1994) (see also Henningson (1995, 1996) and Schmid and Henningson (2001)) stressed that the possibility of transient growth of small enough disturbances obeying linear dynamic equations is necessary for the transition of a given flow to turbulence. This conclusion follows from consideration of the energy balance of disturbances. Let an incompressible fluid fill a spatial domain V, which is either finite or bounded by solid walls or is bounded in some directions but unbounded in the directions of one or two coordinate axes. Assume further that there is a flow in domain V with the velocity and pressure fields $\boldsymbol{U}(\boldsymbol{x}) + u(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}) + p(\boldsymbol{x}, t)$. Here $\{U(\boldsymbol{x}), P(\boldsymbol{x})\}$ is a steady solution of the N-S equations satisfying the "no-slip" boundary conditions at solid walls and independent of the coordinates with respect to which the domain V is unbounded, while $\{\boldsymbol{u}(\boldsymbol{x},t),\ p(\boldsymbol{x},t)\}$ are the velocity and pressure of a disturbance (of an arbitrary size) which is periodic in directions of unboundedness. Consider the kinetic energy of the disturbance

$$E(t) = \frac{1}{2}\int\limits_{V'}\sum_{j=1}^{3} u_j^2 dx \tag{3.73}$$

where $d\boldsymbol{x}$ is an element of volume in the three-dimensional space of points $\boldsymbol{x}$ and V' is the whole domain V, if it is bounded in all directions, or the part of V' bounded by one disturbance period in directions of unboundedness. Then using the N-S equations it is easy to obtain for $E(t)$ the following equation for the energy balance:

$$\frac{dE(t)}{dt} = -\int\limits_{V'} u_j u_i \frac{\partial U_j}{\partial r_i} d\mathrm{x} - \nu \int\limits_{V'} \sum_{j,i=1}^{3} \left(\frac{\partial U_j}{\partial r_i}\right)^2 dx \tag{3.74}$$

where, as always, summation is carried out over three values of indices occurring twice in the same term (see, e.g., Monin and Yaglom (1971), Sect. 2.9, or Joseph (1976), Sect. 3). Equtaion (3.74) was first derived by Reynolds (1894) and was then used by Orr (1907); at present it is usually called the *Reynolds-Orr* (or R-O) *equation*.

Both terms on the right-hand side of Eq. (3.74) are of the second order with respect to the disturbance velocities. The second of them describes the dissipation of the disturbance energy due to viscosity, and is always negative, while the first term, which describes the exchange of energy between the undisturbed flow and the disturbance, can be of any sign but, as a rule, is positive (the transfer of energy is usually directed from the undisturbed flow to the disturbance). If so, then the relative value of the two considered terms will determine whether the energy of the disturbance decreases or increases. If we transform Eq. (3.74) to dimensionless quantities, measuring distance, velocity, and time in units of characteristic length L, velocity U, and time L/U, respectively, then the dimensional coefficient ν in the second term on the right-hand side will be transformed into the dimensionless coefficient $\nu/UL = 1/\mathrm{Re}$. Hence, if the Reynolds number Re is sufficiently small, the negative second term on the right-hand side will always dominate the positive first term, and the energy of any disturbance will be damped, *i.e.*, the flow will be stable to disturbances of any shape and amplitude. Equation (3.74) in principle makes it possible to obtain certain estimates from below $\mathrm{Re}_{\mathrm{cr\ min}}$, which bounds the range of "sufficiently small" Reynolds numbers, within which the energy of any disturbance can only decrease. This remark is due to Reynolds, who tried to use Eq. (3.74) for estimation of $\mathrm{Re}_{\mathrm{cr\ min}}$ in his paper of 1894 where the quantities Re and $\mathrm{Re}_{\mathrm{cr\ min}}$ first appeared. Later this equation was used many times for the same purpose by a number of authors; some of them will be indicated below, partly in this section and partly in Chap. 4, Sect. 4.1. In cases where the flow is unbounded in some directions and the disturbance is periodic with respect to the corresponding coordinates (say, to x and y with wavelengths $\lambda_1 = 2\pi/k_1$ and $\lambda_2 = 2\pi/k_2$, respectively) the domain V' depends on k_1 and k_2 and hence here the Reynolds number below which the right-hand side of (3.74) is necessarily negative also depends on k_1 and k_2. Let us denote this value by $\mathrm{Re}_1(k_1, k_2)$. Then at $\mathrm{Re} < \mathrm{Re}_1(k_1, k_2)$ the energy density $E(k_1, k_2; t)$ of any disturbance with horizontal wave numbers (k_1, k_2) cannot grow with time (thus, $E(k_1, k_2; t) \leq E(k_1, k_2; 0)$ for any $t > 0$) and hence if $\mathrm{Re} < \min_{k_1,k_2} \mathrm{Re}_1(k_1, k_2) = \mathrm{Re}_{\mathrm{cr\,min}}$ we may be sure that the energy $E(t)$ of *any* disturbance will be damped.

In the above-mentioned paper by Henningson and Reddy (see also Reddy and Henningson (1993); Schmid and Henningson (1994); and Henningson and Alfredsson (1996)) the importance of the fact that the nonlinear terms of the N-S equations for the disturbance velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ drop out, when Eq. (3.74) is derived, was especially emphasized. (These terms play no role since they produce a divergence term in the integrand of the first integral on the right-hand side of Eq. (3.84) and thus by virtue of the boundary conditions this term disappears after application of the Gauss theorem.) Therefore, Eq. (3.74) preserves its form when the full N-S equations are replaced by the linearized Eq. (2.7) describing the time evolution of very small disturbances. This implies that if the energy of some disturbance of any amplitude grows with time in a flow satisfying the above conditions (and such growth is necessary for transition to turbulence since otherwise $E(t) < E(0)$ for any disturbance and any $t > 0$), then certainly the energy of an infinitesimal disturbance of the same shape will be also growing, at least for small values of t. Since for $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ only transiently growing infinitesimal disturbances can exist and the disturbances appearing in real flows are usually quite small at the beginning, it is natural to assume that subcritical transitions to turbulence of laminar flows encountered in engineering and nature begin, as a rule, with the transient algebraic growth of randomly arising small disturbances.) It is easy to show that subcritical transition is impossible for flows where the transient growth of infinitesimal disturbances cannot occur; see Henningson and Reddy (1994)).

According to results by Busse (1969) and Joseph and Carmi (1969) for plane Poiseuille flow, and similar results by Joseph (1966) for plane Couette flow and by Joseph and Carmi (1969) for circular Poiseuille flow, $\mathrm{Re}_{\mathrm{cr\,min}} \approx 49.6$, 20.7, and 81.5, respectively, for these three flows (for more details see Sect. 4.1 in Chap. 4 of this series). Hence in these flows, at values of Re smaller than the given values of $\mathrm{Re}_{\mathrm{cr\,min}}$, energy of a disturbance of any size will decrease monotonically with time, while at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr\,min}}$, disturbances of any size will exist whose energy will grow with time, at least for not too large values of t. Farrell and Ioannou (1993b) carried out a similar computation for the case of an unbounded Couette flow and showed that for disturbances with the given horizontal wave numbers k_1 and k_2 no growth is possible if $\mathrm{Re} = bl^2/\nu \equiv b\pi^2/k^2\nu < 2\pi^2 \approx 19.7$, where b is the velocity shear and $k = (k_1^2 + k_2^2)^{1/2}$. Thus, in this case a range of "sufficiently small" values of ν exists within which the energy of any disturbance with a given value of k decays, but "sufficiently small-scale" disturbances can grow here at any value of viscosity.

In Sects. 3.2–3.3, where transient growth of small disturbances in laminar plane-parallel flows was discussed, two physical mechanisms of such growth were indicated: the direct transfer of the kinetic energy of the undisturbed laminar flow with velocity $U(z)$ to streamwise disturbance energy $u^2/2$ occurring when $uwdU/dz$ takes negative values; and the forcing of the vertical vorticity of disturbance by spanwise variations of vertical velocity w closely related to Landahl's lift-up effect, which produces streamwise high- and low- velocity streaks. Butler and Farrell (1992) suggested the term "vortex tilting" for the second, most effective, growth mechanism since here the growth is due to spanwise tilting of the vertical vorticity. There is, however, also a mathematical explanation of the growth mechanism, which is related to some special features of linearized fluid dynamics equations which have been mentioned briefly in Sect. 2.5.

Let us rewrite the lienarized dynamic equations for the disturbance velocity $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},\ t)$ in the operator notations:

$$\frac{\partial \mathbf{u}}{\partial t} = -iL\mathbf{u}. \tag{3.75}$$

where L is a time-independent linear operator (easily derived from the N-S equations) in the space of divergence-free vector functions $\boldsymbol{u}(\boldsymbol{x}),\ \boldsymbol{x} \in \boldsymbol{V},$ satisfying the boundary and periodicity conditions indicated above. (The factor $-i$ on the right-hand side is added to make the eigenvalues of L equal to the complex frequencies ω of the normal modes entering Eq. (2.8) in Sect. 2.5.) To use the highly developed mathematical theory of linear operators in Hilbert spaces we must extend Eq. (3.75) to the space of complex vector-functions $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$ satisfying the above conditions and having finite norm $||\boldsymbol{u}(\boldsymbol{x},\ \boldsymbol{t})|| < \infty$, where

$$||u(x,t)||^2 = \tfrac{1}{2}\int_{v'} |\mathbf{u}(\mathbf{x},t)|^2 d\mathbf{x} = E_u$$

is the kinetic energy (or, for unbounded flows, kinetic energy density) of a disturbance. Then the disturbance velocity at any value of t will belong to the Hilbert space H of vector functions $\boldsymbol{u}(\boldsymbol{x})$ with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \tfrac{1}{2}\int_{v'} [\mathbf{u}(\mathbf{x}) \cdot \mathbf{v}^*(\mathbf{x})] d\mathbf{x}$$

where the asterisk denotes complex conjugation. This allows us to consider the evolution operator L on the right-hand side of Eq. (3.75) as an operator in H, and define for it the adjoint operator L^* as a linear operator satisfying the condition $(L\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, L^*v)$ for any $\boldsymbol{u}$ and $\boldsymbol{v}$. The best studied are the *self-adjoint* operators L satisfying the condition $L^* = L$. According to the spectral theory of self-adjoint operators, any such operator L has a real spectrum (which can be discrete, continuous, or mixed) and every element of H can be expanded into eigenvectors of L (or "generalized eigenvectors" corresponding to points of a continuous spectrum) where the eigenvectors corresponding to different eigenvalues are orthogonal to each other. This important property of eigenvector orthogonality is valid also for the wider class of the *normal* linear operators in a Hilbert space H having the property that $LL^* = L^*L$ (see, e.g., Dunford and Schwartz (1958, 1971); Rudin (1973); Kato (1976, 1982) and Pazy (1983) for more information about this topic[6]). For normal operators the

[6] Note that the properties of self-adjointness and normality of an operator L depend not only on the operator itself but also on the norm (and scalar product) introduced in the space H. Thus, both these properties can be lost (or gained) when the norm is changed.

spectral representation theorem is also valid and according to it every element of H can again be expanded into reciprocally orthogonal eigenvectors ("generalized" for points of a continuous spectrum) of L, but the spectrum of eigenvalues in this case is placed in the complex plane. However, the *evolution operator* L entering the linearized N-S equations written in the form(3.75) is very often not only non-self-adjoint but also non-normal (in particular, it is non-normal in all cases of parallel shear flows considered in this chapter[7]). Nevertheless, as was indicated in Sect. 2.5, the existence of a complete system of eigenfunctions and the corresponding eigenfunction expansion theorem has also been proved for all these cases (and for some more general cases too), in particular by Di Prima and Habetler (1969), Yudovich (1965, 1984) and Herron (1980, 1982, 1983). It was also noted in the indicated section that for non-normal operators eigenvectors corresponding to different eigenvalues are as a rule not orthogonal. This last circumstance is very important and it has direct relation to the strong transient growth of small disturbances in parallel shear flows.

Let us normalize all the eigenvectors making their norms equal to one. Then in the expansion of a vector $\boldsymbol{u}$ of the Hilbert space H into orthogonal eigenvectors the squared norm $\|\boldsymbol{u}\|^2$ (*i.e.*, the energy, if in accordance with the above agreement the "energy norm" is used) is equal to half the sum of the squares of all the expansion coefficients. Therefore, the square of any coefficient is here definitely not greater than twice the energy. However, when the eigenvectors are not orthogonal, the energy is given by some complicated positive-definite quadratic form of the coefficients and in this cases individual coefficients can take very large values. This is especially so when the operator L is "strongly non-normal," which means that its eigenvectors corresponding to different eigenvalues are not only non-orthogonal but in some cases even nearly linearly dependent. Then only small parts of some new eigenvectors will provide really new contributions to the linear combination of all previous eigenvectors. Suppose for definiteness that at $t = 0$ the initial condition $\boldsymbol{u}(\boldsymbol{x}, 0)$ is a vector-function of unit norm so that $E_{\boldsymbol{u}}(0) = 1$. Let us now represent the solution $\boldsymbol{u}(\boldsymbol{x}, t)$ of Eq. (3.75) in the form of eigenfunction expansion

$$\boldsymbol{u}(x,t) = \sum_j a_j \boldsymbol{u}_j(\boldsymbol{x}) e^{-i\omega_j t}$$

where a_j are expansion coefficients, while $\boldsymbol{u_j}(\boldsymbol{x})$ are the normalized eigenvectors and ω_j the eigenvalues of operator L. If L is "strongly non-normal" then coefficients a_j corresponding to nearly linearly dependent eigenvectors must be quite large to provide significant values of the "really new contributions" from these eigenvectors. At the same time, these coefficients must lead to the near-cancellation of linearly dependent parts of various vectors to yield the initial value $\boldsymbol{u}(\boldsymbol{x}, 0)$ of unit norm

[7] The situation is different in the case of stability problems four Couette-Taylor flow between two rotating coaxial cylinders and for convection in a layer of fluid heated from below. This fact leads to a fundamental difference between these two problems and stability problems for parallel shear flows, and means that the initial-problem approach is not useful in the case of the former two problems.

in spite of large values of a_j. For moderate values of t, the sum still consists of large terms, even if $\Im\text{m}\ \omega_j < 0$ for all subscripts j and hence all the exponentials are decaying functions. However, since the time-dependent factors $\exp(-i\omega_j t)$ differ from each other, the cancellation that occurs at $t = 0$ need not occur later and therefore the norm of the solution $\boldsymbol{u}(\boldsymbol{x}, t)$ at positive but not too large values of t can exceed considerably its initial value $\|\boldsymbol{u}(\boldsymbol{x}, 0)\| = 1$. At still greater values of t, the decaying factors $\exp(-i\omega_j t)$ begin to play the main part making the growth of the norm of $\boldsymbol{u}(\boldsymbol{x}, t)$ only transient.

Given here the "mathematical explanation" of the transient growth of small disturbances, whose evolution is described by non-normal dynamic operators, was given by Henningson (1991) who first found that the coefficients of an eigenfunction expansion of a small disturbance in a subcritical plane Poiseuille flow sometimes take surprisingly high values (so, some of the expansion coefficients for the initial disturbance with $E_u(0) = \text{O}(1)$ in a flow with $\text{Re} = 3000$ were found to be of the order of 10^3). Later Reddy et al. (1993) developed a method for crude estimation of the order of expansion coefficients and showed that this order grows very rapidly as Re increases: according to their estimates, for two-dimensional disturbances to a plane Poiseuille flow with $E_u(0) = 1$ the coefficients may have the order of 10^8 if $\text{Re} = 10000$, of 10^{10} if $\text{Re} = 15000$, and of 10^{16} if $\text{Re} = 40000$. Very large expansion coefficients definitely show that the corresponding evolution operator is "strongly (even enormously, if Re is relatively high) non-normal" and hence must produce very great transient growth of disturbance energy. On the other hand, rapid transient growth of small disturbances shows by itself that the corresponding evolution operator L must be strongly non-normal. The given "mathematical explanation" of the reason of the rapid transient growth of small disturbances clearly has no relation to the search for the physical mechanisms producing such growth; it only explains how these mechanisms affect the form of the evolution operator L.

Let us emphasize in conclusion that the study of the possibility of large transient growth of small disturbances is only a part of the comprehensive investigation of transition of laminar flows to turbulence. In the majority of papers considered above, most attention was given to "optimally growing disturbances" or, at least, to disturbances whose growth is "nearly-optimal". However, most often the disturbances appearing in flows encountered in real life are rather far from optimal ones. So, in the future it will be quite desirable to combine the study of transient growth with the study of the sources producing disturbances in real flows (of the flow *receptivity* to various disturbing factors in the terminology of Morkovin (1969); cf. Sect. 2.9.2 in Chap. 2) and of characteristics of the resulting disturbances.

Let us now consider briefly (omitting all technical details) some mathematical tools used in studies of growth potential for a given non-normal evolution operator L. (For proofs of the statements given below and the additional details see, *e.g.*, the books by Kato (1976, 1982), Pazy (1983), and the accounts by Reddy et al. (1993) and Trefethen (1996)). As is known, the range of possible rates for exponential growth or decay of disturbances as $t \to \infty$ is given by the spectrum Λ of the evolution operator L. In particular, the exponential growth is impossible (*i.e.*, the flow is stable according to the normal-mode stability theory) if and only if all points of Λ are in the lower half-plane of the complex-variable plane **C** (*i.e.*, have non-positive imaginary parts). The spectrum Λ can also be defined as the set of complex numbers z such that the resolvent $R_z = (zI - L)^{-1}$ of the operator L (here$I$is the identity operator) has an infinite norm. (The norm $\|A\|$ of a linear operator A in the Hilbert space H is defined as $\sup_{u \in H, ||u||=1} ||A\mathbf{u}||$ where sup denotes the maximum or, if it does not exist, the least upper bound.) However, in fact, behavior of the solutions $\boldsymbol{u}(\boldsymbol{x}, t)$ of Eq. (3.75) is not determined by the spectrum Λ of L alone but depends also on the region in complex plane **C** where the norm $\|R_z\|$ of the resolvent is "very large". Therefore it is natural to consider, parallel with the spectrum Λ of L, the greater set of complex numbers Λ_ε which includes not only Λ but also all such numbers z that $||(zI - L)^{-1}|| > \varepsilon^{-1}$ (or, what is the same, that $\|\Lambda \boldsymbol{u} - z\boldsymbol{u}\| \leq \varepsilon$ for some $\boldsymbol{u} \in H$ with $\|\boldsymbol{u}\| = 1$), where ε is a given (usually small) positive number. The set Λ_ε is called the *ε-pseudospectrum* of L; under wide conditions it can also be defined as a set of eigenvalues of all the operators of the form $L + K$ where K is a "small" operator with $\|K\| < \varepsilon$. The concept of a pseudospectrum was independently introduced in the 1970s and 1980s by a number of authors who often used different names for it, and at present the applications of this concept are very numerous and diverse. For more details about this topic, see recent surveys by Trefethen (1992, 1996) containing many examples and additional references; this author is also now writing a book on this subject.

If L is a normal operator, its pseudospectra have simple shape since here

$$||R_z|| \equiv ||(zI - L)^{-1}|| = \frac{1}{\operatorname{dist}\{z, \Lambda\}} \tag{3.76}$$

for any $z \notin \Lambda$, where Λ is the spectrum of L and dist $\{z, \Lambda\}$ is the distance from the point z of the plane **C** to the set Λ. Therefore, in this case the ε-pseudospectrum is simply the union of disks of radius ε centered at all points of Λ. However, if L is non-normal, then the equality in (3.76) is replaced by $\geq$, and the norm of the resolvent R_z may be quite large even if the point z is far from the spectrum Λ. Therefore, the figure showing the pseudospectra of L also gives some information about its degree of non-normality, which is characterized by the excess of the depicted pseudospectra over their shapes given by Eq. (3.76).

The pseudospectra of the evolution operator L characterize its "degree of non-normality" and thus have definite relation to the transient growth possible for the solutions of Eq. (3.75). Another set in the complex plane **C** also useful for estimation of this growth is the *numerical range* of L which is the set $\Gamma$of all complex numbers which can be represented as scalar products $(\boldsymbol{Lu}, \boldsymbol{u})$ where $\boldsymbol{u} \in H$ and $\|u\| = 1$. It is clear that all the eigenvalues of L are contained in Γ. Moreover, under some wide conditions (which are fulfilled for evolution operators L of steady parallel fluid flows and below will be always assumed to be satisfied) the spectrum Λ lies as a whole in the closure of the set Γ(consisting of the points of $\Gamma$and all limits of sequences of such points). The ε-pseudospectra Λ_ε also lie not far from the numerical range Γ; it may be proved that Λ_ε lies in the set $\Gamma + \Delta_\varepsilon$ formed by the union of disks of radius ε with centers in all points of Γ.

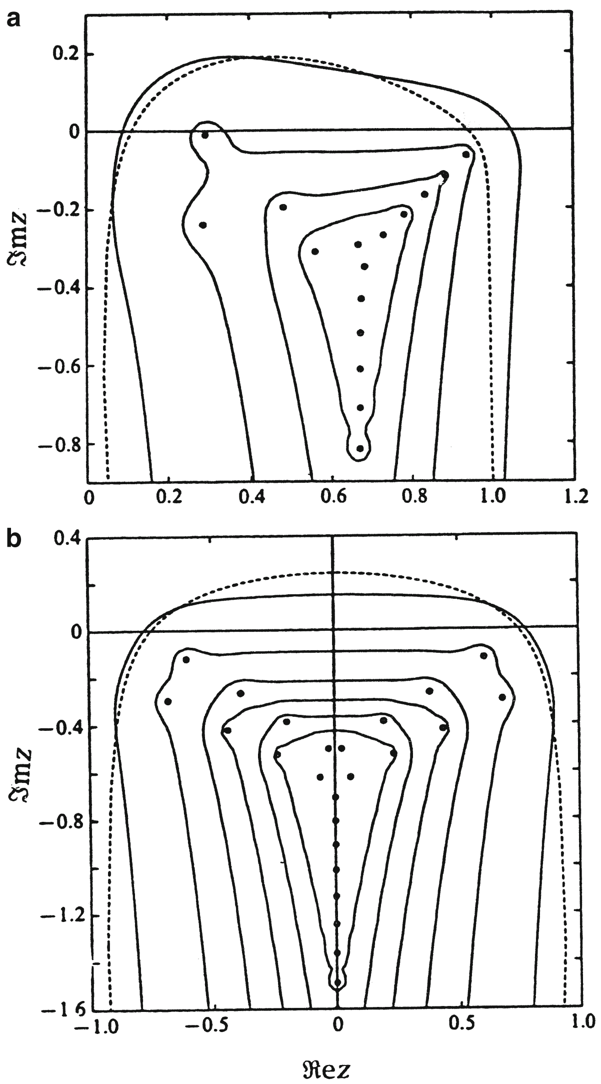


Fig. 3.22 Pseudospectra and the numerical range for operators L determining the evolution of antisymmetric wave disturbances with $k_1 = 1$ and $k_2 = 0$ in plane Poiseuille flow with Re = 3000 (**a**), and plane Couette flow with Re = 1000 (**b**) (after Reddy et al. (1993) and Reddy and Henningson (1993)). • - eigenvalues (in the case of ε-pseudospectra they are represented by discs of radius ε); *dotted lines* represent boundaries of the numerical ranges; *solid lines*, from *outer* to *inner* are the boundaries of the ε-pseudospectra for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$ in Figure (**a**) and for $\varepsilon = 10^{-1}, 10^{-2}, \ldots, 10^{-6}$ in Figure (**b**)

The importance of the numerical range of L for the analysis of behavior of disturbances $\boldsymbol{u}(\boldsymbol{x}, t)$ satisfying Eq. (3.75) is due to the following *Hill-Yosida theorem* (see, e.g., Pazy (1983)): $\|\boldsymbol{u}(\boldsymbol{x}, t)\| \le \|\boldsymbol{u}(\boldsymbol{x}, 0)\|$ for all initial values $\boldsymbol{u}(\boldsymbol{x}, 0)$ and all $t \ge 0$ if and only if Γ lies in the closed lower half-plane of **C** (*i.e.*, $\Im\text{m}\, z \le 0$ for all $z \in \Gamma$). The last condition may also be formulated in terms of ε-pseudospectra; namely, it is equivalent to the condition that $\Im\text{m}\, z \le \varepsilon$ for all $z \in \Lambda_\varepsilon$ and any $\varepsilon \ge 0$. It is easy to show that the conditions given here of absence of any disturbance growth are equivalent to the condition following from the energy-balance equation (3.74).

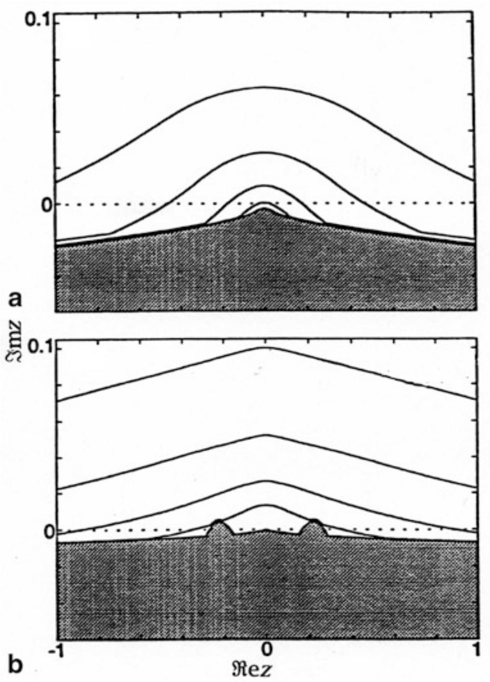


Fig. 3.23 The spectra (represented by shaded regions) and the upper boundaries of the ε-pseudospectra (the *solid lines*, from *outer* to *inner*, correspond to $\varepsilon = 10^{-2}$, $10^{-2.5}$, 10^{-3}, and $10^{-3.5}$) for plane Poiseuille flow at Re = 1000 **a**, and Re = 10000 **b**. (After Trefethen et al. (1993))

The estimation of the constant $C = \max_{\mathbf{u(x,0)},t\geq 0}[||\mathbf{u}(\mathbf{x},t)||/||\mathbf{u}(\mathbf{x},0)||]$, which determines the maximum possible growth of the disturbance energy, is more difficult than the determination of conditions guaranteeing that $C = 1$. A definite estimate can be obtained if norms $||R_z^k|| = ||(zI - L)^{-k}||$ are known for all integers $k > 0$ and all z lying in the upper half-plane of $\mathbf{C}$ (see, *e.g.,* Pazy (1983) and Reddy et al. (1993)), but this estimate is rather complicated and it will not be presented in this book. It can also be proved that if $C' = \max_{\varepsilon>0}[\max_{z\in\Lambda_\varepsilon}\Im z/\varepsilon] > 1$, then $C > C'$, but this estimate of C is not simple enough and moreover is less precise than the numerical estimates of the maximum possible growth of the disturbance energy described in Sects. 3.33 and 3.34.

Many graphs showing the shapes of computed pseudospectra Λ_ε and numerical ranges Γ for dynamic operators L corresponding to various steady parallel flows were published, together with some comments about connection of pseudospectra with transient growth of disturbances, by Trefethen et al. (1993), Reddy et al. (1993), Reddy and Henningson (1993), A. Trefethen et al. (1994), Schmid and Henningson (1994), and Trefethen (1996). Some typical examples of such graphs are presented in Figs. 3.22 and 3.23. In Fig. 3.22 the pseudospectra Λ_ε and numerical ranges Γ are shown for the evolution operators L determining the development of two-dimensional disturbances, antisymmetric with respect to the channel midplane and with dimensionless wave number $k_1 = 1$, in plane Poiseuille flow with Re = 3000 and plane Couette flow with Re = 1000. The figure makes it clear that the operator

L is here non-normal (the pseudospectra are much greater than the union of disks centered at the eigenvalues). Note that the greatest excess in pseudospectrum areas is observed near the intersection of eigenvalue branches; this makes it clear that the eigenvectors corresponding to eigenvalues in this region are especially far from being reciprocally orthogonal. In Fig. 3.23 the upper boundaries of the spectrum and several pseudospectra are depicted for plane Poiseuille flows with $\mathrm{Re} = 1000$ and 10000. The eigenvalues $z = \omega_j$ and complex numbers z belonging to pseudospectra clearly depend on the disturbance wave numbers k_1 and k_2 ; in Fig. 3.23 they are collected for all values of wave numbers and therefore in place of discrete spectra shown in Fig. 3.22 we have here continuous spectral regions symmetric with respect to the axis $\Re e z = 0$. These regions lie wholly in the lower half-plane if $\mathrm{Re} < \mathrm{Re}_{cr}$, but they have bumps extended into the upper half-plane if $\mathrm{Re} > \mathrm{Re}_{cr}$.

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Chapter 4
Stability to Finite Disturbances: Energy Method and Landau's Equation

The main part of Chap. 2 and the whole of Chap. 3 were devoted to topics of linear stability theory dealing with the evolution of very small flow disturbances satisfying the linearized fluid dynamics equations. In Chap. 2 it was shown that the classical normal-mode method of the linear theory of hydrodynamic stability often leads to results which strongly disagree with experimental data. It was also indicated there that these disagreements are apparently due to nonlinear effects, which make linearization of the equations of motion physically unjustified. In Chap. 3 it was explained that the necessity for consideration of the full nonlinear dynamic equations often follows from the fact that many solutions of the initial-value problems for linearized fluid dynamics equations grow considerably at small and moderate values of the time t even in the cases when the normal-mode analysis shows that these solutions decay asymptotically (i.e. at $t \to \infty$).

The nonlinear theory of hydrodynamic stability has achieved a high level of development. Although the theory is still far from being completed, it has elucidated many formerly mysterious properties of fluid flows which are interesting for physicists and important for engineers. There is now an enormous literature on this subject and only a small part of it, dealing with relatively simple flows of incompressible fluids, will be considered in this book. In the present Chapter two topics from the nonlinear stability theory will be discussed: the energy method of stability analysis (short introductory consideration of this method was included in Sect. 3.4 above) and Landau's approach to the weakly nonlinear stability theory which described the initial period of the nonlinear development of flow disturbances.

4.1 The Energy Method of Stability Analysis and its Generalisations

4.1.1 Flows of Fluids of Constant Density

Remember first of all what was said about the energy method in Sect. 3.4 of Chap. 3. There, a flow of an incompressible constant-density fluid in a domain V was

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considered, where V is either bounded by solid walls or is unbounded in the directions of some coordinate axes x_j. It was assumed that the velocity and pressure fields of the flow are of the some of the form $\mathbf{U}(\mathbf{x})+\mathbf{u}(\mathbf{x}, t)$ and $P(\mathbf{x})+p(\mathbf{x}, t)$, where $\{\mathbf{U}(\mathbf{x}), P(\mathbf{x})\}$ are the velocity and pressure of some steady 'undisturbed flow'(which, in the case of unbounded flow, has the property that $\mathbf{U}$ and ∇P do not depend on those coordinates x_j that correspond to directions of flow unboundedness), while $\mathbf{u}(\mathbf{x}, t)$, $p(\mathbf{x}, t)$ are the velocity and pressure of some disturbance of arbitrary size (which in the case of unbounded V is periodic, with given periods $l_j = 2\pi/k_j$, with respect to the coordinates x_j). Hence $\{\mathbf{U}(\mathbf{x}), P(\mathbf{x})\}$ and $\{\mathbf{U}(\mathbf{x})+\mathbf{u}(\mathbf{x}, t), P(\mathrm{x})+p(\mathbf{x}, t)\}$ both satisfy the Navier–Stockes (for short, N-S) equations with "no-slip" boundary conditions at solid walls. Note also that the derivation of the energy-balance equation is unchanged if the undisturbed flow is unsteady and spatially periodic (with periods l_j) in directions in which V extends to infinity; moreover, the walls bounding the domain V can be moving, and V can depend on t.

The energy-balance equation for a flow disturbance follows easily from the equations of motion for $\mathbf{u} = (u_1, u_2, u_3)$, which are the differences between the N-S equations for $U_i + u_i$ and those for U_i alone, for $i = 1, 2, 3$:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{4.1a}$$

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{4.1b}$$

where, as usual, the summation is carried out over all three values of any indices which occur twice in single-term expressions ("repeated indices"). Let us now multiply Eq. (4.1a) by u_i, sum the equations obtained for $i = 1, 2, 3$, and then integrate the sum over the region V', where V' coincides with V if V is bounded, while if V is unbounded then V' includes only one period l_j in directions in which V extends to infinity. It is easy to see that the result of the integration can be written in the form

$$\frac{dE(t)}{dt} = -\int\limits_{V'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} - \nu \int\limits_{V'} \sum_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x} \tag{4.2}$$

where $d\boldsymbol{x}$ is an element of volume in the three-dimensional space of points $\boldsymbol{x}$ and

$$E(t) = \frac{1}{2}\int\limits_{V'} \sum_{j=1}^{3} u_j^2 d\mathbf{x} \tag{4.3}$$

is either the total kinetic energy of a disturbance (if V is bounded) or the energy density per wavelength (the unimportant factor ρ representing the constant density of the fluid is here omitted for simplicity). Equations (4.2) and (4.3) are just Eqs. (3.74) and (3.73) of Sect. 3.4 and the first of them is just the *Reynolds-Orr* (or R-O) *equation of the energy balance.* This was first derived more than a hundred years ago by Reynolds (1894) (who took $\mathbf{U}$ as the average flow velocity and $\mathbf{u}$ as the deviation of the velocity at a point from the average) and was later studied and used by Orr (1907) (whose interpretation of the velocities **U** and **u** was the same as that given above). It was noted in Sect. 3.4 that the single nonlinear term of Eq. (4.1a) for the velocity u_i—the last term of the left-hand side—makes no contribution to Eq. (4.2), since it produces a divergence term which drops out after the integration by virtue of boundary conditions. As a result, all the terms of the R-O equation turn out to be quadratic in the disturbance velocities u_i; therefore, the sign of the left-hand side of the R-O equation does not change when the velocity $\mathbf{u}(\mathbf{x}, t)$ is multiplied by some factor (i.e., this sign does not depend on the disturbance intensity). It was also noted in Sect. 3.4 that changing to dimensionless quantities transforms the energy-balance Eq. (4.2) into an equation of the same form but with dimensionless coordinates and velocities (measured in appropriate length and velocity units L and U) and with the dimensional factor ν replaced by the dimensionless combination $\nu/UL = 1/\text{Re}$.

From the R-O Eq. (4.2), where all the velocities and coordinates are now assumed to be non-dimensionalized, it follows that if $UL/\nu = \text{Re}$ takes a value which is greater than the value of the ratio

$$\frac{\left[\int\limits_{V'} \sum\limits_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x}\right]}{\left[-\int\limits_{V'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x}\right]} = R[\mathbf{u}(\mathbf{x})] \tag{4.4}$$

for a given solenoidal (zero-divergence) vector field $\mathbf{u}(\mathbf{x}) = \{u_1(\mathbf{x}),\ u_2(\mathbf{x}),\ u_3(\mathbf{x})\}$ which satisfies all the necessary boundary and periodicity conditions, and if $\mathbf{u}(\mathbf{x}, \mathrm{t} = 0) \equiv \mathbf{u}(\mathbf{x})$, then $dE(t)/dt > 0$ at $t = 0$. On the other hand, if Re is smaller than (or equal to) the greatest value of $R[\mathbf{u}(\mathbf{x})]$ accessible for the class of solenoidal fields $\mathbf{u}(\mathbf{x})$ satisfying the necessary boundary and periodicity conditions, and if the undisturbed flow is a steady one, then for any shape and size of the initial field $\mathbf{u}(\mathbf{x}, \mathrm{t} = 0)$ of disturbance velocity for derivative dE/dt will be nonpositive at any $t \geq 0$. Therefore, we may conclude that *the minimal Reynolds number* $Re_{cr\,min}$, which first appears in the paper by Reynolds (1883), *coincides with the minimum value of* $R[\mathbf{u}(\mathbf{x})]$ *over all solenoidal vector fields* $\mathbf{u}(\mathbf{x})$ *representing possible initial values of the disturbance velocity.* Such a definition of $\text{Re}_{\text{cr min}}$ implies that at $\text{Re} < \text{Re}_{\text{cr min}}$ the undisturbed flow considered is *globally* (i.e., *unconditionally*) and *monotonically stable* (for more details about these concepts see, e.g., Joseph (1976); Manneville (1990); Dauchot and Manneville (1995), and Chap. 2 in Godreche and Manneville (1998)). Later it was also shown that if the flow region V is bounded in at least one spatial direction (and hence can be contained between some pair of parallel planes), then for any $\text{Re} < \text{Re}_{\text{cr min}}$ there exists a positive constant $\Lambda = \Lambda$ (Re) such that $E(t) \leq E(0)\,\exp(-\Lambda \mathrm{t})$ for any $t > 0$; therefore, in this case the disturbance energy falls off exponentially with time (see, e.g., Serrin (1959), and also the books by Joseph (1976), Sect. 4, Galdi and Rionero (1985), Chap. 1, Georgescu (1985), Sect. 1.1.5, and Straughan (1992), Chap. 3).

If Re^* is the smallest value of $R[\mathbf{u}(\mathbf{x})]$ corresponding to some subset of all admissible disturbance velocities $\mathbf{u}(\mathbf{x})$, then the inequality $\text{Re}_{\text{cr min}} < \text{Re}^*$ is clearly valid, and hence Re* is an estimate of $Re_{cr\ min}$ from above. Reynolds (1894) used his version of Eq. (4.2) for just such an estimate from above of $Re_{cr\ min}$ for plane Poiseuille flow. For this purpose he determined the minimum value of $R[\mathbf{u}(\mathbf{x})]$ for one special family of admissible two-dimensional vector fields $\mathbf{u}(\mathbf{x}) = \{u(x, z), 0, w(x, z)\}$ depending on two numerical parameters (not counting the amplitude whose value is unimportant) and thus proved that in this case $Re_{cr\ min} \leq 517$, where Re is based on the distance H between the walls and the mean velocity of the undisturbed flow $U_m = 2U_{max}/3$. Later Sharpe (1905) carried out a similar computation for a quite different two-parameter family of two-dimensional disturbances $\mathbf{u}(\mathbf{x})$, and in this way found a considerably lower estimate, $Re_{cr\ min} \leq 167$, of the minimum Reynolds number for plane Poiseuille flow. Sharpe also applied this method to estimation from above of $Re_{cr\ min}$ for the circular Poiseuille flow in a round tube; here the value of $\min_{u(x)} R[\mathbf{u}(\mathbf{x})]$ for a particular two-parameter family of axisymmetric velocity disturbances gave $Re_{cr\ min} < 470$, where Re is based on the tube diameter D and the undisturbed mean velocity U_m. Then Lorentz (1907), computed the value of $\min_{u(x)} R[\mathbf{u}(\mathbf{x})]$ for a class of 'elliptic whirls' disturbing a plane Couette flow and found that for this case $Re_{cr\ min} \leq 288$ where $Re = HU/\nu$, H is the flow thickness and U is the velocity of the moving wall.

It was already clear to Orr (1907) that only very crude estimates of $Re_{cr\ min}$ can be found from investigations of special low-parametric subsets of disturbance velocities $\mathbf{u}(\mathbf{x})$. For this reason Orr did not consider any such subsets, but set up the variational problem of finding the solenoidal vector field $\mathbf{u}(\mathbf{x})$ which satisfies the required boundary conditions (and periodicity conditions, if V is unbounded), and minimizes the functional (4.4) where $\mathbf{U}(\mathbf{x})$ is a given undisturbed velocity field. Orr noted that he tried to solve this problem for three-dimensional vector fields u(x) but found it to be too difficult (remember that this was written in 1907). Therefore he considered only two-dimensional disturbances $\mathbf{u}(\mathbf{x}) = \{u(x, z), 0, w(x, z)\}$ (or, in the case of tube flow, $\{u_x(x, r), 0, u_r(x, r)\}$ assuming that such disturbances must be less stable than three-dimensional ones. For two-dimensional disturbances the solenoidal vector field u(x) may be represented in terms of the scalar stream function $\Psi(x, z)$ (or $\Psi(x, r)$) and substituted in this form into Eq. (4.4). In particular, for a plane-parallel undisturbed flow with velocity profile $U(z)$, the functional $R[\mathbf{u}(\mathbf{x})]$ in the case of a two-dimensional disturbance can be written as

$$R[\Psi(x,z)] = \frac{\iint (\Delta\Psi)^2 dx dz}{\iint \frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial z}\frac{dU}{dz} dx dz}; \tag{4.5}$$

and a similar equation may be obtained for tube flow. Therefore, determination of the value of $Re_{cr\ min}$ corresponding to two-dimensional disturbances can be reduced to the variational problem of finding the minimum value of functional (4.5) (or some similar functional for an axisymmetric primary flow) over the set of twice-differentiable functions $\Psi(x, z)$ (or $\Psi(x, r)$) satisfying the appropriate boundary conditions (in particular, conditions $\partial\Psi/\partial z = \partial\Psi/\partial x = 0$ at plane solid walls). This was just the variational problem Orr tried to solve for the cases of plane Couette flow and plane and circular Poiseuille flows. Unfortunately, his assumption that two-dimensional disturbances are the most unstable was later found to be incorrect, and also some of his numerical methods proved to be not sufficiently precise. However, this fact does not diminish Orr's main achievement, the first accurate formulation of the general variational problem of the energy method of stability theory.

Subsequent investigations of the stability of some simple parallel flows by the energy method were carried out during the 1910s and 1920s, in particular by Hamel (1911); Havelock (1921), and von Kármán (1924) (the first publication of the author's results of 1910, presented at the time only in a lecture). These workers also considered only two-dimensional disturbances, and used rather crude approximate solutions of Orr's variational problem. The papers by Tamaki and Harrison (1920) and Harrison (1921) were devoted to the study of the stability of circular Couette flow by the energy method, but the first of these papers was erroneous, while in the second the extremum was sought only among a rather special and narrow set of disturbances. However, for many years the inaccuracy of these calculations seemed to be an insufficient explanation of the fact that all estimates of the critical Reynolds numbers obtained by this method (often being 'estimates from above') turned out to be considerably lower than both the values of $\mathrm{Re}_{\mathrm{cr}}$ given by the normal-mode method of linear stability theory and the experimentally observed values of Re corresponding to transition of real flows to turbulence. This circumstance gave rise to extensive criticism of the energy method by a number of authors, proclaiming that, even in principle, this method can give only serious underestimates of $\mathrm{Re}_{\mathrm{cr}}$. The observed inadequacy of the method was usually explained by the fact that the minimization of the functional (4.4) (or (4.5)) was carried out over a set of disturbance velocities (or stream functions) satisfying only the required boundary and incompressibility conditions, while the equations of motion were not taken into account at all. Critical remarks of this kind can be found, e.g., in the books by Lin (1955), p. 59, Monin and Yaglom (1971), p. 152 (here a reference is given to the paper by Petrov (1938), where it was allegedly shown 'that the value of $\Psi(x, z)$ minimizing the functional (4.5) cannot generate a dynamically possible motion'), and Hinze (1975), p. 77, and also in papers by Serrin (1959), p. 4, and Joseph (1966), pp. 181–182 (these two papers will be considered below at greater length). However, this criticism is in fact unjustified; the energy method considers only the flow conditions at one instant of time, and at fixed time t the velocity $\mathbf{u}(\mathbf{x}, t)$ can take any value satisfying the above boundary and incompressibility conditions. (This fact was stressed by Lumley (1971) who also analyzed the arguments by Petrov (1938) to show their inconsistency). Since in the energy method $\min_{\mathrm{u(x)}} R[\mathrm{u(x)}]$ is taken over all possible instantaneous values of disturbance velocity, then—if the undisturbed flow is steady—at any Re below this minimum $dE(t)/dt$ will be negative at any non-negative value of t, i.e., the energy of the disturbance will decay monotonically with time for any intensity and shape of the initial disturbance.

Let us stress, however, that the validity of the inequality $\mathrm{Re} \leq \mathrm{Re}_{\mathrm{cr\ min}} = \min_{\mathrm{u(x)}} R[\mathbf{u}(\mathbf{x})]$, which guarantees the monotonic decrease of the disturbance energy with time, is only a *sufficient* (but not necessary) *condition for flow stability*. On the other hand, the validity of the opposite inequality $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr\ min}}$ is a *necessary* (but not sufficient) *condition for flow instability*. Remember also that in Chap. 2 it was noted

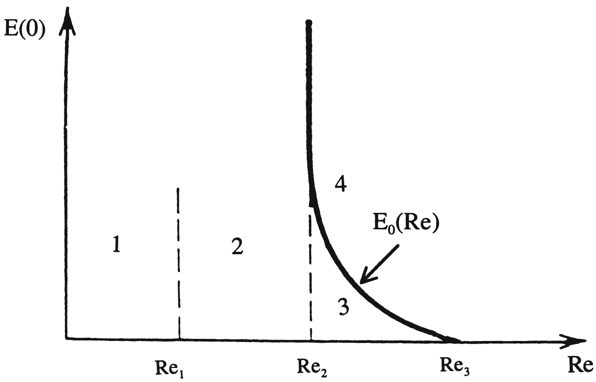


Fig. 4.1 Schematic representations of various stability regions of a given flow in the (E(0), Re)-plane. (After Joseph (1976)) $\mathrm{Re}_1 = \mathrm{Re}_{\mathrm{cr\,min}}$; $\mathrm{Re}_2 = \mathrm{Re}_{0,\mathrm{cr}}$; $\mathrm{Re}_3 = \mathrm{Re}_{\mathrm{cr}}$; *1*-the region of global and monotic stability; *2*-the region of global nonmonotonic stability; *3*-the region of conditional stability; *4*-the region of instability

that the normal-mode method of the linear stability theory gives the value of $\mathrm{Re}_{\mathrm{cr}}$ such that $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ is a *sufficient* (but not necessary) *condition for flow instability* (while the opposite condition $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ is a *necessary*, but not sufficient, *condition for flow stability*). Therefore, the value of $\mathrm{Re}_{\mathrm{cr\,min}}$ can be quite different from both the value or $\mathrm{Re}_{\mathrm{cr}}$ of the linear stability theory and the value of Re characterizing real transition to turbulence. Thus it is only natural that, even when sufficiently precise computations of $\min_{\mathrm{u(x)}} R[\mathbf{u}(\mathbf{x})] = \mathrm{Re}_{\mathrm{cr\,min}}$ are used, the energy method often leads to values of Re which are far below the Reynolds numbers observed at transition of flow to turbulence (and below the linear-theory values of $\mathrm{Re}_{\mathrm{cr}}$ which are usually higher than Reynolds numbers at laminar-turbulent transition). Let us stress again in this respect that $\mathrm{Re}_{\mathrm{cr\,min}}$ determines only the threshold value of the Reynolds numbers corresponding to *global* (unconditional) *monotonic stability* of the flow considered, i.e., the validity of the condition $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr\,min}}$ is both necessary and sufficient for being sure that any initial disturbance will decay *monotonically* tending to zero as $t \to \infty$. However, certain range $\mathrm{Re}_{\mathrm{cr\,min}} < \mathrm{Re} < \mathrm{Re}_{0,\mathrm{cr}}$ of Reynolds number exceeding $\mathrm{Re}_{\mathrm{cr\,min}}$ can exist, having the property that if Re belongs to it then any disturbance will necessarily decay to zero as $t \to \infty$ but the energy of some disturbance will transiently grow during some finite time intervals. This range corresponds to *global* (but *nonmonotonic*) *flow stability* and it is clear that transition to (undamped) turbulence cannot happen at $\mathrm{Re} < \mathrm{Re}_{0,\mathrm{cr}}$. The range of Reynolds numbers corresponding to *conditionally stable flows* adjoins the globally-stable-flow range $0 \le \mathrm{Re} \le \mathrm{Re}_{0,\mathrm{cr}}$; at values of Re from this range the disturbances satisfying some definite condition necessarily decay to zero while others can grow indefinitely. The most usual conditions guaranteeing the decay of disturbances have the form of energy limitations: the disturbance necessarily decays as $t \to \infty$ if its initial energy $E(0)$ does not exceed some threshold value $E_0(\mathrm{Re})$ depending on Re. The value of $E_0(\mathrm{Re})$ clearly must decrease monotonically with the increase of Re apparently tending to zero at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ (where $\mathrm{Re}_{\mathrm{cr}}$ is the critical value of the linear stability theory dealing with the infinitesimal disturbances) and to infinity at $\mathrm{Re} = \mathrm{Re}_{0,\mathrm{cr}}$ (see schematic Fig. 4.1; additional information may be again found in Joseph (1976); Manneville (1990); Dauchot and Manneville (1995), and Godrèche and Manneville (1998)). As to the transition to turbulence, it occurs most often at some Reynolds number intermediate between $\mathrm{Re}_{0,\,\mathrm{cr}}$ and $\mathrm{Re}_{\mathrm{cr}}$.

Earlier in this section some very early papers on the energy method of the stability theory were mentioned. In the 1930s, 1940s, and early 1950s this theory did not attract much attention; note, however, two remarkable papers by Sorokin (1953, 1954) which will be discussed later in this section. Slightly later the important paper by Serrin (1959) appeared, stimulating a number of authors to resume stability investigations by the energy method. This resulted in a great number of new publications relating to many different problems on hydrodynamic stability.

Serrin began with an accurate derivation of the fundamental R–O Eq. (4.2) under rather general conditions (he considered a general unsteady flow in the presence of an external force in the region V bounded by walls which could be moving). Then he formulated the variational problem by Orr as a problem of finding the maximum of the functional $\prod[\mathbf{u}(\mathbf{x})] = \int\limits_{V'} u_j u_i(\partial U_j/\partial x_i)d\mathbf{x}$ under the following conditions: $D[\mathbf{u}(\mathbf{x})] = \int\limits_{V'} \sum\limits_{i,j=1}^{3} (\partial u_j/\partial x_i)^2 \,\mathrm{d}\,\mathbf{x} = 1$ and div $\mathbf{u}(\mathbf{x}) = 0$, where $\mathbf{u}(\mathbf{x})$ satisfies the necessary boundary and periodicity conditions. Serrin wrote down the Euler-Lagrange (E-L) equations corresponding to this variational problem, which included Lagrange multipliers (since a conditional extremum was sought). He also showed that the equations obtained can be easily transformed into an eigenvalue problem for a system of partial differential equations similar to the N-S equations of fluid dynamics. (A slightly different derivation of these E-L equations, under slightly more general conditions, was given by Lumley (1971), while the corresponding eigenvalue problem was also considered by Galdi and Rionero (1985), Chap. 1, Geovgescu (1985), Sects. 1.1.2 and 1.3.1, and Straughan (1992), Chap. 3.) However, in the 1950s the determination of the exact solution of the eigenvalue problem seemed to be very difficult. Therefore Serrin concentrated his main efforts on the derivation of some approximate results, based on some relatively crude general inequalities.

In particular, Serrin showed that in the case of an arbitrary bounded region V with smooth enough boundary and a maximum diameter D the following inequality holds:

$$\int\limits_{V} \sum_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x} \geq \frac{\mathrm{a}}{\mathrm{D}^2} \int\limits_{V} \sum_{\mathrm{i}=1}^{3} \mathrm{u}_{\mathrm{i}}^2 \mathrm{d}\mathbf{x}, \tag{4.6}$$

where $a = \frac{3+\sqrt{13}}{2}\pi^2 \approx 32.6$, for any solenoidal vector field $\mathbf{u(x)}$ in V vanishing on the boundary of V (this is a particular case of the known Poincaré inequality; see, e.g., Straughan (1992)). Using then the obvious relations

$$-\int\limits_{V} u_j u_{\mathrm{i}} \frac{\partial U_j}{\partial x_i} d\mathbf{x} = \int\limits_{V} \frac{\partial u_j}{\partial x_i} u_i U_J d\mathbf{x},\ \frac{\nu}{2} \int\limits_{V} \sum_{i,j} \left(\frac{\partial u_i}{\partial x_i}\right)^2 d\mathbf{x} - \int\limits_{V} \frac{\partial u_j}{\partial x_i} u_i U_j dx$$

$$+ \frac{1}{2\nu} \int\limits_{V} \sum_{i=1}^{3} u_i^2 \sum_{j=1}^{3} U_J^2 d\mathbf{x} \geq 0,$$ Serrin obtained the inequality

$$-\int_V u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} \le \frac{\nu}{2} \int_V \sum_{i,j} \left(\frac{\partial u_i}{\partial x_i}\right)^2 d\mathbf{x} + \frac{U_{\max}^2}{2\nu} \int_V \sum_{i=1}^{3} u^2 d\mathbf{x}, \tag{4.7}$$

where $U_{\max}$ is the maximum of the modulus of the undisturbed velocity $\mathbf{U(x)}$. Equations (4.2), (4.3), (4.6) and (4.7) easily imply that

$$E(t) \le E(0) \exp\left(\frac{U_{\max}^2}{\nu} - \frac{a\nu}{D^2}\right), \ i.e., \mathrm{Re}_{\mathrm{cr\ min}} = \left(\frac{{}^{U}\max{}^{D}}{\nu}\right)_{\mathrm{cr\ min}} \ge \sqrt{a} \tag{4.8}$$

where $\sqrt{a} = [(3+\sqrt{13})/2]^{1/2}\pi \approx 5.71$. This 'estimate from below' of the critical Reynolds number may seem to be too low but we must remember that the value of $\mathrm{Re}_{\mathrm{cr\ min}}$ can be much smaller than that of $\mathrm{Re}_{\mathrm{cr}}$ found from transition experiments, and we should also take into account that the result (4.8) is based on rather crude inequalities and is very universal, being applicable to any bounded region of diameter D or less and to any flow in this region.

Serrin obtained similar estimates for flows in arbitrary straight channels of variable width not exceeding D (i.e., with width $H(y)$ which can depend on y and satisfies the condition $\max_y H(y) \le D$) and straight tubes of arbitrary cross section with diameter not exceeding D. Serrin proved that the inequalities (4.6) and (4.7) are valid for these cases too (with region V replaced by V'), except that the constant a in Eq. (4.6) is equal to π^2 in the case of a straight channel and to $2\pi^2$ in the case of a straight tube. Therefore, the new universal stability estimates have now the forms: $\mathrm{Re}_{\mathrm{cr\ min}} = (U_{\max} D/\nu)_{\mathrm{cr\ min}} \ge \pi \approx 3.14$ for channels of maximum width D and $\mathrm{Re}_{\mathrm{cr\ min}} = (U_{\max} D/\nu)_{\mathrm{cr\ min}} \ge \sqrt{2}\pi \approx 4.43$ for tubes of maximum diameter D.

Finally Serrin applied analogous arguments to a circular Couette flow between concentric cylinders of radii R_1 and R_2 rotating with angular velocities Ω_1 and Ω_2 (where index 1 relates to the inner cylinder and $\Omega_1 > 0$). Here (see e.g. Eq. (2.10), Sect. 2.6) the undisturbed velocity is given by

$$\mathbf{U(x)} = \mathbf{U}(r,\varphi,z) = \{U_r, U_\varphi, U_z\} = \left\{0, Ar + \frac{B}{r}, 0\right\},$$
$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \quad B = \frac{R_1^2 R_2^2(\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}. \tag{4.9}$$

Using these equations it is easy to show that

$$-\int_{V'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} \le |B| \int_{V'} \sum_{i=1}^{3} u_i^2 \frac{d\mathbf{x}}{r^2} \tag{4.10a}$$

$$\int_{V'} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j}\right)^2 d\mathbf{x} \ge b \int_{V'} \sum_{i=1}^{3} u_i^2 \frac{d\mathbf{x}}{r^2}, \quad b = \left[\frac{\pi}{\log(R_2/R_1)}\right]^2 \tag{4.10b}$$

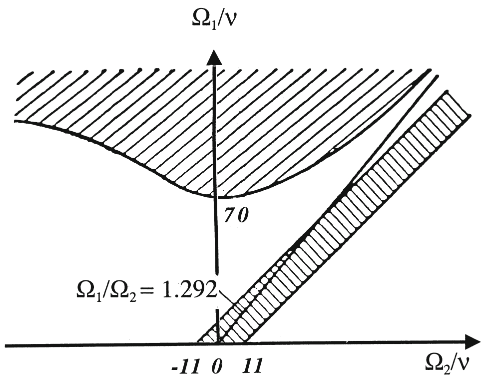


Fig. 4.2 Position of the region of instability to infinitesimal disturbances, and the region of stability to any finite disturbance, for Couette flow between rotating cylinders, studied by Taylor (1923). The *upper dashed* region corresponds to instability to infinitesimal disturbances, while flows corresponding to points of the shaded strip are definitely stable to any finite disturbance. (After Serrin (1959)) the continuous *straight line* in the figure is the boundary of the region of instability for the case of an inviscid fluid (Chap. 2)

where, as usual, V' is a part of the flow region having a width in the z direction equal to one disturbance wavelength (which can take an arbitrary value). Combining inequalities (4.10) with the R-O Eq. (4.2), Serrin obtained the relation

$$\frac{dE(t)}{dt} \leq (\,|B| - b\nu) \int\limits_{V'} \sum_{i=1}^{3} u_i^2 \frac{d\mathbf{x}}{r^2} \tag{4.11}$$

which, together with the expressions for the coefficients B and b, implies that circular Couette flow is stable to arbitrary disturbances if

$$\frac{|\Omega_2 - \Omega_1|}{\nu} \leq (R_2^2 - R_1^2)\left[\frac{\pi}{R_1 R_2 \log{(R_2/R_1)}}\right]^2 \tag{4.12}$$

(another proof of condition (4.12) was given by Joseph (1976), Sect. 37). The region (4.12) of the $(\Omega_1,\, \Omega_2)$-plane is usually only a small part of the region of stability to infinitesimal disturbances (see, for example, Fig. 4.2 for the case where $R_1 = 3.55$ cm, $R_2 = 4.03$ cm, so that $R_2/R_1 = 1.13$, which was studied experimentally by Taylor (1923); his results for this case were presented in Chap. 2, Fig. 2.10). The smallness of the region of universal stability in comparison to that of stability to infinitesimal disturbances may seem to be only natural, but let us also note that the result (4.12) is far from being exact, being based on the rather crude inequalities (4.10).

Shortly after the appearance of Serrin's paper of 1959, Velte (1962) improved the possible values of the coefficient a in the Poincaré inequalities (4.6) relating to the particular classes of fluid flows considered by Serrin. Namely, he showed that this coefficient is in fact not less than $6\pi^2$ in the case of flows in bounded regions of diameter D, not less than $3.74\pi^2$ for flows in straight channels of bounded width, and not less than $4.7\pi^2$ for flows in straight tubes of bounded diameter. These results imply the following sharpening of Serrin's estimates: $\text{Re}_{\text{cr min}} \geq \sqrt{6}\pi \approx 7.7$ for flows in bounded regions, $\text{Re}_{\text{cr min}} \geq \sqrt{3.74}\pi \approx 6.1$ for flows in straight channels, and $\text{Re}_{\text{cr min}} \geq \sqrt{4.7}\pi \approx 6.8$ for flows in straight tubes, where Reynolds number is based on the maximum flow velocity and the maximum diameter or width of the flow region.

The indicated estimates from below of $\text{Re}_{\text{cr min}}$ may be made more precise if their 'universality' is relaxed, i.e. if they are sought in more restricted sets of spatial regions V and/or velocity fields $\mathbf{U}(\mathbf{x})$. One of the first such attempts was made by Payne and Weinberger (1963) who considered the special case where V is a sphere of diameter D. These authors found that in this case the maximal possible value of the coefficient a in Eq. (4.6) is $4a_1^2$ where a_1 is the lowest positive root of the equation $\tan a_1 = a_1$. It follows from this that $a \approx 80$ and hence $\text{Re}_{\text{cr min}} \geq 8.94$ in the case considered. Since it is clear that the coefficient a cannot decrease when the region is shrinking, the last result gives the final improvement of Serrin's estimate of $\text{Re}_{\text{cr min}}$ for bounded regions of fixed maximal diameter, admitting no further corrections.

Later Sorger (1966a) (see also Joseph (1976), Sects. B7 and B8) independently considered the more general case of a region V bounded by two concentric spheres of radii R_1 and R_2, (where $0 \leq R_1 < R_2$, and $2R_2 = D$) and proved that here $\sqrt{a} = 4a_1^2$ where a_1 is the minimal zero of some combination of the Bessel functions of the first and second kinds, of order 3/2, taken at arguments a_1 and ηa_1 where $0 \leq \eta = R_1/R_2 < 1$ (for $\eta = 0$ this combination of Bessel functions becomes a function proportional to $\tan a_1 {}_- a_1$, as it must do according to the result of Payne and Weinberger). Sorger (1966a) also found exact analytical solutions of the variational problem of determininig the largest possible value of a for two-dimensional flows in a planar region V, bounded by a circle of diameter D or by two concentric circles of radii R_1 and $R_2 = D/2 > R_1$; he used these solutions to determine relatively narrow ranges for the true values of a (and thus also for values of $\text{Re}_{\text{cr min}} = a^2$) in the cases of flows in a circular tube or in a circular channel between two concentric cylinders. Some energy-method estimates of $\text{Re}_{\text{cr min}}$ for flows in unbounded regions which cannot be confined between a pair of parallel planes were given by Galdi and Rionero (1985), Chaps. 2 and 3; see also Chap. 5 of Straughan's book (1992).

In the second half of the 1960s, numerical methods began to be widely applied to solution of the main variational problem of the energy theory of hydrodynamic stability, for a number of primary flows U(x) given in various spatial regions V (some of these methods were considered by Straughan (1992), pp. 217–224). This allowed the determination, with good accuracy, of values of the stability bounds $\text{Re}_{\text{cr min}} = \min_{\text{u(x)}} R[\mathbf{u}(\mathbf{x})]$ (cf. Eq. (4.4)) for many important flows, both of homogeneous fluids of constant density ρ and of inhomogeneous fluids of variable density $\rho(\mathbf{x}, t)$ (dependent, for example, on the temperature $T(\mathbf{x}, t)$). The main results obtained in the late 1960s and early 1970s were summarized in the two-volume book by Joseph (1976). Let us recall in this respect that in Chap. 3, Sect. 3.4, it was mentioned that both Busse (1969) and Joseph and Carmi (1969) solved numerically the general (three-dimensional) variational problem of the energy stability theory for plane Poiseuille flow and found that $\mathrm{Re}_{\mathrm{cr\,min}} = 49.6$, while Joseph and Carmi simultaneously found that $\mathrm{Re}_{\mathrm{cr\,min}} = 81.5$ for circular Poiseuille flow in a round tube, and Joseph (1966) calculated that $\mathrm{Re}_{\mathrm{cr\,min}} = 20.7$ for plane Couette flow in a layer bounded by two parallel walls. (For Poiseuille flows Re is formed with the maximal velocity $U_{\max}$ and the channel half-width H_1 or the tube radius R, while in the case of Couette flow the half-difference of wall velocities U_0 and half-distance between walls H_1 are used as velocity and length scale). The paper by Joseph and Carmi also contains the energy-method determination of the value of $\mathrm{Re}_{\mathrm{cr\,min}}$ for Poiseuille flow (produced by a constant axial pressure gradient) in the annuli between two concentric round cylinders of different radii, while Joseph (1966) considered in addition the case of stratified Couette flow between parallel walls where the temperature of the lower wall is higher than that of the upper one (his main result for this case will be presented later). Note in conclusion that all the above-mentioned papers include the determination of the 'most dangerous' disturbances which correspond to the maximum value of $R[\mathbf{u}(\mathbf{x})]$ (i.e., are the most unstable). The results presented here, and also many results of the energy method of stability theory for more complicated flows (e.g., the pressure-gradient flows in annuli between concentric cylinders which are either sliding with respect to each other or rotating, or flows between rotating concentric spheres) can be found in the book by Joseph (1976). However, we wil not linger here to consider these more complicated flows. Instead, we will return to the applications of the energy method to the classical stability problem of Couette flow between concentric rotating cylinders.

Above, we mentioned the early, rather inaccurate, papers of Tamaki and Harrison (1920) and Harrison (1921) devoted to this problem, and also described the derivation by Serrin (1959) of the important universal stability condition (4.12). Note now that in the same paper Serrin supplemented the exact inequality (4.12) by some stronger but not fully rigorous conclusions. Namely, he assumed without proof that Orr's variational problem in the case of a Couette flow between rotating cylinders has an axially symmetric solution of the form $\mathbf{u}(\mathbf{x}) = \hat{\mathrm{u}}(r)e^{ikz}$ where cylindrical coordinates r, ϕ, z are now used and the wave number k takes arbitrary real values. Then the system of Euler-Lagrange differential equations determining the solution of the energy-method variational problem relating to such disturbances can be reduced to an eigenvalue problem for a linear system of two ordinary differential equations, with unkown functions $\hat{u}_r(r)$, $\hat{u}_\phi(r)$. Serrin could not solve this problem in the general case but he showed that in the case of a 'small gap' between the cylinders, i.e. where $R_2 - R_1 \ll (R_2 + R_1)/2$, his system of differential equations can be approximated by a pair of simpler equations, leading to an eigenvalue problem whose solution is known from previous work on hydrodynamic stability. Then the smallest eigenvalue of the problem studied (which depends on k so that the minimum over all real values of k must be considered) will determine the new stability criterion (with respect to axisymmetric disturbances) valid in the small-gap case. According to Serrin it has the form

$$\frac{|\Omega_2 - \Omega_1|}{\nu} \leq \frac{2\sqrt{1708}}{\sqrt{R_1 R_2}(R_2 - R_1)}. \tag{4.13}$$

This somewhat tentative criterion gives the 'stability region' in the form of a strip similar to that presented in Fig. 4.2 but having much greater width.

Later Sorger (1966b, 1967) proved, under rather wide conditions, the existence of an axially symmetric solution of the form $\mathbf{u}(\mathbf{x}) = \hat{\mathbf{u}}(r)e^{ikz}$ for Orr's variational problem for the case of circular Couette flow. He also developed a method to determine numerical values of the function $\hat{u}_r(r) = \hat{u}_r(r;k)$, and of the corresponding critical Reynolds number $\mathrm{Re}_{\mathrm{cr\ min}}\,(k)$, where $\mathrm{Re} = U(R_1)(R_2 - R_1)/\nu$, for various values of k and $\eta = R_1/R_2$. (Reynolds number $\mathrm{Re}_{\mathrm{cr\ min}}\,(k)$ determines the boundary of stability with respect to axisymmetric disturbances with wave number k).The results obtained were then compared with those obtained from the linear theory of hydrodynamic stability, and used to determine the dependence of the value of $\mathrm{Re}_{\mathrm{cr\ min}} = \min_{0 \le k \le \infty} \mathrm{Re}_{\mathrm{cr\ min}}\,(k)$ on the value of η.

More detailed study of the stability of circular Couette flow was carried out by Hung (1968) and Joseph (see Joseph and Hung (1971) and Joseph (1976), Chap. 5). In particular, Hung (1968) solved numerically the general (three-dimensional) Orr's variational problem relating to the circular Couette flow for a number of values of $\eta = R_1/R_2$, A and B (see Eq. (4.9)). The found solution determined the region of 'universal stability' of Couette flow to arbitrary disturbances. According to Hung's results (partially presented by Joseph (1976) in Sect. 37) the stability region in all cases studied was of the form $|\Omega_2 - \Omega_1|/\nu \le \tilde{R}_{\mathrm{cr}}(\eta)(R_2^2 - R_1^2)/(R_1R_2)^2$, where $\tilde{R}_{\mathrm{cr}}(\eta)$ is some universal function of η. We see that here again the stability region has the shape of a strip, similar to that presented in Fig. 4.2, whose width depends on values of R_1 and R_2. It was also found that the disturbances which first become unstable when $|\Omega_2 - \Omega_1|/\nu$ is increasing are axisymmetric in all the cases studied, and similar to the Taylor vortices described in Sect. 2.6. Remember that in this section it was also noted that according to experimental results over a wide range of flow conditions, when circular couette flow becomes unstable the appearing unstable disturbance mode is a set of axisymmetric Taylor vortices. These results stimulated Joseph and Hung to began a more complete energy-balance investigation of the stability of Couette flow to axisymmetric disturbances.[1]

Joseph and Hung integrated the equations of motion for the squares $u_r^2 = w^2, u_\phi^2 = v^2$ and $u_z^2 = u^2$ of the velocity components of an axisymmetric disturbance over the spatial region V' (whose span in the z-direction is equal to the wavelength of the disturbance), and considered equations for $\frac{1}{2}\frac{d}{dt}\langle w^2 + u^2\rangle = \frac{d}{dt}E^{(1)}(t)$ and $\frac{1}{2}\frac{d}{dt}\langle v^2\rangle = \frac{d}{dt}E^{(2)}(t)$ (where angle brackets denote the integrals over V') neither of which contains pressure terms (since $\partial p/\partial\phi = 0$ in the case of axisymmetric disturbances). Summing these two equations one will obtain the R-O energy Eq. (4.2) for $E(t) = E^{(1)}(t) + E^{(2)}(t)$, which is used in the energy method of stability theory. However, it is easy to see that the convergence to zero of $E(t) = \frac{1}{2}\left\langle \sum_{i=1}^{3} u_i^2 \right\rangle$ is only

[1] Joseph and Hung's paper of 1971 in fact represented a continuation of the work, unknown to them, of Pritchard (1968) who studied the same problem by the same method but restricted himself to consideration of linearized dynamic equations. (Pritchard's paper will be described at greater length in Sect. 4.12).

one of many consequences of the decay to zero of disturbance velocity u(x*, t*) as $t \to \infty$. Since the half-sum of squared velocity components has an important physical meaning, the requirement that $E(t) \to 0$ is a very attractive stability condition. Nevertheless, instead of this we may in principle require that the spatial average of some other nondegenerate positive-definite quadratic form of velocity components converges to zero as $t \to \infty$. Such convergence also shows that the disturbance decays to zero (and quite often it implies also the convergence to zero of $E(t)$); hence the absence of an explicit physical meaning of the selected quadratic form cannot be considered as a radical defect of the new method. These arguments clearly allow one to suggest a great number of modifications of the classical energy method of stability analysis.

Joseph and Hung (1971) (see also Joseph (1976), Sect. 40) proposed to use the condition $d[E^{(1)}(t) + \lambda E^{(2)}(t)]/dt = dE_\lambda(t)/dt < 0$, where λ is some positive constant and the disturbance velocity field u(x) is axisymmetric, as a new condition of stability with respect to axisymmetric disturbances (replacing the more usual requirement of negativity of $dE(t)/dt$). The new condition implies that $E_\lambda(t) = E^{(1)}(t) + \lambda E^{(2)}(t)$ decays to zero monotonically as $t \to \infty$; since $E(t) \le \min[1, \lambda]\ E_\lambda(t)$, the energy $E(t)$ also decays to zero in this case. However, if $\lambda \ne 1$, the set of primary Couette flows, and of initial disturbances **u**(**x**) for which $dE_\lambda(t)/dt < 0$, does not coincide with the similar set corresponding to the condition $dE(t)/dt \equiv dE_1(t)/dt < 0$. Therefore, the study of exact conditions guaranteeing that $dE_\lambda(t)/dt < 0$ for $\lambda \ne 1$ leads to the possibility of finding some new classes of stable disturbances of circular Couette flows.

The Reynolds–Orr energy-balance Eq. (4.2) implies that the given class $\breve{\mathbf{D}}$ of velocity disturbances **u**(**x**) is certainly stable (and what is more, its kinetic energy decays monotonically as $t \to \infty$), if $\mathrm{Re} < \min_{u(x)\subset\breve{\mathbf{D}}} R[\mathbf{u}(\mathbf{x})] = \mathrm{Re}_{\mathrm{cr\ min}}$, where $R[\mathbf{u}(\mathbf{x})]$ is given by Eq. (4.4) and all lengths and velocities are measured in the units L and U used in the definition of Re. (Below it will be assumed that class $\breve{\mathbf{D}}$ consists of all axisymmetric velocity fields, therefore the value $\mathrm{Re}_{\mathrm{cr\ min}}$ will refer to axisymmetric disturbances only). However the balance equation for the 'modified energy' $E_\lambda(t)$ with $\lambda \ne 1$ differs from Eq. (4.2); therefore, conditions guaranteeing that $dE_\lambda(t)/dt < 0$ must also differ from conditions guaranteeing the negativity of $dE(t)/dt$. The most important difference between the balance equations for $E_\lambda(t)$ and for $E(t)$ is due to the fact that the nonlinear terms of the N–S equations for u(x, t) do not contribute to the dynamic equation for $dE(t)/dt$ but do affect the value of $dE_\lambda(t)/dt$. It was noted above that the absence from Eq. (4.2) of the terms produced by the nonlinear terms in these N–S equations means that all terms on the right-hand side of this equation are of second order with respect to the velocity components u_i. Therefore, the ratio $R[\mathbf{u}(\mathbf{x})]$ of such terms does not depend on the intensity ('amplitude') of the disturbance **u**(**x**). However, if $\lambda \ne 1$, then the equation for $dE_\lambda(t)/dt$ contains a term of third order in the velocity components (equal to $(1-\lambda)\,(wv^2/r)$). This makes the sign of $dE_\lambda(t)/dt$ dependent not only on ν (i.e. on Reynolds number Re), the parameters R_1, R_2 and Ω_1, Ω_2 of the primary Couette flow, and the shape of the initial disturbance **u**(**x**), but also on some characteristic of the

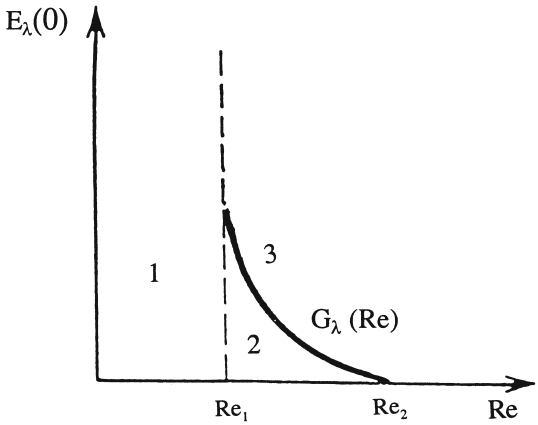


Fig. 4.3 Schematic representation of Joseph and Hung's (1971) results for stability of a flow between rotating cylinders to axisymmetric disturbances. $Re_1 = Re_{cr\ min}$; $Re_2 = Re_{\lambda,cr}$; *1*-the region of global stability and monotonic decay of $E(t)$; *2*-the region of conditional stability and monotonic decay of $E_\lambda\ (t)$; *3*-combined region of nonmonotonic stability and instability

intensity of $\mathbf{u}(\mathbf{x})$ (it is convenient to use the value of $E_\lambda(0)$ as such characteristic). As a result the main conclusion derived by Joseph and Hung from the study of conditions guaranteeing the negativity of $dE_\lambda(t)/dt$ has the form of a theorem about the *conditional stability* of axisymmetric disturbances in a circular Couette flow, determining a new stability region for such disturbances. The new stability region is a part of the (Re, E_λ)-plane which consists of such points that at the Reynolds number Re the 'generalized energy' $E_\lambda(t)$ of any axisymmetric disturbance with the initial 'energy' $E_\lambda\ (0) < E_\lambda$ decays monotonically to zero as $t \to \infty$ (and hence the energy $E(t)$ also decays to zero but its decay can be nonmonotonic). Note that for some nonnegative values of λ the new stability region can perfectly well include some points where $Re > Re_{cr\ min}$ and hence the energy $E(t)$ will not decay monotonically. In this case the new result represents an informative specification of the general statement about the possible existence of conditionally stable flows illustrated in Fig. 4.1 (see schematic Fig. 4.3 which represents graphically just this case of the Joseph and Hung theorem).

We will not give here the exact formulation of the theorem by Joseph and Hung but only its general character. The role of the functional (4.4) is now played by the functional $R_\lambda[\mathbf{u}(\mathbf{x})] = \langle D_\lambda[\mathbf{u}(\mathbf{x})\rangle / \langle P_\lambda[\mathbf{U}(\mathbf{x}), \mathbf{u}[\mathbf{x}]\rangle$, where $\langle D_\lambda[u(x)]\rangle$ is the sum of the viscous terms in the equation of motion for $E_\lambda\ (t) = \langle u^2 + \lambda\ v^2 + w^2 \rangle /2$, divided by the kinematic viscosity v (more exactly, by $(Re)^{-1}$ since the equation for E_λ (t) is now assumed to be non-dimensionalized) while $\langle P_\lambda\ [\mathbf{U}(\mathbf{x}),\ \mathbf{u}(\mathbf{x})]\ \rangle$ is the sum of production terms, linear in the undisturbed velocity gradient $dU(r)/dr$. The new functional $R_\lambda\ [\mathbf{u}(\mathbf{x})]$ is a natural replacement for the functional (4.4) when the 'modified kinetic energy' $E_\lambda\ (t)$ is considered instead of the energy $E(t)$. Let $Re_{\lambda,cr}$ denote $\min_{u(x)} R_\lambda\ [\mathbf{u}(\mathbf{x})]$, where the minimum is taken over the whole class of disturbance velocities considered (i.e., over the class of velocities of all axisymmetric disturbances). Then, if $Re < Re_{\lambda,cr}$, the sum of all right-hand-side terms of the equation for $dE_\lambda(t)/dt$ which are of second order in the components u_i will be negative. However, this does not mean that the derivative $dE_\lambda(t)/dt$ will necessarily be negative, since the term of the equation for $dE_\lambda(t)/dt$ which is of third order in the components u_i is not taken into account here. Remember that the size of this term, relative to the terms which are quadratic in u_i, depends on the intensity of the disturbance **u(x)** (which can be measured, e.g., by the modified kinetic energy $E_\lambda(0)$) and increases with increase of this intensity. Therefore, a condition guaranteeing the negativeness of $dE_\lambda(t)/dt$ must in some way restrict possible values of the initial disturbance intensity and thus diminish the possible influence of the third-order term.

These circumstances explain the following final form of the theorem found by Joseph and Hung (1971): *if* $\mathrm{Re} < \mathrm{Re}_{\lambda,\mathrm{cr}}$ and $E_\lambda(0) < G\,(\mathrm{Re}_{\lambda,\mathrm{cr}} - \mathrm{Re}, \lambda, R_1, R_2, \Omega_1, \Omega_2)$, *where G is a definite function of given arguments proportional to* $(\mathrm{Re}_{\lambda,\mathrm{cr}} - \mathrm{Re})^2$, *then* $dE_\lambda(t)/dt < 0$ *for any nonnegative value of t, and* $E_\lambda(t)$ *decays to zero monotonically and not slower than expotentially* (hence $E(t) = \min[1, \lambda]\, E_\lambda(t)$ also decays to zero not slower than expotentially). This theorem clearly makes sense only if $\mathrm{Re}_{\lambda,\mathrm{cr}} > \mathrm{Re}_{\mathrm{cr\,min}} = \min_{\mathbf{u}(\mathbf{x})} R[\mathbf{u}(\mathbf{x})]$ and also $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr\,min}}$, since at $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr\,min}}$ the energy of any axisymmetric disturbance decays monotonically to zero. However, if $\mathrm{Re}_{\mathrm{cr\,min}} < \mathrm{Re} < \mathrm{Re}_{\lambda,\mathrm{cr}}$, then Joseph and Hung's theorem contains valuable information: it proves that here the 'generalized energy' $E_\lambda(t)$ of any axisymmetric disturbance, with an initial amplitude so small that $E_\lambda(0) < G = G_\lambda(\mathrm{Re})$ (for the sake of simplicity other arguments of the function G are here omitted) decays monotonically to zero in a circular Couette flow. This means that for Re within this interval, axisymmetric disturbances are *conditionally stable* (namely, stable under the condition that $E_\lambda(0) < G_\lambda(\mathrm{Re})$). Since the value of G is proportional to $(\mathrm{Re}_{\lambda,\mathrm{cr}} - \mathrm{Re})^2$, it vanishes at $\mathrm{Re} = \mathrm{Re}_{\lambda,\mathrm{cr}}$ and hence at this value of Re the theorem can be applied only to infinitesimal disturbances (see again the schematic Fig. 4.3 illustrating the Joseph–Hung theorem). In this figure $\mathrm{Re}_2 = \mathrm{Re}_{\lambda,\mathrm{cr}}$ represents the smallest Reynolds number at which there exists an axisymmetric disturbance having arbitrarily small value of $E_\lambda(0)$ and such that its 'generalized energy' $E_\lambda(t)$ does not decay monotonically to zero as $t \to \infty$. The value $\mathrm{Re}_{\lambda,\mathrm{cr}}$, which clearly must be greater than $\mathrm{Re}_{\mathrm{cr}}$, depends on the choice of 'energy' $E_\lambda(t)$(and of the class of considered disturbances). This value differs from the Reynolds number $\mathrm{Re}_1 = \mathrm{Re}_{\mathrm{cr\,min}}$, determining the threshold below which the energy $E(t)$ of any axisymmetric disturbance decays monotonically, and can exceed this number. Figure 4.3 refers just to this case.

It was noted above that λ can be chosen as any positive number. Note now that the usefulness of the Joseph-Hung theorem increases as the number $\mathrm{Re}_{\lambda,\mathrm{cr}}$ and the function $G_\lambda(\mathrm{Re})$ shown in Fig. 4.3 increase, leading to enlargement of the region of stable disturbances indicated in this figure. Joseph and Hung showed that when $\mathrm{Re} < \mathrm{Re}_{\lambda,\mathrm{cr}}$ is fixed, the value of $G_\lambda(\mathrm{Re})$ increases without limit as $R_1/R_2 \to 1$ or $\lambda \to 1$ (the last result agrees well with the known fact that no restriction of the disturbance amplitude is needed at $\lambda = 1$). Moreover, these authors also considered the problem of determination of the optimum value λ_0 of λ corresponding, at given values of R_1, R_2, Ω_1 and Ω_2, to the maximum possible value of $\mathrm{Re}_{\lambda,\mathrm{cr}}$. They proposed a relatively simple numerical method for computation of λ_0. Especially simple results were obtained for the case where $R_2^2\Omega_2 < R_1^2\Omega_1$ and $\Omega_2/\Omega_1 > 0$. In this case an analytic approximation of high precision was found for the optimal value λ_0. Using this approximation it was possible to compute quite accurately the values of $\mathrm{Re}_{\lambda_0,\mathrm{cr}}$ (i.e., of the maximum value of Re at which Joseph and Hung's theorem makes sense). It was found that here the values of $Re_{\lambda_0,\mathrm{cr}}$ practically coincide with the values of the critical Reynolds numbers $\mathrm{Re}_{\mathrm{cr}}$ given by the linear theory of hydrodynamic stability. This coincidence may be considered as being natural, since the critical values $\mathrm{Re}_{\mathrm{cr}}$ and $Re_{\lambda_0,\mathrm{cr}}$ both apply here only to infinitesimal axisymmetric disturbances $\mathbf{u}(\mathbf{x})$ (because for the class of Couette flows studied by Joseph and Hung the linear stability theory shows that $\mathrm{Re}_{\mathrm{cr}}$ is just the boundary of stability with respect to axisymmetric disturbances). Neverthelss, the coincidence is interesting, since it connects results obtained by two different approaches to the same problem. Note also that the approach by Joseph and Hung inspired many subsequent studies of various stability problems which will be considered at the end of Sect. 4.13.

An even more surprising coincidence relating to the same problem was found slightly earlier by Busse (1970). He considered the classical energy method of Reynolds and Orr, and compared stability results given by this method with those following from the linear theory of hydrodynamic stability. He analyzed the 'narrow gap' approximation, where $(R_2 - R_1)/(R_2 + R_1) \ll 1,$ assuming that the relative difference of angular velocities also asymptotically vanishes simultaneously, so that $(\Omega_2 - \Omega_1)/(\Omega_2 + \Omega_1) \ll 1.$ Busse found that then, if in addition $(\Omega_2 - \Omega_1)/(\Omega_2 + \Omega_1) = -4(R_2 - R_1)/(R_2 + R_1),$ the Reynolds-Orr energy method leads to an eigenvalue problem which coincides exactly with the eigenvalue problem (2.17–2.17′) arising in the linear theory of hydrodynamic stability for circular Couette flow. Therefore, in this case the stability boundaries (the critical Reynolds numbers, Re, or Taylor numbers, $\mathrm{Ta} = \Omega_1^2 R_1 (R_2 - R_1)^3/\nu^2$ often used instead of Re) given by the linear stability theory for the case of infinitesimal disturbances and by the energy method for disturbances of arbitrary size are exactly the same (and hence $\mathrm{Re}_{\mathrm{cr}} = \mathrm{Re}_{\mathrm{cr\ min}}$, $\mathrm{Ta}_{\mathrm{cr}} = \mathrm{Ta}_{\mathrm{cr\ min}}$). A similar result was obtained by Busse for a plane Couette flow rotating around the y-axis with some definite angular velocity; in this case it was again found that $\mathrm{Re}_{\mathrm{cr}} = \mathrm{Re}_{\mathrm{cr\ min}}$. Some other examples (dating as far back as the 1950s) of flows where the critical values of the dimensionless flow parameter given by the linear stability theory and by the energy method coincide with each other will be considered in the following subsection; see also the paper by Wahl (1994) which contains further examples.

The discovery of flows where $\mathrm{Re}_{\mathrm{cr}} = \mathrm{Re}_{\mathrm{cr\ min}}$ evidently refutes the opinion, which was popular in the first half of the twentieth century, that the stability region given by the energy method must in principle be much smaller than the stability region determined by the linear theory of hydrodynamic stability. This discovery was then supplemented by the development by Joseph and Hung (1971) of the method which enlarged the region of validity of 'energy stability' results by introduction of the concept of 'conditional stability' and replacement of the energy density $E(t)$ by some other positive-definite functional of disturbance variables. This work led to a considerable revival of interest in the energy (and generalized-energy) methods of stability theory. Many of the papers devoted to this subject concerned motion of fluids with varying temperature (and hence also density) in a gravitational field producing a significant buoyancy effect. Therefore it will be reasonable to begin the next subsection by considering energy-method investigations of flow stability for a fluid with variable temperature.

4.1.2 Stability of Convective Motions and Related Stability Problems

In Subsect. 4.11 much attention was given to the classical Taylor problem of stability of Couette flow between coaxial rotating cylinders. However, there is another classical problem which also played a very important part in the early development of the linear theory of hydrodynamic stability. This is the famous Bénard–Rayleigh problem of stability of a stationary horizontal layer of fluid heated from below, which was considered in Sect. 2.7. Now we will turn to the applications of the energy method to this and to some other stability problems where buoyancy forces are of great importance.

In the case of motion of a fluid of variable temperature under gravity, the N–S dynamic equations must be replaced by some more general equations. Under rather general conditions, which in this book will be assumed to be always valid, we can neglect density variations except in the buoyancy term and, as in Sect. 2.7, use the Boussinesq equations. Let $\mathbf{U}(\mathbf{x})$ be the primary velocity field (it can also depend on time t but we will not consider this case) and $T(\mathbf{x})$ be the undisturbed temperature field. Then the nonlinear Boussinesq, continuity and heat conduction equations for the disturbances u_i and ϑ of the velocity and temperature will have the following form:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_j}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + \delta_{i3} g \beta \vartheta, \quad i = 1,2,3, \tag{4.14a}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{4.14b}$$

$$\frac{\partial \vartheta}{\partial t} + u_i \frac{\partial T}{\partial x_i} + U_i \frac{\partial \vartheta}{\partial x_i} + u_i \frac{\partial \vartheta}{\partial x_i} = \chi \nabla^2 \vartheta, \tag{4.14c}$$

where p is the deviation of the pressure field from the undisturbed pressure P, g is the acceleration due to gravity, and β is the coefficient of thermal expansion of the fluid. The boundary conditions on stationary solid walls at constant temperature have a very simple form: $\mathbf{u}(\mathbf{x}, t) = \vartheta(\mathbf{x}, t) = 0$. More complicated boundary conditions must be used in the cases of moving walls, solid walls of non-constant temperature (i.e., those which have fixed finite thermal conductivity, or are characterized by fixed heat flux normal to the wall), and free surfaces of liquids; see e.g. the discussion of this question in Sect. 2.7 of this book, and in Sect. 55 of Joseph's book (1976) where some additional references relating to this subject can also be found. However, in the discussion below most attention will be given to the simplest case of zero boundary conditions for velocity and temperature disturbances at the walls.

The Boussinesq Eq. (4.14a) differ from the N-S equations only by the additional term $g\beta\vartheta$ in the equation for $u_3 = w$. This term produces an extra term in the energy-balance Eq. (4.2) which now takes the form

$$\frac{dE(t)}{dt} = -\int\limits_{v'} u_j u_i \frac{\partial U_j}{\partial x_i} d\mathbf{x} + g\beta \int\limits_{v'} u_3 \vartheta d\mathbf{x} - \nu \int\limits_{v'} \sum_{j,i=1}^{3} \left(\frac{\partial u_j}{\partial x_i}\right)^2 d\mathbf{x}$$

$$= -\left\langle u_j u_i \frac{\partial U_i}{\partial x_i} \right\rangle + g\beta \left\langle u_3 \vartheta \right\rangle - \nu \left\langle \sum_{j,i=1}^{3} (\frac{\partial u_j}{\partial x_i})^2 \right\rangle = \mathbf{P}_1 + \mathbf{P}_2 - \nu \mathbf{D} \quad (4.15)$$

which differs from the R-O Eq. (4.2) by the extra term $\mathbf{P}_2 = g\beta \left\langle u_3 \vartheta \right\rangle$ on the right side. This term can be easily estimated by the following crude inequality

$$g\beta\langle u_3\vartheta\rangle \leq g\beta\langle |u_3\vartheta|\rangle \leq g\beta\langle u_3^2\rangle\langle\vartheta^2\rangle^{1/2} \leq 2g\beta[E(t)E_T(t)]^{1/2} \quad (4.16)$$

where $E_T(t) = 0.5\langle\vartheta^2\rangle$ is an integral measure of the intensity of temperature disturbance (while $\langle\vartheta^2\rangle$ is often called the 'temperature variance'). Moreover, the heat-conduction Eq. (4.14c), together with the boundary conditions given above, leads to the following balance equation for the temperature-disturbance intensity $E_T(t)$

$$\frac{dE_T(t)}{dt} = -\left\langle \vartheta u_i \frac{\partial T}{\partial x_i} \right\rangle - \chi \left\langle \sum_{i=1}^{3} \left(\frac{\partial \vartheta}{\partial x_i}\right)^2 \right\rangle = \mathbf{P}_T - \chi \mathbf{D}_T. \quad (4.17)$$

Here the first term on the right-hand side is clearly less than or equal to $2\gamma[E(t)E_T(t)]^{1/2}$ where $\gamma = \max_{\mathbf{x}\in v} |\nabla T(\mathbf{x})|$, while the following analog of the inequality (4.6) can be proved for the factor $\mathbf{D}_\mathrm{T}$ in the second term

$$\mathbf{D}_\mathrm{T} = \left\langle \sum_{i=1}^{3} \left(\frac{\partial \vartheta}{\partial x_i}\right)^2 \right\rangle \geq \frac{a_T \pi^2}{D^2} \langle\vartheta^2\rangle = \frac{2a_T\pi^2}{D^2} E_T(t). \quad (4.18)$$

In Eq. (4.18) $a_T = 3$ if V is a bounded region of diameter D, and $a_T = 1$ for a horizontal layer of maximal thickness D. Combining the balance Eqs. (4.15) and (4.17) with the estimates of the terms of these two equations given above, we may obtain for the derivative $d[\sqrt{E(t)} + \lambda\sqrt{E_T(t)}]/dt$ (where the dimensional factor λ has a positive value) an inequality of the form

$$\frac{d[\sqrt{E(t)} + \lambda\sqrt{E_T(t)}]}{dt} \leq \lambda_1\sqrt{E(t)} + \lambda_2\sqrt{E_T(t)} \quad (4.19)$$

where expressions for the coefficients λ_1 and λ_2 include the dimensional constants λ; $\gamma = \max|\nabla \mathrm{T}|$; and the coefficients entering Eqs. (4.6), and (4.15–4.18). If $\lambda = \sqrt{(a - \mathrm{Re}^2)g\beta\nu/2a_T\pi^2\gamma\chi} = \lambda_0\sqrt{g\beta\nu/\gamma\chi}$, where $Re = U_{max}D/\nu, \lambda_0^2 =$

$(a - \mathrm{Re}^2)/2a_T\pi^2$ is a dimensionless constant, and it is assumed that $a > \mathrm{Re}^2$, then the inequality (4.19) takes an especially useful form. In this case $\lambda_1 = -\xi[\sqrt{(a - \mathrm{Re}^2)a_T\pi/2} - \sqrt{Ra}] = -\xi(\lambda_0 a_T\pi^2 - \sqrt{Ra}), \lambda_2 = \lambda\lambda_1$, where $\xi = \lambda_0\nu/D^2$ if $\lambda_0(\mathrm{Pr})^{1/2} \leq 1$ and $\xi = \chi/\lambda_0 D^2$ if $\lambda_0(\mathrm{Pr})^{1/2} > 1$, and where $\mathrm{Ra} = g\beta\gamma D^4/\nu\chi$ and $\mathrm{Pr} = \nu/\chi$ (cf. Joseph (1965)). It follows from this result that the convective motion will be universally (in other words, unconditionally or globally) stable to any disturbance of the velocity and/or temperature if

$$0 \leq \mathrm{Ra} < \frac{a_T\pi^2(a - \mathrm{Re}^2)}{2} \tag{4.20}$$

since under this condition, for the value of λ indicated above, we have

$$\sqrt{E(t)} + \lambda\sqrt{E_T(t)} \leq [\sqrt{E(0)} + \lambda\sqrt{E_T(0)}] \exp\left\{-\xi[\sqrt{a_T\pi^2(a - \mathrm{Re}^2)/2} - \sqrt{\mathrm{Ra}}]t\right\}. \tag{4.21}$$

The results (4.20–4.21) (obtained by Joseph(1965, 1966) in slightly different from) are similar to the Serrin-Velte-Sorger results of 1959–1967, derived for constant-density (non-convective) flows: they do not depend on any specific details of the flow geometry or on the distributions of the primary velocity and temperature fields. For the special case of a stationary horizontal fluid layer (for which $a_T = 1, a = 3.7\pi^2$, and $\mathrm{Re} = 0$) we obtain the result: $\mathrm{Ra}_{\mathrm{cr\ min}} > 1.85\pi^4 \approx 180$. The last result can easily be improved; in fact the inequality (4.7) is clearly unsatisfactory in the case of stationary fluid where its left-hand side is equal to zero. If we simply omit the first term on the right-hand side of (4.15) and then repeat all the arguments, we obtain twice as good an estimate: $\mathrm{Ra}_{\mathrm{cr\ min}} > 360$ (which is still much smaller than the value $\mathrm{Ra}_{\mathrm{cr}} = 1{,}708$ given by linear theory). A similar improvement can also be made in the estimate (4.20) of the boundary of the universal stability region in the (Ra, Re)-plane if one uses a different estimate of the first term on the right-hand side of Eq. (4.15) (giving zero for fluid at rest) and another definition of Reynolds number (see Joseph (1965)). Note however that in the case of primarily stationary fluid all the results obtained in this way were much weaker than the older results of Sorokin (1953, 1954) and several other workers who studied conditions for the appearance of convection in fluids at rest.

Sorokin considered the stability problem for a stationary fluid in a given spatial region V (he assumed it to be bounded but his arguments can be applied to many unbounded regions too). Using Eqs. (4.14a–c) he proved that under very general conditions (which are satisfied in almost all situations of practical interest) $\mathrm{Ra}_{\mathrm{cr\ min}} = \mathrm{Ra}_{\mathrm{cr}}$ where $\mathrm{Ra}_{\mathrm{cr}}$ is the critical Rayleigh number determined by the linear theory of hydrodynamic stability, while $\mathrm{Ra}_{\mathrm{cr\ min}}$ is the stability boundary given by the energy method. (This means that for $\mathrm{Ra} < \mathrm{Ra}_{\mathrm{cr\ min}}$ both $E(t)$ and E_T(t) decay monotonically with time).

Morcover, Sorokin also proved that the principle of exchange of stabilities is valid here, i.e. that the eigenfrequency ω corresponding to the most unstable mode, if such a mode exists, is real (and that all other eigenfrequencies ω_j are also real here). (Remember, that for the Bénard problem, where V is an infinite horizontal layer, the principle of exchange of stabilities was first proved by Pellew and Southwell (1940); see Sect. 2.7). At first, these important papers by Sorokin did not attract much attention, and some of his results were later independently rediscovered by a number of authors (in particular, by Ukhovskii and Yudovich (1963); Howard (1963); Sani (1964), and Platzman (1965)). Then Joseph (1965, 1966) also independently derived Sorokin's results, and some of their generalizations, by a new method and under more general conditions than those used in the previous publications. His derivation was later described in the book by the same author (see Joseph (1976), Chap. VIII) which played a very important part in the revival of interest in energy methods. Therefore only Joseph's approach will be outlined below.

Seeking the stability boundary in the (Ra, Re)-plane, Joseph investigated conditions guaranteeing the decay with time of the quantity $E_\lambda(t) = E(t) + \lambda E_T(t)$ where, as above, λ is a dimensional factor having positive value. According to Eqs. (4.15) and (4.17), the right-hand side of the equation for $dE\lambda(t)/dt$ includes three "production terms", $\mathbf{P}_1$, $\mathbf{P}_2$ and $\lambda\mathbf{P}_\mathrm{T,}$ and two "dissipation terms", $-\nu\,\mathbf{D}$ and $-\lambda\chi\mathbf{D}_\mathrm{T}$. The production terms can take positive values and they then describe the growth of the intensity of the velocity and temperature disturbances, caused by the interaction of flow disturbances with the primary flow. As to the dissipation terms, they are always negative and represent the decay of disturbance velocity and temperature fields caused by molecular viscosity and heat conductivity. Therefore, for decay of the 'modified energy' $E\lambda(t)$ of a disturbance with given velocity and temperature fields $\{\mathrm{u(x)},\vartheta(\mathrm{x})\}$, the sum of the absolute values $\nu\,\mathbf{D}$ and $\lambda\chi\ \mathbf{D}_\mathrm{T}$ of the dissipation terms must be greater than $\mathbf{P}_1 + \mathbf{P}_2 + \lambda\mathbf{P}_\mathrm{T}$.

Joseph made the balance Eqs. (4.15), (4.17) and the equation for $dE\lambda(t)/dt$ dimensionless, replacing the dimensional independent and dependent variables $x_i, t, U_i, u_i, T, \vartheta, E, E_T$, and the coefficient λ by $x_i^+ = x_i/L, t^+ = t\nu/L^2, U_i^+ = U_i/U_0, u_i^+ = u_i L/\nu, T^+ = T/\Theta_0, \vartheta^+ = \vartheta(\chi g\beta L^3/\nu^3\Theta_0)^{1/2}, E^+ = E/\nu^2 L, E_T^+ = E_T\chi g\beta/\nu^3\Theta_0$, and $\lambda^+ = \lambda(\Theta_0/g\beta L)$ where L, U_0 and θ_0 are typical length, velocity and temperature scales of the primary flow (these scales must be chosen in a reasonable way for every specific problem). It is easy to verify that Eqs. (4.15) and (4.17) then take the following forms

$$\frac{dE^+(t^+)}{dt^+} = -\,\mathrm{Re}\left\langle u_j^+ u_i^+ \frac{\partial U_j^+}{\partial x_i^+}\right\rangle + \sqrt{\mathrm{Ra}}\langle u_3^+\vartheta^+\rangle - \left\langle \sum_{i,j=1}^{3}\left(\frac{\partial u_i^+}{\partial x_j^+}\right)^2\right\rangle, \qquad (4.15\mathrm{a})$$

and

$$\mathrm{Pr}\,\frac{dE_T^+(t^+)}{dt^+} = -\sqrt{\mathrm{Ra}}\left\langle \vartheta^+ u_i^+ \frac{\partial T^+}{\partial x_i^+}\right\rangle - \left\langle \sum_{i=1}^{3}\left(\frac{\partial \vartheta^+}{\partial x_i^+}\right)^2\right\rangle \qquad (4.17\mathrm{a})$$

where $\mathrm{Re} = U_0 L/\nu, \mathrm{Ra} = g\beta\Theta_0 L^3/\nu\chi, \mathrm{Pr} = \nu/\chi$, and angular brackets now denote integration, with respect to dimensionless coordinates x_i^+, over the region V'.

Also, changing to dimensionless variables makes the quantity $E_\lambda(t) = E(t) + \lambda E_T(t)$ proportional to $E^+(t^+) + \lambda^+ \mathrm{P}_T E_T^+(t^+)$. In the rest of this section we will use only dimensionless variables, and for simplification of notation we will omit the superscript 'plus' signs. Hence, the symbol $E_\lambda(t)$ will now denote the sum $E(t) + \lambda\,\mathrm{Pr} E_T(t)$, and according to Eqs. (4.15a)and (4.17a) the balance equation for this quantity has the form

$$\begin{aligned}\frac{\partial E_\lambda(t)}{\partial t} &= -\mathrm{Re}\left\langle u_i u_j \frac{\partial U_j}{\partial x_i}\right\rangle + \sqrt{\mathrm{Ra}}\left(\langle u_3 \vartheta\rangle - \lambda\left\langle \vartheta u_i \frac{\partial T}{\partial x_i}\right\rangle\right)\\ &\quad -\left\langle \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \lambda \sum_{i=1}^{3}\left(\frac{\partial \vartheta}{\partial x_i}\right)^2\right\rangle\\ &= \mathrm{Re}\,\mathbf{P}_1 + \sqrt{\mathrm{Ra}}(\mathbf{P}_2 + \lambda \mathbf{P}_T) - \mathbf{D} - \lambda \mathbf{D}_T. \end{aligned} \tag{4.22}$$

Equation (4.22) takes an especially simple form in the case of stationary fluid, where $\mathrm{Re} = 0$. Here the critical Rayleigh number of the energy theory, $\mathrm{Ra}_{\mathrm{cr\,min}}$, can be determined from the equation

$$\mathrm{Ra}_{\mathrm{cr\,min}} = \left[\max_\lambda \min_{\mathbf{u(x)},\vartheta(\mathbf{x})} \frac{\mathbf{D} + \lambda \mathbf{D}_T}{\mathbf{P}_2 + \lambda \mathbf{P}_T}\right]^2 \tag{4.23}$$

where the minimum is taken over all solenoidal vector fields $\mathbf{u(x)}$ and scalar fields $\vartheta(\mathbf{x})$ satisfying the boundary conditions appropriate to the problem considered, and the maximum over all nonnegative values of λ (thus, the value of λ is varied in the search for the highest estimate of $\mathrm{Ra}_{\mathrm{cr\,min}}$). It can be shown that in the case where $\lambda = 1$ the Euler–Lagrange equations corresponding to the variational problem of finding the minimum in the right-hand side of (4.23) can be reduced to the same eigenvalue problem that appears in the linear stability theory applied to a given volume of stationary fluid with given temperature field $T(\mathbf{x})$. The critical Rayleigh number of linear stability theory, $\mathrm{Ra}_{\mathrm{cr}}$, is determined by the solution of this eigenvalue problem for the case of zero frequency (i.e. the eigenvalue $\omega = 0$) in exactly the same way that $\mathrm{Re}_{\mathrm{cr\,min}}$ is determined by the solution of the eigenvalue problem derived from the Euler-Lagrange equations.This means that $\mathrm{Re}_{\mathrm{cr\,min}} = \mathrm{Re}_{\mathrm{cr}}$ in this case, and that the optimal value of λ in Eq. (4.23) is $\lambda = 1$ (since the value of $\mathrm{Re}_{\mathrm{cr\,min}}$ clearly cannot be greater than $\mathrm{Re}_{\mathrm{cr}}$). In particular, for the Bénard-Rayleigh problem of stability of a horizontal layer of stationary fluid heated from below we find that $\mathrm{Re}_{\mathrm{cr\,min}} \approx 1{,}708$ in the case of two rigid walls at constant temperatures, while $\mathrm{Ra}_{\mathrm{cr\,min}} \approx 1{,}101$ for one rigid and one free boundary and $\mathrm{Ra}_{\mathrm{cr\,min}} \approx 657$ for the idealized case of two free boundaries, if the values $L = H$ (the distance between two walls) and $\Theta_0 = \Delta T$ (the difference between lower-wall and upper-wall temperatures) are used in the definition of the Rayleigh number (see Sect. 2.7).

In the more general case of a flow satisfying the Boussinesq equations and having given velocity and temperature fields $\mathbf{U(x)}$ and $T(\mathbf{x})$, the energy method can be used to find the stability boundary in the (Re, Ra)-plane, determining the region of (Re, Ra)-values that guarantes the decay of any initial disturbance regardless of its size. Such decay will clearly occur if, for at least one positive value of λ, the sum of dissipation terms $\mathbf{D} + \lambda\mathbf{D}_\mathrm{T}$ for any initial disturbance $\{\mathbf{u}(\mathbf{x}), \vartheta(\mathbf{x})\}$ is greater than the sum of production terms Re $\mathrm{P}_1 + \sqrt{\mathrm{Ra}}(\mathbf{P}_1 + \lambda\, \mathbf{P}_T)$. Hence the stability boundary in the (Re, Ra)-plane will now coincide with the boundary of the largest region in this plane in which, for some positive value of λ, the ineaquality $(\mathbf{D} + \lambda\mathbf{D}_\mathrm{T}) \geq (\mathrm{Re}\mathbf{P}_1 + \sqrt{\mathrm{Ra}}(\mathbf{P}_2 + \lambda\mathbf{P}_T))$ is valid for any values of $\{\mathrm{u}(\mathrm{x}), \vartheta(\mathrm{x})\}$ satisfying the boundary conditions of the problem considered.

Determination of the boundary curve in the (Re, Ra)-plane is a more difficult problem than in the case of fluid at rest, when only a boundary point on the Ra-axis must be found. Joseph (1966) proposed to assume at the beginning that that $\mathrm{Re}/\sqrt{\mathrm{Ra}} = \mu$ is fixed. Then at fixed values of λ and μ the boundary value of Ra (and hence also of $\mathrm{Re} = \mu\sqrt{\mathrm{Ra}}$) may be found from the equation

$$Ra(\lambda, \mu) = \left[\min_{\mathbf{u}(\mathbf{x}),\partial(\mathbf{x})} \frac{\mathbf{D} + \lambda\mathbf{D}_T}{\mu\mathbf{P}_1 + \mathbf{P}_2 + \lambda\mathbf{P}_T}\right]^{\mathbf{2}}. \tag{4.24}$$

It follows that, at a fixed value of μ, the optimal value of Ra (i.e., the value of $\mathrm{Ra}(\mu)_{\mathrm{cr\ min})}$ is equal to $\max_{\lambda > 0}$ Ra(λ, μ). Then $(\mu\sqrt{\mathrm{Ra}_{\mathrm{cr\ min}}(\mu)}, \mathrm{Ra}_{\mathrm{cr\ min}}\ (\mu))$ is a point of the boundary curve in the (Re, Ra)-plane and the set of all such points corresponding to nonnegative values of μ forms the whole of this curve.

As an example Joseph considered the case of a plane Couette flow heated from below, i.e., of a Couette flow in a layer between rigid planes at $z = 0$ and $z = H$ having different temperatures T_0 and $T_1 = T_0 - \Theta_0$ where $\Theta_0 > 0$. The determination of the boundary curve in the (Re, Ra)-plane can be simplified here, since it can be proved that the optimal value of λ is 1 (hence $\max_{\lambda < 0}$ Ra(λ, μ) = Ra(1, μ) at any μ). (This is connected with the fact that the temperature gradient ∇T is directed everywhere along the negative z-axis, i.e., has the same direction as the acceleration due to gravity; see Joseph (1976), Sect. 61, and Straughan (1992), pp. 60–61). Moreover, it can be shown that the most-unstable disturbance $\{\mathbf{u}(\mathbf{x}), \vartheta(\mathbf{x})\}$ minimizing the functional on the right-hand side of Eq. (4.24) has the form of streamwise rolls independent of the horizontal coordinate x. This allows the search for the minimum in the right side of (4.24) to be reduced to an eigenvalue problem for a system of ordinary differential equations. It turns out that this system may be transformed, by simple replacement of parameters, into a system equivalent to that appearing in the linear stability theory of a stationary layer of fluid heated from below. Using the known results of this theory, Joseph proved that any disturbance $\mathbf{u}(\mathbf{x})$, $\vartheta(\mathbf{x})$ will decay in a plane Couette flow heated from below if the values (Re, Ra) satisfy the inequality $4\mathrm{Re}^2 + \mathrm{Ra} < 1{,}708$, where Ra has the same meaning as above and $\mathrm{Re} = U_0 H_1/\mathrm{v}$ where U_0 is the half-difference of the two wall velocities and $H_1 = H/2$ is the half-width of the channel (this is the definition of Re for a plane Couette flow already used in Sects. 2.1, 3.3 and 3.4). This result shows again that for the case of fluid at rest (when Re = 0) the energy method gives the estimate $\mathrm{Ra}_{\mathrm{cr\ min}} = 1{,}708$, the same as the critical value given by linear stability theory. At the same time, for the unstratified problem (when Ra = 0) the estimate found ($\mathrm{Re}_{\mathrm{cr}}\ \mathrm{min} = \sqrt{1{,}708}/2 \approx 20.7$) is much smaller, not only smaller than the critical value $\mathrm{Re}_{\mathrm{cr}} = \infty$ given by the normal-mode method of linear stability theory but also smaller than the minimal values Re_1 of Re at which the instability of a plane Couette flow has been observed in the most accurate modern experiments and numerical simulations. (Recall that according to results presented in Sect. 2.1 of Chap. 2, Re_1 lies in the range from 320 to 370. Note also the conclusion by Hamilton et al. (1995) that turbulence cannot be sustained in a plane Couette flow at $Re \leq 300$, and the results of recent experiments by Bottin et al. (1998a, b) and Bottin and Chaté (1998), and numerical simulations by Barkley and Tuckerman (1998, 1999) according to which $Re_1 \approx 325$). It is, however, incorrect to say, as it often is, that it follows that the energy method is exact in the case of the pure convection problem but gives very poor results when applied to the non-convective Couette flow. In fact the results show only that for the Bénard-Rayleigh problem $Ra_{cr\,min} = Ra_{cr}$ (which is an exception), while for a plane Couette flow $Re_{cr\,min}$ is much smaller than $Re_{cr,}$ while the minimal value of Re at which instability is observed is here greater than $Re_{cr\,min}$ but smaller than Re_{cr} (this may be considered as being normal).

The methods for determination of the stability boundaries by the energy method and its modifications developed by Joseph (1965, 1966) can be applied to many other fluid-dynamic problems. A number of such problems was considered in Joseph's book (1976). Thus, for example, stability was studied for flows of a liquid with density depending on disturbed fields both of temperature, $T(\mathbf{x}) + \vartheta(\mathbf{x}, t)$, and of concentration of some admixture (e.g., salinity), $C(x) + c(x, t)$. The Boussinesq approximation was assumed to be valid here too but now it leads to an equation for $u_3 = w$ which includes a term proportional to c; therefore the diffusion equation must now be added to Eq. (4.14). In this case, Joseph replaced the function $E_\lambda(t) = E(t) + \lambda E_T(t)$ by the function $E_{\lambda_1,\lambda_2}(t) = E(t) + \lambda_1 E_T(t) + \lambda_2 E_C(t)$, where $E_C(t) = \langle c^2 \rangle$. It was shown that if the liquid is stationary, while the temperature gradient is directed downwards and the salinity gradient is directed upwards ('heating from below and salting from above') the critical parameters obtained from the linear and energy theories coincide, as in the case where only heating from below takes place. However, if both gradients ∇T and ∇C are directed downwards (the case of heating and salting from below) the two gradients produce opposite effects and here quite new solutions can appear. Other applications of energy methods considered in Joseph's book include, in particular, the cases of Boussinesq fluids with internal heat sources; convection in spherical layers, in porous media heated from below and in some non-Newtonian fluids; and stability of magneto-hydrodynamic flows. For more details relating to these and other applications of energy methods see, e.g., the papers by Joseph and Shir (1966); Joseph and Carmi (1966); Shir and Joseph (1968); Joseph (1970, 1988); Bhattacharyya and Jain (1971), and Ayyaswami (1971), and the numerous publications on this subject appearing in the 1980s and early 1990s. These newer publications include special monographs by Straughan (1982, 1992) and Galdi and Rionero (1985), a collection of papers edited by Galdi and Straughan (1988), an extensive survey paper by Galdi and Padula (1990) (these sources contain several hundred references), and a great number of research papers only a small part of which will be referred to below.

In the more recent literature on the energy method in hydrodynamic stability, most effort has been devoted to the extension of the classical Reynolds–Orr method of nonlinear stability analysis. Remember that in some of the above-mentioned stability investigations conditions were considered for the decay, not of $E(t)$ but of some other positive functions $E_\lambda(t)$ (or $E_{\lambda_1,\lambda_2}(t)$). Thus, the stability criteria were based, not on the kinetic energy of disturbance but on some other positive-definite quadratic forms of disturbance variables. It was therefore only natural that later some authors began the search for possible improvements of known results of the energy method by replacing the energy functional $E(t)$ by another integrated positive definite quadratic form. Some of these methods of stability analysis were called *weighted energy methods* while the name *generalized energy methods* was often applied to all such methods. However, even more often they are called *Lyapunov methods* since in fact they represent an application to fluid mechanics of the well-known direct (or second) Lyapunov method of stability analysis. (This method forms the most important part of the general theory of stability of motion developed by Lyapunov (1892) in his doctoral dissertation[2]). The direct Lyapunov method later gained wide popularity and was expounded in a great number of textbooks, special monographs, and collections of papers (see, e.g., Zubov (1957); LaSalle and Lefschetz (1961); Kazda (1962); Hahn (1963); Yoshizawa (1996), and Rouche et al. (1977)). In the first half of the twentieth century this method was mostly used to study the stability of dynamic systems having a finite number of degrees of freedom and described by ordinary differential equations; later, however, some of its applications to systems described by partial differential equations were also considered, e.g., by Zubov (1957); Movchan (1959); Knops and Wilkes (1966), and Lakshmikantham and Leela (1969). In the 1960s the first applications of the Lyapunov method to fluid mechanics appeared, quite independently of work based on the R-O Eq. (4.2). Later, Lyapunov's approach to stability of fluid motion underwent considerable development, and in fact formed a new branch of hydrodynamic stability theory having many points of contact, but nevertheless not merging, with work on generalizations of the classical energy method of Reynolds and Orr.

4.1.3 Applications of the Direct Lyapunov Method and Generalized Energy Functionals. Arnol'd's Variational Method

Lyapunov's stability was mentioned in Sects. 3.21 and 3.23, when the papers by Dikii (1960a, b) were considered. As was explained in Sect. 3.21 (see in particular footnote no. 1 there) Lyapunov's stability presupposed that some norm $\| \bullet \|$

[2] About 25 years later, in 1918, this brilliant Russian scientist, a member of the Russian Academy of Sciences, died at the age of 61 from hunger and lack of appropriate medical help in the city of Odessa enveloped in a civil war between bolsheviks and their opponents.

was introduced in the phase space H of the dynamical system considered, making **H** *a linear normed space*[3]. In problems on hydrodynamic stability*, Lyapunov's stability* of the 'primary flow' $\mathbf{U}_0(t)$, $0 \le t < \infty$ (where $\mathbf{U} = \mathbf{U}(\mathbf{x})$ is a collection of hydrodynamic fields uniquely determining the flow), means that for any $\varepsilon > 0$ there exists such a number $\delta(\varepsilon) > 0$ that the inequality $\|\mathbf{U}(0) - \mathbf{U}_0(0)\| < \delta(\varepsilon)$ implies that $\|\mathbf{U}(t) - \mathbf{U}_0(t)\| < \varepsilon$ for any nonnegative t. (Sometimes it is also additionally required that $\|\mathbf{U}(t) - \mathbf{U}_0(t)\| \to 0$ as $t \to \infty$, either for any $\mathbf{U}(0)$ or under the condition that $\|\mathbf{U}(0) - \mathbf{U}_0(0)\| < d$ for some given $d > 0$; if so then Lyapunov's stability is called *asymptotic*). The phase space H is here the functional space of all possible values of $\mathbf{U}$(x) (in the cases where the velocity field uniquely determines the flow, **H** is the space of all solenoidal vector fields **u(x)** satisfying the appropriate boundary conditions). The norm in such a space is usually given by the square root of the integral, over the set of points **x**, of some non-degenerate positive-definite quadratic form of components $U_1(\mathbf{x})$, $U_2(\mathbf{x})$, $U_n(\mathbf{x})$ of the vector function U(x). Then $\|\mathrm{U}(\mathbf{x})\|^2$, the square of the norm of $\mathbf{U}(\mathbf{x})$, is a function of the functional argument $\mathbf{U}(\mathbf{x})$. Functions of functional arguments in mathematics are called *functionals;* therefore $\|\mathbf{U}(\mathbf{x})\|^2 = L[\mathbf{U}(\mathbf{x})]$ is a functional in the space H. The Lyapunov condition for stability (representing the main theorem of Lyapunov's second method) in application to stability of the primary flow $\mathbf{U}_0(\mathbf{x}, t)$ has the following form: If $\mathbf{U}(\mathbf{x}, t) = \mathbf{U}_0(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t)$ (so that $\mathbf{u}(\mathbf{x}, t)$ is a disturbance of the flow $\mathbf{U}_0(\mathbf{x}, t)$), then the flow $\mathbf{U}_0(\mathbf{x}, t)$ will be stable with respect to the norm $\| \bullet \|$ if $\mathrm{d}L[\boldsymbol{u}(\boldsymbol{x}, \mathrm{t})]/dt' < 0$ for any $\mathbf{u}(\mathbf{x}, t) \in H$ and any $t > 0$. The functional $\mathrm{L}[\mathrm{u}(\mathrm{x}, t)]$ satisfying the given conditions is called the *Lyapunov functional* (in the case of dynamic systems with a finite number of degrees of freedom the simpler name *Lyapunov function* is used). Some other formulations of conditions characterizing Lyapunov's functionals, and much additional information about the direct Lyapunov method of the study of stability, can be found in the literature on this subject mentioned above. Note only that since the definition of the norm $\| \bullet \|$, the existence of such a functional does not guarantee the stability of the given flow with respect to norms different form $\| \bullet \|$; in fact, a flow which is stable with respect to one norm can perfectly well be unstable with respect to some other norm.(Some examples of this phenomenon will be considered later in this subsection). Note also that, unfortunately, "there are no clear guidelines of how to choose Lyapunov's functionals; what is required is a little experience and a lot of luck" (this remark is due to Payne (1975); see also Rionero (1988)). However, Lyapunov's method of stability analysis has nevertheless proved to be very useful in many applications, and has been repeatedly applied to problems of hydrodynamic stability.

One of the first applications of Lyapunov's method to problems of hydrodynamic stability was due to Dikii (1960a, b), who did not indicate this explicitly but in fact investigated precisely the Lyapunov stability of the flows he considered. Since this author used only linearized dynamic equations, his results were given in Chap. 3 of

[3] The definition of such spaces and description of their main properties can be found, for example, in the book by Kolmogorov and Fomin (1957).

this book, the present Chapter being nominally on nonlinear methods. Dikii studied the stability of two-dimensional disturbances of plane-parallel inviscid flows; therefore, here a scalar field of the stream function $\Psi(x, z, t) = \Psi(z, t)e^{ikx}$ (or of the vertical velocity $w = \partial\Psi/\partial z$) of a disturbance could be used as the field of functions $\mathbf{U}(\mathbf{x}, t) - \mathbf{U}_0(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t)$. In Dikii's paper (1960b) the flow of homogeneous fluid between two solid walls was investigated and it was proved that under certain conditions the values of $|\Psi(x, z, t)|$ (where x and z are fixed, but t can take any nonnegative value) are bounded by a constant decreasing to zero when the initial values of the function Ψ, and of its spatial derivatives of the first and second orders, tend to zero. It is clear that this means that the flow is stable according to Lyapunov, with respect to a norm $\| \Psi(z,t) \|$ given by the square root of the integral with respect to z of a linear combination of $|\Psi|^2$, $|\Psi|^2$ and $|\Psi''|^2$ where primes denote d/dz (the appropriate norm is given by Eq. (2.75); see also Dikii (1976)). In the paper (1960a) the flow of an inhomogeneous fluid with the density profile $\rho(z) = \rho_0\exp(-az)$, where $0 \le z < \infty$, was studied; here the Lyapunov stability was considered for a norm given by the square root of the integral with respect to z of a linear combination of $|\Psi|^2$ and $|\Psi'|^2$ only. The Lyapunov stability of the flows considered was proved by Dikii for the same conditions under which their asymptotic stability (i.e., asymptotic decay of the function $\Psi(x, z, t)$ as $t \to \infty$) was independently proved in the papers by Case (1960a, b) (see Chap. 3 for additional details).

Later Pritchard (1968) applied Lyapunov's method to a study of the two most famous problems of hydrodynamic stability—the Rayleigh-Bénard problem of convection in a layer of stationary fluid heated from below, and the Taylor-Couette problem of stability of flow between coaxial rotating cylinders. Like Dikii, he considered only linearized dynamic equations but took into account the effects of molecular viscosity and thermal diffusivity neglected by Dikii. In Sect. 2.7 it was shown that, in the case of the Rayleigh-Bénard problem, linearized equations for the disturbance $\mathbf{u}(\mathbf{x}, t)$, $\vartheta(\mathbf{x}, t)$ can easily be transformed into a system of two equations with unknowns $u_3 = w$ and ϑ; therefore here the space of pairs of scalar functions $\{w(\mathbf{x}, t), \vartheta(x, t)\}$, periodic with respect to coordinates $x_1 = x$ and $x_2 = y$ and satisfying definite boundary conditions at $x_3 = z = 0$ and $z = H$, can be taken as the space H. (The boundary conditions are naturally different for the cases of two rigid, two free, and one rigid and one free surfaces considered by Pritchard; see the discussion of this topic in Sect. 2.7). In the case of Taylor–Couette flow, only disturbances that were axisymmetric (independent of ϕ) and periodic in the z-direction were studied in Pritchard's paper. Therefore, here $\boldsymbol{H}$ was the space of functions $\{u(r, z, t), v(r, z, t), w(r, z, t)\}$ satisfying the axisymmetric continuity equation $r^{-1}\partial(ru)/\partial r + \partial w/\partial z = 0$, periodic with respect to z and vanishing on the walls at $r = R_1$ and $r = R_2$. The Lyapunov functional $L = \| \bullet \|^2$ in $\boldsymbol{H}$ in the case of the Rayleigh–Bénard problem was chosen to have the form $L[w, \vartheta] = \| (w, \vartheta) \|^2 = \int_{V'} [w^2 + k^{-2}(\partial w/\partial z)^2 + \lambda Pr\vartheta^2] d\mathbf{x}$, where $k = k_3$ is the wavenumber, $\mathrm{Pr} = \nu/\chi$ is the Prandtl number and λ is a positive constant whose value can be varied in search of the strongest stability criterion. In the case of the Talylor–Couette problem, Pritchard assumed that $L[u, v, w] = \|(u, v, w\|^2 = \pi \int_{V'} (u^2 + \lambda v^2 + w^2)\, r dr dz$, i.e. the norm $\|(u, v, w)\| =$

$[E\lambda(t)]^{1/2}$ was used which was independently applied, slightly later, to the nonlinear extension of the same problem by Joseph and Hung (1971). (Remember that these authors also considered only disturbances which were axisymmetric and periodic with respect to z). To find conditions guaranteeing the negativity of the derivative $dL(t)/dt$, where $L(t) = L[w(\mathbf{x}, t), \vartheta(\mathbf{x}, t))$ or alternatively $L(t) = L[u(r, z, t), v(r, z, t), w(r, z, t)]$, Pritchard derived a number of new integral inequalities. Using them he found that the inequality $dL/dt < 0$ is valid for values of the dimensionless primary-flow parameters Ra (the Rayleight number) or Ta (the Taylor number) smaller than some value of $\mathrm{Ra_c}$ or $\mathrm{Ta_c}$, depending on λ and on the wavenumber $k = (k_1^2 + k_2^2)^{1/2}$or $k = k_3$. The maximum values, $\mathrm{Ra_{cr}} = \max_{\lambda,k} \mathrm{Ra_c}$ and $\mathrm{Ta_{cr}} = \max_{\lambda,k} \mathrm{Ta_c}$ are then just the critical values given by the version of the Lyapunov stability theory considered. Pritchard found that these critical values of Ra and Ta (and also the critical wave numbers k_{cr} (corresponding to them) agreed quite well with the critical values given by the normal-mode method of the linear stability theory. This clearly agrees with the earlier finding that the linear stability theory and the energy method lead to the same value of $\mathrm{Ra_{cr}}$ in the case of the Rayleigh–Bénard problem, and also agrees with subsequent results by Joseph and Hung (1971) relating to small disturbances in circular Couette flow.

Dikii's and Pritchard's applications of the Lyapunov method produced no appreciable repercussions. However the use of a related method by Arnol'd (alias Arnold) (1965a, 1966a, b, c) attracted much more attention which led to a definite revival of interest in the subject (see, e.g., the books by Arnol'd (1989a, Appendix 2); Marsden and Ratiu (1994); Marchioro and Pulvirenti (1994), and Arnol'd and Khesin (1998) and the references therein). Arnol'd considered two-dimensional disturbances in steady planar flows of inviscid ('ideal') fluid, but in contrast to Dikii and Pritchard he used in his studies the full nonlinear dynamic equations, not their linear approximation. Here we will pay most attention to the simplest case of two-dimensional disturbances having velocities $\mathbf{u}(\mathbf{x}, t) = \{u(x, z, t), w(x, z, t)\} = \{-\partial\Psi/\partial z, \partial\Psi/\partial x\}$ in a plane-parallel channel flow with velocity profile $U(z) = -d\Psi_0(z)/dz$, and only later will briefly describe the general results by Arnol'd relating to steady curvilinear plane fluid motions. Let us assume that all lengths are made dimensionless with a characteristic length L_0 and all velocities with a characteristic velocity U_0; then all quantities may be considered nondimensional (which means that we may take arbitrary functions of them, and add together any two quantities). The functions $\psi(x, z, t)$, $\psi_0(z)$ and $\psi(x, z, t) = \psi_0(z) + \psi(x, z, t)$ are non-dimensional stream functions of the disturbance, the undisturbed flow and the instantaneous disturbed flow, respectively, so $\Delta\psi = \partial w/\partial x - \partial u/\partial z$, $\Delta\psi_0 = d^2\psi_0/dz^2$ and $\Delta\psi$ are the corresponding vorticities. The nonlinear Euler equations of motion here reduce, as is well known, to a single equation for the conservation of vorticity $\Delta\psi$:

$$\frac{\partial}{\partial t}\Delta\psi - \frac{\partial\psi}{\partial z}\frac{\partial\Delta\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial z} = 0. \tag{4.25}$$

As usual, we will assume that disturbances are periodic in the coordinate x and that the period can take any value. Since the total energy of an inviscid flow is conserved in time and the vorticity $\Delta\Psi$ is also conserved, it is clear that both the integrals

$$E = \frac{1}{2}\int\int_{V'} (\nabla\psi)^2 dxdz \quad \text{and} \quad J_\Phi = \int\int_{V'} \Phi(\Delta\psi)dxdz$$

(where Φ is an arbitrary function of a single variable and V' is the rectangular region in the (x, z)-plane with width and length equal to the width of the channel and the length of disturbance period, respectively) are independent of time (i.e., are invariants of the disturbed motion). (Their invariance may easily be deduced from Eq. (4.25)). Following Arnol'd, let us consider the invariant functional $G = E + J_\phi$ of the stream function Ψ. It is easy to see that then the first variation of the functional G (i.e., the main part of the increment $\delta G = G[\psi_0 + \Psi] - G[\psi_0]$ for a small disturbance Ψ) may be represented in the form

$$\delta G[\psi]|_{\psi=\psi_0} = \int\int_{V'} [\Phi'(\Delta\psi_0) - \psi_0]\Delta\Psi dxdz \tag{4.26}$$

where Φ' is the derivative of the function Φ. Now let us assume that the velocity profile $U(z) = -d\psi_0(z)/dz$ has no inflection points. Then $d^2U(z)/dz^2 = -d^3\psi_0(z)/dz^3 \neq 0$ for all z, so that $\Delta\psi_0 = d^2\psi_0/dz^2$ is a monotonic function of z. This means that $\Delta\psi_0$ may be used as a new transverse coordinate instead of z. Hence, in particular, the stream function $\psi_0 = \psi_0(z)$ may also be considered as a function of $\Delta\psi_0$, i.e. it satisfies the equation

$$\psi_0 = \phi(\Delta\psi_0) \tag{4.27}$$

for some function ϕ. (Arnol'd showed that in fact Eq. (4.27) is also valid under a number of other conditions; in particular, under Fjørtoft's condition mentioned below). If now Φ is so chosen that $\Phi' = \phi$, then, according to (4.26) and (4.27), $\delta G[\Psi_0] = 0$, i.e., $\Psi = \Psi_0$ will be the stationary value of the functional $G[\psi]$. It is known that in the case of a function of a finite number of variables, the stationary points most often encountered are the points of its local maxima and minima. Now let $\Psi = \Psi(t)$ describe some dynamic system in a finite-dimensional space, with Ψ_0 a local extremal point of a time-invariant function $G[\Psi(t)]$ and Ψ_0 a disturbance of the initial value $\psi(t) = \psi_0$. The values of $G[\Psi_0 + \Psi(t)]$ corresponding to various disturbances Ψ_0 will clearly belong to the contour surfaces $G[\Psi_0 + \Psi(t)] = G[\Psi_0 + \Psi_0] =$ constant of the function $G(\Psi)$. At small values of the initial disturbance Ψ_0 the contour surfaces topologically have the appearance of the surfaces of small ellipsoids surrounding the extremal point Ψ_0. Therefore, if $\Psi_0 = \Psi(0)$ is small, then the values of $\Psi(t)$ will remain small at all values of t. This finite-dimensional analogy illustrates visually the main idea of the theory of Arnol'd. To make these arguments rigorous, we must now describe conditions guaranteeing that Ψ_0 is an external point of $G[\Psi]$, determine the strict sense of the statement that $\Psi = \Psi(x, z, t)$ is small and, finally, present a strict proof of the assertion for the case of an infinite-dimensional space of functions $\Psi(x, z, t)$.

The stationary point Ψ_0 of the functional $G[\Psi]$ will be a local extremum if, in some inertial system of coordinates, the second variation $\delta^2 G[\Psi_0]$ is either positive or negative definite, i.e. has the same sign for all disturbances $\Psi(x, z, t)$. It is easy to see that in the case considered the second variation of $G[\Psi_0]$ has the form

$$\delta^2 G[\psi]|_{\psi=\psi_0} = \int\int_{v'} \left[\left\{\frac{U(z)}{U''(z)}\right\}(\Delta\Psi)^2 + (\nabla\Psi)^2\right] dxdz \tag{4.28}$$

where $U''(z)$ denotes the second derivative of $U(z)$. It is clear that if $U''(z) \neq 0$ for all z (i.e., if Rayleigh's condition given in Sect. 2.82 is valid) then it is possible to choose an inertial coordinate system such that $U(z)/U''(z)$ will be positive everywhere, and hence Ψ_0 will correspond to a local minimum of the functional $G[\Psi]$. The same conclusion will also be true if there exists a constant K such that $[U(z) - K]/U''(z) \geq 0$ for all z, i.e., if the more general condition of Fjørtoft (given in the same section) is valid. The main stability theorem proved by Arnol'd states that the positive-definiteness of the quadratic form in the integrand on the right-hand side of Eq. (4.28) implies the Lyapunov stability of the flow with respect to the functional $L[\Psi(x, z, t)]$ on the right-hand side. (The proof of this statement can be found, e.g., in Monin and Yaglom (1971, 1971), pp. 158–160 of Vol. 1 and p. 853 of Vol. 2, while Arnol'd (1965a, 1966a, 1989a, App. 2) outlined the proof for a more general case of arbitrary steady planar motions). Note also that in the above-mentioned cases the ratio $U(z)/U''(z)$ (or, respectively, $[U(z) - K]/U''(z)]$ is bounded from above and from below. Therefore in these cases the Lyapunov functional $L[\Psi(x, z, t)] = L(t)$ given by the right-hand side of Eq. (4.28) may be replaced by an equivalent but simpler function of the form

$$L(t) = ||\Psi||^2 = \int\int_{v'} [(\nabla\Psi)^2 + (\Delta\Psi)^2] dxdz = \int\int_{v'} [\mathbf{u}^2 + (\nabla \times \mathbf{u})^2] dxdz \tag{4.29}$$

representing the sum of integrated squares of velocity and of vorticity (i.e. kinetic energy and enstrophy). Arnol'd's stability theorem gives rigorous quantitative sense to the qualitative assertion in the paragraph preceding Eq. (4.28), and shows that the 'size' of the disturbances considered must be measured by the norm given by Eq. (4.29).

Let us now pass on to the general case of an arbitrary steady planar flow with the velocity field $\mathbf{U}(x, z) = \{U(x, z), W(x, z) = \{-\partial\psi_0(x, z)/\partial z, \partial\psi_0(x, z)/\partial x\}$, where $\mathbf{x} = \{x, z\} \in D$, D is an arbitrary (bounded or unbounded) two-dimensional domain with smooth impermeable boundaries (if they exist). Using the arguments similar to given above, Arnol'd (1966a, 1989a) (see also Marchioro and Pulvirenty (1994), Sect. 3.2, and Arnold and Khesin (1998), Sect. II.4) showed that in this case *if the condition* (4.27) *is valid and there exist two constant c and C such that*

$$0 < c \leq \frac{\nabla\psi_0}{\nabla\Delta\psi_0} \leq C < \infty, \tag{4.30}$$

then under sufficiently wide conditions the steady planar flow considered is stable in the Lyapunov sense with respect to the norm (4.29). (The inequalitites (4.30) make sense since for any steady flow in two dimensions the gradient vectors of the stream function and of its Laplacian are collinear; in particular, $\nabla\psi_0/\nabla\Delta\psi_0 = U(z)/U''(z)$ in the case of a plane-parallel flow with velocity profile U(z)). The statement printed in italics is the First Stability Theorem of Arnol'd. His second Stability Theorem is relating to the case where the ratio $\nabla\psi_0/\nabla\Delta\psi_0$ takes negative values. Here condition (4.30) must be replaced by the condition

$$0 < c \leq -\frac{\nabla\psi_0}{\nabla\Delta\psi_0} \leq C < \infty. \tag{4.30a}$$

The Second Theorem states that if inequalities (4.30a) are valid, then under all the other conditions guaranteeing the validity of the First Stability Theorem and one rather general additional condition the two-dimensional steady flow considered will be again stable in the Lyapunov sense with respect to a norm of the same type as the norm (4.29).

Marchioro and Pulvirenti (1994) noted that Arnol'd's condition (4.30) cannot be fulfilled in domains D without boundary. However, these authors also showed that the stability theorem is often valid for flows in such domains too, if the domain D and the primary flow in it possess some symmetry properties (see Sect. 3.3 in their book). Moreover, they showed that the inequality $c > 0$ can be replaced in the conditions of the First Stability theorem by the weaker inequality $c \geq 0$. Slight weakening of the conditions included by Arnol'd in the formulation of his Second Stability Theorem was indicated by Wolansky and Ghil (1996).

Arnol'd's results have a direct relation to the important question of the *admissibility of linearization* in the investigation of hydrodynamic stability. This question concerns the extent to which the stability (or instability) of solutions of linearized equations of fluid mechanics entails also the stability (or instability) of the corresponding solutions of the full nonlinear equations of motion. Before answering this question it is of course necessary to define exactly when the solution of the nonlinear system is called 'stable'. The most appropriate such definition is precisely that given by Lyapunov, who himself bore in mind this use of it (in application to motions described by systems of ordinary differential equations).

In the case of a finite-dimensional dynamic system described by the nonlinear vector equation $d\mathbf{x}(t)/dt = \mathbf{f}(\mathbf{x})$, the admissibility of linearization means that a one-to-one relationship exists between the stability in the sense of Lyapunov of a time-invariant solution $\mathbf{x_0}$ of this equation and its linear stability (the condition for the latter being that none of the eigenvalues of the equation linearized in the neighborhood of the point $\mathbf{x_0}$ shall have a positive imaginary part). For this case the existence of the one-to-one relation was proved under sufficiently general conditions of Lyapunov himself (indeed, the method of linearization is just Lyapunov's first method of stability analysis). However, in the case of dynamic systems in functional spaces described by nonlinear *partial* differential equations the situation is more complicated.

Let us remember that Luyapunov stability of fluid motion depends on the chosen norm in the functional space of fields of the flow quantities considered; therefore, there are in fact many different types of such stability. The proof of the admissibility of linearization must indicate which type of Lyapunov stability of solutions of the nonlinear system proves to be equivalent to the usual ('normal-mode') stability of solutions of the linearized equations, i.e., to the absence of normal-mode frequencies ω with positive imaginary parts. It easy to give simple arguments supporting the idea that in cases where the flow is unstable according to the linear theory (i.e., where there is an eigenvalue ω with $\Im m\omega = \omega^{(i)} > 0$), Lyapunov's stability conditions is usually also untrue. In fact, if the initial disturbance is chosen very small then it will evidently be well described by linearized equations. Hence in the cases considered a small initial disturbance may be chosen, such that for small t it grows proportional to $\exp[\omega^{(i)}t]$, where $\omega^{(\mathrm{i})} > 0$. Then, as the disturbance becomes relatively large, the linear approximation ceases to apply and the nonlinear terms changes the character of evolution of the disturbance (usually diminishing at first the rate of its growth and in many cases later even halting the growth entirely; see Sect. 4.21, below). If we now decrease the size of the initial disturbance (keeping its form), we merely achieve a longer time interval during which the linear theory is a suitable description of the flow, the subsequent fate of the disturbance being the same. Thus the maximal values achieved by the disturbance cannot be changed by diminishing its initial amplitude, and therefore it seems very likely that the flow considered must be unstable in the sense of Lyapunov. However, the rigorous proof of this assertion proved to be a far from easy matter.

It is quite plausible that under sufficiently broad conditions the reverse implication also holds—from the stability of a solution of linearized equations it follows that the corresponding solution of the complete non-linear system of equations is stable in the sense of Lyapunov. The assumption that linearization of equations of motion is possible for stability investigations has just this sense. In hydrodynamic stability theory this assumption is usually taken on trust (see, e.g., Lin (1955), Sect. 1.1, or Drazin and Reid (1981), Sect. 3; however, the book by Georgescu (1985) is an exception to this rule), but in most cases it is not at all easy to prove this rigorously. (Moreover, such a proof must clarify what disturbance norm provides Lyapunov stability of a flow in the case where all normal modes of linearized equations are decaying—this rather subtle question is also usually ignored in texts on hydrodynamic stability). The work of Arnol'd discussed above gives just such a proof for some particular cases. Remember, that Rayleigh's and Fjørtoft's conditions were introduced in Sect. 2.82 as sufficient conditions for the absence of unstable normal modes of the corresponding Rayleigh equation. Now we see that these conditions also guarantee the Lyapunov stability with respect to the norm (4.29) for two-dimensional solutions of the corresponding nonlinear equations. Arnol'd also showed that Fjørtoft's condition (which is weaker than Rayleigh's) can be replaced in his theorem on Lyapunov stability by some even weaker conditions which are valid, in particular, for velocity profiles which do not satisfy the Fjørtoft condition but, according to Tollmien (1935), nevertheless guarantee stability for solutions of linearized equations (again see Sect. 2.82). Thus, it was proved that, here again, linear stability implies Lyapunov instability for solutions of nonlinear equations.

Let us however emphasize that only two-dimensional disturbances of inviscid plane fluid flows were considered in the above-mentioned papers by Arnol'd. In fact, for three-dimensional disturbances of a flow (and three-dimensional flows), the reasoning presented above proved to be insufficient. Arnol'd (1965b) and Dikii (1965a) found only a few partial results relating to these cases, which do not resolve the question of interrelation between linear stability and nonlinear Lyapunov stability of three-dimensional disturbances. Abarbanel and Holm (1987) also tried to apply Arnol'd's method to nonlinear stability analysis of three-dimensional inviscid flows but they also found that the method does not work so successfully here as in the case of flows in two dimensions. Since the Squire theorem of the linear stability theory, given in Chap. 2, Sec. 2.8, cannot be generalized to the case of nonlinear stability theory (where only some much weaker statements are valid; cf. Sect. II.5.D in the book by Arnol'd and Khesin (1998)), the search for sufficiently general conditions of instability with respect to three-dimensional finite-amplitude disturbances presents a problem of considerable importance. Some arguments suggesting that the method developed by Arnol'd for investigation of stability of planar flows with respect to two-dimensional disturbances must be inadequate in the case of hydrodynamics in three dimensions were briefly noted by Arnol'd in the early paper (1966c); later this conclusion was explained more clearly by Arnol'd (1989a, App. 2); Rouchon (1991); Sadun and Vishik (1993) and in Sect. II.5.G of Arnol'd and Khesin's book (1998). Note however that, as early as the late 1960s and early 1970s, it was discovered that Arnol'd's variational approach (presented in the general form in his paper (1966b), which surprisingly linked up with some early ideas by Kelvin (1887)) can be successfully applied to studies of nonlinear Lyapunov stability for many types of disturbances encountered in a number of inviscid flows of practical interest. Such methods were first widely applied in geophysics; the works by Dikii (1965b, 1976); Blumen (1968, 1971); Dikii and Kurganskii (1971); Pierini and Salusti (1982); Benzi et al. (1982); Holm et al. (1983); Grinfeld (1984); Abarbanel et al. (1986), and Kurganskii (1993) are just typical examples. Somewhat later the same methods were used in many studies of stability magnetohydrodynamic flows and plasma oscillations. These new applications led, in particular, to the appearance of the excellent extensive survey by Holm et al. (1985) of the modern state of nonlinear stability investigations by methods developed by Arnol'd, which contains more than 150 references. For further examples of applications of this approach to the theory of hydrodynamic stability see, e.g., the books by Marsden (1992) and Marsden and Ratiu (1994), and papers by McIntyre and Shepherd (1987); Davidson (1998) and Vladimirov and Ilin (1998, 1999). Many other references to modern developments of the approach considered above can be found in Chap. II of the book by Arnol'd and Khesin (1998); here we will only mention the paper by Vladimirov (1990) where the direct Lyapunov method is applied to stability studies for some flows of viscous liquids affected by surface tension.

The question of the admissibility of linearization is also quite important in stability studies relating to steady flows of viscous fluids. In the case of viscous flows in smooth bounded domains one part of the linearization principle states that *if* all eigenfrequencies ω_j of the linearized dynamic equations corresponding to a given flow have negative imaginary parts, *then* the flow is also stable in the sense of Lyapunov (with respect to the norm (4.31), below). This was proved under sufficiently general conditions by Prodi (1962) (see also the detailed exposition of his proof by Georgescu (1985), Sect. 2.4.2, where a number of additional references relating to this topic can be found). The Lyapunov norm ||•|| used by these authors is given by the equation

$$||\mathbf{u}(\mathbf{x})||^2 = \int_v [\sum_{i=1}^{3} u_i^2(\mathbf{x}) + \sum_{i,j=1}^{3} \{\partial u_i(\mathbf{x})/\partial x_j\}^2] d\mathbf{x}. \quad (4.31)$$

The rigorous proof of the other part of the linearization principle (also for viscous flows in bounded domains) was briefly sketched in a note by Yudovich (1965) and was later given in detail in his special monograph (see Yudovich (1984)). A more elementary proof of admissibility of linearization for viscous flows in bounded domains, under slightly less general conditions, was given by Sattinger (1970). A quite different approach to the linearization principle was developed within the framework of the modern bifurcation theory (this theory will be briefly discussed in Sect. 4.22 and will be also mentioned in some subsequent parts of this book). Bifurcation theory allowed one to obtain some rather general conditions under which the solutions of the linearized equations certainly approximate faithfully the phase-space dynamics of a flow disposed in the vicinity of the steady primary flow. (The phase space has here the same meaning as in Sect. 2.3). These conditions are given by the so-called Hartman-Grobman theorem (see, e.g., Sect. 1.3 in Guckenheimer and Holmes (1993)), but they are based on the use of some new concepts which cannot be considered here.

Yudovich's monograph (1984) also contains a discussion of many other aspects of the general linearization problem, requiring the introduction of a number of different Lyapunov norms in functional spaces and the use of quite sophisticated mathematical techniques. Yudovich showed, in particular, that different norms are often needed for different purposes, and the answer to the question whether a flow is stable or unstable in Lyapunov's sense depends on the selection of the norm which is most appropriate for the given purpose. To illustrate the possibility of paradoxical disturbance behavior, Yudovich considered the simple case of a two-dimensional disturbances of an inviscid plane Couette flow. Here the velocity and vorticity of the disturbance remain bounded, but the vorticity derivatives grow unboundedly with time. Therefore in this case the vorticity $\Delta\Psi$ at large times t is reminiscent of a continuous but nowhere-differentiable Weierstrass function, and the flow is clearly unstable with respect to any norm which includes the square of the vorticity derivative. In the case of three-dimensional disturbances in the same flow, the velocity vector remains bounded but the vorticity vector grows unboundedly (see also Sect. 3.21, where related results were obtained for some other steady plane-parallel inviscid flows); hence the flow considered is unstable with respect to any norm including the square of the vorticity vector. However, there is no space for us to discuss the results in Yudovich's monograph in more detail. Let us only remember, in connection with the last remarks, the results by Arnol'd (1972) presented in Sect. 3.21 (and expounded in more detail in Sect. II.5 of Arnol'd and Khesin's book (1998)), which show that in three-dimensional inviscid flows disturbances can sometimes have extremely paradoxical asymptotic behavior, making the flow unstable with respect to rather simple norms.

Let us now return to the remark made at the end of Sect. 4.12, that Joseph's book (1976) prompted the appearance of a number of works investigating the possibility of improving the known energy-theory stability results by replacing the traditional energy functional $E(t)$ by some 'generalized energy' (i.e., by some new Lyapunov functional $L(t)$). We postponed the discussion of this remark until now, since the method used in the majority of these investigations is not in fact the traditional Lyapunov method considered above in this subsection. To explain this it is necessary to refer to particular examples. One of the first problems investigated in the above-mentioned way was that of convection in a horizontal fluid layer of thickness H, heated from below and rotating around a vertical axis with angular velocity Ω. Since the Coriolis force is orthogonal to the velocity and hence does no work, rotation does not change the energy-balance Eq. (4.2). Therefore the energy-method stability results are identical in the cases of rotating and non-rotating convection. However, the computations of the corresponding normal modes by linearized dynamic equations, carried out long ago by Chandrasekhar (1953, 1961), and the subsequent experiments by Rossby (1969) and some other workers (see, e.g., the survey by Bubnov and Golitsyn (1995)) both showed that the critical Rayleigh number $\mathrm{Ra}_{\mathrm{cr}}$, increases considerably with rotation rate (measured, e.g., by the so-called Taylor number $\mathrm{Ta} = \Omega^2 H^4/\nu^2$) and also depends on the Prandtl number Pr. Thus, while in the case of stationary layers of fluid the linear normal-mode theory and the energy method give the same value of $\mathrm{Ra}_{\mathrm{cr}}$, in the case of rotating layers, the values of $\mathrm{Ra}_{\mathrm{cr}}$ given by the energy method prove to be considerably smaller than those predicted by the linear stability theory or observed in experiments. Consequently, Joseph (1966) noted that the stabilizing influence of rotation on the emergence of convection in a fluid cannot be explained by the energy method of stability theory.

Later, however, some authors tried to replace Joseph's 'energy functional' $E_\lambda[\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t)] = E(t) + \lambda \,\mathrm{Pr}\, E_T(t) = 0.5[\langle \mathbf{u}^2\rangle + \lambda \,\mathrm{Pr}\, \langle\vartheta\rangle^2]$ by another Lyapunov functional ('generalized energy') $L[\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t)]$ in the hope of obtaining a larger value of the energy stability boundary $\mathrm{Ra}_{\mathrm{cr}} = \mathrm{Ra}_{\mathrm{cr}}$ (Ta) for rotating flows. In one of the first such attempts Galdi and Straughan (1985a) (see also the subsequent works by Mulone and Rionero (1989); Galdi and Padula (1990), and Straughan (1992), Sect. 6.1) tried to use a Lyapunov functional of the following form: $L[\mathbf{u}(\mathbf{x},t), \vartheta(\mathbf{x},t)] = \langle \mathbf{u}^2\rangle + \lambda_1 \,\mathrm{Pr}\, \langle\vartheta\rangle^2 + \lambda_2 \left\langle(\zeta + \lambda_3 \,\mathrm{Pr}\, \partial\vartheta/\partial z)^2\right\rangle + \lambda_4 \left\langle(\nabla\mathbf{u})^2 + \lambda_5 \,\mathrm{Pr}\, (\nabla\vartheta)^2\right\rangle$, where $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vertical vorticity and λ_{i}, $i = 1, \ldots, 5$, are adjustable constants. Of course, the equation for $dL[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)]/dt = dL(t)/dt$ will then also include terms which are cubic in the disturbance fields u_i, $i = 1, 2, 3$, and ϑ. To deal with the resulting nonlinear problem, all the above-mentioned authors used the approach by Joseph and Hung (1971), i.e., they neglected the cubic terms at first and only later calculated corrections to their results due to the nonlinearity of the system studied. Thus, the stability results obtained in the first stage of these investigations were only conditional, i.e., guaranteeing stability only for disturbances having very small norm $||(\mathbf{u}, \vartheta)|| = \{L[\mathbf{u}, \vartheta]\}^{1/2}$.

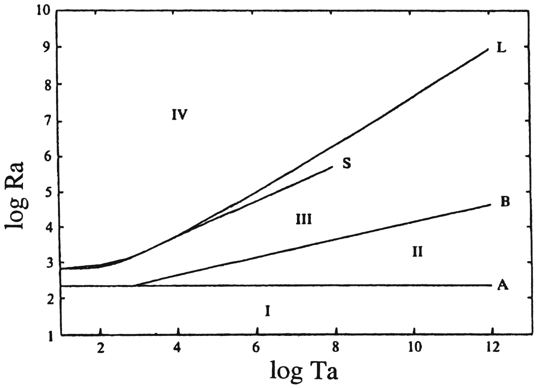


Fig. 4.4 Bounds in the (Ta, Ra)-palne of four stability regions for a rotating layer of fluid with $Pr \geq 1$ heated from below and bounded by two free surfaces. (After Malkus and Worthing (1993)) L: Chandrashekhar's neutral curve of the linear stability theory; S-the boundary of the maximal region of conditional stability (i.e., of the limit as $\alpha \to 0$ of the regions of stability with respect to disturbances with the nondimensionalized 'initial amplitude' A_0 satisfying the inequality $A_0 < \alpha$); B: the boundary of the region of stability with respect to disturbances with $A_0 < 10^{-6}$; A: the energy-theory stability boundary of the region of global monotonic stability

However, for disturbances with such a small norm that the cubic terms of the equation $dL/dt = 0$ can be neglected, it was found that the values of coefficients λ_I can be chosen in such a way that the stability region in the (Ra, Ta)-plane (i.e., the region where $dL/(t)/dt < 0$), turns out to be very close to the region determined by the linear theory of hydrodynamic stability (see, for example, Fig. 4.4 below in this section). This result is clearly analogous to the previously-mentioned results of Joseph and Hung (1971) relating to the Taylor–Couette stability problem. Similar results for the case of fluid layers heated from below (and also for some such layers of constant temperature) which are rotating with horizontal angular velocity $\Omega = \{\Omega_x, \Omega_y, 0\}$ were obtained by Wahl (1994), who used the ordinary energy norm but a special representation of divergence-free velocity field $\mathbf{u}(\mathbf{x}, t)$. The same representation of $\mathbf{u}(\mathbf{x}, t)$ was also used by Wahl (1994) and Kagel and Wahl (1994) in studies of Lyapunov stability (with respect to some particular Lyapunov functionals $L[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)]$ including derivatives of fields $\mathbf{u}$ and ϑ) of arbitrary steady solutions of Boussinesq equations describing possible stationary disturbances in a horizontal fluid layer heated from below (see also the related paper by Schmit and Wahl (1993) where Lyapunov functionals of this type were used in detailed study of the onset of convection in a stationary layer of fluid heated from below).

Another stability problem which has often been studied by the method of Lyapunov is the magnetic Bénard problem of convection in a horizontal layer of a fluid conductor in the presence of a homogeneous vertical magnetic field; see, e.g., Galdi (1985); Rionero (1988); Rionero and Mulone (1988), and Galdi and Padula (1990). Here the main results found were similar to those obtained in the cases of the Taylor-Couette and rotational Bénard problems, but it was also shown that in this case the

Lyapunov functional can be chosen so that, for some range of flow parameters, the linear and Lyapunov nonlinear stability bounds coincide with each other. Note that in earlier studies of stability of magnetohydrodynamic flows by the energy method, carried out by Rionero (1967, 1968); Carmi and Lalas (1970); Bhattacharyya and Jain (1971), and Joseph (1976, Addendum to Chap. IX), a linear combination of integrated kinetic and magnetic energies was used as Lyapunov functional L, such that all cubic terms cancelled in the equation for dL/dt. However, this condition was not fulfilled in the cases of the more complicated Lyapunov functionals L used in publications appearing in 1980s and 1990s. Therefore in this later work the stability boundaries obtained were valid only under the condition that disturbances were small enough. This relates to the general conditions guaranteeing the coincidence of the critical parameters given by the linear and Lyapunov nonlinear stability theories, whose discussion plays a very important part in the work of Galdi and Straughan (1985b); Galdi and Padula (1990), and Straughan (1992). In fact, as a rule these conditions use only the linear parts of the differential equations determining the time evolution of flow disturbances, and hence presuppose the smallness of the latter—unless the cubic terms cancel in the equation for dL/dt.

The above-mentioned stability results, derived by the Lyapunov direct method employing Lyapunov's functionals L where dL/dt contains cubic terms, concern conditional stability only, and this clearly diminishes the practical usefulness of these results. This was specially emphasized in the review by Malkus and Worthing (1993) of the book by Straughan (1992). The reviewers considered the popular example of convection in a rotating horizontal layer of fluid. They illustrated the importance of amplitude restriction of results on conditional stability by supplementing curves L and S, shown in Fig. 6.2 of the book by Straughan (1992) (and relating to the case of a rotating layer of fluid with $\mathrm{Pr} \geq 1$ bounded by two free surfaces), by two additional curves A and B (see Fig. 4.4). The straight line A represents Joseph's (1966) energy-theory stability boundary $\mathrm{Ra}_{\mathrm{cr}} \approx 657$, which is independent of Ta and Pr. Hence points of region I in Fig. 4.4 correspond to flows stable with respect to disturbances of any size. The curve L is the linear stability curve computed by Chandrasekhar (1961) (and hence the region IV corresponds to instability with respect to arbitrarily small disturbances and the region below curve L—to stability with respect to infinitesimal disturbances). S is the boundary of the maximal region of conditional stability (corresponding to condition $L(0) = 0$) calculated by Galdi and Straughan (1985a) starting from the form of the functional $L[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)]$ given above, with the optimal values of coefficients λ_i. Therefore, flows corresponding to points between curves L and S are linearly stable (i.e., exponentially-growing infinitesimal wave-like disturbances do not exist in these flows) but nevertheless $L[\mathbf{u}(\mathbf{x}, t), \vartheta(\mathbf{x}, t)] = L(t)$ can grow here with time for disturbances with an arbitrarily small value of $L(0)$. On the other hand, in the case of flows represented by points below curve S, such growth is impossible if $L(0)$ is small enough. However, the meaning of the words "small enough" was not explained in the book by Straughan. Trying to do this, Malkus and Worthing used an equation in the paper by Galdi and Straughan (1985a) which determines the maximal value $G(\mathrm{Ra}, \mathrm{Ta}, \mathrm{Pr})$ of the dimensionless initial 'energy' $L(0)$ of disturbances, which certainly do not destabilize a flow with given values of Ra, Ta, and Pr. This equation allowed them to compute the boundary B of the region of stability with respect to disturbances having initial 'amplitude' $[L(0)]^{1/2}$ less than10^{-6} times the appropriately defined 'unit amplitude' (so that the region III corresponds to flows unstable to at least one disturbance with initial dimensionless amplitude equal to 10^{-6} but stable to all smaller disturbances). We see that this region is rather large; therefore Malkus and Worthing were in doubt whether the curve S can be considered as a real boundary for the 'region of nonlinear stability', giving their opinion that, in almost all practical situations, even the curve B (bounding the region of stability to disturbances with dimensionless amplitudes not exceeding 10^{-6}) will be not useful as such a boundary.

Above, some applications of the "generalized energy method" determining the "conditional-energy bounds" were listed, and at the end we considered the review by Malkus and Worthing (1993) which sharply criticizes the usefulness of some of the results obtained by this method. (Note that this review also contains formulations of several interesting unsolved problems which are worth investigating by traditional and generalized energy methods). Let us now stress that the energy (and more general Lyapunov's methods) have already yielded some important new results concerning stability of fluid flows. The classical Reynolds–Orr energy-balance Eq. (4.2) and its generalization to the case of convective flows led to the discovery, for many cases, of exact or almost exact minimal-critical values of dimensionless global characteristics of laminar flows (e.g., of $\mathrm{Re}_{\mathrm{cr\ min}}$ or $\mathrm{Ra}_{\mathrm{cr\ min}}$) determining the boundary of the region of 'absolute' (i.e., 'unconditional' or 'global') stability of a flow to disturbances of any size. Such bounds, which have already been mentioned in Sect. 2.1, clearly have considerable theoretical and practical value. Energy methods also showed that there exist two quite different types of fluids flows. The first type consists of flows where the region of the normal-mode stability with respect to infinitesimal disturbances coincides with the region of energy stability with respect to disturbances of arbitrary size, while for flows of the second type the latter of these two regions covers only a small part of the first region. It is clear that the nonlinear development of disturbances and transition to turbulence must have quite different forms in flows of these two types. Moreover, Lyapunov's generalized energy method led to the discovery of a great number of explicit conditions for both nonlinear and linear flow stability, often concerning flows of great practical importance; see in this connection the survey by Holm et al. (1985) mentioned above, and the papers and books by Arnol'd (1965a, b; 1966a, b, c; 1989a); McIntyre and Shepherd (1987); Marsden and Ratiu (1994); Marchioro and Pulvirenti (1994); Arnol'd and Khesin (1998), and Davidson (1998). As to results relating to conditional Lyapunov stability, they imply physically-observable stability diagrams of the type shown in Fig. 4.3, where for given 'energy' E_λ the exact shape of the curve in the diagram can be determined from the equations of generalized energy theory. The possible extension of the region of conditional stability by means of replacement of the Reynolds–Orr energy functional $E(t)$ by some Lyapunov functional $L(t)$ also clearly leads to extension of the range of Re (or Ra, Ta, etc.) numbers covered by such a diagram. In particular cases where the Lyapunov method yields the same critical numbers that follow from the linear normal-mode theory, the diagram in Fig. 4.3 covers the whole range between the region of unconditional (global) stability with respect to arbitrarily large disturbances and the region of absolute instability with respect to arbitrarily small (infinitesimal) disturbances.

An important feature of the energy methods is their ability to determine the most-unstable types of disturbance, which capture the energy of the primary flow most efficiently and hence grow faster than all the others. In this connection Lumley (1971) conjectured that some modifications of the classical energy method might also be useful in investigations of developed turbulence. As an example, he tried to apply such a method to the study of the near-wall region of a turbulent boundary layer. Within this region he replaced the constant molecular viscosity ν by an empirical function $\nu_m(z)$ describing, with reasonable accuracy, the combined influence of the molecular viscosity and small-scale turbulent fluctuations on the mean flow and the accompanying large-scale structures. Then he appropriately modified the R–O energy-balance Eq. (4.2) and with its help determined the most unstable longitudinal (i.e., x-independent) disturbances. It was found that these disturbances agreed satisfactorily with the longitudinal structures actually observed in the near-wall regions of turbulent flows along flat plates. Later Poje and Lumley (1995) further developed the same idea, suggesting the use of the energy-balance method to identify the large-scale organized ('coherent') structures which, according to data accumulated during the second half of the twentieth century, exist everywhere in turbulent flows and play a rather important role in them. However, we cannot linger here on this subject which clearly lies outside the content of the present chapter.

4.2 Landau's Equation, its Generalizations and Consequences

4.2.1 The Landau Equation for the Amplitude of a Disturbance

The energy method of stability analysis deals with general (quite arbitrary) flow disturbances; the highly-developed linear theory of hydrodynamic stability is not used at all here. This theory suggests that in the case when the initial disturbance is rather weak its most-unstable normal-mode component (or the least stable, if unstable normal modes do not exist) will play the main part in the primary disturbance development. Therefore the study of the development of a normal-mode disturbance is important for understanding the behavior of disturbed flows, and such a study must take into account the influence of the nonlinear terms of the equations of motion, which clearly affect the disturbance evolution if the disturbance is not very small. The results obtained will be of interest both in the case where $\mathrm{Re} < \mathrm{Re}_{cr}$, where Re_{cr} is the critical Reynolds number[4] defined from the linear stability theory (in this case an investigation of the nonlinear normal-mode development can yield the

[4] For simplicity, we shall speak only of Reynolds number, although in some cases the initiation of instability will be determined by transition through a critical value of some other dimensionless control parameter of the same type.

critical Reynolds number for finite disturbances of fixed amplitude) and in the case where $\mathrm{Re} > \mathrm{Re}_{cr}$ (in this case the nonlinear results describe further evolution of weak disturbances, which increase exponentially according to linear theory).

The great importance of nonlinear effects in the development of flow disturbances was already fully appreciated by Reynolds in 1883, and some attempts to incorporate these effects into theoretical analysis were also made very early (in particular, by Noether (1921) and Heisenberg (1924)). However, the first really significant step towards the creation of the nonlinear theory of hydrodynamic stability was taken in a short note by Landau (1944) whose contents was described also in the books by Landau and Lifshitz (1944), Sect. 24; (1958), Sect. 27; and (1987), Sect. 26 (in the last of these, the presentation was partially changed to reflect more recent developments of the theory which will be considered later in this book). Landau's arguments were quite general and did not use any specific form of the equations of motion.

Landau considered simply the development of a normal-mode disturbance in a steady laminar flow. He was especially interested in the evolution of an unstable (exponentially-growing) wave-like mode of very small initial amplitude (which may be considered as being infinitesimal) at a slightly supercritical value of Re (i.e., only a little larger than Re_{cr}). However his reasoning can be equally well applied to slowly-decaying infinitesimal normal-mode disturbances at slightly subcritical $\mathrm{Re} < \mathrm{Re}_{cr}$; hence we will consider both these cases here. To Landau, it was only important that the velocity field of the mode considered could be represented in the form

$$\mathbf{u}(\mathbf{x}, t) = A(t)\mathbf{f}(\boldsymbol{x}), \tag{4.32}$$

where $\mathbf{f}(\mathbf{x})$ is the eigenfunction of the corresponding eigenvalue problem while $A(t)$ is the complex disturbance amplitude, which can be represented in a form $A(t) = e^{-i\omega t} = e^{\gamma t - i\omega_1 t}$ for values of t at which the linear stability theory is valid. Here $\omega_1 = \Re e\, \omega$ and $\gamma = \Im m \omega$ so that $\gamma > 0$ for growing waves, $\gamma < 0$ for decaying waves and $\gamma \to 0$ as $\mathrm{Re} \to \mathrm{Re}_{cr}$ (and therefore $|\gamma| \ll |\omega_1|$ for sufficiently small $|\mathrm{Re} - \mathrm{Re}_{cr}|$ if $\omega_1 \neq 0$). The form of $A(t)$ given above makes it clear that the real disturbance amplitude $|A(t)|$ satisfies the equation

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2. \tag{4.33}$$

However, Eq. (4.33) is correct only within the framework of linear stability theory. If $\mathrm{Re} > \mathrm{Re}_{cr}$ and $A(t)$ increases, there will inevitably come a point at which this theory is no longer valid and must be replaced by a more complete one, which takes into account those terms in the equations of motion that are nonlinear in the disturbances. Then the right side of Eq. (4.33) must be considered as the first term of the expansion of $d|A|^2/dt$ in a series of powers of A and A^* (where as usual the asterisk denotes the complex conjugate). In the case where $\mathrm{Re} < \mathrm{Re}_{cr}$, $A(t)$ is a decreasing function and here Eq. (4.33) is true for all t, but only in cases where the initial amplitude $A(0)$ is small enough. If, however, $A(0)$ is not sufficiently small, then at small values of t this equation represents only the first term of the expansion in powers of A and A^*.

If $|A(t)|$ is small, but not small enough for all the higher-order terms of the above-mentioned expansion to be neglected, then it is necessary to take into account the terms of the next order of the series, i.e. the third-order terms. However, it must also be remembered that the motion (4.32) is accompanied by periodic oscillations in the expression for $A(t)$, rapid in comparison with the characteristic time $1/|\gamma|$ of an appreciable change in the value of $|A(t)|$), and described by the factor $e^{-i\omega_1 t}$, where $|\omega_1| \gg |\gamma|$. These periodic oscillations do not interest us; hence to exclude them, it is convenient to average the expression $d|A|^2/dt$ over a period of time that is large in comparison with $2\pi/|\omega_1|$ (but small in comparison with $1/|\gamma|$). Since third-order terms in A and A^* will inevitably contain a periodic factor, they will all disappear during the averaging.[5] In the case of the fourth-order terms, there will remain, after averaging, only one term, which is proportional to $A^2A^{*2} = |A|^4$. Thus, retaining terms of no higher than fourth order, we will have an equation of the form

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \delta|A|^4. \tag{4.34}$$

Since the period of averaging is much less than $1/|\gamma|$ the terms $|A|^2$ and $|A|^4$ will be practically unchanged by averaging, so that Eq. (4.34) may be considered as an exact equation for the amplitude of the averaged disturbance. (In the case where $\omega_1 = 0$ the third-order terms also often disappear because of the symmetry properties of the problems considered, and hence Eq. (4.34) is valid here too; certain examples of this kind will be considered below). Equation (4.34) is called the *Landau equation,* and its coefficient δ, which can be either positive or negative (and can also be zero, but only in exceptional cases), is the *Landau constant*. Positive values of δ show that nonlinear effects stabilize the disturbance considered, decreasing the growth of its amplitude, while negativity of δ means that nonlinear effects destabilize the disturbance.

Equation (4.34) can be also rewritten as the following linear equation in $|A|^{-2}$

$$\frac{d|A|^{-2}}{dt} + 2\gamma|A|^{-2} = \delta, \tag{4.35}$$

whose general solution is easily seen to be

$$|A(t)|^2 = \frac{A_0^2 e^{2\gamma t}}{\left(1 - \frac{\delta}{2\gamma}A_0^2\right) + \frac{\delta}{2\gamma}A_0^2 e^{2\gamma t}} \tag{4.36}$$

Where $A_0 = |A(0)|$ is the initial amplitude of the disturbance. From Eq. (4.36) it follows that if $\delta > 0$, if the initial disturbance is sufficiently small, and if $\gamma > 0$ (i.e. $\mathrm{Re} > \mathrm{Re}_{cr}$ and the evolution of an unstable mode is studied), the amplitude $A(t)$ will

[5] To be more exact, we must say that third-order terms do not fully disappear after averaging but generate some terms of the fourth order which can be included in the fourth-order terms of the expansion considered.

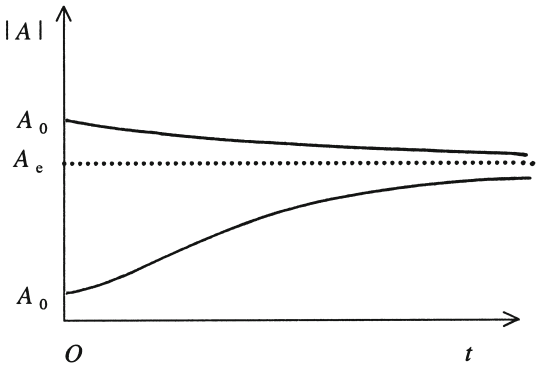


Fig. 4.5 The dependence of the disturbance amplitude $|A(t)|$ on time t in the case where $\delta > 0$ and $\mathrm{Re} > \mathrm{Re}_{cr}$ (and hence $\gamma > 0$) for disturbances with the initial amplitude $A_0 < Ae = (2\gamma/\delta)^{1/2}$ and $A_0 > A_e$ (but $A_0 - A_e$ small) according to Landau's Eq. (4.34)

first increase exponentially (in accordance with the linear theory), but then the rate of the increase slows, and as $t \to \infty$ the amplitude will tend to a finite 'equilibrium value' $A(\infty) = A_e = (2\gamma/\delta)^{1/2}$ independent of $A(0)$ (see the lower part of Fig. 4.5). Note now that γ is a function of the Reynolds number which becomes zero at $\mathrm{Re} = \mathrm{Re}_{cr}$ and may be expanded as a series in power of $\mathrm{Re} - \mathrm{Re}_{cr}$ (the latter fact may be deduced from the small-disturbance theory) while $\delta \neq 0$ for $\mathrm{Re} = \mathrm{Re}_{cr}$. Thus $\gamma \approx b\,(\mathrm{Re} - \mathrm{Re}_{cr})$ at small enough values of $|\mathrm{Re} - \mathrm{Re}_{cr}|$ where b is a positive constant. Consequently, $A(\infty) = |A|_{max} \propto (\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ for $\delta > 0$ and small positive values of $\mathrm{Re} - \mathrm{Re}_{cr}$ (see Fig. 4.7a below). Hence $A(t)$ remains small at all values of t if $\mathrm{Re} - \mathrm{Re}_{cr}$ is small enough (therefore, even the inclusion in Eq. (4.34) of higher-order terms, for example one proportional to $|A|^6$, will not qualitatively change the behavior of the function A*(t)*). In the case where $A_0 > A_e = (2\gamma/\delta)^{1/2}$ but is nevertheless small (this is possible when $\mathrm{Re} - \mathrm{Re}_{cr}$ is small) Eqs. (4.34) and (4.36) can again be used as a reasonable first approximation; the corresponding behavior of $A(t)$ is shown in the upper part of Fig. 4.5. We see that here, at Re slightly exceeding Re_{cr}, any disturbance containing the unstable component transfomrs the primary laminar flow into a new laminar flow which is practically independent of the initial conditions. (In fact this new flow can turn out to be unstable to some disturbances neglected in the fluid-dynamic derivation of Landau's equation considered below. However, here we will not linger on this topic). If, however, $\delta > 0$ but $\mathrm{Re} < \mathrm{Re}_{cr}$ and hence $\gamma < 0$, then Eq. (4.36) shows that the disturbance decays monotonically and in accord with the linear theory (*i.e.*, $A(t) \propto e^{\gamma t}$ as $t \to \infty$). Here evidently neither the last term on the right side of Eq. (4.33), nor the terms of higher order omitted from this equation, significantly affect the disturbance evolution.

Let us now consider briefly the case where $\delta < 0$. If in this case $\gamma < 0$ (i.e., $\mathrm{Re} < \mathrm{Re}_{cr}$), then for $A_0 < (2\gamma/\delta)^{1/2}$ the solution $|A(t)|$ decays monotonically to zero (see the lower part of Fig. 4.6); hence in this case too the inclusion of the higher-order terms of the amplitude equation will not change the behavior of $A(t)$ qualitatively. If $\delta < 0$, $\gamma < 0$, but $A_0 = (2\gamma/\delta)^{1/2}$, then $A(t) = A_0$ at any $t > 0$; however; for $A_0 > (2\gamma/\delta)^{1/2}$ the function $A(t)$ grows with t (see again Fig. 4.6) and here the inclusion of higher-order terms in Eq. (4.34) becomes necessary at moderate

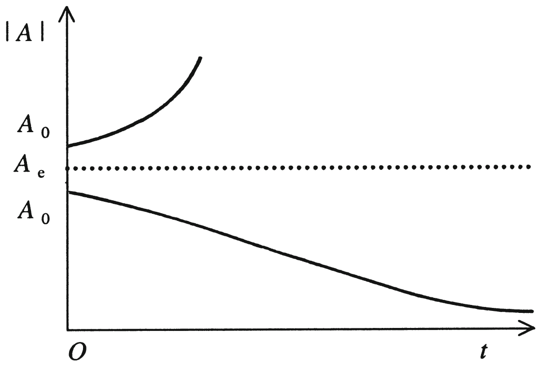


Fig. 4.6 The dependence of the amplitude $A(t)$ on t in the case where $\delta < 0$ and $\mathrm{Re} < \mathrm{Re}_{cr}$ (i.e., $\gamma < 0$) for disturbances with the initial amplitude $A_0 < A_e = (2\gamma/\delta)^{1/2}$ and $A_0 > A_e$ according to Landau's equation

positive values of t. The possible influence of such terms will be illustrated later by a simple example; for now, we merely note that, according to the above argument, if $\delta < 0$ and $\mathrm{Re} < \mathrm{Re}_{cr}$ then very small disturbances decay, but some disturbances which are not small enough grow with time; this is the *subcritical instability* of finite-amplitude disturbances. If now $\delta < 0$ but $\gamma > 0$ (i.e., $\mathrm{Re} > \mathrm{Re}_{cr}$) then, for any $A_0 > 0$, solution (4.36) quickly becomes infinite; hence in this case the behavior of the amplitude $A(t)$ as $t \to \infty$ cannot be determined from Eq. (4.34) for any initial value A_0. To obtain a sensible result we must take into account the next term of expansion in the power of A and A^* and to assume it to be negative. Let the next term be $-\beta|A|^6$ where *β > 0*. Then, neglecting all terms of higher than the sixth order we obtain

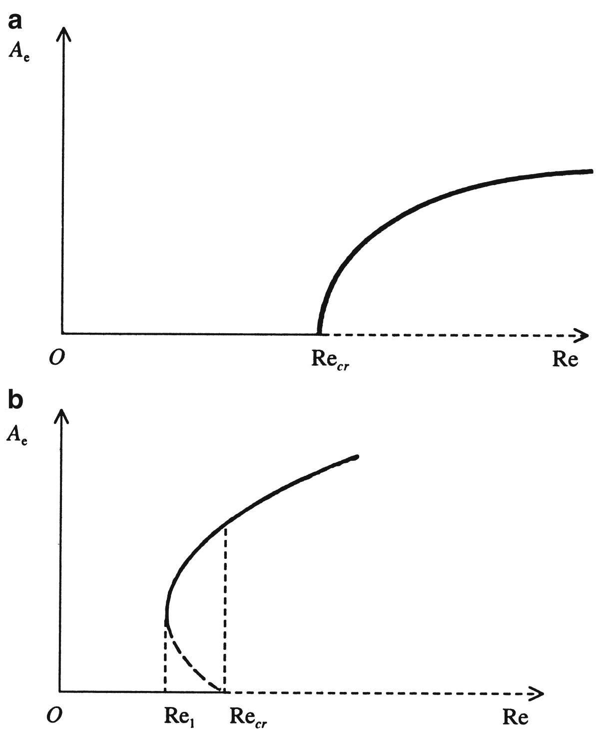
$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 + |\delta||A|^4 - \beta|A|^6, \tag{4.37}$$

and hence

$$|A|^2_{\max} = \frac{|\delta|}{2\beta} \pm \left[\frac{|\delta|^2}{4\beta^2} + \frac{2\gamma}{\beta}\right]^{1/2} \tag{4.37a}$$

where $|A|^2_{\max}$ is the value of $|A|^2$ at which $d|A|^2/dt = 0$ and $\gamma \approx b(\mathrm{Re} - \mathrm{Re}_{cr})$. The relation (4.37a) is shown in Fig. 4.7b, while Fig. 4.7a. corresponds to the case where $\delta > 0$. (The dotted lines in this figure correspond to amplitudes of unstable waves). In Fig. 4.7b two values $|A_1|^2_{\max} \approx \frac{|\delta|}{\beta} + \frac{2b}{|\delta|}(\mathrm{Re} - \mathrm{Re}_{cr})$ and $|A_2|^2_{\max} \approx \frac{2b}{|\delta|}(\mathrm{Re}_{cr} - \mathrm{Re})$, given by Eq. (4.37a), are shown for the case where $\mathrm{Re} < \mathrm{Re}_{cr}$. (If $\mathrm{Re} > \mathrm{Re}_{cr}$, then only the first of these is meaningful, while the second becomes negative and must therefore be replaced by the value $|A_2|^2_{\max} = 0$ which also corresponds to a vanishing right-hand side of Eq. (4.37)). Since $d|A|^2/dt < 0$ for $|A| > |A_1|_{\max}$ and $|A| < |A_2|_{\max}$, while $d|A|^2/dt > 0$ if $|A_2|_{\max} < |A| < |A_1|_{\max}$, we see that for $\delta < 0$, *β > 0* the primary flow is unconditionally stable only for Re < Re'$_{cr}$ (where $\mathrm{Re}'_{cr} \approx \mathrm{Re} - |\delta|^2/8b\beta$ is the value of Re at which two roots (4.37a) coincide). For Re'_{cr} x $< \mathrm{Re} < \mathrm{Re}_{cr}$ this flow is 'conditionally stable', i.e., stable with respect to small disturbances with

Fig. 4.7 The dependence of the equilibrium amplitude $|A(t)| = A_e$, satisfying the equation $d|A(t)|/dt = 0$, on the Reynolds number Re in the cases where either $\delta > 0$ (**a**), or $\delta < 0$ but the amplitude equation has the form (4.37) with $\beta > 0$ (**b**) Re_{cr} the critical Reynolds number; $\text{Re}_1 = \text{Re}_{\text{cr}}$ the threshold of subcritical instability. The *solid* and *dotted lines* represent amplitudes of stable and unstable equilibrium disturbances, respectively



$A_0 < |A_2|_{\text{max}}$, but if $A_0 \geq |A_2|_{\text{max}}$ then the disturbance amplitude grows rapidly to the 'equilibrium value' $|A_1|_{\text{max}}$ (this conclusion makes more precise the above statements about the possibility of subcritical finite-amplitude instability when $\delta < 0$). For $\text{Re} > \text{Re}_{\text{cr}}$ the primary flow is unstable to disturbances of any amplitude and the normal-mode disturbance grows to the value corresponding to the point on the solid line (of course, this is correct only if $|\text{Re} - \text{Re}_{\text{cr}}|$ is small enough to justify the expansion in powers of A and A^* up to the approximation (4.37)).

The above results describe only a part of the contents of Landau's paper (1944). Landau, assuming that $\delta > 0$, considered the development of flow structures with further increase of Re beyond Re_{cr}. It was natural to assume that at some higher value of the Reynolds number, $\text{Re}_{2,\text{cr}} > \text{Re}_{\text{cr}}$, the oscillatory stable flow (with frequency ω_1) arising from the primary steady flow at $\text{Re} = \text{Re}_{\text{cr}}$ may itself become unstable to small disturbances, transforming it to a new stable oscillatory motion which includes oscillations of two frequencies ω_1 and ω_2 and therefore has two degrees of freedom. (Steady laminar motion is fully determined by the general flow conditions and hence has no degrees of freedom; in the case of oscillatory motion with fixed frequency ω_1 the phase θ_1 can take any value and hence this motion has one degree of freedom; while quasi-periodic oscillations with two periods $2\pi/\omega_1$ and $2\pi/\omega_2$ possess two degrees of freedom). This new motion in its turn becomes unstable at $\text{Re} = \text{Re}_{3,\text{cr}} > \text{Re}_{2,\text{cr}}$ generating a motion with three degrees of freedom, and so on. Several short series of such successive transformations of a steady flow into an oscillatory one, and then into more complicated oscillations, were in fact observed after 1944 in some particular flows when the corresponding value of Re (or of another appropriate dimensionless control parameter) was increased step by step; some of these series will be mentioned later in this chapter. However Landau also assumed that, as Re increases, the intervals between consecutive critical Reynolds numbers $Re_{n,\,cr}$ and $Re_{n\,+\,1,\,cr}$ will become smaller and smaller, so that at large, but not excessively large, values of $Re - Re_{cr}$ the number of degrees of freedom of the resulting motion will reach a very high value. According to Landau, the complicated and disordered motion appearing in this way just represents the fully developed turbulent flow. This Landau's (or, as it is also often called reflecting the contribution of Hopf (1948), Landau-Hopf's) scenario of transition to turbulence seemed at first to be physically quite convincing, and during many years it was considered by the majority of experts as being correct in its main features even though it was often stressed that its validity was not proved rigorously and that it cannot be universal; see, e.g., Monin and Yaglom (1971), p. 165, or Drazin and Reid (1981), p. 370. However later it was found that Landau's theory of transition to turbulence is far less satisfactory than was thought earlier and must be radically revised; this conclusion was based on some amazing new developments which will be described later in this book. These new results concern Landau's ideas about the development of irregular fluctuations at $Re \gg Re_{cr}$, but they do not diminish the importance of his equation for the description and explanation of the initial stage of evolution of small disturbances at values of Re close to Re_{cr}.

The coefficient γ of Landau's Eq. (4.34) is equal to the imaginary part of the eigenvalue $\omega = \omega_1 + i\gamma$ corresponding to the normal-mode disturbance considered (originally Landau assumed that this disturbance was the one with the greatest imaginary part of ω). So, to determine this coefficient one need merely solve the eigenvalue problem of linear stability theory (in the case of a plane-parallel flow this is the famous Orr-Sommerfeld eigenvalue problem). Solutions of this eigenvalue problem may nowadays be calculated rather easily. However Landau's derivation of Eq. (4.33) gave no instructions about possible methods for determination of the numerical value of δ. It was clear from the outset that here the full nonlinear equations of motion must be used, but at first it was not known how to do this. Three-dimensionality of the Navier-Stokes equations complicates the problem considerably; therefore in the book by Eckhaus (1965) (which was the first one on nonlinear stability theory) much attention was given to simplified model problems in one-dimensional space (a related model was considered also in Sect. 50 of Drazin and Reid's book (1981)) and then only two-dimensional disturbances of two-dimensional flows were studied. The first, still imperfect, attempts to estimate the numerical value of Landau's constant δ for some particular flows with the help of the equations of motion were made by Meksyn and Stuart (1951) and Stuart (1958). In the first of these papers much attention was given to nonlinear effects leading to distortion of the primary velocity profile by disturbances in a plane Poiseuille flow, while in the second paper an approximate estimate of the value of δ for two-dimensional plane waves in a plane Poiseuille flow, and axisymmetric wave-like disturbances in a circular Couette flow, was based on the assumption that the disturbance's shape is preserved during its evolution. In both papers it was assumed that the Reynolds number Re has a slightly supercritical value and that the disturbances studied are unstable according to linear stability theory. (Note also that $\omega_1 = \Re e\ \omega$ differs from zero in the case of a plane Poiseuille flow but is equal to zero in a circular Couette flow). Meksyn and Stuart (1951) used the full Navier–Stokes system to compute the velocity distortion, and came to the conclusion that δ can have either sign. However, Stuart (1958) found that, according to the assumptions he made, the single Reynolds-Orr energy Eq. (4.2), which is only a particular consequence of the N-S system, implies Landau's Eq. (4.34) for disturbance amplitude $A = A\ (t)$ with a definite value of δ which is always positive.

Since some of the conclusions obtained by Meksyn and Stuart (1951) and by Stuart (1958) contradicted each other, Stuart (1960) (see also his survey papers (1962a, 1971)) and Watson (1960a) developed more precise methods to compute Landau's constant for small two-dimensional normal-mode disturbances in a plane Poiseuille flow. Stuart took into account that in the nonlinear development of a two-dimensional disturbance with given wave number k (i.e., having at $t = 0$ an initial velocity field of the form $\mathbf{u}(\mathbf{x}, 0) = \{u(z), 0, w(z)\}\ e^{ikx}$), higher harmonics (proportional to e^{inkx}, $n = 2$, 3,...) will also be generated. Therefore, he represented the disturbance velocity field $\mathbf{u}(\mathbf{x}, t)$ for $t > 0$ in the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(z, t) + \mathbf{u}_1(z, t)e^{ikx} + \mathbf{u}_2(z, t)e^{i2kx} + \dots . \tag{4.38}$$

Here $\mathbf{u}_\mathrm{n}(z, t)e^{inkx}$, $n = 0, 1, 2, \dots$, are two-dimensional solenoidal vectors (in general complex; remember that the true velocity is equal to the real part of the given expression) depending on t, and the term $\mathbf{u}_0(z,\ t)$ describes the distortion of the laminar Poiseuille-flow velocity profile by the disturbance. Further, it was assumed that as $t \to 0$, only the term on the right-hand side of Eq. (4.38) which is proportional to e^{ikx} is conserved, while for very small $t > 0$, this term becomes the solution $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_1(z)e^{i(-\omega t + kx)} = \mathbf{u}_1(z)e^{ik(x - ct)}$ of the Orr-Sommerfeld equation describing a growing or damped wave-like disturbance. Then, for slightly greater, but nevertheless small, positive t the first harmonic will be leading term on the right-hand side and $\mathbf{u}_1(z, t)$ may be written as

$$\mathbf{u}_1(z, t) = A(t)\mathbf{u}_1(z) + \text{higher-order terms}. \tag{4.39}$$

Stuart (1960) substituted Eqs. (4.38) and (4.39) into the nonlinear Navier–Stokes equations (which he replaced by the equivalent non-linear equation for the stream function $\Psi(x, z, t)$] corresponding to the velocity field $\mathbf{U} + \mathbf{u}(\mathbf{x}, t)$, where $\mathbf{U} = \{U(z), 0, 0\}$ is the Poiseuille-flow velocity (instead of using only Eq. (4.2) as in his 1958 paper). Assuming now that $|\gamma| = |\Im m\omega|$ is a small quantity (i.e., considering a disturbance with small amplification or damping corresponding to a point in the (k, Re)-plane close to the neutral-stability curve) and using expansion in powers of this quantity, he obtained for the complex amplitude $A(t)$ an approximate Landau-type equation of the form

$$\frac{dA}{dt} = -i\omega A - \tfrac{1}{2}l|A|^2 A \tag{4.40}$$

where ω is the same complex frequency as above and $l = \delta + i\delta'$ is another complex coefficient. (Later Fujimura (1989); Dušek et al. (1994), and Park (1994) reconsidered the derivation of Eq. (4.40) from the Navier–Stokes equations and indicated several sets of assumptions implying its validity, while Zhou (1991) indicated that in some cases the computation of the second term on the right-hand side of (4.39) is necessary for obtaining the satisfactory agreement with the experimental data). Equation (4.40) is usually called either the *complex Landau equation* or the *Stuat-Landau equation* (see e.g., Kuramoto (1984)) and l is the *complex Landau constant*. Representing the complex amplitude $A(t)$ as $|A(t)|\, e^{i\phi(t)}$, it is easy to show that the real part of Eq. (4.40) is equivalent to Landau's Eq. (4.34) for $|A|^2 = AA^*$, where $\gamma = \Im m\omega$ and $\delta = \Re el$. On the other hand, the imaginary part of Eq. (4.40) can be written as the following equation for the phase $\phi(t)$, supplementing Landau's equation:

$$\frac{d\phi}{dt} = -\omega_1 - \tfrac{1}{2}\delta'|A|^2 \tag{4.34a}$$

where $\omega_1 = \Re e\omega$, $\delta'' = \Im ml$.

According to Stuart's results, $\delta = \delta_1 + \delta_2 + \delta_3$ where the three terms correspond to three different physical processes affecting the nonlinear development of a wave-like disturbance. He also noted that only the term δ_1 (which is always positive) was taken into account in his paper of 1958 (hence the conclusion of this paper that δ was positive was an inevitable consequence of the assumptions made); and only terms δ_1 and δ_3 were considered (and imprecisely estimated) by Meksyn and Stuart (1951). For all three terms Stuart obtained explicit expressions, which were however rather cumbersome and contained the eigenvalues and eigenfunctions of the corresponding Orr-Sommerfeld equation (and also of the adjoint equation) in a complicated manner. These expressions clearly depend on k and Re; however, the numerical calculation of them (and of their sum δ) seemed to a very difficult problem in the early 1960s.

In the paper by Watson (1960a) accompanying that by Stuart a more complete Fourier representation of the disturbance velocity was used and the technique, traditional for the disturbance theory, of expansion into powers of the amplitude (instead of the powers of $\gamma = \Im m\omega$ considered by Stuart) was applied to the fluid-dynamic equations describing disturbance development. (However, expansion in powers of $|\gamma|$ was also used here and hence $|\gamma|$ was assumed to be small in Watson's derivations too). As a result, Watson obtained a new and more rigorous reformulation of Stuart's theory, leading to the generalized Landau equation of the form

$$\frac{d|A|^2}{dt} = |A|^2 \sum_{m=0}^{\infty} a_m |A|^{2m} \tag{4.41}$$

for the squared amplitude $|A|^2$. Here evidently $a_0 = 2\gamma, a_1 = -\delta$, while expressions for the coefficients a_m with $m > 1$ were found to be much more complex than for the Landau coefficient δ. Another rigorous analytical method allowing the investigation of streamwise periodic solutions of nonlinear equation for two-dimensional disturbances in a plane Poiseuille flow, which supplemented the expansion in powers of the small quantity $|\gamma| = |\Im m\omega|$ by expansion of all relevant functions of z in terms of the eigenfunctions of the linear O-S equation, was developed by Eckhaus (1965); it also led to confirmation of Stuart's (1960) results. Note, however, that the assumptions about the smallness of $|\gamma|$ used by Stuart, Watson, and Eckhaus made their theories inapplicable in principle to plane Couette and circular Poiseuille flows (for example), where unstable normal modes do not exist and therefore $|\gamma|$ cannot be very small. Therefore Ellingsen et al. (1970) and Itoh (1977a, b), who wanted to apply Stuart-Watson's theory to just these two exceptional flows, were forced to modify this theory to a form where only the smallness of the amplitude A was assumed. It was found in these papers that in fact the smallness of the disturbance amplitude is sufficient for the possibility of rigorous derivation of the Landau equation from the Navier-Stokes equations. More detailed analysis of assumptions utilized in the rigorous derivations of Eq. (4.41) was undertaken in particular by Herbert (1983b) and Fujimura (1989, 1991, 1997) whose papers will be discussed later in this subsection.

Stuart (1960) and Watson (1960a) investigated only the temporal nonlinear development of a two-dimensional wave disturbance in a steady plane Poiseuille flow. Two-dimensionality of the waves significantly simplified the theory, and could be justified to a certain degree by the results of Watson (1960b) and Michael (1961) mentioned in Chap. 2, they showed that, in the framework of the linear stability theory, there always exists for any steady plane-parallel flow a range of supercritical values of Re, $\mathrm{Re}_{cr} < \mathrm{Re} < \mathrm{Re}_1$, within which the most rapidly growing normal-mode disturbance is necessarily two-dimensional. However, Benney and Lin (1960) (see also Benney (1961, 1964)) indicated that when the nonlinear development is studied, interactions between two- and three-dimensional waves must be also of great importance. In this context Stuart (1962b) (see also his surveys (1962a, 1971)) generalized his and Watson's weakly-nonlinear disturbance theory of 1960 to the case of the evolution in plane Poiseuille flow of a disturbance which is composed of a two-dimensional and a three-dimensional plane wave with the same streamwise number k_1. Assuming that both disturbances are slowly growing or decaying, it is permissible, for relatively small values of t, to represent the velocity field of the disturbance considered in the form

$$\mathbf{u}(\mathbf{x}, t) = A_1(t)\mathbf{u}_1(z)e^{ik_1x} + A_2(t)\mathbf{u}_2(z)e^{i(k_1x+k_2y)} + \text{higher-order terms} \qquad (4.42)$$

including two time-dependent amplitudes $A_1(t)$ and $A_2(t)$. Then, using the expansion technique given in Stuart's and Watson's papers of 1960, Stuart obtained, for both amplitudes A_1 and A_2, two generalized Landau-type equations differing from (4.41) by the presence of their right-hand sides of the sums of composite terms $a_{m,n}|A_1|^{2m}|A_2|^{2n}$. In the lower non-linear approximation the "amplitude equations" for real amplitudes $A_1(t)$ and $A_2(t)$ (obtained when the complex exponential functions in Eq. (4.42) are replaced by real trigonometric functions) had the following form:

$$
\begin{aligned}
\frac{dA_1}{dt} &= \gamma_1 A_1 - (\delta_1 A_1^2 + \beta_1 A_2^2)A_1, \\
\frac{dA_2}{dt} &= \gamma_2 A_2 - (\beta_2 A_1^2 + \delta_2 A_2^2)A_2,
\end{aligned}
\tag{4.43}
$$

which, for $A_2 = 0$ or $A_1 = 0$, clearly yield an equation which is equivalent to Landau's Eq. (4.34) for the amplitude of a single wave disturbance. The system (4.43), under the condition that $\gamma_1/\delta_1, \gamma_2/\delta_2$ and the ratios of bilinear combinations of the coefficients entering Eq. (4.44) below are positive, evidently has the following four steady solutions:

$$
\begin{aligned}
&\text{(I)} \quad A_1 = A_2 = 0, \\
&\text{(II)} \quad A_1 = 0,\ A_2 = (\gamma_2/\delta_2\ )^{1/2}, \\
&\text{(III)} \quad A_1 = (\gamma_1/\delta_1\ )^{1/2},\ A_2 = 0, \\
&\text{(IV)} \quad A_1 = (\gamma_1\delta_2 - \gamma_2\beta_1)^{1/2}\,(\delta_1\delta_2 - \beta_1\beta_2)^{1/2}, \\
&\qquad\quad A_2 = (\gamma_2\delta_1 - \gamma_1\beta_2)^{1/2}\,(\delta_1\delta_2 - \beta_1\beta_2)^{1/2}.
\end{aligned}
\tag{4.44}
$$

The stability of these solutions, which may be verified by known methods of stability theory of nonlinear differential equations (or nonlinear oscillations), is of considerable interest, and it was only natural that Stuart considered this question, paying special attention to cases where solution (IV), which represents an equilibrium state consisting of a combination of two- and three-dimensional wave oscillations, is stable. Stuart's two-mode weakly-nonlinear theory of 1962 was developed further by Itoh (1980) who supplemented it by some numerical examples illustrated by graphs.

In all the above-mentioned papers devoted to rigorous derivation of amplitude equations of the Landau type, only the nonlinear temporal development of wave-like disturbances with fixed wave numbers was considered. However, it was explained in Chap. 2 of this book that, in the case of steady flows with significant streamwise velocity $U(z)$ (e.g. boundary layers along flat plates or plane Poiseuille flows), the model of a streamwise developing disturbance of fixed real angular frequency ω corresponds better to observations in real experiments on flow instability, and therefore seems to be more appropriate. Taking this into account Watson (1962) modified the theory developed in his paper (1960a) assuming that a two-dimensional wave-like disturbance in a plane Poiseuille flow has fixed real frequency ω but complex streamwise wave number $k = k_1 + ik_2$, determined from the Orr-Sommerfeld eigenvalue problem with fixed real ω and unknown complex eigenvalue k. Then, according to the weakly nonlinear stability theory, the leading term of the evolving disturbance will have the form $\mathbf{u}(x,t) = A(x)\mathbf{u}(z)e^{i\omega t}$, where $\mathbf{u}(z)$ is the eigenfunction of the spatial O-S eigenvalue problem and $A(x) = e^{ikx}$ for very small values of x. Then, representing the velocity field $\mathbf{u}(\mathbf{x},t) = \mathbf{u}(x,z,t)$ (or the streamfunction field $\Psi(x,z,t)$) for $x > 0$ as a Fourier series in powers of $e^{i\omega t}$ (instead of the spatial Fourier series (4.38)) and applying appropriately-modified arguments from his paper

(1960a), Watson obtained for the spatially evolving amplitude $A(x)$ an equation of the form

$$\frac{d|A|^2}{dx} = |A|^2 \sum_{m=0}^{\infty} b_m |A|^{2m} \tag{4.41a}$$

which is completely similar to Eq. (4.41) (and turns into the spatial version of Landau's Eq. (4.34) when only the first two terms on the right-hand side are retained). It is clear that the value of $b_0 = -2k_2 = -2\Im mk$ can be now calculated by numerical solution of the spatial O-S eigenvalue problem (which is somewhat more complicated than the corresponding temporal problem but nevertheless accessible to computation; see Sect. 2.92). However, the expression found by Watson for the coefficients b_m with $m > 0$ turned out to be much more complex than the—also rather complicated—expressions for the corresponding coefficients a_m; therefore in the early 1960s their evaluation seemed to be impossible. But somewhat later Itoh (1974a, b) showed that by that time the values of the 'spatial Landau constant' $\delta_s = -b_1$ might already have been calculated with satisfactory accuracy for some important plane-parallel flows (see Figs. 4.11 and 4.17 below).

Note that Stuart (1960; a, 1962a,b) and Watson (1960a, 1962) used the fluid dynamics equations only for rigorous derivation of amplitude equations, and did not try to determine numerical values of the coefficients of the latter. Simultaneously, Stuart stressed that the early estimates of the value of δ by Meksyn and Stuart (1951) and Stuart (1958) are not trustworthy. Therefore it was natural to think that Stuart's and Watson's papers would stimulate other authors to find, at last, some accurate estimates of Landau's constant and of other coefficients of amplitude equations. And in fact papers devoted to such estimation began to appear soon after those mentioned above. We will now pass on to results of this subsequent work.

4.2.2 Evaluation of Coefficients of Amplitude Equations and Equilibrium Disturbances for Plane Poiseuille Flows

One of the first attempts to find a more or less reliable value for the Landau constant δ was made by Davey (1962) for the case of the growth of axisymmetric Taylor vortices in a Couette flow between rotating cylinders. Davey reformulated for this case all the arguments of Stuart (1960) and found that Stuart's equation $\delta = \delta_1 + \delta_2 + \delta_3$, where the three terms δ_i have the same physical meaning as in the case of plane Poiseuille flow, also appears here. He also found that in this case the expressions for these terms are again rather complicated but are nevertheless accessible to numerical computation. So, he calculated the value of δ for three particular combinations of the ratios $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2$. The values found turned out to be positive in all cases considered, for all vertical wave numbers k and Reynolds numbers Re, and these values agreed satisfactorily with the then-available experimental data. However, we will not linger here on these results of Davey, since nonlinear stability of circular

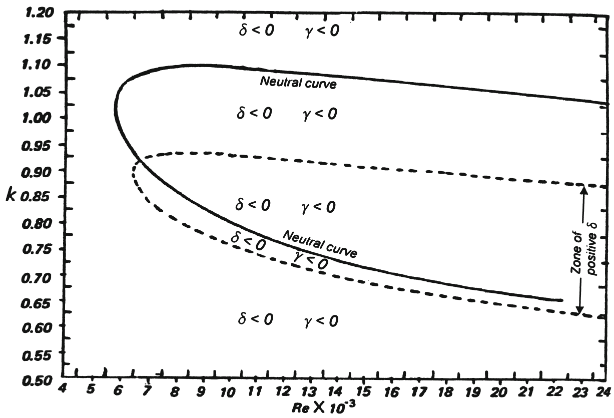
Couette flow will be considered separately later in this book. So now we will turn again to cases of plane-parallel (or nearly plane-parallel) primary flows.

Let us revert first of all to the model case of *plane Poiseuille flow.* Recall that Stuart's and Watson's nonlinear-stability papers of 1960 were both devoted to just this case, which was also considered rather early by Meksyn and Stuart (1951), then by Stuart (1958), and later by Eckhaus (1965). This was only natural, since plane Poiseuille flow is a classical example of steady, strictly plane-parallel, laminar flow having a very simple velocity profile, and had been extensively investigated within the framework of the linear theory of hydrodynamic stability. Thus, it was not surprising that relatively accurate estimates of the values of the Landau constant for disturbances in a plane Poiseuille flow were among the first applications of the Stuart-Watson theory to appear.

The above-mentioned estimates were calculated independently and almost simultaneously by Reynolds and Potter (1967) and Pekeris and Shkoller (1967). Reynolds and Potter used some extension and modification of the Stuart-Watson approach where determination of the equilibrium disturbances, introduced in application to another problem by Malkus and Veronis (1958), played a very important part, while Pekeris and Shkoller based their computations on the Eckhaus eigenfunction-expansion method. In both papers the computations were carried out for two-dimensional normal-mode disturbances corresponding to the unstable (or, if $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$, to the least stable) solution of the Orr-Sommerfeld eigenvalue problem under the condition that $|\gamma| = |\Im m\omega|$ is sufficiently small. However, Reynolds and Potter also included in their paper some remarks relating to three-dimensional disturbances, and presented some numerical results for the more general case of plane Couette–Poiseuille flows (these results will be discussed in Sect. 4.23). For plane Poiseuille flow Reynolds and Potter calculated values of δ at five different points of the neutral stability curve in the (k, Re)-plane (including the critical point (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$)), and at two points in the neighborhood of the neutral curve, while Pekeris and Shkoller evaluated the coefficient $\delta = \delta(k, \mathrm{Re})$ for an extensive region of the (k, Re)-plane (using equations which are in fact reasonable only in the vicinity of the neutral curve). The results of these two papers do not coincide numerically (one reason being that they used different normalizations and somewhat different definitions of the amplitude $|A|$, besides which some of the assumptions and approximations taken for granted in the two papers were different), but both results have the same general behavior and imply close agreement for ratios of the values $\delta = \delta$ (k, Re) at different points of the (k, Re)-plane.

In Fig. 4.8, results by Pekeris and Shkoller (agreeing, in general, with Reynolds and Potter's conclusions) are presented, including the neutral curve but without numerical values for γ and δ. (As to the values of Re and k, it is here assumed, as usually, that $\mathrm{Re} = U_{\max} H_1/\nu$ and k is made dimensionless by multiplication by H_1). These results show, in particular, that at the critical point (the point of the neutral curve farthest to the left), and at all points of the upper branch of the neutral curve, δ is negative. Some unstable two-dimensional disturbances of finite amplitude with wave number k must correspond to values of (k, Re) at points lying close to the neutral curve in the region where $\delta < 0$ and $\gamma < 0$; this means that at these values of (k, Re),

Fig. 4.8 The regions of positive and negative values of the coefficients γ and δ in the (k, Re)-plane for the case of plane Poiseuille flow. (After Pekeris and Shkoller (1967))



subcritical finite-amplitude instabilities exist in plane Poiseuille flow. Therefore, in the region where $\delta < 0$, the neutral curve (which bounds the set of points (k, Re) corresponding to unstable two-dimensional disturbances) shifts, in the case of finite disturbances, from the neutral-stability curve of linear stability theory (which relates to infinitesimal disturbances) and takes the shape shown in the schematic Fig. 4.9. On the other hand, for points (k, Re) in the region where $\delta < 0$, $\gamma > 0$ the negativity of δ means that supercritical finite-amplitude equilibrium states are rather unlikely to be observed here. Figure 4.8 shows also that $\delta > 0$ on the main part of the lower branch of the neutral curve. At the points (k, Re) close to this part the subcritical finite-amplitude instability does not exist for disturbances with $\gamma < 0$; however, if

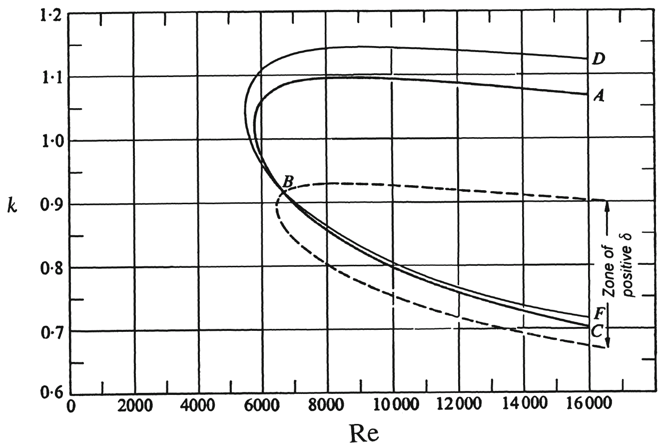


Fig. 4.9 Schematic form of the neutral-stability *curve DBF* for wave disturbances of plane Poiseuille flow having a fixed finite amplitude A. (After Pekeris and Shkoller (1969b)) the *curve ABC* is the neutral curve for infinitesimal disturbances where $\gamma = 0$, and the dotted *curve* represents points where $\delta = 0$

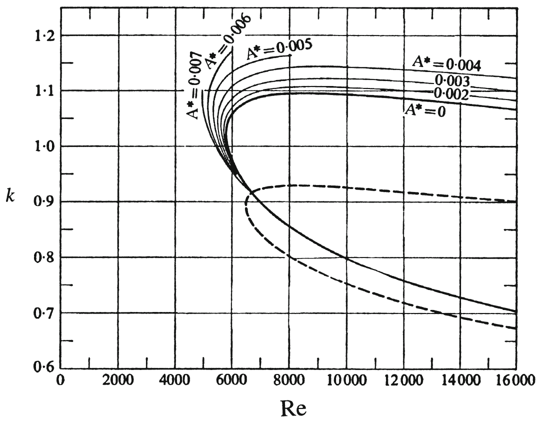


Fig. 4.10 Deviations of the neutral *curves* for wave disturbances of plane Poiseuille flow having finite amplitudes (characterized by the value of some dimensionless 'amplitude parameter' A^*) from the neutral curve for infinitesimal disturbances (corresponding to $A^* = 0$) in the region where $\delta < 0$, computed by Pekeris and Shkoller (1969b). The *dotted curve* have the same meaning as in Figs. 4.8 and 4.9

$\gamma = 0$ and $\delta > 0$ for a small but not infinitesimal disturbance, then this disturbance will decay according to Eq. (4.34). Thus, the neutral curve for finite disturbances corresponding to points where $\delta > 0$ must shift into the supercritical region where $\gamma > 0$, and hence finite-amplitude equilibrium states must exist.[6]

Reynolds and Potter's and Pekeris and Shkoller's papers stimulated the appearance of many subsequent papers on the nonlinear evolution of wave disturbances in a plane Poiseuille flow. These later papers, only some of which will be referred to below, include various amendments, modifications and revisions of results presented in the publications of 1967. In particular, Pekeris and Shkoller (1969a,b; 1971) computed some approximate solutions of the nonlinear initial-value problem for the least-stable Tollmien–Schlichting (T–S) wave with given wave number k, i.e., for the two-dimensional disturbance having the initial stream function of the form $\Psi(x,z,0) = Af_1(z)e^{ikx}$ where $f_1(z)$ is the normalized first (least stable) O-S eigenfunction of the plane Poiseuille flow and A is a disturbance amplitude which is finite (but small enough, since an expansion in powers of amplitude was used here). Using the computed results Pekeris and Shkoller tried to estimate quantitatively the shifts of the neutral curves for finite-amplitude disturbances of the form given above, for various values of A (see Fig. 4.10, taken from their paper (1969b)), and to determine the value of the finite-amplitude critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}(A)$ (which corresponds to the point which is farthest to the left on the neutral curve for disturbances of amplitude A). The same problem was studied by Georg and Hellums (1972) and Georg et al. (1974) who considered another initial form of disturbance (i.e. they did not use the traditional approach of considering the least-stable T–S wave) and another method of numerical solution of the nonlinear initial-value problem (which used neither the Eckhaus expansion into O-S eigenfunctions nor the expansion in powers of the amplitude, and hence was applicable to disturbances of any initial size).

[6] According to Eq. (4.34) and Fig. 4.8, the lower branch of the neutral curve in the case of finite disturbances must shift upward (to points where $\gamma \approx \delta|A|^2/2$). This very small shift is exaggerated in Fig. 4.9 to simplify its representation in the figure but later it will be neglected.

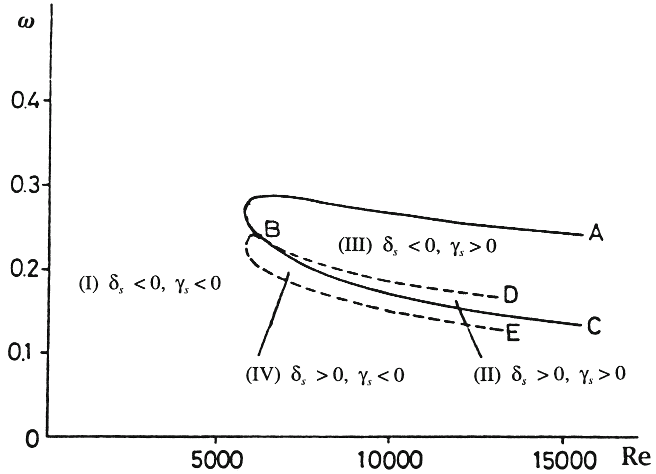


Fig. 4.11 The regions of positive and negative values of the coefficients γ_s and δ_s in the (ω,Re)-plane for the case of plane Poiseuille flow. (After Itoh (1974a))
ABC: the *curve* $\gamma_s(\omega, \text{Re}) = 0$ (the spatial neutral-stability *curve* of the linear stability theory bounding the region where $\gamma_s > 0$); *DBE*: the *curve* $\delta_s(\omega, \text{Re}) = 0$ bounding the region where $\delta_s > 0$

Georg et al. (whose estimates of the values of critical numbers $\text{Re}_{cr}(A)$ were later found by Orszag and Kells (1980) to be too high because of the use of a non-optimal initial form of the disturbance) compared their results with those of several previous papers (including those by Reynolds and Potter and Pekeris and Shkoller). They found that the quantitative results of different authors sometimes do not agree adequately well, but all of them demonstrate the same general tendency. One more method for approximate determination of the neutral curve for two-dimensional finite-amplitude wave disturbances in a plane Poiseuille flow was proposed by Struminskii and Skobelev (1980), who used for this purpose the generalized Landau equation of the form (4.37). Later Luo (1994) reexamined the previously used methods of determination of complex coefficients ω and l in the Stuart-Landau Eq. (4.40). He suggested some improvements and showed that in the case of plane Poiseuille flow they lead to values of coefficients which agree well with those given by numerical simulation of disturbance evolution in this flow.

Itoh (1974a) studied the development of a spatially-evolving two-dimensional disturbance of frequency ω in a plane Poiseuille flow, using the theory by Watson (1962) modified by accounting more accurately for distortion of the mean flow by the disturbance. Using the modified version of Watson's theory, he computed approximate shapes of the curves $\gamma_s(\omega, \text{Re}) = 0$ and $\delta_s(\omega, \text{Re}) = 0$ (where $\gamma_s = b_0/2$ and $\delta_s = -b_1$ are coefficients of the 'spatial Landau equation', and ω is non-dimensionalized by multiplication by $H_1/U_{\max}$) on the (ω, Re)-plane. These curves are shown in Fig. 4.11; they determine location of the regions of positive and negative values of γ_s and δ_s in the (ω, Re)-plane and proved to be qualitatively similar to Pekeris and Shkoller's curves in Fig. 4.8 which correspond to temporally-evolving disturbances in the same flow.

Related computations were performed by Herbert (1976, 1977, 1978) (see also his review (1983a)) who used a quite different method. This author followed the approach initiated by Zahn et al. (1974) (and outlined in rudimentary form as far back as Noether (1921) and Heisenberg (1924)), and studied approximate numerical solutions of the nonlinear initial-value problem for stable Tollmien-Schlichting waves (i.e., those which are exponentially damped according to the linear stability theory), which at small values of t are represented by the O-S eigensolutions. He paid most attention to equilibrium solutions (i.e., to wave disturbances satisfying the condition that $d|A|^2/dt = 0$) at various values of k and Re. To find the value of the stream function $\Psi(x, z, t)$ corresponding to an evolving T–S wave, both Zahn et al. and Herbert represented Ψ by a strongly truncated Fourier series of the form (4.38), and then solved numerically a system of coupled nonlinear equations for the corresponding Fourier coefficients, simplifying this system greatly for the case of equilibrium solutions.

Herbert found numerous equilibrium two-dimensional disturbances in a plane Poiseuille flow which are periodic in the streamwise direction and have finite amplitudes. His results agree well with results of preceding numerical studies by Zahn et al. (1974), and of subsequent more accurate computations by Orszag and Kells (1980); Orszag and Patera (1980, 1981); Milinazzo and Saffman (1985); Ehrenstein and Koch (1991); Balakumar (1997); Hewitt and Hall (1998), and some others (see also the survey by Bayly et al. (1988)). Measuring the size of a two-dimensional wave disturbance by the ratio E of its kinetic energy (per unit length of the channel) to the energy of primary Poiseuille flow (E is clearly a single-valued function of A and is proportional to $|A|^2$ with good accuracy), Herbert determined the shape of the *neutral surface* (corresponding to the set of all two-dimensional equilibrium waves) in the three-dimensional (E, k, Re)-space; this surface is shown schematically in Fig. 4.12. (See also Ehrenstein and Koch (1991) and Sect. 2.8.3 in Godrèche and Manneville (1998) where a slightly different presentation of this surface is given. Two intersections of this surface with the plane Re = const. will be shown in Sect. 4.2.3 in Fig. 4.14a, b where, however, U_{max} is replaced by $U_{ave} = 2U_{max}/3$ in the definition of Re; some of its other intersections with planes Re = const. and k = const. can be found in Sect. 2.8.3 of Godrèche and Manneville (1998) and in the paper by Hewitt and Hall (1998)). The intersection of the neutral surface with the plane $E = 0$ clearly coincides with the Poiseuille-flow neutral curve of linear stability theory (shown, in particular, in Figs. 2.22 and 4.8), while the intersections of this surface with the planes E = const. (where also A = const.) coincide with the neutral-stability curves for finite-amplitude disturbances with given value of E (or A; cf. Figs. 4.9 and 4.10). The projection of the whole neutral surface in (E, k, Re)-space on the (k, Re)-plane is also indicated in Fig. 4.12; this projection determines the region of the (k, Re)-plane corresponding to unstable two-dimensional waves of any amplitude. This region is clearly much larger than the region of unstable infinitesimal waves, which is bounded by the neutral curve of linear theory. The projection of the leftmost point of the neutral surface in (E, k, Re)-space on the (k, Re)-plane determines the lowest Reynolds number Re^*_{cr} at which there exist also undamped two-dimensional waves of any amplitude, and the critical wave number k^*_{cr} corresponding to the undamped wave at Re $= Re^*_{cr}$. Similarly,

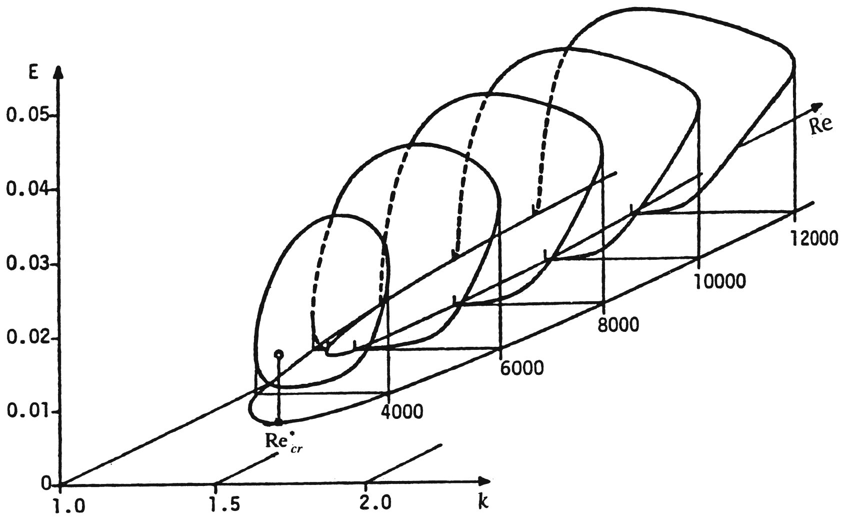


Fig. 4.12 Schematic shape of the nonlinear neutral surface in the three-dimensional (*E, k,* Re)-space corresponding to the set of all two-dimensional equilibrium waves in plane Poiseuille flow. (After Herbert (1977, 1978, 1983a))

the leftmost points of the neutral curves for waves with fixed energy E (and amplitude A) determine the critical Reynolds numbers $\mathrm{Re}_{\mathrm{cr}}(E)$ (or $\mathrm{Re}_{\mathrm{cr}}(A)$) for waves of fixed energy (and amplitude) and their wave numbers $k_{\mathrm{cr}}(E)$ (or $k_{\mathrm{cr}}(A)$). According to Herbert's approximate computations, $\mathrm{Re}^*_{\mathrm{cr}} \approx 2935$ (as usual, channel half-thickness and Poiseuille-flow maximum velocity are used here as length and velocity scales) and to this corresponds the critical wave number $k^*_{\mathrm{cr}} \approx 1.32$. Later Herbert's results were confirmed also by Orszag and Kells (1980); Ehrenstein and Koch (1991), and Balakumar (1997).

Note that the 'nonlinear critical Reynolds number' $\mathrm{Re}^*_{\mathrm{cr}}$ is considerably smaller than the ordinary (linear) critical Reynolds number $\mathrm{Re}_{\mathrm{cr}} \approx 5772$ which relates to infinitesimal wave disturbances. However $\mathrm{Re}^*_{\mathrm{cr}}$ is much greater not only than the value $\mathrm{Re}_{\mathrm{cr\,min}} \approx 50$ which is given by the energy method and applies to disturbances of any shape and size, but also much greater than the value $\mathrm{Re}_1 \approx 1{,}000$ which, according to data by many authors (e.g., by Davies and White (1928); Patel and Head (1969); Kao and Park (1970); Nishioka and Asai (1985), and Alavyoon et al. (1986); see also Sect. 2.1) is typical for transition to turbulence in laboratory experiments on plane Poiseuille flow. During the 1980s and early 1990s a number of authors (in particular, Orszag and Kells (1980); Orszag and Patera (1982, 1983); Saffman (1983); Herbert (1983c, 1984, 1986); Soibelman and Meiron (1991); see also the surveys by Bayly et al. (1988) and Herbert (1988)) suggested the idea that the difference between values of $\mathrm{Re}^*_{\mathrm{cr}}$ and of Re_1 can be explained by secondary instability of stable two-dimensional waves to small three-dimensional disturbances at values of Re smaller than $\mathrm{Re}^*_{\mathrm{cr}}$. To verify this idea these (and some other) authors performed a number of numerical simulations (i.e., solutions of the corresponding nonlinear initial-value problems for the N-S equations) of development, in a plane Poiseuille flow, of stable two-dimensional waves of finite amplitude in the presence of small three-dimensional disturbances with the same streamwise wave number. The results obtained showed that three-dimensional disturbances often destabilize two-dimensional waves, and cause rapid growth of two-mode disturbances at Reynolds numbers of the order of 700–1,000, much smaller than Re_{cr}^* (and close to Re_1). (The simulations by Pugh and Saffman (1988) and the subsequent study by Barkley (1990) showed that the instability of two-dimensional equilibrium waves with respect to superimposed three-dimensional disturbances has a more complex character than was assumed earlier. Moreover, there were several attempts to explain the secondary instability of two-dimensional waves by triad interactions of such a wave with two three-dimensional ones, and these attempts also led to critical Reynolds number close to Re_1; see, e.g., Goldshtik et al. (1983, 1985); Craik (1985); Ehrenstein and Koch (1991), and Ehrenstein (1994). And still later Reddy et al. (1998) considered some quite different scenarios of the primary and secondary instabilities of a plane Poiseuille flow where two-dimensional waves play no part at all. However, we will not consider all these works in this section). The secondary instability of two-dimensional periodic waves usually generates, not a new equilibrium cellular state but a very complicated three-dimensional structure reminiscent developed turbulence; see, e.g., Saffman (1983); Rozhdestvensky and Simakin (1984); Bayly et al. (1988), and Jiménez (1987, 1990). In some of this work several successive transitions of Poiseuille flow to more and more complex behavior were also simulated numerically (more details of this will be given later).

Numerically-simulated equilibrium and developing wave disturbances in a Poiseuille flow may in principle be used to get some information about the values of Landau's constant and other coefficients of the amplitude equations for one-mode or composite two-mode waves. The estimates of δ implied by the results of Herbert's and Orszag and Kell's numerical simulations of two-dimensional equilibrium waves in a plane Poiseuille flow proved not to contradict values found earlier by Reynolds and Potter (1967); Pekeris and Shkoller (1967), and other authors by the quite different methods initiated by Stuart, Watson, and Eckhaus.

Quite another approach was applied to study of development of two-dimensional finite-amplitude waves in a plane Poiseuille flow by Andreichikov and Yudovich (1972) and Chen and Joseph (1973). This approach was based on the general theory of bifurcations, which is a special part of nonlinear science closely connected with stability problems. The word *bifurcation* means here the appearance of a supplementary solution of a given nonlinear 'dynamic equation' (or system of equations), describing the evolution of a definite object, when some dynamic parameters vary. The 'dynamic equation' (or equations) may be here algebraic, ordinary differential, partial differential or any other type. *Bifurcation theory* deals with the most typical features of the nonlinear evolution, namely, with the frequent occurrence of qualitative changes of the object's behavior corresponding to small variation of some dynamic parameters. Drazin and Reid (1981), p. 403, reasonably noted that this theory arose from particular early work by Poincaré and Lyapunov on figures of equilibrium of rotating self-gravitating masses of fluid, but the sphere of its applications has broadened enormously. Therefore it is natural that in recent years this theory attracted much attention, and gave rise to extensive and quite diverse literature. As typical examples we may mention here the books and survey papers by Sattinger (1973); Marsden (1978); Mittleman and Weber (1980); Seydel (1988); Arnol'd (1989b); Iooss and Joseph (1990); Baker (1991); Hale (1991); Iooss and Adelmeyer (1992); Guckenheimer and Holmes (1993); Arnol'd et al. (1994), and Field (1996).

An example of an instability-generated bifurcation in a fluid flow is given in Fig. 4.7a, where the dependence on Reynolds number Re of the equilibrium amplitude $A_e = |A|_{max}$ of a normal-mode disturbance in a primary steady flow is presented for the case where $\delta > 0$. Here $A_e = 0$ for $\mathrm{Re} < \mathrm{Re}_{cr}$; however, if $\mathrm{Re} > \mathrm{Re}_{cr}$ (but $\mathrm{Re} - \mathrm{Re}_{cr}$ is small), then $\gamma \propto \mathrm{Re} - \mathrm{Re}_{cr} > 0$, and the amplitude of a small disturbance tends to the equilibrium value $A_e = (2\gamma/\delta)^{1/2} \propto (\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ (see the upper part of Fig. 4.5). Thus, the flow consisting of the primary flow and a superimposed two-dimensional periodic wave of amplitude A_e bifurcates at $\mathrm{Re} = \mathrm{Re}_{cr}$ from the pure primary flow. Figure 4.7b corresponds to the case where $\delta < 0$ and shows another type of bifurcation: here, according to the figure the 'secondary solution' which includes a finite-amplitude wave appears at $\mathrm{Re} = \mathrm{Re}'_{cr} < \mathrm{Re}_{cr}$ but transition from the primary steady solution to this new solution can be caused only by a wave disturbance with amplitude exceeding $|A_2|_{max}$. Let us now consider the complex Landau amplitude $A(t)$ which satisfies Eq. (4.40) and describes the time-dependence of the leading term of the disturbance velocity $\mathbf{u}(\mathbf{x}, t)$. As we know, here $A(t) = |A(t)|\, e^{i(-\omega t+\theta)}$, where $|A(t)|^2$ satisfies the Landau Eq. (4.34) and the constant θ depends on the initial disturbance $\mathbf{u}(\mathbf{x}, 0)$. According to Fig. 4.7a, if $\delta > 0$, then for $\mathrm{Re} < \mathrm{Re}_{cr}$ the complex amplitude $A(t)$ for any initial value $A(0)$ tends to zero (i.e., to the origin of the complex-variable plane) as $t \to \infty$. In other words, for $\mathrm{Re} < \mathrm{Re}_{cr}$ all trajectories $A = A(t)$ in the complex-variable plane corresponding to various solutions of the complex Landau Eq. (4.40) are attracted to a focus at the origin. If, however, $\mathrm{Re} > \mathrm{Re}_{cr}$, then $|A(t)| \to A_e \propto (\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ as $t \to \infty$ and hence the trajectory $A(t) = |A(t)|\, e^{i(-\omega t+\theta)}$ is here attracted to the circle of radius A_e in the complex-variable plane which makes up the *limit cycle* of the two-dimensional dynamical system corresponding to dynamic Eq. (4.40) (i.e., to a system of two Eqs. (4.34) and (4.34a) for real and imaginary parts of $A(t)$). This is just a specific case of the so-called *Hopf bifurcation*[7], where a periodic solution bifurcates from a steady

[7] This term reflects the contribution by Hopf (1942) to this subject. However sometimes its use meets objections since such bifurcations were in fact explicitly studied by A. A. Andronov (partially in collaboration with A. A. Vitt) in the early 1930s and were described at length in the book by Andronov and Khaikin (1937). It was also sometimes noted that the so-called 'Hopf bifurcation' first appeared in fact in the works of Poincaré; therefore, Marsden and McCracken (1976) wrote in the preface to their book that apparently the term 'Poincaré-Andronov-Hopf bifurcation' would be the most just. However, the short term 'Hopf bifurcation' is now universally accepted; so it will be used in this book too.
Note in conclusion that the classical book by Andronov and Khaikin was in fact written by three authors. Only in the late 1950's it was permitted to S. E. Khaikin, the only one author who was then

one when the latter becomes unstable. Hopf bifurcations form the most elementary class of bifurcations, which are encountered very often in various applied fields (see, e.g., the books by Marsden and McCracken (1976); Hassard (1981), and Moiola and Chen (1996) specially devoted to such bifurcations); some more complicated bifurcations will also be discussed later in this book.

Above, for the sake of simplicity, we discussed only bifurcations of solutions of Landau's amplitude equations (which are ordinary and not partial differential equations). In fact only ordinary differential equations were considered in the early works on bifurcations by Poincaré, Andronov, Vitt, and Hopf. The general theory of periodic flow bifurcations from a steady solution of Navier–Stokes equations was developed independently by Yudovich (1971, 1972); Iooss (1972) and Joseph and Sattinger (1972) (see also Chaps. 9 and 9A in the book by Marsden and McCracken (1976), and references to early examples of such fluid-dynamic bifurcations in the book by Drazin and Reid (1981), p. 407). The papers mentioned contain, in particular, definite conditions under which such bifurcation necessarily occur. Then Andreichikov and Yudovich (1972) and Chen and Joseph (1973) showed that the results of the above-mentioned papers lead to definite assertions about the uniqueness, stability and properties of the two-dimensional periodic solutions which bifurcate from the steady Poiseuille flow at points of the corresponding neutral-stability curve. These assertions proved to be in good qualitative (and in satisfactory quantitative) agreement with the conclusions about disturbance development obtained earlier by other authors who used quite different, and often less rigorous, arguments based on the Stuart–Watson theory and its modifications.

Let us briefly discuss now results of some further work concerning the nonlinear evolution of normal-mode wave disturbances in plane Poiseuille flow. Recall that approximate estimates of the numerical values for the Landau constant for two-dimensional wave disturbances spatially evolving in a plane Poiseuille flow were first given by Itoh (1974a). Early comparisons of the available theoretical estimates with the experimental data by Nishioka et al. (1975), referring to development of waves generated by a vibrating ribbon in a laboratory channel flow, seemed to support both the results by Itoh (1974a) and the conclusions of Herbert (1977). However, subsequent more careful analysis detected some appreciable discrepancies between theory and experimental data, apparently connected with three-dimensional effects affecting measurements by Nishioka et al. and with some inaccuracies of Itoh's calculations; see, e.g., Zhou (1982); Herbert (1980, 1983a), and Sen and Venkateswarlu (1983). Another method for calculation of Landau's constant was proposed by Itoh (1977a); as was indicated by Davey (1978) and Herbert (1983b), this method differs from that of Reynolds and Potter (1967) only by rearrangement of the terms in some

alive, to publish the revised edition of the book as a book by Andronov et al. (1959) with a strange remark in the Preface (which was repeated in the English translation of 1966 too) that 'the name of one of the authors was by an unfortunate mistake not noticed on the title page of the first edition'. The 'unfortunate mistake' was due to the fact that A. A. Vitt, a young talented scientist, was arrested in 1937 by Stalin's notorious secret police (which chose its victims for reasons incomprehensible to any normal mind) and died in prison the next year.

infinite series, and is appropriate only for the case where $A = A_e$ is the equilibrium amplitude of a disturbance.

The accuracy of estimates of the value of Landau's constant is clearly affected by the absence of a unique, universally-accepted definition of the disturbance amplitude A and by possible influence of further terms of Eqs. (4.41) and (4.41a) which were neglected in most of the early papers. (The first attempts to estimate, for a plane Poiseuille flow, the values of two coefficients a_m, $m = 1$ and 2, of Eq. (4.41) (in other words, of the coefficients δ and β of Eq. (4.37)) and of the corresponding complex coefficients λ_m, $m = 1$ and 2, of Eq. (4.41b) presented below were due to Gertsenshtein and Shtemler (1997) and Shtemler (1978). These authors applied the modified method of Reynolds and Potter (1967) to compute the values of the coefficients a_1, a_2, and λ_1, λ_2 for several points (k, Re) of the plane-Poiseuille-flow neutral curve and then, assuming that $A = A_e$, studied the influence of the terms with $m = 2$ on the values of the equilibrium amplitude A_e and the shape and stability of the equilibrium waves). Later it was stressed by Herbert (1980, 1983b) that many theories leading to determination of the higher-order terms do not exclude equally-justified alternative methods of computation, leading to changes in the values of these terms. In the paper of 1980 Herbert developed a consistent method of perturbation expansion for solution of the Navier-Stokes equations which included a unique definition of the real amplitude $A(t)$ and led to Eq. (4.41) with unique values of the Landau constants a_m of all orders. Then he showed how the values of these constants can be determined, and he calculated, for plane Poiseuille flow, the values of the first seven constants a_m at the critical point (k_{cr}, Re_{cr}) of the (k, Re)-plane and at one subcritical point with $\mathrm{Re} < \mathrm{Re}_{cr}$. The results obtained showed that the coefficients a_m increase rapidly with m. Therefore Eq. (4.41) is in fact useful only in the case of a very small amplitude A. In Herbert's paper (1983b) a survey and also a comparison of various expansion methods based on different assumptions was presented, and the ranges of applicability and shortcomings of these methods were discussed. In particular he showed that the method of Watson (1960a) is exact only at points of the neutral curve where $a_0 = 2\gamma = 0$, while if $\gamma \neq 0$, then Watson's value of δ differs from the value given by the more rigorous method of Herbert (1980) (see Fujimura (1987) for a more detailed analysis of this matter). Later Crouch and Herbert (1993) proposed a new general method for determination of the complex Landau constants λ_m of all orders $m \geq 0$ entering the equation for the complex disturbance amplitude A

$$\frac{dA}{dt} = A \sum_{m=0}^{\infty} \lambda_m |A|^{2m} \tag{4.41b}$$

which is a simple generalization of both the Stuart-Landau Eq. (4.40) and the Watson-Landau Eq. (4.41) (where $a_m = 2\Re e \lambda_m$). The same problem was also considered by Sen and Venkateswarlu (1983) and Fujimura (1989, 1991, 1997) whose papers will be discussed below.

Zhou (1982) developed an improved version of the classical Stuart-Watson method of 1960, assuming that both the amplitude $A(t)$ and the angular frequency $\omega_1(t)$ of the unstable wave disturbance vary with time. He expanded the derivatives *dA/dt* and $d\omega_1/dt$ in powers of a suitable small parameter ε (which represents the order of magnitude of nonlinear corrections), and numerically computed the solutions of the resulting system of coupled differential equations for the terms of ε-expansions up to fourth order. In this way Zhou obtained a much more detailed representation of the nonlinear development of subcritical (unstable) wave disturbances, for values of Re from 1,000 up to 5,500 and several values of k. It was found that the accuracy of the method decreases with increasing $\mathrm{Re} - \mathrm{Re}_{cr}$, but the experimental observations by Nishioka et al. (1986) concerning the terminal equilibrium states of disturbances at relatively small values of $\mathrm{Re}–\mathrm{Re}_{cr}$ are represented more satisfactorily by the new results than by the results presented in the preceding papers.

Weinstein (1981) applied Watson's (1960a) method to calculate values of the Landau constants $a_m = a_m$ (k, Re) up to $m = 3$ for the Poiseuille-flow wave disturbances corresponding to small values of both $|\mathrm{Re} - \mathrm{Re}_{cr}|$ and $|k - k_{cr}|$. His main purpose was to compare results following from his version of Watsons method with those given by quite another method, the so-called *method of multiple scales* first applied to some turbulent-flow calculations by Stewartson and Stuart (1971) (for other applications of the method see, e.g., Cole (1968); Kevorkian and Cole (1981); Nayfeh (1981), or Godrèche and Manneville (1998)). This method uses two different time scales (the 'slow' and 'rapid' ones) which allow the slow evolutionary processes to be isolated from the rapid high-frequency oscillations. (In Landau's original derivation of Eq. (4.34) averaging over a time period intermediate between 'slow' and 'rapid' time scales was used to the same end). Weinstein found that in the cases he considered both methods lead to exactly the same results; however, no numerical data were presented in this paper.

New calculations of the higher-order Landau coefficients for nonlinear wave disturbances in a plane Poiseuille flow, corresponding to both subcritical and supercritical regions of the (k, Re)-plane, were carried out by Sen and Venkateswarlu (1983) by both the Reynolds and Potter (1967) and the Watson (1960a) methods. It was found that in the supercritical region the results of both methods are relatively close (in the subcritical region the majority of the computations performed was based on the use of the R-P method). The authors supplemented the results of Pekeris and Shkoller shown in Fig. 4.8, by new lines separating the regions of positive and negative values for Landau's constants a_2 and a_3 in the (k, Re)-plane (see Fig. 4.13, based on their results). They also investigated the region of convergence of the Landau–Watson series (4.41) (it was found that the radius of convergence is rather short here, which agrees with the conclusions of Herbert (1980)) and indicated the summation methods appropriate for computations in the cases of slow convergence (or slow divergence) of this series. The equilibrium amplitudes and equilibrium velocity distributions were also determined for the subcritical region, and the values of a great number of complex Landau coefficients $\lambda_m = \lambda_m(k, \mathrm{Re})$ were presented for some particular cases. Some comparisons of the results obtained with experimental data by Nishioka et al. (1975) were discussed in the paper and were found to be encouraging. Note however that Reynolds and Potter's method and the original Watson methods, considered by Sen and Venkateswarlu, are not of high precision, and many

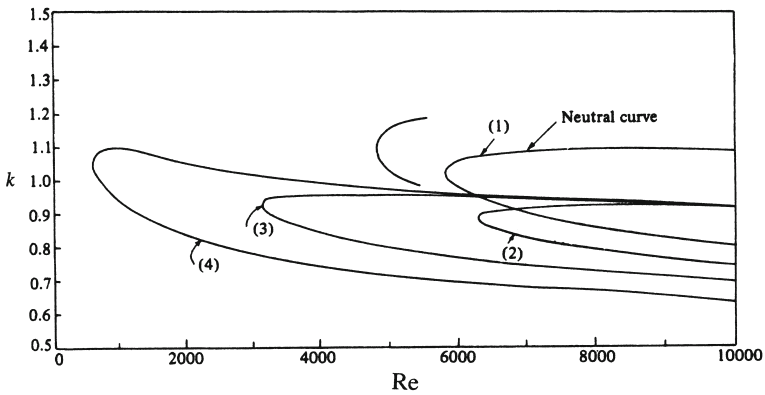


Fig. 4.13 Stability diagram showing the regions of positive and negative values of coefficients γ, δ, and the next two Landau's constants a_2 and a_3 in the (k, Re)-plane for the case of plane Poiseuille flow. (After Sen and Venkateswarlu (1983)) (*1*) *curve* where $\gamma = 0$ (neutral *curve* of the linear stability theory, inside it $\gamma > 0$); (*2*) *curve* where $\delta = 0$ (inside it $\delta > 0$); (*3*) *curve* where $a_2 = 0$ (inside it $a_2 > 0$); (*4*) *curve* where $a_3 = 0$ (inside it $a_3 < 0$)

researchers even supposed that they are inapplicable at points (k, Re) which are far from the neutral curve.

Fujimura (1989) compared two different methods of derivation of the general Landau Eq. (4.41b) for the complex disturbance amplitude $A(t)$ from the Navier-Stokes equations—his own modification of the amplitude-expansion method of Watson and the above-mentioned method of multiple scales (which can be applied to derivation of Eq. (4.41b) if a whole hierarchy of longer and longer time scales is introduced). He began by stressing that the results obtained by both methods depend essentially on the strict definition of the amplitude $A(t)$. Then he showed that if this definition is based on a special normalization condition for the fundamental mode, then in the case of slight supercriticality the method of multiple scales gives results equivalent to those which follow from the modified amplitude-expansion procedure (but not from its original form proposed by Watson). Some results of computations by both methods of the values of the first four complex Landau constants for slightly supercritical wave disturbances in a plane Poiseuille flow are also presented in this paper.

Later Fujimura (1991, 1997) studied one more method of derivation of the complex Landau Eq. (4.41b) from the equations of fluid motion. Note that this equation represents a crucial reduction of the infinite-dimensional dynamical system of flow disturbances evolving in time, to a one-dimensional system fully determined by its amplitude $A(t)$. On the other hand, the modern development of the dynamical system theory led to the appearance of a promising new method of the dimension reduction (i.e., reduction of the numbers of degrees of freedom), called the *method of center manifold* (see, e.g., the books by Carr (1981); Wiggins (1990); Manneville (1990), and Guckenheimer and Holmes (1993)). The method is based on the concept of a *center manifold*—a part S of the phase space R of all possible states of the consid-

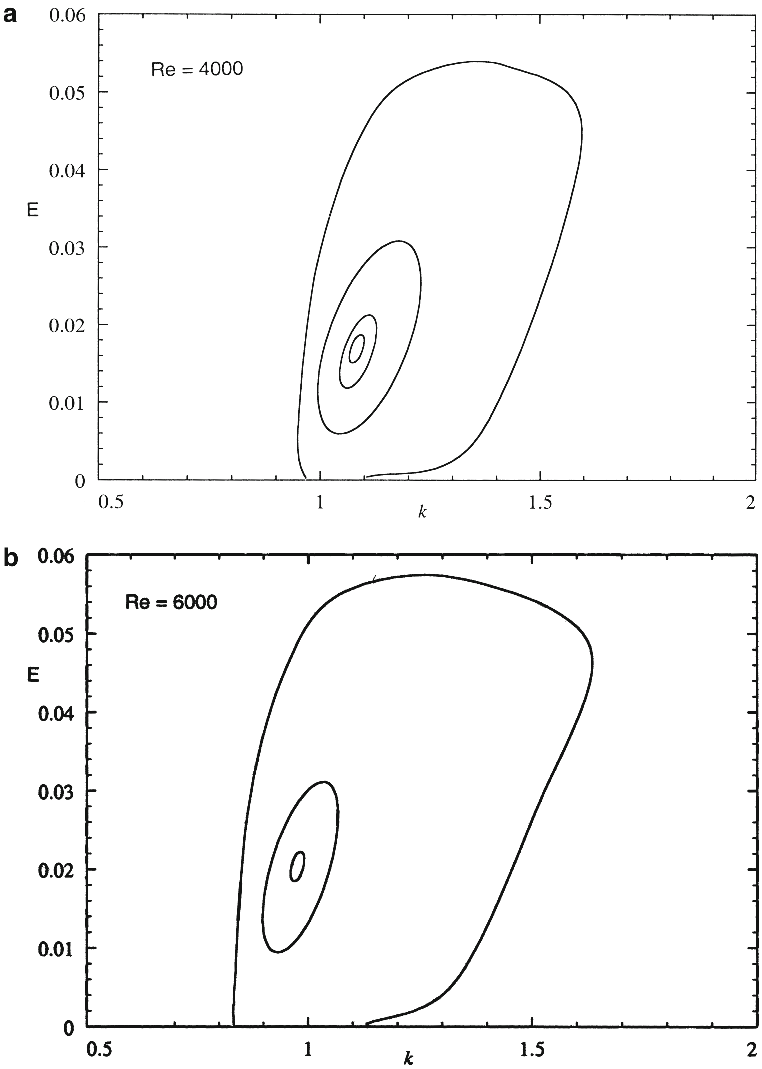


Fig. 4.14 Intersections of the nonlinear neutral surface in the three-dimensional (E, k, Re)-space with the plane Re = 4,000 for Couette-Poiseuille flows with $\hat{A} = 0$, 0.12, 0.144, and 0.147 (**a**), and with the plane $\hat{A} = 6{,}000$ for C-P flows with $\hat{A} = 0$, 0.2, and 0.218 (**b**). (After Balakumar (1997)). Here $\hat{A} = \hat{A}_B = U_W/[\frac{4}{3}U_{max} + U_w]$, $Re = U_{ave}H_1/\nu = [\frac{4}{3}U_{max} + U_w]H_1/2\nu$, and the increase of $\Re$ corresponds to s hrinking of the closed *curves* in the figure

ered system having some special properties. These properties imply, in particular, that any phase trajectory (the curve in R describing time evolution of the system), whose point at time t_0 belongs to S will remain in S also at any $t > t_0$, while a trajectory which is outside of S at time t_0 enters S, under rather general conditions, at some subsequent time moment. (For more information about specialized subspaces of a phase space and their properties see, e.g., Kelley (1967) In the case of a fluid flow the phase space R consists of all admissible values of the main fluid dynamical fields; see Sect. 2.3). The *center manifold reduction* consists in projection of the full phase space R, together with trajectories of a dynamical system lying in R, into some center manifold S of smaller dimension than that of the space R. In the extreme case when the dimension is reduced to one, the state of the nonlinear system becomes fully determined by the amplitude A, and hence the evolution of the system is described by the function $A(t)$ which under a wide range of conditions satisfies an equation of the Landau type. Note however that the center manifold (and thus also the center manifold reduction) may not be unique (see, e.g., Guckenheimer and Holmes (1993)). This nonuniqueness is analogous to the nonuniqueness of Landau's constants because of their dependence on the selected definition of the disturbance amplitude.

Some examples of derivations of Landau's amplitude equations for nonlinear systems with infinite dimensions by the method of center manifold may be found in Carr (1981) and Carr and Muncaster (1983); a number of applications of this method to fluid mechanical problems were considered by Guckenheimer and Knobloch (1983); Iooss (1987); Laure and Demay (1988); Renardy (1989); Manneville (1990); Cheng and Chang (1990, 1992, 1995); Chen et al. (1991), and Chossat and Iooss (1994), among many others. Fujimura at first considered (in the paper of 1991) the most common scheme of the center manifold reduction, applicable to infinite-dimensional systems arising from the partial differential equations (exemplified by the Navier–Stokes system). He applied the method to the classical example of the disturbance development in plane Poiseuille flow, which was also investigated by the center manifold method, in passing, by Renardy (1989) (whose paper was mainly devoted to more general problems). Renardy evaluated the Landau constant δ for a plane Poiseuille flow by this method, and compared her results with those found by Pekeris and Shkoller (1967) and Reynolds and Potter (1967). However, her comparisons had a serious deficiency, indicated by Fujimura (1991) who also showed that her value of δ was identical with that implied by the original Watson's method and hence, according to Fujimara's (1989) conclusion, was different from the value given by the method of multiple scales. Moreover, he also noted that, when applied to derivation of higher-order Landau equations, Renardy's reduction scheme leads to values of the higher Landau constants differing even from those given by Watson's original method. Therefore, Fujimura (1991) carried out a new careful evaluation of the complex Landau constants λ_{m}, with $m = 0$, 1, 2 and 3, for a plane Poiseuille flow by the methods of center manifold and of multiple scales, compared the results obtained by these two methods, and explained how the disturbance amplitude must be defined to make the results of two methods equivalent to each other.

In the paper of 1997 Fujimura applied, to the derivation of Landau's Eq. (4.41b), another center manifold reduction scheme (called by him "the reduction scheme of the second category"), which starts with an infinite, or finite, system of ordinary differential equations (in the cases where original equations are partial-differential, this system can be derived by means of a Galerkin projection or/and a normal-mode expansion). Such reduction scheme was used, in particular, in the above-mentioned papers by Guckenheimer and Knobloch, Cheng and Chang, and Chen et al. The main objecjtive of Fujimura (1997) was to prove the equivalence of Landau's equations, as given by this reduction scheme, to those derived by the method of multiple scales. To reduce the Navier–Stokes equations to a system of ordimary differential equations, a double expansion of flow fields in Fourier series and in eigenfunctions of the linear stability theory was used. Then the first and second Landau constants λ_2 and λ_3 were evaluated by the second-category method of center manifold, for plane Poiseuille flow and for two other simple fluid dynamical problems. Comparison of the values obtained with those given by the method of multiple scales showed that in all three cases the values of λ_2 and λ_3, computed by this version of the method of center manifold, approach their values given by the method of multiple scales as the truncation level of the eigenfunction expansion increases. Hence the three papers by Fujimura (1989, 1991, 1997), taken together, show that Landau's Eq. (4.41b) given by two versions of the center manifold reduction scheme, the method of multiple scales, and the modified Watson amplitude-expansion method are equivalent to each other if the disturbance amplitude is defined in a consistent way.

Stewartson and Stuart (1971) considered the propagation, in plane Poiseuille flow, of a group of two-dimensional waves undergoing both spatial and temporal development. In this case the disturbance amplitude A depends on both the time and the streamwise coordinate, i.e., $A = A\ (t, x)$. Therefore for small positive values of $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$, weakly nonlinear theory now leads to a nonlinear parabolic partial differential equation for $A\ (t, x)$, differing from the complex Landau Eq. (4.40) by an additional term proportional to $\partial^2 A/\partial\xi^2$, where $\xi = x - c_g t$, c_g being the streamwise group velocity. (This equation is now usually called *the Ginzburg–Landau equation* since it appeared in a quite different connection in the paper by Ginzburg and Landau (1950) on the theory of superconductivity. We will meet some other equations of the Ginzburg–Landau type in Sect. 4.24, parts (b) and (d)). To derive this equation, Stewartson and Stuart used the above-mentioned *multiple scale analysis*. Results similar to those by Stewartson and Stuart were found independently by DiPrima et al. (1971) while Hocking and Stewartson (1972) studied some exact solutions of the Ginzburg–Landau equations. Weinstein (1981) extended Stewartson and Stuart's theory, supplementing their amplitude equation by two more terms of higher order in A (and in addition showed that this equation may also be obtained by Watson's (1960a) method). A theory of Stewartson and Stuart's type, referring to groups of three-dimensional waves in a plane Poiseuille flow, was developed by Davey et al. (1974) but in this case it leads to a more complicated pair of coupled partial differential equations for the disturbance amplitude and for some characteristic of the pressure-gradient.

4.2.3 *Amplitude Equations and Equilibrium Disturbances in Other Parallel and Nearly Parallel Wall Flows*

4.2.3.1 Plane Couette-Poiseuille Flows

Passing to other parallel and nearly parallel fluid flows we will begin with the case of strictly-parallel *plane Couette-Poiseuille flows.* The combined *Couette-Poiseuille* (briefly C-P) *flows* are simpler in some respects than pure Couette flows, since unstable infinitesimal wave disturbances and a finite neutral-stability curve exist sin such combined flows, if only in cases where the relative strength of the Couette component is not too high, but they never exist in pure plane-Couette flows (see Sect. 2.91). Therefore methods developed by Stuart, Watson, Eckhaus, Reyunolds and Potter, and Pekeris and Shkoller, which are applicable only at (k, Re)-points close to the neutral curve, can be applied at least to some C-P flows, but are always inapplicable in the case of a Couette flow. Reynolds and Potter (1967), who were the first to investigate weakly nonlinear stability of C-P flows, considered only those relative strengths of the Couette component for which unstable infinitesimal disturbances exist. In these cases, the neutral curve in the (k, Re)-plane can be determined, and on this curve the critical point (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$) can be found. Reynolds and Potter carried out nonlinear stability analysis only for neutrally-stable wave disturbances with $\gamma = 0$, corresponding to critical points at various values of the relative strength of the Couette component. In this analysis they used the same method they applied to disturbances in plane Poiseuille flow. According to the results obtained, δ is negative at the critical point (and hence finite-amplitude instabilities exist at subcritical values of (k, Re) close to the critical point (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$)) in all C–P flows where there are unstable infinitesimal disturbances (and hence $\mathrm{Re}_{\mathrm{cr}}$ is finite though it can be arbitrarily large). This result makes it probable that some finite-amplitude disturbances are unstable in C-P flows, even in cases where all infinitesimal disturbances are stable (and hence $\mathrm{Re}_{\mathrm{cr}} = \infty$).

Shtemler (1978) supplemented Reynolds and Potter's computations by the estimation of the next-order coefficients a_2 and λ_2 of the generalized real and complex Landau Eqs. (4.41) and (4.41b) at the leftmost points (k_{cr}, $\mathrm{Re}_{\mathrm{cr}}$) of the neutral curves of C–P flows for a number of values of the relative strength of the Couette components corresponding to flows having finite values of $\mathrm{Re}_{\mathrm{cr}}$. A more detailed investigation of both linear and weakly-nonlinear stability of C–P flows was carried out by Cowley and Smith (1985) and Balakumar (1997). Studying the linear stability Cowley and Smith discovered that in a C-P flow the stability diagram of the linear theory can have a more complex form than was supposed by Potter (1966), Hains (1967), and Reynolds and Potter (1967). In these early papers it was assumed that if the streamwise wave number k is given, then at any values of Re and of the relative strength $\hat{A}$ of the Couette component either there exists one unstable two-dimensional normal mode or there are no such modes at all. Therefore, the above-mentioned authors thought that if the neutral curve in the (k, Re)-plane, which corresponds to the set of all neutrally-stable waves, exists in a C-P flow (and for this the inequality $\hat{A} < \hat{A}_{\mathrm{cr}}$ must be valid where $\hat{A}_{\mathrm{cr}}$ is some critical value of the relative strength $\hat{A}$), then this

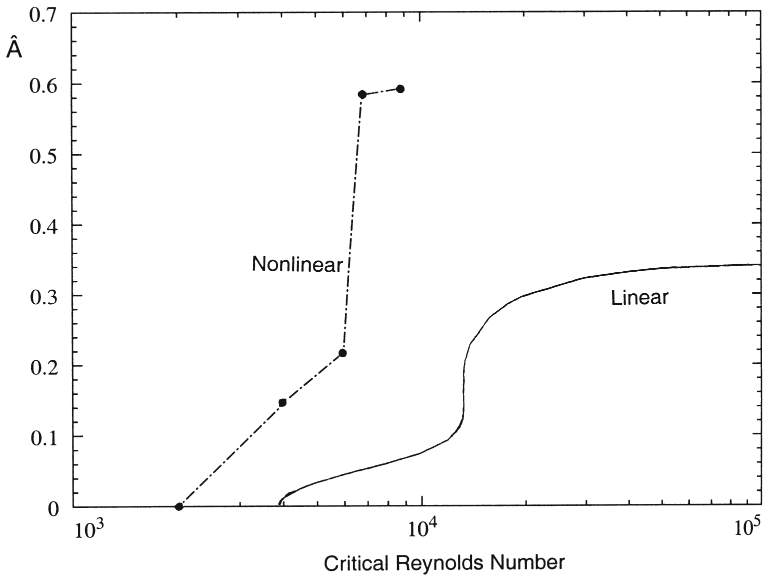


Fig. 4.15 Comparison of the nonlinear critical Reynolds numbers $\mathrm{Re}^*_{\mathrm{cr}}$ (Â), where $\hat{A} = \hat{A}_B$, for C-P flows with various values of Â with the ordinary (*linear*) critical Reynolds numbers $\mathrm{Re}_{\mathrm{cr}}$ (Â) for the same C–P flows. (After Balakumar (1997))

curve will have qualitatively the same form as the neutral-stability curve of a plane Poiseuille flow. However Cowley and Smith found that in a C–P flow with a relative strength of the Couette component that is sufficiently small (appreciably smaller than $\hat{A}_{\mathrm{cr}}$), but non-vanishing, several neutral-stability curves (two or even three, if $\hat{A}$ takes very small values), corresponding to several unstable two-dimensional normal modes can exists simultaneously. This means that in addition to the critical value $\hat{A}_{\mathrm{cr}}$ there also exist in C–P flows the critical values $\hat{A}_{2,\mathrm{cr}} < \hat{A}_{\mathrm{cr}}$ and $\hat{A}_{3,\mathrm{cr}} < \hat{A}_{2,\mathrm{cr}}$ corresponding to the appearance of additional unstable modes (growing more slowly than the most unstable mode appearing at $\hat{A} = \hat{A}_{\mathrm{cr}}$); these new critical numbers clearly signify qualitative changes in the shape of the stability diagram. Balakumar (1997), in his study of the linear stability of C–P flows, considered only the most unstable modes; for them he computed, very accurately, first the value of $\hat{A}_{\mathrm{cr}}$ then the neutral curves in the (k, Re) and (c, Re) planes (where $c = \omega/k$ is the phase velocity of a neutral wave) at a number of values of $\hat{A}$ in the range $0 \leq \hat{A} < \hat{A}_{\mathrm{cr}}$, and finally the shape of the functions $\mathrm{Re}_{\mathrm{cr}}(\hat{A})$ and $k_{\mathrm{cr}}(\hat{A})$ (the first of these functions is shown in Fig. 4.15).

Note also that the relative strength $\hat{A}$ of the Couette component was defined differently by different authors, who also often used different forms of the C-P velocity profile $U(z)$ and different length and velocity scales L_0 and U_0. So, Potter, Hains, and Balakumar defined $U(z)$ as the sum of a parabolic profile $U_{\mathrm{p}}(z)$ of a Poiseuille flow with maximal velocity $U_{\mathrm{P}}(H/2) = U_{\max}$ and a linear Couette's profile $U_C(z)$ growing from the value $U_{\mathrm{C}}(0) = 0$ up to the value $U_{\mathrm{C}}(H) = U_w$, while both Reynolds and

Potter, and Cowley and Smith assumed that $U_C(0) = -U_W/2$ and $U_C(H) = U_W/2$. Potter (1966) and Hains (1967) used the channel thickness H as the scale L_0, while in the papers by Reynolds and Potter (1967), Cowley and Smith (1985), and Balakumar (1997) L_0 was taken as $H_1 \equiv H/2$. Moreover, Hains assumed that $U_0 = U(H/2) = U_P(H/2) + U_C(H/2)$, Potter that $U_0 = U_P(H/2) = U_{\max}$, and Reynolds and Potter, Cowley and Smith, and Balakumar that $U_0 = U_{ave}$ (where U_{ave}, the averaged C-P velocity $U(z)$, clearly depends on the selected Couette-component profile $U_{\text{C}}(z)$). Thus, Hains (1967) measured the relative strength of the Couette component by the value of $\hat{A}_{\text{H}} = U_{\text{W}}/U(H/2) = U_{\text{W}}/[U_{\max} + \frac{1}{2}U_{\text{W}}]$ (this measure was used also in Sect. 2.91 where it was denoted as A); while Potter (1966) assumed that $\hat{A} = \hat{A}_{\text{P}} = U_{\text{W}}/U_{\max}$; and Reynolds and Potter (1967); Cowley and Smith (1985), and Balakumar (1997) defined $\hat{\text{A}}$ as $U_{\text{W}}/2U_{\text{ave}}$. Complying with this definition and with the accepted form of the profile $U_{\text{C}}(z)$, Reynolds and Potter, and Cowley and Smith used the measure $\hat{A} = \hat{A}_{\text{RP}} = \hat{A}_{\text{CS}} = 3U_{\text{W}}/4U_{\max}$, and Balakumar the measure $\hat{\text{A}} = \hat{\text{A}}_{\text{B}} = U_{\text{W}}/[\frac{4}{3}U_{\max} + U_{\text{W}}]$. It is easy to see that the measures $\hat{\text{A}}_{\text{H}}, \hat{\text{A}}_{\text{P}}, \hat{\text{A}}_{\text{RP}} = \hat{\text{A}}_{\text{CS}}$ and $\hat{\text{A}}_{\text{B}}$ of the relative Couette-component strength are in fact simple one-valued functions of each other so that the value of any of them determines the values of all the others. Moreover, the seemingly different critical values found by the above-mentioned authors, namely $\hat{A}_{\text{cr}} \approx 0.55$ (Hains; see also Sect. 2.91 of this book), $\hat{\text{A}}_{\text{cr}} \approx 0.7$ (Potter), $\hat{\text{A}}_{\text{cr}} \approx 0.528$ (Reynolds and Potter, and Cowley and Smith), and $\hat{A}_{\text{cr}} \approx 0.3455$ (Balakumar) only indicate that $\hat{\text{A}}_{\text{H,cr}} \approx 0.55, \hat{\text{A}}_{\text{P,cr}} \approx 0.7, \hat{\text{A}}_{\text{RP,cr}} = \hat{\text{A}}_{\text{CS,cr}} \approx 0.528$, and $\hat{\text{A}}_{\text{B,cr}} \approx 0.3455$; one may verify easily that these values agree rather satisfactorily with each other (only Hains' estimate is overstated by about 7 %).

As to the weakly nonlinear stability of the C-P flows, Cowley and Smith showed, in particular, that at all values of $\hat{\text{A}}$ which are close enough, above or below, to the critical value $\hat{\text{A}}_{\text{cr}}$, $\delta(k, \text{Re})$ is negative for the least-stable two-dimensional wave disturbances corresponding to some parts of the stable region of the (k, Re)-plane. Therefore, equilibrium wave disturbances of small but finite amplitudes can exist in a C-P flow with any such value of $\hat{\text{A}}$. (These results by Cowley and Smith also agree with conclusions by Milinazzo and Saffman (1985) who independently found that a family of two-dimensional equilibrium waves of finite amplitude exists in the C–P flows). In the case of a subcritical C-P flow, where $\hat{\text{A}} < \hat{\text{A}}_{\text{cr}}$, unstable disturbances correspond to periodic solutions of the N–S equations bifurcating from the steady C–P solutions at points of the neutral curve. However in the case of a supercritical flow with $\hat{\text{A}} > \hat{\text{A}}_{\text{cr}}$ the neutral curve does not exist at all. Therefore it is clear that the usual form of bifurcation theory, which requires the existence of a point of loss of stability at which the bifurcation begins, cannot be applied here. (A similar conclusion was also reached simultaneously by Milinazzo and Saffman). In this connection Cowley and Smith recalled rather exotic *bifurcations from infinity* of solutions of nonlinear equations which were considered by Rosenblat and Davis (1979) in their search of a possible origin of finite-amplitude equilibrium flow disturbances, observed in flows where stable infinitesimal disturbances do not exist. This recollection proved to be quite appropriate: Cowley and Smith (1985) succeeded in showing that just such a 'bifurcation from infinity' occurs in the supercritical C–P flows with $\hat{A} > \hat{A}_{cr}$ (where the 'critical value' may be considered as the infinite one, since one may assume that $(\hat{A} - \hat{A}_{cr})^{-1}$ and not $\hat{A}$ is the true stability parameter). Their results stimulated subsequent studies by Cherhabili and Ehrenstein (1995, 1997) and Nagata (1997) of some other types of bifurcations from infinity relating to finite-amplitude equilibrium states in C–P flows. Results of the last-named authors and also of the paper by Rosenblat and Davis (1979), where 'bifurcations from infinity' first appeared, will be discussed later in this section.

Balakumar (1997) did not use bifurcation theory at all in his studies of the nonlinear stability of C-P flows. He concentrated his attention on computations of finite-amplitude equilibrium two-dimensional waves at different values of $\hat{A} = U_W/2U_{ave} = U_W/[\frac{4}{3}U_{max} + U_W]$. His computations were based on application to C–P flows of the method outlined in the early papers by Noether (1921) and Heisenberg (1924), and then used by Zahn et al. (1974) and Herbert (1976, 1977, 1978) in their investigations of nonlinear stability of plane Poiseuille flows (for more details see Sect. 4.22 above). The main objective of Balakumar was to determine the evolution with $\hat{A}$ of the 'nonlinear neutral surface' in the three-dimensional (E, k, Re)-space consisting of points corresponding to two-dimensional equilibrium waves (here E and k have the same meaning as in Fig. 4.12 in Sect. 4.22, and $\mathrm{Re} = U_{ave}H_1/\nu$). Some of his results are presented in Figs. 4.14 and 4.15. Figure 4.14 shows the intersections of the neutral surfaces in (E, k, Re)-spaces corresponding to C–P flows with several values of $\hat{A}$ with the planes Re = 4,000 and Re = 6,000. (For $\hat{A} = 0$ these intersections clearly coincide with those shown in Fig. 4.12, but the values of Re in Fig. 4.14 are equal to 2/3 of the values $\mathrm{Re} = U_{max}H_1/\nu$ used in Fig. 4.12). All the intersections shown (whose boundaries represent the nonlinear 'neutral curves in the (E, k)-plane') have similar shapes but they gradually shrink in size with increasing $\hat{A}$ and, as Balakumar's extensive computations showed, completely disappear at $\hat{A} \approx 0.1472$ when Re = 4,000 and at $\hat{A} \approx 0.2182$ when Re = 6,000. However similar computations for Re = 7,000 showed that at this high Reynolds number the 'neutral curves in the (E, k)-plane' have the shape similar to that in Figs. 4.12 and 4.14 only for $\hat{A} < 0.2$, and when $\hat{A}$ increases further their shapes change very rapidly and begin to include a second loop at low values of E and k.

Figure 4.15 shows the dependence on $\hat{A} = U_W/2U_{ave}$ of the 'nonlinear critical Reynolds number' $\mathrm{Re}^*_{cr} = \mathrm{Re}^*_{cr}(\hat{A})$ the lowest Reynolds number at which unstable two-dimensional waves of finite amplitude exist in the C–P flow with relative strength $\hat{A}$ of the Couette component. For comparison the same figure includes also the computed values of a function $\mathrm{Re}_{cr} = \mathrm{Re}_{cr}(\hat{A})$ where Re_{cr} is the ordinary (linear) Reynolds number indicating the lowest value of Re at which there exist infinitesimal unstable two-dimensional waves (the computations of Re_{cr} are simpler than those of Re^*_{cr} and allow more precise results to be obtained). We see that $\mathrm{Re}^*_{cr}(\hat{A})$ is always much smaller that $\mathrm{Re}_{cr}(\hat{A})$, as it must be. The value of the linear critical Reynolds number $\mathrm{Re}_{cr}(\hat{A})$ increases significantly as $\hat{A}$ grows from zero (where $\mathrm{Re}_{cr} \approx 2 \times 5772/3 = 3848$) up to a value of about 0.1, then it remains almost constant until $\hat{A} \approx 0.3$, and later increases sharply to infinity as $\hat{A}$ approaches the value of $\hat{A}_{cr} \approx 0.3455$. The value of the nonlinear critical Reynolds number $Re^*_{cr}(\hat{A})$ increases as $\hat{A}$ increases from zero (where $Re^*_{cr} \approx 2 \times 2935/3 \approx 1957$) up to value of about 0.2, then remains approximately constant until $\hat{A} \approx 0.58$ and after this again begins to increase with the relative strength $\hat{A}$ of the Couette component. As to the higher values of $\hat{A}$, Balakumar found no steady two-dimensional waves in any C-P flow with $\hat{A} \geq 0.59$. Then he remembered that earlier several authors (in particular, Orszag and Kells (1980) and Milinazzo and Saffman (1985)) were unsuccessful in their attempts to simulate two-dimensional finite-amplitude equilibrium waves in a plane Couette flow, where $\hat{A} = \infty$, and they concluded that apparently such waves cannot exist in this flow. Therefore he assumed that there exists the nonlinear critical relative strength $\hat{A}$ (close to 0.59) above which equilibrium two-dimensional wave cannot exist in a C-P flow (and hence $Re^*_{cr}(\hat{A}) = \infty$). However, the real situation is not so simple, since Cherhabili and Ehrenstein (1995, 1997) (whose work was apparently unknown to Balakumar) found that even in a pure plane Couette flow (i.e. at $\hat{A} = \infty$) two-dimensional finite-amplitude equilibrium states exist if Re exceeds the critical value close to 1500, but these states are not of the form of traveling nonlinear two-dimensional waves, as considered by Balakumar, but are stationary, spatially localized (solitary-like) waves (more details of this will be presented below). Therefore the question of the possible two-dimensional equilibrium states in C-P flows with relatively high strength of the Couette component requires further investigation.

4.2.3.2 Plane Couette and Circular Poiseuille Flows

Now we will turn to the cases of plane Couette and circular Poiseuille flows. It is known that these flows are stable at any Re with respect to infinitesimal disturbances, i.e. are similar in this respect to C-P flows with $\hat{A} > \hat{A}_{cr}$. Rosenblat and Davis (1979) noted that in plane Couette and circular Poiseuille flows there exist sets of infinitesimal disturbances whose decay rates tend to zero as $Re \to \infty$. Therefore, they suggested that perhaps the value $Re = \infty$ may be regarded here as a bifurcation point in the following sense: a branch of finite-amplitude solutions of the complete nonlinear disturbance equations which, according to experimental data definitely exists in these cases, may have the property that for $Re \to \infty$ these solutions tend to coalesce with the primary ('basic') laminar solution of dynamic equations. Rosenblat and Davis proposed to say in such cases that the corresponding finite-amplitude solutions *bifurcate from infinity.* Then they showed that at least for some model nonlinear differential equations containing a real parameter μ, having the property that bifurcation of a steady solution cannot occur at any finite value of μ, such 'bifurcation from infinity' (i.e., at $\mu = \infty$) can really occur.

Let us begin with the case of *plane Couette flow* (briefly PCF). Since all infinitesimal wave disturbances are stable here (i.e., decay as $t \to \infty$), those methods for rigorous derivation of Landau's equations and evaluation of their coefficients which use the assumption that the disturbance studied corresponds to a point of the (k, Re)-plane lying near the neutral-stability curve cannot be applied here. On the other hand, studies of the nonlinear stability of PCF are of particular interest, since only nonlinear theory can explain the striking contradiction between the prediction of the linear stability theory about the stability of PCF and the experimental data which definitely show that, when Re is increasing gradually, PCF becomes unstable at Reynolds numbers Re (based upon half the channel depth H_1 and half-difference of the wall velocities U_1) in the range from 320 to 370, while with further increase of Re it rapidly becomes turbulent (see Sect. 2.1, pp. 16–17, and the discussion of stability of Couette flow heated from below in Sect. 4.12). These arguments stimulated a number of attempts to develop such nonlinear stability theory which can be applied to PCFs.

Relatively crude attempts of this type undertaken by Kuwabara (1967) and Lessen and Chiefetz (1975) will be briefly described later. However, we will first consider the papers by Ellingsen et al. (1970) and Coffee (1977), and, related to their results, the remarks about stability of PCF by Davey and Nguyen (1971); Itoh (1977a, b) and Davey (1978), who paid most attention to the nonlinear stability of circular Poiseuille flow in tubes. All these authors tried to apply, to computation of the development of two-dimensional normal-mode finite-amplitude disturbances in the PCF, some modifications of the Reynolds-Potter method which take into account that in the case considered $|\gamma|$ cannot be assumed to be small. Remember that in this method the determination of threshold ('equilibrium') amplitudes $A_e = A_e(k, \mathrm{Re})$ of the wave disturbances, having the property that $dA_e/dt = 0$, plays the main part.

Ellingsen et al. (1970) showed that in the case of a PCF with a high enough value of Re, a slightly modified Reynolds–Potter method yields Landau's Eq. (4.34) for the amplitude of the least damped two-dimensional wave with given wave number k and also yields an equation for the coefficient δ allowing its numerical computation. The computations showed that δ is negative over a large region of the (k, Re)-plane. Therefore, subcritical instability is possible here and hence the PCF is unstable with respect to finite-amplitude disturbances. Later Itoh (1977a) showed that the results of Ellingsen et al. may also be obtained by another more rigorous method without some of the simplifying assumptions of the latter authors.

Davey and Nguyen (1971) considered slightly different modifications of Reynolds and Potter's method. They applied this modification mainly to the study of nonlinear stability of a tube Poiseuille flow at high values of Re, but also presented some results of calculations for PCF, giving the dependence of the threshold disturbance energy E_e (corresponding to amplitude A_e) on k and Re. According to their results $E_e(\mathrm{Re})^{3/2}$ practically depends only on $k/(\mathrm{Re})^{1/2}$ in PCF with $\mathrm{Re} \geq 500$, and has a minimum value at $k/(\mathrm{Re})^{1/2} \approx 0.13$ (we recall that k is made dimensionless by multiplication by H_1). This means that in PCF two-dimensional waves with $k \approx 0.13(\mathrm{Re})^{1/2}$ are the most unstable. Later Davey (1978) found that the results of his paper with Nguyen relating to disturbances in PCF are very close to those which follow from the application to the same problem of another method of the same type proposed by Itoh (1977b). More detailed calculations of the values of $A_e(k, \mathrm{Re})$ and $E_e(k, \mathrm{Re})$ for numerous values of the arguments (k, Re), also based on a version of the Reynolds–Potter method, were made by Coffee (1977), whose results agree satisfactorily with earlier estimates by Ellingsen et al. and Davey and Nguyen. Since $\gamma < 0$ for all

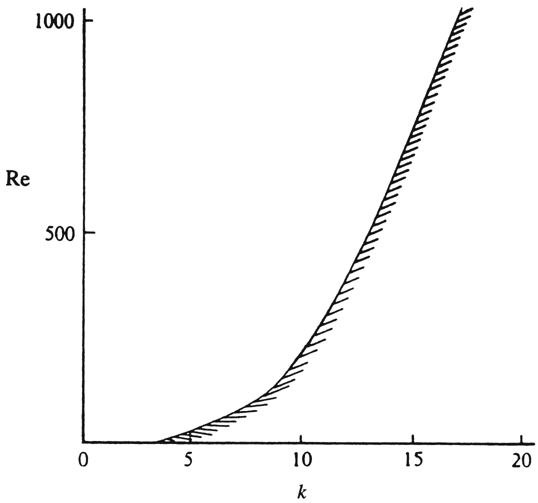


Fig. 4.16 Approximate location of the curve in the (k, Re)-plane separating the regions of positive and negative values of δ for plane Couette flow. (After Coffee (1977)) according to Coffee's calculations, $\delta < 0$ to the left of the given curve but $\delta > 0$ to the right of it

wave disturbances in a PCF, finite values of E_e and A_e show that $\delta < 0$, while for $\delta > 0$ the approximate theory based on Landau's equation leads to the conclusion that $E_e = \infty$, so that the flow is stable to disturbances of any size. Coffee's calculation implies that the region of the (k, Re)-plane where $E_e = \infty$ (and hence $\delta > 0$) is given approximately by the inequality $\mathrm{Re} \leq 1.7k^2$; his graph of the curve dividing the region where $\delta < 0$ from that where $\delta > 0$ is shown in Fig. 4.16 (here this curve replaces the dotted curve in Fig. 4.8 showing the points where $\delta = 0$ in the case of plane Poiseuille flow). Negativity of both γ and δ at a point (k, Re) means that a two-dimensional equilibrium wave with the wave number k can exist at this value of Re. Recall, however, that Orszag and Kells (1980) and Milinazzo and Saffman (1985) were unsuccessful in their attempts to stimulate two-dimensional equilibrium waves in PCF at any Re while Cherhabili and Ehrenstein (1995) found that some, quite specific, waves nevertheless exist in PCF, but only for Re close to 1,500 or even higher (this circumstance was mentioned in part (a) of this section and will be considered at greater length slightly later). Moreover, in Sect. 4.11 it was indicated that it follows from the Reynolds-Orr energy-balance Eq. (4.2) that disturbances of any shape and size must decay monotonically in a PCF if $\mathrm{Re} \leq 20.7$. These facts show that the early modifications of the Reynolds-Potter method discussed above, which had the object of making it applicable to linearly stable flows without a neutral curve, are apparently inaccurate and deserve no credit.

Let us say now a few words about the papers by Kuwabara (1967) and Lessen and Cheifetz (1975). Kuwabara's theory was based on crude assumptions, introduced by Meksyn and Stuart (1951), which do not require the smallness of the damping rate $|\gamma|$. Moreover, he also used some supplementary hypotheses which seemed dubious to some later authors (see, e.g., Lessen and Cheifetz (1975)). Kuwabara found that his assumptions imply the existence of some equilibrium two-dimensional finite-amplitude disturbances (and hence the positiveness of δ) in PCF if Re is high enough.

According to his calculations, $\mathrm{Re}^*_{cr} \approx 45,000$ in the case of PCF. Such a high value of Re^*_{cr} clearly disagrees with the experimental data and makes one suspect that the assumptions made are invalid.

A quite different 'quasilinear' theory, also strongly influenced by Meksyn and Stuart's arguments, was proposed by Lessen and Chiefetz (1975). They took into account only the distortion of the mean motion by a disturbance. This distortion affects the solutions of the Orr-Sommerfeld equation which determines the shapes of infinitesimal normal-mode disturbances. A rather crude finite-difference integration in time of the coupled equations for the distorted mean flow, and the least-stable disturbance corresponding to it, suggested a slow convergence of the disturbed Couette flow to some stable state.

Above, we mentioned some papers where the determination of the amplitudes for possible equilibrium two-dimensional finite waves in a plane Couette flow (PCF) played an important part. However, attempts to simulate such two-dimensional equilibrium waves numerically were unsuccessful for a long time. In this connection Orszag and Kells (1980); Patera and Orszag (1981a); Orszag and Patera (1981); Milinazzo and Saffman (1985), and Balakumar (1997) especially stressed that two-dimensional finite-amplitude equilibrium waves can be easily simulated in plane Poiseuille flow and combined Couette–Poiseuille (C–P) flows with not-too-high relative strength $\hat{\mathrm{A}}$ of the Couette component, but apparently such waves do not exist in plane Couette flow. (As was said above, Balakumar even tried to determine the upper bound of $\hat{\mathrm{A}}$-values at which such equilibrium waves exist in a C-P flow; see Fig. 4.15 and explanations relating to it in the text at the end of Sect. 4.2.3.1).

Recall now that Andreichikov and Yudovich (1972) and Chen and Joseph (1973) showed that finite-amplitude periodic waves in a plane Poiseuille flow bifurcate from the steady laminar solutions of the Navier-Stokes equations at the points of the neutral-stability curve, and Cowley and Smith (1985) found that bifurcations of the same type occur in C-P flows with $\hat{\mathrm{A}} < \hat{\mathrm{A}}_{cr}$. Since such a curve does not exist in a PCF, bifurcations of this type are impossible here, and this fact was sometimes used to explain the non-existence of finite-amplitude wave solutions of the equations of motion in the case of pure Couette primary flow. However, when discussing the problem of equilibrium waves in combined Couette-Poiseuille flows we mentioned that in the 'supercritical' cases, where $\hat{A} > \hat{A}_{cr}$ so that a neutral curve does not exist, such waves can be produced by a 'bifurcation from infinity'. Hence it is natural to think that such bifurcations can also lead to appearance of finite-amplitude equilibrium wave solutions in the case of primary plane Couette flow.

Apparently the first attempt to find some finite-amplitude solutions of the equations for disturbances in PCF which correspond to a 'bifurcation from infinity' was due to Nagata (1990). He applied such a bifurcation to find three-dimensional finite-amplitude standing waves in PCF. In order to find some finite-amplitude disturbance in PCF corresponding to 'bifurcation from infinity', one must first of all determine a family of auxiliary flows which i) depend on some parameter Λ and tend to PCF as $\Lambda \to \Lambda_0$, and ii) have the property that a neutral-stability curve corresponds to an auxiliary flow with a certain value of the parameter Λ, and that at a point on the neutral curve some finite disturbance bifurcates from the solution of the N–S equations describing this auxiliary flow. Then it is often possible to extend the 'composite solution' thus obtained (which includes the auxiliary flow and the finite disturbance superimposed on it) varying the value of Λ. Assuming now that $\Lambda \to \Lambda_0$ one will obtain the required finite-amplitude disturbance in PCF.

Nagata considered the family of primary flows between infinite concentric co-rotating cylinders (i.e., having angular velocities, Ω_1 and Ω_2 of the same sign). Here the steady solution (describing 'circular Couette flow') and solutions corresponding to flows appearing after the most common first supercritical bifurcation ('Taylor vortex flows') are well known, and the three-dimensional steady solutions which bifurcate from the Taylor vortices as Re increases further have also been investigated (in particular, by Nagata (1986, 1988)). Assuming that the dimensionless 'Coriolis parameter' $\mathrm{Co} = (\Omega_1 + \Omega_2)(R_2 - R_1)^2/\nu$ tends to zero (i.e., $d = R_2 - R_1 \to 0$), Nagata (1990) found numerically a branch of three-dimensional finite-amplitude steady solutions ('standing waves') in the limiting plane Couette flow which, according to his computations, appear abruptly at a Reynolds number $\mathrm{Re} = U_1 H_1/\nu$ around 125.

Nagata's paper led to a strong revival of interest in finding new finite-amplitude equilibrium disturbances in PCF arising from bifurcations from infinity. Nagata used the family of ciruclar Couette flows as auxiliary flows satisfying the above-mentioned conditions (i) and (ii), but shortly afterwards Clever and Busse (1992) considered, instead of this, the family of plane Couette flows between lower and upper walls at different temperatures. They began by considering the well-studied longitudinal convective rolls in a layer of motionless fluid heated from below, then passed to the wavy rolls that bifurcate from two-dimensional rolls when the latter become unstable, and finally replaced the motionless fluid layer by a layer having a linear velocity profile (the stability of convection rolls in such a flow was studied earlier by Clever et al. (1977)). Assuming now that $\mathrm{Ra} \to 0$ (where Ra is the Rayleigh number) Clever and Busse determined a family of finite-amplitude three-dimensional standing waves (of the same type as those found by Nagata) relating to the limiting (non-buoyant) case of PCF. At the same time Clever and Busse (see also Busse and Clever (1996a, b)) studied many interesting three-dimensional disturbances in a wide class of unstably-stratified Couette flows which are of great interest in geophysical fluid dynamics. And later Nagata (1996) considered disturbances in PCF in a conducting fluid, in the presence of a transverse magnetic field (which destabilizes the fluid motion and at large enough intensity makes the flow linearly unstable, i.e., a definite neutral-stability curve appears here, with a finite value of Re_{cr}; see Kakutani (1964)). Using such hydromagnetic auxiliary flows and then letting the intensity of the magnetic field tend to zero, Nagata again found three-dimensional standing waves of finite amplitude in PCF, as first found by him in 1990, and even succeeded in considerably improving the accuracy of computation of their characteristics.

Cherhabili and Ehrenstein (1995) tried to apply the same method to find two-dimensional equilibrium states in PCF. They began by considering the family of two-dimensional equilibrium traveling waves of finite amplitude in plane Poiseuille flow found by Herbert (1977, 1978). Then, adding a Couette component to the primary Poiseuille flow, they numerically extended the Poiseuille-flow wave solutions to the combined Couette-Poiseuille (C-P) flow and then also to the limiting case of pure Couette flow. The limiting Couette-flow solutions unexpectedly proved to have the form of spatially-localized two-dimensional standing waves which can exist at Reynolds numbers (defined in terms of the channel half-thickness and half-difference of wall velocities) exceeding the 'critical value' of about 1,500. The authors suggested that previous attempts to compute finite-amplitude two-dimensional waves in PCFs failed because everybody looked for the usual traveling waves whereas only standing two-dimensional waves exist in PCF. Later Cherhabili and Ehrenstein (1997) investigated stability of these two-dimensional equilibrium states with respect to secondary two-dimensional and three-dimensional disturbances. The authors found that the three-dimensional disturbances are the most destabilizing ones; they give rise to some specific three-dimensional stationary equilibrium states (spanwise-periodic but streamwise-localized, and thus differing from the three-dimensional states found by Nagata and by Clever and Busse), bifurcating at points of the neutral-stability surface corresponding to equilibrium two-dimensional waves of finite amplitude. These new equilibrium states were found at values of Re close to 1,000.

In 1997 Nagat noted that none of the available experimental data relating to disturbances in PCF confirmed the existence of time-independent two- and three-dimensional waves of finite amplitude corresponding to the solutions found numerically by Cherhabili and Ehrenstein (1995, 1997) and by Nagata himself (see his papers (1990, 1996)). Therefore he returned to computations of various finite-amplitude solutions of equations for disturbances in C–P flows, and of their limits when the relative intensity of the Poiseuille component Q (which can be, e.g., set equal to the ratio $U_{max}/U(H)$ of the maximal velocity of the Poiseuille component to the velocity of the upper wall) tends to zero. This time the main attention was given to traveling-wave solutions (well known in plane Poiseuille flows). Nagata (1997) showed that in C-P flows at not too high values of Q there exist two different branches of finite-amplitude three-dimensional traveling-wave solutions. Only the first of them was considered in Nagata's paper (1990); as $Q \to 0$ these solutions tend to time-independent ('standing') three-dimensional waves discovered and studied in his papers (1990, 1996) (and also found in PCF by Clever and Busse (1992)). However, there is also a second branch of three-dimensional traveling-wave disturbances in C–P flows, which was unknown earlier. It was found now that this second branch may also be located over a wide range of values for Q, and as $Q \to 0$ it turns into two branches of finite three-dimensional shape-preserving traveling waves. These waves represent a new class of finite equilibrium wave disturbances which can appear in PCF if its Reynolds number $U_1 H_1/\nu$ exceeds 150.

Clever and Busse (1997) (see also Busse and Clever (1996a) and the more general earlier discussion of this matter by Busse (1991)) stressed that the steady three-dimensional equilibrium disturbances found in PCF by Nagata (1990, 1996) and by themselves (1992), which are also present in circular or stratified Couette flows, correspond to tertiary solutions of the equations of motion, arising from the solution describing a steady laminar flow after two subsequent bifurcations. The two-dimensional steady waves found by Cherhabili and Ehrenstein (1995) correspond to secondary solution, but the three-dimensional streamwise localized equilibrium states discovered by Cherhabili and Ehrenstein in 1997 (and also the three-dimensional traveling waves found in 1997 by Nagata) again represent tertiary solutions. Clever and Busse noted the large difference between the 'critical Reynolds number' $\mathrm{Re} \approx 125$ corresponding to Nagata's three-dimensional steady equilibrium states (and also the 'critical value' $\mathrm{Re} \approx 150$ found by Nagata's (1997) for three-dimensional finite-amplitude traveling-wave solutions), and the values of Re in the range from 1,000 to 1,500 which determine the thresholds above which Cherhabili and Ehrenstein (1995, 1997) found the two- and three-dimensional streamwise-localized steady equilibrium solutions of equations of motion. Nagata's 'critical values' are appreciably smaller than the results of experiments and numerical simulations for the lowest Reynolds numbers Re_{cr} at which some disturbances are not decaying in a PCF but produce persistent turbulent spots there, and also much smaller than the smallest values of Re at which the turbulence can be sustained in a PCF (those values do not differ much from Re_{cr}). At the same time the 'critical Reynolds numbers' found by Cherhabili and Ehrenstein are much greater than all observed values of Re_{cr}.

These facts forced Clever and Busse (1997) to consider anew the data relating to the tertiary steady three-dimensional states (having the form of wavy rolls similar to those often observed in the case of convection) found by Nagata (1990, 1996) and by themselves in 1992. They recalled that instability of these states had already been proved by Clever and Busse (1992); Nagata (1993) and Busse and Clever (1996a), and noted that because of this it was important to study the quaternary solutions bifurcating from the tertiary ones. Then they found that some interesting quaternary solutions bifurcate from the tertiary ones at Reynolds numbers not too much exceeding the 'critical Reynolds numbers' at which steady tertiary solutions start to exist. These quaternary solutions have the form of oscillatory way rolls, basically differing from tertiary steady waves only by the time variation of their amplitudes. The comparison of the solutions found with available experimental and numerically simulated data relating to instabilities in PCFs is not an easy matter, but the authors noted that some features of the quaternary solutions are similar to those of the longitudinal vortices found in Couette-flow experiments by Dauchot and Daviaud (1995) and Dauchot et al. (1996) and in Couette-flow simulations by Bech et al. (1995) and Hamilton et al. (1995). (Results of more detailed experimental investigations of the instabilities in PCFs by Bottin et al. (1997, 1998a, b) and Bottin and Chaté (1998), and numerical simulations by Barkley and Tuckerman (1998, 1999) appeared only later. These papers showed very convincingly the leading role of streamwise vortices in transition of PCFs to turbulence, and Bottin et al. (1998a), noting some qualitative differences between the structures detected by the indicated authors and the equilibrium solutions of Navier–Stokes equations found by Nagata, Busse and Clever, and Cherhabili and Ehrenstein, nevertheless related these two types of vortical formations with each other). The possible relation of sequences of three bifurcations, each of which decreases flow symmetry and makes the flow structure more complicated, to final transition to turbulence was also discussed in the papers by Clever and Busse (1993) and Busse and Clever (1996a); moreover, then Busse and Clever (1996b, 1998) considered also some tertiary and quaternary equilibrium states in plane Couette flows between differently heated horizontal walls. However, at present there are not enough data to make the situation clear. Note also that the numerical methods used for the study of solutions produced by several subsequent bifurcations are very complex and their complexity increases greatly with any loss of symmetry properties; therefore, the accuracy of the current computations of higher-order states may not be very good.

Now we will pass to the case of *circular Poiseuille flow* (CPF) in tubes. It has been already noted in Sect. 2.94 that, in many respects relating to stability, this flow is similar to plane Couette flow but is much more complicated. Its greater complexity is reflected, in particular, in the fact that the strict proof of the linear stability of plane Couette flow was found as long ago as the 1970s, while for CPF such a proof is unknown up to now although there is no doubt that this flow is linearly stable. Greater complexity also explains why studies of the nonlinear stability of CPF are appreciably less numerous than those relating to plane Couette flow and mostly deal only with axisymmetric disturbances; moreover, the same complexity has led to some contradictions between the results of different authors.

One of the first attempts to investigate the nonlinear stability of the CPF to axisymmetric wave disturbances, and to estimate the value of the corresponding Landau constant δ, was made by Davey and Nguyen (1971). They applied Reynolds and Potter's (1967) method to this problem and found that δ takes negative values for a wide range of wave numbers k and Reynolds numbers Re. This means that nonlinearity destabilizes the flow. Hence the tube flow must be unstable to finite axisymmetric disturbances, and evaluation of δ allows the determination of the equilibrium amplitudes $A_e = A_e$ (k, Re) and of the neutral-stability surface in the three-dimensional (A, k, Re)-space. Itoh (1977b), who also considered only axisymmetric disturbances, developed another method of stability analysis. His theory showed that the spatial Landau constant δ_s is positive for all values of Re and of frequency ω considered by him. (Itoh studied spatial, not temporal, development of disturbances; therefore in his work the frequency ω replaced the wave number k, and δ_s replaced δ). Thus, according to Itoh's theory, nonlinear effects stabilize CPF and therefore finite-amplitude instabilities and equilibrium disturbances cannot exist in this flow (at least to the approximation that neglects higher powers of amplitude A). The evident contradiction between Davey and Nguyen's and Itoh's conclusions clearly cannot be due only to the difference between temporal and spatial stability analysis, and in fact Itoh easily showed that his results directly contradict those of Davey and Nguyen.

In this connection Davey (1978) reconsidered the derivations of the equations for the Landau constant proposed in his 1971 paper with Nguyen and in Itoh's paper (1977b). He found that slightly different approximations were used in these papers and this led to some difference in the final equations; however, according to Davey, a special investigation was needed to determine which approximations are more accurate. He noted also that in applications to plane Couette flow the two theories imply almost identical results, and the difference is also relatively small for some particular axisymmetric disturbances in CPF which were not considered by Itoh; but in applications to disturbances in CPF which were actually studied in both papers, the results of the two theories prove to be contradictory. To clarify the situation, Patera and Orszag (1981b) applied direct numerical simulation to development of axisymmetric disturbances in CPF, i.e. they solved the corresponding nonlinear (N-S) initial-value problems numerically. They paid particular attention to those disturbances which were found to be undamped either by Davey and Nguyen (1971) or by Davey (1978) (who mentioned some axisymmetric disturbances which tended to equilibrium states according to both the theory proposed by Davey and Nguyen and that of Itoh). Numerical simulation showed, however, that in fact all these disturbances (and also all the other axisymmetric disturbances considered by Patera and Orszag) are damped. Therefore, Patera and Orszsag concluded that apparently all axisymmetric disturbances decay in CPF and that the methods used by Davey and Nguyen (1971) and Itoh (1977b) are probably both inapplicable to CPF. (Remember however that the remark by Orszag and Patera (1980) about the nonexistence of two-dimensional equilibrium waves in plane Couette flow was dismissed by Cherhabili and Ehrenstein (1995)).

Another method for the study of nonlinear stability of CPF, applicable to small but finite, and in general non-axisymmetric, disturbances in high-Reynolds-number tube flow was proposed by Smith and Bodonyi (1982). Their theory further develops the approach initiated independently by Benney and Bergeron (1969) and Davis (1969), applied to two-dimensional disturbances in plane-parallel flows and then used in a large number of subsequent papers (see, e.g., discussion of this topic in the book by Drazin and Reid (1981), Sect. 52.5, and more recent survey papers by Maslowe (1986) and Churilov and Shukhman (1995)). Benney and Bergeron, and Davis noted that if $\mathrm{Re} \gg 1$ and $\mathrm{A} \ll 1$ (where A is the dimensionless amplitude of the disturbance), then the linear stability theory (i.e., the linear Orr-Sommerfeld equation) is applicable only when $\lambda = A(\mathrm{Re})^{2/3} \ll 1$. However, if $\lambda \gg 1$ or $\lambda \approx 1$, then some specific nonlinear effects play an important part in the vicinity of the 'critical layer' where the phase velocity c of a normal-mode disturbance coincides with the undisturbed flow velocity $U(z)$. Smith and Bodonyi considered the time evolution of a normal-mode disturbance with velocity of the form $\mathbf{u}(x,t) = A \exp[i\{k(x-ct)+n\phi\}]\mathrm{F}(r)$, where all independent and dependent variables are non-dimensionalized by using the maximal Poiseuille-flow velocity U_0 and the tube radius R as units of velocity and length, $\mathbf{F}(r)$ is an $O(1)$ vector function (having all components of the order of one) and A is a small amplitude factor which determines the order of magnitude of the true amplitude (whose definition is not unique, though this topic was not considered in the paper). For the sake of simplicity it was also assumed here that $A = \mathrm{Re}^{-2/3}$, although it was noted that the majority of the conclusions obtained is also valid in the case where $1 \gg A \gg \mathrm{Re}^{-2/3}$. The authors looked for equilibrium (neutrally-stable) solutions and hence the dimensionless phase velocity c (which varies with k, n, and Re) was assumed to be real; moreover, they also accepted that $0 < c < 1$. Careful analysis of the dynamic equations for the disturbance velocities showed that here (exactly as in the problems studied by Benney and Bergeron (1969) and Davis (1969)), the nonlinear terms prove to be quite important in the thin 'nonlinear critical layer' (whose thickness is determined just by this condition) where $U(z) \approx c$. It was also found that neutrally-stable disturbances of the form considered here exist in CPF for $0.284 < c < 1$ and $n = 1$ (and the shapes of these disturbances were also determined by Smith and Bodonyi); at the same time, arguments were presented suggesting that no neutral solutions of this form exist for other values of c and n. The existence of neutrally stable disturbances implies that the Landau constant δ is negative, and hence unstable disturbances of finite amplitude can exist here. Thus, Smith and Bodonyi proved that at large values of Re the CPF is unstable to some small non-axisymmetric disturbances of finite amplitude.

Slightly later the nonlinear stability of the CPF was investigated by Sen et al. (1985), who used in their work the same version of the equilibrium-amplitude method of Reynolds and Potter (1967) that was applied by Sen and Venkateswarlu (1983) to the problem of the stability of plane Poiseuille flow. Sen et al. disagreed with the popular opinion that Reynolds and Potter's method has an acceptable precision only at points (k, Re) near the neutral curve (this opinion prompted Itoh (1977b) to announce that the indicated method is inapplicable to CPF). Therefore they tried to use it to study the stability of tube flow to both axisymmetric (with the azimuthal wave number $n = 0$) and non-axisymmetric (with $n = 1$) least-stable central normal-modes of disturbance (i.e., the modes with the disturbance energy concentrated mainly near the tube axis; it was for this mode with $n = 0$ that the results by Davey and Nguyen (1971) and by Itoh (1977) proved to be contradictory). As in all versions of Reynolds and Potter's method, it was assumed beforehand that there exists the equilibrium state of the normal mode considered, with the time-independent finite amplitude A_{e} (i.e., the existence of undamped finite-amplitude disturbances was postulated). Then the disturbance stream function $\Psi(x, r, t)$ if $(n = 0)$ or velocity and pressure $\mathbf{u}(x, r, \phi, t)$ and $p(x, r, \phi, t)$ if $(n = 1)$ where expanded in powers of $e^{i[n\phi + k(x-ct)]}$ (where n and k are the azimuthal and streamwise wave numbers and c is the phase speed of the normal wave given by the linear stability theory) and the terms of the series obtained were represented as the appropriate powers of amplitude multiplied by the normalized disturbance functions. When such forms of the flow fields were substituted into the equations of motion and the boundary conditions, the solvability conditions for the equations for different terms of power series allowed successive determination of the values of the complex Landau constants $\lambda_m(k, \mathrm{Re}), m = 1, 2, 3, ...$, and then to evaluate the equilibrium amplitude A_{e} from the real part (4.41) of Eq. (4.41b).

According to the numerical results of Sen et al. there is, for axisymmetric or non-axisymmetric disturbances and at any Re, a definite range of wave numbers k for which a finite equilibrium amplitude A_{e} exists, showing that there are some undamped finite-amplitude disturbances. As Re$\to \infty$, $A_{\mathrm{e}} \to 0$ as $\mathrm{Re}^{-4/3}$, and hence the velocities of the equilibrium disturbances tend to zero as $\mathrm{Re}^{-2/3}$. Some examples of the radial velocity distributions for equilibrium disturbances, of the dependences of amplitudes of velocity components on Re, and of the numerical values of about ten Landau constants λ_m for some specific values of k and Re, and for $n = 0$ and 1, are also presented in the paper. The authors stressed that their analysis had a number of limitations (relating, e.g., to the ranges of k and n studied, and to the choice of normal modes), and was based on very complicated calculations which used a number of approximations; therefore, a check of these results by other methods, and their further extension, would definitely be worthwhile. However, apparently no attempts to carry out such a check were undertaken up to now.

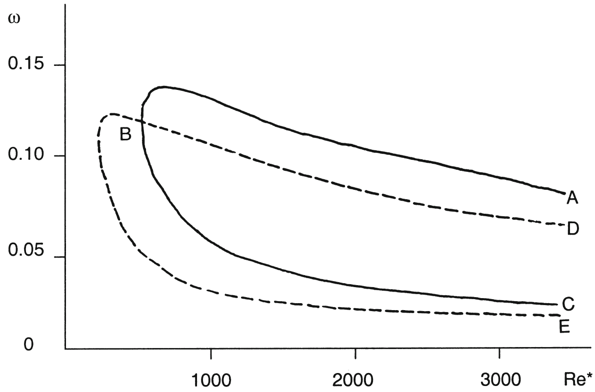
4.2.3.3 Nearly Plane-parallel Boundary-layer Flows

In the constant-pressure *boundary-layer flow* (BLF) on a flat plate a neutral-stability curve, and unstable modes of infinitesimal disturbances, certainly exist; in this respect the stability properties of the flow are simpler to investigate than those of the linearly-stable plane Couette and circular Poiseuille flows. However in some other respects BLF is more complex than the two flows mentioned above. First of all, BLF is not strictly parallel; here the primary steady flow has non-zero vertical velocity $W(x, z)$ and the conventional thickness of BLF, $d(x)$,[8] is not constant but increases in the streamwise direction Ox. Moreover, BLF extends to infinity in the vertical direction Oz; therefore the Orr-Sommerfeld eigenvalue problem (corresponding to a simplified plane-parallel flow model) now has a spectrum which includes both discrete and continuous components. Infinite vertical extent of the flow also complicates the upper boundary conditions for the BLF disturbances. It was explained in Chaps. 2 and 3 that in linear-stability studies of boundary layers the parallel-flow approximation is usually used, i.e., the real BLF with the thickness $d(x)$ is usually replaced by a definite plane-parallel model (usually by the so-called *parallel Blasius model* of a flow in the half-space $0 \le z < \infty$ with the velocity field of the form $\{U(z), 0, 0\}$, where $U(z)$ is the standard Blasius profile corresponding to some fixed boundary-layer thickness $d(x_0)$ which does not depend on the x coordinate). Such a parallel-flow approximation is often used in studies of the nonlinear stability of BLF too, but here it has a much more narrow domain of applicability (therefore Stuart in his highly authoritative review (1971) of nonlinear stability theory expressed doubt about the validity of the approximation in this case). All these reasons complicate the determination of the Landau equations for disturbance amplitudes in BLF.

Apparently the first investigation of the Landau equation for boundary-layer flow was carried out by Itoh (1974b) (some of his results were previously announced by Tani (1973)). Exactly as in his Poiseuille-flow paper (1974a), Itoh studied the spatial development of disturbances, using a modification of Watson's (1962) theory combined with an extension of the Stuart–Watson approach of 1960. He took it for granted that at large values of x and Re the streamwise variation of the flow conditions is of minor importance, and hence the streamwise growth of the boundary-layer thickness $d(x)$ (and of $\mathrm{Re}(x) = U_0 d(x)/\nu$ where $U_0 = U(\infty)$ is the free-stream velocity outside the boundary layer) may be taken into account rather crudely. Thus, he introduced the contracted streamwise coordinate $\xi = (x - x_0)/\varepsilon$, where x_0 corresponds to a point far from the leading edge of a plate at $x = 0$ and $\varepsilon = d(x_0)/x_0$ is

[8] Since the letter δ is now used to denote the Landau constant, the boundary-layer thickness will be denoted in this (and only in this) subsection as $d = d(x)$. Similarly, the displacement thickness of the BLF, which is the most widely used vertical length scale of this flow, will be denoted here as $d^* = d^*(x)$.

Fig. 4.17 The regions of positive and negative values of the coefficients γ_s and δ_s in the (ω, Re*)-plane for the constant-pressure boundary-layer flow. (After Itoh (1974b)) *ABC*: the *curve* γ_s (ω, Re*) = 0 (the spatial neutral-stability *curve* of the linear stability theory, bounding the region where $\gamma_s > 0$); *DBE*: the *curve* $\delta_s(\omega$, Re*) = 0 bounding the region where $\delta_s > 0$



a small parameter. Then he treated the flow in the neighborhood of the point x_0 as homogeneous with respect to the coordinate ζ, and neglected the terms in the equations of motion which are of order ε^2 or higher. The assumption used (which is close to the plane-parallel approximation) allowed Itoh to evaluate both coefficients of the spatial Landau equation (corresponding to a two-dimensional wave-like disturbance proportional to $e^{i(k\xi-\omega t)}$ where ω is real but k is complex) by a method similar to that applied in his paper (1974a) to strictly plane-parallel Poiseuille flow. He thus determined the spatial neutral-stability curve $\gamma_s(\omega, \mathrm{Re}^*) = 0$ in the (ω, Re*)-plane (where $\mathrm{Re}^* = d^*\ U_0/\nu;\ U_0$ and d^*—the displacement thickness of the BLF—will now be used as velocity and length units in all considerations of the results relating to this flow making in all physical quantities dimensionless). Then the curve $\delta_s(\omega, \mathrm{Re}^*) = 0$ was also computed (recall that $\gamma_s = b_0/2$) and $\delta_s = -b_1$ are coefficients of the 'spatial Landau equation'). The curves obtained are shown in Fig. 4.17; they are of the same general shape as the curves for plane Poiseuille flow in Fig. 4.11, and again show that the spatial Landau constant δ_s is negative along the upper branch of the neutral curve but positive on the main part of the lower branch. Finally Itoh tried to compare his theoretical results with the experimental data of Klebanoff et al. (1962), relating to disturbances generated by a vibrating ribbon located not far from the leading edge of a plate in a wind tunnel. However he found that his theory could explain the behavior of real periodic disturbances only in the case of disturbances with rather small initial amplitude.

Independetly of Itoh, Herbert (1975) also studied Landau's equations for the BLF. However, he considered not spatial but temporal development of two-dimensional disturbances of given wave number k, i.e. he tried to evaluate coefficients of the temporal Landau Eqs. (4.34) and (4.40) and the corresponding equilibrium amplitudes $A_e = (2\gamma/\delta)^{1/2}$. The results obtained were then used to determine the curves γ (k, Re*) = 0 and δ (k, Re*) = 0 in the (k, Re*)-plane. Herbert's computations were based on an approximation of the same type as that introduced by Itoh (1974b) and naturally led to quite similar results. Similar approximation was used also by Gertsenshtein and Shtemler (1997), who applied it to computation of the real coefficients

a_1 and a_2 and complex ceofficients λ_1 and λ_2 of Eqs. (4.41) and (4.41b) at the points of the BLF neutral curve in (k, Re)-plane.

The results of Itoh (1974b) provoked Smith's (1979b) distrust, since in his paper (1979a) Smith found that nonparallelism of BLF appreciably affects the disturbance development. Therefore Smith (1979b) proposed a quite different asymptotic theory of the nonlinear evolution of two-dimensional disturbances in BLF. He considered the case where Re is very high and the disturbance amplitude A is sufficiently small and, using the results of his paper (1979a), derived new values of the coefficients of the spatial Landau equation for the disturbance amplitude. This derivation will not be considered here at length; note only that Smith's computations, relating to a nonparallel model of BLF, confirmed Itoh's (1974b) conclusion that the spatial Landau constant δ_s, corresponding to two-dimensional disturbances, takes positive values near the main part of the lower branch of the BLF neutral curve. At the same time Gajjar and Smith (1985), who used similar methods which also took into account the flow nonparallelism, found that the influence of nonparallelism does not change the conclusion, obtained for the parallel model of BLF, according to which δ_s is negative near the upper branch of the BLF neutral curve. Let us remind the reader in this respect that in Chap. 2 it was noted that direct numerical simulations by Fasel and Konzelmann (1990) and Bertolotti et al. (1992) of the disturbance development in BLF, and also the careful measurements of this development by Klingmann et al. (1993), led to the conclusion that the actual effect of nonparallelism of the BLF is apparently considerably smaller than was suggested in many previous theoretical papers on this subject (which often contradicted each other). The comparison of the results of Itoh (1974b); Herbert (1975); Smith (1979b), and Gajjar and Smith (1985) with each other shows that this conclusion is at least qualitatively (when only the signs of quantities are taken into account) applicable to values of Landau's constants of the BLF too. The same conclusion also follows from the results of a study by Itoh (1984) of the values of Landau's constants in the BLF, supplementing his earlier investigation (1974b).

Trying to improve the simplified treatment of flow non-parallelism used in his paper (1974b); Itoh (1984) referred to his paper (1977a) where a more accurate approach to derivation of Landau's equations for two-dimensional normal-mode disturbances was suggested. He stressed that this approach is applicable only to subcritical (i.e., linearly stable) disturbances, and therefore proposed a new modification of the Stuart–Watson method, which leads to results similar to those found in his paper (1977a); this modification made the results applicable to the cases where supercritical (linearly unstable or neutral) disturbances are studied. Simultaneously, he also developed a more accurate method for taking the slight flow nonparallelism into account. Using these modifications he re-evaluated the neutral-stability curve in the (k, Re*)-plane for two-dimensional temporally-evolving infinitesimal wave disturbances and computed a new the location of the maximum-growth-rate line of the supercritical region in this plane, and also the values of $\mathrm{Re}^*_{\mathrm{cr}}$, k_{cr} and ω_{cr} (he found that $\mathrm{Re}^*_{\mathrm{cr}} \approx 519$, $k_{\mathrm{cr}} \approx 0.30$, and $\omega_{\mathrm{cr}} \approx 0.12$). Then he computed the values of the complex Landau coefficient $l = \delta + i\delta'$ of Eq. (4.40) at the points of the neutral-stability curve and of the maximum-growth-rate line of the supercritical region (consisting of the points (k, Re*) where $\gamma(k, \mathrm{Re}^*) = \max_{k'} \gamma(k', \mathrm{Re}^*)$ and is positive). Computations of the values of l at points on the maximum-growth-rate line were carried out by two different methods, the first of which used a version of the parallel-flow approximation while the second took the flow nonparallelism into account more accurately. The results given by both methods showed that the real Landau constant δ takes positive values on the main part of the line considered, and that corrections due to the more accurate accouinting for nonparallelism are inessential at points far from the neutral-stability curve, but become significant at points near the 'critical point' ($k_{\mathrm{cr}}, \mathrm{Re}^*_{\mathrm{cr}}$) where this line intersects the neutral-stability curve. Therefore the computations of the values of l on the neutral curve were now performed only by the second ('non-parallel') method. The new computations led to negative values of δ at all points of the upper branch of the neutral curve, and to positive values of δ at almost all points of the lower branch (except only the 'critical point' and its small surroundings, where δ takes slightly negative values). These results agree with Itoh's previous results shown in Fig. 4.17, and with the above-mentioned results of Herbert (1975); Smith (1979b), and Gajjar and Smith (1985), showing also that the temporal and spatial Landau constants $\gamma(k, \mathrm{Re}^*)$ and $\gamma_s(k, \mathrm{Re}^*)$ apparently usually have the same signs.

Numerical values of l and $\delta = \Re e\, l$ clearly depend on the definition of the complex amplitude A. In the earlier discussions, the approach developed in the papers by Stuart and Watson of 1960 was always used, and therefore it was assumed that $A(t)$ represents the numerical factor entering the leading term of the Fourier expansion of the initially-infinitesimal normal-mode disturbance satisfying the Orr-Sommerfeld equation (see, e.g., Eqs. (4.38) and (4.39)). However Itoh (1984) used in the beginning of his paper another particular definition of the disturbance amplitude, based on the distribution of the vertical velocity $w(x, z, t)$. This definition is mathematically convenient but it is difficult to measure the corresponding amplitude A in laboratory experiments and thus to compare the proposed theory with experimental data. Therefore Itoh later repeated the computation, now using as A some typical value of the streamwise disturbance velocity u at the height $z = d^*/2$. The new values of l were approximately four times greater than the old ones, but the form of their dependence on Re* proved to be practically the same. Itoh also computed the values of the Landau constant l for three-dimensional disturbances of a special type, namely, for some special wave packets composed of three-dimensional plane waves. In this case the values of real and imaginary parts of l proved to be much smaller than the values corresponding to two-dimensional waves.

Recall that when two coefficients of Landau's equation (either temporal, or spatial) are of the same sign, they determine the value of the amplitude A_e of the equilibrium disturbance (subcritical if $\gamma < 0$, $\delta < 0$, and supercritical if $\gamma > 0$, $\delta > 0$). In the theories where some higher-order real Landau constants a_m, $m \geq 2$, are also taken into account, the equilibrium amplitude A_e can be determined as the smallest positive root of the appropriately-truncated Eq. (4.41), if such root exists. On the other hand according to Reynolds and Potter (1967), the existence of an equilibrium disturbance can considerably simplify the derivation of the corresponding Landau's equation from the equations of motion. Sen and Vashist (1989) applied the method of Reynolds and Potter to derivation of the higher-order complex Landau equations for two-dimensional normal-mode wave disturbances in the plane-parallel model of the Blasius boundary layer. This derivation was carried out quite similarly to those of Sen and Venkateswarlu (1983) and Sen et al. (1985) for two-dimensional wave disturbances in plane and circular Poiseuille flows. Sen and Vashist again considered the unstable (or the least stable) two-dimensional wave corresponding to given values of k and $\mathrm{Re} = U_0 d/\nu$ (or $\mathrm{Re}^* = U_0 d^*/\nu$—they used both definitions of the Reynolds number) and computed the values of the complex coefficients λ_m, $m = 1, 2, \ldots, 8$, for a number of values of Re and k. Then they determined the nonlinear neutral curve in the (k, Re*)-plane corresponding to their nonlinear model of the eighth order. It was shown that the nonlinear effects decrease the value of Re_{cr} and increase the values of k_{cr} approaching the non-linear neutral curve to the experimental data then available. However, at that time the authors had no accurate enough experimental data for quatitative comparison with their theory, and they did not try to estimate the influence of the non-parallelism of BLF, which they neglected in the computations.

Note now that in the case of a strictly parallel flow, equilibrium disturbances can also be computed by direct numerical simulation (DNS), i.e., by numerical solution of the corresponding N-S equations (see, e.g., Herbert's work (1976, 1978, 1983a) relating to plane Poiseuille flow). However, in the case of BLF an additional difficulty arises from the fact that the Blasius boundaryu layer is not an exact solution of the N-S equations with standard boundary conditions. Moreover, the boundary-layer thickness $d = d(x)$ depends on x and hence the disturbance cannot be assumed to be proportional to e^{ikx}, with $k = \text{const}$. Therefore, to compute the equilibrium two-dimensional wave-like disturbances in the BLF, Milinazzo and Saffman (1985) supplemented the N–S equations by a fictitious counter-streamwise 'force' which suppresses the boundary-layer growth and makes the two-dimensional flow with velocity $\{U(z)\ 0,\ 0\}$, where $U(z)$ is a standard Blasius profile, an exact solution of the equations of motion considered. (The authors noted that the inclusion of such a force is an 'old well-known idea' which apparently was due originally to L. Prandtl). Later Fischer (1995) used the same modification of the equations of motion for careful evaluation of the Landau constants $\delta(k, \mathrm{Re})$ and equilibrium amplitudes A_e corresponding to a plane-paralle model of the Blasius BLF. On the other hand Lifshits and Shtern (1986); Lifshits et al. (1989), and Koch (1992) also used the plane-paralle approximation in their calculations of the BLF equilibrium solutions, but modified, not the equations of motion but the boundary conditions. Note also that local parallelism of the flat-plate boundary layer and streamwise periodicity of the disturbances were simply assumed to be valid in the important studies of BLF nonlinear stability of Laurien and Kleiser (1989) and Zang and Hussaini (1990).

Milinazzo and Saffman (1985) and Lifshits et al. (1989) considered some particular examples of equilibrium two-dimensional disturbances in BLF (Lifshits et al. also presented some examples of special periodic-halving bifurcations). Lifshits and Shtern (1986) and Koch (1992) tried to determine the neutral surface in the thre-dimensional (E, k, Re)-space similar to that computed by Herbert (1978, 1983a) for plane Poiseuille flow (see Fig. 4.12 above). However, in their computations of equilibrium solutions Lifshits and Shtern (1986) used only the terms of orders zero and one in the Fourier expansion of the disturbance stream function $\Psi(x, z, t)$ similar to (4.38), while Koch discovered in 1992 that in the case of BLF such severe truncation of Fourier series leads to results which can even be qualitatively incorrect. Therefore Koch also showed results of neutral-surface computations where the second harmonic was included in truncated Fourier series for $\Psi(x, z, t)$, and additionally presented graphs of several cross-sections of this surface by the planes $\mathrm{Re} = \text{const.}$ and $k = \text{const.}$ computed with the help of Fourier series truncated after the nth harmonic, where n varied from 1 to 6. The results obtained gave much information about the complicated shape of the neutral surface in the case of BLF and also allowed an estimate of what truncation is sufficient for obtaining the necessary degree of precision. Then Koch passed to the important problem of secondary instability of two-dimensional equilibrium disturbances to small three-dimensional disturbances. His results relating to this topic, and also the results of simultaneously-published papers by Stewart and Smith (1992) and Smith and Bowles (1992), provide a very valuable supplement to the survey of the same subject by Herbert (1988) and shed additional light on the process of boundary-layer transition.

4.2.4 Amplitude Equations for Disturbances in Free Flows in an Unbounded Space

4.2.4.1 Plane Mixing Layers and Jets

Now we will pass to consideration of parallel (or nearly parallel) free flows in the unbounded space $-\infty < z < \infty$ and begin with the case of a strictly plane-parallel *plane mixing layer* between two parallel flows in contiguous half-spaces $-\infty < z < 0$ and $0 < z < \infty$, having constant but different velocities $\{-U_0, 0, 0\}$ and $\{U_0, 0, 0\}$ where U_0 is positive. In Sect. 2.93 it was mentioned that a very convenient and widely-used analytic approximation to the mixing-layer profile is the hyperbolic-tangent profile: $U(z) = U_0 \tanh(z/H)$ where H characterizes the mixing-layer thickness. Therefore we will also use this approximation.

In Sect. 2.93 it was explained that $\mathrm{Re}_{cr} = 0$ for the hyperbolic-tangent mixing layer, i.e. this flow is linearly unstable at any value of $\mathrm{Re} = U_0H/\nu$. The corresponding neutral-stability curve in the (k, Re)-plane was shown in Fig. 2.35; it suggests that in an inviscid fluid, where $\mathrm{Re} = \infty$, this flow must be linearly unstable with respect to two-dimensional wave-like disturbances if $kH < 1$. This is in fact so, as was proved long ago by Tatsumi et al. (1964) (see also Sect. 31.10 in the book by Drazin and Reid (1981)). Assuming that the influence of viscosity must be insignificant at large values of Re, Schade (1964) tried to calculate the value of the Landau constant δ for the neutral two-dimensional disturbances with $kH = 1$ in inviscid flow with a hyperbolic-tangent velocity profile. He based his calculation on the method of Stuart (1960) but supplemented it by some simplifying assumptions (in particular, he neglected the mean-flow-distortion effect on δ). To overcome the difficulty arising from the singularity of the inviscid Rayleigh Eq. (2.48) (see Sect. 2.82) at the 'critical level' where $U(z) = c$, Schade introduced viscosity in some of his equations (but, as we will see below, this was insufficient for obtaining the correct results). His calculation led to the conclusion that $\delta > 0$ (equal to $32/3\pi$, if U_0 and H are taken as the velocity and length units) at $kH = 1$, and hence small unstable disturbances in the mixing layer with wave numbers slightly smaller than $(H)^{-1}$ must tend to a finite equilibrium state as $t \to \infty$. This conclusion also agreed with the results of Stuart's (1967) study of equilibrium finite-amplitude disturbances in various inviscid laminar mixing layers (including the hyperbolic-tangent one). Later Maslower (1977a), who used a method quite similar to that of Schade (1962), computed, for a hyperbolic-tangent mixing layer with finite value of Re, the values of δ corresponding to two-dimensional disturbances with wave numbers k which are equal to or slightly smaller than the wave number k_0 of the neutrally-stable disturbance. His results for neutral disturbances with $k = k_0$ agreed with Schade's result for the case where $\mathrm{Re} = \infty$, and showed that the value of δ is positive at any Re and decreases with decreasing Re. Simultaneously Maslowe also noted at the end of his paper, that the effect of the mean-flow distortion, which was neglected in his and Schade's studies, apparently also affects the value of δ but he did not elaborate on this remark.

Maslowe (1977a) apparently did not know at the time about the paper by Gotoh (1968) who also calculated values of the Landau constant δ and of the equilibrium amplitude $A_e = (2\gamma/\delta)^{1/2}$ for small finite disturbances in viscous mixing layers, with very large but finite values of Re and with values of k near the neutral-stability curve. Gotoh gave special consideration to the contribution of the 'nonlinear critical layer' (which also included the effect of the mean-flow distortion) and found that in the case of a hyperbolic-tangent mixing layer, δ is positive at all the values of Re and k he considered, and is given by equations

$$\delta = 16.35(\mathrm{Re})^{1/3}\left[1 + 0.25\left(\frac{\gamma}{k}\right)\mathrm{Re}\right], \quad \text{if}\ \frac{\gamma}{k} < (\mathrm{Re})^{-1/3}, \tag{4.45}$$

and

$$\delta = \frac{0.5k^4}{\gamma^3}, \quad \text{if}\ \frac{\gamma}{k} > (\mathrm{Re})^{-1/3}, \tag{4.45a}$$

where δ, γ and k are non-dimensionalized in the usual way. (Note also that γ is a smooth function of k, vanishing at the wave number k_0 of the neutral disturbance; therefore, under the natural assumption that this function is differentiable with nonzero first derivative, $\gamma \propto (k_0 - k)$ near the neutral curve). These results clearly disagree with Schade's conclusion relating to $\mathrm{Re} = \infty$. However, Michalke (1972), who supported Gotoh's criticism of the results of Schade (1964), asserted at the same time that Gotoh's results are also erroneous, since the value of δ must be independent of the value of γ. As will be indicated below, this conclusion by Michalke was later found to be groundless; nonetheless, it was possibly one of the reasons why Gotoh's paper was totally forgotten for a number of years, while the incorrect result by Schade was repeatedly reproduced, as a particular case of some more general results, in papers by Stuart (1967); Benney and Maslowe (1975); Maslower (1977a,b), and Huerre (1977) (Fig. 4.18).

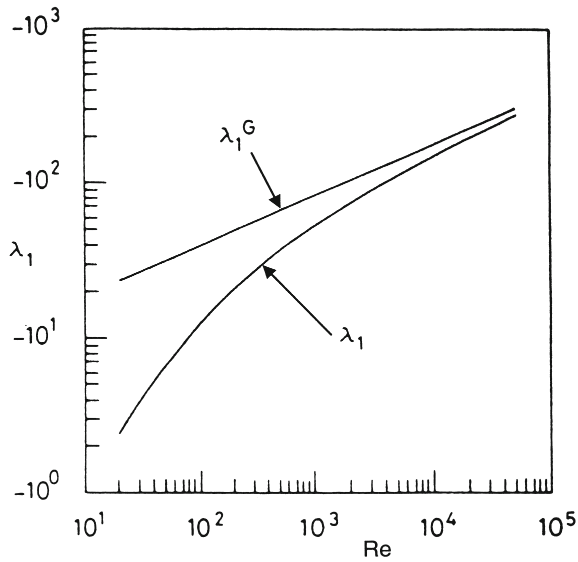


Fig. 4.18 Comparison of the values of $\lambda_1(\text{Re}) = -\delta(\text{Re})/2$ for the neutral wave disturbances in a hyperbolic-tangent mixing layer, computed by Fujimura, with Gotoh's asymptotic equation $\lambda_1 = \lambda_1^G = -8.177\text{Re}^{1/3}$ for $\text{Re} \gg 1$ (After Fujimura (1988))

Huerre (1980) tried to develop a new theory of the nonlinear stability of small free-shear-layer disturbances, based on the approach applied by Benney and Bergeron (1969); Stewartson and Stuart (1971) and Benney and Maslower (1975) to studies of the space-time development of wave packets with amplitude $A = A\ (t, x)$ in high-Reynolds-number parallel shear flows and leading to a nonlinear parabolic partial differential equations of the Ginzburg–Landau type for the function $A\ (t, x)$ (see the last paragraph of Sect. 4.22 and the related paper by Huerre and Scott (1980)). In his paper of 1980 Huerre came to the incorrect conclusion that in the hyperbolic-tangent mixing layer Landau's constant $\delta = \delta\ (k, \text{Re})$ is negative for wave numbers k near the neutral curve, and hence no equilibrium finite-amplitude states can exist here for waves which are slightly unstable according to linear theory. However later he found an error in his paper of 1980, whose correction (presented in Huerre (1987)) led him to the conclusion that δ is positive at all large enough values of Re and small values of $k_0 - k$, and is proportional to $\text{Re}^{1/3}$ (with the same coefficient of 16.35 which was earlier found by Gotoh) in cases when Re is sufficiently large and $k = k_0$ (i.e. $\gamma = 0$). The inaccuracy of Huerre's paper (1980) was discovered independently by Churilov and Shukhman (1987) who also solved the same problem more accurately and obtained, under the condition that $\text{Re} \gg 1$ and $(\text{Re})^{-1} \ll \gamma \ll (\text{Re})^{-1/3}$, the same Eq. (4.45) for δ which was found by Gotoh (1968). (Apparently neither Churilov and Shukhman, nor Huerre, knew in 1987 about Gotoh's paper of 1968). Finally Fujimura (1988) applied the general amplitude expansion proposed by Herbert (1983b) to the computation of Landau's constant δ for small, slightly-unstable, disturbances in a hyperbolic-tangent mixing layer. He found that, for large enough values of Re, numerical values of δ obtained in this way agree well with Gotoh's asymptotic Eqs. (4.45) and (4.45a), and computed also the Landau constants a_2 and a_3 of the next two

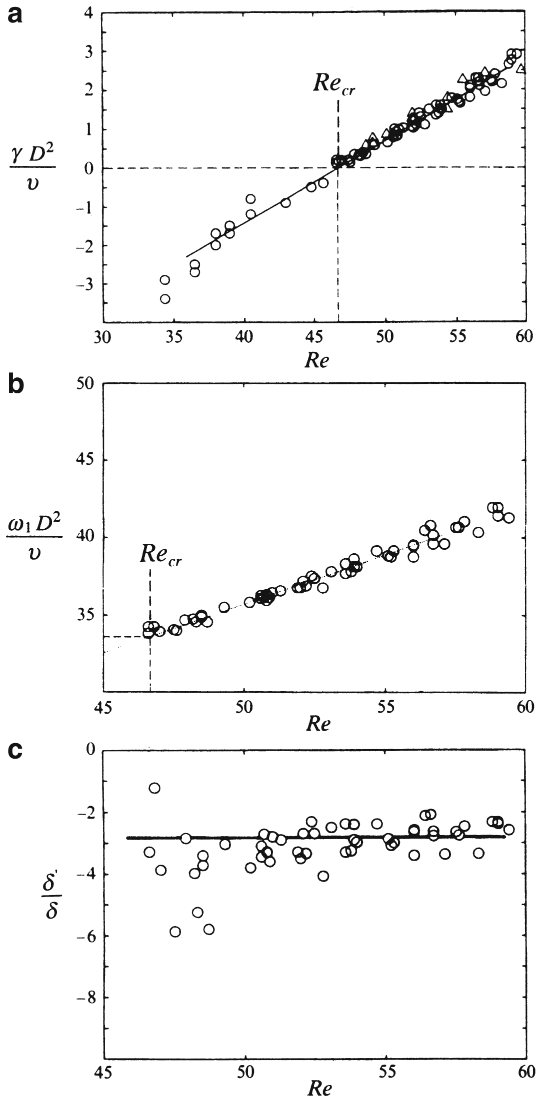


Fig. 4.19 The dependence on $\mathrm{Re} = U_0 D/\nu$ of the values of $\gamma D^2/\nu$ (**a**), $\omega_1 D^2/\nu$ (**b**), and δ'/δ (**c**) in the circular-cylinder wake according to measurements of the development of controlled wake oscillations at the point $\mathbf{x} = \{x, y, z\} = \{8D, 0, 4D\}$ behind the cylinder. (After Schumm et al. (1994)). *Symbols* O and Δ correspond to two different methods of wake-oscillation control

orders, showing in particular that a_2 is negative (i.e., stabilizes the flow) in a wide ragne of Re and k values. An example of Fujimura's results is given in Fig. 4.19 where computed values of $\lambda_1 = -\delta/2$ for neutral disturbances with $k = k_0$ (i.e., with $\gamma = 0$) and Reynolds numbers in the range $20 \le \mathrm{Re} \le 50{,}000$ are compared with Gotoh's asymptotic equation $\lambda_1^G = -\delta_G/2 = -8.177(\mathrm{Re})^{1/3}$, valid at $\mathrm{Re} \gg 1$.[9] Note also

[9] Fujimura found numerically, and also proved analytically, that for neutral two-dimensional disturbances for the difference $\delta - \delta_G$ in fact tends to a constant $\Delta\delta$ as $\mathrm{Re} \to \infty$. According to both his computations and his analytical results $\Delta\delta$ is close to 57.

that still later, in their extensive survey of work on the influence of the critical layer on nonlinear development of small disturbances in weakly supercritical shear flows, Churilov and Shukhman (1995) also indicated that the old paper by Gotoh (1968) gave correct results. Simultaneously they showed that the critical-layer contribution alone often leads to very high values of the Landau constant δ, and sometimes produces amplitude equations of a form quite different from that proposed by Landau in 1994.

Comparision between experimental data and the above theoretical estimates for the values of Landau's constant in a mixing layer (or similar estimates relating to other parallel flows in unbounded space) is rather difficult. In fact the the theory considered above deals mainly with slowly-growing wave disturbances in such flows corresponding to (k, Re)-points near the neutral curve, while in real life the most important role in instability phenomena is played here by modes which are maximally (or almost maximally) amplified, and hence far from neutral. The rapidly-growing most-unstable waves later generate subharmonic waves with half the frequency of the dominant mode, and the interaction of the dominant mode with subharmonic ones and with the mean flow cannot be described by the Landau-type theory (see, e.g., the old survey of appropriate experimental data by Miksad (1972) and the more recent paper by Monkewitz (1988a) containing many additional references). However, this does not mean thtat Landau's theory is useless for quantitative description of instability phenomena in free shear flows; see in this respect the discussion of wake-flow instabilities below.

Plane jets in an unbounded space represent type of plane-parallel flows having some similarities with the parallel mixing layers. It was mentioned in Sect. 2.93 that the most widely-used model of the corresponding velocity profile $U(z)$ is the so-called *Bickley jet profile* $U(z) = U_0 \operatorname{sech}^2 (z/H)$, where $-\infty < z < \infty$ and H characterizes the jet thickness (see Eq. (2.87), unlike the hyperbolic-tangent approximation for the mixing layer this profile is an exact analytical solution of the boundary-layer equations). The problem of nonlinear evolution of normal-mode disturbances in the Bickley jet has attracted less attention than the same problem for the hyperbolic-tangent mixing layer and we will not discuss it in detail. Note only that Gotoh (1968), in parallel with his work on development of disturbances in the mixing layer, considered the same problem for the case of Bickley's jet and found that here again equations of the form (4.45) and (4.45a) are valid. However, now the numerical coefficients 16.35 and 0.25 in Eq. (4.45) must be replaced by coefficients 2.19 and 1.5., while the new value of the coefficient in Eq. (4.45a) was not indicated by Gotoh. His results relating to the Bickley jet, unlike his results for the hyperbolic-tangent mixing layer, have not yet been confirmed by other authors but one may conjecture that they are valid too. Some remark about the instabilities of round jets will be made at the very end of this section.

4.2.4.2 Wake Flows: the Case of a Circular-cylinder Wake

Let us now consider nearly plane-parallel *wake flows* with velocity profiles of the type shown in Fig. 2.31c. In Chap. 2 it was indicated that the 'Gaussian' velocity profile (2.89) describes, accurately enough, the velocity distribution in the laminar wake behind a thin flat plate parallel to the free stream. Some remarks about the nonlinear instability of wakes behind flat plates will be made in the part (c) of this section, but most attention will be paid here to the most important and most widely studied *plane wakes behind long cylindrical bluff bodies* of constant cross-section, in uniform flows with free-stream velocity U_0. constant and normal to the body length. As in Sect. 2.93 it will be assumed below that the axis Oy is parallel to the cylinder axis (and defines the 'spanwise' direction), while the axis Ox is directed along the direction of the oncoming uniform flow and the midpoint of the body is chosen as the origin of coordinates.

Let us begin with the case of *circular-cylinder wakes,* while the wakes behind some other spanwise homogeneous bodies will be briefly considered later. It is well known that when Reynolds number $\mathrm{Re} = U_0D/\nu$ is gradually increased the flow around a circular cylinder of diameter D undergoes a whole series of remarkable transformations produced by a number of instability phenomena (see, e.g., Sects. 3.3 and 17.8 in the textbook by Tritton (1988), the survey by Coutanceau and Defaye (1991), the nice old survey paper by Morkovin (1964), and-for more details-the recent book by Zdravkovich (1997), vol. 1 of which (vol. 2 has not appeared at the time of writing) is about 700 s long and contains a huge bibliography which, however, does not intersect too much with that at the end of this chapter). In the present section devoted to Landau's equation, the first two transformations of the cylinder wake are the most interesting. The first of them takes place at $\mathrm{Re} \approx 4$ (this $\mathrm{Re}_{cr} = \mathrm{Re}_{0,\,cr}$ corresponds to the origin of linear instability of a laminar wake) and leads to a steady wake flow of a new type characterized by the appearance of the recirculation zone just behind the cylinder, the size of which slowly increased with Re and which consists of two symmetrical stationary vortices attached to the rear of the cylinder (for more data about this flow see, e.g., Coutanceau and Bouard (1977) or Zdravkovich (1997)). The second transformation leads to the formation at some $\mathrm{Re} = \mathrm{Re}_{1,\,cr}$ above 40 of the *von Kármán* (or, as it is also sometimes called, the *Bénard-von Kármán*) *vortex street*[10], consisting of a double row of opposing vortices, convected downstream and producing wake oscillations (see, e.g., the excellent Photos 94–98 in the album by Van Dyke (1982)). The appearance of the vortex street is due to the 'shedding' of vortices periodically torn away from the back of a cylinder with a frequency f coinciding with the frequency of the wake oscillations. The next transition to a three-

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[10] These names mark the contributions by Kármán (1911) (and Kármán and Rubach (1912)) and by Bénard (1908) to the investigation of this phenomenon. Note, however, that in fact the formation and subsequent 'shedding' of vortices behind bluff bodies was observed and repeatedly sketched by Leonardo da Vinci about the year 1,500 (one of his brilliant drawings opens Zdravkovich's book of 1997) and has been studied at least from the days of Strouhal (1878) who, in particular, first measured the frequency f of arising wake oscillations.
Experimental data show that critical Reynolds number $\mathrm{Re}_{1,cr}$ depends on the cylinder aspect ratio L/D (where L is the length of the cylinder) and boundary conditions at the cylinder ends; usually this number takes values between 40 and 50. It was however noted that under some special conditions a short vortex street (which is not stable and is wholly located in a region near the cylinder) can be excited at smaller values of Re between 22 and 40 (see, e.g., Plaschko et al. (1993)).

dimensional flow regime occurs usually at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} \approx 170$–190; it will be briefly considered at the end of the present part b of this subsection.

The Reynolds number $\mathrm{Re}_{1,\mathrm{cr}}$ (below it will often be simply denoted as $\mathrm{Re}_{\mathrm{cr}}$) is the threshold value for the appearance of instability of the steady wake flow arising at $\mathrm{Re} = \mathrm{Re}_{0,\mathrm{cr}}$, which leads to its transition to a new oscillating regime. Such a transition clearly represents a Hopf bifurcation. The corresponding value of $\mathrm{Re}_{\mathrm{cr}}$ was theoretically evaluated by a number of researchers-in particular, by Zebib (1987); Jackson (1987); Morzyński and Thiele (1991, 1992, 1993), and Noack and Eckelmann (1992, 1994a) whose results do not differ too muich from each other, from the available experimental data, or from estimates of this number given by numerical simulations. The methods used by by these authors were different from those described in Sects. 2.8 and 2.9, since here *non-parallel stability analysis* was used (i.e. the flow around the cylinder was not assumed to be plane-parallel). However, as a rule this flow was assumed to be two-dimensional (independent of the spanwise y coordinate) and was given as the steady solution of the two-dimensional Navier-Stokes equations satisfying the appropriate boundary conditions. The use of the two-dimensionality assumption clearly means that here only the centrial part of the wake behind a long cylinder with large enough value of L/D (where L is the cylinder length) is considered. As to the temporal development of the wake oscillations occurring at $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$, it was successfully described by Landau's equations in a number of papers which will be considered below. Note that in contrast to the above discussion of the cases of plane mixing layers and jets, these papers concentrated, not on the mathematical evaluation of the Landau coefficients for some given primary velocity profiles $U(z)$, but on the investigations of the disturbance development in real wake flows. Therefore below Landau's equations will not be applied to the idealized neutral or nearly-neutral normal modes, corresponding to points of the (k, Re)-plane neighboring the neutral curve, but to the most-unstable disturbances, which suppress all the others and play the dominant part in the observed disturbance development. This implies, in particular, that the coefficients of these equations will now depend on Re but not on k, since the value of Re uniquely determines the wave number of the most unstable wave disturbance.

Apparently Mathis (1983) and Mathis et al. (1984) where among the first experimenters to show that the 'shedding of vortices' and formation of the vortex street in a flow around a long circular cylinders represents a Hopf bifurcation which can be described by Landau's equation. Therefore the complex Landau Eq. (4.40). (which, as mentioned above, is also often called the Stuart–Landau equation), having the complex coefficients ω and l, was introduced here for the complex amplitude, $A(t) = |A(t)|\, e^{i\phi(t)}$, of the 'vertical' (i.e. 'transverse' or z-wise) velocity $w\,(t)$ of wake oscillations at a fixed point inside the wake (namely, at the point with coordinates ($5D$, 0, 0)). The complex equation for $A(t)$ was then replaced by two real equations for the functions $|A(t)|$ and $\phi(t)$ (both of which have been already given in Sect. 4.21):

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \delta|A|^4, \tag{4.34}$$

$$\frac{d\phi}{dt} = -\omega_1 - \tfrac{1}{2}\delta'|A|^2 \tag{4.34a}$$

where $\omega_1 + i\gamma = \omega$, $\delta + i\delta' = l$.

Laser-Doppler-anemometer measurements by Mathis, and Mathis et al., of the velocity w (t) in the wakes of a number of cylinders plaed in a wind-tunnel were made at various values of Re and confirmed that $\gamma \approx b(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ at small values of Re–$\mathrm{Re}_{\mathrm{cr}}$ where $\mathrm{Re}_{\mathrm{cr}} \approx 47$ and $b = \mathrm{const.} \approx \nu/5D^2$ if the aspect ratio L/D is large enough. (At small values of L/D, $\mathrm{Re}_{\mathrm{cr}}$ takes greater values—this observation by Mathis et al. agreed with results of some preceding experiments and later it was confirmed, in particular, by Lee and Budwig (1991) and Norberg (1994)). At the same time the coefficients δ, δ' and ω_1, in contrast to γ, do not vanish at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, and their values at small values of Re – $\mathrm{Re}_{\mathrm{cr}}$ may be approximated by two-term relations:

$$\begin{aligned} &\delta \approx \delta_0 + \delta_1(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \quad \delta' \approx \delta'_0 + \delta'_1(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \\ &\omega_1 \approx \omega_{10} + \omega_{11}(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \end{aligned} \tag{4.46}$$

where δ_0, δ'_0 and ω_{10} are the values of these coefficients at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, and δ_1, $\delta\,'_1$ and ω_{11} are their derivatives with respect to Re at this point. As has been already repeatedly noted above, it follows from Eq. (4.34) that if $\delta > 0$, then a Hopf bifurcation of the disturbed flow occurs at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ and, at slightly supercritical conditions (i.e., when $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ but Re – $\mathrm{Re}_{\mathrm{cr}}$ is small), a small initial disturbance tends to a equilibrium state with the amplitude $A_e = (2\gamma/\delta)^{1/2} \approx (2b/\delta_0)^{1/2}$ $(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{1/2}$. The existence of equilibrium amplitude A_e in supercritical wake flows was confirmed by the experimental data of Mathis et al. (and of many other authors); thus, the data definitely show that $\delta > 0$ in the case of the most unstable disturbance in the wake behind a circular cylinder. The data show also that the relation $A_e \propto (\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{1/2}$, which corresponds to the first term of the Taylor-series expansion of $(2\gamma/\delta^{1/2})$ in powers of Re – $\mathrm{Re}_{\mathrm{cr}}$, is valid even when Re – $\mathrm{Re}_{\mathrm{cr}}$ is not too small. Hence the derivative δ_1 is rather small in absolute value and may usually be neglected. (The same conclusion follows from the validity of the relation (4.47), below, over a wide range of Reynolds numbers). Moreover, measurements of the values of δ/δ' at various Reynolds numbers, which will be described below (see, in particular, Fig. 4.19c) show that this quantity also is independent of Re over a considerably range of supercritical Reynolds numbers. Hence the derivative δ'_1 may also be neglected, and both coefficients δ and δ' may be considered as being independent of Re.

Note that $(1/2\pi)(d\phi/dt) = f$ is just the local frequency of oscillations of the wake amplitude $A(t)$, while peak-to-peak value $2|A|$ of these oscillations is equal to the double equilibrium $2A_e$. Hence Eqs. (4.34) and (4.34a) (the first of which determines the value of A_e) together with the equation $\gamma = b(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ imply the equation

$$\mathrm{Ro} = a\mathrm{Re} - a_1, \tag{4.47}$$

Where $\mathrm{Ro} = fD^2/\nu$ is the so-called *Roshko number*, $a = -[(b(\delta'/\delta) - \omega_{11}]D^2/2\pi\nu$, and $a_1 = \{\omega_{10} - [b(\delta'/\delta) - \omega_{11}]\mathrm{Re}_{\mathrm{cr}}\}D^2/2\pi\nu$. The dimensionless quantity

fD^2/ν was introduced by Roshko (1953, 1954), who also showed that over a wide range of Reynolds numbers its dependence of $\mathrm{Re} = U_0 D/\nu$ is given by an equation of the form (4.47) with constant coefficients a and a_1. Therefore Eq. (4.47) is often called the *Roshko equations* though in fact the same equation, written in the form

$$St = a - a_1/\,\mathrm{Re}, \tag{4.47a}$$

where $\mathrm{St} = fD/U_0 = \mathrm{Ro}/\mathrm{Re}$ is the so-called *Strouhal numbers* was employed by Rayleigh (1915) (see also Rott (1992) and Williamson (1995, 1996a)). Thus, the empirical 'Ro-Re' and 'St-Re' relations (4.47–4.47a) are fully compatible with Landau's equation.

The experimental data by Mathins et al. (1984), for cylinders with not too small values of the aspect ratios L/D, agreed with Roshko's eqution (4.47) with constant coefficients a and a_1 only in a limited range of Reynolds numbers from $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}} \approx 47$ to $\mathrm{Re} \approx 90$. When the value of Re was increased further, the character of the wake oscillations changed discontinuously and then the values of a and a_1 also changed. Mathis et al. noted that abrupt changes of the regime of wake oscillations found by them agree with earlier results of Tritton (1959, 1971) and Gaster (1971). Later the nature of these changes, their dependence on the value of L/D and on the end conditions at $y = \pm L/2$, and possible methods for getting rid of the changes were discussed by a number of authors; see, eg., Slaouti and Gerrard (1981); Lee and Budwig (1991); Szepessy (1993), and the subsequent discussion of this topic after Eq. (4.49) where additional references will be given.

More detailed experimental studies of disturbance behavior in wakes behind circular cylinders were later carried out both by the group with which Mathis collaborated (see Provansal et al. (1987); Provansal (1988)) and by some other researchers (see, e.g., Strykovski (1986), whose dissertation covered much the same ground as that of Mathis (1983); Sreenivasan et al. (1987); Strykovski and Sreenivasan (1990); Schumm (1991); Schumm et al. (1994); Park (1994), and the survey by Monkewitz (1996)). These authors also based their studies on the Landau model and performed a number of careful measurements which allowed them to determine, at some points of the cylinder wake, the values of all coefficients of Landau's Eqs. (4.34) and (4.34a) at various values of Re. These determinations used methods of wake control allowing the wake oscillations (always existing if $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$) to be switched off (completely or partially) and then switched on again very rapidly. Observing, at different values of Re, the rate of growth with time of the amplitude $|A(t)|$ of the disturbance velocity from the initial small value to the final equilibrium value A_e, one may find the coefficients γ and δ of Eq. (4.34) and their dependence on Re. In this way it was found that δ is usually independent of Re, while γ satisfies the relation $\gamma = b(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ where the values of $\mathrm{Re}_{\mathrm{cr}}$ and b can also be determined from experimental data. Moreover, measurements of the frequency f of equilibrium wake oscillations at various Reynolds number determined the dependence of $\mathrm{Ro} = fD^2/\nu$ on Re, verified the Roshko Eq. (4.47) and gave the values of coefficients a and a_1. Using these values, and also the values of δ, b and $\mathrm{Re}_{\mathrm{cr}}$ given by the results of amplitude measurements, one may also determine the values of $b\delta'/\delta - \omega_{11}$ and ω_{10}. On the other hand, one may observe, at various values of Re, the increase with time of the frequency f of wake oscillations from the moment of their switching on (when $|A| = 0$ and hence $f = -\omega_1/2\pi$) to the final equilibrium conditions (when $|A| = A_e$). Such observations make it possible to determine the dependence of ω_1 on Re (and the values of coefficients ω_{10} and ω_{11}) and to chek the value of Re_{cr} already found. When this is done, the coefficients a and a_1 may be computed a new, to compare their new values with those implied by the experimental verification of the Rayleigh-Roshko laws (4.47) and (4.47a).

The method of control used by Strykovski, and Sreenivasan et al. (and also by Mathis, and Mathis et al). consisted of the quick reduction, perhaps to zero, of the free-stream velocity, with a subsequent quick return to its initial value U_0 (corresponding to given $\text{Re} > \text{Re}_{\text{cr}}$). Schumm, Schumm et al., and Park, also employed several other control methods such as bleeding of fluid from the rear part of the cylinder, wake heating, or forced vertical vibrations of a cylinder with a small amplitude $a_0 \ll D$. (All these operations at supercritical $\text{Re} > \text{Re}_{\text{cr}}$ strongly suppress vortex shedding; see, e.g., Monkewitz's surveys (1993, 1996) and the papers on wake control by Roussopoulos (1993); Schumm et al. (1994); Park et al. (1993, 1994); Park (1994); Roussopoulos and Monkewitz (1996); Gunzburger and Lee (1996), and Gillies (1998) containing many additional references). However, the above-mentioned control methods are applicable only at supercritical Reynolds numbers and can provide no information about the values of coefficients of Eqs. (4.34–4.34a) at $\text{Re} < \text{Re}_{\text{cr}}$. To obtain such information Sreenivasan et al., Schumm et al., and Park used some methods of 'subcritical wake control', i.e. of artificial forcing of the vortex shedding and wake oscillations of the appropriate frequency at subcritical conditions characterized by the given value of Re which is smaller than Re_{cr}. Applying this forcing, and then switching it off rapidly and observing the subsequent damping of oscillations, one may obtain data relating to values of the Landau coefficients at subcrtical Reynolds numbers.

Sreenivasan et al. (1987) measured (by both hot-wire and laser-Doppler anemometers) wake velocity fluctuations behind the central parts of three cylinders with aspect ratios $L/D = 60$, 27 and 14 at several values of x/D and z/D and values of Re in the range $35 < \text{Re} < 100$. They found (as Mathis et al. did earlier) that the characteristics of wake oscillations vary (though not too much) with the cylinder aspect ratio, and most attention was paid to the case where $L/D = 60$, in the hope that the results would also be representative of greater values of L/D. It is natural to think that the complex constant $\omega = \omega_1 + i\gamma$ is simply the most unstable eigenvalue (i.e., that having the greatest imaginary part) of the Orr-Sommerfeld equation corresponding to the plane-parallel model of the wake velocity profile. If so, then this constant is a *global stability characteristic* which does not depend on the point in the wake at which observation is carried out (see the closing paragraph in Sect. 2.93, and the supplementary discussion of this topic at the beginning of the subsequent small-type text). However, the constants δ and δ' are apparently position-dependent and depend also on the choice of the measured flow characteristic and the definition of the amplitude A. (However Sreenivasan et al. found that the in the range $3 < x/D < 7$ the spatial variations of these constants are small and may be neglected). As to the ratio δ'/δ, it affects the values of coefficients a and a_1 of the Roshko equation and hence must be independent of both the point of observation and the value of the Reynolds number. According to measurements by Sreenivasan et al., $\mathrm{Re}_{cr} \approx 46$ in the wake behind a circular cylinder with $L/D > 60$ and

$$\gamma D^2/\nu \approx 0.20(\mathrm{Re} - \mathrm{Re}_{cr}), \quad \omega_1 D^2/\nu \approx -34.3 - 0.7(R - \mathrm{Re}_{cr}), \tag{4.48a}$$

$$\delta D^2/\nu \approx 134, \quad \delta' D^2/\nu = -404 \tag{4.48b}$$

(so that $\mathrm{Ro}_{cr} = (-\omega_1 D^2/2\pi\nu)_{cr} \approx 5.45, \delta'/\delta \approx -2.90$). Note that Sreenivasan et al., who did not know about the work of Mathis (1983) and Mathis et al. (1984), found exactly the same dependence of $\gamma \mathrm{D}^2/\nu$ on $\mathrm{Re} - \mathrm{Re}_{cr}$ as the latter authors and nearly the same value of Re_{cr}. Results of more numerous and careful measurements by Schumm (1991) and Schumm et al. (1994), who investigated wakes behind several circular cylinders with $L/D \geq 50$ and applied several different methods of wake control, prove to be very close to that found by Sreenivasan et al.: according to Schumm et al.

$$\mathrm{Re}_{cr} = 46.7 \pm 0.3, \quad \gamma D^2/\nu = [0.21 \pm 0.005](\mathrm{Re} - \mathrm{Re}_{cr}), \tag{4.49a}$$

$$\frac{\omega_1 D^2}{\nu} = -[33.6 \pm 0.3] - [0.64 \pm 0.02)(\mathrm{Re} - \mathrm{Re}_{cr}), \quad \frac{\delta'}{\delta} = -[2.90 \pm 0.45] \tag{4.49b}$$

(see Fig. 4.19 where results of their measurements of coefficients of Eqs. (4.34) and (4.34a) at different values of $\mathrm{Re} = U_0 D/\nu$ are shown). Close results were obtained also by some other researchers; for example, Albarède and Monkewitz (1992) came to the conclusion that $\delta'/\delta = -3 \pm 0.6$, while numerical simulations of wake flows by Dusék et al. (1994) led to the estimate $\delta'/\delta \approx -2.7$, and according to laboratory measurements by Albarède and Provansal (1995) $\delta'/\delta = -2.6 \pm 0.7$.

Sreenivasan et al. (1987) noted that the values of coefficients a and a_1, implied by their estimate of Re_{cr} and of the Landau coefficients (4.48), do not differ too much from empirical values of a and a_1 recommended by Roshko (1954), while Monkewitz (1996) more methodically compared values of a and a_1 given by estimates (4.49) with values which agree best with empirical St-Re relations. Such a comparison is not an easy matter, since the Strouhal number in a cylinder wake depends on a number of factors. As was shown by Gerrard (1978) and Williamson (1989, 1995, 1996a), the empirical forms on the St-Re relation for the wakes behind circular cylinders, collected over a period of more than one hundred years (beginning with the frequency data of Strouhal (1878)), are very scattered. This scatter evidently cannot be explained by errors of measurements since both Re and St numbers can be easily measured with a high accuracy. (Oscillations of two-dimensional wakes have the unique frequency $f = f(\mathrm{Re})$ coinciding with the frequency of vortex shedding; in the case of three-dimensional wakes, several discrete oscillation frequencies and even the continuous frequency spectrum often exists, but at small and moderate values of Re here too the unique dominant frequency f can be measured accurately by means of numerical or instrumental spectral analysis). Therefore, the scatter must have another explanation. Recall now that, according to the above discussion of the experimental data by Mathis et al. (1984), the values of a and a_1, which correspond to these data depend on both the cylinder aspect ratio L/D and the range of Reynolds numbers considered, and this dependence was also found to be in agreement with results of some earlier observations of wakes behind circular cylinders. Let us add to this that more recent experimental and numerically-simulated data both show that the character of the vortex street behind a cylinder strongly depends on the boundary conditions at the cylinder's ends, and that usually the ordinary 'parallel shedding' is replaced at some $\mathrm{Re}_s > \mathrm{Re}_{cr}$ by 'oblique shedding' at some angle θ to the cylinder axis (see the papers mentioned at the end of the first new paragraph after Eq. (4.47a), and the papers by Williamson (1988a, 1989, 1995, 1996a); Norberg (1994); Persillon and Braza (1998) and references therein). The data show also that the frequency of wake oscillation f and the Roshko and Strouhal numbers Ro and St, which are proportional to it, in the case of oblique shedding depend on the 'shedding angle' θ. It is clear that such dependence must affect the Ro-Re and St-Re relations violating their universality. Moreover, the data presented in the above-mentioned papers (and in those by Williamson (1988b, 1996b, c); Coutanceau and Defaye (1991); Konig et al. (1990, 1992, 1993); Hammache and Gharib (1993); Brede et al. (1994); Zhang et al. (1995); Thompson et al. (1996); Henderson (1997), and Leweke and Williamson (1998), among many others) show that at some greater value of Re the primary mode of 'oblique shedding' is replaced by another three-dimensional mode, which in its turn can be replaced by a more complicated flow regime in the range of still greater Re numbers.

In the late 1980s and 1990s it was also proved that at moderate values of Re the oblique shedding is always due to 'end effects' caused by finite length L of a cylinder and that the shedding angle θ depends on the spanwise boundary condition at cylinder ends which play a very important part even at large aspect ratios $L/D > 100$. This dependence allows the value of θ to be changed by a proper modification of either the cylinder end conditions (dependent on the method of supporting the cylinder) or the flow near the cylinder ends. Therefore one may pass to the parallel regime of vortex shedding by appropriate change of flow configuration near the cylinder ends. In particular, it was found that the parallel regime may be caused by small increase of the undisturbed velocity $U_0 = U_0(y)$ near $y = \pm L/2$, or by suction of small amounts of fluid from just downstream of the ends of a cylinder, whereas without any manipulation affecting boundary conditions, parallel shedding at relatively large values of Re can be attained only at an aspect ratio wellover 1000 (see again the papers referred above and also those by Eisenlohr and Eckelmann (1989); Hammache and Gharib (1989, 1991); Albarède and Monkewitz (1992); Norberg (1994); Miller and Williamson (1994); Monkewitz (1996), and Monkewitz et al. (1996)).

Since the changes of boundary conditions may often be achieved in laboratory experiments by means of some simple mechanical devices, and can also be easily incorporated in numerical simulations, a number of high-quality frequency determinations was carried out during the last decade, in circular-cylinder wakes near mid-span, under conditions guaranteeing the regime of 'parallel vortex shedding'. The results obtained in numerous experiments were collected by Williamson (1988a,

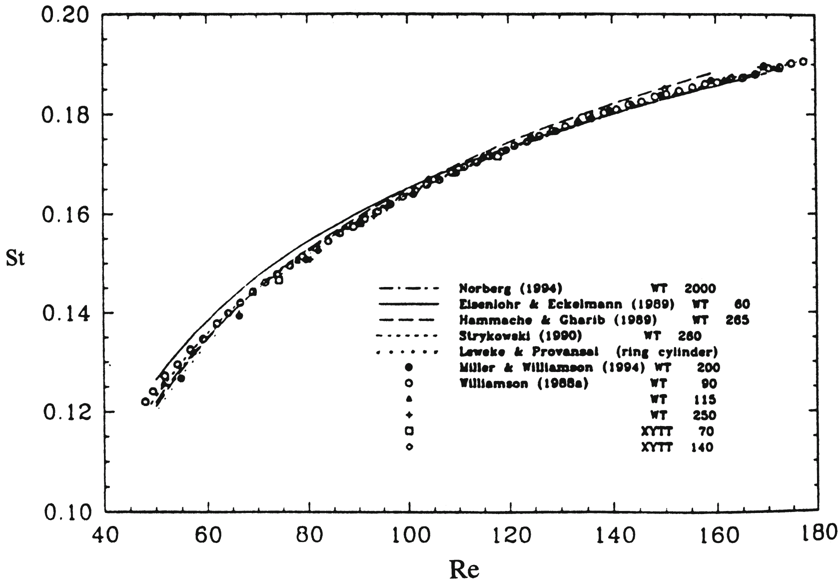


Fig. 4.20 'Universal St-Re relation' found by Williamson for the case of purely parallel vortex-shedding regime of the wake behind a circular cylinder at moderate Reynolds numbers, and its comparison with available experimental data. (After Williamson (1996a)). Various *symbols* represents experimental data by Williamson and his coworkers, various curves–the data of other authors; *WT*: wind-tunnel data; *XYTT*: results obtained in a special water tank facility. The numbers after the facilty marks indicated the cylinder aspect ratios *L*/*D*

1989, 1995, 1996a) in the form of 'universal St-Re relation' shown in Fig. 4.20. The data in this figure include the measurements made both in wind tunnels and water-tank facilities, by a number of different techniques, and covering the Re-range from $\mathrm{Re}_{\mathrm{cr}} \approx 50$ and to Re$\approx$ 180. Very similar results were also found in numerical simulations of the cylinder wake by Karniadakis and Triantafyllou (1989); Thompson et al. (1996); Persillon and Braza (1998), and some others. Moreover, Prasad and Willaimson (1997) showed also that, by the appropriate adjustment of boundary conditions at cylinder ends, one can make vortex shedding parallel also in the case of wakes characterized by high Reynolds numbers much exceeding the values considered in Fig. 4.20. However, in this case the parallel-shedding flow regime quickly becomes three-dimensional and its St-Re relation is no longer universal (this matter will be discussed at greater length at the end of this part of Sect. 4.24).

According to Williamson the empirical St-Re relation shown in Fig. 4.20 may be best approximated by a three-term equation of the form $\mathrm{St} = a - a_1/\mathrm{Re} + a_2\mathrm{Re}$ (where $a = 0.1816$, $a_1 = 3.3265$, and $a_2 = 1.6 \times 10^{-4}$). However, Monkewitz (1996) found that two-term approximation (4.47a), with coefficients $a = 0.199$ and $a_1 = 3.94$, which corresponds to estimates (4.49), in indistinguishable at small and moderate values of $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ from approximation suggested by Williamson, and only at Re$\approx$ 100 this two-term equation leads to results which fall slightly below those giving by Williamson's approximation. Thus, we must conclude tha the Landau model gives quite a good description of the data relating to wake oscillations generated by 'parallel shedding' under the conditions of small and moderate (but not great) supercriticality, and that the empirical estimates (4.49) give, with good accuracy, the values of coefficients of the corresponding complex Landau equation. In fact, it is quite surprising that Landau's equation, with coefficients computed under that conditions that $\mathrm{Re} - \mathrm{Re}_{cr}$ is small, leads to results which agree so well with experimental data for $\mathrm{Re}/\mathrm{Re}_{cr}$ up to 1.5.

Note that time-amplified global oscillations of the entire near wake are intimately connected with local *absolute* (in contrary to *convective*, see the closing part of Sect. 2.93) instability of the wake flow. In fact, the wake flow is not strictly plane-parallel, and hence its local velocity profile, and the Orr-Sommerfeld eigenvalues depending on it, vary slowly with the streamwise coordinate x. Hence, the local values of all coefficients of the complex Landau (i.e., Stuart-Landau) Eq. (4.40) here on x (and this dependence becomes more significant with the increase of non-parallelism of the flow). This means, in particular, that the local oscillation frequency $f = -\omega_1/2\pi$ slowly changes with increase of distance from the cylinder. However the observation definitely show that the near wake, having a considerable streamwise extent, usually oscillates as a whole with constant frequency f, somehow selected from the collection of weakly varying local values. Such 'oscillation as a whole' characterizes the *global instability mode*, which occurs in the wake behind a solid body only in the cases where a considerable regions of the absolute flow instability exists near a body. Thus, one may say that the *Bénard-von Kármán vortex street is due to the absolute instability of the flow in the near wake.* Just this circumstance stimulated numerical investigations of wake regions of local absolute instability, typified by the papers of Koch (1985); Huerre and Monkewitz (1985); Monkewitz and Nguen (1987); Monkewitz (1988b, c); Yang and Zebib (1989); Hannemann and Oertel (1989), and Delbende and Chomaz (1998). The complex amplitude of global wake oscillations can depend on the spatial coordinates (on x and z in the case of a two-dimensional vortex street, and on three coordinates in more complicated cases) but its dependence on t in the case of a non-steady regime of global mode development will satisfy Landau's equation with the same coefficient $\omega = \omega_1 + i\gamma$ at all points $\mathbf{x}$. The data relating to the spatial distribution of the oscillation amplitudes $A(\mathbf{x}, t)$ will be considered at greater length below, for more details and additional references concerning the general properties of the global instability modes of nearly plane-parallel flows see, e.g., the papers by Triantafyllou et al. (1987); Karniadakis and Triantafyllou (1989); Huerre and Monkewitz (1990); Monkewitz (1990, 1996); Chomaz et al. (1991); Monkewitz et al. (1993); Le Dizès (1994), and Le Dizès et al. (1996).

As to the problem of 'oblique shedding', williamson (1988a, 1989, 1995, 1996a) showed that, in the cases where the 'shedding angle' θ is fixed, the 'universal St-Re relation' of Fig. 4.20, which corresponds to parallel shedding, is valid with good accuracy for 'modified Strouhal number' $\mathrm{St_m} = \mathrm{St}/\cos\theta$. This Williamson's 'cosine law' of oblique vortex shedding was confirmed in a number of experimental papers (see, e.g., König et al. (1993); Miller and Williamson (1994), and Monkewitz et al. (1996)) but its theoretical explanation requires the use of some special analytical techniques. Since it was shown that 'oblique shedding' is strongly affected by the 'spanwise boundary conditions' at $y = \pm L/2$, the 'cosine law' can be derived theoretically only from a model which takes into account the influence of the flow configuration near the cylinder ends on the oscillations of the middle part of the wake. This simplest way to achieve this is to introduce a y-dependent oscillation amplitude $A\ (y, t)$ and replace the complex Landau Eq. (4.40) by the more general *complex Ginzburg–Landau* (G–L) *equation* for this amplitude, having the form

$$\frac{\partial A}{\partial t} = -i\omega A + \mu \frac{\partial^2 A}{\partial y^2} - \frac{1}{2} l |A|^2 A \tag{4.50}$$

where ω, μ, and l are three complex coefficients and the second term on the right-hand side describes the spanwise diffusion of oscillations. (For more information about this equation see, e.g., the extensive survey by Cross and Hohenberg (1993) containing a comprehensive bibliography, Chap. 5 of the book of Bohr et al. (1998), the paper by van Saarloos (1995) and other papers in Cladis and Palffy-Muhoray (1995) where a number of modifications, generalizations, and various applications of Eq. (4.50) are collected. A typical example of the useful generalization of Eq. (4.50) is provided by the 'quintic G-L equation' containing an additional term proportional to $|A|^4 A$; this equation was used, in particular, by Shtemler (1978) and Bottin and Lega (1998), who applied it to stability studies relating to plane Poiseuille and Couette flows, and by Iwasaki and Toh (1992), who based on this equation their model description of turbulence structures at high Reynolds numbers). Equation. (4.50) and some other related nonlinear model equations were applied to description of the spanwise-varying cylinder wakes, in particular, by Albarède et al. (1990); Albarède (1991); Noack et al. (1991); Park and Redekopp (1992); Albarède and Monkewitz (1992); Triantafyllou (1992); Chiffaudel (1992); Albarède and Provansal (1995), and Monkewitz et al. (1996). Models by Albarède and Monkewitz, Triantafyllou, Monkewitz et al., and some others lead to results which explain the approximate validity of the 'cosine law'. However, this was not the primary purpose of introduction of these models.

The point is that according to available experimental data of a number of authors (e.g., of Williamson (1988a, 1989, 1992, 1995, 1996a, b); Ohle and Eckelmann (1992); König et al. (1992, 1993); Brede et al. (1994), and Miller and Williamson (1994)), wakes behnd circular cylinders at relatively low Reynolds numbers often have rather complicated spanwise structure. It was found, in particular, that at moderately subcritical values of Re spanwise cell structures frequently appear in such wakes, i.e., several spanwise regions with constant shedding frequency are formed which are separated by the so-called 'nodes' where the frequency changes discontinuously and vortex dislocation is observed. In the cases of 'perfectly symmetric' boundary conditions at the two ends of the cylinder and at large values of Re (and sometimes at relatively small Re but not too small values of x), symmetrical V-shaped (downstream-pointing) 'chevron' structures are often observed, i.e., the vortices on both sides of the cylidner midpiot have shedding angles of equal magnitude but opposite sign. The search for an explanation of these strange features of the observed wakes behind circular cylinders stimulated the introduction of the G–L model (4.50) and its investigation by Albarède et al. (1990); Albarède and Monkewitz (1992); Albarède and Provansal (1995), and Monkewitz et al. (1996) (see also Monkewitz's survey (1996)).

The G-L model can in principle describe the influence of the end conditions on the angle of obliquie shedding and explain the experimental result that the oblique shedding can be converted back into the parallel shedding by changing the flow configuration near the cylinder ends. However, to derive even qualitative conclusions from the G-L model, it is necessary first of all to determine the values of all the coefficients of Eq. (4.50). Since the complex coefficients ω and l have the same meaning here as in Eqs. (4.40) and (4.34–4.34a), it seems natural to make, as a first approximation, the assumption that these two coefficients of Eq. (4.50) do not depend on y, and have the same values as in the case of strictly parallel vortex shedding where oscillations are spanwise homogeneous. This simplifying assumption was accepted in the above-mentioned papers, where the empirical estimates of $\omega = \omega_1 + i\gamma$ and $l = \delta + i\delta$ quite close to the above estimates (4.48) and (4.49) were used. However the third coefficient $\mu = \mu_r + i\mu_1$ of Eq. (4.50) is a new one, and it can be determined only from data of measurements relating to the dependence of cylinder wakes on the spanwise end conditions.

Albarède and Monkewitz (1992) tried to use for this purpose the data for the dependence of Re_{cr} on the aspect ratio L/D of the cylinder generating the wake. If the oscillation amplitude A depends on y and satisfies Eq. (4.50), then the growth of A from the initial infinitesimal value will be described, not by the linearized Landau Eq. (4.32), but by the linearized G-L equation, which differs from Eq. (4.50) by the absence of the cubic term on the right-hand side. Also the measured rate of amplitude growth at $Re > Re_{cr}$ must evidently be equal, in this case, to the rate of growth of the most unstable spanwise-inhomogeneous mode. The normal modes are now given by the eigenfunctions of the linearized Eq. (4.50), which depend on the boundary conditions at $y = \pm L/2$. However, it seemed natural to assume that, at large values of L/D, the boundary conditions will not very essentially affect the rate of growth of normal modes. Therefore Albarède and Monkewitz used the simplest boundary conditions $A\ (y, t) = 0$ at $y = \pm L/2$, hoping that their use could hardly lead to very significant errors. The above arguments allow Re_{cr} to be determined approximately, as the smallest value of Re at which the imaginary part of at least one eigenvalue of the linearized G–L equation is not negative but equal to zero. Re_{cr} clearly depends on the aspect ratio L/D and of μ (recall that ω and l are assumed known); hence $Re_{cr} = Re_{cr}\ (L/D, \mu)$. Therefore, the measured values of Re_{cr} at various values of L/D may be used for estimation of the value of μ.

Albarède and Monkewitz at first attempted to use the results of the measurements by Mathis et al. of the values of Re_{cr} at a number of values of L/D but found that their data were insufficiently accurate eand complete. Therefore they carried out additional careful measurements of the values of Re_{cr} at various aspect ratios L/D and the results led them to the conculsion that $\mu_r/\nu = 32 \pm 6$. To find the imaginary part μ_i of the complex coefficient μ, two different methods were used by Albarède and Monkewtiz, both based on data for the angular frequency ω_1 of the most unstable mode at different values of Re and L/D. The two methods led to not-too-different results, and showed that apparently $(\mu_i/\mu_r) = -0.3 \pm 0.6$. Later Albarède and Provansal (1995) arranged a more careful determination of the values of the various coefficients of Eq. (4.50) (first of all of μ_r). They used somewhat modified boundary conditions, and carried out more complete and accurate measurements of the dependence of characteristics of steady cylinder wakes on Re and L/D. As a result they obtained the new estimate $\mu_r/\nu = 10 \pm 4$ for Re < 100, which differs considerably from the preceding estimate by Albarède and Monkewitz. (This great difference was apparently mainly due to the change of boundary conditions, which were found to be more important than it was assumed earlier). The value of μ_i was unimportant for the majority of applications considered by Albarède and Provansal; in rare cases where it was needed they used the estimate by Albarède and Monkewitz.

A quite different method of determining the values of μ_r and μ_i was used by Monkewitz et al. (1996). Here, special experiments were arranged in which nonsymmetric time-dependent boundary conditions were realized at the cylinder ends. The coefficients of the G-L model were then determined from both the steady shedding data (the only data used previously) and the data of measurements of the 'spanwise wave number shocks', i.e. abrupt increases in shedding angle across the span of a cylinder initiated by appropriate impulsive changes of ends conditions. The observed gradual reduction of the shedding angle θ along the Oy axis was then compared with predictions of the G-L model. Under the condition that the G-L model with coefficients independent of y is valid, this comparison allowed the values of μ_r/ν and μ_i/μ_r to be determined with considerably greater accuracy than was achieved in the previous investigations. Monkewitz et al. published the results obtained for Re = 100, 120 and 140; the values of μ_i/μ_r proved to be practically independent of Re and close to -1, while all values of μ_r/ν were found to be fairly close to 20, growing slightly with Re (from 18.7 at Re = 100 to 25.6 at Re = 140).

Albarède and Monkewitz (1992) found that their version of the G-L model describes, quite well, many phenomena observed in cylinder wakes in the laboratory. The model led to correct dependence of Re_{cr} on L/D and showed, in full agreement with the experimental data, that after the impulsive switching on of an external stream of constant velocity, vortex shedding always starts as the parallel mode while the regions of 'oblique shedding' develop from the cylinder ends and, in the case of symmetric end conditions, lead to steady-state 'chevron patterns'. The possibility of forcing the transition from the 'oblique' to 'parallel' vortex shedding by means of change of flow configuration at the cylinder ends can also be derived from the G-L model considered. Moreover, the plan views (in the (x, y)-plane) of cylinder wakes observed in flow visualizations agree well with results of model computations. Albarède and Provansal (1995) showed that their improvements of the previous version of the G-L model gives a theoretical explanation of a number of even more subtle features of wake development. In addition to this, Monkewitz et al. (1996) demonstrated that the samc G-L model satisfactorily describes many surprising non-steady wake phenomena which can be produced in laboratory experiments where non-symmetric, impulsive (i.e., time-dependent) spanwise boundary conditions are realized. Note however the remark by Leweke and Williamson (1998) indicating that the explanation of the loss of stability of a two-dimensional cylinder wake at supercritical values of Re proposed by Leweke and Provansal (1995), which was based on the G-L model, disagrees with some known properties of the observed cylinder-wake instability. On the other hand, while Landau's Eqs. (4.34) and (4.40) were derived from Navier–Stokes equations as long ago as the early 1960s by Stuart, Watson, and Eckhaus (and then more thoroughly by Fujimura (1989) and Duśek et al. (1994)), who used for this purpose definite asymptotic expansion procedures (see Sect. 4.21 above), apparently no rigorous derivation of this type has yet been given for the G-L Eq. (4.50) (the references of the G-L equations at the end of Sect. 4.22 concerned quite different flows and other equations of the Ginzburg-Landau type). Thus, the problem of the strict derivation of this equation and the accurate determination of conditions for its validity remains unsolved.

Equation (4.50) is the 'transverse' Ginzburg-Landau equation, taking into account the spanwise 'diffusion' of wake oscillations which often becomes apparent in laboratory experiments and numerical simulations. As to the spatial development of these oscillations, it was always neglected above, i.e., it was assumed that none of their characteristics depends on the streamwise coordinate x. This assumption was based mainly on the fact that, according to the available wake observations, the oscillation frequency f is practically the same within a large spatial region, as it must be in the case of a global instability mode. However visualisations of wake flows clearly show that some local characteristics of the oscillations vary considerably when coordinates of the observation point are changed. In particular, it will be explained below that the local oscillation amplitude at the point $(x, 0, 0)$ first gorws with the value of x but then reaches a maximum and begins to decrease when x increases further. Recall that when discussing the experiments by Sreenivasan et al. (1987) we noted (just above Eq. (4.48)) that the assumption about complete streamwise homogeneity of oscillations is just a convenient simplification, applicable only to regions of short streamwise extent.

To take into account the possible dependence of wake oscillations on the streamwise coordinate x one must use some new analytical models differing from the Landau and transverse Ginzburg-Landau models (4.40) and (4.50) by the presence of terms describing the streamwise variability of the flow characteristics. One of the simplest methods of accounting for the streamwise variability is to replace the Stuart–Landau Eq. (4.40) by the 'longitudinal' Ginzburd-Landau equation for the streamwise-dependent oscillation amplitude $A(x, t)$. The simplest version of this G-L equation includes, instead of the transverse-diffusion term of Eq. (4.50), a streamwise-diffusion term proportional to $\partial^2 A/\partial x^2$. Then the streamwise advection may be taken into account by the inclusion of the term $U\partial A/\partial x$ on the left-hand side of th G-L equation and/or by the replacement of the simple second derivative $\partial^2 A/\partial x^2$ on the right-hand side by $\partial^2 A/\partial \xi^2$, where $\xi = x - Ut$. As was indicated at the end of Sect. 4.22, the longitudinal G-L equation has definite theoretical grounds, and it has been repeatedly used in studies of the weakly nonlinear instability of plane-parallel and nearly plane-parallel flows. Some attempts to apply the longitudinal G-L equation to the study of plane wake flows were briefly considered by Park and Redekopp (1992) (in the initial part of their paper), Le Dizès et al. (1993, 1996) and

Hunt (1993). In additional, Park et al. (1993) used the longitudinal G-L equation for the quantitative analysis of control methods for a two-dimensional x-dependent global mode of circular-cylinder wake oscillations, and Xiao et al. (1998) briefly outlined a new application of the longitudinal G-L model to development of control methods regulating the value of the amplitude $A\ (x,\ t)$.

A more complete two-dimensional Ginzburg–Landau equation for anoscillation amplitude $A = A\ (x,\ y,\ t)$ dependent on two spatial coordinates was applied to wake flows by Park and Redekopp (1992) and Chiffaudel (1992), and Roussopoulos and Monkewitz (1996). Park and Redekopp considered the G–L equation of the form

$$\frac{\partial A}{\partial t} + U\frac{\partial A}{\partial x} = -i\omega A + \mu_1\frac{\partial^2 A}{\partial x^2} + \mu_2\frac{\partial^2 A}{\partial y^2} - \frac{1}{2}l|A|^2 A, \tag{4.51}$$

while Chiffaudel used a more complicated model equation which included also the third- and fourth-order derivatives of the amplitude. (The fourth-order G-L equation, containing amplitude derivatives up to the fourth order, was also studied by Raitt and Riecke (1995); however, this model will not be considered in the present book). Generally speaking all four coefficient ω, μ_1, μ_2 and l of Eq. (4.51) can be complex and dependent on the two coordinates x and y (and the real advection velocity U can also depend on x and y), but Park and Redekoppp restricted themselves to the model where $U = \text{const.}$, μ_1, μ_2 and l are complex constants while $\omega(x, y) = i\gamma(x, y)$ is purely imaginary (i.e., γ is real) and has the form $\omega(x, y) = i[c_0(y) - c_1(y)x]$ where c_0 and c_1 are real functions of one variable and $c_1(y) < 0$ at any y. Analyzing solutions of Eq. (4.51) in the region $0 \le x < \infty, -L/2 \le y \le L/2$, under the boundary conditions $A(0, y, t) = 0, A(x, -L/2, t) = F_1(x, t), A(x, L/2, t) = F_2(y, t)$, and choosing reasonable values of constants μ_1, μ_2 and l and functions c_1, c_2, F_1 and F_2, the authors determined the $(x,\ y)$-region of the absolute instability of the wake flow considered, and showed that many observed features of the spatial and temporal development of circular-cylinder wake oscillations (e.g., the observed interrelation of parallel and oblique sheddings and formation of 'chevron patterns') can be explained if one assumes that oscillation amplitude satisfies Eq. (4.51). Roussopoulos and Monkewitz, who studied the feedback control of oblique vortex shedding for Reynolds numbers close to $\mathrm{Re}_{\mathrm{cr}}$ considered another model: they assumed that the oscillation amplitude $A_1\ (x,\ y,\ z,\ t)$ can be represented as a product $A\ (x,\ y,\ t)\ B(z)$ where $A\ (x,\ y,\ t)$ satisfies the G-L Eq. (4.51) in which $U = U\ (x)$ depends linearly on x, $\omega = \omega(x)$ is a complex function quadratic in x, and μ_1, μ_2 and l are complex constants. Then the authors used the results of the stability theory for circular-cylinder wakes and the data of wake oscillation measurements presented in Monkewitz's paper (1988b) to evaluate approximately all coefficients of Eq. (4.51). To apply the G-L amplitude equation to description of wake-oscilltion control methods, Roussopoulos and Monkewitz added to the right-hand side of Eq. (4.51) a function $F(x,\ y,\ t)$ representing the effect of the feedback control. Then solving numerically the obtained equation under the appropriate initial and boundary conditions and varying the values of $F(x,\ y,\ t)$ they could calculate the influence of various control actions on the wake oscillations and compare the calculation results with conclusion following from their laboratory measurements of control effects.

Another method of investigating the dependence of cylinder wake flows on streamwise coordinate x was used by Dušek et al. (1994) and Dušek (1996). Dušek et al. systematically studied the interrelation between the coupled nonlinear equations for the spatially-varying temporal Fourier components (corresponding to expansion of the disturbance velocity $u(\mathbf{x}, t)$ in powers of $e^{i\omega_1 t}$ where ω_1 is the oscillation frequency) and the local Landau equations for oscillation amplitudes A of the dominant harmonic at various points $\mathbf{x}$. They found, in particular, that for validity of the Landau equation the shape of the unstable mode must vary much more slowly than its amplitude. Then Dušek et al. considered the application of the results obtained to a cylinder wake flow, and compared the conclusions implied by direct numerical simulation of this flow, at Re slightly above the first Hopf bifurcation threshold $\mathrm{Re}_{\mathrm{cr}}$, with predictions based on approximate amplitude equations. Later Dušek (1996) used the results of the above-mentioend paper of 1994 to develop a numerical method for computing the spatially-varying temporal Fourier coefficients of velocity components in the cylinder wake. He evaluated the spatial structure of several terms of the Fourier series (the zeroth term describing the distortion of the primary steady flow by a disturbance, the first one which usually corresponds to the dominant harmonic, and a few subsequent terms describing higher harmonics) at two different supercritical values of Re, and showed that far downstream all harmonics behave like parallel traveling waves. Dušek also found that global characteristics of the dominant wave (its frequency, wavelength and phase velocity) agreed well with the experimental data of Williamson (1989). However, he did not try to compare the results of his computations with more complete experimental data for the spatial structure of the cylinder wake since very few such data were then available. Nevertheless, some experimental and numerically-simulated data on the spatial structure of two-dimensional wakes were obtained in the mid 1990s and these data, which will be considered below, agree in general with numerical results by Dušek et al. (1994) and Dušek (1996).

Let us begin with the paper by Goujon-Durand et al. (1994) who investigated the velocity oscillations at various spatial positions behind a spanwise homogeneous bluff body placed in a water tunnel. (In this paper a cylinder with the trapezoidal cross-section shown in Fig. 4.21a, and not a circular cylinder, was used for generation of the wake, but the general features of wake oscillations are similar in this case to those in a circular-cylinder wake). The authors measured the transverse flow velocity $w(\mathbf{x}, t)$ at a number of points $\mathbf{x}$ and numerous Reynolds number $\mathrm{Re} = U_0 D/\nu$ (where D is the 'trapezoid thickness' indicated in Fig. 4.21a) ranging from $\mathrm{Re}_{\mathrm{cr}} \approx 58$ to $2\mathrm{Re}_{\mathrm{cr}}$. Instead of characterization the disturbance intensity by the value of the equilibrium amplitude A_e of velocity oscillations at a fixed spatial point $\mathbf{x}$, Goujon-Durand et al. measured the peak to peak amplitudes $A(x)$ at a number of points $(x, 0, 0)$ and then analyzed the values of the maximal amplitude $A_{\max} = \max_{x > 0} A(x)$ and of the distance from the body, $x_{\max}$, at which the amplitude $A_{\max}$ was observed. They found that in the range of Reynolds numbers from $\mathrm{Re}_{\mathrm{cr}}$ to about $1.6\mathrm{Re}_{\mathrm{cr}}$, power laws of the form $\mathrm{A}_{\max} \propto (\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ and $x_{\max} \propto (\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})^{-1/2}$ are valid. In the same range of Reynolds numbers the local oscillation amplitude $A(x)$ satisfies the following similarity law: $A(x)/A_{\max} = \mathrm{F}(x/x_{\max})$ where $\mathrm{F}(\zeta)$ is an universal function which does not deped on Re.

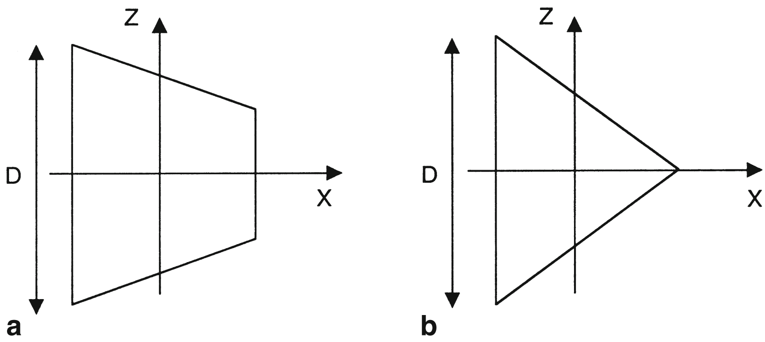


Fig. 4.21 **a** Trapezoidal cross-section of the cylinder used in experiments by Goujon-Durand et al. (1994) and Wesfreid et al. (1996). **b** Equilateral triangular cross-section of the cylinder used in the numerical simulations of a cylinder wake by Zielinska and Wesfreid (1995) and Wesfreid et al. (1996)

Simple similarity laws found by Goujon-Durand et al. for A_{max}, x_{max} and $A(x)/A_{max}$ had not yet received a theoretical explanation. Moreover, the relation $A_{max} \propto (\mathrm{Re} - \mathrm{Re}_{cr})$ seems strange, since it is known that the equilibrium amplitude A_e at a fixed point **x** is proportional to $(\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ over a considerable range of positive values of $\mathrm{Re} - \mathrm{Re}_{cr}$. Therefore, Zielinska and Wesfreid (1995) tried to verify these laws from the results of a numerical simulation of the purely two-dimensional wake behind a cylinder with a cross-section in the form of an equilateral triangle with the apex pointing upstream (see Fig. 4.21b). Their data were based on the analysis of numerical solutions of the two-dimensional Navier-Stokes equations describinb the flows in the (x, z)-plane around an impenetrable equilateral triangle; the solutions were computed for various values of $\mathrm{Re} = U_0 D/\nu$ (where D is the length of triangle sides, and U_0 is the velocity of the uniform flow upstream of the body). The solutions gave the values of vertical and horizontal velocity oscillations $w(x, z, t)$ and $u(x, z, t)$ and of the mean-flow distortion (i.e., of the zeroth harmonic $\Delta \mathrm{u}(x, z, t)$ of the streamwise disturbance velocity) at a number of the points (x, z) where x ran through a set of positive values, while z took two values, $z = 0$ and $z = 0.5\mathrm{D}$. (Note that the oscillations $w(x, z, t)$ and $u(x, z_1, t)$, where z can take arbitrary values but $z_1 \neq 0$, mainly represent the contributions of the dominant first harmonic with frequency ω_1, while the main contribution to the value of the streamwise velocity $u(x, 0, t)$ at the symmetry axis $z = 0$ is due to the second harmonic with doubled frequency $2\omega_1$; see, e.g., Stuart (1960); Hannemann and Oertel (1989); Dušek et al. (1994) and Dušek (1996)). Then the values of the peak-to-peak oscillation amplitudes A_w, A_u and $A_{\Delta u}$ of the two velocity components w and u, and of the mean flow distortion at the chosen points, were computed for a number of Re values. The results showed that the flow undergoes a Hopf bifurcation at $\mathrm{Re} = \mathrm{Re}_{cr} \approx 38$, which can be described by Landau's Eq. (4.40) with a coefficient ω which depends only on Re (and represents a linear function of $\mathrm{Re} - \mathrm{Re}_{cr}$ at small values of $|\mathrm{Re} - \mathrm{Re}_{cr}|$) and a coefficient l depending on **x**. The maximum values of $A_{w,\,max}$, $A_{u,\,max}$ and $A_{\Delta u,\,max}$ of the three amplitudes on the lines $z = 0$ and $z = 0.5D$ and streamwise coordinates $x_{w,\,max}$ etc. of the points corresponding to these maximum amplitudes were also determined by Zielinska and Wesfreid.

Zielinska and Wesfreid then showed that the normalized streamwise and transverse velocity amplitudes $A_u(x)/A_{u,\,max}$ and $A_w(x)/A_{w,\,max}$ (where both the local and maximum amplitudes correspond to the wake oscillations at points with $z = 0$) are represented in the case considered by two different universal functions $F_u(x/x_{u,\,max})$ and $F_w(x/x_{w,\,max})$ of the normalized coordinate x/x_{max}. These conclusions clearly agree with those of Goujon-Durand et al. (1994) for the transverse velocity oscillations in a slightly different but related wake flow. As to the distances x_{max} from the bluff body to the points where the oscillation amplitudes take maximal values, it was shown that the values of $x_{w,\,max}$, corresponding to lines $z = 0$ and $z = 0.5D$, and of $x_{u,\,max}$, corresponding to the line $z = 0.5D$, are proportional to $(\mathrm{Re} - \mathrm{Re}_{cr})^{-1/2}$ in the range of supercritical Reynolds numbers extending up to about $1.3\mathrm{Re}_{cr}$. This result agrees with the similar conclusion found by Goujon-Durand et al. by analysis of the experimental data. However, the values of x_{max} corresponding to oscillations of the streamwise velocity u, and of the mean-flow distortion $U_0 - u$ on the symmetry axis $z = 0$, which are unrelated to the dominant harmonic of the velocity field, depend on $\mathrm{Re} - \mathrm{Re}_{cr}$ in a more complicated manner which cannot be described by a simple power law. Moreover, according to numerical simulations of Zielinska and Wesfreid, the maximal oscillation amplitudes $A_{w,\,max}$ at the axis $z = 0$ and $A_{u,\,max}$ at the line $z = 0.5D$, which characterize the dominant first harmonic of the wake velocity, are both proportional to $(\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ (and not to $(\mathrm{Re} - \mathrm{Re}_{cr})$, as Goujon-Duran et al. claimed) at $\mathrm{Re}_{cr} < \mathrm{Re} < 1.3\mathrm{Re}_{cr}$.

Since some of the results found by Goujon-Durand et al. (1994) and by Zielinska and Wesfreid (1995) contradicted to each other, it was decided to repeat the corresponding measurements and the analysis of the numerically-simulated data, to extend to span of the investigation and to improve its accuracy. Results of this new work were presented in the paper by Wesfreid et al. (1996). The new experiments used the same trapezoidal bluff body and water tunnel as before, but now a laser-Doppler anemometer was used to scan the values of the streamwise velocity $u(x, y, z, t)$ in the central part of the wake (near $y = 0$ where no variations of the oscillation frequency were fond) and the (x, z)-region extendind from $x = 0.7D$ to $x = 25D$ and from $z = 0$ to $z = 2.8D$. The time series of $u(\mathbf{x}, t)$ was fed to a spectrum analyzer to determine the frequency and amplitude of the dominant harmonic of velocity oscillations. The measurements covered the range of Reynolds numbers from $1.1\mathrm{Re}_{cr}$ to $1.6\mathrm{Re}_{cr}$, where this time it was found that $\mathrm{Re}_{cr} = 60.8$. The numerical simulation repeated the previous computations of two-dimensional wake oscillations behind a triangular cylinder with the cross-section shown in Fig. 4.21b. However, now the fluctuations $u(x, z, t)$ of the streamwise velocity was evaluated for the region $0.7D < x < 25D$, $0 < z < 2.75D$ of the (x, z)-plane, and the range of Reynolds numbers was from $\mathrm{Re} = 1.016\ \mathrm{Re}_{cr}$ to $\mathrm{Re} = 1.6\mathrm{Re}_{cr}$ (where $\mathrm{Re}_{cr} = 36.2$). The measured values of $u(x, y, z, t)$ and calculated values of $u(x, z, t)$ were both used to find the amplitude $A(x, z, \mathrm{Re})$ of the u-velocity oscillations at various points (x, z) and various values of Re. Then the maximal amplitude $A_{max}(\mathrm{Re}) = \max_{x,z} A(x, z, \mathrm{Re})$ was determined for various values of Re and the Re-dependent point (x_{max}, z_{max}) was found where the amplitude A_{max}

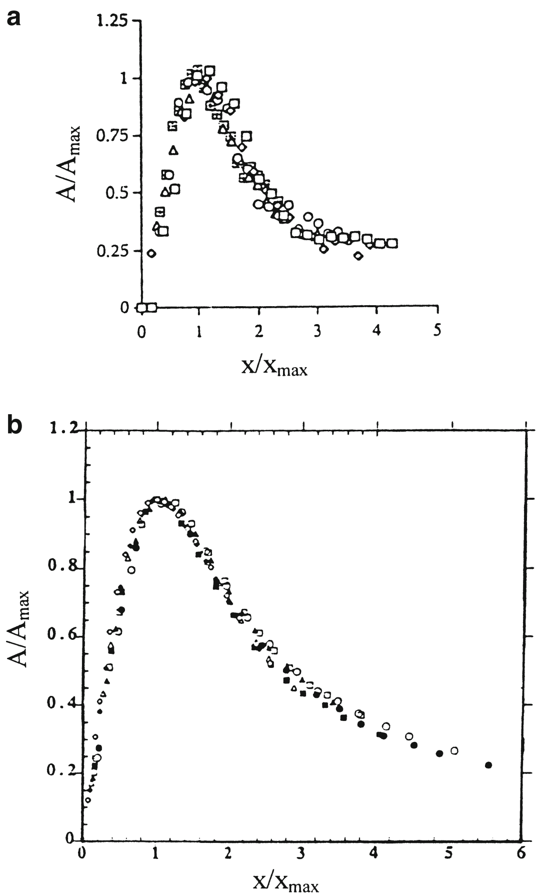


Fig. 4.22 Universal representation of the dependence of the normalized amplitude $A(x, z, \mathrm{Re})/A_{\max}(z, \mathrm{Re})$ of cylinder-wake oscillations on the coordinate x. (After Wesfreid et al. (1996)) (**a**) Values of $A(x, z, \mathrm{Re})/A_{\max}(z, \mathrm{Re})$ corresponding to velocities $u(x, y, z, t)$ measured in the wake behind a cylinder of trapezoidal cross-section at $y \approx 0$ and $z = z_{\max} \approx 0.7\mathrm{D}$; (**b**) values of $A(x, z, \mathrm{Re})/A_{\max}(z, \mathrm{Re})$ corresponding to numerically simulated velocities $u(x, z, t)$ in the two-dimensional wake behind a triangular cylinder at $z = 0.5\mathrm{D} < z_{\max}$. The various *symbols* correspond to different values of Re in the ranges $1.21\mathrm{Re}_{cr} \leq \mathrm{Re} \leq 1.59\mathrm{Re}_{cr}$ (**a**) and $1.02\mathrm{Re}_{cr} \leq \mathrm{Re} \leq 1.31\mathrm{Re}_{cr}$ (**b**)

is reached. The experimental and numerical results had the same general character and both showed that, at given values of z and Re, the amplitude $A(x, z, \mathrm{Re})$ increases with x at small values of x, takes a maximal value $\mathrm{A}_{\max}(z, \mathrm{Re})$ at some point $x_{\max}(z, \mathrm{Re})$ and then begins to decrease as x increases further. The values of $A_{\max}(\mathrm{Re})$ and $A_{\max}(z, \mathrm{Re})$ for $z > 0$ increase with Re in proportion to $(\mathrm{Re} - \mathrm{Re}_{cr})^{1/2}$ over a wide range of Re values (this conclusion agrees with results of Zielinska and Wesfreid (1995)), $x_{\max}(\mathrm{Re}) \propto (\mathrm{Re} - \mathrm{Re}_{cr})^{-1/2}$ in the same range of Re values, but $z_{\max}(\mathrm{Re})$ changes very little when Re is changing. Finally, according to both the experimental and the numerical data, the normalized amplitude values $A(x, z, \mathrm{Re})/A_{\max}(z, \mathrm{Re})$ are represented rather accurately by universal functions of $x/x_{\max}(z, \mathrm{Re})$, both for a fixed arbitrary value of z and $z = z_{\max}$ where $A_{\max}(z, \mathrm{Re}) = A_{\max}(\mathrm{Re})$; see, e.g., Fig. 4.22. This result clearly extends the conclusions found earlier by Goujon-Durand et al. and Zielinska and Wesfreid.

Above we considered the wake flow behind a circular cylinder only in the restricted range of Reynolds numbers from $\mathrm{Re}_{cr} \approx 47$ up to about 100–170 or even less (see, e.g., Figs. 4.19–4.20). This was quite natural, since we were interested in the regime of wake oscillations which an be described by the simple Landau Eq. (4.40). Generated by a Hopf bifurcation at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, the two-dimensional regime of parallel vortex shedding is often then transformed into a three-dimensional regime of oblique shedding by the influence of spanwise end conditions, but, as indicated above, one may prevent this transformation (and thus return to a two-dimensional wake regime) by some modification of the flow conditions at the cylinder ends. However, as Re increases, the wake flow inevitably acquires three-dimensional features. This circumstance was discovered rather early, in particular, by Roshko (1953, 1954); Hama (1956) and Bloor (1964) and was later studied and described (often together with descriptions of some subsequent wake bifurcations at still larger values of Re) in numerous sources dealing with either experimental or numerically-simulated data (see, e.g., Williamson (1988b, 1995, 1996a, b, c); König et al. (1990, 1993); Coutanceau and Defaye (1991); Karniadakis and Triantafyllou (1992); Tomboulides et al. (1992); Hammache and Gharib (1993); Roshko (1993); Norberg (1994); Mansy et al. (1994); Brede et al. (1994, 1996); Willams et al. (1995); Zhang et al. (1995); Mittal and Balachandar (1995a, b); Thompson et al. (1996); Wu et al. (1996a, 1966b); Zdravkovich (1997); Henderson (1997); Persillon and Braza (1998), and Leweke and Williamson (1998)). Results of different authors sometimes contradict each other in detail, but all show that at some $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$, in the range $150 < \mathrm{Re}_{2,\mathrm{cr}} < 200$, the regime of parallel vortex shedding becomes unstable with respect to some spanwise-periodic modes of disturbance, and transforms into a three-dimensional vortical regime. A number of the cited papers also include information about the appearance, at a Reynolds number of around 160 (clearly exceeding the threshold value for the primary instability of a two-dimensional wake), of the second three-dimensional unstable mode, which has smaller spanwise period and different symmetry properties. The existence of these two unstable modes was pointed out by Williamson (1988b, 1989) and was later confirmed in the experiments of Mansy et al. (1994); Williams et al. (1995); Brede et al. (1996); Wu et al. (1996a, b) and of some other researchers, and also in a number of direct numerical simulations of circular-cylinder wakes (e.g., those by Karniadakis and Triantafyllou (1992); Mittal and Balachandar (1995a, b); Zhang et al. (1995), and Thompson et al. (1996)). At present these modes are suaully referred to as modes A and B (see, e.g., the papers by Williamson (1996a, b, c); Henderson (1997) and Leweke and Williamson (1998) where the symmetry properties of these modes and the physical mechanisms of their instability are discussed in detail; in particular, Henderson also considered the Landau constants corresponding to development of the A and B modes). Both modes A and B oscillate with the same dominant frequency coinciding with the frequency of vortex shedding (i.e., they are periodic in time with period T equal to the shedding period). However, the simultaneous existence, at large enough values of Re, of two unstable modes makes transitions between them possible, producing discontinuities in the frequency of mode oscillations and the appearance of oscillations of double period $2T$ (or having even period mT of higher multiplicity); see, e.g., the general theory presented by Ioos and Joseph (1990) and the specific examples of period doubling of wake oscillations found by Tomboulides et al. (1992); Karniadakis and Triantafyllou (1992); Mittal and Balachandar (1995a) and Thompson et al. (1996).

The experimental and numerical-simulation data which illustrate wake transition to three-dimensional regimes of vortex shedding are characterized by considerable scatter in the observed values of the transition Reynolds number $Re_{2,cr}$. According to Roshko (1953, 1954) and Tritton (1959) $Re_{2,cr} = 150$ (though in the later survey by Roshko (1993) the higher estimate $Re_{2,cr} = 180$ was proposed), while Zhang et al. (1995) found that $Re_{2,cr} = 160$, Norberg (1994)–that $Re_{2,cr} = 165$, Williamson (1989)–that $Re_{2,cr} = 178$ (but in the survey of 1995 the latter author gave the much higher estimate $Re_{2cr} = 205$, and in the survey (1996a) he came to the conclusion that $Re_{2,cr} = 194$ is the best estimate). In parallel, Williamson (1996a, b) stated that the next transition, leading to the emergence of the mode B, takes place at $Re = Re_{3,cr}$ in the range between 230 and 260. The scatter of experimental values of $Re_{2,cr}$ and $Re_{3,cr}$ can be explained by the influence of free-stream turbulence, the difference btween between parallel and oblique vortex shedding, and/or the influence of the variability of end conditions (see Williamson, 1996a). Less scattered resutls are given by the careful theoretical investigations of the linear stability of parallel-shedding flows by Noack et al. (1993); Noack and Eckelmann (1994a, b); Barkley and Henderson (1996) and Henderson and Barkley (1996). These stability papers prove that at $Re = Re_{2,cr}$ lying between 170 and 190 the two-dimensional Kàrmàn street generated by paralle vortex shedding becomes unstable with respect to small three-dimensional disturbances with a spanwise wavelength equal to a few cylinder diameters. According to the most precise computations by Barkley and Henderson, $Re_{2,cr} = 188.5$ and the spanwise wavelength $\lambda_{y,\,cr} = 2\pi/k_{y,\,cr}$ of the mode A, the three-dimensional disturbance losing stability at this Re, is close to $4D$ (the authors suggested the even more precise estimate $\lambda_{y,\,cr} = 3.96\mathrm{D}$). Barkley and Henderson computed neutral-stability curves in the (λ_y, Re)-plane, corresponding to neutrally-stable wave disturbances in a two-dimensional Kármán-street flow; these curves are shown in Fig. 4.23. The upper curve in this figure bounds the region of A-mode instability, while the lower curve represents the neutral-stability curve for the mode B of three-dimensional disturbances which becomes unstable at round $Re = Re_{3,cr} \approx 260$ and at this Re has the spanise wavelength $\lambda_{2,y,cr} \approx D$ (more precisely $Re_{3,cr} \approx 259$, $\lambda_{2,y,cr} \approx 0.82D$). The results of Barkley and Henderson for mode A agree, to high accuracy, with williamson's (1996b, c) laboratory measurements. As to the results of Barkley and Henderson for mode B, the validity of their comparison with experimental data may raise some doubts, since these results were obtained by application of the linear stability theory for two-dimensional wake flows to conditions in which the two-dimensional wake is always unstable and where nonlinear effects are inherent. Therefore the fact that the main features of the observed second instability mode do not deviate much from those given by the application of the linear stability theory to a two-dimensional primary flow may be considered as somewhat surprising. However, the agreement of the linear theory developed for the second unstable mode of three-dimensional disturbances in the two-dimensional wake with the experimental data for mode B was confirmed by many authors, and it

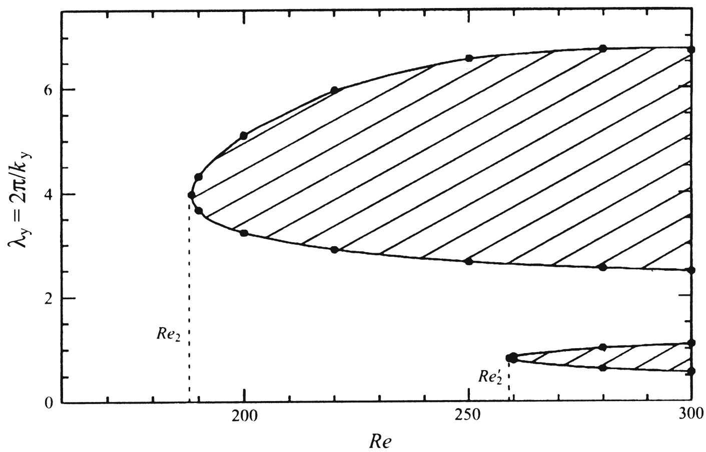


Fig. 4.23 The neutral-stability curves in the (λ_y, Re)-plane, where $\lambda_y = 2\pi/k_y$ is the spanwise wavelength (and U_0 and D are used as velocity and length units), which correspond to two types of neutral three-dimensional wave disturbances in the two-dimensional circular-cylinder wake. (After Barkley and Henderson (1996)). The upper shaded region corresponds to unstable A modes, while points of the lower shaded region correspond to unstable B modes; $\mathrm{Re}_2 \equiv \mathrm{Re}_{2,cr} \approx 188.5$, $\mathrm{Re}_2' \equiv \mathrm{Re}_{3,cr} \approx 260$

will be shown later that a similar situation occurs also in the study of wakes behind a square cylinder and a sphere.

The final transition to fully turbulent wake flow apparently takes place after several successive transformations, at higher and higher values of Re, into more and more asymmetric flow regimes. Breaking of symmetry properties leads not only to more complicated spatial patterns but also to increasingly complex dynamics, i.e., makes the flow more and more tangled (see, e.g., Crawford and Knobloch (1991); Dangelmayr and Knobloch (1991), and Hirschberg and Knobloch (1996)). Some of these further transformations possibly represent Hopf bifurcations which increase by one the number of degrees of freedom of the considered flow and may be described by modified Landau's equations.

Experimental data relating to circular-cylinder wake oscillations at very high Reynolds numbers showed quite early that here the standard definition of the Strouhal number does not allow a universal form of the St-Re relation to be obtained. Therefore Roshko (1961) concluded that at such values of Re the cylinder diameter D cannot be used as an appropriate length scale entering the definition of St; instead, he recommended using the wake thickness H as a length scale and changing the definition of the velocity scale (let us recall in this respect that just H was used as the length scale in the linear stability theory of wake flows considered Sect. 2.93). Later Bearman (1967) and Griffin (1981), trying to obtain the universal form of the St-Re relation, suggested some other choices of length and velocity scales to make the wake-oscillation frequency f dimensionless. Still later Adachi et al. (1966) measured, in a range $1.5 \times 10^4 < \mathrm{Re} < 10^7$ of Reynolds numbers $\mathrm{Re} = U_0D/\nu$, the vortex-shedding frequencies f for eight rough circular cylinders of a fixed diameter D with surfaces covered by homogeneous roughnesses with the heights h of roughness elements satisfying the inequalitites $4.54 \times 10^{-6} < h/D < 2.5 \times 10^{-3}$. Then they calculated for all round frequencies f four different dimensionless combinations $\mathrm{St} = fL/V$ (differing by the used length and velocity scales L and V; the definitions of St proposed by Roshko, Bearman and Griffin were included in their list) and analyzed the dependence of the obtained values of St on h/D and Re. They found that at $h/D < 5 \times 10^{-4}$ the roughness of the cylinder does not affect the wake characteristics and that at such values of h/D the St-Re relation has the most universal form when Bearman's definition of St is used (such St preserves practically the same value in the whole studied range of Reynolds numbers).

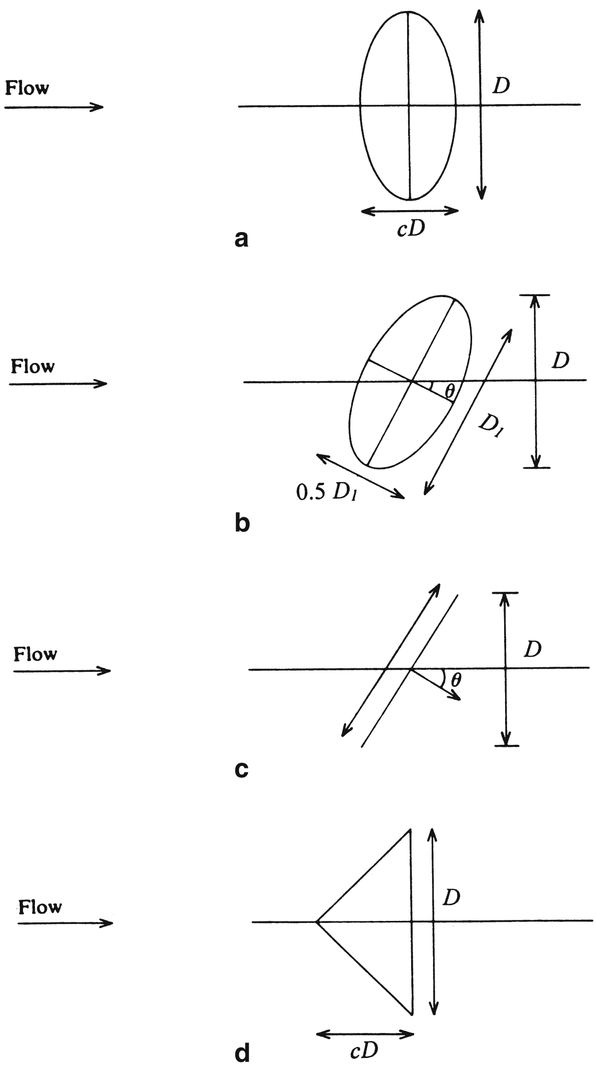
4.2.4.3 Wakes Behind Non-Circular Cylinders and Rectangular Plates

Above, wakes behind circular cylinders were considered almost exclusively. The only exceptions were brief remarks about the two-dimensional wakes behind two particular non-circular cylinders: one which was studied in experiments by Goujon-Durand et al. (1994) and Wesfreid et al. (1996) and the other which was numerically simulated by Zielinska and Wesfreid (1995) and Wesfreid et al. (1996) (see Fig. 4.21a, b). In the remarks it was stated that these wakes are similar in many respects to the circular-cylinder wake and, exactly like the latter, undergo a Hopf bifurcation at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ of the order of a few tens. Now these remarks will be supplemented by brief considerations of some other results relating to *wakes behind non-circular cylindrical bodies.*

Let us beign with the results of Jackson (1987), who calculated the points of onset of vortex shedding in flows past a whole collection of non-circular cylinders. He considered only purely two-dimensional wakes (i.e., the wake flows were assumed to be independent of spanwise coordinate y) and did not try to apply time-consuming direct numerical simulation to this problem. Instead, Jackson used a modification of the simple method of direct location of the Hopf-bifurcation points outlined by Griewank and Reddien (1983) (who in their turn relied on some ideas presented in the collection edited by Mittelmann and Weber (1980)). This method deals with dynamical systems described by systems of ordinary differential equations, and employs some general properties of bifurcating solutions at the Hopf-bifurcation points to compute the position of these points without solving the given equations and computing their eigenvalues.

In the case of flow around a cylindrical body the equations of motion depend on the parameter Re, and its threshold value above which the periodic solution exists is just the critical value $\mathrm{Re}_{\mathrm{cr}} = (U_0 D/\nu)_{\mathrm{cr}}$, symbolizing the emergence of a Hopf bifurcation. Jackson's method allows this value $\mathrm{Re}_{\mathrm{cr}}$ to be computed directly, together with the coordinate $i\omega_1$ of a point of the imaginary axis where, at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$, the spectrum of the Navier–Stokes eigenvalues crosses this axis indicating the appearance of flow instability. The value of ω_1 determined the shedding frequency $f_{\mathrm{cr}} = -\omega_1/2\pi$ and the Strouhal and Roshko numbers $\mathrm{St}_{\mathrm{cr}} = f_{\mathrm{cr}} D/U_0$ and $\mathrm{Ro}_{\mathrm{cr}} = f_{\mathrm{cr}} D^2/\nu = \mathrm{St}_{\mathrm{cr}}\mathrm{Re}_{\mathrm{cr}}$ at

Fig. 4.24 Cross-sections of the non-linear cylinders for which Jackson (1987) determined the points of onset of the vortex shedding from the cylinder body. (**a**) ellipses oriented along the flow; (**b**) ellipses oriented at angles θ to the flow; (**c**) flat plates with normals at angles θ to the flow; (**d**) isosceles triangles with apexes directed upstream



$\mathrm{Re} = \mathrm{Re}_{cr}$. (Here U_0 is the constant velocity of the oncoming flow and D is the cross-stream 'thickness' of this body indicated in Fig. 4.24). Jackosn made calculations for cylindrical bodies with the following cross-sections: (a) ellipses with a principal axis of length D perpendicular to the flow and a principal axis of length cD along the flow direction, where c varies from 10^{-4} to 2; (b) ellipses with the major axis twice as long as the minor axis, oriented at various angles θ to the flow direction where $0° \leq \theta \leq 90°$; (c) straight segments of finite length at various orientations θ to the flow where $0° \leq \theta \leq 60°$ (here 'cylindrical bodies' turn into thin flat plates, at $\theta = 0$ such a plate does not differ in fact from ellipse (a) with $c = 10^{-4}$); (d) isosceles

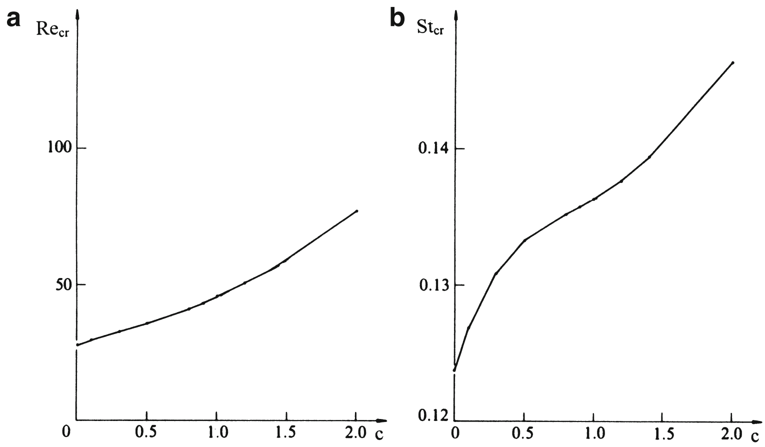


Fig. 4.25 Dependence of the critical Reynolds number Re_{cr} (**a**), and the critical Strouhal number St_{cr} (**b**) on the parameter c for wakes behind elliptic cylinders with cross-sections shown in Fig. 4.24(**a**). (After Jackson (1987))

triangles with base of lengh D perpendicular to the flow, the apex toward the flow and the height h of length cD where $0 \le c \le 2$ (see again Fig. 4.24). For all these bodies the values of Re_{cr} and St_{cr} were computed, and their dependence on the parameters c and θ was presented in the form of tables and graphs (as an example, Fig. 4.25 shows the graphs for elliptic cylinders (a)). It was noted that in the case of a circular cylinder (corresponding to the shape (a) with $c = 1$) the results agree well with those of the previous experiments and numerical simulations by various authors (cf. the similar remark on p. 110 where some references to earlier papers were given). The results relating to some other elliptic cylinders (shapes (a) with $c \neq 1$) were later verified by Morzyński and Thiele (1991, 1992) who used another numerical method and obtained the results close to those by Jackson. (Direct numerical simulation of flows past some elliptic cylinders were carried out, in particular, by Mittal (1994) and Mittal and Balachandar (1995b, 1996); here the values of St were determined for several supercritical values of Re sometimes also exceeding the threshold value $Re_{2,cr}$ for wake transition to three-dimensionality). For the case of the equilateral triangular cross-section Jackson found that $Re_{cr} \approx 35$; this estimate proved to be slightly lower than the estimate $Re_{cr} \approx 38$ found for this case by Zielinska and Wesfreid (1995) but it agreed somewhat better with the subsequent results by Wesfreid et al. (1996) according to which $Re_{cr} = 36.2$.

Jackson determined values of Re_{cr} and St_{cr} making use of the nonlinear bifurcation theory, but these results relate to the linear stability theory and hence they might as well have been discussed in Chap. 2. (However, in Chap. 2, as a rule, only results obtained in the framework of the parallel-flow approximation were considered while Jackson's and Morzyński and Thiele's computations dealt with the two dimensional but non-parallel model). Similar stability computations were performed

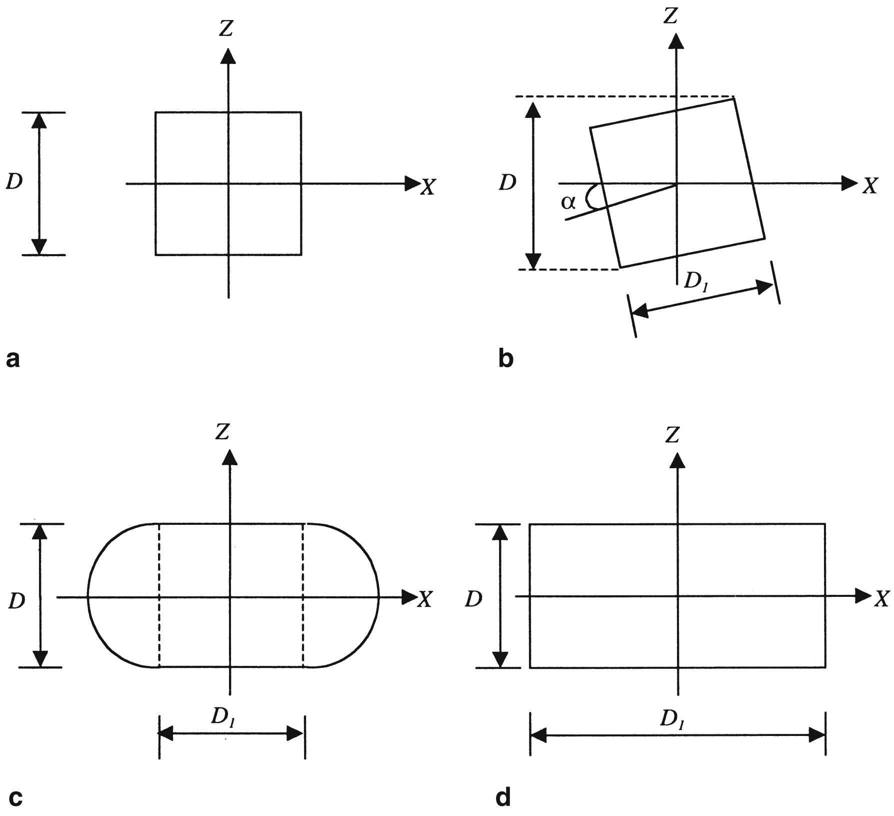


Fig. 4.26 Cross-sections of the square cylinder at zero incidence (**a**); of the square cylinder at nonzero incidence angle α (**b**); of the oblong cylinder (**c**); and of the rectangular cylinder (**d**)

by Kelkar and Patankar (1992) for the case of a flow around a *square cylinder* at zero incidence (i.e., with a plane face perpendicular to the stream; see Fig. 4.26a). These authors calculated the solutions of two-dimensional Navier–Stokes equations describing laminar steady flows around a square cylinder having constant velocity $\mathbf{U} = \{U_0, 0, 0\}$ far from this cylinder and corresponding to several moderate values of $\mathrm{Re} = U_0 D/\nu$. Then the onset of unsteadiness (i.e., the emergence of a Hopf bifurcation) was determined by numerical solution of the linear stability problem for the computed laminar flows. Thus, the values of $\mathrm{Re}_{\mathrm{cr}}$ and $\mathrm{St}_{\mathrm{cr}}$, corresponding to the beginning of vortex shedding, were found for the wake behind a square cylinder placed normal to an uniform flow (in particular, it was found that $\mathrm{Re}_{\mathrm{cr}} = 53$). Values of $\mathrm{St} = \mathrm{St}(\mathrm{Re})$ computed by Kelkar and Patankar for values of Re close to $\mathrm{Re}_{\mathrm{cr}}$ were compared with the results of Okajima's (1982) laboratory measurements of Strouhal numbers of the square-cylinder wake, and it was found that the numerical simulation leads to results which agree well with the experimental ones.

Table 4.1 Critical values Re_{cr} and Ro_{cr} corresponding to the start of vortex shedding from a square cylinder at incidence, versus incidence angle α. (After Sohankar et al. (1997, 1998))

α	0°	10°	20°	30°	45°
Re_{cr}	51.2	51.0	48.7	44.0	42.0
Ro_{cr}	5.9	6.2	6.1	5.4	5.2

Later Sohankar et al. (1997, 1998) and Sohankar (1998) carried out the numerical simulation of flows around square cylinders at variable incidence (with $0° \le \alpha \le 45°$ where α is the angle of incidence shown in Fig. 4.26b) at a number of values of Re and deduced the dependence on the angle α of $\mathrm{Re}_{cr} = (U_0D/\nu)_{cr}$ and $\mathrm{Ro}_{cr} = (fD^2/\nu)_{cr} = \mathrm{St}_{cr}\mathrm{Re}_{cr}$ (where $D = (\cos\alpha + \sin\alpha)\ D_1$ is the cross-stream 'thickness' of the cylinder and D_1 is the length of the square side; see again Fig. 4.26b). Found by them values of Re_{cr} and Ro_{cr} at $\alpha = 0°$ proved to be close enough to Kelkar and Patankar's results; they are presented in Table 4.1 together with the results for other values of α.

The method used by Sohankar et al. to obtain these results will be described at greaterlength a little later, but now we will return to some results of Schumm et al. (1994) which were omitted in discussion of this paper earlier in this section. The point is that the results relating to the vortex-shedding flow behind a circular cylinder (which were summarized in Eq. (4.49) and Fig. 4.19 above) were supplemented by Schumm et al. by results of similar experimental studies of wakes behind some non-circular cylinders. Namely, together with the case of a circular-cylinder wake, Schumm et al. investigated also the vortex shedding from an *oblong cylinder* without sharp corners, with the cross-section sketched in Fig. 4.26c (where $D = 0.69$ mm, $D_1 = 1.68$ mm), and two *rectangular cylinders* (thick plates parallel to the flow direction) with cross-sections of the shape shown in Fig. 4.26d. The wake behind a piezoceramic oblong cylinder with the same cross-section was first studied by Berger (1964, 1967) (see also Berger and Wille (1972)) who paid most attention to the influence of cylinder oscillations on the wake flow. Schumm et al. used the same cylinder and measured the oscillations of the transverse ('vertical') velocity $w(x, y, z, t)$ at the point $(x/D, y/D, z/D) = (10, 0, 1)$ of its wake at different values of $\mathrm{Re} = U_0D/\nu$ and different stages of oscillation development. These measurements allowed the calculation, in exactly the same way as for a circular-cylinder wake, of the values of Re_{cr} and of all the coefficients in the corresponding Landau Eqs. (4.34) and (4.34a). It was found that here $\mathrm{Re}_{cr} \approx 79.2$; $\gamma D^2/\nu \approx 0.116\ (\mathrm{Re} - \mathrm{Re}_{cr})$; $\omega_1 D^2/\nu \approx 58.1$; $\delta'/\delta \approx -1.85$. Moreover, it was also shown that here the growth rate γ and the oscillation frequency ω_1 have the same values at all points $(x/D, 0, z/D)$ with $z/D = 1$ and $10 \le x/D \le 40$. These facts confirm that at $\mathrm{Re} = \mathrm{Re}_{cr}$ a Hopf bifurcation occurs, leading to a global mode of oscillations satisfying the Landau equation (see the final paragraph of Sect. 2.93 and the beginning of the small-type text above).

Similar, but less complete, experimental results were obtained by Schumm et al. for the wake behind a *rectangular plate* with $D = 4$ mm, $D_1 = 60$ mm, and the spanwise length $L = 200$ mm. (The second rectangular plate used by the authors was noticeably larger, and its wake was studied only at greater values of Re which will not be considered here). It was shown that in the case of the first plate $\mathrm{Re}_{cr} = (U_0D/\nu)_{cr} \approx 135$, and $\gamma D^2/\nu \approx 0.083\,(\mathrm{Re} - \mathrm{Re}_{cr})$. The authors remarked that the coefficient 0.083 in the latter equation proved to be close to the value obtained by Hannenmann and Oertel (1989) for the numerically-simulated two-dimensional wake behind the rectangular plate (values of D and D_1 were especially chosen by Schumm et al. so that the ratio D_1/D had the same value 15 as in the numerical simulation). However, the value of Re_{cr} in the numerically-simulated wake was considerably smaller than the value found in the laboratory experiment. Schumm et al. assumed that this discrepancy can be due to deviation of the experimentally-produced wake from the idealized purely two-dimensional numerical model of Hannemann and Oertel.

For the case of *square cylinders at zero incidence* (where $D_1 = D_2 = D$) more detailed investigations of wakes at moderate Reynolds numbers were carried out by Sohankar et al. (1995, 1997, 1998, 1999) and Robichaux et al. (1999) (some parts of this work were also considered in detail in Robichaux's (1997) and Sohankar's (1998) theses). These authors based their work mainly on the analysis of DNS ('direct numerical simulation') data but Sohankar et al. also included in their papers experimental results of Norberg (partially presented in his paper of 1989), which were also compared with the data of Okajima (1982, 1995) and of a few other experimenters. It has been already stated above that Sohankar et al. (1997, 1998) used numerical simulations of flows around square cylinders at various angles of incidence α to determine the dependence of the critical values Re_{cr} and Ro_{cr} on the value of α (the results were presented in Table 4.1). Now a little more will be said about this work. Referring to Schumm et al. (1994) and Park (1994); Sohankar et al. stressed that the onset of flow oscillations, caused by a Hopf bifurcation, can be described by the Stuart-Landau Eq. (4.40) for the complex disturbance amplitude $A(t)$ (or, what is the same, by two real Eqs. (4.34) and (4.34a) for th real amplitude $|A|$ and phase ϕ). Following Park (1994), they chose the amplitude of the lift force on the cylinder to be the amplitude $A(t)$ (recall that the wake oscillations produced by vortex shedding are due to a global instability mode where the values of $\gamma(\mathrm{Re})$ and $\omega_1(\mathrm{Re})$ do not depend on the choice of amplitude A). Then they investigated the growth of $|A(t)|$ with t at various values of Re and α (in the ranges $45 < \mathrm{Re} < 200$, $0° \leq \alpha \leq 45°$) and determined the growth rate $\gamma(\mathrm{Re}, \alpha)$ (representable as $b(\alpha)[\mathrm{Re} - \mathrm{Re}_{cr}(\alpha)]$ at small and moderate values of $\mathrm{Re} - \mathrm{Re}_{cr}$) and the Landau constant $\delta(\mathrm{Re}, \alpha)$. Values of $\gamma(\mathrm{Re}, \alpha)$ were used to determine the function $\mathrm{Re}_{cr}(\alpha)$ while values $\mathrm{St}_{cr}(\alpha) = -\omega_{1,cr}(\alpha)D/2\pi U_0$ were found with the help of Eq. (4.34a).

Moreover, the wake-flow simulations and/or measurements at supercritical Reynolds numbers $\mathrm{Re} > \mathrm{Re}_{cr}$ allowed determination of the dependence of a number of a physical characteristics of vortex shedding (the Roshko and Strouhal numbers Ro and St are typical examples) on Re and α. In particular, it was shown by Sohankar et al. (1997, 1998) (who based their conclusion on the unpublished experimental data of Norberg, supplemented by some new DNS data) that at $\alpha = 0°$ (i.e., for square cylinders with one plane side facing the flow) the dependence of the Roshko number $\mathrm{Ro} = \mathrm{St} \times \mathrm{Re} = fD^2/\nu$ on $\mathrm{Re} = U_0D/\nu$ at $\mathrm{Re}_{cr} < \mathrm{Re} < 200$ is described with reasonable accuracy by the Roshko law (4.47) (which agrees well with the Stuart–Landau

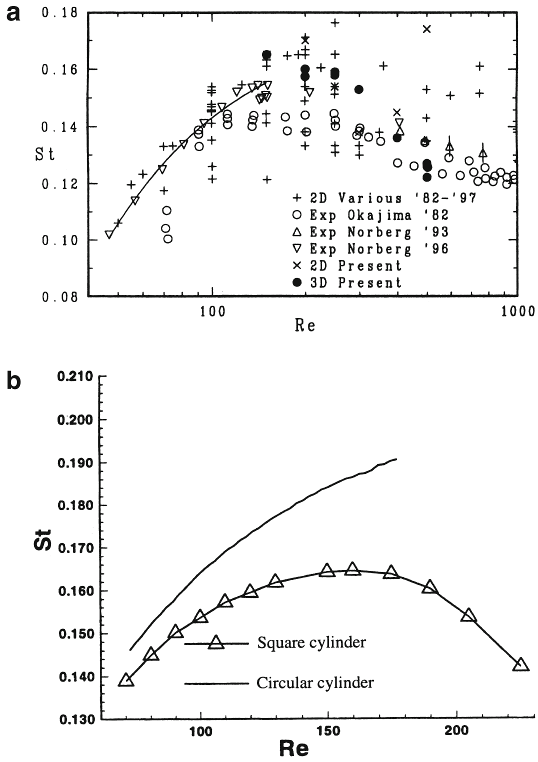


Fig. 4.27 The dependence of the Strouhal number St $= fD/U_0$ on the Reynolds number Re $= U_0D/\nu$ for wakes behind square cylinders at zero incidence according to various experimental and numerically-simulated data. (After Sohankar et al. (1997, 1999) and Robichaux et al. (1999)) (**a**) Summary graph by Sohankar et al. collecting various experimental (*Exp*) and numerically simulated (relating to a two-dimensional (*2D*) or a three-dimensional (*3D*) wake model) data. The solid line represents the empirical law (4.52): St = 0.18-3.7/Re. (**b**) Numerically simulated data (Robichaux et al. 1999) corresponding to a 3D wake model, and their comparison with the experimental data of Williamson (1996b) for circular-cylinder wake oscillations

Eq. (4.34) and (4.34a)). According to Sohankar et al. this law here as the form:

$$\text{Ro} = 0.18\,\text{Re} - 3.7 \tag{4.52}$$

(and hence $a = 0.18$, $a_1 = 3.7$ for wakes behind square cylinders at zero incidence). This conclusion was repeated for the indicated range of Reynolds numbers in the next paper by Sohankar et al. (1999); see Fig. 4.27a reproduced in the papers of 1997 and 1999. Then Robichaux et al. (1999) independently determined from their two-dimensional DNS data the St-Re relation for the square-cylinder wake in the range $70 < \text{Re} < 230$; their result is presented in Fig. 4.27b together with the similar curve for the circular-cylinder wake (this curve is based on the results collected by Williamson (1995, 1996a, b, c) and it extends only slightly the 'universal St-Re relation' shown in Fig. 4.20). One can see that St-Re data in Fig. 4.27b relating to square-cylinder wakes are much less scattered than those in Fig. 4.27a, where data from a number of quite different sources (having different accuracy) sources are collected, but on the whole data of Robichaux et al. for $70 < \text{Re} < 200$ do not disagree greatly with the results presented in Fig. 4.27a and with Eq. (4.52).

As to the comparison of St-Re relations for square—and circular-cylinder wakes, Robichaux et al. noted that the considerably smaller values of St, and their non-monotonic dependence on Re, in the case of a square cylinder may be explained by the fact that such a cylinder is a much bluffer body than the circular one. The blunt upstream face of the square cylinder, and its sharp edges, lead to flow separation and the formatin of recirculation regions on the top and bottom faces. These features lead to increase of the effective cross-stream thickness of the body. Therefore, the flow upstream of the cylinder actually sees a body with the increased 'effective thickness' $D^* > D$. Since $\mathrm{St} = fD/U_0$ contains the body thickness as a factor, the use of an underestimated value of the effective thickness leads to an underestimate of the prompted by physical arguments value of St and, since this underestimate increases with the growth of Re, it can lead to non-physical decrease of St as Re increases. Robichaux et al. introduced a plausible Re-dependent estimate of the 'effective thickness' D^* of the square cylinder and showed that replacement of D by D^* in the expression for St implies an St-Re relation for a square cylinder which does not differ much from the relation for a circular cylinder. It will be shown later that similar reasoning can be used to explain the form of the measured St-Re relation for wakes of flat plates parallel to the stream direction.

Let us stress, however, that the study of the St-Re relation for the square-cylinder wake in a limited range of moderate Reynolds numbers was not the main purpose of the papers by Sohankar et al. (1999) and Robichaux et al. (1999). Both groups of authors took into account the available results of investigations of circular-cylinder wakes, which showed that the simple two-dimensional wake transforms into a more complicated three-dimensional form at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} \approx 190$, while at still greater Re the wake even contains two different three-dimensional modes, A and B, having specific symmetry properties (see the end of part (b) of this section). Therefore, they decided to check whether or not a similar transition to three-dimensionality takes place in the square-cylinder wake. With that end in view, Sohankar et al. collected and analyzed numerous results of measurements in air and water flows and of two- and three-dimensional (2D and 3D) direct numerical simulations of unsteady flows around a square cylinder at zaero incidence for a wide ragne of Reynolds numbers, $\mathrm{Re} = 150 - 1{,}000$ (see Fig. 4.27a). Note that the analyzed data included the experimental and 2D an 3D simulated results of the authors themselves at $\mathrm{Re} = 150 - 500$; this range also extends well above the circular-cylinder critical value of $\mathrm{Re}_{2,\mathrm{cr}}$. The data in Fig. 4.27a show that $\mathrm{Re} = 200$, which was the highest value of Re inspected in the papers of 1987 and 1988, is close to the upper bound of the Re-region where Eq. (4.52) is valid. At higher values of Re this equation is clearly incorrect, and there the results of 3D numerical simulations agree much better with the experimental data than the results of 2D simulations. (The incorrectness, at large values of Re, of the results of 2D simulations of flows around cylindrical bodies was also noted by Tamura et al. (1990)). The 3D numerical simulations performed by Sohankar et al. also showed that the two-dimensional square-cylinder-wake flow becomes unstable and undergoes transition to a three-dimensional form at some Re between 150 and 200. It was also shown that three-dimensional wake flow includes both the three-dimensional instability modes, A and B, which were observed in circular-cylinder wakes, and in a square-cylinder wake these modes have spatial structures similar to those of circular-cylinder modes A and B. There are, however, also some new features specific to square-cylinder wakes; e.g., at $\mathrm{Re} = 200 - 300$ in such wakes some low-frequency lift force pulsations were detected, which apparently do not exist in circular-cylinder wakes. At the same time the Strouhal numbers and mean drag values given by 3D numerical simulations were found to be in satisfactory agreement with experimental results (for St, the validity of this conclusion is seen in Fig. 4.27a).

Robichaux et al. (1999) performed only 2D numerical simulations of the square-cylinder wake and considered a restricted range of Reynolds numbers, $70 \leq \mathrm{Re} \leq 300$. However, they then applied to the simulated two-dimensional models of wake flows a three-dimensional linear theory of hydrodynamic stability of the same type as that used by Barkley ad Henderson (1996) on a 2D model of the circular-cylinder wake. That is, they investigated the stability of 2D wake flows to infinitesimal 3D disturbances depending periodically on the spanwise coordinate y. This investigation showed that at $\mathrm{Re} \equiv \mathrm{Re}_{2,\mathrm{cr}} \approx 160$ (more precisely, at some Re in the range 162 ± 12) the 2D square-cylinder wake becomes unstable to 3D disturbances with a spanwise wavelength (non-dimensionalized by the side length D) $\lambda_{y,\,\mathrm{cr}} \approx 5.22$. The corresponding three-dimensional unstable mode oscillates with a frequency equal to that of the vortex shedding and has a spatial structure similar to that of mode A of the circular-cylinder wake; therefore it was natural to call it mode A too. The second 3D unstable mode ('mode B'), with the same frequency as the first one and a spatial structure similar to that of mode B of the circular-cylinder wake, was also discovered in the square-cylinder wake by stability analysis of Robichaux et al.; it becomes unstable at a slightly greater Reynolds number $\mathrm{Re}_{3,\mathrm{cr}} \approx 190$ (more precisely, 190 ± 14) and has dimensionless spanwise wavelength $\lambda_{2,y,\,\mathrm{cr}} \approx 1.2$. Moreover, Robichaux et al. found that in the square-cylinder wake there also exists a third mode of unstable 3D disturbances (having specific spatial structure) which apparently does not exist in the wake of a circular cylinder; this mode (which was called 'the mode S' by the authors) becomes unstable at $\mathrm{Re} \equiv \mathrm{Re}_{4,\mathrm{cr}} \approx 200$ (more precisely, 200 ± 5), which differs very little from $\mathrm{Re}_{3,\mathrm{cr}}$, and has dimensionless wave length $\lambda_{3,y,\,\mathrm{cr}} = 2.8$ intermediate between $\lambda_{y,\,\mathrm{cr}}$ and $\lambda_{2,y,\,\mathrm{cr}}$. However, this new mode is subharmonic, with an oscillation period twice the shedding period of the primary two-dimensional state (and hence with half the shedding frequency). The discovery of modes A and B by the linear stability analysis of Robichaux et al. confirmed the corresponding results by Sohankar et al. found by a quite different method, specifically a fully-nonlinear three-dimensional DNS, while the discovery of the subharmonic mode S by Robichaux et al. had something in common with the discovery by Sohankar et al. of low-frequency oscillations of the DNS data. (The lack of complete coincidence of the results of two groups seems only natural since the methods used were too different; in particular, the 3D DNS results depend on the choice of the spanwise aspect ratio L/D, which took values of only 6 and 10 in the simulations of Sohankar et al., while the two-dimensional primary flow of the stability analysis correspond to $L/D = \infty$).

Let us now return to the *elongated rectangular cylinders* with $D_1/D = 15$ used in the wake studies of Hannemann and Oertel (1989) and Schumm et al. (1994).

These cylinders represent some examples of rectangular plates of finite thickness placed parallel to the flow direction. Other examples of such plates were considered by Nakayama et al. (1993) who performed 2D numerical simulations of the wakes behind plates, parallel to the flow, of thickness D with values of D_1/D varying from 3 to 10 for two Reynolds numbers $U_0D/\nu = 200$ and 400. For both values of Re it was found that the Strouhal number, $\mathrm{St} = fD/U_0$, varies when the value of D_1/D changes. More detailed numerical simulation of velocity oscillations in the wake behind a rectangular plate of finite thickness were performed by Hammond and Redekopp (1997) who used another idealized two-dimensional model of such a wake. Namely, these authors assumed that the plate, of thickness D, is semi-infinite (filling the volume $-\infty < x \leq 0$, $-\infty < y < \infty$, $-D/2 < z < D/2$) and that along opposite sides of this plate two independent plane-parallel streams are flowing in the Ox direction, with the same (nominally Blasius) velocity profile corresponding to given velocity U_0 outside the boundary layer. (The authors also investigated the case of an asymmetric wake where the limiting velocities U_1 and U_2 outside the upper and lower boundary layers differ from each other; however, we will not linger on the results of this case). Hammond and Redekopp studied the oscillations of the streamwise and transverse ('vertical') velocity components $u(x, z, t)$ and $w(x, z, t)$ at the point $(x/D, z/D) = (1, 0.5)$ and found that at not too large positive values of $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ (where again $\mathrm{Re} = U_0D/\nu$) the amplitudes of both these oscillations satisfy, with high accuracy, the same Landau Eqs. (4.34) and (4.34a) with the coefficients: $\gamma \mathrm{D}/U_0 \approx 0.0078(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ where $\mathrm{Re}_{\mathrm{cr}} \approx 120$ (this value is greater than that found by Hannemann and Oertel and does not differ too much from experimental value of Schumm et al)., $\omega_1(\mathrm{Re}_{\mathrm{cr}})D/U_0 \approx -0.61$, and $\delta'/\delta \approx -1.37$. It was also verified that the values of these coefficients were independent of position over a large region of the (x, z)-plane. Thus we see that this numerical simulation also confirms the fact that at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}}$ a Hopf bifurcation occurs in the flow behind a rectangular plate, and leads to the appearance of a global mode of oscillation with a complex amplitude $A(t)$ that satisfies the Landau Eq. (4.34–4.34a).

Plates of rectangular section, whose wakes were investigated by Hannemann and Oertel (1989); Nakayama et al. (1993); Schumm et al. (1994) and Hammond and Redekopp (1997), can be considered as models of an idealized infinitely thin flat plate parallel to the flow direction. It was indicated in Sect. 2.93 and recalled again on p. 108 of the present section that the laminar wake behind such a plate has the 'Gaussian' velocity profile of Eq. (2.89). The results of linear stability analysis were presented in Fig. 2.34, and from these the values of $\mathrm{Re}_{\mathrm{cr}}$, k_{cr} and $\omega_{1,\mathrm{cr}}$ for the wake of a thin flat plate can be evaluated. However, selection of the most appropriate length and velocity scales is not a trivial matter in this case, since it is clear that the very small 'thickness' of the plate cannot be used now as a reasonable length scale. In Sect. 2.93 and Fig. 2.34 the half-width of the laminar wake was used as the length scale H (the increase of the width with x was neglected) and the difference between U_0 and the velocity at the laminar wake center-line was chosen as the velocity scale, but both these scales are irrelevant when wake behavior at supercritical Reynolds numbers $\mathrm{Re} > \mathrm{Re}_{\mathrm{cr}}$ is considered. Therefore, when Eisenlohr and Eckelmann (1988) investigated, in a wind tunnel, the wakes behind eight different thin plates with blunt

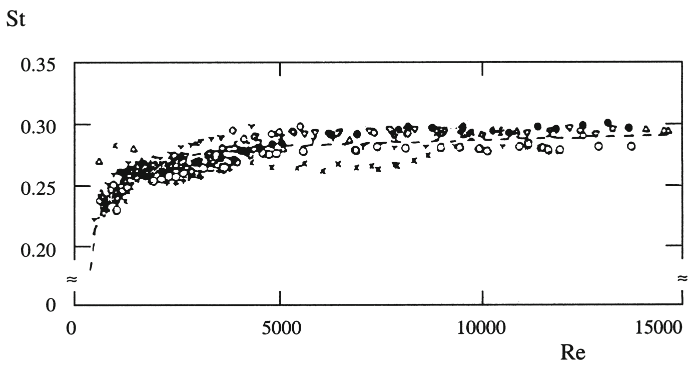


Fig. 4.28 The dependence of the Strouhal number $\mathrm{St} = fD^+/U_0$ on the Reynolds number $\mathrm{Re} = U_0D^+/\nu$, where $D^+ = D + 2\delta^*$, in the case of wake oscillations behind thin plates of thickness D. (After Eisenlohr and Eckelmann (1988)) the *dotted line* represents the empirical relation (4.52a): $\mathrm{St} = 0.286 - 39.2/\mathrm{Re}$. Different *symbols* (which are often superimposed on each other) correspond to different plates

trailing edges (and having a thickness D varying from 1 to 8 mm, with spanwise width L and streamwise length D_1 in the ranges from 280 to 500 mm and from 200 to 800 mm, respectively), they utilized quite different scales for reduction of wake characteristics to dimensionless form. Namely, they used the undisturbed velocity U_0 of the oncoming stream as the velocity scale while the sum $D^+ = D + 2\delta^*$, where δ^* is the displacement thickness of the upper or lower boundary layer near the trailing edge of the plate, was taken to be the length scale. (the length D^+, which was first introduced by Bauer (1961), evidently characterizes the real 'height' of a barrier restraining the flow. This length is similar in many respects to the length scale D^* used by Robichaux et al. (1999) for reduction of the great difference between the St-Re relations for circular-cylinder and square-cylinder wakes; see Fig. 4.27b above and explanations relating to it in the text). Eisenlohr and Eckelmann showed that this definition of the length scale leads to a universal value of the critical Reynolds number, $\mathrm{Re}_{\mathrm{cr}} = (U_0D^+/\nu)_{\mathrm{cr}} \approx 140$, and to a universal form of the general *flat-plate Roshko law* (4.47):

$$\mathrm{Ro} = 0.286\,\mathrm{Re} - 39.2 \tag{4.52a}$$

(where $\mathrm{Ro} = fD^{+2}/\nu$ and f is the frequency of wake oscillations) which was found to be valid with quite satisfactory accuracy for all the plates and all the considered (rather large) values of Re (see Fig. 4.28). Since the boundary-layer thickness δ^* grows with the stream length D_1 of the plate, one may try to use Eq. (4.52a) to explain of the dependence of the values of $\mathrm{St_D} = fD/U_0$ at fixed $\mathrm{Re_D} = U_0D/\nu$ on D_1/D found by Nakayama et al. (1993), but Nakayama et al. did not do this. However later Hammond and Redekopp (1997) recalculated some of their results, in which the plate thickness D had been used as the basic length scale, by including the displacement thickneses of the two boundary layers in the length scale. They found that, with this normalization, their numerically-simulated data led to dimensionless values of f which agreed quite satisfactorily (with a difference of about 4 %) with those observed by Eisenlohr and Eckelmann (1988) at the same values of Re (calculated using the new length scale). This agreement clearly provides additional validation of the results of both these papers.

As was explained above, the law (4.47) (whose particular cases include (4.52) and (4.52a)) is in fact a consequence of the Landauf-Stuart Eqs. (4.34–4.34a) and Eq. (4.46) Since in the case of a flat-plate boundary layer $\delta^* \propto (U_0)^{-1/2}$ (more precisely, $\delta^* \approx 1.73(\nu L_1/U_0)^{1/2}$; see, e.g., Eq. (1.56) in the book by Monin and Yaglom (1971)), Eq. (4.52a) explains the old observation by Taneda (1958), who discovered that in the wake behind a thin flat plate parallel to the flow the oscillation frequency f grows with the flow velocity U_0, not linearly (as in the case of a circular-cylinder wake where the linear relation between f and U_0, i.e. the constancy of St, was established, for a wide range of Reynolds number, by Strouhal (1878) and Rayleigh (1894) and at large values of Re follows from Eq. (4.47a)), but in proportion to $(U_0)^{3/2}$. In fact, when the thickness D of a plate is much smaller than the boundary-layer thickness δ^*, $D^+ = D + 2\delta^*$ is practically proportional to $(U_0)^{-1/2}$ while $\mathrm{Re} = U_0 D^+/\nu \propto (U_0)^{1/2}$. Then Eq. (4.52a) implies that $f \propto (U_0)^{3/2}$ at large values of U_0—this is just the result found by Taneda (1958) which was confirmed by the data by Eisenlohr and Eckelmann shown in Fig. 4.28. However, in flows around circular cylinders δ^* is much smaller than the cylinder diameter D; therefore here the boundary-layer thickness may be neglected and hence $f \propto U_0$ approximately.

Computations of Hannemann and Oertel, Nakayama et al., and Hammond and Redekopp, and also the measurements by Schumm et al. and Eisenlohr and Eckelmann, concern two-dimensional flat-plate wakes only. However Meiburg and Lasheras (1988) and Lasheras and Meiburg (1990) have demonstrated, both experimentally and by numerical simulations, that two different three-dimensional vorticity modes can be generated at moderate values of Re in the two-dimensional wake behind a thin flat plate, by introducing spanwise-varying disturbances in the flow near the trailing edge of the plate. The authors described the symmetry properties of these two modes, which later proved to be practically coincident with the symmetries of the modes A and B in circular-cylinder wakes, first discovered at approximately the same time (in particular, by Williamson (1988b)), but investigated in detail only later. According to Julien et al. (1997) both these modes can also occur in the undisturbed flat-plate wake (apparently at greater values of Re). Therefore, one may surmise that the evolution of the wake of a thin flat plate with increasing Re is similar to that of the wake behind a circular cylinder. Let us recall in this respect that the same similarity to circular-cylinder wakes was discovered by Sohankar et al. (1999) and Robichaux et al. (1999) for wakes behind square cylinders facing the flow.

A thin flat plate parallel to the flow direction corresponds to the special case of a rectangular cylinder with cross-section shown in Fig. 4.26d where $D \ll D_1$ (and hence it is possible to consider the limiting case where $D/D_1 \to 0$). Another interesting limiting case occurs when $D \gg D_1$; it corresponds to flows around long thin plates of finite width D placed in a uniform stream of velocity U_0 but this time normal to the stream direction. The two-dimensional vortex-shedding regime of the wake behind such a plate was briefly considered by Jackson (1987) (the case

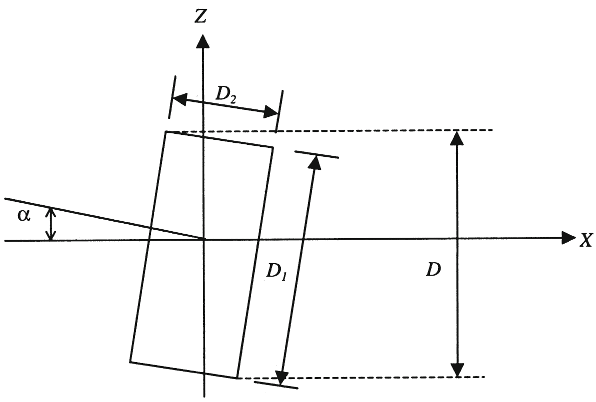


Fig. 4.29 General view of the cross-section of a rectangular cylinder at angle of incidence α

of the cross-section shown in Fig. 4.24c corresponding to $\theta = 0$); according to this computations the transition from a steady wake regime to an oscillating, vortex-shedding, regime occurs here at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}} \approx 27.77$ and the frequency of oscillations arising at this Re corresponds to a Strouhal number $\mathrm{St}_{\mathrm{cr}} \approx 0.1237$. More detailed investigations of the normal-plate wake regime at higher values of Re were carried out by many researchers; here we will mention only Roshko's (1993) survey paper and the short announcement, and rather long subsequent paper, by Najjan and Balachandar (1996, 1998) devoted to discussion of the recent DNS results and also containing (in the paper of 1998) an extensive list of references relating to this subject.

The cited papers on square-cylinder and flat-plate wakes represent only a few examples of numerous studies of wakes behind square and non-square rectangular cylinders, placed along the spanwise axis Oy, in a uniform stream at different angles of attack α between 0 and 90° (see Fig. 4.29). Many characteristics of such wakes (in particular, frequencies of wake oscillations, fluctuating velocities at various points, and pressure, drag and lift forces) were measured by Okajima (1982); Okajima and Sugitani (1984); Knisely (1990), and Norberg (1993), among others, while papers by Davis and Moore (1982); Davis et al. (1984); Franke et al. (1990); Okajima (1990, 1995); Okajima et al. (1992); Li and Humphrey (1995); Sohankar et al. (1995, 1997, 1998, 1999), and some other authors concentrated mainly on analysis of numerical-simulation data but often included supplementary experimental results and cited many additional references. Below we will briefly consider only a small part of the material presented in the above list of papers, which is itself very far from being complete.

Franke et al. (1990) numerically simulated square-cylinder wakes at zero incidence and $40 < \mathrm{Re} < 300$, and compared the resulting St-Re relation with the experimental data of Okajima (1982) and the experimental and numerical data of Davis and Moore (1982) and Davis et al. (1984). They found relatively large discrepancies between the results, and came to the conclusion that there were apparently some significant uncertainties in both experiments and simulations. Knisely (1990) performed numerous measurements (both in a wind tunnel and a water channel) of characteristics of wakes behind square and non-square rectangular cylinders with side ratios D_2/D_1 ranging from 0.04 to 1 and with angles of attack α from 0 to 90° (the data for $\alpha = 0$ and 90° were naturally the most numerous) and supplemented his experimental results by an informative review of similar data from other researchers. In particular, Knisely presented many graphs showing the dependence of the Strouhal number $St = fD/U_0$ (where f is the frequency of wake oscillations, U_0 is the free-stream velocity, and $D = D_1 \cos \alpha + D_2 \sin \alpha$ is the apparent thickness of the rectangle seen from the front, as indicated in Fig. 4.29) on the angle of attack α, for wakes of cylinders with various D_2/D_1 (but Re was often not held constant in his experiments). Norberg (1993) measured, in a wind tunnel, the values of the Strouhal numbers St and pressure forces for wakes behind rectangular cylinders of high aspect ratio $L/D_1 > 50$ (where L is the spanwise length of a cylinder) having various side ratios D_2/D_1 (in the range from 1 to 5), and placed at various angles of attact α in streams corresponding to various Reynolds numbers $Re = U_0 D/\nu$. The values of St were first of all measured for the case where $\alpha = 0^{\circ}$ is fixed but the ratios D_2/D_1 and Reynolds numbers Re take various values. This allowed Norberg to determine the dependence of the number St on D_2/D_1 at different values of Re, and on Re (in the range $400 \leq Re \leq 3 \times 10^4$) at a number of values of D_2/D_1. Then the values of St were measured at various values of all three parameters Re, D_2/D_1 and α and the dependence of St on α was graphically presented at a number of values of Re and D_2/D_1. Li and Humphrey (1995) analyzed the numerically-simulated data on the St-Re relation for wakes behind square cylinders at various orientations and $100 < Re < 1{,}000$.

The examples of rectangular-cylinder-wake studies presented here should give a general idea of this extensive field of research, which is quite important in practice. The studies of the wakes behind non-circular and non-rectangular cylindrical bodies are much less numerous than those for the cases of circular and rectangular cylinders, and here only two typical examples of such studies will be mentioned. Eibeck (1990) compared, for $Re = U_0 D/\nu = 1.3 \times 10^5$, the data of circular-cylinder wake measurements with results of similar measurements behind a cylinder with the tapered cross-section having a circular (of diameter D) upstream part turning smoothly into a triangular downstream part with a sharp angle at the apex (so that the streamwise length D_1 of the considered cylindrical body was almost 2.5 times greater than its thickness D). He found that the vertical structures differed appreciably in two compared wakes. Breier and Gatzmanga (1995) measured, in a wide range of Reynolds numbers, the St-Re relations for wakes behind cylindrical bodies of rectangular, triangular, trapezoidal, and a more complicated combined cross-sections, trying to determine in which case St is practically independent on Re in the most wide range of Reynolds numbers. Their purpose was to find the cross-section guaranteeing that the wake-oscillation frequency is proportional to flow velocity U_0 in a wide ragne of velocities, and hence the velocity measurements can be replaced by more simple frequency measurements. (The utilization of wake-frequency measurements for determination of flow velocity was first suggested by Roshko (1953, 1954) and later was practiced on a large scale; see, e.g., the discussion of this subject by

Takamoto (1987)). Some recommendations relating to this matter are included in the Breier and Ganzmanga's paper."

4.2.4.4 Wakes Behind Tapered Cylinders and Circular Rings

Now we will turn to wakes behind bluff bodies nonhomogeneous in the 'spanwise' direction (in contract to 'spanwise homogeneous bodies' considered above). We will begin with the case of vortex shedding from *linearly tapered cylinders* of length L with diameters D_1 and $D_2 < D_1$ of two ends at the points with coordinates (0, 0, 0) and (0, L, 0) (if $D_2 = 0$, the cylinder clearly becomes a cone). As in the cases considered above, the axis Oy (directed along the cylinder or cone axis) is assumed to be orthogonal to the stream direction Ox, but now circular cross-sections of a cylinder have diameters diminishing linearly with y. The study of vortex shedding from tapered cylinders was initiated by two papers by Gaster (1969, 1971) who investigated the wakes behind such cylinders placed in a water tunnel at first (in the paper of 1969) for the cases of the taper ratios $R_T = L/(D_1 - D_2)$ equal to 36 and 18 and then (in 1971) for the case of a more mildly tapered cylinder with $R_T = 120$ (the wake behind a circular cylinder was also studied in the latter paper which has been already referred to above in this connection). Later, further measurements of vortex shedding from linearly tapered cylinders and cones, with different values of R_T (ranging from 13 to about 600) and $\phi = \tan^{-1}[(D_1 - D_2)/2L]$ were obtained, in particular, by Piccirillo (1990); Van Atta and Piccirillo (1990); Noack et al. (1991); Papangelou (1991, 1992), and Piccirillo and Van Atta (1993), while Jespersen and Levit (1991) carried out a numerical simulation of the flow past a tapered cylinder with $R_T = 100$.

In 1969 Gaster found that wake oscillations behind a tapered cylinder do not have one dominant frequency f but are characterized by a combination of two quite different main frequencies f_1 and $f_2 \ll f_1$ (the frequency f_2 modulates the high-frequency wake oscillations and depends only on $(U_0)^2/\nu$ but not on the body length scales). In the second paper (1971) his measurements at $R_T = 120$ showed that the wake oscillations have a definite cellular nature, i.e. are composed of spanwise cells with a constant dominant shedding frequency which changes from cell to cell. Later such cells were discovered in all the wakes of tapered cylinders and cones considered in the above-mentioned papers, whenever $\mathrm{Re}_{max} = U_0 D_1/\nu$ was not too large. It was found that the cells often have clear boundaries and quite definite dominant frequencies (see, e.g., a typical example shown in Fig. 4.30a). Recall that cellular structure was also found by many authors in the wakes of circular cylinders, but there the cells usually depended essentially on conditions at the cylinder ends, while in the cases of tapered cylinders no influence of the end conditions on the cell structure was found. (In this respect the cells behind tapered cylinders are similar to cells of circular-cylinder wakes in shear flows with undisturbed velocity $U_0 = U_0(z)$ having a constant velocity gradient dU_0/dz, studied, e.g., by Griffin (1985) and Woo et al. (1989)).

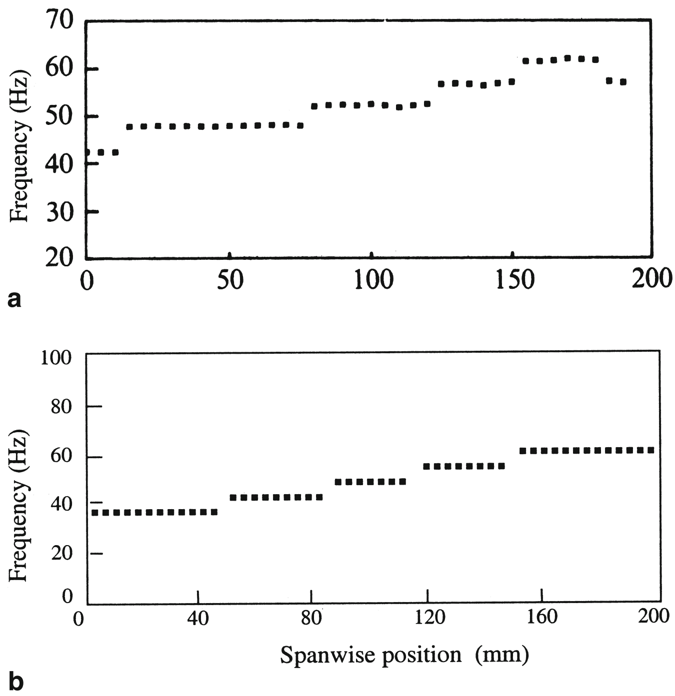


Fig. 4.30 (**a**) The dependence of the measured dominant frequency f of wake oscillations at points $\{x, 0, y\}$ (where $x \approx 15$ mm is fixed) behind the tapered cylinder on the spanwise coordinate y. (After Papangelou (1991, 1992)). The cylinder with the end diameters $D_1 = 2.57$ mm, $D_2 = 1.55$ mm and the length $L = 202$ mm was placed normal to the air flow of velocity U_0 such that $\mathrm{Re}_{max} = U_0 D_1/\nu = 123$, (**b**) Values of the frequencies $f(y)$ computed by the Ginzburg–Landau model Eq. (4.50) with the appropriately chosen coefficients. (After Papangelou (1991, 1992))

The dependence of the cell lengths and frequencies on the values of the taper ratio R_T and of the maximal and mean Reynolds numbers Re_{max} and $\mathrm{Re}_{mean} = U_0(D_1 + D_2)/2\nu$ was investigated carefully by Van Atta and Piccirillo (1990); Piccirillo and Van Atta (1993) and Papangelou (1991, 1992). It was found in particular that the difference in shedding frequencies between adjacent spanwise cells is a constat, coinciding with Gaster's modulation frequency f_2, and that the spanwise length of a cell divided by the cylinder diameter at the cell midpoint D_{cm} multiplied by R_T is also constant if $\mathrm{Re}_{cm} = U_0 D_{cm}/\nu > 100$. Piccirillo and Van Atta (1993) also found that the dependence of the cell Strouhal number $\mathrm{St}_c = f_c D_{cm}/U_0$, where f_c is the frequency of cell oscillations, on the cell Reynolds number $\mathrm{Re}_c = U_0 D_{cm}/\nu$ may be approximated with reasonable accuracy by the Rayleigh-Roshko law (4.47a) with constant coefficients $a \approx 0.195$ and $a_1 \approx 5.0$.

The Roshko law is a consequence of the Landau Eq. (4.34) and (4.34a) but now St_c and Re_c vary with the spanwise coordinate y. Therefore an analytic model describing wake oscillations behind tapered cylinders and cones must include the dependence on y in some way. The first, rather crude, model of this type was proposed by Gaster (1969) who described the wake oscillations $u(\mathbf{x}, t)$ by a system of coupled equations representing nonlinear van der Pol oscillators (with a coupling described by a spanwise-diffusion term proportional to $\partial^2 u/\partial y^2$) and corresponding to different spanwise coordinates y. This model was later refined by Noack et al. (1991), who applied their modification of Gaster's model to describe the cellular structure of wakes behind both untapered and tapered circular cylinders. However Papangelou (1992) found that the model of Noack et al. successfully describes only the appearance of spanwise cells, not their quantitative characteristics. Therefore he tried to utilize the Landau–Ginzburg model (4.50) for this purpose. The estimates (4.48) of complex coefficients $\omega = \omega_1 + i\gamma$ and $l = \delta + i\delta'$ given by Sreenivasan et al. (1987) were used in Papangelou's model, together with their estimate $\mathrm{Re}_{\mathrm{cr}} \approx 46$ of the critical Reynolds number (but now the values of D and $\mathrm{Re} = U_0 D/\nu$ were dependent on y) while the coefficient μ was assumed to be real and positive (contrary to the applications of Eq. (4.50) to modeling of wakes behind non-tapered cylinders described above, where μ was always assumed to be complex). Solutions of the corresponding Eq. (4.50) with various positive values of μ showed that this value may be chosen in such a way that the solution will satisfactorily describe many (though not all) quantitative features of the observed cell structure (see, e.g., Fig. 4.30b).

Tapered cylinders and cones with axes orthogonal to the stream direction represent only one special class of spanwise-inhomogeneous bluff bodies. Now we will turn to another class of such bodies, to whose wakes the G-L model (4.50) was also applied with definite success. Recall first of all that this model, supplemented by the appropriate boundary conditions at cylinder ends, allows a number of important characteristics of wakes behind circular cylinders to be calculated with satisfactory accuracy. However the experiments show that the flow regime of such a wake depends very substantially on the details of flow conditions near the cylinder ends, and this circumstance essentially complicates the determination of the boundary conditions which are 'appropriate' for a given experiment. Therefore as a rule, calculations based on the G-L model use some artificial boundary conditions selected by the requirement to produce results consistent with the available data. Because of this, Leweke et al. (1993a, b) and Leweke and Provansal (1994, 1995) applied the same model to the case where a cylinder of finite length was curved into a torus (a circular ring) so that no end conditions were needed.

Roshko (1953, 1954) was apparently one of the first researchers to study the *wake behind a circular* (*toroidal*) *ring* placed perpendicular to a uniform stream of velocity U_0. He showed that for $L/D \geq 10$ (where L is the ring outside diameter and D is its cross-section diameter) and for a not-too-small value of $\mathrm{Re} = U_0 D/\nu$, vortices are shed from a ring in almost the same way as from a straight cylinder, and form an annular vortex street. Later it was realized that frequency measurements in such wakes can be successfully used for the flow-velocity determinations (see the remark at the end of the previous part (c) relating to this matter) and this fact stimulated more detailed experimental investigations of wakes behind rings by Takamoto and Izumi (1981); Monson (1983); Takamoto (1987) and Bearman and Takamoto (1988) (in the two latter papers, wakes behind circular rings of trapezoidal cross-section were studied in detail, and wakes of some rings with rectangular and triangular cross-sections were also considered in passing), and finally by Leweke and his coworkers, whose studies of 1994 and 1995 of wakes behind toroidal rings contain the most interesting experimental data for the subject considered here. It was found in these works that the vertical structure behind a ring of circular shape can have a number of different forms: the wake can consist of an array of counter-rotating vortex rings parallel to the central plane of the toroidal solid ring, or of counter-rotating inclined vortex rings (i.e., shed at some angle θ with respect to the plane of symmetry of the torus), or of a pair of counter-rotating helical vortices (i.e., any inclined vortex after one 'round' connects to the next one) with discrete helix steps of $2\pi n/k$ (where n is an integer and k is a fixed streamwise wave number of wake oscillations), or of groups of interwoven helical vortices, and so on. Thus, a number of different normal modes can exist in the ring wakes. The number n (which can take either sign) also determines the dependence of the phase Φ of the wake velocity oscillations on the 'spanwise coordinate' $y = L\phi/2$ (where ϕ is the angular coordinate of the cylindrical coordinate system (x, r, ϕ) with the origin at the center of symmetry of the toroidal ring). Namely, as y increases from $y = 0$ (at an arbitrary point of the ring) to $y = \pi L = L_1$ (where L_1 is the length of the outer circle of the torus), the difference $\Phi(y) - \Phi(0)$ changes from zero to $2\pi n$. If $n = 0$, the vortex rings are parallel to the torus midplane and hence correspond to 'parallel shedding' with $\theta = 0$, while the wake structures with $n \neq 0$ are produced by 'oblique shedding', with shedding angle $\theta \neq 0$ depending in a definite way on n, D, L and k. Leweke and Provansal (1995) constructed graphs representing the St-Re relations (where again $\mathrm{St} = fD/U_0$, $\mathrm{Re} = U_0 D/\nu$ and f is the frequency of wake oscillations) for various values of n and showed that Williamson's 'cosine law' of oblique shedding is valid here too, with high accuracy. (This means that if St is the Strouhal number corresponding to oblique shedding at angle θ, the $\mathrm{St_m} = \mathrm{St}/\cos\theta$ practically coincides with the value of St corresponding to parallel shedding, i.e. $n = 0$, at the same values of Re and L_1/D). Generally speaking, the values of St depend on three variables-the aspect ratio L_1/D, n and Re, if $n = 0$ the St-Re relation for the ring wake tends, as $L_1/D \to \infty$, to the straight-cylinder relation shown in Fig. 4.20 (see Fig. 4.31).

The experiments also show that in the wake of a ring, every mode of wake oscillations is characterized by its own critical Reynolds number $\mathrm{Re_{cr,\,n}}$, so that at $\mathrm{Re} < \mathrm{Re_{cr,\,n}}$ the nth-mode disturbances cannot exist at all (any such disturbance dies down to zero whatever the initial amplitude). The Reynolds numbers $\mathrm{Re_{cr,\,n}}$ are ring-wake equivalents of the critical Reynolds number $\mathrm{Re_{cr}} \approx 46$ characterizing the beginning of the vortex shedding in the circular-cylinder wake, but now the transition Reynolds number depends on the number n of the emerging vertical mode, and hence the whole family of integers n must be considered. Moreover, the nth mode is itself stable only for a definite Reynolds-number range $\mathrm{Re_{cr,n}} < \mathrm{Re} < \mathrm{Re^*_{cr,n}}$ while at $\mathrm{Re} > \mathrm{Re^*_{cr,n}}$ this mode becomes unstable to small disturbances and therefore transforms into a different, more complicated, vortical structure. (Reynolds number $\mathrm{Re^*_{cr,n}}$ is the nth-mode ring-wake equivalent of the Reynolds number $\mathrm{Re_{2,cr}} \approx 190$ characterizing the beginning of instability of the two-dimensional Bénard-Kármán vortex street produced by parallel vortex shedding; now it also depends on the mode

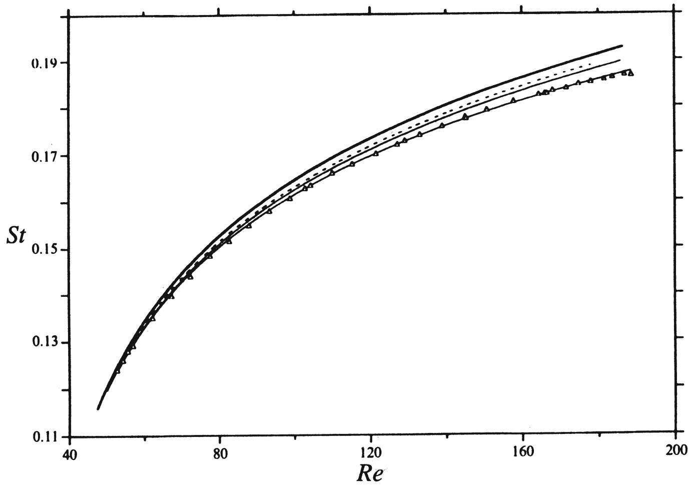
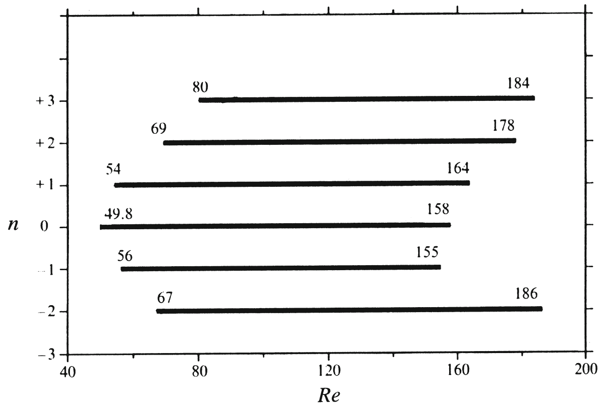


Fig. 4.31 Comparison of the St-Re curves for parallel vortex shedding (i.e., $n = 0$) from rings of different aspect ratios L_1/D with the curve for a straight cylinder: $L_1/D = 99.5$ (- - -), 59.0 (- - - - -), and 31.5 (- - Δ - -); straight cylinder (——) (After Leweke and Provansal (1995))

Fig. 4.32 Stability domains $\text{Re}_{\text{cr,n}} < \text{Re} < Re^*_{\text{cr,n}}$ of periodic-vortex-shedding modes with different values of n for the wake of a ring with aspect ratio $L_1/D = 59.0$. (After Leweke and Provansal (1995)). The numbers indicate the critical Reynolds numbers $\text{Re}_{\text{cr,n}}$ and $\text{Re}^*_{cr,n}$



number n). The 'stability regions' $\text{Re}_{\text{cr,n}} < \text{Re} < Re^*_{\text{cr,n}}$ corresponding to various modes of ring-wake oscillations often overlap (see the typical Fig. 4.32 showing some experimental data of Leweke and Provansal (1995)). Therefore for many values of Re several normal modes are stable simultaneously. Apparently the initial conditions alone determine which mode will dominate the wake oscillations in such cases. It was also shown that the vortical structure of the wake depends significantly on the aspect ratio L_1/D. In particular, for aspect ratios smaller than about 20, the ring wake behaves similarly to the wake of a solid disk. On the other hand, for $20 < L_1/D < 100$ the ring curvature plays relatively minor role, and locally the wake has an appearance similar to that of the wake of a straight long cylinder. (However the minimal critical Reynolds number $\mathrm{Re}_{\mathrm{cr}} = \mathrm{Re}_{\mathrm{cr},0}$ depends here on the body-curvature parameter $K = D/L_1$ and increases nearly linearly with K; see again the paper by Leweke and Provansal (1995)). Referring to the similarity of the ring wake to that of straight cylinder, Leweke et al. (1993a, b) applied the G-L Eq. (4.50) to the wake of a ring, with the same numerical coefficients as were used successfully in the case of the wake of a straight circular cylinder. However, later Leweke and Provansal (1994, 1995) carried out a direct experimental determination of some coefficients of Eq. (4.50) for ring wakes.

Leweke and Provansal used the fact that the boundary conditions for the amplitude $A(y, t)$ of wake oscillations in the case of the wake of a ring have a very simple form: here evidently $0 \le y \le L_1$ and $A(0, t) = A(L_1, t)$ for any $t \ge 0$. It follows from this that the amplitude $A(y, t)$ can be represented as a sum of Fourier components of a form $A_\mathrm{n}(y, t) = B_\mathrm{n}\exp\{i[\Omega_\mathrm{n} t + Q_\mathrm{n} y]\}$, whre n takes integer values (only components with $|n| = 0$, 1, 2 and 3 were in fact detected in their experiments), $Q_\mathrm{n} = 2\pi n/L_1$, and the real amplitudes B_n and angular frequencies Ω_n can be determined from the G-L Eq. (4.50). In particular, the real and imaginary parts of Eq. (4.50) imply that the equilibrium values of amplitude B_n and angular frequency Ω_n (which do not depend on t) are given by the following expressions

$$B_n = \left[\frac{2(\gamma - \mu_r Q_n^2)}{\delta}\right]^{1/2}, \quad \Omega_n = -\left(\omega_1 + \gamma\frac{\delta'}{\delta}\right) - \left(\mu_i - \mu_r\frac{\delta'}{\delta}\right) Q_n^2 \tag{4.53}$$

where, as usual, $\omega_1 + i\gamma = \omega$, $\delta + i\delta' = l$, and $\mu_r + i\mu_i = \mu$. (Equation (4.53) generalized the known equations determining the equilibrium amplitude A_e and Strouhal frequency $f = \Omega_0/2\pi$ which follow from Landau's Eqs. (4.34) and (4.34a) and correspond to parallel shedding where $\mu = \mu_r + i\mu_\mathrm{I} = 0$). Leweke and Provansal used for γ and δ'/δ the values $\gamma = 0.2(\nu/D^2)(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ and $\delta'/\delta = -3.0$ which were obtained earlier from data of circular-cylinder wake experiments; this means that the effect of the ring curvature was neglected (relying on measurements by the authors which show that this effect does not play an important part if D/L_1 is small enough; see, e.g., Fig. 4.31). However to find μ_r Leweke and Provansal used the equations

$$B_0^2 = \frac{0.4\nu}{\delta D^2}(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}), \quad \frac{B_n^2}{B_0^2} = 1 - \frac{4\pi^2\mu_r/\nu}{0.2(L_1/D)^2(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})}n^2 \tag{4.54}$$

which follows from the first Eq. (4.53) and the expression for γ given above. Eq. (4.54) were verified by measurements of $(B_0)^2$ and $(B_n/B_0)^2$, where B_0 and B_n are the amplitudes of the zeroth and n th modes of the streamwise-velocity oscillations, in the wake behind a ring of outer diameter $L = 56.9$ mm and cross-sectional diameter $D = 3.03$ mm (so that the aspect raito $\pi L/D = L_1/D$ was 59.0). The oscillations of the streamwise velocity $u(\mathbf{x}, t)$ were measured at the point with coordinates (7D, 0, −2D) in a coordinate system with the origin at the ring center and the Ox axis pointing in the downstream direction. Measured values of the squared normalized amplitude $(B_0 D/\nu)^2$ of the zeroth oscillation mode at various values of $\mathrm{Re} = U_0 D/\nu$ are shown in Fig. 4.33; they confirm the proportionality of $(B_0)^2$ to $\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}$ over

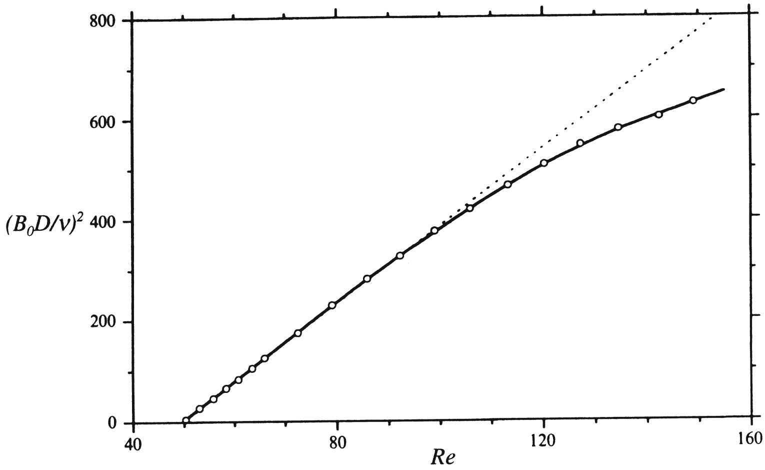


Fig. 4.33 Dependence of the normalized square of streamwise velocity fluctuations, $(B_0D/v)^2$, on $\mathrm{Re} = U_0D/v$ for the parallel shedding mode with $n = 0$ at the point {x, y, z} = {7*D*, 0, −2D} behind a ring with aspect ratio $L_1/D = 59.0$. (After Leweke and Provansal (1995))

a considerable range of Re, and allow the critical Reynolds number $\mathrm{Re}_{\mathrm{cr}}$ and the value of δ corresponding to streamwise-velocity oscillations at the measurement point to be estimated with good accuracy. The measured values of $(B_nB_0)^2$, where n took the values 1, 2, and 3, proved to be more scattered than the values of $(B_0)^2$, but on the whole they agreed with the second Eq. (4.54) and led to the conclusion that $\mu_r/v \approx 10$ over a range of not-too-high values of Re – $\mathrm{Re}_{\mathrm{cr}}$. Above $\mathrm{Re} = 100$, however, μ_r/v begins to increase with Re. Moreover, the second Eq. (4.53) allows $\mu_i - \mu_r(\delta'/\delta) = \mu_r[(\mu_i/\mu_r) - (\delta'/\delta)]$ to be determined from measurements of the difference of two angular frequencies $\Omega_n - \Omega_m$ (or of two ordinary frequencies $f_n - f_m = (\Omega_n - \Omega_m)/2\pi$) corresponding to two different oscillation modes of the ring wake. Leweke and Provansal (1994) measured the differences $f_n - f_m$ for a number of integer values of n and m and various values of Re, and deduced the dependence of $(\mu_r/v)[(\mu_i/\mu_r) - (\delta'/\delta)]$ on Re over a wide range of Reynolds numbers. The results were compared with estimates of $(\mu_r/v)[(\mu_i/\mu_r) - (\delta'/\delta)]$ from measurements of wake oscillations behind circular cylinders made by Williamson (1989) and Monkewitz, Williamson and Miller (whose results were known in 1994 but were published only in 1996). The comparison showed that the estimates derived from data on wake oscillations behind straight cylinders and behind rings agree rather satisfactorily with each other. Analyzing the data of both types Leweke and Provansal recommended in 1994, for a wide range of not too high supercritical values of Re, the estimate: $(\mu_i/\mu_r) - (\delta'/\delta) = 2.9 \pm 0.8$, but in 1995 they replaced this by two separate estimates: $\mu_i/\mu_r \approx -0.65, \delta'/\delta \approx 3.0$. Then they showed that the G–L Eq. (4.50) with the above values of coefficients describes, quite satisfactorily, the general development of wakes behind rings placed normal to the flow and also many observable features of such wakes. Let us recall, however, the remark by Leweke and

Williamson (1998), which has been already mentioned at the end of the discussion of the transverse Ginzburg-Landau Eq. (4.50). They commented that the application by Leweke and Provansal (1995) of the G-L model for the determination of the instability threshold for the wake flow implied a type of instability differing from that observed in laboratory experiments or numerical simulations of wake flows.

4.2.4.5 Wakes Behind Spheres and Other Axisymmetric Bodies

Wakes behind circular rings placed normal to the flow represent a special example of wakes behind axisymmetric bluff bodies. However in the above discussion of ring wakes, we emphasized first of all their similarity to wakes behind straight circular cylinders, paying only secondary attention to their axial symmetry. Now we will consider some other axisymmetric wakes, concentrating mainly on the consequence of axisymmetry.

Axisymmetry wakes appear behind any body of revolution submerged in an uniform stream directed along the body axis. Vortex shedding from the downstream parts of such bodies, and global oscillations of the resulting wakes, have been observed by many researchers. It was found that these features are related to the existence in the wakes behind axisymmetric bodies, in the cases when the Reynolds number Re is not too small, of zones of absolute instability with respect to non-axisymmetry disturbances with azimuthal wave number $n=1$ (see, e.g., Monkewitz (1988c)). Since the sphere is a prototype axisymmetric body, the *wakes behind spheres* are clearly the most significant axisymmetric wakes. Flows past spheres can be easily produced in the laboratory and are encountered in some engineering devices and natural phenomena; therefore sphere wakes began to attract attention very early and were studied quite extensively. In Sect. 2.2 it was mentioned that the dependence of the drag of a sphere submerged in a fluid flow on the Reynolds number Re was studied long ago by Eiffel (1912) and Prandtl (1914) (in fact there were also many other early studies of sphere drag); all these studies inevitably included the consideration of sphere wakes. The formation of vortices behind a sphere and vortex shedding from spheres were described in the 1930s in particular by Winny (1932); Foch and Chartier (1935), and Möller (1938), while later the vortical structures and quantitative characteristics of sphere wakes were studied by Taneda (1956, 1978); Torobin and Gauvin (1959); Magarvey and Bishop (1961a, b); Magarvey and MacLatchy (1965); Goldburg and Florsheim (1966); Zikmundova (1970); List and Hand (1971); Calvert (1972); Masliyah (1972); Achenbach (1972, 1974); Nakamura (1976); Pao and Kao (1977); Perry and Lim (1978); Kim and Durbin (1988); Sakamoto and Haniu (1990, 1995); Berger et al. (1990); Bonneton and Chomaz (1992); Wu and Faeth (1993); Provansal (1996); Provansal and Ormières (1998); Ormières et al. (1998); Ormières and Provansal (1999), and many other experimenters. Nevertheless experimental data for sphere wakes continue to be scattered and sometimes contradictory. The scatter can be explained by a number of factors complicating the wake measurements, such as the influence of the sphere supports, the effect of free-stream turbulence, and the weakness and slowness of wake oscillations at values of Re near the instability threshold. The influence of support devices can be diminished or even annulled by the use of spheres towed through, or freely falling or rising in, quiescent fluid but here some other complications often emerge. However the general features of sphere wakes are now known rather well, and many of them are quite similar to those of wakes behind circular cylinders.

The available data show that in the case of uniform external stream the flow around a sphere at low Reynolds numbers is steady, axisymmetric, and attached to the whole sphere body. At some greater value of Re, flow separation occurs and an axisymmetric, toroidal recirculation eddy, which is attached to the segmental area on the downstream side of the sphere, appears. According to experiments by Taneda (1956), the separation is first observed at $\mathrm{Re} = \mathrm{Re}_{0,\mathrm{cr}} \approx 24$ (where $\mathrm{Re} = U_0 D/\nu$ is based on the sphere diameter and free stream velocity). This estimate agrees with the results of some relatively early theoretical investigations of flows around a sphere, using either analytical or numerical approximations of the corresponding solutions of the Navier-Stokes equations (see, e.g., the summary of a number of such studies by Pruppacher et al. (1970) which implies that $\mathrm{Re}_{0,\mathrm{cr}} \approx 20$). There were also some experimenters who obtained different estimates of $\mathrm{Re}_{0,\mathrm{cr}}$ (e.g., Nakamura (1976) found that $\mathrm{Re}_{0,\mathrm{cr}} \approx 10$, and this estimate was also given by Wu and Faert (1993) who, however, made no measurements at so small value of Re). On the other hand, numerical simulations of flow past a sphere by Shirayama (1992), and the subsequent more careful and explicit simulations by Tomboulides (1993) (see also Tomboulides et al. (1993)) and Johnson (1996) (see also Johnson and Patel (1999)), which will be discussed at greater length later, confirmed the old estimates of $\mathrm{Re}_{0,\mathrm{cr}}$ given by Taneda and Pruppacher et al. (all of them show that $\mathrm{Re}_{0,\mathrm{cr}} \approx 20$). As Re increases further, the flow remains axisymmetric and steady, but the downstream extent of the recirculating wake zone, and the separation angle which determine the sphere segment adjoining to this zoen, progressively increase. The increase with Re of the streamwise length of the recirculating zone and of the separation angle were measured in Taneda's and Nakamura's experiments and were also determined from the numerically-simulated data by Pruppacher et al. (1970); Fornberg (1988); Shirayama (1992); Tomboulides (1993); Magnaudet et al. (1995), and Johnson (1996) who found that the numerical results agree quite well with each other and with the experimental ones (with the sole exception of Taneda's values of the length of recirculating zone at large values of Re, which were obviously underestimated; see, e.g., Fig. 4.34 and the above-mentioned papers by Shirayama, Tomboulides et al., and Johnson and Patel). The same is also true for values of the drag coefficient of a sphere which were also computed by Shirayama, Tombulides, and Johnson in a range of Reynolds numbers not too far above the critical value $\mathrm{Re}_{0,\,\mathrm{cr}}$; here again the computed values agree excellently with values given by Ross and Willmarth (1971) who accurately measured the sphere drag and compared their results to those of numerous previous drag studies. However, at some $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}$ in the range between 100 and 350, the steady axisymmetric wake flow becomes unstable, and this leads to an abrupt change of the wake structure. At this value of Re a new wake regime emerges which, according to the results of many recent studies, is non-axisymmetric and steady, while in some older work it was found to be non-axisymmetric and oscillating (more will be said about this

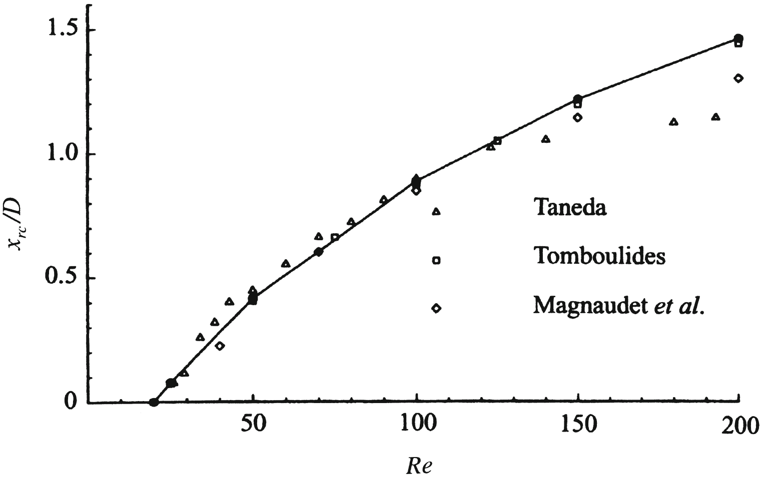


Fig. 4.34 Comparison of the dependence of the dimensionless streamwise length x_{rc}/D of the recirculation zone (measured on the wake axis) on Re given by Johnson and Patel's computations (the *solid line* with *black dots*) with results of Taneda's experiments and of earlier computations by Tomboulides, and Magnaudet et al. (After Johnson and Patel (1999))

below). The transition to an oscillating flow regime means that the periodic shedding of vortices begins at this Re, and signifies a Hopf bifurcation which may be described analytically by the complex Landau Eq. (4.40); while the replacement of one steady flow by another is a regular (non-Hopfian) bifurcation whose description does not require consideration of a complex amplitude equation.

In many cases, values of the critical Reynolds number $\mathrm{Re}_{1,\mathrm{cr}}$ (which for the sake of simplicity will often be denoted below by $\mathrm{Re}_{\mathrm{cr}}$) given by different experimenters disagree with each other. Recently Johnson and Patel (1999) stated that the observed onsets of the oscillatory shedding regime of a sphere wake covers the range $290 < \mathrm{Re}_{\mathrm{cr}} < 400$; however, if all the results indicated below were taken into account, then this range would be expanded to at least $130 \leq \mathrm{Re}_{\mathrm{cr}} < 400$. According to the experiments of Möller (1938), who towed a sphere through water, $170 < \mathrm{Re}_{\mathrm{cr}} < 200$. Later Taneda (1956) found that a weak oscillation with a long period appears in the sphere wake at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}} \approx 130$. A value of $\mathrm{Re}_{\mathrm{cr}}$ close to this was also found by Zikmundova (1970), who concluded from her observation of aluminum spheres dropped through the solutions of glycerol and water that $130 < \mathrm{Re}_{\mathrm{cr}} < 150$. Taneda's value of $\mathrm{Re}_{\mathrm{cr}}$ was accepted by some other authors (e.g., by Fornberg (1988)) but almost all recent data show that Taneda's and Zikmundova's estimates of this value were appreciably too low. (Note, however, that Provansal and his coworkers, whose work will be discussed at the end of this paragraph, found in the late 1990s that at $\mathrm{Re} \approx 150$ the sphere wake undergoes a bifurcation, but of a different type from that found by Taneda and Zikmundova). Magarvey and MacLatchy (1965), who made rather accurate observations of the wakes behind freely-falling solid spheres, found that the recirculation zone becomes unstable, and the wake begins to oscillate, only at $\mathrm{Re} \approx 300$. The start of wake oscillations at $\mathrm{Re} \approx 300$ (accompanied by an abrupt change of the vortical wake structures leading to the appearance of hairpin-shaped vortex loops) was later detected also by Levi (1980) and Sakamoto and Haniu (1990). Magarvey and Bishop (1961a, b) presented a number of photographs of wakes produced by liquid drops settling in a immiscible liquid; these photos show that the wake became non-axisymmetric at $\mathrm{Re} \approx 210$ but lost its steadiness only at $\mathrm{Re} \approx 270$. Goldburg and Florsheim (1966) also studied the wakes behind freely-falling solid spheres at moderate values of Re, and found that the dependence of the Strouhal numbers $\mathrm{St} = fD/U_0$ of wake oscillations on Reynolds number is described, with good accuracy, by the Rayleigh–Roshko Eq. (4.47a) with $a \approx 0.387$ and $a_1/a \approx 270$ over a considerable range of Re. As was shown above, Eq. (4.47a) follows from Landaus equation for the complex amplitude of wake oscillations. These values of coefficients a and a_1 show that the oscillatory wake regime was observed at $\mathrm{Re} > 270$, but Goldburg and Florsheim also noted that in their experiments the wake lost its axisymmetry at $\mathrm{Re} \approx 210$. Ross (1968) and Roos and Willmarth (1971) stated that their observations of spheres towed through water showed that $215 < \mathrm{Re}_{\mathrm{cr}} < 290$. According to Nakamura's (1976) experiments with falling spheres, some change in the nature of the wake occurs at $\mathrm{Re} = 190$, but the change was not described in detail and therefore Kim and Pearlstein (1990) interpreted it as a transition to non-axisymmetric oscillating wake regime while Natarajan and Acrivos (1993) took it as the loss only of the axisymmetry, but not the steadiness, of the wake flow. Shirayama (1992) described some experiments according to which $\mathrm{Re}_{\mathrm{cr}} \approx 250$, but he paid his main attention to flow simulation for $\mathrm{Re} = 500$. Then Wu and Faeth (1993) towed a polished plastic ball through a rectangular bath filled with quiescent water and glycerol mixture, visualized the flow near the towed sphere, and measured by laser velocimeter the mean streamwise velocities and root-mean-square velocity fluctuations at a number of points. Their measurements cover the range of Reynolds number $\mathrm{Re} = U_0D/\nu$ from 30 to 4,000, but for the topic discussed in this subsection the range $30 \leq \mathrm{Re} \leq 400$ represents the main interest. According to the results of these authors, the recirculation region on the downstream side of the sphere was steady and axisymmetric at $\mathrm{Re} < 200$, steady but non-axisymmetric at $200 < \mathrm{Re} < 280$, and unsteady with vortex shedding at $\mathrm{Re} > 280$. Still later the French researchers (Provansal (1996); Provansal and Ormières (1998); Ormières et al. (1998) and Ormières and Provansal (1999); see also beautiful photos presented by Leweke, et al. (1999)) used flow visualization in a water channel to observe the flow behind a fixed sphere (held by a thin upstream metallic pipe with three holes allowing to inject the dye into the water), and laser-Doppler and hotwire anemometers to measure velocities in a wind tunnel flow behind another sphere held inside the tunnel by four thin wires. According to their data, the sphere wake is steady and axisymmetric at $\mathrm{Re} < 150$, while at $\mathrm{Re} \approx 150$ its axisymmetry breaks and for $150 < \mathrm{Re} < 180$ the wake is non-axisymmetric but remains steady and has the vortical structure including a single linear vortical thread. At $\mathrm{Re} \approx 180$ this structure changes and becomes more complicated (begins to include a pair of vortical threads) but at $180 < \mathrm{Re} < 280$ the wake continues to be steady and non-axisymmetric. However, if Re grows further, then at $\mathrm{Re} = \mathrm{Re}_{\mathrm{cr}} \approx 280$ the sphere wake begins to oscillate with a frequency f which

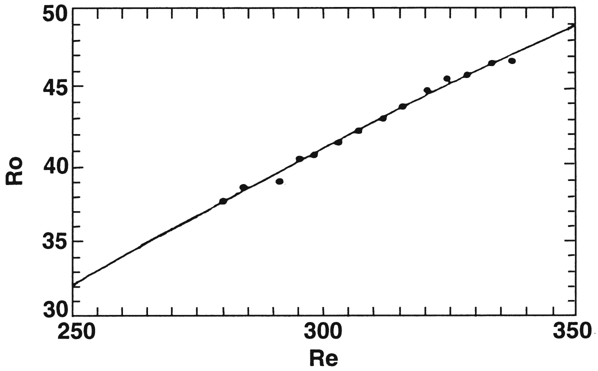


Fig. 4.35 Dependence of $\mathrm{Ro} = fD^2/\nu$ on $\mathrm{Re} = U_0 D/\nu$ in the wake behind a sphere at supercritical Reynolds numbers $\mathrm{Re} \geq \mathrm{Re}_{cr}$ according to measurements of Ormières et al. (1998)

does not depend on the point of observation, and corresponds to a Roshko number $\mathrm{Ro} = fD^2/\nu \approx 38$. Spectral analysis of streamwise velocity fluctuations was then used to measure the values of Ro in the Reynolds-number range $280 < \mathrm{Re} < 340$, corresponding to the periodic vortex-shedding regime. Data by Ormièdes et al. (see Fig. 4.35) show that in this range the Ro-Re relation can be approximated by the linear Eq. (4.47), while according to the 1998 and 1999 papers by Provansal and Ormières even higher precision can be reached if Eq. (4.47) will be replaced by the three-term equation $\mathrm{Ro} = a\mathrm{Re} - a_1 + a_2\mathrm{Re}^2$ where $a = 0.391$, $a_1 = -48.2$, and $a_2 = -3.6 \times 10^{-4}$ (recall that an equation of such form was earlier proposed by Williamson for the Ro-Re relation in the supercritical circular-cylinder wake; see the explanation relating to Fig. 4.20 in part (b) of this section).

The experimental results listed above (which clearly do not exhaust all the available results) must be supplemented by consideration of a few attempts to compute the value of Re_{cr} by applying linear stability theory to the axisymmetric steady flow around a sphere. The first such attempt was due to Kawaguti (1955), but his results ($\mathrm{Re}_{1,cr} = 51$, corresponding to instability of the steady sphere wake to unsteady axisymmetric disturbances) contradicts all other available results of stability computations (and also of experiments or simulations), and must therefore be disregarded. However the paper by Kim and Pearlstein (1990), whose results are apparently also incorrect, signified a more serious attack on the problem. Modifying Fornberg's (1988) approach, the authors computed a new the axisymmetric solution of the Navier–Stokes equations corresponding to the laminar flow past a sphere in a free stream with constant velocity $\mathrm{U}_0 = \{\mathbf{U}_0, 0, 0\}$. Then they investigated, in the framework fo the linear theory of hydrodynamic stability, the stability of this solution to infinitesimal disturbances proportional to $\exp[i(n\phi - \omega t)]$, where ϕ is the angular cylindrical coordinate, $n = 0, 1, 2..$, and possible values of ω are determined by the eigenvalue problem of linear stability theory. (Hence, both axisymmetric (azimuthal wave number $n = 0$) and non-axisymmetric ($n \neq 0$) disturbances were considered by the authors). The analysis showed that as $\mathrm{Re} = U_0 D/\nu$ increases the disturbance which becomes unstable first of all has the azimuthal wave number $n = 1$. According to Kim and Pearlstein's computations, the instability of disturbances with $n = 1$ emerges at

$\mathrm{Re} = \mathrm{Re}_{cr} = 175.1$ and leads to a non-axisymmetric oscillating flow regime. As to disturbances with other values of n, the authors did not find instability for any of them in the whole investigated range of Reynolds numbers. Let us recall from Sect. 2.94 that the linear-stability-theory results of Batchelor and Gill (1962) and some other authors showed that the disturbances with $n = 1$ are the most unstable in a number of other axisymmetric jet and wake flows, and that Monkewitz's (1988c) results also indicated the paramount role of disturbances with $n = 1$ in formation of the global instability modes in axisymmetric spatially-developing flows.

Kim and Pearlstein compared their theoretical results with the results of previous experimental work and concluded that the agreement of their theory with the experimental data is more or less satisfactory. However later Natarajan and Acrivos (1993), who solved the same stability problem by a more advanced numerical method leading to different results, reconsidered Kim and Pearlstein's conclusion. The new authors applied to the computation of the solution of the equations of motion, describing the steady axisymmetric flow past a sphere, the numerical procedures developed for other purposes by Fornberg (1991) and Natarajan et al. (1993). This allowed them to describe the flow more explicitly than was possible earlier. Then Natarajan and Acrivos applied a new numerical method to solution of the linearized equations describing the evolution of small disturbances in the flow past a sphere. This method confirmed the result of Kim and Pearlstein, according to which the disturbances which become unstable at the smallest value of Re have the azimuthal wave number $n = 1$ (and instability to disturbances with $n \neq 1$ was again not found for any Re). However, the new computations showed that the unstable disturbance with $n = 1$ first appears at Reynolds number $\mathrm{Re}_{1,cr} \approx 210$, greater than was found by Kim and Pearlstein, and the disturbance differs qualitatively from the unstable disturbance of Kim and Pearlstein's theory. Namely, according to Natarajan and Acrivos the disturbance which becomes unstable at $\mathrm{Re} = \mathrm{Re}_{1,cr}$ is non-axisymmetric but also nonoscillatory, i.e., it corresponds to a purely imaginary eigenvalue $\omega = i\gamma$ with the imaginary part γ (determining the growth rate of the disturbance) proportional to $(\mathrm{Re} - \mathrm{Re}_{1,cr})$ which is negative for $\mathrm{Re} < \mathrm{Re}_{1,cr}$ but positive for $\mathrm{Re} > \mathrm{Re}_{1,cr}$ (see Fig. 4.37b below). Hence the critical Reynolds number $\mathrm{Re}_{1,cr}$ signifies a regular bifurcation (not of the Hopf type), the replacement of the axisymmetric steady flow by a new steady flow which includes a non-axisymmetric velocity mode with azimuthal wave number $n = 1$. This means that the transition of the axisymmetric wake regime to instability here proceeds through a steady state, corresponding to zero eigenvalue $\omega = 0$, i.e., it is of the same "exchange of stabilities" type which was encountered in this book when the instabilities of the Taylor-Couette flow between two rotating cylinders and of an immovable fluid layer heated from below were considered (see Sects. 2.6 and 2.7).

Natarajan and Acrivos computed, for the stability problem relating to small non-axisymmetric disturbances with $n = 1$, not only the eigenvalue $\omega = \omega_0$ with the greatest imaginary part γ_0 (reducing to zero at $\mathrm{Re} = \mathrm{Re}_{1,cr}$) but also a number of other complex eigenvalues $\omega_j = -\omega_{1,j} + i\gamma_j$, $j = 1, 2, \ldots$. Some of these eigenvalues are represented in Fig. 4.36 which shows that besides the eigenvalue ω_0 (which at $\mathrm{Re} = \mathrm{Re}_{1,cr}$ crosses the imaginary axis at the zero point), there is another eigenvalue (which will be temporarily denoted as ω_1) whose imaginary part

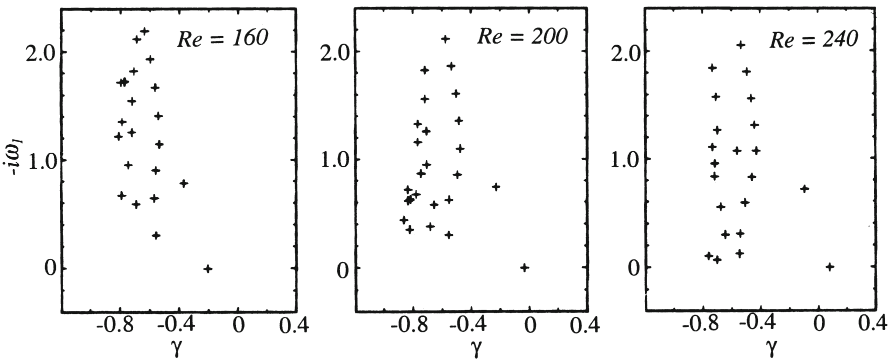


Fig. 4.36 Values of complex eigenvalues $i\omega = \gamma - i\omega_1$ corresponding to the eigenvalue problem of the linear stability theory for non-axisymmetric disturbances with $n = 1$ in the steady axisymmetric flow of velocity U_0 past a sphere of diameter D. The eigenvalues are made dimensionless by using U_0 and D as velocity and length units and are presented for three values of $\mathrm{Re} = U_0D/\nu$. (After Natarajan and Acrivos 1993)

also approaches zero (only slightly more slowly than that of ω_0) as Re increases. Accurate computations showed that the eigenvalue ω_1 crosses the imaginary axis at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} \approx 277.5$. Figure 4.36 showed that ω_1 has nonzero real part; according to the computations, this eigenvalue crosses the imaginary axis at the point where $-\Re e\omega_1 = \omega_{1,1} \approx 0.710U_0/D$. Thus, Natarajan and Acrivos found that the axisymmetric steady sphere wake loses its axisymmetry (but not steadiness) and acquires the azimuthal wave number $n = 1$at $\mathrm{Re} \approx 210$, and at $\mathrm{Re} \approx 277.5$ the second unstable mode of disturbance, which is also non-axisymmetric with $n = 1$ but is unsteady, appears in the flow. This could mean that at $\mathrm{Re} > 277.5$ the flow preserves the azimuthal wave number $n = 1$ but begins to oscillate with the frequency $f = \omega_{1,1}/2\pi \approx 0.113U_0/D$. If so, then the wake transformation at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$ clearly represents a Hopf bifurcation produced by the emergence of periodic shedding of vortices from the sphere; the wake oscillations arising at this Re correspond to a Strouhal number $\mathrm{St}_{\mathrm{cr}} \approx 0.113$.

The cautious description (using the expression "could mean...") of the result relating to $\mathrm{Re}_{2,\mathrm{cr}}$ is due to the fact that the theory only shows that at $\mathrm{Re} > 277.5$ the axisymmetric flow past a sphere becomes unstable with respect to non-axisymmetric oscillatory disturbances. However, the theory also shows that at some lower value of Re axisymmetric flow becomes unstable to infinitesimal disturbances of another type. The situation here is quite similar to that in the case of the stability studies for the circular-cylinder wake performed in 1996 by Barkley and Henderson, and Henderson and Barkley. As was noted in part (b) of this section, the critical Reynolds number $\mathrm{Re}_{3,\mathrm{cr}} \approx 260$ (and the whole lower stability curve in Fig. 4.23) found by these authors was also obtained by application of the linear stability theory to an obviously-unstable primary flow. It was explained, however, that the resulting value of $\mathrm{Re}_{3,\mathrm{cr}}$ nevertheless agrees well with the experimental threshold for the appearance of the second unstable mode B. A similar situation apparently occurs in the case of the square-cylinder wake (see the discussion of the paper by Robichaux et al. (1999) in part (c) of this section). As will be indicated below, the value of $Re_{2,cr}$ determined by Natarajan and Acrivos by means of linear stability analysis also agrees well with the available experimental data.

Natarajan and Acrivos noted that the loss of axisymmetry of the sphere wake, at a smaller value of Re than that at which the wake becomes unsteady, was observed in experiments by Magarvey and Bishop (1961a, b) and Goldburg and Florsheim (1966). Moreover, the values of the two critical Reynolds numbers, indicating the thresholds of the two bifurcations, which were found by these authors, are quite close to the values of $Re_{1,cr}$ and $Re_{2,cr}$ given by Natarajan and Acrivos' stability calculations. The experimental results by Nakamura (1976) may also be considered as being in good agreement with the calculated results, if one assumes that the bifurcation observed by this author corresponds to the loss of wake axisymmetry but not steadiness. Thus, Natarajan and Acrivos concluded that the available experimental data are substantially more favorable to their results than to those of Kim and Pearlstein. Note however that, during the preparation of the paper of 1993, Natarajan and Acrivos did not know about the paper by Wu and Faeth (1993), which contains a very convincing experimental confirmation of their theoretical results[11]. In fact, the latter authors observed both bifurcations predicted by Natarajan and Acrivos and gave, quite independently, the estimates $Re_{1,cr} \approx 200$ and $Re_{2,cr} \approx 280$ for the two critical Reynolds numbers, which are very close to the values computed by Natarajan and Acrivos. The later experimental results of Provansal and coworkers (Provansal (1996); Provansal and Ormières (1998) Ormières et al. (1998) and Ormières and Provansal (1999)), and the results of the sphere-wake observations by Johnson and Patel (1999), also show that the sphere wake loses its axisymmetry at a smaller value of Re than its steadiness and begins to oscillate only at $Re \approx 280$. The results of Johnson and Patel in fact agree in many other details with Natarajan and Acrivos' theoretical predictions. These new discoveries increase considerably the cogency of the statement made by Natarajan and Acrivos, that the available experimental data agree much better with their results than with Kim and Pearlstein's stability computations.

Of course, Natarajan and Acrivos did not analyze all the available experimental data relating to sphere wakes which, as mentioned above, are rather scattered. Moreover, they also did not look for a possible error in Kim and Pearlstein's complicated and tedious computations, which could explain the difference between the conclusions of two papers devoted to the same problem. However, Natarajan and Acrivos indicated one more very important confirmation of their results: namely, they stressed that their results agree very well with the results of independent stability computations for three-dimensional flows past a sphere, by a completely different method, carried out by Tomboulides (1993) and Tomboulides et al. (1993) practically simultaneously with Natarajan and Acrivos' investigation.

[11] Experimental results presented by Wu and Faeth (1993) are described at greater length in the thesis by Wu (1994). In addition Wu's thesis also contains descriptions of experimental studies of wakes behind spheres placed in a uniform stream where considerable velocity disturbances are presented; see in this respect also the papers by Wu and Faeth (1994, 1995).

Tomboulides, and Tomboulides et al., did not use the linear theory of hydrodynamic stability as in the papers discussed above but used a nonlinear approach based on direct numerical simulation (i.e., on numerical solution of the full nonlinear equations of fluid dynamics). First of all, they numerically solved the nonlinear equations describing the steady flow of a uniform stream with a constant velocity $\mathbf{U}_0 = \{U_0, 0, 0\}$ past a sphere of diameter D. The solutions found were axisymmetric, and the results for the Reynolds-number dependence of the streamwise length of the recirculation zone, the separation angle, and the sphere drag coefficient, discussed above, were computed from just these solutions. After this, at Reynolds numbers 200, 220, 250, 270, 285, and 300 a non-axisymmetric velocity disturbance with the azimuthal wave number $n = 1$ was added to initial conditions corresponding to the computed axisymmetric solution of equations of motion. The disturbance had the total energy equal to 10^{-8} of the energy of the axisymmetric flow and was randomly generated in such a way that its initial energy was distributed over all the eigenmodes with $n = 1$. Then the full equations of motion were solved numerically for the new initial conditions and the energy of all modes with $n = 1$ was traced in time.

It was found that the energy of the initial non-axisymmetric disturbance decayed in time, and the flow eventually returned to full axisymmetry, only at $\mathrm{Re} = 200$. For all other inspected values of Re the energy of the disturbance grew and asymptoted to a finite constant value. This showed that $200 < \mathrm{Re}_{1,\mathrm{cr}} < 220$. The observed dependencies of the disturbance amplitudes $A(t)$ on time (exemplified in Fig. 4.37a) allowed easy detection of the regions of initial exponential decay or growth, yielding the initial growth rate (positive or negative) $\gamma = \Re\mathrm{m}\,\omega$ (where ω is the corresponding eigenvalue of the linear stability problem) of the least-stable mode with $n = 1$. Values of γ obtained in this way are shown in Fig. 4.37b, together with the same quantity computed by Natarajan and Acrivos from linear stability theory. One may see that the agreement between the results of linear and nonlinear computations is remarkable. Note also that the data shown in Fig. 4.37b agree well with the approximate equation $\gamma \approx b\,(\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}})$ which according to Landau's theory must be valid at small values of $|\mathrm{Re} - \mathrm{Re}_{\mathrm{cr}}|$.

As to the exact value of $\mathrm{Re}_{1,\mathrm{cr}}$, Natarajan and Acrivos found that $\mathrm{Re}_{1,\mathrm{cr}} = 210$ while a thorough investigation of this question by Tomboulides led to the conclusion that $\mathrm{Re}_{1,\mathrm{cr}} = 212$; the difference between these two estimates is clearly negligible. The above-mentioend experimental estimates by Magarvey and Bishop (1961a, b) ($\mathrm{Re}_{1,\mathrm{cr}} \approx 210$) and Wu and Faeth (1993) ($\mathrm{Re}_{1,\mathrm{cr}} \approx 200$) of the threshold Reynolds number signifying the transition to non-axisymmetric wake regime are also very close to the corresponding results of Tomboulides, and Tomboulides et al. The recent experiments of Provansal (1996); Provansal and Ormières (1998); Ormières et al. (1998), and Ormières and Provansal (1999), confirmed that the sphere wake loses its axial symmetry at a value of Re below the onset of wake oscillations, and implied an estimate of the oscillation threshold $\mathrm{Re}_{2,\mathrm{cr}}$ which is very close to that found by Natarajan and Acrivos and by Tomboulides. However this recent work led to results relating to the value of $\mathrm{Re}_{1,\mathrm{cr}}$ which deviate from the conclusions of Natarajan and Acrivos' linear and Tomboulides' nonlinear stability theory. In fact, according to Provansal and his coworkers there exist two different bifurcations, both leading

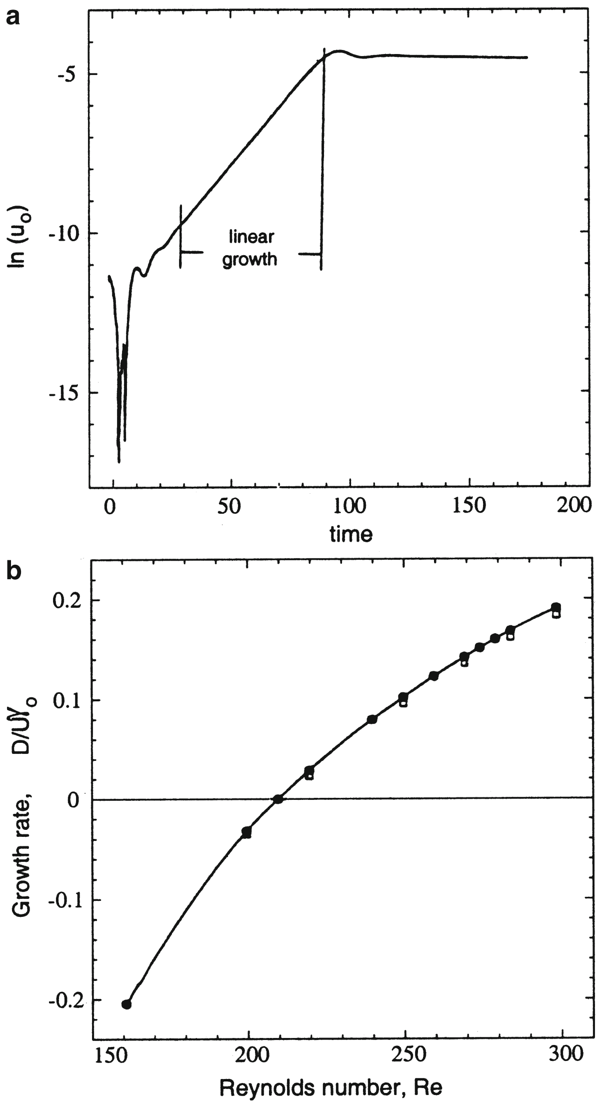


Fig. 4.37 **a** Dependence on time (measured in conventional units) of $ln(u_\phi)$ where u_ϕ is the non-dimensionalized azimuthal disturbance velocity at the point $\{x, r\} = \{D, 0.2D\}$ of the wake behind a sphere at Re = 250. (After Tomboulides 1993) **b** The dependence of the dimensionless growth rate $\gamma D/U_0$ of the least stable mode with $n = 1$ on Re in the wake behind a sphere. (After Tomboulides 1993 and Tomboulides et al. 1993). Circles: results following from the linear stability analysis of Natarajan and Acrivos, filled circles: values given by nonlinear direct numerical simulation of Tomboulides

to the emergence of some non-axisymmetric steady flow regimes, which occur at smaller Re than the value $Re_{1,cr} \approx 210$ at which the wake regime first becomes non-axisymmetric according to Natarajan and Acrivos' and Tomboulides' computations. However, only the second of the found by the French researchers non-axisymmetric regimes conforms to the non-axisymmetric steady wake regime predicted by the linear and nonlinear stability theories and the corresponding to it critical Reynolds number $Re \approx 180$ does not differ very much from the value $Re_{1,cr}$ implied by the above-mentioend stability studies. As to the another non-axisymmetric sphere-wake regime detected by the French experimenters at $150 < \mathrm{Re} < 180$, it apparently required further investigation. Without it this specific result (which may have been affected by the influence of the sphere supports) can hardly outweigh the available data supporting Natarajan and Acrivos' and Tomboulides' conclusions.

The linear-theory results of Natarajan and Acrivos determine only the character of the initial evolution of a very small disturbance in the sphere-wake flow. However, the results of Tomboulides' nonlinear analysis lead to asymptotic values $A\infty$ of the disturbances as $t \to \infty$ and hence to values of the Landau constants $\delta = 2\gamma/A_\infty^2$ (proved to be positive) at various Reynolds numbers and positions in the sphere wake. (Note that Landau's equation describing the emergence of sphere-wake oscillations was considered by Ormières et al. (1998) and Ormières and Provansal (1999) who, in particular, showed that the increase of the energy of the streamwise velocity fluctuations with the Reynolds number Re is linear at small supercritical values of Re, as it must be according to Landau's theory). It was also found by Tomboulides that the non-axisymmetric steady wake structure emerging at $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}}$ preserves planar symmetry with respect to some plane parallel to the flow direction. Such symmetry, which is weaker than the axial symmetry but not incompatible with azimuthal wave number $n = 1$, was also observed by Magarvey and Bishop (1961a) and Levi (1980), and was later found in the numerically-simulated supercritical sphere wakes computed by Shirayama (1992) and Johnson and Patel (1999). The time history of velocity disturbances described by numerical solution of the nonlinear equations of motion computed by Tomboulides show that the mode with $n = 1$ begins to oscillate at some value of Re in the interval $270 \leq \mathrm{Re} \leq 285$. Recall that according to the linear theory by Natarajan and Acrivos the transition to an oscillating wake regime takes place at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}} = 277.5$ while according to the experimental results by Magarvey and Bishop $\mathrm{Re}_{2,\mathrm{cr}} \approx 270$, and both Wu and Faeth, and Provansal and his coworkers (who used wake control to observe the time evolution of velocity disturbances at subcritical and supercritical values of Re close to the critical value $\mathrm{Re}_{2,\mathrm{cr}}$) found that $\mathrm{Re}_{2,\mathrm{cr}} \approx 280$. We see that here again the conclusions of nonlinear numerical stability analysis agree very well with results given by linear stability theory and by several trustworthy experimental investigations. Let us also re-emphasize that the computations by Natarajan and Acrivos, and by Tomboulides, and experimental investigation by Wu and Faeth were carried out practically simultaneously and independently from each other. Therefore the remarkable coincidence of the results obtained independently by three very different methods gives good reason to trust them. It is now possible to support this conculsion by reference to results of a subsequent successful numerical simulation of the flow past a sphere at a number of moderate Reynolds numbers, combined with visual observation of the wake regimes at different values of Re.

Let us mention at first a numerical simulation by Gebing (1994) of the flow of a compressible fluid past a sphere at Reynolds numbers from 20 to 1,000 and a Mach number of 0.4. This simulation also showed the existence of two subsequent transition of the same type as those found for incompressible flows—a loss of axial symmetry at $\mathrm{Re} \approx 300$ and the emergence of an oscillatorty wake regime at $\mathrm{Re} \geq 400$. However, up to now compressible flows have not been considered at all in this book; therefore, the main attention will be given below to the results of accurate numerical simulations of the incompressible flows past a sphere at $20 \le \mathrm{Re} \le 300$ performed by Johnson (1996). He employed a numerical method differing from that of Tomboulides, and presented the final version of his results in the paper by Johnson and Patel (1999), where numerically simulated data were accompanied by results of dye-injection observations of the wake flow behind a sphere towed through a water tank. Both the simulations and visual observations showed that, at Reynolds numbers from 20 to approximately 210, the wake flow is steady, axisymmetric, and does not undergo any substantial topological transformations. As has been noted above, the length of the recirculation zone, the separation angel, and the drag coefficient computed for Re-values in this range coincided very well with many previous experimental and numerical results. However, at a Reynolds number of 211 the calculated solution of the equations of motion becomes non-axisymmetric, but preserves planar symmetry with respect to some plane parallel to the flow direction, and remains steady.

Steady non-axisymmetric numerical solutions were found for all investigated values of $\mathrm{Re} \ge 211$ up to $\mathrm{Re} = 270$. However at $\mathrm{Re} = 280$, which was the next higher Reynolds number considered, the solution, obtained for the initial conditions leading at $\mathrm{Re} = 270$ to steady non-axisymmetric solution, was found to be oscillating with a fixed frequency f. Hence, Johnson's numerical simulations show that $\mathrm{Re}_{1,\mathrm{cr}} = 211$ and $270 < \mathrm{Re}_{2,\mathrm{cr}} \le 280$. These results agree excellently with those found in numerical studies of Natarajan and Acrivos (1993); Tomboulides (1993), and Tomboulides et al. (1993), and in experiments of Magarvey and Bishop (1961a, b) and Wu and Faeth (1993) (and in the part relating to the onset of wake oscillations at $\mathrm{Re} = \mathrm{Re}_{2,\mathrm{cr}}$ also with experimental results of Provansal and his group). Wishing to understand (and to explain) the physical mechanisms leading to the loss of wake axisymmetry at $\mathrm{Re} = 211$ and the transition to unsteady vortex-shedding regime at $\mathrm{Re} \approx 275$, Johnson and Patel analyzed very thoroughly all the numerical and visualization data relating to $\mathrm{Re} = 250$ and $\mathrm{Re} = 300$, and presented in their paper of 1999 an extensive collection of graphs, photos, and model pictures illustrating the properties of the wake regimes at these two Reynolds numbers.

The collected data gave reasons to associated the transition to a non-axisymmetric steady regime at $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}} = 211$ with an azimuthal instability of the low-pressure core of the toroidal vortex, emerging at $\mathrm{Re} = \mathrm{Re}_{0,\mathrm{cr}} \approx 20$ and then growing with Re, becoming more unstable with the decrease of the role of viscosity. Relying on this general idea, Johnson and Patel proposed a physical mechanism describing the transition process. This mechanism allowed them to interpret physically their visualization results for $\mathrm{Re} = 250$ and 300, and to explain the appearance at $\mathrm{Re} > \mathrm{Re}_{1,\mathrm{cr}}$ behind a sphere of two streamwise vortices extending downstream and forming two parallel vortical threads. (These vertical threads were first observed in the liquid-drop experiments by Magarvey and Bishop, whose results were later confirmed by visualization experiments of Levi (1980), Provansal and his coworkers (who found the two-thread regime for $180 < \mathrm{Re} < 280$), and Johnson and Patel, and by numerically-simulated data of Shirayama (1992), Tomboulides, and Johnson and Patel). The value of $\mathrm{St} = fD/U_0$ at $\mathrm{Re} = 300$ computed by Johnson was equal to 0.137, and coincided almost exactly with the result $\mathrm{St} = 0.136$ of Tomboulides' computations and with the value given by the experimental form of the Roshko law (4.47) given by Provansal and Ormièdes in their papers of 1998 and 1999. (The experimental values of St for vortex shedding from a sphere found by Johnson and Patel were slightly higher than the corresponding numerical results but they agree well with experimental values of St found by Sakomoto and Haniu (1990, 1995) for nearby values of Re). The calculated drag coefficient at $Re = 300$ was also close enough to previous experimental and numerical results.

A similar physical mechanism was also proposed for explanation of the transition to unsteadiness at $Re = Re_{2,cr} \approx 275$. This mechanism explains not only the observations of Achenbach (1974); Perry and Lim (1978), and Sakamoto and Haniu (1990, 1995) of periodic shedding, at $Re > Re_{2cr}$, of hairpin vortices of consistent orientation, but also the shedding of previously-unrevealed oppositely-oriented hairpin vortices which were seen in the new visualizations of the sphere wake and, according to Johnson and Patel, may have a rather simple physical origin. However, space limitations forbid more detailed discussion of this subject.

The results of Natarajan and Acrivos, Tomboulides (and Tomboulides et al)., and Johnson (and Johnson and Patel) may be applied in principle to determination of the coefficients γ and δ of the real Landau Eq. (4.34), describing the bifurcation at $Re = Re_{1,cr}$ of the steady axisymmetric wake flow observed at smaller values of Re (see, in particular, Fig. 4.37b where data relating to the coefficient $\gamma = \gamma(Re)$ are presented). The transition at $Re = Re_{2,cr}$ of a steady non-axisymmetric wake flow to a non-axisymmetric oscillating vortex-shedding regime represents a Hopf bifurcation and requires the use of a complex Landau Eq. (4.40) (or, what is the same, two real Eqs. (4.34) and (4.34a)) for its theoretical interpretation. Coefficients γ and δ in both cases can be estimated if some method of control of wake development is used, so that the time history of the real amplitude of some appropriately chosen characteristics of the wake flow can be observed from the initial instant of this development (cf. the discussion of Eqs. (4.48) and (4.49) in part (b) of this section). This procedure was applied to the study of the sphere wake at Re near $Re_{2,cr}$ by Ormières et al. (1998); Provansal and Ormières (1998), and Ormières and Provansal (1999) who, in particular, determined the values of $\gamma(Re)$ (it was found that $\gamma D^2/\nu \approx 0.9(Re - Re_{2,cr})$ at small values of $Re - Re_{2,cr}$) and the coefficients of the Ro-Re relation corresponding to the vortex-shedding regime of the sphere wake. As to the coefficients ω_1 and δ', which are needed for the description of a Hopf bifurcation at $Re = Re_{2,cr}$, the first coincides with the real part of the corresponding complex eigenvalue (denoted by ω_1 in the above discussion of the paper by Natarajan and Acrivos), while the second can be easily determined from the values of γ, δ, and the frequency f of the observed wake oscillations. The values of f were given for a number of values of Re by both Tomboulides (1993) and Johnson (1996), who used their own numerical simulations for this purpose, and by Provansal and his group who used spectral analysis of measured velocity fluctuations in the wake; the results of all studies were practically the same. Moreover, the French researchers also measured the dependence of the energy of streamwise-velocity oscillations E (more exactly, of the normalized energy E/E_{max}) on the streamwise coordinate x of the observation point in a sphere wake and of Re, and the dependence on Re of the coordinate x_{max} at which the amplitude of the velocity oscillations takes the greatest value. The results of these measurements were found to be similar in many respect to the results of Wesfreid et al. (1996) relating to spatial variations of velocity oscillations in the wake of a circular cylinder.

Many details of the flow past a sphere at larger values of Re can be found, in particular, in the papers by Achenbach (1974); Pao and Kao (1977); Perry and Lim (1978); Taneda (1978); Kim and Durbin (1978); Sakamoto and Haniu (1990, 1995); Shirayama (1992); Bonneton and Chomaz (1992); Wu and Faeth (1993); Tomboulides (1993), and Tomboulides et al. (1993). The values of the wake-oscillation frequency f and of the Strouhal number St at many values of Re were determined, in particular, by Achenbach (1974); Taneda (1978); Kim and Durbin (1988), and Sakomoto and Haniu (1990, 1995); the last-named of them includes a general sketch of the shape of the St-Re relation for a wide range of Re, both for a sphere in a constant-velocity stream and in streams with various constant transverse velocity gradients. Note, however, that at large enough values of Re wake oscillations often have the shape of supperpositions of several harmonics of different frequencies. (For example, Shirayama (1992) found that at $\mathrm{Re} = 500$ two frequencies, corresponding to different Strouhal numbers, are clearly seen in the spectrum of sphere-wake oscillations). With further growth of Re the number of different spectral components of wake oscillations increases and the transition to turbulence leads to the appearance in the wake of a continuous frequency spectrum. In the above-mentioned papers, many topological transformations of the vorticity field of sphere wakes are described; however, these high-Re wake transitions will not be considered in this chapter.

Let us now say a few words about the wakes behind some other axisymmetric bodies. We will begin with the *wakes behind flat circular disks* perpendicular to a uniform steady flow. Such wakes were studied in experiments by Schmiedel (1928) (who considered spheres and round disks freely falling in a liquid), Marshall and Stanton (1931); Fail et al. (1957) (here wakes behind circular plates were considered, together with those behind some other plates perpendicular to the flow), Carmody (1964); Willmarth et al. (1964); Calvert (1967a, b) (who also studied wakes behind cones with axes parallel to the stream direction and flat disks non-orthogonal to the stream); Roos (1968); Roos and Willmarth (1971); Fuchs et al. (1979); Takamoto (1987); Bearman and Takamoto (1988); Berger et al. (1990); Lee and Bearman (1992); Cannon et al. (1993); Provansal (1996) (who indicated that he had studied wakes behind discs and cones parallel to the stream, together with the sphere wakes discussed above, but mentioned only one specific result relating to cones), Miau et al. (1997), and some other researchers. However, the results of this work are much less definite than those relating to sphere wakes. The vortical structures in disk wakes were investigated at various Reynolds numbers and by various experimental methods, in particular, by Fuchs et al., Berger et al., Lee and Bearman, Cannon et al., and Miau et al., but the results obtained are still very scattered. Apparently the only attempt to calculate the stability characteristics of wakes behind circular disks was due to Natarajan and Acrivos (1993), using the same method as in their study of the stability of sphere wakes. They found that, as in the case of a sphere wake, a steady axisymmetric disk wake loses its stability first of all to a nonoscillatory non-axisymmetric disturbance with $n = 1$. According to their calculations, this loss occurs at $\mathrm{Re} = \mathrm{Re}_{1,\mathrm{cr}} = 116.5$ (where Re is based on the disk diameter and free-stream velocity). However, the non-axisymmetric steady flow past a circular disk emerging at this Re loses its stability at only slightly lager Reynolds number, $Re_{2,cr} = 125.6$, when a new oscillating non-axisymmetric wake regime (again with $n = 1$) appears with a frequency of oscillation corresponding to $St_{cr} \approx 0.125$. Natarajan and Acrivos noted also that the results of experiments by Willmarth et al. (1964), who oserved the behavior of freely falling circular disks, can be interpreted as a crude confirmation of their theoretical results.

A number of observations of wake regimes behind axisymmetric bodies differing from spheres and round disks can be also found in the literature, but only a few quantitative conclusions can be obtained from the results. It was noted above that Calvert (1967a) studied wakes behind cones with axes parallel to the flow direction, apexes directed upstream, and various apex angles. Such wakes were investigated in more detail by Goldburg and Florsheim (1966) who observed wakes behind freely falling cones (with apexes directed downwards) together with wakes behind falling cone-spheres (hemispheres attached to the base of cones). They showed that the Rayleigh-Roshko law (4.47a) with constant coefficients a and a_1 is valid for the oscillation frequencies of these wakes (in particular, in the case of a cone with 20° apex angle, $a \approx 0.454$ and $a_1/a \approx 160$; this means that periodic vortex shedding from such a cone is observed for $Re > 160$). Provansal (1996) indicated that, according to his experiments, $Re_{cr} \approx 185$ determined the threshold of a periodic vortex-shedding regime behind an upstream-pointing cone, but gave no further details. Zikmundova (1970) (whose results relating to the value of $Re_{1,cr}$ for the sphere wake gave rise to doubt) studied, together with the sphere wake, the wake behind a spheroid, and also Masliyah (1972) observed both sphere wakes and wakes behind several oblate spheroids. Hama and Peterson (1976); Hama et al. (1977), and Peterson and Hama (1978) studied the wakes behind slender bodies of revolution, and found that here instability appears at much greater Reynolds numbers (based on the body diameter) than in the cases of bluff bodies (such as disks, spheres and cones). The number of references to papers dealing with wakes behind various axisymmetric bodies can be easily increased, but we will not linger on this subject here.

4.2.4.6 Axisymmetric Jet Flows

At the beginning of this section a short remark was made relating to the Landau constant δ for the plane Bickley jet. Now, in conclusion of the present section we will mention several papers dealing with amplitude equations for unstable disturbances in axisymmetric jets issuing from a circular orifice into a space filled with a fluid at rest. Let us recall that at the end of Sect. 2.93 it was indicated that if the fluid in a jet does not differ from the fluid in the surrounding space, then only convectively (but not absolutely) unstable disturbances can exist in jet flow, while in the case of a jet which is heated (or for some other reason has appreciably smaller density than that of the surrounding fluid) absolute instability can take place. It has been mentioned several times in this section that the presence of regions of absolute instability is necessary for the excitation of the global mode of self-sustained oscillations in a nearly-parallel fluid flow. Therefore the development of disturbances in non-heated jets must inevitably differ from the same process observed in wakes or heated jets above the Hopf-bifurcation threshold.

Danaila et al. (1997) applied direct numerical simulation to investigate the spatial disturbance development in round (unheated) jets with some widely-used initial velocity profiles from the list given by Michalke (1984) (see also Sect. 2.9.4 in Chap. 2) and several initial Reynolds numbers U_0D/ν (where U_0 is the typical jet velocity at the orifice and D is the orifice diameter). It was shown that at relatively small, slightly-supercritical Reynolds numbers 'helical modes' with $n = \pm 1$ are most unstable (i.e. their amplitudes grow most quickly) while at highly-supercritical Reynolds numbers the axisymmetric mode with $n = 0$ becomes the most amplified. At some stage of the disturbance development in a slightly-supercritical round jet a Hopf-like bifurcation was detected which however led to a quasiperiodic (and not purely periodic) final state. In the subsequent paper by Danaila et al. (1998) the nonlinear disturbance development of a Hopf bifurcation leading to the production of oscillating helical modes with $n = \pm 1$ was studied by analysing the corresponding amplitude equations. Since here amplitudes of two modes, with $n = 1$ and $n = -1$, must be considered and the higher harmonics (whose frequencies are multiples of the dominant frequency) also play a definite role (cf. the discussion of papers by Dušek et al. (1994) and Dušek (1996) in part (b) of this section), these amplitude equations are more complicated than the simple Landau equation and may be considered as its generalizations (of the same type as Stuart's Eqs. (4.43) which were considered in Sect. 4.21).

Let us now pass to the case of heated round jets. At the end of Sect. 2.93, literature was cited, in which it was proved that absolute instability can emerge under certain conditions in heated jets, and some conditions making such emergence possible were indicated. (Sect. 2.93 was devoted to plane free flows in an unbounded space but in discussion of heated jets it was specially noted that the statements made are valid for both plane and round jets). Some examples of experimental confirmation of results relating to the absolute instability of heated jets with negligible buoyancy effects can be found in papers by Monkewitz and Sohn (1988) and Sreenivasan et al. (1989) referred to in Sect. 2.93; valuable supplementary data of the same type can be found in papers by Monkewitz et al. (1989, 1990). In the case of a heated jet the regime of jet oscillations depends on two dimensionless parameters: the Reynolds number $\mathrm{Re} = U_0D/\nu$ and the ratio $\rho_0/\rho_\infty = S$ of the density of fluid issuing from the orifice to the ambient density far from the jet. (Here we again assume that the influence of buoyancy can be neglected in comparision with the influence of the inertia of moving fluid. This assumption is usually true near the orifice; the case where buoyancy is essential was considered by Krizhevsky et al. (1996) but will be not treated here). The above-mentioned experiments show that over a wide range of Reynolds numbers strong global oscillations of the 'jet column' arise automatically, if the value of the parameter S lies below $S_{\mathrm{cr}} \approx 0.62$. It follows from this that at such values of S a Hopf bifurcation takes place, which corresponds to Landau's equation with positive Landau constant $\delta > 0$. Just this situation will be considered below in line with the presentation given by Raghu and Monkewitz (1991).

Raghu and Monkewitz analyzed the experimental data for a jet of hot air issuing from a round nozzle of diameter $D = 15$ mm into unheated still air (this arrangement and experimental conditions were practically the same as used by Monkewitz et al. (1990)). Since wake oscillations were observed only at $S < S_{cr} \approx 0.62$, it was possible to suppress the oscillations by extending the length of the nozzle by another 15 mm and then cooling the nozzle extension to reduce the air temperature, in this way increasing the density ratio S above the critical value S_{cr}. Thus, jet control could be realized by regulating the jet temperature. By switching off the cooling system it was possible to return S quickly to its initial low value $S_0 < S_{cr}$ and hence to create conditions promoting the excitation of jet oscillations. After this the researchers could observe, at a selected point of the jet, the transient growth of the complex oscillations amplitude $A(t) = |A(t)|e^{i\phi(t)}$ from zero up to its equilibrium value corresponding to the selected position of the observation point and the value $S_0 < S_{cr}$ of the parameter S. This transient growth is determined by the complex Landau (otherwise, Stuart–Landau) Eqs. (4.34) and (4.34a); the observations described allow evaluation of all four real coefficients γ, ω_1, δ and δ'' of these equations as in the papers by Mathis et al. (1984); Provansal et al. (1987); Sreenivasan et al. (1987) and Schumm et al. (1994) on cylinder-wake oscillations, described in part (b) of this section.

Let us replace the complex amplitude $A(t)$ by the normalized amplitude $A(t)/A_e = R(t)e^{i\vartheta(t)}$, where $Ae = (2\gamma/\delta)^{1/2}$ is the real equilibrium amplitude and $R(t) = |A(t)|/A_e$ is the real normalized amplitude which can vary in the range $0 \le R(t) \le 1$ (it is assumed here that $S < S_{cr}$ and therefore $\gamma > 0$ and $\delta > 0$). Then Eqs. (4.34) and (4.34a) can be rewritten in the following form:

$$\frac{1}{R}\frac{dR}{dt} = \gamma(1 - R^2), \tag{4.55}$$

$$\frac{d\phi}{dt} = -\omega_1 - \frac{\delta'\gamma}{\delta}R^2. \tag{4.55a}$$

Switching off the cooling system at first and then switching it on again, one could measure, at a given point of observation, values of the real amplitude $|A(t)|$ (gradually growing from zero at $t = 0$ to its equilibrium value A_e at large t) together with the jet oscillation frequency $(1/2\pi)(d\phi/dt) = f(t)$. Values of $|A(t)|$ and A_e determine $R(t)$, and in Fig. 4.38 the values of $[dR/dt]/R$ and $d\vartheta/dt$ measured by Raghu and Monkewitz at the point with coordinates $(x, r) = (1.3D, 0.5D)$ (where x is the streamwise coordinate measured from the jet orifice and r is the radial cylindrical coordinate indicating the distance from the jet axis) are presented in their dependence on the value of R^2, varying from zero to one, for the case where $S = 0.546$. We see that the experimental data agree well with the linear dependence of both presented in Fig. 4.38 quantities on R^2, predicted by Eqs. (4.55) and (4.55a), and allow evaluation of all coefficients of these equations for the given observation point, value of S, and flow conditions. (Recall that in the case of the global mode of wake oscillations, the values of γ and ω_1 do not depend on the observation points and that the results of similar cylinder-wake observations presented in part (b) of this section showed that the ratio δ'/δ is also practically constant over a large spatial region). Resutls of Raghu

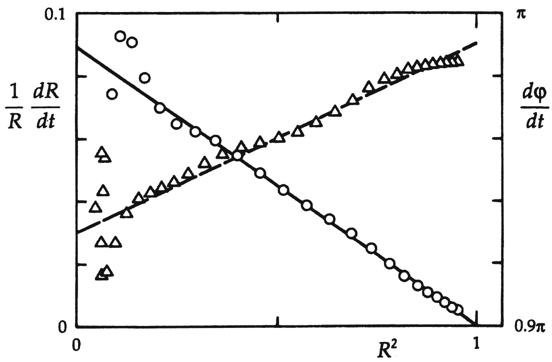


Fig. 4.38 Dependence of $[dR/dt]/R$ (o) and of $d\phi/dt$ (Δ) on R^2 at the point $\{x, r\} = \{1.3D, 5D\}$ of a heated circular jet with $S = 0.546 < S_{cr}$. (After Raghu and Monkewitz 1991)

and Monkewitz's measurements at different values of the parameter S showed that the global oscillations of the heated jet come to an end at $S = S_{cr} \approx 0.62$ (this value is slightly less than the estimate $S_{cr} \approx 0.63$ found by Monkewitz et al. in 1990). More precisely, Raghu and Monkewitz found that at their chosen point of observation the critical value S_{cr} and the coefficients of Landau's Eqs. (4.55) and (4.55a) for the heated jet take the following values:

$$S_{cr} = 0.62 \pm 0.01, \quad \gamma D/U_0 = [1.15 \pm 0.15](S_{cr} - S), \tag{4.56a}$$

$$\omega_1 D/U_0 = -[0.68 + 0.01] - [0.88 + 0.02](S_{cr} - S), \quad \delta'/\delta = -2.5 + 0.6. \tag{4.56b}$$

We see that the measurements of the transient growth of the jet oscillations confirm the emergence of a Hopf bifurcation at a critical density ratio $S = S_{cr}$, and yield rather accurate estimates of the values of S_{cr} and of coefficients of Landau's Eqs. (4.55–4.55a).

This example will conclude the present section of the book, devoted to various applications of the real and complex Landau equations (and in some cases also of more general Ginzburg–Landau equations) to description of the nonlinear instabilities of fluids flows.

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Chapter 5
Further Weakly-Nonlinear Approaches to Laminar-Flow Stability: Blasius Boundary-Layer Flow as a Paradigm

Landau's equation and its generalizations considered in Sect. 4.2 represent a particular weakly-nonlinear approach to the study of flow stability, based on the assumption that the disturbance amplitude A is small enough to justify the expansion of solutions of fluid-dynamic equations in powers of A. However this approach has a severe limitation: only the evolution of one isolated mode of disturbance is traced, while its interaction with all other modes is only roughly characterized by the values of real or complex Landau's constants of various orders.

A comprehensive nonlinear theory of hydrodynamic stability must include a more direct description of interdependencies between disturbance modes. The complexity of the problem does not permit a universal analytical treatment. However, there is a vast number of approximate methods applicable to one or another particular case. Recall in this respect that some such approximate methods were briefly mentioned in Sect. 4.2 when the papers by Benney and Lin (1960); Benney (1961, 1964); Stuart (1962a, b); Itoh (1980), and Danaila et al. (1998) were cited. In these papers the simultaneous development of two or more modes of disturbance was considered and, therefore, instead of one Landau's equation, more general systems of differential equations for amplitudes of these modes were used (a typical example of such a system is given by Eq. (4.43)). However, in Sect. 4.2 no details and/or applications of these multimode analyses were presented.

In contrast to this, in the present chapter and the next some approximate methods for the study of multimode weakly-nonlinear flow instabilities will be considered at greater length, together with a number of applications of these stability theories to development of disturbances in some steady laminar flows of great practical importance. However, since the amount of material relating to this subject accumulated up to now is really enormous, a rather strict selection of topics has been necessary here, and even then it has been impossible to include in the present chapter an adequate description of results of weakly-nonlinear instability theory for a wide range of laminar flows. At first, herefore, only results relating to one such flow will be considered at full length, but this will allow us to shorten considerably the presentation of analogous results for other flows. As to the choice of the primary example, it was made easy by the quite exceptional place occupied in fluid mechanics by the Blasius boundary layer growing on a flat plate aligned with a parallel flow with

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constant velocity U_0 (and hence with zero pressure gradient). This model laminar flow is quite a good approximation to many flows met in nature and in engineering facilities, which makes it one of the most important laminar flows. Moreover, this flow has a rather simple structure, and it has been carefully studied by a number of outstanding scientists who obtained many interesting results about it, often directly relating to weakly-nonlinear stability. Note also that these results show very convincingly that in the case of the Blasius boundary layer the multimode type of instability plays an especially important part, and in fact determines the development of disturbances leading to transition of this flow to turbulence. Therefore it seems natural to devote the present chapter entirely to the study of weakly-nonlinear multimode instability of the Blasius boundary layer flow and only after this to consider some other laminar flows.

5.1 Resonance Mechanisms of Wave-Disturbance Growth; Two-Wave and Three-Wave Resonances

In physics the term 'resonance' is most often used to describe the rapid growth of the amplitude of a steady-state periodic oscillation of a physical system affected by an external oscillating force, when the frequency of the force oscillations approaches the fundamental frequency of the system considered (or one of these frequencies if there are many of them). The same term was also met in Sect. 3.32 of this book where, however, it had a slightly different meaning—there, the growth of flow disturbances produced by the degeneracy of the system of eigenfrequencies of the linear stability problem was called the 'resonance growth'. It was explained in Sect. 3.32 why the work 'resonance' is appropriate here—if there are two eigenfrequencies taking the same value ω_0 and both the corresponding flow oscillations are excited, then either of them may be considered as a force affecting the other and producing resonance growth of the oscillation amplitude. Since in Sect. 3.32 only the linear stability theory, dealing with disturbances of very small amplitude, was considered, resonance conditions were there formulated in terms of eigenvalues of linearized wave equations, and no attempts to evaluate the resonance growth of amplitudes were made. However, a more general formulation states that the 'resonance mechanism of disturbance growth' means that there are several modes of disturbance such that their interactions efficiently excite some (or all) of them leading to rapid increase of the corresponding amplitudes. According to this formulation, a resonance mechanism includes reciprocal interactions among two or even more modes, and hence it cannot be studied in the framework of the one-mode Landau weakly-nonlinear theory. However, the general weakly-nonlinear approach, based on the assumption that initial amplitudes of all disturbances considered are small (but not infinitesimal) can be used here too. This section will be devoted entirely to the weakly-nonlinear theory of resonances and other intermode interactions appearing in fluid flows.

In Sect. 3.32 we considered only the particular *two-wave resonances* which are due to the coincidence of the frequencies of two wave modes of infinitesimal disturbance and lead to power-law growth of the amplitudes of these modes. It was explained there that such resonances are rather numerous, and can occur for both two-dimensional (2D) and three-dimensional (3D) wave disturbances of steady plane-parallel or axisymmetric-parallel flows. The use of the adjective 'infinitesimal' means that in Sect. 3.32 only linearized equations of motion were considered, and hence all the resonances studied were of the elementary linear type. (The possible importance of nonlinear resonance effects was mentioned only once in Chap. 3, with reference to the paper by Benney and Gustavsson (1981) but was not discussed there). At the end of Sect. 3.32 it was also stressed that the resonance growth rates of amplitudes found in the papers discussed were always much smaller than the growth rates of disturbances observed in laboratory experiments and numerical simulations. This discrepancy clearly shows that there are some other growth mechanisms, more efficient than the linear resonance mechanism.

Let us now consider a more complicated situation relating to the manifestation of resonances in nonlinear physical systems (exemplified by a viscous fluid flow consisting of a steady primary flow with a *finite* disturbance superimposed on it). Note, first of all, that the nonlinear resonance is much more versatile than the linear one. In the simplest case of a one-dimensional oscillation $u(t)$ the quadratic term ($\propto u^2$) of the oscillation equation leads to the appearance, in addition to the harmonic oscillation of fundamental angular frequency ω_0, of the second harmonic proportional to $\exp(2i\omega_0 t)$; therefore, the system may resonate here also if the external force has a frequency close to $2\omega_0$. Higher harmonics $\exp(ki\omega_0 t)$, $k = 3, 4, \ldots$, also appear in many nonlinear systems together with the primary oscillation. In general, the response of a nonlinear system to a sinusoidal external disturbance may be highly nontrivial and lead to exceedingly complicated behavior; see, e.g., Chirikov's survey (1979), Sects. 3 and 4, and the book by Rabinovich and Trubetskov (1989), Chap. 13. In particular, the phase space of a nonlinear system, even a one-dimensional one, can include a number of different resonance bands which can overlap, complicating the situation considerably. However, this topic will not be considered in this book where the main attention will be paid to other aspects of the nonlinear resonance.

The possibility of nonlinear resonance produced by the interaction of a primary oscillation of frequency ω_0 with an external force of double that frequency, $2\omega_0$, (or of frequency $k\omega_0$, $k > 2$) means that in a nonlinear system the simultaneous appearance of two oscillations with frequencies ω_0 and $k\omega_0$, where k is an integer, may also sometimes produce rapid amplitude growth. From this one may deduce that, for example, in a two-dimensional steady fluid flow the interaction of a pair of two-dimensional Tollmien–Schlichting (T-S) waves of finite amplitude can lead to resonance not only in the case considered in Sect. 3.32, where both waves have the same frequency ω_0 (i.e., ω_0 is a degeneracy point of the corresponding eigenvalue spectrum), but also in cases where these two T-S waves have frequencies ω_0 and $k\omega_0$ where k is an integer. A similar increase in the number of possible resonance effects is produced by the nonlinearity of the equations of motion when three-dimensional (3D) wave disturbances, instead of simple two-dimensional T-S waves, are considered. (Such 3D resonances were also analyzed in Sect. 3.32, in the framework of the linear stability theory). We see that in the case of wave disturbances of finite amplitude there are many more possibilities for two-wave resonances than in the case of waves of infinitesimal amplitude. Moreover, since the product of harmonic oscillations of frequencies ω_1 and ω_2 may be represented by a linear combination of two harmonic oscillations of frequencies $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$, the interaction of two oscillations of frequencies ω_1 and ω_2 in a nonlinear system may cause a 'resonance growth' of a third oscillation of frequency $\omega_1 + \omega_2$ (or $\omega_1 - \omega_2$), if such an oscillation is also present. In other words, the nonlinear resonance may be produced in a nonlinear system with quadratic nonlinearities by a triad of small (but finite) harmonic oscillations with frequencies ω_1, ω_2, and ω_3 (which can be incommensurable with each other), such that

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{5.1}$$

for some choice of the signs of the frequencies considered (the sign of the frequency of a sinusoidal oscillation may be equally correctly considered as being positive or negative). Similarly, nonlinear resonances may also be produced by n-tuples of harmonic disturbances, where $n > 3$, with frequencies $\omega_1, \omega_2, \ldots, \omega_n$ of any signs whose sum is equal to zero. Note, however, that condition (5.1) and the other frequency relations indicated above imply only the possibility of resonance but are not sufficient for its occurrence. In practice the emergence of a resonance and the rate of the corresponding resonance growth of amplitude depend on the general structure of the nonlinear system considered, and on the numerical values of its characteristics. Note also that in the cases of exponentially growing or decaying harmonic oscillations the variables ω in Eq. (5.1) designate the real physical frequencies—as in the many other relations dealing with exponentially growing or decaying oscillations which we shall meet below. (Thus, for T-S waves corresponding to points of the (k, Re)-plane away from the neutral curve, the symbol ω will as a rule designate the real part, $\omega^{(r)} = \Re e\omega$, of the complex eigenvalue ω of the Orr–Sommerfeld eigenvalue problem. As for the imaginary parts $\omega^{(i)} = \Re e\omega$, they determine exponential factors $\exp(-\omega^{(i)}t)$ which will usually be included in the corresponding amplitudes $A(t)$). Moreover, if one takes it that the frequencies are positive by definition, then the + signs in Eq. (5.1) and the similar relations must, of course, be replaced by $\pm$ signs.

Above, only the case of a one-dimensional oscillation $u(t)$ satisfying some nonlinear ordinary differential equation was considered (although wave disturbances depending on spatial coordinates were sometimes mentioned as examples). Let us now discuss the case of oscillations relating to fluid mechanics at slightly greater length. Here the oscillating disturbances always have the form of vector fields $\mathbf{b}(\mathbf{x}, t) = \{u(\mathbf{x}, t), v(\mathbf{x}, t), w(\mathbf{x}, t), p(\mathbf{x}, t)\}$ (where u, v, w, and p are the three velocity components and pressure) depending on time t and coordinates $\{x, y, z\} = \mathbf{x}$ and satisfying the Navier-Stokes (N-S) partial differential equations. In such a case the study of resonance conditions and of possible types of resonance effect represents a complicated problem, where it is difficult to expect that practically useful results can be found for general disturbances. A very important particular class of disturbances, which played the central part in Chaps. 2 and 3 of this book (and has already been mentioned occasionally in the present section too), is the class of *wave disturbances* having the form $\mathbf{b}(\mathbf{x}, t) = A\ (t)\ \mathbf{B}e^{i}(kx - wt)$ (where $\mathbf{B} = \{U, V, W, P\}$ is a constant vector, and

$A(t)$ is the amplitude, which is slowly changing with t) or some other related wave form. Therefore, it is natural to suppose that investigation of the nonlinear wave resonance must represent an important part of the nonlinear stability theory. The remark above about 'other related wave forms' implied that in some cases it is convenient to consider only one- or two-dimensional waves, where the three-dimensional 'spatial wave factor' $e^{i\mathbf{k}\mathbf{x}}$ is replaced by e^{ikx} or $e^{i(k_1x+k_2y)}$, and the vector **B** may depend on spatial coordinates not entering the given exponents. (In the spatial formulation of the problem of hydrodynamic stability the amplitude A is assumed to be dependent not on t, but on the spatial coordinate or coordinates (most often, on the streamwise coordinate x)). In some cases it is also reasonable to assume that $A = A\,(\mathbf{x}, t)$ is a slowly varying function of both the time and spatial coordinates; see, e.g., the discussion of the corresponding one-mode stability problems in Sect. 4.22 and 4.24(b) and of the three-wave resonances of waves with amplitudes $A(\mathbf{x}, t)$ in Craik (1985), Chap. 5). As to the four-dimensional vector $\mathbf{b} = \{u, v, w, p\}$, in the case of a plane-parallel flow of incompressible fluid it may often be replaced by the two-dimensional vector $\mathbf{w} = \{w, \zeta\}$ (where w is the vertical velocity and $\zeta = \zeta_3$ is the vertical vorticity; see., e.g., Sect. 3.33), while in the case where only two-dimensional wave disturbances are studied it is enough to consider only the scalar stream-function field $\psi = \psi\,(x, z, t)$ (a similar change of arguments must then also be applied to the vector **B**). For the sake of simplicity, we will first consider scalar waves of the form $u(\mathbf{x}, t) = A(t)U e^{(kx-wt)}$ (or of the related one- and two-dimensional forms) representing one component of the vector $\mathbf{b}(\mathbf{x}, t)$ and only later will pass to more general vector waves. Let $u(\mathbf{x}, t) = A(t)U e^{i(kx-wt)}$ be a wave of small enough amplitude which satisfies some nonlinear wave-propagation equation including a nonlinear quadratic term. A very important particular case is that in which $(\mathbf{k}, \omega)$ are the eigenvalues of the corresponding linearized equation (i.e., where ω is the eigenfrequency of the eigenvalue problem corresponding to a given value $\mathbf{k}$ of the wave vector or, if spatial disturbance development in a plane-parallel flow is studied, the streamwise component k_1 of the vector $\mathbf{k} = \{k_1, k_2\}$ is the eigenvalue corresponding to given values of ω and of k_2 or k_2/k_1). In this case the nonlinear equation may be used for approximate evaluation of the effect of nonlinearity on the evolution of an initially very small (practically infinitesimal) wave disturbance. Quadratic nonlinearity entering the equation will produce a term proportional to $exp(2i(\boldsymbol{kx} - \omega t))$, representing a wave with doubled frequency and wave number. As a rule the values $(2\mathbf{k}, 2\omega)$ will not be the eigenvalues of the linearized problem; then the nonlinear equation for the amplitude $A(t)$ will be reducible to an equation of Landau's type, as considered in Sect. 4.2. However, in some exceptional cases both $(\mathbf{k}, \omega)$ and $(\mathbf{2k}, 2\omega)$ will be eigenvalues of a linearized problem and here resonance may occur. In fact, in this case a very small wave proportional to $e^{2i\,(kx-wt)}$ may be generated in the flow by background "noise" (including turbulence and not only acoustic waves) of environmental origin, and then its interaction with the square of the first wave disturbance produced by quadratic nonlinearity of the wave equation will lead to resonant growth of a disturbance component with double the frequency and wave number. Thus, in the case of finite disturbances *a two-wave resonance may be possible in a fluid flow if there is an eigenvalue* (k, ω) *of the linearized equation of motion such that* $(2\boldsymbol{k}, 2\omega)$ *is also an eigenvalue*. (Of course, resonance may also be possible if ($\mathbf{k}$, ω) and ($l\mathbf{k}$, $l\omega$), where l is an integer exceeding 2, are linear eigenvalues. However, this is a higher-order resonance, in which growth terms are proportional to higher powers of small initial amplitudes, and it will not be considered in this book).

The condition printed above in italics is valid only rather rarely. However, *three-wave resonances* may also occur in fluid flows, and the conditions making such a resonance possible can often be satisfied more easily than conditions for the two-wave resonance. Suppose that ($\mathbf{k}_1$, ω_1) and ($\mathbf{k}_2$, ω_2) are both eigenvalues of the linearized equation, determining the infinitesimal wave modes of a disturbance. Then the waves with wave-vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and angular frequencies ω_1 and ω_2 may be simultaneously excited and their interaction (described by the part of the nonlinear term proportional to the product of two waves) will generate waves with wave vectors and frequencies $(\mathbf{k}_3, \omega_3) = (\mathbf{k}_1 + \mathbf{k}_2, \omega_1 + \omega_2)$ and $(\mathbf{k}_4, \omega_4) = (\mathbf{k}_1 - \mathbf{k}_2, \omega_1 - \omega_2)$. The arguments, which were summarized above for the case of harmonic oscillations in a nonlinear system (and which led to Eq. (5.1)) now show that in the case of waves of small (but not infinitesimal) amplitudes satisfying a quadratically non-linear wave equation, the three-wave nonlinear resonance may occur if, together with the waves with wave number and frequency ($\mathbf{k}_1$, ω_1) and ($\mathbf{k}_2$, ω_2), a third wave is present which has the same ($\mathbf{x}$, t)-periodicity as one of the waves produced by nonlinear interaction of the above-mentioned waves, i.e., If $(\mathbf{k}_3, \omega_3) = (\mathbf{k}_1 + \mathbf{k}_2, \omega_1 + \omega_2)$ (or $(\mathbf{k}_1 - \mathbf{k}_2, \omega_1 - \omega_2)$). Hence here the conditions making the resonance possible may be written in the form

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0, \tag{5.2}$$

where it is assumed that signs of the frequencies and wave vectors can be chosen arbitrarily (if this assumption is not accepted, then Eq. (4.2) must be written in the form

$$\omega_1 \pm \omega_2 = \omega_3, \quad \mathbf{k}_1 \pm \mathbf{k}_2 = \mathrm{k}_3; \tag{5.2a}$$

see, e.g., Phillips (1960, 1974a, b). Phillips assumed that the three waves considered have small amplitudes $A_i(t)$, $i = 1, 2, 3$, of the same order of magnitude, and, substituting the sum of three waves into the nonlinear wave-propagation equation (whose form depends upon the nature of the waves considered), he obtained, for the case of three real (sinusoidal) steady waves satisfying the conditions

$$\omega_1 + \omega_2 = \omega_3, \quad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \tag{5.2b}$$

the following approximate equations determining, to the order of the squares of the initial amplitudes, the rates of change of the three amplitudes:

$$\frac{dA_1}{dt} = C_1 A_2 A_3', \quad \frac{dA_2}{dt} = C_2 A_3 A_1, \quad \frac{dA_3}{dt} = C_3 A_1 A_2, \tag{5.3}$$

where C_1, C_2 and C_3 are three interaction coefficients dependent on the particular wave motion considered, and on the wavenumbers involved and their geometrical configuration. (For more details see the paper by Bretherton (1964), the book by Craik (1985) specially devoted to wave interactions in fluid flows, and the papers and books cited below in this section, many of which contain rigorous derivations of these equations for a number of particular cases). Craik considered the complex wave disturbances in a plane-parallel fluid flow where the amplitudes A_1, A_2 and A_3, and also the frequencies ω_1, ω_2 and ω_3, may be complex (as they are, e.g., in the case of unsteady T-S waves). Here the first condition (5.2b) takes the form $\Re_e(\omega_1 + \omega_2) = \Re e\, \omega_3$ (while the second does not change), and Eq. (5.3) take the form:

$$\frac{dA_1}{dt} = \omega_1^{(i)} A_1 + C_1 A_2^* A_3, \quad \frac{dA_2}{dt} = \omega_2^{(i)} A_2 + C_2 A_1^* A_3,$$
$$\frac{dA_3}{dt} = \omega_3^{(i)} A_3 + C_3 A_1 A_2, \tag{5.4}$$

where $\omega_n^{(i)} = \Im m \omega_n, n = 1, 2, 3$, and the asterisk denotes a complex conjugate. In the case of a spatial formulation of the parallel-flow stability problem the frequencies ω_1, ω_2 and ω_3 take real values but the streamwise components, k_{11}, k_{21} and k_{31}, of the vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ may be complex. Therefore in this case the first condition (5.2b) remains unchanged, while the second condition must be replaced by equation $\Re e(\mathbf{k}_1 + \mathbf{k_2}) = \Re e \mathbf{k}_3$ (where the symbol $\Re e$ relates only to the streamwise components of the wave vectors), and Eq. (5.4) must be written as

$$\frac{dA_1}{dx} = -k_{11}^{(i)} A_1 + B_1 A_2^* A_3, \quad \frac{dA_2}{dx} = -k_{21}^{(i)} A_2 + B_2 A_1^* A_3,$$
$$\frac{dA_3}{dx} = -k_{31}^{(i)} A_3 + B_3 A_1 A_2, \tag{5.4a}$$

where $k_{n_1}^{(i)} = \Im\, \mathrm{m}\, k_{n_1}, n = 1, 2, 3$, and the B_n are new interaction coefficients. Note also that, as in the case of complex frequencies, the superscript (r) in the symbols for the real parts of streamwise wave vectors representing exponentially growing or decaying waves will often be omitted, and the real parts will be denoted by symbol k_1 (or k, if the wave is two-dimensional).

Equations (5.4) and (5.4a) represent the nonlinear approximation of lowest order in the weakly-nonlinear stability theory. In the approximation of the next order with respect to wave amplitudes, additional third-order terms appear on the right-hand sides of Eqs. (5.4) and (5.4a); see e.g. Eq. (5.11) below, and the papers by Usher and Craik (1975) and Goncharov (1981), and the book by Craik (1985), Sects. 16.3 and 25–26. The quadratic terms of amplitude equations characterize only resonant modes, while for the more usual nonresonant modes cubic terms follow the linear ones; see, e.g., Landau-Stuart's Eq. (4.40) and Stuart's Eq. (4.43). Presence of quadratic terms clearly means that for resonant modes nonlinearity begins to be important 'sooner' (i.e., at smaller amplitudes) than for nonresonant modes.

The computation of the interaction coefficients C_1, C_2 and C_3 is a complicated problem, for which a number of special methods (applicable to one or other particular wave-interaction situation) have been developed (see e.g. the important early paper by Simmons (1969) and the discussion of this topic in Craik's book). The problem is significantly complicated by the fact that real physical media are very often *dispersive*. This means that the wavenumber $\mathbf{k}$ and the frequency ω of waves in the medium cannot be chosen arbitrarily but must satisfy a definite *dispersion relation*

$$\omega = D(\mathbf{k}), \tag{5.5}$$

where the function $D(\mathbf{k})$ (one- or many-valued) may depend on physical parameters affecting wave propagation and on the dimensionless characteristics of the corresponding physical regime (e.g., on the Reynolds number: see e.g. Eq. (2.90) in Chap. 2 and, for more details, Karpman's monograph (1975)). Therefore in many cases it is not easy to find triads of wave vectors $\mathbf{k}_i$ and frequencies ω_i satisfying both Eq. (5.2b) (or a related equation differing by the signs of some terms and/or by replacement of ω_i by $\Re e\omega_i$) and (5.5), since such triads (if they exist at all) represent only some rare exceptions. In particular, Phillips (1960, 1961) was one of the first to look for three-wave resonances in fluid flows, in his study of inviscid gravity waves in deep water (where $\mathbf{k}$ is a two-dimensional vector, and the dispersion relation has the form $\omega^2 = g\ |\mathbf{k}|$ where g is the acceleration due to gravity). He found that here the condition (5.2b) cannot be fulfilled at all. (note that such a dispersion relation evidently prevents two-wave resonance also). Hence Phillips was forced to pass to four-wave resonances of quadruples of waves satisfying the conditions $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$, $\omega_1 + \omega_2 = \omega_3 + \omega_4$. He found that such quadruples of gravity waves really exist, and hence the four-wave resonances may occur here and produce unsteadiness of the gravity waves. Then he determined the corresponding amplitude equations which, in the case of four-wave resonance contain, in the lowest-order nontrivial approximation, terms of third order in wave amplitudes on the right-hand sides; see also Craik (1985), Sects. 8 and 23. In some plasma and geophysical wave problems the five-wave resonances produced by a coupled pair of resonant wave triads having one member in common are also of importance, and are discussed in Craik (1985), Sect. 16.2, while for examples of resonances of this type appearing in fluid flows see, e.g., the small-type text in the final part of Sect. 5.3 of the present book). Three-wave resonances may be important in many physical situations (the case of gravity waves on a deep-water surface may be considered as an exception) and therefore the literature devoted to study of such resonances is quite extensive. In particular, it was found by McGoldrick (1965) that such resonances may occur in the case of capillary-gravity waves and ripples on the water surface (which are affected by both the gravity and the surface tension, and which have a dispersion relation of the form $\omega^2 = g|\mathbf{k}| + \gamma\mathbf{k}|^3/\rho$, where γ is the coefficient of surface tension and ρ is the density of water). In the case of gravity waves in a heavy liquid beneath a solid plate, the surface tension plays no role but here the waves are affected by the elastic properties of the plate, and this leads to a dispersion relation which again makes three-wave resonance possible; see e.g. Marchenko (1991, 1999). It was also found that three-wave resonances may occur among effectively-inviscid internal gravity waves in stratified flows with density depending on the vertical coordinate; among various large-scale geophysical waves (e.g. those depending on Earth's angular velocity, such as Rossby waves in the troposphere and plasma waves in the ionosphere at much greater heights); and among many other types of interacting nonlinear waves. Note also that the three-wave resonance may be realized in triads of waves of different types (in particular, in triads consisting of two gravity waves on the surface of a stratified liquid and one internal wave in the same liquid, or of two short capillary-gravity waves and one longer purely-gravity wave unaffected by surface tension). A number of theoretical and experimental studies of three-wave resonances in fluid flows may be found in papers by McGoldrick (1965, 1970a, b, 1972); McGoldrick et al. (1966); Phillips (1966, 1967, 1974a, b, 1977, 1981); Longuet-Higgins and Smith (1966); Longuet-Higgins and Gill (1967); Craik (1968); Nayfeh (1971); Brekhovskikh et al. (1972); Loesch (1974); Ripa (1981); Banerjee and Korpel (1982); Yuen and Lake (1982); Hogan (1984); Henderson and Hammack (1987); Perlin et al. (1990); Christodoulides and Dias (1994); Trulsen and Mei (1996) and many others (these publications and the book by Craik (1985) also contain many supplementary references relating to this topic). Since nonlinear dispersive waves may occur in quite different media and situations, the nonlinear wave-resonance theory has many applications to problems outside classical fluid mechanics; in such cases the theory has often been developed independently of studies of waves in ordinary fluids. As typical examples of publications dealing with three-wave resonances relating to waves of other origins, we may mention the papers by Jurkus and Robson (1960) on nonlinear electronics, by Khokhlov (1961) on electromagnetic wave propagation in dispersive conductors, and by Dimant (2000) on nonlinear interactions among ionospheric waves; the books and papers by Armstrong et al. (1962); Bloembergen (1965, 1968) and Akhmanov and Khokhlov (1972) on nonlinear optics, by Davidson (1972); Weiland and Wilhelmsson (1977) and Turner and Boyd (1978) on plasma waves; and the general survey by Kaup et al. (1979) (containing an extensive bibliography and supplemented by Kaup's paper (1981)). However, for present purposes only waves in an incompressible Navier-Stokes fluid are of interest, and in this chapter only the case of waves in plane-parallel and nearly plane-parallel fluid flows will be investigated.

In almost all the papers and books cited above which deal with wave resonances in fluid mechanics, waves in immovable fluids (where there is no basic flow) were considered. In this case the total energy of any group of waves interacting with each other must be conserved. (This means, in particular, that if the wave energy $E \propto |A|^2$ is always positive, the coefficients C_i, $i = 1, 2, 3$, of Eq. (5.3) cannot all have the same sign. More complicated cases, in which the excitation of waves lowers the total energy of the system so that the wave energy must be considered as negative, are often encountered in plasma physics, and have also been considered in application to fluid mechanics, e.g. by Cairns (1979); Craik and Adam (1979), and Craik (1985), Sects. 2.3 and 14.3; however, they will not be discussed in this book). Energy conservation implies that the growth of one wave may be achieved only as a result of energy exchange between various waves, leading to energy redistribution and the attenuation of some other wave (or waves). Such energy redistribution changes the wave amplitudes (and also the wave shapes, which become distorted by the growth of supplementary waves extracting energy from the primary one) producing unsteadiness and hence making the waves unstable. Developing unsteadiness of waves is often of oscillatory type (in which the energy of one of the waves decreases for a time because of transfer to other waves, but then begins to grow anew when the energy transfer changes sign; see e.g. Fig. VII.3 in Phillips (1974b)). Such unsteadiness clearly represents an interesting physical phenomenon which differs strongly from the monotonic growth of disturbance amplitudes, which was studied in previous chapters in connection with transition of laminar flows to turbulence. If the viscosity of the fluid cannot be neglected, then the redistribution of energy between interacting waves will be accompanied by their viscous decay, but here too the growth of one wave amplitude must necessarily be accompanied by simultaneous attenuation of others.

The difference between the wave instabilities observed in immovable fluids and the flow instabilities studied in Chaps. 2–4 above is due first of all to the fact that in these chapters only instabilities of steady laminar *shear flows* (most often of plane-parallel flows with nonuniform velocity distributions $U(z)$) were considered. In such flows the most important mechanism of disturbance growth is connected with the transfer of energy from the primary flow to the disturbance (reverse energy transfer is also possible in principle but it occurs much more rarely). This mechanism plays the leading role in the majority of cases of transition to turbulence, and also in all the flow instabilities studied in Chaps. 2–4 (see in this respect Sect. 4.1 on the energetics of instability phenomena). Therefore it is natural to suppose that the same mechanism may have an essential effect on the resonant growth of wave disturbances in steady shear flows, and thus lead to some new interesting and important instability phenomena.

Apparently Raetz (1959) (see also the discussion of this work by Stuart (1962a) and in Raetz's survey (1964)) was the first to suggest that three-wave nonlinear resonances may play an essential part in the transition to turbulence of a laminar boundary layer with, say, the Blasius velocity profile $U(z)$. Shortly after this Benney and Niell (1962) expressed doubts about the possibility of a nonlinear resonance leading to a large growth of some wave amplitudes; however, later their doubt was found to be groundless (and the importance of nonlinear resonance was then stressed, in particular, by Benney and Gustavsson (1981)). As will be shown below, the main idea of Raetz proved to be correct and very important; for this reason Raetz's unpublished report of 1959 stimulated the appearance of a great number of papers further developing this idea. This matter will be discussed at greater length in the next section; first, however, the results of two relatively old (but quite typical) papers relating to some special cases of nonlinear resonance of waves in shear flows will be briefly considered, as illustrations of the general tendency of nonlinear-resonance studies.

Kelly's paper (1968) was devoted to the search for resonant interactions of waves in two particular plane-parallel inviscid shear flows—in a Bickley jet, where $U(z) = U_0 \operatorname{sech}^2(z/H)$ in $-\infty < z < \infty$, and in a stably stratified plane mixing layer with the velocity profile $U(z) = U_0 \tanh(z/H))$ and the density profile $\rho(z) = \rho_0 \exp(-\beta \tanh^3(z/H))$. In this paper only two-wave resonances, involving pairs of neutrally-stable two-dimensional waves, were considered. For waves proportional to $\exp\{i(k_j x - \omega_j t)\}$,

$j = 1, 2$ (where k_j and ω_j are real and positive), nonlinear resonance is possible if $k_2 = 2k_1$, $\omega_2 = 2\omega_1$; it was found that to identify waves satisfying these conditions, the data collected by Drazin and Howard in their survey (1966) of stability of parallel flows in inviscid fluids are very useful.

According to Drazin and Howard, a Bickley jet can support a pair of neutral two-dimensional waves with the following wave numbers and frequencies: $k_1 = 1/H$, $\omega_1 = (2/3)U_0/H$, and $k_2 = 2/H$; $\omega_2 = (4/3)U_0/H$ (the stream-function vertical profiles $\psi_1(z)$ and $\psi_2(z)$ of these waves were also given by Drazin and Howard). Thus, these two waves may interact resonantly. As for the stratified mixing layer, Miles (1963) showed that there can exist an infinite number of two-dimensional neutral modes depending on the value of the overall Richardson number $\mathrm{Ri}^* = g\beta H/(U_o)^2$ (which characterizes the flow stability). Results given by Drazin and Howard show that at $\mathrm{Ri}^* = 4/3$ the resonance conditions $k_2 = 2k_1$, $\omega_2 = 2\omega_1$ are satisfied for the first two modes; while for $\mathrm{Ri}^* > 4/3$ other resonant cases may occur which also involve higher modes (in particular, at $\mathrm{Ri}^* = 10/3$ a three-wave resonance may occur among the first three modes). Kelly studied the interactions of these pairs of two-dimensional waves in the Bickley jet and in the stratified mixing layer, and found that, at least in the stably stratified mixing layer at $\mathrm{Ri}^* = 4/3$, two-wave resonance can occur, leading to the rapid temporal growth of a wave disturbance with a fixed spatial periodicity. This continuous growth shows that in this case the nonlinear interaction of the waves with each other and with the primary flow leads to transfer of primary-flow energy to the waves.

Slightly later Craik (1968) examined the possibility of resonant gravity-wave interactions in a horizontal liquid layer with the linear velocity profile $U(z) = -u_1 z$, $0 \geq z > -\infty$. (The condition that $|U(z)| \to \infty$ as $z \to -\infty$ is not essential here, because the gravity-wave motions involve only a thin upper layer of liquid). It was indicated above that two-wave and three-wave resonant interactions cannot occur among gravity waves in a liquid at rest, while such interactions among quadruples of waves can occur here but cannot produce continuous growth of any of the waves. Craik found that in a shear flow with a linear velocity profile, two-wave and three-wave resonant interactions are also impossible among two-dimensional gravity waves, but three-wave resonant interactions are now possible among two- and three-dimensional gravity waves. He did not try to examine all possible resonant triads of such gravity waves but limited himself to consideration of triads comprising one two-dimensional wave proportional to $\exp(i(kx - \omega t))$, and two symmetric oblique waves which are proportional to $\exp(i(k_1 x \pm k_2 y - \omega_1 t)$ and thus have inclination angles $\theta_{1,2} = \pm \tan^{-1}(k_2/k_1)$, with the same absolute value but opposite signs. The frequencies ω and ω_1 and the wave numbers k, k_1 and k_2 were assumed to be real, i.e., all the waves considered were neutrally stable. However, these frequencies and wave vectors could not take arbitrary real values, but had to satisfy a definite dispersion relation. Craik showed that, in a homogeneous shear flow with constant shear u_1, a gravity wave with frequency ω and wave vector $\mathbf{k} = (k_1, k_2)$ must satisfy a dispersion relation of the form

$$\omega^2 - (k_1 u_1/|\mathbf{k}|)\omega = g\,|\mathbf{k}|\,,\ \ |\mathbf{k}| = \left(k_1{}^2 + k_2{}^2\right)^{1/2}. \tag{5.6}$$

Relation (5.6) allows us to examine conditions under which a triad of plane waves with wave numbers and frequencies ($\mathbf{k}_1$, ω_1), ($\mathbf{k}_2$, ω_1) and ($\mathbf{k}_3$, ω), where $\mathbf{k}_1 = (k_1, k_2)$, $\mathbf{k}_2 = (k_1, -k_2)$ and $\mathbf{k}_3 = (k, 0)$, may satisfy conditions (5.2b). Since $\omega(k_1, k_2) = \omega(k_1, -k_2)$ by virtue of Eq. (5.6), these conditions now take the very simple form:

$$k_1 = k/2, \qquad \omega_1 = \omega/2. \tag{5.7}$$

Thus the values of k, k_2 and ω must be chosen so that Eq. (5.6) will be valid for the following two wavevector-frequency combinations: (i) (k, 0, ω) and (ii) ($k/2$, k_2, $\omega/2$). Assuming, without loss of generality, that k and ω are positive, Craik showed that such values of k, k_2 and ω exist only under the condition that u_1 is also positive and large enough that

$$\frac{u_1}{(gk)^{1/2}} \geq \frac{[7 + (48)^{1/2}]}{[8 + (48)^{1/2}]^{1/2}} \approx 3.60. \tag{5.8}$$

This means that three-wave resonant interactions, which are completely impossible for gravity waves in a motionless liquid, may be possible for such waves in a homogeneous shear flow for wave triads of special form, but only in the cases where the shear u_1 is positive and large enough. Craik also showed that, under condition (5.8), to every permissible value of $u_1/(gk)^{1/2}$ there correspond two permissible values of $k_2 > 0$ and of the angle $\theta = \tan^{-1}(2k_2/k)$. Moreover, here the two values of θ coincide with each other, and are close to 74° when $u_1/(\mathrm{g}\mathrm{k})^{1/2}$ takes its minimum permissible value (close to 3.6), while when the value of $u_1/(gk)^{1/2}$ increases one of the two values of θ is continuously growing and the other is continuously decreasing, tending to values 90° and 60° as $u_1/(gk)^{1/2} \to \infty$.

The subsequent part of Craik's paper is devoted to a lengthy approximate evaluation of the growth rates for triads of interacting plane waves satisfying the resonant conditions (5.2b). Assuming that the viscosity v is very small and that the initial complex amplitudes $A_1(0)$, $A_2(0)$ and $A_3(0)$ of the three surface gravity waves considered have small (but not infinitesimal) absolute values, Craik derived, under rather general conditions, a system of equations for the functions $A_1(t)$, $A_2(t)$ and $A_3(t)$, of the form (5.4). Here the first terms on the right-hand sides (which describe the viscous decay of the waves) can usually be neglected, while for the leading terms of the expressions for the coefficients C_i, $i = 1, 2, 3$, the following order-of-magnitude estimates were obtained: $C_1 = \mathrm{O}\ (\omega^2/kv)$, $C_2 = \mathrm{O}\ (\omega^2/kv)$, but $C_3 = \mathrm{O}\ (\omega k)$. It is remarkable that at small values of v (i.e. large Re) the coefficients C_1 and C_2, determining the growth rates of amplitudes of the two oblique waves, turn out to be much greater than the coefficient C_3. Hence here the oblique waves grow very rapidly, while the amplitude of the two-dimensional wave changes much more slowly. This shows that in this case a very strong resonant interaction of the three waves takes place, and the oblique waves effectively extract energy from the primary flow while the amplitude of the two-dimensional wave changes only a little. Of course, since these estimates of the wave growth rates were based on the assumption that all the amplitudes are small, the estimates are valid only during a restricted time interval.

In conclusion Craik briefly considered also the resonant-interaction problem for interfacial gravity waves in a two-layer flow where for $z>0$ and $z<0$ the liquid has different densities ρ_1 and $\rho_2>\rho_1$ and the flow has constant but different velocity gradients $dU/dz=u_1$ and u_2. He found that here the condition for three-wave resonance can often be satisfied by much smaller values of the velocity gradients than those given by Eq. (5.8). However we will not linger on this special problem in this book.

5.2 Resonance and Secondary-Instability Manifestation in Boundary-Layer Development

The title of this chapter and the short introduction indicated that the chapter will be devoted to weakly-nonlinear mechanisms of instability development in a steady laminar boundary layer in zero pressure gradient (a "flat plate" boundary layer). Consideration of this flow alone was justified by the prevalence and great practical importance of boundary-layer flows in nature and in industry. It was also noted that properties of the laminar flat-plate boundary-layer flow (often called the 'the Blasius flow' since its velocity profile was computed by Blasius as long ago as 1908; cf. Chap. 2) have been intensively studied by both theorists and experimenters during many years. These studies led to many interesting results which, unfortunately, have not solved all the problems relating to boundary-layer flow instability and transition to turbulence, but nevertheless have considerably clarified the situation and had a great effect on the whole weakly-nonlinear theory of hydrodynamic instability.

The main topics of the present chapter are the nonlinear resonance among three wave-like disturbances with small amplitudes of the same order of magnitude appearing in the boundary-layer flow, and the secondary instabilities of two-dimensional waves of small but finite amplitudes with respect to wave disturbances of other types. However, some other multimode weakly-nonlinear theories of hydrodynamic instability will also be briefly considered. Let us stress again that although enormous amounts of research effort were devoted during the whole twentieth century to the study of instability and transition to turbulence of flat-plate boundary layers, our understanding of these processes is still far from being complete. This statement repeats the remark made more than thirty years ago by Tani (1969), which the work of the intervening years has not disproved. One of the first puzzles relating to instability was produced by the discovery, in the classical experiments on boundary-layers stability by Schubauer and Klebanoff (1956) (see also Schubauer (1958)), Klebanoff and Tidstrom (1959) and Klebanoff et al. (1962), of the fact that the development in a boundary layer of an initially small two-dimensional disturbance always leads to the appearance slightly later of some fast-growing, spanwise-periodic three-dimensional structures. (This fact was later confirmed by many other authors; see, for example, the papers by Tani (1967) and Komoda (1967), which preceded a great number of more detailed experiments and numerical simulations, some of which will be discussed below). The streamwise development of these structures was thoroughly studied by

Klebanoff et al. (and then also by Tani, Komoda, and many others). All the above-mentioned authors followed Schubauer and Skramstad (1947) in using a vibrating ribbon to excite waves in the boundary layer. However Klebanoff et al., and then Tani, Komoda, and some others, modified this technique by inserting spacers (typically small pieces of adhesive tape) at regular intervals beneath of the ribbon, thus producing a weak spanwise periodicity of the disturbance. The spanwise wavelength then depended on the distance between adjacent pieces of tape, and hence could be varied; in the above-mentioned experiments it was always chosen to be equal to that appearing in experiments without any spanwise forcing. Therefore here the usual 3D periodicity was present at the start of the excited region.

The appearance of flow three-dimensionality evidently contradicted the known results of linear stability theory (see, e.g., Sect. 2.81) according to which the most unstable small disturbances in any plane-parallel flow of viscous fluid have the form of two-dimensional wave independent of the spanwise coordinate y. This contradiction could evidently be explained only by some nonlinear effects which were neglected in the linear theory. However, to find an explanation it was necessary to go beyond the Landau approach where only the evolution of disturbance amplitude, but not the change of its shape, was considered.

Benney and Lin (1960) and Benney (1961, 1964) were among the first who attempted to explain theoretically the growth of three-dimensionalities in disturbed plane-parallel flows. For this purpose they applied second-order weakly-nonlinear theory (which preserves only terms of the first two orders in disturbance amplitudes) to the simultaneous development, in a plane-parallel shear flow, of two rather small disturbances: a two-dimensional (2D) wave proportional to $\exp(ikx - i\omega t)$ and a three-dimensional (3D) wave proportional to $\exp(ikx - i\omega_1 t)\cos(k_1 y)$ where ω and ω_1 are complex parameters having the same real parts, $\omega_0 = \Re e\omega = \Re e\omega_1$. (In the first two papers the case of a flow in an unbounded space with a hyperbolic-tangent velocity profile was considered, while in the third paper the simplified model of a plane-parallel boundary-layer flow having the piecewise-linear velocity profile shown in Fig. 3.1a was used as the primary flow). Although the velocity profiles studied differed from the real boundary-layer profile, some of the features of the predicted wave developments recalled phenomena observed in the boundary-layer stability experiments. However, the agreement with experimental data was only qualitative and quite incomplete, and the subsequent attempt by Nakaya (1980) to repeat the calculations using the Blasius boundary-layer velocity profile instead of some simplified model of it did not lead to more satisfactory results. Moreover, Stuart (1962a) noted that the assumption used in the above-mentioned papers, that 2D and 3D waves have the same (real) frequency, contradicted the available experimental data, and Craik (1971) stressed that in these papers the spanwise wavelength was chosen quite arbitrarily while experiments show that it has a definite preferred value. (In fact, Klebanoff et al. and then also Anders and Blackwelder (1980) and some other early experimenters found that this wavelength always takes the same value; however later it was shown that this statements is incorrect). Antar and Collins (1975) relaxed Benney and Lin's, and Benney's, assumptions and accepted that $\Delta\omega_0 = \Re e\omega - \Re e\omega_1$ may differ from zero (and then used in their computations, relating to both Blasius and Falkner-Skan boundary-layer velocity profiles, values of $\Delta\omega$ given by numerical-simulation results). Then Nelson and Craik (1977) considered another relaxation of Benney and Lin's and Benney's models, assuming that $\Delta\,\omega_0 = 0$ but accepting that the streamwise wave numbers of 2D and 3D waves may take different values, while Herbert and Morkovin (1980) studied the superposition of a 2D wave, with the wave vector $\mathbf{k}_1 = (k_1,\ 0)$, and a spanwise wave with wave vector $\mathbf{k}_2 = (0, k_2)$. These generalizations of the previous models, which will not be considered at length here, yielded a somewhat better (but not too good) agreement with the experimental data available in the 1970s (see Craik's paper (1980), especially devoted to comparison of various theoretical models with the experimental data of Klebanoff et al. (1962) and Kovasznay et al. (1962)).

Stuart (1962a, b) supplemented his criticism of the Benney and Lin model by a sketch of a somewhat different approach to the study of development of three-dimensionality in plane-parallel flows. Namely, he applied a weakly-nonlinear analysis of Landau's type to the time evolution of a pair of interacting small wave disturbances (one 2D and the other 3D) having finite real amplitudes, arbitrary complex frequencies ω_1, ω_2 and real wave vectors $\mathbf{k}_1 = (k_1, 0)$ and $\mathbf{k}_2 = (k_1, k_2)$. Neglecting terms of higher than third order with respect to wave amplitudes, Stuart showed that here Landau's Eq. (4.34) is replaced by the system (4.43) of two coupled nonlinear equations for the amplitudes $A_1(t)$ and $A_2(t)$ of the two waves. However he did not try to compute the coefficients of these equations in preparation for solving these equations for any particular plane-parallel flow. Instead, he confined himself to a description of the equilibrium finite-amplitude solutions of these equations, and discussion of the stability of the resulting equilibrium states (see Sect. 4.21 above). Later Itoh (1980) carried out an approximate evaluation of the coefficients of Eq. (4.43) for a plane Poiseuille flow with $\mathrm{Re} = HU_0/\nu$ varying in the range from 4,000 to 8,000, which covers both subcritical and supercritical conditions (as usual, here H and U_0 are the half-distance between the walls and the maximum velocity of the undisturbed flow). It was also assumed by Itoh that $k_1H = 1$ while k_2H takes a number of values not exceeding 1. Moreover, the popular assumption that the contribution of the least stable eigenmode of the linearized equations of motion must dominate the eigenfunction expansions of both 2D and 3D nonlinear waves was also accepted and used to simplify the computations. The coefficients of Eq. (4.43) were found to be dependent on the phase difference of the two waves considered, but this dependence could be eliminated by averaging the solutions over the period of 'fast oscillations' produced by the difference of the primary frequencies of 2D and 3D waves. Using such averaging Itoh found some conditions for stability of a two-dimensional wave, superimposed on a plane Poiseuille flow, to three-dimensional wave disturbances, and he estimated the threshold value of the 2D-wave amplitude A_1 above which the three-dimensional waves are growing continuously. Using numerical values for the coefficients in the equation, the equilibrium solutions (4.44) for Poiseuille flows could be found and their stability characteristics determined. Even earlier, these stability characteristics were studied by Volodin and Zel'man (1977) for the simpler case of two interacting two-dimensional T-S waves in the Blasius boundary-layer flow at supercritical values of Re.

Let us recall that in the 1960s, when Benney and Lin began to study the time evolution of two-mode disturbances in plane-parallel flows, they assumed that the streamwise wave numbers of a two-dimensional and a three-dimensional wave have the same value of k. This assumption (which was later rejected by Nelson and Craik (1977)) evidently excluded the possibility of including the 2D and 3D waves in a resonant triad of wave disturbances. However, even before the appearance of these studies, Raetz (1959) stated that according to his computations for the plane-parallel model of a Blasius boundary layer, there exist some triads of three-dimensional wave disturbances which satisfy the conditions (5.2b). These conditions imply that the corresponding waves may interact resonantly, producing rapid growth of wave amplitudes, and Raetz suggested that such resonant instability of wave triads may play an important part in the transition of boundary layers to turbulence.

Raetz considered only neutral 3D waves, corresponding to real values of the eigenfrequency ω. Because his report of 1959 did not contain a complete description of the computations, Stuart, preparing his survey lecture (1962a), computed a new one more resonant triad for a Blasius boundary layer at a supercritical value of Re (this triad also consisted of three-dimensional neutrally-stable waves; according to Raetz resonant triads do not exist among purely two-dimensional waves). However, neither Raetz nor Stuart computed the corresponding growth rates (determined by the values of the three coefficients C_i in Eq. (5.3)). This was, of course, a necessary step; recall that in general conditions (5.2b) are necessary for the resonant character of three-wave interactions, but do not guarantee that resonance will actually occur in all cases where these conditions are valid.

A much more detailed study of resonant three-wave interactions in boundary-layer flows was carried out by Craik (1971). He noted that Raetz's and Stuart's limitation to neutrally-stable waves wrongly restricts the class of resonant wave triads to be studied, since such triads can in principle also exist among both subcritical and supercritical waves. (In the case of non-neutral waves the real frequencies ω_I must of course be replaced, in the first condition (5.2b), by the real parts $\Re\mathrm{e}\omega_\mathrm{i}$ of the corresponding complex eigenvalues $\omega_\mathrm{i} = c_i/k_1$ of the Orr–Sommerfeld Eq. (2.41), but no limitations are imposed by these conditions on their imaginary parts). However Craik did not consider the general case of arbitrary triads of three-dimensional Tollmien–Schlichting (T-S) waves with any wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$. As in his paper of 1968 on gravity waves, discussed at the end of the previous subsection, he examined only special triads, comprising one 2D wave propagating in the streamwise direction (proportional to $\exp(i(kx - \omega t))$) and two 3D waves proportional to $\exp(i(k_1 x \pm k_2 y - \omega_1 t)$ (i.e., inclined at equal but opposite angles $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1)$ to the flow direction). Hence he assumed that $\mathbf{k}_1 = (k, 0)$, $\mathbf{k}_2 = (k_1, k_2)$ and $\mathbf{k}_3 = (k_1, -k_2)$ and thus the resonance conditions (5.2b) took the very simple form $k_1 = k/2$, $\Re\mathrm{e}\omega_1 = \Re\mathrm{e}\omega/2$ (cf. 5.7). The reason for giving much attention to such special triads was connected with the fact that here the three waves have the same phase velocity $c = \Re\mathrm{e}\omega/k = \Re\mathrm{e}\omega_1/k_1$ and hence there is only one critical layer, at the height z_0 where $U(z_0) = c$. Since it is known that the most intensive interaction of a small wave with the primary steady flow takes place in the vicinity of the critical layer (see, e.g., the discussion of the role of a critical layer in nonlinear resonant interactions by Mankbadi (1990, 1991, 1994); Mankbadi et al. (1993), and Goldstein (1995)), it is natural expect that in the case where three interacting waves have the same critical layer these waves may extract energy from the primary flow in a particularly powerful manner. This expectation was confirmed by Craik (1968) for the case of gravity waves on the surface of a liquid shear layer; the results of his paper of 1971 (see below) also agreed with the stated expectation rather well (a small correction of the conclusion that in this case the oblique-wave growth must take a maximal value was discovered later, and will be discussed in Sect. 5.4: see, in particular, Figs. 5.13, 5.14 and 5.15 and the text relating to these figures).

In the case of T-S waves in a plane-parallel primary flow the search for possible resonant triads may be facilitated considerably by the use of the Squire theorem presented in Sect. 2.81. According to this theorem $\omega_j(k_1, k_2, \mathrm{Re}) = \omega_j(|\mathbf{k}|, 0, k_1\mathrm{Re}/|\mathbf{k}|)$, where $\omega_\mathrm{j}(k_1, k_2, \mathrm{Re}) = c_\mathrm{j}(k_1, k_2, \mathrm{Re})/k_1$ is the jth eigenfrequency of the general O-S Eq. (2.41) corresponding to the wave vector $\mathbf{k} = (k_1, k_2)$ and Reynolds number Re, while $|\mathbf{k}| = ({k_1}^2 + {k_2}^2)^{1/2}$. Thus here we need only the eigenvalues $\omega_\mathrm{j}(k, 0, \mathrm{Re})$ of the two-dimensional O-S Eq. (2.44) for various values of k and Re. One convenient method for determination of such resonant triads in the case of a plane-parallel flow with a given value of Re begins with the plotting of the curves $\Re\mathrm{e}\,\omega(k_1, k_2)/k_1 \equiv c_\mathrm{r}(k_1, k_2) = \mathrm{const.}$ and $\Im\mathrm{m}\,\omega(k_1, k_2)/k_1 \equiv c_\mathrm{i}(k_1, k_2) = \mathrm{const.}$ (where $c(k_1, k_2)$ is the eigenvalue of the O-S eigenvalue problem (2.41–2.42) with the greatest imaginary part) in the (k_1, k_2)-plane (the Squire theorem is, of course very useful here). Then one may select an arbitrary point $(k, 0)$ on the k_1-axis, plot a curve $c_\mathrm{r} = \mathrm{const.}$ passing through this point, and then determine the two intersections (symmetric with respect to the k_2 axis) of the line $k_1 = k/2$ with the plotted curve. These intersections determine the values k_2 and $-k_2$ such that the oblique waves with wave vectors $\mathbf{k}_2 = (k_1, k_2)$ and $\mathbf{k}_3 = (k_1, -k_2)$ together with the 2D wave with the wave number $\mathbf{k}_1 = (k, 0)$ make up a resonant triad. Therefore, for any wave number k, a resonant triad may normally be found which consists of a 2D wave with wave vector $\mathbf{k}_1 = (k, 0)$ and two symmetric oblique waves with wave vectors $\mathbf{k}_2 = (\mathrm{k}/2, k_2)$ and $\mathbf{k}_3 = (k/2, -k_2)$.

Craik (1971) applied this method to find a number of resonant triads for two-dimensional Blasius boundary layers with various values of $\mathrm{Re} \equiv \mathrm{Re}^* = U_0\delta^*/\nu$. (Here U_0 is the free-stream velocity and δ^* is the displacement thickness. Below, these velocity and length scales will be used to make dimensionless all physical parameters relating to the Blasius boundary layer; therefore, the symbol k will now denote the dimensionless combination $k\delta^*$, the symbol ω the combination $\omega\delta^*/U_0$ and so on). Figure 5.1a shows one of Craik's examples, for $\mathrm{Re}^* = 882$. Here two resonant triads are denoted by dotted arrows; the first of them with $k = 0.254$ includes the linearly-most-unstable 2D wave as its first component while its second and third components are linearly stable, and the second triad with $k = 0.46$ includes the linearly-stable 2D wave but linearly-unstable oblique waves. Later F. Hendriks (see his Appendix at the end of Usher and Craik's paper (1975)) extended Craik's calculations to the six additional resonant triads (with $0.1 \le k \le 0.5$) in the Blasius boundary layer with $\mathrm{Re}^* = 882$; some comparisons of the results of Craik's and Hendriks' calculations with experimental data were made by Craik (1980). A number of other examples of resonant triads of the same type in Blasius boundary layers with

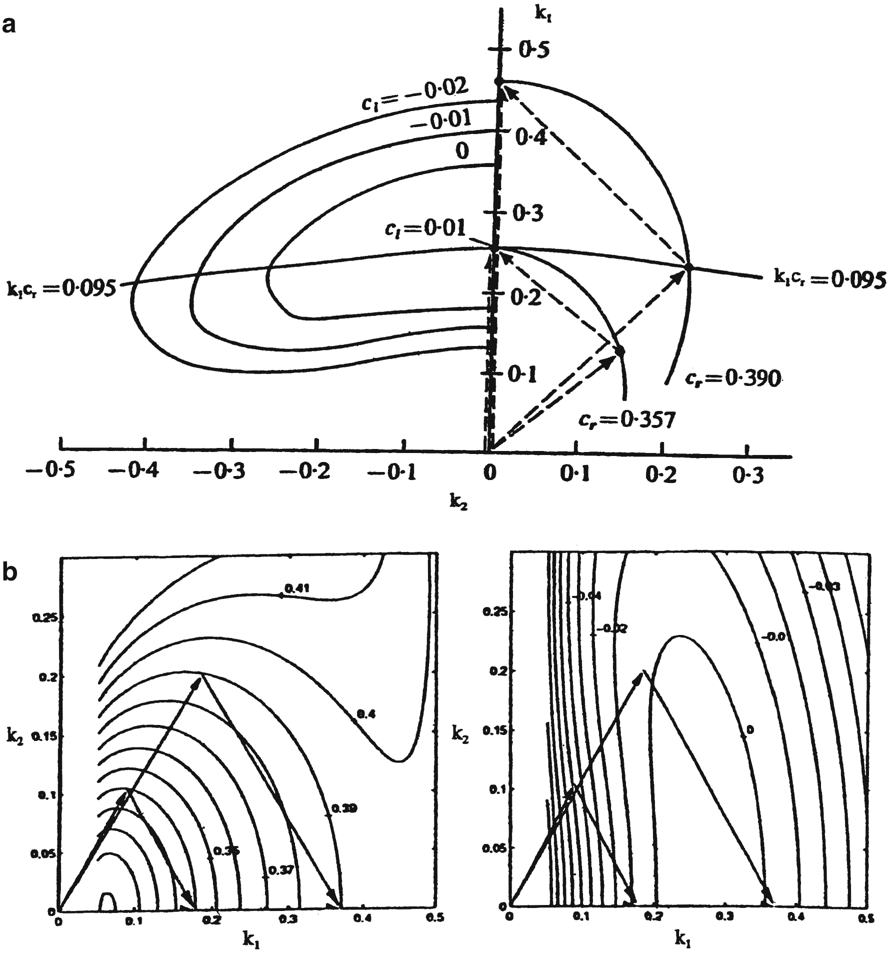

Fig. 5.1 Examples of calculated resonant wave triads in the Blasius boundary layers with $\mathrm{Re}^* = 882$ and $\mathrm{Re}^* = 750$. (**a**) Contours in the (k_1, k_2)-plane of phase-velocity values $c_r = \hat{A}ec = \hat{A}e(\omega/k_1)$ and values of $c_i = \acute{A}mc = \acute{A}m(\omega/k_1)$ (determining the growth, or decay, rate $k_1 c_i$) for temporally-evolving T-S waves in a boundary layer with $\mathrm{Re}^* = 882$. Since $c_r(k_1, k_2) = c_r(k_1, -k_2)$ and $c_i(k_1, k_2) = c_i(k_1, -k_2)$, contours for c_i are shown only for $k_2 \leq 0$, and those for c_r only for $k_2 \geq 0$. Two resonant triads with wave vectors $\mathbf{k}_1 = (k, 0)$, $\mathbf{k}_{2,3} = (k/2, \pm k_2)$, satisfying the condition $c_r(k, 0) = cr(k/2, \pm k_2)$, are shown by arrows. (after Craik (1971)). (**b**) Contours in the (k_1, k_2)-plane of phase velocity $c_r = \hat{A}ec = \hat{A}e(\omega/k_1)$ (the left diagram) and of $c_i = \acute{A}mc = \acute{A}m$ (ω/k_1) (the right one) for temporally evolving T-S waves in a boundary layer with $\mathrm{Re}^* = 750$. Two examples of resonant triads are shown by arrows. (After Schmid and Henningson (2001))

various values of Re^* and k may be found, in particular, in Volodin and Zel'man's paper (1978) and the book by Schmid and Henningson (2001). As an example, two triads computed by Schmid and Henningson for $\mathrm{Re}^* = 750$ are presented in Fig. 5.1b; here the first triad (where $k \approx 0.37$) comprises three waves which have negative growth

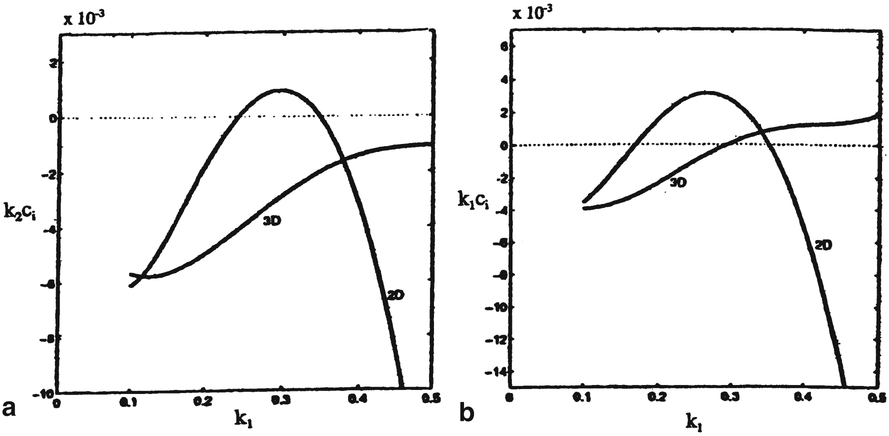

Fig. 5.2 Temporal growth (or decay) rates $k_1 c_i = \Im m\omega$ for 2D and 3D components of resonant triads of Craik's type with various values of streamwise wavenumber k_1 of 2D waves in Blasius boundary layers with (**a**) Re* = 600, and (**b**) Re* = 1,000. (After Schmid and Henningson 2001)

rates close to zero according to linear stability theory, while all members of the second triad (where $k \approx 0.18$) are essentially stable (have appreciably negative rates of growth). In Figs. 5.2a, b, also taken from Schmid and Henningson's book, the linear stability properties of all resonant triads of Craik's type with $k \leq 0.5$ are shown for two Reynolds numbers, Re* = 600 and 1,000. One may see that for the lower value of Re* there is a range of wavenumbers k where the 2D component of a resonant triad is linearly unstable, while the 3D components are linearly stable at any k; on the other hand, at Re* = 1,000 both 2D and 3D components may be simultaneously unstable. Note, however, that for linearly-stable oblique T-S waves entering a resonant triad, the rates of their resonant growth usually greatly exceed the rates of their decay given by the linear theory of stability. Therefore the linear stability of such waves in fact plays no part here.

An essential part of Craik's paper of 1971 was devoted to approximate evaluation of the coefficients C_1, C_2 and C_3 of Eq. (5.4) for the amplitudes of three resonant waves. It was based on the use of nonlinear equations for the velocity components of a steady primary flow disturbed by three T-S waves of small amplitude. Craik's main attention was paid to asymptotic results for large values of kRe. He found that if the critical layer is located far from flow boundaries, then, under rather general conditions for fairly large (but finite) values of the Reynolds number, the magnitudes of the coefficients are $C_1 = \mathrm{O(Re)}$, $C_2 = \mathrm{O(Re)}$, and $C_3 = \mathrm{O}(1)$, for an arbitrary velocity profile $U(z)$ of the primary flow (here it is assumed that the 1st and 2nd waves are the oblique ones while the 2D wave has number 3). This shows that, again, the amplitudes of two oblique waves grow very fast at the expense of the energy of the primary flow, while the amplitude of the two-dimensional wave changes much more slowly.

Since Craik's results relating to the Blasius velocity profile $U(z)$ and triads of waves shown in Fig. 5.1a were found to be very complicated and cumbersome, he also considered the simpler model of a piecewise linear velocity profile of the form

$$U(z) = bz, b > 0, \text{ for } 0 \leq z \leq H, U(z) = bH = U_0 \text{ for } z > H \tag{5.9}$$

(shown in Fig. 3.1a in Chap. 3). For this profile, Craik was able to find explicit asymptotic equations for the coefficients of Eq. (5.4) which showed at once that in this case $C_1 = \text{O(Re)}$, $C_2 = \text{O(Re)}$ and $C_3 = \text{O}(1)$ at large values of $\text{Re} = U_0H/\nu$.

Later Reutov (1990) examined resonant wave interactions in the model of a boundary-layer flow with the velocity profile (5.9), assuming that $\nu = 0$ and hence $\text{Re} = \infty$ (note that stability with respect to infinitesimal disturbances of such an inviscid flow was thoroughly investigated by Tietjens (1925), whose results were used extensively by Craiks). Reutov's idea was to show that results similar to those found by Craik may be obtained more easily for the simpler case of an inviscid fluid. In an inviscid flow with the velocity profile (5.9) the dispersion relation determining the frequency ω of a three-dimensional wave proportional to $\exp(i(k_1x + k_2y - \omega t)$ has the form

$$\omega = k_1U_0\left[1 - \frac{1}{2\,|\mathbf{k}|\,H}(1 - \exp(-2\,|\mathbf{k}|\,H)\right], \quad |\mathbf{k}| = \left(k_1^2 + k_2^2\right)^{1/2}. \tag{5.10}$$

(Eq. (5.10) was also obtained by Tietjens (1925); it is, of course, much simpler than the dispersion relations for viscous plane-parallel flows where the possible values of ω at given $\mathbf{k}$ are given by the eigenvalues of the corresponding O-S problem (2.41–2.42)). Like Craik, Reutov considered only those wave triads consisting of a two-dimensional wave proportional to $\exp(i(kx - \omega t))$ and two oblique waves proportional to $\exp(i(k_1x \pm k_2y - \omega_1 t)$. Conditions (5.2b) then take the form (5.7), and by virtue of Eq. (5.10) these conditions will be valid here if, and only if, $k/k_1 = 2$ and $k_2/k_1 = \sqrt{3}$. Hence here the possible resonant triads consist of a 2D wave with wave number k (which can take any value) and two oblique waves with streamwise wave number $k/2$ which are inclined at angles $\pm 60°$ to the primary-flow direction. Investigating the nonlinear interaction between three plane waves of small amplitudes, Reutov paid his main attention to the subject studied in his earlier paper (Reutov (1985)), namely the most important contribution to this interaction, which is produced in the vicinity of the critical layer where $U(\text{z}) = \text{c}$. He was able to show that at small positive values of t, when it is sufficient to keep only the terms of first and second order in the amplitudes, oblique waves grow exponentially while the amplitude of a two-dimensional wave remains practically constant. Thus here also a strong nonlinear resonance takes place, and leads to very effective transfer of energy from the inviscid steady flow to a pair of oblique waves; the two-dimensional wave plays the role of a catalyst, stimulating this process but preserving practically constant amplitude.

For the case of nonlinear development of resonant wave disturbances in viscous plane-parallel (or nearly plane-parallel) fluid flows, Zel'man (1974) proposed to take an average of the equations of motion over an 'intermediate region' which is much smaller than typical scales of change of the most important 'slow variations' of wave amplitudes but much greater than the wave lengths and periods of unimportant rapid oscillations, which considerably complicate the required solution. This method generalizes Landau's approach (1944) considered at the beginning of Sect. 4.21, and also has many features in common with the popular method of multiple scales, which was mentioned several times in Chap. 4 and will be mentioned again later in this chapter. This method of averaging has many applications to various physical problems relating to nonlinear oscillations and waves (see e.g. Chap. 11 of the book by Rabinovich and Trubetskov (1989)). In particular, after 1974 this method was often applied to fluid-dynamic equations, where it facilitated the evaluation of interaction coefficients in the equations for amplitudes of resonant wave systems. One of the first examples of its use was due to Volodin and Zel'man (1978) who applied this method to the study of the spatial, rather than temporal, development of resonant wave triads of Craik's type in a Blasius boundary-layer flow. (For further applications of the method of averaging to development of disturbances in boundary layers see Zel'man and Kakotkin (1985); Zel'man (1991) and Zel'man and Maslennikova (1993a)). Volodin and Zel'man based their analysis on the numerical integration of the averaged nonlinear equations for the vertical velocity w and the vertical vorticity $\zeta = \zeta_3$ of the disturbed flow (i.e. Eqs. (3.44) and (3.54) supplemented by nonlinear terms). These equations are equivalent to the Navier-Stokes equations for velocity components (since components u and v may be determined if w and ζ are known) but the velocity-vorticity equations do not contain the pressure; therefore they are more convenient and are used very frequently (see e.g. the review paper by Gatski (1991)). The computational procedure used by the above authors allowed them to determine the values of the interaction coefficients $B_{1,2}$ and B_3 in Eq. (5.4a), relating to wave amplitudes $A_\mathrm{i}(x)$, $i = 1, 2, 3$, for a number of values of Re* in the range from 650 to 1,300, and of the non-dimensionalized wave number $k_1 = k/2$ in the range from 0.19 to 0.5. Recall that in the case of a two-dimensional less-stable T-S wave with $k_2 = 0$, values of Re* and k uniquely determine the value of ω). By virtue of the 3D Orr–Sommerfeld Eq. (2.41), if the values of Re* and ω are fixed then the value of k_1 determines the value of k_2 for the most unstable wave and therefore also determines the inclination angles $\theta_{1,2} = \pm \tan^{-1} (k_2/k_1)$ of the oblique components of the wave triad (for more details see Kachanov and Michalke (1994, 1995) and Kachanov (1996)). Volodin and Zel'man found that in the spatial formulation of the stability problem the ratio $|B_{1,2}|/|B_3|$ also takes fairly large values, which increase appreciably with increasing Re* (the coefficients B_1 and B_2 coincide here for reasons of symmetry). They also applied the approximate method of Bouthier (1973) to incorporate the effect of streamwise variation of the boundary-layer flow into the computation; it was found that this effect does not invalidate the important conclusion formulated above. This conclusion was later confirmed by the analytical investigations of resonant-triad development in a streamwise-growing boundary layer by Smith and Stewart (1987); Nayfeh (1987a, b) (who criticized some of the assumptions of Smith and Stewart which were also called in question by the work of Mankbadi et al. (1993) and Wu (1993, 1995); see also Healey's (1995) critical discussion of various approximations used in derivation of multimode amplitude equations) and the papers by Khokhlov (1993, 1994) (who used a slight modification of Smith and Stewart's assumptions), and Zel'man and Maslennikova (1984, 1993a) (some of whose results will be discussed below).

Craik (1971) considered, at the end of his paper, several exact solutions of some particular amplitude equations of the form (5.4) (for the simplest case of Eq. (5.3) with constant coefficients C_i, exact solutions, represented in terms of elliptic functions, were found independently by Jurkus and Robson (1960); Armstrong et al. (1962) and Bretherton (1964)). Craik's solutions also include examples where amplitudes of some waves become infinite after a finite time. (These singularities apparently show that the wave energy grows faster than exponentially; of course, the second-order Eq. (5.4) cease to be valid in such cases before the predicted 'infinite instability burst' occurs). Later Craik and Adam (1978) and Craik (1978, 1985) also considered three-wave resonances, for wave with amplitudes depending on both spatial coordinates and time. In this case the left-hand sides of Eq. (5.3a) must be supplemented by the terms $(\mathbf{v}_i\nabla)A_i$ where $\mathbf{v}_i$ is the appropriately-defined velocity of the ith wave. Wave systems of such types are met in a number of diverse physical problems; therefore the exact solutions of some of the corresponding amplitude equations found by Craik may have many applications. However, this subject will not be discussed here at any length.

Craik (1971) found also that 'direct computation' of the interaction coefficients C_i with the help of the Navier-Stokes equations was quite complicated and labor-consuming. Therefore Usher and Craik (1974) tried to replace the ordinary form of N-S equations in this computation by the variational formulation suggested by Bateman and presented in the textbook by Dryden et al. (1956). This attempt was stimulated by the fact that in the case of a similar problem for capillary-gravity waves a variational analysis by Simmons (1969) proved to be much more simple and elegant than the 'direct evaluation' of the interaction coefficients by McGoldrick (1965) by means of Euler's inviscid equations of motion. According to Usher and Craik the viscous terms and the non-self-adjointness of the N-S equations considerably complicate the derivation of an appropriate variational formulation of these equations. Nevertheless, such a formulation can be found, and it indeed allows computation of the interaction coefficients more simply than by Craik's method of 1971. However, the subsequent rapid increase in the power of digital computers, combined with the development of improved numerical methods, soon made applications of the variational method unnecessary.

A more complete, but still weakly-nonlinear, theory of resonant three-wave interactions, which takes into consideration terms of third order in wave amplitudes, was developed by Usher and Craik (1975). Recall that the Landau and Stuart-Landau Eqs. (4.34) and (4.40) for the amplitude of a single mode, and Stuart's Eq. (4.43) for amplitudes of a pair of non-resonantly interacting modes, both include terms of the third order (but second-order terms are absent). Hence the amplitude equations, which include terms up to the third order, generalize both the one-mode and two-mode equations of Landau and Stuart, and Eq. (5.4) for the three-wave resonant interactions. The third-order amplitude equations for three-wave resonant interactions derived by Usher and Craik (and later by Weiland and Wilhelmsson (1977) and

Goncharov (1981); see also Craik (1985), Sects. 16.3 and 25–26) have the form

$$\frac{dA_1}{dt} = \omega_1^{(i)} A_1 + C_1 A_2^* A_3 + A_1 \left(c_{11}|A_1|^2 + c_{12}|A_2|^2 + c_{13}|A_3|^2\right),$$
$$\frac{dA_2}{dt} = \omega_2^{(i)} A_2 + C_2 A_1^* A_3 + A_2 \left(c_{21}|A_1|^2 + c_{22}|A_2|^2 + c_{23}|A_3|^2\right), \quad (5.11)$$
$$\frac{dA_3}{dt} = \omega_1^{(i)} A_3 + C_3 A_1 A_2 + A_3 \left(c_{31}|A_1|^2 + c_{32}|A_2|^2 + c_{33}|A_3|^2\right).$$

(In the case of non-resonant three-wave interactions, third-order amplitude equations have the same form but with $C_1 = C_2 = C_3 = 0$; therefore, non-vanishing of the latter coefficients shows that the wave interactions are resonant).

Compared with Eq. (5.4), the new equations include nine additional unknown coefficients c_{ij}. Usher and Craik gave their main attention to the case of a resonant triad of Craik's type, consisting of two symmetric oblique waves and one plane 2D wave. As in Craik (1971), they assumed that numbers 1 and 2 correspond to the oblique waves while number 3 corresponds to the 2D wave; then the oblique-wave symmetry implies that $c_{11} = c_{22}$, $c_{13} = c_{23}$, $c_{12} = c_{21}$, and $c_{31} = c_{32}$. Therefore, in this case only five new coefficients need evaluation. Nonlinear N-S equations for the velocity components lead to some lengthy expressions for these coefficients, showing that at large values of Re all coefficients take large values (proportional to some positive powers of Re). These asymptotic estimates force one to conclude that at large values of Re the second-order Eq. (5.4) may be valid only for waves with rather small amplitudes.

Craik (1975) studied equilibrium solutions of the third-order three-wave amplitude Eq. (5.11) and the stability of these solutions. Recall that the third-order Landau and Stuart-Landau Eqs. (4.34) and (4.40) imply that if Landau's constant $\delta > 0$, then at small supercritical values of $\mathrm{Re} > \mathrm{Re}_{cr}$ there is an equilibrium periodic solution of Eq. (4.40) which separates from the steady primary flow by a Hopf bifurcation. On the other hand, if $\delta < 0$, then an equilibrium solution exists under slightly subcritical conditions $\mathrm{Re} < \mathrm{Re}_{cr}$ where a periodic wave of finite amplitude bifurcates from the primary flow if its initial amplitude exceeds a small, but finite, critical value (proportional to $(\mathrm{Re}_{cr} - \mathrm{Re})^{1/2}$). Craik used Eq. (5.11) for investigation of the stability of equilibrium solutions of Eq. (4.40) with respect to pairs of symmetric oblique waves of small amplitude, and for determination of conditions making possible a second bifurcation, leading to the appearance in the flow of a resonant triad, consisting of the same two-dimensional wave as that entering the primary equilibrium solution together with symmetric oblique waves of finite amplitudes. These results of Craik are relevant to the results by Herbert (1984a, 1985, 1986, 1987, 1988a, b) relating to the secondary-instability mechanism of boundary layer instability which will be considered slightly later in this subsection.

Let us now revert to discussion of Craik's (1971) resonant triads consisting of one two-dimensional T-S wave with the angular frequency ω and wave vector $(k, 0)$ and two fully symmetric oblique waves with the same frequency $\omega/2$ and wave vectors $(k/2, \pm k_2)$. Following Craik we will assume that three waves of a triad have small amplitudes of the same order of magnitude. Assume that the value of

ω is determined by the conditions of the experiment and is therefore known. (This condition is fulfilled, in particular, if the 2D plane wave is excited by some device oscillating with a fixed frequency—e.g., by the vibrating ribbon used by Schubauer and Skramstad (1947) and then by many others; or by an acoustic radiator used, among others, by Morkovin and Paranjape (1971); Yan et al. (1988), and a number of authors cited by Nishioka and Morkovin (1986); or by a heating element with periodically varying temperature used, in particular, by Liepmann et al. (1982); or by localized periodically-alternating blowing and suction of fluid considered by Konzelmann et al. (1987)). Among the flow disturbances produced by a device oscillating with frequency ω, a dominant role is played by the least-stable T-S wave having this frequency; such a wave is always two-dimensional and its wavenumber k can be uniquely determined from the two-dimensional O-S eigenvalue problem (2.44), (2.42). If this wave is linearly unstable, then it will grow, and some time later will excite a pair of oblique waves forming, with the primary T-S wave, Craik's resonant triad (or, maybe, a fast-growing triad close to this—such a possibility will be discussed later).

In the case when Craik's triad alone is excited (here, only this case will be considered) the spanwise wavenumbers $\pm k_2$ of the oblique waves with frequency $\omega/2$ and streamwise wavenumber $k/2$ may be determined from the three-dimensional O-S eigenvalue problem (2.41), (2.42). (A number of results relating to computation of the 3D T-S waves by numerical solution of this eigenvalue problem (where k_1, and not ω, is considered as a complex eigenvalue), supplemented by comparison of the results obtained with the available experimental and numerical-simulation data, were presented by Kachanov and Michalke (1994, 1995) and briefly discussed by Kachanov (1996)). Consider now another case, where the disturbances penetrate into the boundary layer from an unsteady external stream generating background "noise", including an extensive collection of weak fluctuations. Then among the boundary-layer waves produced by these fluctuations, the two-dimensional T-S wave with the greatest linear growth rate will naturally dominate the initial stage of disturbed-flow development. At a given value of Re, such a wave has definite values of ω and k which may be determined with the help of Eqs. (2.44) and (2.42) (see Figs. 2.23 and 2.26); hence here also the values of ω and k may be considered as known. Knowledge of ω and k (and hence also of $\omega/2$ and $k/2$) again allows the spanwise wavenumbers $\pm k_2$ of the oblique components of the resonant triad to be determined uniquely, from Eqs. (2.41) and (2.42). We see that in the framework of Craik's model the spanwise periodicity of the 3D structure may usually be determined uniquely.

However, the uniqueness of the value of k_2 at given values of Re, ω and k is contradicted by the data of Saric and Thomas (1984); Saric et al. (1984), and Kozlov et al. (1984) who found that in their experiments (where a 2D wave was excited by a vibrating ribbon) the observed value of k_2 depended not only on ω and k but also on the initial amplitude of the excited 2D wave (for more details see part (b) of this section). On the other hand, the exact symmetry of oblique waves entering Craik's triad clearly requires the initial real amplitudes $|A(0)|$ and phases θ (0) (where $|A(0)|\, e^{i\theta(0)} \equiv A(0)$ is the initial complex amplitude) of two oblique waves to coincide with each other.

This requirement appreciably restricts the Craik model of development of three-dimensional disturbances in a boundary layer. These two circumstances led Herbert (1983a, 1984a) and Saric and Thomas (1984) to doubt the universal applicability of Craik's model of resonant-triad generation of three-dimensionality in steady plane-parallel (or nearly plane-parallel) shear flows and to attribute some of the observed 3D structures in such flows to the secondary-instability mechanism.

The suggestion that the discrepancy between Craik's theory and experiment, and the apparent restrictions of this theory, required it to be replaced by a more universal secondary-instability approach was not unanimously supported. In particular, Zel'man and Maslennikova (1985, 1989, 1990, 1993a) (see also Maslennikova and Zel'man (1985)) showed that exact symmetry of oblique waves (which implies that the initial amplitudes of two oblique waves must take the same value), and exact equality of the oblique-wave real frequency and streamwise wavenumber to half those of the accompanying plane wave, are not necessary for the rapid resonant growth of the two oblique waves entering the wave triad. They considered wave triads where the frequencies and wave vectors of the plane wave and the two oblique waves are $\{\omega, k, 0\}$ and $\{\omega/2, k_1 \approx k/2, \pm k_2\}$, respectively, but with $k/2 - k_1 \neq 0$ (for given values of ω and k_2 the values of k and k_1 may be uniquely determined with the help of the O-S Eqs. (2.44) and (2.41); therefore only the values of ω and k_2 can be chosen arbitrarily). According to the results of Zel'man and Maslennikova's computations, if the initial amplitudes $|A_2(0)|$ and $|A_3(0)|$ of two oblique waves are not equal and k_1 does not coincide exactly with $k/2$, the growth of oblique-wave amplitudes nevertheless remains much larger than the growth of the plane-wave amplitude. Moreover, the nonlinear interactions usually lead to rapid equalization of amplitudes $|A_2(t)|$ and $|A_3(t)|$ and to recovery of oblique-wave symmetry, and after a short time these two amplitudes become greater than the amplitude of the 2D wave. Still later, fast-growing oblique waves start to influence the plane wave very strongly, and cause its explosive growth, which is more rapid than the exponential growth of the oblique waves.

The above formulation is the temporal one, but in fact Zel'man and Maslennikova considered the spatial, not temporal, growth of boundary-layer waves, which is more convenient for comparison with the experimental data. Therefore they used amplitude equations of the form (5.4a), not (5.4), and determined the corresponding interaction coefficients B_n, $n = 1, 2, 3$, by a method generalizing that used by Volodin and Zel'man (1978) in the study of spatial development of strictly symmetric wave triads. A typical example of results more general than those found in 1978 is shown in Fig. 5.3, taken from Zel'man and Maslennikova's paper (1993a). In this figure the dependence of wave amplitudes $|A_i(x)|$, $i = 1, 2, 3$, on the streamwise coordinate x is replaced by their dependence on Reynolds number $\mathrm{Re}^+ = U_0\delta^+/\nu = (U_0 x/\nu)^{1/2} = (\mathrm{Re}_x)^{1/2}$, where $\delta^+ = (\nu x/U_0)^{1/2} \approx 0.58\delta^*$ is a new scale of the boundary-layer thickness which is often used instead of δ^* (it was used, in particular, in Chap. 2). Figure 5.3 corresponds to some definite values of the dimensionless frequency $F = \omega\nu/U_0{}^2$ and spanwise wavenumber $K_2 = \nu k_2/U_0$ (the frequency F was already used in Chap. 2—see Figs. 2.26 and 2.27) and to the case where initially $|A_1| \gg |A_2| \gg |A_3|$ (where $|A_1|$ is the plane-wave amplitude) and the

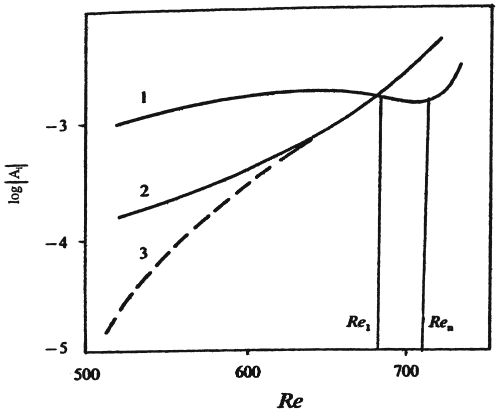

Fig. 5.3 Calculated dependence of the wave amplitudes $|A_i(x)|$, $i = 1, 2, 3$, on $\mathrm{Re} = (U_0x/\nu)^{1/2} \propto x^{1/2}$ for the wave triad consisting of a plane wave 1 of frequency ω and wave vector $\mathbf{k}_1 = (k, 0)$ and oblique waves 2 and 3 of frequency $\omega/2$ and wave vectors $\mathbf{k}_{2,3} = (k_1, \pm k_2)$ in the case when initial amplitudes $|A_{i,0}|$ and phases $\phi_{1,0}$ (at Re = 525) satisfy the conditions: $|A_{1,0}| \gg |A_{2,0}| \gg |A_{3,0}|$, $\phi_{1,0} = \phi_{2,0} + \phi_{3,0}$. It was assumed here that $F = \omega\nu/U_0^2 = 115 \times 10^{-6}$ and $K_2 = \nu k_2/U_0 = 0.18 \times 10^{-3}$; the values of k and k_1 were then determined from the Orr-Sommerfeld Eq. (2.44) and (2.41) which showed that $k_1 \approx k/2$. (After Zel'man and Maslennikova 1993a)

initial phases of the waves are matched. Logarithmic scaling of the amplitudes allows us to see clearly the region of exponential growth of oblique-wave amplitudes and the explosive growth of plane-wave amplitude at $\mathrm{Re} > \mathrm{Re}_n$ (for simplicity the usual notation Re will be used to denote the particular Reynolds number Re^+). Zel'man and Maslennikova (1993a) also presented figures showing examples of the plane-wave and oblique-wave amplitude-growth curves for (a) a fixed value of the initial oblique-wave amplitudes $|A_2|(\mathrm{Re}_0) = |A_3|(\mathrm{Re}_0)$ and three different initial plane-wave amplitudes $|A_1|$ (Re_0), (b) fixed values of both $|A_2|$ $(\mathrm{Re}_0) = |A_3|$ (Re_0) and $|A_1|(\mathrm{Re}_0)$, but with three different values of initial phase mismatch, and (c) a fixed value of $|A_1|$ (Re_0), and three different values of $|A_2|(\mathrm{Re}_0) = |A_3|(\mathrm{Re}_0)$ (it was assumed here that $\mathrm{Re}_0 = 500$, $|A_1|(\mathrm{Re}_0) > |A_2|(\mathrm{Re}_0) = |A_3|(\mathrm{Re}_0)$, and that in cases (a) and (c) there is no phase mismatch). The figure corresponding to case (a) illustrated the existence of a threshold value of $|A_1|$ below which the plane wave cannot excite the rapid growth of three-dimensional oblique waves which in turn produces the later explosive growth of the plane wave itself.

Zel'man and Maslennikova (1990, 1993a) stated that a more general and accurate version of the three-wave-resonance theory described in their papers showed that the three-wave resonance could be considered as the universal dominant mechanism of the so-called subharmonic (S-type or, alternatively, N-type—the latter name will be used consistently in Sects. 5.3 and 5.4) instability development in boundary layers (for more detailed discussion of this type of instability development see Sect. 5.3). However, this statement also was not universally accepted. Moreover, it did not imply that the other possible mechanisms are worthless; the authors only insisted on the possibility of interpreting the subharmonic disturbance development in the framework of the appropriately-modified three-wave-resonance theory in all cases. However it will be shown below that the three-wave-resonance approach often leads to results which are very close to those given, e.g., by the secondary-instability theory, which presupposes that the wave modes have amplitudes of two different orders of magnitude (see Fig. 5.16a). Note also that in some cases the secondary-instability computation allows the derivation of the required results to be simplified considerably. Moreover, the secondary-instability mechanism seems to be the most appropriate one in the widely-studied cases where a primary plane wave of finite amplitude is produced by a vibrating ribbon and later excites some secondary waves which are initially very weak. In addition, this mechanism is important in itself since it has many applications to problems unrelated to three-wave resonances. On the other hand, Mankbadi (1990, 1991, 1993a, b); Mankbadi et al. (1993), and Wu (1993, 1995) in their approximate evaluation of the resonant growth rates of two symmetric oblique waves (with frequencies and streamwise wavenumbers which are close, but not necessary equal, to half of those corresponding to the 2D wave of the triad) used quite another method (based on the idea that the dominant part of the nonlinear wave interactions is concentrated in the neighborhood of the critical layer; see, e.g., Mankbadi (1990)). Apparently this new method could in some cases replace both the resonant-triad and the secondary-instability methods, but its range of applicability is not clear up to now (cf. Healey (1995); for more details see the end of the present subsection, printed in small type). Note also that Jennings et al. (1995) considered the most general resonant triads consisting of three oblique waves (recall that just such triads were earlier discussed by Raetz (1959) and Stuart (1962a, b)) and showed that rapid growth of oblique-wave amplitudes is possible in this case also. The paper by Jennings et al. supplemented Zel'man and Maslennikova's results, showing that the three-wave resonance mechanism of generation and development of three-dimensional structures in boundary layers has a much wider domain of applicability than was assumed in the 1970s; however, this does not exclude the possibility that other mechanisms may also play important parts in some cases of boundary layer transition to turbulence and are therefore worth studying.

Let us also stress that the available experimental data relating to evolution of Blasius laminar boundary layers disturbed by a two-dimensional 'primary wave' (some of these data will be discussed in Sect. 5.3 below) definitely show that very different three-dimensional structures may appear in the course of this evolution. Therefore, it seems natural to suppose that there exist many different mechanisms of generation of such structures. Having this in mind, and also recalling remarks above relating to the *secondary-instability mechanism* of generation of flow three-dimensionality, we will now pass to discussion of this mechanism.

The secondary-instability approach to development of flow instabilities is based on a simple two-stage model. The first stage consists of the growth of some relatively simple small disturbance in accordance with the linear hydrodynamic-stability theory considered in Chap. 2. When this 'primary disturbance' becomes strong enough, it becomes unstable with respect to some disturbances of a quite different form, and then the second stage of disturbance development begins. Recall that in Sect. 4.22, the secondary instability of the two-dimensional equilibrium disturbances of a plane Poiseuille flow was briefly discussed on p. 72, where also a number of references touching upon this subject was presented, while on pp. 92–93 of the next subsection 4.23 even some tertiary and quaternary flow instabilities were mentioned, and a few references relating to such instabilities were indicated. Now we will consider a model where the superposition of some two-dimensional T-S wave of a finite amplitude A on the Blasius boundary-layer flow is considered as the 'primary flow', whose stability with respect to three-dimensional background ('environmental') waves of small amplitudes must be investigated. (For the sake of simplicity all amplitudes will now usually be assumed to be real and the possible effect of the 'phase mismatch' will as a rule be ignored). Thus here the 'primary flow' has the velocity $\mathbf{V}_1(x, z, t) = \mathbf{V}(z) + A\mathbf{v}_1(z)e^{i(kx-\omega t)}$ where $\mathrm{V}(z) = \{U(z), 0, 0\}$ (and $U(z)$ is the Blasius velocity profile at streamwise distance x, if the locally-plane-parallel model of the boundary-layer flow is used), while $\mathbf{v}_1(z)$ is the velocity profile of the selected T-S wave, normalized in a reasonable way, and A is its amplitude. (Normalization of the vector-function $\mathbf{v}_1(z) = \{u(z), v(z), w(z)\}$ is necessary to give meaning to the amplitude A. In particular, if $\mathrm{v}_1(z)$ is normalized by the condition that $\max_z u(z)/U_0 = 1$ where U_0 is the free-stream velocity, then A measures the maximal streamwise velocity of the T-S wave as a fraction of U_0). Note also that the representation of V_1 used here involves some other conventional approximations of the linear stability theory, excluding local parallelism; in particular, the velocity-profile distortion by disturbances is here neglected for both the steady Blasius boundary layer and the periodic T-S wave within it (for more details see Herbert' surveys (1988a, b). The primary flow with velocity $\mathbf{V}_1$ is disturbed by a 'secondary disturbance' of velocity $\mathrm{v}_3(x, y, z, t)$, where $|\mathbf{v}_3| \ll |\mathbf{V}_1|$. The last condition makes it possible to apply linear stability theory, i.e. to base the stability analysis on the N-S equations for the velocity field $\mathbf{V}_1(x, z, t) + \mathbf{v}_3(x, y, z, t)$ linearized with respect to the velocity and pressure ($\mathbf{v}_3$, p_3) of the disturbance. Thus, in contrast to the theory of three-wave resonance, where the amplitudes of all three waves are assumed to be of the same order of smallness and the equations of motion are expanded into subsequent powers of all amplitudes, in the secondary-instability theory the amplitude A of the 2D wave is considered as a fixed finite parameter and only the amplitude of the supplementary 3D disturbance is assumed to be small.

The papers of the 1980s on secondary instability of steady shear flows cited in Chap. 4 contain much material directly relating to the present topic (in fact this instability of laminar boundary layers was briefly discussed even earlier, by Görtler and Witting (1958) and Maseev (1968a, b). In solving the secondary-instability problem it is convenient to use, instead of a stationary frame, a frame moving in the Ox direction with the phase velocity c of the T-S wave having velocity $A\mathbf{v}_1(z)e^{i(kx-\omega t)}$, i.e., to replace x by the variable $x' = x - ct$. In this frame the primary flow is independent of time and periodic in x, i.e., here $\mathbf{V}_1(x, z, t = \mathbf{V}_1(x', z)$, where $\mathbf{V}_1(x', z) = \mathbf{V}_1(x' + \lambda_x, z)$, $\lambda_x = 2\pi/k$. Therefore, the frame transformation reduces the secondary-stability problem to the study of the linear stability of a steady but streamwise-periodic, locally-plane-parallel flow. Numerical investigation of this linear stability problem for the plane-parallel model of a Blasius boundary-layer flow (and also for some other flows) was carried out, in particular, by Orszag and Patera (1983) who obtained some interesting new results which were later confirmed by other authors. However, a much more explicit study of the secondary instability of the primary flow considered here was accomplished by Herbert (1983b, 1984a, 1985, 1986, 1987, 1988a, b) (see also Herbert and Santos (1987); Herbert et al. (1987) and Crouch and Herbert (1993)); therefore we will mainly discuss the latter work.

Herbert used the fact that the linear stability analysis of steady periodic flow with respect to a small three-dimensional disturbance may be reduced to study of a Floquet system of linear differential equations with periodic coefficients. The main properties of such systems may be found, e.g., in Coddington and Levinson's textbook (1955); various applications of Floquet theory to hydrodynamic stability were considered, in particular, by Kelly (1967); Clever and Busse (1974); Davis (1976); Barkley and Henderson (1996), and Schulze (1999) (see also Craik (1995) and references therein). However, Floquet theory was primarily developed in relation to the study of nonlinear periodic oscillations, and therefore in fluid mechanics it was most often applied to investigations of stability of time-periodic primary flows. Since Herbert considered, instead of this, the case of spatially periodic primary flow, it is reasonable to present here some details of his method.

The normal-mode concept, which was widely applied in Chap. 2 to problems relating to the linear stability theory for steady non-periodic flows, may now be used in exactly the same form for description of the dependence of the disturbance on the variables y and t. Here it leads to the representation of the disturbance velocity $\mathbf{v}_3(x', y, z, t)$ in the form of a superposition of modes depending on parameters k_2 and Ω (and admitting separate study) of the form

$$\mathbf{v}_3(x', y, z, t) = e^{i(k_2 z - \Omega t)}\mathbf{v}_4(x', z). \tag{5.12}$$

As in Sect. 2.5, the spanwise wave number k_2 may be assumed real (by virtue of the spanwise homogeneity of the primary flow), while (again exactly as in Chap. 2) the parameter Ω is generally complex: $\Omega = \Omega_r + i\Omega_i$. (Note that here Ω_r characterizes the frequency shift of the 3D disturbance with respect to the frequency ω of the primary T-S wave; modes with $\Omega_r = 0$ travel with the primary flow of velocity $\mathbf{V}_1(x, z, t)$). As to the dependence of $\mathbf{v}_4(x', z)$ on the streamwise coordinate x', the Floquet theory implies that it may be represented in the form

$$\mathbf{v}_4(x', z) = e^{\gamma x'}\mathbf{v}_5(x', z), \tag{5.13}$$

where $\gamma = \gamma_r + i\gamma_i$ is a complex *characteristic exponent* of the problem and $\mathbf{v}_5(x', z)$ is a periodic function of x': $\mathbf{v}_5(x' + \lambda_\xi, z) = \mathbf{v}_5(x', z)$. The periodicity of $\mathbf{v}_5(x', z)$ allows to it to be expanded in a Fourier series and thus to obtain the following general form of the three-dimensional disturbance $\mathbf{v}_3(x', y, z, t)$:

$$\mathbf{v}_3(x', y, z, t) = e^{\gamma x' + i(k_2 y - \Omega t)} \sum_{ms} \hat{\mathbf{v}}_m(z) e^{imkx'}, \quad -\infty < m < \infty, \tag{5.14}$$

where wave numbers k and k_2 are real, and constants γ and Ω are complex.

The additional complex parameter γ leads to the appearance here of new possible forms of disturbance. Note first of all that the values γ and $\gamma + ink$ of this parameter, where n is an integer of either sign, lead to the same collections of functions (5.14), differing only in numbering of the Fourier coefficients. Therefore, it is possible to assume that $-k/2 < \gamma_i \le k/2$. Moreover, it is also reasonable to subdivide the set of all disturbances of the form (5.14) into three classes of more special disturbance modes:

a. *Fundamental modes,* $\gamma_\mathrm{i} = 0$. Here

$$\mathbf{v}_3(x', y, z, t) = e^{\gamma_r x' + i(k_2 y - \Omega t)} \sum_m \hat{\mathbf{v}}_m(z) e^{imkx'}, \; -\infty < m < \infty. \tag{5.14a}$$

b. *Subharmonic modes,* $\gamma_i = \mathrm{k}/2$. Here

$$\mathbf{v}_3(x', y, z, t) = e^{\gamma_r x' + i(k_2 y - \Omega t)} \sum_m \hat{\mathbf{v}}_m(z) e^{imk_1 x'},$$
$$k_1 = k/2, \; m = 2n + 1, \; -\infty < n \; < \infty, \tag{5.14b}$$

c. *Detuned modes*, $0 < |\gamma_i| < k/2$. Here, if $2\gamma_\mathrm{i}/k = \varepsilon$, then $0 < \varepsilon < 1$ and

$$\mathbf{v}_3(x', y, z, t) = e^{\gamma_r x' + i(k_2 y - \Omega t)} \sum_m \hat{\mathbf{v}}_m(z) e^{i(m+\varepsilon)k_1 x'},$$
$$k_1 = k/2, \; m = 2n, -\infty < n < \infty. \tag{5.14c}$$

The word ‘detuend’ simply implies a streamwise wave number somewhere between the fundamental and the subharmonic modes. The terms corresponding to $m = \pm 1$ are the dominant ones on the right side of Eq. (5.14a) describing the 3D fundamental modes. These terms show that the primary 2D mode having the streamwise wavenumber k may excite resonant 3D waves with the same streamwise wavenumber (in the temporal presentation of the theory it means that a 2D wave of frequency ω may excite 3D waves of the same frequency). This process is associated with the so-called *primary resonance* in a Floquet system. In Eq. (5.14b) the main terms are also those with $m = \pm 1$; they correspond to subharmonic 3D modes having streamwise wavenumber $k_1 = k/2$ (or, in temporal presentation to subharmonic modes of frequency $\omega/2$). The resonant excitation in a Floquet system of 3D waves with streamwise wavenumber (or frequency) equal to half of the corresponding characteristic of the primary 2D wave represents a phenomenon which is often called the *principal parametric resonance* (the adjective ‘parametric’ is used because in many real physical systems the primary oscillation of frequency ω represents oscillatory variations of some physical parameter affecting the system). Real detuned modes must include on the right-hand side of Eq. (5.14c) two complex-conjugate summands with opposite detuning parameters $\pm\varepsilon$. Herbert (in (1988a, b) and some other papers) called real detuned modes the *combination modes*; and said that they participate in the *combination resonances* (see Santos and Herbert (1986); Herbert and Santos (1987) and Herbert et al. (1987); cf. also the surveys by Nayfeh (1987a, b).

As will be shown later, all the above-mentioned types of secondary-instability resonances can participate in the development of fluid-flow instability. However here only the evident similarity of the principal parametric resonance to Craik's three-wave resonance will be emphasized. This similarity makes the principal parametric resonance especially interesting for the analysis of boundary-layer instabilities. Note in this respect that resonances of such type occur also in many other physical systems. Apparently the first description of such phenomenon in scientific literature is due to Faraday (1831), who discovered that when a vessel containing liquid is made to vibrate vertically, some vibrations of the free surface of the liquid have a frequency equal to only half of that of the vessel. This seemingly unusual *Faraday resonance* (or *Faraday waves*, *Faraday instability*) attracted much attention and was later studied by many authors both theoretically and experimentally (in particular, Rayleigh (1883a, b) participated in both kinds of studies). Nevertheless, a satisfactory theory of this resonance was developed only in the second half of the twentieth century and its study is not yet complete; see, e.g., the papers by Benjamin and Ursell (1954); Miles (1984, 1993); Guthart and Wu (1994); Friedel et al. (1995); Wright et al. (2000), and the survey by Miles and Henderson (1990) containing many supplementary references (cf. also the paper by Schulze (1999) indicating some conditions under which the principal parametric resonance cannot occur).

Let us now return to a description of Herbert's work. The imaginary part Ω_r of the parameter Ω determines the time growth of the 3D-disturbance amplitude, which is proportional to $\exp(\Omega_i t)$. However, this is correct only for amplitudes at fixed points of the frame of reference moving with velocity c, while amplitudes at fixed points x of the stationary frame will be proportional to $|\exp(-i\Omega t + \gamma\,(x - ct))| = \exp(\gamma_1\, x)\exp((\Omega_i - \gamma_r c)t)$. Models of purely temporal growth of disturbances (which were the main objects of investigation in all early theoretical studies and continue to be widely studied; see, e.g., Sect. 2.92) correspond to the assumption that $\gamma_r = 0$, while purely spatial growth in the laboratory frame corresponds to the condition $W_i = \gamma_r \mathrm{c}$.

Substitution of the above expressions for the disturbance modes (5.14a, b and c) (assumed to be real) into linearized N-S equations for the velocity disturbances leads to infinite systems of coupled linear differential equations for the functions $\hat{\mathbf{v}}_m(z)$. A numerical solution may be obtained if the Fourier series are truncated, making the infinite systems finite. Numerical studies by Herbert (1984a, b; 1985, 1986, 1988b); Herbert and Santos (1987); Herbert et al. (1987), and Crouch and Herbert (1993) (see also Herbert's survey (1988a)) showed that reasonable accuracy may be achieved even when truncation is very severe—in the case of subharmonic modes it is often enough to preserve only the terms with $m = -1$ and $m = 1$, while for fundamental modes the truncation of all terms with $|m| > 1$ (i.e., inclusion in the analysis only the terms with $m = -1$, 0 and 1) gives satisfactory accuracy in many cases. (This conclusion, which confirms the above statements about the dominant terms of

Eqs. (5.14a, b), also agrees with the results of subsequent numerical investigations of secondary instability of a Blasius boundary-layer by Wang and Zhao (1992) and Ustinov (1994)). The resulting systems depend on the boundary-layer and primary-wave velocity profiles $U(z)$ and $\mathrm{v}_1(z)$ and include parameters A, k, ω, k_2, γ_r, γ_i, Ω_r, Ω_i, characterizing the primary T-S wave, and 3D disturbances interacting with this wave. (Strictly speaking, ω takes a complex value if the primary T-S wave is not neutral, while if spatial, and not temporal, development of disturbances is considered, then ω is real but k is complex. However, we will follow Herbert and assume that the T-S wave varies slowly in comparison with the 3D disturbance; hence the T-S amplitude A may be assumed to be locally constant and both parameters ω and k will be real. More general models where k or ω may be complex were considered by Wang and Zhao (1992) but will not be discussed here). Parameters A, k and ω characterize the primary T-S wave and may be assumed to be known; as to the other parameters mentioned, the majority can take any values, which may be chosen on the basis of available data or physical arguments. This, however, is not true for all parameters, since, as in the case of the Orr–Sommerfeld equation, the systems of equations for functions $\hat{\mathbf{v}}_m(z)$ with appropriate boundary conditions define eigenvalue problems—their solutions exist only for special values of some of the parameters ('eigenvalues', which depend on the chosen values of the other parameters). And, exactly as in the case of the O-S equation (where the eigenvalues are the real and imaginary parts of ω or, if a spatial formulation of the stability problem is used, of k), only two of the above real parameters must be treated here as eigenvalues determined by the requirement of solubility of the system. Note also that in the case of the spanwise wavenumber k_2 it is natural to suppose that the value to which the highest growth rate of the wave amplitude corresponds should be just the wavenumber that is most likely to appear in experiments. This assumption (which is entirely similar to that used in the linear stability theory for determination of the value of k in the O-S Eq. (2.44)) provides a criterion for determination of the preferred spanwise periodicity. Some of the results obtained in this way will be considered, together with the appropriate experimental and numerically-simulated data, in the next part of this subsection. It will be also shown there that numerical solutions of the amplitude equations for resonant waves in a boundary layer, and for the disturbance modes (5.14a, b and c) of its secondary instability, allows many observable characteristics of the boundary-layer instability to be determined, yielding information about the most appropriate instability models and values of the corresponding parameters. As will be seen, in spite of the essential differences between resonance and secondary-instability mechanisms, the quantitative consequences of the two theories sometimes (though not always) lead to results which are very close to each other. Note in this respect that both theories were independently proposed at a time when almost no reliable data existed for comparison with theoretical predictions. In the case of the secondary-instability theory the early (and nowadays rarely cited) papers by Görtler and Witting (1958) and Maseev (1968a, b) are worth mentioning in this respect. It is curious to note that both the German authors and the Russian one (in the first of his two papers) independently chose practically the same title, which was later used also by Herbert (1988a). Herbert noted in this paper that Maseev's papers (the first being published in

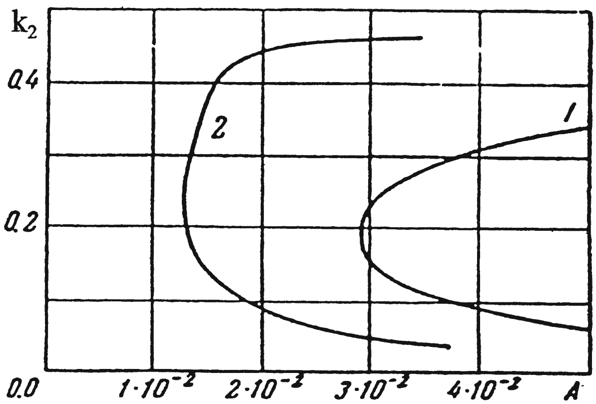

Fig. 5.4 Threshold amplitude A of the plane T-S wave with streamwise wavenumber k_1 in a Blasius boundary layer for the onset of three-dimensionality with spanwise wavenumber k_2. (After Maseev 1968b) curve (*1*): $\mathrm{Re}^* = 1203$, $k_1 = 0.43$; curve (*2*): $\mathrm{Re}^* = 519$, $k_1 = 0.27$. All the quantities are non-dimensionalized by scales δ^* and U_0

Russian in a small-circulation collection of papers written by lecturers from a Moscow engineering college, while the second was translated into English but is very short and not entirely clear) apparently contained some new, nontrivial correct ideas about the role of the secondary instability in boundary-layer transition to turbulence (in fact, these ideas had something in common with the contents of the earlier paper by Görtler and Witting). In particular, Maseev gave, without explicit proof, some reasonable estimates of the threshold amplitudes of the 2D wave needed for the generation of three-dimensionality with a given spanwise wavenumber k_2 (see Fig. 5.4) (the estimates are compatible with the data of Klebanoff et al. (1962)). A similar schematic graph was given by Görtler and Witting who did not indicate scales but stated that their graph agrees with the experimental data of Schubauer (1958).

Let us now say a few words about the papers of Mankbadi (1990, 1991, 1993a) and some related work. In the 1990 and 1991 papers Mankbadi considered fully-resonant triads, where all waves have small amplitudes and the same phase velocity c. For these conditions he analyzed the role of the critical layer, where $U(z) = c$, in triad development. He found that the main contribution to the growth rates of wave amplitudes is due to wave interactions in the neighborhood of the critical layer, and at large values of Re this neighborhood is the only flow region where nonlinearity strongly affects the wave dynamics. Mankbadi (1993a) considered a more general triad, in which frequencies and streamwise wave numbers of oblique waves were close, but not necessarily equal, to half those of the 2D wave. According to Mankbadi, in this case too the oblique- and plane-wave growth rates G_o and G_p at large values of Re are determined with high accuracy by the contributions of the neighborhood of the critical layers (which in this case are clearly close to each other for all three waves). Based on this, Mankbadi carried out an asymptotic evaluation of the growth rates, and found that if the initial amplitude of the plane wave is much greater than the oblique-wave amplitudes and Re is large enough, then $G_o \gg G_p$ and oblique waves with quite different spanwise wavenumbers k_2 can grow rapidly, extracting energy very efficiently from the undisturbed flow (i.e. a three-wave resonance of some sort takes place for a wide range of k_2-values, and the plane wave then plays the role of a catalyst stimulating growth of oblique waves). The positive growth rates G_o depend on the plane-wave amplitude and the values of Re* and k_2; hence $k_{2,\,pr} = k_{2,\,pr}$ (A, Re*) where $k_{2,\,pr}$ is the preferred value of k_2 corresponding to the maximal value of G_o. Dependencies of G_o on A, Re* and k_2, and of $k_{2,\,pr}$ on A and Re*, computed by Mankbadi were in good agreement with the available experimental and numerical data (see Figs. 5.15a, b in Sect. 5.4). This agreement clearly increases confidence in Mankbadi's results but since the problems solved by him are quite involved, a supplementary check of all his arguments remains desirable.

A more complicated asymptotic theory of the spatial development of resonant triads in a Blasius boundary layer at large values of Re was developed by Mankbadi et al. (1993). Here the wave triads considered included one plane wave and a pair of symmetric oblique waves, having frequencies ω and $\omega/2$ and streamwise wave numbers k and $k_1 \approx k/2$, respectively. (Such triads were also analyzed, by a quite different method, by Zel'man and Maslennikova (1993a); the interest of theoreticians in them was stimulated by papers by Corke and Mangano (1988, 1989) describing experimental investigations of development of such wave triads in a boundary layer). Since the value of k_1 could vary, the spanwise wavenumbers $\pm k_2$ and the inclination angles $\theta_{1,2} = \pm \tan^{-1}(k_2/k_1)$ could also take different values. Mankbadi et al. estimated the wave growth rates of G_o and G_p by a somewhat refined method of critical-layer analysis which took into account the nonlinear critical-layer effects which lead to the appearance, in the amplitude equations, of nonlinear integral terms which account for the influence of the upstream wave history. Their main attention was paid to the case where the plane wave is linearly unstable while the oblique waves are linearly stable (i.e., decaying according to the linear stability theory), and where amplitudes of all three waves are small but the initial amplitudes of the oblique waves are much smaller than that of the plane wave. It was shown that at first the plane wave causes fast growth of oblique waves, while the plane wave itself continues growing for some time at a rate close to that given by the linear stability theory (this growth rate is much smaller than the simultaneous growth rates of the oblique waves). Later, when amplitudes of the oblique waves become considerably greater than the plane-wave amplitude, nonlinearity begins to affect the evolution of the plane wave as well. At this stage the self-interaction of oblique waves becomes important and considerably changes the law of their growth, leading to oscillations of their growth rates, at first around their earlier high growth rate and then around the zero growth rate corresponding to the final saturation stage. These conclusions agree with some experimental results by Corke and Mangano (1988, 1989) (for more information about their work see Sect. 5.3) but in general there are not enough data to confirm the results; moreover, it was noted by Healey (1995) that the assumptions used by Mankbadi et al. may be valid only at unrealistically large Reynolds numbers. Some results supplementing those discussed here were presented, in particular, by Goldstein (1994, 1995) and Wu (1995) but we have no space to discuss them here.

5.3 K and N Regimes of Instability Development in Boundary Layers; Experimental Studies of the N Regime

Experimental data which could be compared with the weakly-nonlinear theories considered above appeared only relatively recently. Therefore, it is no wonder that for some time these theories did not attract much attention. Recall that at the beginning of Sect. 5.2 the classical papers of Schubauer and Klebanoff (1956); Klebanoff and Tidstrom (1959) and Klebanoff et al. (1962) were cited as the primary source of experimental information about the nonlinear development of 3D wave disturbances in boundary-layer flows. In particular, the last-named has for many years been referred to very frequently by experts in the flow-stability theory. However, it has already been mentioned that the experimental data contained in these papers agreed only qualitatively with the early theoretical models by Benney and Lin (1960) and Benney (1961, 1964) of the two-mode disturbance development. The point is that in these theoretical papers it was assumed that two-dimensional and three-dimensional modes have the same frequency, while according to the results of Klebanoff and his co-authors this is not the case. However, these results disagree even more strongly with Craik's model of a resonant-triad interaction, where the frequency ω_1 of the two three-dimensional waves is taken equal to one-half of the frequency ω of the two-dimensional wave. Klebanoff and his co-workers studied the development of disturbances produced by a vibrating ribbon in a flat-plate boundary layer and found three-dimensional flow oscillations, but their frequency ω_1 differed only slightly from the fundamental frequency ω_0 of ribbon oscillations and of the 2D wave produced by it. These 3D oscillations appeared at relatively small values of x (i.e., soon after the origin of the 2D wave) and later, at larger values of x, these regular oscillations were transformed into irregular bursts of high-frequency fluctuations (so-called 'spikes'; see Fig. 5.22) which preceded the formation of turbulent spots and final transition to turbulence (cf. the short description of boundary-layer instability in Sect. 2.1; for more detailed characterization of the boundary-layer instability considered here see Sect. 5.5). However, no subharmonic waves with half the fundamental frequency were found in these experiments.

It is now clear that these experimental results did not prove the incorrectness of Craik's model but only showed that the nonlinear development of boundary-layer disturbances observed by Klebanoff et al. was not due to Craik's resonance mechanism. Note in this respect that Morkovin and Reshotko (1990) reasonably remarked that even in the cases of similar flow geometries and initial velocity fields there is no universality in the instability and transition process; because of the wide variety of external-flow disturbances feeding this process, and the large number of permissible nonlinear developments of them there is a very wide range of possible behavior. This remark (which was also made in a less definite form by Herbert and Morkovin (1980) and was often repeated by later authors; Shaikh and Gaster's paper (1994) is just a typical example) describes excellently the conclusion following from numerous experimental results collected during the whole twentieth century. So it may also quite convincingly explain the reason for the deviation of Klebanoff's experimental results from the predictions of Craik's theory.

For a number of years after Craik's theory of 1971, no subharmonic disturbances of frequency $\omega_0/2$ were observed in boundary layers where a two-dimensional 'fundamental wave' of frequency ω_0 was generated by some means (although *two-dimensional* subharmonics of the 'fundamental frequency' ω_0 were repeatedly found in mixing layers with antisymmetric velocity profiles, e.g., by Sato (1959); Browand (1966) and Miksad (1972), and also in plane and circular jets—see, e.g., Wehrmann and Wille (1958)). Therefore, it was usually assumed during these years that Craik's theory was inapplicable to real boundary-layer instabilities. Apparently the first work in which it was shown that subharmonic waves of frequency $\omega_1 = \omega_0/2$ do indeed sometimes appear in a constant-pressure boundary layer perturbed by a ribbon vibrating with the frequency ω_0 (corresponding, at a given value of Re, to a 2D wave unstable according to the linear stability theory) was that of Kachanov et al. (1977) in Novosibirsk, Russia. These authors made hot-wire anemometer measurements of the streamwise velocity fluctuations $u(x, y, z, t)$ (the deviations of instantaneous streamwise velocities from the undisturbed velocity $U(z)$) in a ribbon-excited boundary layer. Then they determined normalized amplitudes $A = u'/U_0$ of these fluctuations (where, as above, u' is the appropriately defined[1] real amplitude of u-fluctuations and U_0 is the free-stream velocity; note that in the experiments of Kachanov et al. the initial values of A were much smaller than in the experiments of Klebanoff et al.). Kachanov et al. measured the frequency spectra $P_u(f)$ (where $f = \omega/2\pi$ is the frequency measured in Hz) of the streamwise-velocity fluctuations $u(t)$ (describing the spectral composition of these fluctuations) at various points (x, y, z). They found that, together with the main spectral peak at the frequency f_0 of the ribbon oscillations and higher harmonics of frequencies $2f_0$ and $3f_0$ (which are typical for any nonlinear wave development and were seen almost everywhere in the flow), velocity fluctuations with frequencies much below f_0 were also observed at large enough values of x. Moreover, at such values of x subharmonic fluctuations of frequency $f_1 = f_0/2$ were also detected at all points of observation (as a typical example see Fig. 5.5a, where peaks at frequencies $2f_0$ and $3f_0/2$ are produced by nonlinear interactions of the primary wave of frequency f_0 with itself and with the subharmonic of frequency $f_0/2$, and where peaks at $3f_0$ and $5f_0/2$ are due to interactions of the same primary wave and its subharmonic with the second harmonic). Another example of the same type is shown in Fig. 5.5b, taken from the paper by Kachanov and Levchenko (1984); here a relatively wide low-frequency range of amplitude fluctuations with a peak at $f = f_0/2$ is seen at both values of x and f_0. The appearance of subharmonic fluctuations in the experiments by Kachanov et al. (1977) coincided with the onset of three-dimensionality, producing appreciable spanwise variations of flow characteristics. These results strongly suggested to the authors that Craik's three-wave resonance took place at the corresponding values of x.

[1] It is often convenient to define the fluctuation amplitude as the root-mean-square value (i.e., as the square root of the temporal mean value of squared fluctuations). This definition is widely used, in particular, in studies of turbulent flows.

a

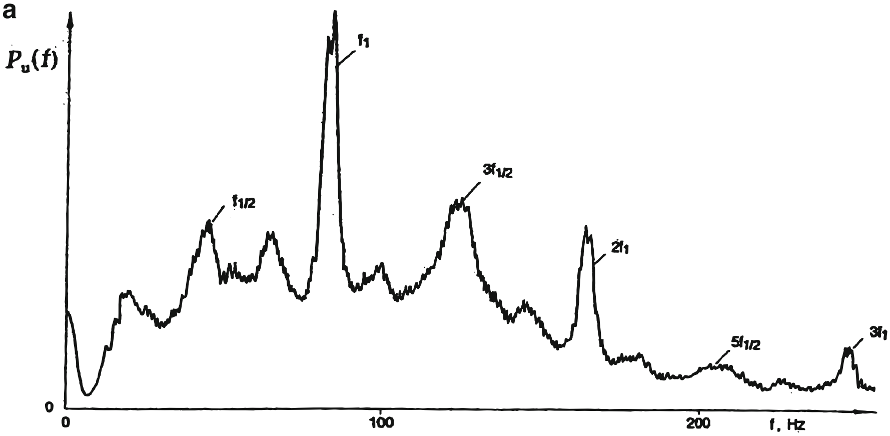

Fig. 5.5 Examples of the amplitude spectra $P_u(f)$ of streamwise-velocity fluctuations $u(t)$ in a laboratory flat-plate boundary layer disturbed by a ribbon vibrating with frequency f_0. (**a**) Typical form of spectrum $P_u(f)$ measured by Kachanov et al. (1977). (After Kachanov 1994a) peaks denoted as f_1, $f_{1/2}$, $3f_{1/2}$, $2f_1$, $5f_{1/2}$ and $3f_1$ correspond to frequencies f_0, $f_0/2$, $3f_0/2$, $2f_0$, $5f_0/2$ and $3f_0$. (**b**) Spectra $P_u(f)$ measured inside a boundary layer at two values of frequency f_0 and coordinate x (measured from plate leading edge) but fixed values of y and z: *(1)* $f_0 =$ 96.4 Hz ($F_0 = 2\pi f_0 \nu / U_0^2 = 109 \times 10^{-6}$), $x = 600$ mm ($\text{Re} = (U_0 x/\nu)^{1/2} = 608$); *(2)* $f_0 = 111.4$ Hz ($F_0 = 124 \times 10^{-6}$), $x = 640$ mm (Re = 633). (After Kachanov and Levchenko 1984)

The results found by Kachanov et al. in 1977 were later confirmed, supplemented by many details, and expounded in research papers and surveys both by members of the Novosibirsk group (see, e.g., Kachanov et al. (1978, 1980, 1982); Kachanov and Levchenko (1982, 1984); Kachanov (1987, 1994a, b); Boiko et al. (1999)), and by other scientists, partially in collaboration with those from this group (see, e.g., Thomas and Saric (1981); Saric et al. (1981); Saric and Thomas (1984); Santos and Herbert (1986); Thomas (1987); Yan et al. (1988); Corke and Mangano (1988, 1989); Corke (1989, 1990, 1995); Saric et al. (1984); Kozlov et al. (1984), and Bake et al. (1996, 2000)). It was also noted by Saric and Thomas (1984) and Herbert (1988a) that some related results (which will be described later) had been observed in early flow-visualization studies by Knapp and Roache (1968) which did not attract much attention at the time.

Comparison of the results of the above-mentioned papers with those found by Klebanoff and his co-workers clearly shows that there exist at least two different routes of boundary-layer transition to turbulence. The first of these transition regimes, whose study was initiated by Klebanoff's work, usually corresponds to relatively large initial amplitude of a two-dimensional wave disturbance (with values of u'/U_0 of the order of 1 % or more, where u' is the amplitude of streamwise-velocity fluctuations at the distance from the wall where this amplitude is a maximum). Herbert and Morkovin (1980) proposed to call this regime the *K-Regime* (for Klebanoff); their proposition was widely accepted and will be used in this book too. As was indicated

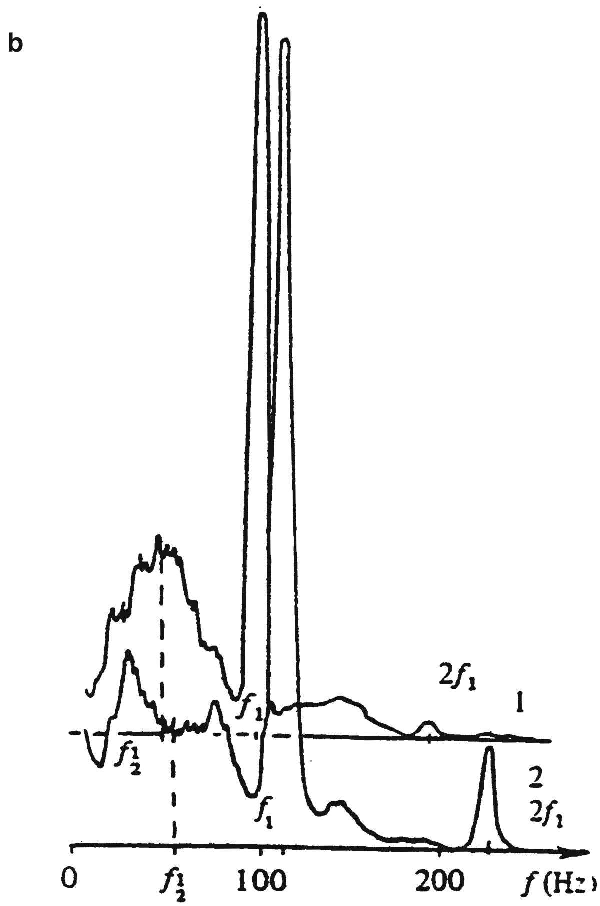

Fig. 5.5 (Continued)

earlier, the K-regime includes the formation of three-dimensional structures leading to appearance of bursts of high-frequency fluctuations which are later transformed into separate turbulent spots; these spots multiply and grow with time, then start merging with each other, and finally occupy the whole boundary layer. Only the first stage of this regime was studied by Klebanoff et al. (1962) and only this regime was briefly considered in Sect. 2.1. The second regime, discovered in experiments of the Novosibirsk group, is often called the *N-Regime* (see, e.g., Kachanov's survey papers (1987, 1994a, b), and below we will normally use this name); other names found in the literature are *Subharmonic Regime* and *S-Regime* (the latter two names stress the importance here of the subharmonic resonance). The N-regime of disturbance development does not lead to the appearance of 'turbulent spots' (localized regions of very strong fluctuations), and usually occurs only under some special initial conditions (in particular, at initial values of u'/U_0 appreciably smaller than 1 %) and is rarely realized in natural and engineering flows (therefore, it was not mentioned in Sect. 2.1). In particular, the emergence of the N-regime requires that the boundary layer contains a two-dimensional T-S wave with rather small initial amplitude u'/U_0 (but not smaller than about 0.3 %; this last condition was first mentioned in qualitative form by Görtler and Witting (1958), was independently presented, together with the quantitative (but numerically incorrect) Fig. 5.4 by Maseev (1968a, b) and later was proved by different theoretical methods by Orszag and Patera (1983); Herbert (1984a, 1985, 1988a) and Zel'man and Maslennikova (1984, 1993a)). According to many authors, the N-regime may begin either with a nonlinear wave resonance of Craik's type or with a secondary-instability phenomenon. Saric and Thomas (1984), who found that the spanwise periodicity and the character of the observed nonlinear wave development can depend on the initial value of u'/U_0, even recommended distinguishing these two origins of boundary-layer three-dimensionality by introducing the attributes 'C-type' (for Craik) and 'H-type' (for Herbert) (the data motivating their proposal will be considered later). However, later it was shown that the nonlinear resonance may have many different forms, and often it cannot easily be distinguished from the secondary-instability development of flow disturbances.

Before detailed consideration (in this and the next section) of the results relating to the N-regime of disturbance development in a boundary layer and then (in Sect. 5.5) of the main features of the K-regime, it is worth making some general remarks about this subject. Note that both the regimes were discovered in experiments where a ribbon vibrating with a constant angular frequency ω was used to generate the primary disturbance. Hence, we consider here only the so-called 'normal transition scenarios', which begin with the emergence in the flow of a linearly-unstable (or linearly-stable but transiently growing) Tollmien–Schlichting wave. However, it was noted in Chap. 2 (Sect. 2.9.2) that, both in laboratory experiments and in real life, external-stream disturbances can be large enough for 'by-pass transition' to occur, with no observable small-amplitude T-S waves at the beginning of the process as in 'normal transition'.

In fact, the N-regime of disturbance development can really occur only in cases of rather low levels of external disturbances (of background or environmental origin). In such cases it is often even unimportant whether only one periodic T-S wave or a more complicated disturbance appears first. It will be explained below that even in the case of a single primary plane wave the N-regime development quickly leads to a disturbance spectrum of rather complicated form. The K-regime corresponds to the cases of boundary layers with a higher level of external disturbances; here also the primary disturbance may not necessarily have the form of a single T-S wave. Usually the K-regime leads to the scenario of transition to turbulence through the stage of 'turbulent spots' (see Sect. 2.1); therefore, the final stages of the K-regime may also be realized in 'by-pass transition'. Note also that when the N-regime of boundary-layer evolution develops without further disturbance for a long enough time, it may gradually acquire some features of the K-regime; see in this respect the discussion of papers by Bake et al. (1996, 2000) at the end of this section and in Sect. 5.5.

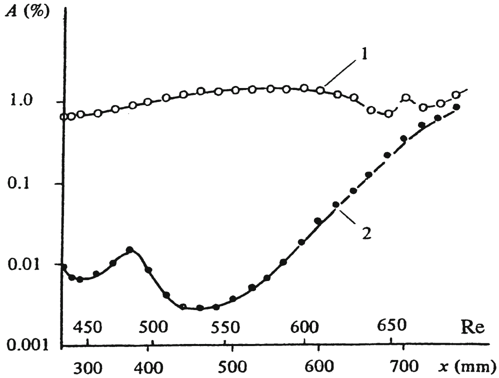

Fig. 5.6 Dependence of the dimensionless amplitudes $A = u'/U_0$ of the primary plane wave (*1*) and subharmonic oblique waves (*2*) on the coordinate x (and $\mathrm{Re} = (U_0 x/\nu)^{1/2}$) at $y = -2.5$ mm, $z/\delta = 0.26$ (where δ is the boundary-layer thickness), according to measurements by Kachanov and Levchenko (1984)

Let us now consider at greater length the data relating to the first stage of the N-regime. Kachanov et al. (1977) in their experiments showed only that an appreciable subharmonic component of velocity fluctuations with frequency $f_1 = f_0/2$ appeared simultaneously with the onset of flow three-dimensionality. This observation gave reason to suggest that Craik's three-wave resonance may have been present but, of course, it could not be considered as a proof of such resonance. Therefore a much more detailed study of the instability phenomenon observed in 1977 was carried out by Kachanov and Levchenko (1982, 1984). Here frequency spectra of streamwise velocity fluctuations in the constant-pressure boundary layer (identical to that studied by Kachanov et al. (1977)) where measured at a number of distances x from the plate leading edge and heights z above the plate (one of the results obtained is shown in Fig. 5.5b). Then narrow-band frequency filters were used to isolate (*a*) the 'primary wave' of velocity fluctuations produced by ribbon vibrations of frequency f_0, and (*b*) the subharmonic waves of half that frequency. The phase ϕ and streamwise wavenumbers k_1 of the primary and subharmonic waves were measured, and it was shown that the phase synchronism required for resonance (usually reducing to the condition $\phi_{1,0} = \phi_{2,0} + \phi_{3,0}$, where $\phi_{i,0}$ is the initial phase of the ith wave, and $i = 1$ for the primary wave) actually occurred, and that the resonance condition $k_1 = k/2$ of Eq. (5.7) was satisfied with high accuracy. It was also found that the amplitude of the subharmonic wave of frequency $f_0/2$ grew rapidly with x (from the viewpoint of a fluid element, with time t measured from the moment of wave excitation by the vibrating ribbon) over a considerable range of x, while the amplitude of the primary wave changed only a little in this range (see Fig. 5.6, and also Fig. 5.3 which shows subsequent calculated results relating to more general initial conditions). All this supports very convincingly the suggestion by Kachanov et al. (1977) that the three-wave resonance predicted by Craik was really observed in their experiments.

Kachanov and Levchenko also measured the spanwise distributions of the amplitude and phase for both the primary wave and the subharmonics; one typical result of such measurements is shown in Fig. 5.7. These measurements confirmed that

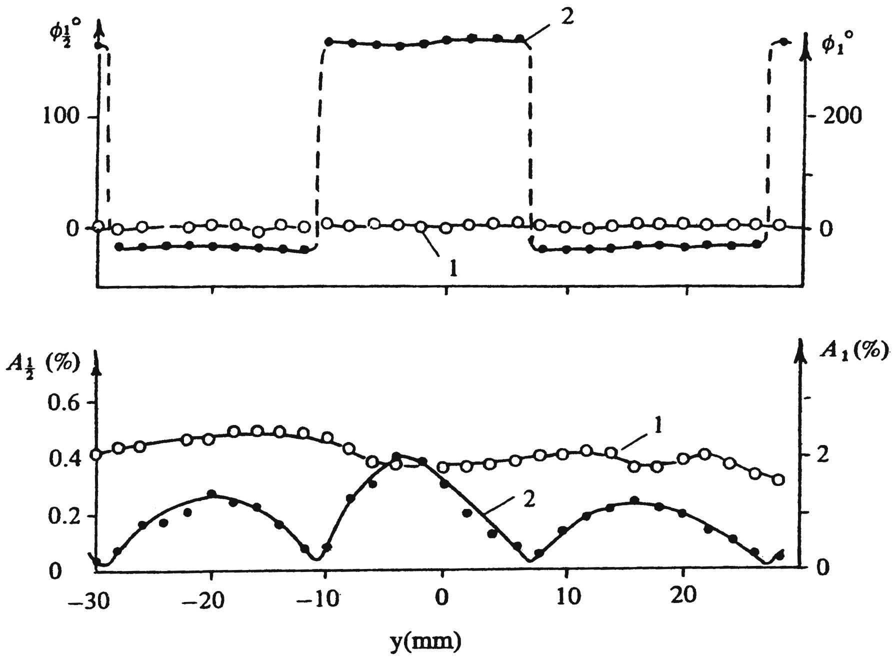

Fig. 5.7 Measured dependence of phases ϕ_1 and $\phi_{1/2}$, and amplitudes A_1 and $A_{1/2}$, of the primary plane wave (*1*) and subharmonic waves (*2*) on the spanwise coordinate *y*. (After Kachanov and Levchenko 1984)

the primary wave is two-dimensional, while the subharmonic of frequency $f_0/2$ is three-dimensional and the dependence of its amplitude on y is close to that of the function $B\cos(k_2y)$ (corresponding to a pair of symmetric oblique waves with spanwise wavenumbers $\pm k_2$), where B depends on x and z (and also on the frequency f_0 of the primary wave). Experimental data of the type presented in Fig. 5.7 were used by Kachanov and Levchenko to determine the spanwise wavenumber k_2 and the angles $\theta_{1,2} = \pm\tan^{-1}(k_2/k_1)$ between the propagation directions of the plane 2D wave and of the two subharmonic oblique waves. According to the results obtained, $|\theta_{1,2}| \approx 63 - 64°$ in the main part of the region where strong three-wave resonance was observed. These values differ from the theoretical estimate $|\theta_{1,2}| \approx 50°$ obtained by Volodin and Zel'man in 1978 (when there were no experimental data to compare with predictions) for a version of Craik's three-wave-resonance model of disturbance development. However, Kachanov and Levchenko did not pay too much attention to this discrepancy, which did not shake their confidence in the discovery of Craik's resonant structure. Subsequent theoretical studies, which will be considered later, showed that Kachanov and Levchenko were right, since Volodin and Zel'man's estimate of the angle $|\theta_{1,2}|$ was based on an oversimplification of the problem.

Kachanov and Levchenko's data also included the measured values of vertical (z-wise, normal-to-wall) profiles of the real amplitude $|A|$ and the phase ϕ (where

$|A|e^{i\phi}=A$ is the complex amplitude[2]) for both the primary 2D wave and its subharmonics of half the primary frequency. The profile measurements were made in the flow region where strong resonant interactions take place among waves of these two types. Results obtained for amplitude $A(z)$ and phase $\phi(z)$ of the primary-wave streamwise velocity fluctuations $u(x, z, t) = A(z)\exp(i\{kx - \omega t + \phi(z)\})$ were found to be very close to the corresponding conclusions of linear stability theory relating to the two-dimensional T-S wave considered. As to the measured vertical profile of subharmonic-wave real amplitude A, its accuracy was also confirmed by Corke and Mangano's (1989) measurements, which will be considered slightly later. It will be also noted later in this section that, according to the experimental results of Corke and Mangano (1989) and Corke (1995), the vertical profile of subharmonic-wave amplitude found by Kachanov and Levchenko is close to the profiles corresponding to subharmonic waves entering more general resonant triads, which satisfy the resonant conditions (5.7) not exactly but only approximately. Moreover, in his survey Kachanov (1994a) compared vertical profiles of the subharmonic-wave amplitude and phase presented in Kachanov and Levchenko (1982, 1984) with some theoretical and numerically-simulated estimates of these profiles, and showed that their experimental data agree excellently with these estimates (for more details see Figs. 5.16a, b and the text in Sect. 5.5 relating to these figures, including that in small type).

Continuing the consideration of experimental data relating to the N-regime of nonlinear disturbance development in boundary-layer flow, we note the visualization studies of boundary-layer instabilities carried out in the early 1980s by Saric and his co-authors (who in fact began with independent repetition of the early observations by Knapp and Roache (1968)). These studies showed that three-dimensional vortical structures, which appear in the Blasius boundary layer in the course of nonlinear development of an initially two-dimensional Tollmien–Schlichting wave, differ considerably in the cases of the K-regime and the N-regime of laminar-flow breakdown. In both cases nonlinear effects produce some regular process of distortion of the primary 2D wave into three-dimensional vortices reminiscent of the Greet letter Λ, with tips directed downstream (so-called 'Λ-vortices'). In the case of the K-regime, these vortices form an ordered vortical structure of peak-valley splitting in which the successive peaks are spatially in phase and follow regularly behind one another (see a typical flow-visualization picture in Fig. 5.8a). On the other hand, in the case of the N-regime the structure consists of spanwise rows of Λ-vortices, where successive rows are out of phase and the peaks of one row are aligned with the valleys in the next row (see Fig. 5.8b). Just such a 'staggered vortical structure' was first observed in visualizations of disturbed boundary-layer flow by Knapp and Roache (1968); this structure clearly corresponds to twice the streamwise wave length (i.e., half the wave number) of the ordered K-regime structures in Fig. 5.8a. Later, structures of both types were independently found and described by Thomas and Saric (1981) and Saric et al. (1981) who applied to boundary-layer flows the technique of air-flow visualization by smoke developed by Corke et al. (1977). More detailed analysis of the data

[2] Below, in cases where complex amplitude is not considered, the real amplitude $|A|$ will usually be denoted simple as A.

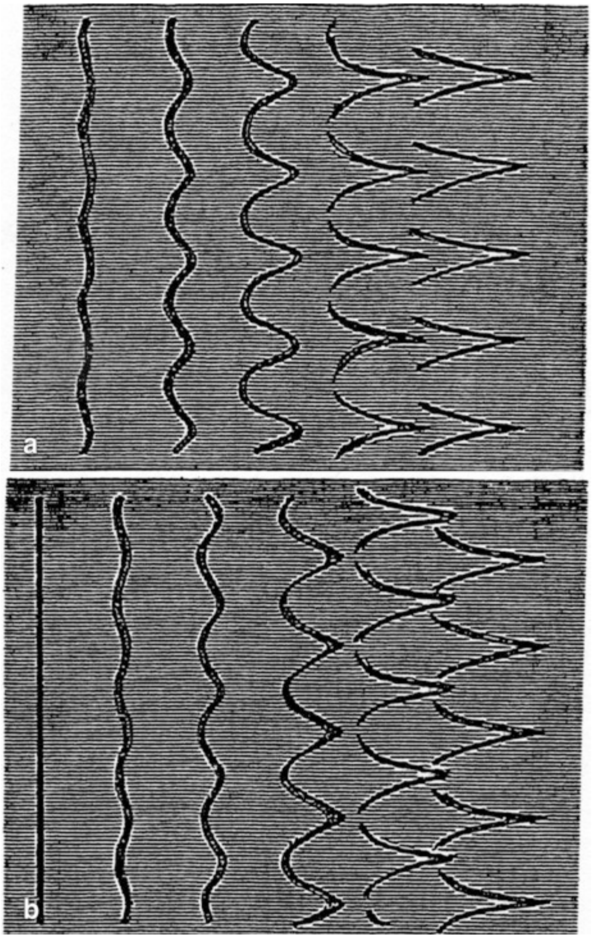

Fig. 5.8 **a** Regular system of Λ-vortices typical of the K-regime of disturbance development in a boundary layer. **b** Staggered system of Λ-vortices typical of the N-regime of disturbance development. The figures show the flow streaklines appearing when the disturbed flow is visualized by a spanwise smoke wire. (After Herbert et al. 1987)

of Saric and his co-workers was presented by Saric and Thomas (1984); Saric et al. (1984); Kozlov et al. (1984); Craik (1985); Thomas (1987); Herbert et al. (1987); Herbert (1988a, b), and Nayfeh (1987a, b); in these publications numerous flow-visualization pictures were presented (Figs. 5.8a, b represent just one such example). The first high-quality pictures were published by Saric and Thomas (1984), who used flow visualization to observe the nonlinear wave development in a zero-pressure-gradient boundary layer disturbed by a vibrating ribbon, at different values of the disturbance level u'/U_0 (where, as above, u', observed not far from the ribbon, is the maximum with respect to z of the amplitude of the streamwise-velocity oscillations in the excited plane T-S wave, and U_0 is the free-stream velocity). At $u'/U_0 = 0.7$ %, Saric and Thomas found the usual K-type nonlinear development which was earlier observed by Schubauer, Klebanoff and his co-authors, and a number of other experimenters. However, for $u'/U_0 < 0.5$ % the character of the picture changed, and instead of an ordered peak-valley vortical structure corresponding to that in Fig. 5.8a a staggered structure of the type shown in Fig. 5.8b was observed. Moreover, Saric and Thomas also found that some important details of the staggered structure depended critically on the initial value of u'/U_0. At $u'/U_0 = 0.3$ % they obtained a picture which agreed excellently with Craik's fully-resonant triad: here the angular frequency of the 3D oblique waves was equal to $\omega/2$, with high precision, and the streamwise wavenumber of these waves was practically equal to $k/2$, where k is the wavenumber of the two-dimensional T-S wave excited by the vibrating ribbon. At the same time, the spanwise wavenumber k_2 of oblique 3D waves found at this value of u'/U_0 agreed well with the value given by the general O-S Eq. (2.41) for a three-dimensional T-S wave with angular frequency $\omega/2$ (the angular frequency of ribbon oscillations is now denoted by ω) and streamwise wavenumber $k/2$, while the vertical profile of the 3D-wave amplitude $u_3{}'(z)$ measured by a hot-wire anemometer agreed with amplitude calculations based on Craik's (1971) resonant-triad theory. However, at a slightly higher disturbance level, $u'/U_0 = 0.4$ %, the value of k_2 given by flow-visualization data was more than twice as large as that corresponding to a three-dimensional T-S wave with frequency $\omega/2$ and streamwise wavenumber $k/2$. The results of Saric et al. (1984) also showed that spanwise periodicity of the 3D structures depended very significantly on the disturbance level. These results, which have already been mentioned in part (a) of this subsection, clearly showed that the observed vortical structure could not always be due to the simple Craik mechanism of three-wave resonance, which has the same form at any value of the 2D-wave amplitude.

Important subsequent experimental studies of the N-regime of wave development in Blasius boundary-layer flow were carried out by Corke and Mangano (1988, 1989); Corke (1989, 1990, 1995), and Bake et al. (1996, 2000). These authors produced controlled wave disturbances in a boundary layer by means other than the old but still-popular vibrating ribbon. In particular, Corke and his group used the method proposed by Liepmann et al. (1982) and then refined by Robey (1987). Instead of the usual vibrating ribbon, Liepmann et al. used a heating wire, placed in the initial part of a water boundary layer and excited electrically to give a temperature varying periodically with given frequency $f = \omega/2\pi$. They used a single wire which was stretched spanwise from wall to wall of the test rig; since wire-temperature variations generate local changes of flow viscosity (and local buoyancy forces), the spanwise heating wire excites a 2D wave of frequency f in the flow. Robey noted that this technique lends itself to 3D forcing since the heater geometry can be prescribed arbitrarily. He used a heater array consisting of 32 rectangular surface elements separated by narrow gaps. In Robey's experiments, individual elements were aligned in a single spanwise row, but by varying the distribution of the phase and/or amplitude of the temperature fluctuations across the span of the array he could produce many different 3D disturbances. In the experiments of Corke's group, this method was applied to the air boundary layer in a wind tunnel where a single heating wire, whose temperature fluctuated with frequency f, was supplemented by a spanwise array, at a fixed x-location close to that of the first wire, of individual heating segments of fixed spanwise length s, again separated by narrow gaps. The temperature of the heating segments oscillated with a fixed frequency (most often with the subharmonic frequency $f_1 = f/2$); moreover, these authors also introduced a definite phase shift ϕ between temperature variations at any two adjacent segments. This arrangement generated time-periodic and spanwise-periodic variations of flow velocity which excited a pair of symmetric oblique waves. These waves propagated streamwise, and their spanwise wavenumbers $\pm k_2$ and inclination angles $\theta = \pm \tan^{-1} (k_2/k_1)$ depended on f, ϕ and s and hence could be changed by changing values of some of these parameters. (Usually the values of k_2 and θ were adjusted by changing the phase shift ϕ). The amplitudes of the plane and oblique waves depended on the amplitudes of heating-wire and heating-array temperature oscillations; hence both wave amplitudes could be arbitrarily varied. Thus, the heating method had an important advantage over the vibrating ribbon, since here all the important parameters of both the 2D plane and 3D oblique waves could be prescribed by experimenters. Results were recorded by smoke-flow visualization and by hot-wire measurements of all three velocity components.

Recall that at the beginning of this subsection and the preceding one, it was noted that Klebanoff et al. (1962) also artificially generated spanwise periodicity of the boundary-layer disturbances, but the purpose of this procedure was then quite different. In the old work of 1962 and in all repetitions of it by other authors, spanwise forcing was used only to shorten the time needed for the natural appearance of spanwise variations of the nominal 2D disturbance. Therefore, the experiments by Corke's group, where the amplitudes, frequencies, streamwise and spanwise wavelengths of all waves of a triad, and also the degree of phase synchronism between plane and oblique waves could be prescribed beforehand by the investigators, were much more informative than those of Klebanoff et al. and their successors.

Corke and Mangano (1988, 1989) began their experiments with boundary-layer observations in the absence of any heating-wire forcing. They found that then the boundary-layer velocity profile preserved the Blasius shape down the whole length of the wind-tunnel test section, and among the observed weak disturbances induced by background noise the least–stable T-S wave played the dominant part. Then the authors switched on the wall-to-wall heating wire, using two different temperature-oscillation frequencies f corresponding to values $F \times 10^6 = 88$ and $F \times 10^6 = 79$ of the dimensionless frequency $F = 2\pi f \nu / U_0^2 = \omega\nu / U_0^2$ (used in Figs. 2.26, 2.27, in Sect. 2.92 and also in Fig. 5.3). At the position of the heating elements exciting the waves, both frequencies corresponded to linearly-unstable T-S waves (in these experiments $x_1 = 45$ cm and $\mathrm{Re}^+ = (U_0 x_1/\nu)^{1/2} = 430$ at the position of the heating wire, where x_1 is measured from the beginning of the test section upon whose wall the boundary layer was developed). At larger values of x_1 (where Re increased because of boundary-layer growth) these T-S waves became stable to infinitesimal disturbances. The hot-wire measurements showed that, even for a very small initial amplitude of the wave excited by the heating wire (much smaller than the initial wave amplitudes used in all studies of the K-regime of boundary-layer breakdown), the T-S wave corresponding to the frequency F of the excitation was easily detected against the background of much weaker external noise. Moreover, the initial exponential growth and later decay of this T-S wave, predicted by the linear stability theory, was also found in the experiments of Corke and Mangano. This agreement with the linear theory showed that the locally-plane-parallel approximation used in the theory was sufficiently accurate. However, conclusions based on the linear stability theory were in fact unimportant here, since the linear-theory rates of wave growth and decay were negligibly small in comparison with the rates of change due to nonlinear interactions, which were the main object of the study.

As to the experiments where both 2D and 3D waves were artificially excited, Corke and Mangano considered only cases where $f_1 = f/2$, and restricted themselves to the study of three special cases. In two of these cases the dimensionless frequency took the value $F \times 10^6 = 79$ (and hence $F_1 \times 10^6 = 2\pi f\nu/U_0{}^2 \times 10^6 = 39.5$) and the phase shift ϕ took either the value corresponding to oblique-wave inclination angles $\theta_{1,2} = \pm 45°$, or a value such that $\theta_{1,2} = \pm 59°$ (cases 1 and 2, respectively), while in the third case the values were $F \times 10^6 = 88$, $F_1 \times 10^6 = 44$ and $\theta_{1,2} = \pm 61°$. In all three cases flow visualization showed a 'staggered vortical structure' of the type presented in Fig. 5.8b. For case 3 the spanwise distributions of the amplitude $A = u'_{\max}/U_0$ (where as before $u'_{\max}$ is the value of u at the height z where it is a maximum) and the phase ϕ of the primary 2D wave of frequency $F = 79 \times 10^{-6}$, and of the sum of 3D oblique waves with half this frequency, are shown in Fig. 5.9. (These distributions were determined from hot-wire measurements at points with different values of y, fixed $x = 150$ cm (measured from the location of the array of heaters) and a value of z corresponding to the critical layer where the mean velocity $U(z)$ is equal to the phase velocity c of the 2D wave). Figure 5.9 confirms that the amplitude and phase of the primary wave have uniform spanwise distributions, as must be the case for a plane wave, while for subharmonic oscillations of half the frequency these distributions are consistent with the sum of two symmetric oblique waves with spanwise wavenumbers $\pm k_2$. Similar results were obtained by Corke and Mangano for two other cases; cf. also Fig. 5.7 showing the results of Kachanov and Levchenko (1984).

The measurements of the streamwise velocity fluctuations u at a number of points on the centerline ($y = 0$), with different values of the coordinates x and z corresponding to the maximum amplitude of these fluctuations, allowed Corke and Mangano to determine the downstream development of the streamwise-velocity amplitude $u'_{\max}$ of both the plane wave (having frequency F) and the subharmonic oblique waves (with frequency $F_1 = F/2$); see Fig. 5.10. Figure 5.10a shows that the rates $G = dA/Adx$ of the downstream growth of the oblique-wave amplitude A differ in the three cases considered, but in all of them these rates considerably exceed those given by the linear stability theory, over a wide range of x-values (i.e., of times t measured from the moment of wave excitation). Corke and Mangano showed also that the rates $G = G(x)$ in all three cases change strongly with x—at first increasing with x to some maximal value $G_{\max}$ (different in the three cases and also occurring at different values of x) and then decreasing with x. On the other hand, Fig. 5.10b shows that the amplitude of the 2D plane wave changes much more slowly. Recall that Fig. 5.6 showed similar behavior of the amplitudes of 2D and 3D waves; however, it represented the results of Kachanov and Levchenko's experiments where only

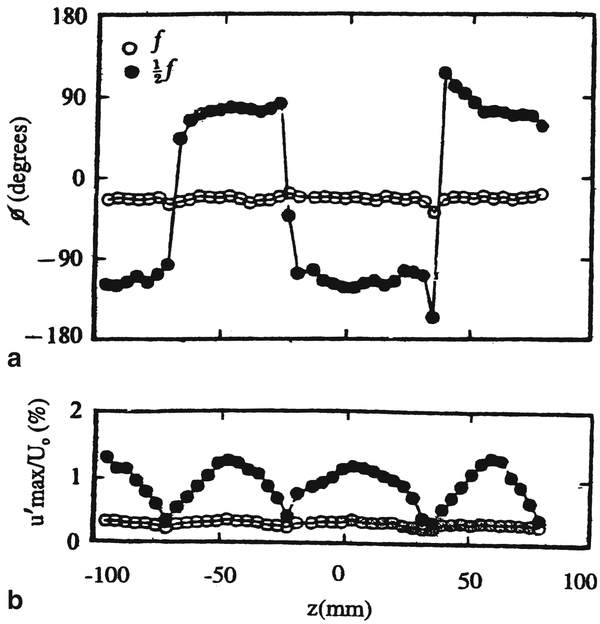

Fig. 5.9 Spanwise distributions of phases ϕ (**a**) and maximum (with respect to z) amplitudes A (**b**) for the primary plane wave of frequency $f(0)$ and subharmonic oblique waves of frequency $f/2$ (0) for case 3 of Corke and Mangano's measurements. (After Corke and Mangano 1989)

the 2D wave was artificially excited, while oblique 3D waves were mainly due to background noise. Hence it was natural to suppose that the observed 3D waves are just those with the highest rate of growth in the presence of the excited plane wave. Therefore, it was assumed that the excited 3D waves, together which, according to Craik's theory, extracts energy from the undisturbed flow in the most powerful way. As to Fig. 5.10, here all three waves of the triads were artificially excited and their frequencies, wave vectors and amplitudes could be chosen by the experimenters; therefore, it was not clear beforehand whether they would or would not satisfy Eq. (5.7) representing Craik's conditions of strict resonance.

Since the frequencies $F{=}\omega\nu/U_0^2$ and $F_1 = \omega_1\nu/U_0^2$ were chosen so that $F_1 = F/2$, the second condition (5.7) was valid in all three cases studied by Corke and Mangano. However, the first condition, which concerns the wavenumbers and guarantees that the primary 2D wave and subharmonic 3D waves have exactly the same phase velocity, was not automatically satisfied in their experiments. Under the conditions of these experiments k could be determined with the help of the O-S Eq. (2.44) as the streamwise wave number of the leasT-Stable plane T-S wave in the Blasius boundary layer having the given frequency $\omega = FU_0^2/\nu$, and the agreement with directly-measured values of k was usually rather close. The experimental data also allowed the value

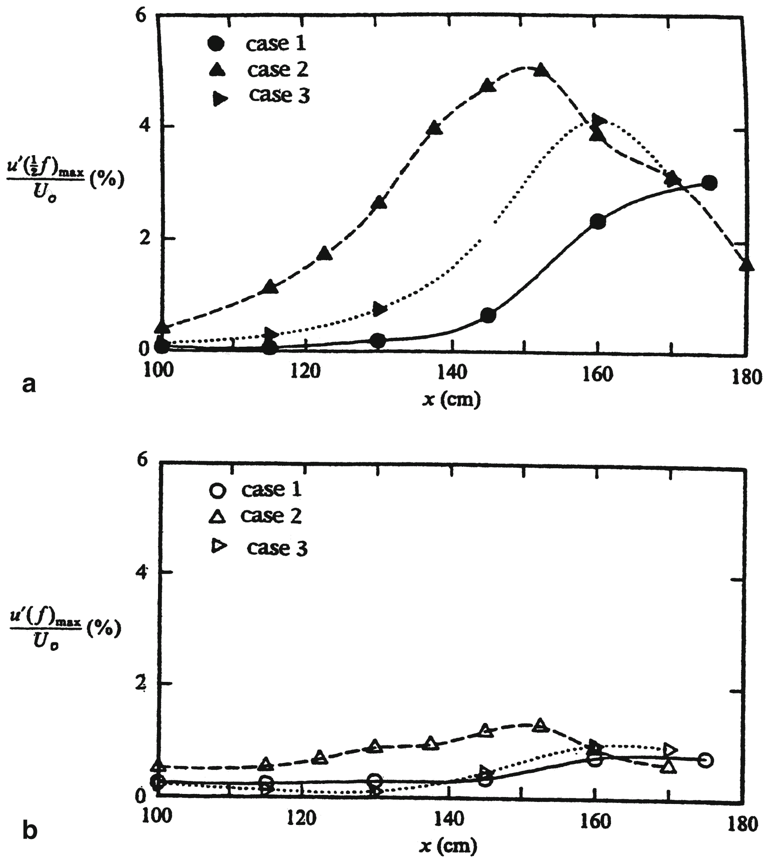

Fig. 5.10 Streamwise development of maximum amplitudes of streamwise velocity fluctuations for (**a**) subharmonic waves of frequency $f/2$, and (**b**) primary waves of frequency f in cases 1, 2, and 3. (After Corke and Mangano 1989)

of k to be deduced directly. The authors used both methods and found that usually they led to very similar values of k.

Values of ω and k determine the phase velocity $c = \omega/k$ of the primary T-S wave. As to the phase velocity c_1 of the 3D oblique waves, knowledge of $k_2/k_1 = \tan\theta$ (or of the value of k_2 which could be determined from Fig. 5.9 and similar figures for the two other cases studied) allowed k_1 to be computed from the three-dimensional O-S Eq. (2.41) (This equation has the same form as the 2D Eq. (2.44) and satisfies the same boundary conditions (2.42), but it determines only the vertical profile of the vertical velocity amplitude $W(z)$. For discussion of the computations of the horizontal velocity components see the papers by Kachanov and Michalke (1994) and Kachanov (1996), and also the earlier papers by Chen and Bradshaw (1984) and Tang and Chen (1985) demonstrating the use of 2D linear stability computations for determination of eigenvalues and eigenfunctions of the 3D liner stability problem). Calculations of k_1 with the help of the O-S equations led to values of $c_1 = k_1/\omega_1$ according to which the condition $c = c_1$ was satisfied with high accuracy in Corke and Mangano's case 1, while in cases 2 and 3 it was not satisfied, although the differences between the two phase velocities were not large. Values of c_1 determined from the experimental data led to much closer agreement with Craik's resonant conditions, in all three cases, than did the values computed from the linear O-S equations. Corke and Mangano therefore concluded that in the presence of the primary 2D mode the 3D subharmonic modes reach phase-velocity synchronization with the primary mode in the course of their development, whatever the initial conditions, and noted that this conclusion agrees with Herbert's secondary-instability theory but disagrees with Craik's theory of fully-resonant triads. This topic will not be further discussed here; note only that, according to Fig. 5.10, considerable growth of 3D waves was observed in all three cases studied (but was different in different cases). Because of this one may suppose that a subharmonic resonance of some form occurred in all these cases.

Corke and Mangano carried out a more detailed investigation of their wave triads, and found that all properties observed in their experiments agreed well with the predictions by Herbert (1983b, 1988a) and Herbert et al. (1987) (and also with subsequent results of Crouch and Herbert (1993)) relating to evolution of secondary-instability waves in boundary layers (see in this respect Fig. 5.15a in Sect. 5.4, which is taken from Corke and Mangano's paper). However, as was noted above, some of the properties observed in cases 2 and 3 were found to be inconsistent with those of fully-resonant triads. Therefore Corke and Mangano concluded that the C-type and H-type of nonlinear development of subharmonic waves in the N-regime of boundary-layer instability growth may be distinguished in practical situations, and that in their experiments case 1 corresponded to C-type development, while cases 2 and 3 corresponded to H-type development, while cases 2 and 3 corresponded to H-type development. Note, however, that later Zel'man and Maslennikova (1993a) generalized Craik's concept of the fully-resonant triad and stated that their version of the three-wave-resonance theory admitted deviations of wave characteristics from the strict-resonance conditions (5.2b) and led to results which also agreed very well with Corke and Mangano's data. Furthermore, Fig. 5.15b shows that the method proposed by Mankbadi (1991, 1993a) for approximate evaluation of the growth rates of oblique waves entering symmetric wave triads gives results which agree excellently with the experimental data of Corke and Mangano in all three cases studied.

A more thorough analysis of Corke and Mangano's data, supplemented by results of a few additional experiments of the same type, was carried out by Corke (1987, 1989, 1990, 1995). In his papers the main attention was paid to the spectra of the velocity fluctuations and the explanation of their origin. In this respect Corke investigated the spatial development of various harmonics generated by nonlinear interactions of 2D and 3D waves with themselves and with each other, and by higher-order interactions of these 'harmonics of the lowest order' among themselves and with the primary 2D and 3D waves. In the 1990 and 1995 papers the effect of 'mode detuning' (noncoincidence of the frequency f_1 of an artificially-excited 3D-wave with the 'resonant frequency' $f/2$) was specially studied. Corke (1995) used the same combination of heating elements as Corke and Mangano (which allowed the frequencies of 2D and 3D waves to be set to any values) to excite a pair of symmetric oblique waves with dimensionless frequency $F_1 \times 10^6 = 39.5$ (corresponding to $f_1 = 16$ Hz) and $\theta_{1,2} = \pm \tan^{-1}(k_2/k_1) = \pm 59°$ together with a 2D (plane) T-S wave whose dimensionless frequency F took different values in the five successive experiments. The values of $F \times 10^6$ used were: 79 (this is the 'tuned case' where $F_1 = F/2$) and 81, 84, 86 and 88 (they correspond to frequencies $f = 32$, 32.8, 33.5, 34.75 and 36 Hz). In all Corke's wave triads the streamwise wavenumbers k and k_1 of the 2D and 3D primary waves satisfied the 'wavelength resonance condition' $k_1 = k/2$ with high accuracy, but the 'frequency resonance condition' $\omega_1 = \omega/2$ was satisfied only in the 'tuned case'. Measurements of the spectra of streamwise-velocity fluctuations downstream of the heating elements showed, in all cases, numerous 'higher-order waves', produced by nonlinear interactions among existing waves and having frequencies and wave vectors equal to differences or sums of those of the pre-existing waves. Recall that in the case of simple fully-resonant triads quite similar 'oscillations and waves of higher orders' were observed by Kachanov et al. (1977) and Kachanov and Levchenko (1982, 1984) and some of them are shown in Fig. 5.5a.

Corke's results corresponding to the 'tuned case', where $f = 2f_1 = 32$ Hz, agreed excellently with those found by Corke and Mangano (1989), while among the 'detuned cases' (where $f \neq 2f_1$) only some representative results for the 'most-detuned' case where $F \times 10^6 = 88$ (i.e., $f = 36$ Hz) were described at length in his paper of 1995. In this 'most-detuned' case the artificially-excited 2D wave with frequency $f = 36$ Hz and wave number k, together with 3D oblique waves with frequency $f_1 = 16$ Hz and wave vectors $(k/2, \pm k_2)$, generated a number of supplementary 3D wave harmonics with 'combined' frequencies and wave numbers (in particular, with frequencies $20 = 36 - 16$, $4 = 20 - 16$, $32 = 16 + 16$, and $24 = 20 + 4$ Hz). Among these 'higher-order harmonics', the lowest order had 3D waves with frequency $f_2 = 20\,\text{Hz} = f_1 + \Delta f$, $\Delta f = 4$ Hz, and wave vectors $\mathbf{k} = (k/2, \pm k_2)$ produced by nonlinear interactions of primary 2D and 3D waves. These waves are especially interesting since, together with the original 2D and 3D waves, they form a 'five-wave resonant system' consisting of two 'detuned resonant triads' with frequency-wavevector combinations $(f, k, 0)$, $(f_1, k/2, k_2)$, $(f_1 + \Delta f, k/2, -k_2)$, and $(f, k, 0)$, $(f_1, k/2, -k_2)$, $(f_1 + \Delta f, k/2, k_2)$ (cf. the related 'tuned five-wave resonances' mentioned in Sect. 5.1. and considered by Craik (1985), Sect. 16.2). The corresponding 'detuned resonances' explain well the rapid growth observed by Corke (which began immediately after the appearance of the wave of frequency 20 Hz) of both the primary 3D wave of frequency 16 Hz and 3D harmonics of frequency 20 Hz (see Fig. 5.11). Note that in the early stages of disturbance development the 'harmonics' had smaller amplitude than the primary 3D wave; this was, of course, natural since 'harmonics' did not exist at the very beginning and had to be generated by interaction of the primary 2D and 3D waves. However, after their appearance the harmonics began to grow faster than the primary 3D wave, and some time later their amplitudes overtook that of the slowly-growing 2D wave. This situation is entirely

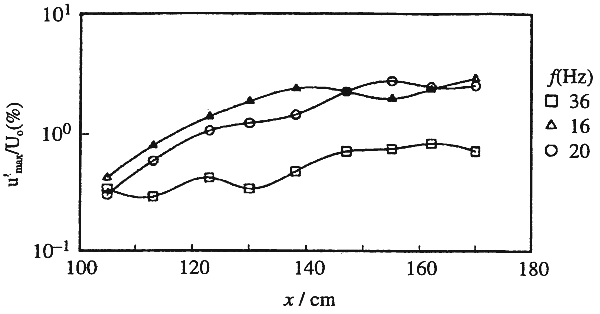

Fig. 5.11 Streamwise development of maximum amplitudes of an artificially- excited plane wave of frequency 36 Hz and an oblique wave of frequency 16 Hz, together with development of the 3D wave of frequency 20 Hz produced by their nonlinear interaction. (After Corke 1995)

similar to that predicted by Zel'man and Maslennikova for the modified cases of Craik's fully-resonant triad, where two oblique waves have initially different amplitudes (see Fig. 5.3 above). Using the data of some preliminary experiments of Corke's group, Mankbadi (1993b) proposed some approximate equations describing the dependence of the amplitudes of the 2D wave, and of two pairs of symmetric oblique waves entering 'a pair of detuned resonant triads', on Re (i.e. on the streamwise coordinate x determining the value of Re). The equation given for two oblique-wave amplitudes included cubic terms (more general than those in Eq. (5.11) for the 'fully-resonant case') which allowed the saturation of the oblique wave to be determined. Mankbadi's amplitude equations were simplified by Corke (1995), who presented them in the form of three equations for the three amplitudes; these equations contained eleven constant coefficients requiring special determination. In this context Corke also discussed some data from his amplitude measurements which will not be considered here.

According to Corke, both oblique waves (with frequencies 16 and 20 Hz) of the 'detuned triad' had practically the same phase velocity (and hence the same 'critical layer'). They also had the same normalized vertical amplitude profile $|A(z/\delta^*)|/A_{max}$, which did not differ much from the amplitude profile of the oblique components of Craik's 'tuned' resonant triad with $f_1 = f/2$, $k_1 = k/2$, which was measured both by Kachanov and Levchenko (1982, 1984) and by Corke and Mangano (1989) (the results found by these two groups were rather close to each other; they are shown in Fig. 5.16a and will be discussed below. At the same time, Corke and Mangano's results showed that in their cases 2 and 3, where k_1 took values close, but not equal, to $k/2$, the normalized profiles of subharmonic-wave amplitudes did not differ much from those observed in the 'fully-resonant' case 1). On the other hand, the amplitude profile of the 'higher-order harmonic' with low frequency $f = 4$ Hz differed considerably from that in Fig. 5.16a, while as a rule the mean value of the amplitude

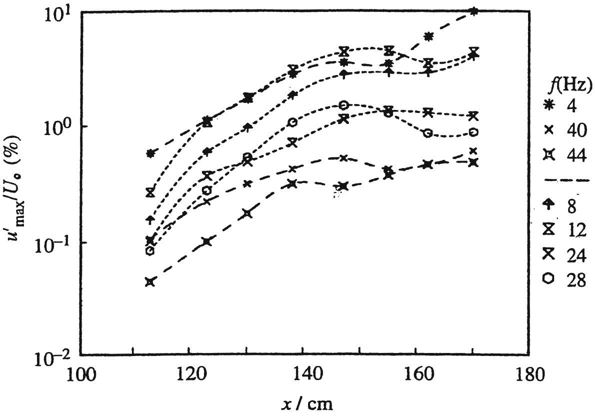

Fig. 5.12 Streamwise development of a number of waves produced by nonlinear interactions of waves from an artificially-excited 'detuned resonance triad'. (After Corke 1995)

$|A(z)|$ of this (and other) higher-order harmonic components of velocity fluctuations grew significantly as x increased.

Corke also showed that in the course of disturbance development new wave components were repeatedly generated by numerous nonlinear interactions among existing components. Thus, the detuned-triad resonance studied in his paper led to the appearance of a borad range of streamwise-growing discrete modes at intervals equal to the lowest difference frequency (equal to 4 Hz in the case considered here). An example of Corke's observations of the downstream growth of a number of such higher-order harmonic components is shown in Fig. 5.12. Let us recall that Figs. 5.5a, b show that frequency spectra of the nonlinearly-developing disturbances in a Blasius boundary layer perturbed by a vibrating ribbon are in fact very far from the pair of discrete lines at frequencies ω_0 and $\omega_0/2$ corresponding to a resonance triad of Craik's type. And detuned resonances generated by background noise, with low detuning Δf, may be one of the mechanisms producing the rapid growth of energy of low-frequency fluctuations and thus leading to formation of spectra of the type presented in Fig. 5.5a, b.

Another method of controlled wave excitation, proposed by Gaponenko and Kachanov (1994), was used by Bake et al. (1996) and Bake et al. (2000). These authors carried out their experiments in a wind tunnel at the Technical University of Berlin, having an axisymmetric test section with a diameter of 441 mm and a total length of 6,000 mm. The boundary layer studied developed on the wall of the test section. At a free-stream velocity $U = 7.2$ m/s the boundary-layer thickness δ at the position of excitation was close to 6 mm (with $\delta^* \approx 2$ mm), and the undisturbed normalized velocity profile $U(z/\delta^*)/U_0$ had practically the same Blasius form (which corresponds to flat-plate boundary layers) at all streamwise and spanwise measurement positions. The wave disturbances were introduced into the boundary layer by a 'slit generator' consisting of a long narrow slit (with 0.5 mm width, 5 mm depth and 260 mm length in spanwise—i.e. circumferential—direction) cut into the inner wall, and a set of 32 small tubes (with a spanwise spacing of 8 mm) placed under the slit and connected to three loudspeakers. The loudspeakers were fed by three different time-periodic signals which combined with each other inside the slit generator forming, near the outlet of the slit, a field of flow fluctuations corresponding to a 2D or 3D disturbance of any type of interest to the investigators.

Bake et al. used the primary frequency $f = 62.5\,\text{Hz}$ (corresponding to $F = 2\pi f\nu/U_0^2 = 115.5 \times 10^{-6}$ and to subharmonic frequencies $f_1 = f/2 = 31.25\,\text{Hz}$ and $F_1 = 57.8 \times 10^{-6}$) and studied four cases of excited wave disturbances:

1. The primary 2D wave of frequency f and large amplitude A is excited simultaneously with a pair of oblique subharmonics of frequency f_1 and low amplitude $A_1 \ll A$. The spanwise wavenumbers of the oblique waves $\pm k_2$ were determined by the spanwise spacing of the tubes feeding the slit generator, but the phases of primary and subharmonic waves could be prescribed by the experimenters and were chosen to be close to values which, according to previous data, are most favorable for the subharmonic resonance.
2. Only the pair of subharmonic waves with the same characteristics as in case I was excited.
3. Only the primary wave (the same as in the case I) was excited.
4. The same three waves as in case I were excited, but the phase shift between the fundamental and subharmonic waves was selected to be *least* favorable for the subharmonic resonance.

In cases II-IV no indication of resonance was found; therefore only results for case I will be discussed below. Results relating to the initial stage of the disturbance development (for $\Delta x = x - x_s < 250\,\text{mm}$, where x_s is the streamwise length of the disturbance source) as a rule agreed well with those of the previous investigations. It was found that at the chosen values of f, k_2, A, A_1 and the phase shift between primary and subharmonic waves, the resonance conditions (5.7), guaranteeing the equality of phase velocities of three waves, were satisfied with good accuracy. Hence it was only natural that for $\Delta x < 250$ mm the results of Bake et al. for spanwise distributions of the amplitudes and phases of the primary and subharmonic waves, for the normalized vertical profiles of the same amplitudes and phases, and for the 'growth curves' representing the dependence of the amplitudes of three waves on the streamwise coordinate x, did not differ much from the values of the same characteristics found for fully-resonant wave triads, e.g., by Kachanov and Levchenko (1984); Saric et al. (1984); Kozlov et al. (1984); Thomas (1987), and Corke and Mangano (1989), who mostly also restricted themselves to not-too-large values of Δx (some results from these papers are shown in Figs. 5.6, 5.7, 5.9, and in Figs. 5.16 and 5.17 to be discussed in Sect. 5.4).

Note, however, that the wind tunnel used by Bake et al. had a very long test section and the region $\Delta x < 250$ mm is only a small part of it. In fact, the main purpose of the investigators was to study the N-regime of wave-disturbance developments over a downstream range much greater than any explored previously. They found that at large values of x the wave development which began as the N-regime unexpectedly acquired some features which were previously considered as typical only for the K-regime. However, these results can be discussed only with those relating to the

K-regime of disturbance development, and this discussion must be postponed until Sect. 5.5.

5.4 Comparison of Theoretical Predictions for the N-Regime with Experimental and Numerical Data

Let us begin this section with a discussion of the remark made in Sect. 5.3 that the discrepancy between Kachanov and Levchenko's (1982, 1984) experimental value of the inclination angle $\theta = |\theta_{1,2}|$ of observed subharmonic oblique components of a resonant wave triad, and the theoretical estimate of this angle by Volodin and Zel'man (1978), may be explained by some defects of Volodin and Zel'man's theory. The first hint indicating that, contrary to the conclusion of this theory, the value of the angle $\theta = |\theta_{1,2}|$ is apparently not universal but depends on the value of the plane-wave amplitude A_1 was given by Zel'man and Maslennikova (1984). In subsequent more explicit studies (1989, 1990, 1993a) thse authors proved that there is a direct link between the values of θ and A_1. This proof confirmed the experimental results of Saric and Thomas (1984) and Saric et al. (1984), which have already been mentioned in Sect. 5.3 and will be considered at greater length slightly later. Moreover, the proof clearly implies that the unique value of θ given by Volodin and Zel'man in 1978 cannot be universal.

The point is that in 1978 Volodin and Zel'man followed Craik's paper of 1971 and considered only 'fully-resonant triads' consisting of one plane and two symmetric oblique T-S wave exactly satisfying Eq. (5.7), where k and k_1 are real parts of streamwise wave-numbers of plane and oblique waves, and ω and ω_1 are the real parts of the corresponding frequencies. (For the sake of brevity, the words 'real parts' are applied here to both wavenumbers and frequencies. Of course, in the overwhelming majority of actual stability problems only one of these two wave characteristics takes complex values). Later Zel'man and Maslennikova (1989, 1990, 1993a) generalized Craik's model admitting, in particular, that two oblique waves may not be strictly symmetric (e.g., the amplitudes of these waves may differ from each other) while the two Eq. (5.7) may be valid not exactly but only approximately. According to the results of these papers (some of which have already been mentioned in Sect. 5.2) in the cases of these more general wave triads rapid resonant growth of the oblique waves also occurs quite often; see, e.g., Fig. 5.3 taken from the paper (1993a) and also Figs. 5.10 and 5.11 showing some experimental data confirming this conclusion. Now we will continue the discussion of the corresponding theoretical and experimental results.

Figure 5.3 is only one example illustrating the general results given by Zel'man and Maslennikova (1993a). According to these results, if the value of $\mathrm{Re}^+ = U_0\delta^+/\nu = (U_0x/\nu)^{1/2}$ (or of $\mathrm{Re}^* = U_0\delta^*/\nu \approx 1.73\ \mathrm{Re}^+$) is given, then under rather general conditions there exists, for a given plane T-S wave of frequency ω, streamwise wavenumber k, and amplitude A_1, a large set of pairs of oblique 3D-waves of frequency $\omega_1 \approx \omega/2$ inclined at angles $\pm\theta$ to the undisturbed-flow direction; together with the primary plane wave these form 'resonant triads'. (These triads, as a rule, do not satisfy Eq. (5.7) exactly, but nevertheless they are 'resonant' since the corresponding amplitude equations include resonant quadratic terms. Therefore, here the growth rates $G_o = dA_{2,3}/A_{2,3}dx$ of the oblique-wave amplitudes $A_2 = A_3$ strongly exceed the growth rate G_p of the plane-wave amplitude, which remains close to that given by linear stability theory). For a number of such generalized resonant triads Zel'man and Maslennikova computed the interaction coefficients B_1, B_2 and B_3 of Eq. (5.4a) by a method similar to that used by Volodin and Zel'man (1978); a few results of these computations are shown in Fig. 5.3. In the cases where two oblique waves entering the triads had the same amplitude, the same frequency $\omega/2$ and matched phases, these waves had variable streamwise wavenumbers $k_1 \approx k/2$ and values of spanwise wavenumbers $\pm k_2$ filling a rather wide range. For the existence of a collection of pairs of oblique waves resonantly excited by a given plane wave, it is only necessary that Re (according to any suitable definition) be high enough and that A_1 be not too small. Note in this respect that the existence of the threshold value A_{tr} of A_1, below which no growing 3D waves can be excited, was predicted quite early by Görtler and Witting (1958) and Maseev (1968a, b), and that Maseev's Fig. 5.4 implies also that at any $A > A_{tr}$ there is a finite range of k_2 values corresponding to 3D waves growing in the presence of the given plane wave.

The ranges of admissible values of k_2 and k_2/k_1, corresponding to positive growth rates G_0, and also the preferred values of k_2 and k_2/k_1, $(k_2)_{pr}$ and $(k_2/k_1)_{pr}$, (corresponding to the greatest value of G_0), depend on A_1, ω and Re, while the value of G_0 itself depends on A_1, ω, Re and k_2/k_1. (Note that for given values of Re, $\omega/2$, and k_2, the streamwise wavenumber k_1 of the corresponding most-unstble oblique wave may be determined uniquely with the help of the O-S Eq. (2.41). However, the strict equality $k_1 = k/2$ will be valid here only for one special value of k_2). Figure 5.13, which is based on the results of Zel'man and Maslennikova's computations, shows a typical example of the dependence of G_0 (non-dimensionalized with δ^+ as the unit of length) on $K_2 = k_2\nu/U_0$ (and also on $k_2/k_1 = \tan\theta$) for some definite values of the dimensionless parameters $\mathrm{Re}^+ = U_0\delta^+/\nu$ and $F_1 = \omega_1\nu/U_0^2$ and a number of values of the amplitude A_1 (measured as fractions of U_0). This figure shows that here the preferred value of k_2/k_1, which must be met often in real boundary-layer flows, is not constant but grows with the value of A_1. According to the results of Zel'man and Maslennikova (1993a) (only partially represented in Fig. 5.13) the value of $(k_2/k_1)_{pr}$ depends very little on Re^+ and F_1, while at values of A_1 only slightly above the threshold value (which makes possible the resonant growth of some oblique waves), $(k_2/k_1)_{pr} \approx 1$ (and $|\Delta k| = |k_1 - k/2|$ is very small, i.e., the resonant triad are here close to Craik's conditions of perfect resonance).With an increase of the amplitude A_1, the range of values of k_2/k_1 corresponding to resonance conditions where $G_0 > 0$ also increases, the value of $(k_2/k_1)_{pr}$ grows and approaches 2 and $|\Delta k|$ also grows (but, nevertheless, $|\Delta k|/k$ remains relatively small). Hence, contrary to the expectation of Craik (1971), among the triads including one 2D and two symmetric 3D waves the growth rate of 3D waves usually attains its greatest value for a triad satisfying only approximately, but not exactly, the resonance conditions (5.2b).

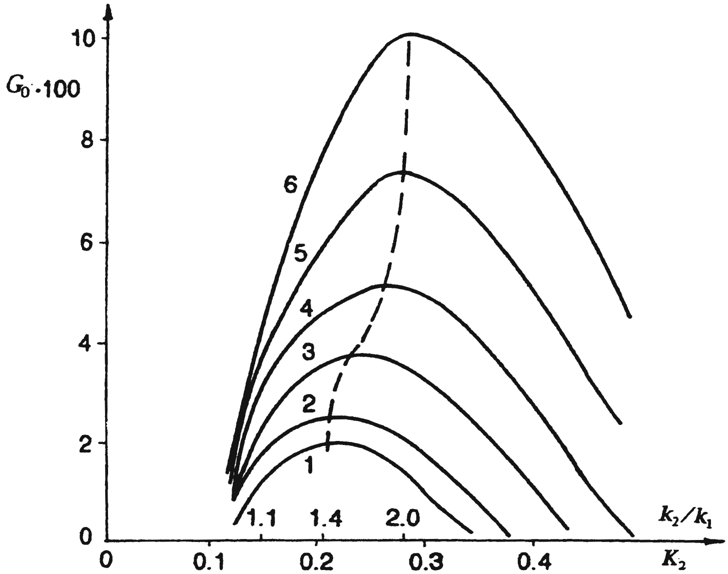

Fig. 5.13 Dependence of $K_2 = k_2\nu/U_0$, and on k_2/k_1, of the amplification rate $G_0 = dA(x)/Adx$ of the oblique-wave amplitude $A \equiv A_2 = A_3$ of a resonant-wave triad, for different values of the plane-wave amplitude A_1 (for $F_1 \equiv \omega_1\nu/U_0^2 = 115 \times 10^{-6}$ and $\mathrm{Re}^+ = (U_0x/\nu)^{1/2} = 640$). All dimensional quantities are non-dimensionalized by scales $\delta^+ = (\nu x/U_0)^{1/2}$ and U_0. Curves 1, 2, . . . , 6 correspond to $A_1 = 0.14$, 0.21, 0.28, 0.40, 0.53, 0.72 %, and the dotted line shows the dependence of the optimal values $(k_2/k_1)_{\mathrm{pr}}$ and $(K_2)_{\mathrm{pr}}$ on A_1. (After Zel'man and Maslennikova 1993a and Kachanov 1994a)

Recall that Kachanov and Levchenko (1982, 1984) stressed that in their experiments the symmetric wave triads appearing in the flow (and hence apparently corresponding to the most rapid growth of oblique waves) were those exactly satisfying Eq. (5.7). However, this statement apparently shows only that in these experiments $|\Delta k|/k$ was so small that it was difficult to distinguish from zero. As to the results of Corke and Mangano (1989) shown in Fig. 5.10, according to which three wave triads with different values of the oblique-wave angle $\theta = \tan(k_2/k_1)$ also have different rates of oblique-wave growth, they clearly conform to the results just discussed. Having this in mind, Zel'man and Maslennikova (1993a) compared the growth rates G_0 measured by Corke and Mangano for three pairs of excited oblique waves with the results of their own computations. This comparison showed that the experimental values of G_0 found by Corke and Mangano for three different values of k_2/k_1 (corresponding to two values of F_1 relatively close to each other, and to known values of Re and A_1 which differed very little in the three cases) agree very well with the computed values of G_0. As will be shown below in Figs. 5.15a, b, the closeness of the oblique-wave amplifications measured by Corke and Mangano to theoretical estimates was also confirmed by Corke and Mangano themselves, using a quite different theoretical model, and then by

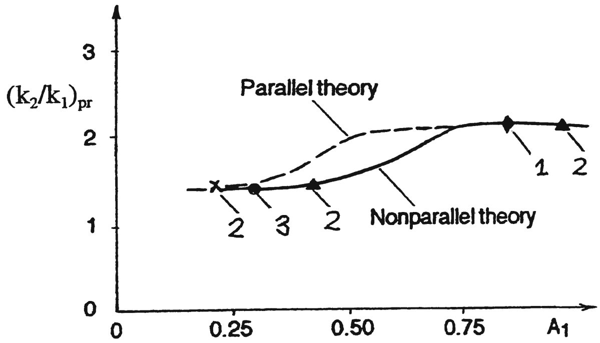

Fig. 5.14 Dependence on the plane-wave amplitude A_1 of the value of $(k_2/k_1)_{pr}$ corresponding to the most-amplified oblique subharmonics of a plane wave. Numbered symbols correspond to laboratory observations: (*1*) Kachanov and Levchenko (1982, 1984); (*2*, *3*) Saric et al. (1984); (*4*) Saric and Thomas (1984). (After Zel'man and Maslennikova 1993a; Kachanov 1994a)

Mankbadi (1993a) who made comparisons with results of computations based on the use of one more theoretical model. The good agreement found by the above-mentioned authors between one set of experimental data and the results of three different theories apparently shows that these three theories in fact differ much less than appears at first sight.

Let us now pass to Fig. 5.14 which is also based on results by Zel'man and Maslennikova (1993a). This figure shows values of k_2/k_1 observed in some recent experiments where A_1 took different values. It is natural to believe that the values of k_2/k_1 observed in experiments are just those which correspond to maximal growth rates of oblique waves (recall that a similar assumption has been widely used in comparisons of the results from linear stability theory with experimental data). Therefore in Fig. 5.14 the values of k_2/k_1 observed in experiments are compared with theoretical estimates of $(k_2/k_1)_{pr}$. Two types of these theoretical estimates are shown in the figure: the simplest ones derived for the plane-parallel model of Blasius boundary layer, and the improved estimates based on the theory of Zel'man and Kakotkin (1982) which took into account the non-parallelism of the boundary layer. As can be seen, the latter estimates agree excellently with the available data, confirming the idea that the values of k_2/k_1 observed in experiments where only a plane wave is artificially excited are very close to $(k_2/k_1)_{pr}$.

Similar results were obtained by Herbert (1983b, 1984a, 1988a, b) (and by Herbert and Bertolotti (1985)) in studies of the secondary-instability mechanism of generation of three-dimensional structures in a boundary layer by means of a principal parametric resonance of oblique waves (see Eq. (5.14b) and the text relating to it). In these studies it was also found that at a given value of Re (Herbert used the Reynolds number Re^+)

and for a given plane T-S wave of frequency ω and not-too-small amplitude A_1, there is usually a wide range of values of k_2 (and hence of k_2/k_1) corresponding to pairs of fast-growing 3D waves of frequency $\omega/2$. This range widens, and the growth rates G_0 increase, with an increase of A_1 above some rather small threshold value, while the preferred values, $(k_2)_{pr}$ and $(k_2/k_1)_{pr}$, of k_2 and k_2/k_1 corresponding to the greatest possible value of G_0 increase monotonically (but relatively slowly) with an increase of the amplitude A_1 or of the Reynolds number Re^*. Herbert's results also agree entirely satisfactorily with some experimental and numerical-simulation data (see e.g. Fig. 5.15a, c). Mankbadi (1993a) also tried to estimate theoretically the growth characteristics of the oblique components of a resonant triad at different values of parameters A_1, Re^*, and k_2 (Note that $|\theta_{1,2}| = \tan^{-1}(k_2/k_1)$ may of course be used instead of k_2). He applied for this purpose his 'nonlinear-critical-layer method', which was briefly discussed at the end of Sect. 5.2. Like the other authors mentioned above, Mankbadi found that at given not-too-small values of A and Re^*, positive values of the oblique-wave growth rate G_0 occur over a wide range of values of k_2 and this range widens (and the maximum value of G_0 increases) when A_1 and/or Re^* increase. His quantitative results agreed quite satisfactorily with the experimental data of both Kachanov and Levchenko (1984) and Corke and Mangano (1989), and with the results of Spalart and Yang's (1987) numerical simulation of disturbance development in Blasius boundary-layer flow disturbed by a vibrating ribbon (see, in particular, Figs. 5.15b, c). Thus, three different methods of computation of the resonant-triad development in the boundary layer led to results which are close to each other and agree satisfactorily with both experimental and numerical-simulation data. At the same time, all the above-mentioned results clearly contradict the early conclusion of Volodin and Zel'man (1978) based on their use of the original model of Craik (1971).

Let us now pass to comparison of the measurements by Kachanov and Levchenko (1984), of the vertical profiles of thes amplitude $A_{1/2}(z)$ and phase $\phi_{1/2}(z)$ of the subharmonic oblique waves entering the resonant triad, with the available theoretical results. (Now notations $A_{1/2}$ and $\phi_{1/2}$ are used instead of the notations A_2, A_3, $A_{2,3}$, and ϕ_2, ϕ_3 used above). It was mentioned above that Kachanov (1994a) compared these profiles with several theoretical and numerically-simulated estimates. The first theory used by him for this purpose was the well-known three-wave-resonance theory. However, the initial form of this theory proposed by Craik (1971) was too crude to give sufficiently accurate values of the subharmonic-wave amplitude $A_{1/2}(z)$; therefore the refinements of Craik's theory by Zel'man and Maslennikova (1989, 1990, 1993a), briefly described above, were used by Kachanov to determine curve 3 in Fig. 5.16a. Moreover, Kachanov also considered the results of Herbert's (1984a) theory of secondary instability of the primary plane wave, which relate to computation of profiles of $A_{1/2}$; these results led to curve 1 in the same figure. Finally, we can also use the results of numerical solutions of the Navier-Stokes equations describing the downstream propagation of a wave triad in the boundary-layer—such a solution was computed, in particular, by Fasel et al. (1987) and led to the results presented as

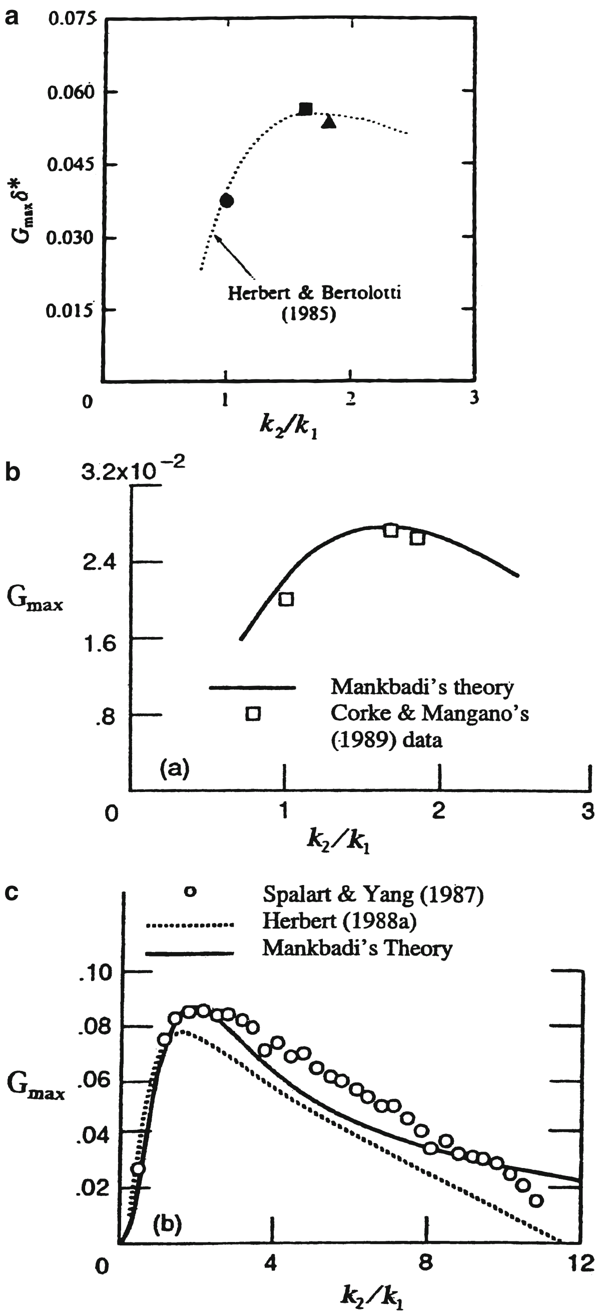

Fig. 5.15 Comparison of the maximum values G_{max} of the oblique-wave amplification rate, from Corke and Mangano's (1989) experiments and Spalart and Yang's (1987) numerical simulation, with some theoretical estimates of the dependence of G_{max} on k_2/k_1. **a** Comparison of Corke and Mangano's values of $G_{max}\delta^*$ (filled symbols) with corresponding results of the secondary-instability theory of Herbert (1983b, 1988a); Herbert and Bertolotti (1985). (After Corke and Mangano 1989). **b** Comparison of Corke and Mangano's data (with a special normalization) with a theoretical estimate of the dependence of G_{max} on k_2/k_1 given by Mankbadi's theory of critical-layer nonlinearity. (After Mankbadi 1993a). **c** Comparison of Mankbadi's estimate of dependence of G_{max} on k_2/k_1 with the corresponding theoretical estimate by Herbert (1988a) and results of numerical simulation by Spalart and Yang (1987) of a boundary layer disturbed by a vibrating ribbon; for $F \equiv \omega\nu/U_0{}^2 = 58.8 \times 10^{-5}$ and initial conditions $A_1(0) = 1.4\ \%$ and $\mathrm{Re}^+(0) = 950$. (After Mankbadi 1993a)

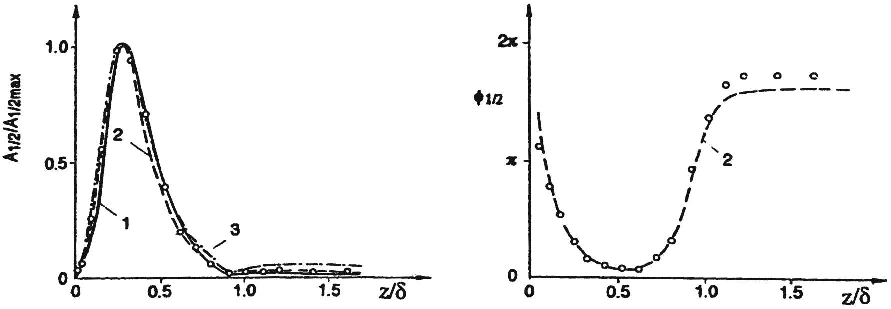

Fig. 5.16 Measured (*symbols*) and calculated (*curves*) vertical profiles of the amplitude $A_{1/2}(z)$ (*left*) and the phase $\phi_{1/2}(z)$ (*right*) of the subharmonic 3D wave of frequency $\omega/2$ resonantly amplified in the N-regime of instability development in a Blasius boundary layer. Experimental data by Kachanov and Levchenko (1982). Calculations: (*1*), secondary-instability theory of Herbert (1984a); (*2*), numerical simulation of Fasel et al. (1987); (*3*), resonant-triad theory of Zel'man and Maslennikova (1989, 1990, 1993a). (After Kachanov (1994a)

curves 2 in Figs. 5.16a, b.[3] One may see that, once more, two different theoretical approaches and the numerical simulation all give results which agree very well with the experimental data and with each other.

Herbert (1984a, 1986); Herbert and Santos (1987); Herbert et al. (1987); Crouch and Herbert (1993); Zel'man and Maslennikova (1993a), and Kachanov (1994a) showed also that the results of measurements by Kachanov and Levchenko (1982, 1984); Saric et al. (1984), and Corke and Mangano (1989) for the dependence of the primary-wave and subharmonic-wave amplitudes A on x (or on Re $\propto x^{1/2}$), which are presented, in particular, in Figs. 5.6 and 5.10, agree excellently with the results of available computations of the spatial amplitude growth. The accuracy achieved was found to be practically the same for computations based on Herbert's secondary-instability analysis and on the three-wave-resonance theory of Zel'man and Maslennikova. The same, if not better, accuracy was found in comparisons with numerical solutions by Fasel et al. of the initial-value problem for Navier-Stokes equations, describing development of a three-wave disturbance in the Blasius boundary layer. Some results confirming the statements made here are collected in Figs. 5.17a–c. Let us also note that Mankbadi (1991, 1993a) compared amplitude-

[3] Some other attempts at numerical simulation of the N-regime of boundary layer instability developments were carried out by Spalart and Yang (1987) and Laurien and Kleiser (1989) (one result of the former authors is shown in Fig. 5.15c). However, in both these papers the less-accurate temporal, and not spatial, simulation was performed (see the small-type text below for discussion of the difference between these two approaches and the remarks about this topic in the next footnote 4) and the results found were less complete than those of Fasel et al. Therefore, except for Fig. 5.15b, these results will not be considered here. On the other hand, Rist and Fasel (1995) improved somewhat on the numerical method of Fasel et al.; however, as to the results relating to N-regime, the paper of 1995 contains only the indication that here "the quantitative agreement between numerical results and experiments was at least as good or even better than that achieved by Fasel et al. (1987)".

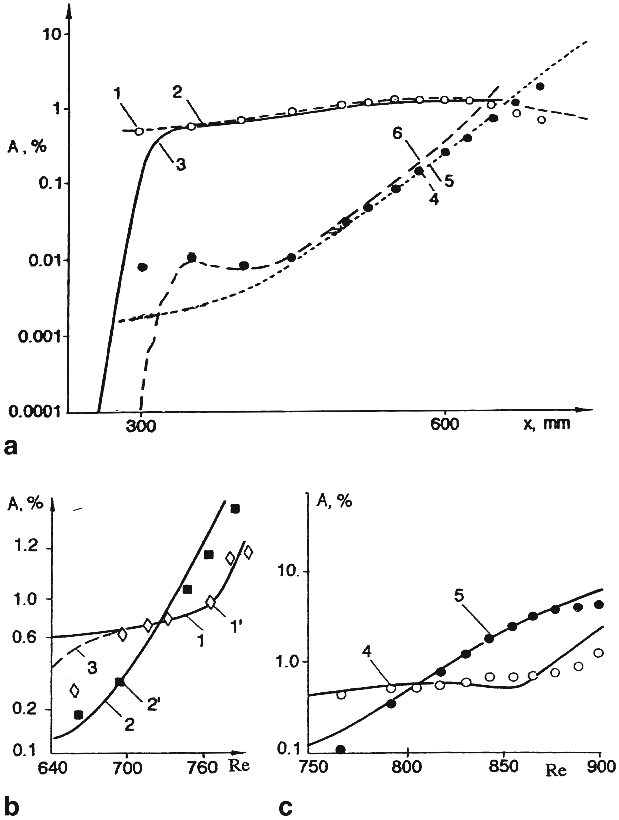

Fig. 5.17 **a** Resonant streamwise amplifications of the plane wave amplitude $A_1(x)$ (results *1*, *2*, *3*) and of the amplitude $A_{1/2}(x)$ of the two subharmonic waves of twice smaller frequency (results *4*, *5*, *6*) during the initial stage of the N-regime of instability development. Experimental data (points *1* and *4*) by Kachanov and Levchenko (1982); calculations (*curves*); *2* and *5*—Herbert's (1984a) theory; *3* and *6*—numerical simulations by Fasel et al. (1987). **b** and **c** the same resonant amplifications at later stages of the N-regime when values of $A_{1/2}(x)$ (*2*, *2′* and *5*) overtake those of $A_1(x)$ (*1*, *1′*, *3* and *4*); $\mathrm{Re} = (xU_0/\nu)^{1/2} \propto x^{1/2}$. **b**: experimental data (points *1′* and *2′*) by Saric et al. (1984), calculated curves *1* and *2*—theory by Maslennikova and Zel'man (1985) and Zel'man and Maslennikova (1993a); *dotted curve 3*—theory taking non-parallelism into account. **c** experimental data (points *4* and *5*) by Corke and Mangano (1989); theoretical calculations (*curves*) by Crouch and Herbert (1993). (All figures after Kachanov 1994a)

growth data for the primary and subharmonic waves in a boundary layer, found in experiments by Kachanov and Levchenko (1984) and Corke and Mangano (1989), with results of his theoretical calculations by the nonlinear-critical-layer method and with the appropriate numerical-simulation results; his comparison showed yet again that there is good agreement between the available experimental, theoretical and numerical data (see Figs. 5.15b, c).

Figures 5.13–5.17 require some comments. Let us note first of all that the numerical-simulation data presented in Fig. 5.15c were obtained by numerical simulation of temporal (and not spatial) disturbance development. This means that the authors assumed that the disturbance studied was streamwise-periodic, and then used the N-S equations for computation of its evolution in time. This assumption presupposed that the parallel-flow approximation was used, but this corresponds to the real experimental conditions somewhat more poorly than the spatial-growth approximation, where the disturbance is assumed to be time-periodic (with a prescribed frequency) while its dependence on coordinate x has to be computed with the help of the N-S equations (cf. a similar comparison of temporal and spatial solutions of the Orr-Sommerfeld eigenvalue problem in Chap. 2). Moreover, the influence of boundary-layer growth can be, at best, only crudely taken into account in the framework of the temporal approach[4], while in the case of a spatial numerical simulation the dependence of the primary flow on x offers no difficultly. However, for temporal simulations much less computer resources (memory and computation time) are needed and the determination of the appropriate outflow boundary conditions at the downstream end of the computation domain is much easier than in the case of spatial simulation; therefore it is not surprising that the temporal approach to flow simulations has been very popular. In addition to Spalart and Yang (1987), temporal numerical simulations of boundary-layer instability development have been carried out by Wray and Hussaini (1984); Zang and Hussaini (1985, 1987, 1990); Laurien and Kleiser (1989); Zang (1992), and some others (see also the description of some related numerical-simulation results in Sect. 5.5). In particular, Zang (1992) showed that results of temporal numerical simulation agree well with the data of Corke (1990) relating to the effect of mode detuning on wave triad development in boundary layers. However Fasel et al. (1987), whose data are shown in Figs. 5.16 and 5.17, carried out a spatial numerical simulation of boundary-layer instability, and the spatial approach was also discussed and used by Murdock (1986); Fasel (1990); Fasel and Konzelmann (1990); Konzelmann (1990); Rist (1990, 1996); Kleiser and Zang (1991); Kloker (1993); Rai and Moin (1993) (who studied the case of a compressible boundary layer with a high level of external disturbances); Joslin et al. (1993); Reed (1994); Rist

[4] The simplest way of doing this is based on the supplementation of the N-S equations by an artificial 'force term' guaranteeing the existence of a solution describing the plane-parallel Blasius boundary layer with time-dependent thickness $\delta(t)$, growing at a rate equal to that registered by an observer who moves streamwise with a reasonably chosen velocity. According to Gaster's (1962) arguments, the group velocity c_g of a packet of T-S waves (which depends only weakly on the vertical coordinate z) may be chosen as such a 'reasonable velocity'. Then the corresponding time-dependent plane-parallel boundary layer may be considered as a temporal model of the real streamwise-growing boundary layer (cf. a remark in Chap. 4, about a similar method of numerical simulation of the steady plane-parallel model of a Blasius boundary layer). This method of approximate allowance, in temporal numerical simulations, for the spatial (streamwise) growth of a boundary layer was used, in particular, by Spalart and Yang (1987) and later gained great popularity.

and Fasel (1995), and Rist and Kachanov (1995), while the corresponding outflow boundary conditions were discussed by Kloker et al. (1993).

Now let us pass to other subjects. In some of the above-mentioned papers by Herbert it was indicated that the secondary instability of a plane T-S wave in a laminar boundary-layer flow may manifest itself in plane-wave instability with respect to some 3D Squire (Sq) waves, which satisfy Eq. (2.46) of Chap. 2 and the conditions indicated there. (Recall that all Sq waves are rapidly damped and hence decay as $t \to \infty$; however, as pointed out in Chap. 3, these waves may nevertheless make a large contribution to the transient growth of very small disturbances). The instability with respect to Sq, and not T-S, wave disturbances was first considered by Herbert (in short, H) in his studies (1983a, 1984b) of the secondary instability of a plane T-S wave in a plane Poiseuille flow, where the midplane symmetry of the undisturbed velocity profile greatly reduces the possibility of plane-wave secondary instability with respect to oblique T-S waves (more will be said about this in the next chapter of this book; cf. also Wu (1996)). Based on his experience of plane-channel secondary instability, H stated in the papers (1983b, 1984a, 1988a) on the secondary instability of the Blasius boundary-layer flow that here the instability with respect to 3D Squire waves may also occur, in principle.

Later Zel'man and Maslennikova (in short, Z-M) in the paper (1993a) criticized Herbert's conclusion, stating that for the triad comprising a plane T-S wave and a pair of Sq waves with half the streamwise wavenumber, the resonance frequency condition (5.7) is strongly violated. According to Z-M, this shows that resonance among one T-S plane and two 3D subcritical Sq waves is impossible; thus, a T-S mode cannot stimulate fast growth of some Sq modes. Furthermode Z-M indicated that the form of the vertical profile of the subharmonic-wave amplitude computed by Herbert (1984a) (see Fig. 5.16a) clearly showed that here the subharmonic wave was represented by a three-dimensional T-S wave and not by a Sq wave which has a quite different amplitude profile. Therefore, Zel'man and Maslennikova (1993a) considered only amplitude equations of the forms (5.4) and (5.4a) corresponding to wave triads comprising three T-S waves. According to their results, numerical solutions of such equations agreed well with all available data for the initial stage of the N-regime of boundary layer development. In particular, the results of their computations agreed very well with the data of Kachanov and Levchenko (1982, 1984) for the profile of the subharmonic-wave amplitude (see again Fig. 5.16a) and of Saric et al. (1984) relating to the streamwise-growth curves for amplitudes of primary and subharmonic waves (see Fig. 5.17a).

However, Herbert (1983b, 1984a, 1988a) did not assert that the excitation by a plane T-S wave of two Sq waves really plays an important part in the development of three-dimensional structures in the Blasius boundary-layer flow; he only indicated that this mechanism must also be considered. In fact, results presented in his papers (1984a, 1988a) clearly show that interactions among triads of T-S waves play the dominant part in the development of three-dimensionality in boundary layers.[5] On

[5] However, the Squire waves also possibly made some contribution to the secondary disturbances computed by Herbert and his co-authors (such a possibility was explicitly mentioned by Crouch

the other hand, the assertion by Zel'man and Maslennikova (1993a) about the impossibility of strong excitation in a boundary-layer flow of oblique Sq waves by a plane T-S wave was not correct. The point is that, even earlier, Nayfeh (1985) proved that a strong interaction of a T-S wave with a pair of Sq waves is quite possible in the Blasius boundary layer. Slightly later, and independently, this results was confirmed by Zang and Hussaini (1990). These authors computed several solutions of N-S equations describing the downstream propagation, in a plane-parallel flow with Blasius velocity profile, of wave triads consisting of a linearly-unstable plane T-S wave and a pair of symmetric 3D Sq waves with half the streamwise wavenumber. Growth curves for amplitudes of one plane T-S and two oblique Sq waves determined by Zang and Hussaini had the same form as the growth curves in Figs. 5.17a–c, and thus clearly showed that a plane T-S wave may stimulate rapid growth of two symmetric Sq modes. The computations by Zang and Hussaini also showed that a resonance triad consisting of one T-S and two Sq waves produces in a boundary layer a vortical structure, which depends on the value of the plane-wave amplitude in exactly in the same way as was found in the experiments of Saric and Thomas (1984). However, Zang and Hussaini did not try to compare their results quantitatively with any experimental data relating to the N-regime of boundary-layer instability development. Therefore, their work cannot be used for a reliable determination of the physical mechanism which produced the N-regime of boundary-layer development observed in this or that specific experiment.

To identify this mechanism, it is necessary to use the results of comparisons of specific experimental data with the predictions of various theoretical models. Let us consider from this point of view the results shown in Figs. 5.13–5.17. All these figures illustrate the excellent agreement of the experimental results with the calculations. Among the theoretical models considered, those developed by Z-M were most often used to produce these figures. These models generalize Craik's model of a resonant triad comprising three T-S waves (one plane and two oblique, but the strict symmetry of the oblique waves and precise fulfillment of the resonance conditions are not now required). Excellent agreement of the model predictions with the observed data allows one to conclude that the general three-wave-resonance model describes one of the instability mechanisms which can produce the N-regime of disturbance development in boundary-layer flows. In other words, the Craik-type resonance among three T-S waves satisfying, exactly or approximately, the resonance conditions (5.7) may quite satisfactorily explain the observed features of the N-regime.

On the other hand, the good agreement of Herbert's (1984a) and Crouch and Herbert's (1993) computational results with experimental data, demonstrated by Figs. 5.15a, 5.16a and 5.17a, c show that secondary instability of a primary plane T-S wave with respect to a pair of symmetric oblique T-S (not Sq—Fig. 5.16a proves

and Herbert (1993) in their study of nonlinear development of secondary disturbances in boundary layers). It is also possible that some small Squire-wave contribution was present even in some of the computational results of Zel'man and Maslennikova; as was pointed out by E. Reshotko (personal communication) Squire waves sometimes appear quite unexpectedly in numerical solutions of the Navier-Stokes equations.

this quite definitely) waves may also be a mechanism leading to the development of 3D structures in boundary layers, as observed by several groups of experimenters. The fact that both the three-wave-resonance theory and the secondary-instability theory lead to results which equally well describe the available experimental data does not seem surprising. The point is that both theories relate to practically the same situation of downstream propagation of a triad of T-S waves approximately satisfying the resonance conditions. The only difference is that in the secondary-instability theory the amplitudes A_1 and A_2 of the two oblique waves are assumed to be much smaller than the plane-wave amplitude A_0, while in the three-wave-resonance theory these three amplitudes are assumed to be of the same order of magnitude (but both theories are restricted to cases in which all three waves have sufficiently-small amplitudes). In such situations it seems natural to suppose that there must be an intermediate range of ratios A_1/A_0 and A_2/A_0 within which both theories will be applicable. In principle, the secondary-instability theory must be considered as the more justified in cases where the plane T-S wave has already been growing for some time, so that its amplitude has reached a finite value, while oblique T-S waves have only recently been produced and therefore have very small amplitude; the opposite opinion seems natural in cases where all three waves have already been growing for some time and have more nearly equal amplitudes. However, it is known that in the physical sciences approximate equations very often turn out to be applicable over a wider range of conditions than those under which they were derived. So it is quite possible that the close agreement between the results of the secondary-instability and three-wave-resonance theories over a wide range of amplitude conditions is just one more illustration of this fact.

The numerical-simulation results shown in Figs. 5.16–5.17 also support the above statement that the N-regime of boundary-layer instability development is due to strong interaction among triads of T-S waves. Let us begin with Figs. 5.16a and 5.17a, which show excellent agreement between the results of the numerical simulation of Fasel et al. (1987), the experimental data of Kachanov and Levchenko (1982, 1984) and the theoretical work of H (1984a) and of Zel'man and Maslennikova (1990, 1993a). Recall again that the theory of Z-M is based on the assumption that the main features of the N-regime of boundary-layer development are due to the appearance in the flow of a resonant triad, comprising one plane T-S wave of relatively small amplitude and two symmetric oblique T-S waves of approximately half the frequency. Therefore, Fig. 5.16a apparently implies that both the numerical results of Fasel et al. and the theory of H also relate to situations where resonant triads including three T-S waves play the dominant role. In the case of the simulation data, this assumption also agreed well with the description of the computations. In fact, Fasel et al. considered the model of a laminar plane-parallel boundary-layer flow disturbed by vertical (normal to the wall) velocity oscillations produced by periodic blowing and suction of fluid through a narrow strip in the upstream part of the plate (see also the description of this disturbance model by Konzelmann et al. (1987), which may be compared to the description of five different models of this type by Berlin et al. (1999)). The vertical velocity fluctuations were represented in the simulation of Fasel et al. by the sum of a spanwise-independent component proportional to $\sin(\omega_0 t)$, and a spanwise-periodic component proportional to $\sin(\omega_1 t)\cos(k_2 y)$. It was assumed here that $\omega_1 = \omega_0/2$ while the values of ω_0, k_2 and the amplitudes of the two components of the disturbance could be varied. It is natural to expect that such disturbances will generate a plane T-S wave of frequency ω_0 and a pair of oblique T-S waves having frequency $\omega_0/2$ and opposite spanwise wavenumbers $\pm k_2$; moreover, the amplitude of the plane wave could be chosen within the amplitude range corresponding to the N-regime. However, it seems highly improbable that vertical velocity oscillations produced by blowing and suction of fluid could generate Squire waves, which have zero vertical velocity.

Note in conclusion that Ustinov (1994) also tried to compare some results that follow from three different theoretical models of the nonlinear development in a boundary-layer flow of resonant triads comprising three T-S waves. The models he considered were Craik's three-wave model leading to amplitude equation of the form (5.4), a DNS model based on numerical solutions of the N-S equations describing the downstream propagation of resonant T-S-wave triads (here the approximations applied by Ustinov (1993) to computations of a plane-channel flow were used), and Herbert's secondary-instability model (where Ustinov did not suppose that Sq waves would play any role). According to computations for the cases where $A_0 \gg A_1 = A_2$ (here A_0, A_1 and A_2 have the same meaning as above) Herbert's theory lead to results which agree very well with numerical solutions of N-S equations, while Craik's approach leads, if the initial amplitude of the 2D wave is not small enough, to results differing considerably from those of the other two models. These results apparently show that the question of the accuracy of different theories of the N-regime of boundary-layer instability development cannot be considered to have been fully answered at present.

Most of the results considered above in this section, and almost all the figures (Figs. 5.5a, b and 5.12 being exceptions), are related to the study of development in a boundary layer of plane and oblique Tollmien–Schlichting waves entering a resonant (but not necessary fully-resonant) wave triad. As to Figs. 5.5a, b and 5.12, they clearly show that a disturbed boundary layer usually includes not just one resonant wave triad but a great variety of disturbances of different types. Moreover, if the nonlinear development of disturbances is studied as the initial stage of laminar-flow transition to turbulence, then one has no right to consider only isolated wave triads, since such flow conditions are very far from real pre-transition situations. Therefore it is reasonable to mention here some other scenarios of disturbance development in a boundary layer which may also play a significant role in transition processes. Note, however, that there are many different scenarios which may be realized under one or another combination of flow conditions. Below, only a few typical examples of such scenarios will be briefly considered; some other examples (which are far from exhausting all the possibilities) will be considered in Sect. 5.6.

Let us first cite the study by Zel'man and Smorodsky (1990) of the influence of resonant interactions on the downstream propagation in a boundary layer of a narrow packet of three-dimensional T-S waves. However, this work will not be discussed here, since propagation of wave packets will be separately considered in Sect. 5.6, and for now attention will be paid only to disturbances consisting of a finite number

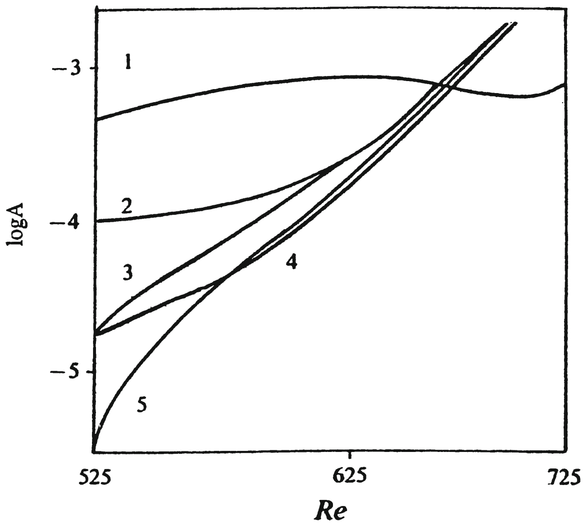

Fig. 5.18 Downstream-growth curves for amplitudes of five-wave disturbance system in a Blasius boundary layer. The system includes the plane wave *1* of frequency ω and wave vector $\{k, 0\}$ and oblique-wave pairs *2-3* and *4-5* with frequency-wave vector values $\{\omega/2, k_1, \pm k_2\}$ and $\{\omega/2, k_1{}^*, \pm k_2{}^*\}$ where $F = \omega\nu/U_0{}^2 = 230 \times 10^{-6}$, $K_2 = k_2\nu/U_0 = 0.171 \times 10^{-3}$, $K_2{}^* = k_2{}^*\nu/U_0 = 0.15 \times 10^{-3}$; $\mathrm{Re} = (U_0 x/\nu)^{1/2}$. (After Zel'man and Maslennikova 1993a)

of individual T-S waves. Recall in this respect that in Corke's (1990, 1995) experiments the development of a artificially-produced resonant triad was accompanied by the appearance of a great number of secondary waves. (In fact, many such waves were observed by Kachanov and Levchenko (1984) as well; see also Kachanov (1994a)). According to Corke, superposition of primary and secondary waves often included, in particular, the 'five-wave resonant systems' consisting of two 'resonant triads' (maybe of a detuned type) which both include the same primary 2D wave. Zel'man and Maslennikova (1993a) independently computed the time evolution of a 'five-wave resonant system' comprising a primary plane wave with frequency ω and wavenumber k, and two pairs of nonsymmetric oblique waves (i.e. having different initial amplitudes) with frequency-wavevector combinations $\{\omega/2, k_1, \pm k_2\}$ and $\{\omega/2, k_1{}^*, \pm k_2{}^*\}$. (In the above combinations, k_2 and $k_2{}^*$ are arbitrarily chosen parameters while k_1 and $k_1{}^*$ may then be computed with the help of the 3D O-S equation). Accurate determination of all interaction coefficients entering the five amplitude equations corresponding to this system, and subsequent numerical integration of these equations, allowed Z-M to determine the streamwise development of all five waves for different initial conditions and different values of the parameters Re, ω, k_2 and $k_2{}^*$ affecting disturbance development. Figure 5.18 shows a typical example of their results (the dependence of wave amplitudes on x is replaced here by their dependence on $\mathrm{Re}^+ = (U_0 x/\nu)^{1/2}$). One may see that, as in the results for one strictly-resonant wave triad shown in Fig. 5.3, the amplitudes of the four subharmonic 3D waves grow rapidly with x; moreover, their amplitudes quickly become almost equal and their growth curves cross the 2D-wave growth curve together.

Similar results were obtained by Zel'man and Maslennikova (1993b) for wave systems comprising more than five individual waves. Such systems may be used for modeling, by discrete wave combinations, the process of filling-in of low frequencies of the velocity-fluctuation spectra in a disturbed boundary layer (this process leads to the formation of the low-frequency band clearly seen in Figs. 5.5a, b). Moreover, results for many-wave systems are needed to describe Corke's (1995) observations of a great number of secondary, tertiary and quaternary 3D waves in a boundary-layer flow. Z-M began attempts to explain, by the weakly-nonlinear instability theory, the process of spectrum filling peculiar to the N-regime in their (1990, 1992) papers, and the work was continued in the (1993b) paper. They considered the case of a laminar boundary layer which is disturbed at time $t = 0$ by a plane, linearly-unstable T-S wave having frequency ω_0, wave vector $\mathbf{k}_0 = (k, 0)$, and very small amplitude. This wave will begin to grow in accordance with the results of linear stability theory. When the wave amplitude becomes large enough, interaction with the permanently-existing background noise will start, resulting in the extraction from the noise of two fast-growing secondary oblique T-S waves with frequency $\omega_1 \approx \omega_0/2$ and wave vectors $\mathbf{k}_1 = (k_1, k_2)$ and $\mathbf{k}_2 = (k_1, -k_2)$, where $k_1 \approx k/2$ (recall that Z-M considered only T-S, but not Sq, waves). During this stage of disturbance development the primary plane wave will continue to grow at a rate close to that given by the linear stability theory (which is much smaller than the growth rate of oblique waves). When the amplitudes of all three waves become approximately equal, the oblique waves will strongly affect the plane wave, leading to its explosive growth (cf. Fig. 5.3 where the evolution of one non-symmetrical resonant triad is shown). However, even before the beginning of the explosive growth of the primary 2D wave, but at some value of x where the oblique waves of the first order are already rather large, the evolved first-order waves will begin to excite two new pairs of symmetric oblique waves (again at the expense of the energy of the background fluctuations) having frequency $\omega_2 \approx \omega_o/4$ and wave vectors $\mathbf{k}_3 = (k_1', k_2')$, $\mathbf{k}_4 = (k_1', -k_2')$, $\mathbf{k}_5 = (k_1'', k_2'')$, $\mathbf{k}_6 = (k_1'', -k_2')$, where $k_1' \approx k_1'' \approx k_1/2$. These two pairs of the 3D waves of the second order will form, together with two oblique waves of the first order, two new resonant triads. Then the same process may be repeated with respect to waves of frequency ω_2 and so on. As a result of cascade transfer of the energy to 3D waves of lower and lower frequency will take place, filling the low-frequency part of the spectrum (of course, direct nonlinear interactions between all the generated waves will also contribute substantially to the filling of this spectral range).

For the locally plane-parallel model of a Blasius boundary layer Zel'man and Maslennikova (1993b) studied quantitatively the first two steps of the cascade process of spectrum filling. To do this, they determined a system of 7 differential equations for the amplitudes A_i, $i = 0, 1, \ldots, 6$, of 7 interacting waves: amplitude A_0 of the primary plane wave, amplitudes A_1, A_2 of two secondary oblique waves, and amplitudes $A_3, \ldots, A_6$ of the four tertiary 3D waves. One typical example of the computed dependencies of the amplitudes of these seven waves on $\mathrm{Re}^* = U_0\delta^*/\nu \approx 1.73(U_0 x/\nu)^{1/2} \propto x^{1/2}$ is shown in Fig. 5.19. Here it has been assumed for simplicity that $\omega_1 = \omega_0/2$ and $\omega_2 = \omega_0/4$, while the initial amplitudes $A_1, \ldots, A_6$ of the six oblique secondary and tertiary waves (normalized by division by the free-stream velocity U_0) were taken to be equal to 10^{-5} (this value represented, in the model considered, the relative intensity of background noise with a flat frequency spectrum, but it was found that the results were almost the same for a wide range of

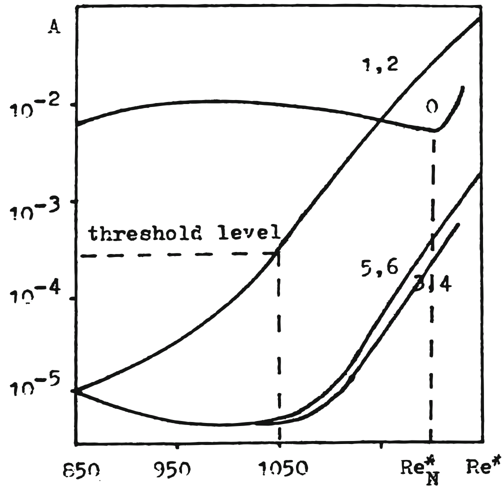

Fig. 5.19 Amplification curves for seven-wave system including the primary plane wave 0 with frequency-wave vector (f-w) combination $\{\omega_0, k, 0\}$, a pair of secondary oblique waves *1–2* with f-w combination $\{\omega_0/2, k_1, \pm k_2\}$ and two pairs of tertiary oblique waves *3–4* and *5–6* with f-w combinations $\{\omega_0/4, k_1', \pm k_2'\}$ and $\{\omega_0/4, k_1'' \pm k_2''\}$. Here $F_0 = \omega_0 \nu / U_0^2 = 122 \times 10^{-6}$, $k_2/k_1 = 2$, $k_2'/k_1' = 2.8$, $k_2''/k_1'' = 3.44$, $\mathrm{Re}^* = (U_0 \delta^* / \nu)^{1/2}$. (After Zel'man and Maslennikova 1993b)

these values) and the initial value of A_0 was chosen to be much greater than 10^{-5}. The computations were carried out for a number of values of ω_0, k_2'/k_1' and k_2''/k_1'' while k_2/k_1 was chosen to be equal to 2 which, according to Zel'man and Maslennikova (1990), is the value corresponding to maximal growth-rate of the amplitudes $A_1 = A_2$. Figure 5.19 shows the results for a specific value of ω_0 and for values of k_2'/k_1' and k_2''/k_1'' which lead to the fastest growth of the amplitudes $A_3 = A_4$ and $A_5 = A_6$ of the tertiary waves. The figure shows that subharmonic waves of frequency $\omega_0/2$ begin to grow from the moment of their appearance (corresponding, in the case considered here, to the value of x for which $\mathrm{Re}^* = 850$), while the primary-wave amplitude is almost unchanged at first (cf. similar results in Figs. 5.3, 5.6 and 5.10, where the results for resonant triads including only waves of frequencies ω_0 and $\omega_0/2$ were presented). As to the amplitudes of the tertiary subharmonics of frequency $\omega_0/4$, they even diminish slightly at first. However, beginning from a value of x corresponding to $\mathrm{Re}^* = 1{,}050$, when $A_1 = A_2$ reaches some threshold level, the amplitudes $A_3 = A_4$ and $A_5 = A_6$ also begin to grow rapidly (all amplitudes $A_1, \ldots, A_6$ are then growing approximately as exponential functions of the streamwise coordinate x and thus also of the time t) while the amplitude A_0 continues to change very slowly. Only later, where Re^* reaches a value $\mathrm{Re}_N{}^* \approx 1{,}200$, the amplitude of the primary plane wave begins to grow very rapidly (faster than exponentially) while all the subharmonics continue to grow exponentially with time. Z-M assumed that Re^*_N must be close to the empirical value of the Reynolds number, Re_{tr}, characterizing the transition of the boundary layer to turbulence, and they derived from this assumption some results relating to transition prediction; however in this chapter the later stages of transition to turbulence will not be discussed.

Zel'man and Maslennikova (1993b) also considered the amplitude equations for cases where a number of detuned (i.e., having frequency-ratios differing from the simple values 2 and 4 considered above) two- and three-dimensional waves of various amplitudes were introduced into the boundary-layer flow at some initial value of the coordinate x. In particular, they studied the case where the 2D T-S wave and two pairs of 3D waves, with frequencies and wave vectors of the form (ω_0, k_0, 0), (ω_1, k_1, $\pm k_2$) and ($\omega_0 - \omega_1$, k_1', $\pm k_2'$), were simultaneously introduced into a boundary layer flow, and investigated the dependence of the characteristics of the corresponding instability developments on the 'detuning parameter' $1–2\omega_1/\omega_0$. They also considered the development of a complicated wave system, comprising two 2D and ten 3D detuned waves close to those actually observed by Corke (1990). In this case they found many coincidences between the wave behavior given by their theory and that observed in the laboratory experiment. Some other results of Z-M allowed them to interpret, in a natural way, some observations by Yan et al. (1988) who also observed the cascade process of filling in the velocity-fluctuation spectrum in the course of instability development in a boundary layer. The methods used by Z-M can in principle be applied also to interpretation of Corke's (1995) results presented in Fig. 5.12, but the corresponding computations are rather complicated and apparently have not yet been carried out. Nevertheless, the results discussed above definitely show that the multimode weakly-nonlinear stability theory may be very useful for the quantitative theoretical description of many phenomena observed during the initial stage of transition of the boundary-layer flow from laminar to turbulent flow regime.

Let us now consider the investigation by Nayfeh and Bozatli (1979a) of the possibility that a primary plane T-S wave of frequency and wavenumer (ω, k) in a Blasius boundary layer can excite, by means of the principal parametric resonance of secondary-instability theory, a *plane*, 2D T-S wave with frequency and wavenumber close to half those of the primary wave. Recall that at the beginning of Sect. 5.1 it was indicated that nonlinear resonance may occur among two waves with frequency-wavenumber combinations (2ω, $2\mathbf{k}$) and (ω, $\mathbf{k}$); therefore, in principle, such resonance in a Blasius boundary layer seems to be probable. Moreover, since the two 2D waves considered will have critical layers which are close to each other, it seems natural to expect that their nonlinear interaction will be rather powerful. Nayfeh and Bozatli analyzed the spatial development of disturbances, i.e., they considered the primary wave with real frequency ω and with wavenumber k which may be complex, and assumed that the 2D wave excited by the primary wave has the frequency-wavenumber combination ($\omega/2$, k_1) where k_1 may also be complex but is such that $\Re\mathrm{e}(k/2 - k_1) = \Delta k_1$ is a small detuning parameter. (Frequencies and wavenumbers are assumed here to be made dimensionless by using the displacement thickness δ^* and free-stream velocity U_0 as length and velocity scales). To compute the interaction between the primary and secondary waves the authors used the *method of multiple scales* (see the book by Nayfeh (1981) for a description of this method and a number of its applications). The computations were performed for three values of the dimensionless frequency $F \times 10^6 = \omega\nu/U_0^2$, namely 60, 52 and 40, and a wide range of Reynolds numbers. However, the results were rather disappointing: they showed that to trigger the parametric instability in a Blasius boundary layer and achieve rapid growth of the secondary 2D wave, the amplitude (peak value) of the primary plane wave must exceed a critical value close to 29 % of the free-stream velocity U_0. Since it is known that in a boundary layer secondary instabilities of many other types become significant at considerably smaller amplitudes of the primary wave, it became clear that this instability mechanism cannot play a significant role.

Later Healey (1994, 1995, 1996) turned anew to the study of a possible two-wave resonance between a pair of two-dimensional waves in a Blasius boundary layer with frequency-wavenumber combinations (ω, k) and (ω_1, k_1), where ω_1 and k_1 have real parts twice as large as those of ω and k. (Note that the subscript 1 now refers to the wave with larger frequency and wavenumber). Healey somewhat changed Nayfeh and Bozatli's problem formulation by admitting that both parameters ω and k (and naturally ω_1 and k_1 too) may take complex values. Recall that Nayfeh and Bozatli assumed that ω and ω_1 are real, $\omega_1 = 2\omega$, while k and k_1 are complex and such that the real part of k_1 is close to twice the real part of k. The assumptions used allowed Nayfeh and Bozatli to choose values of ω and Re almost arbitrarily; moreover, they spoke only about *closeness* of the real parts of k_1 and 2k, since in 1979 it was believed that at real values of ω and $\omega_1 = 2\omega$ the condition $\Re e k_1 = 2\Re e k$ could not be satisfied exactly. As will be explained below, it was found recently that this assumption is incorrect, but this discovery does not invalidate Nayfeh and Bozatli's reasoning.

Nayfeh and Bozatli used the traditional spatial formulation of the stability problem inspired by the experiments of Schubauer and Skramstad (1947), and many of their followers, where a plane wave of fixed frequency was artificially produced in the initial part of a laminar boundary layer, and the subsequent development of this wave and any further instability phenomena generated by it were studied. The admittance by Healey of complex values for both the frequency and the wavenumber clearly expanded considerably the class of plane waves considered, and simultaneously forced Healey to change the resonance conditions, giving them the form of two equalities: $\Re e\omega_1 = 2\Re e\omega$ and $\Re e k_1 = 2\Re e k$. Augmenting the set of waves considered of course meant that a new physical situation, which led to a new stability problem, was being studied. Healey's problem formulation corresponded to the case where the primary plane wave had an amplitude which was not constant but was modulated as $A(t) = A_0 \exp(-\omega^{(i)} t)$. To illustrate the importance of the instability phenomena produced by such a wave, Healey referred to the remark by Gaster (1980) who pointed out that the amplitude threshold above which a flow disturbance leads to boundary-layer breakdown and transition to turbulence is often several times lower in the case of a modulated wave-packet disturbance than in the case of a disturbance having the form of a sinusoidal plane wave. Healey also noted the results of subsequent experiments by Shaikh and Gaster (1994) on randomly-modulated wavetrains, which again showed that modulation enhances the nonlinear effects of a disturbance. These facts stimulated Healey's study of the instability of a boundary layer disturbed by an amplitude-modulated wave.

Healey (1994) investigated whether there exist complex eigenvalues $k = k^{(r)} + ik^{(i)}$ and $k_1 = k_1^{(r)} + ik_1^{(i)}$ of two O-S eigenvalue problems (2.44), (2.42) (where $c = \omega/k$ and $U(z)$ is the Blasius velocity profile) with complex parameters $\omega = \omega^{(r)} + i\omega^{(i)}$ and $\omega = \omega_1 = \omega_1^{(r)} + i\omega_1^{(i)}$ respectively, satisfying the condition $\omega_1^{(r)} = 2\omega^{(r)}$, which are such that $k_1^{(r)} = 2\mathrm{k}^{(r)}$. Performing some complicated computations, Healey showed that such pairs (ω, k) and (ω_1, k_1) exist at all high enough values of Re, and that it is also possible to find more special pairs (ω, k) and (ω_1, k_1) of complex frequency-wavenumber combinations where not only $\omega_1^{(r)} = 2\omega^{(r)}$ and $k_1^{(r)} = 2\mathrm{k}^{(r)}$ but even $\omega_1 = 2\omega$ and $k_1 = 2\mathrm{k}$. In particular, Healey found that at $\mathrm{Re}^* = 2{,}100$ the latter equalities are valid if $\omega = 0.04318 - 0.01819i$ and $k = 0.1433 - 0.0600i$ (here as usual $\delta*$ and U_0 are used as length and velocity units). In Healey's (1995, 1996) papers many results supplementing those given by Healey (1994) are presented. In particular, in the paper (1995) he analyzes the dependence on Reynolds number of the location in the complex plane of the resonant pairs (ω, k) and (2ω, $2k$), and also shows that if the condition $k_1 = 2k$ is replaced by the less restrictive condition $k_1^{(r)} = 2k^{(r)}$, then it is possible to satisfy the condition, together with the condition $\omega_1 = 2\omega$, by a combination of real ω and complex k and k_1. (For example, at $\mathrm{Re}^* = 2{,}000$ these conditions are satisfied for $\omega = 0.0817$, $k = 0.256 - 0.0101i$, $\omega_1 = 2\omega = 0.1634$, $k_1 = 0.512 + 0.225i$). However, the results relating to resonant wave pairs with a real value of ω (i.e., corresponding to the traditional problem of spatial wave development) do not completely undermine the early belief that such pairs do not exist in practice. The point is that in the wave pairs found by Healey one of the two frequency-wavenumber combinations considered necessarily belongs to the higher-order O-S eigenvalues describing rapidly-damped higher modes, which earlier were never taken into account. Nor do the new results contradict those of Nayfeh and Bozatli (1979a), since Healey showed only that there exist pairs of waves for which resonance interaction is in principle possible, but said nothing about the efficiency of this interaction. At the same time it seems physically doubtful that interactions including higher-order modes may really play an essential part in boundary-layer instability development.

Healey also considered the equations for the complex amplitudes $A_1(x)$ and $A_2(x)$ of two two-dimensional waves, with complex values of ω and k satisfying the conditions given above for resonance interaction to be possible. According to his results these equations, accurate to the order of the quadratic nonlinearities, have the form

$$\frac{dA_1}{dx} = -k^{(i)}A_1 + b_1 A_1^* A_2, \qquad \frac{dA_2}{dx} = -k_{10}^{(i)}A_2 + b_2 A_1^2 \tag{5.15}$$

where b_1 and b_2 are the interaction coefficients corresponding to the situation considered. (Here again nonvanishing of these coefficients shows that the interaction is a resonant one). For details of the derivation of Eq. (5.15) and evaluation of their coefficients see Healey (1995), where a small wavenumber detuning of two waves is also allowed (cf. also Dangelmayr (1986)). Some of the results implied by these equations were verified by Healey in some specially-arranged wind-tunnel experiments where development of modulated waves, and also the influence of the phase difference between two waves (which according to Eq. (5.15) must be rather significant) were measured. The experimental data confirmed, to sufficient accuracy, the theoretical results (including, in particular, the detection of resonances under just the conditions indicated by the theory). However, the full clarification of the role of modulated waves in real boundary-layer breakdown and transition processes evidently requires much further work.

Let us now continue the description of the work by Nayfeh and Bozatli. In their papers (1979b, 1980) these authors used the method of multiple scales to study the nonlinear interactions between two 2D T-S waves of different frequencies and wavenumbers, and also between three such waves of frequencies ω_1, $\omega_2 > \omega_1$, and $\omega_2 - \omega_1$. They found that a 2D wave of moderate amplitude has little influence on its subharmonic: that is, a plane wave of frequency ω and moderate amplitude cannot generate a fast-growing wave of frequency $\omega/2$ (this result clearly confirms the conclusion of the paper 1979a). However, a plane wave has a strong influence on its second harmonic, so a moderate-amplitude wave of frequency ω may generate a secondary wave of frequency 2ω. Moreover, waves of frequencies ω_1 and ω_2 have a strong influence on a wave of frequency $\omega_2 - \omega_1$, often making it unstable (i.e., growing in time). Many of Nayfeh and Bozatli's results were verified in experiments by Saric and Reynolds (1980) in which a vibrating ribbon in a boundary layer was used to excite either one plane wave of fixed frequency ω_1 or two plane waves of frequencies ω_1 and ω_2. (This experiment was stimulated by the similar one by Kachanov et al. (1980) where oscillations of two frequencies were introduced in a boundary layer by two separate ribbons). In particular, the experiments showed that a primary plane wave of frequency ω_1 may generate a plane wave of frequency $2\omega_1$ with an amplitude approximately twice that of the primary wave, but no cases of generation of subharmonic waves with frequency $\omega_1/2$ were detected. When waves of two frequencies ω_1 and ω_2 were introduced into the flow, secondary waves of frequencies $\omega_1 - \omega_2$ (and also $2\omega_1 - \omega_2$) were detected, but the streamwise development of their amplitudes did not follow the predictions of Nayfeh and Bozatli. The experimental data agreed satisfactorily with some of Nayfeh and Bozatli's theoretical results but strongly disagreed with others; hence revision of the theory seems necessary. However, this subject will not be discussed further here, since all the instabilities considered in Nayfeh and Bozatli's paper led to much smaller growth rates than those corresponding to the three-wave resonances, and therefore these instabilities can hardly play an important part in transition of a boundary layer to turbulence.

Still later, Nayfeh (1985) showed that if a two-dimensional primary T-S wave in a Blasius boundary layer is disturbed by a single secondary T-S wave which has a frequency and streamwise wavenumber equal to half of those of the primary wave but is three-dimensional, with spanwise wavenumber k_2 larger than some small critical value, then the principal parametric resonance become very effective and leads to fast growth of the secondary wave. This results clearly agrees well with those considered earlier in this subsection.

The secondary-instability problem considered by Nayfeh (1985) (and also those studied by Herbert 1983b, 1984a; Herbert et al. 1987; Bertolotti 1985) deals with the principal parametric resonance in a boundary-layer flow, which leads to the appearance of subharmonic 3D waves and of the staggered vertical structure shown in Fig. 5.8b. Recall now that in the K-regime of the evolution of a disturbed boundary layer observed by Klebanoff and his co-authors an ordered, and not staggered, vortical structure was observed. Trying to simulate this regime, Nayfeh and Bozatli (1979c) (see also Nayfeh 1987a, b) introduced a four-wave instability model. In the Nayfeh-Bozatli (N–B) model four different O-S waves interact with each other in a boundary-layer flow: they are the primary plane wave with frequency and wave vector (ω, k, 0); its second harmonic, a plane wave of frequency 2ω and wave vector ($k_1 \approx 2$k, 0); and two oblique waves with frequency and wave vectors (ω, k, $\pm k_2$). The downstream propagation of the N–B wave system was analyzed by Zel'man and Maslennikova (1984, 1989, 1993a), who determined the values of all interaction coefficients of the corresponding system of four amplitude equations, and performed numerical integration of this system for a number of initial conditions. Here, in fact, two different resonances are simultaneously realized—resonant growth of the second harmonic stimulated by the first one, and fast growth of two oblique waves, produced by Craik's three-wave resonance interaction of the amplified second harmonic with a pair of oblique waves of half the frequency. The spatial development of the four-wave system leads to generation of an ordered system of vortices, sketched in Fig. 5.8a and typical of the K-regime of instability development. Since in this section we are discussing only the N-regime, and not the K-regime, the N–B model will not be considered here in any detail (but it will be mentioned in the next Section, Sect. 5.5, devoted to study of the K-regime).

5.5 Weakly-Nonlinear Instabilities in the K-Regime of Boundary-Layer Development

The K-regime of the boundary-layer instability development was discovered and explored in the late 1950s and early 1960s by Klebanoff and his co-authors. In this book, these results were very briefly considered in Sect. 2.1 (where even the name 'K-regime', which marks Klebanoff's contribution, was not mentioned) and, in a little more detail, were presented in the beginning of Sect. 5.2 with a brief mention in the beginning of Sect. 5.3. Below some recent studies of this regime will be described at greater length; therefore, it is appropriate to make here some additional remarks about its main features.

In Sect. 5.2 it was indicated that Klebanoff et al. (1962) studied the downstream evolution of the three-dimensional structures which, according to the results of a number of earlier investigations, regularly appear in a laminar boundary layer at some distance downstream of a vibrating ribbon exciting a linearly-unstable two-dimensional T-S wave. Since Klebanoff et al. were interested first of all in the spatial development of these 3D structures, they artificially generated a weak spanwise periodicity of the amplitude of the ribbon vibrations (with the same spanwise wavelength λ_y which was earlier observed in boundary layers excited by ribbon vibrations with y-independent amplitude). Then the amplitude of the y-periodicity of the streamwise disturbance velocity was measured at different values of the streamwise coordinate x. The results (some of which are shown in Fig. 5.20) showed that the spanwise modulation of the disturbance velocity grows rapidly with x, producing a specific peak-valley wave structure with a constant 'fundamental spanwise wavelength' λ_y. As was indicated in Sect. 5.3, it was later found by other authors that this structure consists of a strictly ordered collection of streamwise 'Λ-vortices', and that two

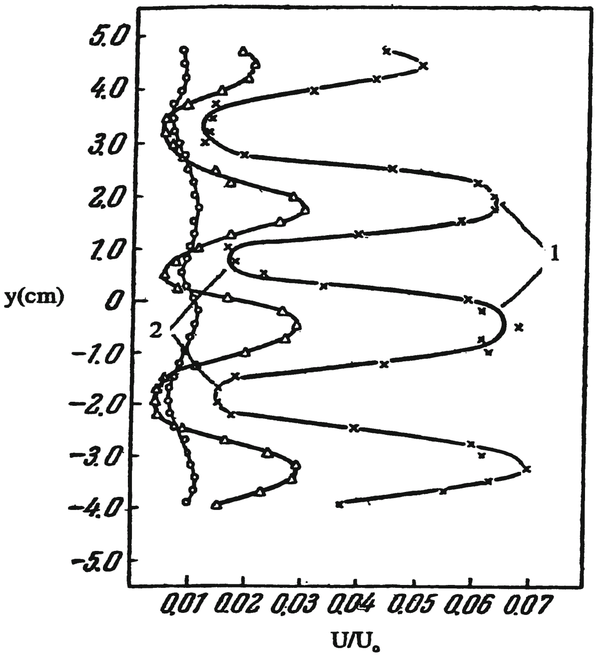

Fig. 5.20 Downstream growth of spanwise modulation of the amplitude u' of streamwise disturbance velocity in a boundary layer disturbed by vibrating ribbon. o – data for $x = 7.6$ cm; – for $x = 15.2$ cm; × – for $x = 19$ cm, where x is measured from the trailing edge of the ribbon; *1* – modulation 'peaks', *2* – 'valleys'. (After Klebanoff et al. 1962)

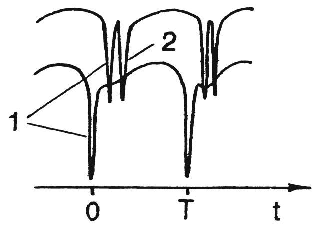

Fig. 5.21 Typical single and double spikes in a boundary layer flow (after Klebanoff et al. 1962). *1* – 1st spike; *2* – 2nd spike; T – fundamental period of spike repetitions

quite different orderings of vortices are realized in the cases of large and small amplitudes of the initial two-dimensional T-S wave (see Figs. 5.8a, b and the related text). Klebanoff et al. considered only the case of relatively large initial T-S waves, and therefore they dealt only with a regularly-ordered vortical structure of the type shown in Fig. 5.8a (but they did not use flow visualization and therefore could not observe the ordering of the vortices). However, their hot-wire measurements showed that, in the case of large T-S wave amplitude, short-duration bursts of high-frequency velocity oscillations regularly appear at spanwise maxima of the velocity distribution (and are then repeated in each period of the primary T-S wave). These high-frequency bursts were called "spikes" by Klebanoff et al. since some spikes are seen in the traces of disturbance velocity against time (see Fig. 5.21).

In the course of their downstream evolution, the spike structures are doubled (see Fig. 5.21 again), then tripled and so on. Klebanoff et al. associated the spikes with the formation of a family of small hairpin-shaped vortices, produced by the inflectional instability of high-shear layers formed around the large-scale vortical structures. The 'legs' of hairpin vortices may be gradually converging in the course of their evolution; this process may explain, in particular, the formation of 'ring vortices' which are also sometimes observed in the later stages of the K-regime. According to Klebanoff et al. a breakdown of medium-size vortices into smaller and still smaller vortical structures leads at first to the appearance of spikes and then to transformation of spikes into wholly irregular "turbulent spots" which are the precursors of the final transition to turbulence. Because of the connection with irregular turbulent spots the spikes were long regarded as irregular ("random") embryos of the future spots. Moreover, in accordance with the point of view of Klebanoff et al., it was long assumed that spikes arise from inflections of the disturbed Blasius velocity profile, local in time and space. In fact, it can be shown that near the inflection points produced by a low-frequency disturbance, a strong instability to high-frequency oscillatory disturbance must develop (this statement was due to Betchov (1960), and its support by Klebanoff et al. (1962) made it quite popular afterwards; see, e.g., numerous references to subsequent studies of the 'local high-frequency secondary instability' (briefly, LHSI) in reviews by Nayfeh (1987a) and Kachanov (1991a, 1994a) and the paper by Kachanov et al. (1993)). However, as will be explained later in this section, in the late 1980s and early 1990s it was discovered that spikes apparently have quite another origin; they are not 'wholly irregular', and their transformation into "turbulent spots" does not occura to once but only after some specific intermediate stages. Moreover, some more recent results of experiments and numerical simulations give the impression that the origin of ring vortices may differ from that sketched above; this topic will also be discussed later in the present section.

Among the first experimental results relating to the K-regime were those of Nishioka et al. (1980); Nishioka and Asai (1985a, b) and Asai and Nishioka (1989) who performed detailed measurements of the instability development in a plane-channel flow. This flow is usually modeled as plane Poiseuille flow, but it has many features similar to those of Blasius boundary-layer flow. In particular, Nishioka and his coworkers found that a strictly ordered collection of streamwise 'Λ-vortices' followed by 'spikes' may be observed in plane-channel flow too, while in similar experiments by Kozlov and Ramazanov (1981, 1983, 1984a, b) and Ramazanov (1985) it was shown that plane-channel flow may also undergo K- and N-regimes of instability development. Nisshioka et al., studying the K-regime of disturbance growth in a channel flow, obtained the first experimental corroboration of the fact that LHSI may really take place during the K-regime of the channel-flow development. However channel-flow instabilities will be considered at length in the next chapter, Chap. 6[6]; so now we will pass to the original research and survey papers by Kachanov et al. (1984, 1985, 1989); Borodulin and Kachanov (1988, 1989, 1994, 1995); Kachanov (1987, 1990, 1991a, 1994a, b); Dryganets et al. (1990); Bake et al. (1996, 2000); Lee (1998, 2000) and Lee et al. (2000) (see also the recent books by Boiko et al. 1999; Schmid and Henningson 2001) where many results of recent experimental studies of the K-regime in boundary layers are presented.

[6] Editors addendum: before he died, Prof. Akiva Yaglom was working on Chapter 6.

The experiments described and discussed in the above-mentioned papers and books were often (but not always) based on the technique which was first used much earlier by Klebanoff et al. (1962). Here again the laminar boundary layer on a flat plate placed in a wind tunnel was disturbed by a spanwise-oriented vibrating ribbon, and simultaneously a weak spanwise nonhomogeneity of the resulting disturbance was artificially produced by an array of identical pieces of tape placed beneath the ribbon. However, in contrast to the earlier experiments by Kachanov et al. (1977, 1978, 1980), in all the experiments considered here the amplitude of ribbon fluctuations was chosen to be so great that it guaranteed the realization of the K (and not N) regime of disturbance development. Moreover, the authors tried to make the experimental conditions as close as possible to those of the experiment by Klebanoff et al. (1962). But the new experiments differed from the early studies of Klebanoff's group by more sophisticated measurement techniques and by more careful investigation of the frequency (and spanwise-wavenumber) composition of the velocity fluctuations at various points $\mathbf{x} = (x, y, z)$ of the boundary layer.

Passing to the consideration of these more recent experimental studies of boundary-layer instability development, one must note first of all the results of Borodulin and Kachanov (1988). These authors showed that LHSI does in fact occur in boundary layers but leads to some special nonlinear effects, which must be distinguished from the production of spikes. The point is that spikes usually appear at considerably greater 'height' (distance from the wall) than the velocity-profile inflection (which is expected to be the site of any quasi-inviscid instabilities) and have amplitudes exceeding those of LHSI-produced formations. Borodulin and Kachanov (1988) (see also the subsequent discussion of their results in the surveys by Kachanov 1990, 1991a, 1994a, b; Borodulin and Kachanov 1994, 1995 and the theoretical papers by Zel'man and Smorodsky 1991a, b; Kachanov et al. 1993) often observed both types of nonlinear formations at the same values of coordinates x and y but quite different values of the vertical coordinate z. The lower formations were always observed just at the heights of velocity-profile inflections, and the measurements agreed very well with theories of local high-frequency secondary instability (see e.g. the discussion of this matter in Kachanov et al. 1993 and the papers cited there by Smith, and by Smith and co-authors on this subject). However the spikes (whose importance for boundary-layer instability development was demonstrated quite early Klebanoff et al. 1962) certainly have an origin unrelated to LHSI.

Spectral analysis of velocity fluctuations performed by Kachanov and his co-authors showed that in the K-regime of boundary layer development numerous higher harmonics of the primary 2D wave, with frequencies $\omega_n = n\omega$, $n = 2, 3, \ldots$, and values of n up to several tens, always exist in the flow together with the oscillations of frequency $\omega_1 = \omega$ equal to that of the ribbon vibrations and of the primary plane wave produced by them. Thus, an amplitude of the primary wave larger than that leading to the N-regime leads to an intensity of high-harmonic generation much greater than in the N-regime. Klebanoff et al. did not observe so many higher harmonics of 2D velocity oscillations and did not pay much attention to them, but according to Kachanov et al. (1984, 1985, 1989) these harmonics are highly important in the K-regime. Therefore, the latter authors concluded that Klebanoff et al. underestimated

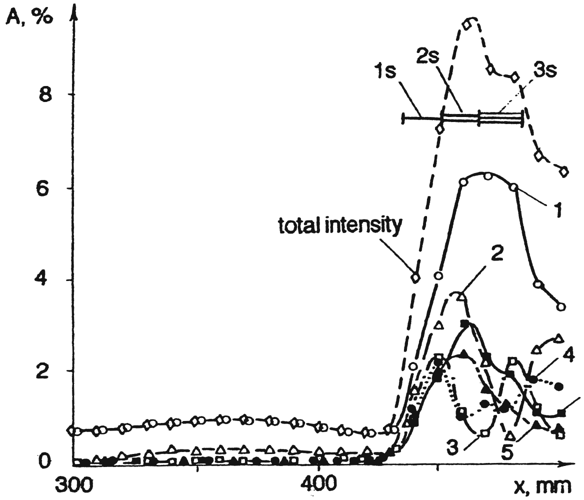

Fig. 5.22 Amplification of total intensity of streamwise disturbance velocity $u(\mathbf{x}, t)$ and amplitudes of its harmonics with frequencies $\omega_1, \omega_2, \ldots, \omega_6$ (*symbols* and *curves 1, 2, ..., 6*; ω_1—fundamental frequency of primary T-S wave, $\omega_n = n\omega_1$) observed at y corresponding to peak position of spanwise modulation, fixed value of z and variable x-coordinate. Streamwise intervals *1s*, *2s*, *3s*—places of formation of the 1st, 2nd and 3rd spike. (After Kachanov et al. 1984; Kachanov 1994a)

the role of higher harmonics of the primary wave. However, Rist and Fasel (1995), who performed careful numerical simulation of the K-regime of boundary-layer instability development as observed by Kachanov et al. (using in this simulation the same model of disturbance generator as that used by Fasel et al. (1987), whose work was discussed in Sect. 5.4), disagreed with the above-mentioned conclusion. Rist and Fasel indicated that although Kachanov et al. tried to repeat experiment by Klebanoff et al. very accurately, there were nevertheless some small differences in experimental conditions. These differences led, in particular, to a considerable greater initial value (measured just downstream of the vibrating ribbon) of the ratio A_1/A_0 of the 3D-wave amplitude to that of the primary 2D wave in the experiments of Kachanov et al. than in the similar experiments of Klebanoff et al. This explains why fewer higher 2D harmonics were significantly excited in the experiments of Klebanoff's group, and there these harmonics really were of somewhat smaller importance. However, almost all the experimental results of Kachanov et al. (1984, 1985) were confirmed, with high accuracy, by Rist and Fasel's numerical-simulation data (see also the papers by Rist and Kachanov (1995) and Rist (1996), where some supplementary numerical data are presented).

Kachanov et al. measured downstream-growth curves for the amplitudes of various higher harmonics of the primary oscillation, and found that these amplitudes begin to grow rapidly at approximately the same value of x at which the primary-plane-wave amplitude begins to grow faster than predicted by linear stability theory. The streamwise growth of amplitudes of the primary wave and its higher harmonics is arrested (and is sometimes replaced by a decrease) just in the region where spikes appear (see Fig. 5.22 where data for the spatial growth of the first 6 harmonics are presented together with data relating to growth of the total disturbance intensity; amplification curves for 17 harmonics may be found in Borodulin and Kachanov

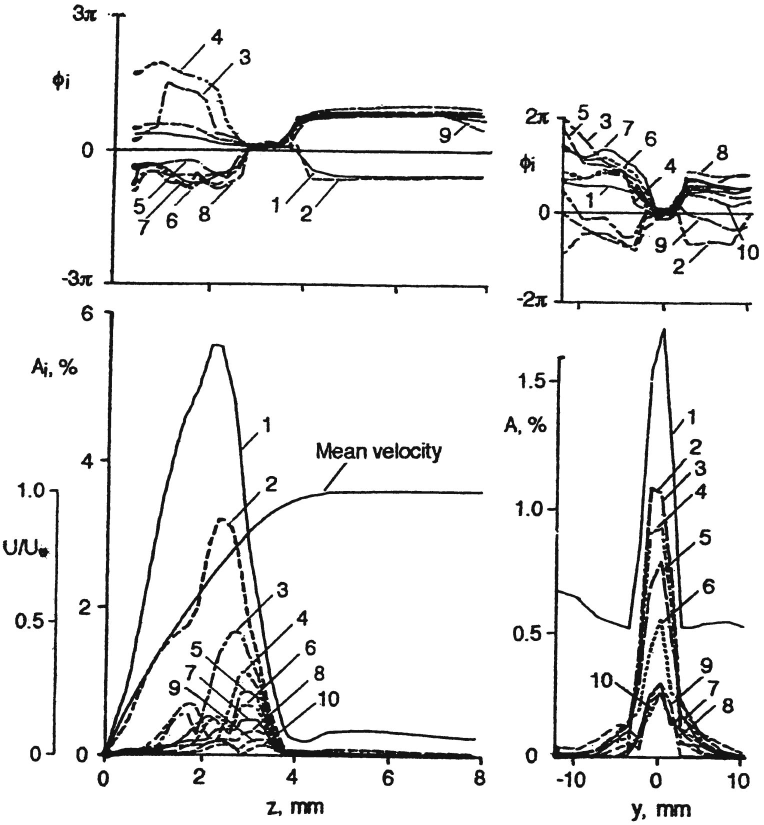

Fig. 5.23 Vertical (*left*) spanwise (*right*) profiles of the amplitudes $A_i =$ (*bottom*) and phases ϕ_i (*top*) of streamwise-velocity harmonics with frequencies $\omega_1, \omega_2, \ldots, \omega_{10}$ at the stage of developed spikes (*curves 1, 2, ..., 10*). (After Borodulin and Kachanov 1992; Kachanov 1994a). Mean-velocity profile is added to vertical amplitude profiles to show the boundary-layer thickness

1988; Kachanov 1994a). According to Kachanov et al., the spike, i.e. the short-term highly-localized outbreak of high-frequency oscillations, is produced, not by a sudden rapid increase of intensity of all higher harmonics, but by phase synchronization, in a small range of y and z, of all harmonics shown in Fig. 5.23, leading to strong amplification of the observed oscillations. (As indicated above, the spanwise coordinate y of a spike was found in all cases to be close to a peak position of the spanwise wave shown in Fig. 5.20). Therefore, the new theory considered spikes not as random formations but as regular structures, naturally produced by deterministic evolution of the Fourier composition of the upstream flow disturbances. The experimental data shown in Figs. 5.22 and 5.23 were later confirmed by observations of the Novosibirsk group, and also agree very well with the results of thorough direct numerical simulations of the K-regime by Kloker (1993); Rist and Fasel (1995) and

Rist (1996) (see also the paper by Rist and Kachanov 1995 where new numerical-simulation results were compared with the new measurements from Novosibirsk).[7] The regular character of spike structures was also confirmed in careful experiments (which will be discussed in Sect. 5.52) by Breuer et al. (1997) devoted to study of development of some localized disturbances in a boundary layer; see also the survey paper by Bowles (2000). Thus, the data presented in Figs. 5.22 and 5.23, and the new explanation of the origin of spikes following from them, may now be considered as reliable.

Using the data of the Novosibirsk experiments (of which Figs. 5.22 and 5.23 represent only a small part), Kachanov (1987) (see also his papers 1990, 1991a, 1994a, b) proposed a wave-resonance theory of the K-regime of boundary-layer instability development. This theory assumes that K-regime leads to the emergence of a cascade of successive four-wave resonances, generalizing the four-wave resonance studied by Nayfeh and Bozatli (briefly, N–B) in the paper (1979c). Recall that N-B resonance includes 2D and 3D waves with frequency-wavevector combinations (2ω, k, 0), (ω, k', 0), and (ω, k'', $\pm k_2$), where $k \approx 2k'$, $k'' \approx k'$, and $k_2 = k_0$ corresponds to the spanwise periodicity of 3D disturbances observed in experiments by various authors (i.e., to the fundamental wavelength λ_y of spanwise waves seen in Fig. 5.20). It was also noted in Sect. 5.4 that this resonance generates the vortical system typical of the K-regime of boundary-layer development. According to Kachanov, there is a cascade of resonances leading to the rapid growth of 3D structures in the K-regime comprising resonances among quadruples of waves with frequencies and spanwise wavenumbers ($n_1\omega$, 0), ($n\omega$, 0) and ($n\omega$, $\pm mk_0$). Here n_1, n and m are integers, $n_1 = 1, 2, 3, \ldots, n \approx n_1/2$, $mk_0 \approx k_2$, where k_2 is the spanwise wavenumber corresponding to spanwise periodicity of small-scale disturbances while k_0 is the 'primary' (or 'fundamental') spanwise wavenumber mentioned above, describing the spanwise waves which appear in experiments, either naturally or as the result of artificial disturbances such as pieces of tape under a vibrating ribbon.

These wave quadruples may be produced by nonlinear interactions of waves of the same type but with smaller values of n_1, n and m. The ensuing interactions among the quadruples may be resonant and similar to those taking place in the case of an N-B quadruple where $n_1 = 2$ and $n = m = 1$. Kachanov (1987) showed that the above-mentioned cascade of resonances may lead to the appearance of spikes at the locations where they were actually observed in experimental studies of the K-regime. Slightly later, a numbers of waves which may participate in 'Kachanov's resonances' were identified in observations of the K-regime by Borodulin and Kachanov (1989, 1994) (see also Kachanov et al. 1989; Kachanov 1990, 1991a, 1994a, b). Borodulin and

[7] In Sect. 5.4, attempts by Wray and Hussaini (1984); Zang and Hussaini (1985, 1987, 1990). Murdock (1986); Laurien and Kleiser (1989); Kleiser and Zang (1991); Zang (1992) and some others to simulate numerically the boundary-layer instability development were mentioned. These papers contain a number of results relating to the K-regime of such development and almost all of them agree satisfactorily with available experimental data. However, these results are less accurate and less complete than those by Rist and Fasel (1995) and Rist and Kachanov (1995); therefore results of the earlier numerical simulations of the K-regime will not be considered in this book.

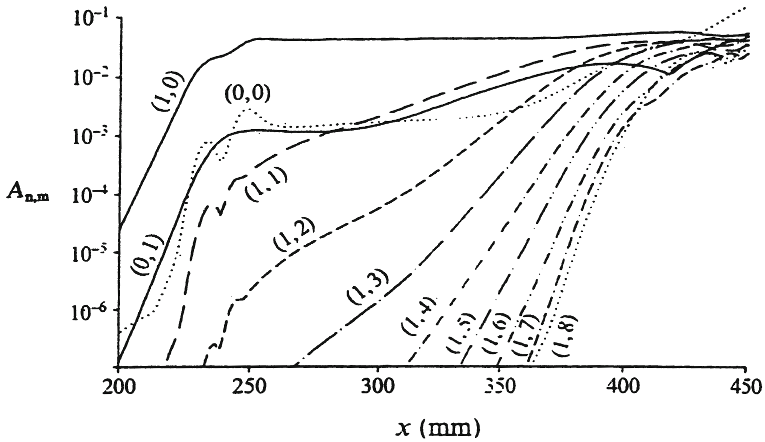

Fig. 5.24 Streamwise development of values of amplitudes $A_{n,m}$ of (n, m)-Fourier components of velocity $u(\mathbf{x}, t)$ at the heights z where these amplitude take maximal values (after Rist and Fasel 1995) (n, m)-Fourier component corresponds to frequency $\omega_n = n\omega_1$ and spanwise wavenumber $k_{2,m} = mk_0$ (where ω_1 – fundamental frequency of primary waves, k_0 – wavenumber of the fundamental spanwise periodicity $(0, 1)$ shown in Fig. 5.21). Symbols $(0, 0)$ and $(1, 0)$ correspond to amplitudes of the 'mean flow correction' and 'primary 2D wave'

Kachanov found that for some Kachanov's wave quadruples, the phase velocities of the four waves were quite close to each other, making strong four-wave interaction quite probable. They also stated that at $n = 1$ and 2 the most rapid growth of the oblique waves present in these quadruples is reached for $m \approx 4$ to 7. Kachanov's cascade of resonances clearly fills the high-frequency and high-wavenumber parts of the frequency and spanwise-wavenumber spectra (these parts correspond to small-scale oscillations of 'spike type') but it does not generate 'genuine subharmonics' corresponding to large-scale oscillations.

Kachanov's wave-resonance theory did not seriously contradict the numerical-simulation results of Rist and Fasel (1995) who found that the higher spanwise harmonics of the primary 2D wave, which correspond to the 3D $(n\omega_1, \pm mk_0)$-modes with $n = 1$, $m = 1, 2, \ldots, 8$, appear in the flow successively and then begin to grow rapidly with downstream distance x, while their initial growth rates increase with m, reaching a maximum for $m = 7$ and 8 (see Fig. 5.24; supplementary data may be found in Rist (1996), where similar growth curves are also given for some (n, m)-modes where $n = 0$ or 2). This figure shows that all the modes considered reach approximately the same saturation level at $x = 420$ mm, which is close to the position where spikes first appear. However, a more thorough treatment of the results of a subsequent, more refined numerical simulation of the same type carried out by Rist led to a conclusion differing from that formulated above. New numerical results, presented in Rist and Kachanov (1995) and Rist (1996), give a clearer picture of the flow than that derived from previous experiments and simulations. As pointed out in these papers, the new results showed large amplification rates of spanwise modes

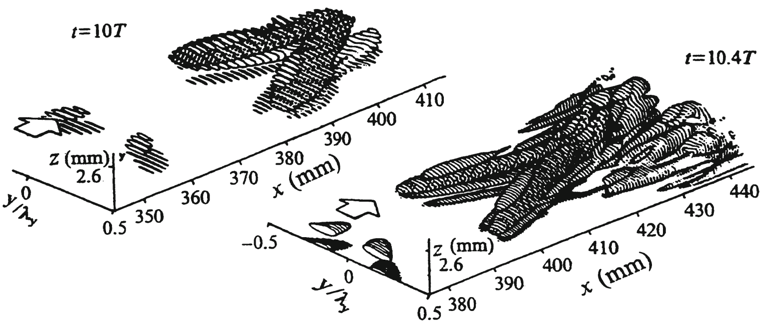

Fig. 5.25 Two examples of Λ-vortices appearing at two time instants in the numerical simulation of instability development in a flat-plate boundary layer by Rist and Fasel. The three-dimensional Λ-shaped structures are bounded by surfaces $|\eta_x| =$ constant, where η_x is the x-component of vorticity; and λ_y is the primary spanwise wavelength clearly seen in Fig. 5.20. (After Rist and Fasel 1995)

with high values of m (cf. also the related earlier results by Zang 1992), which cannot be explained by the resonances considered by Kachanov (1987). According to the papers of 1995 and 1996, the modes corresponding to $m = 2, 3, \ldots, 8$, are apparently just higher harmonics of the (1, 1)-mode produced by non-resonant nonlinear interactions. If so, then their amplification with x must be of the same origin as the amplification of higher temporal harmonics of the primary T-S mode and of other products of non-resonant two-wave interactions (cf. the amplification curves in Fig. 5.12). It is clear, however, that a final solution of the numerous problems relating to the origin and subsequent evolution of higher 2D and 3D instability modes in the K-regime of boundary-layer development requires much additional work.

Note in conclusion that Rist and Fasel (1995); Rist and Kachanov (1995), and Rist (1996) also used numerical-simulation results for the preliminary investigation of various 3D vortical structures appearing in the K-regime, and compared the computed structures with experimental data. Particular attention was paid here to the study of the Λ-shaped structures ('Λ-eddies') observed in numerous flow-visualization experiments (see, e.g., the visualization pictures in Fig. 5.8a; two examples of Λ-structure given by numerical-simulation results are shown in Fig. 5.25). Rist and Kachanov also noted that, according to the new simulation data, at a late stage of flow development (which corresponds to the appearance and subsequent multiplication of spikes) ring-like vortices connected to spikes emerge, pinching off from the downstream 'legs' of the pre-existing Λ-vortices. (This mechanism of generation of ring vortices clearly differs from the previous suggestions discussed earlier in this Section). The numerical results showed the appearance of 'spikes' at the same points, and with the same amplitudes and durations, as those which were observed in wind-tunnel experiments; one such example is shown in Fig. 5.26.

Recall now that when the results of Bake et al. (1996, 2000) relating to the N-regime of boundary-layer instability development were briefly reviewed at the end of Sect. 5.3, it was promised that similarities between some features of the N- and

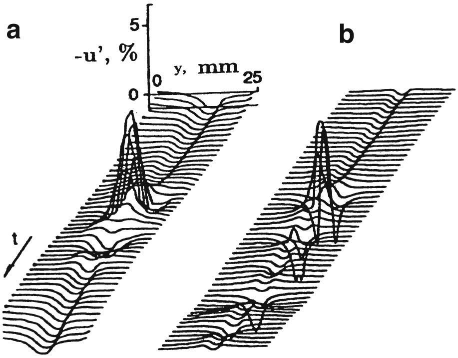

Fig. 5.26 Comparison of the 'spike signal' in (u', t, y)-space (where u' is the streamwise disturbance velocity as a fraction of U_0) appearing in the numerical simulation of boundary-layer instability development by Rist and Kloker at $x = 500$ mm, $z = 8$ mm (**a**) with the 'spike signal' observed at the same values of x and z in the corresponding laboratory experimental of Kachanov and Borodulin (**b**). (After Rist and Kachanov 1995)

K-regimes found by these authors would be described in Sect. 5.5 It was noted in Sect. 5.3 that the wind tunnel used by Bake et al. had a very long test section and only the results relating to the initial part of it agreed well with experimental data from previous investigations of the N-regime. (It was said in Sect. 5.3 that such agreement was observed for $\Delta x < 250$ mm, where Δx is the streamwise distance between the point of measurements and the disturbance generator. In fact the first deviations from the ordinary N-regime were observed by the authors as early at $\Delta x = 220$ mm, but they were weak enough to be ignored for the purposes of Sect. 5.3). The main study of the structure of developed disturbances far from the disturbance generator was made by Bake et al. at $\Delta x = 380$ mm and here the behavior was quite different from that observed earlier in the N-regime; in fact, the measurements represented some mixture of features typical of the N- and K-regimes. At $\Delta x = 380$ mm strong spanwise modulation of streamwise disturbance velocity, of the type shown in Fig. 5.20, was clearly seen (faint signs of such modulation were present at $\Delta x = 220$ mm) and, what is especially important, Klebanoff 'spikes' very similar to those observed repeatedly in the K-regime were also present. The spikes had the same shape as in the case of the K-regime, and again they appeared in the outer part of the boundary layer at spanwise peaks of disturbance velocity and could be doubled and tripled. However, in contrast to the K-regime, they now appeared periodically in time with the subharmonic period $T_1 = 2\pi/\omega_{1/2} = 4\pi/\omega_1$ and not with the primary-wave period $T = 2\pi/\omega_1$ (where ω_1 is the 'fundamental frequency' of the primary 2D wave). Moreover, the vortical structure generated by the developing disturbances again consisted of Λ-vortices, but they were now positioned in space in the staggered order shown in Fig. 5.8b, and not regularly as in Fig. 5.8a. However, in spite of these differences, the subsequent development of spikes and vortices was very similar to that observed in late stages of K-regime development. Therefore, there is reason to assume that prolonged N-regime development may lead to transition of a boundary layer to turbulence by the

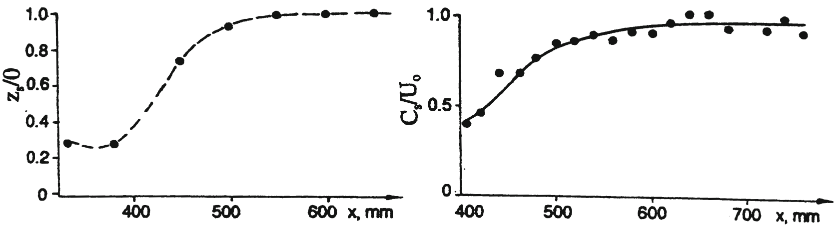

Fig. 5.27 **a** Dependence of the dimensionless height z_s/δ (where δ is the boundary-layer thickness) of the center of a spike on the streamwise coordinate x **b** Dependence on x of the streamwise velocity c_s of a spike. (After Borodulin and Kachanov 1994)

same process that takes place at large amplitudes of the primary wave leading to the K-regime of boundary-layer development.

Let us now consider briefly the results relating to the long-time evolution of spikes appearing in the K-regime of instability development. This topic differs from the subjects considered above, since spikes have some features which invalidate standard methods of weakly-nonlinear stability theory. Observations by Kachanov et al. (1984, 1985, 1989); Borodulin and Kachanov (1988, 1989, 1994, 1995) and some others showed that spikes include a great number of phase-synchronized 2D and 3D modes strongly interacting with each other (cf. Figs. 5.22–5.24). Therefore, ordinary systems of equations for mode amplitudes are of little use in this case. Recall now that spikes are localized in small spatial domains (spanwise localization is especially strong; see e.g. Fig. 5.23). Observations also showed that a newly-formed spike at first moves away from the wall but on reaching the upper part of the boundary layer it moves downstream at practically constant z, and with practically constant velocity close to that of the external stream (see Fig. 5.27a, b and their discussion by Kachanov et al. (1993), accompanied by some supplementary data; similar results were obtained by Acarlar and Smith (1987) for evolution in a boundary layer of "harpin vortices", which are similar to spikes in many respects). During its downstream travel a spike preserves its shape (and also its spatial size and temporal extent), i.e. it does not disperse as do, for example, the ordinary wave packets in which individual waves have different phase velocities determined by the dispersion law (5.5). Not only the spatial form but also the spectral composition and the amplitude of a spike are in the main preserved during its convection downstream. This circumstance was first stressed by Borodulin and Kachanov (1988) and was confirmed by their subsequent experimental studies; see also Kachanov's surveys (1991a, 1994a). It allows spikes, once fully-formed, to be considered as *coherent structures*, i.e. flow formations with a definite degree of ordering which is preserved during long time intervals. The term 'coherent structure' appeared in fluid mechanics only in the second half of the twentieth century and was not at once universally recognized (for example, it was not used at all by Monin and Yaglom 1971, 1975), but now it is clear that such structures play a very important part in the mechanics of turbulence (see, e.g., the book by Holmes et al. 1996). Note, however, that coherent structures of many different types are met in fluid mechanics, especially in fully-turbulent flows, and spikes represent a very special type of such structures. Spikes appear in laminar flows at a relatively late stage of instability development; they are strongly localized, mobile, and have definite boundaries, and thus may be associated with the notion of *solitons*.

The term 'soliton' was apparently first introduced by Zabusky and Kruskal in their paper (1965) devoted to plasma waves, but in fact it has a long history, being directly connected with the observation by J. Scott Russell in 1834 of a strange *solitary wave* in the Edinburgh to Glasgow canal. The wave was produced by a suddenly stopped boat and had the form of a rounded well-defined heap of water elevated above the mean level and for a long time rapidly moving forward (i.e., in the direction of boat motion before the stop) without any change of form or speed—Russell pursued it on a horse for more than a mile. This observation stimulated subsequent attempts by Russell to generate such waves in the laboratory, and led to publication in 1844 of his report to the British Association for the Advancement of Science devoted to this subject. Russell's solitary wave attracted considerable attention, but only in the second half of the twentieth century was it discovered that it represents a particular case of a wide class of flow phenomena which are met in many parts of quite different physical sciences, and have numerous important applications. At present soliton studies form a special science to which an enormous and very diverse literature has been devoted (here it will be enough to name only the relatively small introductory books by Lamb 1980; Drazin 1983; Drazin and Johnson 1989). Up to now there is no universally recognized strict definition of the soliton; to follow Drazin's books one may say that this word means usually a solution of a nonlinear equation or system of equations which describes a wave or collection of waves of a conservative form which is spatially localized, mobile, and may strongly interact with other objects of the same type, retaining its identity after the interaction.

Kachanov and his group stressed the similarity of spikes to solitons mainly on the basis of their localization and conservation of form. However, the relation of spikes to coherent structures was also emphasized by this group; therefore Kachanov suggested applying to spikes the new name 'CS-solitons' (CS for 'coherent structure'). This name indicates the special place of spikes in both collections—of coherent structures and of solitons. As indicated above, solitons usually represent some special solutions of a definite nonlinear equation or equations (in particular, the strict theory of Russell's 'solitary waves' emerged when Korteweg and de Vries (1895) discovered the nonlinear equation for surface waves in a liquid of finite depth and proved that this equation has solitary-wave solutions). Therefore, the identifications of spikes with a special kind of soliton seemed incomplete without a nonlinear equation to describe them.

The first attempts to develop an analytical theory of the soliton-like formations in flat-plate boundary layers were made independently by Zhuk and Ryzhov (1982) and Smith and Burggraf (1985). In both papers by boundary-layer disturbances considered were those which, in the case of small enough amplitude and large streamwise length scale, may be described with good accuracy by the so-called Benjamin-Ono (briefly, B-O) equation, a nonlinear integro-differential equation of the form

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial^2 A/\partial \xi^2}{\xi - x} d\xi + \varphi(t, x) \tag{5.16}$$

where A = A (x, t) is the unknown amplitude of the disturbance (replaced in the integrand by $A(\zeta, t)$), the integral, if divergent, is understood as the Cauchy principal value, while $\mu(t, x)$ is the 'source term', which may be absent in some cases but in others may have different origins and forms. Equation (5.16) (without the source term) was derived by Benjamin (1967) and Ono (1975) (and used by Davis and Acrivos 1967[8]) to describe the variation of amplitude of two-dimensional long internal waves of small amplitude in stratified fluid of great depth, and it was shown by Benjamin and by Ono that this equation has soliton solutions of the same form as those known for the Korteweg-de Vries equation. Later it was discovered that the same equation may also be applied to many other nonlinear waves of large streamwise lengthscale and small amplitude in steady shear flows bounded by a wall (the above-mentioned papers by Zhuk and Ryzhov, and Smith and Burggraf and also those by Goncharov 1984; Romanova 1984; Demekhin and Shkadov 1986; Benjamin 1992; Matsuno 1996 are just typical examples). However in these papers neither the K-regime of boundary-layer transition nor the spikes were considered explicitly.

Application of the B-O equation to the development of strongly nonlinear disturbances in a boundary-layer flow was studied, in particular, by Rothmayer and Smith (1987). However here only a rather special one-parameter family of soliton solutions of the B-O equation was considered, and these solutions proved to be inappropriate to describe the spikes observed in the K-regime. Then Zhuk and Popov (1989) found some new soliton solutions of Eq. (5.16) (with non-zero 'source term'), while Ryzhov (1990) investigated a more general three-parameter family of soliton solutions of the homogeneous B-O equation (some important features of this solution are shown in Fig. 5.28). Ryzhov's investigation was continued by Kachanov et al. (1993); Ryzhov (1994), and Bogdanova-Ryzhova and Ryzhov (1995) who applied this and some related soliton solutions to a description of real fluid-mechanics instabilities (see also Kachanov's survey papers (1991a, 1994a) and an interesting survey by Ryzhov and Bogdanova-Ryzhova (1998) containing a long list of references). In particular, Bogdanova-Ryzhova and Ryzhov (1990) studied the soliton solutions of the inhomogeneous B-O equation describing the evolution of disturbances in a boundary layer on a rough wall (where the effect of roughness elements may be described by a definite form of the source term $\varphi(t, x)$). These authors also cited some papers in which the same equation was applied to development of atmospheric and oceanic waves affected by a mountain ridge or by large bottom irregularities, which also generated source terms, of a form different from that applying to the rough wall. (for more details see Ryzhov and Bogdanova-Ryzhova's survey 1998). A detailed

[8] Therefore instead of the name 'Benjamin-Ono (or B-O) equation' the name 'Benjamin-Davis-Acrivos (or BDA) equation' or 'Davis-Acrivos-Benjamin-Ono (or DABO) equation' is sometimes used.

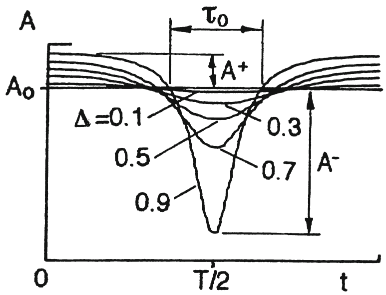

Fig. 5.28 Schematic form of a family of Ryzhov's three-parameter soliton solutions of the Benjamin-Ono Eq. (5.16) corresponding to various values of the amplitude parameter Δ and fixed values of the other two parameters (determining scales of the dependence of A on t and x). Here $A_0 = T^{-1} \int_0^T A(t)dt$ is the mean amplitude, A^-, A^+ and τ_0 are some numerical parameters, and T is the fundamental spike period. (After Kachanov et al. 1993)

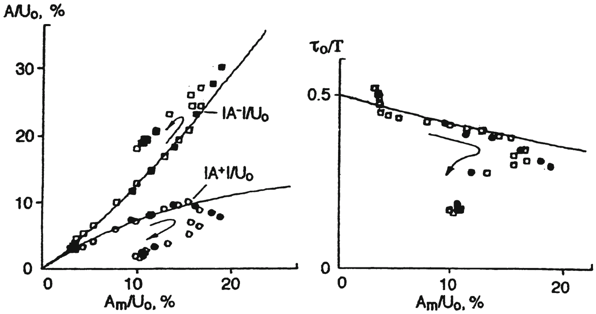

Fig. 5.29 Experimental (*points*) and theoretical (*curves*) dependencies of the soliton form characteristics A^-, A^+ and τ_0 on the soliton magnitude $A_m = (A^+ + A^-)/2$. Theoretical correspond to the solution of the B-O equation shown in Fig. 5.28, with appropriately chosen parameter values; points—data of Borodulin and Kachanov (1988). (After Kachanov et al. 1993)

study of the solutions of the inhomogeneous B-O equation and their application to water-wave problems was also carried out by Matsuno (1996), whose paper contains many supplementary references relating to this topic.

Further, Ryzhov (1990) and Kachanov et al. (1993) showed that the determining parameters of Ryzhov's family of soliton solutions may be chosen so that the general form of these solutions, and a number of their numerical characteristics, are very close to those found by Kachanov (1991b); Borodulin and Kachanov (1988, 1994, 1995) and in some other experiments on Klebanoff's spikes in the early stage of their downstream evolution (see, e.g., Fig. 5.29). These results may be considered as the confirmation of the soliton nature of spikes. The deviations in Fig. 5.29 of the experimental results for far-downstream points of observation from the theoretical curves may be explained by the fact that B-O equation deals only with two-dimensional disturbances; therefore it represents a two-dimensional spike model which is inapplicable to the later, essentially three-dimensional, stages of spike evolution. Many details of these later stages of spike development were discussed by Kachanov et al. (1993) and studied experimentally by Borodulin and Kachanov (1995). Detailed analysis of the nonlinear evolution in boundary layers of two-dimensional T-S waves, which, in particular, described some of the limitations of applying the B-O equation to this problem, was performed by Moston et al. (2000). A generalization of the B-O equation to the case of three-dimensional near-wall disturbances was proposed by Shrira (1989) in connection with the study of 3D waves in the upper layer of the ocean. Later Abramyan et al. (1992) proved that Shrira's equation has three-dimensional soliton solutions which may possibly be used to describe spikes in the three-dimensional stage of their evolution.

It has already been mentioned that the time evolution of spikes leads finally to their transformation into 'turbulent spots'. Such spots (one of which is shown in Fig. 2.2) represent the spatial regions where the flow becomes truly turbulent, i.e. it becomes irregular, is accompanied by random ('stochastic') fluctuations, and therefore cannot be studied mathematically without the use of probability-theory concepts. Recall that it was long assumed that 'stochastization'/randomization of the boundary-layer flow takes place when spikes (considered as irregular formations) first appear. However it was found later that spikes themselves are regular structures which may be described by deterministic equations of motion, while random velocity fluctuations emerge only at the later stages of spike development. The process of gradual development of the 'flow randomness' associated with spikes in an initially laminar boundary layer disturbed by a two-dimensional T-S wave was studied experimentally by Dryganets et al. (1990) whose results were discussed by Kachanov (1994a); see also the descriptions of experiments by Breuer et al. (1997) in Sect. 5.62. An analytical model of the gradual randomization of a spike and subsequent formation of a spot was briefly outlined by Smith in Kachanov et al. (1993) and then developed further by Smith (1995). Bogdanova-Ryzhova and Ryzhov (1995) considered the model of randomization of a soliton by a wall hump and then returned to the problem of the possible connection between solitons and the onset of random flow disturbances in Ryzhov and Bogdanova-Ryzhova (1998). Note in this respect that many different mechanisms may be responsible for the appearance of random fluctuations in real boundary-layer flows; a definite part may be played also by 'local high-frequency secondary instability' (LHSI: discussed above) of a flow disturbed by a T-S wave, and by penetration into this flow of background (environmental) disturbances in the form of random 2D and 3D T-S waves or wave packets, corresponding to the continuous part of the spectrum of the boundary-layer Orr-Sommerfeld eigenvalue problem. However a detailed analysis of the appearance of randomness in a laminar boundary flow lies outside of the contents of this chapter on weakly-nonlinear stability theory.

Note in conclusion that recent experimental studies of the K-regime of boundary-layer instability development by Lee (see Lee 1998, 2000; Lee et al. 2000 and references therein) lead to some results differing from those considered above. Lee studied disturbance development in the boundary layer on a flat plate mounted in the low-turbulence water channel at Beijing (Peking) University. In these experiments a wave disturbance was excited in the flow by periodic pumping of water in and out of the boundary layer through a narrow spanwise slit near the leading edge of the plate. Then the disturbance development was recorded by hot-wire measurements at a number of downstream positions and by numerous photographs of the evolution of flow structures visualized by hydrogen bubbles. Lee used the name ‘CS-solitons’ to denote some new flow structures, which occupy the whole thickness of the boundary layer but have quite different forms in the near-wall region (where long streaks occur at the ‘peak positions’ of the spanwise velocity modulation), in the middle part of the boundary layer, and in its upper part (where Kachanov’s ‘CS-solitons’ travelled most of the time). According to Lee, the upper part of CS-solitons is produced by short chains of ringlike vortices appearing periodically (with the same frequency as the primary T-S wave) at the tips of Λ-vortices whose breakdown generates spikes. Lee’s CS-solitons differ from Kachanov’s, but both these formations are strongly localized spanwise and preserve their main features up to final breakdown (leading to the appearance of turbulent spots). Lee noted that some of his results are similar to those observed by Hama and Nutant (1963); as to the disagreement of some of Lee’s conclusions with those of Kachanov, they were partly resolved in their joint work (see, e.g. Lee et al. (2000) and references in Lee (2000)). Nevertheless, at present it seems that Lee’s results require further careful investigation, and that his claim of the possible finding of the ‘universal transition scenario’ is questionable. However, since ‘transition scenarios’ are only indirectly connected with the main content of the present chapter, Lee’s results will not be further considered here.

5.6 Some Other Scenarios of Instability Development in Boundary Layers

The N- and K-regimes of boundary-layer instability development considered in the previous section of this chapter have a very important common feature—in both of them the instability process starts with the appearance in the flow of a linearly-unstable two-dimensional Tollmien-Schlichting wave. (This T-S wave is often identified with that solution of the O-S eigenvalue problem (2.42), (2.44) corresponding to the eigenvalue ω, or k, which has the maximal imaginary part. Such identification is then justified by the assumption that any disturbance to excite a T-S wave may enter the boundary layer from the disturbed free-stream flow, and hence the most-unstable T-S wave must play the dominant role in boundary-layer evolution). The simplest case, the instability regime initiated by a sole plane T-S wave, was investigated in the famous experiments by Schubauer and Skramstad (1947) and in numerous subsequent similar boundary-layer stability studies (including all the experimental studies of the N- and K-regimes considered above) which used a vibrating ribbon (or some other periodically-oscillating device) for the excitation in the flow of a weak 2D wave of fixed frequency ω. Recall however the remark made in Sect. 2.92 (p. 118) that, in the majority of boundary-layer transitions to turbulence met in wind- and water-tunnel experiments and in real life, the appearance in the flow of an isolated linearly-unstable T-S wave of small amplitude growing in accordance with the laws of linear stability theory, is not observed at all, i.e. this stage of instability development is *by-passed*. Therefore, the scenarios of the boundary-layer instability development which begin with the appearance in the flow of the most-unstable plane T-S wave of small amplitude are inapplicable to the majority of real-life boundary-layer-transition phenomena. Note that the term *by-pass transition* is often used in engineering practice to describe response to such high levels of free-stream turbulence that transition starts at Reynolds numbers far below the critical value predicted by linear instability theory, so that no stage of the route to randomness discussed above, nor the even the behavior of the simple finite-amplitude subcritical modes discussed in Sects. 5.3 and 5.4 have any relevance.

In this book no attempts will be made to consider all scenarios of boundary-layer instability development and transition to turbulence met in practice. However, at least some of the regimes of instability development differing from the N and K regimes considered above must, clearly, be discussed here.

5.6.1 Oblique and Streak-Breakdown Transition Scenarios

Let us begin with a remark about the paper by Goldstein and Choi (1989). These authors considered the case of a plane-parallel shear layer ("mixing layer") between two parallel streams with uniform velocities U_1 and $U_2 \neq U_1$. Then they studied the evolution in this flow of a pair of linearly-unstable symmetric oblique waves of the same amplitude A and frequency ω, with two-dimensional wave vectors $\mathbf{k}_1 = (k_1, k_2)$ and $\mathbf{k}_2 = (k_1, -k_2)$. The waves were assumed to be harmonic in time (i.e., ω is real) but streamwise-growing (k_1 is complex while k_2 is real). It was found that the two waves interact strongly with each other and, as in the case of Craik's resonant triads satisfying conditions (5.7), strong nonlinear wave interaction is concentrated in the neighborhood of the common critical layer of these two waves. Using known methods of approximate asymptotic analysis of the critical-layer contribution to nonlinear wave interactions (see, e.g., the review by Maslowe (1986) and the subsequent related paper by Goldstein (1995)), Goldstein and Choi derived an equation for the amplitude $A = A(x)$. This equation proved to be integro-differential and cubically nonlinear and implied rapid streamwise growth of A. A similar method was applied by Wu et al. (1993) to the study of disturbance development in a near-wall fluid layer above a horizontal plate oscillating sinusoidally in the x-direction; here again the nonlinear interaction between a pair of symmetric oblique waves leads to rapid growth of flow disturbances. (Note that later Wu and Stewart (1995) showed that rapid growth of three-dimensional disturbances in a plane shear layer may also be produced by the interaction between another pair of T-S waves having the same critical layer—namely, one two-dimensional T-S wave and one oblique wave having the same phase velocity. However this instability mechanism will not be considered in this chapter).

The development in a plane-channel flow (plane Poiseuille flow) of disturbances initiated by the appearance of a pair of oblique waves having amplitude A, frequency ω and wave vectors $\mathbf{k}_1 = (k_1, k_2)$ and $\mathbf{k}_2 = (k_1, -k_2)$ was apparently first studied by Lu and Henningson (1990) and Schmid and Henningson (1992a, b) who performed temporally-developing direct numerical simulations of this development. The authors considered the oblique-wave development in a steady laminar flow as the first step en route to transition of this flow to turbulence, an alternative to the N- and K-routes. To describe this new route the term *oblique transition* was used by these authors, while in the book by Schmid and Henningson (2001) the name *O-type transition* was also used. As to the boundary-layer flows, the development of a pair of oblique waves was first considered as a possible transition mechanism in the case of compressible flow; see, e.g., Thumm et al. (1989, 1990); Chang and Malik (1992, 1994); Fasel et al. (1993), and Sandham et al. (1994). (Special interest in the compressible case was stimulated by the fact that Squire's theorem of Sect. 2.81 is not valid here and, in contrast to incompressible boundary layers, the most-unstable wave in compressible shear layers is usually an oblique one; see, e.g., Reshotko (1976)). However, since in this book compressible flows are not considered, these papers will not be discussed here.

A spatially-developing numerical simulation of the oblique transition in the incompressible Blasius boundary layer was apparently first carried out by Joslin et al. (1993) who solved the exact Navier-Stokes equations together with the approximate 'parabolic stability equations' (see Sect. 2.92, p. 117), and by Berlin et al. (1994) whose numerical simulation covered a greater number of flow-development stages than that of Joslin et al. The paper by Berlin et al. (1994) contains the first outline of Berlin's extensive numerical study of development of oblique waves in a boundary-layer flow, while the final results of this study were summarized in Berlin's doctoral thesis (1998) and in the paper by Berlin et al. (1999). The experimental part of the work, which was also included in the latter paper, was based on results from the doctoral dissertation of Wiegel (1996). One more recent doctoral dissertation devoted to experimental study of oblique transitions in plane Poiseuille and Blasius boundary-layer flows was presented by Elofsson (1998b); his results relating to boundary-layer instability are given also in Elofsson (1998a) and Elofsson and Alfredsson (2000). Then Schmid and Henningson (2001) presented results of somewhat different temporally-developing numerical simulations of the plane-Poiseuille and boundary-layer oblique transitions; these results will be considered a little later.

Numerical simulations of the 'oblique-transition regime' (or the 'oblique-transition scenario' as this regime is often called) may be realized by solving the N-S equations for the disturbance velocity in a given laminar flow under the condition that at some $x = x_0 > 0$ there is a 'disturbance generator' which generates a pair of symmetric oblique waves propagating streamwise. This means that here the oblique waves are included in the 'inflow boundary condition' stating that the 'inflow velocity' at $x = x_0$ is represented by the Blasius velocity profile plus velocity profiles of two symmetric oblique waves. In the case of spatial simulation the frequency ω, spanwise wavenumbers $\pm k_2$, amplitude A and phase ϕ of oblique waves are real constants which may be chosen arbitrarily. If the plane-parallel model of a boundary layer is used, then the streamwise wavenumber $k = k_1$ may be determined as the complex eigenvalue, having the smallest imaginary part, of the corresponding Orr-Sommerfeld Eq. (2.41) with given values of ω and k_2, and $c = \omega/k_1$. In the more general case where a locally-plane-parallel approximation is used, k_1 is a slowly-changing complex function of x which is given by the eigenvalue, with the smallest imaginary part, of the local Orr–Sommerfeld Eq. (2.41) (corresponding to the Blasius velocity profile $U(z) = U(z, x)$ at the streamwise coordinate x). In temporal simulations the wavenumbers k_1 and k_2, amplitude A and phase ϕ are real and may take arbitrary values; while the frequency ω is the complex eigenvalue of the corresponding O-S equation, with given values of k_1 and k_2, that has the greatest imaginary part. In experimental studies of oblique transition the 'disturbance generator' must be realized, of course, as some device exciting the oblique waves with prescribed values of ω, A and k_2. In the boundary-layer experiments by Wiegel (1996); Elofsson (1998a, b) and Elofsson and Alfredsson (2000) this device was similar to the 'wave generator' proposed by Gaponenko and Kachanov (1994) and then applied by Bake et al. (1996, 2000), while in the channel-flow experiments of Elofsson and Alfredsson (1995, 1998) (see also Elofsson (1998b)) the oblique waves were produced by a pair of 'oblique ribbons' vibrating with the frequency ω and amplitude A and placed at equal and opposite angles to the mean-flow direction. This latter method of oblique wave generation was also used by Elofsson and Lundbladh (1994) who, simultaneously with their experiments, carried out a numerical simulation of this transition where as 'disturbance generator' a pair of vibrating oblique ribbons was simulated.

The early temporally-developing direct numerical simulations of the disturbance development in a plane-channel flow disturbed by a pair of small (but not infinitesimal) symmetric oblique waves performed by Schmid and Henningson (1992a, b) showed that strong nonlinear interaction between two waves arises almost at once, and produces a rapid growth of the disturbance energy and the appearance of a number of new disturbance structures, essentially accelerating transition to turbulence. Subsequent more complete spatial numerical simulations by Berlin et al. (1994, 1999) (see also Henningson et al. (1995)), and temporal numerical simulations by Schmid and Henningson (2001), of the analogous development of a Blasius boundary layer disturbed by a pair of oblique waves, revealed many important features of the process, and substantiate Schmid and Henningson's idea of the possible importance of the oblique-wave mechanism in laminar-turbulent transition.

In papers by Berlin et al. it was again shown that a strong nonlinear interaction of a pair of symmetric oblique waves develops much more rapidly than in the case of a single 2D Tollmien–Schlichting wave initiating the nonlinear N- and K-routes to transition. This circumstance can be explained by the fact that both the N- and K-routes begin with the exponential growth of an unstable T-S wave according to linear stability theory, and only when the amplitude of this T-S wave becomes large enough does the nonlinear resonant-triad interaction begin to play an essential part. However, the linear-theory prediction of the growth rate of a wave corresponding to eigenfunctions of the Orr–Sommerfeld equation is very small in comparison, not only with the growth of wave disturbances produced by their nonlinear interactions

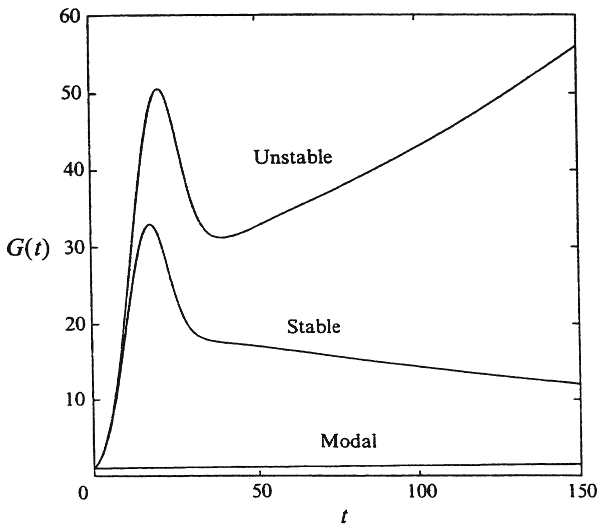

Fig. 5.30 The dependence on t of the growth curves $G(t) = E(t)/E(0)$ for the energy E of plane-wave disturbances with $k_1 = 1$ (all physical quantities are non-dimensionalized by scales $H_1 = H/2$ and $U_0 = U(H/2)$) in a plane Poiseuille flow between walls at $z = 0$ and $z = H$. The curves labelled 'Unstable' and "Stable" correspond to the 'optimal' 2D wave having the greatest transient growth in linearly-unstable Poiseuille flow with $\mathrm{Re} = U_0 H_1/\nu = 8{,}000$ and linearly-stable flow with $\mathrm{Re} = 5{,}000$, respectively, while the 'Modal' curve shows the growth of the unstable solution of the Orr–Sommerfeld eigenvalue problem with $k_1 = 1$ and $\mathrm{Re} = 8{,}000$. (After Reddy and Henningson 1993)

but even with the non-modal transient disturbance growth due to non-normality of the linearized Navier-Stokes equations (see in this respect Chap. 3, where the meaning of 'non-normality' was explained, and also the expressive Fig. 5.30 taken from the paper by Reddy and Henningson (1993)). Recall that in order to eliminate the stage of very slow growth of disturbances following linear-theory laws, Klebanoff et al. (1962) and many of their followers artificially excited three-dimensional disturbances in the vicinity of a spanwise vibrating ribbon. This was necessary since otherwise the test section of a low-turbulence wind tunnel would usually be too short for the most interesting stages of flow development to be reached.

Berlin et al. (1994) carried out a spatial numerical simulation of a Blasius boundary-layer flow with a pair of oblique waves in it, and used the simulation results to study the appearance and subsequent growth of a number of new wave structures produced by nonlinear interactions between the primary oblique waves. The inflow conditions specified at $x = x_0$ corresponded to a Blasius boundary layer with $\mathrm{Re}^* = U_0\delta^*/\nu = 400$ (where U_0 is the free-stream velocity and δ^* is the displacement thickness at the inflow; recall that at such a low Re* unstable T-S waves do not exist in a boundary-layer flow) plus a pair of oblique waves with frequency $\omega_0 = 0.08$ (this and all other quantitative characteristics of this simulation discussed below are non-dimensionalized by the scales U_0 and δ^*), spanwise wavenumber

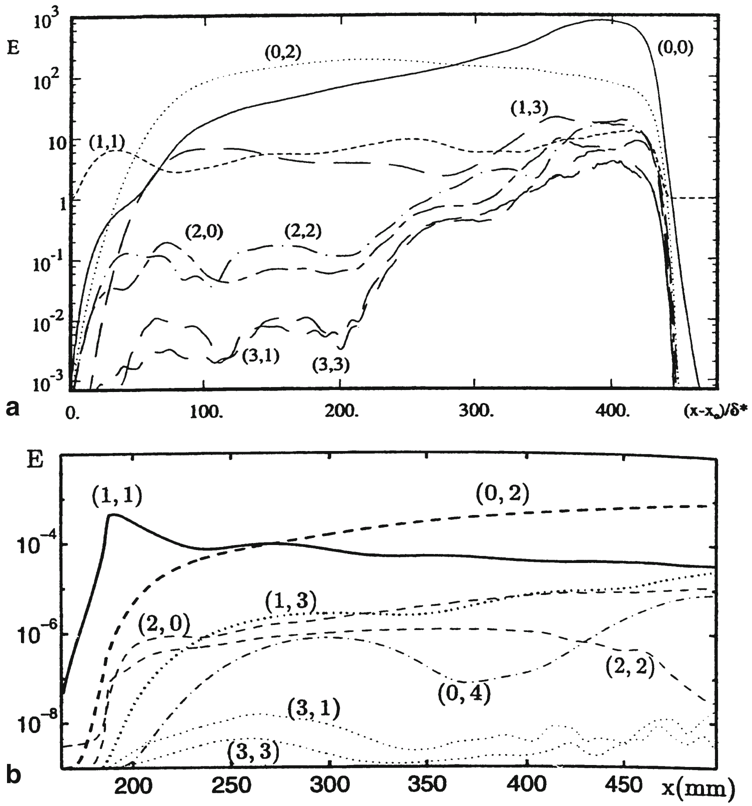

Fig. 5.31 The dependence of the energies E of a number of (n, m)-Fourier components (i.e. waves with frequencies and spanwise wavenumbers $(n\omega_0, mk_{2,0})$) on x in a Blasius boundary layer with a pair of oblique waves with frequencies and spanwise wavenumbers $(\omega_0, \pm k_{2,0})$. **a** Dependence of E on $(x - x_0)/\delta^*$ in the case when the initial energy of the (1, 1)-mode is equal to 1 (after Berlin et al. 1994). (**b**) Dependence of E (measured in some conventional units) on x (in mm). The position $x = x_0$ of the 'wave generator' is here close to 186 mm. (After Berlin et al. 1999)

$k_{2,0} = 0.192$ (the value of k_1 was then determined from the O-S eigenvalue problem) and amplitude $A = 0.01$. A more extensive and careful numerical simulation of the same type (where five different models of inflowing oblique waves were considered) was carried out by Berlin et al. (1999). Here somewhat different values of Re*, ω_0, $k_{2,0}$ and A, and of the range of x-values studied, were chosen to achieve a satisfactory match with the conditions of Wiegel's experiments. In Fig. 5.31a, b results from the two papers by Berlin et al. are presented, showing the dependence on $(x - x_0)/\delta^*$ (in Fig. 5.31a) or on x in mm (in Fig. 5.31b) of the energies E of a number of (n, m)-Fourier components with frequencies and spanwise wavenumbers $(\omega, k_2) = (n\omega_0, mk_{2,0})$. (The numbers n and m may be always assumed to be nonnegative since the symmetry of the (ω, k_2) and $(\omega, -k_2)$ modes means that modes with negative values of k_2 need not be considered explicitly). In Fig. 5.31a the energies are divided by the inflow energy of the primary (1, 1) mode, and hence here $E(0) = 1$ for the (1, 1) mode and is zero for all other modes; in Fig. 5.31b energies E are measured in some conventional dimensional units, and the coordinate x_0 of the 'wave generator' was here close to 186 mm.

It is easy to see that the quantitative results of the 1994 and 1999 simulations do not coincide; the differences are apparently due to the use of different numerical methods, models of inflowing waves, and outflow conditions (Fig. 5.31a clearly corresponds to conditions annihilating waves at the outflow end of the computational domain). Qualitatively however, the two collections of results are sufficiently close to each other. Both simulations show that the energy of the primary oblique waves does not change much with the streamwise coordinate x (in Fig. 5.31a it grows slightly at first and then remains almost constant, while in Fig. 5.31b it begins to decrease slowly immediately after the peak at the wave-generation point, but in both cases the energy changes for this mode are small in comparison with those for the other modes). Figure 5.31a shows the generation of a rather energetic (0, 0) mode describing the distortion of the mean velocity profile by nonlinear waves, but this effect does not appear in Fig. 5.31b. However, according to both figures the most important feature of the oblique-wave interaction is the rapid growth of the (0, 2) mode, greatly exceeding the growth of all other modes and quickly making this mode the most important disturbance structure. The (0, 2) mode does not oscillate and has half the spanwise wavelength of the primary oblique waves oscillating with frequency ω_0. Thorough analysis of the results of the numerical simulations by Berlin et al. (1999) and the flow visualizations performed by Wiegel (1996); Elofsson (1998a, b) and Elofsson and Alfredsson (2000) showed that in the case of oblique transition the (0, 2) mode represents a periodic array of pairs of counter-rotating streamwise vortices with spanwise wavelength half that of the primary oblique waves.

Recall now that, according to results presented in Chap. 3,[9] arrays of streamwise vortices are just the structures which are subjected to the greatest transient growth produced by the so-called lift-up effect studied, in particular, by Landahl (1975, 1980, 1990) and discussed in Sects. 3.22, 3.32 and 3.33. Therefore, after the generation of the (0, 2) mode by the direct nonlinear interaction between primary modes (1, 1) and (1, −1) its subsequent growth is due to two different factors: the quadratically-nonlinear interactions among existing oblique waves and the linear lift-up effect. The combined action of two growth mechanisms explains naturally the excess of the growth rate of the (0, 2) mode over those of the (2, 0) and (2. 2) modes, which are also produced by direct nonlinear interactions of primary waves. As indicated by Landahl, the lift-up effect leads to the transformation of the streamwise vortices into a spanwise-periodic collection of horizontal streaks of fluid with alternating low and high streamwise velocity. Such streaky structures are in fact clearly seen in flow-

[9] Note that the results of Butler and Farrell (1992) presented in Chap. 3 were obtained for a simplified, strictly plane-parallel model of the Blasius boundary layer. The optimally-growing disturbance structures for the more accurate model of a streamwise-thickening boundary layer were studied by Andersson et al. (1999) and Luchini (2000) but will not be considered here.

visualization photos of boundary-layer flows by Wiegel, Elofsson and Elofsson and Alfredsson, and on contour plots of disturbance velocity and vorticity determined from the data given either by appropriate numerical simulations or by detailed hot-wire-anemometer measurements (see, e.g., Berlin et al. (1994, 1999); Berlin (1998); Elofsson (1998a, b) and Elofsson and Alfredsson (2000)).

In a range of x-values where distinct horizontal streaks are seen, the streak amplitude A_s at a fixed value of x depends on the amplitude A of the primary oblique waves and, for not-too-large values of A, it is proportional to A^2 as it must be in the case of streak generation by quadratic interaction of two oblique waves. With increasing x, the amplitude A_s grows approximately linearly at first and then saturates but, if the primary forcing amplitude A is too low, then after the initial growth A_s begins to decrease and the streaks gradually disappear (these facts were first discovered by Joslin et al. (1993) and then confirmed in other papers mentioned above). According to results of both the numerical simulations and the experiments, if A is not too low then streaks of saturated large amplitude A_s become unstable with respect to high-frequency oscillations, and this instability leads at first to oscillations of streaks and then to their breakdown and transformation into collections of irregular small-scale vortices forming the turbulent flow regime. Such a scenario of transition to turbulence was studied for both plane-channel and boundary-layer flows, in particular, by Henningson et al. (1995); Schmid et al. (1996); Alfredsson and Matsubara (1996); Reddy et al. (1998); Berlin et al. (1999); Brandt et al. (2000), and Elofsson and Alfredsson (2000); see also Schmid and Henningson's book (2001). Most of these studies were based on the analysis of numerical-simulation data (which Brandt et al. supplemented by some stability-theory computations), but Alfredsson and Matsubara, and also Elofsson and Alfredsson, made direct experimental studies of the streak-breakdown process in flat-plate boundary layers. However, in this chapter the transition to turbulence is not discussed; therefore here only a few remarks about this transition scenario will be given.

Let us begin with a short consideration of the results of numerical simulations of oblique transition in a boundary-layer flow, carried out by Henningson, Schmid and their coworkers and described in the papers by Henningson et al. (1995) and Schmid et al. (1996), and in the book by Schmid and Henningson (2001). (These papers and the book contain also results of similar simulation of the oblique transition in a plane Poiseuille flow which will be briefly discussed in the next chapter). As was indicated earlier in this Section, Schmid and Henningson performed temporal, not spatial, numerical simulations which differed in some respects from the earlier spatial simulations by Berlin et al. (1994, 1999). In their temporal simulations the authors used the same model of a strictly plane-parallel Blasius boundary layer, with thickness $\delta(t)$ growing with time, that was used by Spalart and Yang (1987) and was briefly described in footnote 4. As to the above-mentioned difference from the approach by Berlin et al., it is connected with the inclusion of weak random background disturbances (supplementing a pair of oblique waves of much greater amplitude) in the numerical model of disturbed boundary-layer flow used by Schmid and Henningson. To model an oblique boundary-layer transition these authors disturbed the Blasius boundary layer, not only by a pair of primary symmetric oblique waves of

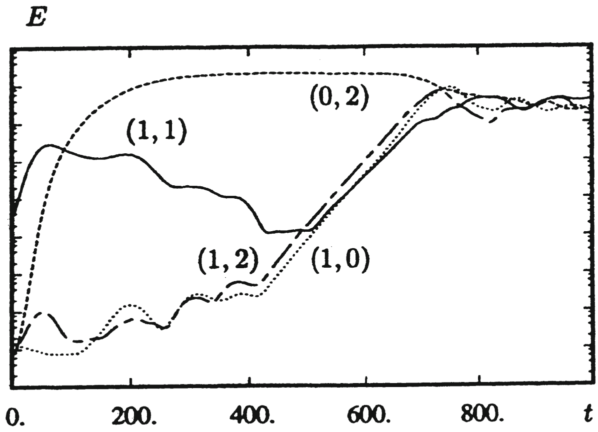

Fig. 5.32 Computed energy-growth curves for a number of (n, m)-waves with streamwise and spanwise wavenumbers $(nk_{1,0}, mk_{2,0})$ in a boundary layer disturbed by 'primary oblique waves' with wavenumbers $(k_{1,0}, \pm k_{2,0})$ and 'weak noise' consisting of supplementary small-amplitude (n, m)-waves with $n = 0, 1, 2$, and $m = 0, 1, 2$. (After Schmid and Henningson 2001)

finite amplitude with wave vectors $(k_1, \pm k_2)$, but also by random 'noise waves' of much smaller amplitudes having the following 'neighboring multiple wave vectors'; (0,0) (random 'mean-velocity correction'), $(k_1, 0)$, $(2k_1, 0)$, $(0, \pm k_2)$, $(2k_1, \pm k_2)$, $(0, \pm 2k_2)$, $(k_1, \pm 2k_2)$, and $(2k_1, \pm 2k_2)$. Hence, contrary to the previous numerical models where all 'higher modes' were produced entirely by nonlinear interactions of the two primary oblique waves among themselves and with their higher harmonics, here weak random higher modes were assumed to exist right from the start, and could grow by extracting energy from the much more energetic primary waves. In Fig. 5.32 an example, computed by Schmid and Henningson, of dependencies on the time t of the energies E of the primary mode (1, 1) and of three selected 'higher modes' is shown. (Here t and E are measured in some conventional units, and the numbers in parentheses indicate the ratios $(K_1/k_1, K_2/k_2)$ of the streamwise and spanwise wavenumbers of the mode to those of primary waves). The modes represented in Fig. 5.32 are not the same as in Figs. 5.31a, b (where moreover spatial, and not temporal, wave amplification was simulated), but nevertheless the qualitative results of the initial part of Fig. 5.32 (say, until $t \approx 500$) are reminiscent of those given in Figs. 5.31a, b. However, at larger values of t the horizontal-streak array (generated by streamwise vortices which are also (0, 2)-mode structures) becomes unstable with respect to local high-frequency fluctuations, begins to oscillate in disorderly fashion, and then breaks down. As a result, the flow becomes turbulent, containing a large collection of various finite-amplitude structures. (This process is partially reflected in the right-hand part of Fig. 5.32; cf. also Waleffe (1995) and Hamilton et al. (1995) where the streak breakdown and the following stages of disturbance development were also considered; it was shown that, at $\mathrm{Re} > \mathrm{R}_{cr}$, streak breakdown leads to regeneration of roll structures and may be a part of a self-sustaining process forming a steady near-wall turbulent regime). The numerical results presented in Figs. 5.31a, b and 5.32 may be supplemented by figures in Elofsson and Alfredsson (2000) showing the dependence on x of the amplitudes of various (n, m) modes in Blasius boundary-layer flow; however we will not discuss the latter results here.

Note that the N- and K-regimes of disturbance development in a boundary layer both begin with the growth of a primary linearly-unstable two-dimensional Tollmien-Schlichting wave. Then this wave stimulates the appearance in the flow of some three-dimensional T-S waves (different in the two regimes) forming, together with the primary wave, an unstable wave system (this instability is clearly the secondary one). Thus, both regimes may be interpreted as initial stages of the *TS-wave-secondary-instability transition scenario*; as explained earlier, whether the N-regime or the K-regime will be realized depends only on the primary-wave amplitude. Quite another route to boundary-layer transition is represented by the *oblique-transition scenario* (or 'O-regime') considered above, where the first stage consists of the development in the flow of a pair of symmetric oblique T-S waves. In parallel with these two scenarios Schmid et al. (1996); Reddy et al. (1998) and Schmid and Henningson (2001) considered also a third *streak-breakdown transition scenario* which does not include the stage of growing T-S waves (i.e., represents some particular type of the *by-pass transitions* considered by Morkovin (1969); cf. Chap. 2). This third scenario has many features in common with the oblique-transition scenario but it completely disregards the first stage of the latter regime, connected with TS-wave development, and begins with a collection of streamwise vortices which is a (0, 2)-mode structure, while in the oblique transitions (0, 2)-structures are produced by nonlinear development of a pair of symmetric oblique waves.

At the very beginning of this subsection it was noted that if one assumes that any T-S wave may penetrate the boundary-layer from the disturbed free-stream flow, then it seems natural to suppose that the most unstable of such waves must dominate the initial stage of the development of flow instability. However, if the free-stream flow is so disturbed that all possible T-S waves are present there and can penetrate the near-wall flow region, then similar penetration must also be possible for many non-modal disturbances (i.e. those differing from T-S waves) existing in the boundary-layer environment. It seems equally natural to assume that the initial stage of boundary-layer instability development will be dominated, not by the most-unstable T-S wave but by the optimally-growing disturbance of non-modal type which, during the initial stage of disturbance development, grows much faster than any T-S wave (again see Fig. 5.30). As explained in Chap. 3, if the transient, rather than the asymptotic disturbance growth (as $t \to \infty$), is considered and the disturbances are assumed to be so small that their development may be described by linear instability theory, then the optimally-growing disturbance will be non-modal, and in the case of a boundary-layer flow will have the form of a spanwise-periodic array of streamwise vortices. Therefore, it seems reasonable to suppose that a disturbance development starting with the appearance in the boundary layer of streamwise vortices of small amplitude may also be a quite important mechanism of real boundary-layer transition to turbulence. Exactly this mechanism was called the 'streak-breakdown transition scenario' in the papers mentioned in the previous paragraph.

Is it possible to estimate quantitatively, if only roughly, the relative likelihood of various transition regimes for different flows met in practice? It is clear that for this it is necessary, first of all, to estimate somehow the probabilities of the appearance of disturbances of various types, with various amplitudes, frequencies and wave vectors. However, such an estimate is impossible without detailed knowledge of the qualitative and quantitative characteristics of free-stream turbulence and other environmental 'noise' while in practice these characteristics can seldom be determined with satisfactory accuracy. Thus, the problem of likelihood estimation cannot have a general solution and may be solved, even partially, only in some exceptional cases. Hence it is only natural that Schmid et al. (1996); Reddy et al. (1998), and Schmid and Henningson (2001) did not try to study the problem in its general form but considered only two special subproblems, having definite relevance to a rough assessment of the likelihoods of different routes to transition.

It has already been indicated earlier in this section that a pair of symmetric oblique waves may lead to 'oblique transition' of the boundary-layer flow only if the initial amplitude A of these waves is not too small. Otherwise the waves will at best only begin to grow and later they (and also the streamwise vortices, if they were generated by primary waves) will begin to decay and finally disappear. This means that there exists some threshold amplitude A_{tr} of the oblique waves, oblique transition being possible only if $A > A_{tr}$. (Of course, the threshold amplitude may take different values for oblique waves with different values of (k_1, k_2) or (ω, k_2); below, the symbol A_{tr} will always be applied to 'optimal waves' corresponding to the greatest threshold amplitude). Recall now that in Sect. 5.2 it was indicated that a threshold amplitude exists also in the case of resonant-triad interactions: at too small an amplitude of the primary plane T-S wave, resonant growth of the oblique wave becomes impossible. In Sect. 5.2 only the stage of resonant growth of oblique waves was considered; it is clear, however, that for the full realization of the TS-wave-secondary-instability transition scenario the initial amplitude A of the linearly unstable plane T-S wave must exceed a definite threshold value A_{tr}, which is apparently greater than the threshold value determining the possibility of a transient growth of oblique-wave amplitude. Finally, a definite threshold value A_{tr} of the initial amplitude of the streamwise vortices must also exist, and determine whether or not an array of such vortices can be transformed into a periodic array of streamwise streaks and then disintegrate into a collection of disordered ('turbulent') vortical structures. Hence, for all transition scenarios considered above, there exists an initial threshold amplitude A_{tr} determining whether the corresponding initial (oblique) disturbances may or may not lead to transition. The value of A_{tr} does not determine the likelihood of this transition scenario but it is clear that a decrease in this value increases the chances that the scenario will be realized in practice. Therefore the evaluations of amplitudes A_{tr} may be quite useful for the assessment of the likelihood of various transition regimes.

Another problem, also having relation to attempts to determine which of the scenarios is the most likely, is the problem of estimation of the 'transition time' T_0 (or streamwise distance) which is necessary for completion of the transition to turbulence (if it may be achieved) by the route considered. The point is that if the time T_0 is large, then there is a real chance that during this time some extraneous disturbances will begin to interfere with the normal flow development and will disrupt the transition process. Hence, an increase in T_0 diminishes the likelihood of the scenario.

For the case of a Blasius boundary-layer flow, an approximate estimate of the values of A_{tr} and T_0 corresponding to the three transition scenarios listed above was made by Schmid et al. (1996) (see also Schmid and Henningson's book (2001)). This estimate was based on the results of the temporal numerical simulations of the three transition scenarios described above, performed by the same authors. All the simulations were of the same type as the simulation of the oblique transition which was briefly described above, and led to the results shown in Fig. 5.32 (here too the boundary layer was assumed to be plane parallel with thickness $\delta(t)$ growing with time, and the initial disturbances included 'random noise' whose energy was about 1 % of the energy of the primary disturbance). The primary disturbances—a plane T-S wave, or a pair of symmetric oblique waves, or a spanwise-periodic array of streamwise vortices—were always chosen to be close to the optimal ones (those which grow most rapidly with time), but the initial amplitudes A of these disturbances were varied, and, in all cases in which transition to turbulence was found to be possible, the simulations were continued up to transition. These computations yielded, for the three scenarios, the dependence of T_0 on the value of the initial amplitude A, and thence the value of A_{tr}, being the greatest value of A at which the transition could not be reached (corresponding to $T_0 = \infty$). Figure 5.33 shows results obtained by Schmid et al. for the Blasius boundary layer with the initial Reynolds number Re* = 500. Here the initial amplitude A is replaced by the initial energy of the primary disturbance $E = \frac{1}{2V}\int_W (u^2 + v^2 + w^2)d\mathbf{x}$, where W is the periodic box domain of the computations, V is its volume, and E and the other dimensional quantities are non-dimensionalized using $\delta*$ and U_0 as length and velocity scales. It is seen that the threshold energy E_{tr} takes its lowest value for the oblique transition, and the highest for the streak-breakdown regime which begins with the appearance of an array of streamwise vortices. The TS-wave-secondary-instability regime (which may be either of N- or of K-type) takes an intermediate place, but at high values of the initial energy E it develops more slowly (leading to a greater value of T_0) than the streak-breakdown regime, and this increases the competitiveness of the latter regime.

5.6.2 Linear and Nonlinear Development of Localized Disturbances

The three transition scenarios considered above all begin with the appearance of some spatially-unbounded disturbance in a laminar Blasius boundary layer. However, it was noted in Chap. 3, that real disturbances appearing in various natural, engineering and laboratory flows are as a rule initially localized in some finite fluid volume. In this respect several papers which were cited in Chap. 3 were devoted to studies of the temporal evolution of localized disturbances in wall-bounded shear flows. Most of these papers dealt with inviscid flows, which are not considered in this chapter (an important exception is the paper by Henningson et al. (1993), some results of which will be discussed below). Moreover, in Chap. 3 only results relating to initial disturbances of very small amplitudes, whose evolution may be described by

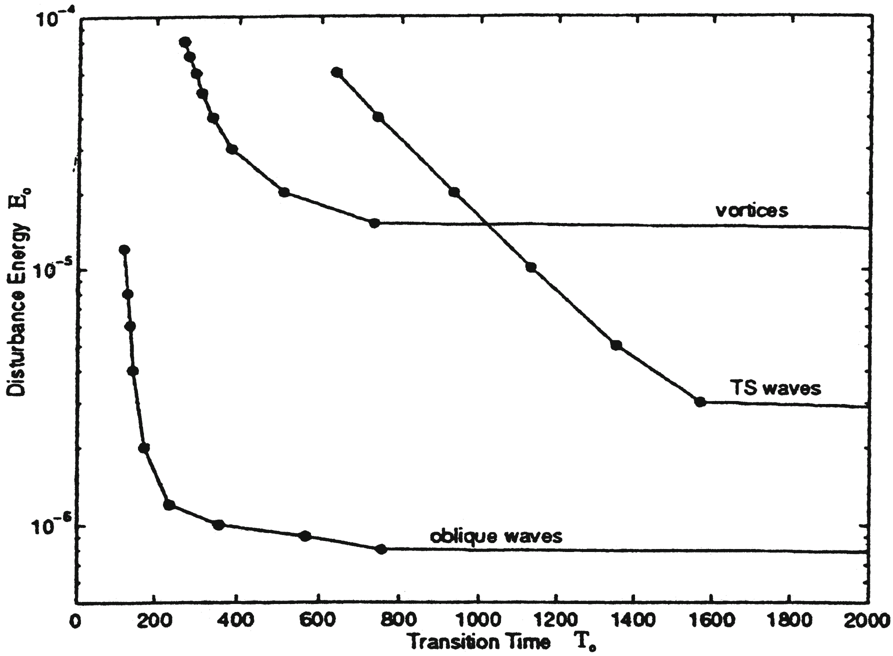

Fig. 5.33 Dependence of the transition time T_0 on the disturbance initial energy E_0 (which is proportional to $A_0{}^2$) for three transition scenarios: (i) 'oblique transition' (labelled 'oblique waves'), (ii) T-S-wave-secondary-instability scenario' (K-regime, labelled 'T-S waves') and (iii) 'streak-breakdown scenario' (labelled 'vortices' since streaks are produced by streamwise vortices which are (0, 2)-structures). The results are for a temporally-growing Blasius boundary layer with initial Re* = 500. (After Schmid et al. 1996)

linearized Navier-Stokes equations, were studied. On the other hand, Gaster and Grant (1975) and Breuer and Haritonidis (1990) (these papers were considered in Chap. 3), who tried to describe the results of their wind-tunnel observations of the evolution of localized disturbances in a boundary-layer flow in the framework of linear stability theory, both found that the deductions from this theory agree with observations only during some initial time interval and become invalid at later times. Hence it is clear that the linear theory is insufficient for a satisfactory description of the development of localized disturbances.

As well as the above-mentioned work by Gaster and Grant, and by Breuer and Haritonidis, other attempts to study evolution of artificially produced localized disturbances in laboratory flat-plate boundary layers have been made; the papers by Gaster (1984, 1990); Tso et al. (1990); Cohen et al. (1991) and Breuer et al. (1997) are just typical examples of such work. Gaster, and Tso et al. paid their main attention to a late stage of the disturbance evolution directly connected with formation of turbulent spots and transition to turbulence; since this chapter is devoted mainly to the weakly-nonlinear effects mentioned above, their papers will be mentioned only occasionally below. The results of Cohen et al. and Breuer et al. will be described below at greater length; first, however, data of quite a different origin will be considered.

The purely theoretical results available at present cannot satisfactorily describe the weakly-nonlinear stage of localized-disturbance development, but results of numerical simulations are more informative. Apparently one of the first attempts to apply nonlinear numerical simulation (i.e., the numerical solution of the appropriate initial-value problem for the nonlinear Navier-Stokes equations) to the study of the evolution of a localized disturbance in a boundary-layer flow was made by Breuer (1988). His numerical-simulation results were then carefully analyzed by Breuer and Landahl (1990). The numerical solution of the nonlinear initial-value problem considered in Breuer's dissertation (1988), and his paper with Landahl, related to the evolution in the Blasius flow (assumed to be plane-parallel but with thickness $\delta(t)$ growing with time) of a localized disturbance initially having the form shown schematically in Chap. 3, Fig. 3.2; see also Eqs. (5.19), (5.22) and Fig. 5.38a. (As noted in Chap. 3, the same model of the initial disturbance was used in stability computations by Russell and Landahl (1984); Henningson (1988); Breuer and Haritonidis (1990) and Henningson et al. (1993); later it was also accepted as one of the three initial conditions considered by Bech et al. (1998)). Breuer and Landahl's paper represented a continuation of the work of Breuer and Haritonidis (1990), where the same initial-value problem was solved for the inviscid linearized N-S equations; some of the results obtained there were shown in Fig. 3.3. These results agreed satisfactorily with Breuer and Haritonidis' wind-tunnel data (relating to a flat-plate boundary layer where localized disturbances of a shape close to that shown in Fig. 3.2 were artificially produced) but only for small and moderate values of dimensionless time $\tau = tU_0/\delta*$. However, for larger values of τ the numerical results of Breuer and Landahl agreed better with the available experimental data than those of Breuer and Haritonidis.

Breuer and Landahl (1990) (and also Landahl et al. 1987) stressed that both Breuer's computational results and the experimental data of Breuer and Haritonidis showed that the disturbance evolving from a strongly-localized initial disturbance in a boundary layer consists of two very different parts. Recall that in Chap. 3, it was pointed out that in the case of small disturbances in a plane-parallel steady inviscid flow, the general solution of the corresponding linear initial value problem includes terms of two different types. (In Chap. 3 this result was attributed to Gustavsson (1978), but in fact it was already mentioned by Case (1960) for the case of two-dimensional disturbances). The first type is formed by the so-called 'convective components' (the adjective 'convective' is sometimes replaced here by 'transient'); these components are convected streamwise with the local flow velocity $U(z)$ and they often undergo considerable transient growth followed by a rapid decline. (In connection with the phenomenon of 'transient growth', much attention was paid in Chap. 3 to these components). The disturbance components of the second type are 'dispersive waves', i.e., waves with phase velocities depending on their frequencies and wave numbers. In Chap. 3 it was stressed that in the case of an 'ideal' (inviscid) fluid the phase velocities c of the wave component do not coincide with the discrete eigenvalues of Rayleigh's eigenvalue problem. However, in the case of fluids with non-zero viscosity the phase velocities of the 'dispersive waves' are just the eigenvalues of the Orr-Sommerfeld eigenvalue problem and the 'dispersive component' of any evolving disturbance is represented by some collection of T-S waves.

The discussion of the 'convective' (i.e. 'transient') and 'dispersive' flow disturbances in Chap. 3 related only to inviscid fluids and to very small (regarded as 'infinitesimal') disturbances. However, the closing sentence of the last paragraph implies that the same notions may also be applied to disturbances in viscous flows. (Recall that small transiently-growing disturbances in viscous fluid flows were in fact considered at length in Sect. 3.3). Also, the results of the above-mentioned papers by Breuer, Landahl, and Haritonidis (and the experimental results of Tso et al.) confirmed that the division of the set of all disturbances in steady plane-parallel (or nearly plane-parallel) flows into 'convective' and 'dispersive' parts is fully appropriate in the case of finite-amplitude disturbances in viscous flows, where the two types can often be easily distinguished. These results also showed that convective disturbances are really transient ones—they grow considerably during a short initial time (or streamwise) intervals but then begin to decay rapidly and as a rule entirely disappear shortly afterwards. Therefore, in studying the long-time evolution of localized disturbances leading to transition to turbulence, it is reasonable to pay attention mainly to dispersive wave disturbances.

One of the results found by Breuer and Landahl is shown in Fig. 5.34; it is similar in many respects (although not identical) to that presented in Fig. 3.3b. Note, in particular, that both figures show the appearance of a strong tilted shear layer between low-speed and high-speed regions produced by the lift-up effect; this result was confirmed later by numerical simulations, both linear and nonlinear, of the development of a localized disturbance in plane Poiseuille and boundary-layer flows by Henningson et al. (1993) (one of the linear results of these authors was shown in Fig. 3.17). However, the linear and nonlinear instability of the boundary layer to two-dimensional waves ($k_2 = 0$) with high values of k_1, found by Breuer and Landahl, strongly contradicted the results of Henningson et al. but, as the latter authors showed, this was due to the insufficient resolution of Breuer and Landahl's computations in the wall-normal direction. However, many other results of these two groups of authors agree quite well with each other (and were confirmed also by the results of careful experiments by Cohen et al. (1991) and Breuer et al. (1997) which will be discussed later). Breuer and Landahl found that nonlinear effects strongly influence the temporal evolution of disturbance structures and the behavior of disturbances at large values of dimensionless time τ. Two-dimensional spatial spectra of the normal-to-wall disturbance velocity $w(x, y, z, t)$ at $z/\delta^* = 1.05$, computed by them for a number of values of τ; show that at $\tau = 43$ the spectrum contours have smooth oval shapes and there is a unique spectral peak at the common center of these ovals. This simple spectral shape is close to that corresponding to the initial conditions in Fig. 3.2. However at larger values of τ the shape becomes much more complicated, and a number of new spectral peaks emerge at points $(k_1\delta^*, k_2\delta^*)$ corresponding to

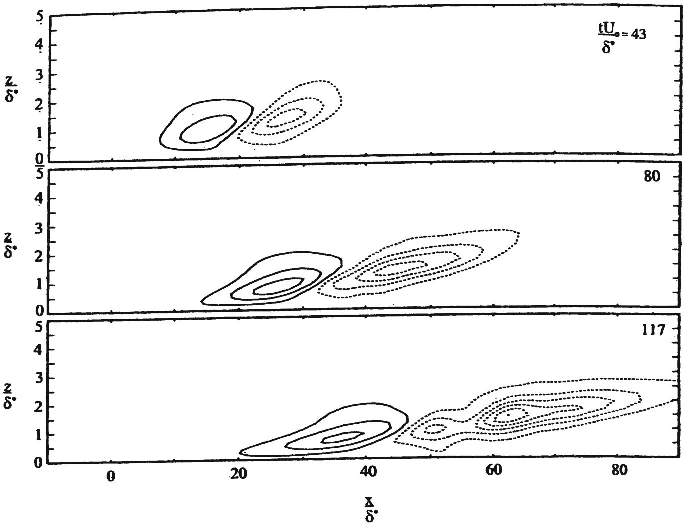

Fig. 5.34 Computed contours in the (x, z)-plane of the streamwise disturbance velocity $u(x, y, z, t)$ at $y = 0$ and several values of t, for a localized disturbance of finite amplitude with given value at $t = 0$. *Solid* and *dotted lines* represent positive and negative velocity values; contour spacing is 2 % of U_0. (After Breuer and Landahl 1990)

larger values of spanwise wavenumber k_2. In particular, at $\tau = 136$ peaks were found at $(k_1\delta^*, k_2\delta^*) \approx (0, 0.7), (0.1, 1.3), (0.2, 2.0)$, and this recalls the series of harmonics of increasing order produced by nonlinear interactions. Henningson et al. showed analytically that, after the appearance of the peaks of the energy distribution at wave vectors $(\pm k_1, \pm k_2)$, the nonlinear interactions give rise to new peaks at $(\pm 2k_1, 0)$ and $(0, \pm 2k_2)$ (the latter will be the most rapidly growing), and also at $(\pm k_1, \pm 3k_2)$, $(0, \pm 4k_2)$, etc., corresponding to propagation of energy to higher spanwise wavenumbers. Moreover, both groups of investigators found that solutions of the nonlinear initial-value problem imply the generation, at later stages of the instability-development process, of a system of long spanwise-alternating streaks of high- and low-speed fluid (see for example Fig. 5.35 by Hennington et al.; similar figures were also presented by Breuer and Landahl, and Bech et al.). These streaks then form streamwise-elongated vortical structures, recalling the streamwise Λ-vortices observed in other regimes of boundary-layer transition, and still later produce turbulent spots which are the precursors of full transition to turbulence (these stages of instability development were more explicitly described by Cohen et al. and Breuer et al.). Therefore, the results support Morkovin's idea of the ordinariness of so-called 'by-pass boundary-layer transitions' whose late stages do not differ much from those for transitions initiated by primary T-S waves. Moreover, they allow the *localized-disturbance scenario* of boundary-layer transition to be added to the other three transition scenarios considered above.

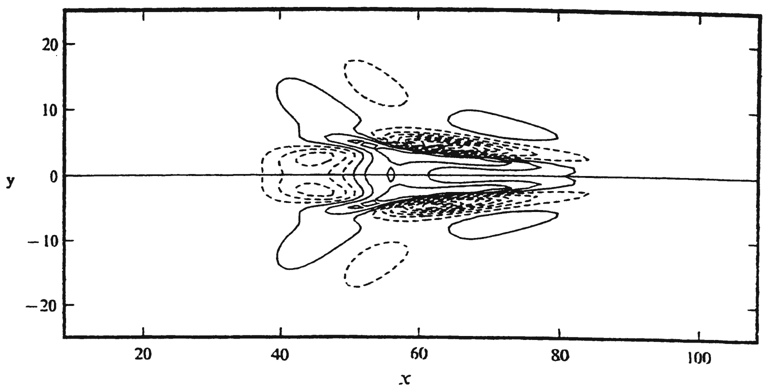

Fig. 5.35 Contours in the horizontal (x, y)-plane of the vertical velocity $w(x, y, z, t)$ at $z = 0.99$, $t = 117$ (all quantities are non-dimensionalized by scales δ^* and U_0). *Solid* and *dotted lines* represent positive and negative velocity values; controur spacing is 0.001. (After Henningson et al. 1993)

The nonlinear interactions play an important part in the temporal evolution of high- and moderate-amplitude wave packets consisting of a collection of two- and three-dimensional T-S waves. Let us recall that in Sect. 3.31 it was mentioned that such wave packets were used by a number of researchers as natural models of localized disturbances in a boundary layer. In particular, Gaster (1975) used a wave-packet model to describe quantitatively the results of Gaster and Grant's (1975) experiments on the development of a localized disturbance, produced by a short acoustic pulse, in the boundary layer on a flat plate. The streamwise evolution of such disturbances was investigated by hot-wire measurements of the streamwise disturbance velocity u at $z = 3.2\delta^*$ (i.e., slightly above the boundary layer) with various values of x and y. As stated in Sect. 3.31, Gaster modeled this evolution by representing the values of the streamwise-velocity disturbances $u(x, y, z, t)$ at positive values of x in the form:

$$u(x, y, z, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(k_2, \omega; z) \exp\left[i\{k_1(k_2, \omega)x + k_2 y - \omega t\}\right] dk_2 d\omega \qquad (5.17)$$

(this equation appeared in Sect. 3.31 as Eq. (3.52)). Here $u(k_2, \omega; z)$ is the Fourier transform, with respect to y and t, of the initial value of the streamwise velocity disturbance at $x = 0$ and a fixed value of z, and $k_1(k_2, \omega)$ is the complex eigenvalue with the smallest imaginary part appearing in the spatial 3D Orr-Sommerfeld eigenvalue problem (2.41), (2.42) (corresponding to given values of k_2 and ω).[10]

[10] In both papers it was assumed that the boundary layer is plane-parallel but in the treatment of data relating to a given value of x, values of δ^* and Re* corresponding to this x were used. (A more precise analysis of some data of Cohen et al., which took into account the streamwise growth of the boundary layer, was developed by Cohen (1994).) Measurements by Cohen et al. and Breuer et al. showed that in their studies the pressure gradient in the boundary layer was slightly negative, and therefore the function U(z) was slightly closer to a Falkner-Skan profile for $\beta \approx 0.01$ (see Chap. 2,

Gaster and Grant dealt with the supercritical ($Re > Re_{cr}$) boundary layer to which a small disturbance was introduced at $x = 0$. Hence there existed that plane wave which grows most rapidly with x, having frequency $\omega = \omega_0$ and the streamwise wavenumber $k_1 = k_1(0,\omega)$ given by the O-S eigenvalue with numerically-greatest negative imaginary part. Moreover, the waves with (k_2, ω)-values close to (0, ω_0) and $k_1 = k_1(k_2, \omega)$ are also spatially growing in this case, and their rate of growth is only slightly smaller than that of the most unstable wave. There is also a larger region of the (k_2, ω)-plane which corresponds to the collection of all spatially-growing waves. Gaster described the evolution of the localized disturbance by an approximate numerical value of the integral in Eq. (5.17) in which only spatially-growing waves were taken into account. Thus, the approximate solution of the initial-value problem he considered has the form of a superposition of spatially-growing two- and three-dimensional T-S waves each of which is the least stable of the waves with the same values of k_2 and ω and, being governed by linear stability theory, does not interact with the other waves.

In Gaster and Grant's experiments the amplitude of a wave packet took rather low values and they found that in this case the theoretical model (5.17) led to results which agreed well with their observations at the majority of the measuring stations. However, they noted that the data obtained at the largest value of x disagreed with the predictions of Eq. (5.17). The authors explained this discrepancy by the influence of nonlinear effects at large x. This explanation is evidently confirmed by the results presented above, relating to transition scenarios starting with the growth of T-S waves. In fact, these results show that even when there is only one such wave whose amplitude exceeds a relatively small threshold value, it necessarily begins to interact at once with the background disturbances ("noise") that always exist in practice. Moreover, in the case of a group of growing T-S waves, their nonlinear interactions must necessarily become apparent after quite a short period of independent development. Therefore model (5.17) of wave-packet development can represent only an approximation applicable to packets of small initial amplitude during some limited initial period of time.

Cohen et al. (1991) and Breuer et al. (1997) repeated the experiments by Gaster and Grant (1975) using a low-turbulence wind tunnel with a test section about 6 m long. Cohen et al. made hot-wire measurements of the mean-velocity profile $U(z)$ (depending only on the local value of δ^*) and of the three disturbance-velocity components u, v and w at a great number of points $\mathbf{x} = (x, y, z)$ inside the boundary layer, while Breuer et al. measured only the streamwise disturbance velocity u but with a rake of hot-wire probes to make simultaneous measurements of u at eight values of z. The long wind-tunnel test section made possible the observation of boundary-layer development over a large range of x. Moreover, the disturbance generator (which produced short sinusoidal air pulses of acoustic origin) allowed the amplitude A of the initial localized disturbance to be varied easily. The Reynolds number $Re^* = U_o\delta^*/\nu$ at the location of this generator was close to 1,000 (well above the critical value) in these experiments.

p. 119) than to the Blasius profile corresponding to $\beta = 0$. However, the Blasius approximation was found to be accurate enough to be usable in the analysis of the experimental data.

Cohen et al. and, later, Breuer et al. found that at small enough values of A three different stages of streamwise development of wave packets may be observed. The first stage (called the *linear stage* by these authors) corresponded very well to Gaster and Grant's observations; here Gaster's Eq. (5.17) (based on the linear stability theory) described the disturbance evolution with high accuracy. Spectral analysis of the velocity fluctuations showed that during this stage the disturbance included both two- and three- dimensional T-S waves corresponding to ranges of dimensionless frequencies $\omega = 2\pi f\delta^*/U_0$ (where f is the dimensional frequency measured in Hz) and spanwise wavenumbers $K_2 = k_2\delta^*$ centered around the values $\omega = \omega_0$ and $K_2 = 0$ corresponding to the most rapidly growing T-S wave (which is two-dimensional, i.e. with $K_2 = 0$, by virtue of Squire's theorem—see Chap. 2). (Symbols ω, K_2 and K_1 ($= k_1\delta^*$) will now denote dimensionless frequencies and wavenumbers). One example of the (K_2, ω)-spectrum found in the linear stage of localized-disturbance development is shown in Fig. 5.36a. In full accordance with Gaster's model (5.17), during the linear stage the values of $K_1 = K_1(K_2, \omega)$ could be determined for all waves considered by means of the O-S equation, as those corresponding to the most-unstable wave with given values of ω and K_2. It was also found that as the wave packet moved downstream all wave components evolved according to the O-S equations (and hence independently from each other). Therefore, in the linear stage of disturbance development the most rapidly growing T-S wave, and a group of T-S waves with frequency and wave number close to the most rapidly growing wave (and hence with values of K_2 and ω close to $K_2 = 0$ and $\omega = \omega_0$ corresponding to the most unstable T-S wave), gained energy most effectively. As a result, a relatively narrow band of two- and three-dimensional T-S waves centered at the most-amplified wave quickly began to play the dominant role in the evolution of the wave packet. In the initial series of experiments by Cohen et al. the first (linear) stage was observed from $x = 160$ cm to $x = 220$ cm (the disturbance generator being placed at $x = x_0 = 81$ cm from the plate leading edge). The amplitude A of the wave packet was close to 0.3 % of U_0 at $x = 160$ cm and grew to 0.46 % of U_0 at $x = 220$ cm (i.e., during the linear stage it continued to be quite small).

At $x = 220$ cm the second stage of wave-packet development began. Here, in addition to the spectral peak at (0, ω_0) two additional spectral peaks of the two-dimensional (K_2, ω)-spectrum appeared at the points ($K_{2,1}$, ω_1) and ($-K_{2,1}$, ω_1) which corresponded to a definite pair of symmetric oblique waves (see Fig. 5.36b). Cohen et al. discovered that the peak frequency ω_1 was equal to $\omega_0/2$, i.e. half the frequency of the most-amplified 2D wave. (This fact agrees with Gaster's 1990 discovery of spectral peaks at frequencies $\omega/2$ and $3\omega/2$ (the latter was clearly due to secondary nonlinear interactions) in the wave packet produced in the boundary layer on a flat plate by a sinusoidal acoustic signal of frequency ω). Cohen et al. also found that to the peak frequency ω_1 and spanwise wavenumber $K_{2,1}$ there corresponded the T-S wave with complex streamwise wavenumber $K_1 = K_1(K_{2,1}, \omega_1)$ having the real part $\Re e K_1(K_{2,1}, \omega_1)$ close to half the real part $\Re e K_1(0, \omega_0)$ of the complex streamwise wavenumber $K_1(0, \omega_0)$ of the most unstable 2D wave. Hence the two new spectral peaks appearing in the second stage of the localized-disturbance development together with the existing spectral peak at the point

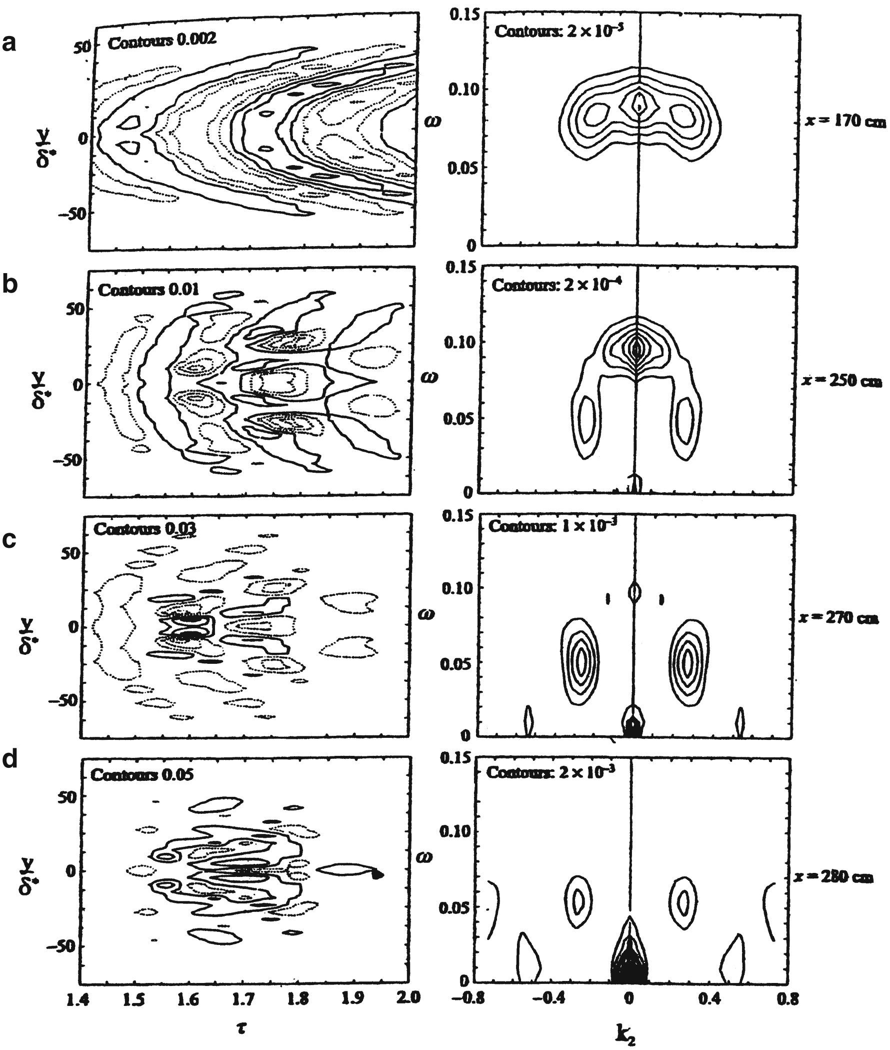

Fig. 5.36 Overall view of the dependence of velocity $u(x, y, z, t)$ on y and $\tau = (t - T_0)U_0/(x - x_0)$ (*left-hand column*) and of the dependence of the corresponding wavenumber-frequency spectra on ω and k_2 (*right-hand column*) at $z/\delta^* = 0.5$ and four different values of x (data for separate x-values are noted by marks (**a**), (**b**), (**c**) and (**d**)). *Solid* and *dotted lines* show positive and negative values, respectively. (After Breuer et al. 1997)

$(0, \omega_0)$ corresponded to a Craik's resonant triad of T-S waves. To the spectral regions surrounding two peaks at points $(K_{2,1}, \omega_1)$ and $(-K_{2,1}, \omega_1)$ there corresponded two symmetric bands of subharmonic oblique waves with frequencies close to $\omega_0/2$, recalling the band of subharmonic oblique waves appearing during the N-regime of instability development initiated by the primary unstable plane T-S wave (this band is clearly seen in Figs. 5.5a, b). During the second stage of wave-packet development the two bands of subharmonic oblique waves gained energy very effectively, so that the oblique waves experienced rapid growth, exceeding considerably the growth of waves corresponding to the primary peak centered at the point $(0, \omega_0)$. Cohen et al. suggested that this gain was due to a number of three-wave resonances. They also found that the primary band of waves, with frequencies close to ω_0 and small values of $|K_2|$, began to lose its energy somewhere in the initial part of the second stage (where the growth of its waves turns into decay) and disappeared entirely in the third stage (see Figs. 5.36c, d). and Breuer et al. (1997) called the second stage of the wave-packet development the *subharmonic stage*. In the first series of experiments considered above, published in 1991, this stage was observed between $x = 220$ and $x = 300$ cm, and within this range the amplitude A of the wave packet increased from 0.46 % of U_0 to 5.2 % of U_0 (this growth evidently considerably exceeds that observed in the first stage). Cohen et al. (1991) found that in this stage the weakly-nonlinear stability theory, which disregards the higher-order terms of the amplitude-power expansions, may be applied to computation of disturbance development.(The attempt by Zel'man and Smorodsky (1990) to describe a wave-packet evolution by a system of amplitude equations relates to just this stage).

The third and final stage of wave-packet development is strongly nonlinear. Here a number of new spectral peaks, representing sums and differences of spectral characteristics of primary and secondary waves and due to the nonlinear interactions of the latter, appear in the disturbance spectra. In particular, the (0, 0)-mode corresponding to velocity-profile distortion also emerges from such interactions, and may lead to the appearance of local profile inflections, producing quasi-inviscid flow instabilities and high-frequency small-amplitude velocity oscillations. These oscillations have random phases and may later contribute to the formation of turbulent spots, indicating the imminence of transition to turbulence (see, however, the closing part of Sect. 5.5 where the appearance of turbulent spots is connected with the evolution of Klebanoff's spikes, which have an origin other than inflection-generated oscillations). In the series of experiments by Cohen et al. considered above, the third stage was observed between $x = 320$ and $x = 350$ cm and was accompanied by rapid growth of disturbance energy leading at $x = 350$ cm to a very high value of amplitude A, close to 27 % of U_0. To study the second and third stages of the wave-packet development, Cohen et al. performed a number of experiments with larger initial values of A to shift these stages upstream and thus make observations easier. A more detailed experimental study of the late stage of wave-packet development was carried out by Breuer et al. (1997), while Cohen (1994) published some theoretical considerations relating to the initial stage of wave-packet development, and compared his theoretical results with the experimental data of Cohen et al. (1991).

The theoretical results of Cohen (1994) were based on an improved linear model of the evolution of waves in a laminar boundary-layer flow. This model took into account the nonparallelism of the flow (i.e. the weak dependence of δ^* on x) by an approximate method developed by Saric and Nayfeh (1975) and Nayfeh and Padhye (1979). Cohen extended Gaster's (1975) model to the case of a slightly nonparallel boundary layer and then calculated anew the time evolution of amplitudes for a large number of two- and three-dimensional T-S components of the wave packet studied by Cohen et al. (1991). Data obtained in the latter work for the evolution of amplitudes of individual waves were then compared with the evolution predicted by the extended linear stability theory. Cohen found that the results of the linear stability theory relating to the most rapidly growing two-dimensional T-S wave of frequency ω_0 or to any T-S waves with values of (K_2, ω) close to $(0, \omega_0)$ and high rates of the 'linear' spatial growth, agreed very well with the experimental data within the whole first stage of wave-packet development and a considerable part of the second stage. However, the subharmonic oblique modes with frequencies close to $\omega_0/2$ begin to grow much faster than predicted by linear stability theory, before the end of the first stage of wave-packet development (this was not observed in experiments since at corresponding values of x the subharmonic modes were still rather weak). Thus Cohen (1994) concluded that in the case of wave propagation in a laminar boundary layer, nonlinear effects often become significant at appreciably smaller values of x (measured from the leading edge of the plate) than was assumed earlier, and these effects make the linear stability theory inapplicable to subharmonic wave modes for all but rather small values of x.

The measurements carried out by Breuer et al. (1997) were confined to the streamwise velocity components $U(z)$ and $u(x, y, z, t)$ but they were made in a very dense grid of spatial points and times, and provided the authors with a vast amount of numerical data. (In particular, a great number of repeated observations yielded large ensembles of data, guaranteeing the accuracy of statistical characteristics). The results found by Breuer et al. supported, and made more precise, the conclusions of the paper by Cohen et al. As an example of the new results, Fig. 5.36 shows the constant-velocity contours in the (τ, y)-plane and the corresponding two-dimensional (K_2, ω)-spectra for velocities $u(x,y,z,t)$ at $z/\delta^* = 0.5$ and for four values of x relating to the first, second, and third (two x-values) stages of wave-packet development.[11] Here $\tau = (t - T_0)U_0/(x - x_0)$ is non-dimensionalized time, t is dimensional time of the measurement (counted from the moment of air-pulse ejection by disturbance generator), x is the x-coordinate of the measurements counted from the leading edge of the plate, $x_0 = 81$ cm is the x-coordinate of the disturbance generator, while T_0 is the delay time, proportional to $(x - x_0)/U_0$ with a proportionality coefficient chosen to make the origin of the time τ close enough to the time when the leading edge of the wave packet reaches the measurement coordinate x.

The two upper diagrams in Fig. 5.36, labeled as Fig. 5.36a, are for $x = 170$ cm, within the first (linear) stage of disturbance development (the wave-packet amplitude A was here close to 0.6 % of U_0). At this value of x the u-velocity contours had the form of smooth swept-back crescents, which was also the form of the wave-packet observed by Gaster and Grant at points far from the disturbance generator (closer to the generator, Gaster and Grant's wave packet had an oval shape). The $(K_2,$

[11] The measurements by Breuer et al. discussed here related to waves excited by an acoustic pulse with a different amplitude from that used in the experiments by Cohen et al. (1991). Therefore the streamwise locations of the three stages of wave-packet development mentioned in our discussion of the results of Cohen et al. are not the same as those in the series of experiments considered here.

ω)-spectrum shows that, at this x, most of the wave-packet energy is concentrated in the band of 2D modes (and 'almost 2D' modes with $|K_2| \ll 1$), centered at the mode with $(K_2,\omega) = (0,0.09)$ which is just the most-unstable T-S wave at the Re* corresponding to $x = 170$ cm. There are also two much smaller spectral peaks at points $(K_2, \omega) \approx (\pm 0.25, 0.085)$, which apparently represent weak 'oblique-wave contributions' to the disturbance energy at x-values corresponding to the first stage of wave-packet development, as noted by Cohen et al. (1991).

At $x = 250$ cm, in the second (subharmonic) stage of disturbance development, the nonlinear effects were much more influential and this is clearly seen in Fig. 5.36b. In particular, two 'side spectral peaks' appeared here, at the frequency $\omega_1 \approx \omega_0/2$ and spanwise wavenumbers $\pm K_{2,1} \approx \pm 0.25$. These peaks acquired their energy from preexisting 'background noise' and the values of ω_1 and $K_{2,1}$ implies that 3D waves corresponding to them have a phase velocity close to that of the most-unstable 2D wave. This means that these 3D waves, together with the most-unstable 2D wave, form a resonant triad (but not necessarily of Craik's 'fully-resonant' type where Eq. (5.7) are exactly valid). Thus the growth of subharmonic modes corresponding to the side peaks and to spectral regions adjacent to them may be due to three-wave resonance or to secondary instability of the primary waves to subharmonic disturbances—cf. the discussion of the N-regime of boundary-layer development in Sects. 5.3 and 5.4. The velocity contours at $x = 250$ cm show that some streamwise elongated structures appeared, with some similarity to streamwise Λ-vortices. Note also the appearance of a group of waves, apparently produced by nonlinear wave interactions, with (K_2, ω)-values belonging to the 'low-K_2, low-ω' spectral region near the mean-flow distortion mode with $(K_2, \omega) = (0, 0)$.

The two lower pairs of diagrams in Fig. 5.36 (Figs. 5.36c and d) correspond to streamwise coordinates $x = 270$ cm and $x = 280$ cm, in the third, strongly-nonlinear stage of wave-packet development. The corresponding velocity contours show the formation at $x = 270$ cm of a system of elongated structures including spanwise-alternating streaks of fluid having alternately higher and lower streamwise velocity than the mean $U(z)$. At $x = 280$ cm this system is more compact and gives the impression of approaching the 'turbulent spot' stage (other experimental results given in the paper by Breuer et al. (1997) allowed the authors to suggest that the formation of turbulent spots actually began near $x = 282.5$ cm). The velocity spectrum at $x = 270$ cm shows that the primary band of waves centered at the most-unstable $(0, \omega_0)$-wave has practically disappeared here, but the subharmonic bands with frequencies close to $\omega_0/2$ became considerably more pronounced. The spectral peak at the coordinate origin, and the adjacent region of 'low-K_2, low-ω' points corresponding to mean-flow distortions and nearly-2D low-frequency waves, also grew considerably in comparison to those at $x = 250$ cm. In addition two small spectral bands appeared near the peaks at points $(\pm 2K_{2,1}, 0)$, produced by nonlinear interactions of $(\pm K_{2,1}, \omega_1)$ and $(0, \omega_0) = (0, 2\omega_1)$ modes. At $x = 280$ cm the primary spectral band adjacent to the point $(0, \omega_0)$ completely disappeared, and bands around the subharmonic peaks at $(\pm K_{2,1}, \omega_1)$ became less energetic than at x = 270 cm, while bands near points (0,0) and $(\pm 2K_{2,1}, 0)$ became much more pronounced and other peaks appeared near the points $(\pm 3K_{2,1}, \omega_1)$. (According to Breuer et al., diagrams more detailed than Fig. 5.36d show additional spectral peaks at x = 280 cm, at points (0, ω_1), (0, 3ω_1), and near the points (3$K_{2,1}$, 2ω_1) and (3$K_{2,1}$, 3ω_1)). These results clearly agree with observations by Breuer and Landahl (1990) and Henningson et al. (1993) of the 'propagation of the disturbance energy along the K2-axis'.

Further results of Breuer et al. (1997) describe in more detail the spatial and spectral structures accompanying the wave-packet development. In Fig. 5.37, contours in the (ω, z)-plane of the frequency spectra P(ω; x) = P(ω; x, y, z) of velocity oscillations u(x, t) at points $\mathbf{x} = (x, y, z)$ are shown for $y/\delta^* = 4.7$ and four values of x corresponding to the different development stages. These contours again illustrate that at $x = 170$ cm (i.e., in the linear stage) the disturbance energy is concentrated near the frequency $\omega_0 = 0.09$ corresponding to the most-unstable T-S wave, and that near $x = 250$ cm (in the subharmonic stage) an additional band of oscillations, with frequencies close to $\omega_0/2 = 0.045$, appears. These figures also show vertical profiles of different spectral components, which agree well with the results of linear stability theory at $x = 170$ cm, while by $x = 250$ cm they have become more complicated. However the data for $x = 270$ and 280 cm, relating to the strongly-nonlinear third stage, show rather energetic high-frequency components of u-fluctuations which are absent from Fig. 5.36. The reason for this discrepancy is apparently that Fig. 5.36 shows spectra of the ensemble-averaged velocity fields, and if the high-frequency oscillations have random phases they will be canceled by ensemble averaging. However, spectral contours in Fig. 5.37 were obtained from spectra computed for individual observations by subsequent ensemble averaging. It is clear that here the contributions to various individual spectra from oscillations with the same frequency but different phases will be added to each other in the sum of individual spectra, and will be represented by the ensemble-averaged contributions in the averaged spectra of Fig. 5.37. Therefore, the high-frequency velocity oscillations shown in Figs. 5.37c, d (but absent from Figs. 5.36c, d) must be real. They may be connected, e.g., with local velocity-profile inflections due to distortions of the local mean-velocity profiles by strongly-amplified disturbances; such local profile inflections were also observed by Breuer et al.

The appearance of high-frequency velocity fluctuations with random phases clearly means that the flow has acquired disorderly features typical of turbulence. Hence the observations summarized in Figs. 5.37a–d have a direct bearing on studies of the onset of randomness in boundary-layer flows. Note that Breuer et al. also consistently observed, during the late stages of instability development, the appearance of 'spike disturbances' of the same type as found by Klebanoff et al. (1962), and later by many others, in boundary layers excited by a vibrating ribbon. Therefore, the experiments of Breuer et al. proved very convincingly that spikes are a rather general phenomenon, unrelated to any special mechanism of disturbance generation. Moreover, since the authors repeated their observations many times, collecting an ensemble of observations under identical conditions, they were able to show that spikes are quite repeatable—they regularly appear at practically the same points and preserve the same main features in all repetitions. Hence the observations by Breuer et al. confirmed the earlier statement of Borodulin and Kachanov about the

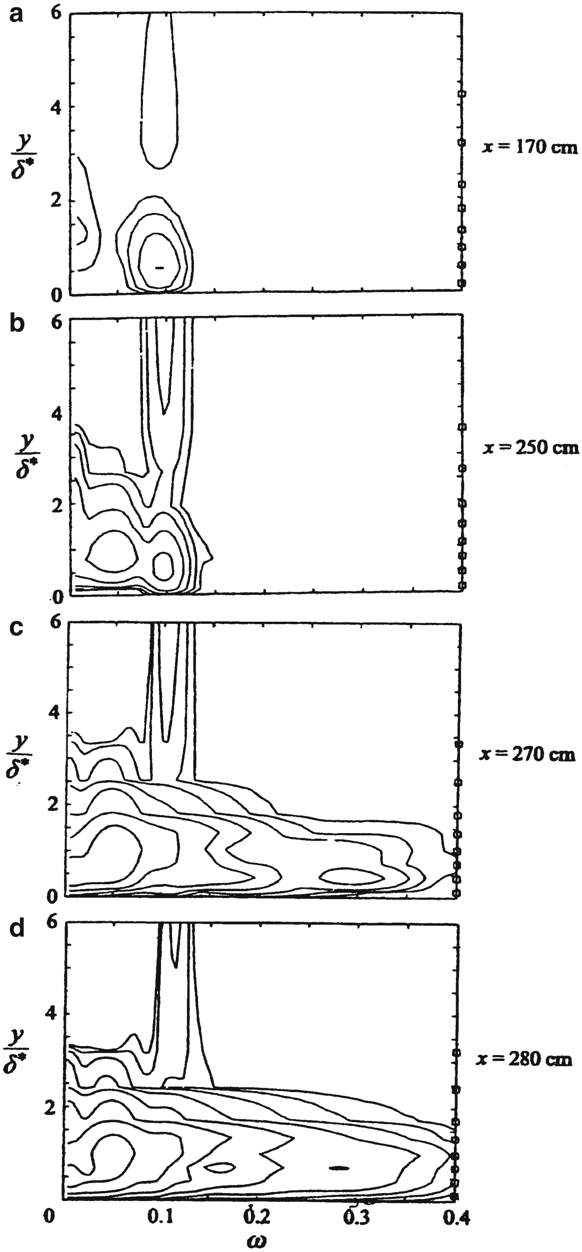

Fig. 5.37 Contours in the (ω, z)-plane of averaged frequency spectra $P_z(\omega)$ of streamwise velocity fluctuations $u(x, y, z, t)$ at $y/\delta^* = 4.7$, for four different values of x. Spectra $P_z(\omega)$ were computed for a number of independently-observed velocity fields and then were averaged over the ensemble of made observations. (After Breuer et al. 1997)

regular, non-random nature of spikes. On the other hand, LHSI-produced small-amplitude high-frequency oscillations have random phases and amplitudes, and thus these disturbances may generate the early flow randomness.

The final part of the paper of Breuer et al. is devoted entirely to discussion of the late-stage transformations of the wave-packet studied. These transformations lead at first to the appearance of 'turbulent spots' (as noted above, the authors found that their formation begins near $x = 282.5$ cm; recall that it is connected also with the latest stages of spike development discussed at the end of Sect. 5.5) and then to the onset of the laminar-flow breakdown to a chaotic ('turbulent') state. (Quite another approach to the study of these transformations was sketched by Waleffe (1995) in the paper cited above; see also the recent survey by Bowles (2000)). Additional information about the 'breakdown-stage' of wave-packet development is contained in Gaster's (1990) description of the results of his experiments; less detailed observations relating to this stage were described by Tso et al. (1990). However, this final stage of instability development is beyond the scope of the present chapter.

In the above discussion of the (temporal or spatial) development of localized disturbances in a laminar boundary layer it was usually assumed that the initial disturbance had a form close to that sketched in Fig. 3.2 of Chap. 3. This rather special assumption was accepted here, since it was widely used in numerical simulations of this development performed by various researchers. Therefore, even in the analysis of experiments where the initial disturbance was produced by some 'disturbance generator' and clearly did not coincide with that in Fig. 3.2, the data were often compared with numerical results for this special initial form of disturbance.

One of the purposes of the recent numerical-simulation work by Bech et al. (1998) was just the verification of the influence of the initial form of a localized disturbance on its subsequent development. The authors also made an attempt to verify the accuracy of the approximate method of temporal numerical simulation of disturbance development, used in this and almost all previous simulation studies. Finally, apparently the main object pursued by the authors was the determination of the influence of non-zero pressure gradient dp/dx on the development of localized disturbances in a laminar boundary layer.

To determine the influence of the initial form of a disturbance, Bech et al. chose three different forms and solved the corresponding initial-value problems numerically for the full Navier-Stokes equations. All chosen forms of the initial velocity field $\mathbf{u}(\mathbf{x}) = \{u(\mathbf{x}), v(\mathbf{x}), w(\mathbf{x})\}$, where $\mathbf{x} = (x, y, z)$, corresponded to 'localized disturbances', with values of $\mathbf{u}(\mathbf{x})$ differing noticeably from zero only in a bounded region surrounding the coordinate origin. Moreover, all these disturbance forms could be represented in terms of a scalar streamfunction $\psi(x, y, z)$ which for the three cases considered had the forms:

$$\psi = Axyz^3 \exp\left(-\left[x^2 + y^2 + z^2\right]\right), \tag{5.18}$$

$$\psi = Axz^3 \exp\left(-\left[x^2 + y^2 + z^2\right]\right), \tag{5.19}$$

and

$$\psi = 0.5Ar^2sz^3 \exp\left(-\left[r^2 + z^2\right]\right), \quad r^2 = x^2 + y^2. \tag{5.20}$$

Here the coordinates x, y, z are assumed to be non-dimensionalized by length scales l_x, l_y and l_z (which Bech et al. defined separately for the three models), A is a disturbance amplitude, small in comparison with the free-stream velocity U_0, and the velocity fields $\{u, v, w\}$ in the three cases are expressed in terms of the function ψ by the following three different equations:

$$\{u, v, w\} = \left\{0, -\frac{\partial\psi}{\partial z}, \frac{\partial\psi}{\partial y}\right\}, \tag{5.21}$$

$$\{u, v, w\} = \left\{\frac{\partial\psi}{\partial z}, 0, -\frac{\partial\psi}{\partial x}\right\}, \tag{5.22}$$

and

$$\{u, v, w\} = \left\{-\left(\frac{\partial\psi}{\partial z}\right)xr^{-2}, -\left(\frac{\delta\psi}{\partial z}\right)yr^{-2}, \left(\frac{\delta\psi}{\delta r}\right)r^{-1}\right\}. \tag{5.23}$$

The simulations were carried out with $\mathrm{Re}^* = 950$ at $x = 0$, and all the lengths, velocities and times relating to process of disturbance development were made dimensionless by the length δ^* at $x = 0$ and the free-stream velocity U_0.

The first model of Eqs. (5.18) and (5.21) just corresponds to the form sketched in Fig. 3.2 of Chap. 3; the equations given here for the initial velocity field agree exactly with those used by Henningson et al. (1993), and are almost identical to those used by Breuer and Haritonidis (1990) and Breuer and Landahl (1990). (Recall the remark in Chap. 3, that in this model disturbance the initial streamwise velocity $u(\mathbf{x})$ is everywhere equal to zero, but it undergoes rapid transient growth and soon becomes greater than the other two velocity components). A schematic form of the initial vertical-velocity contours for this model is shown in Fig. 5.38a. In the second model (5.19), (5.22) the initial spanwise velocity $v(\mathbf{x})$ is equal to zero; this model describes a wave packet where the energy is mainly concentrated in plane 2D waves. The initial velocity contours for this model are shown in Fig. 5.38b. The third model (5.20), (5.23) has already been used in numerical simulations by Henningson et al. (1993); here it is assumed that $l_x = l_y$ and hence the initial disturbance is axisymmetric with respect to the vertical z-axis (see Fig. 5.38c).

The numerical simulations of the disturbance development in a boundary layer presented in the main part of the paper by Bech et al. were temporal ones, i.e., they were based on the assumption that the flow is plane-parallel and the spatial Fourier-components of disturbance velocities evolve in time. (As is now usual, the parallel-flow assumption was supplemented by the assumption that the boundary-layer thickness δ is not constant but grows with time; cf. footnote 4). To verify

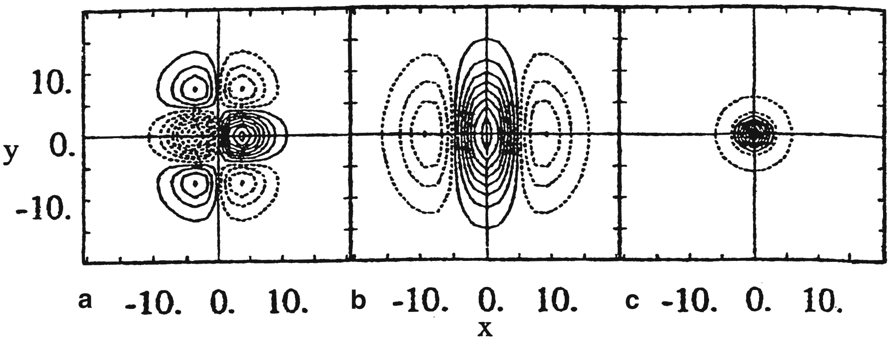

Fig. 5.38 Contours in the (x, y)-plane of the initial vertical velocity $w(x, y, z)$ at $z/\delta^* = 1.5$ for three selected models of the initial velocity field. The marks (**a**), (**b**) and (**c**) correspond to the first, second and third models. (After Bech et al. 1998)

the accuracy of the somewhat simplified temporal approach, one of the temporal simulations was repeated, using a spatial approximation which assumes that the flow is steady but may be nonparallel, and that disturbances are time-periodic and spatially evolving. The latter approximation is evidently more accurate than the temporal one but it is also more complicated and more expensive in computer time. Comparison of the results of the two simulations revealed some small inaccuracies of the temporal-simulation results, but also showed that these inaccuracies appear only at rather late stages of wave-packet development, while the overwhelming majority of predictions of the temporal simulations agreed quite satisfactorily, qualitatively and quantitatively, with those of the approximate spatial simulation. Thus, it was concluded that the results of temporal simulations were sufficiently reliable to be investigated in detail.

At the beginning of the paper of Bech et al. some simpler small-amplitude results were considered. Here the authors analyzed numerical solutions of the linear initial-value problem (corresponding to linearized N-S equations) for three chosen forms of the initial velocity field, where the streamfunction amplitude A was chosen to make the maximum $|w_0|$ of the initial vertical velocity equal to $10^{-5}U_0$. (Solutions were computed for two values of the pressure gradient, but for now only the case of a Blasius boundary layer, with $dp/dx = 0$, will be discussed). 'Linear' (given by the linear stability theory) temporal growths of the disturbance energy and of the maximal values of streamwise and vertical velocities were computed for various values of the time t. It was found that results for the three initial conditions described differ considerably from each other, as must be the case since both the partition of the developing disturbance into convective and dispersive components, and also the T-S-wave composition of the dispersive component, were different in the three cases. Then the flow patterns arising from the three chosen initial conditions were reconstructed, for the later stages of the 'linear development', from simulation results. Figures 5.39a–f show velocity contours of u and w for the dimensionless time $t = 300$. Figures 5.39a and c show that in cases 1 and 3 (corresponding to initial streamfunctions (5.18) and (5.20)) the elongated streaky structures, composed of alternating streaks

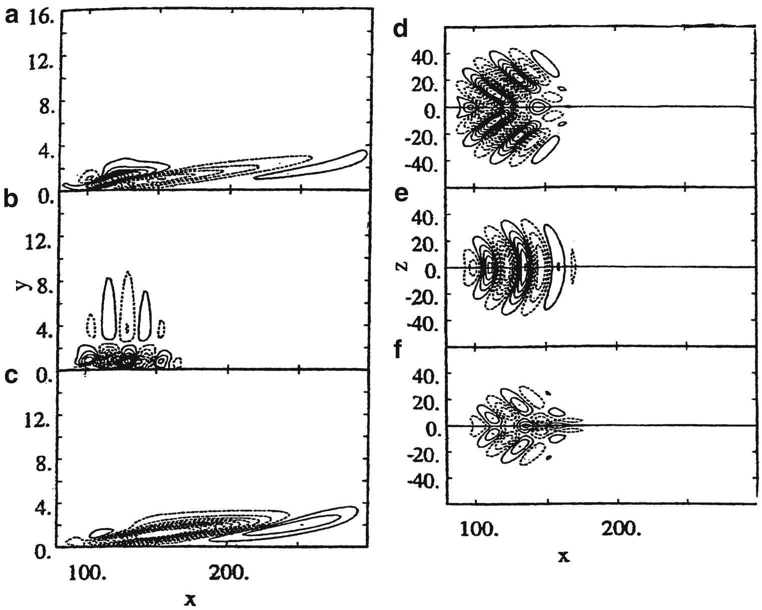

Fig. 5.39 **a**–**c** Contours in the (x, z)-plane of the velocity $u(x, y, z, t)$ at $y = 0$ and $t = 300$ for three selected models of the initial velocity field. Labels (**a**), (**b**) and (**c**) have the same meaning as in Fig. 5.38. **d**–**f** Contours in the (x, y)-plane of the vertical velocity $w(x, y, z, t)$ at $z = 1$ and $t = 300$ for three selected models of the initial velocity field. The labels (**d**), (**e**) and (**f**) correspond to the first, second and third models. The results represent computations at $\text{Re}^* = 950$ and such amplitude A that $\max_x |w(\mathbf{x}, 0)| = 10^{-5}$. All quantities are made dimensionless by scales δ^* and U_0

of low and high streamwise velocities, emerged in the flow before $t = 300$. It seems evident that these streaky structures were produced here by the transiently-growing part of the disturbance, subjected to Landahl's (1980) mechanism of streamwise elongation. This mechanism affects only the velocity u; therefore the contours of vertical velocity w in Figs. 5.39d–f, which again are quite similar to each other in cases 1 and 3 but have somewhat different forms in case 2, represent typical wave-packet structures corresponding to the dispersive part of the developing disturbance. Recall that in case 2 (initial streamfunction (5.19)) the initial disturbance had vanishing spanwise velocity and contained no $(0, k_2)$-modes, so that the generation of spanwise inhomogeneity played an important part in the formation of streaky structures. For this reason the initial disturbance (5.19), (5.22) produced no streaky structures by $t = 300$ and Fig. 5.39b is quite different from Figs. 5.39a and c.

Bech et al. also analyzed the appearance of weakly-nonlinear effects on disturbance development. They first of all supplemented the computations with initial amplitude A corresponding to the condition $|w_0|/U_0 = 10^{-5}$ with computations for larger values of A, corresponding to $|w_0|/U_0 = 5 \times 10^{-5}$ and 10^{-4}. Then the authors studied the expansions of their solutions in powers of the amplitude A, and extracted from these expansions the linear terms (describing the results of the linear stability theory) and the weakly-nonlinear quadratic (proportional to A^2) and cubic (proportional to A^3) terms. This procedure allowed them to isolate contributions of some nonlinear interactions to the developed disturbances; in particular, the component corresponding to wave vector $(0, 2k_{2,0})$ (and marking the beginning of the energy transport to higher spanwise wavenumbers) was detected in the quadratic part of the disturbance, accompanied by the most energetic T-S wave of the linear theory, with wave vector $\mathbf{k} = (k_{1,0}, k_{2,0})$.

To study strongly-nonlinear effects on wave-packet propagation, the authors further extended the range of values of the initial disturbance amplitude, and in addition to the above-mentioned values they carried out numerical simulations for cases where $|w_0|/U_0 = 10^{-3}, 5 \times 10^{-3}$, and 10^{-2}. A preliminary study of numerical-simulation results for disturbances with $|w_0|/U_o = 5 \times 10^{-3}$ showed that in the case of the third model (Eqs. (5.20), (5. 20a)) strongly nonlinear effects develop more slowly than in the cases of the other two models of the initial disturbance. Therefore, it was found that for complete analysis of the nonlinear development of the disturbance (5.20), (5.23), the range of investigated values of t should be considerably extended. For this reason the authors decided to study strongly nonlinear effects only for the first and second models of the initial disturbance.

For these two models, the authors were able to cover, in their numerical simulations, all the stages of nonlinear development of a localized disturbance in the Blasius boundary layer found in the experimental and numerical-simulation studies of earlier authors. In particular, the subharmonic disturbance growth produced by secondary subharmonic instability of primary waves was detected in data relating to case 2, with the initial conditions (5.19), (5.22), and appeared here much earlier than in the experiments of Cohen et al. (1991) with a considerably smaller initial disturbance amplitude. Also in case 2, when the subharmonic growth of oblique waves began, the generation of the streaky structures, absent from Fig. 5.39b, also began and took practically the same form as in the case of the other two initial conditions and in the experiments by Cohen et al. (1991) and Breuer et al. (1997). This means, in particular, that exactly as in the experiments, nonlinear effects led to cascade transfer of energy to higher spanwise wavenumbers. The numerical simulations of Bech et al. also show that the streaky structures sometimes reach breakdown only with a very high amplitude of disturbance velocity—e.g., in the case of the initial conditions (5.18), (5.21) with $|w_0| = 5 \times 10^{-3}$ the streaks continue to exist at an amplitude of streamwise-velocity oscillations close to 30 % of U_0. This demonstration of the high value of velocity amplitude needed for breakdown of streaky structures agrees, in particular, with results by Reddy et al. (1998) relating to streaks in a plane-channel flow. Nevertheless, breakdown of the streaky structures, and emergence of the disorderly high-frequency fluctuations accompanied by rapid increase of the disturbance kinetic energy and of the maximal values of velocity fluctuations[12], were also

[12] Note that the growth of the disturbance kinetic energy does not necessary imply the growth of disturbance velocities. For example, in the case of transient growth of localized disturbances in plane shear flows studied by Landahl (1980), the growth of disturbance energy due to the "lift-up effect" described by him is due to elongation of the disturbance increasing its volume, and not to the growth of velocities of individual fluid particles.

detected in the numerical simulations for both the high-amplitude initial conditions, if the value of $|w_0|$ (and hence also of A) was large enough. Bech et al.'s large-amplitude simulations of disturbance development also revealed many other details of streaky-structure breakdown, transition to the unordered flow regime, and the accompanying flow phenomena. However, again these results are outside the scope of this chapter.

As noted above, a considerable part of the paper by Bech et al. is devoted to the study of the development of localized disturbances in boundary layer with an adverse pressure gradient $dp/dx > 0$ decelerating the fluid motion. Boundary layers with non-zero pressure gradients are met very often in practical applications of fluid mechanics, and have therefore attracted much attention by investigators. Therefore, it is only natural that the nonlinear instability of pressure-gradient boundary layers is considered in a great number of publications; the papers and dissertations by Bertolotti (1985); Herbert and Bertolotti (1985); Wubben et al. (1990); Goldstein and Lee (1992); Kloker (1993); Zel'man and Maslennikova (1993a); Kosorygin (1994); Kloker and Fasel (1995); van Hest (1996); van Hest et al. (1996); Corke and Gruber (1996); Liu (1997); Liu and Maslowe (1999); and Borodulin et al. (2000) represent only a small part of this work. Bech et al. were interested in boundary layers with adverse pressure gradient since here, at a not-too-small absolute value of the Falkner-Skan parameter β (see Chap. 2) the profile $U(z)$ has a pronounced inflection point where $U''(z) = 0$, and is inviscidly unstable with respect to small-amplitude disturbances according to the classical results of Rayleigh (see Chap. 2). This increased linear instability of a laminar boundary layer in adverse pressure gradient (in comparison to the case of a boundary-layer with zero pressure gradient) must also strongly influence the nonlinear instability effects and produce some additional phenomena worth special study. Bech et al., who performed simulations for $\beta = 0$ (i.e. $dp/dx = 0$) and $\beta = -0.155$, in fact detected a number of interesting differences between disturbance developments in these two flows. However, volume limitations give no possibility for discussion in this book of results for boundary layers with non-zero pressure gradients.

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