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VOLUME 15

CONFLICT, COMPLEXITY AND MATHEMATICAL SOCIAL SCIENCE

Series Editor MANAS CHATTERJI *Professor, School of Management, Binghamton University, US*
Guest Professor, Peking University, Beijing, China

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CONFLICT, COMPLEXITY AND MATHEMATICAL SOCIAL SCIENCE

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FOREWORD

Peace is as old as mankind. The gospel of peace and harmony is inherent in all religions and expressed through philosophy, literature, art, and music. However, in contemporary academic disciplines, peace has been addressed differently. In sociology, individual and group conflicts that also involve psychological conflicts have been a matter of discussion for years. For the last two or three decades, international, interregional, and intraregional conflicts have received a prominent place. The study of conflict in economics is a recent phenomenon. In management, industrial relations and labor relations cover this topic through contract negotiations, mediation, and arbitration. Extensive studies are now available in many applied areas such as energy, water resources, and the environment.

Until recently, these studies have been descriptive and case oriented. Researchers have avoided more quantitative and theoretical approaches. As in economics, more and more theoretical and quantitative formulations of conflict, peace, and war are being attempted now. An interdisciplinary branch of social science called Peace Science has emerged. It approaches conflict analysis without political, social, financial, or nationalistic bias. It uses tools, methods, and a theoretical framework drawn from the social and natural sciences, law, engineering, and other disciplines and professions. Peace Science and Peace Studies are complementary. The impetus of Peace Science has come mostly from mathematical economics.

This book offers a preview of Peace Science and mathematical social science. Peace is a complex phenomenon. This book combines Peace Science with an analysis of its complexity covering a wider area of economics, psychology, and political science on an individual and group level. It also shows how Peace Science is related to System Science.

Manas Chatterji
Series Editor

ACKNOWLEDGMENTS

This book is very much the product of the Conflict Research Society (CRS). I first joined the CRS in 1982 and was strongly influenced by the small group of people who formed the core of CRS activity: Cedric Smith, Paul Smoker, Chris Mitchell, Keith Webb, Tony de Reuck and of course Michael Nicholson who was at that time Director of the Richardson Institute of Conflict Research. Two decades later I was beginning to pull my ideas together and asked the CRS if it was willing to sponsor a series of seminars which I would present on modelling social conflict. Herb Blumberg very kindly arranged that these would be jointly hosted with London University's Goldsmiths College. And indeed I received warm support from all the members of the CRS Council: Herb Blumberg, Jim Bryant, Dennis Bury, Marwan Darweish, Peter Emerson, Nick Gale, Steve Hills, Athina Karatzogianni, Julie Lloyd, David Maxwell, Hugh Miall, Isabel Phillips, Oliver Ramsbotham, Elizabeth Rosenthal, Tom Woodhouse, Martin Wright and Steve Wright. It has been a huge boost to feel that they are all cheering me on. Even so this would not have occasioned a book had it not been for Manas Chatterji taking an interest in my endeavours and providing encouragement and support to write this book. Closer to home I have been particularly fortunate in having as colleagues and close friends Geoff Moss and Hossein Zand who have given me strong support and with whom I have hugely enjoyed discussion over a wide range of topics. Finally, back in 1970 I explained the χ^2 test to Catherine and ever since she has been teaching me much about social reality.

CHAPTER 1

INTRODUCTION AND OVERVIEW

> While I do not believe that world peace will be achieved by the solution of some appropriate set of differential equations, I do believe that the development of a rigorously developed international theory will help in that direction.
>
> – Nicholson (1989, p. xi)

> I am enjoying [the book on physics] and the dynamic introduction contains many interesting things, but it makes not the slightest attempt to comply with the requirements which you (and I as well) would expect from a properly grounded theory. . . .
>
> – Bohr, cited in Moore (1967, p. 22)

This book seeks to present a foundational mathematical approach with rigorously developed, properly grounded theory as advocated by Nicholson and Bohr. Each of the concepts in the title is grounded in and developed from a system of foundational ideas. The concept of conflict is grounded in and developed from a system of ideas relating to value. The concept of complexity is grounded in and developed from a system of ideas relating to a set of individuals moving through time and space. The topics of mathematical social science are grounded in and developed from a foundational account of mathematical science.

History and life are characterised by the pursuit of value and yet the achievement of value can sometimes be elusive, and so the pursuit of value must at best be considered speculative.

Values are often in conflict and conflict often destroys value. The value of economic growth is in conflict with the value of environmental sustainability. There is a puzzle in that in the richer countries of the world economic growth does not seem to have been accompanied by a corresponding increase in subjective satisfaction. The trajectory of value fluctuates and can exhibit dramatic swings. Social value has an unclear relationship with the set of values for individual people, and the value experienced by a person has an unclear relationship with the set of values for individual ideas in that person's mind. Conflict within a set of values makes social and individual choices difficult. The ethical value of equality contrasts with the reality of inequality.

Seeking to understand how society changes over time, we are led to abstract mathematical models of systems which change over time. Although some systems can have simple trajectories, there is a growing awareness that trajectories can exhibit complexity of various kinds. One way in which this complexity can arise is through the interactions between the components of a multiple-entity system, and of course this is precisely the situation with a society of interacting individuals or a mind of interacting ideas. In our reflections on the history of societies and the lives of individuals, we shall repeatedly be referring back, sometimes explicitly and sometimes implicitly, to a basic framework of 10 models with increasingly specific assumptions:

(1) History as a trajectory of states
(2) History as a function-driven trajectory of states
(3) History as an action-driven trajectory of states
(4) History as an alternative-selection-driven trajectory of states
(5) History as a value-dependent alternative-selection-driven trajectory of states
(6) History as a value-consequence value-dependent alternative-selection-driven trajectory of states
(7) History as a thought-driven trajectory of states
(8) History as a rational-choice-driven trajectory of states
(9) History as a parameter-driven trajectory of states which themselves include the parameters
(10) History as a probabilistic parameter-driven trajectory of states which themselves include the parameters

To understand social reality well, we need powerful theory and powerful evidence. Mathematics provides the powerful theory, and science provides the powerful evidence. This view is not uncontested. Some would argue that social understanding derives from ordinary language accounts – both the accounts which are provided by the people participating directly in social reality and the accounts provided in literature and in history. Others would argue that any project which has the concept of social reality is fundamentally misconceived. In response to those views, the book argues that these arguments in particular and ordinary language in general can be analysed in terms of mathematical science and that the validity of ordinary language statements should rest on their mathematical science validity. (Here I am talking about validity in relation to the understanding of social reality.) No doubt a large amount of ordinary language discourse does have mathematical science validity but there is much that does not. Objects and structures in reality are to be modelled by mathematical objects and structures; and discourse about real objects and structures is to be carried out in terms of mathematical definitions, theorems, proofs and theories. Claims about social reality need to be appraised in terms of evidence, they need to be guided by the rules of valid inference and they need to have an awareness that the amount of evidence required may be substantial and an awareness of the threats to valid inference. As we shall see, these features are characteristic of a substantial literature in what might be called mathematical social science: mathematical psychology, mathematical sociology, mathematical political science, mathematical economics and the use of mathematical models in the field of international relations and peace and conflict research.

OVERVIEW OF THE CHAPTERS

Set Theory and Social Reality

Set theory provides a foundational approach to mathematics, and mathematics provides an abstract way of looking at social reality. The first section presents some of the elementary concepts of set theory. The second section presents a variety of examples of social reality and shows how the abstract features of reality can be modelled by set theory. The third section shows how set theory can provide a way of looking at the accounts of social reality presented in humanities disciplines such as history and literature. The fourth section briefly indicates how set theory and the concept of a structure provide a foundational approach to mathematics. The fifth section looks at the debates surrounding realism and, albeit warily, espouses mathematical social science realism.

Mathematics, Logic, Artificial Intelligence and Ordinary Language

Language is an important aspect of social reality, and mathematics provides an abstract way of looking at all aspects of language. The intention of the language in this book – indeed the intention of the language in all of mathematical science – is to make meaningful, true, precise and mathematical statements about reality. This dedicated use of language stands in contrast to the more varied usage which we find in ordinary language. Not all ordinary language is meaningful, not all ordinary language is truth-intending, not all ordinary language is precise and not all ordinary language is mathematical. Turning now to mathematics, there is an important distinction between the thinking process involved in mathematical inquiry and the completed mathematical knowledge which is the product of that inquiry. The structure of mathematical knowledge includes a structure of concepts, a structure of statements (propositions), a structure of arguments (proofs) and a structure of contexts. The context–proposition structure is of particular interest. Mathematical logic takes an abstract look at mathematics. The standard account of mathematical logic discusses first propositional calculus and then predicate calculus. Propositional calculus focuses on the relationship between propositions and provides a conceptual foundation for an account of the logical structure of argument. Predicate calculus focuses on what is being said inside the proposition and on the nature of the mathematical objects being discussed, and hence links back to the discussion of mathematical structure in Chapter 2. In a given context, for each theory there is a set of realities for which the theory is true, and for each reality, there is a set of theories which are true of that reality. Language is a medium supporting thought and action, with ordinary language supporting most human thought and action, and mathematical language supporting logical thought and action. Artificial intelligence and its application involve building systems which think or act, and for this, the notions of mathematical structure and mathematical logic are foundational.

Possibility and Probability: Value, Conflict and Choice

Complete knowledge of a particular world involves knowing the truth values of all the propositions concerning that world. Complete ignorance involves knowing nothing at all. The first major step beyond complete ignorance is knowing the set of all possibilities within which the particular world might occur. Between complete knowledge and knowing just the set of possibilities, there is partial knowledge which can be expressed in terms of the probability of events. In some situations, it is appropriate to assume a priori that each (elementary) event is equally likely.

Possibilities and probabilities are to the fore in the discussion of value, conflict and choice. In the absolute notion of value, value may be binary, ordinal or quantitative.

In the comparative notion of value, a preference is expressed in relation to each pair of objects. Multidimensional value arises when values are placed on a set of objects by a set of individuals – either by individual people or by individual criteria. The likelihood of value consensus decreases and the likelihood of value conflict increases as the number of individuals and the number of options increases.

How should we choose? All of five sets – options, methods, criteria, choosers and situations – need to be thought about when addressing the fundamental problem of social choice: there is conflict between choosers, methods and criteria in that in some situations the different choosers, methods and criteria select different options. For example, there are situations where the Condorcet majority principle, the De Borda ranking principle and the welfare principle select different options. Also, there are situations where a voting cycle occurs – a result which provides the basis for Arrow's general impossibility theorem. The likelihood of those undesirable situations is considered.

An individual may seek success or power (or influence or decisiveness). 'Success equals power plus luck'. The likelihood of success and the likelihood of power depend on the social choice function, and both likelihoods decrease as the number of individuals and the number of options increases. In their discussion of possible voting rules for the European Council of Ministers, Laruelle, Martinez, and Valenciano (2006) ask whether states seek power or success, and argue that states which are concerned about their sovereignty and states which are concerned to deepen integration are likely to press for different voting rules.

Theory, Evidence and Reality: The Mean and Median Ideals of Competing Groups

Mathematical truth requires consistency with axioms, whereas scientific truth requires correspondence between theory and reality. Theory may be more restricted than reality, and reality can be more restricted than theory. Sometimes reality can be represented by a simple equation and at other times a complex structure of context-dependent equations may be required. Evidence from social science investigations often requires us to consider empirical probabilities and approximations.

The notion of value in this chapter is that objects can be located in a continuous space and that preferences for objects are single-peaked or Euclidean. There is some evidence that the population distribution of peaks (or ideals) is itself sometimes peaked. Under certain circumstances, these features eliminate the possibility of voting cycles and give the median ideal or the mean ideal as the majority winner. If the outcome is the mean ideal, then it can be shown that an individual's power decreases as the size of the population increases. Larger groups have greater power. In the absence of equal democratic power, the outcome may be modelled as the weighted mean ideal with overall power being a combination of egalitarian power and non-egalitarian power. The presence of non-egalitarian power can be detected by looking at the social outcome in terms of the relationship between the overall mean (or median) and the means (or medians) of competing groups. Wiseman and Wright's (2008) investigation of evidence of partisan policy in the US Senate is used as a case study.

Social Design, Ethics and the Amount of Value

Ethics is a complex subject, and here the focus is on a specific ethical criterion, the utilitarian social welfare function. The ideas are relevant to other values besides welfare, and the maximisation of total welfare may under certain circumstances be associated with the minimisation of inequality. The notion of value in this chapter is that an object can have a certain amount of value for an individual. Limitations on social value are noted. There are tensions between competing options. The provision of more than one option allows some relaxation of these limitations and tensions. If the option space is continuous, then the social value function can take a variety of specific forms. The notion of value-generating power is introduced. Given certain assumptions, the mean social value is a maximum at the mean ideal. Sub-optimal social value can arise as a result of the following factors: a sub-maximal value of the best option, population variation in ideals, the distance of the provided option from the best option and sensitivity to deviation from the ideal. Practical social design requires attention to a variety of design dimensions and knowledge about people's values regarding these dimensions. This knowledge may not be known in advance and so the design process can be usefully informed by the identification of design dimensions and by obtaining evidence about people's values regarding these dimensions. An application of these ideas to educational design is described.

Change, Multiple-entity Systems and Complexity

The concept of change is of fundamental importance. To understand change is to understand why things are the way they are, where things came from and where things are heading. A system may consist of a single unidimensional entity or a multidimensional or multiple-entity system. In the latter case, the linkage between micro and macro attributes and indeed the composition of attributes is of interest.

It is helpful (and possibly misleading!) to make a distinction between information systems and behavioural systems: the former looking towards computing theory and mathematical logic and the latter looking towards dynamic systems and statistical dynamics, and the former having various notions of computational complexity and the latter concerned with the complexity of trajectories. The generation of language and mathematical objects by information systems is discussed. Models of systems may be classified according to whether or not they possess or emphasise the following properties: discrete/continuous space, one-dimensional/multidimensional, static/dynamic, discrete/continuous time, deterministic/probabilistic, linear/non-linear, single entity/multiple entities, single attribute/multiple attributes, homogeneity/heterogeneity, interactive/non-interactive, based on choice/influence and individual/structural. The discussion here is selective, focusing on the trajectories of a single entity in one-dimensional space, the trajectory of multiple interacting entities, multiple entities with non-identical probabilities and the most probable trajectory of a macro parameter.

Mathematical Psychology

Psychology addresses the questions, 'who am I?', 'what do I spend my time doing?' and 'reflecting on my life, where have I come from and where am I going?'. Models of life as a trajectory have as their elementary component a model of a single step in the trajectory. The single step may be modelled either as a stimulus input giving rise to a response output (Chapter 8) or as a choice (Chapter 9). In psychophysics, the basic model involves an input space X, an output space Y and a response function f from X to Y. A variety of procedures are used to obtain measurements, sometimes indirectly, of X and Y. An important example of stimulus–response is the common phenomenon of the imitation of the behaviour of one individual by another individual. Turning now to psychometrics, consider a response involving an individual in a situation. Looking at a set of individuals and a set of situations, the response depends on the individual and the situation. For example, high performance depends on the high ability of the individual and on the high easiness of the situation. Typically, these variables are multidimensional. Although controlled investigations enable a balancing of situations and individuals, observation in naturalistic settings often does not. Turning now to methodological issues, some puzzlement is expressed concerning measurement theory. Just as Chapter 4 identified some problems with choice, here I find some problems with model selection. An important distinction is between models of the average individual and models of specific individuals. Finally, Falmagne's (2005) survey of four decades of mathematical psychology research is noted.

Models of Choice

Although Chapter 8 viewed behaviour as the response to an input stimulus situation, here in Chapter 9, behaviour as the outcome of a choice is viewed. Of course I have already discussed choice in Chapters 4–6 where the social choice depended on the choices of individuals. There, however, individual choice was a very simple and rather automatic matter: each individual had a set of options and placed a value on each option or expressed a preference between each pair of options, and then chose the best option. This indeed may be what happens at the point of making the choice but what this account ignores is the process which leads up to that end point. It is a process of some complexity and a wide variety of psychological models have sought to capture the essence of the process. Classical models of the choice process can be classified according to whether they are static or dynamic and whether they envisage the chooser as a single unit or as a system of multiple units. The dynamic multiunit models offer the potential for the phenomena of complexity theory to manifest themselves. To reach the end point, a chooser needs to identify the set of options and establish the value of these options. The psychological process may or may not involve a process of conscious reasoning and the reasoning may or may not be exact and mathematical. In the end state, the option which is chosen may not be the best option even though the chooser thinks so at the time. One model of life is to see it as a trajectory of choice points where the path chosen is chosen in pursuit of value. The possible sources of limitation on the value of the chosen option are the individual, the situation, the set of options, the value function, the valuation of options, the option selection and the experienced, recalled and reported value. If the value of the options is multidimensional, then the limitations discussed in Chapters 4–6 also apply.

Mathematical Sociology

Society consists of a set of interacting entities. The entities are either people or activity units. The life of an individual is a trajectory of participation and value in a structure of social activity units. The history of a social activity unit is a population flow of individual trajectories characterised by population participation and value parameters. In pursuit of value, populations of individuals flow through the structure, selectively participating and differentially performing. Ordinary living involves a participation of people in the social activities of family, leisure and holidays, shopping, work and travel. Activity within a unit is structured by relationships and choices, rules, rituals and randomness. Ordinary living also involves the participation of cultural ideas and artefacts in social activities. Trajectories of value are exemplified in the religious conversions of William Wilberforce and in the common patterns identified in cultural stories. Population flows of participation and performance are illustrated using an educational case study.

Mathematical Political Science and Game Theory

A game is a structure of actions chosen within the rules. Politics is the game of choosing the rules. In any social activity, the participants have action options, and these action options have value consequences. The value consequences for any individual of that individual's actions may or may not be dependent on other participants' actions. If the value consequences are not dependent on others' actions, then the individual may proceed to make his/her choice in the manner discussed in Chapter 9. However, if the value consequences are dependent on others' actions, then the situation has a structure which forms the basis for game theory. In some situations, game theory allows an unambiguous identification of the set of best actions for all participants. However, there are some situations where it is not clear what the set of best actions for the participants is. In some situations, there is a tendency for conflict rather than cooperation. It may be that participants can learn or evolve so that cooperation is more likely. Beyond the simple two-person game, a variety of additional features have been added to introduce more realistic complexity to the models, and there is growing interest in exploring this complexity using computer simulation.

Social activities are governed by rules and a supporting rule system. A rule partitions the set of all action options into a subset of rule-conforming options and a subset of rule-breaking options. The social choice, that is the option selected, is influenced by both individual preferences and rules and a consideration of the rule system. For any social activity A, there is a social meta-activity concerning the rules for social activity A. Politics is the social activity which focuses on rules: making representations about rules, and making, implementing and applying rules. There are conflicting views about what is the ideal social activity structure, and this conflict is played out in the political sphere with actors engaging in strategic interaction within the constraint of rules which are themselves part of the social activity structure. Note that much of the discussion of social choice and social welfare in Chapters 4–6 is relevant to politics.

The Mathematical Economics of Social Participation: Complexity

Economics is about the production and consumption of social activity participation – about the set of participation possibilities, the constraints on these possibilities, the value of these possibilities and the selection of the outcome. Important constraints are time and money. An important type of social participation is the social exchange of quantities of objects, in particular economic exchange where consumers obtain goods and services supplied by producers. The exchange may be governed by a temporal and monetary price. Value, constraint and price drive the demand for and supply of participation. Price dynamics depends on supply and demand. Social participation brings a stream of values and is engaged in due to speculative anticipation of that stream of values. Speculative anticipation may be based directly on relevant information and/or on opinion communicated in a social network. The dynamics of social participation is based on the dynamics of relevant information and the dynamics of social opinion. Social participation in the stockmarket is discussed in terms of the efficient market hypothesis and the complexity models of multiagent theories.

Life and History: The Speculative Pursuit of Value

The history of society and the lives of people are characterized by the speculative pursuit of value. The phenomenon goes beyond our own society and beyond the pursuit of a higher standard living – indeed beyond the pursuit of happiness. The starting and end points and the process of pursuit itself may have positive or negative value. Investigations of social well-being have been carried out for almost half a century. Although large differences in real income relate to differences in well-being, the relationship is weak for smaller differences in real income. Moreover, in rich countries, despite decades of growth in real income, there has been little or no corresponding increase in social well-being. Contributions to social well-being also come from non-economic sources. Participation in high-value locations of the social structure such as being married or being unemployed makes significant contributions to social well-being. Utility functions may be based on comparisons which are temporal or social (or both) and this can be counter-productive. The findings of this chapter in conjunction with the findings of earlier chapters identify a variety of limitations on the truth and the value which people experience in their lives. This necessarily makes people's pursuit of value speculative. Society places differential speculative values on the array of social activity projects and these values have a complex dynamics.

World History: The Growth and Distribution Dynamics of Power, Truth and Value

The central notion of this chapter is that of the growth and distribution dynamics of a system in space and time. Very briefly the notion is applied to the growth and distribution, first of the physical universe and then of the biosphere on earth. I then turn to the growth of human society drawing on ideas from the new theory of economic growth. The impact on the natural environment of the growth of human society is noted. Next, I consider the distribution dynamics of power, truth and value, noting how this reflects the history of the rise and fall of dominant powers. This reflects a system of interacting territorial units – states. Power resides in resources and relationships. In pursuit of value, resources are allocated between different types of production activity: between internal (domestic) activity and external (international) activity, externally between different states, between cooperation and competition, between non-military activity and military activity and between peace and war. The levels and types of interaction relate to geographical proximity and cultural proximity. Arms production and wars are social reciprocation processes, and their level can be modelled using either systems of differential equations or game theory. States are not the only actors on the world stage. There are inter-state actors, intra-state actors and trans-state actors. The complex structure of actors can be characterised by the set of all actors, a membership relation specifying which actors are members of which other actors and a specification of the territorial base of each actor. The world map of actors changes over time: existing actors disintegrating and new actors integrating, the initiation and cessation of an actor's memberships, the recruitment and loss of members by an actor and the territorial base of an actor changing. Just as states are not the only actors on the world stage, so political and military actions are not the only type of actions and the pursuit of self-interest not the only criterion for action.

Debating the Mathematical Science Approach to International Relations

The mathematical science approach to the study of social affairs has been much debated not least among scholars of international relations. Alongside this debate about methodology, there has been a debate about the topics which international relations research tends to focus on. Criticisms tend to take one of the following forms: 'I don't like the fact that your subject matter is S' or 'I don't like your representation of the subject matter S'. There are errors of omission and errors of commission. There is under-representation and over-representation. Smith's (2004) wide-ranging critique of international relations theory research provides an agenda for checking whether there are important areas which this book neglects. First, philosophy: Chapter 2 provides some discussion of the debate between realism and constructivism, but the rest of the book has preferred to go ahead and develop the mathematical science approach. Second, ethics: This topic has been specifically discussed in Chapter 6 and the concept of value has been central throughout the book. Third, power: The concept of power is first introduced in Chapter 4 and appears throughout the book, and Chapter 14 provides a discussion of power as a resource and power as a relationship. A useful notion is the power-weighted mean ideal. (From this point on the chapter develops a model to incorporate certain of the features identified by Smith.) Fourth, the impact of power on value: The deviation of the social outcome from optimality depends on the bias magnitude, the variation in individuals' ideals and the correlation between the bias vector and the vector of ideals. Fifth, ethics, identity and social ideals: Ethics and identity can be incorporated into the model by using interpersonal welfare weights to define an individual's social ideal. Sixth, the social production of ideals: This can be modelled by a process of influence in a social network in the manner of the complexity theory models discussed in Chapters 12 and 13. Seventh, the complex structure of actors and actions: This links back to the Chapter 14 discussion of how the centrality of states in certain theories of international relations can be replaced by a model which represents the complex structure of actors, actions and criteria on the world stage.

CHAPTER 2

SET THEORY AND SOCIAL REALITY

> The fundamental work of mathematics is to examine structures.
>
> – Cori and Lascar (2000, p. 113)

> Physics is an attempt conceptually to grasp reality.
>
> – Einstein, cited in Moore (1967, p. 406)

> On this view everything in the physical universe is indeed governed in completely precise detail by mathematical principles [although not necessarily today's mathematical principles] . . . even our own physical actions would be entirely subject to such ultimate mathematical control . . .
>
> – Penrose (2004, pp. 18–19)

Set theory provides a foundational approach to mathematics and mathematics provides an abstract way of looking at social reality. The first section presents some of the elementary concepts of set theory. The second section presents a variety of examples of social reality and shows how the abstract features of reality can be modelled by set theory. The third section shows how set theory can provide a way of looking at the accounts of social reality presented in humanities disciplines such as history and literature. The fourth section briefly indicates how set theory and the concept of a structure provide a foundational approach to mathematics. The fifth section looks at the debates surrounding realism and, albeit warily, espouses mathematical social science realism.

THE ELEMENTARY CONCEPTS OF SET THEORY

In this section we shall give a brief account of the elementary concepts of set theory – elementary in the sense that they are concepts which students might meet in the first week of a mathematics foundation course at a university; and elementary also in the sense that everything else in mathematics is dependent on them. The concepts are at once both extremely simple and extremely abstract.

We start with the notion of an *element*. In order to refer to an element we give it a label, say '*a*'. We can represent the element in a diagram using a single point (Fig. 2.1).

Next is the notion of a *set* of elements. In order to refer to a set we give it a label, say '*S*'. The set can be described by listing its elements, say '*a*', '*b*' and '*c*'. We can represent the set in a diagram using a box, with the elements as points within the box (Fig. 2.2).

Next is the notion of a *function*. In order to refer to a function we give it a label, say '*f*'. With a function we have two sets, *S* and *T*. Suppose *S* has elements, '*a*', '*b*' and '*c*'; and that *T* has elements, '1', '2' and '3'. The function specifies a linkage between the elements in *S* and the elements in *T*. We can represent the function in a diagram using two boxes, and lines linking elements in one set with the corresponding elements in the other (Fig. 2.3).

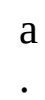

Fig. 2.1. An Element.

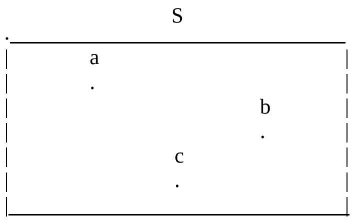

Fig. 2.2. A Set of Elements.

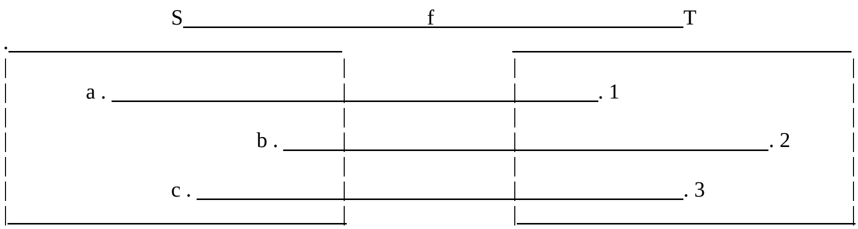

Fig. 2.3. A Function.

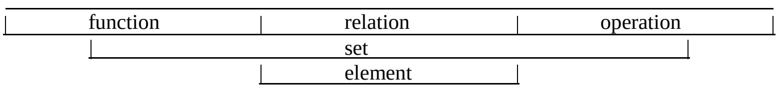

Fig. 2.4. The Systematic Development of Concepts.

Next is the notion of a *relation*. In order to refer to a relation we give it a label *R*. With a relation we have a set *S*. The relation *R* is then the set of pairs of elements of *S* which have some specified relation. Indeed a relation is defined to be a set of ordered pairs of elements of *S*. Suppose that the elements of *S* are people and that *a* is a parent of *c* and that *b* is also a parent of *c*. The relation here is 'is a parent of'. We represent the relation by a set containing just the two pairs (*a*, *c*) and (*b*, *c*).

Next is the notion of an *operation*. In order to refer to an operation we give it a label, *O*. With an operation we have a set *S*. The operation *O* is then the association of pairs of elements of *S* with some third element of *S*. Suppose again that the elements of *S* are people and that *a* is a parent of *c* and that *b* is also a parent of *c*. In other words *c* is the offspring of *a* and *b*. The operation is 'is an offspring of' and consists of the association of the pair (*a*, *b*) with the element *c*.

Extremely simple, extremely abstract – and, you might be thinking, extremely laboured. The rationale for it being laboured is that we mathematicians feel that a careful specification enables us to feel confident that we have a strong and secure structure of concepts. For the same reason care is taken in the order in which the concepts are developed. Here we started with the concept of an element. We then defined the concept of a set in terms of the concept of an element. We then defined the concept of a function in terms of the concepts of an element and a set . . . and the concepts of relation and operation also in terms of the concepts of an element and a set. This is common in mathematics: starting with a very simple concept then proceeding step-by-step defining each new concept in terms of the concepts which have gone before. In summary, mathematics is simple and abstract and involves careful specification and the progressive development of concepts (Fig. 2.4).

SET THEORY MODELS OF SOCIAL REALITY

We now present a variety of examples of social reality and show how the basic abstract features of reality can be modelled by set theory. Examples (1)–(3) are standard fare for the application of mathematics to social reality.

Examples (4)–(8) seek to show that mathematics can be applied much more widely than some people imagine: to situations represented by ordinary language. This is a prelude to the next section, where mathematics is applied to history and literature.

Example 1: Data

Organisations and researchers routinely collect and store information about people. The data can be presented in a table and on the computer this can take the form of an Excel sheet. The key features of a table are that there are rows, columns and entries in the matrix cells. The data in a table can be thought of in abstract in the following way: a set I of individual cases (the rows); a set J of variables (the columns); a set X of possible values in the data matrix; the set $I \times J$ of pairs, pairing individual i with variable j; and the function from $I \times J$ to X, specifying the cell entry x_{ij} in the matrix corresponding to individual i and variable j.

Example 2: Individual Choice

Life involves choices. A choice situation involves a set A of options and a preference relation P on the set of option which specifies for each pair of options (a_1, a_2) whether or not it is the case that $a_1 < a_2$. (Here $a_1 < a_2$ means that a_1 is less preferred than a_2.)

Example 3: Game Theory

In the United Kingdom, I drive on the left hand side of the road but in many other countries I drive on the right. My best option depends on what other people are doing. This is the province of game theory. 'A game in strategic (or normal) form has three elements: the set of players $i \in I$, which we take to be the finite set $\{1, 2, \ldots, I\}$, the pure-strategy space S_i for each player i, and payoff functions u_i that give player i's von Neumann–Morgenstern utility $u_i(s)$ for each profile $s = (s_1, \ldots, s_I)$ of strategies' (Fudenborg & Tirole, 1993, p. 4).

Example 4: Language

In linguistics a proper noun is a noun which refers to a specific object, e.g. 'Edinburgh Castle'; and a common noun is a noun which refers to any one of a class of objects, e.g. 'castle'. Consider a set R of objects in reality; a set N of nouns in a language and a function from N to R specifying for each noun n in N the set R_n of objects that the word refers to. If R_n contains just one object then n is a proper noun, otherwise n is a common noun.

Example 5: Early Lexical Development in Children

Barrett (1995) reports research on young children's speech. An example of underextension is where the word 'dog' is used to refer only to the family's own dog. An example of overextension is where the word 'dog' is used to refer to any four-legged animal. An example of overlap is where the word 'dog' is used to refer to any one of the family's pet animals (the cat, the dog and the rabbit) and not anything else. An example of mismatch is where the word 'dog' is used to refer to cats and nothing else. What is involved in all of this is the following: a word w, the set A of objects denoted by w according to adults and the set C of objects denoted by w according to the child's use of the word w. Underextension, overextension, overlap and mismatch correspond to the following relationships between the two sets A and C: A contains C, C contains A, A contains only part of C and C contains only part of A, and finally, A and C contain no objects in common.

Example 6: Conflict and Comparison between Groups

When groups are in conflict, comparisons between the groups are often made and often carry importance. For example the statement, 'women are more cautious than men', was the subject of some debate back in 2009 (see Chapter 3). Linguistically this statement appears quite simple but mathematically and scientifically it is quite complex. Mathematically there is a set P of people partitioned into two subsets males, M, and females, F. There is a cautiousness function c from P to a set of quantities Q. This function is defined for elements but can be extended to a mean cautiousness function defined on subsets. There is an inequality relation such that the mean cautiousness of the female subset F is greater than the mean cautiousness of the male subset M. Scientifically, one needs to consider how one might obtain evidence in support of the statement. This involves a variety of notions such as sampling and significance testing. So although group comparison statements have been stated in ordinary language for millennia, only in the past century has the proper mathematical science basis for them been known.

Example 7: A Nursery Rhyme

When our two daughters were very small I recall reading them this nursery rhyme:

> My sister Mollie and I fell out
> And what do you think it was about?
> She liked coffee and I liked tea
> That was the reason we couldn't agree.

Despite its simplicity the nursery rhyme provides a coherent account of a conflict. First it notes that conflict action is taking place: the two sisters have fallen out – they cannot agree. The rhyme invites the listener to think about what gave rise to the situation and proceeds to supply the reason: there has been a conflict of interest. So the rhyme illustrates how conflicts of interest give rise to conflict actions. It is worth spelling out the conflict of interest in more detail.

(1) There is a set of two people: Mollie and 'I'. Let us refer to them using the labels, 'M' and 'I'.
(2) There is a set of two objects: coffee and tea. Let us refer to them using the labels, 'C' and 'T'.
(3) There is a set of two valuations: like and not like (the latter is only implied). Let us refer to them using the labels, 'L' and 'N'.
(4) There is a relationship between people, objects and valuations. This is represented in Table 2.1. For example the entry 'L' in row 'M' and column 'C' indicates that Molly Likes Coffee.

Table 2.1. The Valuations Placed on Different Objects by Different People.

People	Objects	
	C	T
	Valuations	
M	L	N
I	N	L

The people–object–valuation relationship portrayed in Table 2.1 is a foundational concept in the analysis of social conflict. A powerful approach to the analysis of such situations is provided by social choice theory, which is discussed in a later chapter.

Example 8: Honour Killing

Laura Blumenfeld's (2002) 'Revenge' fills me with despair (I am only at page 207). Here is just one of the stories. A brother loved his sister and valued his honour. The sister decide to leave home, she left with a boy but was brought back home by her uncle 12 h later. The family asked a doctor and police to arrange to check her virginity but they refused. The relatives argued the case. The brother was unable to live with the shame of it all. To restore his honour he killed his sister.

There are four possible 'states of the world' here: HD, honour and sister alive; HK, honour and sister killed; ND, honour lost and sister alive; and NK, no honour and sister killed. The brother has a preference relation on this set: HD > HK > ND > NK, where '>' means 'is preferred to'. The story starts with state HD, the brother's most preferred state. The sister's action brings about state ND, the brother's third most preferred state. The brother's action brings about state HK, the brother's second most preferred state, an improvement on ND according to the brother's preferences. His confession:

> I murdered her. Because she shamed me for the rest of my life and if I didn't murder her, I would die every day. Because everything in life can be replaced, but if honor is lost, it never returns. . . .
>
> If it happened to you, what would you do? Wouldn't you do the same?
>
> (Blumenfeld, 2002, pp. 19–22)

SET THEORY MODELS OF HISTORY AND LITERATURE

The examples (4), (5) and (7) show how set theory can be applied to language, language development in children and children's nursery rhymes. To what extent though can set theory be applied to more sophisticated discourses such as those found in history and literature?

History provides a rich and powerful account of life and society. So what exactly is history? I would like to suggest that history can usefully be understood in terms of mathematical science. In support of this suggestion

Table 2.2. The Sections and Authors Cited in Fergusson's Essay.

Introduction
Lewis Namier

Divine intervention and predestination
Scientific determinism: materialism and idealism
Isaac Newton, Leibniz, David Hume, Hempel, Laplace, Kant, Hegel, Comte, Mill, Buckle, Tolstoy, Marx, Darwin, Engels and Ranke

Contingency, chance and the revolt against causation
Thomas Carlyle, Dostoevsky, Fisher, Trevelyan, A. J. P. Taylor, Meinecke, Collingwood, Oakeshott and Bertrand Russell

Scientific history – continued
E. H. Carr, E. P. Thompson, Eric Hobsbawm, Gramsci, Weber, 'Annales', Marc Bloch, Braudel, Stone, Paul Kennedy and Geertz

Narrative determinism: why not invent paths?
Barzun

The garden of forking paths
Robert Musil, Jorge Luis Borges and Robert Frost

Chaos and the end of scientific determinism
Einstein, C. S. Pierce, Richard Dawkins, Stephen Jay Gould, Ian Stewart, Poincaré, Feigenbaum, Mandelbrot, Lorenz, May and Roger Penrose

Towards Chaostory
Popper, Frankel, Gallie, Hart, Honoré, Braithwaite, Gardiner, Isaiah Berlin, Marc Bloch, Trevor-Roper, Collingwood, Dray and Ranke

I refer you to Niall Fergusson's (1997) extended essay 'Virtual history: towards a 'chaotic' theory of the past'. The essay is 90 pages long and includes the sections shown in Table 2.2. The section titles provide explicit reference to science and a key issue is the nature of the mathematical science model which is appropriate in the study of history: should it be the determinism of Newtonian mechanics? or the probabilistic model of quantum mechanics? or the chaotic dynamics of complexity theory? Fergusson's account has an intellectual breadth and depth that I make no attempt to imitate, but the display conveys its flavour by providing a roll call of the authors he cites.

These names suggest that the nature of history and its relation to mathematical science involves deep philosophical issues. In contrast, this section is extremely pedestrian. My aim is simply to think about how the elementary ingredients of history might be represented in abstract by the elementary concepts of mathematics. I do this by looking at just tiny fragments of history: looking in turn at local history, and at the history of England in the early fourteenth century.

Local History

Let me start by telling you about my home town of Newport Pagnell. Sometimes I take a walk up to our local church. This takes me past the war memorial with the names of those who died. As I enter the church, a board on the opposite wall lists the names of the incumbent vicars of the church back to 1236.

The elementary ingredients of this little fragment of local history can be represented in abstract by the elementary concepts of element, set and function. There are numerous elements: the church, the war memorial, the board on the wall, each individual name, each individual date and so on. The fragment also involves the mathematical concept of a set of elements, for example the set of people named on the war memorial. The fragment also involves the mathematical concept of a function. Here the function takes the form of a temporal sequence of elements, namely the list of dates and people on the board linking each date to the rector appointed in that year. This function is an example of the first of the ten models of history mentioned in Chapter 1: history as a trajectory of states.

Models of History: England in the Fourteenth Century

From local history we now turn to the history of England in the early fourteenth century. Consider the following brief extract from my encylcopedia:

> *Isabella of France* (1292–1358) [was] the wife (1308–27) of Edward II of England; [and] the daughter of Philip (IV) the Fair. Increasingly isolated by Edward's favourites, she left England in 1325. She became the mistress of Roger Mortimer and together they returned to overthrow and murder Edward (1327). In 1330 her son Edward III executed Mortimer and confined Isabella to a nunnery.
>
> The Macmillan Encyclopedia (1981)

Here too we have an example of the first of the 10 models of history mentioned in Chapter 1: history as a trajectory of states. There is an ordered set T of points in time, a set S of states and a function f_1 from points in time

Table 2.3. History as a Trajectory of States.

Time *t*	State *s*
1326	Edward II rules
1328	Isabella and Roger de Mortimer rule
1331	Edward III rules

to states. The function can be represented by a table with the times in one column and the states in another column (Table 2.3).

The next step in our conceptualisation of history involves the important concept of change from one state to another. History might be viewed as a sequence of changes. Each change or transition can be represented by a pair of states, 'before' and 'after': (Edward II rules; Isabella and Roger de Mortimer rule) and (Isabella and Roger de Mortimer rule; Edward III rules).

It is the explanation of these changes which is introduced in the second of the models of history mentioned in Chapter 1: history as a function-driven trajectory of states. In the chapters ahead this provides a powerful model. Here, however, it is more natural to proceed to consider the third of the models of history: history as an action-driven trajectory of states. This views history as a sequence of actions bringing about changes in the situation, the actions themselves being responses to the previous situation. For this we introduce four more ingredients: a set A of actions, a function f_2 specifying the occurrence of an action over each time interval, a function f_3 specifying how the action is brought forth by the situation and a function f_4 specifying how the action generates the change. This third model then has the following ingredients: $(T, S, f_1; A, f_2, f_3, f_4)$. In the extract: the change from Edward II being king to Isabella and Mortimer being regents is brought about by the action of the overthrow and murder of Edward. The action itself was a response to Edward II being king and Isabella and Mortimer's desire to be regents.

At this stage we can introduce some of the notions discussed by Fergusson. Virtual history allows us to ask: what if Isabella and Mortimer had not tried to overthrow Edward II?; had not succeeded in overthrowing him?; or had not murdered him? Do these represent possibilities that might actually have happened? Was it inevitable that Edward II was murdered or was it partly a chance matter?; could it have been predicted? Is the historical process deterministic, probabilistic or characterised by complexity theory dynamics?

In the preceding model of history it is action that drives history, but this merely prompts the question, what dictates the action? This is an extremely interesting question which has been much debated. Are individuals free to choose their actions? Or are they constrained to act the way they do because of social rules? Fergusson (1997, p. 86) regards this issue as crucial. At any given moment in history there are real alternatives, namely the alternatives which were considered by the people living at that time. Historians should consider *plausible* alternative histories, namely '*those alternatives which we can show on the basis of contemporary evidence that contemporaries actually considered*'.

In the remaining models of history (models 4–10 in Chapter 1) we try to acknowledge both that actions are the outcomes of considering the values of competing alternatives and also that this happens under the partial constraint of the rules of the prevailing social systems. For example economists routinely model individual actions as choosing the most valued options within a money constraint, money and private property being rules of the prevailing social system.

This combination of structure and agency is well illustrated by England in the fourteenth century. The feudal system involved a quite rigid structure of roles and rules but even so individuals were able to make choices not only within the structure but also by breaking out of the structure. A colourful portrayal of this is Ken Follett's (2007) novel 'World without end'. In one of the novel's sub-plots, Earl Roland dies 'of old age' and is succeeded by Earl William. Then Earl William dies of the plague but is survived by his wife Lady Philippa. The King has the role of appointing a new earl who would then be obliged to marry Lady Philippa. The Black Death has killed off many would-be aspirants. The dastardly knight, Ralph, wants to be the new earl and also wants to marry Lady Philippa. However, Ralph is married to Tilly. So Ralph kills his wife Tilly. He also performs a service to the king to gain the king's support. In order to have Lady Philippa accept him in marriage, Ralph threatens instead to marry her daughter. In this way Ralph weaves a course of action through the rule structure of the feudal system in pursuit of what he values.

I think you can see why Ken Follett sells more books than I do! His novels exemplify how history and literature provide rich and powerful accounts of life and society. In contrast, my models seem impoverished and weak. Nonetheless, I hope you feel that I have made some small first steps in the spirit of the views of some peace scientists who feel that some engagement across methodological boundaries might be fruitful. Levy (2008, p. 1) notes that 'scholars within different research communities have long worked in isolation from each other'. As an alternative to this isolation he believes in 'the utility of multi-method research'. Likewise Stohl (2007, p. 263) advocates 'drawing together the area specialists and historians with the quantitative and behavioural social scientists'.

SET THEORY, STRUCTURES AND MATHEMATICS

In this short section I merely wish to indicate how set theory and the concept of structure provide a foundational approach to mathematics. Mathematics has been referred to as the study of abstract pattern. This loose characterisation finds a more formal expression in the concept of structure. Cori and Lascar (2000) (hereafter denoted 'CL'), in their book on mathematical logic, observe that 'the fundamental work of mathematics is to examine structures, to suggest properties that might pertain to these and to ask whether these properties are satisfied or not' (p. 113). Later they note that 'the word structure is generally understood in mathematics to mean a set on which a certain number of functions and relations (or internal operations) are defined' (CL, p. 130). Typically systematic accounts of mathematical topics start with these basic ideas and use them to construct more complicated mathematical objects. For example Stone's (1973) introductory text on discrete mathematics starts with an account of the basic concepts of set, graph, function, relation and operation and then proceeds to define an algebraic structure and then discusses specific types of algebraic structure such as groups, rings and fields, semi-groups, monoids and finite state machines, Boolean algebras and lattices. The concept of a structure can be developed as follows. Let A be a set, F a family of finitary operations on A and R a family of finitary relations on A. Then a universal algebra or algebra is a pair (A, F), a relational system is a pair (A, R) and a structure is a triplet (A, F, R) (Gratzer, 1968, pp. 8, 223).

REALITY AND REALISM

In using the word 'reality' I am intentionally adopting a particular stance, namely that of philosophical realism. It is a stance which I take for granted throughout this book. Here in this section, however, we take time out to look at the debates surrounding realism and, albeit warily, espouse mathematical social science realism. The key statements of this section are as follows:

(1) Reality exists independently of us.
(2) Reality corresponds well with the way it is described by common sense and scientific knowledge.
(3) Reality contains us – individually as individuals, and collectively as a society – and our knowledge of reality can handle that.
(4) Scientific realism holds that science provides a road to knowledge about reality.
(5) Social scientific realism holds that science provides a road to knowledge about social reality.
(6) 'Mathematical science realism' holds that the real world can be entirely modelled by mathematics.
(7) 'Mathematical social science realism' holds that social reality can be entirely modelled by mathematics.
(8) There is some wariness about claiming too much for mathematical science realism: your ideas, my ideas, do not quite capture the essence of the thing.

Issues of ontology and epistemology – what exists and how we know it – have long preoccupied philosophers and have produced a variety of views regarding knowledge and truth, without any one view being successful in demonstrating its monopoly of the truth against the contenders (Audi, 2003; Engel, 2002). The position adopted in this book is that of realism. Devitt (1991, pp. 44–45) distinguishes 'fig-leaf realism' from 'a realism worth fighting for'. The former involves 'a commitment merely to there being something independent of us'. The latter is characterised by the belief that: 'tokens of most common sense, and scientific, physical types objectively exist independently of the mental'. This doctrine embraces common sense realism which is concerned with observables and scientific realism which is in addition concerned with unobservables. In part following Devitt, Wendt (1999, p. 51) argues in favour of the realist position, defining it in terms of three principles: '(1) the world is independent of the mind and language of individual observers; (2) mature scientific theories typically refer to this world; (3) even when it is not directly observable'.

Part of this can be recast using set theory. Three objects exist: the world (W), references to the world (R) and observations about the world (O). Viewed as sets, W contains R, and R contains O. There is a relation r from R to W (and hence from O to W) namely the relation of reference. There is another relation c from W to itself whereby one part of W 'depends on' another part of W.

Even those who are happy to accept realism with respect to events in the physical world may be less happy to accept it with respect to the social world. How can I know about the world when I am myself (allegedly) part of this world? Much of this is well expressed in the discussion by Cox (2005, pp. 115–116) of Minsky (1968): 'there is W, the world and M, the modeller who exists in the world. The model of the world is referred to as W^*. W^* is used to understand and answer questions about the world. So to answer questions about oneself in the world, it must also be the case that there exists within the model of the world, W^*, a model of the modeller, termed M^*. One should conceive of W^* simply as the agent's knowledge of the world, and likewise, M^* as the agent's reflective knowledge of itself in the world. ... one must have a model of one's model of the world, or W^{**}, in order to reason about and answer questions concerning its own world knowledge, ... Finally, M^{**} represents the agent's knowledge of its self-knowledge and its own behaviour, including its own thinking'.

Scientific realism holds that science provides a road to knowledge about reality. Capturing the essence of reality is precisely what science seeks to do. Certainly that was Einstein's view. 'Physics' he said, 'is an attempt conceptually to grasp reality' (Moore, 1967, p. 406). Those who are happy to accept scientific realism with respect to events in the physical world may be less happy to accept it with respect to the social world. A belief that scientific realism is applicable to the social realm – what might be called social science realism would appear to be the position taken by Wendt (1999, pp. 38–40), a position to which Smith (2004. p. 502) would appear to take exception.

However this book wishes to advocate a stronger version of realism, what might be referred to as 'mathematical science realism'. This is the belief that the real world is entirely modelled by – even controlled by – mathematics. This would appear to be the position of Penrose (2004, pp. 17–19). He proposes three worlds: the physical world W, the mental world of ideas I and the mathematical world M. Penrose suggests that the mathematical world is in principle knowable completely. The relationship between mathematics and the world is central. Not all mathematics refers to the world. However, 'everything in the physical universe is governed in completely precise detail by mathematical principles [although not necessarily today's mathematical principles] ... even our own physical actions would be entirely subject to such ultimate mathematical control ... '. In the latter part of this quotation there is the implication of mathematical social science realism.

Yet there is some wariness about claiming too much for realism. A somewhat different view from Einstein was taken by his intellectual twin, Niels Bohr. ‘It is wrong’ he said ‘to think that the task of physics is to find out how nature is. Physics concerns what we can *say* [my italics] about nature’ (Moore, 1967, p. 406). This fundamental debate between Einstein and Bohr started in the 1920s and was conducted over the following three decades until their deaths in 1955 and 1962 respectively.

Bohr’s more ‘philosophical’ position is consistent with that of his fellow-countryman, Søren Kierkegaard. Kierkegaard’s position might be summarized as ‘your ideas, my ideas, don’t quite capture the essence of the thing – the essence of things is elusive’. For example, for Kierkegaard, the church does not quite capture the essence of faith. Kierkegaard offers the following satirical image. ‘A man begins to wonder whether he is truly a Christian. His wife responds, “You are Danish, aren’t you? Doesn’t the geography book say that the predominant religion in Denmark is Lutheran-Christian? You aren’t a Jew are you, or a Mohammedan? What else would you be then?”’ (Westphal, 1998, pp. 115–116). Just as, for Kierkegaard, the church does not quite capture the essence of faith, so art does not quite capture the essence of reality (Pattison, 1998, p. 79), knowledge does not quite capture the essence of truth (Evans, 1998, p. 169) and the self does not quite capture the essence of selfhood (Hannay, 1998, p. 335).

CHAPTER 3

MATHEMATICS, LOGIC, ARTIFICIAL INTELLIGENCE AND ORDINARY LANGUAGE

> The ordinary languages we converse and write in are very rich in meaning. Vocabularies in natural languages run to hundreds of thousands of words, and even this is not enough: each word has its numerous shades of meaning and connotations. The richness of our language reflects the richness of our experience, and it is much of this same richness of experience which social science attempts to capture and codify. This may be one reason why sociology is perhaps the last of the empirical sciences in which the main stream of effort is as yet almost wholly discursive and nonmathematical.
>
> – Coleman (1964, p. 1)

> There is no important theoretical difference between natural languages and the artificial languages of logicians; indeed, I consider it possible to comprehend the syntax and semantics of both kinds of languages within a single, natural and mathematically precise theory.
>
> – Montague's thesis, cited in Cann (1993, p. 2) and Montague (1974, p. 222)

> It has been clear for some time that [Montague's thesis] can be extended to include programming and specification languages from computer science, and representation languages in artificial intelligence.
>
> – Van Benthem (2002, p. 71)

Language is an important aspect of social reality and mathematics provides an abstract way of looking at all aspects of language. The intention of the language in this book – indeed the intention of the language in all of mathematical science – is to make meaningful, true, precise and mathematical statements about reality. This dedicated use of language stands in contrast to the more varied usage which we find in ordinary language. Not all ordinary language is meaningful, not all ordinary language is truth-intending, not all ordinary language is precise and not all ordinary language is mathematical. Turning now to mathematics, there is an important distinction between the thinking process involved in mathematical inquiry and the completed mathematical knowledge which is the product of that inquiry. The structure of mathematical knowledge includes a structure of concepts, a structure of statements (propositions), a structure of arguments (proofs) and a structure of contexts. The context–proposition structure is of particular interest. Mathematical logic takes an abstract look at mathematics. The standard account of mathematical logic discusses first propositional calculus and then predicate calculus. Propositional calculus focuses on the relationship between propositions and provides a conceptual foundation for an account of the logical structure of argument. Predicate calculus focuses on what is being said inside the proposition and on the nature of the mathematical objects being discussed – and hence links back to the discussion of mathematical structure in Chapter 2. In a given context, for each theory there is a set of realities for which the theory is true; and for each reality there is a set of theories which are true of that reality. Language is a medium supporting thought and action, with ordinary language supporting most human thought and action and mathematical language supporting logical thought and action. Artificial intelligence and its application involve building systems which think or act and for this the notions of mathematical structure and mathematical logic are foundational.

ORDINARY LANGUAGE

Early Lexical Development in Children

With two grandsons aged five and three I have been fascinated and privileged to observe – and to participate in – their language development. The variety of roles which language can play can be seen from early on and development consists of an increasing structural sophistication with increasingly closer approximation to adult language. Theories of language can usefully be checked by asking whether they are true for my grandsons! For example, if a serious-minded academic like myself should imagine that all language is about the solemn pursuit of truth, then they need to account for the following exchange. (The reader will have to imagine the cheeky smile for themselves.)

Grandpa: 'Is your name Robert?'
Robert: 'No!'

In his discussion of early lexical development in children Barrett (1995) notes that utterances can be of the following types: an expression of an internal state, a response to a specific context, a social-pragmatic utterance and a referential utterance. A referential utterance can be thought of in the following way: the thing referred to, a mental representation, a word representation and a word sound. An utterance may refer to an object, an action, an attribute or an event. Some utterances are used as the names of classes of objects while other utterances are used as the proper names of individual objects. Looking at this in abstract we might say that, in the early years of childhood, language is used to refer to elements, sets, functions and relations – in other words to the mathematical structures which were discussed in Chapter 2. Of course although early language is used to refer to mathematical objects, the character of the language itself takes the form of ordinary language.

Ordinary Language and Non-Mathematical Discourse

Almost 50 years ago, Coleman (1964, p. 1) opened his book, '*Introduction to mathematical sociology*', with the remarks quoted at the beginning of this chapter. It is worth spending a little time reflecting on Coleman's contrast between ordinary language and mathematics, and on his observation that non-mathematical discourse is predominant. When people refer to social reality they almost invariably use a non-mathematical discourse. This is particularly so in those areas which are held by most people to be the areas of the greatest importance for their lives, sometimes literally sacred areas – religion, poetry, the arts, literature, history, the person and personality, family, community, practice; these areas, it is claimed, are what life is really all about.

Sometimes people's discourse about social reality is a representation of direct experience – sometimes it is a representation of what someone else has said. It is helpful to think of a chain of representations. Starting with the social reality itself we can move step by step to representations which are increasingly distant in some sense from the reality they seek to represent. The chain runs as follows: direct experience of reality (as a participant), direct action in reality (as a participant), direct observation of reality (as an observer), contemporary accounts (as in the news media), primary and secondary sources of history, history, autobiography, biography, historical novels, literature and drama, commentaries on literature, etc. All these might be said to come within the humanities. All these representations are almost entirely non-mathematical.

Focusing now on the academic world, non-mathematical discourse is typically found in the humanities, social science and applied social studies faculties, the last of these covering management, education, health and welfare. (Mathematics is not totally absent from these areas, and indeed is quite extensively used in certain areas such as economics, but for the most part discourse is non-mathematical). Moreover, humanities can sometimes disavow social science, and within social science the claim to be scientific is debated. Outside the academic world, most of social life, most of ordinary language and practice, are held by many people to be beyond the reach of academic theory let alone mathematics. Finally there is another location of non-mathematics which is often overlooked: mathematical scientists and others spend a lot of time discussing mathematical science in a non-mathematical way.

Thus non-mathematics overwhelmingly dominates people's experience. Most people most of the time operate in a non-mathematical context. A non-mathematical context is one which is neither experienced as having, nor studied using, any overt mathematics (by those who are dominant in those contexts). In other words, the official and popular view of these areas is that they are non-mathematical. Necessarily then, from the perspective of those who inhabit these contexts, the mathematical approach which I am offering in this book may seem rather unusual!

Truth-Intending and Non-Truth-Intending Discourses

Not all discourses aspire to make true statements. In order to appreciate the varieties of truth-intending and non-truth-intending discourses, consider a statement S made by individual A to individual B. Let us assume that the statement has a unique precise meaning and so may be true or false. The speaker may intend to make a true statement or they may intend to make a false statement, in other words the speaker is respectively either truth-intending or non-truth-intending. This gives four possibilities:

(1) Truth-intending success: The speaker intends to speak the truth and succeeds.
(2) Truth-intending failure: The speaker intends to speak the truth but fails.
(3) Falsehood-intending failure: The speaker intends to speak a falsehood but fails.
(4) Falsehood-intending success: The speaker intends to speak a falsehood and succeeds.

In each of these cases the speaker may or may not 'really' know the truth. If the former, then there may be correct execution of the intention or there may be a slip, an error of execution. If the latter, then the speaker lacks knowledge – they are in ignorance. In ignorance the speaker may guess and the guess may be right or wrong.

In the case of the intentional falsehood, the speaker may or may not intend to be believed – may or may not intend to deceive. In the former case, the speaker may or may not wish the listener harm: the act may be a harmful lie or a white lie, a bluff, a joke or a tease. In the case where there is no attempt to deceive, speaker and listener share an awareness that a falsehood is being spoken. This happens in the case of euphemism, exaggeration, irony, joking, fiction and fantasy; in the case of non-literal truth, metaphor, poetic truth and aesthetic truth; and in the case of a counterfactual and a supposition 'for the sake of argument'. In the case of the intentional truth, the speaker may or may not intend to be believed. Although the former is the normal occurrence, an example of the latter is the double-bluff.

Although non-truth-seeking discourse does not itself seek the truth it is nonetheless possible – as the preceding remarks demonstrate – to provide a truth-seeking account of non-truth-seeking discourse. Indeed conflict theory contains a variety of truth-seeking accounts of non-truth-seeking discourse and of failed attempts at truth-seeking. Failed attempts at truth-seeking can be a cause of conflict through misunderstanding, misperception and false 'images of the enemy' or false stereotypes. The pursuit of conflict can be accompanied by non-truth-seeking discourse. Conflict participants can seek advantage in the conflict by propaganda, rhetoric, deception, bluff and double-bluff. Psychological conflict can involve a conflict between truth-seeking and other discourse processes – as exemplified by Kuran's (1995) book '*Private truths, public lies*'; by the truth-seeking of cognitive therapy; and arguably by Freud's notion of a conflict between a truth-seeking ego and a potentially dubious id and super-ego.

Meaning and Precision … Meaninglessness and Vagueness

Some discourses have precise meaning, some do not. To appreciate this, consider the field of semantics (Cann, 1993). There is a distinction between natural languages, namely those languages which are spoken or written as the native languages of human beings, and formal languages, namely logical and mathematical languages. Semantics is the study of meaning in abstract. What have been abstracted out are: particular occasions, particular individuals, particular speech communities, the intentions of speakers, their psychological states, the socio-cultural context and the pragmatic aspects of an utterance.

Linguistic semantics is the study of meaning as expressed in languages. Formal semantics is an approach to linguistic semantics which makes greater use of mathematical techniques and places reliance on logical precision. It might be thought that formal semantics is appropriate for formal languages and that general linguistic semantics is appropriate for natural languages. However, Montague propounded the view that formal semantics could address both types of language.

What is meaning? A first basic task of semantics is to provide an account of the nature of meaning of the linguistic expressions. A linguistic expression is said to denote particular sorts of (possibly extra-linguistic) objects. With respect to the first, Frege's principle of compositionality states that 'the meaning of an expression is a function of the meaning of its parts'. There are a number of competing theories of meaning. The theory adopted here is that extensional denotation constitutes the core of meaning. In other words, the meaning of an expression is the relation between the expression and the object that expression is extensionally denoting.

What does a given expression mean? A second basic task of semantics is to provide an account of the relation between linguistic expressions and the things that they can be used to talk about. In order to study the link between natural language and the world two objects are interposed between them: the logical language and a model. The logical language removes the ambiguities and underdeterminacies of natural language. The model is the representation of the world.

natural language___logical language___model___world

A given expression may be perceived on first inspection to have a precise, well-defined meaning or multiple meanings or no discernible meaning. (Also, the same meaning can be expressed by many utterances.) Multiple meanings are a problem for truth-seeking discourse, and multiple utterances an inefficiency. However, for entertainment purposes, multiple meanings and multiple utterances provide richness and amusement. Multiple meanings found in vague discourse are also a means of avoiding making precise statements – either because the precise truth is not known or because the speaker does not want the precise truth to be known.

Linguistics: The Study of Ordinary Language

Linguistics is the study of ordinary language. Some of its sub-fields focus on language as an aspect of social interaction, while others focus on language as an aspect of the representation of meaning. Some of its sub-fields adopt a formal mathematical approach, while others do not. On the whole, sub-fields focusing on social interaction tend to be more discursive, while sub-fields focusing on meaning tend to be more formal. The following quotations from Aronoff and Rees-Miller (2001, pp. 374, 244, 296, 369, 319, 394, 428, 563) give the briefest of glimpses at some of these sub-fields. Sociolinguistics, discourse analysis, pragmatics and functional linguistics are four sub-fields which address the social interaction aspects of language:

(i) 'Sociolinguistics is the empirical study of how language is used in society'
(ii) 'Discourse analysis is concerned with the contexts in and the processes through which we use ... language to specific audiences, for specific purposes, in specific settings'.
(iii) Pragmatics: 'how do we ever understand each other?'
(iv) Functional linguistics focus on the fact that 'language is used for communication'.

The sub-fields of syntax, generative grammars, formal semantics and computational linguistics all involve formal mathematical representations of ordinary language:

(i) 'Formal semanticists seek ... precise mathematical models of the principles that speakers use to define those relations between expressions in a natural language and the world which support meaningful discourse'.
(ii) 'Generative grammars use recursive functions to generate complex sentences from a base of grammatical components. Inspired by earlier work in mathematical logic and the foundations of computer science, Chomsky [proposed] ... that we think of grammars as devices that put pieces of sentences together according to precise rules, thereby 'generating' well-formed sentences'.

MATHEMATICAL INQUIRY AND COMPLETED MATHEMATICAL KNOWLEDGE

There is an important distinction between the process of mathematical inquiry and the completed mathematical knowledge which is the product of that inquiry. Completed mathematical knowledge consists of a coherent structure of concepts, context-dependent propositions and the proofs of these propositions. The process of mathematical inquiry is a stumbling through a landscape of concepts, contexts and propositions not all of which are relevant or true. (Some of us stumble more than others!)

Mathematical logic is an attempt to express the nature of completed mathematical knowledge. Attempts to portray the tentative nature of mathematical inquiry include Hadamard's (1954) '*The psychology of invention in the mathematical field*', Polya's (1971) '*How to solve it*' and Newell and Simon's (1972) '*Human problem solving*', and the subsequent literature which these books fostered. The main concern of this section is with completed mathematical knowledge. However, the following brief section is provided in order to illustrate something of the flavour of mathematical inquiry.

The Process of Mathematical Inquiry: An Elementary Illustration

Here is a little bit of mathematical thinking which occurred to me today. It sort of just happened without me intending it. It is not at all mathematically impressive but does give a glimpse of what can happen in mathematical thinking (by someone like myself).

I wanted to give you an example of a simple statement and hit upon '$2+3=5$'. I also wanted a statement which contained a variable and so hit upon the generalisation 'for all whole numbers x, $x+(x+1)=2x+1$'. I realised that this could be rendered as 'the sum of every pair of consecutive whole numbers is odd'.

After our New Year's Day walk round Emberton Park, I thought again about these two examples. Aimlessly it occurred to me that the sum of every pair x and $(x+2)$ would be even. This prompted the question in my mind of what one could say about the pair of numbers x and $(x+n)$. Rather carelessly and wrongly it occurred to me that the pair might be divisible by n. Then it occurred to me that the pair would be odd or even depending on whether n is odd or even. Then I got a bit more formal and noted that $x+(x+n)=2x+n$. Noting that $2x+n=r(2x/r+n/r)$, it follows that $2x+n$ is divisible by r if r divides n and either r divides 2 or r divides x. If n is even then $r=2$ will do the trick; otherwise n and x must have a common divisor, r. Then I realised my thinking was a bit defective. Was the direction of implication going both ways or just one way? Also, I had gone too quickly to $2x+n=r(2x/r+n/r)$, without considering the more general notion, $2x+n=r((2x+n)/r)$. Also, I wanted a statement which covered a particular x and n, or a statement which covered a particular n for all x.

Let us reflect on what I was doing in all of this. I was engaged in a process where I reflected on a variety of contexts sometimes not having knowledge about the context, sometimes having a hypothesis but not being sure it was true and sometimes knowing that a particular statement was true – knowing because I knew I could prove it to be true. This imperfect disorganised process stands in contrast to the well-organised structure of completed mathematical knowledge.

Completed Mathematical Knowledge: Concepts, Propositions and Contexts

The structure of mathematical knowledge includes a structure of concepts, a structure of statements, a structure of arguments and a structure of contexts. A simple example of a structure of concepts is Fig. 2.4, with relationships between the concepts of element, set, function, relation and operation. The structure of arguments is discussed in the following section. Here I focus on the structure of contexts and propositions.

Consider the set of all contexts and the set of all propositions. In a particular context, some propositions are true and some propositions are false. Likewise, any given proposition will be true in some contexts and false in others. These thoughts lead to the idea of what might be called the context–proposition truth function.

> **Definition.** Given a set C of contexts and a set P of propositions the context–proposition truth function t maps each pair (c, p) in $C \times P$ into the set T of truth values, $\{0, 1\}$, where $t(c, p) = 1$ if p is true in c; and $t(c, p) = 0$ if p is false in c.

Some propositions are true in many contexts and some are true in just a few. A proposition which is true in all contexts is called a universal proposition (in mathematical logic it is a tautology). A proposition which is false in all contexts is called a universally false proposition (in mathematical logic it is a contradiction). Let $c(p)$ to be the set of contexts in which p is true. A proposition p has greater generality than proposition p' if $c(p)$ contains $c(p')$.

Propositions which are true in a large set of contexts are powerful and valuable. The search for these powerful propositions characterises much of mathematics, and indeed much of physics. In 1985, physicists at the Niels Bohr Centennial Conference (Ambjørn, Durhuus, & Petersen, 1985) reflected on the nature of physics: ‘physicists have ... tried to formulate general “laws of nature" to cope with the apparent diversity of natural phenomena' (Englert, 1985, p. 39) ... 'throughout the history of physics the goal has been to explain ever larger circles of phenomena in terms of ever smaller sets of rules, i.e., natural laws, which are in turn to be explained by even more fundamental laws' (Nielsen, Bennett, & Brene, 1985, p. 263). Twenty years later in 2004, Roger Penrose's book was revealingly entitled: '*The road to reality. A complete guide to the laws of the universe*'. Penrose introduces it thus: 'The purpose of this book is to convey to the reader some feeling for what is surely one of the most important and exciting voyages of discovery that humanity has embarked upon. This is the search for the underlying principles that govern the behaviour of our universe' (2004, p. xv).

I now provide an example of a context–proposition structure. Consider geometry and its concepts of a space with points, straight lines, intersections, line segments, polygons, lengths and angles, etc. What statements can we make? Suppose we are interested in triangles and the relationship between the lengths of the three sides, in particular how the longest side z depends on the other two sides. Let the lengths of the sides be $x \leq y \leq z$. Let θ be the 'included' angle between x and y. Pythagoras' theorem is common knowledge: 'the square on the hypotenuse is equal to the sum of the squares on the other two sides', but this is true only in the special case of a right-angled triangle (and assuming that the triangle is in Euclidean space). However, different equations hold true for other types of triangles – for other contexts.

Table 3.1 pairs each type of triangle with an equation which holds for that type of triangle. The proposition that a triangle is of a certain type implies that the equation takes a particular form, and vice versa. All the sets are subsets of T; IR is a subset of both I and R; M and G are subsets of R; and O is a subset of M and R. Likewise, all the propositions may be deduced from the proposition for T: $\theta = 0$ gives R, $x = y$ gives I, etc.

Table 3.1. Context and Proposition Structure.

Set	Context	Proposition
T	Triangle	$z^2 = x^2 + y^2 - 2xy \cos \theta$
R	Right-angled triangle	$z^2 = x^2 + y^2$
I	Isosceles triangle	$z^2 = 2y^2 (1 - \cos \theta)$
IR	Isosceles right angled triangle	$z = (\sqrt{2})y$
M	Right-angled triangle where $x = my$	$z = (\sqrt{(m^2 + 1)})y$
G	Right-angled triangle where $x = \sqrt{[f^2(y) - y^2)]}$	$z = f(y)$
O	Triangle where the smallest side has zero length	$z = y$

The context G has been included to highlight the point that the proposition depends on the context – and that strange or unanticipated propositions may be true in specific contexts.

Let us also anticipate here the use of dummy variables in regression equations. What the dummy variable does is to signal a specific context and modify the proposition accordingly. The set of propositions can be represented in the universal formulation below, where s_i is a dummy variable which takes the value of 1 when the context i holds and the value of 0 at other times. Both $s_i = f_i(x, y, \theta)$ and $z_i = g_i(x, y, \theta)$ are the functional relationships which hold in context i. Finally, the set I is an index set representing a set of exclusive and exhaustive context types.

$$z = \sum_i s_i z_i, \quad \text{where } s_i = f_i(x, y, \theta) \text{ and } z_i = g_i(x, y, \theta);\ s_i \in \{0, 1\}$$

Completed Mathematical Knowledge: Proofs

Having considered a structure of concepts and a structure of statements in the previous section I now consider the structure of an argument. For illustration I consider a letter which appeared in the *Financial Times* (2009) under the heading 'Claims that women are inherently more cautious are deeply troubling'. The letter consists of non-mathematical discourse. It has the status of appearing in the *Financial Times* and its authors are professors at three world-class business schools. The subject matter is important in that it relates to the economic meltdown in 2008 and to arguments about the relative merits of men and women. There is disagreement about the statements being made. There is concern about the quality of debate.

I now wish to identify the logical structure of the argument used in the letter. There are four key arguments considered in the letter. For the purpose of this presentation I caricature them as follows:

A: Men cause downturns.
B: Men do badly in downturns.
C: Men do well in healthy economic times.
D: Best talent does best.

The overall thesis of the letter is: 'arguments A, B and C are false, but argument D is true'. Associated with arguments A, B, C and D are alternative world arguments A*, B*, C* and D*.

It so happens that the arguments all have a similar logical structure. For example, consider argument A. Argument A consists of the following five statements:

(1) Women are more risk-averse or cautious or prudent than men. (G)
(2) The business world has been mostly run by men. (H)
(3) The business world has been run with a lack of caution. (H) (1) (2)
(4) A lack of caution can cause meltdown. (G)
(5) The business world experienced meltdown. (H) (3) (4)

Notice that some of the statements concern historical events (H), while other statements concern general principles (G). Notice how certain statements can be derived from other statements: (3) from (1) and (2); and (5) from (3) and (4). Although statements (3) and (5) are derived from other statements, statements (1), (2) and (4) are not. All this can be represented by the derivation tree for the argument (Fig. 3.1).

Argument A then moves on to consider an alternative world argument A^*. In the alternative world argument, statements (1) and (4) still hold. However, statement (2) is replaced by statement (2^*). As the derivation tree might suggest, this leads to a replacement of statements (3) and (5) by statements (3^*) and (5^*).

(1) Women are more risk-averse or cautious or prudent than men. (G)
(2^*) The business world is mostly run by women. (AW)
(3^*) The business world is run with a degree of caution. (AW) (1) (2^*)
(4) A lack of caution can cause meltdown. (G)
(5^*) The business world does not experience meltdown. (AW) (3^*) (4)

These arguments yield the 'current speculation that the meltdown might have been averted had more women been running the business world'.

A similar analysis can be carried out for arguments B, C and D.

A complete summary of the propositional structure is given in Table 3.2. Look at the column for statement (2). There are four pairs of arguments.

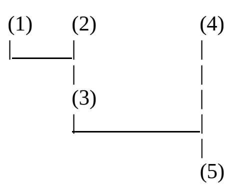

Fig. 3.1. The Abstract Derivation Tree for the Argument.

Table 3.2. A Complete Summary of the Propositional Structure.

Argument	Statements				
A	(1)	(2)	(3)	(4a)	(5a)
A*	(1)	(2*)	(3*)	(4a)	(5a*)
B	(1)	(2)	(3)	(4b)	(5b)
B*	(1)	(2*)	(3*)	(4b)	(5b*)
C	(1)	(2)	(3)	(4c)	(5c)
C*	(1)	(2*)	(3*)	(4c)	(5c*)
D	(1d)	(2)	(3d)	(4d)	(5d)
D*	(1d)	(2**)	(3d*)	(4d)	(5d*)

In each pair of arguments the first argument considers the present world where (it is claimed): (2) the business world has been mostly run by men. The second argument considers an alternative world where: either (2*) the business world is mostly run by women (AW); or, in the case of argument D*, (2**) the business world is run by equal numbers of men and women (selected by talent).

Look at the column for statement (1). The first three pairs of arguments assume that (1) women are more risk-averse or cautious or prudent than men. This assumption is denied in the fourth argument pair, D and D*, where it is held that (1d) women and men have similar distributions of talent. It is the debate between statements (1) and (1d) that constitute the central issue of the letter.

Look at the column for statement (3). The statements there are logical consequences of the statements in columns (1) and (2). In the present world: (3) the business world has been run with a lack of caution. In an alternative world: (3*) the business world is run with a degree of caution. In the fourth pair of arguments, in the present world: (3d) the business world has not been run with the best talent; whereas, in an alternative world: (3d*) the business is run with the best talent.

Look at each of columns for statements (4) and (5). Each pair of arguments is somewhat differently focused, A on causing a downturn; B on problems in a downturn; C on healthier economic times; and D on causing problems.

MATHEMATICAL LOGIC

There is an important distinction to be made between mathematics and logic. Cori and Lascar (CL; 2000), in their book on mathematical logic, note that ‘logic originates from reflecting upon mathematical activity’ and observe self-deprecatingly that ‘the common gut-reaction of the mathematician is to ask: “what is all that good for? We are not philosophers and it is surely not by cracking our skulls over modus ponens or the excluded middle that we will resolve the great conjectures or even the tiny ones”’ (CL, p. v).

The standard account of mathematical logic discusses first propositional calculus and then predicate calculus. Propositional calculus focuses on the relationship between propositions and provides a conceptual foundation for the above discussion of the logical structure of argument. Predicate calculus focuses on what is being said inside the proposition and on the nature of the mathematical objects being discussed – and hence links back to the previous discussion of concepts, propositions and contexts. In what follows, I provide an extended illustration of the key ideas in propositional calculus and this is followed by an illustration of the key ideas in the predicate calculus.

Propositional Calculus

In the previous section, the truth of one statement was sometimes dependent on the truth of other statements. A systematic account of how the truth values of statements in an argument are related is provided by the propositional calculus. I shall illustrate the propositional calculus using a very simple example. Here is the opening line of a poem by William Wordsworth:

> ‘My heart leaps up when I behold a rainbow in the sky!’

This is a compound statement made up of the two simple propositions, L and R:

L: My heart leaps up.
R: I behold a rainbow in the sky.

Given these two simple propositions we now ask what are the possible situations to which these propositions might refer? One possible situation would be where ‘My heart leaps up’ is true but where ‘I behold a rainbow in the sky!’ is false.

L: My heart leaps up.	True (1)
R: I behold a rainbow in the sky.	False (0)

However, that is just one possibility. In fact there are four possible situations, each resulting from a different assignment of true or false to the two simple propositions.

L: True (1)	R: True (1)	Leaps and rainbow
L: True (1)	R: False (0)	Leaps and no rainbow
L: False (0)	R: True (1)	No leap and rainbow
L: False (0)	R: False (0)	No leap and no rainbow

Each of the four possible situations corresponds to a different compound proposition. This prompts the question: what sort of compound propositions can one have? Compound propositions can be systematically generated by apply the logical connectives 'not', 'or' and 'and' to the two simple propositions to generate four compound propositions. We can repeat the procedure to obtain more compound propositions, and so on, indefinitely.

Does this mean that there are an infinite number of compound propositions that we can make? Yes indeed it does. Fortunately, however, there is a sense in which a lot of these propositions are the same. We might say that there are classes of equivalent propositions. For example, 'my heart leaps up' is the same as 'it is not the case that my heart does not leap up' and so on. Now that is just one class of equivalent propositions. How many proposition classes are there? It turns out that there are just 16 proposition classes: $1 + 4 + 6 + 4 + 1 = 16$.

L or not-L

|

L or R; not-(L and R); R or not-(R or L); L or not-(R or L);

|

nR; L; (L and not-R) or (R and not-L); R; nL; not-(L or R)

|

(L and not-R); L and R; (R and not-L); not-L and not-R

|

L and not-L

It is worth noting that proposition classes occur in pairs, X and not-X. So, for any given situation, eight proposition classes are true and eight are false.

Often we are interested in a particular world and we want a complete account of that world – a complete theory. By this we mean a set of statements from which the truth of all the other statements can be deduced. Here the set of two statements R and L provide a complete theory. Moreover, the set of any two logically independent statements provide a complete theory. By a set of logically independent statements we mean that no statement can be deduced from the other statement(s).

Our argument so far can be summed up as follows. We started with two simple propositions. These propositions could be either true or false, thus allowing four possible worlds. Using 'not', 'or' and 'and' we could construct an infinity of compound propositions. However, many of these propositions were equivalent to one another. Indeed there were just 16 classes of equivalent propositions. Propositions occur in pairs, for example, proposition L and proposition not-L. So, in any given world, there are eight true proposition classes and eight false ones. A theory is a set of propositions. We can use rules of inference such as modus ponens in order to deduce other propositions. A complete theory can deduce the status – true or false – of all the other propositions.

Predicate Calculus

Later Cori and Lascar elaborate on the nature of mathematical activity: 'the fundamental work of mathematics is to examine structures, to suggest properties that might pertain to these and to ask whether these properties are satisfied or not. Predicate calculus is in some way the first stage in the formalisation of mathematical activity' (CL, p. 113). As has been noted in the previous chapter: 'The word structure is generally understood in mathematics to mean a set on which a certain number of functions and relations (or internal operations) are defined' (CL, p. 130). If a language L is to express mathematical structures with sets and their elements, functions and relations then it needs:

(1) Symbols
Constant symbols referring to sets and their elements, function symbols referring to functions and relation symbols referring to relations.

(2) Terms
Terms serve to denote 'objects' in the structure. Terms are defined recursively: constants are terms, and the functions of terms are terms.

(3) Atomic formula
Atomic formulae serve to denote statements of facts about the objects in the structure. Atomic formulae are relations of terms.

Consider, for example, the statement '$2+3=5$'. It consists of five symbols: the constant symbols 2, 3 and 5; the function symbol $+$; and the relation symbol $=$. The constant symbols are terms. The function $+$ of the terms 2 and 3 is also a term, $2+3$. The atomic formula is the relation $=$ between the two terms $2+3$ and 5.

(4) Symbols for variables
Variables are also terms. And so the language allows the construction of terms such as $x+1$ and atomic formulae such as $x+(x+1)=2x+1$.

(5) The universal quantifier and the existential quantifier, 'for all x ...', 'there exists an x ...'.
These quantifiers combine with formulae to produce other formulae. And so the language allows the construction of formulae such as 'for all x in the set of real numbers, $x+(x+1)=2x+1$'.

(6) The connectives used in propositional calculus.
Here also the connectives can be used to connect formulae to produce other formulae. And so the language allows the construction of formulae such as 'A if B and (C or D)', with A '$2x+n$ is divisible by r'; B 'r divides n'; C 'r divides 2'; D 'r divides x'.

It is worth noting that statements of the form 'for all x in the set of positive integers, $x+(x+1)=2x+1$' are in the context–proposition form discussed earlier.

The Relationship between a Language and a Reality

The predicate calculus addresses the question of how one can use a language to refer to a reality. Consider the three basic questions: what is the nature of a reality?; what is the nature of a language?; and what is the nature of a relationship between a language and a reality? Let us start with a simple initial answer. Let a language be a set L of elements; let a reality be a set R of elements; and let a relationship between L and R be a one–one function f identifying each linguistic element with a specific element of reality. An example would be the set L of proper names, the set R of objects and the function f linking each proper name to the object for which the proper name is a label. This account links back to the discussion of meaning in section 'Meaning and Precision ... Meaninglessness and Vagueness' and the notion that extensional denotation constitutes the core of meaning.

Suppose now that reality R has structure in the sense that elements can be combined to form other elements. The ways of combining the elements of L need to correspond to the ways of combining the elements of R. Typically compound elements are made up of simple elements. For example, in ordinary language, words are made up of a sequence of letters. These remarks lead us to the concept of an abstract language as a set of elements which can be combined. Of particular interest is the case where there is a one-to-one correspondence between language and reality – between aspects of the language and aspects of reality, each aspect of reality being referred to as the interpretation of the associated aspect of the language.

We now apply these general remarks to the case of the first-order predicate calculus. Consider first the interpretation of terms. Each constant symbol maps to an element, each function symbol maps to a function and each relation symbol maps to a relation. The interpretation of terms which contain only constants follows quite naturally – however, when a term contains variables, the interpretation of the term varies with the interpretation of the variable.

Consider now the interpretation of formulae. If an atomic formula contains only constants then the truth value of the formula depends on whether the referred to relation holds. Otherwise the truth value of the atomic formula varies depending on the values of the variables. The same points apply to any statement which is not atomic. Note that if a formula contains a free variable (one not governed by a quantifier) then its truth value varies with the interpretation of the variable. Thus the truth value of a formula is only definite if the formula is closed (one containing no free variables).

We define a theory in a language L to be a set of formulae. A theory is consistent if it has at least one model (one reality in which it is true, that is the set of all formulae in the theory are true). A theory is complete if and only if (i) it is consistent and (ii) the theory implies either the truth or the falsity of every formula in L.

A language can refer to many possible alternative realities, and within a language many theories can be expressed. For each theory there is a set of realities for which the theory is true and for each reality there is a set of theories which are true of that reality. This theory–reality linkage is an extension of the proposition–context linkage discussed earlier.

The field of mathematical logic runs deep and somewhere down the line one encounters Gödel's theorem. In view of the fact that the complexity literature contains references to Gödel (Byrne, 1998, pp. 54, 56, 59–60), it is worth noting Penrose's (2004, p. 377) caution:

> There is a common misconception that Gödel's theorem tells us that there are 'unprovable mathematical propositions', and that this implies that there are regions of the 'Platonic world' of mathematical truths ... that are in principle inaccessible to us. This is very far from the conclusion that we should be drawing from Gödel's theorem. What Gödel actually tells us is that whatever rules of proof we have laid down beforehand, if we already accept that those rules are trustworthy (i.e. that they do not allow us to derive falsehoods), then we are provided with a new means of access to certain mathematical truths that those particular rules are not powerful enough to derive.

ARTIFICIAL INTELLIGENCE

In the section 'Mathematical Inquiry and Completed Mathematical Knowledge', I noted a distinction between the process of mathematical inquiry and the completed mathematical knowledge which is the product of that inquiry. There is a similar distinction between ordinary language output and the process which generates the ordinary language output. In general, there is an important distinction between process and product. Much of our discussion so far has been on the product – completed knowledge about triangles, the letter to the *Financial Times* and the lines of Wordsworth's poem. There has been rather little discussion of process, a neglect which I now seek to remedy.

The link between ordinary language, mathematical logic, computing languages and artificial intelligence is noted in the reference to Montague's thesis and the Van Benthem quotation given at the beginning of the chapter. Russell and Norvig (1995) define artificial intelligence to be the building of systems of one of the four types:

Systems that think like humans (cognitive modelling)
Systems that act like humans (the Turing test)
Systems that think rationally (the laws of thought)
Systems that act rationally (the rational agent)

Their textbook on artificial intelligence has the following main parts:

Artificial intelligence
Problem-solving
Knowledge and reasoning
Acting logically
Uncertain knowledge and reasoning
Learning
Communicating, perceiving and acting

Artificial intelligence draws on ideas from a variety of other disciplines: philosophy, mathematics, psychology, computer engineering and linguistics.

Its history has exhibited changing views about what is the best way forward. The early work on general-purpose methods for problem solving gave way to a realisation that specific domain knowledge was important. The focus here is on 'Knowledge and Reasoning'. How might an intelligent agent have knowledge about the world and reason about it? A key issue is how to represent knowledge. The authors focus on first-order logic. The importance of mathematical structure (objects and relations) and mathematical logic (first-order logic) to artificial intelligence is indicated by the following remarks in Russell and Norvig (1995, p. 185):

> First-order logic has been so important to mathematics, philosophy, and artificial intelligence precisely because these fields – and indeed much of everyday existence – can usefully be thought of as dealing with objects and relations between them. We are not claiming that the world really is made up of objects and relations, just that dividing up the world that way helps us reason about it.
>
> There are many different representation schemes in use in artificial intelligence, ... Some are theoretically equivalent to first-order logic and some are not. But first-order logic is universal in the sense that it can express anything that can be programmed. We choose to study knowledge representation and reasoning using first-order logic because it is by far the most studied and best understood scheme yet devised.

Having discussed first-order logic, the authors then note that a logic does not offer any guidance as to what facts should be expressed, nor what vocabulary should be used to express them. The process of building a knowledge base is called knowledge engineering. This involves investigating a particular domain and determining what concepts are important in that domain and creating a formal representation of the objects and relations in that domain (Russell & Norvig, 1995, p. 217).

Callan (2003, pp. 10, 364) suggests, 'the whole science of AI has largely revolved around issues of knowledge representation'. With particular reference to ordinary language he comments as follows: 'There is a great deal of inherent structure in natural language. The classical approach to natural language processing attempts to represent this structure to extract meaning. A parser is used to analyse the [grammatical] structure of a sentence ... Semantic analysis is concerned with extracting the meaning of a sentence. ... A challenge is the considerable amount of general world knowledge that [a general conversational artificial agent] would need to possess'.

Davis (2005, pp. 81, 84) presents 'a theory of informative communications among agents that allows a speaker to communicate to a hearer truths about the state of the world'. What is of interest in this theory is its ability to represent ordinary language statements and arguments, its grounding in mathematical logic and its addressing of certain substantive topics. Davis notes that his theory covers 'the occurrence of events, including other communicative acts; and the knowledge states of any agent – speaker, hearer, or third parties – any of these in the past, present, or future – and any logical combination of these, including formulas with quantifiers'. He uses 'a situation-based, branching theory of time; an interval-based theory of multi-agent actions; and a possible-worlds theory of knowledge'.

He uses a sorted first-order logic with equality, where the sorts are clock-times, T; situations, S; Boolean fluents, Q; actions, E; agents, A; and actionals, Z. An actional is a characterisation of an action without specifying the agent. Times and situations are ordered. There is a function from a situation to its clock time. An action occurs from one situation to another – the action of an agent doing an actional. A fluent holds in a given situation.

Developments in artificial intelligence have fed into the development of computer languages and the associated systems engineering – for example, object-oriented programming and design: 'in the field of artificial intelligence, developments in knowledge representation have contributed to an understanding of object-oriented abstractions' ... 'object-oriented technology is more than just a way of programming ... it is a way of thinking abstractly about a problem using real-world concepts, rather than computer concepts' (Booch, 1991, pp. 35, ix).

The engineering process is discussed in terms long familiar in systems analysis: the four stages of problem formulation, analysis/modelling, design and implementation. Our interest here is with the modelling stage. Rumbaugh, Blaha, Premerlani, Eddy, and Lorenson (1991) distinguish between object modelling, dynamic modelling and functional modelling. Odell (1998) emphasises the importance of basic mathematical conceptualisation in the field of object-oriented analysis. From a mathematician's point of view an object-oriented model is a mathematical structure which is described using object-oriented labels rather than the long-established mathematical labels. Object modelling involves 'classes' of objects' (sets of elements), values of attributes of objects (functions from sets of one type to sets of another type), operations and methods (another type of function) and links and associations (relations). More advanced concepts are derived from these – such as homomorphism maps between two associations. Dynamic modelling involves states and events. The functional model describes what happens, the dynamic model specifies when it happens and the object model specifies what it happens to. (Filman, Elrad, Clarke, & Aksit, 2005; pp. 6–7) observe:

> The current state-of-the-art in programming is object-oriented (OO) technology. ... Objects are not the last word in programming organization. This book is about an emerging candidate for the next step in this progression, aspect-oriented software development.

Ideas from artificial intelligence have also influenced the development of expert systems engineering. There are four main stages in the development of an expert system: problem analysis, creation of the formal model, implementation and testing. There are various ways of eliciting data about expertise – usually, however, the data are in verbal form. There follows a stage of analysis leading to a specification of the knowledge in terms of concepts, relations and procedures ... main problems, a hierarchy of sub-problems and a hierarchy of strategies. Expertise resides in the extensiveness of knowledge, not in its sophistication (apparently!?) ... it resides in practice and ability to guess well in situations which are uncertain. The functions performed by expert systems are: classification, monitoring, design, planning and prediction. Systems may be advisory, dictatorial or criticising.

Knowledge representation may be by first-order logic, production systems, semantic nets or frames. Production systems consist of: rules of the form if P then Q; a database of facts consisting of propositions; and a rule interpreter which does pattern matching and conflict resolution. Various strategies are forward or backward strategies ... depth or breadth strategies and meta-rules. Semantic nets involve 'isa' and 'instance' relations in a hierarchy supporting a taxonomy. (The dangers of not being aware of basic mathematical conceptualisation is illustrated by some early systems which failed to distinguish between properties of members of a category and properties of the category as a whole (Russell & Norvig, 1995, p. 317).) Frames are like a pro forma containing declarative and procedural information.

The procedure for carrying out an expert systems analysis allows us to make sense of fuzzy logic which otherwise looks a bit strange. Again the motivation is the desire to have a precise machine representation of human control systems. Although human experts express their expertise using fuzzy ordinary language, the fuzzy systems engineer represents this fuzziness mathematically in a fuzzy model which involves input variables, control variables and output variables. A precisely defined control response is obtained from the fuzzy model by a process referred to as defuzzification.

CHAPTER 4

POSSIBILITY AND PROBABILITY: VALUE, CONFLICT AND CHOICE

> Social choice theory is concerned with the evaluation of alternative methods of collective decision making, as well as with the logical foundations of welfare economics. In turn, welfare economics is concerned with the critical scrutiny of the performance of actual and/or imaginary economic systems, as well as with the critique, design and implementation of alternative economic policies.
>
> – Suzumura (2002, p. 1)

Complete knowledge of a particular world involves knowing the truth values of all the propositions concerning that world. Complete ignorance involves knowing nothing at all. The first major step beyond complete ignorance is knowing the set of all possibilities within which the particular world might occur. Between complete knowledge and knowing just the set of possibilities, there is partial knowledge which can be expressed in terms of the probability of events. In some situations it is appropriate to assume a priori that each (elementary) event is equally likely.

Possibilities and probabilities are to the fore in the discussion of value, conflict and choice. In the absolute notion of value, value may be binary, ordinal or quantitative. In the comparative notion of value a preference is expressed in relation to each pair of objects. Multidimensional value arises when values are placed on a set of objects by a set of individuals – either by individual people or by individual criteria. The likelihood of value consensus decreases and the likelihood of value conflict increases as the number of individuals and the number of options increases.

How should we choose? All of five sets – options, methods, criteria, choosers and situations – need to be thought about when addressing the fundamental problem of social choice: there is conflict between choosers, methods and criteria in that in some situations the different choosers, methods and criteria select different options. For example, there are situations where the Condorcet majority principle, the De Borda ranking principle and the welfare principle select different options. Also there are situations where a voting cycle occurs – a result which provides the basis for Arrow's general impossibility theorem. The likelihood of those undesirable situations is considered.

An individual may seek success or power (or influence or decisiveness). 'Success equals power plus luck'. The likelihood of success and the likelihood of power depend on the social choice function; and both likelihoods decrease as the number of individuals and the number of options increases. In their discussion of possible voting rules for the European Council of Ministers, Laruelle et al. (2006) ask whether states seek power or success and argue that states which are concerned about their sovereignty and states which are concerned to deepen integration are likely to press for different voting rules.

POSSIBILITY AND PROBABILITY

The Set of All Possibilities and the Set of All Propositions

It is quite common for the discussion of a topic to start with a consideration of the set of all possibilities. In Chapter 3 we considered the set of all propositions and the associated set of all possible worlds. We considered two simple (or basic) propositions and noted how different patterns of truth values for these propositions gave rise to four possible worlds. Let us recap the argument here for the cases of one, two and three basic propositions. The results are presented in Table 4.1.

(1) Suppose there is just one basic proposition: *A*. There are two possible truth values for the proposition: true or false – two possible worlds

Table 4.1. The Relationship between the Number of Elementary Events and the Corresponding Numbers of Other Aspects.

	Number of basic propositions (required)							
	(1)	1	(2)	2	(3)	...	3	...
	Number of elementary events, elementary event propositions, possible worlds							
	1	2	3	4	5	...	8	...
Size of power set (the number of events or propositions)	2	4	8	16	32	...	256	...
The number of events in each world; truths in each world	1	2	4	8	16	...	128	...

$(2 = 2^1)$. There are an infinite number of compound propositions but only 4 classes of equivalent propositions $(4 = 2^2)$. In any given world 2 of the proposition classes are true and 2 are false. In one of the worlds *A* and '*A* or not-*A*' are true while not-*A* and '*A* and not-*A*' are false. In the other world not-*A* and '*A* or not-*A*' are true while *A* and '*A* and not-*A*' are false. So in any given world 2 of the proposition classes are true and 2 are false.

(2) Suppose there are two basic propositions *A* and *B*. There are two possible truth values for each proposition: true or false. There are four possible combinations of truth values for the two propositions – four possible worlds $(4 = 2^2)$. There are an infinite number of compound propositions but only 16 classes of equivalent propositions $(16 = 2^4)$. In any given world 8 of the proposition classes are true and 8 are false.

(3) Suppose there are three basic propositions *A*, *B* and *C*. There are two possible truth values for each proposition: true or false. There are eight possible combinations of truth values for the three propositions – eight possible worlds $(8 = 2^3)$. There are an infinite number of compound propositions but 'only' 256 classes of equivalent propositions $(256 = 2^8)$. In any given world 128 of the proposition classes are true and 128 are false.

Consider now the set of all possible events that can occur in a given context. There is a distinction between an elementary event and a compound event. The set of elementary events is exhaustive, exclusive and elementary: the elementary events cover all the possible events; no two of them can occur at the same time; and all other events are constituted by compounds of these. Denoting the set of all elementary events by E, the set of all (possibly compound) events is the power set of E, S^E. The set of events, S^E, consists of pairs of events: for each event e there is its complementary event not-e; and for the event not-e there is its complementary event not-(not-e) = e. In any given world only one event of any complementary pair can occur.

For example, consider the set of all possible events that can occur when throwing a dice cube with six faces numbered from 1 to 6. The set of all possible elementary events has six elements $\{1, 2, 3, 4, 5, 6\}$. An example of a compound event is the dice falling on an even number. This can occur whenever either of the following elementary events occurs: a '2' or '4' or '6' is thrown. The event of an even number has, as its complementary event, the event of an odd number.

The set of all possible events relates to the set of all possible propositions in an obvious way. The possible event of e occurring corresponds to the possible truth of the proposition 'the event e occurs'. Corresponding to the set E of elementary events there is a set P of elementary event propositions. Corresponding to the set S^E of all possible events there is a set S^P of all possible event propositions, that is, propositions of the form 'event e occurs'.

If there are n elementary events then there are n elementary event propositions; and 2^n events and 2^n event propositions. In any given world only one event of any complementary pair can occur; and only one proposition of any complementary pair can be true. So in any given world there are 2^{n-1} events which occur and 2^{n-1} event propositions which are true.

In the previous chapter we found that two basic propositions corresponded to four possible worlds determined by the four possible patterns of truth values for the two propositions. In general, if $n = 2^p$ then n elementary events can be specified in terms of p basic propositions. If $n \neq 2^p$ then n elementary events can be specified in terms of p basic propositions where p is such that N is the lowest number greater than n such that $N = 2^p$. These various points are summarised in Table 4.1.

Sometimes the set of elementary events has a structure. The total number of elementary events then depends on the number of elements in the components of the structure. For example, the set E may equal the Cartesian product of two other sets $E_1 \times E_2$. Using $n(S)$ to denote the number of elements in a set S, we have $n(E) = n(E_1) \cdot n(E_2)$. Another example is where E is the set of all functions from a set E_1 to a set E_2. Here $n(E) = n(E_2)^{n(E_1)}$. Another example is the set E of all possible orderings of n elements. Here $n(E) = n!$. This is the topic of permutations and combinations.

Probability Theory

Between complete knowledge and knowing just the set of possibilities, there is partial knowledge. There is a variety of approaches to partial knowledge. There is a distinction between quantitative and qualitative approaches. Qualitative approaches include modal logic, non-monotonic logic, plausible reasoning and heuristic search. Quantitative approaches can be one-variable or two-variable or set-based. The prime example of one-variable approaches is probability theory, and another example involves 'certainty factors' which can be modified so as to refer to probabilities. In two-variable approaches, a proposition is given an upper and a lower estimate of its uncertainty. Set-based methods include fuzzy set theory, rough set theory and probability enhanced by correlations. From this great variety of approaches we shall restrict our attention to just one, namely probability theory.

In addition to knowing the set of possible events we want some way of talking about the likelihood of events. In probability theory this involves attaching numbers to events. Following Pollard (2002), we conceptualise the set of events as a sigma field, and introduce the notion of a measure space as a way of attaching numbers to sets and the notion of a probability space as a measure space in which the total measure is unity. Finally, we note that in some situations it is appropriate to assume a priori that each (elementary) event is *equally likely*.

Consider the set X and its power set, $A = S^X$. These are the basic ingredients of the definition of a sigma field. A *sigma field* is a pair (X, A), where X is a set and A is a class of subsets of X with the following properties: A contains the set X and also the empty set; and A is closed under complementation, countable union and countable intersection.

A *measure space* is a quadruple $(X, A, R+; m)$, where (X, A) is a sigma field and m is a measure – a function from A to $R+$, the non-negative real numbers. The measure m maps the null set to zero; and is such that the measure of a countable collection of pairwise disjoint sets is equal to the sum of the measures of the individual sets.

A *probability space* is a measure space with $m(X) = p(X) = 1$. Suppose that $p(X) = p(\{e_1\}) + p(\{e_2\}) + \cdots + p(\{e_n\}) = 1$, where the $\{e_i\}$ are elementary events. Suppose also that if S_1 and S_2 are disjoint sets then $p(S_1 \cup S_2) = p(S_1) + p(S_2)$. So p is a probability measure on X.

What might be called a *number space* can be defined by setting $m(S) = n(S)$, where we associate each set S with the number $n(S)$ of elements it contains. With this definition $n(S)$ is indeed a measure because if S_1 and S_2 are disjoint sets then $n(S_1 \cup S_2) = n(S_1) + n(S_2)$. If e is an element then $n(\{e\}) = 1$. If X has N elements then $n(X) = N$.

In some situations it is appropriate to assume a priori that each (elementary) event is *equally likely*. If e is an element then $p(\{e\}) = 1/N$. It follows that $p(S) = n(S)/N$, where $N = n(X)$. If S_1 and S_2 are disjoint sets then $p(S_1 \cup S_2) = p(S_1) + p(S_2)$. Also, $p(X) = 1$.

VALUE, CONFLICT AND CHOICE

Value and Choice

Although there is a natural association between value and choice, the two concepts are distinct: objects may have value even when no choice is required between them; a choice may be made between objects without the value of the objects being taken into account.

In the literature there are two distinct notions of value: an absolute notion and a comparative notion. The absolute notion is that an object has a value; and the comparative notion is that one object has either more value than, or less value than, or equal value with, another object – in other words there is a preference between a pair of objects. Although the simplest notion is that of absolute value, the literature sometimes takes preference (comparative value) as the starting point. In the absolute notion of value, value may be binary, ordinal or quantitative. The binary notion is that an object either has value or it does not. For example, 'Sarah likes chocolate', 'Bertrand likes truth'. The ordinal notion is that an object can take one of a number of degrees of value and this is sometimes expressed in terms of ordinary language. For example, opinion surveys might ask people if they are 'very satisfied', 'fairly satisfied', 'not very satisfied' or 'not at all satisfied'. The quantitative notion is that an object has a certain amount of value. For example, opinion surveys might ask people, 'on a scale from 0 to 10 how satisfied are you?' In the comparative notion a preference is expressed in relation to each pair of objects. For example, 'I prefer Gordon to Nick; Nick to David; and Gordon to David'.

The notion of choice between a pair of objects is that given a pair of objects, person A chooses object X over object Y. Another notion is that of the criterion for choice: given a pair of objects, person A chooses object X over object Y as a result of applying criterion C. A possible link between these notions is the notion that value provides the criterion for choice: given a pair of objects, person A chooses object X over object Y as a result of applying criterion C and the criterion is to choose the object with the greater value. The literature commonly assumes that the criterion of choice is that the object with greater value is chosen.

Multidimensional Value: The Probability of Consensus or Conflict

Consider a set of *n* individuals. The set of individuals may be a set of individual people or a set of individual criteria. Consider a situation where there are *m* objects. If each individual places value on the objects then the set of individual values constitutes the placing of a multidimensional value on the objects. More specifically, suppose each individual has a preference ordering on the m objects from the most preferred object to the least preferred object. A specific set of preference orderings for the set of individuals is referred to as a preference pattern. An individual's most preferred option is referred to as the individual's ideal.

Preference consensus is said to occur if the individuals have identical preference orderings; and ideal consensus is said to occur if the individuals have identical ideals. Preference consensus implies ideal consensus but not vice versa. Preference conflict occurs if there is no preference consensus; and ideal conflict occurs if there is no ideal consensus. (Note that one might refer to unanimity rather than consensus.)

Consider all possible patterns of preferences. The number of preference orderings is $m!$. The number of preference patterns is $(m!)^n$. The number of ideals is m. The number of ideal patterns is m^n.

What is the likelihood of consensus and conflict? If the social preference patterns are equally likely to occur then the relative frequency corresponds to the probability of occurrence. Table 4.2 presents the results. The answer depends on the number of options and the number of individuals. As the number of individuals and the number of outcomes increases, the probability of ideal consensus tends to zero and the probability of ideal conflict tends to one. (Other more qualified definitions of consensus and conflict can be made. It seems plausible that, for many of these definitions also, the probability of consensus tends to zero and the probability of conflict tends to one, as the number of options or the number of individuals increases.) Note that because preference consensus implies ideal consensus but not vice versa, the probability of ideal consensus is greater than the probability of preference consensus. In summary, ideal conflict is highly likely.

Table 4.2. The Probability of Ideal Consensus or Ideal Conflict.

No. of individuals	1	1	2	2	n	n	2	∞
No. of objects	2	m	2	m	2	m	∞	2
Total possibilities	2	m	4	m^2	2^n	m^n	∞	∞
Probablity of ideal consensus	1	1	0.5	$1/m$	$1/2^{n-1}$	$1/m^{n-1}$	0	0
Probability of ideal conflict	0	0	0.5	$(m-1)/m$	$1-(1/2^{n-1})$	$1-(1/m^{n-1})$	1	1

Notes:

(a) The probability of preference consensus is $1/m$, which tends to 0 as m tends to ∞.

(b) So the probability of preference conflict is $(m-1)/m$, which tends to 1 as m tends to ∞.

(c) The probability of ideal consensus is $1/m^{n-1}$, which tends to 0 as m tends to ∞.

(d) So the probability of ideal conflict is $1-(1/m^{n-1})$, which tends to 1 as m tends to ∞.

Social Choice and Social Welfare

The previous subsection shows that ideal consensus is highly unlikely. So when there is a need to choose between options different individuals will want to choose different options. So what choice should be made? This is the question addressed by social choice theory and social welfare theory (see the quotation at the start of this chapter). Suzumura (2002, p. 10) (hereafter referred to as 'S') provides an account of its history citing the earlier work of Condorcet, Borda, Dodgson, Black and others and then singles out Kenneth Arrow's *Social Choice and Individual Values* as '[elevating] social choice theory to a stage which is qualitatively different altogether' (S, p. 10). Whereas the previous writers had focused on specific social choice mechanisms, Arrow 'developed an analytical method which allowed him to treat all conceivable voting schemes simultaneously within one unified conceptual framework' (S, p. 10).

How should we choose? The word 'choose' implies a set of possible options. The word 'how' implies a set of possible methods. The word 'should' implies a set of criteria for judging the choice. The word 'we' implies a set of choosers. Finally, because we want to provide a general answer to the question we need to consider the set of possible situations. All these five sets – options, methods, criteria, choosers and situations – need to be thought about when addressing the fundamental problem of social choice: there is conflict between choosers, methods and criteria in that in some situations the different choosers, methods and criteria select different options.

Let A be a set of choice situations; B_a be a set of options in situation a; and C be a set of choice methods. Let d be the choice situation of choosing which choice method to apply (from the set C of choice method options); and let E be a set of choice methods for choosing the choice method to apply to A. We might refer to A as the set of primary choice situations and to d as the meta-choice situation. (Clearly we could proceed to consider the meta-meta-choice situation and so on.) The multidimensional value of the primary options consists of different people placing different values on the options whereas the multidimensional value of the meta-options consists of different criteria placing different values on the meta-options (namely the different choice methods).

The Condorcet Majority Principle, the De Borda Ranking Principle and the Welfare Principle

To make all this concrete, consider the persuasive example given by Pattaniak (2002, p. 363). Consider a committee of seven members needing to select one option from a set of options. The first four members share the same set of preferences having option A as the first preference, and the other three members share a different set of preferences having option B as a first preference. So the two groups are in conflict. How should the conflict be resolved? What choice should be made?

One attractive criterion is the Condorcet (1785) majority principle: the option preferred by the majority should be chosen. Applying this criterion, the option A should be chosen. A more precise definition of the Condorcet principle is that the option which has a majority over each other option is the option which should be selected. For example, suppose that there are just three options and that the preferences held by the first four members are $A>B>C$; and by the other three members are $B>C>A$. Under Condorcet's majority voting between each pair of options, A defeats B and A defeats C, in each case by four votes to three votes, (and B defeats C by seven votes to zero). So option A is the majority or Condorcet winner.

Notice that the preferences in the previous paragraph are such the second preference of the first group is the first preference of the second group but the first preference of the first group is the third preference of the second group. In other words the minority group gives the majority winner a low ranking. De Borda's (1781) positional voting criterion is designed to capture this aspect of the preference pattern. In the case of three options, a score of 3 is assigned to a first preference, a score of 2 is assigned to a second preference and a score of 1 is assigned to a third preference. So in our example, option A scores 15, option B scores 17 and option C scores 10. Option B has the highest score. Applying this criterion, the option B should be chosen.

Although the De Borda pays attention to the ranking of the options it does not consider the amount of value associated with any of the options. For example, are the minority appalled by option A or do they think it is just slightly less valued than option C? The amount of value is an aspect which has been studied in social welfare theory. In the history of welfare economics, the utilitarian ideas of Bentham were developed by later economists and synthesised by Pigou. This assumed that values could be given a quantity and that comparisons could be made between individuals' values. In rejection of this approach attempts were made to base values on ordinal interpersonal non-comparable preferences making use of the Pareto principle, compensation criteria and social welfare functions. Of these attempts, the Bergson–Samuelson social welfare function has found favour. However here it is the earlier Pigou welfare principle which we apply to our example.

Consider the utility of the options for each person. Suppose the utilities equal the De Borda scores. Necessarily then the De Borda total equals the total utility. So the option which maximises the Pigou welfare principle of maximising the total utility is the same as the option which maximises the De Borda scores: namely option B. Instead, suppose that the utilities do not equal the De Borda scores. Suppose the utilities held by the first four members are $A = 4$, $B = 2$ and $C = 1$; and by the other three members are $B = 3$, $C = 2$ and $A = 1$. Option A scores 19, option B scores 17 and option C scores 10. So option A has the highest score and so is social welfare winner.

So in this situation the Condorcet principle and the De Borda principle produce different results and the Pigou welfare principle may produce one or other result depending on the precise utilities of the options. The three methods all are intuitively appealing. Ideally we would like them to produce the same result in all situations. The example shows that this is not possible. There is a meta-conflict between the methods. None of the methods is perfect (because every method sometimes fails to produce the winner produced by the other attractive methods).

Arrow's List of Principles and his General Impossibility Theorem

We have found that each method has the defect that in some situations it fails to select the option recommended by other attractive methods. Worse still, a method can be defective in that in some situations it is unable to fulfil all of its own principles. Arrow focused on the Condorcet principle and combined it with a number of other desirable principles. In his celebrated impossibility theorem Arrow showed that the Arrow set of principles was impossible to implement in all situations: there were some situations where at least one of the principles did not hold.

In particular, in some situations there are voting cycles (one of Arrow's principles was that there should be no voting cycles). A voting cycle of length three occurs if option A is socially preferred to option B, option B is socially preferred to option C and option C is socially preferred to option A. The simplest example of this is where there are three people with preference orderings (A, B, C), (B, C, A) and (C, A, B) and the social ordering is defined by majority voting. Option A defeats B by two votes to one, option B defeats C by two votes to one and option C defeats A by two votes to one.

The Probability of Things Going Wrong

In summary, what we have found is that it is possible for things to go wrong. Moving on now from possibilities to probabilities, we can ask: how likely is it that things will go wrong?

As an example suppose there are three people voting between three options, A, B and C. Suppose without loss of generality that the preference ordering for the first person is (A, B, C). There are six possible preference orderings for the second person; and quite independently of this there are six possible preference orderings for the third person. So in combination there are 36 possible preference patterns.

How likely is a voting cycle? In this situation where the first person has (A, B, C), a voting cycle only arises if the other two patterns are (B, C, A) and (C, A, B). There are only two ways of assigning these two patterns to the other two people. So a voting cycle arises in just 2 of the 36 cases. If all possibilities are equally likely then the probability of a voting cycle is $2/36 = 1/18 = 0.0555 \ldots$.

Of course the example looks at only the most simple case with three individuals and three options. Things are more likely to go wrong if there are more individuals and more options. Let n be the number of individuals and m be the number of options. Denoting $Q(m, n)$ to be the proportion of situations where there are no Condorcet winners it has been found that $Q(3, 3) = 0.0555$; $Q(3, n)$ tends to 0.0877 as n tends to infinity; and $Q(9, n)$ tends to 0.4545 as n tends to infinity (Gaertner, 2002, p. 143).

SUCCESS EQUALS POWER PLUS LUCK

The equation, 'success equals power plus luck', captures much of the essence of social life. Success is accompanied by scenes of great rejoicing – whether it be the success of an explorer, an army, a political party, a sports team or a jackpot winner. In marked contrast scenes of deep despondency accompany failure. Whichever is the case it is difficult for the bystander not to be carried along in the flood of emotions. Moreover, the notions of power and luck play a key role in stories of success and failure. Sometimes success is explained in terms of the virtues and powers of the winners and failure is explained in terms of the fatal flaws of the losers – as in, for example, the concept of the tragic hero in literature. At other times the winner gains no praise and the loser receives no blame – instead success and failure are attributed to chance circumstances. The cold equation, success equals power plus luck, contains much of the heat of social life.

In the study of politics the concept of power is central – as is the concept of success. The concept of success is at the core of rational choice theory which can be viewed as a theory of success-seeking. So it is not surprising that the equation 'success equals power plus luck' originates in the political science literature. It was first used by Barry in 1980 and has attracted continued interest – see for example, Laruelle et al. (2006).

The discussion of the set of possibilities in section 'Possibility and Probability' enables us to have our first attack on the statement 'success equals power plus luck'. For the purposes of this section we shall interpret this to mean 'success is due to power or luck (but not both)'. This reformulation is prompted by Laruelle et al.'s (2006, p. 191) comment that 'irrelevance' might be a more suitable term than 'luck' and that strictly speaking what is involved is 'success without decisiveness'. The equation now becomes 'success equals success with decisiveness plus success without decisiveness'.

This can be proved as follows. Consider the set of all events. Let S denote events which contain success; and D denote events where success occurs and is due to decisiveness. Clearly, $S \geq D$. The set difference $S-D$ corresponds to events where success occurs without decisiveness. (The sets D and $S-D$ are disjoint.) The set S is the union of D and $S-D$: $S = D \cup (S-D)$. In terms of the corresponding propositions, 'S is D or $(S-D)$'. In other words 'success is due to success with decisiveness or success without decisiveness'. Notice that in making this argument, there has been no need to consider the meanings of success or of decisiveness. It depends merely on the fact that whatever the contents of sets A and B it is always true that if $A \geq B$ then $A = B \cup (A-B)$.

We now consider the probability of these events. If $S = D \cup (S-D)$ then $n(S) = n(D) + n(S-D)$. If all the events are equally likely then $p(S) = p(D) + p(S-D)$ where $p(Z) = n(Z)/N$ with N being the total number of events. If the events are not all equally likely then it is still the case that $p(S) = p(D) + p(S-D)$, but here $p(Z) = \Sigma_Z p_e$. Notice that again in making this argument, there has been no need to consider the meanings of success or of decisiveness. It follows from basic results in set theory and probability theory and from the fact that the event of success corresponds to the disjoint union of two sets.

The Probability of Success and Power

In the section 'Social Choice and Social Welfare', we considered the probabilities of consensus and conflict. We now follow this up with a consideration of the probabilities of success and power. First let us consider the probability of success. The probability of success depends on the number n of individuals, the number m of options, the pattern of preferences and the option selected. The following argument shows that the probability of success given a random choice of option is $1/m$ and the maximum success is $S^*/100$ if the option x^* is chosen.

(i) Suppose there is ideal consensus. If the option selected is everyone's ideal, then there is 100% success. If the option selected is not everyone's ideal, then there is 0% success. Noting that there are m options, if the option is chosen at random then the probability of success $p = 1(1/m) + 0((m-1)/m) = 1/m$.
(ii) Suppose there is ideal conflict. Suppose each option x is the ideal for just S% of the individuals. Notice that $\Sigma S\% = 100\%$. Suppose x^* is the option for which S% is a maximum, S^*. If the option selected is x, then there is S% success. If the option selected is x^*, then there is S^*% success. Noting that there are m options, if the option is chosen at random then the probability of success $p = \Sigma S\%/m = 100\%/m = 1/m$.

The preceding analysis is based on the assumption that only one of the options actually occurs. If more than one option occurs and people can select which option they obtain then success is more likely. Indeed success is certain if all m options are provided! ... or indeed, which might be fewer, if all the options which anyone has as an ideal are provided. Thus the success probability increases from S^*% to 100% as the number of options provided increases.

The probability of power is always less than or equal to the probability of success. Given a fixed number of options, m, the probability of success is as we have just seen $1/m$. So the probability of power is less than or equal to $1/m$. Moreover, the probability of power tends to zero as the number of individuals increases.

Finally, a variety of measures can be defined for a particular pattern of preferences. For each outcome, the popularity of an outcome can be defined as the percentage of the population for whom that outcome is their most preferred outcome. For each pair of outcomes A and B there are three percentages: a is the percentage preferring A to B; b is the percentage preferring B to A; and n is the percentage who are neutral between A and B. One measure of the social tension between A and B can be given by the product ab, which has a maximum when $a = b = 0.5$.

Success Versus Decisiveness: Three Voting Rules for the European Council of Ministers

Many of the ideas discussed in this chapter are well illustrated in a recent article by Laruelle et al. (2006, p. 185). The authors are concerned with the appraisal of voting rules. They note that there is a substantial literature on voting power with a variety of power indices being proposed. Moreover, the notion of being decisive or pivotal in a decision is widely focused on. Some authors have questioned this focus on power and have suggested that the notion of satisfaction or success is more relevant. Laruelle et al. seek to articulate the distinction between the concepts of success and decisiveness and to justify 'the relevance of the notion of success or satisfaction for the normative assessment of voting rules'. In their work they cite Barry's equation 'success equals decisiveness plus luck'. As their case study, Laruelle et al. study the process leading up to the adoption of a voting rule for the European Council of Ministers. They consider three voting rules that were proposed for the enlarged Council of 25 Member States.

Table 4.3. Nice Weights and Populations in 2000 for the Member States.

Country	Nice Weights	Population in 1000s
Germany	29	82165
UK	29	59623
France	29	58747
Italy	29	57680
Spain	27	39442
Poland	27	38654
The Netherlands	13	15864
Greece	12	10546
Czech Republic	12	10278
Belgium	12	10239
Hungary	12	10043
Portugal	12	9998
Sweden	10	8861
Austria	10	8092
Slovakia	7	5399
Denmark	7	5330
Finland	7	5171
Ireland	7	3775
Lithuania	7	3699
Latvia	4	2424
Slovenia	4	1988
Estonia	4	1439
Cyprus	4	755
Luxemburg	4	436
Malta	3	380

(1) The rule proposed at the Intergovernmental Conference in Nice in 2000 was that a proposal would be accepted if a simple majority of states voted for it *and* if a weighted majority of states voted for it. The simple majority requires at least 13 votes. The weighted majority requires the weighted sum of votes in favour to be at least 232. This constitutes 72% of the total weights which sum to 321. The Nice weights are given in Table 4.3.
(2) The rule proposed by the convention which was set up after the summit of Laeken in 2001 was that a proposal would be accepted if a simple majority of states voted for it *and* if a weighted majority of states voted for it. The simple majority requires at least 13 votes. The weighted majority requires the weighted sum of votes in favour to be at least 60%. The weights are proportional to population – see Table 4.3.
(3) The rule which was finally adopted in the European Council in Brussels in June 2004 was that a proposal would be accepted if either condition A held *or* condition B held. Condition A was that an overwhelming majority, 22 out of the 25 states, supported the proposal (88%). Condition B was that more than a majority of states voted for it *and* a weighted majority of states voted for it. The 'more than a majority' requirement was that the proposal should receive at least 15 votes. The weighted majority requires the weighted sum of votes in favour to be at least 65%. The weights are proportional to population – see Table 4.3.

Laruelle et al. suggest that whereas the academic literature is preoccupied with power, what the European states were concerned about was success. They might be concerned with:

(a) the probability of any given proposal being accepted
(b) the probability of a proposal which they accepted being accepted
(c) the probability of a proposal which they rejected being accepted

States which are concerned about their sovereignty are likely to want to keep (c) low. States which are concerned to deepen integration are likely to want to keep (a) and (b) high.

CHAPTER 5

THEORY, EVIDENCE AND REALITY: THE MEAN AND MEDIAN IDEALS OF COMPETING GROUPS

> We show that the median legislator in the US House is unambiguously closer to the majority party median than to the minority party median. An important implication of this finding is that the median legislator is predisposed to support the majority party's policy agenda. ... We conclude that partisan and floor majority, or median, theories of lawmaking are more often complementary than conflicting, and that party activities in the *electoral* arena have implications for *legislative* partisanship.
>
> – Wiseman and Wright (2008, p. 5)

Mathematical truth requires consistency with axioms whereas scientific truth requires correspondence between theory and reality. Theory can be more restricted than reality and reality can be more restricted than theory. Sometimes reality can be represented by a simple equation and at other times a complex structure of context-dependent equations may be required. Evidence from social science investigations often requires us to consider empirical probabilities and approximations.

The notion of value in this chapter is that objects can be located in a continuous space and that preferences for objects are single-peaked or Euclidean. There is some evidence that the population distribution of peaks (or ideals) is itself sometimes peaked. Under certain circumstances these features eliminate the possibility of voting cycles and give the median ideal or the mean ideal as the majority winner. If the outcome is the mean ideal then it can be shown that an individual's power decreases as the size of the population increases. Larger groups have greater power. In the absence of equal democratic power the outcome may be modelled as the weighted mean ideal with overall power being a combination of egalitarian power and non-egalitarian power. The presence of non-egalitarian power can be detected by looking at the social outcome in terms of the relationship between the overall mean (or median) and the means (or medians) of competing groups. Wiseman and Wright's investigation of evidence of partisan policy in the US Senate is used as a case study.

THEORY, EVIDENCE AND REALITY

Are the statements which we make true? ... and, if they are true, how do we know that they are true? To address these two questions let us first note that this book is interested in two types of truth: mathematical truth and scientific truth. Mathematical truth applies to statements within a mathematical theory. A statement is true within a theory either if it is one of the axioms of the theory or if it can be deduced from the axioms of the theory (see Chapter 3). Scientific truth applies to statements about the real world. A statement about the real world is true if it corresponds to what happens in the real world. A theory about the real world is true if all of its statements correspond to what happens in the real world. Given a mathematical theory which is consistent (i.e. true within itself) or a specific statement which is true within the theory, we can enquire whether or not the theory or statement is true in relation to reality.

For example within the theory of Euclidean geometry, Pythagoras theorem (which applies to the set of right-angled triangles) is mathematically true, because it can be deduced from the axioms of the theory. However, in reality Pythagoras theorem is demonstrably false when applied to large-scale right-angled triangles on the surface of our planet. Although the earth appears flat on a small scale, because the earth is a globe there are right-angled triangles in which all three sides are equal and all three angles are right-angled. To see this imagine the following: start at the North Pole and fly south until you reach the equator; turn at right angles and fly west along the equator for an equal distance; again turn at right angles and fly north back to the North Pole, thus completing the triangle. Not only is Pythagoras theorem false on the large-scale surface of our planet but it necessarily follows that the theory of Euclidean geometry as a whole is also false on the large-scale surface of our planet.

The history of Euclidean geometry over the past two centuries is helpfully viewed through Buhler's (1981, pp. 99–103) biography of Gauss: 'Kant in his critique of pure reason had asserted that the Euclidean concept of space was an essential component of our mental framework'. However, there had always been debate concerning the axiom of parallels (roughly, do the angles of a triangle add up to 180°?). There were two aspects to this question: 'Gauss was not interested in the philosophical question of the independence of the axiom of parallels; much more interesting was the actual geometric nature of physical space'. In the mid-nineteenth century Bolyai and Lobachevsky published their accounts of non-Euclidean geometries. In the early twentieth century Einstein's theory of special relativity postulated that physical space was non-Euclidean and this has some current empirical support (Penrose, 2004). Thus, the history started from an initial situation where a particular conceptual geometry, that is Euclidean geometry, was assumed to represent physical geometry. However, there were conceptual problems with the conceptual geometry. Conceptual work led to the development of alternative conceptual geometries. A theory of the world was produced postulating that real-world geometry was represented by one of the alternative geometries and not by the original geometry. Empirical work produced some evidence in favour of this (although the issue still remains open).

The example of Pythagoras theorem and Euclidean geometry illustrates that there can be a mismatch between theory and reality when the theory fails to envisage some aspect of reality. In a sense the theory is more restricted than reality. However, it can also happen that reality is more restricted than the theory. The theory may envisage possibilities which are not encountered in reality. The set of events in reality may be only a subset of the set of possible events envisaged by the theory. If the events in reality are more restrictive in this way then a more restrictive theory or statement may apply and be more powerful.

For example, the choice theories in this chapter are more restrictive than the choice theories in the previous chapter and the choice theories in the next chapter are more restrictive than the choice theories in the present chapter. However, the archetypal example of a restrictive theory is the second law of thermodynamics. In reality air always rushes out of a blown-up balloon, it never rushes in, even though both the former and the latter would be possible under the theory of Newtonian dynamics applied to the gas molecules. Reality is more restrictive than Newtonian theory. 'Never?' – well, hardly ever: cf. 'once the gas has found its way out of R, it is ridiculously unlikely that it will find its way back again into R (at least not within any time-scale that is not "utterly ridiculously long")' (Penrose, 2004, p. 697). In this context, reality is almost certain to be more restrictive than Newtonian theory.

For some situations a simple theory or statement suffices whereas for others a much more complex statement is required – and indeed it may be that there is no analytic equation which completely covers the situation. In order to find an analytic equation one may need to restrict attention to a limited set of situations. Even within the set of situations for which an analytic equation does exist, knowledge may consist of a complex structure of context-dependent formulae. An extended example of this is provided in the section 'The median ideals of competing groups' of this chapter.

In order to establish whether a theory corresponds to reality we need to appeal to empirical evidence about reality. This prompts three questions: how does the evidence relate to theory?; how does the evidence relate to reality? and what does the evidence then tell us about how the theory relates to reality? In all of this there are three sets of contexts. The first is the set of (theoretical) possibilities. The second is the set of contexts which occur in reality. The third is the set of contexts which occur in a body of empirical evidence about reality. The relationship between these three sets is important in that the equations appropriate for one set of contexts may not be appropriate for another set of contexts.

Evidence about reality can arise either naturally or through special investigation, the latter using a controlled experiment or using a quasi-experiment or using some less controlled approach. In this chapter, we take a special interest in a type of evidence-based knowledge which is commonly found in social science. The conclusion to investigations quite commonly takes the form: 'the results suggest that the probability is 0.95 that the real value of y is within two standard deviations of the value of y estimated from the equation $y = f(x)$'. This type of conclusion has five ingredients. There is reference to an approximating equation $y = f(x)$; the approximating equation gives an estimated value; the estimated value is an approximation to the real value; the error in the approximation is not known with certainty; and there is a specific probability that the error is within a specific range. The three subsections which follow pick up on three of the issues involved: probability, approximation and evidence.

A Priori Probability and Empirical Probability

In the previous chapter we adopted an a priori equal likelihood approach to probability. We considered the set of all possibilities, we made the a priori assumption that these were equally likely and from this we deduced the probability of certain types of events: the probability of success, the probability of being powerful, the probability of there being a voting cycle, etc.

In the present chapter we adopt an empirical approach – we observe the frequencies of different types of events in the real world and use the relative frequency as an estimate of the probability of that type of event occurring in the real world. As an example of the contrast between these two approaches consider the probability that a human baby will be a boy. One might reason that there is no particular reason to suppose that a girl is more likely than a boy and so one assumes that the a priori probability is 0.5. Alternatively one might collect data about births and find out the relative frequency of girl births, taking that number to be the probability of a girl birth. One of the disadvantages of the a priori approach is that reality might not exhibit the set of all possibilities but rather a quite restricted subset of these. For example, if reality exhibits only single-peaked preferences then the probability of voting cycles is zero. Thus, reality-based probabilities reflect the fact that reality contains restricted subsets of events.

Error and Approximation

Many social science investigations apply standard statistical models to the empirical data to obtain an equation which provides a statistical approximation to the data and hence also to reality. But what is the status of this approximating equation? How does the approximating equation relate to the true equation? It is these questions which motivate the discussion in this and the following section.

Consider a situation where a variable takes a certain value, say 2. The statement 'the value is 2' is true and any other statement of the form 'the value is x' is false. We can characterise the false statements in terms of the size of the error, the absolute error or the relative error: $(2-x)$, $|2-x|$ or $|2-x|/2$. A false statement provides an approximation with a certain amount of error. Some false statements are better than other false statements in the sense that they have lower error.

Consider a situation where a variable is specified by the equation $z = f(y)$. Any equation $z = g(y)$ is false unless $g(y) = f(y)$. We can characterise the false equations in terms of the size of the error, the absolute error or the relative error: $(f(y)-g(y))$, $|f(y)-g(y)|$ or $|f(y)-g(y)|/f(y)$. A false equation provides an approximation with a certain amount of error. Some false equations are better than other false equations in the sense that they have lower error – they are better approximations. For any true equation there may be many possible approximating equations. The form of an approximating equation may be very different from the form of the true equation – even while still giving reasonably bounded approximations.

Let us illustrate these points using a situation where we know the true equations. We return to the discussion of triangles in Chapter 3. There it was found that different equations applied exactly to different types of triangles. Suppose now that we tried to apply the wrong equation to a certain type of triangle. What would happen?

(a) Consider the set of all triangles. The true equation is $z^2 = x^2 + y^2 - 2xy\cos\theta$. Suppose we used $z^2 = x^2 + y^2$ as an estimating equation. The error is $-2xy\cos\theta$; the absolute error is $|2xy\cos\theta|$ and the relative error is $2(x/z)(y/z)|\cos\theta| < 2|\cos\theta|$. The error is quite small if θ is near $\pi/2$, that is if the triangle is almost right-angled.

(b) Consider the set of right-angled triangles. The true equation is $z^2 = x^2 + y^2$. Suppose we used $z = 1.25y$ (with y as the larger of the other two sides) as an estimating equation. Writing $x = ry$ where $0 \leq r \leq 1$, the true equation becomes $z = y\sqrt{(1 + r^2)}$. So the error is $y[\sqrt{(1 + r^2)} - 1.25]$. This ranges from $-0.25y$ to $+0.16y$, for $0 \leq r \leq 1$. We can also derive an average error if we suppose that the possible values of the errors were equally likely. In this case we would expect a positive error of $0.08y$ and a negative error of $-0.125y$ – in other words the expected relative magnitude of the error is approximately 10%, not too bad for such a crude approximation. (But do not let your builder use it!)

(c) Consider again the set of right-angled triangles. The true equation is $z^2 = x^2 + y^2$. Suppose we used $z = y + 0.5x$ as an estimating equation. This will be considered in the following subsection.

Approximating equations can be derived by a process of plausible reasoning. For example the approximating equations in (b) and (c) above can be derived in the following way. Consider the following question: in a right-angled triangle, how does the length of the longest side relate to the lengths of the other two sides? Consider a right-angled triangle. Let the lengths of the sides be $x \leq y \leq z$. The longest side z must be less than the sum of the other two sides. So we have the following two inequalities for the longest side z: $y \leq z \leq x + y$. These inequalities suggest a plausible approximation for z: if z is halfway between the two bounds then $z = y + 0.5x$, case (c) above. Furthermore $0 \leq x \leq y$. If x is halfway between the bounds then $x = 0.5y$. Combining the two we obtain $z = 1.25y$, case (b) above.

Sometimes plausible reasoning does not take us to a specific equation but rather to a class of equations. An important stage is the identification of relevant variables, of dependent and independent variables and of the direction of influence of the latter on the former. For example

(1) Other things being equal, increasing the length of the shortest side x will increase the length of the longest side z.
(2) Other things being equal, increasing the length of the middle-lengthed side y will increase the length of the longest side z.

So z is an increasing function of x and y. The simplest equation would be the linear equation: $z = ax + by + c$, with $a > 0$ and $b > 0$. Notice that our theory has nothing to say about the precise values of a, b and c. This is in contrast with the theory in case (c) which also took the form $z = ax + by + c$ but with the specification $a = 0.5$, $b = 1$ and $c = 0$. Also the theory in case (b) took the form $z = ax + by + c$ but with the specification $a = 0$, $b = 1.25$ and $c = 0$. One might say that cases (b) and (c) used stronger methods of plausible reasoning.

Evidence about Reality

How can we justify theoretical claims about reality? One approach is to appeal to empirical evidence about reality. In looking at reality we may have one, many or no specific models in mind. If we have many models in mind then we wish to see which model best matches the evidence – for example in terms of minimising the error. If we are only interested in an approximation then there may be a number of equations which are satisfactory, and the less precision we demand the greater the number of equations which are satisfactory.

The methodology for obtaining and using evidence about reality has a massive literature. Cook and Campbell (1979) is particularly penetrating about the status of evidence obtained from field settings. Here we merely wish to continue our reflections about the relationship between an approximating equation and the true equation. We seek to obtain evidence in support of the theory we developed that in a right-angled triangle, $z = ax + by + c$.

Consider the following sample of five right-angled triangles: the triangle with sides of length 3, 4 and 5 units; the triangle with sides 5, 12 and 13 units; the triangle with sides 10, 10 and 14.142 units; the triangle with sides 1, 1.732 and 2 units and the triangle with sides 2, 9 and 9.22 units.

Using this data we can now carry out a regression analysis. We take the longest side (the hypotenuse) as the dependent variable; and the shortest and middle-lengthed sides as the two independent variables. The regression gives the following equation.

$$z = -0.035 + 0.52x + 0.89y$$

$$R^2 = 99.9\%,\ p = 0.001;\ \ \text{se} = 0.23,\ 0.04,\ 0.04;\ \ p = 0.9,\ 0.007,\ 0.002$$

The equation explains 99.9% of the variance. So the theory that $z = ax + by + c$, with $a > 0$ and $b > 0$, receives strong empirical validation. The theory has no comment to make on the specific values −0.035, 0.52 and 0.89. Looking at this from another point of view, the virtue of the empirical methodology is that it obtains a precise equation even when a theoretical statement is weak, absent or non-existent.

In contrast the theory in case (c) did specify precise values. The empirical coefficients of −0.035, 0.52 and 0.89 are quite close to the theoretical coefficients of 0, 0.5 and 1. Thus, the approximating equation $z = 0.5x + y$ has a plausible rationale and receives empirical support. It is able to do this even though it is not the true equation. Also, because we know the true equation, we can investigate the relationship between the approximating equation and the true equation and this has shown the approximating equation to provide a fair approximation. Despite all of this the form of the approximating equation is not the same as the form of the true equation.

CONTINUOUS SPACES AND SINGLE-PEAKED OR EUCLIDEAN PREFERENCES

We continue the discussion of social choice started in the previous chapter but whereas there we said nothing about the nature of the set of options (and hence the statements applied to all possible sets of options), here in the present chapter we restrict our attention to sets of options which can be represented in a continuous or ordered space. There are two reasons for this restriction. The first is that we may be able to make stronger statements about the more restricted domain. The second is that some sets of options in reality are of this form.

The Median Ideal and the Mean Ideal

In the previous chapter we noted that no method of social choice was perfect. In particular, one of the main methods, the majority voting principle, had the defect that voting cycles occurred. Over the past few decades this problem has given rise to an extensive literature which explores the possibility that good social choice functions may exist in more restricted domains (Gaertner, 2002).

The most direct approach is to declare by fiat that patterns of preference which give rise to voting cycles are to be excluded from the analysis. Note that a voting cycle arises with options $\{x, y, z\}$ when each option appears once as most preferred, once in the middle and once as least preferred. So let us simply rule that these patterns of preference are inadmissible. We say that a pattern has value restriction if for every triple of options there is one option which is either never most preferred or never the middle option or never least preferred. Sen's majority decision theorem says that: if n is odd and pattern p is value restricted then majority rule is transitive at p. (Campbell & Kelly, 2002, pp. 39, 66 [hereafter referred to as 'CK']).

Not only can voting cycles be excluded by fiat, there are also certain types of well-defined situations where voting cycles do not occur. Moreover, these types of situations are thought to occur sometimes in reality. The types of situation involve special types of option sets and special types of preference patterns. Suppose there is a mapping of the set of options onto the real line such that each individual's preferences are single-peaked. In other words, an individual has an ideal, a most preferred option, which is represented on the real line; this is a single peak in the sense that the further left of the ideal an option is the less preferred it is; and similarly to the right. Looking at the set of individual ideals on the real line, there is a median ideal and an individual (referred to as the median individual) whose ideal corresponds to the median. Black's theorem states that the majority winner is the median ideal.

In general, Black's theorem does not extend to spaces of more than one dimension unless certain restrictions are made: the preference ordering needs to be Euclidean; the distribution of ideal points over the population needs to be compact; and the proportion of voters required for a majority needs to be of a certain size. The details are as follows. Consider a space of l dimensions. A preference ordering is said to be Euclidean if there is an ideal point x^* such that y is preferred to z if $d(x^*, y) < d(x^*, z)$. (I prefer the phrase 'ideal point' to 'bliss point'.) 'Caplin and Nalebuff (1991) prove that for any compact subset of X of l-dimensional Euclidean space, a lower bound of $1-[l/(l+1)]^l$ on the proportion of voters needed for a majority guarantees the existence of an undefeated alternative. As l increases, the bound increases monotonically to $1-1/e$, which is almost 64%. The undefeated alternative is the mean voter's bliss point. Ma and Weiss (1995) demonstrate that this outcome is not always invariant to transformations of the parameters of the individual utility functions, even when the transformations do not change the individual's underlying preference ordering' (CK, p. 69).

The results of the previous paragraphs introduce the notions that the voting outcome can be sometimes the median ideal and sometimes the mean ideal. The rest of the chapter will focus on that, albeit acknowledging that voting cycles are avoided only under certain conditions.

Definitions. Consider a set I of individuals, a set of options in ordered or continuous space and single-peaked individual preferences. The *individual ideal* for individual i is the option $x = x_i$ which is most preferred by i. The *mean ideal* is the mean of the individual ideals and the *median ideal* is the median of the individual ideals. The *range of ideals* is $[a, b]$ with $a = \min\{x_i\}$ and $b = \max\{x_i\}$.

Result 5.1. If the distribution of ideals is symmetric then the mean ideal equals the median ideal. If the option set is one-dimensional and the dimension is transformed into an ordinal space, the order of ideal preferences preserved and with equal distances between each individual ideal, then the mean ideal equals the median ideal.

Result 5.2. The set of Pareto optima equals $[a, b]$. There is consensus that points in $[a, b]$ are preferred to points outside $[a, b]$; and conflict regarding the preference between any pair of points within $[a, b]$.

Result 5.3. The mean and median ideals each have certain optimal properties. The mean ideal is the point which minimises the sum of squares of the distances between the ideals and any given point. The median ideal is the point which minimises the sum of squares of the ordinal distances between the ideals and any given point. These optimal properties have virtue to the extent that preferences correspond with distances. However, it may be that the mean or median ideal might provide a useful social outcome function even when the required conditions are not fully met.

The Distribution of Ideal Points: Representative Democracy as Social Choice

In the section 'The median ideal and the mean ideal' some of the results require the distribution of ideal points over the population to be compact. Schofield (2002, pp. 435, 440 and 445) reports on the positions or ideal points of voters, parties and presidents in the United States, Israel and the United Kingdom. The UK data (1979) shows that the distribution of voter's ideals is single-cone-shaped in two dimensions. The Israel data (1992, 1996) shows that the distribution of voter's ideals is ridge-shaped in two dimensions, with two peaks emerging on the ridge in 1996. In these cases then there is a degree of centralising tendency in voters' preferences rather than being randomly scattered over the option space.

Despite this centralising tendency in the voter distribution candidates are not located at the centre but at some distance from it. In the United States, Democrat presidents Kennedy, Johnson, Carter and Clinton are significantly to the left of centre and Republican presidents Eisenhower, Nixon, Reagan, Ford and Bush were significantly to the right of centre. In the United Kingdom in 1979, the Liberal party was closest to the centre with Labour and Conservative parties equidistant to the left and right respectively, Labour being more distant from the centre than the Conservatives. In Israel, there are many parties and they tend to be scattered along the ridge. Schofield discusses the coalition-building process between these variously located parties. When there are just two parties theory predicts that both parties converge on the centre. With more than two parties a variety of strategic considerations may prevent convergence to the centre.

The Weighted Mean Ideal and Power

We now continue the Chapter 4 discussion of power. How much power do individuals and groups have when the outcome is the mean ideal? The following statements apply and are amplified first informally and then in a more formal manner.

When the social outcome is determined by the mean ideal how much power might an individual be said to have? One measure of power of the individual might be the change in the outcome, the mean ideal, dependent on a change in the individual's ideal, namely the derivative dx^*/dx_i, where x^* is the mean ideal and x_i is the ideal of individual i. By this definition the power of each individual is $1/n$ where n is the number of individuals. All the individuals have the same power and the sum of powers equals 1. How much influence does an individual have relative to the rest of the individuals collectively? As in the previous chapter the greater the number of individuals the lesser the power of any one individual and indeed the power tends to zero. Although the power of the individual is $1/n$, the power of the rest of the individuals collectively is $(n-1)/n$. So it is not surprising that in large collectives of individuals, each individual can feel quite powerless. To summarise:

(1) Individuals have equal weights; and so each individual has power $1/n$.
(2) The collective power of the rest of the individuals is $(n-1)/n$.

(3) An individual's power decreases and the collective power increases as n increases.

How much power does a group have? One measure of power of the group might be the change in the outcome, the mean ideal, dependent on a change in the group's ideal, namely the derivative dx^*/dx_g, where x^* is the mean ideal and x_g the ideal of group g. By this definition the power of the group is g/n where g is the number of individuals in the group. The greater the relative size of the group the greater the power. How much influence does a group have relative to the rest of the individuals collectively? Although the power of the group is g/n, the power of the rest of the individuals collectively is $(n-g)/n$. If the population is partitioned into two groups, A and B, the overall mean ideal equals the weighted sum of the two group mean ideals, the weights being the relative sizes of the two groups. If A is in the majority then the overall mean ideal is closer to the group A's mean ideal than it is to group B's mean ideal. Although, as a group, group A has more power than group B, each individual in A has the same power as each individual in B. If the population is partitioned into r groups, the overall mean ideal equals the weighted sum of the r group mean ideals, the weights being the relative sizes of the r groups. To summarise:

(4) The greater the relative size of a group the greater its relative power.
(5) Each individual has the same individual power whichever group they belong to.
(6) The overall mean ideal equals the weighted sum of the group mean ideals, the weights being the relative group sizes.

Suppose that the outcome is determined not by the mean ideal but by a weighted mean ideal. In general, individuals have different power. This can be decomposed into an egalitarian power $1/n$ and an inegalitarian power, w. An important question is whether the outcomes reflect democratic equal power or undemocratic unequal power. It is not enough to show that the outcome favours a particular group, since that may simply reflect the fact that the given group is larger than the rest. To summarise:

(7) In the weighted mean ideal different individuals have different weights.
(8) Each individual has power $p = w/n$, where $1/n$ is the egalitarian power and w the inegalitarian power.

We now express the foregoing results more formally. Consider a set I of n individuals; a continuum of possible outcomes X; a set of individuals' preference orderings and the associated set of individuals' ideal outcomes

($\underline{x}_1$, $\underline{x}_2$, ..., $\underline{x}_n$), where $\underline{x}_i$ is the ideal outcome vector for individual *i*; a set of individuals' weights (w_1, w_2, ..., w_n), where $\sum_i w_i = 1$; and a social outcome function which specifies that the outcome $\underline{x}$ is the weighted mean of the individuals' ideals: $\underline{x} = \sum_i w_i \underline{x}_i$. We refer to this as the weighted mean social outcome function.

Result 5.4. Given a partition of *I* into subgroups, under a weighted mean social outcome function, the outcome can be expressed as the weighted mean of the group ideals: $\underline{x} = \sum_g w_g \underline{x}_g$.

The group ideal is defined as the weighted sum of the ideals of the individuals in the group, $\underline{x}_g = \sum_{i \text{ in } g} w_{ig} \underline{x}_i$, where the weights are given by $w_{ig} = w_i / w_g$, where $w_g = \sum_{i \text{ in } g} w_{ig}$.

Corollary. The result holds for an outcome space of any dimension – and in particular for a one-dimensional space.

If the weights are equal then each individual weight is $w_i = 1/n$; and each group weight is given by $w_g = p_g = n_g/n$, where n_g is the number of individuals in group *g*.

Defining the intrinsic weight w_g^* of a group so that $wg = p_g w_g^*$ we have $\underline{x} = \sum_g w_g^* p_g \underline{x}_g$.

From $\underline{x} = \sum_g w_g \underline{x}_g$ it follows that $0 = \underline{x} - \underline{x} = \sum_g w_g (\underline{x}_g - \underline{x})$.

In the case where there are just two groups we have: $w_1(\underline{x}_1 - \underline{x}) + w_2(\underline{x}_2 - \underline{x}) = 0$ and so $w_1|(\underline{x}_1 - \underline{x})| = w_2|(\underline{x}_2 - \underline{x})|$. So if $w_1 < w_2$ then $|(\underline{x}_1 - \underline{x})| > |(\underline{x}_2 - \underline{x})|$.

The more weighty group mean is closer than the less weighty group mean to the overall mean.

The majority group mean is closer than the minority group mean to the overall mean.

THE MEDIAN IDEALS OF COMPETING GROUPS

The section 'The median ideal and the mean ideal' noted that the majority winner, if there is one, is sometimes the mean ideal and sometimes the median ideal. The section 'The distribution of ideal points: representative democracy as social choice' has established a number of important results for the mean ideal, and done so in a very straightforward manner. In this section, we find that the situation is much more complicated for the median ideal. There is no one simple formula and an empirically based approximation may be the best we can do.

The Legislative Median and Partisan Policy

Wiseman and Wright (2008) [hereafter referred to as 'WW'] look at the Democrats and Republicans in the US House of Representatives. One or other party has a majority. Each person has a policy position which can be represented as a point on a one-dimensional continuum, 'left' to 'right'. In this type of situation the median person is of particular interest because the median voter theorem states that the policy outcome will be the policy preferred by the median person. An interesting question is therefore how the overall median, *m*, in the US House is related to the Democrat median *m*(D) and the Republican median *m*(R) (Fig. 5.1).

In their abstract the authors say:

> We show that the median legislator in the US House is unambiguously closer to the majority party median than to the minority party median. An important implication of this finding is that the median legislator is predisposed to support the majority party's policy agenda. Thus, in the event that the majority party organization exerts no influence over the legislative process, and in the event that all policies then default to the legislative median, policy outcomes will still substantially favour the majority party over the minority. We demonstrate that the legislative median moves predictably toward the majority party in response to changes in majority control and the size and ideological homogeneity of the two parties. Consequently, the median legislators' partisan predisposition increases and decreases in response to electoral change. We conclude that partisan and floor majority, or median, theories of lawmaking are more often complementary than conflicting, and that party activities in the *electoral* arena have implications for *legislative* partisanship. (WW, p. 5)

The article is in three main parts: 'partisan models of legislative politics' (WW, pp. 7–11); 'dynamics of the median' (WW, pp. 12–21) and 'partisanship and the median' (WW, pp. 21–25). However, our sole interest here is in the second section which discusses the 'dynamics of the median'. The key question is: what determines the legislative median?

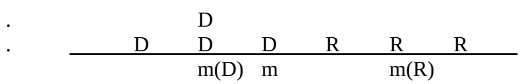

Fig. 5.1. Illustration of Overall, Democrat and Republican Medians.

In their theoretical discussion the authors proceed to offer informal mathematical arguments in support of the following observations (WW, pp. 16–17):

Observation 1. Ceteris Paribus, the legislative median will shift to the right (left) as the party control of the chamber switches from Democrat (Republican) to Republican (Democrat).

Observation 2. Ceteris Paribus, the legislative median will shift to the right (left) as the size of the Republican (Democrat) majority increases.

Observation 3. Ceteris paribus, increasing heterogeneity in the Republican Party will shift the House median left-ward, while increasing heterogeneity within the Democratic Party will shift the House median right-ward.

In all three of these 'observations', the dependent variable is the legislative median. It depends on four variables: *R*, a dummy indicating that the Republicans are in the majority; *p*, the proportion of Republicans and the Republican and Democrat heterogeneities respectively.

The authors then report on an empirical investigation designed to test these 'observations' (WW, pp. 17–21). Four models (listed below) are considered, all producing very high levels for the percentage of explained variance. In this way the evidence produced by the empirical investigation confirms the points made in the theoretical discussion.

(a) Model 4 uses just two predictors – *p*, the proportion of Republicans and *R*, a dummy indicating that the Republicans are in the majority – and explains 83% of the variance.
(b) Model 3 adds in two extra predictors – dummy variables indicating a third- and a fourth-party system respectively – and explains 89% of the variance.
(c) Model 2 adds in two extra predictors – the Republican and Democrat heterogeneities respectively – and explains 91% of the variance.
(d) Model 1 adds in two extra predictors – the interaction of the majority dummy with the Republican and Democrat heterogeneities respectively – and explains 96% of the variance.

As we discussed in the section 'Theory, evidence and reality', there is an important distinction to be made between a statement about the evidence and a theoretical statement. When Wiseman and Wright say 'we show that the median legislator in the US House is unambiguously closer to the majority party median than to the minority party median', they are referring to a statement about the evidence. We may recast this as follows: 'in the context of the empirical evidence about the history of the US House of Representatives, looking at legislators' policy positions, the overall median is closer to the majority group median than it is to the minority group median'. Removing the context, we obtain the universal theoretical statement, 'the overall median is closer to the majority group median than it is to the minority group median'.

Is the theoretical statement true? From the work of the previous section we know that the corresponding statement about means is true. However, the following counterexample shows that the statement for medians is not always true. Suppose that the Democrats have a majority of four to three. Suppose that the Democrats have policy positions 1, 2, 2 and 9 and that the Republicans have policy positions 10, 11 and 12. The Democrat median is 2, the Republican median is 11 and the overall median is 9. So the overall median is closer to the minority (Republican) median. This counterexample prompts the question: in terms of theory, what can we say about the relationship between the overall median and the component medians? This is the question which the following section seeks to answer.

How Does the Overall Median Relate to the Component Medians?

We forget about the specific political context and pose the abstract question: what is the relationship between the overall median of a population and the component medians of the two subpopulations? In what follows only the results will be presented – the proofs are given in a background paper.

Let $f_1(x)$ and $f_2(x)$ be two probability density functions. Given a proportion p such that $0 \leq p \leq 1$ then the function $f(x) = (1-p)f_1(x) + pf_2(x)$ is also a probability density function. We refer to f as the overall distribution and to f_1 and f_2 as the two component distributions. Let m, m_1 and m_2 be the medians of f, f_1 and f_2 respectively. The question we wish to address is: what is the relationship between the overall median m and the two component medians, m_1 and m_2?

Our strategy for addressing this question is as follows. Because it is simpler to do so we work much of the time with the distribution functions rather than with the density functions. The first key result is that the overall median lies between the two component medians. In other words m is in the interval $M = [m_1, m_2]$. (Without loss of generality we assume that $m_1 \leq m_2$.)

We now consider how the overall median depends on the proportion p and on the characteristics of the two component distributions. The second key result is that, for a given pair of distributions (and this is an important qualification), the overall median m is an increasing function k of the proportion p. The function maps the interval [0, 1] to $[m_1, m_2]$. In particular when $p = 0$ the overall median is identical with the lower median; and when $p = 1$ the overall median is identical with the upper median. Note that all four of Wiseman and Wright's models have a term where m is an increasing function of p.

We now consider special types of situation. For a certain class C of pairs of distributions, the function k is such that $k(0.5) = (m_1 + m_2)/2$, in other words, when the sub-populations are of equal size and hence $p = 0.5$, the overall median is the average (the mean) of the two component medians. For class C, it follows that the overall median is always closer to the median of the majority distribution. This is the claim made by Wiseman and Wright. What we are saying here is that the claim is true only to class C situations.

Furthermore there is a subclass A of C of pairs of distributions where the function is such that the overall median equals the proportion-weighted sum of the two component medians. The subclass includes pairs of uniform distributions with equal variance.

Result 5.5. Given a pair of uniform distributions with *equal* variance

$$m = m_1 + p(m_2 - m_1) = (1 - p)m_1 + pm_2$$

[Note that this formulation specifies m as a linear function of p with gradient equal to $(m_2 - m_1)$. The figures reported by Wiseman and Wright suggest a value of around 0.7 for the difference between the medians. All four of Wiseman and Wright's models have a term where m is a linear function of p with coefficients 0.76, 0.9, 0.86 and 0.68 respectively – all four numbers not too far from (the separately obtained) 0.7. So the theoretical formula *may* provide an explanation of the specific numerical values reported by Wiseman and Wright. This is an example of a theory which makes quantitative predictions of the model parameters.]

We have now reached a rather interesting point in the argument. It concerns the distinction between the median and the mean. Our interest here is with the median – yet usually statisticians work with the mean. The mean is much simpler to work with. For example it is *always* true that the overall mean is equal to the proportion-weighted sum of the two component means. In contrast, the median is much harder to work with – as we are discovering: the equation for the medians in Result 5.5 is true only for a subclass and only for a fixed pair of distributions. The theory of medians is much more complicated because there is no simple equation which covers all cases.

Instead what is needed is a taxonomy of situations and a theory of what happens in each of the situations.

Although Result 5.5 covers the case of a pair of uniform distributions with *equal* variance, to cover the case of a pair of uniform distributions with *unequal* variance a more general formula is required. Here the overall median equals the spread-and-proportion weighted sum of the components medians. By the word 'spread' here, I am referring to the standard deviation – to what Wiseman and Wright refer to as 'heterogeneity'. Note that it is in Models 1 and 2 that the heterogeneities are introduced – albeit only as additive factors or in interaction with the majority dummy. The formula in Result 5.6 has a more structured form than do the WW models.

Result 5.6. Given a pair of uniform distributions with *unequal* variance,

$$m = \frac{[(1-p)/\sigma_1]m_1 + [p/\sigma_2]m_2}{[(1-p)/\sigma_1] + [p/\sigma_2]}$$

However, an important fact about these formulae has not been mentioned. They apply only when the overall median is in the domain overlap between the two distributions. Some distributions, such as the uniform distribution, only have a finite effective domain – and, in the case of a finite sample, it is *always* the case that the effective domain is finite. So it can happen that there are regions where the two distributions do not overlap and it can happen that the overall median m lies in one such region. This is important because the type of region determines the nature of the formula. This introduces an extra complication and our systematic theory now needs to take this into account (Fig. 5.2).

Result 5.7. If the overall median occurs within the domain overlap [denoted B] then its position depends on the proportion p and on the characteristics of *both* distributions.

Result 5.8. If the overall median occurs outside the domain overlap [the left and right regions being denoted D_1 and D_2, respectively] then the

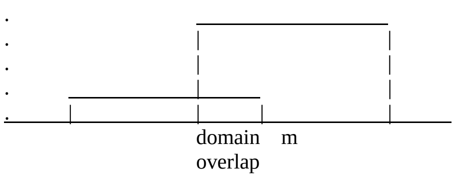

Fig. 5.2. Domain Overlap.

Table 5.1. The Theory for a Pair of Uniform Distributions, Part 1.

Given a pair of distributions and a proportion p	
If the pair of distributions are uniform	
If the region D_1, D_2, B and N exists	
If m is in D_1:	$m = \mu_1 + \sqrt{3}\sigma_1(1/p - 1)$
If m is in D_2:	$m = \mu_2 + \sqrt{3}\sigma_2(1 - 1/p)$
If m is in B:	$m = [[(1-p)m_1/\sigma_1] + [pm_2/\sigma_2]]/[[(1-p)/\sigma_1] + [p/\sigma_2]]$
If m is in N (and so with $p = 0.5$):	$m = N$

Table 5.2. The Theory for a Pair of Uniform Distributions, Part 2.

The existence of the regions:
N exists if and only if $(m_1 + a_1) < (m_2 - a_2)$
B exists if and only if N does not exist
D_1 exists if and only if $m_1 < (m_2 - a_2)$
D_2 exists if and only if $(m_1 + a_1) < (m_2 - a_2)$
$[a_i = \sigma_i\sqrt{3}]$

Table 5.3. The Theory for a Pair of Uniform Distributions, Part 3.

The location of m in each region:
If N exists then m is in D_1, N or D_2 depending on $p \leq 0.5$, $p = 0.5$ or $p \geq 0.5$
If B exists then m is in D_1, B or D_2 depending on $p \leq p_1$, $p_1 \leq p \leq p_2$ or $p \geq p_2$
where $p_1 = a_1/[(m_2 - m_1) + (a_1 - a_2)]$ and $p_2 = a_2/[(m_2 - m_1) + (a_2 - a_1)]$
$[a_i = \sigma_i\sqrt{3}]$

overall median is in the effective domain of the distribution of the majority population and its position depends *only* on the proportion p and on the characteristics of just one of the distributions, namely the majority distribution.

Result 5.9. If there is no domain overlap – in other words if there is a gap between the two distributions – and if $p = 0.5$ then the overall median might be said to occur throughout the gap [N], with one possibility being to define the median as the midpoint of the gap [N].

Note that these results may be the explanation for the presence of the majority dummy variable in Wiseman and Wright's models.

Table 5.4. The Logic of the General Situation.

What determines the overall median?

Given a pair of distributions and a proportion p

if the pair of distributions are of a certain type with parameter vectors $\underline{\mu}_1$ and $\underline{\mu}_2$;

then the existence of regions D_1, D_2, B and N depend on $\underline{\mu}_1$ and $\underline{\mu}_2$;

the regional location of m depends on $\underline{\mu}_1$ and $\underline{\mu}_2$ – and on p;

If m is in D_1:	$m = f(\underline{\mu}_1, p)$
If m is in D_2:	$m = g(\underline{\mu}_2, p)$
If m is in B:	$m = h(\underline{\mu}_1, \underline{\mu}_2, p)$
If m is in N (and so with $p = 0.5$):	$m = N$

For example the theory for a pair of uniform distributions is given in Table 5.1. Note that there are different formulae. Which formula applies depends on the situation, namely in which region the overall median m lies; and this in turn depends on whether that particular region exists and finally the whole set of results applies just in the case of uniform distributions.

The theory also needs to specify when each region exists and when the overall median lies in each region. Uniform distributions take the form $f(x) = 1/2a$ over the interval $[m-a, m+a]$ and zero elsewhere. The median equals m and the standard deviation $\sigma = a/\sqrt{3}$. The existence of the regions D_1, D_2, B and N, depend on the parameters of both distributions (Table 5.2).

Likewise the location of m within one of these regions depends on p and also on the parameters of both distributions (Table 5.3).

We are now ready to give our answer to the question: what determines the overall median? The logic of the general situation is given in Table 5.4.

CHAPTER 6

SOCIAL DESIGN, ETHICS AND THE AMOUNT OF VALUE

> Social ethics addresses the question 'what should be done in society?'
>
> – Kolm (1998, p. 3)

> Pigou thought that welfare economics was a potent instrument for the bettering of human life.
>
> – Suzumura (2002, p. 26)

> These choices along these various dimensions are indeed relevant to us and we will be seeking over the next period to find these balances.
>
> – A social design practitioner (see the case study in this chapter)

Ethics is a complex subject and here I focus on a specific ethical criterion, the utilitarian social welfare function. The ideas are relevant to other values besides welfare, and the maximisation of total welfare may under certain circumstances be associated with the minimisation of inequality. The notion of value in this chapter is that an object can have a certain amount of value for an individual. Limitations on social value are noted. There are tensions between competing options. The provision of more than one option allows some relaxation of these limitations and tensions. If the option space is continuous then the social value function can take a variety of specific forms. The notion of value-generating power is introduced. Given certain assumptions, the mean social value is a maximum at the mean ideal. Sub-optimal social value can arise as a result of the following factors: a sub-maximal value of the best option; population variation in ideals; the distance of the provided option from the best option; and sensitivity to deviation from the ideal. Practical social design requires attention to a variety of design dimensions and knowledge about people's values regarding these dimensions. This knowledge may not be known in advance and so the design process can be usefully informed by the identification of design dimensions and the obtaining of evidence about people's values regarding these dimensions. An application of these ideas to educational design is described.

ETHICS AND THE UTILITARIAN SOCIAL WELFARE FUNCTION

According to Kolm (1998, p. 3), social ethics addresses the question 'what should be done in society?' The topic of justice constitutes a very large part of social ethics although other virtues are also important. Kolm distinguishes between macro-justice and micro-justice. For the former, Kolm proposes 'a combination of the three rationales of rights and duties about capacities: process-freedom, partial income equalisation by efficient means, and the satisfaction of basic needs and the alleviation of deep suffering'. Sen (1992, pp. ix, 21–22, 150) argues that 'a common characteristic of virtually all the approaches to the ethics of social arrangements that have stood the test of time is to want equality of something – something that has an important place in the particular theory'. For example, even libertarian thinkers such as Nozick who are perceived as being anti-egalitarian place importance on people having liberty and hence that equality of liberties is important. Sen's own capability approach 'has something to offer both to the evaluation of well-being and to the assessment of freedom'.

The proposals of Kolm and Sen reveal the complexity of the literature on ethics. In contrast the focus in this chapter is on the quite simple notion of a utilitarian social welfare function.

> The utilitarian form is by far the most common and widely applied social welfare function in economics. Under a utilitarian rule, social states are ranked according to the linear sum of utilities. (Jehle & Reny, 2001, p. 255)

In its selective focus the chapter ignores a variety of issues. Thus despite its ethical importance, inequality is not be explicitly discussed. However, noting Cowell's (1995, p. 21) list of measures of inequality (range, relative mean deviation, variance, coefficient of variation, Gini coefficient and log variance) and Rawls' concern for the welfare of society's worst-off (Jehle & Reny, 2001, p. 252), we shall be interested to see under what circumstances maximising total utility also minimises these inequality criteria.

Social welfare theory has a quite specific focus on the welfare consequences of the options for individuals, 'welfarist-consequentialism' (Suzumura, 2002, pp. 23–25). Both the welfare and the consequentialist part of this can be challenged. It can be argued as noted above that options have value for individuals quite apart from the value associated with welfare consequences – for example, the values of individual liberty, social primary goods, resources, capabilities ... conferment and realisation of rights, etc. However, much of the formalism of social welfare theory can be applied equally well to any type of value and so much of the discussion of this chapter may have relevance beyond the issues of social welfare.

Next, it might be argued that certain aspects of social value are not connected to the value for individuals. While some might argue that truth and beauty were subjective (and hence related to the values of individuals) others might say that they were absolute (and hence not related to the values of individuals).

Finally, the utilitarian social welfare function assumes that objects have a certain amount of value. This is an issue which we now turn to.

THE AMOUNT OF VALUE

Although the previous two chapters have discussed value they have not conceived of value as an amount. Instead they have conceived of value either in binary terms (an object either has value or it does not) or in ordinal terms or in terms of preference. Much of the literature regards preference as the primary concept and has reservations about the concept of an amount of value. Do people in reality perceive an object as having an amount of value? People might be willing to say that they preferred A to B, but would they be willing to say that A had a certain amount of value and B had a certain amount of value? Even if people were willing to associate amounts of value with objects, how do we know that when two people say that they give the same amount of value to an object that in fact they do place the same amount of value on it? This issue is referred to as the personal inter-comparability of values.

Despite these concerns the concept of an amount of value is very attractive and perhaps not as unrealistic as suggested in the previous paragraph. In order to define such a concept, the social welfare literature introduces a number of additional assumptions to those made when discussing preferences. For example, Jorgenson (1997) offers the following discussion, leading from the weaker assumptions underlying Arrow's result to the stronger assumptions required for various classes of social welfare function. Arrow's result makes rather weak assumptions about individual preferences – he assumes only ordinal non-comparability. Arrow's result still holds if cardinal non-comparability is assumed. However, if cardinal comparability is assumed then there exists a class of social orderings which can be represented by certain types of social welfare functions: cardinal unit comparability yields utilitarian social welfare functions; and cardinal full comparability yields a class of social welfare functions which are the sum of two components, an average of the individual welfare functions plus a measure of dispersion in individual welfare levels (Jorgenson, 1997, pp. 3, 63–67). Further assumptions yield other classes of welfare functions (Jorgenson, 1997, pp. 3, 67–72).

The Individual and Social Value of Objects and Attributes

In this section I provide a formal treatment of the following ideas. The basic notion is that an individual regards an object as having a certain amount (or quantity) of value. The object has certain attributes and the value of the object for the individual depends on the value of the attributes for the individual. There is a set of individuals and the social value of the object for the set of individuals depends on the values which the individuals place on the object. A common assumption about these dependences is that they are additive. For example, the utilitarian social welfare function assumes that individual values can be added to form the social value. Sometimes interest centres on total social value and sometimes on mean social value – the two measures order states in the same way when the number of individuals is constant.

Here we confine our attention to the utilitarian social welfare function. In Eq. (6.1), x is a social state, U is the social welfare function based on the vector $\underline{\boldsymbol{u}}(x)$ of individual welfare functions $u_i(x)$ for individuals $i = 1, \ldots, N$, and the 'welfare weights' w_i are constants with $\sum_{i=1}^{N} w_i = 1,\ w_i \geq 0$.

$$U[\underline{\boldsymbol{u}}(x)] = \sum_{i=1}^{N} w_i u_i(x) \tag{6.1}$$

The set of social states may be discrete but here we make the assumption that the set of social states is a subset of a multi-dimensional real space. We say a utility function is separable if it can be expressed as the sum of functions, with each function dependent on just one dimension of the social state space. There are many situations where the utility functions are not separable. However, here we focus on the simpler case of separable utility functions.

In the additive version of multi-attribute utility theory, the utility of an option is a weighted sum of the utilities of its attributes, the weights being the sensitivities of the dimensions. This has been found to be quite a robust model (Borcherding, Schmeer, & Weber, 1995; Diederich, 1995). Suppose the attributes are labelled $j = 1, \ldots, M$. The social state $\underline{x}$ has co-ordinates $\{x_j\}$;

and w_{ij} denotes the sensitivity attached to attribute j by individual i.

$$u_i(\underline{x}) = \sum_{j=1}^{M} w_{ij} u_i(x_j) \quad \text{with} \sum_{j=1}^{M} w_{ij} = 1, \; w_{ij} \geq 0 \tag{6.2}$$

Substituting this in Eq. (6.1) we can obtain two equivalent expressions for the social welfare function, either as the weighted sum of the individual utilities $u_i(\underline{x})$ or as the weighted sum of the social welfares U_j generated by each dimension j.

$$U[\underline{\boldsymbol{u}}(\underline{x})] = \sum_{i=1}^{N} w_i u_i(\underline{x}) \tag{6.3}$$

$$U[\underline{\boldsymbol{u}}(\underline{x})] = \sum_{j=1}^{M} W_j U_j \tag{6.4}$$

where $U_j = \sum_{i=1}^{N} a_{ij} u_i(x_j)$ and $W_j = \sum_{i=1}^{N} w_i w_{ij}$, $a_{ij} = (w_i w_{ij}/W_j)$ and $\sum_{i=1}^{N} a_{ij} = 1$. This follows from $U[\underline{\boldsymbol{u}}(\underline{x})] = \sum_{i=1}^{N} w_i \sum_{j=1}^{M} w_{ij} u_i(x_j) = \sum_{j=1}^{M} \sum_{j=1}^{N} w_i w_{ij} u_i(x_j)$.

Limitations on the Value of a Social Design

Given a choice between social options, social design seeks to maximise social value but there are certain limitations on social value which need to be recognised. In particular, mean value is limited because individuals have conflicting utility functions. There may be substantial conflict between two options with the same mean value.

The value v of a social outcome lies within a nest of intervals. Firstly it must lie in the interval V_1 between the minimum possible value and the maximum possible value. Next, it must lie in the interval V_2 between the mean of individuals' minimum values and the mean of individuals' maximum values. Next it must lie in the interval V_3 between the mean of the option with the lowest value and the mean of the option with the highest value.

So reality falls short of perfection! The value can be expressed as the sum of three gaps or deficits from the maximum possible value. Let us start with perfection – the maximum possible value. The first problem is that even when an option is the one which is best for an individual, still the value may fall short of the maximum possible value. The second problem is that different individuals may have different ideal options. As a result, even the best option may have less value than the mean value of individual best values. The third problem is that a given option may not be the best option.

$$v = v_{1\,\max} + (v_{2\,\max} - v_{1\,\max}) + (v_{3\,\max} - v_{2\,\max}) + (v - v_{3\,\max})$$
$$v = v_{1\,\max} + g_{12} + g_{23} + g_3$$

Even when options have the same mean value, there may be considerable conflict between the options. Comparing any two options A and B, we can identify three groups. In one group with n_1 individuals, the group's means are $m_{A1} > m_{B1}$; in a second group with n_2 individuals, the group's means are $m_{A2} < m_{B2}$; and in a third group with n_3 individuals, the group's means are $m_{A3} = m_{B3}$. If the option selected is A then the first group experiences a total gain of $n_1(m_{A1} - m_{B1})$; the second group experiences a total loss of $n_2(m_{A2} - m_{B2})$; and the third group experiences indifference – all of these results in comparison with the value of option B. The total gain equals the total loss if and only if the two options have the same mean value. So even though two options have the same value there may be substantial losses and gains by different groups depending on the option chosen.

Finally, thinking about the different measures of inequality, we note a relationship between the criterion C of maximising the minimum value and the criterion C' of minimising the range of values. If, for every option, the maximum value is always achieved by some individual then an option which secures criterion C also secures criterion C'.

SOCIAL VALUE FUNCTIONS ON A CONTINUOUS SPACE

Now the ideas of the previous chapter are introduced, namely that the set of options may be a continuous or ordered space. With this assumption we now consider in greater detail the form of the social welfare function. From Eq. (6.1), the form of the social welfare function depends on the form of the individual utility functions. Because the utility functions are separable, it is sufficient for some purposes to consider just one dimension. First, note that there are a number of results which apply to all types of utility functions. Secondly some utility values may not be systematically related to the ordering of the dimension. For functions that have a systematic relationship, following Gottfried and Weisman (1973, pp. 1–21), the functions can be classified according to whether they are unimodal (single-peaked) or multi-modal (multi-peaked); symmetric about the peak(s) or not; convex or concave or a mixture of these two types; continuous or not (in the continuous case, a convex function has a decreasing derivative); their degree; one-variable or multi-variable; and simple or compound (i.e., a function of variables which are themselves functions of variables).

Single-Peaked Functions

If the social welfare function is multi-peaked then local improvements may not move the situation towards the global optimum. The situation is simpler if the social welfare function is single-peaked. For this to happen it is sufficient that the individual utility functions are single-peaked with decreasing derivatives – see Theorem 6.1. First though we mention a few preliminary results.

Consider just one dimension, x. Suppose that, for individual i, the utility on this dimension is a single-peaked function $u_i(x)$, attaining a maximum when $x = x_i$. We refer to x_i as the individual ideal (for that individual). So utility decreases with distance either side of the individual ideal. Consider now a set I of individuals. Let $a = \min\{x_i\}$, $b = \max\{x_i\}$ and $d = b - a$.

Result 6.1. Suppose the individual utility functions are single-peaked and identical apart from a lateral shift. Then the criterion of maximising the minimum welfare is achieved at a point x such that the utility functions with peaks at a and b are equal, $u_a(x) = u_b(x)$, and x is in the closed interval $[a, b]$.

Theorem 6.1. *Suppose the individual utility functions are single-peaked. If the individual utility functions are also differentiable with a derived function which is decreasing, then the social welfare function is also single-peaked. Where the maximum social welfare occurs is referred to as the welfare ideal.*

Proof. By assumption, the $u_i(x)$ are differentiable and single-peaked and the $u_i'(x)$ are decreasing functions of x. Using $U[\underline{\boldsymbol{u}}(x)] = \sum_{i=1}^{N} w_i u_i(x)$, we differentiate with respect to x to obtain: $U' = \sum_{i=1}^{N} w_i u_i'(x)$. So U' is a decreasing function of x. For $x < a$ all the $u_i(x)$ are positive and for $x > b$ all the $u_i(x)$ are negative. So $U'(a - \varepsilon) > 0 > U'(b + \varepsilon)$ for all $\varepsilon > 0$. So there exists x^* such that $U'(x^*) = 0$ where $a \leq x^* \leq b$.

Distance Functions (Symmetric about the Peak)

Single-peaked functions can be further classified into those which are symmetrical about the peak and those which are not. Symmetry is equivalent to value being a decreasing function of distance from the peak.

Result 6.2. Suppose the individual utility functions are single-peaked and symmetric and identical except for a lateral shift. Then the criterion C of maximising the minimum welfare is achieved at a point x such that $u_a(x) = u_b(x)$ and $x = (a+b)/2$.

Result 6.3. If it is possible to supply more that one design and for individuals to choose their best design then the designs which optimise criterion C of maximising the minimum welfare are placed at:

$(a+d/4)$ and $(a+3d/4)$ where two designs are provided;
$(a+d/6)$, $(a+3d/6)$, and $(a+5d/6)$ where three designs are provided; and so on.

The previous chapter obtained results relating power to the mean ideal and the median ideal. Attention was paid to the distances between the ideal and other points in the option space. An individual, or group, was thought to be better placed if the outcome was nearer the individual's ideal. In particular, the case where the majority party was closer than the minority party to the outcome was discussed. These results have an additional interpretation if the value functions are identical distance functions, namely that being closer to one's ideal means experiencing greater welfare. For example, one might define value-generating power as $\mathrm{d}u_i/\mathrm{d}x_i$ as the change in value for i dependent on the change in the ideal of i. Somewhat schematically we have $\mathrm{d}u_i/\mathrm{d}x_i = (\mathrm{d}u_i/\mathrm{d}x^*)(\mathrm{d}x^*/\mathrm{d}x_i) = k_i m_i w_i$, the product of a constant, the individual's value sensitivity to change in the option dimension and the power as defined in the previous section.

The two simplest types of symmetrical single-peaked function are the modulus function and the quadratic function and both types are commonly found as assumptions in the literature.

The Modulus Function

The modulus utility function specifies that the utility decreases linearly with distance from the ideal. In Eq. (6.5a), the parameter c_i represents the utility ceiling and the parameter m_i represents the sensitivity of utility to changes in the social state.

$$u_i(x) = -m_i(|x - x_i|) + c_i, \quad m_i \geq 0 \tag{6.5a}$$

The Quadratic Function

It is possible to construct a quadratic Taylor approximation to the utility function near the ideal. Here we simply assume that the utility function *is* quadratic. In Eq. (6.5b), the parameter c_i represents the utility ceiling and the parameter m_i represents the sensitivity of utility to changes in the social state.

$$u_i(x) = -m_i(x - x_i)^2 + c_i, \quad m_i \geq 0 \tag{6.5b}$$

In this situation, the utilitarian social welfare function in the form of Eq. (6.1) can be re-expressed in terms of the ideals of the individuals in the population.

Theorem 6.2. *Suppose that the set of social states is one-dimensional and that each individual has a quadratic utility function – as in Eqs. (6.5a) and (6.5b). Suppose also that all individuals have the same sensitivity. Then the welfare ideal x^* is the weighted sum of the individual ideals. The social welfare of a situation depends on the population sensitivity m, the population-weighted variation V, the deviation D of the situation from the welfare ideal and the welfare ceiling C.*

$$x^* = \sum_{i=1}^{N} w_i x_i \tag{6.6}$$

$$U[\underline{\boldsymbol{u}}(x)] = -m(V + D^2) + C \tag{6.7}$$

$$V = V(\{x_i\}) = \sum_{i=1}^{N} [w_i(x_i - x^*)^2]$$

$$D = (x - x^*)$$

$$C = \sum_{i=1}^{N} w_i c_i$$

Corollary of Theorem 6.2. *For a given population, social welfare U is maximised and is at the level, $-mV + C$, when the social state is identical with the welfare ideal, $x = x^*$. If, in addition, all the population share the* *same ideal, $x_i = x^*$, and so $V = 0$, then the social welfare is simply C. Taking the derivative of U with respect to x indicates the impact on social welfare of a unit change in design: $dU/dx = -2mD$.*

Proof.

$$U[\underline{\boldsymbol{u}}(x)] = \sum_{i=1}^{N} w_i u_i(x) = \sum_{i=1}^{N} w_i[-m(x-x_i)^2 + c_i]$$
$$= -m\sum_{i=1}^{N} w_i[(x-x_i)^2] + \sum_{i=1}^{N} w_i c_i$$
$$= -m\sum_{i=1}^{N} w_i[(x-x_i)^2] + C \tag{6.8}$$

Differentiating with respect to x and setting to zero gives $x - \sum_{i=1}^{N} w_i x_i = 0$. So the maximum social welfare is at the welfare ideal, $x^* = \sum_{i=1}^{N} w_i x_i$.

We now consider the first of the two terms on the right of the Eq. (6.8).

Note that $(x-x_i)^2 = (x - x^* + x^* - x_i)^2 = (x-x^*)^2 + (x^* - x_i)^2 + 2(x-x^*)(x^*-x_i)$

Taking each of these three parts in turn we have:

$$\sum_{i=1}^{N}[w_i(x-x^*)^2] = (x-x^*)^2$$
$$\sum_{i=1}^{N}[w_i(x^*-x_i)^2] = V(\{x_i\}), \quad \text{the population-weighted variation}$$
$$\sum_{i=1}^{N}[w_i(x-x^*)(x^*-x_i)] = (x-x^*)\sum_{i=1}^{N} w_i(x^*-x_i) = 0$$

Theorem 6.3. *Suppose that each individual has a quadratic utility function with individual sensitivity m_i. Then the welfare ideal x^* is the weighted sum of the individual ideals, the weights being a combination of the welfare weights and the sensitivities. Eq.* (6.7) *still applies but with m, V and D redefined as follows:*

$$x^* = \sum_{i=1}^{N} \omega_i x_i \qquad \textit{the weights are now } \omega_i$$
$$\omega_i = \frac{m_i w_i}{\sum_{i=1}^{N} m_i w_i} \qquad \textit{where } \sum_{i=1}^{N} \omega_i = 1$$

$$m = \sum_{i=1}^{N} m_i w_i \qquad \textit{the weighted mean sensitivity}$$

$$V(\{x_i\}) = \sum_{i=1}^{N} [\omega_i (x_i - x^*)^2] \qquad \textit{the weights are now } \omega_i$$

$$D = (x - x^*) \qquad \textit{with } x^* = \sum_{i=1}^{N} \omega_i x_i$$

$$C = \sum_{i=1}^{N} \omega_i c_i \qquad \textit{as before}$$

The proof follows the pattern of that for Theorem 6.2. Note that now the welfare ideal depends on the individual sensitivities so that the individual ideals of more sensitive individuals receive greater weight.

The multi-dimensional individuality utility function, with separable dimensions, analogous to Eqs. (6.5a) and (6.5b), is:

$$u_i(x) = \sum_{j=1}^{M} W_j(-m_{ij}(x_j - x_{ij})^2 + c_{ij}) \quad m_i \geq 0 \tag{6.9}$$

Theorem 6.4. *If the individual utility functions have the form given in Eq. (6.8), then the social welfare of a situation is the weighted sum of the one-dimensional social welfares given in Theorem 6.2. The welfare ceiling is $C = \sum_{j=1}^{M} W_j C_j$. The multi-dimensional welfare ideal is the vector of the one-dimensional welfare ideals.*

$$\begin{aligned} U[\underline{\boldsymbol{u}}(\underline{x})] &= \sum_{j=1}^{M} W_j U_j \\ &= \sum_{j=1}^{M} W_j[-m_j\{V_j + D_j^2\} + C_j] \\ &= \sum_{j=1}^{M} W_j[-m_j\{V_j + D_j^2\}] + \sum_{j=1}^{M} W_j C_j \end{aligned} \tag{6.10}$$

Finally, it is worth noting that the specific results in this section are in accordance with the earlier remarks on the limitations to value. Here $v = (-m(V + D^2) + C)$, $v_{2\max} = C$ and $v_{3\max} = (-mV + C)$; and the gaps due to sub-maximal ceilings, variation in ideals and sub-optimal design are: $g_{12} = (C - v_{1\max})$, $g_{23} = -mV$ and $g_3 = -mD^2$.

$$v = v_{1\max} + (C - v_{1\max}) + ((-mV + C) - C) + ((-m(V + D^2) + C) - (-mV + C))$$

A PRACTICAL APPLICATION

The theory has discussed how the best design depends on individuals' values. In practice, individuals' values may not be known and so a special investigation is needed to find out this information.

Ten Dimensions of Educational Design

Quite recently I have started asking my colleagues at the Open University about the dimensions of educational design which are of particular concern to them. What has been interesting has been my colleagues' readiness to identify specific dimensions, and also the strength of their concern about these dimensions. They often seem to be saying: 'this is something I really want to sort out ... this is an important choice that we need to get right'. Sometimes it is a choice between what they have been doing in the past and what they would like to do in the future. Sometimes it is a choice between what *they* want to do and what *their colleagues* want to do.

Firstly there is quite a lot of concern about student support. How much student support should there be? Should our contact with students be mainly or entirely by electronic means – or should we maintain our tradition of face-to-face contact? Should we be proactive in our contact with students – or should we wait for them to come to us? Given our limited resources, should these be devoted to the provision of tutorials or to the provision of feedback on assignments?

Assignment policy itself is of course an extremely important aspect. What should the weighting be between tutor-marked and computer-marked assignments? Should we provide formative assignments as well as summative?

Student workload too is a critical area. Not least because we are anxious about our retention rates and are afraid that if we overload the courses the students will drop out.

Finally, the curriculum and teaching. Should we just give the students the content and let them get on with it – or should we provide some teaching? ... and, if the latter, how much teaching should we give? Should we restrict the students to the topics we have selected – or should we

Table 6.1. Dimensions of Educational Design.

Curriculum and teaching	
	Curriculum balance
	Balance between content and teaching
	Freedom
Workload	
	Weekly study time
Assignments	
	Assignment policy
	The weighting of the computer-marked assignment
	The percentage of assignments being formative
	The balance between tutor contact and assessment feedback
Tutorial support	
	The amount of tutorial support
	The balance between face-to-face and other forms of tutorial support
	The number of proactive tutor contacts

allow them some freedom to choose topics to suit their own interests. In particular, should we restrict the content to pure academic knowledge – or should we include content which relates to the student's current or future workplace?

These then are the dimensions which I have been looking at (Table 6.1).

One Dimension

To illustrate my methodology I now look at the results for just one dimension, namely the provision of formative assignments. Formative assignments are those which the students are invited to do, but which are not compulsory and do not count towards the students' final grade. I designed a survey which contained the question shown in Table 6.2. A sample of 200 students was sent the questionnaire and 65 students responded (Table 6.2).

The Results

My discussion of the results is in three parts. First I look at the response of the average student. Then I consider the differences which exist between students. Finally, I look at the implications for educational design.

Table 6.2. The Question about Formative Assignments.

Imagine that you have the option to include in your course a proportion of formative assignments. Six options are listed below, ranging from 'no formative assignments' to '50% formative assignments'. Imagine your reactions to each of these options.
Please say on a scale of 0–10 how satisfied you think you would be to receive each of these options:
None of the assignments is formative
Around 10% of the assignments are formative
Around 20% of the assignments are formative
Around 30% of the assignments are formative
Around 40% of the assignments are formative
Around 50% of the assignments are formative

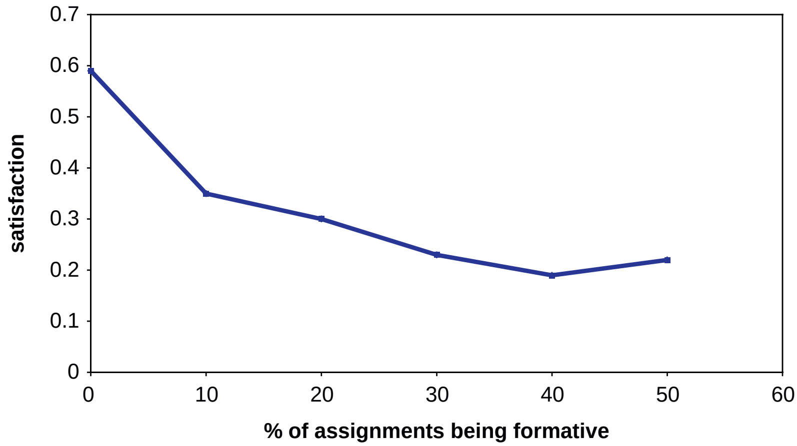

Fig. 6.1. The Mean Function.

The Average Student: What does the Mean Function Look Like?

We start by looking at the mean function, in other words the means of the students' responses for each option. This is given in Fig. 6.1. The horizontal scale gives the percentage of assignments being formative. The vertical scale gives the mean satisfaction and runs from zero to one. Zero is no satisfaction and one is extremely high satisfaction. If there are no formative assignments then satisfaction is 0.59 and this is the peak option. A quite sharp fall in satisfaction occurs if even just 10% of the assignments are formative. As the percentage of formative assignments rises the satisfaction continues to fall. The moderate level of peak satisfaction, 0.59, arises because students have different preferences. To see this we now look at the responses for three different students.

Differences Between Students

Three Different Students

One student prefers an option on the left; another student prefers a middle option; and a third student prefers an option on the right (Fig. 6.2). The consequence of this is that the mean function never reaches the peak satisfaction attained by any of the individual students.

Of course, that is just three students and three peaks – a left peak, a middle peak and a right peak. Where are the peaks – the ideal points – for the other students?

Students have Different Ideals

Fig. 6.3 presents the distribution of ideal points. For a majority of students, just over 50%, the ideal is to have no formative assignments. At the other extreme, almost a quarter of students would like half of their assignments

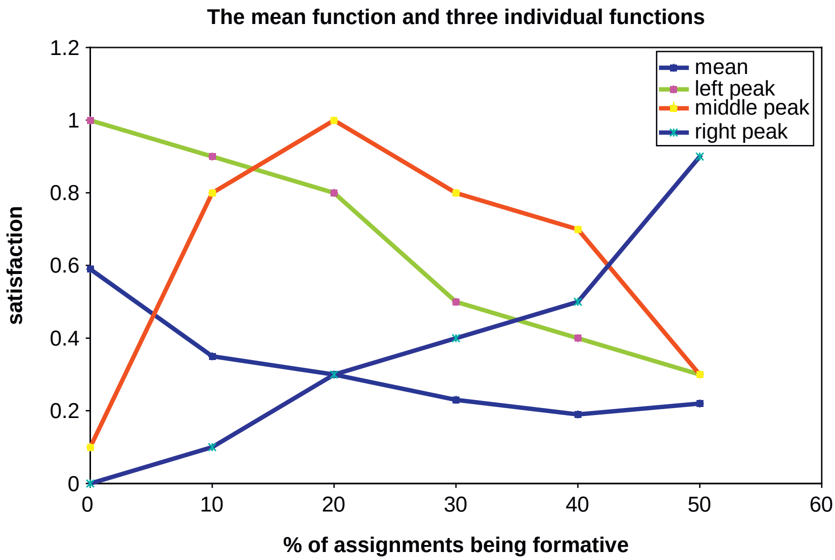

Fig. 6.2. The Individual Functions for Three Students.

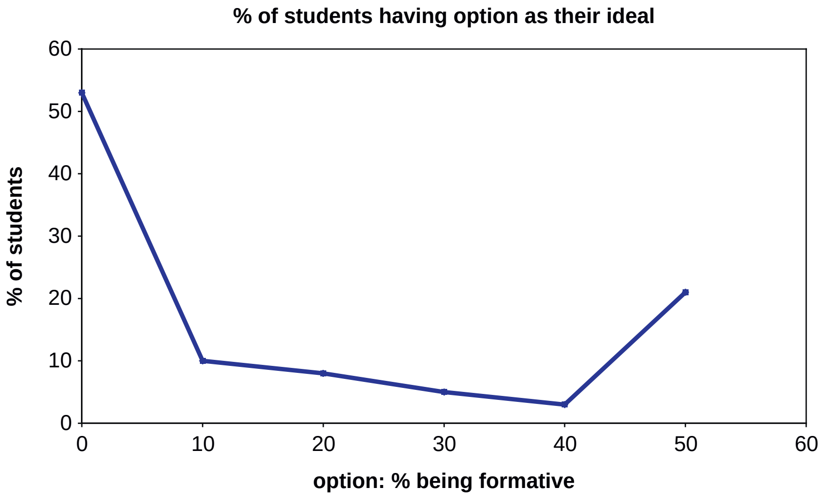

Fig. 6.3. The Distribution of Ideal Points.

```
Strongly single-peaked          Weakly single-peaked
   x                                      xxx
 x  x                                   x     x
x      x                              x          x

Separate multiple peaks          Point function
   x     x                                  x
 x  x  x  x
x      x      x                        xxx  xxxxx
```

Fig. 6.4. Types of Individual Utility Functions.

to be formative. Only a minority of students want any particular option in between the two extremes. These then are the ideal points of the individual students. However, knowing only the ideal points tells us rather little about the shape of the individual student's responses. We refer to a student's responses as the utility function of that student.

What do the Individual Utility Functions Look Like?

So, what do the individual utility functions look like? There are a number of possibilities (Fig. 6.4). They might be strongly single-peaked or weakly single-peaked or they might have separate multiple peaks. A special case of strongly single-peaked is the point function.

It turns out that almost all (98%) of the students' utility functions are 'weakly single-peaked' – in other words once the function has started to fall it never rises. Furthermore, 86% of the utility functions are strongly single-peaked (with the peak occurring at just one option) and 12% are flat-peaked (with the peak occurring at more than one option).

About a third of the students had point functions (i.e., a constant baseline and a single maximum point) ... 22% had point functions consisting of a constant baseline of zero and a single maximum point of one. In a further indication of the sharpness of some peaks, 46% of the students used none of the first four scale positions below their peak rating.

The results we have obtained so far have certain implications. In particular, they impose limitations on the mean value of any option. As educational designers, in an ideal world we would want to provide an option with which everybody would be extremely satisfied.

Implications for Educational Design

Limitations on the Mean Value

However, reality falls short of perfection! There are three distinct components to this. Let us start with perfection – a score of 1.0. The first problem is that even when an individual student is given their best option, still their satisfaction may fall short of 1.0. In fact the mean of individuals' maximum satisfaction is 0.83. The second problem is one which we have already noted. Different individuals have different ideal points. As a result, even the best option has a mean satisfaction of only 0.59. The third problem is that a given option may not be the best option. If individuals are very unlucky they get the worst option and a mean satisfaction of only 0.19!

The Social Tension in the Best Option

So, perfection is unattainable. The best we can do is to go for the best option. In other words, we provide that option which elicits the highest mean satisfaction. Of course, we cannot please all of the people all of the time. Some individuals will prefer an option other than the one we have chosen.

So there is social tension even in the best option. We can measure this social tension in a variety of ways. Let us consider the pair of options *zero formative assignment* (option A) and *10% of the assignments being formative* (option B).

One indicator of social tension is the proportion of people favouring either of the two options. Here 53% favour option A and 32% favour option B with 16% being neutral.

Another indicator of tension is the amount of satisfaction which people gain or lose depending on which option is selected. Those favouring option A stand to lose 0.63 if option B is chosen. Those favouring option B stand to lose 0.27 if option A is chosen.

An alternative indicator weights these results according to how many people are involved. *Overall mean satisfaction* is reduced by 0.33 when those favouring option A are provided with option B. *Overall mean satisfaction* is reduced by 0.09 when those favouring option B are forced to take option A. The gap between the satisfactions with these two options is 0.24.

How Else Might We Define 'Best'?

We have rather easily slipped into referring to the option with the highest mean satisfaction as the 'best' option. However, there are a variety of other criteria which can be used for evaluating options and for choosing between them (Cowell, 1995).

First consider the voting procedures. With all options on offer at the same time, the zero option would take the most votes. In pairwise voting, the zero option would defeat all others, in other words it is the Condorcet winner. The median voter favours the zero option.

The zero option is a Pareto optimum. However, every other option is also a Pareto optimum – illustrating the well-known weakness of the Pareto optimum as a selection criterion.

Turning to measures of inequality, we find that the zero option is no longer always the most preferred. The 30% and 40% options are 'best' in terms of minimising range and standard deviation – and minimax. However, the zero option is 'best' in terms of the coefficient of variation and maximin.

To Increase Satisfaction, Increase Choice

So, not only does the best option have only a modest mean satisfaction but also it is characterised by a fair amount of social tension and a high degree of inequality according to some measures. Is there not some way of doing better?

Well, yes there is – just as long as we are allowed to change our definition of the situation. Suppose that we are able to provide all the options. In that case, individuals can choose their best option, and the mean satisfaction will

Table 6.3. Mean Value Depends on the Number of Options Offered.

One option	0.59
Two options	0.70
Three options	0.80
Four options	0.80
Five options	0.83
All six options	0.83

then be increased to 0.83, the mean of the individuals' maximum satisfaction (Table 6.3). However, it may be that we cannot afford to provide all the options. How much can we boost satisfaction by providing just a few options? What Table 6.3 indicates is that offering three options provides almost as much mean satisfaction as offering all six options.

CHAPTER 7

CHANGE, MULTIPLE-ENTITY SYSTEMS AND COMPLEXITY

> What sorts of laws shape the universe with all its contents? The answer provided by practically all successful physical theories, from the time of Galileo onwards, would be given in the form of a dynamics – that is, a specification of how a physical system will develop with time, given the physical state of the system at one particular time. ... How its state might develop from moment to moment, in accordance with some dynamical law.
>
> – Penrose (2004, p. 686)

> ... in practice one has little knowledge of the behaviour of the individual ingredients of a system ... An important issue, therefore, is whether or not a good initial knowledge of such averaged 'overall' parameters will, in practice, suffice for determining the dynamical behaviour of the system to an adequate degree.
>
> – Penrose (2004, p. 686)

> What is complexity? A great many quantities have been proposed as measures of something like complexity. In fact, a variety of different measures would be required to capture all our intuitive ideas about what is meant by complexity and by its opposite, simplicity.
>
> – Gell-Mann (1995)

The concept of change is of fundamental importance. To understand change is to understand why things are the way they are, where things came from and where things are heading. A system may consist of a single uni-dimensional entity or a multi-dimensional or multiple-entity system. In the latter case the linkage between micro and macro attributes and indeed the composition of attributes is of interest.

It is helpful (and possibly misleading!) to make a distinction between information systems and behavioural systems, the former looking towards computing theory and mathematical logic and the latter looking towards dynamic systems and statistical dynamics; and the former having various notions of computational complexity and the latter concerned with the complexity of trajectories. The generation of language and mathematical objects by information systems is discussed. Models of systems may be classified according to whether or not they possess or emphasise the following properties: discrete/continuous space, one-dimensional/multi-dimensional, static/dynamic, discrete/continuous time, deterministic/probabilistic, linear/non-linear, single entity/multiple entities, single attribute/multiple attributes, homogeneity/heterogeneity, interactive/non-interactive, based on choice/influence, and individual/structural. The discussion here is selective, focusing on: the trajectories of a single entity in one-dimensional space; the trajectory of multiple interacting entities; multiple entities with non-identical probabilities; and the most probable trajectory of a macro parameter.

SINGLE-ENTITY, MULTIPLE-ENTITY AND COMPOSITE SYSTEMS

I am interested in a set X of entities and refer to this as a multiple-entity system unless the set contains just a single entity. Each entity can be characterised either by a single attribute or by many attributes. In general, then we have a system of n entities with m attributes, giving nm attributes in all. A model of a system usually focuses on the variables associated with the attributes. So a model for a unitary entity with nm attributes, a model for a system of nm entities each with just one attribute and a model for a system of n entities with m attributes may be all formally identical with one another.

Apart from attributes relating to the entities there may also be attributes relating to the composite, the system as a whole. These might be referred to as the micro attributes and the macro attributes, respectively. A core issue is the micro–macro linkage: how the micro attributes relate to the macro attributes.

Sometimes the macro attributes are the same as the micro attributes. The simplest and also most important examples of macro attributes are the total (or aggregate) and the mean (or some other characteristic of the distribution of individual attribute values).

Sometimes the macro attributes are different in kind from the micro attributes – but nevertheless related. For example, the pressure, volume and temperature at a macro level relate to the mass and velocity of gas molecules in a closed space at a micro level.

To illustrate, consider a gas molecule of mass m and velocity v in a cube of sides of length x, bouncing perpendicularly first against one face and then against the opposite face. The time between bounces is x/v. The change of momentum when the particle bounces is $2mv$. The pressure of the gas is equal to the change of momentum per unit time per unit area of the cube, and is $P = [(2mv)/(6x^2)][v/x] = [(mv^2)/(3x^3)]$. The volume of the gas in the cube is $V = x^3$. The total kinetic energy of the system is $E = mv^2/2$. So $PV = mv^2/3 = 2/3E$.

Of course there are more complex and more realistic models. The Krönig–Clausius model (1856/1857) has six equally numerous uniform beams of molecules moving in each of the six coordinate directions. The Maxwell model (1860) has the velocity components of the molecules in each of the three coordinate directions being independently, symmetrically and identically distributed and depending only on the magnitude of the velocity. In a second model he also considered an alternative to independence, by taking into account collisions between molecules. Boltzmann's model (1872) made the assumption of molecular chaos, which concerned the joint distribution of velocities of a pair of molecules. Despite their vastly increased complexity all these models derive the same relationship, $PV = 2/3E$. Experimentally, temperature is defined as $PV = NkT$, where N is the number of molecules, k is Boltzmann's constant and T is the absolute temperature. (Thompson, 1972, pp. 2–4, 4–6, 6–16).

Corresponding to the set X of entities, there is the power set $A = S^X$ of subsets of X. Just as the entities have micro attributes and the set X has macro attributes, so the subsets have subset attributes. Suppose an attribute of the entities can be characterised in terms of some measure m. Then, following Chapter 4, the quadruple $(X, A, R+; m)$ is a measure space, where m is a measure – a function from A to $R+$, the non-negative real numbers. The measure m maps the null set to zero; and is such that the measure of a countable collection of pairwise disjoint sets is equal to the sum of the measures of the individual sets. One might introduce the notion of a multi-dimensional measure space $(X, A, R+; m)$, where m is a measure vector corresponding to measured attributes of the entities. An application of this notion is by Faden (1977), who applies measure theory widely across social science.

INFORMATION SYSTEMS

Information systems have in a sense already been discussed in Chapter 3 on mathematics, logic, artificial intelligence and ordinary language. The Chapter 4 discussion of possibilities and probabilities is also relevant in that measures of uncertainty are related to measures of information. For example, Gell-Mann (1995) considers 'the algorithmic information content (or AIC) of a string of bits defined as the length of the shortest program that will cause a standard universal computer to print out the string of bits and then halt' but considers it unsuitable as a measure of complexity. Computational complexity refers to the time or number of steps or amount of space required for a specific computation. However, it is not the aim of this section to say anything more about computational complexity – the aim is the much less ambitious one of simply introducing some of the basic ideas relating to the abstract foundations for computing and artificial intelligence as discussed in Chapter 3.

The basic concern here is with a processor carrying out a process dictated by a procedure such that a specific action is carried out on a specific object with a specific result. The section starts with the notion of a process as a simple sequence of elements. The distinction is made between a specific sequence, the set of all possible sequences and a focal subset of sequences. This leads into a discussion of machine theory. Roughly speaking, the machines carry out just one step at a time but are designed in such a way that the accumulation of single steps meets the requirements. The following section revisits the processes discussed in mathematical logic: the generation of true statements, the generation of references to mathematical objects and the generation of the mathematical objects themselves.

A Process as a Simple Sequence

A discrete process can be characterised by a sequence of elements, for example (a, b, b). The length of the sequence is the number of elements in it, here three. The set for the sequence is the set of elements which the sequence contains, here $S = \{a, b\}$.

We may be interested in a specific sequence such as (a, b, b). We may be interested in the set of all sequences of a specific length (say three) with elements drawn from a specific set (say S):

$$(a, a, a); (a, a, b); (a, b, a); (a, b, b); (b, a, a); (b, a, b); (b, b, a); (b, b, b)$$

We may be interested not in all sequences but in a subset of sequences which possess some property. We shall refer to this as the focal subset. For example, we may be interested in sequences which involve only alternations of elements:

$$(a, b, a); (b, a, b)$$

Finally, we may be interested in the procedure for generating the focal subset:

1. Start with any element.
2. Take the sequence constructed so far and add an element different from the most recent element.
3. Repeat step 2 unless the sequence is of length three (in which case stop).

Symbol Processing and Symbol Processors

The concept of a focal subset is important. For example, we are not interested in the set of all strings of letters, only in the subset of strings which make up words. We are not interested in the set of all strings of words, only in the subset of strings which make up grammatical sentences. We are not interested in the set of sentences, only in the subset of sentences which are true.

The generation of and identification of members of the focal set is thus important. In this section, I seek to formalise this notion of generation by conceiving of a processor carrying out processing. In the literatures that I have examined, (special classes of) processes are sometimes referred to as algorithms, and (special classes of) processors are referred to as machines or automata.

Finite automata are special kinds of algorithms for deciding the membership of languages. A language L over an alphabet A is defined as a subset of the set of all strings of symbols, based on some set of symbols – the alphabet. In general, a language processor involves transitions between states and the transformation of input strings into output strings. A language is a subset of a construction with the set of alphabetical symbols as base set, the operation of concatenation and the set of all strings (Lawson, 2004, pp. 21–22; Cori & Lascar, 2000, p. xviii).

Generative Processes

We now revisit the processes discussed in mathematical logic: the generation of true statements; the generation of references to mathematical objects; the generation of an abstract language and the generation of the mathematical objects themselves. In general, certain structures are generated by a sequence of operations on a set of initial generators. The machines discussed in the previous section identify members of the focal set. Here we consider in abstract the generation of sets.

Sometimes a set A can be generated by the repeated application of a set of operations O, firstly to an initial generator set $G = G_0$ and then to a new generator set G_1 and so on until we have $G_n = A$. Each $G_{i+1} = G_i \cup O(G_i)$, where $O(G_i)$ is the set of elements generated by applying O to G_i. The operators O are defined on A. Also, A is closed under each of the operators in O. We might refer to this as a generation and represent it by the triple (G, O, A).

Example 7.1. Take G as any set and O as the union of two sets and A as the power set of G.

Example 7.2. Take G to be the set of generators of some group A and O to be the group operation.

Generalising the concept one might have a sequence of generations giving rise to a sequence of sets $A_0, A_1, \ldots, A_n$, with each set serving as the generator for the next set by the application of a sequence of sets of operations $O_0, O_1, \ldots, O_{n-1}$.

Stone (1973) discusses group graphs based on generator relations (p. 64), semi-group graphs based on generator relations (p. 191), state graphs for finite-state machines (p. 208) and Boolean algebra as a lattice (p. 364). In general we can have the graph for a generative process.

A construction produced by a generative process can be defined 'from above' – 'the smallest subset of a fixed set A that includes a given subset and is closed under certain operations defined on A'; or 'from below' – generating the set A one level at a time, starting with the given subset, applying the operators to this subset to obtain the next level, and so on recursively (Cori & Lascar, 2000, p. 11). Using the latter approach, it is clear that each element of A corresponds to some sort of sequence – and corresponds to a word in some language and also corresponds to some process on some finite state machine.

BEHAVIOURAL SYSTEMS

The Trajectories of a Single Entity in One-Dimensional Space

> The differential equation $\mathrm{d}x/\mathrm{d}t = ax$ is the simplest differential equation. It is also one of the most important. ... some of the most basic ideas of differential equations are seen in simple form. ...
>
> – Hirsch and Smale (1974, p. 1)

Consider a space X and time T. A trajectory through space X over time T is a function f from T to X, $x = f(t)$. Consider the set of all such functions and the subset of differentiable functions. For each differentiable function f there is a corresponding derived function f'. For each derived function there may be many functions which correspond to it. An initial condition specifies one point on the trajectory $x_0 = f(t_0)$. Given a derived function f' and an initial condition (t_0, x_0), there is a unique function f which corresponds to the derived function and satisfies the initial condition. The pair f' and (t_0, x_0) specify an initial value problem and f its solution.

Consider now a family of initial value problems, where both the derived function and initial condition are specified in terms of parameters which can vary. If the parameters change then the problem changes and so does the solution. We can ask a number of questions:

How do the solutions change in response to changes in the parameters?
What are the different types of trajectory?
How many equilibria are there: none, one, many …?
Is an equilibrium a single point or an orbit?
When can there be a change between the different types of trajectory?
Are the equilibria stable in relation to changes in the parameters?

Example 7.3. Consider first Hirsch and Smale's 'simplest differential equation'. Given the differential equation $\mathrm{d}x/\mathrm{d}t = ax$ and an initial condition $x(0) = K$, then there is a unique function $f(t) = Ke^{at}$ which satisfies these conditions. Different initial conditions – different values of K – give different functions as solutions. However, all these different functions are similar in type. In contrast, different values of a not only give different functions as solutions but moreover give different types of functions as solutions. If a is positive the value of x tends to ∞ as t increases; if a is negative the value of x tends to 0 as t increases; and if a is zero the value of x is constant. In other words if $a>0$ then there is no equilibrium; if $a<0$ there is a single equilibrium at $x = 0$; and if $a = 0$ then there is a single equilibrium at $x = K$. In those cases where an equilibrium exists, it is stable in terms of variation of x. The value $a = 0$ represents a boundary between the two main types of functions. It is unstable in the sense that slight changes in the value of a from zero will tip the solution so that it becomes one of the main types. Other values of a are stable in the sense that slight changes in the value of a will not lead to a change in the type of solution.

Example 7.4. Now consider a more advance differential equation which is discussed in Chapter 12. It is used by Lux (1995, pp. 885–886) in his model for herd behaviour: $\mathrm{d}x/\mathrm{d}t = v[(1-x)e^{ax}-(1+x)e^{-ax}]$. If $\mathrm{d}x/\mathrm{d}t = 0$ then either $x = 0$ or $ve^{-ax}[(1-x)e^{2ax}-(1+x)] = 0$. The second equation has no solutions if $a \leq 1$; but has two solutions, $x = \pm(1/a)\ln\sqrt{[(1+x)/(1-x)]} = \pm b$, if $a > 1$. Moreover, if $a \leq 1$ then the sole equilibrium $x = 0$ is stable whereas if $a > 1$ then the equilibrium $x = 0$ is unstable and the two other equilibria $x = \pm b$ are both stable. The value $a = 1$ represents a boundary between the two main types of situation. It is unstable in the sense that slight increases in the value of a from $a = 1$ will tip the solution so that it becomes the second type. Other values of a are stable in the sense that slight changes in the value of a will not lead to a change in the type of solution. Lux identifies a as a herding parameter – if a is above 1 then the herd will move to one or other of the two extremes.

The Trajectory of Multiple Interacting Entities

This section provides an illustration of how the interaction between individuals in a social network results in the patterned flow of a population through space. In terms of the earlier classification of models, the model here is dynamic, deterministic, non-linear, interactive, based on influence (not choice) and individual (not structural). In this particular example the individuals converge to a common pattern.

The Keynote Address to the European Complex Science Society Conference in 2007 was given by Steve Smale. The subject matter of his address reminded me of the old adage: 'birds of a feather flock together'! We all have seen flocks of starlings flying in rapidly changing directions but still flying together as a group. Can we develop a model of this? In an earlier paper (Cucker & Smale, 2005, p. 1 ['CS' hereafter]) note:

> It has been observed that under some initial conditions, for example on their positions and velocities, the state of [a flock of birds] converges to one in which all the birds fly with the same velocity. A goal of this paper is to provide some justification of this observation. To do so, we will postulate a model for the evolution of the flock and exhibit conditions on the initial state under which a convergence as above is established. In case these conditions are not satisfied, dispersion of the flock may occur. There has been a large amount of literature on flocking, herding and schooling. Much of it is descriptive, most of the remaining proposes models, which are then studied via computer simulations.

So, 'a flock of birds moving together'. This rather vague phrase can be formulated more precisely in a variety of ways, each way giving a somewhat different model.

(1) *Time* may be discrete or continuous (CS give a model for each).
(2) *Space* may have one, two or three dimensions (CS focus on three, but refer to two and provide an example using one dimension).
(3) *Movement* may be characterised in terms of heading or in terms of velocity (CS focus on velocity but cite a reference on heading).
(4) *Social network*: Does the bird take account of all the other birds; or just a subset of them, possibly just the neighbouring birds? ... if the latter, how large is a neighbourhood?
(5) *Social influence*: Is each other bird taken account of in the same way, say by taking the average of their velocities; or does the influence between a pair of birds depend on the distance between them? ... and if so how precisely is this formulated?

The core model of CS runs as follows. At each time t each bird i has a position x_i and a velocity v_i. The influence of bird j on bird i is specified in terms of an adjacency matrix $\mathbf{A}_x = [a_{ij}]$ where each entry depends on y, the square of the distance between the two birds, in the following way:

$$[a_{ij}] = \eta(y) = \frac{K}{(s^2 + y)^b} \quad \text{for some fixed } K,\ s > 0 \text{ and } b \geq 0$$

The model consists of the system of differential equations below. The second equation represents the fact that each bird adjusts its velocity by taking account of the velocities of the other birds, towards the average of its neighbours' velocities, where $\boldsymbol{x}$ and $\boldsymbol{v}$ are the vectors of the birds' position and velocity vectors and $L_{\boldsymbol{x}} = \mathbf{D}_{\boldsymbol{x}} - \mathbf{A}_{\boldsymbol{x}}$. [The Laplacian L of a non-negative, symmetric, $k \times k$ matrix $\mathbf{A}$ is defined as $L = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is a diagonal matrix with entries $d_i = \sum_{j=1}^{k} a_{ij}$. When $\mathbf{A}$ is the adjacency matrix of a graph G, many of the properties of G can be read out from L.]

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{v} \\ \boldsymbol{v}' &= -L_x \boldsymbol{v} \end{aligned}$$

'The first two results give conditions to ensure that the birds' velocities converge to a common one and the distance between birds remain bounded'. ... The result holds whenever $b < 1/2$ and holds for $b >= 1/2$ provided the initial conditions satisfy certain conditions.

Cucker and Smale also consider the situation of a linguistic population evolving with time. The state of the population is given by a vector of positions $\boldsymbol{x}(t)$ and a vector of languages $\boldsymbol{f}(t)$. Agents tend to move towards other agents using languages close to theirs (and therefore communicating better). Also, languages evolve by the influence from other agents' languages and this influence decreases with distance (for instance, because of a decrease in the frequency of linguistic encounters). The model consists of the system of differential equations:

$$\begin{aligned} \boldsymbol{x}' &= -L_f \boldsymbol{x} \\ \boldsymbol{f}' &= -L_x \boldsymbol{f} \end{aligned}$$

Multiple Entities with Non-Identical Probabilities

I now consider multiple entity systems. Homogeneity models of these systems assume that all the individuals (i.e., entities) are identical. Heterogeneity models allow that the individuals may be different from one another. As one might expect, homogeneity models are easier to work with although they may correspond less well with reality. Here I consider a set of binary probabilistic events and contrast the homogeneous model's prediction of a binomial distribution for the results with the heterogeneous model's prediction of a heavier tailed distribution. (This of course is reminiscent of complexity theory models which predict fat-tailed distributions in contrast to classical theory models which predict normal distributions. Note, however, that here the situation is static in contrast to the dynamic situation of complexity theory.)

I consider a set I of n individuals and a set $X_i = \{0, 1\}$ of states for each individual. The set X of states of the population consists of vectors $\boldsymbol{x}$ which has as its ith component $\boldsymbol{x}_i$. The state of the population can also be represented by the state distribution (n_0,n_1) where $n_0 + n_1 = n$. The proportions in the two states (P_0, P_1), where $P_0 + P_1 = 1$ are $P_0 = n_0/n$ and $P_1 = n_1/n$. Introductory treatments of the analysis of binary data such as this start with simple models. The simplest model would be that each individual has a propensity p of being in state 1; and a probability $q = 1-p$ of being in state 0. Given this model, the expected distribution is (nq,np). An estimate of p from the data is $p = n_1/n$.

As Cox and Snell (1989, pp. 106, 109) note 'this relatively simple formulation can need elaboration in various ways'. Cox and Snell proceed to discuss cases of 'anomalous dispersion'. Anomalous dispersion concerns the fit between the model and the data. The model predicts a certain level of dispersion whereas the data display a different level of dispersion. Both over-dispersion and under-dispersion can arise but the former is more common. The possibility of over-dispersion in the analysis of binary or categorical data has long been recognised and receives discussion in standard texts as a complication to or problem for the assumptions used in the simpler models of the data (McCullagh & Nelder, 1983/1989, pp. 124–135; Cox & Snell, 1989, pp. 106–115; Agresti, 1990, p. 42; Le, 1998, pp. 123–126). It occurs when the data display more variation than that predicted by the model adopted. Over-dispersion is common in practice and 'unless there are good external reasons for relying on the binomial assumption, it seems wise to be cautious and to assume that over-dispersion is present to some extent unless and until it is shown to be absent' (McCullagh & Nelder, 1983/1989, p. 125). 'Overdispersion … causes concerns because the implication is serious; the analysis which assumes the logistic model often underestimates standard error(s) and, thus wrongly inflates the level of significance' (Le, 1998, p. 123).

Anomalous dispersion arises whenever one of the two assumptions of the model – independence and homogeneity of propensities – is broken. In many applied situations population heterogeneity is a very natural assumption to make. Agresti (1990, pp. 42, 74) mentions the case of insects surviving a low dose of insecticide: 'extra variation could also occur when an insect has probability π of surviving, but the value of π varies for insects in the batch according to some distribution'. One way in which heterogeneity can arise is through the clustering of the population with each cluster having a characteristic propensity. Indeed heterogeneity of probabilities is routinely demonstrated by logistic regression studies where the probabilities are shown to vary as the independent variables vary. Since a study rarely manages to include all the relevant independent variables then it follows that heterogeneity is likely to be even greater than that suggested by the study.

Dependence of propensities can arise with either positive or negative correlation between pairs of individual responses. Although within-group heterogeneity gives rise to under-dispersion, across-group heterogeneity gives rise to over-dispersion. If dependences exist between propensities then negative correlation gives rise to under-dispersion whereas positive correlation gives rise to over-dispersion.

I consider three models: the simple model A, a more complex model B and a data model D. I wish to know whether model D is significantly different from model A. I am interested in the variance V of the frequency n_1. There are three estimates of this, two from the theoretical models V_A and

V_B and one from the data V_D. The 'excess variance' arising from model B in comparison with model A is given by $E = V_B - V_A$ and written as $e = E/V_A$. So $V_B = V_A + E = (1+e)V_A$.

Cox and Snell (1989, pp. 108–110) present a simple model for the case of dependence of propensities and obtain a value for e of $(m-1)\rho$ where m is the population size and ρ is the correlation between responses; and also a model for the case of between-group heterogeneity and obtain a value for e of $(m-1)\gamma$ where m is the population size and γ is a parameter which depends on the mean and variance of the group propensities. The case of within-group heterogeneity is discussed more fully below.

Empirical Evidence of Within-Group Heterogeneity of Propensities

Perhaps the most obvious source of evidence for within-group heterogeneity of propensities comes from models where other variables exhibit a significant effect on propensities. Within-group variation in the explanatory variables implies within-group variation in the propensities. An alternative source of evidence comes from repeated observations on the same population. This evidence has the virtue of providing an immediate estimate of the variation of propensities.

For each individual we are interested in the proportion P of 1s recorded in the n repeated measures. We are then interested in the distribution of P over the population. If propensities are homogeneous and equal to p the expected distribution is the binomial (n, p). If propensities are heterogeneous then the expected distribution is more polarised than this. Table 7.1 reports the results of a reanalysis of 11 studies and shows that polarisation is present in at least some of the cases. The studies in the table are numbered and the author cited, each study being discussed below where the complete citation is given.

Collett (2003, pp. 285–287) discusses data from a clinical trial involving 59 epileptics. For the purposes of illustration Collett condenses the data. A binary response variable denotes whether ('yes') or not ('no') a patient experienced at least five seizures in a two-week period. The data concern four consecutive two-week periods. Overall there were 49% 'yes' events. We now consider the number of 'yes' events for each of the 59 patients. If all patients had the same propensity to have a 'yes' event then we would expect 6% of the patients to have no occurrences, 25% of the patients to have one occurrence, 38% of the patients to have two occurrences, 25% of the patients to have three occurrence and 6% of the patients to have four occurrences. In contrast, the data reveal the bipolar distribution of occurrences given in Study 7 in Table 7.1.

Table 7.1. Distribution of the Frequencies of Occurrence (11 Studies).

	Percentage of Individuals Having *N* Occurrences									
	0	1	2	3	4	5	6	7	8	9
Binomial $p = 0.5$	12	38	38	12						
1 Bonney	1	6	30	63						
2 Conaway	23	25	28	24						
3 Little & Rubin	0	34	30	36						
4 Little & Rubin	10	10	13	68						
5 Yang et al.	5	35	17	43						
6 Yang et al.	45	13	15	28						
Binomial $p = 0.5$	6	25	38	25	6					
7 Collett	29	19	10	14	29					
8 Bonney	1	3	17	26	53					
9 Conaway	13	27	25	18	17					
10 Burt	35	5	3	3	2	2	4	11	34	
11 Gediga & Duntsch	7	10	16	6	16	13	10	8	8	6

We have also investigated other cases in the literature. Little and Rubin (1987, pp. 4–5) discuss data from a longitudinal study of coronary risk factors in schoolchildren. Obesity is recorded on three successive occasions. However, an obesity record is sometimes missing. For our Study 3, we treat this as our binary response. Restricting our attention to the cases where a full obesity record is available, we note that the obesity variable is binary, 'obese' or 'not obese' and this we treat as our Study 4. Bonney (1987) presents data on spontaneous abortions in successive pregnancies. Study 1 concerns the data for the first three pregnancies and Study 8 concerns the data for the first four pregnancies. Conaway (1990, p. 322) presents the data from a longitudinal study involving binary responses reported by Duncan (1985) – Study 2. Conaway (1990, p. 325) also presents the data on four binary items in an Armed Services Test – Study 9. A study by Yang, Goldstein and Heath (2000) present data concerning participation in the longitudinal study which consisted of three consecutive stages – Study 5. The study concerned voting intention – Conservative or not-Conservative-at three points in the UK electoral cycle, namely in the years 1983, 1986 and 1987 – Study 6. Burt (2003) reports the submission by students of a sequence of eight assignments – Study 10. Gediga and Duntsch (2002) present data on soldiers' physical reactions to the dangers of battle in the form of a nine-point Guttman scale – Study 11. The distributions of occurrence for the studies are presented in Table 7.1. Studies 2, 6, 7, 9, 10 and 11 exhibit much greater variation than would be expected from a binomial distribution, suggesting the presence of heterogeneous rather than homogeneous propensities.

The Most Probable Trajectory of a Macro Parameter

In this section a statistical dynamics approach is adopted. Balescu (1997) presents an account of classical non-equilibrium statistical mechanics which includes a number of non-standard subjects which 'require a more direct appeal to probabilistic concepts'. Helbing (1995) applies these ideas to 'sociodynamics'. The aim is to provide an account of a social system by establishing a link between the dynamic probabilistic behaviour of individuals and the dynamic probabilistic behaviour of the population as a whole. So there are two levels involved: at the micro level there are individuals and at the macro level there is the population as a whole. There is also a micro–macro linkage between the two levels. At the micro level it is useful to make a distinction between the behaviour of each individual and the processes within each individual which give rise to the behaviour (see Table 7.2).

The nature of the micro–macro linkage is as follows. The aim is to provide an account of the macro behaviour of a system in terms of its micro behaviour. Suppose we are interested in some macro parameter and wish to discover its trajectory. However, because of uncertainty all we can do is discover the most probable trajectory of the parameter. This most probable trajectory is derived from the dynamics of the probability distribution of the parameter. This in turn is derived from the dynamics of the probability distribution of system states. This latter dynamics is basic to the model and the corresponding equation is referred to as the master equation. The master equation specifies the dynamics in terms of transition rates – which refer to transitions between states of the system. Transitions between states of the system can be derived from transition probabilities between states of individuals (see Table 7.3).

Table 7.2. Macro, Micro and Micro–Macro Linkage.

Macro	The behaviour of the population as a whole
Micro–macro linkage	The behaviour of each individual
Micro	The process underlying the behaviour of each individual

Table 7.3. Deriving Macro Dynamics from Micro Dynamics.

Macro
What is the trajectory dynamics of the parameter?
What is the most probable trajectory dynamics of the parameter?
Dynamics of the probability distribution of the parameter
Dynamics of the probability distribution of system states
The master equation
Transition rates between states of the system
Micro
Transition probabilities between states of each individual

A Toy Example

Before moving on to discuss the real thing, let us first consider a toy example which illustrates the following features: the system has multiple entities; the system is 'macro' and each entity is 'micro'; each entity has a micro behaviour; the macro behaviour of the system is an aggregate of the micro behaviour of the entities; the micro behaviour is probabilistic and so there are *a* set of possible micro trajectories each with a certain probability; the macro behaviour is probabilistic and so there are *a* set of possible macro trajectories each with a certain probability; some macro trajectories are more probable than others; finally, there is a most probable macro trajectory ... but this does not correspond to the most probable *type* of trajectory.

Consider two coins which are tossed three times. The 'micro' state of each coin after each toss is either H or T, each occurring with probability 1/2. The 'micro' trajectory of each coin can be any of the following: HHH, HHT, HTH, THH, TTH, THT, HTT, TTT, each occurring with probability 1/8. The system as a whole consists of the two coins. The 'macro' state of the system after each toss is either 0, 1 or 2, corresponding to the number of heads shown by the two coins, these possibilities occurring with probabilities of 1/4, 1/2 and 1/4, respectively. We denote the more probable occurrence, namely 1, by A; and denote either of the less probable occurrences, namely 0 or 2, by B. The 'macro' trajectory of the system can be any of the following types:

AAA with probability 0.125; the most probable trajectory, $p = 0.125$, is (1, 1, 1)

AAB with probability 0.375; the next most probable trajectories, each $p = 0.0625$, are (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1)

ABB with probability 0.375; the next most probable trajectories, each $p = 0.03125$, are (1, 0, 0), (1, 0, 2), (1, 2, 0), (1, 2, 2); (0, 1, 0), (0, 1, 2), (2, 1, 0), (2, 1, 2); (0, 0, 1), (0, 2, 1), (2, 0, 1), (2, 2, 1)
BBB with probability 0.125; the least probable trajectories, each $p = 0.015625$, are (0, 0, 0), (0, 0, 2), (0, 2, 0), (0, 2, 2); (2, 0, 0), (2, 0, 2), (2, 2, 0), (2, 2, 2)

Quantitative Sociodynamics
Helbing (1995) discusses in turn a variety of stochastic models – see Fig. 7.1.

He repeats the basic argument of Table 7.3 in each of his chapters – see also Table. 7.4. The details of the argument vary mathematically according to whether the state space is discrete or continuous, and whether the set of states is a simple set, a set of states for each element or a 'configuration' of elements. The argument varies substantively depending on whether opinions or actions are being considered.

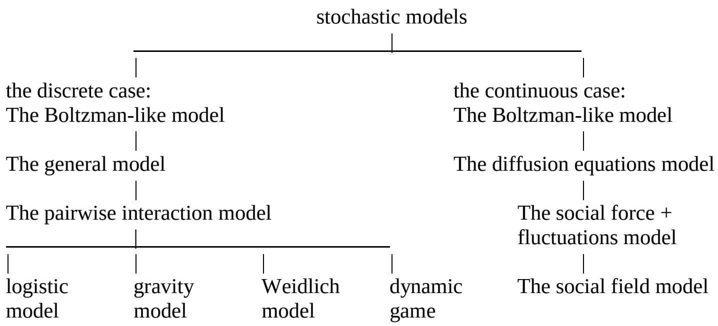

Fig. 7.1. A Conceptual Map of Helbing's (1995) Stochastic Models.

Table 7.4. The Components of the Pairwise Interaction Model – A Rough Guide.

Rate of change of behaviour probability vector = inflow−outflow
Flow = sum of component flows
Component flow = transition rate × population probability
Transition rate = spontaneous rate + pairwise induced rate + 3-wise rate + ..
Pairwise rate = pair interaction rate × pair readiness rate × pair probability
Pair probability = imitation + avoidance
Imitation (avoidance) = imitation (avoidance) frequency × source probability
Readiness = exponential of utility difference/distance

CHAPTER 8

MATHEMATICAL PSYCHOLOGY

> If mathematical psychology is possible, it must be founded on the basis of material phenomena ... There is nothing, however, to stop us from considering the materialistic phenomena that underlie a given psychical event as a function of the psychical event and vice versa.
>
> – Fechner (1851, p. 373; cited by Laming, 1973, p. ii)

Psychology addresses the questions, 'who am I?', 'what do I spend my time doing?' and 'reflecting on my life, where have I come from and where am I going?'. Models of life as a trajectory have as their elementary component a model of a single step in the trajectory. The single step may be modelled either as a stimulus input giving rise to a response output (Chapter 8) or as a choice (Chapter 9). In psychophysics, the basic model involves an input space X, an output space Y and a response function f from X to Y. A variety of procedures are used to obtain measurements, sometimes indirectly, of X and Y. An important example of stimulus–response is the common phenomenon of the imitation of the behaviour of one individual by another individual. Turning now to psychometrics, consider a response involving an individual in a situation. Looking at a set of individuals and a set of situations, the response depends on the individual and the situation. For example, high performance depends on the high ability of the individual and on the high easiness of the situation. Typically these variables are multi-dimensional. Although controlled investigations enable a balancing of situations and individuals, observation in naturalistic settings often does not. Turning now to methodological issues, some puzzlement is expressed concerning measurement theory. Just as Chapter 4 identified some problems with choice, here we find some problems with model selection. An important distinction is between models of the average individual and models of specific individuals. Finally, Falmagne's survey of four decades of mathematical psychology research is noted.

PSYCHOLOGY

The subject matter of psychology can be illustrated by addressing the questions, 'who am I?', 'what do I spend my time doing?' and 'reflecting on my life, where have I come from and where am I going?'. I am a human being and also more generally an animal. I am a biological entity and also a psychological entity. I have an identity and a personality. I am an individual and also a member of a social system. I am similar to other human beings but also different. In some respects I am normal and in other respects abnormal.

I spend my time sleeping and waking. Sleeping, I sometimes dream. Waking, sometimes I think, sometimes I do or act and sometimes I sense or perceive. Sometimes one particular type of activity predominates. For example, I may be sitting thinking, doing the housework or watching television. Sometimes I am doing a mixture of these things: perceiving *and* thinking; thinking *and* doing; perceiving *and* doing (e.g. driving the car automatically without thinking) and perceiving, thinking *and* doing. I am motivated to act and I experience emotion.

Reflecting on my life I see it as 'a personal journey through changing social landscapes' (the phrase I introduced in the first chapter). From my birth to the present day, I have learned and developed – and no doubt will continue to do so.

All of the aspects mentioned in the preceding paragraphs correspond to major areas of inquiry in psychology – and have done so for almost a century or more (cf. McDougall, 1912; Colman, 1994).

MATHEMATICAL PSYCHOLOGY

I now turn to mathematical psychology itself. The use of mathematics in psychology falls into four broad categories. Statistical modelling and inference is the most common use of mathematics and is covered by standard textbooks on statistics. Psychometrics relates to testing and individual differences and is covered by the journals, *Psychometrika* and *Educational and Psychological Measurement*. Information processing models of humans and of machines is covered by journals such as *Cognitive Science* and *Artificial Intelligence*. Finally, what might be thought of as 'classical' mathematical psychology involves the use of classical mathematics to model the substantive concepts of psychology and is covered by the *Journal of Mathematical Psychology* (JMP).

In similar vein, Meyer (2005, p. 110) makes the following contrast: 'formal theoretical modelling in psychological science has taken two complementary approaches: one based on models expressed in terms of standard mathematics (e.g. abstract algebra, geometry, differential equations, probability theory, stochastic processes, etc.) and another based on models expressed in terms of symbolic computational algorithms'. The JMP focuses on the former. Yet the latter may be appropriate 'when cognition and action involve complex, numerous knowledge structures and information-processing operations'. I have provided some discussion of information processing in Chapter 3 and focus in this and the next chapter on the standard mathematics approaches.

The opening quotation from Fechner shows that the notion of mathematical psychology goes back at least 160 years. A number of sources provide brief comment on the history of mathematical psychology (Luce, Bush, & Galanter, 1963, Vol. I, p. v; Atkinson, Bower, & Crothers, 1965, pp. 19–24; Coombs, Dawes, & Tversky, 1970, pp. 1–2; Laming, 1973, p. 1), whereas Miller (1963) is cited as providing a more detailed account. Authors start by citing the work of Fechner in the mid-nineteenth century. There follows the work by Ebbinghaus on memory and by Thorndike on learning at the turn of the nineteenth/twentieth centuries. The first half of the twentieth century saw major developments in psychometric testing and also the beginnings of a mathematical learning theory. After the Second World War, a number of developments constituted 'modern' mathematical psychology.

The JMP was first published in 1964, edited by Dick Atkinson with a board comprising Bob Bush, Clyde Coombs, Bill Estes, Duncan Luce, Bill McGill, George Miller and Pat Suppes. Falmagne (2005) presents a survey of articles published in the JMP over the subsequent four decades. The frequencies of different topics are given in Table 8.1.

Falmagne also looks at how these numbers have changed over time. There are two major trends. In the beginning, chapters about learning constituted almost half the total but by 1975 there were scarcely any such chapters. There are a number of possible reasons for this: problems arose with the mathematical models which were being developed; chapters on mathematical learning theory became acceptable for publishing in general psychology journals; and learning theory is now pursued in the fields of artificial intelligence and experimental economics.

In contrast, after a slow start, mathematics and methodology become the top category. Falmagne is concerned that JMP articles have become 'technically – rather than substantively – oriented', authored by

Table 8.1. Number of Papers in Each Area Published in the JMP between 1964 and 2004.

Content Areas	Number of Papers
Mathematics, methodology and statistics	224
Games, choice, utility and decision	198
Perception and psychophysics	156
Measurement theory	147
Learning	104
Memory, cognition and human information processing[a]	89
Response latencies and timing	59
Neural mechanisms and AI	45
Multidimensional scaling	44
Concept formation (including identification)	20
Others	18
Social and political phenomena	16
Psychometrics	11
Psycholinguistics	10

[a]Many knowledge spaces papers included in this category.

non-psychologists such as mathematicians or economists. He suggests that the mathematical education of psychology students has declined and that 'the most difficult and interesting problems that should normally, in view of its history, be in the province of psychology tend to be taken over by computer science, neuroscience or engineering'. He presents two proposals:

1. *Joining with other departments in the creation of combined graduate programmes, such as cognitive computational sciences or psychological economics.*
2. *Refocus the field of mathematical psychology by concocting an ambitious list of major unsolved problems.*

Iverson (2006) expresses a similar concern that 'the training of contemporary cognitive psychologists exhibits a trend away from quantitative sophistication' and proposes 'a series of MASS (mathematics and social sciences) publications aimed at attracting good undergraduate students to the field'.

Roughly speaking, the picture presented is one of psychologists with insufficient mathematics and mathematicians with insufficient psychology!

THEORY, EVIDENCE AND REALITY

Two of the topics in Table 8.1 – measurement theory, and mathematics, methodology and statistics – relate to the subject matter of Chapter 5, 'Theory, evidence and reality'. Measurement theory is concerned with the nature of the scale we use when measuring a particular variable. Consider, for example, the nature of the scale used in Chapters 4–6 to measure value. Although Chapters 4 and 5 used a preference ordering, an ordinal scale, Chapter 6 talked about an amount of value, implying an interval scale. Turning now from models of value to the value of models, a major concern of methodology is to try to ensure that our models are good. However, the work of Chapter 4 tells that if there are multiple criteria by which the goodness of models is judged then problems may arise.

Measurement Theory

Measurement theory is often the first topic dealt with in introductory texts on mathematical psychology. There is an excellent reason for this. The empirical validity of the models discussed in mathematical psychology depends on obtaining measurements of the variables in the model. It is important therefore that we understand the measurement process.

However, I must confess to being puzzled by measurement theory. There would appear to be rules for what you can and cannot say. We are told: 'you can't add apples and oranges'. But surely you can: two apples and three oranges gives five pieces of fruit! Somewhat similar scepticism is expressed by Suppes and Zinnes (1963, p. 3):

> Although measurement is one of the gods modern psychologists pay homage to with great regularity, the subject of measurement remains as elusive as ever. A systematic treatment of the theory is not readily found in the psychological literature. For the most part, a student of the subject is confronted with an array of bewildering and conflicting catechisms, catechisms that tell him whether such and such a ritual is permissible or, at least, whether it can be condoned. To cite just one peculiar, yet uniformly accepted, example, as elementary science students we are constantly warned that it 'does not make sense' (a phrase often used when no other argument is apparent) to add numbers representing distinct properties, say, height and weight.

The authors discuss the standard classification of scales and proceed to put them on a rigorous foundation. But even of this I am sceptical. I keep returning to certain key points:

1. Any given physical property can be measured by a variety of alternative scales.
2. Formulae specify how to convert one scale to another.

3. Properties of one scale may or may not correspond to properties of the other scale.
4. Any given physical law can be expressed by a formula using any of the scales – and the formula depends on the scales being used.

The property of temperature provides a useful illustration of these points. Temperature can be measured by many different scales, for example the Fahrenheit (*F*), Centigrade (*C*) and Kelvin (*K*) scales. Conversion between the scales is provided by the formulae mentioned below.

$$C = \frac{100(F - 32)}{180}$$

$$K = C + 273.4$$

The three scales give the same ordering of temperatures. Moreover, equal intervals on one scale remain equal intervals on any other of the two scales. However, the three scales have different zero points. Also temperature ratios in one scale do not equal those on another.

Even the equal intervals property fails to correspond if we postulate a fourth scale based on Stefan's law. This expresses radiancy in terms of the fourth power of temperature (Kelvin). So radiancy *E* (total energy emitted per unit time per unit area from a black body) can be taken as a scale for temperature.

$$E = \sigma K^4$$

The ideal gas law can be expressed in each of the following ways.

$$PV = RK$$

$$PV = R(C + 273.4)$$

$$PV = R\left[\left(\frac{100(\mathrm{F} - 32)}{180}\right) + 273.4\right]$$

$$PV = R(E/\sigma)^{1/4}$$

Clearly the law is most simply expressed using the Kelvin scale. However, there is nothing actually wrong in using any of the other scales.

Mathematics, Methodology and Statistics

Model Selection

'The main objective of the scientific enterprise is to find explanations for the phenomena we observe. Such explanations can oftentimes be couched in terms of mathematical models. In the field of psychology, one may for instance wonder what mathematical models best describe or explain distributions of response times, forgetting curves, changes in categorisation performance with learning, etc.'

With these words, Wagenmakers and Waldorp (2006, pp. 99–100) introduce a special issue of JMP devoted to 'Model selection: theoretical developments and applications'. They refer back to the earlier special issue in June 2000 on the same theme. Model selection refers to a situation where there is a set of alternative models and one wants to select the best model. To understand the logic of this problem, it is helpful to keep in mind three different sets of models.

A: the set of all possible models for the situation
B: the set of all available models for the situation
C: the set of models currently being considered

By definition it is set *C* which is being considered, but conclusions regarding set *C* need to be qualified by an acknowledgement that sets *B* and *A* may provide better models. The word 'better' implies a preference ordering on the set of models. For the preference ordering to be principled, it needs to be derived from a preference criterion. Typically, there are more than one criteria. Typically, the different criteria give different orderings of the models. Two common criteria are simplicity and goodness of fit. There is often a trade-off to be made between them. In keeping with the philosophy of model selection, there are several different models of model selection!

Models of Individuals and Models of Sets of Individuals

Consider a set of individuals. Suppose that we have a micro-model of each of the individuals and also a macro-model of the set of individuals as a whole. How does the set of micro-models relate to the macro-model? This question is relevant when we have data relating to a set of individuals and then apply statistical analysis to the data. Typically the results provide a macro-model, for example a model of the average individual – and yet this may conceal important aspects of the set of micro-models of the individuals.

For example, Brusco and Cradit (2005) consider 'the common practice in applications of multi-dimensional scaling ... for the analyst to apply the selected model to a single proximity matrix produced by ‘pooling’ proximity scores ... the resulting ‘group’ matrix reflecting a simple averaging of the individual scores’. The authors propose and apply a method of identifying groups of individuals with similar proximity matrices. In their application, the set of 64 individuals were found to consist of 33 individuals with an understandable and discriminating set of judgements, 4 individuals with a puzzling and discriminating set of judgements and 27 individuals with an understandable but grossly simplified set of judgements. Of course whether these groupings reflect the individual’s judgements per se or simply the task they were set is an open question.

MODELS OF LIFE AND MODELS OF SINGLE STEPS IN LIFE

One of the three questions raised at the start of the chapter was ‘reflecting on my life, where have I come from and where am I going?’ and it was suggested that life is ‘a personal journey through changing social landscapes’. To model this journey, we look back to the 10 models of history in Chapter 1 and simply adapt them to give 10 models of life:

(1) Life as a trajectory of states.
(2) Life as a function-driven trajectory of states.
(3) Life as an action-driven trajectory of states.
(4) Life as an alternative-selection-driven trajectory of states.
(5) Life as a value-dependent alternative-selection-driven trajectory of states.
(6) Life as a value-consequence value-dependent alternative-selection-driven trajectory of states.
(7) Life as a thought-driven trajectory of states.
(8) Life as a rational-choice-driven trajectory of states.
(9) Life as a parameter-driven trajectory of states which themselves include the parameters.
(10) Life as a probabilistic parameter-driven trajectory of states which themselves include the parameters.

What needs to be noted in all of this is that ‘life’ is a system where an individual interacts with their environment. So the state of the system consists of the state of the individual and the state of the environment; and the action of the system consists of the internal action of the individual, the internal action of the environment and the external interaction between the

Table 8.2. Different Models.

Chapter 8
Stimulus input and response output
Stimulus input, individual attribute and response output
Chapter 9
Choice as stimulus response
Choice as the outcome of a reasoning process
Choice as the outcome of a process of mathematical reasoning
Choice: the establishment of the values of the options

individual and the environment, namely action by the individual on the environment and action by the environment on the individual. What also needs to be noted is that the states of the individual and the environment have a complex structure. So what we have is a trajectory of a system of interacting complex structures. However, the trajectory is made up of individual steps and the structure is made up of individual elements, and it is this micro-level which much of mathematical psychology focuses on and which will be the concern of this and the following chapter.

I consider two broad ways of modelling an individual step. The first approach is characteristic of behaviourism and views an individual's action as a response to a specific situation. The second approach is characteristic of choice models and views an individual's action as a selection from a set of options. The present chapter focuses on stimulus–response models, whereas the following chapter looks at choice models – of course these are models of individual choice in contrast to the models of social choice in Chapters 4–6. Within each of the two approaches there are further sub-divisions and these can be illustrated using the following educational test item.

$$\text{Question} \quad (6+4-7)-((9+8)-2)/(3\times 5)=1 \quad \text{True/False?}$$

There are a number of points which can be made about an individual answering questions of this kind. Each of the following points relates to one of the models in Table 8.2.

(1) The more complex the arithmetical expression, the less likely the student is to answer the question correctly.
(2) A student's performance on a question depends on the difficulty of the question and the ability of the student.
(3) Answering the question involves making a choice.

(4) Answering the question involves a reasoning process.
(5) Answering the question correctly involves a rational choice.
(6) Answering the question involves establishing the values of the options. (It is not always the case that the correct answer is the answer which has value for the individual.)

Perception and Psychophysics

Consider our everyday sensations. The following questions relate to the senses of sight, hearing, smell, taste, touch, pain, temperature and the perception of time:

Shall I switch on the light?
Am I talking too loudly?
Is that the toast burning?
Has the skimmed milk gone off?
This isn't my train – its roof is too rough?
Where about does it hurt?
Is it too hot in here?
How long have we been in this room?

In each case, a physical stimulus gives rise to a psychological sensation. When the stimulus varies, the sensation also varies. What we want to know is how the sensation depends on the stimulus. In mathematical terms, we need to measure the magnitude x of the stimulus, measure the magnitude y of the sensation and then express the nature of the relationship in an equation, $y = f(x)$. Here f is a mapping f from a stimulus space X to a perception space Y. Each of the spaces may be discrete or continuous. Each of the spaces X and Y may be represented in a variety of different ways. Classical psychophysics assumes a particular form for this situation. Each space is multi-dimensional – an Euclidean space. Moreover, there is the following correspondence between the spaces: each stimulus dimension x_i corresponds to one and only one perceptual dimension y_i – a 'physical correlate' corresponds to a specific 'sensory magnitude'. There is a relationship $y_i = f_i(x_i)$, where f_i is monotonic and smooth, possibly being a power or logarithmic function.

For the physical stimulus, we can use the customary physical measurement scales for light, sound, concentration of substances, pressure, temperature, etc. The measurement of the psychological sensation poses greater problems. Although an individual experiencing a sensation can directly observe it, an external observer cannot. So researchers arrange for subjects to make an experimentally invoked external response. Several procedures are available. One type of procedure – which might be called the direct procedure – involves presenting the subject with just one stimulus. There are three variants: the subject is asked to estimate the physical magnitude of the stimulus, or to place the magnitude on a numerical scale or to place the magnitude on a verbal scale. A second type of procedure – the choice procedure – involves presenting the subject with a pair of stimuli. The subject is asked to compare the magnitudes of the two stimuli. A third type of procedure – the reproduction procedure – involves presenting the subject with a pair of stimuli. Here there are two variants: the subject is asked either to adjust one magnitude to be equal to the other or to reproduce the magnitude. What follow are brief sketches of three recent articles on psychophysics.

Perception of the Duration of Time

The method of reproducing the magnitude of a stimulus was used by Wackermann (2006). He was investigating how humans represent the duration of time. Subjects were presented with a stimulus which lasted s units of time and then asked to reproduce that time duration. Let us use r to denote the reproduced duration. If the subject gives a correct representation then $r = s$, in other words $r/s = 1$, a constant. What is found is that as s increases, r/s decreases.

Weber's Law

Another approach involves the discrimination between a fixed standard stimulus and a varying one. As we see in the next chapter, the response is probabilistic. Suppose the standard is x. The 'just noticeable difference' $j(x)$ is based on two levels of stimulus, a and b. These two levels are selected so that at a the probability of noticing the difference is 1/4; and at b the probability of noticing the difference is 3/4. I define $j(x)$ as $(b-a)/2$. It is found that $j(x) = wx$, where w is a constant. This is referred to as Weber's law. Roughly speaking, the size of error is proportional to the size of the standard stimulus. In certain situations, there is a slight but systematic departure from this law. The departures can be well fitted by either $w = Cx^{\alpha}$ or $w = Kx^{\beta}-1$. This is referred to as 'the near miss to Weber's law'. Doble, Falmagne, Berg, and Southworth (2006) argue that the value of the exponent depends on the discrimination criterion.

The Perception of Colour
Dzhafarov and Colonius (2005) provide ‘a purely psychological theory of Fechnerian scaling in continuous stimulus space’ which is applicable to a non-classical physical stimulus such as colour. The authors argue that the classical psychophysics approach is not applicable to the perception of colour. Instead they propose that a local discriminability measure can be computed from the $g(a, b)$, the probability that stimuli a and b are judged to be different from each other.

Imitation: Response Reproduction

> in the earlier ‘Theory of Moral Sentiments’, [Adam Smith] asserted forcefully that emulation is the most pervasive of human drives.
>
> (Kindleberger, 1989, p. 244)

This briefest of sections is very much an afterthought. A common phenomenon is the imitation of the behaviour of one individual by another individual. Its importance is indicated by the above-mentioned quotation and is made use of in complexity theory models of herd behaviour in the stock market (which I discuss in Chapter 12). In my present life, it is very much to the fore as I try to replicate in real-time the movements of my neighbours on the line-dancing floor!

Let X be the set of behaviours. Let x_{A} be the behaviour of person A and x_{B} be the behaviour of person B, x_{A} preceding x_{B}. Then imitation occurs if $x_{\mathrm{A}} = x_{\mathrm{B}}$ (to some degree of approximation) and if the occurrence of x_{B} is the result of an imitation process. The imitation process consists of the following stages.

A does behaviour x_{A}.
This produces a physical process which represents x_{A}.
This produces in B a sensory process which represents x_{A}.
This produces in B a motor process which represents x_{A}.
This produces in B a behaviour x_{B} which represents x_{A}.

Psychometrics: Individuals, Situations and Events

Consider an event involving an individual in a situation. How should we explain the event? Does the event occur because of the nature of the individual or because of the nature of the situation? If the event depends on the nature of both the individual and the situation, how do the two work together to produce the event? These questions arise frequently. For example, is successful performance in a task due to the ability of the individual or the easiness of the task? How do individual ability and task easiness combine? How does individual personality combine with situation characteristics to affect behaviour? These questions have typically been studied within the field of psychometrics.

Although the previous section had just two variables, an input stimulus and an output response, here we add a third variable, the individual. Rather than input stimulus we refer to 'situation'; and rather than output response we refer to 'event' or 'attribute'. Here we regard the individual and the situation as inputs and some event attribute as an output. Consider all possible combinations of individuals and situations. We consider a set I of individuals and a set J of situations with some attribute y_{ij} for each individual i and each situation j. How does y_{ij} depend on i and j – how does y_{ij} vary across individuals and situations?

$$y_{ij} = f_i(j)$$

We now consider what possible forms this function might take. It may be that, for any particular individual i, the variable y can be expressed in terms of a situation variable x. For example, in psychophysics, the psychological sensation is a function of the physical stimulus.

$$y_{ij} = f_i(x_j)$$

The above-mentioned equation is for a particular individual i. From these individual functions, we may derive a mean function:

$$y_{i\ \text{mean}\ j} = f_{i\ \text{mean}}(x_j)$$

It may be that the variable y can be expressed in terms of both an individual variable a and a situation variable x.

$$y_{ij} = f(x_i, a_j)$$

There are several areas where this kind of situation is discussed. For example, the analysis of variance investigates whether a data matrix $[y_{ij}]$ can be represented by the sum of an individual effect and a situation effect. Deviations from the model may be interpreted either as the operation of an interaction variable or as error. Some literatures refer to aptitude–treatment interactions.

In some cases, the y variable is a valued attribute – for example performance. It is meaningful to refer to the y-ability or the y-effectiveness of the individual and the y-facility (as opposed to y-difficulty) of the situation. Perhaps the word 'propensity' can be used to cover the general notion. It is helpful to highlight this issue with the following statement:

> high y-value is due to high y-propensity of the individual and high y-propensity of the situation.

Multi-dimensional Individuals and Situations

Another important issue is whether the individual effect is unidimensional or multi-dimensional. Personality, intelligence and educational achievement – are they one-dimensional or multi-dimenstional? In personality measurement, Cattell and others used factor analysis to arrive at equations where the response depended on an individual's personality vector and a situation characteristic vector. Indeed the development and design of a test often has the deliberate aim of ensuring that subsets of items measure distinct dimensions of the individual effect. Equations of a similar form were developed in the area of intelligence testing.

The situation in educational measurement is somewhat different. Tracing the history of educational measurement over the past 100 years, Keeves and Masters (1999, pp. 8–10) describe how classical test theory now has to compete with generalisability theory and item response theory. All three approaches express the probability p of a test item score y as a function f of the item's characteristics x and the student's characteristics a.

$$p(y) = f(x, a)$$

In classical test theory, each student has a true score and this is added to by measurement error to give the observed score. In generalisability theory (or item sampling theory), a student's score is the summation of an overall mean, the student's true score, the item difficulty and the error or the student-by-item interaction effect. This model corresponds to the random effects analysis of variance design (Shavelson & Webb, 1991). In item response theory, a logistic function is used and a key role is played by the ability of the student relative to the difficulty of the task (Hambleton, Swaminathan, & Rogers, 1991; Sirotnik, 1987, pp. 32–73; Keeves & Alagumalai, 1999, pp. 23–42).

All three approaches assume a single dimension of ability or achievement – a single 'true score' for each student. 'The use of procedures of measurement for the assessment of a particular characteristic demands that characteristic is unidimensional or involves the presence of one or more traits that are operating in unison with each other. However, situations arise in practice where the requirement of unidimensionality is not met since more than one dimension is involved in the responses obtained' (Keeves & Masters, 1999, pp. 2–3). The unidimensionality requirement in classical test theory and in item response theory is further discussed by Keeves and Alagumalai (1999, pp. 30–32). Discussing latent trait measurement models, Swaminathan (1999, p. 51) writes 'the assumption that the latent space is unidimensional has been a source of concern for many practitioners and psychometricians'.

Dissatisfaction with unidimensionality has created pressure for the development of multi-dimensional models. However, 'unlike the factor analysis situation where a multi-dimensional extension of the Spearman model has been operationalised and developed fully, the multi-dimensional extension of the item response model is still in its infancy' (Swaminathan, 1999, pp. 51–52). Such a model specifies the probability of a correct response in terms of the multi-dimensional ability of students and the multi-dimensional difficulty of items. Different functional forms are used depending on how the relative abilities combine – for example, whether they are mutually compensatory or not.

But is multi-dimensionality enough of an advance on unidimensionality? Both seek to explain the scores in terms of general abilities or traits – at the macro-level. Yet a look at test items (such as the one we discussed earlier) suggests that an item score may be due to quite specific knowledge and skills at the micro-level. This leads us naturally to look at the literature on knowledge spaces (Doignon & Falmagne, 1998) and to the artificial intelligence models of Chapter 3.

Naturalistic and Controlled Allocation of Individuals to Situations

In the study of individuals and situations, what many psychologists do is different from what many sociologists do. Many psychologists design their investigation so that they control the allocation of individuals to situations. Sometimes they randomly assign individuals to different 'treatment' situations. Sometimes they make all the individuals participate in all the situations. The evidence arising from these experiments allows a much stronger identification of individual effects and situation effects. In contrast, many sociologists study people in naturalistic settings – which is what we all do in our everyday lives! Unfortunately – from the point of view of interpreting the evidence – what happens naturally in social systems is that individuals are selectively assigned to situations. When sociologists see group A in situation A behaving differently from group B in situation B, is it because of the difference between situation A and situation B or because the people in group A are different from the people in group B? This point is picked up in Chapter 10 on mathematical sociology where divergent population flows through situations is selective in terms of individuals' characteristics.

CHAPTER 9

MODELS OF INDIVIDUAL CHOICE

To be, or not to be, that is the question :-
Whether 't is nobler in the mind, to suffer
The slings and arrows of outrageous fortune;
Or to take arms against a sea of troubles,
And by opposing end them? – To die,-to sleep,
No more :- and, by a sleep, to say we end
The heart-ache, and the thousand natural shocks
That flesh is heir to,-'t is a consummation
Devoutly to be wished. To die,-to sleep :-
To sleep! Perchance to dream :- ay, there's the rub;
For in that sleep of death what dreams may come,
When we have shuffled off this mortal coil,
Must give us pause ...

...
For who would bear the whips and scorns of time,

...
But that the dread of something after death, -

...
Thus conscience does make cowards of us all;
And thus the native hue of resolution
Is sicklied o'er with the pale cast of thought;
And enterprises of great pith and moment
With this regard their currents turn awry,
And lose the name of action.

– Hamlet (Act II, Scene I, pp. 56–89)

Although Chapter 8 viewed behaviour as the response to an input stimulus situation, here in the present chapter I view behaviour as the outcome of a choice. Of course I have already discussed choice in Chapters 4–6 where the social choice depended on the choices of individuals. There however individual choice was a very simple and rather automatic matter: each individual had a set of options and placed a value on each option or expressed a preference between each pair of options – and then chose the best option. This indeed may be what happens at the point of making the choice, but what this account ignores is the process which leads up to that end point. It is a process of some complexity, and a wide variety of psychological models have sought to capture the essence of the process. Classical models of the choice process can be classified according to whether they are static or dynamic and whether they envisage the chooser as a single unit or as a system of multiple units. The dynamic multi-unit models offer the potential for the phenomena of complexity theory to manifest themselves. To reach the end point, a chooser needs to identify the set of options and establish the value of these options. The psychological process may or may not involve a process of conscious reasoning, the reasoning may or may not be exact and the reasoning may or may not be mathematical. In the end state, the option which is chosen may not be the best option even though the chooser thinks so at the time. One model of life is to see it as a trajectory of choice points where the path chosen is chosen in pursuit of value. The possible sources of limitation on the value of the chosen option are: the individual, the situation, the set of options, the value function, the valuation of options, the option selection and the experienced, recalled and reported value. If the value of the options is multi-dimensional, then the limitations discussed in Chapters 4–6 also apply.

THE NATURE OF THE CHOICE PROCESS

Choice as the Outcome of a Reasoning Process

In his soliloquy, Hamlet reasons about the choice of whether to end his life or not. Our own experience of choice contains many such instances of choice as the outcome of a reasoning process. This aspect of choice is not discussed in this book although I would look to Chapter 3 on mathematics, logic, artificial intelligence and ordinary language to provide a route into investigating this aspect.

Choice as the Outcome of a Mathematical Reasoning Process

In some situations, an individual is presented with an explicitly posed mathematical problem, the solution to which involves either establishing the truth of a statement or selecting from a set of options the one option which has maximum value. In deterministic situations, the value is the utility, and in probabilistic situations, the value is the expected utility.

In some experiments, the problem is quite mathematically demanding, and it is not surprising to find that people get it wrong and do not choose what is objectively the best option. Nevertheless, it may be that they are 'doing the best they can': they are optimising within the constraints imposed by the situation they are in and by their limited capabilities. Thus, in certain situations one can think of them as maximising their subjective expected utility or as using heuristics. For example, Bearden, Murphy, and Rapoport (2005) consider the Sultan's dowry problem (how many princesses to interview before choosing one of them as a bride) or the secretary problem (how many candidates to interview before choosing one of them as secretary).

The authors compare the optimal solution according to mathematical theory with the practice of subjects in an experimental situation. They find that the subjects make sub-optimal decisions.

Inexact Reasoning, Heuristics and Error

As indicated in the previous section, it is found that people sometimes choose an option which is different from the option indicated by exact mathematical reasoning. So what are people doing and what are the consequences? It is often suggested that people use heuristics. For example, Katsikopoulos and Martignon (2006) ask: in what circumstances does a given heuristic select the best option?; and is one heuristic better than another and in what circumstances? They consider a task where the subject has to say which of the two objects, A or B, has the higher value on some criterion, given the values of the two objects on a set of binary variables. One approach to the task is to use heuristics, for example 'tallying', 'take the best' or 'minimalist'. Under certain conditions, certain procedures are optimal.

The consequence of using heuristics can be that the best option is not chosen. Poulton (1994, p. 7) has provided a useful account of some of the types of sub-optimal heuristics which people use (Table 9.1).

Choice: The Establishment of the Options and Their Value

The values of the options are a key aspect of many models of choice. In some situations, the options and their value are already known to the decision-maker, and some models assume this to be the case. However, there are some situations where the set of options is not known. In this case, a search process is required to identify the possible options. The Sultan's

Table 9.1. Heuristic or Complex Biases in Dealing with Probabilities.

Complex Bias	Normative Rule	Heuristic Bias
Apparent overconfidence	Use objective probability	Use probability of related knowledge
Hindsight bias	Avoid using it	Use it
Small sample fallacy	Small samples are not as representative	Small and large samples are equally representative
Conjunction fallacy	A conjunction is less probable than component	A conjunction is more representative than is its less probable component
Regression fallacy	Future scores regress towards average	Future score should be maximally representative of past scores, and so should not regress
Base rate neglect	Use Bayes method	Give less weight to the prior probability or base rate
Availability and simulation fallacies	Use objective measures of frequency of probability	Judge frequency or probability from availability in memory or ease of simulation
Anchoring and adjustment biases	Avoid using anchors	Make use of anchors
Expected utility fallacies	Maximise expected utility	Choose certain gains, unless probability very low
Bias by frames	Avoid	Use

dowry problem or secretary problem discussed in the section 'Choice as the outcome of a mathematical reasoning process' concerns just such a search process. Even when the options are known their values may not be known and so the decision-maker has to establish what the values are. This may not be easy and one problem which we consider here may be the large number of options available.

van Rooij, Stege, and Kadlec (2005) aim at using mathematical theories of computational complexity to evaluate the a priori plausibility of subset choice models'. The authors 'show how the theory of computational complexity (including the theory of nondeterministic polynomial time (NP)-completeness and fixed parameter tractibility) can be used to evaluate the psychological plausibility of such models under different assumptions of processing speed, parallelism and size of problem parameters'. Theoretical analysis can suggest cognitive structures and mechanisms, inform the definition and assessment of parsimony of a model and specify theoretical limits of models. The authors' paper addresses the latter, using the following example as a case study.

Suppose you go to a restaurant and decide you would like a pizza. What toppings would you like on it? … tuna, salmon, mozarella? The first point to note is that you (I assume) had to spend a bit of time thinking about the value of the options – you did not know them in advance. You may also feel that you would not know the true value until you have actually tried the pizza! However, the issue which I want to focus on is the complexity of the task. Even with just three choices of toppings, there are eight options. Usually pizza restaurants have around a dozen or more options. For example, online I can customise from 29 toppings on one of three types of base. This gives a rather large number of options. For each topping, there are two choices: you can either have it or not. In general, a list of N toppings gives rise to 2^N options. To choose the best option, we need to know the values of the 2^N options – or know the preferences between the $2^N(2^N-1)/2$ pairs (e.g. the situation mentioned in the preceding text with 3 toppings had 8 values and 28 preference pairs).

How might the task be simplified? The simplest case would be if the value of each topping separately was known and if the value of the combination of toppings depended only on the separate values of the ingredients. Perhaps the value of the combination of the toppings is the sum of the values of the individual toppings. If this is so, then the value of a pizza is an example of a measure space! Rather than the sum, the product may be required to ensure that the inclusion of a forbidden topping (due to allergy or religion) immediately nullifies the value of any combination containing the forbidden topping. It may not be enough to consider just the values of the toppings separately, it may be necessary to consider the value of pairs of toppings, or triples of toppings and so on.

CLASSICAL MODELS OF CHOICE

Static, Single-Unit Models

Table 9.2 presents a framework for the different models in this section. The first model simply posits a direct relationship between the input stimulus and the choice response. The second type of model posits an intervening

Table 9.2. A Framework for the Different Models.

1	Input		Choice response	
2	Input	Intervening response	Choice response	
3	Hypotheses	Evidence	Judgement	Tests
4	States of the world	Realisations	Actions	Decisions

response. There is an analogy here with the standard hypothesis testing model in the field of statistics and this in turn has a correspondence with the fourth model, statistical decision theory.

The choice set in this section has just two options and so the choice response is a binary variable, denoted by b. The input is denoted by x and the intervening response by y. In model 1, $b = f(x)$. In the other models, $b = g(y)$ and $y = h(x)$. So $b = g[h(x)] = f(x)$. Typically, a criterion variable c is involved. The equations then become $b = f(x, c)$ and $b = g(y, c)$, $y = h(x)$ and $b = g[h(x), c] = f(x, c)$. In probabilistic models, it is usual for h to be probabilistic and g deterministic.

The Yes-or-No Threshold Model

Let us start with a simple threshold model of hearing. If I shout you can hear me and if I whisper you cannot. This suggests that hearing is all or nothing. There is some threshold value of loudness such that if the loudness of a sound is above the threshold you can hear and if the loudness is below the threshold you cannot.

In the following equation, x denotes the loudness of the sound, c the value for the hearing threshold and b is a binary variable which equals 1 if the sound is heard and 0 if the sound is not heard. The function $b = F(x, c)$ is specified in the following equation (Fig. 9.1).

$$b = 1 \quad \textit{if } x \geq c$$
$$b = 0 \quad \textit{if } x < c$$

The Probabilistic Threshold Model

However, experimental results go against the concept of a yes-or-no threshold. Consider the following experiment. A sequence of events is presented to a person. Each event involves either the presence or the absence

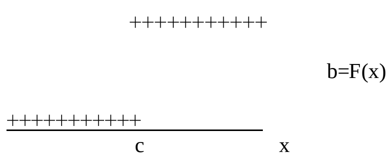

Fig. 9.1. The Yes-or-No Threshold Model.

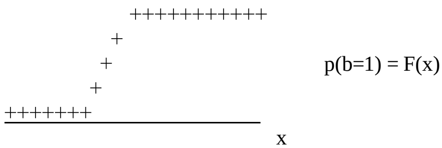

Fig. 9.2. The Probabilistic Threshold Model.

of a sound. Sometimes a sound is presented, sometimes not. The loudness of the sound (when presented) varies from one event to another. After each event, the person is asked to say whether or not a sound was present. It is found that the louder the sound, the more likely the person is to say a sound was present. This suggests a probability model: the probability p of hearing a sound (i.e. the probability that $b = 1$) is an increasing function F of the loudness x of the sound (Fig. 9.2). (What happens when the event involves the absence of sound? This question is addressed in signal detection theory which is one of the later models.)

$$p(b = 1) = F(x)$$

The Variable Experience Model

Laming (1973, p. 53) comments:

> This gradual transition in the frequency of seeing a near threshold stimulus is a very general phenomenon. It is found with both absolute and differential thresholds and in every psychophysical dimension. It therefore reflects some fundamental property of sensory discrimination and requires explanation in its own right. Accordingly, with the sole exception of Luce's choice model, every model for psychophysical thresholds incorporates in some way or other the idea that *the stimulus, as experienced by the subject, is intrinsically variable.*

In other words, the person's experience y of the sound depends on the physical magnitude x of the sound, and this dependence is probabilistic, taking the form of a probability density function $h_x(y)$.

The Three-State Threshold Model (One Stimulus)

We now combine the variable experience model with the binary threshold model. It is helpful at this point to envisage a three-state (two-step) process (cf. Table 9.1). The three states are: the sound itself, experiencing

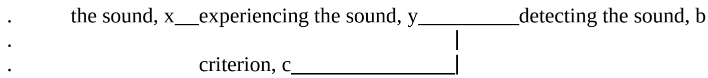

Fig. 9.3. The Three-State Threshold Model.

the sound and detecting the sound (deciding that the sound has occurred). Corresponding to these three states, we have three variables: a stimulus magnitude x, the experienced magnitude y and the detecting response b. The variables x and y are continuous, whereas the variable b is binary (as in the earlier discussion). The variable response model gives the relationship between x and y as a probability density function $h_x(y)$. The binary threshold model, applied now to the relationship between y and b and using a criterion threshold c to decide whether or not a sound has occurred, gives $b = f(y)$, where $b = 1$ if $y \geq c$ and $b = 0$ if $y < c$ (Fig. 9.3).

Given x, the cumulative distribution function for y is $H_x(y)$. The probability that y is less than the criterion c is $H_x(c)$ and the probability that y is more than the criterion c, and hence that the sound is detected, is $F_x(c) = [1 - H_x(c)]$. Notice that the higher the threshold, the less likely it is that y is above it and so the less likely it is that the sound will be detected.

Note that the preceding paragraph concerns some given x. Consider now that x can take different values. There will be a different cumulative distribution $F_x(c)$ for each x. In other words, the probability depends both on the physical magnitude x and on the threshold value c. To reflect this, we write $F(x, c)$. The experimental evidence suggests that as x increases, the probability of *detecting* it increases. In other words, $F_c(x)$ is an increasing function of x for fixed c.

One possibility would be that $F(x)$ is the cumulative distribution function of a normal distribution with mean x and some standard deviation σ. If x is fixed, then the probability of hearing the sound is a decreasing function of c; and if c is fixed, then the probability of hearing is an increasing function of x – as we require. More generally:

> Typically, $[F(x(c))]$ is a sigmoid function, such as the Gaussian or logistic distribution function. In case of the logistic distribution, the psychometric function ... can be characterised by two parameters $\delta(b)$ and $\mu(b)$ which refer to the slope and threshold ... respectively. $Pb(a) = 1/(1+\exp(-\delta(b)(a-\mu(b))))$. Note that when $a = \mu(b)$, $p = 1/2$, in other words $\mu(b)$ is the threshold. Thus $\delta(b)$ is the slope around a.
>
> – Augustin (2005, p. 71)

The Three-State Threshold Model (Two Stimuli)

The work of the previous section concerns detecting the presence or absence of a stimulus. We now consider the detection of a difference between two stimuli. Sometimes I talk loudly and sometimes I talk quietly – but do you notice the difference? It seems reasonable to suppose that the greater the difference in the physical magnitudes of two stimuli, the more likely it is that the difference will be identified.

This is illustrated by the results of an experiment by Guilford in 1931, where a subject was presented with two objects and asked which was the heavier. The weights used were 185, 190, 195, 200, 205, 210 and 215 g respectively. One subject compared each pair of objects in both time orders a total of 200 times. The results are presented in Table 9.3 which is derived from Laming (1973, Table 2.1, p. 16). When the first weight is heavier, the second weight is less likely to be judged heavier, and when the second weight is heavier, the second weight is more likely to be judged heavier. In general, the heavier the second weight is in comparison to the first, the more likely it is to be judged heavier. In particular, when both weights are equal, there is a 50% chance of the second weight being judged heavier.

Table 9.3. An Example of Paired-Comparison Data: The Proportion of Times that the Second Presented Weight was Judged Heavier than the First Presented Weight.

Difference in Weight (g) (Second Weight – First Weight)	Proportion of Times Second Weight is Pronounced Heavier*
−30	0.01
−25	0.05
−20	0.07
−15	0.14
−10	0.22
−5	0.35
0	0.50
5	0.64
10	0.78
15	0.87
20	0.93
25	0.94
30	0.99

Note: Each figure is the complement of the median (for a particular value of the difference) of the figures reported in Laming.
Source: This is derived from Laming (1973, Table 2.1, p. 16).

This suggests the following model: the probability p of one weight A being judged heavier than another weight B is an increasing function of the difference $x = (x_A - x_B)$ between the magnitudes of the two weights. By analogy with the previous model, we can develop a three-state (two-stage) process model here too. The three state variables are: a difference x, the experienced difference y and the detecting response b. So we can use the same model. However, the present situation has certain specific features. The criterion c is zero: if the experienced difference y is positive then A is pronounced heavier; and if y is negative then B is pronounced heavier. If the experienced weights y_A and y_B have normal distributions $N(x_A, \sigma)$ and $N(x_B, \sigma)$, then the difference $y = (y_A - y_B)$ has a normal distribution $N((x_A - x_B), \sigma\sqrt{2})$. With suitable σ the corresponding cumulative distribution is predicted to provide at least a qualitative fit to the data in the Guilford experiment.

Signal Detection Theory

Signal detection theory applies to the situations which we have been looking at but it conceptualises the situations in a somewhat different manner and also introduces new variables which affect the outcome. Up till now we have been asking, 'what is the response to the given stimulus?'. We now ask 'what is the response to the possibility of a given stimulus?'. The situation is that either event A or event B may occur; and a person may think event A occurs or think event B occurs. The two events and the two responses combine to give four possibilities. There are two possibilities for a correct response and two possibilities for an incorrect response – listed below. In some situations, the event A is of primary interest, for example when A is the occurrence of an important 'signal'. Special labels are then given to three of the possibilities: a hit, a miss and a false alarm.

The correct identification that A has occurred. [a 'hit']
The failure to identify that A has occurred. [a 'miss']
The correct identification that B has occurred.
The failure to identify that B has occurred. [a 'false alarm']

Such situations are an everyday occurrence. For example, in our house the telephone is in the other room. When Catherine and I are watching television, there are four types of event which can occur:

The telephone rings and we hear it.
The telephone rings but we do not hear it.

The telephone is not ringing and we do not think it is ringing.
We think the telephone is ringing but go through and find it is not ringing.

The person bases their response (a or b) on their sensation y of the environment. The person also associates a prior probability (p_A or p_B) to the occurrence or non-occurrence of a signal. Finally, the person associates a value with each of the four possibilities ($v_{a|B}$, $v_{b|B}$, $v_{a|A}$ and $v_{b|A}$). With this information, the person uses some decision rule to decide how to respond. We now postulate that the choice is 'rational'. By this we mean that the response with the higher expected value is chosen (Coombs et al., 1970, pp. 166–175). It can be shown that this is equivalent to applying the following likelihood ratio (LR) criterion.

Definition. The choice is said to be rational if response a occurs when $E(v_a|y) \geq E(v_b|y)$, and response b occurs otherwise.

Theorem 9.1. *Rational choice is achieved by applying a likelihood ratio (LR) criterion,* $LR \geq c$, *where* $LR = p_{y|A}/p_{y|B}$ *and* $c = -[(p_B/p_A)][(v_{a|B} - v_{b|B})/(v_{a|A} - v_{b|A})]$.

Proof.

$$E(v_a|y) = v_{a|A}p_{A|y} + v_{a|B}p_{B|y}$$

$$E(v_b|y) = v_{b|A}p_{A|y} + v_{b|B}p_{B|y}$$

$$\text{So } E(v_a|y) - E(v_b|y) = (v_{a|A} - v_{b|A})p_{A|y} + (v_{a|B} - v_{b|B})p_{B|y}$$

$$= (v_{a|A} - v_{b|A})p_{y|A}\frac{p_A}{p_y} + (v_{a|B} - v_{b|B})p_{y|B}\frac{p_B}{p_y}$$

Suppose $E(v_a|y) - E(v_b|y) \geq 0$.

$$\text{So } (v_{a|A} - v_{b|A})p_{y|A}\frac{p_A}{p_y} + (v_{a|B} - v_{b|B})p_{y|B}\frac{p_B}{p_y} \geq 0$$

$$(v_{a|A} - v_{b|A})p_{y|A}\frac{p_A}{p_y} \geq -(v_{a|B} - v_{b|B})p_{y|B}\frac{p_B}{p_y}$$

$$\frac{p_{y|A}}{p_{y|B}} \geq -\frac{(v_{a|B} - v_{b|B})p_B}{(v_{a|A} - v_{b|A})p_A}$$

$$\frac{p_{y|A}}{p_{y|B}} \geq -\left[\frac{p_B}{p_A}\right]\left[\frac{v_{a|B} - v_{b|B}}{v_{a|A} - v_{b|A}}\right]$$

The adoption of a particular rule entails a certain level of frequency for each of the four possibilities. Particular attention is given to the hit rate (HIR) and the false alarm rate (FAR). (The other two rates are the complements of these.) So each decision rule is characterised by a pair of numbers (FAR, HIR) which can be represented as a point on a graph, the 'receiver operating characteristic point'. We have seen that the threshold value depends on the prior probabilities and on the values. Varying these variables will cause c to vary and hence the point (FAR, HIR) to vary. Indeed it is necessarily the case that as the false alarm rate increases so does the hit rate. This relationship is referred to as the receiver operating characteristic (ROC) curve.

The ROC curve refers to a particular situation. Different situations have different ROC curves. Some situations are easier than others and their ROC curves lie above those for more difficult situations. This comparison between situations in difficulty can be explained in terms of the distributions of y. Suppose that the distribution of y has a mean μ_A and a standard deviation σ_A when the stimulus is A, and a mean μ_B and a standard deviation σ_B when the stimulus is B. Then the distribution of the difference has mean $(\mu_A - \mu_B)$ and standard deviation $\sqrt{(\sigma_A^2 + \sigma_B^2)}/2$. A measure of the easiness of signal detection in the situation is given by $z = 2(\mu_A - \mu_B)/\sqrt{(\sigma_A^2 + \sigma_B^2)}$. [Sometimes the standard deviation is taken as that of the 'noise' – see Coombs et al. (1970, pp. 177–178).]

Up till this point in our discussion of the responses we have considered only whether or not the response is correct. We have not considered the time required to make the response. There is nothing in the model which involves response time. Nevertheless, a common model of response time is to explain it as some inverse function of the modulus of z:

> recently, models of perceptual classification which have been successful in describing choice probabilities, have been augmented with processing time assumptions that have allowed them to be equally successful in predicting response times in these tasks as well . . .
>
> The first model of the decision time . . . to be explored is the signal detection model (SDT) that is augmented with a RT-distance hypothesis [RT = response time] . . .
>
> Assume a two-choice task in which one of two (unidimensional) signals, S1 and S2, is presented on a trial and the observer is to identify it. The classic SDT assumes that the internal representation of each stimulus is stochastic, usually unidimensional normal with mean μ_i and standard deviation σ_i for stimulus S_i, and that the observer sets a criterion on a perceptual continuum that divides it into two response regions, one for each stimulus. The RT-distance hypothesis states that on a trial the time to classify the stimulus as an S1 or S2 is an inverse function of the distance between the stimulus' percept on the continuum and the observer's criterion . . . in most applications an

exponential function is used so that for a percept, x, the decision time, $T = a \exp(-b|x-c|)$ where a and b are scaling parameters and c is the decision criterion ...'

– Thomas (2006, pp. 441, 443)

Hypothesis Testing

The problem of hypothesis testing is one of deciding whether or not some hypothesis that has been formulated is plausible. By a test of a hypothesis we simply mean a rule which, on the basis of the relevant evidence, specifies whether that hypothesis should be accepted or rejected.

– Open University (1977, p. 50)

Hypothesis testing involves a process with three stages: hypotheses, patterns of evidence and judgements. We consider the set X of all hypotheses, the set Y of all possible patterns for the evidence and the set Z of all possible judgements. We now consider mappings between these three sets.

The first mapping is straightforward enough. For each hypothesis x, there is a judgement z that hypothesis x should be accepted. Thus, there is a one-to-one correspondence between hypotheses and judgements.

The second mapping t is what is referred to in the above-mentioned quotation as a test, a rule. What a test does is to map each evidence pattern y into a judgement z in favour of one particular hypothesis. The test t partitions the set of patterns of evidence into subsets. Each subset corresponds to the acceptance z of some hypothesis x. (The complement of that subset corresponds to the rejection of the hypothesis x.)

The third mapping p is from the set of hypotheses to the set of patterns of evidence. Each hypothesis is associated with a probability distribution on the set of patterns of evidence. In other words, given the truth of a particular hypothesis x then there is a certain probability $p(y/x)$ that the pattern of evidence y will occur (Fig. 9.4).

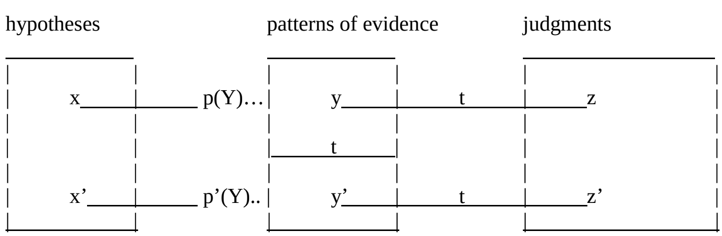

Fig. 9.4. Hypothesis Testing.

The standard approach to hypothesis testing involves the consideration of two mutually exclusive hypotheses. Let us denote them by A and B, where B is logically equivalent to not-A. Typically one of the hypotheses is privileged (usually because it is the simpler of the two hypotheses) and is referred to as the null hypothesis. The other hypothesis is referred to as the alternative hypothesis. Here we take A to be the null hypothesis.

Corresponding to these two hypotheses are two (mutually exclusive) judgements about the hypotheses: *a*, the acceptance of A and *b*, the acceptance of B. The combination of hypotheses and judgements gives rise to four possibilities:

$a\|\mathrm{A}$	The correct identification that the null hypothesis is true	
$b\|\mathrm{A}$	The failure to identify that the null hypothesis is true	Type I
$b\|\mathrm{B}$	The correct identification that the null hypothesis is not true	
$a\|\mathrm{B}$	The failure to identify that the null hypothesis is not true	Type II

$b|\mathrm{A}$ is referred to as the Type I error and $a|\mathrm{B}$ the Type II error.

(Using signal detection terminology, we can think of a Type I error as a miss and a Type II error as a false alarm.)

The judgement about the hypotheses is made on the basis of evidence. We use Y to denote the set of all possible patterns of evidence – and y to denote a specific pattern of evidence.

A test or decision rule prescribes what judgement should be made. The rule partitions Y into two subsets Y_{A} and Y_{B}, such that if y is in Y_{A} then judgement a is given, namely hypothesis A is accepted; and if y is in Y_{B} then judgement b is given, namely hypothesis B is accepted.

A perfect test is one which generates judgements which are always correct. Usually, tests are not perfect and have only a certain probability of generating a correct judgement. Thus, each particular test or decision rule gives rise to a set of probabilities for the occurrence of the four possibilities aforementioned. So each test gives rise to Type I and Type II error probabilities. The probabilities satisfy

$$p(a|\mathrm{A}) + p(b|\mathrm{A}) = 1$$

$$p(a|\mathrm{B}) + p(b|\mathrm{B}) = 1$$

Judgement errors arise because the link between hypotheses and evidence is probabilistic. The probability of evidence pattern y, given that hypothesis A is true, is denoted by $p(y|\mathrm{A})$, and the probability of evidence pattern y, given that hypothesis B is true, is denoted by $p(y|B)$. In other words, each hypothesis is associated with a probability distribution on the set of patterns of evidence.

A good test is one with low error probabilities, that is low $p(b|A)$ and low $p(a|B)$. Clearly, we want to choose a test which makes both types of error as small as possible.

Unfortunately, this is usually not possible. Reducing the Type I error increases the Type II error and vice versa. Knight (2000, pp. 355–356) shows how this can occur. Consider two tests t and t' such that $U < V$ in Y. Then $p(b|A, u) < p(b|A, v)$ and $p(a|B, u) > p(a|B, v)$. In other words, if tests t and t' are such that $U < V$, then test u has a lower Type 1 error but test v has a lower Type II error.

So the selection of a test becomes a question of how to make the trade-off between the two types of error. Normally we privilege the simpler of the two hypotheses, namely the null hypothesis. We choose a test such that Type I error is small because we do not want to 'miss' the simpler model when it is true: '... it is conventional practice to assign an upper bound [α] to the type I error probability, and to attempt to minimise the other error probability [β] subject to this condition' (Open University, 1977, p. 53).

We refer to α as the size or level or significance level of the test and to $1-\beta$ as the power of the test. For the significance level we have selected, we want a test which has maximum power. The Neyman–Pearson Lemma suggests an important principle for finding such tests, as follows. Accept the alternative hypothesis B if the likelihood of the data given the alternative hypothesis, $p(y|B)$, is greater than (some multiple of) the likelihood of the data given the null hypothesis $p(y|A)$. To cover more general cases, a test involving the likelihood ratio, $LR = [p(y|A)/p(y|B)]$, can be sought. This is the likelihood ratio which formed the basis for signal detection theory.

Bayesian Hypothesis Testing

Bayesian hypothesis testing involves an extra ingredient. The situation is as before but with the additional information that each hypothesis has an a priori probability of being true. The a priori probabilities are denoted as $p(A)$ and $p(B)$ respectively. This information allows us to use Bayes theorem to relate the probabilities of the evidence conditional on the hypotheses to the probabilities of the hypotheses conditional on the evidence. This too was an aspect of signal detection theory.

Theorem 9.2. *(Bayes):*
$p_{A|y}/p_{B|y} = p_{y|A}p_A/p_{y|B}\, p_B$

Proof.
$p(\text{A and } y) = p_{y|\text{A}}p_\text{A} = p_{\text{A}|y}p_y$

So $p_{\text{A}|y} = p_{y|\text{A}}(p_\text{A}/p_y)$
Similarly, $p_{\text{B}|y} = p_{y|\text{B}}(p_\text{B}/p_y)$

Question. A man is about to toss a coin. I have three hypotheses: the coin is fair, the coin is double-headed or the coin is double-tailed. Suppose the coin falls heads. What is the likelihood of this datum given each of the three hypotheses?

Statistical Decision Theory

In comparison with hypothesis testing and Bayesian hypothesis testing, statistical decision theory involves a further ingredient. Each of the four possibilities $a|\text{A}$, $b|\text{A}$, $b|\text{B}$ and $a|\text{B}$ has a value: $v_{a|\text{A}}$, $v_{b|\text{A}}$, $v_{b|\text{B}}$ and $v_{a|\text{B}}$ respectively. This is referred to as the loss function. With this extra ingredient, we now have all the constituents of signal detection theory (historically signal detection theory was developed by drawing on statistical decision theory).

The situation is as before but with the additional information that each hypothesis has an a priori probability of being true. The a priori probabilities are denoted as $p(\text{A})$ and $p(\text{B})$ respectively. This information allows us to use Bayes theorem to relate the probabilities of the evidence conditional on the hypotheses to the probabilities of the hypotheses conditional on the evidence.

> Before proceeding further, let us first note that the main topics considered in inference – namely, estimation and hypothesis testing – are special cases of the general decision problem structure outlined [below].
>
> – Open University (1977, p. 8)

Statistical decision-making involves a process with three stages: 'states of the world', 'realisations of a random variable' and actions. We consider the set X of all states of the world, the set Y of all possible realisations of the random variable and the set Z of all possible decisions. A decision d is a mapping from Y to Z. The correspondence with hypothesis testing is clear – refer back to Table 9.1.

I now introduce the extra ingredients, the conditional values and consider a decision rule *d*. This gives rise to probabilities for the four possibilities. Combining values and probabilities, we can obtain the expected value yielded by each decision rule conditional on the states of the world, A or B. This is referred to as the risk function of *d*.

$$R(\mathrm{A}, d) = \sum_x v_{d(x),\mathrm{A}}\ p(x|\mathrm{B})$$
$$R(\mathrm{B}, d) = \sum_x v_{d(x),\mathrm{B}}\ p(x|B)$$

We want to choose a decision rule which yields the maximum expected value. However, the expected value varies depending on what the state of the world is. A decision rule may have high value if A is the case but low value if B is the case. There are two approaches here:

(1) maximise the minimum value and
(2) take account of your prior knowledge about the likelihood of the states of the world, A and B.

Following the latter approach, suppose that the prior probabilities of A and B are $p(\mathrm{A})$ and $p(\mathrm{B})$ respectively. Then the Bayes risk $r(d)$ is defined as

$$r(d) = R(\mathrm{A}, d)p(\mathrm{A}) + R(\mathrm{B}, d)p(\mathrm{B})$$

The Bayes decision function d^* is chosen as to optimise $r(d)$, that is to maximise value (minimise loss). As in signal detection theory, the decision depends on the prior probabilities and values of the possible outcomes.

More Than Two Options: Classification into Three Classes

The discussion so far has concerned choosing between two options. What if the number of options is greater than two? Edwards and Metz (2006) comment that 'decision rules, in particular ideal observer decision rules, increase rapidly in complexity with the number of classes [options] involved'. They are concerned with the case of three options. They are 'attempting to develop a fully automated mass lesion classification scheme for computer-aided diagnosis (CAD) in mammography', cases being classified as malignant lesions, benign lesions and non-lesions. The authors consider three different types of decision rule and note the problems associated with each.

Static, Multi-Unit Models

In the preceding section, all the models at least implicitly envisaged a single unit processing the information and making the choice. In this briefest of sections, we note the existence of models which envisage multiple units each of which in a sense makes its own choice and a final stage where the final choice is made on the basis of the choices of the multiple units. In the case where the multiple units are people, this is covered in the field of social choice which we have already discussed extensively in Chapters 4–6. Covering the case where the multiple units are criteria is the substantial literature on multi-criteria decision-making.

Dynamic, Single-Unit Models

In all the preceding models, there have been a once-and-for-all calculation of the criterion value and the choice has been made on the basis of that value. We now turn to models where there is a process of updating the criterion value as fresh information comes in. One way of thinking about this is to say that the dynamic models incorporate a threshold model of procrastination. A famous example of procrastination is of course Hamlet's speech quoted at the start of the chapter.

A Threshold Model of Procrastination

It is quite straightforward to adapt our earlier threshold model to incorporate procrastination. There are now three options: judging in favour of option A, judging in favour of option B and postponing the decision. The choice between these three options again depends on the evidence value y.

In the following equation, $y(t)$ denotes the evidence value at time t. There are now two threshold values, a and b. If $y(t)$ is above a, then option A is selected ($z = 1$). If $y(t)$ is below $-b$, then option B is selected ($z = 0$). Otherwise the decision is postponed. The sequence of evidence $y(t)$ continues

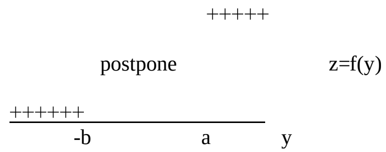

Fig. 9.5. A Threshold Model of Procrastination.

just as long as $y(t)$ remains within the interval $(-b, a)$ (Fig. 9.5).

$$z = f(y)$$

$$\begin{aligned} z &= 1 && \text{if } y \geq a \\ z &= \text{postpone} && \text{if } -b < y < a \\ z &= 0 && \text{if } y \leq -b \end{aligned}$$

Stochastic General Recognition Theory

Thomas (2006) comments:

> Though evidence has been marshalled in support of the RT-distance hypothesis ... other aspects of the data are incompatible with this basic representation ... hence a preferred alternative ... is the stochastic general recognition theory of Ashby (2000). When restricted to the one-dimensional case, I will refer to this model as stochastic signal detection theory.

Ashby's stochastic general recognition theory runs as follows.

The perception of stimulus i is modelled as a sequence of elements $\{x_i(t)\}$ over time. The distribution of such sequences given i, is $f_i(\underline{x})$. The elements may be multi-dimensional and may have, say, a multi-variate normal distribution.

At each point in time t, a criterion $y_i(t)$ is formed, depending on the sequence up to t:

$$y_i(t) = h[\underline{x}(t)]$$

For example, there may be a linear decision bound. It is a stochastic process with initial value $y_i(0)$ and mean $t\mu_i$ and standard deviation $t\sigma_i$, where μ_i is the drift rate.

$$y_i(t) = \underline{b}'\underline{x}(t) + c + e(t)$$

Suppose the observer sets up an upper and lower criterion a and $-b$. At each time t, the observer has an accumulated percept $y_i(t)$. This is used as the criterion for the choice:

If $y_i(t) > a$, then A is chosen.
If $y_i(t) < -b$, then B is chosen.
If $-b < y_i(t) < a$, then a decision is postponed and the sampling of percepts continues.

This model allows us to predict the response probabilities and the expected response times – and also the distribution of response times (Ashby, 2000, p. 199).

$$p(\mathrm{A}|i) = \frac{1-w}{1-u}$$
$$p(\mathrm{B}|i) = \frac{w-u}{1-u}$$

$$E(T|\mathrm{A}_i) = \left(\frac{a+b}{\mu_i}\right)\left(\frac{1+u}{1-u} - \frac{b}{a+b}\frac{1+w}{1-w}\right)$$
$$E(T|\mathrm{B}_i) = \left(\frac{a+b}{\mu_i}\right)\left(\frac{1+u}{1-u} - \frac{a}{a+b}\frac{1+v}{1-v}\right)$$

$$u = g(a+b);\ \ v = g(a);\ \ w = g(b)$$
$$g(z) = \exp\frac{-2\mu_i}{\sigma_i^2}$$

Dynamic, Multi-Unit Models

There are a variety of models which envisage multiple units and some of these envisage interaction between the units. For example, Thomas considers a race between stored percepts – the exemplar-based random walk. McMillen and Holmes (2006) offer a connectionist model for choice among multiple alternatives. A stimulus is presented and the subject has to say which of n alternatives it is. There are n units each of which is associated with one of the alternatives. Over time t, each unit i accumulates evidence x_i in favour of alternative i. The evidence changes due to stimulus input S_i, decay at rate k and a Wiener process W_i, reflecting random fluctuations in the signal, intrinsic accumulator noise and unmodelled inputs.

An additional feature means that there is interaction between units: the evidence is also reduced due to suppression from other units by a factor w. The authors proceed to explore the consequences of this model.

$$dx_i = (-kx_i - w\sum_{j\neq i} x_j + S_i)dt + c\,dW_i$$

Dynamic multi-unit models such as this with their notion of interacting units offer the potential for the phenomena of complexity theory to manifest themselves.

LIMITATIONS TO THE VALUE OF INDIVIDUAL CHOICE

One model of life is to see it as a trajectory of choice points where the path chosen is chosen in pursuit of value. Consider an individual in a situation entertaining a set of options, attaching values to the option in the set, choosing one of the options, experiencing value from the consequences of the chosen option and in later life recalling and reporting the experienced value. Kahneman, Wakker, and Sarin (1997) stress the distinction between utility as the choice criterion and utility as experienced satisfaction. Considering all of this in terms of the pursuit of value several points are worth noting.

(a) Some individuals have the potential for value and others do not.
(b) Some situations have the potential for value and others do not.
(c) The set A of options considered may not be the set A^* of all possible options and so the best option in A may be inferior to the best option in A^*.
(d) The value function which the individual uses to judge the values of the options may be problematic.
(e) Given the value function the values associated with the option may be in error.
(f) The option selected may not be the best option.
(g) The experienced value may not correspond to the value anticipated at the choice point.
(h) The recalled value may not correspond to the experienced value.
(i) The reported value may not correspond to the recalled value.

All these features can be the source of limitations on the value experienced. In addition to this, the values of the objects may be multi-dimensional and so the discussion in Chapters 4–6 can be applied regarding limitations to social value (here applied to multi-dimensional value).

CHAPTER 10

MATHEMATICAL SOCIOLOGY

> Mathematical sociology is not statistics; having said that, it's a little difficult to define exactly what it *is* ...
>
> – Bonacich (2008?)

> Relational analysis, even if not metrical, may be mathematical, in the sense that it will apply non-quantitative, relational mathematics. The mathematics which will be required ultimately for a full development of the science of society will not be metrical, but will be that hitherto comparatively neglected branch of mathematics, the calculus of relations, which, I think, is on the whole more fundamental than quantitative mathematics.
>
> – Radcliffe-Brown (1957, p. 69); cited by Freeman (2004, p. 102)

> Diffusion in incomplete social structures
> ... One of the most pervasive processes in the study of social behaviour has been the process of diffusion: diffusion of ideas, of technology, of cultural traits, of rumours, of opinion, of fads and fashions, and of population itself. Some social theorists have, in fact, made diffusion their central mechanism of social change ... and many others have examined empirical cases of diffusion ... The empirical investigations have covered a remarkably wide range of topics, from hybrid corn ... to automobiles ... to the practice of boiling drinking water ...
>
> – Coleman (1964, p. 492)

Society consists of a set of interacting entities. The entities are either people or activity units. The life of an individual is a trajectory of participation and value in a structure of social activity units. The history of a social activity unit is a population flow of individual trajectories characterised by population participation and value parameters. In pursuit of value, populations of individuals flow through the structure, selectively participating and differentially performing. Ordinary living involves a participation of people in the social activities of family, leisure and holidays, shopping, work and travel. Activity within a unit is structured by relationships and choices, rules, rituals and randomness. Ordinary living also involves the participation of cultural ideas and artefacts in social activities. Trajectories of value are exemplified in the religious conversions of William Wilberforce and in the common patterns identified in cultural stories. Population flows of participation and performance are illustrated using an educational case study.

BACKGROUND TO MATHEMATICAL SOCIOLOGY

The Wikipedia (2008) entry for mathematical sociology cites four books with 'mathematical sociology' in the title: Coleman (1964), Fararo (1973), Leik and Meeker (1975) and Bonacich (2008). Fararo (1973, pp. 764–766) provides a guide to the literature in mathematical sociology covering journals, bibliographies, reviews and expository essays, readers, texts, original monographs and research papers. Many of the references are either broader than mathematical sociology, for example, concerning the behavioural sciences in general, or narrower, dealing with a particular topic within sociology, or concerning a related field such as social psychology. Three classical original monographs are identified: Dodd (1942), Zipf (1949) and Rashevsky (1951). Included in a second generation of monographs is Coleman's (1964) '*An Introduction to Mathematical Sociology*'. Could it be that this is the first use of the phrase 'mathematical sociology'?

Coleman, Fararo and Bonacich cover the same general areas. Firstly the subject matter is defined. Coleman (1964, pp. 9–53), in his opening chapter identifies five uses of mathematics in sociology:

descriptive statements of observed behaviour
mathematical techniques for arriving at 'disposition' properties
quantitative empirical generalisations
use of mathematics in theory construction
prediction models

Secondly the relevant topics of mathematics are discussed: mathematical logic; sets, relations and functions; vectors and matrices; Markov chains; and dynamic models.

Thirdly there is some discussion of probability and statistics. However, by and large, the assumption appears to be that, although important, probability and statistics is a subject matter which is dealt with elsewhere. Fourthly there is a discussion of measurement (Chapter 2 in Coleman, 1964, and Chapter 7 in Fararo, 1973). Coleman distinguishes between two types of variables: directly observable variables and derived latent variables.

Fifthly substantive topics in mathematical sociology are dealt with. There is a major distinction made between social structure and social dynamics – this corresponds to a notional distinction in the above list between mathematical structure and mathematical dynamics. Both Fararo and Bonacich end with game theory (Chapters 20–25 and Chapters 12–14, respectively). Bonacich's four chapters on structure deal with: the strength of weak ties; the critical mass; balance theory and compound relations; and prescriptive marriage systems and permutation matrices. His chapters on dynamics are on dynamic models of influence and demography (plus three chapters on Markov chains). There is a distinction here between deterministic and probabilistic dynamics. Finally as the extended quotation at the beginning of the chapter indicates there has been a long-standing interest in what Coleman refers to as 'diffusion in incomplete social structures' thus bringing together the notions of structure and dynamics.

Finally a major aspect of mathematical sociology is social network analysis (Wasserman & Faust, 1994), and Freeman (2004, p. 10) provides a fascinating account of its historical development. He characterises social network analysis as 'an approach to social research that displays four features: a structural intuition, systematic relational data, graphic images and mathematical or computational models'.

PEOPLE PARTICIPATION IN SOCIAL ACTIVITIES

Life: A Trajectory through Social Structure

Life is a social activity. The event of birth, the event of death, the event of illness and the event of marriage – all take place within a complex social activity. The essential routine event of eating a meal can itself be a social activity and is the outcome of a complex social activity of food production – and is paid for by earnings from the activity of work for which qualifications from the activity of education may be required. Life outside work can involve participation in a variety of social activities. Participating in all of this involves travel, again a social activity. In summary, life is a journey – a trajectory – through space and time participating in a complex structure of social activities.

In order to illustrate the key features of a social life, I shall use myself as an example! Over the past 65 years I have lived in a sequence of homes in a sequence of places, and participated in a sequence of educational institutions, a sequence of work situations and a sequence of leisure pursuits. My homes have been in Dunfermline, Edinburgh, Corby, London, Exeter, Great Linford, London and, for the past 36 years, Newport Pagnell. Dunfermline and Edinburgh provided my first family home. Following that I stayed in a company's hall of residence, then with relatives, then in a university hall of residence, then in a shared house, and finally in my second family home.

I went to school in Dunfermline then Edinburgh and to university in Edinburgh for a mathematics degree and later to Exeter for a psychology degree. My first job was as an Organisation & Methods officer at a steelworks and this was followed briefly by a job for a small 'programmed learning' company. For the past 40 years I have been a lecturer at the Open University. I dislike shopping. In the 1980s and 1990s I became involved in politics but now my participation in the political process is minimal. I read books – and watch television regretfully (I was brought up on radio). My childhood upbringing in religion gave way in adulthood to an attachment to humanism. Chess has been the leisure pursuit I am best at. Walking and the outdoors have always been a delight. There were a variety of sports in my youth. I like to sing – but just by myself. Folk dancing was where Catherine and I met. My 50-year struggle with ballroom dancing has been recently renewed but line dancing is better because you can't stand on anyone's toes. Just a few weeks ago Catherine and Rona and I went to a laughter workshop! Relatives are mostly far away but Rona's family is just around the corner. On foot, by bus, by train, by car and by ferry as a small boy, and then by bicycle as an older boy, was how I travelled – and only later by aeroplane.

Mine has been a fairly ordinary life with a pattern and with features common to many other people's lives. Of course, there are other patterns of life which are also quite common and indeed certain individuals have a dramatically different pattern of life. Whatever the pattern, what is common to each life is that it is a trajectory through a (changing) social structure. This prompts the questions: to what extent can one represent a life mathematically; what does a life look like in abstract?

Family and Kinship

> The scientific spirit forbids us to regard society as composed of individuals. The true social limit is certainly the family – reduced, if necessary, to the elementary couple which forms its basis.
>
> – Comte, translation by Martineau (1853/2000, v. II, p. 234)

> I was struck by the importance of the historical role played by kinship in social network analysis. An anthropologist might have taken that for granted, but sociologists less often think of kinship as referring to their field. The fact is that by far the earliest example of a systematic network approach was focused on kinship.
>
> – Freeman (2004, p. 160)

As the quotations indicate, family and kinship are fundamental aspects of society and as such have long been the focus of social research. Although much of this work has adopted a social network approach, the focus here is somewhat different. Again I shall use my family and myself as an example.

My first family home, h_{21}, contained a wife, a husband, a daughter and myself (the son) – denoted w, h, d and s. The occupants of this home changed over time. The sequence starts with my parents marrying and sharing a home, and this was followed by the birth of my sister, the birth of myself, my sister leaving home, myself leaving home, the death of my mother and finally the death of my father. This trajectory can be represented as follows: $T(h_{21}) = (wh, whd, whds, whs, wh, h, \ldots)$. My second family home, h_{31}, started with me marrying my wife and has followed the trajectory $T(h_{31}) = (wh, whd_1, whd_1d_2, whd_2, wh)$. So my life has been a trajectory of membership of family homes, (h_{21}, h_{31}).

Representing the home trajectory of my relatives in similar fashion we have the patterns listed below where h_{11} and h_{12} are my parents' first homes; h_{21}, h_{22}, h_{23} and h_{24} are my parents' and my uncles' second homes (one uncle remained a bachelor, denoted 'b'); h_{31} is my second home; and h_{41} and h_{42} are my two daughters' second homes.

$T(h_{11}) = (wh, whs, whsd_1, whsd_1d_2, whd_1d_2, hd_1d_2, d_1d_2, d_1, .)$
$T(h_{12}) = (wh, whd, whds_1, whds_1s_2, whds_1s_2s_3, whs_1s_2s_3, whs_1s_2, \ldots s_1s_2, s_1,.)$
$T(h_{21}) = (wh, whd, whds, whs, wh, h, .)$
$T(h_{22}) = (b)$
$T(h_{23}) = (wh, h, .)$
$T(h_{24}) = (wh, h, .)$
$T(h_{31}) = (wh, whd_1, whd_1d_2, whd_2, wh)$
$T(h_{41}) = (wh, whs_1, whs_1s_2)$
$T(h_{42}) = (wh)$

In general, there is a set *I* of individuals, a set *H* of homes and a home membership relationship. The set of individuals have a set *R* of kinship relationships between pairs of individuals. A family home is a home in which most of the members are kinship related. All this changes over time. The life of an individual is a trajectory of membership of homes. The history of a home is a trajectory of sets of members. The social history is a trajectory of live individuals and live homes (and their memberships). So the life of an individual is a trajectory of membership of homes and of an individual location in changing relationship configurations in each home. The history of a home is a trajectory of sets of members, and of relationship configurations.

The state of the home changes either by gaining or by losing roles/individuals. Individuals are gained by birth or by immigration. Individuals are lost by death or by emigration. Emigration from an existing home is a major event and the move can be to set up a new family home (e.g., as a married or co-habiting couple), or to take up further education or to take up work and a 'non-family' home. In this way change in education and work can lead to changes in homes. The formation of a couple prior to setting up in a new home can occur through meeting in education or in work or in leisure. Naturally, education and work has a particular geographical location and so the meeting requires the two individuals to be in the same geographical location at least at the point of meeting.

Leisure and Holidays

In the previous section, our study of the family did not consider the detailed nature of the activity which takes place within the home. This neglect is compensated for here in our brief consideration of leisure. There is a set I of individuals, a set L of leisure activity units and a participation relationship indicating the participation of individuals in leisure activity units. The set of individuals have a set R of leisure activity relationships between pairs of individuals. The activity which takes place within a leisure activity unit has a structure. For example, a packaged holiday has a very explicit structure as indeed do non-packaged holidays. More specific leisure events also have a structure – for example, consider the performance of a play in a theatre.

The activity has a temporal structure (indeed some of it is timetabled): before, during and after the play; during the play there is alternation between the 'acts' and the intervals; and within the 'acts', a succession of 'scenes'. Participants have one of the following roles: audience, actors, supporting staff – related to one another in a triangular structure of role relationships. The activity specification is different for the different roles. The audience receives a programme outlining the plot, giving the acts and scenes and which actors play which parts. The behaviour of the audience is highly constrained by implicit and explicit rules – they must sit still and watch and listen (and have their mobile phones switched off!) and when appropriate respond. Of course, each individual in the audience is free to mentally experience the performance in their own way in accordance with their psychology. The actors themselves are working to the script for the play. It is highly constraining and yet allows freedom to the artistry of the actors and director – and allows variations in the quality of the actor's performance of their part. The supporting staff also have their specification, for example, the ice-cream sellers have to march into the auditorium at the interval to the designated ice-cream selling locations!

Shopping

Viewed as an economic activity, going to the theatre is rather unusual in that the production by the cast is transmitted directly to the audience for their simultaneous 'consumption'. More commonly, the product is transmitted by the producer at a particular time and place and the consumption takes place later and in a different place; and the producer and consumer are separated by intermediaries and intermediary processes. So usually what the consumers find themselves doing is shopping.

When, where and how people shop has changed over time. Changes in transport and domestic storage technology has meant that people can travel further for their shopping. In the UK the development of out-of-town supermarkets has over the past decades put continual pressure on 'high street shops'. People drive to supermarkets rather than walk to the high street. When shops open has been influenced by economic, religious and political considerations. Holidays (holy days) were decreed by religions and governments and shopping was not permitted on certain of those days. In the UK, despite resistance from the Lord's Day Observance Society and some trade unions concerned about the shopworkers' interests, shops have opened on Sundays albeit in some cases only from 11 a.m. to 4 p.m.

Work and the Organisation: Rules or Choice?

Work and leisure often take place within the setting of an organisation. The sociological literature on organisations contrasts with certain economic theories of organisations. This is the same contrast that exists between the sociological spirit of this chapter and the strong focus on choice in the Chapters 4–6, 8 and 9 on social choice and individual choice.

The Classical Economic Versus Behavioural Theories of the Firm

Classical economic theory defines rationality as taking that action which has the best consequences. It proposes that actors should and do act rationally. An account of the classical economic theory of the firm is given by Cohen and Cyert (1975). In contrast, behavioural studies find that people and

Table 10.1. Key Ideas in the Epilogue, Chapter 9, of Cyert and March (1992).

9.1	Core ideas in the behavioural theory of the firm bounded rationality; imperfect environmental matching; unresolved conflict
9.2	Developments in economic theories of the firm theory of teams; control theories of the firm; transaction cost economics; agency theory; evolutionary theories;
9.3	Developments in behavioural studies of organisational decision making
9.3.1	Decision making as intentional, consequential action ambiguous preferences; risk taking; conflict among rational actors;
9.3.2	Decisions as rule-based action
9.3.3	The ecological structure of decision making networks; attention mosaics
9.3.4	Decisions as artifacts

organisations do *not* act rationally. In 1963, Cyert and March (hereafter referred to as 'CM') published a book rejecting the classical economic theory of the firm and focused instead on the actual behaviour of firms. In the epilogue to the 1992 edition of their book, Cyert and March provide 'a new statement of some of the more significant modern ideas for understanding the firm as an organization'. Table 10.1 presents the contents of this epilogue.

Rationality, Rules, Religion or Randomness?

A rational choice is one which intentionally selects the choice which has the best consequences. In situations of uncertainty it is the selection of the option which maximises expected utility. If, in addition, the individual's knowledge is subjective rather than objective then rationality involves the maximisation of subjective expected utility (Carter & Ford, 1972). Where a sequence of choices is involved rationality involves sequential Bayesian estimation (Cyert & DeGroot, 1987).

A fundamental distinction in philosophy is that between deontic and consequentialist criteria for action. This distinction corresponds to that made by Cyert and March between 'decision making as intentional, consequential action' and 'decisions as rule-based action' (CM, pp. 230–232). Theories of rational choice 'underestimate both the pervasiveness and the intelligence of an alternative decision logic – the logic of appropriateness, obligation, duty and rules. Much of the decision making we observe reflects the routine way in which people do what they believe they are supposed to do. Much of the behaviour in organisations is specified by standard operating procedures, professional standards, cultural norms, and institutional structures. Decisions in organisations, as in individuals, seem often to involve finding appropriate rules to follow. The terminology is one of duties, scripts, identities, and roles rather than anticipatory, consequential choice'.

A further possibility is that action is taken at random or at least haphazardly. This can arise as the result of a 'garbage-can' decision process where there are flows of problems and solutions which are largely independent of one another (CM, p. 235).

Finally it may be that the process of decision making is quite unrelated to the actual decision: decision making is not about making decisions! (CM, pp. 235–237). What matters is the process of decision making, not the decision-making outcome. The process provides 'an arena for developing and enjoying an interpretation of life and one's position in it. A business firm is a temple and a collection of sacred rituals . . . the rituals of choice tie routine events to beliefs about the nature of things . . . the role of decisions and decision making [is] in the development of meaning and interpretation. The focus has shifted form the "substantive" to the "symbolic" components of decisions. What is important is the process not the outcome – for it is the process which gives meaning. The reason that people involved in decision making devote so much time to symbols, myths and rituals is that they (appropriately) care more about them'.

A Social Process

Conceiving of the application of the criterion as a social process, the theory of teams deals with the problem of co-ordination which exists even when we consider the firm as having a coherent preference function. Cases of unresolved conflict are considered in agency theory (CM, pp. 221–223), transaction cost economics (CM, pp. 219–221), game theory and classical administrative theory (CM, pp. 228–229). Agency theory envisages 'a cascade of principal-agent relations in a firm beginning with stockholders and passing through the board of directors and the top management to lower levels of employees' (CM, p. 222). Transaction cost economics envisages bounded rationality and 'opportunism' – the notion that 'participants will lie, cheat and steal in their own self-interest if they can' (!) (CM, p. 220).

Travel

Life's activities occur in different places and so it is necessary to journey between activity locations. Complex transport activity structures exist.

There is public transport and private road transport. Even private road transport is structured.

Consider the activity of driving a motorised vehicle on a public road. The specification of this activity contains the following: definitions of a public road, a motorised vehicle and driving; specification of the attributes an individual needs to have to participate in the activity (i.e., needs to be a qualified driver); specification of the side of the road to drive on (different countries have different specifications); and much more besides.

CULTURAL PARTICIPATION IN SOCIAL ACTIVITIES

Religious Conversion as a Trajectory of Value

For many people, religion constitutes the most important part of their lives and, for those who have experienced it, their conversion to religion the most important passage in their lives. Consider William Hague's (2007) (hereafter denoted 'H') biography of William Wilberforce, the great anti-slave trade campaigner. Our primary interest is in Wilberforce's conversion to Evangelical Christianity at the age of 27 which is discussed in Chapter 4, 'Agony and purpose' (H., pp. 70–93).

In 18th century Britain, the dominant religion was a lax Church of England faith (CE) and this was in contrast to a variety of more strict faiths such as Methodism (M), Evangelical Christianity (EC) and others. Wilberforce was born in Hull in 1759 into a CE family (H., p. 8) but then in 1768, on the death of his father, he moved to London where he lived with his aunt and uncle who introduced him to Methodism (H., p. 7). In 1771 his mother 'rescued' him from this and brought him back to Hull and CE (H., p. 14). In 1785, at the age of 27, he went through a period of uncertainty (from May to November) and finally became an Evangelical Christian which he remained for the rest of his life (H., pp. 70–93).

During the months of his conversion to Evangelical Christianity, he placed extreme low valuation on a wide variety of aspects of himself and others. He expressed despair at the universal corruption and profligacy of the times (H., p. 77); disapproval of a variety of public habits (H., p. 76); and 'The deep guilt and black ingratitude of my past life ... I condemned myself for having wasted my precious time, and opportunities and talents' (H., pp. 77–78).

Let me very briefly sketch an abstract model for this. The social landscape offers three religious states: CE, M and EC. Wilberforce's personal journey through this social landscape may be characterised by the sequence of states: CE, (U), M, (U), CE, (U), EC, where 'U' indicates the uncertainty of the conversion process.

Focusing now on the period of his conversion, we may describe it as a crisis of confidence, a sudden collapse of value in his old self and world and the emergence of value in a new self and new world. In other words, there is a collapse of an old equilibrium and a switch to a new equilibrium. This is the kind of behaviour which can be modelled by complexity theory. Here we have the dynamics of multiple ideas in the mind interacting with one another. This dynamics of multiple ideas in the mind has an analogy with multiple individuals in society. In particular, complexity models of stock market crashes might well provide insight into crises of religious conversion.

The Trajectories of Value in Cultural Stories

Much as a script drives the production of a play, and 'scripts' in organisations provide a frame for organisational activities there are some who argue that what drives society are the overarching stories or myths of its culture. These stories are told in a variety of forms including fiction. What might we say about such stories? One answer to this question is to follow Dr Samuel Johnson who thought he might write 'a work to show how small a quantity of REAL FICTION there is in the world; and that the same images, with very little variation, have served all the authors who have ever written'. Citing Johnson, Booker (2004) follows through on that thought with his book, '*The seven basic plots*'. These are:

Overcoming the monster
Rags to riches
The quest
Voyage and return
Comedy
Tragedy
Rebirth

A rather simpler view is that there are only two types of story, those with happy endings and those with sad endings. In this view stories are about value and moreover value is binary. This prompts the following idea. It may be that just as Chomsky has identified a generative grammar for sentences so there might be a generative grammar for stories. This would have three

Table 10.2. Generator, Initial Conditions and Stories Generated.

Story generator: the generating rule is that if an episode is happy then the next episode should be sad – and vice versa.

A set of initial conditions: the first episode is sad.

The set of stories produced by applying the story generator to the initial conditions:

Number of episodes:	Stories:
1	(sad)
2	(sad, happy)
3	(sad, happy, sad)
4	(sad, happy, sad, happy)

components: a story generator, a set of initial conditions and a set of stories generated by the generator from the set of initial conditions (see Table 10.2).

Of course, if the same story generator is applied to the initial condition that the first episode is happy, then the set of stories produced is {happy}, {happy, sad}, {happy, sad, happy}, etc. Whatever the initial condition, all stories are binary value processes.

This may seem simplistic but if we assume that peace is happy and war is sad then stories of alternations of war and peace (cf. Tolstoy) would correspond to stories of alternations of sadness and happiness. Looking back to our account of William Wilberforce's life, there is an alternation between religious conviction and uncertainty. If we assume that conviction is happy and uncertainty sad then Wilberforce's life was a story of alternations of sadness and happiness. So stories and lives are about the alternation between binary states and these binary states map onto binary values. Let us label this the binary state-value alternation model.

All of this follows Dr Johnson's view that all stories conform to a few simple forms. Looking in more detail at stories might reveal that states were not binary but continuous and not unidimensional but multi-dimensional . . . and that time was continuous. Snapshots of this continuous trajectory might then take the same form as the daily pages of stock market prices. This suggests that models of stock market prices might provide insights into the stories of a culture and the societal trajectory.

Technology Development and the Trajectory of Value

Social participation includes participation in technology. The quotation by Coleman at the beginning of this chapter emphasises the importance of the diffusion process by which participation in new technology spreads through society. An illustration of the complex dynamics of the technology adoption process is provided by Cowan and Gunby's (1996, p. 806) discussion of technological development. Cowan and Gunby note the consequence of adopting the optimal policy: 'the optimal policy will, with positive probability, lead to the inferior technology being adopted infinitely often and the superior technology being adopted only a finite number of times'. Rosenberg (1994, p. 5) observes that ' ... historical analysis supports the view that technological change often takes place in quite information-poor and uncertain environments'.

Population Flows of Participation and Performance in Education

Burt (2003) presents a mathematical theory of participation and performance, and illustrates the theory with an educational case study. The theory covers a situation where a population of individuals participates and performs in a structured multi-stage environment. In the case study, a population of students engage with the 'programme structure' of their course, submitting a series of assignments, their performance on these assignments being related to the knowledge structure of the course. The mathematical theory has wide application beyond the case study – for example, to the progression of students through the stages of educational systems in general, to the employment history of a population, to the dynamics of voter participation in elections and to the dynamics of participation in religions and in communities.

The motivation for the theory is that we want to know what determines participation – and what determines performance. What the theory proposes in answer to these two questions is that there is a structured dynamic interaction between participation, performance and the state of the environment. Firstly the present depends on the past: present participation depends on past participation, present performance depends on past performance and the present state of the environment depends on the past state of the environment. Secondly there are interactions between the main variables: past performance affects present participation, past participation affects present performance and the environmental state affects both participation and performance. Thirdly structure is a major influence: the content of individuals' states and the content of the environment's states have a structure and this content structure – for example, its prerequisite structure – determines what happens in the dynamic interaction.

The Case Study
The case study concerns the population of 7,053 students who were initially registered for the UK Open University's Science Foundation Course, S102, in 1995. The data consist of the students' submissions of and scores on the eight tutor-marked assignments (TMAs), the nine computer-marked assignments (CMAs) and the exam.

The course has a multi-level 'programme structure'. There are four disciplines: physics and general science, earth sciences, chemistry and biology – presented in that order. Each discipline has two blocks (three in the case of physics and general science). Each block has a CMA and a TMA (except for Block 9 which has only a CMA). Each block has three or four units. Each assignment covers three or four units. In order to pass the course, a student needs to submit the assignments and sit the exam and achieve a certain level of performance.

Participation and Population Models of Propensity Dynamics
Different types of models for an individual's propensity to participate give rise to associated properties of the population process. We study the properties of the assignment submission data in order to identify the model which is operating. We look at the following key features: the overall and subgroup participation rates at each stage; the distribution at each stage of the frequencies of participation up to that stage; the one-step transition rates; the two-step transition rates; and the history tree of transition rates. From the properties of the assignment data, we infer that propensities are dependent from one stage to the next and that they are dynamic rather than fixed. There is a strong order effect and although this might be consistent with a deterministic S-shaped propensity transformation, we prefer to interpret the data as the expression of a stochastic dynamic propensity model. In particular, the evidence points to the decreasing-gap auto-dependent model and the parameters for this model are crudely estimated.

(1) The overall participation rates at each stage

There are 7,053 students at initial registration. The participation rates for the eight assignments and the exam are in sequence: 62%, 58%, 55%, 51%, 47%, 50%, 45%, 46% and 52%.

(2) The one-step transition rates

Of the students who submit the current assignment, 90% submit the next assignment. Of the students who do not submit the current assignment, 6% submit the next assignment. (These figures represent the broad pattern. The actual figures range between 87% and 93%; and between 3% and 11%.)

(3) The two-step transition rates

Of the students who submit the two most recent assignments, 88% submit the next assignment. Of the students who submit the most recent assignment but not the previous assignment, 66% submit the next assignment. Of the students who did not submit the most recent assignment but did submit the previous assignment, 31% submit the next assignment. Of the students who submit neither of the two most recent assignments, 1% submit the next assignment. (These are the figures for the submission of the fourth assignment. The figures for other assignments have a broadly similar pattern although sometimes less well-defined.)

(4) Subgroup participation rates at each stage

The population can be divided into subgroups according to their history of participation. Of the students who submit both the first two assignments, 92%, 88% and 87% submitted the third, fourth and fifth assignments. Of the students who submit the second but not the first assignment, 68%, 52% and 42% submitted the third, fourth and fifth assignments. Of the students who submit the first but not the second assignment, 22%, 16% and 15% submitted the third, fourth and fifth assignments. Of the students who submit neither the first two assignments, 1%, 0% and 1% submitted the third, fourth and fifth assignments.

(5) The distribution at each stage of the frequencies of participation up to that stage

The distribution D_1 after the first assignment was $D_1 = (38, 62)$, in other words 38% of the students do not submit the assignment and 62% do. The distribution D_2 after the second assignment was $D_2 = (36, 9, 56)$, in other words 36% of the students submit zero assignments, 9% submit one and 56% submit two. The subsequent distributions are $D_3 = (35, 6, 8, 51)$; $D_4 = (35, 5, 4, 9, 47)$; ... ; and $D_8 = (35, 5, 3, 3, 2, 2, 4, 11, 34)$.

(6) The history tree of transition rates

We now consider the node transition rates for TMA submissions up to TMA5. History dependence is now expressed in terms of the entire history of previous assignment submission. The tree gives the full information about assignment submission by the population. The rates are displayed in the branches of the history tree in Fig. 10.1. (Not shown in the figure are the transition rates for the non-submission of the most recent TMA – this is simply one minus the corresponding transmission rate for submission. For example, the transition rate to the node 0 is $0.38 = 1-0.62$.)

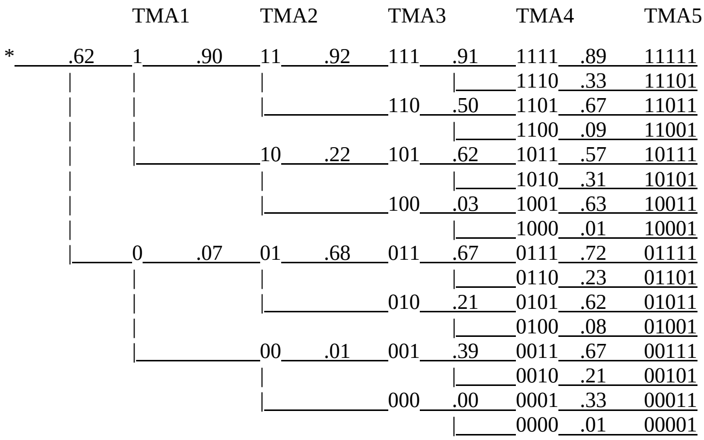

Fig. 10.1. History Tree Displaying Node Transition Rates.

The pattern exhibited by the history tree of node transition rates is consistent with the propensity of each individual following a decreasing-gap one-step auto-dependent process and a population with heterogeneous initial propensities.

The equation for the one-step auto-dependent process decomposes f into two functions g and h. In the decomposition of f, the functions g and h apply depending as $b = 1$ or $b = 0$, indicating participation and non-participation at stage t, respectively.

$$p_{t+1} = f(p_t, b_t) = b_t g(p_t) + (1 - b_t)\, h(p_t) = b_t(g(p_t) - h(p_t)) + h(p_t)$$

The equation $p_{t+1} = b_t(0.6{+}0.4p_t){+}(1{-}b_t)(0.4p_t)$ provides a simple illustration. Suppose $p_1 = 0.6$. Then the propensities p_2 for histories {0, 1} are {0.24, 0.84}, respectively; the propensities p_3 for histories {00, 01; 10, 11} are {0.10, 0.70; 0.34; 0.84}, respectively; and the propensities p_3 for histories {000, 001; 010, 011; 100, 101; 110, 111} are {0.04, 0.64; 0.28, 0.88; 0.06, 0.66; 0.34, 0.94}, respectively. If we order the histories in reverse binary order {000, 100, 010, 110, 001, 101, 011, 111} then the corresponding propensities are increasing: {0.04, 0.06, 0.28, 0.34, 0.64, 0.66, 0.88, 0.94}. The simple model compares with the data as indicated below. The broad pattern is similar but the asterisked data points deviate from the simple model.

History:	{000, 001, 010, 011, 100, 101, 110, 111}
Simple model:	{0.04, 0.64, 0.28, 0.88, 0.06, 0.66, 0.34, 0.94}
Data:	{0.00, 0.39*, 0.21, 0.67*, 0.03, 0.62, 0.50*, 0.91}

History:	{000, 100, 010, 110, 001, 101, 011, 111}
Simple model:	{0.04, 0.06, 0.28, 0.34, 0.64, 0.66, 0.88, 0.94}
Data:	{0.00, 0.03, 0.21, 0.50*, 0.39*, 0.62, 0.67*, 0.91}

We refer to the reverse binary ordering of a history as its recency pattern. So the results may be summarised by saying that the higher the recency pattern of the history the higher the propensity for participation at the next stage.

Student dropout is an issue of concern for higher education. In particular, an institution like the UK Open University with its open access policy and its commitment to widening participation is particularly concerned to ensure that maximum number of students reap the full benefit from their learning experience. A major component of student dropout at the UK Open University is students starting a course but failing to sit the exam at the end of the course. Retrospective studies can identify those characteristics which distinguish student dropouts from students who continue their studies. However, remedial action requires a predictive model. If we can predict which students are unlikely to sit the exam then we may be able to intervene and encourage the students to sit the exam after all. One type of prediction is based solely on whether the most recent assignment has been submitted. Predictions based on additional information are superior but only marginally so. This is because most students either submit most assignments – or submit none or very few. This in turn follows from the fact that submission of assignments has a stochastic dynamic propensity of the type just discussed.

The Criterion for Participation and its Stochastic Dynamics

What determines the propensity for participation? Here we explain it in terms of a criterion variable. Participation takes place if the criterion is positive; otherwise not. If the criterion changes in a stochastic manner then the criterion also exhibits a recency pattern effect and this explains the recency pattern effect for the propensity.

At time t, a criterion variable takes the value c_t and the participation variable takes the value b_t. Participation b depends on criterion variable c.

$$b_t = \{1 \text{ if } c_t \geq 0; \text{ and } 0 \text{ if } c_t < 0\}$$

At time $t+1$, the criterion has changed in accordance with the equation.

$$c_{t+1} = f(c_t) + a_{t+1} \quad t \geq 0 \quad \text{where } a_{t+1} \text{ is some random variable.}$$

Consider the distribution f of c_{t+1}. Splitting the distribution at $c_{t+1} = 0$ gives the probabilities $p(c_{t+1} > 0)$ and $p(c_{t+1} > 0)$, and the expectations

$v_{t+1} = E(c_{t+1}|c_{t+1} > 0)$ and $w_{t+1} = E(c_{t+1}|c_{t+1} < 0)$. Note that the two conditions are equivalent to $b_{t+1} = 1$ and $b_{t+1} = 0$. Writing $v_{t+1} = v(c_t)$ and $w_{t+1} = w(c_t)$ and assuming that v and w are increasing and that $v > 0$ and $w < 0$ we have:

$$E(c_{t+1}|b_{t+1}) = b_{t+1}\ v(c_t) + (1 - b_{t+1})\ w(c_t)$$

If $b_t = 1$ and $b_{t+1} = 1$ then $c_t \geq 0$ and $c_{t+1} \geq 0$ and $E(c_{t+1}|b_{t+1}) = v(c_t)$ and $0 < e_{01} < v(c_t)$.
If $b_t = 0$ and $b_{t+1} = 1$ then $c_t < 0$ and $c_{t+1} \geq 0$ and $E(c_{t+1}|b_{t+1}) = v(c_t)$ and $0 < v(c_t) < e_{01}$.
If $b_t = 1$ and $b_{t+1} = 0$ then $c_t \geq 0$ and $c_{t+1} < 0$ and $E(c_{t+1}|b_{t+1}) = w(c_t)$ and $e_{00} < w(c_t) < 0$.
If $b_t = 0$ and $b_{t+1} = 0$ then $c_t < 0$ and $c_{t+1} < 0$ and $E(c_{t+1}|b_{t+1}) = w(c_t)$ and $w(c_t) < e_{00} < 0$.

So when $t = 1$, histories {00, 10, 01, 11} partition the values of $E(c_{t+1}|b_{t+1})$ into $[-\infty, e_{00}]$, $[e_{00}, 0]$, $[0, e_{01}]$ and $[e_{01}, \infty]$.

So when $t = 2$, histories {000, 100, 010, 110, 001, 101, 011, 111} partition the values of $E(c_{t+1}|b_{t+1})$ into $[-\infty, e_{000}]$, $[e_{000}, e_{100}]$, $[e_{100}, e_{010}]$, $[e_{010}, 0]$, $[0, e_{001}]$, $[e_{001}, e_{101}]$, $[e_{101}, e_{011}]$, and $[e_{011}, \infty]$.

In summary, after each stage the expected value of the criterion depends on the participation history and the higher the recency pattern of the history the higher the expected value of the criterion. The result of the previous section follows by noting that the higher the expected value of the criterion the higher the expected propensity at the next stage.

THE CONCEPTUAL FRAMEWORK

Society consists of a set E of interacting entities. The set E consists of a set I of individuals and a set U of activity units. There is a class C of activity types. The class of activity types includes family, leisure, work and travel. The set U consists of a set H of homes, a set L of leisure units, a set W of work units and a set M of (movement) travel units. There is a set G of geographical locations and a relationship indicating the location of individuals and units.

There is a set A of actions by entities which has as subsets: sole actions by an entity x, pairwise interactions between entities x and y, etc. There is a participation relationship indicating participation of individuals in units. The activity in a unit may follow an activity specification. The activity in a unit has space–time structure which includes a temporal structure either as a sequencing of events or as a timetabling of events. The activity in a unit may involve participants in a structure of role relationships. The activity within a unit may be governed by rules or by choice or some mix of these.

Participation and performance are fundamental features of society. In pursuit of their goals social institutions – businesses, professions, educational systems, political parties, religions, nations and communities – control individuals. Social institutions sometimes promote and prolong the participation of individuals and sometimes restrict and terminate participation. Social institutions also provide a reward structure for certain performances by individuals. From the viewpoint of individuals, society presents a complex structure of social institutions.

All this changes over time. Individuals and units are born and they die. The life of an individual is a trajectory of participations in units. The history of a unit is a trajectory of participations. This trajectory can be looked at either as a trajectory of individual participations or as a flow of participations. In pursuit of their own individual goals, populations of individuals flow through the structure, selectively participating and differentially performing.

CHAPTER 11

MATHEMATICAL POLITICAL SCIENCE AND GAME THEORY

> A game is a structure of actions chosen within the rules. Politics is the game of choosing the rules.

> Callum: 'Hurrah! All these cards are mine!'
> Grandpa: 'No, you didn't shout 'Snap!'. The rules say you've got to shout 'Snap!'.'
> Callum: 'But I am playing Uncle Carsten's rules.'

> 'They civilize left and civilize right
> Till nothing is left and nothing is right.
> They civilize freedom till nothing is free
> Except perhaps by coincidence me!'
>
> (the musical and film 'Paint Your Wagon', sung by Lee Marvin)

In any social activity, the participants have action options and these action options have value consequences. The value consequences for any one individual of that individual's actions may or may not be dependent on other participants' actions. If the value consequences are not dependent on others' actions then the individual may proceed to make their choice in the manner discussed in Chapter 9. However, if the value consequences are dependent on others' actions then the situation has a structure which forms the basis for game theory. In some situations game theory allows an unambiguous identification of the set of best actions for all participants. However, there are some situations where it is not clear what the set of best actions for the participants is. In some situations there is a tendency for conflict rather than cooperation. It may be that participants can learn or evolve so that cooperation is more likely. Beyond the simple two-person game a variety of additional features have been added to introduce more realistic complexity to the models; and there is growing interest in exploring this complexity using computer simulation.

Social activities are governed by rules and a supporting rule system. A rule partitions the set of all action options into a subset of rule-conforming options and a subset of rule-breaking options. The social choice, the option selected, is influenced by both individual preferences and rules and a consideration of the rule system. For any social activity A there is a social meta-activity concerning the rules for social activity A. Politics is the social activity which focuses on rules: making representations about rules and making, implementing and applying rules. There are conflicting views about what is the ideal social activity structure and this conflict is played out in the political sphere with actors engaging in strategic interaction within the constraint of rules which are themselves part of the social activity structure. Note that much of the discussion of social choice and social welfare in Chapters 4–6 is relevant to politics.

GAME THEORY

The Simple Game

Definitions. A game consists of a set of players I, a pure-strategy space S, a value space V and a value function f from the strategy space to the value space.

In the illustration of this definition consider two players A and B. Player A can adopt strategy a or a' and player B can adopt strategy b or b'. Strategy space then consists of all four possible pairs of individual strategies (a, b), (a, b'), (a', b) and (a', b'). The strategy pair (a, b) results in value v for A and value w for B. The value pair (v, w) is a point in value space. Similarly the other strategy pairs result in other points in value space.

What is missing from this definition – what is missing from game theory – is a satisfactory criterion function, a criterion by which the players can decide what option to choose. The criterion of choosing the action which has the best value consequence works for some game situations but not for all. This is because the value consequence for A of one of A's strategies depends on what strategy B adopts.

The lack of a universal 'solution concept' is a key problem for game theory as a device for predicting or recommending action. An appealing candidate for a solution concept is the Nash equilibrium, and indeed this can be applied in a variety of situations. However, there are some games which have either no such equilibrium or a multiplicity of them. A further problem with the Nash equilibrium is that it may not be a Pareto optimum. In the Prisoners' Dilemma game the Nash equilibrium involves both players defecting but this is Pareto-dominated by both players cooperating.

These and other problems with the model have been voiced by Kreps (1990) and Binmore (1990, 1992). More generally there have been concerns that game theory fails to capture social reality – concerns which were expressed by Schelling as early as 1960. By 1990, Kreps and Binmore were suggesting that evolutionary game theory might resolve some of the problems of game theory, although in 1998 Mailath was concerned that evolutionary game theory was still too stylised. Binmore (1992, pp. 396–412) discusses 'adjusting to circumstances' through economic libration (learning on the job) and the possibility that this might lead to a Nash equilibrium. 'Unfortunately, matters do not always work out so well' and Binmore notes that in the dynamic processes of adjusting to circumstances 'there are no good reasons at all for assuming that the trajectories of a dynamic process will behave nicely. It is something that needs to be checked out'. In the long history of attempts to overcome these problems the work of Axelrod has an important place.

ADDING COMPLEXITY TO GAME THEORY

Why is there a need to add complexity to game theory? The reason is that there is a gap between game theory on the one hand and social reality and social theory on the other. In his book, *The Complexity of Cooperation*, Axelrod (1997) proposed bridging that gap. He notes that the title illustrates the dual purposes of his book:

> One meaning of "the complexity of cooperation" refers to the addition of complexity to the most common framework for studying cooperation, namely the two-person iterated Prisoner's Dilemma ... The second meaning of "the complexity of cooperation" refers to the use of concepts and techniques that have come to be called complexity theory.

Axelrod had been investigating conflict of interest since the 1960s (Axelrod, 1970). His 1984 book (Axelrod, 1984) applied the paradigm of the two-person iterated Prisoner's Dilemma and had the theme that 'cooperation based on reciprocity can evolve and sustain itself even among egoists provided there is sufficient prospect of a long-term interaction'. Because of its simplicity the paradigm has become a standard in diverse fields, has allowed a large variety of studies to be undertaken in a common framework and has allowed diverse fields to talk to one another and offer cross-fertilising insights. Axelrod's 1997 book is a sequel to his 1984 book but goes beyond it. Axelrod provides added complexity covering the following aspects: the set of actors, the number of options, the strategy mechanism, the nature of the strategy, the time structure of the action, the population dynamics and the evolutionary mechanism.

The contrast between a simple one-shot two-person game and the complexity introduced in Axelrod (1997) is indicated in Table 11.1. The one common feature is that both have value payoffs. A simple game has two players whereas a complex game may have many players. Where there are many players the simple situation is where interactions are pairwise – more complex situations may involve interactions between many actors. A further complexity may be that groups or coalitions of individuals may form. The set of options or possibilities may be two – or many. Knowledge may or may not be perfect. The choice or strategy mechanism may be based on rationality – or may be myopic, adaptive or random – or may be based on a certain number of previous historic stages or on a certain number of anticipated future stages. The strategies may be common to all individuals or may vary across individuals. The strategies may be independent of other players'

Table 11.1. Contrast between Simple Game and Complex Model (Axelrod, 1997).

	Simple	Complex
Value pay-offs	Yes	Yes
Population size	2	Many
Interaction size	Pair	Many
Coalitions	No	Yes
Options	2	Many
Strategy mechanism 1	Rationality	Myopic, adaptive, other and random
Strategy mechanism 2	–	1/2/3/Many stage history rule
Strategy mechanism 3	–	1/2/3/Many stage future rule
Perfect knowledge	Yes	No
Strategy 1	Homogeneous	Heterogeneous
Strategy 2	Other-independent	Other-dependent
Strategy 3	Time-independent	Time-dependent
Time 1/action	Simple play	Compound play (sequential)
Time 2	One play	Repeated plays
Time 3	One generation	Repeated generations
Population dynamics	Fixed members	Evolving
Population	Fixed frequency	Changing frequency
Evolution mechanism 1	–	Selection
Evolution mechanism 2	–	Creation: mutation and crossover
	Determined	Noise
	–	History
Fitness	–	Payoffs

strategies or dependent and may or may not vary with time. There may be just one play of the game or repeated plays. A single play may be simple or compound, consisting of a sequence of actions. Only one generation of individuals may be considered – or several generations. The population may be fixed or evolving, with categories having fixed or changing frequencies. The evolutionary mechanism may involve selection only or may also include creation by mutation or crossover. The payoffs may correspond to the fitness. The events may be determined or there may be noise. With the introduction of these features the complexity of the situation increases significantly. In particular with the introduction of a dynamics of many interacting individuals the possibility of applying complexity theory arises.

SIMULATION: THE EVOLUTION OF COOPERATION

We now present an investigation which is very much in the Axelrod tradition. It uses simulation to understand the complexity of the evolution of cooperation. As a prelude to a discussion of the simulation, we briefly consider the following four questions:

Is cooperation rational?
Can one learn to be cooperative?
Is cooperation successful in a diverse society?
Is cooperation successful in evolutionary terms?

Is Cooperation Rational?

The attractive notion that cooperation is rational is confounded by the logic of collective action. The Prisoner's Dilemma game is often used to illustrate this point. In the Prisoner's Dilemma game each of the two players has two strategies which we refer to as cooperation (C) and defection (D) and the payoffs are as follows:

If both players cooperate then both have moderate gains.
If one player cooperates and the other defects then the cooperator has a large loss and the defector has a large gain.
If both players defect then both have moderate losses.

Each player has an incentive to defect to avoid the damaging possibility of cooperating at the same time as the other player is defecting. Therefore, it might be argued, it is rational for a player to defect. This is by no means the end of the story. But it is sufficient for our purposes here to register that there is a problem with a straightforward claim that cooperation is rational.

Can One Learn to be Cooperative?

Perhaps cooperation can be rescued if we think of two players repeatedly playing the prisoner's dilemma game. Perhaps, once they have experienced the consequences of both defecting then the players will learn to cooperate. The topic of repeated games has a substantial literature (Mailath & Samuelson, 2006).

Is Cooperation Successful in a Diverse Society?

In society each individual encounters a number of different people – it's not a case of just playing the one person all the time. Perhaps cooperation is successful in this situation. This idea was famously explored by Axelrod in the 1980s. He ran a computer tournament between different strategies. One strategy was to cooperate with everyone all the time; another strategy was to defect against everyone all the time and there were other more complicated strategies. The most successful strategy was 'tit-for-tat' – in other words start by cooperating then cooperate in response to cooperation and defect in response to a defection. Roughly speaking this allows one to get on well with cooperators and to avoid the costs of 'giving in' to defectors.

Is Cooperation Successful in Evolutionary Terms?

Axelrod's initial experiment involved a fixed population of individuals with fixed strategies. What happens if we allow the populations and the strategies to evolve? – is cooperation a successful strategy in evolutionary terms?

This is the question addressed by a very complex simulation carried out by Majeski (2004). Here we shall confine ourselves to a discussion of four of the aspects in his model: the evolution of a population in a territory, the game situations, the game strategies, and the initial strategy mix and the subsequent results.

Majeski set up a population of individuals occupying territorial squares. Each individual could die either by chance or because they had low resources or because they had reached the maximum permitted lifespan. Each individual had a distinctive strategy. Each individual could reproduce a clone of themselves, that is they produced another individual possessing the same strategy. Mutation could, however, lead to the offspring having a somewhat different strategy from the parent. At each point in time the individuals played a variety of games with each other and added the gains from these games to their resources.

Five game situations are considered. In each situation there are four possible action pairs: both cooperate (CC), both defect (DD), the first cooperates and the second defects (CD) or the first defects and the second cooperates (DC). The situations can be put in order of increasing conflict they are: assurance, stag, prisoner's dilemma, chicken and deadlock. Least conflictual are assurance and stag, where there is one pair of strategies which leads to the best outcome for both players.

Note:

the first number is the payoff to each if both cooperate;
the second number is the payoff for a cooperator if the other defects;
the third number is the payoff for a defector if the other cooperates and the fourth number is the payoff to each if both defect.

Assurance: (CC, CD, DC, DD) = (1, −3, −2, 0) respectively.
Stag: (CC, CD, DC, DD) = (1, −3, 0, −1) respectively.
Prisoners dilemma: (CC, CD, DC, DD) = (1, −3, 3, −1) respectively.
Chicken: (CC, CD, DC, DD) = (1, −1, 3, −3) respectively.
Deadlock: (CC, CD, DC, DD) = (−1, −3, 3, 1) respectively.

Four strategies are considered. One strategy is an 'exploitive' strategy in which an actor always defects whatever the history of the interaction. The other three strategies are 'cooperative' strategies in that an actor always cooperates on first encountering another actor. However, the three cooperative strategies react differently to a defection by the other player. The strategies vary in punitiveness. The 'always-cooperate' strategy always cooperates – it never punishes defection. The 'grim' strategy never forgives a defection and punishes it for ever more: after the first defection, it always itself defects. The tit-for-tat strategy delivers immediate punishment to any defection but then forgets about it, responding cooperatively to each subsequent cooperation.

The central question being investigated was how an initial population with a given mix of cooperative and exploitative agents evolve. Broadly speaking the results showed that cooperation would evolve. However, this was less likely to happen if the situation was less cooperative, if the strategies were less punitive and if the initial mix was low on cooperation.

POLITICS

Some General Remarks about the State

In Chapter 10, life was portrayed as a journey through a social activity structure. It was noted that society itself shapes the structure. There is a variety of possible options for various aspects of the structure of society. Typically there is at least some conflict of opinion about what the ideal structure is – in particular, some conflict about whether the present structure is ideal. This conflict is played out in the political sphere and the dominant role in this sphere is played by the state. The conflict is governed by certain rules within which actors press for their own viewpoints. The social choice theory discussed in earlier chapters is therefore relevant. The structure is thus under tension and exhibits both stability and change.

A major influence on stability and change is the state. The state acts to maintain certain aspects of the structure and to change certain other aspects. The state influences all social spheres. Within the state unit there are subunits dealing with different social spheres. The laws and policies created by the state constitute the framework for activity specification in all social arenas: birth, death, illness, marriage, home, education, work, leisure and transport.

The state itself is a social activity unit and has its own activity specification – the constitution: laws specifying how state activity should operate. The constitution specifies roles in the process and rules for determining outcomes. In these roles are actors, actions and action criteria. The actors are of different types and within each type belong to different groups. Actors may be individual people or collectives. Some actions may be outside the constitution, the constitution itself may not be well defined, the constitution may be changed by individual actions and the constitution may have provision for constitutional change.

There is a territorial structure of state activity units. Each unit has a territorial range over which its rules apply. For example, the rules of the UK state apply to activities within the territory of the United Kingdom. The UK state relates to state entities outside the territory of the United Kingdom such as the European Community and to the United Nations. The UK state relates to state entities governing national territories within its own boundary: the assemblies of Scotland, Wales and Northern Ireland. The UK state relates to the many local authorities governing local territories within the United Kingdom. Thus, there is a set of state units which maps to a set of territories. The set of states has a partial ordering induced by the subset structure of the territories.

Although the state has the special function of rule-making for a society, it is the society which is the encompassing entity. In other words the society contains the state. There are two broad processes involving the state: the governing process is what the state does to the society and the representation process is what society does to the state. Within the governing process there is the legislative process, the executive process and the judicial process.

The Governing Process

The governing process takes place over time and involves a multiplicity of actors making individual choices in accordance with their preferences and so the ideas in Chapters 4–6 and in the section 'Game theory' of the present chapter apply. For example, Chapter 5 included a discussion of how the legislative median in the US House of Representatives depended on the distribution of opinion within the two main parties. Another example (Fox, 2006, p. 68) provides an interesting combination of the policy space models of Chapter 5 with the game theory models in the present chapter. In the legislature there is a process of considering and passing a sequence of legislative decisions over time. One model of this would be to have each decision made independently of the others. However, Fox observes that 'it has long been noted that legislators can profit by participating in 'log-rolls' whereby a group of legislators give their support on issues of salience to other legislators in exchange for support on issues of salience to themselves'. Sustained cooperation over time has been explained by the use of repeated game models and spatial models of the political process and Fox builds on this and finds that 'when the dimensionality of the policy space is sufficiently large, parameterizations of the model which do not admit cooperation are rare and atypical. ... Furthermore ... legislative cooperation is possible in a one-dimensional policy space'.

The Representation Process

The representative process, like the governing process, takes place over time and involves a multiplicity of actors making individual choices in accordance with their preferences. Again the ideas in Chapters 4–6 and in the section 'Game theory' of the present chapter apply. For example, Chapter 4 provides a discussion of the appropriate representation of member states in the European Council of Ministers. Another example is Wiseman's (2006) model of the electoral process. The voters' action space consists of possible choices of candidate. The candidates' action space consists of offering a policy position and some other action dimensions. Although voter's utility is not increasing in the policy position, it may be that it is increasing in terms of the other dimensions. Depending on the influence of these other dimensions the utility may or may not have a limit. For example there might be limits to the candidates' budgets. There might be diminishing or zero marginal value where value is either voter utility or number of votes. Wiseman's model is a three-stage game of complete and perfect information played between three actors, each acting in turn. Backwards induction yields the equilibrium actions of each actor.

We denote the actors by A, B and C and the actions by '*a*' and the utility functions by '*u*'. There are three utility functions, one for each actor, giving the utility of that actor's actions as a function of the actions of the other actors.

$$u_{\mathrm{A}}(a_{\mathrm{A}}/a_{\mathrm{B}}, a_{\mathrm{C}})$$

$$u_{\mathrm{B}}(a_{\mathrm{B}}/a_{\mathrm{A}}, a_{\mathrm{C}})$$

$$u_{\mathrm{C}}(a_{\mathrm{C}}/a_{\mathrm{B}}, a_{\mathrm{A}})$$

The last actor, actor C, knows the (earlier) actions of the other actors. So C can choose their optimising action directly. It is a function f of the actions of the others.

$$a_{\mathrm{Cmax}} = f(a_{\mathrm{B}}, a_{\mathrm{A}})$$

The second last actor, actor B, now knows the earlier action of the first actor and also the (conditional) optimising action of the third actor. So they choose their optimising action. It is a function g of the actions of the others.

$$a_{\mathrm{Bmax}} = g(a_{\mathrm{Cmax}}, a_{\mathrm{A}})$$

$$a_{\mathrm{Bmax}} = g(f(a_{\mathrm{B}}, a_{\mathrm{A}}), a_{\mathrm{A}})$$

The optimising actions of B and C can now be expressed as functions of the action of A.

$$a_{\text{Bmax}} = k(a_{\text{A}})$$

$$a_{\text{Cmax}} = q(a_{\text{A}})$$

The first actor now knows the (conditional) optimising action of the second and third actors. So they choose their optimising action. It is a function h of the actions of the others. The optimising actions of the others are themselves functions of the action of A. This gives an implicit function for the action of A from which an explicit function can be formed.

$$a_{\text{Amax}} = h(a_{\text{C}}, a_{\text{B}})$$

$$a_{\text{Amax}} = h(k(a_{\text{A}}), q(a_{\text{A}}))$$

CHAPTER 12

THE MATHEMATICAL ECONOMICS OF SOCIAL PARTICIPATION; COMPLEXITY

> The unpredictable rending of confidence is one reason that recessions are so difficult to forecast ... Our economic models have never been particularly successful in capturing a process driven in large part by non-rational behaviour.
>
> – Alan Greenspan, Chairman of the US Federal Reserve (13th February 2001)

> I have considerable belief in microefficiency of liquid organized markets. I am doubtful about any great macroefficiency. [...] When Franco Modigliani sees a mispricing of GM common and preferred, he and others can make profits doing what corrects that discrepancy ... [but] when, in the late 1970s Professor Modigliani opined that the Dow was below 1,000 "irrationally" ... all he could do was write about it. Arguing with the tape by selling the general index short could be costly, and in any case ineffective, while animal spirits were what they were and analysts' shortcomings were what they were.
>
> – Samuelson (1994, p. 23)

> Even as a general assumption of economics, maximization of income by individuals with independent preferences is too simple. Adam Smith gave that theory a big send-off in The Wealth of Nations, but in the earlier Theory of Moral Sentiments, he asserted forcefully that emulation is the most pervasive of human drives.
>
> – Kindleberger (1989, p. 244)

Economics is about the production and consumption of social activity participation – about the set of participation possibilities, the constraints on these possibilities, the value of these possibilities and the selection of the outcome. Important constraints are time and money. An important type of social participation is the social exchange of quantities of objects, in particular economic exchange where consumers obtain goods and services supplied by producers. The exchange may be governed by a temporal and monetary price. Value, constraint and price drive the demand for and supply of participation. Price dynamics depends on supply and demand. Social participation brings a stream of value and is engaged in due to speculative anticipation of that stream of value. Speculative anticipation may be based directly on relevant information and/or on opinion communicated in a social network. The dynamics of social participation is based on the dynamics of relevant information and the dynamics of social opinion. Social participation in the stock market is discussed in terms of the efficient market hypothesis and the complexity models of multi-agent theories.

MICRO-ECONOMICS

The aim of this section is to develop a model of the linkage between the macro-dynamics of price and the micro-dynamics of individual buyers and sellers, drawing on classical micro-economic theory (Jehle & Reny, 2001). Jehle and Reny's book is in three parts. The first part discusses economic agents, namely consumers and firms. The second part discusses markets, in other words what happens when the economic agents interact; and welfare, namely the social value of the outcome. The third part is on strategic behaviour, covering game theory, information economics and auctions and mechanism design. Here, the emphasis will be on the core concepts rather than on the mathematical details. In emphasising the core concepts, attention will be drawn to the fact that these core concepts have a much wider range than simply micro-economics.

The Individual Consumer and Prices

> There are four building blocks in any model of consumer choice. They are the consumption set, the feasible set, the preference relation, and the behavioural assumption. Each is conceptually distinct from the others, though it is quite common even for economists sometimes to lose sight of the fact. This basic structure is extremely general, and so, very flexible. By specifying the form each of these takes in a given problem, many different situations involving choice can be formally described and analyzed. Although we will tend to concentrate here on specific formalizations that have come to dominate economists' view of an individual consumer's behaviour, it is well to keep in mind that 'consumer theory' per se is in fact a very rich and flexible theory of choice.
>
> – Jehle and Reny (2001, pp. 3–4)

Jehle and Reny start their book with two chapters on consumer theory which as they note is essentially a theory of choice. Their book shares with ours the belief that choice and value are foundational concepts. Choice has already been discussed in a number of our earlier chapters. The focus is the stronger conception of value and choice discussed in Chapter 6 where there is a continuous utility function. What is new here is the technical precision of the concepts and the introduction of the notion of constrained optimisation.

My treatment here will emphasise the theory of choice and use consumer theory as an example. The theory envisages a situation of the following type, which can be conceived of either in terms of preferences or in terms of utility:

There is an individual *i*.
There is a set *X* of options.
There is a preference relation *P* on the set *X*.
  *There is a utility function *u* on *X*.
There is a set *B* of feasible options.
  *There is a constraint function *c* on *X*.
There is a selection $x^\wedge$ from the set *X*.
  *There is a selection function *s* on *X*.

The Preference Relation on the Option Set
The theory is developed by specifying certain properties for the option set and for the preference relation. A preference relation *P* is defined as a binary relation which is complete and transitive. The strict preference relation and the indifference relation are defined.

Much of economics is concerned with a consumer consuming a certain quantity of goods. So there is special interest in the case where the option set is restricted to a subset of R^n_+, namely the positive orthant of *n*-dimensional real space. The option set is closed, convex and contains the null set. The preference relation on R^n_+ is taken to have the following properties: continuity, local non-satiation, strict monotonicity, convexity and strict convexity.

Having set up the concept of preferences, we now consider the relationship between preferences and utility. A real-valued function u: $R^n_+ \rightarrow R$ is called *a utility* function *representing the preference* relation *P* if more preferred options always have greater utilities. It can be shown that any continuous preference relation can be represented by a continuous real-valued function. If a utility function *u* represents a preference relation *P* then any positive monotonic transformation of $f(u)$ also represents *P*. Thus a continuous preference relation corresponds to a class of equivalent utility functions. This is strong enough for much of economic theory but stronger assumptions need to be made to discuss certain topics in social welfare theory – see Chapter 6.

Additional properties of the preference relation give rise to additional properties of the utility function and vice versa: strict monotonicity, convexity and strict convexity in the former give rise to strictly increasing, quasi-concavity and strict quasi-concavity in the latter. Also certain properties in the preference relation give rise to differentiability of the utility function.

One way of understanding the relationship between the preference relation and the corresponding class of utility functions is to consider indifference sets. Different properties of the preference relations give rise to corresponding properties of the indifference sets. The ordering of these difference sets is fixed by the preference relation but the quantification of this ordering is arbitrary – and so there is a class of utility functions.

The concept of marginal rates is important. For any differentiable function u, $\mathrm{d}y/\mathrm{d}x = (\mathrm{d}u/\mathrm{d}x)/(\mathrm{d}u/\mathrm{d}y)$. Interpreting u as utility and x and y as quantities of goods we have the result: the marginal rate of substitution of y for x equals the ratio of the marginal utilities of y and x. If certain properties hold, a law of diminishing marginal utilities holds.

The Selection Function

The selection function is a mapping from the characteristics of the situation to a subset of the option space, possibly a unique element of the subspace. The selection function is an optimising function if the image set consists of the most preferred options with the highest values of the utility function (in much of what follows the image set has just one option).

The Constraint

However not all options in the option set are attainable. This is because there is a constraint. So the individual has to maximise their utility subject to the constraint. This situation receives its most general treatment in the literature on constrained optimisation.

> The general optimization problem treated in this book is to locate from within a given subset of a vector space that particular vector which minimizes a given functional. In some problems the subset of admissible vectors competing for the optimum is defined explicitly …; in other cases the subset of admissible vectors is defined implicitly by a set of constraint relations. [In the latter situation, one can either reduce it to the former situation or one can work with the constraint relations as given.]
>
> – Luenberger (1969, p. 213)

The general problem is to maximise $u(x)$ subject to the constraint $h(x)$ in X and $g(x) \leq c$.

We define a constraint set $A(c)$ consisting of all points x such that $g(x) \leq c$.

Note that if $c_1 < c_2$ then $A(c_1)$ is contained in $A(c_2)$.

We define a utility set $U(u)$ consisting of all points x such that $u(x) \geq u$.

Note that if $u_1 < u_2$ then $U(u_1)$ contains $U(u_2)$.

Thus as c decreases the A get smaller and as u increases the U get smaller.

Thus if either c decreases or u increases the intersection of A and U gets smaller – and may eventually consist of just one point.

We define the unique intersection set Q to be the set of all triples $z^* = (u^*, c^*, x^*)$ such that the intersection of $A(c^*)$ and $U(u^*)$ consists of just one point x^*.

Then each unique intersection triple z^* in Q has the following properties:

It is the solution of the following problem: maximise u such that $g(x) \leq c^*$.

It is the solution of the following problem: minimise c such that $u(x) = u^*$.

The tangent plane to $g(x) = c^*$ is identical with the tangent plane to $u(x) = u^*$, and so at z^* $\mathrm{d}u_i/\mathrm{d}u_j = \mathrm{d}g_i/\mathrm{d}g_j$.

The following functions can be defined and related to certain functions in micro-economics. The first two functions apply throughout the set X. Note that y is the constraint variable, usually money. The remaining four functions apply within the unique intersection set Q. The Marshallian demand function expresses how the amount of demand depends on money (when constrained optimality holds). The indirect utility function expresses how the amount of utility depends on money (when constrained optimality holds). The expenditure function expresses how the amount of money depends on utility (when constrained optimality holds). The Hicksian demand function expresses how the amount of demand depends on utility (when constrained optimality holds). Note that what these expressions do not contain is the constraint parameters, in micro-economics the prices.

$u(x)$	Direct utility function
$y(x)$	Budget constraint
$x^*(y^*)$	Marshallian demand function
$u^*(y^*)$	Indirect utility function
$y^*(u^*)$	Expenditure function
$x^*(u^*)$	Hicksian demand function

The constraint can be some total resources which is available to the individual, for example the total amount of money belonging to the individual or the total time available to the individual. A quantity of each activity is ‘bought’ for a certain amount of resource (money or time) and the rate of this exchange depend on the unit price of the activity. A key point then is that an individual consumer's level of activity is determined by price.

The Individual Firm and Prices

Whereas an individual consumer seeks to obtain a bundle of consumer goods to maximise utility subject to an expenditure constraint, a firm seeks to obtain a bundle of input goods to maximise output subject to a cost constraint. From a mathematical point of view the two situations are identical.

Given a production function $y(x)$ and a cost function $c(x)$ there are optimal triples of inputs, outputs and costs, $\{(x^*,y^*,c^*)\}$. This is cost minimisation.

Given a production function $y(x)$, a cost function $c(x)$ and a profit function $p(x)$ then there are optimal quadruples of inputs, outputs, costs and profits, $\{(x^*,y^*,c^*,p^*)\}$. Profit maximisation entails cost minimisation.

Concepts introduced are marginal product, marginal rate of substitution, etc. A number of concepts did not appear in the discussion of the consumer – but could have done so: separable production functions, the elasticity of substitution, linear homogeneous production functions, average product, output elasticity of input, returns to scale, input share, etc. Other concepts are: short run, total variable cost, total fixed cost; marginal revenue product, output supply functions, input demand functions, etc.

As for the individual consumer, the quantity of each activity is 'bought' or 'sold' for a certain amount of resource (money or time) and the rate of this exchange depends on the unit price of the activity. A key point then is that an individual firm's level of activity is determined by price.

Aggregating Consumer Demand and Firm Supply: Price Dynamics

Up till now we have studied the 'micro' behaviour of individuals – individual consumers and individual firms. The individual actor optimises their behaviour in response to a fixed environment (prices) which is unaffected by the individual's behaviour. We now consider sets of individual actors, their aggregate behaviour, the dependence of aggregate behaviour on price and the dependence of price dynamics on aggregate behaviour. It is important to note that in this model there is no direct interaction between individuals only indirect interaction via prices.

The first thing we can do is to consider aggregates of individual attributes and the mean value of individual attributes. For example, the aggregate demand of goods and the aggregate supply of goods. The aggregate for any one particular good depends in general not only on the parameter(s) for that particular good, such as its price, but also on the parameters for the other goods as well.

Aggregation says nothing about the connectedness of the individuals. One key concept is the notion of the balance between aggregate demand and aggregate supply. There are three possibilities: excess demand, excess supply and equality of demand and supply, 'market clearing'. A situation is said to be in equilibrium if the option selected by each individual in that situation is optimal given that situation. In a situation of perfect competition, market clearing constitutes a short-run equilibrium. Market clearing and zero long-term profits constitute a long-term equilibrium.

Given an individual i, the individual's budget M, the individual's utility function u and the market's price vector p, then the individual's utility reaches a maximum value u^* when some quantity vector q^* of goods is bought. The vector q^* is referred to as the individual's demand for goods at market price p.

$$q^* = q^*(M, u, p)$$

$$u(q^*) = u^*$$

Given a set of such individuals the aggregate demand is defined as the sum of the q^*.

$$Q^{\mathrm{d}}(p) = \sum_i q^*$$

Similarly, given a set of firms the aggregate supply is defined as the sum of the q'^*.

$$Q^{\mathrm{s}}(p) = \sum_i q'^*$$

There are three possibilities. It may be that aggregate demand equals aggregate supply. This is referred to as markets clearing. It means that firms can sell all that they optimally produce at the given prices and that individuals can buy all that they optimally demand at the given prices. The market is in equilibrium in that each agent is able to implement their optimal action (at the given set of prices).

However demand and supply may not be in balance. Demand can exceed supply or supply can exceed demand. A measure which covers all three possibilities is the excess demand which is defined as the difference between demand and supply:

$$z(p) = Q^{\mathrm{d}}(p) - Q^{\mathrm{s}}(p)$$

When demand equals supply $\mathrm{d}Q(p) = 0$. A key question in the literature surrounds the existence of this equilibrium. Another key question is what happens when the system is not in equilibrium. It is this latter question which interests us here. The situation is one in which the market is not in equilibrium ($z(p)\neq 0$) at price p but would be at equilibrium ($z(p) = 0$) at price p^*. Thus demand and supply can come into alignment if price p can come into line with price p^*. A possible dynamics is that price change is driven by excess demand

$$\frac{\mathrm{d}p}{\mathrm{d}t} = az(p)$$

and that excess demand can at least be approximated as proportional to the deviation of the price from equilibrium.

$$\frac{\mathrm{d}p}{\mathrm{d}t} = b(p^* - p)$$

$$z(p) = b(p^* - p)/a$$

FINANCIAL ECONOMICS AND THE EFFICIENT MARKET HYPOTHESIS

International Macro-Economics

We now introduce some ideas from international macro-economics as a prelude to a discussion of financial economics. International macro-economics, open-economy macro-economics and international finance are to some extent interchangeable terms.

> One fundamental way open and closed economies differ is that an open economy can borrow resources from the rest of the world or lend them abroad ... Resource exchanges across time are called intertemporal trade. Much of the macroeconomic action in an open economy is connected with its intertemporal trade ...
>
> – Obstfeld and Rogoff (2002, p. 1)

In representative agent models, the nation is regarded as a single entity, making its decisions according to its utility function. Given the above-noted importance of inter-temporal trade, we are interested in inter-temporal preferences and utility functions. Typically a particular form is assumed: 'the assumption of inter-temporally additive preferences with an unvarying period utility function will form the backbone of our formal analysis' (Obstfeld & Rogoff, 2002, p. 13; hereafter denoted 'OR').

The concept of inter-temporal utility is as follows. Consider the utility function $u(\boldsymbol{x})$ of a vector $\boldsymbol{x}$. The utility function is said to be separable if $u = \sum_i a_i u(x_i)$. In particular this applies to a temporal sequence of values of x. A particularly simple form has a constant subjective discount or time preference factor b, $b<1$, giving $u_t = \sum_{s=t}^{\infty} b^{s-t}u(x_s)$. If there is uncertainty then what is of interest is the expected utility,

$$E(u_t) = E\left(\sum_{s=t}^{\infty} b^{s-t}u(x_s)\right)$$

$$u_{t+1} = \sum_{s=t+1}^{\infty} b^{s-(t+1)}u(x_s)$$

$$\begin{aligned} u_{t+1} - u_t &= \sum_{s=t+1}^{\infty} [b^{s-(t+1)} - b^{s-t}]u(x_s) - u(x_t) \\ &= \sum_{s=t+1}^{\infty} b^{s-(t+1)}[1-b]u(x_s) - u(x_t) \\ &= [1-b]u_{t+1} - u(x_t) \end{aligned}$$

$$u_{t+1} = [u_t - u(x_t)]/b$$

Chapter 1 of Obstfeld and Rogoff is entitled 'Inter-temporal trade and the current account balance' and starts by considering a small two-period endowment economy. An individual maximises utility over the two-period lifetime by arranging consumption so that the following inter-temporal Euler equation is satisfied (OR, (3), p. 3):

$$u'(c_1) = (1+r)bu'(c_2)$$

where $u'(c_1)$ and $u'(c_2)$ are the derivatives of utility with respect to consumption, b a fixed preference parameter, called the subjective discount or time-preference factor and r the real interest rate for borrowing or lending in the world capital market.

Chapter 2 of Obstfeld and Rogoff discusses small open economies and includes a discussion of firms, the labour market and investment, in particular focusing on the investment behaviour of the representative consumer. As well as the earlier equation holding, the following equation also holds where v_t and d_t are the market value and dividend at time t (OR, (52–53), p. 101):

$$v_1 u'(c_1) = (v_2 + d_2) b u'(c_2)$$

Combining the two equations 'we see that under perfect foresight, consumers will be indifferent on the margin between foreign assets and shares provided the gross rate of return on shares equals the gross real interest rate' (OR, (53), p. 101):

$$1 + r = (v_2 + d_2)/v_2$$

From this can be derived an equation which shows that a firm's current market value is the present discounted value of the dividends it will pay shareholders over the future. This result assumes that there are no self-fulfilling speculative asset-price bubbles. (OR, (53–57), p. 102)

This latter point is discussed more fully in Appendix 2B (pp. 121–124) which covers speculative asset price bubbles, Ponzi games and transversality conditions. 'some fairly compelling arguments rule out speculative bubbles in the class of infinite-horizon models studies in this chapter ... [these arguments] are related to the transversality requirements for optimality at the level of individual decision makers. However, the appendix to Chapter 3 shows that there are alternative models in which speculative bubbles can arise. Ultimately, therefore, one must appeal to empirical as well as theoretical arguments to rule out bubbles entirely.' Appendix 3A (pp. 191–195) discusses dynamic inefficiency and notes that 'assets without intrinsic value may trade at strictly positive prices in a dynamically inefficient economy' (p. 194). What is being discussed here is financial economics, a topic we now turn to.

Financial Economics

Cuthbertson (1996, p. xiii) provides an introduction to 'some of the theories and empirical methods used by financial economists in the analysis of speculative assets prices in the stock, bond and foreign exchange markets. ... The baseline paradigm throughout the book is the efficient market hypothesis (EMH). If stock prices always fully reflect the expected discounted present value of future dividends (i.e. fundamental value) then the market will allocate funds among competing firms, optimally'. Part 2 of the book covers three broad views about financial markets. As well as discussing the EMH, there is a discussion of rational bubbles and of noise traders.

The Change in the Monetary Value of an Investment over Time
Consider an individual at time t with a sum of money M_t. Suppose that this money is invested and that the investment has monetary value M_{t+1} at time $t+1$. The relationship between the monetary values at times t and $t+1$ can be expressed in a number of ways – see below. Note that return, rate of return and growth factor are each an increasing function of M_{t+1}, and so maximisation of wealth entails maximisation of each of the three factors. Note that the d factor is simply the inverse of the growth factor g; and that the relationship between r and d is $d = 1/(r+1)$.

Return		$M_{t+1} - M_t$
Rate of return	r	$(M_{t+1} - M_t)/M_t$
Growth factor	g	M_{t+1}/M_t
d factor	d	M_t/M_{t+1}

Let us assume that r, and hence d, are constant over time. Then

$$M_t = d^n M_{t+n}$$

Suppose that the sum of money M_t is used at time t to buy stock of value $V_t = M_t$. Suppose that at time $t+1$ the stock now has value V_{t+1} and that a dividend worth D_{t+1} is paid. The total wealth at time $t+1$ is therefore $M_{t+1} = V_{t+1} + D_{t+1}$. So we obtain the following equation which is identical with (OR, (53), p. 101) in section 'International macro-economics'.

$$V_t = d(V_{t+1} + D_{t+1})$$

A similar expression holds for V_{t+1} which can then be substituted back in the earlier equation. And this process can be continued to n terms and to infinity.

$$V_t = d(d(V_{t+2} + D_{t+2}) + D_{t+1})$$

$$V_t = d^n V_{t+n} + \sum_{i=1}^{i=n} d^i D_{t+i}$$

$$V_t = B_t + A_t$$

where

$$B_t = \lim[d^n V_{t+n}] \text{ and } A_t = \lim[\sum_{i=1}^{i=n} d^i D_{t+i}]$$

Suppose now that future values and dividends are not known with certainty – but that it is known with certainty that d is constant. Then, taking expectations at time t, we have

$$V_t = d(E_t V_{t+1} + E_t D_{t+1})$$

$$V_t = d^n E_t V_{t+n} + \sum_{i=1}^{i=n} d^i E_t D_{t+i}$$

$$V_t = E_t B_t + E_t A_t$$

where

$$E_t B_t = \lim[d^n E_t V_{t+n}] \text{ and } E_t A_t = \lim\left[\sum_{i=1}^{i=n} d^i E_t D_{t+i}\right]$$

There are two cases to consider. Case I is where $E_t B_t = 0$ (the so-called transversality condition) and Case II, the general case, where $E_t B_t > 0$. Case I is in accord with the efficient market hypothesis and Case II leads to the rational bubble hypothesis.

[Note that the equations can be divided by the number of shares to represent the value per share and the dividend per share.]

Empirical Challenges to the Efficient Market Hypothesis

For a long time the dominant theory of speculative markets has been the efficient market hypothesis. As we have just seen, according to this theory, rational traders use all relevant information to form expectations about future prices and buy and sell accordingly. This brings current prices into line with future expectations. So the only price change possible is one that is random. However, commentators have found it difficult to reconcile the theoretical assumption of rationality with the behaviour of real stock markets – see the quotations which opened this chapter.

The rationality postulated in the efficient market hypothesis has been challenged by a variety of empirical findings: excess aggregate volatility in the stock market, excess dispersion, the equity premium puzzle, the concentration of portfolios and the overpricing of initial public offerings. Benartzi and Thaler (1995) suggest that the equity premium puzzle arises because investors have loss aversion, and so fail to maximise their expected earnings, their frequent 'myopic' evaluation increasing the experience of loss. Morris (1996) suggests that the overpricing of initial public offerings is a consequence of the variety of information-free priors which investors use to evaluate the initial offering. Bulkley and Harris (1997) suggest that excess dispersion is due to forecasters using an incorrect model. Excess aggregate volatility has been explained in terms of the interaction between heterogeneous traders – for example between sophisticated traders and naïve traders – or noise traders (references to this literature are contained in Lux, 1995). Shiller (1984) and Kindleberger (1989), in particular, have been influential in establishing the view that irrational 'animal spirits' affect stock market prices. Opinions about prices are formed as a result of communication in a social network (see particularly Shiller & Pound, 1989).

COMPLEXITY THEORY MODELS OF FINANCIAL MARKETS

In view of these challenges to the efficient market hypothesis Kirman (1999) has argued that rather than the standard model of social interaction solely via the price system 'a more promising avenue is to look at the economy as an interactive system in which agents interact directly'. The past couple of decades has seen a growing literature on models of financial markets viewed as a system of interacting agents giving rise to the various phenomena discussed in complexity theory. Lux and Marchesi (1999, p. 498) note that

> financial prices have been found to exhibit some universal characteristics that resemble the scaling laws characterizing physical systems in which large numbers of units interact. This raises the question of whether scaling in finance emerges in a similar way – from the interactions of a large ensemble of market participants. ... [The authors] describe a multi-agent model of financial markets which supports the idea that scaling arises from mutual interactions of participants.

'Herd Behaviour, Bubbles and Crashes'

This section provides an overview of the model of the stock market proposed by Lux (1995). We have already discussed aspects of the model in Chapter 7 and here we provide more details. The model has two components, a price dynamics and an opinion dynamics, the first interacting with the second. There are two types of traders: fundamentalists who know the true price p^* and buy or sell accordingly; and noise traders who are moved by the prevailing opinion to buy or sell. The proportions of the two types of traders are denoted f and n respectively.

In the simplest of his three models he envisages naïve traders in the stock market changing their opinions purely on the basis of other traders' opinions. The model can be thought of as consisting of an external communication stage which registers the mean opinion $x^\wedge$ followed by a response stage where individuals adjust their opinion on the basis of the mean opinion. The response stage for each individual is probabilistic: $p(x(t+1)|x(t)) = f(x(t+1), x(t), \exp(x^\wedge), v)$. The parameter v represents the speed of opinion change. Opinion in this case concerns optimism, whether the market price is about to go up ($x = +1$), or pessimism, whether it is about to come down ($x = -1$). The opinions of individual naïve traders can change from optimism to pessimism and vice versa.

The dynamics of $x^\wedge$ are derived by Lux from the transition probabilities between the opinion states. Using the probabilities Lux applies the theory of synergetics to obtain an equation of the form given below.

$$\mathrm{d}x^\wedge/\mathrm{d}t = 2v[\mathrm{Tan}h(z) - x^\wedge]\mathrm{Cos}h(z)$$

Note that in the first of his models, $z = ax^\wedge$. The dynamics of this equation have been discussed in Chapter 7. Although the first of his models is a pure socialisation model involving only a process of contagion of opinion, his second and third models include an adaptive feedback loop from the situation on the stock market. This is captured by the criterion parameter z in the equation. In Lux's second model, z depends not only on mean opinion z but also on the change in prices $\mathrm{d}p/\mathrm{d}t$; and in his third model, z includes price p and price change $\mathrm{d}p/\mathrm{d}t$ as indicators of real returns.

The price dynamics are driven by the demands of the noise traders and the fundamentalists weighted by their proportions. The noise traders pressing for their current opinion and the fundamentalists acting on the deviations between the current price and the fundamental price

(Lux, 1995, Eq. (8), p. 889).

$$\mathrm{d}p/\mathrm{d}t = b[nx^{\wedge} + f(p^{*} - p)]$$

Lux (1995, p. 889) comments on the complex dynamics of the third model: 'taking the randomness which has been suppressed in our derivations into account, the system may undergo transitions between cyclic behaviour and stable steady states leading to an erratic appearance of the overall evolution of prices'. There are basically two forces at work here: a balancing force which tends to even out opinion, and a herding force which reinforces any imbalance which happens to occur. What happens to mean opinion depends on the relative strengths of these two forces. If the balancing force is strong enough relative to the herding force, then there is a single stable central equilibrium with naïve traders split equally between optimism and pessimism. If the herding force dominates the balancing force, then the central equilibrium becomes unstable and there is motion towards one or other of the two stable extremal equilibria: either optimism or pessimism is predominant. (These are the most likely situations. Owing to the stochastic nature of the model there can be variations from these, in particular a jump from one extreme equilibrium to the opposite extreme). Note that the complex behaviour in this model arises from the opinion contagion dynamics and not from the price dynamics. Complexity of price dynamics are discussed by Day and Pianigiani (1991) and Gu (1993).

A Comparative Evaluation of Competing Models of Financial Markets

We have now discussed the classical economics model of financial markets and also a complexity theory model. A useful review of where things currently stand is provided by Lux (2006) (see also Lux & Ausloos, 2002). His arguments can be summarised as follows. The starting point is the empirical data: the time series for each of a variety of financial measures. The data are found to have the properties associated with complexity theory: a cubic power law for large returns; long-range dependence in volatility; temporal scaling of trading volume and multi-scaling of higher moments of returns. A variety of models have been proposed in explanation of these properties of the data. One type of model simply suggests that the time series for returns are determined by the structure of news about fundamentals and hence the properties of the former are determined by the properties of the latter. Other models focus on 'the intrinsic dynamics of speculative interaction in financial markets'. Lux identifies four classes of such models and in each case notes the source of the observed power laws and identifies certain problems in each model. The four classes are: rational expectations bubbles, percolation models, multi-agent models (associated with Lux himself) and discrete choice models. Thus the chapter links together empirical data, properties of the data and models generating the properties. We now proceed to more of the detail on Lux's argument.

What Patterns are Exhibited by the Data?

The first substantive section of Lux's chapter is entitled 'empirical power laws in finance'. The fundamental question here is: what patterns are exhibited by the data? Here the data consists of financial time series. The focus is on the time series for prices $\{p_t\}$ and this basic data is transformed into a time series for 'returns', $\{R_t\}$, where $R_t = \log(p_t/p_{t-1})$, in other words a measure of price changes.

What pattern is exhibited by the distribution of returns? A natural thought might be that the data has a normal distribution – but this is not so. Certain features of the data suggest that the observed distributions are Paretian or Levy stable distributions. However other features of the data do not conform to these distributions. In particular, many studies find that the cumulative density function of returns converge in the tails in a manner which conforms to a power law. Moreover the exponent α in this power law consistently has values approximately 3.

$$P(R > x) \approx x^{-\alpha}$$

The fluctuations ('volatility') in a time series are also of interest. One measure of volatility is the absolute magnitude of the returns, $|R_t|$. Taking the product of two successive measures leads us to the auto-covariance function. This function also exhibits a power law, with the exponent γ typically between 0.2 and 0.3.

$$E[|R_t|.|R_{t-\Delta t}|] \approx \Delta t^{-\gamma}$$

Any power q of the absolute returns also provides a measure of volatility. In this way we can obtain a sequence of measures of volatility. There is some evidence that these measures also exhibit a power law, what Lux refers to as the 'multi-scaling in the temporal dependence structure of financial fluctuations'. Lux comments 'most excitingly, a non-linear dependence of the scaling parameter on the power q is also a key characteristic of turbulent fluids and has motivated the development of so-called multi-fractal models in statistical physics.'

$$E[|R_t|^q.|R_{t-\Delta t}|^q] \approx \Delta t^{-\gamma(q)}$$

Lux emphasises that the equations have application. They can be used to assess the risk of extreme events and to predict price fluctuations.

What Statistical Models would Generate the Patterns Found in the Data? Having identified certain patterns in the data, one can then seek models which would generate the patterns. The simplest model would be a random process with normally distributed errors. However this would yield a normal distribution of returns, contrary to the pattern found in the data.

$$R_t = \sigma \varepsilon_t$$

An extension of this model is the (G)ARCH (Generalised Autoregressive Conditional Heteroskedasticity) class of processes in which the variance varies with time and shows autoregressive dependence.

$$R_t = \sigma_t \varepsilon_t$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^l$$

What Theory-Based Models would Generate the Patterns Found in the Data? The rational actor theories leading to the efficient market hypothesis and the more complex rational bubbles model have already been discussed in the section 'Financial economics and the efficient market hypothesis'. In the simpler model the structure of the time series for returns reflects the structure of the time series for the 'news process'. And this provides a direct explanation for the pattern of returns. In the more complex model we need an account of the bubble dynamics. A fairly general class of processes which do so are the multiplicative stochastic processes. These are 'generic power-law generators'. However Lux suggests that the exponent in the power law is less than one and so inconsistent with the patterns in the data. Lux (2006, p. 12) concludes that 'it seems that economists have to accept deviations from the ideal case of perfect rationality'.

$$B_t = a_t B_{t-1} + \varepsilon B_t$$

Lux compares a range of models seeking to explain the power laws which are observed in financial data. He summarises the position in the following way. According to traditional finance models the source of power laws is to be found in fundamentals – the problem with this is that fundamentals are unobservable. According to another exogenous model the source of power laws is to be found in Zipf's law for investment capital – the problem with this is the neglect of behavioural roots and of empirical validity. According to the rational expectation bubble model the source of power laws is to be found in multiplicative dynamics – the problem with this is that the data indicates sizes of exponents which are unrealistic. According to percolation models the source of power laws is to be found in cluster formation – the problem with this is that the power laws are not robust. According to multi-agent models the source of power laws is to be found in intermittent dynamics – the problem with this is the sensitivity to the number of agents. According to discrete choice models the source of power laws is to be found in switching between attractors – the problem with this is the sensitivity to noise amplitude.

CHAPTER 13

LIFE AND HISTORY: THE SPECULATIVE PURSUIT OF VALUE

> To travel hopefully is a better thing than to arrive, and the true success is to labour.
>
> – Stevenson, El Dorado; cited by Hyman (1962, p. 307)

> We hold these truths to be self-evident: that all men are born equal; that they are endowed by their Creator with certain inalienable rights; that among these are life, liberty and the pursuit of happiness.
>
> – Declaration of American Independence (4 July, 1776); cited by Hyman (1962, p. 139)

> ours is a society in which one of the principal social goals is a higher standard of living ... [This] has great significance for the theory of consumption ... the desire to get superior goods takes on a life of its own. It provides a drive to higher expenditure which may even be stronger than that arising out of the needs which are supposed to be satisfied by that expenditure.
>
> – Duesenberry (1949, p. 28)

The history of society and the lives of people are characterised by the speculative pursuit of value. The phenomenon goes beyond our own society and beyond the pursuit of a higher standard living – indeed beyond the pursuit of happiness. The starting and end points and the process of pursuit itself may have positive or negative value. Investigations of social well-being have been carried out for almost half a century. Although large differences in real income relate to differences in well-being, the relationship is weak for smaller differences in real income. Moreover, in rich countries, despite decades of growth in real income there has been little or no corresponding increase in social well-being. Contributions to social well-being also come from non-economic sources. Participation in high-value locations of the social structure such as being married or being unemployed makes significant contributions to social well-being. Utility functions may be based on comparisons which are temporal or social (or both) and this can be counter-productive. The findings of this chapter in conjunction with the findings of earlier chapters identify a variety of limitations on the truth and the value which people experience in their lives. This necessarily makes people's pursuit of value speculative. Society places differential speculative values on the array of social activity projects and these values have a complex dynamics.

INVESTIGATIONS OF SOCIAL WELL-BEING

> Taken all together, how would you say things are these days – would you say that you are very happy, pretty happy or not very happy? (USA General Social Surveys Question 157)

> On the whole, are you very satisfied, fairly satisfied, or not at all satisfied with the life you lead? (Eurobarometer Survey Series)

For at least half a century questions such as the above have been used to investigate social well-being. In an authoritative review of three decades of well-being research, Diener, Suh, Lucas, and Smith (1999) suggests that social well-being (SWB) is a general area of scientific interest rather than a single specific construct and that the major divisions are pleasant affect, unpleasant affect, life satisfaction and domain satisfactions. The distinction between these divisions is in part one of the scope: affect refers to a single instantaneous reaction to a single contemporary situation; life satisfaction refers to an individual's complete trajectory; and domain refers to one subset of dimensions. Typically a well-being reaction will be based on an aggregation over aspects and over time – thus the use of the phrases 'taken all together', 'on the whole' and 'the life you lead' in the above two questions implies an aggregation over all aspects; and the phrase 'these days' implies an aggregation over some recent – but unspecified – period of time.

What are the domains which might contribute to value (or happiness, or satisfaction, or well-being)? Prompted partly by previous chapters, value might depend on one's biological state, one's psychological state, one's social state, one's cultural state, one's economic state and one's political state.

ECONOMIC GROWTH AND SOCIAL WELL-BEING

When we compare different countries greater happiness is associated with greater real income: in their study of 55 nations, Diener et al. (1995) found a correlation of 0.6 between income and social well-being. However, if we restrict our attention to the richest countries and compare individuals 'the amount of happiness bought by extra income is not as large as some would expect' (Blanchflower & Oswald, 2000, pp. 10 and 11). Even more surprising is the finding about changes to well-being over time. Oswald (1997) has drawn the attention of economists to the seminal work of Easterlin (1974) who was 'one of the first social scientists to study data over time on the reported levels of happiness in the United States and who found that higher income [over time] was not systematically accompanied by greater happiness'. Similarly the subsequent three decades of substantial increases in real income brought no increase in well-being (Blanchflower & Oswald, 2000).

These empirical results are not easily squared with orthodox consumer theory. There utility u is an increasing function of the quantity q of goods possessed. One possible resolution might be to postulate a law of diminishing returns which says that $\mathrm{d}u/\mathrm{d}q$ is a decreasing function tending to zero – and that the richer countries have reached the point where $\mathrm{d}u/\mathrm{d}q$ is almost zero.

There are alternative explanations. For over a century economists have identified forms of consumption which would seem to be at odds with orthodox consumer theory. Veblen (1899) introduced the concept of conspicuous consumption; later authors noted the insatiability of 'second class' needs to keep abreast or ahead of one's fellow being; Galbraith (1958) referred to a 'dependence effect' whereby wants are increasingly created by the process by which they are satisfied; Hirsch referred to positional goods and the social limits to growth (*Veblen, 1899; *Duesenberry, 1949; *Galbraith, 1958, updated 1969, 1976, 1998; Hirsch, 1976; see also Layard, 1980; Frank, 1997; *extracted/cited in Schor & Holt, 2000). Thus, individuals may have counter-productive utility functions.

Indeed counter-productive utility functions may be forced on individuals. Citing Adorno and Horkheimer (1944) and Galbraith (1958), Schor and Holt (2000) ask: what drives consumer society? – do corporations determine consumer wants through their marketing, or do consumer wants determine production? This process may not be restricted to the economic sphere. Much of the communication received by individuals comes from the social activity structure, from public institutions and organisations – the media, companies, government, religious institutions, etc. – and in some cases these sources are very powerful. The core idea is that powerful institutions form opinions which they then propagate among individuals who then incorporate the opinions into their utility functions.

PARTICIPATION IN THE SOCIAL STRUCTURE AND SOCIAL WELL-BEING

> One of the interesting conclusions from an economist's point of view is how influential non-financial variables appear to be in human welfare. (Blanchflower & Oswald, 2000, pp. 10 and 11)

In an earlier chapter I portrayed life as a journey through the social activity structure. It is interesting then to find that participation in this structure is associated with well-being. Being married and being in employment are two of the strongest factors associated with well-being (Blanchflower & Oswald, 2000). Frey and Stutzer (2000) find that 'a higher extent of direct political participation possibilities can be expected to raise citizen's subjective well-being'. In a study of 55 nations, Diener et al. (1995) found a correlation of 0.8 between 'individualism' and social well-being. So changes in participation are associated with changes in well-being. Reduced membership of valued institutions or groups (and consequent increased membership of less valued institutions or groups) might be expected to depress the overall level of well-being.

Relevant to the findings for changes in well-being over time is the fact that participation in the social institutions or groups has changed over time. The proportion of married people in their sample falls from 67% in the United States and 72% in the United Kingdom in the early/mid-1970s to 48% in the United States and 55% in the United Kingdom by the late 1990s (Blanchflower & Oswald, 2000). Generally, there is evidence of declining social participation. Putnam (2000) is '... a minutely documented catalogue of social disengagement of virtually every kind: political apathy, retreat from church attendance, eroding union membership, the decline of bridge clubs and dinner parties, volunteering and blood donation' (Ehrenhalt, 2000). Participation is particularly poor for certain groups in certain areas – attachment to the labour force is low among residents of such areas, but so too is their attachment to marriage, school and obeying the law.

The well-being productivity of particular situations may also have changed. Rising real incomes might be expected to contribute to the well-being productivity of the workplace and of any locations where income is consumed.

Change in the well-being productivity of particular situations can also arise through a reshuffle, namely a changing occupancy even in the case where there is no change in frequency of occupation. For example, a change in the political administration with some politicians losing office and being replaced by other politicians. In general, occupants may change places in a league table. This is a zero-sum game if the gains of the winners are balanced by the losses of the losers. Even worse, experiments suggest that losses may outweigh gains. More competitive times may prompt more frequent and more substantial re-shuffles and thereby incur heavier reshuffle loss.

There may also be an interaction between the well-being of a particular situation and frequency of participation. This may arise from social superiority/inferiority utility or social conformity utility (see next section). As the frequency of participation increases from 0% to 100% the social conformity utility increases – whereas the social superiority utility decreases in the case of a superior location but increases in the case of an inferior location.

The introduction of new locations allows the possibility of gaining the value of the new location. To the extent that this is frequency-dependent then earlier occupancy will be more valuable than later occupancy. However, the value of new locations is especially speculative. The literature on new offerings on the stock market is instructive – as is the literature on technology innovation.

TEMPORAL AND SOCIAL COMPARATIVE UTILITY FUNCTIONS

Utility functions may be based on comparisons which are temporal or social (or both). Social comparisons may place value on being superior to, being the same as or being different from others.

An example of temporal comparison is Fuhrer (2000). He develops a habit formation model in which current consumption is valued relative to previous consumption. Layard (1980) considers a variety of utility functions in which an individual values their own income relative to the income distribution. Abel (1990) explores the equity premium puzzle using a utility function that nests three classes of utility functions: (1) time-separable utility functions; (2) catching up with the Joneses utility functions relative to the lagged cross-sectional average level of consumption and (3) utility functions that display habit formation.

Consider an individual i with budget B_i choosing a basket of goods $\{x_{ij}\}$ priced $\{p_j\}$, and so $B_i = \Sigma_j p_j x_{ij}$. To find the basket which maximises utility u_i we form the Lagrangian $z_i = u_i - LB_i$, take partial derivatives and set to zero to obtain $(\partial u_i/\partial x_{ij}) = Lp_j$.

The utility is separable if $u_i = \Pi_j u_{ij}$, where each u_{ij} is a function of x_{ij} only. In this case $(\partial u_i/\partial x_{ij}) = (\partial u_i/\partial u_{ij})(\partial u_{ij}/\partial x_{ij}) = (u_i/u_{ij})(\partial u_{ij}/\partial x_{ij}) = Lp_j$. So $(\partial u_{ij}/\partial x_{ij}) = Lp_j u_{ij}/u_i$.

In an autonomous utility function, utility depends only on the individual's own basket of goods: $u_i = f(\{x_{ij};\ i \text{ fixed}\})$. For example suppose that

$u_{ij} = U_{ij}x_{ij}^{m_{ij}}$, where U_{ij} is independent of x_{ij}. So $(\partial u_{ij}/\partial x_{ij}) = U_{ij}m_{ij}x_{ij}^{m(ij)-1} = m_{ij}u_{ij}/x_{ij}$. Combining this with the result of the previous paragraph we have $m_{ij}/x_{ij} = Lp_j/u_i$. Rearranging we obtain $p_jx_{ij} = m_{ij}u_i/L$. Applying this to the budget constraint we obtain $B_i = \Sigma_j p_j x_{ij} = u_i\Sigma_j m_{ij}/L$. Dividing these two equations we obtain $p_jx_{ij}/B_i = m_{ij}/\Sigma_j m_{ij}$. In other words:

Result 13.1. The proportion of the budget spent on a good equals the utility productivity of the good as a proportion of the sum of the utility productivities.

We now consider the first type of comparative utility: social superiority. What is valued is the quantity x_{ij} possessed by the individual relative to some standard amount $x_j^\wedge$ such as the mean of the quantities possessed by all the individuals: $x_{ij}/x_j^\wedge$. (Note that utility is monotonically increasing in x_{ij} for fixed $x_j^\wedge$.) Here we use the geometric mean: $x_j^\wedge = (\Pi_i x_{ij})^{1/n}$.

Now consider a composite utility which has autonomous and superiority components weighted exponentially by a and b respectively. We have $u_{ij} = x_{ij}^{a(ij)}(x_{ij}/x_j^\wedge)^{b(ij)} = x_{ij}^{a(ij)+b(ij)-1/n}/x_{j-i}^\wedge$, where $x_{j-i}^\wedge = x_j^\wedge/x_{ij}^{1/n}$ does not involve x_{ij}. For large n the exponent of x_{ij} becomes $m_{ij} = (a_{ij} + b_{ij})$. Applying Result 13.1 we have $p_jx_{ij}/B_i = (a_{ij} + b_{ij})/\Sigma_j(a_{ij} + b_{ij})$.

Social superiority utility can engender a prisoner's dilemma game. In the absence of cooperation, people will allocate their budget towards goods which have higher social superiority utilities. With cooperation present, people can agree mutual reductions in their spending on such goods, enabling more to be spent on goods which each individual values independently of others' possession of that good, thereby enhancing utility.

Consider the simple case of just two goods. The first good is valued quite independently of other people while the second good is valued purely in comparison to other people. So $u_{i1} = x_{i1}^{a1}$ and $u_{i2} = (x_{i2}/x_{i2}^\wedge)^{b2}$, giving $u_i = x_{i1}^{a1}(x_{i2}/x_{i2}^\wedge)^{b2}$. Suppose that competition for superiority merely has the effect that all people buy the same amount of the second good. So $x_{i2} = x_{i2}^\wedge$. So $u_i = x_{i1}^{a1}$.

Result 13.2. In the situation described, utility increases with the proportion of the budget spent on the autonomous good and decreases with the proportion of the budget spent on the social superiority good. Utility is maximised by spending the whole budget on the autonomous good.

We now consider the second type of comparative utility: social conformity. What is valued is how close the individual's quantity x_{ij} is to the social norm x_j^*. (The social norm may simply be the mean.) Thus, social conformity utility is a single-peaked function of quantity bought whereas social superiority utility is an increasing function of quantity bought. In order to illustrate social conformity utility we shall use beta functions because they are single peaked and because the product of beta functions is again a beta function with a single peak which is a weighted sum of the component single peaks. Hence:

Result 13.3. If utility is the product of autonomous and social conformity components then utility has a peak at a point which is the weighted sum of the autonomous peak and the social conformity peak. The weights reflect the relative peakiness of the autonomous and social conformity components.

The points in the previous paragraph are demonstrated as follows. Consider the beta function $B(r,s) = x^r(1-x)^s$ over [0,1]. This has a maximum value at $x = r/(r+s)$. The product B^* of beta functions is a beta function $B^*(r^*,s^*) = x^{r^*}(1-x)^{s^*}$ over [0, 1], where $B^*(r^*,s^*) = \prod_j x^{r_j}(1-x)^{s_j} = x^{\Sigma r_j}(1-x)^{\Sigma s_j}$, and so the maximum is at $x^* = r^*/(r^*+s^*) = \Sigma_j r_j/\Sigma_j(r_j+s_j)$. This is a weighted sum of the component maximum points: $x^* = \Sigma_j r_j/\Sigma_j(r_j+s_j) = \Sigma_j(r_j/(r_j+s_j))((r_j+s_j)/\Sigma_j(r_j+s_j)) = \Sigma_j x_j w_j$ where $x_j = (r_j/(r_j+s_j))$ and $w_j = ((r_j+s_j)/\Sigma_j(r_j+s_j))$. Note that the w_j are a measure of the relative peakiness.

The three components of this model correspond to the three components of Kuran's (1995) model of private and public preferences. Kuran writes:

> I had just immersed myself in modern political economy, having spent my student years studying economic development and microeconomic theory. It struck me as a weakness of the literature that it generally failed to recognise, let alone explain and interpret, that some issues are more open to discussion, and some viewpoints better tolerated, than others. (1995, p. x)

Kuran expresses the composite utility of a response y as the sum of three components of utility: private, reputation and expressed autonomy. The private utility is the value to the individual of the social decision d which depends on his/her publicly expressed choice. The reputational utility is the value to the individual which arises from the reputational consequences of his/her publicly expressed choice. The expressed autonomy utility is the value to the individual which arises from publicly expressing a choice which is close to his/her own privately held value. We suggest that autonomous and social conformity utilities can be considered to correspond to Kuran's intrinsic and reputational utilities respectively. We further suggest that expressed autonomy utility corresponds to the relative peakiness of autonomy as against social conformity.

THE LIMITATIONS ON TRUTH AND THE SPECULATIVE PURSUIT OF VALUE

The findings of this chapter in conjunction with the findings of earlier chapters identify a variety of limitations on the truth and the value which people experience in their lives. This necessarily makes people's pursuit of value speculative.

There are limitations on truth. The discussion in Chapter 5 of theory, evidence and reality indicated the problems of using evidence to establish the truth about reality – what Cook and Campbell (1979) refer to as threats to valid inference. The same point is made by Power (1997, pp. 27–31) when he talks of the 'essential obscurity' of the financial auditing process, arguing that the 'cost-assurance relation [for auditing] is ultimately inscrutable'. Chapter 7 noted how the trajectory of some dynamic models can be complex and difficult to predict and Chapter 12 provided an instance of this in the complexity and unpredictability of financial markets.

There are limitations on value. In Chapter 9, on models of individual choice, it was noted that the option which is chosen may not be the best option even though the chooser thinks so at the time. The possible sources of limitation on the value of the chosen option are: the individual, the situation, the set of options, the value function, the valuation of options, the option selection and the experienced, recalled and reported value. In Chapters 4–6 on social choice, each identified limitations. The likelihood of success and the likelihood of power depend on the social choice function; and both likelihoods decrease as the number of individuals and the number of options increases. If the outcome is the mean ideal then a certain definition of power shows an individual's power to decline as the size of the population increases. There are tensions between competing options. The provision of more than one option allows some relaxation of these limitations and tensions. Sub-optimal social value can arise as a result of a sub-maximal value of the best option, population variation in ideals, the distance of the provided option from the best option and sensitivity to deviation from the ideal.

In the section 'Temporal and social comparative utility functions' of the present chapter we have just seen that utility functions may be based on comparisons which are temporal or social (or both) and this can be counter-productive. The arguments can be applied to two different situations. If the well-being function and the utility function are identical then the utility function is effective in delivering well-being, but the well-being function may be counter-productive because of the problems associated with social superiority and social conformity. If the well-being function and the utility function are not identical then the utility function is ineffective – and particularly so perhaps when associated with social superiority and social conformity. Suppose the utility function weights the social components relatively higher than does the well-being function. The individual acts on the basis of the social components only to find that their well-being reaction is driven by the personal components. This may be particularly the case when the personal component is uncertain and the social component seems assured.

The point is worth emphasising. It is often when the truth is unknown that social communication and social opinion exchange is at its strongest. The result is that speculative social-opinion-driven pursuit of value takes place. The situation is analogous to that in financial markets and so the models discussed in Chapter 12 are likely to be relevant. Society places differential speculative values on the array of social activity projects and these values have a complex dynamics.

CHAPTER 14

WORLD HISTORY: THE GROWTH AND DISTRIBUTION DYNAMICS OF POWER, TRUTH AND VALUE

> [the] growth in the number and magnitude of harms humans inflict on the natural environment
>
> – Mitchell (2002, p. 501)

> The implications of the vast differences in standards of living over time and across countries for human welfare are enormous. The differences are associated with large differences in nutrition, literacy, infant mortality, life expectancy, and other direct measures of well-being. And the welfare consequences of long-run growth swamp any possible effects of the short-run fluctuations that macroeconomics traditionally focuses on.
>
> – Romer (2001, p. 7)

> Power has figured importantly in discussions of international interaction since the time of Thucydides
>
> – Baldwin (2002, p. 177)

> The question that then arises is whether intensified transboundary social interactions that are already at a relatively high level signify a further decline in the importance of nationally defined borders
>
> – Zurn (2002, p. 237)

The central notion of this chapter is that of the growth and distribution dynamics of a system in space and time. Very briefly, the notion is applied to the growth and distribution, first of the physical universe and then of the biosphere on earth. We then turn to the growth of human society drawing on ideas from the new theory of economic growth. The impact on the natural environment of the growth of human society is noted. Next, we consider the distribution dynamics of power, truth and value, noting how this reflects the history of the rise and fall of dominant powers. This reflects a system of interacting territorial units – states. Power resides in resources and relationships. In pursuit of value, resources are allocated between different types of production activity: between internal (domestic) activity and external (international) activity; externally between different states; between cooperation and competition; between non-military activity and military activity; and between peace and war. The levels and types of interaction relate to geographical proximity and cultural proximity. Arms production and wars are social reciprocation processes and their level can be modelled using either systems of differential equations or game theory. States are not the only actors on the world stage. There are inter-state actors, intra-state actors and trans-state actors. The complex structure of actors can be characterised by the set of all actors, a membership relation specifying which actors are members of which other actors and a specification of the territorial base of each actor. The world map of actors changes over time: existing actors disintegrating and new actors integrating; the initiation and cessation of an actor's memberships; the recruitment and loss of members by an actor; and the territorial base of an actor changing. Just as states are not the only actors on the world stage, so political and military action is not the only type of action and the pursuit of self-interest not the only criterion for action.

THE GROWTH AND DISTRIBUTION OF A SYSTEM IN SPACE AND TIME

The Physical Universe: Growth and Distribution

Our physical universe is 1.5×10^{10} years old. It began with the Big Bang. There is some debate about what happened in the first tenth of a second! The first 3×10^{5} years were radiation dominated. Since then it has been matter dominated. (This in accordance with the first law of thermodynamics which states that total mass-energy is conserved.) The universe has continuously expanded in space and in the future either this may continue, or expansion may stabilise at a fixed size or the universe may contract in the Big Crunch (depending on the spatial curvature). At a certain scale the universe is spatially isotropic and homogeneous. Its trajectory exhibits increasing entropy in accordance with the second law of thermodynamics. These statements are in accordance with certain models and empirical data: distant galaxies are receding from us at a velocity proportional to their distance; there is greater spatial uniformity at greater distances from us; there is uniform presence in space of radiation with a temperature of 2.7 K; etc.

Its history and future can be understood in terms of a trajectory through phase space. The phase space of a system is the set of all possible states of the system. We can imagine phase space partitioned into regions of different sizes. If all the possible states are equally likely to occur then larger regions are more likely to occur than smaller regions – and extremely large regions are almost certain to occur. It is almost certain that the trajectory of the universe passes through regions of increasing likelihood of occurrence – of increasing entropy. This is in accordance with the second law of thermodynamics.

The Sun and all other stars have condensed gravitationally from a uniform gas of mainly hydrogen. The Sun sends 'relatively few' high-energy photons to the Earth and the Earth sends 'relatively many' low-energy photons back to space. The Earth emits just slightly more energy than it receives. The sun's photons are absorbed by plants, and hence into the food chain. The sequence of events in this process involved increasing entropy, from lower entropy states to higher entropy states. (Penrose, 2004, pp. 686–781).

The Biosphere: Growth and Distribution

Mathematical biologists have developed an extensive literature on population models. The simplest model is for the population of a single species subject just to a constant growth rate, $dn/dt = an$. This equation was discussed in Chapter 7 and the population trajectory may be constant or exponentially increasing or exponentially decreasing to zero. The next model envisages a single species with a growth rate which is constrained by limited resources. This is modelled by the logistic function, $dn/dt = n(a - bn)$. This can be extended to a multi-species model $d\underline{\boldsymbol{n}}/dt = \underline{\boldsymbol{n}}' \times (\mathbf{A}\underline{\boldsymbol{n}}^{*})$, where $\boldsymbol{n}$ is the vector of population sizes and $\mathbf{A}$ is the matrix of coefficients, indicating the species growth rate and the species limited resources and the inter-species interaction rates corresponding to mutualism, competition and predator–prey relationships. These models have the potential to exhibit complex behaviour (Shone, 1997).

There is an important distinction between the expected behaviour of the population and the probability distribution of the behaviour. A focus on the expected behaviour gives an almost deterministic impression of the situation which is particularly vulnerable when applied to small sample situations. Renshaw (1991, p. xiii) warns that 'popular deterministic ideas … can change markedly when viewed in a stochastic light' and notes that 'we are often asked to infer the nature of population development from a single data set, yet different realizations of the same process can vary enormously'.

As well as simple models exhibiting complex behaviour and the same process having a variety of manifestations, there is a variety of additional features which have been built onto the basic models: birth, death and migration; time lags, overlapping generations and age distribution; spatial dynamics; fluctuating environments; linear and branching architectures; evolutionary demography (Renshaw, 1991; Caswell, 2001).

The Growth of Human Society

In the quotation which opens this chapter, Romer testifies to the enormous importance of long-run growth. Before looking at specific growth models some general remarks are in order. We consider the social activity as a whole. The activity has input x and output y with $y = f(x)$. The growth in output is related to the growth in input according to the equation $\mathrm{d}y/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}x)(\mathrm{d}x/\mathrm{d}t)$. The growth in input may be independent of output, in other words it may be exogenous.

Output can be allocated in a variety of ways. It can be fed back into input, or into consumption value or into waste. Denoting the proportions of these by a, b and c, respectively, with $a + b + c = 1$, we have endogenous growth in input, $\mathrm{d}x/\mathrm{d}t = ay$, and consumption growth $b\mathrm{d}y/\mathrm{d}t$ and waste growth $c\mathrm{d}y/\mathrm{d}t$. The first of these gives a direct expression for the output growth $\mathrm{d}y/\mathrm{d}t = (\mathrm{d}y/\mathrm{d}x)ay$. In this way inputs produce outputs and outputs are fed into inputs.

Economic growth models envisage multiple inputs to social activity: land, natural resources, capital, knowledge and people (labour). Denoting these by the vector $\underline{\boldsymbol{x}}$, we have $y = f(\underline{\boldsymbol{x}})$. Partial derivatives give the local productivities of these inputs $\partial y/\partial x_j$ or $(\partial y/\partial x_j)/x_j$. In some models such as the Solow growth model some of the inputs are exogenous while others are endogenous. In particular, capital is an endogenous input. Implicit in this is the notion of multiple outputs, for example, outputs of consumption goods and capital goods or outputs of physical capital and knowledge capital. Reference in the title of this chapter to growth in power, truth and value can be taken to refer to capital, knowledge and consumption value.

Romer starts with the Solow growth model. The two key concepts are the output per effective worker, y, and the capital per effective worker, k. Capital determines output, $y = f(k)$, and capital grows according to the equation $k' = g(k)$ with a unique stable equilibrium occurring at $k = k^*$.

'Regardless of its starting point, the economy converges to a balanced growth path – a situation where each variable in the model is growing at a constant rate ...' the variables in the model being output, capital, labour and knowledge or the effectiveness of labour. 'The principal conclusion of the Solow model is that the accumulation of physical capital cannot account for either the vast growth over time in output per person or the vast geographic differences in output per person' (Romer, 2001, pp. 6, 16–17).

In contrast, new growth theory treats technological progress as endogenous. Kremer considers population growth and technological change since 1 million BC. He suggests that technological progress is an increasing function of population size and that over almost all of human history until recently, technological progress has led to increases in population rather than increases in output per person. Spatial connectedness is important. 'From the disappearance of the continental land bridges at the end of the last ice age to the voyages of the European explorers, Eurasia-Africa, the Americas, Australia, and Tasmania were almost completely isolated from one another. ... The model predicts that during the period that the regions were separate, technological progress was faster in the regions with larger populations [and hence population grew faster]'. Consistent with this model, the figures for area and estimates of population density in 1500AD give the following area–density pairs: Eurasia-Africa (84, 4.9), the Americas (38, 0.4), Australia (8, 0.03), and Tasmania (0.1, 0.03), measured in million square kilometres and people per square kilometre, respectively (Romer, 2001, pp. 126–130).

Humans and their Natural Environment

The growth of human society has involved strong interaction with the natural environment. The history of the human species has been characterised by an exponential growth in their numbers, by an extension of their territory and by their destruction of existing species and habitats – which then are replaced by humanised species and habitats. The natural environment consists of land, sea, ice, air, space and extra-terrestrial bodies. The ownership of the natural environment is a potent source of conflict. The ownership relation creates a partition of the natural environment although this partition can be and is disputed. The concepts of property, ownership and sovereignty entail a rule which differentiates what the owner can do with the property from what the non-owner can do with the property.

Mitchell (2002, p. 501) refers to the 'growth in the number and magnitude of harms humans inflict on the natural environment' and notes that 'since the time of Malthus, people have recognized that both the carrying capacities of natural systems (the amount and rate at which they can supply human demands) and the magnitude and types of human demands placed on them vary'. Mitchell thus invokes the population models of mathematical biology discussed above. Mitchell (2002, p. 501) continues: 'supply-demand conflicts are exacerbated because capitalist, socialist and communist economies actively create incentives to disregard the environment and passively fail to remedy situations involving Tragedies of the Commons and other externalities, that is situations involving actions that bestow benefits on those who engage in them but impose larger costs on society as a whole'. The Tragedy of the Commons refers to common land and the tragedy is that because it is common land, there is no incentive for anyone to stop using it for grazing and this leads to over-grazing. A similar situation exists today in the over-fishing of the 'common' seas. The problem is: how can one get people to cooperate in order to protect the shared resource? A diversity of actors participate in the environmental issue. The environmental aspects can be modelled by the mathematical biology models discussed above and the social aspects and political process can be modelled by the game theory models of Chapter 11.

The Distribution Dynamics of Power, Truth and Value

In any specific period of world history there is a certain distribution of power, truth and value – or, using the variables in economic growth models, a certain distribution of land, natural resources, capital, knowledge, people (labour), output, welfare and waste. Dominant locations of power correspond to dominant locations of truth and of value in that dominant powers have leading centres of learning, leading centres of culture and also sufficient economic power to support higher welfare for its people – although this observation needs to be qualified by the observation that power can distort truth and value and can be associated with inequality of value.

The distribution of power, truth and value has a dynamics which exhibit the rise and fall of dominant locations. This dynamics has a spatial aspect. The earliest civilisations were centred around Egypt and the river Nile, Babylon and the rivers Tigris and Euphrates, the Indus River and China. There was the Persian empire, the Grecian empire and the Roman empire. Later Christian Europe fought Muslim Middle East and Northern Africa.

There was a succession of European sea-based empires: Spain, Portugal, Netherlands, France and Britain; and European land-based empires: the brief Napoleonic 'empire', Russia, the Austro-Hungarian empire and the Ottoman empire. The latter three collapsed as a result of the First World War. Briefly, just prior to and during the Second World War, Germany and Japan both expanded dramatically the territory under their control, only to lose it all by the end of the war. From the end of the Second World War to the collapse of the Soviet Union the world was seen as bipolar, with the United States and the Soviet Union being the two superpowers. Briefly, the world was viewed as unipolar but at the present time of writing there is a return to a view of the world as bipolar with the United States and China as the two dominant powers, China thus returning to the dominant position it held in earlier times.

STATE, POWER, GEOGRAPHY, CULTURE, ARMS AND WAR

The distribution dynamics of power, truth and value and the rise and fall of dominant powers reflects a system of interacting territorial units – a system of interacting states, sometimes in cooperation and sometimes in competition and conflict. States exercise sovereignty over their territories, have power, produce arms, go to war, experience victory or defeat and gain or lose territory.

Power: Resources and Relationships

Baldwin (2002, p. 177) (hereafter denoted 'B') notes that 'power has figured importantly in discussions of international interaction since the time of Thucydides' and proceeds to discuss how different scholars have tackled the question: 'what is the nature of power?' There is continuing debate around the distinction between 'power as resources' and 'relational power'.

In the 'power as resources' approach – sometimes referred to as the 'elements of national power' approach – power is conceived of as a possession or property of states: 'the power of individual states was conceived to be susceptible of measurement by certain well-defined factors including population, territory, wealth, armies and navies ... [there was an] assumption that it was possible to add up the various elements of power [to form a measure of overall power] ...' (B, pp. 177–178).

In the 'relational power' approach power is conceived of 'as a relationship in which the behaviour of actor A at least partially causes a change in the behaviour of actor B' (B, p. 178). Baldwin himself is an advocate of the concept of 'relational power' and is impatient with the 'power as resources' approach: 'If international relations researchers were to give up the search for a universally valid measure of overall national power, much useful research could be focused on measuring the distribution of power within specified domains' (B, p. 181).

Baldwin's offers suggestions for further research: '(1) the treatment of power as a dependent variable, (2) the forms of power; (3) institutions and power; (4) domestic politics and power; (5) strategic interaction; and (6) power distributions in different issue areas' (B, p. 188).

We now seek to express these ideas in mathematical terms. Two alternative approaches are considered: the development of a general linear model and the application of economic growth theory. First consider the 'power as resources' approach. There is a set S of states, a set X of possible power resource vectors, a function f mapping S to X and a function g mapping X to Y where $y = g(\underline{\boldsymbol{x}})$ is the overall power represented by the power resource vector $\underline{\boldsymbol{x}}$.

Next consider the notion of relational power. There is a pair of states A and B; a behaviour of A, b_A; a change in the behaviour of B, $\mathrm{d}b_B$; and a function f such that $\mathrm{d}b_B = f(b_A)$. An interaction between A and B might then be represented by a pair of equations:

$$\mathrm{d}b_A = f_A(b_B)$$

$$\mathrm{d}b_B = f_B(b_A)$$

One extension of this model would be to suppose that a state's behaviour had its own momentum – in other words a change in the state's behaviour also partly depended on its own behaviour. Thus:

$$\mathrm{d}b_A = f_A(b_A, b_B)$$

$$\mathrm{d}b_B = f_B(b_A, b_B)$$

Corresponding to this equation for the dynamics of behaviour, one can postulate a corresponding equation for the dynamics of power resource. Here the two states are labelled '1' and '2'. Each state has a power resource (x_1 and x_2); each state has a relational power on the other actor (a_{12} and a_{21}) and on themselves (a_{11} and a_{22}). Each state also has another property

(a_{01} and a_{02}). The change in the resource power of each state depends on all these different variables. Characterised in this way the equation succeeds in expressing both resource power and relational power.

$$\mathrm{d}x_1/\mathrm{d}t = a_{10} + a_{11}x_1 + a_{12}x_2$$

$$\mathrm{d}x_2/\mathrm{d}t = a_{20} + a_{21}x_1 + a_{22}x_2$$

Notice that the equations above have resource power as both an independent and a dependent variable, thus relating to the first of Baldwin's suggestions for further research. Baldwin's second suggestion that we should look at different forms of power emphasises that power is multidimensional. From a mathematical point of view this means that the two 'x' power variables become power vectors '$\underline{\boldsymbol{x}}$' and that the '$a$' coefficients become vectors $\underline{\boldsymbol{a}}$ and matrices **A**. Moreover, the matrices **A** provide measures of fungibility, the extent to which one form of power can be converted into another form of power, an important issue in the theorising of power.

$$\mathrm{d}\underline{\boldsymbol{x}}_1/\mathrm{d}t = \underline{\boldsymbol{a}}_{10} + A_{11}\underline{\boldsymbol{x}}_1 + A_{12}\underline{\boldsymbol{x}}_2$$

$$\mathrm{d}\underline{\boldsymbol{x}}_2/\mathrm{d}t = \underline{\boldsymbol{a}}_{20} + A_{21}\underline{\boldsymbol{x}}_1 + A_{22}\underline{\boldsymbol{x}}_2$$

Having introduced a multidimensional power model we now ask whether in some sense it can reasonably be represented by a unidimensional model representing overall power, y:

$$\mathrm{d}y_1/\mathrm{d}t = c_{10} + c_{11}y_1 + c_{12}y_2$$

$$\mathrm{d}y_2/\mathrm{d}t = c_{20} + c_{21}y_1 + c_{22}y_2$$

Having developed a general linear model of power we now consider an alternative approach. Looking back at the dimensions of resource power mentioned by Baldwin they correspond to the standard components of economic growth theory: land, labour, technology, knowledge, etc. Growth theory combines these variables not to form an overall measure of power but rather to form the production function for economic output. From this growth theory for a single unit one can then develop a model of multidimensional growth and also a model of interacting units of economic growth.

A central issue is how productive resources are allocated – and how they should be allocated – between different types of production activity: between internal (domestic) activity and external (international) activity;

externally between different states; between cooperation and competition; between non-military and military; and between peace and war.

States, Sovereignty and Territories

States assert sovereignty over territories. Let A be the set of states. Let G be the set of all geographical points in the world. A division of the world into territories involves a partition of G. We use H to denote the set of territories constituting a specific partition of G. We use h to denote the function from A to H assigning to each state the territory over which it asserts sovereignty. So the quadruple (A,G,H,h) represents a mapping of the world into state territories.

The world map of state territories changes over time. The set A of states changes; the set G changes only very slowly; the set H of territories changes; and the mapping h from A to H changes. The territorial sovereignty of states changes over time. Although some state territories may remain the same, there may be the incorporation of a state territory into a larger one or the secession of a state territory from a larger one; or the integration of several state territories into one large one or the disintegration of one into several.

States have a topological contiguity relationship. State x is related to state y if they have a common boundary. There is thus a network of states based on the contiguity relationship. The basic formulation can be extended to cover international waters and air space, and to cover cases where a state has a territory the parts of which are not connected.

The distance between states can be measured in a variety of ways. One way is in terms of the links in the minimum chain from one state to the next. Another way is to use a metric relationship in terms of distance. Given the extended nature of a state's territory several definitions of distance are possible. Sometimes what may be important is the travel time for a specific means of transport. The greater the distance the greater the time and the greater the economic cost of activity between one country and another.

Geographic Proximity and Interaction: Cooperation or Conflict?

Robst, Polachek, and Chang (2007, p. 1) note that 'closer countries interact more by nature of their proximity'. In general, geography has a role in social interaction and there is a substantial literature on location economics. At this point one might note a linkage between location economics and peace science in the person of Walter Isard, founder of both regional science and peace science. One model of social proximity and social interaction uses an analogy with Newton's laws of gravitation. Consider a set of bodies moving in space. The acceleration of each body equals the net force on the body divided by the body's mass. The net force is the sum of the forces exerted on the body by the other bodies. The gravitational force exerted by a body A on another body B equals the mass of body A multiplied by the mass of the body B divided by the square of the distance between the bodies multiplied by a constant. So the level of interaction between two social units depends on the sizes of the two units and on the distance between them.

In 1999, the Journal of Peace Research had a special issue on trade and conflict. Opening the debate, Barbieri and Schneider (1999) note that: 'the contemporary surge in economic interdependence referred to as "globalization" has evoked rampant speculation about the impact of increased levels of trade and investment on interstate relations. Most leaders still cling to the longstanding belief that expanding economic ties will cement the bonds of friendship between and within nations that make the resort to arms unfathomable. In contrast, realist and Marxist critics reject this liberal view with the same vigor as internationalization sceptics debate the allegedly beneficial or neutral effects of globalization'.

Barbieri and Schneider review the literature identifying three types of formal model: expected utility models, cooperative game models and non-cooperative game models. The main exponent of expected utility models in this context is Polacek and it is the article in 2007 by Robst, Polacek and Chang that I focus on here. The authors elaborate on their point that 'closer countries interact more by nature of their proximity' (p. 1). Greater interaction allows greater levels of activity in a variety of spheres. In particular, greater interaction allows both greater cooperation and greater conflict. Consistent with this point, closer countries have greater cooperation and greater conflict – the latter relationship being a consistent finding in conflict research. Greater interaction also allows greater trade. Consistent with this point, closer countries do indeed have greater trade. In their article, Robst and his fellow authors aim to examine how trade and geographic distance work together to influence international interactions.

The authors make use of a model they had developed earlier (Polachek, Robst, & Chang, 1999). Their model – like the first of the models in the previous section – involves a utility function, a constraint and the optimisation of utility subject to a constraint. A country's utility function (or social welfare function) expresses utility u as a function of total consumption C and the conflict relationship with other countries Z. Consumption consists of production and net imports. There is a balance of payments constraint. The country chooses some conflict vector z^* in order to maximise its own utility subject to its budget constraint. From this model it is deduced that conflict is reduced: if trade, both exports and imports, is increased; if the target country does not impose tariffs on *X*; if target provides foreign aid to *X*; and if there is greater proximity. Also trade with a large target reduces conflict more than trade with a small target.

[Note: There is an analogy here with a consumer buying a basket of goods constrained by a budget. Here a country buys a basket of activities constrained by a budget. The activity categories considered here are cooperation, conflict and trade and the distance can be considered as a component of the price of the activity. Thus when the authors ask how conflict and cooperation depends on trade and distance, they are asking a question analogous to: how does the level of one sort of bought good depend on the levels of other sorts of bought good and on a price component shared by all types of bought good.]

Having developed the model, the authors now test it using multivariate regression. In the regression for conflict both distance and trade have negative coefficients, and these two main effects are modified by a positive interactive term, namely distance multiplied by trade. In the regression for cooperation, also, both distance and trade have negative coefficients, and these two main effects are modified by a positive interactive term, again distance multiplied by trade. The main effect for trade would seem to suggest that trade reduces cooperation 'but [because of the interaction term] this is the case only when the distance between countries is extremely small [638 miles]'.

The modifying effect of the interaction term is given a fair bit of attention by the authors. To what extent though is the interaction term an artefact of the linear model used? The surprising negative effect, noted in the above paragraph, of trade on cooperation for small distances may suggest that some non-linear model is more appropriate and if this is the case this model may supply a rather different interpretation of the effects of trade and distance.

Cultural Proximity and Interaction: Cooperation or Conflict?

Different states have different cultures with differences in language, nationality and ethnicity, religion, social norms or in type of political or economic regime. We can conceptualise these differences in terms of states being located in cultural space and hence being at varying cultural distances from one another. By analogy with the approach of the previous section we can ask: do states which are closer culturally to one another interact more; do states which are closer culturally to one another engage in conflict more; and do victorious states seek to impose their culture on defeated states?

Looking back over the past century the First World War was a war between similar cultures – between proto-democratic European monarchies (furthermore, the rulers having close family relationships with one another). As the war ended, the monarchies of Russia, Germany and Austria also effectively ended. In contrast, the Second World War was a three-cornered contest between capitalist democracies, fascist states and a communist state. The victorious states imposed capitalist democracy and communism on the countries within their respective spheres of influence. The cold war which followed was between capitalist democracies and communist states. In the following two sub-sections we look at two lines of research which, rather than discussing the question in the more abstract way we are doing here, discuss the question in relation to the political culture of democracy.

The Democratic Peace

Over the past two decades, the topic of 'the democratic peace' has been a major topic of research. 'One distinctive feature of the democratic peace research programme is that it has engaged scholars from several different research communities sharing rather different methodological orientations – large-n statistical methods, small-n case study methods, and formal modelling' (Levy, 2002, p. 361). The first of these methods established the key empirical regularities. The second of these explored potentially anomalous cases, the application of definitions to cases and the assessment of the validity or spuriousness of inferred causal connections. The third explored causal paths, incorporating theory and generating some plausible theoretical explanations of empirical regularities.

Commenting on a number of studies in the mid-1980s, it had been suggested that the absence of war between democracies comes as close as anything to an empirical law in international relations. This empirical relationship cannot be explained by the indirect action of other variables such as the geographical distance between states, the level of trade between states, the role of American hegemonic power post-1945, or other economic and geopolitical variables. This has led most scholars to believe the effect is real rather than spurious. Most believe that the direction of causality is such that joint democracy promotes peace. Other related empirical results are as follows.

(a) Democracies are possibly slightly more peaceful than other states (although there is conflicting evidence on this). Democracy dyads are more peaceful than authoritarian dyads and least peaceful are democracy-authoritarian dyads.
(b) In particular, democracies frequently fight imperial wars; in wars between democracies and autocracies, democracies are more likely to be the initiators than the targets; and democracies occasionally use covert action against one another – but engage in more peaceful processes of conflict resolution with one another.
(c) Democracies almost always end up on the same side. They win a disproportionate number of the wars they fight and suffer fewer casualties in the wars they initiate. There is no difference in the duration of wars fought by democracies and other states.
(d) Democracies are possibly slightly more peaceful than democratising states (although there is conflicting evidence on this) – the occurrence of war being rather greater in the initial stages of transition from authoritarian to democratic state.

There are a variety of theories to explain these empirical relationships. The various explanations of why democracies are more peaceful either generally (false) or jointly (true) are as follows. Democracies do not go to war because:

(1) Citizens will not vote to send themselves to war.
(2) Democracies have norms of bounded political competition and peaceful resolution of disputes and extend this to relations between democratic states ... but fear being exploited by non-democratic states.
    Against this:
        they initiate imperial wars against weaker opponents;
        they fight against autocracies with an intensity disproportionate to any plausible security threat.
(3) Shared identity and perceived distinction between self and other.
    For this: democratic hostility to culturally dissimilar non-democratic states.
    Against this: covert action, low levels of military force against one another.
(4) Institutional constraints (checks and balances, dispersion of power and role of free press) constrain leaders, provide debate and require broad base of support.
    Against this: why equally warlike? why imperial wars?
    Are leaders more warlike than public? ... war boosts public support?

(5) Democracies signal their intent and extent of support enjoyed by leadership more clearly – via democratic processes, opposition and free press – and reduces misperception – on both sides – and so dispute settled in advance.

Against this: result for joint democracy.

Should We Really 'Force them to be Free?'

Walker and Pearson (2007) provide an empirical examination of Peceny's liberalising intervention thesis that 'while most US military interventions are not successful in bringing about democracy, those cases of intervention in which the U.S. also pushes for 'free and fair elections' are likely to produce long-term democratic outcomes in target countries' (2007, p. 37). The empirical study considers 160 countries. The dependent variable is whether or not a country is a democracy in 1993. A country is classified as a democracy if it has a score of 6 or above (on a scale that runs from 0 to 10) in the Polity dataset. This definition yields 87 democracies and 73 autocracies in 1993 (Table 14.1).

Can we use earlier events to 'predict' which countries are democracies in 1993? The following are used as independent variables: presence of democracy in 1944; military intervention by United States in 1944–1993; the United States opposed regime before intervention; presence of civil war in 1944–1993; presence of international war in 1944–1993; and United States pushes for free and fair elections during intervention.

Twenty countries had been the target of US 'democratic' interventions, the United States achieving a success rate of 70% in the sense that 70% of the countries were democratic in 1993. Fourteen countries were democratic

Table 14.1. The Number and Percentage of Cases for Which Certain Statements are True.

Statements about the 160 Cases	Cases for which the Statement is True	
	n	%
All countries are autocracies in 1993	73	
All countries are democracies in 1993	87	
$X =$ democratic in 1944 if and only if (IFF) democratic in 1993	95	59
X and IFF 'free and fair elections'	101	63
X plus IFF all variables except free and fair	110	70
X plus IFF all variables	112	70

in 1993: Austria, Cambodia, Dominican Republic, El Salvador, Germany, Greece, Grenada, Honduras, Italy, Japan, Korea, Nicaragua, Panama and Philippines. Six countries were not: Angola, China, Iraq, Laos, Libya, and Vietnam.

Walker and Pearson suggest that this analysis might overestimate the success rate and provide arguments suggesting success rates of 62.5%, 56% and 53%. Moreover, looking at the 140 countries which had not experienced US 'democratic' interventions, 52% had become and remained democratic.

Some technical points about this study need to be mentioned. One point is that 160 cases is not a large sample – the associated error percentages are plus or minus 10%. Another point is that there is significant overlap which makes it difficult to estimate effects. Thus the effect of the 'free and fair' variable adds the same amount to the explanation as all the other variables.

There is also a substantive issue. Should we conceptualise the data as a set of independent cases; or should we conceptualise the data as a set of related events, related by virtue of their places in the overall historical process? The cases appear to relate to the Second World War, the cold war and geopolitical spheres of influence. Of the 14 countries which continued to be democratic, 6 became democracies through US occupation during the Second World War: Austria, Germany, Greece, Italy, Japan and Philippines. Six became democracies as part of US efforts to control its own geopolitical influence in its own 'backyard' of Latin America: Dominican Republic, El Salvador, Grenada, Honduras, Nicaragua and Panama. Only two were in Asia: Cambodia and Korea. Of the six countries which were not democratic, all were in Asia or Africa: Angola, China, Iraq, Laos, Libya and Vietnam.

What we have here is a state participating in the social activity of the political culture of democracy. So the model of social participation in Chapter 10 may apply. Let state i have culture c_{it} at time t. Let p_t be the probability that a state has a certain culture at time $t+1$. In the one-step auto-dependent process the probability reflects recency and frequency effects. The probability of a state being democratic at the next point in time depends on how often the state has been democratic in the past and on how recently it was democratic.

Arms and War

Arms Races: Social Reciprocation Processes

> In the early 1930s, when Quincy Wright in America and Lewis Richardson in England began their respective investigations into the causes of war ... they were, of necessity,

> ... unaware of the radical change that their studies would produce in the field of war and peace research.
>
> – Singer (1981, p. 228)

In the allocation of productive resources a major distinction is between resources for development and cooperation and resources for competition, for war. Elementary student textbooks used to address (and maybe still do) the 'guns versus butter' decision. Some have argued that the allocation of resources for war production is self-reinforcing, in other words an arms race takes place and this thought prompted one of the first uses of mathematical models in peace and conflict research. Richardson's (1990; originally 1919 and 1935) 'linear theory of two nations', a model of the arms race, has the following pair of equations where x_1 and x_2 are the amount of defences of the two sides; a_{12} and a_{21} are the reaction coefficients of each side to the other; a_{11} and a_{22} are cost coefficients restraining each side from spending too fast on defences; and a_{10} and a_{20} are ambition coefficients.

$$\mathrm{d}x_1/\mathrm{d}t = a_{10} - a_{11}x_1 + a_{12}x_2$$

$$\mathrm{d}x_2/\mathrm{d}t = a_{20} + a_{21}x_1 - a_{22}x_2$$

During the cold war, this model was developed and applied to the arms race between the two superpowers. Although the above equations are linear, Saperstein notes that 'the Richardson approach can be nonlinearized ... a chaos stability analysis can also be used to examine the effects of specific military strategies on international stability'.

Note that the negative term reflects the attention to alternative uses of the resources – the 'guns versus butter' issue. It is worth noting too that the arms race model does not contain any element of decision making on the part of the arms racers – the racers simply have their parameters and behave accordingly. Nevertheless, the equations can be the consequence of a reasoning choice process – as indicated by the quotations of statesmen cited by Richardson. And indeed the model is an example of what follows from a social comparison utility function as discussed in an earlier chapter.

Arms Production: Data, Models and Coefficients – The Case of the United States

Two points arise from the Richardson arms race model. The first is how we might test the model using empirical evidence. The second is whether there might not be a better model which incorporates in a more sophisticated way the economic aspects of arms procurement. Brauer (2007) provides a useful illustration of an approach commonly used by economists. As Brauer says, 'this article is an exercise in economic methodology' (p. 55). Brauer considers two published models of US military expenditure. One model is (in a sense) a micro-economics model and the other model is a macro-economics model.

The first model starts with the standard economics model of a consumer, here the government. Suppose the government makes a one-off decision. The government can buy two types of goods: amount M of military goods and amount N of non-military goods (guns versus butter). The prices of these two goods are p_M and p_N, respectively. The government has a total budget of G; and so $p_M M + p_N N = G$. The government chooses M and N so as to maximise utility $u = M^a N^{1-a}$. This optimisation problem can be solved and it is found that $M = \lambda G$: military goods are a certain proportion λ of total expenditure.

Let us now consider the government making a sequence of decisions over time. We interpret the proportion λ as reflecting some overall or long-run relation between military goods M and total government expenditure G. We need to establish the short-run relationship. To do this an error-correction model can be used – see the equation below. The equation includes an interaction effect between the long-run adjustment parameter c and short-run special circumstances. Special circumstances here are: the Vietnam war, the Carter–Reagan military build-up and the post-9/11 year, 2002.

$$\Delta m_t = a + b\Delta g_t + c(g_{t-1} - m_{t-1})(1 + z) + u_t$$

Where $m_t = \log(M_t)$; $g_t = \log(G_t)$; $\Delta m_t = m_t - m_{t-1}$; $\Delta g_t = g_t - g_{t-1}$; u_t is a random error; z is a dummy variable representing the presence of special circumstances; and a, b and c are constants. This model is tested on data and estimates are derived for the parameters.

The second model is a Keynesian macro-economics model. This consists of five equations. The first equation states that national income equals national expenditure. National income is measured by output Q and national expenditure is comprised of consumption C, investment I, net exports X, military expenditure M and non-military government expenditure G. (All of these variables are expressed in natural log terms).

$$Q = C + I + X + M + G$$

Consumption depends on the difference between output Q and taxes T.

$$C = a + b(Q - T)$$

Tax is comprised of a lump sum n and income taxes with tax rate g.

$$T = n + gQ$$

Investment depends on the real interest rate R (non-log).

$$I = e - fR$$

Net exports depend on national output and real interest rates.

$$X = z - mQ - nR$$

From these five equations we can derive the following equation. Output depends on military and non-military expenditure and the real interest rate, together with an error term u. (The α, β, δ and λ are constants.)

$$Q = a + \beta M + \delta G - \lambda R + u$$

Brauer estimates the parameters of these models using two different sources of data. There are differences between the results found when NIPA data are used and the results found when budget data are used. Brauer suggests that, whereas much of the literature uses budget data, 'budgets do not measure the economic resources devoted to a state's defense function' (p. 57). A better measure of the latter would be that obtained in the National Income and Product Accounts.

Brauer considers many different data sets involving different timespans and finds 'frequent sign reversals and considerable instability in the estimated parameter values. ... This it seems to me is a major methodological issue that needs debate ... At issue is what reliable statements we can or cannot make about the effect of military expenditures on an economy' (p. 63).

Explaining Limited Conflicts

We now consider that a war can be prosecuted at a certain level of intensity. Once started, does a conflict continue at the same level or does it worsen – worsen in terms of increasing in scale or worsen in terms of worsening in a qualitative way? Kilgour and Zagare (2007) contrast models which envisage that conflict will almost inevitably worsen with models which envisage decision making being able in certain circumstances to control the level of conflict – hence 'limited' conflict. They use their Perfect Deterrence Theory 'to understand whether a conflict between rational actors can be capped and, if so, under what conditions' (p. 67). Kilgour and Zagare apply their theory to the Gulf War, the Cuban missile crisis, the Fashoda crisis of 1898, the Korean crisis of 1950 and other crises.

They introduce the asymmetric escalation game which involves two players taking action in alternation, with three options: cooperate, continue the conflict at the same level or escalate. Once both players have escalated the situation is that of all-out conflict. The conflict is initiated by a challenger, the second player being the defender.

Pay-offs are associated with the different outcomes. Both players prefer (i) the status quo to (ii) limited conflict to (iii) all-out conflict. However, the challenger prefers (i) defender concedes to (ii) status quo to (iii) challenger wins to (iv) limited conflict to (v) all-out conflict and defender escalates/wins. The defender prefers (i) the status quo to (iii) the defender escalates/wins to (iii) limited conflict and defender concedes to (iv) all-out conflict and challenger wins. These preferences are consistent with the credibility of the players' threats, being associated with their preference to execute it.

REPRESENTING THE COMPLETE CAST OF ACTORS ON THE WORLD STAGE

The section 'State, power, geography, culture, arms and war' has single-mindedly focused on a view of the world which sees it as a system of interacting states each using power in pursuit of their interests, maximising their utility subject to resource constraints. This focus is consistent with a dominant theme in the literature on international relations. However, this exclusive attention to states and great powers has been strongly challenged by those arguing that non-state actors are also important (see Smith, 2004, pp. 504–506). In this section we wish to consider the great variety of types of actors on the world stage, a variety well covered by chapters in Carlnaes, Risse, and Simmons (2002) on nationalism and ethnicity; from interdependence to globalisation; comparative regional integration; transnational actors and world politics; domestic politics and international relations; and feminist perspectives on international relations – and also a chapter which reflects more critically on the nature of the state. What we seek to do here is to develop a model which represents the complex structure of actors in mathematical terms.

How can we represent the variety of actors mentioned in the preceding paragraph? A useful starting point is the discussion in section 'States, sovereignty and territories' of states, sovereignty and territories. There the quadruple (A,G,H,h) was used to represent a mapping of the world into state territories. The notation was: A, the set of states; G, the set of all geographical points in the world; H, the set of territories constituting a specific partition of G; and h, the function from A to H assigning to each state the territory over which it asserts sovereignty.

Some of the actors mentioned above have as their participants subsets of the set of states (bilateral relationships, multilateral relationships, regional or continental groupings, military alliances) or indeed the complete set of states (international organisations, the United Nations). We refer to all of these as the set B of inter-state actors. There is a mapping m from B to the power set P^A of A indicating for each inter-state actor a the set of states which are members of a. This gives the triple (B,P^A,m). Corresponding to the set of participating member states there is a corresponding set of territories belonging to the power set P^H of territories.

We now consider the domestic actors within a state. In much the same way as the set of states gives rise to the set of inter-state actors, so the set I of individuals gives rise to a set C of intra-state actors: (C,P^I,m). Some of the domestic actors have a territorial base which is a subset of the territory of the given state. Finally we note actors which cut across state boundaries, the set D of trans-state actors having a territorial base which intersects the territory of more than one state.

In summary, we have a complex structure of sets of state actors A, inter-state actors B, intra-state actors C and trans-state actors D. This is to define the structure with reference to states. However, we can also define it more generally. Let Z be the set of all actors. Let m be a relationship on Z such that xmy means that actor x is a participating member of actor y. There is a mapping from Z to the power set of Z such that for each actor x there is the set $m(x)$ of actors of which x is a participating member; and there is a set $m^{-1}(x)$ of actors which are participating members of x. For those actors which have a territorial base, there is a mapping h from Z to the power set of G, $h(x)$ denoting the territory of x. So the basic ingredients of the complex structure of actors are given by the quadruple $(Z,G;m,h)$.

The world map of actors changes over time. The set Z of actors changes, existing actors disintegrating and new actors integrating; the set G changes only very slowly; the mapping m of actor memberships changes, both the set of actors of which a given actor is a member and also the set of members of a given actor; and the territorial mapping h from Z to G changes.

SPHERES OF ACTION

Just as states are not the only actors on the world stage, so political and military action is not the only type of action and the pursuit of self-interest not the only criterion for action.

CRITERIA FOR ACTION

> The long-standing divide between those who believed that international rules per se shaped state behaviour and those who saw such rules as epiphenomenal or insignificant has given way to a more nuanced and complex debate.
>
> – (Carlneas et al., 2002, p. 538)

> Human rights are a set of principled ideas about the treatment to which all individuals are entitled by virtue of being human. Over time, these ideas have gained widespread acceptance as international norms ... Human rights norms create a relationship between individual (and very occasionally collective) right holders and other entities (usually states) that have obligations.
>
> – (Carlnaes et al., 2002, p. 517)

In Chapter 11, we noted that social activity involves rules and choices. The rules are a constraint on choice and the rules are a matter of choice. It is helpful to posit a contrast between individual freedom and social control. The contrast is the central element in the classical definition of the state: the state is sovereign in that it exercises social control over its own territory but enjoys individual freedom in the international arena – there is no social control over its actions – and hence the state system is characterised by anarchy. One view of global politics is to see it as a system of states exercising choice each in pursuit of their own national interest unconstrained by rules in their choices and unaccountable to rules for their actions. One view of the global economy, particularly global finance, is to see it as a system of economic actors exercising choice each in pursuit of their own economic interest unconstrained by rules in their choices and unaccountable to rules for their actions.

Concern about the social consequences of repression within states, of wars between states, of economic inequalities and of economic crashes has led to pressure for rules which would constrain actions and prevent these events occurring. '[Most] scholars have come to regard international institutions as sets of rules meant to govern international behaviour', and international regimes are seen as 'rules, norms, principles and procedures that focus expectations regarding international behaviour' (Carlnaes et al., 2002, pp. 193–194).

CHAPTER 15

DEBATING THE MATHEMATICAL SCIENCE APPROACH TO INTERNATIONAL RELATIONS

> A common reaction to the view that major developments in the theory of international relations will be achieved through its progressive mathematisation is an expression of bewilderment.
>
> – Nicholson (1989, pp. 10–12)

> The puzzles and conflicts of international relations cannot be solved in the manner of a logical puzzle; instead they have to be unwrapped and understood from the view points of the actors involved.
>
> – Smith (2004, p. 511)

> If the field does focus on rationalism versus constructivism, then the central debate in IR will not be about international relations but rather about how to study international relations.
>
> – Fearon and Wendt (2002, p. 52)

> When it comes to concrete empirical research it is doubtful if anyone could consistently occupy any one of the positions and still maintain coherence.
>
> – Wight (2002, p. 41)

The mathematical science approach to the study of social affairs has been much debated not least among scholars of international relations. Alongside this debate about methodology there has been a debate about the topics which international relations research tends to focus on. Criticisms tend to take one of the following forms: 'I don't like the fact that your subject matter is *S*'; or 'I don't like your representation of the subject matter *S*'. There are errors of omission and errors of commission. There is under-representation and over-representation. Smith's (2004) wide-ranging critique of International Relations theory research provides an agenda for checking whether there are important areas which this book neglects. Firstly consider philosophy. Chapter 2 provides some discussion of the debate between realism and constructivism but the rest of the book has preferred to go ahead and develop the mathematical science approach. Secondly consider ethics. This topic has been specifically discussed in Chapter 6 and the concept of value has been central throughout the book. Thirdly consider power. The concept of power was first introduced in Chapter 4 and has appeared throughout the book and most recently Chapter 14 provides a discussion of power as a resource and power as a relationship. A useful notion is the power-weighted mean ideal. Fourthly consider the impact of power on value. The deviation of the social outcome from optimality depends on: the bias magnitude, the variation in individuals' ideals and the correlation between the bias vector and the vector of ideals. Fifthly consider ethics, identity and social ideals. Ethics and identity can be incorporated into the model by using interpersonal welfare weights to define an individual's social ideal. Sixthly consider the social production of ideals. This can be modelled by a process of influence in a social network in the manner of the complexity theory models discussed in Chapters 12 and 13. Seventhly consider the complex structure of actors and actions. This links back to the Chapter 14 discussion of how the centrality of states in certain theories of international relations can be replaced by a model which represents the complex structure of actors, actions and criteria on the world stage.

THE DEBATE

The mathematical science approach to the study of social affairs has been much debated not least among scholars of international relations. Wight (2002, p. 37) reviews the current debate – discussing the views of Michael Nicholson and Steve Smith quite extensively – and comments:

> all of this adds up to a very confused picture in terms of the philosophy of science. IR has struggled to incorporate an increasingly diverse set of positions into its theoretical landscape. In general, the discipline has attempted to maintain an unsophisticated and outdated two-category framework based on the science/anti-science issue. ... Currently there are three continuums that the discipline seems to consider line up in opposition to each other. The first of these is the explaining/understanding divide (Hollis & Smith, 1990). The second is the positivism/post-positivism divide (Lapid, 1989; Sylvester, 1993). The third is Keohane's distinction between rationalism and reflectivism (Keohane, 1989). The newly emerging constructivism claims 'the middle ground' in between. (Adler, 1997; Price & Reus-Smit, 1998; Wendt, 1999)

Apart from the brief discussion in Chapter 2, this book has not addressed these philosophical issues, preferring instead to push ahead and present a mathematical science approach. Now that this has been done it is appropriate to pause and consider whether the ideas in this book are vulnerable to the criticisms which are generally made of a mathematical science approach. In his highly controversial presidential address to the International Studies Association in 2003, Steve Smith said:

> What kind of International Relations theory do I want to see in this new millennium? Above all, I want to see a discipline that is open to a variety of issues, subjectivities, and identities rather than taking the agenda of the powerful as the natural and legitimate focus for the discipline. I want to see a discipline that enquires into the meanings and subjectivities of individuals in cultures different to those of the dominant world powers rather than assuming their rationality, interests, and thus identities. I want to see a discipline that admits of many routes to understanding, rather than treating one model of social science as if it was the sole bearer of legitimacy and thus beyond criticism. I want to see a discipline that realizes the limitations on correspondence theories of truth, and instead treats truth not as a property of the world waiting to be discovered, but as a matter for negotiation and interpretation. Finally, I want to see a discipline that does not hide behind the mask of value-neutrality and empiricism. (2004, p. 514)

What Smith is doing here is challenging the dominant rationalist approach to international relations and arguing the case for a constructivist approach. According to Smith, present international theory reflects the interests of the dominant in what are presented as neutral and universal theories. He rejects Weber's distinction between science and politics as a vocation: 'I do not see any possibility (or desirability) of separating ethics from academic study' (Smith, 2004, p. 500). He considers the arguments of Keohane (1989) and others concerning the debates between rationalism, reflectivism and constructivism and notes Keohane's denial of academic legitimacy to post-modernism. This leads Smith to question what kind of social science dominates the academic disciplines of Political Science and International Relations. He identifies the dominance of rational choice theory and in particular game theory. He concedes that rational choice theory is technically very efficient. However, he asks whether the success of the theory is because it is true, or because it becomes accepted as being true; because it is how the world works or because it determines how the world works; because it is how the world works or because it serves dominant interests. He identifies 10 core assumptions which help to construct a discipline that has a culturally and historically very specific notion of violence, one resting on distinctions between economics and politics, between the outside and inside of states, and between the public and private realms. This specific notion of violence is challenged by the United Nations Development Programme (UNDP) and Smith presents some UNDP data on violence to human security, much of it lying outside what has traditionally been regarded as the remit of International Relations theory. Smith asks why the discipline of International Relations only focuses on a small subset of these types of violence. Smith notes that the discipline has focused on explanation rather than understanding and uses the art of Magritte and Velazquez to reflect on the nature of representation. The kind of International Relations theory Smith wants to see considers all actors on the world stage, not just the most powerful ones, it enquires into meanings, subjectivities and identities, it is characterised by methodological pluralism, treats truth not as a property of the world but as a matter for negotiation and interpretation and, finally, it 'does not hide behind the mask of value-neutrality and empiricism'.

With his phrase 'a very specific notion' Smith is saying that he does not like IR theory's representation of the subject of violence. In the last two lines of the quotation Smith uses the word 'subset' and this invites us to make explicit use of the mathematics of set theory! Let S denote the set consisting of the forms of violence referred to in the UN report and let R denote the set consisting of the forms of violence referred to in the discipline of International Relations theory. What Smith claims is that R is a [small] subset of S. More generally let W be the set of all events in the world and let V be the set of events referred to in the discipline of International Relations theory. Smith is claiming that V is a [small] subset of W. So what exactly is the set V of events referred to in the discipline of International Relations theory? Different people have different notions of what V is and also people such as Smith argue that there is a divergence between what V is and what V should be.

DEBATE AND THE REPRESENTATION OF A SUBJECT

However, it is sometimes quite difficult to work out just exactly what is involved in the debate. So at this point I do not wish to go into the details of the debate. Instead I wish to step back and take a quite general look at what is going on when people disagree. My hope is that this general framework will help me later on when I do come to look at the details of the debate.

Consider the following situation. One person A says something and a second person B says that they do not like what A has said. Suppose that what A says is of the form 'I am talking about subject matter S and this R is my representation of it'. The person B may dislike this in two major ways – they may say either: 'I don't like the fact that your subject matter is *S*' or 'I don't like your representation *R* of subject *S*'. We now consider each of these in turn.

I don't like your representation *R* of subject *S*.

There are various ways in which a representation *R* may be a poor account of subject *S*. In order to illustrate these, we consider *R* and *S* first as sets of topics; then as weighted sets of topics and finally as sets of individuals with attributes.

Suppose that both *R* and *S* are sets of topics. The representation may exhibit errors of commission and errors of omission. In other words, the representation may contain topics which are not in the subject matter – and the representation may fail to contain topics which are in the subject matter.

Suppose now that both *R* and *S* are again sets of topics – but with the additional point that each topic has a certain weight. The representation may give certain topics more weight than these topics have in the subject matter – and certain topics less weight. If so the representation of the subject is biased.

Suppose now that both *R* and *S* are sets of individuals with attributes – so each attribute occurs in the population of individuals with a certain frequency. The representation may give certain attributes greater relative frequency than these topics have in the subject matter – and certain attributes lesser relative frequency. If so the representation of the subject is again biased.

This last situation corresponds to that discussed in the literature on sampling theory. A representative sample is one where the relative frequency of different types of individual in the sample is equal to their relative frequencies in the whole population. (A *process* of representative sampling is one in which the chance of a type being sampled is proportional to the relative frequency in the population).

I don't like the fact that your subject matter is *S*.

The issue here revolves around the value of the subject matter *S*. If the value involved is simply a matter of personal taste then the disputants will just have to agree to disagree. However, there may be an implicit notion in the debate that subject matter *S* is in some sense a 'representation' of some (larger) subject matter *U*. The criticism of *S* may then be that *S* is a poor 'representation' of *U* – in the various ways which have been discussed earlier.

MEETING THE POINTS RAISED

I now wish to take the key points raised by Smith and consider the extent to which they have been addressed in this book and where necessary adding some further work in order to go some way towards meeting the points raised.

Philosophy

The first point concerns philosophy. It is a point which perhaps lies at the core of Smith's argument, namely the fundamental debate between philosophical realism and constructivism. I have addressed this issue briefly in section 'Reality and realism' of Chapter 2. My remarks there do not resolve the debate but they state my espousal of mathematical social science realism and provide some rationale for my position. Chapter 3 on 'mathematics, logic, artificial intelligence and ordinary language'; discussion of possibility and probability in Chapter 4 and discussion of theory, evidence and reality in Chapter 5 further articulated the mathematical science approach.

I believe that ordinary language has validity only if it can be grounded in mathematical science. For example ordinary language statements of the form 'we are better than them' have underpinned conflict for millennia. And yet the mathematical science that would be required to support the truth of such a statement – the *t*-test – was only developed about a century ago (see the discussion in Chapter 3 of the statement 'women are more cautious than men'). Let me be a little provocative: my hypothesis is that much of the debate between realism and constructivism lacks validity because it cannot be grounded in mathematical science! Oh dear! – how easily have I fallen on my own sword!

Ethics and Value

The second point concerns ethics and value. Smith (2004, p. 500) argues that ethics is inseparable from academic study. Chapter 1 of this book opens by quoting Michael Nicholson's hope that a rigorously developed international theory might help us move in the direction of world peace. Chapter 6 addresses the issue of ethics and notes that Pigou thought that welfare economics was a potent instrument for the bettering of human life.

The concept of value has been central throughout this book with Chapters 4–6 establishing the abstract conceptual foundations and Chapter 13 reflecting on life as the speculative pursuit of value. It may be that the discussion of value has had a welfarist colouring. However, this is well suited to Smith's concerns. Smith (2004, pp. 508–509) is concerned about all forms of violence, his UNDP statistics covering income, food, health, displacement and death, which all might be seen as dimensions of welfare. Moreover, the concept of social welfare carries with it an implicit demand for the full accounting of welfare and an important part of Smith's critique is precisely that International Relations theory involves only a partial accounting.

In focusing on well-being the model has used a simple representation of what is involved in the concept of ethical. Even in this simple form, various features of the model correspond to various features discussed by Kolm and Sen. For example, the concept of 'population-weighted variation' relates to Sen's (1992, pp. x–xi) stress on the importance of human diversity. Beyond these correspondences, what might be involved in extending the model to consider the more complex aspects of ethics discussed by Kolm and Sen? One possible approach would be to simply re-interpret the existing model. In the re-interpretation, social states would be ordered not according to social well-being but according to ethical value. Instead of an ordering of individual preferences there would be an ordering of the ethical value of different social states for each individual. A key question would then be how overall ethical value related to individual ethical values. Assuming this relationship to have Arrow-like desirable properties and assuming certain properties for individual ethical values then one could define a social ethical function as the weighted sum of individual ethical functions. In this way, making parallel assumptions to those made above in the preceding argument, we would obtain corresponding theorems to the theorems in Chapter 6. The social ethical ideal would be a weighted sum of the ethical ideals for the individuals. The ethical value of a situation would depend on the ethical ceiling, the population sensitivity, the population-weighted variation and the deviation of the situation from the ethical ideal. In this way the mathematical model for individual utility and social welfare may be re-interpreted as a model of a more general ethics of social arrangements.

Power

The third point concerns power. Smith (2004, pp. 507, 510) refers to 'the power of the dominant social forces' and 'the concerns of the white, rich, male world of the power elite'. These notions reflect the view that what happens in the world is not determined by ethics but by powers and interests. The concept of power was first introduced in Chapter 4. It was noted that success equals power plus luck, and that power or decisiveness decreased as the number of people and the number of options increased. Chapter 5 introduced the notion that the social outcome might be the weighted sum of individual ideals and that the weight had an egalitarian component and a non-egalitarian component. The general notion that the collective outcome is the weighted sum of individual attributes occurs at a number of different points in the book: in the kinetic theory of gases in Chapter 7; in the multi-unit model of choice in Chapter 9; in the model of demand and supply in Chapter 12 and of course in the model of power in Chapter 14.

Let us revisit the work of Chapter 6. We can use a set of utility functions in order to define a social outcome function and use this to identify the social outcome. We define a social outcome function, say g, as a mapping from L to X, that is from the set of vectors of individual preference orderings to the set of social outcomes. This function too we wish to define in a reasonable way. To do this we continue with our assumption that the set of social states is a subset of a multi-dimensional real space. We suppose that the social outcome depends on the set of individual utility functions. Denoting the set of parameters of these functions by A, we have $x = g(A)$. Each individual's ideal is a parameter and so we have $x = g(\{x_i\}, A/\{x_i\})$. We now restrict our attention to situations with the following characteristics: (i) g is an increasing function of each of the x_i; (ii) x does not depend on any of the other parameters of the utility function; (iii) if all individuals have the same ideal then the outcome will be that ideal and (iv) g is a linear function – or can be approximated by a linear function. Such linear power equations such as that above have been discussed and investigated by Stokman and Zeggelink (1996). Corresponding to the static outcome equation, $x = \sum_{i=1}^{N} w'_i x_i$, the dynamics of a social process could be modelled by $\mathrm{d}x/\mathrm{d}t = \sum_{i=1}^{N} w'_i (x - x_i)$ with an equilibrium at $x = \sum_{i=1}^{N} w'_i x_i$. Of course complexity theory and the work of Chapters 7 and 12 alert us to the more complicated dynamics which can occur.

Theorem 15.1. *Suppose that $\partial x/\partial x_i \geq 0$; $g(\{x_i = a\}) = a$ for all a; and g is a linear function of the $\{x_i\}$. Then the social outcome is the power-weighted mean ideal:*

$$x = \sum_{i=1}^{N} w'_i x_i \quad \textit{where} \sum_{i=1}^{N} w'_i = 1,\ w'_i \geq 0$$

Proof. If g is linear then $x = g(\{x_i\}) = w + \sum_{i=1}^{N} w'_i x_i$. Each $w'_i = \partial x / \partial x_i > 0$. For each a, $a = w + \sum_{i=1}^{N} w'_i a$. So $w = 0$ and $\sum_{i=1}^{N} w'_i = 1$.

The Impact of Power on Ethical Value

The fourth point is the damage which power does to ethical value. Because the social outcome is determined by power, it deviates from what would be ethically desired. In particular, according to Smith, international theory is aligned with power and so facilitates a social outcome which deviates from what would be ethically desired. One of the key points in Smith's address is the complicity of International Relations theory in social outcomes which deviate from social optimality because of a theoretical bias in favour of the dominant social forces in world society: '[does rational choice theory work] ... because it serves some interests: remember, it is an approach located within a particularly powerful academic community itself based in the dominant power in the world' (Smith, 2004, p. 503). Smith's UNDP statistics cover inequality of income and health – inequality across countries and within countries, and between men and women and adults and children.

We now put together the Chapter 6 account of the social welfare function and account above of the social outcome. Under certain assumptions, both the welfare ideal x^* and the social outcome x can be represented as the weighted sum of individuals' ideals. This makes it straightforward to find an expression for the deviation D of the social outcome from optimality.

As a preliminary in obtaining this expression consider the distribution of individuals' ideals. We use μ_x and σ_x to refer to the mean and standard deviation of this distribution. We introduce the notion of the bias of power for each individual $b_i = (w'_i - w_i)$, in other words the extent to which the individual's power exceeds their weight in the social welfare function. We use μ_b and σ_b to refer to the mean and standard deviation of the distribution of bias over the population. Note that $\mu_b = 0$ (because $\sum(w'_i - w_i) = 1 - 1 = 0$). The correlation between ideals and biases is ρ_{xb}. The covariance between ideals and biases is $\text{cov}(b_i, x_i) = \Sigma b_i x_i / n$ because $\mu_b = 0$.

We now introduce and motivate the concept of the bias magnitude $b^\wedge$. Consider a population with σ_b as the standard deviation of the bias. Now suppose that we have a new population consisting of n replications of the original population. All the weights and hence all the biases will be reduced by a factor $1/n$. Thus, the bias can be eliminated by arbitrarily replicating the population. To overcome this problem we define the bias magnitude, $b^\wedge = n\sigma_b$.

These statistical parameters can also be expressed in terms of vectors. Consider the vector of individuals' ideals, $\underline{\boldsymbol{x}}$. The mean ideal is $\boldsymbol{\mu}_x = \underline{\boldsymbol{x}} \cdot \underline{1}/n$. Writing $\underline{\boldsymbol{\mu}}_x = \boldsymbol{\mu}_x \cdot \underline{1}$ and $(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_x) = \underline{\boldsymbol{d}}_x$, the variance is $\sigma_x^2 = ||(\underline{\boldsymbol{x}} - \underline{\boldsymbol{\mu}}_x)||^2/n = ||\underline{\boldsymbol{d}}_x||^2/n$ and so $\sigma_x = ||\underline{\boldsymbol{d}}_x||/\sqrt{n}$. The vector of biases is $\underline{\boldsymbol{b}}$ with $||\underline{\boldsymbol{b}}|| = \sigma_b\sqrt{n}$.

Theorem 15.2. *The deviation D of the social outcome from optimality depends on: the bias magnitude* $b^\wedge$*; the variation* σ_x^2 *in individuals' ideals and the correlation* ρ_{xb} *between the bias vector and the vector of ideals.*

$$D = n\sigma_b\sigma_x\rho_{xb} = b^\wedge\sigma_x\rho_{xb} = ||\,\underline{\boldsymbol{b}}\,|| \;\; ||\underline{\boldsymbol{d}}_x||\,\rho_{xb}$$

Proof.

$$D = x - x^* = \sum w_i' x_i - \sum w_i x_i = \sum (w_i - w_i') x_i$$
$$= \sum b_i x_i = n\,\mathrm{cov}(b_i, x_i) = n\,\sigma_b\sigma_x\rho_{xb}$$

Theorem 15.3. *Suppose the set of social states is multi-dimensional. The deviation* $\underline{\boldsymbol{D}}$ *of the social outcome from optimality is then a vector. The norm of* $\underline{\boldsymbol{D}}$ *depends on the bias magnitude* $b^\wedge$*; on the variation in individuals' ideals* s_x^2 *; and on* R^2*, a variation-weighted mean of the squares of the correlation between the bias vector and the vector of ideals* ρ_{xb}, $||D|| = bs_xR$.

Proof. $\underline{\boldsymbol{D}} = \underline{\boldsymbol{x}} - \underline{\boldsymbol{x}}^*$. The j component of this vector is $D_j = b^\wedge s_{xj}\,\rho_{xjb}$.

$$||D||^2 = \sum_j D_j^2 = \sum_j (b^\wedge s_{xj}\,\rho_{xjb})^2 = b^{\wedge 2} s_x^2 \sum_j \left(\frac{s_{xj}}{s_x}\right)^2 \rho_{xjb}^2 = b^{\wedge 2} s_x^2\, R^2$$

where $R^2 = \sum_j ((s_{xj}/s_x)^2 \rho_{xjb}^2)$ and $s_x^2 = \sum_j s_{xj}^2$

Ethics, Identity and Social Ideals

The fifth point is about ethics, identity and social ideals. The present section remedies an important problem with the development of the model so far. Ethics in general and the social welfare function in particular are often seen as standing outside the social process (cf. Smith's (2004, pp. 503–504) criticism of the notion that ethics stands outside International Relations theory). Thus, the above discussion of ethics did not consider the part which ethics might play in the social process and in the determination of the social outcome. Nor was ethics considered in the discussion of the social process and the social outcome – instead it was assumed that actors pressed for their own individual ideals. Thus, in the model so far the welfare ideal does not enter into the equation for the social outcome. The only ideals to take part in the process are the individuals' own personal ('self-interested') ideals.

We now allow for the possibility that an individual may exert their power in pursuit of an ideal which is not identical to their own personal ideal. We refer to the former as the individual's demanded ideal. The demanded ideal may well be dependent on other individuals' personal ideals and so from now on we refer to the demanded ideal as the social ideal. The notion is that an individual may exert their power in pursuit of some broader purpose, some social ideal which can give others' personal ideals non-zero weight (this embraces the pursuit of a purely personal ideal as a special case) reflecting their 'social preference'. The social ideal is the combination of a number of factors. The social ideal may in part be externally imposed and it may in part be internalised. Internalised sources of the social ideal include ethical considerations and identity.

Smith (2004, pp. 506, 507, 510, 511) argues that 'international relations has tended to ignore questions of identity'. Wendt (1999, pp. 224–227) defines four types of identity: personal, type, role and collective. 'Collective identity ... is a distinct combination of role and type identities, one with the causal power to induce actors to define the welfare of the Other as part of that of the Self, to be altruistic'. So far the model contains no explicit reference to identity. However, the notion of identity is implicit in the concepts of personal ideal and social ideal. It is proposed that there is an identity component in the personal ideal and also an identity component in the social ideal. In the model, social identity entails identifying to a certain degree with certain other actors in society. The social identity ideal is thus a weighted sum of personal ideals. However, the model in the present section will not distinguish between the different sources of the social ideal. One approach in modelling these sources would be to have the social ideal as a weighted sum of the different component ideals and in this case many of the points which follow about the social ideal would also apply to the component ideals – such as identity.

We define the social ideal x_i^* for each individual i as some weighted sum of the personal ideals of the population: $x_i^* = \sum_k w'_{ik} x_k$. The weight w'_{ik} constitutes the importance which i attaches to k in i's social ideal. Each individual i exerts their direct power w'_i in pursuit of their social ideal x_i^*. One component of this direct power is $w'_i w'_{ik}$ which is the contribution which i makes to the pressure for k's personal ideal. The pressure for the ideal of k, in other words the indirect power of k, w'_k, consists of the sum of these contributions $w'_k = \sum w'_i w'_{ik}$.

Theorem 15.4. *The social outcome x' is the weighted sum of social ideals, with the weights being the direct power of the individuals. The social outcome can also be expressed as the weighted sum of personal ideals, with the weights being the indirect power (that is the sum of contributions from other person's social ideals, weighted by the other persons' direct power).*

$$x' = \sum_i w'_i x^*_i = \sum_k w'_k x_k$$

Proof.

$$x' = \sum_I w'_i x^*_I = \sum_I w'_i \left(\sum_k w'_{ik} x_k \right) = \sum_k \left(\sum_I w'_i w'_{ik} \right) x_k = \sum_k w'_k x_k$$

Is society better off if individuals pursue social ideals rather than simply their own personal ideals? Not necessarily. Any outcome in the convex hull of personal ideals can be achieved by a suitable pattern of social ideals. Some social ideals bring the outcome closer to the welfare ideal, other social ideals are neutral while other social ideals take the outcome further away from the ideal. There is an analogy here with the notion of a tax as progressive, neutral or regressive. Not only can any outcome be achieved by social ideals, the same social outcome can be achieved by many different patterns of social ideals. For example we can achieve the welfare ideal either by each individual adopting the welfare ideal as their social ideal; or by the powerful giving suitable compensation to the weak.

The matrix $W' = [w_{ij}]$ of contributions represents the network of social linkages which are implicit in the set of social ideals. If the population is large then the social matrix is likely to be quite sparse – each individual can only consider a few other individuals. If the linkages occur at random then the effect of social ideals will be neutral. If the linkages are 'assortative' (to use terminology from evolutionary theory) then the amount of bias may be maintained.

An individual may be mistaken as to what their ideal is, or their efforts might have the effect of supporting an outcome different from their ideal. The aggregate error is the sum of the individual errors. The error can be progressive, neutral or regressive. To summarise, the deviation of the social outcome from the welfare ideal is the sum of the personal ideal outcome, the social ideal addition and the error addition – minus the welfare ideal.

The Social Production of Ideals

The sixth point is the social production of ideals. Smith (2004, pp. 502, 503, 514, 515) argues that 'the rational choice theorist models behaviour on the basis of fixed, and pre-given identities and interests' whereas in fact 'we sing our worlds into existence, yet rarely reflect on who wrote the words and the music, and virtually never listening out for, nor recognizing, voices or worlds other than our own until they occasionally force us into silence'.

Certainly so far the model lacks a dynamic aspect. In the model, the social welfare (and the welfare ideal) and the social outcome are the only dependent variables. All the other variables are independent, in particular all the personal ideals, all the welfare weights, all the power weights and the matrix of components of the social ideals. One way in which a dynamic aspect could be introduced into the model could be to regard the equations as referring to one particular stage in a multi-stage process. Formally this can be achieved by introducing subscripts into the equations – as indicated below – in order to indicate how the state at time $(t+1)$ depends on the state at time t. In the equations below both the social welfare U and the welfare ideal x^* each depend on the social state x, the individual ideals, the interpersonal welfare weights w and the person-attribute welfare weights W; and the social outcome x depends on the individual ideals, the powers w' and the linkage matrix W'.

$$U_t = f(\{x, x_{it}, w_{it}, W_t\})$$

$$x_t^* = f(\{x, x_{it}, w_{it}, W_t\})$$

$$x_{t+1} = f(\{x_{it}, w'_{it}, W'_t\})$$

Smith's remarks cited at the start of this section prompt the following question: to what extent can the model in the present article be used to represent the social construction of reality? So far the model takes identities and ideals as given. In order to represent the social construction of reality there is a need to formulate equations which specify how the independent variables are themselves formed. One possible approach is to invoke another set of variables which determine the independent variables. This would then invoke the question as to what would determine these variables – and so on. In order to prevent an infinite regress at some stage one would need to specify that the existing sets of variables determine themselves. The simplest approach is to take this step immediately: the independent variables determine themselves.

In the present chapter the dynamics of the ideals, the welfare weights and of the powers will not be considered, only the dynamics of the social ideals. Consider an individual with a certain social ideal which gives consideration to a certain set *S* of other individuals. In deciding how to change their social ideal the individual might well consider the social ideals of the individuals in the set *S*. By definition each column of W_t sums to 1 and it follows that each column of W_{t+1} also sums to 1.

$$W_{t+1} = W_t \, W_t$$

For example consider two actors, A and B. Consider the social ideal of B and specifically, the weight w_{BA} which B gives to A in their social ideal. How might this change? It is proposed that the new value of w_{BA} is a combination of the (old) weight which B gives to A and the (old) weight which A gives to themselves and that the weights in this combination are the components of the social ideal of B:

$$w_{BA,t+1} = w_{BA,t} \, w_{AA,t} + w_{BB,t} \, w_{BA,t}$$

One might say that the equations show how the outcome depends on the social reality and how social reality is constructed by actors. Thus, the weights not only cause outcomes but they also construct themselves. In this model, regarding weights as ideas, social life is ‘ideas all the way down’ (Wendt, 1999, p. 90; quoted in Smith, 2004, p. 502). Note that in this model it is not the case that ‘a common rationality’ is produced, unlike the case criticised by Smith (2004, p.506): ‘The role of structure in constructing the identity and interests of the actors is linked to the assumption that these actors are therefore forced, via socialization, into accepting a common rationality.’

State-centrism and Politics-centrism: The Structure of Social Actors and the Structure of Social Actions

The seventh and final point is about the representation of actors and actions. Four of the 10 core features of International Relations theory identified by Smith relate to the topic of this section (Smith, 2004, pp. 504–506): ‘the focus on the state and not on humanity as a whole or the individual’; ‘the consequent distinction between the inside and the outside of the state'; 'the absence of consideration of gender and ethnicity from the main theories'; and '[war] is privileged as a form of violence ... [theory] tends to ignore conflicts within states'. The argument here is that International Relations theory involves partitioning the world into states and then considering relations between states – ignoring the fact that the structure of actors in the world is much more complex. So far the model has not addressed the notion of states – it has merely referred to an undifferentiated population of individuals. The same abstract model might be interpreted either as referring to a world of people or a world of states. There is a need to introduce a group structure into the model to reflect the fact that many conflict situations involve conflict not only between groups but also conflict within groups.

In the model so far the population of individuals is undifferentiated. Yet many conflict situations involve a group structure with conflict not only between groups but also conflict within groups. Theorem 15.6 shows that the deviation for the population is the sum of the within-group deviation and the between-group deviation. First we note how any population attribute can be expressed in terms of the attributes of the constituent groups.

Theorem 15.5. *Consider a population attribute z which is a weighted sum of individual attributes z_i. Suppose the population is divided into two distinct groups, G and H. We define the relative powers of G and H, g and h respectively, with respect to z as $g = \Sigma_g w_i$ and $h = \Sigma_h w_i$. So $g + h = 1$. We define the within-group weights: $w_{ig} = w_i/g$ and $w_{ih} = w_i/h$. We define the subgroup attributes: $z_g = \Sigma_g w_{ig} z_i$ and $z_h = \Sigma h_h w_{ih} z_i$. Then $z = gz_g + hz_h$.*

Proof.

$$z = \sum w_i z_i = \sum_g w_i z_i + \sum_h w_i z_i = g \sum_g w_{ig} z_i + h \sum_h w_{ih} z_i = gz_g + bz_h$$

Theorem 15.6. *The deviation for the population is the sum of the within-group deviation D_W and the between-group deviation D_B. The former is the weighted sum of the deviations within each group, D_g and D_h. The latter is the between-group bias b_g times the difference between the group ideals.*

$$D = D_W + D_B \quad D_W = gD_g + hD_h \quad D_B = b_g(x_g^* - x_h^*)$$

Proof.

$$D = x - x^* = (gx_g + hx_h) - (g^* x_g^* + h^* x_h^*)$$

Theorem 15.6 is relevant to the issue of IR theory's focus on states; and Smith's (2004, p. 502) concern that the 'rational choice theorist is not concerned with the internal workings of actors, that is to say for states, in their internal political debates'. To the extent that we can represent the set of states by a partition of the entire population of the world then the world deviation from well-being is the sum of the between-state deviation from well-being and the within-state deviation from well-being. To focus solely on states and neglect looking inside states thus involves a neglect of within-state deviation.

The ideas of this section can be extended. So far the population has been partitioned into groups. Yet the structure of world society is still more complex. Individuals belong to more than just one group ... they belong not just to a state, they also have a gender or ethnicity ... there are cross-border minorities. If one allows that an actor can be a set of individuals then the set of all possible actors corresponds to the set of all subsets of the set of people in the world. In mathematical terms, the set of all subsets of a given set is referred to as the power set. (A related concept is a sigma field). As in Theorem 15.6 above, theory often needs to refer to 'measures' on actors-quantitative attributes of individuals, groups and the whole population. The relevant mathematical theory is measure theory and indeed Faden (1977) has proposed a measure theory foundation for social science.

An analogous situation exists with respect to the structure of activities. One of the 10 core features of International Relations theory identified by Smith relate to the topic of this section (Smith, 2004, pp. 504–506): 'the discipline has historically relied on a clear distinction between economics and politics ... its central concerns have been to explain international political relations, not economic ones'; and 'most central to International Relations is violence caused by military conflict whereas by far the most violence on the planet is economic in origin'. Expressed more generally, Smith wants to consider all spheres, not just the political sphere. In terms of the model this corresponds to an appeal to consider multiple dimensions rather than just one single dimension. Although the model has tended to focus on just a single dimension, on a number of occasions reference has been made to multiple dimensions. In particular the overall social welfare is taken to be a weighted sum of the welfare associated with each dimension. As noted earlier, Smith (2004, pp. 508–509) is concerned about all forms of violence, his UNDP statistics covering income, food, health, displacement and death, which all might be seen as dimensions of welfare. Some dimensions of welfare are very specific whereas others are overarching and therefore we can form the power set of the set of elementary dimensions. In other words there is a structure of dimensions and a corresponding structure of activities generating the outcomes on those dimensions. Welfare is a measure defined on this structure of dimensions and so measure theory is applicable here also.

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