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A. G. Pinus

Boolean Constructions in Universal Algebras

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by

A. G. Pinus

Novosibirsk Institute of Electrical Engineering,
Novosibirsk, Siberia

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TABLE OF CONTENTS

Introduction

In the last few decades the ideas, methods and results of the theory of Boolean algebras have played an ever increasing role in various branches of mathematics and cybernetics. The degree of this influence varies from field to field, but it reveals most distinctly in algebra itself and, if at first the constructions and ideas pertaining to Boolean algebras arose while developing theories of concrete classical algebraic systems: groups, rings, modules and lattices, lately they have obtained a certain universality and are being successfully used for studying algebras of various kinds, i.e., in the theory of universal algebras. At the same time, various restrictions on the application of Boolean constructions when investigating different classes of universal algebras have been elucidated.

This monograph is devoted to studying the fundamentals of the theory of Boolean constructions in universal algebras, to the problems of presenting different varieties of universal algebras with these constructions (Chapter 2) and to the use of Boolean constructions for investigating the spectra and skeletons of varieties of universal algebras (Chapter 3). Chapter 1 is of an introductory character which presents the basic notions and formulates a number of results of the theory of Boolean and universal algebras to be used in the proofs of Chapters 2 and 3. When presenting this material, the author thought it possible to omit the proofs, as at present there is a whole series of monographs dedicated to presenting both the fundamentals of the theory of Boolean and universal algebras, and to various special problems of these theories. As far as Boolean algebras are concerned, we should mention first of all a three-volume edition "Handbook of Boolean Algebras". As for the basic notions of universal algebra, there is a perfect monograph by S.Burris and H.P.Sankappanavar, "A Course of Universal Algebra", as well as a monograph by R.Freese and R.McKenzie, "Commutator Theory for Congruence Modular Varieties", and a monograph by the author "Congruence-Modular Varieties of Algebras", published in Russian. In the application section one can find some results pertaining to the elementary theory of skeletons of varieties, as well as proofs of some statements on Boolean algebras not to be found elsewhere in English literature. Besides, one can also find there fundamentals of the theory of better quasi-orders which is discussed in Chapter 3 and has not yet received a wide recognition in universal algebra.

CHAPTER 1
INTRODUCTION

1.Basic Notions of the Theory of Boolean Algebras

The aim of this section is to recall some basic notions, constructions and results associated with ordered sets and Boolean algebras of the type to be used below. The very definitions of partially, linearly, well-ordered sets and Boolean algebras, their basic properties, the definitions and properties of the algebraic operations on these sets and algebras can be found in practically any textbook on algebra or set theory. Therefore, in the present section these results will be either just mentioned or assumed to be known.

A.General notions on ordered sets and Boolean algebras

Definition 1.1.

(a) A set A characterized by a binary relation $\leq$ is called partially ordered if for any elements $a,b,c \in A$ the following statements are valid:

(1) $a \leq a$;

(2) $a \leq b$ and $b \leq a \rightarrow a = b$;

(3) $a \leq b$ and $b \leq c \rightarrow a \leq c$.

(b) A partially ordered set $< A;\leq >$ is called a linearly ordered set (LOS) if for any $a,b \in A$ one has either $a \leq b$ or $b \leq a$.

(c) A linearly ordered set $< A;\leq >$ is well-ordered if for any nonempty subset $P \subseteq A$ there exists a least element, i.e., an $a \in P$ such that for any $b \in P$ one has $a \leq b$.

(d) A non-singleton ordered set $< A;\leq >$ is said densely ordered if for any

$a,b \in A$ such that $a \leq b$ and $a \neq b$ there is a $c \in A$ such that $a \leq c \leq b$ and $c \neq a$, $c \neq b$. A linearly ordered set is said scattered if it contains no densely ordered subsets.

It should be remarked that for any densely ordered set, there is an ordered set of rational numbers isomorphically embeddable into it and, hence, an ordered set is scattered iff there is no ordered set of rational numbers imbeddable into it.

By the type of an isomorphism of a partially ordered set and, later, of an arbitrary algebraic system, we will mean either a class of all algebraic systems which are isomorphic to the given one, or a certain fixed representative of this class.

All the considerations to follow, unless otherwise specified, will be within the framework of a *ZFC* set-theoretical system and, in particular, an ordinal will be viewed as a fixed representative of the type of an isomorphism of well-ordered sets, i.e., as a transitive set which is well-ordered by the relation of a set-theoretical inclusion $\subseteq$. We will use standard notations $0,1,2,\ldots,n,\ldots$ to denote finite ordinals, $\alpha+1$ to denote the ordinal following the ordinal α (it should be recalled that the family of all the ordinals, *Ord*, is well-ordered by the same relation of the set-theoretical inclusion), ω is the least infinite ordinal, ω_i is the least ordinal of the power $\aleph_i$. A family $\{\aleph_i | i \in Ord\}$ of the powers in the system *ZFC* is also well-ordered by a standard relation of embedding on the sets of the powers considered. Henceforth $\aleph_i$ will be sometimes identified by ω_i, which is the initial ordinal of the power $\aleph_i$. The notions of the initial and of final intervals, as well as those of cofinal and coinitial subsets for partially ordered sets will be assumed known. The notions of a sum, as well as of a Cartesian and lexicographic product of partial orders are defined in a standard way.

Definition 1.2. If $< A;\leq>$, $< B;\leq>$ are partially ordered sets, then:

(a) $< A;\leq> + < B;\leq>$ (assuming $A \cap B = \varnothing$) will be understood as the set $A \cup B$ partially ordered by the relation $\leq_1$ such that for $a,b \in A(B)$, $a \leq_1 b$ iff $a \leq b$ in $< A;\leq>$ (in $< B;\leq>$) and for any $a \in A$, $b \in B$ $a \leq_1 b$;

(b) $< A;\leq> \otimes < B;\leq>$ will be understood as the Cartesian product of the sets $A \times B$ partially ordered by the relation $\leq_1$, which is a Cartesian product of the relations $\leq$ in A and B, i.e., for any $a_1,a_2 \in A$ and $b_1,b_2 \in B$, $< a_1,b_1 > \leq_1 < a_2 b_2 >$ iff $a_1 \leq a_2$ in $< A;\leq>$ and $b_1 \leq b_2$ in $< B;\leq>$;

(c) $< A;\leq> \cdot < B;\leq>$ will be understood as a lexicographical product of $< A;\leq>$ and $< B;\leq>$, i.e., a Cartesian product of the sets $A \times B$ partially ordered by the relation $\leq_1$ in such a way that for any $a_1,a_2 \in A$, $b_1,b_2 \in B$ $<a_1,b_1> \leq_1 <a_2,b_2>$ iff $b_1 < b_2$ or $b_1 = b_2$ and $a_1 \leq a_2$.

The sums and products of large families of partially ordered sets are defined in an analogous way.

Definition 1.3. If for $i \in I$, $<A_i;\leq_i>$ are partially ordered sets and $\leq$ is a partial order on the set I, then:

(a) $\sum_{i \in <I;\leq>} <A_i;\leq_i>$ is understood as a set $\coprod_{i \in I} A_i$ (assuming the sets A_i to be pairwise disjunct) which is partially ordered by the relation $\leq_1$ in such a way that for $a,b \in A_i$ $(i \in I)$, $a \leq_1 b$ iff $a \leq_i b$, and for $a \in A_i, b \in A_j$ $(i \neq j)$, $a \leq_1 b$ iff $i \leq j$ in $<I;\leq>$;

(b) $\prod_{i \in I} <A_i;\leq_i>$ will be understood as a direct product of the algebraic systems $<A_i;\leq_i>$;

(c) $L_{i \in <I;\leq>} <A_i;\leq_i>$ will be understood as a direct product of the sets $\prod_{i \in I} A_i$ partially ordered by the relation $\leq_1$ in such a way that for $f,g \in \prod_{i \in I} A_i$, $f \leq_1 g$ iff there is an $i \in I$ comparable to any other element of I in terms of $\leq$ and such that for any $j < i$ we get $f(j) = g(j)$ and $f(i) < g(i)$, or $f = g$.

It is obvious that if $<A_i;\leq_i>(i \in I)$ and $<I;\leq>$ are linearly ordered, then $\sum_{i \in <I;\leq>} <A_i;\leq_i>$ is also linearly ordered. If, moreover, $<I,\leq>$ is well-ordered, then the set $\mathbf{L}_{i \in <I;\leq>} <A_i;\leq_i>$ is also linearly ordered. If both $<A_i;\leq_i>(i \in I)$ and $<I;\leq>$ are well-ordered, then $\sum_{i \in <I;\leq>} <A_i;\leq_i>$ is also well-ordered. In the case when $<A;\leq>$ and $<B;\leq>$ are well-ordered, then $<A;\leq> \cdot <B;\leq>$ is also well-ordered. Besides, for finite sums and lexicographical products these operations on ordinals coincide with common definitions of the addition and multiplication of ordinals. Let us now recall the definition of an ordinal power. An ordinal γ is called a limit one provided that it has no last element, in which case $\gamma = \sup_{\delta < \gamma} \delta$ in a well-ordered class Ord. Any non-limit ordinal γ can be represented as $\beta + 1$, where $\beta < \gamma$.

Definition 1.4. The ordinal α^β for any ordinals α,β is defined by induction over β: $\alpha^1 = \alpha$; $\alpha^{\gamma+1} = \alpha^\gamma \cdot \alpha$; if γ is limit, then $\alpha^\gamma = \sup_{\delta<\gamma} \alpha^\delta$.

It should be remarked that for any ordinal α and any $n \in \omega$ $\alpha^n = \underset{i<n}{L}\, \alpha_i$ when $\alpha_i = \alpha$ for every $i < n$, but $\alpha^\omega \neq \underset{i<\omega}{L}\, \alpha_i$ at $\alpha_i = \alpha$ for $i < \omega$.

By induction over α one can prove the well-known fact that for any ordinal α there exist uniquely defined $n \in \omega$, ordinals $\gamma_1 > \gamma_2 > \ldots > \gamma_n$ and natural numbers $m_1, \ldots, m_n$ such that $\alpha = \omega^{\gamma_1} \cdot m_1 + \ldots + \omega^{\gamma_n} \cdot m_n$. A representation of this kind is called a normal form of the ordinal α.

Any linearly ordered set $< A; \leq >$ is either scattered or presentable as a sum $\sum_{i \in < I; \leq >} < A_i; \leq_i >$, where $< I; \leq >$ is a densely ordered LOS, while $< A_i; \leq_i >$ are scattered.

It is convenient to introduce the relation $\approx$ on A in the following way: $a \approx b$ iff the interval (a,b) of the LOS $< A; \leq >$ is scattered. Obviously, $\approx$ is an equivalence relation on A, each equivalence class over $\approx$ is a scattered interval of the LOS $< A; \leq >$, while the factor $< A/\sim; \leq >$ (in the case when $< A; \leq >$ is not scattered) is a densely LOS. In this case $< A; \leq > = \sum_{B \in < A/\approx; \leq >} < B; \leq >$, where $< B; \leq >$ are scattered intervals which are equivalence classes on the LOS $< A; \leq >$ in terms of $\approx$.

There is also an inductive process of constructing a class of all scattered LOS. Let $\mathfrak{S}_0$ be a class consisting of no empty and singleton ordered sets. Let us determine a class $\mathfrak{S}_\gamma$ for any $\gamma \neq 0$ in the following way:

$$\mathfrak{S}_{\gamma+1} = \{\sum_{i \in \delta} < A_i; \leq >, \ \sum_{i \in \delta^*} < A_i; \leq > \mid < A_i; \leq > \in \mathfrak{S}_\gamma,\ \delta \in Ord\},$$

while for a limit γ

$$\mathfrak{S}_\gamma = \bigcup_{\delta<\gamma} \mathfrak{S}_\delta .$$

Let $\mathfrak{S} = \bigcup_{\delta \in Ord} \mathfrak{S}_\delta$. The class $\mathfrak{S}$ coincides with the class of all scattered LOS. In order to prove that all scattered LOS are incorporated in the class $\mathfrak{S}$, let us introduce a sequence of equivalences $\equiv_i$ $(i \in Ord)$ on a scattered LOS. $< A; \leq >: a \equiv_0 b$ iff the interval (a,b) is finite. If the relation $\equiv_i$ is defined, then the relation $\equiv_{i+1}$ is a complete preimage of the relation $\equiv_0$ defined on the LOS $< A/\equiv_i; \leq >$ under a natural homomorphism $< A; \leq >$ on $< A/\equiv_i; \leq >$. For a limit $i \in Ord$ the relation $\equiv_i$ is defined on $< A; \leq >$ as the union of relations $\equiv_j$ at $j < i$. One can easily see that for a certain $i \in Ord$ the factor $A/\equiv_i$ of a scattered LOS $<A;\leq>$ will be a singleton set. The least i of this kind is called the rank of a scattered LOS $<A;\leq>$, while the induction over the rank proves the incorporation of any scattered LOS $<A;\leq>$ into the class $\mathfrak{S}$.

Definition 1.5. A set A with a binary relation $\leq$ is called quasi-ordered if for any $a,b,c \in A$ the following statements are valid:

(a) $a \leq a$;

(b) if $a \leq b$ and $b \leq c$, then $a \leq c$.

A natural equivalence on the quasi-ordered set $<A;\leq>$ will be a relation $a \equiv_{\leq} b$ which is valid iff $a \leq b$ and $b \leq a$. One can easily see that the relation $\equiv_{\leq}$ is indeed an equivalence relation over the set A, and for any $a,b,c,d \in A$, $a \leq b$, $c \equiv_{\leq} a$, $d \equiv_{\leq} b$ entail $c \leq d$. $[c]$ will denote an equivalence class in terms of $\equiv_{\leq}$ containing an element c, by $A/\equiv_{\leq}$ the family of all such classes. Let us introduce a relation $\leq$:$[c] \leq [d]$ iff $c \leq d$ over $A/\equiv_{\leq}$. One can easily check that $<A/\equiv_{\leq};\leq>$ will be a partially ordered set; let us call $<A/\equiv_{\leq};\leq>$ a natural partial order related to a quasi-order $<A;\leq>$.

For any quasi-ordered set $<A;\leq>$, $<A;\leq>^*$ will denote the dual of $<A;\leq>$, i.e., the quasi-ordered set $<A;\leq_1>$, where the quasi-order $\leq_1$ is defined in the following way: $a \leq_1 b$ iff $b \leq a$.

An ideal (filter) of a Boolean algebra $\mathcal{B} = <B;\wedge,\vee,\neg,0,1>$ will be, as usual, a nonempty subset $\mathfrak{J}(\mathfrak{F})$ of the basic set B of this algebra with the following properties:

(1) if $a,b \in \mathfrak{J}(\mathfrak{F})$, then $a \vee b \in \mathfrak{J}(a \wedge b \in \mathfrak{F})$;

(2) if $a \in \mathfrak{J}(\mathfrak{F}), b \in B$ and $b < a$ $(a < b)$, then $b \in \mathfrak{J}$ $(b \in \mathfrak{F})$. The ideal (filter) is proper if it is other than the whole of the Boolean algebra. The maximal among the proper ideals (filters) of a Boolean algebra of inclusion is called its maximal ideal (ultrafilter). It should be recalled that for any homomorphism φ of a Boolean algebra $\mathcal{B}$ onto a Boolean algebra $\mathcal{B}_1$ $\varphi^{-1}(0)(\varphi^{-1}(1))$ will be an ideal (filter) of the Boolean algebra $\mathcal{B}$, and, conversely, for any ideal $\mathfrak{J}$ (filter $\mathfrak{F}$) of the Boolean algebra $\mathcal{B}$ there exists a congruence θ on the algebra $\mathcal{B}$:

$$<a,b> \in \theta \leftrightarrow (a \setminus b) \vee (b \setminus a) \in \mathfrak{J} (<a,b> \in \theta \leftrightarrow \neg((a \setminus b) \vee (b \setminus a)) \in \mathfrak{F}),$$

such that under a natural homomorphism $\varphi: \mathcal{B} \to \mathcal{B} / \theta$ the equalities $\mathfrak{J} = \varphi^{-1}(0)$ ($\mathfrak{F} = \varphi^{-1}(1)$) are valid. The Boolean algebra $\mathcal{B} / \theta$ will in this case be denoted by $\mathcal{B} / \mathfrak{J}$ or $\mathcal{B} / \mathfrak{F}$, respectively. It should also be recalled that there is a mutually unique correspondence between filters and ideals: for any ideal $\mathfrak{J}$ of the Boolean algebra $\mathcal{B}$ the set $\{\neg b | b \in \mathfrak{J}\}$ is a filter.

Let us also recall the following well-known and easily verifiable result: if $\mathfrak{F}$ is a proper filter of the Boolean algebra $\mathcal{B}$, then the following conditions are equivalent:

(1) $\mathfrak{F}$ is an ultrafilter;

(2) if $a, b \in \mathcal{B}$ and $a \vee b \in \mathfrak{F}$, then either $a \in \mathfrak{F}$, or $b \in \mathfrak{F}$;

(3) for any $a \in \mathcal{B}$ we have either $a \in \mathfrak{F}$ or $\neg a \in \mathfrak{F}$;

(4) $\mathcal{B} / \mathfrak{F} \cong 2$ (a two-element Boolean algebra).

Definition 1.6. By $St(\mathcal{B})$ we will mean a topological space formed by a family of all ultrafilters of the Boolean algebra $\mathcal{B}$ with a topology, the basis of the open neighborhoods of which is a family of sets of the type $\psi_a = \{\mathfrak{F} \in St(\mathcal{B}) | a \in \mathfrak{F}\}$ for $a \in \mathcal{B}$. The topological space $St(\mathcal{B})$ is called a Stone space of the Boolean algebra $\mathcal{B}$.

One can easily observe that for any $a \in \mathcal{B}$, we have $\psi_{\neg a} = St(\mathcal{B}) \setminus \psi_a$ and, therefore, the basis of the topology of $St(\mathcal{B})$ consists of open-closed sets. One could also easily check the fact that the mapping $\varphi: a \to \varphi_a$ is an isomorphism from a Boolean algebra to the Boolean algebra of open-closed subsets of the topological space $St(\mathcal{B})$. It can be checked that $St(\mathcal{B})$ is a compact, totally disconnected topological space. The converse is also true: for any compact, totally disconnected space X. Let $B(X)$ denote a Boolean algebra of open-closed subsets of X, in which case there is a homomorphism from the space X to a Stone space $St(B(X))$ of the Boolean algebra $B(X)$. This dualism of Boolean algebras and compact totally disconnected topological spaces (sometimes called Boolean spaces) is modified by the following statement.

Theorem 1.1. Let φ be a certain homomorphism from a Boolean algebra $\mathcal{B}$ to a Boolean algebra $\mathcal{B}_1$. Let us define a mapping $S(\varphi)$ from the Stone space $St(\mathcal{B}_1)$ into the Stone space $St(\mathcal{B})$ as $S(\varphi)(p) = \{a \in \mathcal{B} | \varphi(a) \in p\}$ for any

$p \in St(\mathcal{B}_1)$. Let $f: X \to Y$ be a continuous function between compact, totally disconnected spaces X, Y. Let us determine a mapping $B(f)$ from the Boolean algebra $B(Y)$ to $B(X)$ as $B(f)(B) = f^{-1}(B)$ for any $B \in B(Y)$. Let h and ψ respectively denote the natural homomorphism of the spaces $X(Y)$ and $St(B(X))(St(B(Y)))$ and the natural isomorphism of the Boolean algebras $\mathcal{B}(\mathcal{B}_1)$ and $B(St(\mathcal{B}))(B(St(\mathcal{B}_1)))$, mentioned above. In this case the following statements are valid:

(a) $S(\varphi)$ is a continuous function;

(b) if φ is an isomorphic embedding, then $S(\varphi)$ is a mapping from the space $St(\mathcal{B}_1)$ onto the space $St(\mathcal{B})$; if φ is a homomorphism from the algebra $\mathcal{B}$ to the algebra $\mathcal{B}_1$, then $S(\varphi)$ is a homomorphic embedding of $St(\mathcal{B}_1)$ into $St(\mathcal{B})$; if φ is an isomorphism of the algebras $\mathcal{B}$ and $\mathcal{B}_1$, then $S(\varphi)$ is a homomorphism of the spaces $St(\mathcal{B})$ and $St(\mathcal{B}_1)$;

(c) $B(f)$ is a homomorphism;

(d) if f is a homomorphic embedding of X into Y, then $B(f)$ is a homomorphism from the Boolean algebra $B(Y)$ to the algebra $B(X)$; if f is a continuous mapping of the space X onto the space Y, then $B(f)$ is an isomorphic embedding of the algebra $B(Y)$ into the algebra $B(X)$; if f is a homomorphism of the spaces X and Y, then $B(f)$ is an isomorphism of the Boolean algebras $B(X)$ and $B(Y)$;

(e) the following diagrams are commutative:

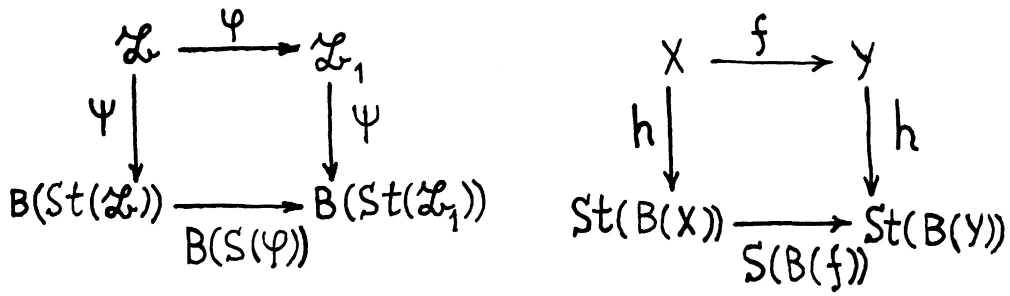

Fig. 1

(f) if, moreover, η is a homomorphism from the Boolean algebra $\mathcal{B}_1$ into the Boolean algebra $\mathcal{B}_2$, and g is a continuous function between the compact, totally disconnected spaces Y and Z, then $S(\eta \cdot \varphi) = S(\varphi) \cdot S(\eta)$ and $B(g \cdot f) = B(f) \cdot B(g)$.

Henceforth for any set I, $P(I)$ will denote the set of all subsets of the set I considered as a Boolean algebra with respect to set-theoretical operations of union, intersection and complement formation.

Let us now return to considering the properties of the sets mentioned above. Let λ be a certain cardinal (it should be recalled that λ is identified with an initial ordinal of the power λ). A subset $C \subseteq \lambda$ is closed if for any $B \subseteq C$ such that $B < x$ for a certain $x \in \lambda$, $\sup B \in C$. A subset $C \subseteq \lambda$ is unlimited provided that for any $x \in \lambda$ there is a $y \in C$ such that $y \geq x$.

Definition 1.7. A subset $S \subseteq \lambda$ is called stationary if the intersection of S with any closed unlimited subset λ is nonempty.

Let λ be a regular uncountable cardinal. One can easily see that for any stationary $S \subseteq \lambda$, the power of S is equal to λ, and the family of stationary subsets of the cardinal λ forms a filter in the Boolean algebra $P(\lambda)$.

One of the most important properties of stationary sets are the following statements.

Theorem 1.2. Let S be a stationary subset of an uncountable regular cardinal λ, and let $f: S \to \lambda$ be a regressive function (i.e., $f(\alpha) < \alpha$ for any $\alpha \in S \setminus \{0\}$). Then there exists a stationary subset $T \subseteq S$ such that f is constant on T.

Theorem 1.3. If λ is a regular uncountable cardinal, and A is a stationary subset of λ, then there are λ subsets $A_i (i \in \lambda)$ of the set A which are pairwise disjunct and stationary in λ.

B. Interval and Superatomic Boolean Algebras

The notion of a Boolean algebra with an ordered basis was first introduced by Mostowski and Tarski [143].

Definition 1.8. A Boolean algebra $\mathcal{B}$ has an ordered basis provided that there is a chain of elements of the algebra $\mathcal{B}$ generating it.

This notion is equivalent to a more descriptive notion of an interval Boolean algebra. For any LOS $<I;\leq>$ $B(I)$ will denote a Boolean algebra of the subsets of the set I generated by intervals of the kind $(a,b]$, where $a,b \in I^+$. By I^+ we denote here a LOS $<\{-\infty\};\leq> \oplus <I;\leq> \oplus <\{\infty\};\leq>$. It is obvious that any element a of the Boolean algebra $B(I)$ can be represented as $\bigcup_{i\leq n}(a_i,b_i]$ for a certain $n \in \omega$ and some elements $a_1,b_1,\ldots,a_n,b_n \in I^+$ such that $-\infty \leq a_1 < b_1 < \ldots < a_n < b_n \leq \infty$. A representation of this kind will be called a canonical representation of an element a of the algebra $B(I)$, by $\sigma(a)$ we will mean a set $\{a_1,b_1,\ldots,a_n,b_n\}$, and by $\overline{\sigma(a)}$ a tuple $< a_1,b_1,\ldots a_n,b_n >$.

Definition 1.9. An interval Boolean algebra is any Boolean algebra of the type $B(I)$, where $<I;\leq>$ is a LOS.

By a^* for $a \in I$ we will mean an element of the Boolean algebra such that $\overline{\sigma(a^*)} = < -\infty, a >$. A family of the elements $a^* (a \in I)$ is an ordered basis of the algebra $B(I)$ and, on the other hand, if the Boolean algebra $\mathcal{B}$ has an ordered basis J (under the assumption that $0,1 \notin J$), then $\mathcal{B}$ is isomorphic to the interval Boolean algebra $B(J)$.

If $<J;\leq>$ is a subset of the LOS $<I;\leq>$ with an induced order, then there is a canonical embedding f of the Boolean algebra $B(J)$ into $B(I)$: for $a \in B(J)$ we set $f(a) = b$, where $b \in B(I)$ and $\overline{\sigma(a)} = \overline{\sigma(b)}$.

Since for any homomorphism f of the interval Boolean algebra $B(I)$ on the Boolean algebra $\mathcal{B}$ we have $\mathcal{B} \cong B(f(I))$, any homomorphic image of an interval Boolean algebra is interval. It is also obvious that there exists an embedding g of the LOS $f(I)$ into the LOS I such that on $f(I)$ a mapping fg is identical and, in particular, $f(I)$ can be identified with a subset of the LOS I. Hence, according to the remarks made above, any homomorphic image of an interval Boolean algebra is isomorphic to a certain subalgebra of this algebra.

On the other hand, subalgebras of interval Boolean algebras need not be interval: a Boolean algebra of finite and co-finite subsets of an ordinal $\omega_i (i \geq 1)$ is a subalgebra of an interval algebra $B(\omega_i)$, not being itself, as can be easily seen, an interval Boolean algebra.

The class of interval Boolean algebras is quite large and, in particular, it includes all not more than countable Boolean algebras. The fact that for any $n \in \omega \setminus \{0\}$ a 2^n-element Boolean algebra is isomorphic to $B(n-1)$ is obvious.

Theorem 1.4. Any countable Boolean algebra is interval.

At the same time, a large number of most important classes of Boolean algebras contain no interval ones. For instance, any infinite countably complete Boolean algebra is not interval. Let us assume the inverse statement: let $\mathcal{B} = B(I)$ be countably complete and infinite, and, in particular, I is an infinite LOS. Therefore, in I one can find a subset $J = \{a_1, a_2, \ldots\}$ ordered either by the type ω or by the type ω^*. Assume, for instance, that $a_1 < a_2 < \ldots < a_n < \ldots$, in which case it is obvious that no element of the algebra $B(I)$ can be equal to an element $\coprod_{i \in \omega}(a^*_{2i+2} \setminus a^*_{2i+1}) \in \mathcal{B}$.

A free $\aleph_i$ – generated Boolean algebra is interval iff $i = 0$. Indeed, since a free $\aleph_0$ – generated Boolean algebra is countable, it is, by theorem **1.4**, interval. In fact it is isomorphic to the algebra $B(\eta)$, since $B(\eta)$ is a countable atomless algebra, which, as is well known, implies the property of being an $\aleph_0$ – free Boolean algebra. Here η is an ordered type of an ordered set of rational numbers. On the other hand, as has been noted above, any homomorphic image of an interval Boolean algebra is interval, and we have seen an example of Boolean algebras of finite and co-finite subsets of the ordinal $\omega_i (i \geq 1)$, which are not interval and have a power $\aleph_i$. Therefore, a $\aleph_i$-free Boolean algebra cannot be interval for $i \geq 1$.

An ideal of a LOS I is any $J \subseteq I$ which has the following property: for any $a \in J$ and $b \in I$ if $b \leq a$, then $b \in J$. A family of all nonempty proper ideals of the LOS I is linearly ordered by inclusion and is called a Dedekind completion of the LOS I. A Dedekind completion of the LOS I^+ will be denoted by iI. A LOS of the type iI is complete for any LOS I, i.e., any subset J of the LOS iI has a least upper and a biggest lower bound in iI. The LOS I itself can be identified with a subset of the LOS iI by putting the ideal $J_a = \{x \in I^+ | x \leq a\} \in iI$ into correspondence with an element $a \in I$. One can also easily notice that the set iI, which has a topology the basis of the open sets of which consists of the intervals of the set iI of the type $(a,b]$ for all $a,b \in I^+$, is homomorphic with a Stone space of the Boolean algebra $B(I)$. For this purpose it is sufficient to show that for any $J \in iI$ the family B_J is an ultrafilter of the Boolean algebra $B(I)$ and, conversely, any ultrafilter of the algebra $B(I)$ has a form B_J for a certain $J \in iI$ (here $B_J = \{a \in B(I) |$ and there exist $a_i, b_i \in \sigma(a)$ such that $a_i \in J$, and either $b_i \notin J$ or $J = J_{b_i}$). Henceforth a topological space defined on iI in the above-mentioned way will be denoted by $(iI)^t$ and, therefore, $(iI)^t$ is homomorphic with $St(B(I))$ for any LOS I.

The notion of a superatomic Boolean algebra was first introduced by Mostowski and Tarski [143] who also defined the primary basic properties of these algebras.

Definition 1.10. A Boolean algebra $\mathcal{B}$ is called superatomic if any of its homomorphic images is atomic.

This property proves to be equivalent to a whole number of others. $I(\mathcal{B})$ will denote a Frechet ideal of an arbitrary Boolean algebra $\mathcal{B}$ which can be represented as finite unions of atoms. Let us determine a sequence of the ideals $I_i(\mathcal{B})$ for any ordinal $i > 0$ in the following way: $I_1(\mathcal{B}) = I(\mathcal{B}); I_i(\mathcal{B}) = \bigcup_{j<i} I_j(\mathcal{B})$ if i is limit; $I_{i+1}(\mathcal{B})$ is a complete preimage of the ideal $I(\mathcal{B}_i)$ under a natural homomorphism from the algebra $\mathcal{B}$ to $\mathcal{B}_i = \mathcal{B} / I_i(\mathcal{B})$. Obviously, $I_i(\mathcal{B})$ is an increasing sequence of the ideals of the Boolean algebra $\mathcal{B}$ and, hence, for a certain i we have $I_i(\mathcal{B}) = I_{i+1}(\mathcal{B})$ and for all j greater than i we have $I_i(\mathcal{B}) = I_j(\mathcal{B})$.

Definition 1.11. An atomic rank of an arbitrary Boolean algebra $\mathcal{B}$ will be the least ordinal i such that $I_i(\mathcal{B}) = I_{i+1}(\mathcal{B})$. The atomic rank of the Boolean algebra $\mathcal{B}$ will be denoted by $at(\mathcal{B})$.

Obviously, for any Boolean algebra $\mathcal{B}$ the algebra $\mathcal{B} / I_{at}(\mathcal{B})$ is either singleton or atomless. By $F_\alpha(\mathcal{B})$ we will mean a filter $\{x \in \mathcal{B} \mid \neg x \in I_\alpha(\mathcal{B})\}$.

Theorem 1.5. The following properties of Boolean algebras are equivalent:

(a) $\mathcal{B}$ is superatomic;
(a^1) no homomorphic image of $\mathcal{B}$ is atomless;

(b) any subalgebra of the algebra $\mathcal{B}$ is atomic;
(b^1) $\mathcal{B}$ contains no atomless subalgebras;

(c) $\mathcal{B}$ contains no infinite free subalgebras;

(d) $\mathcal{B}$ contains no chain of elements which is ordered by the type of η- rational numbers;

(e) $I_i(\mathcal{B}) = \mathcal{B}$ for a certain ordinal i;

(f) any nonempty subspace of a Stone space $St(\mathcal{B})$ of the Boolean algebra $\mathcal{B}$ has at least one isolated point.

This theorem yields a corollary.

Corollary 1.1. Homomorphic images and subalgebras of any superatomic Boolean algebra are superatomic themselves.

It should be remarked that if $\mathcal{B}$ is a superatomic Boolean algebra, then $\mathcal{B} / I_{at}(\mathcal{B})$ is singleton, and for all $i < at(\mathcal{B})$ we have $1 \notin I_i(\mathcal{B})$. Therefore, $at(\mathcal{B})$ is a non-limit ordinal, i.e., it has the form $\alpha + 1$ for a certain ordinal α, in which case $\mathcal{B} / I_\alpha(\mathcal{B})$ is a finite Boolean algebra. Let $\mathcal{B} / I_\alpha(\mathcal{B})$ have exactly n atoms. A pair $< \alpha, n >$ will be called a characteristic of the superatomic Boolean algebra $\mathcal{B}$.

If the interval Boolean algebra $B(I)$ is superatomic, then, by the equivalence of conditions (a) and (d) of theorem 1.5, the LOS I contains no subsets of an ordered type η, i.e., it is a scattered LOS. The converse is also valid: for any scattered LOS I the Boolean algebra $B(I)$ is superatomic. Indeed, by theorem 1.5 it suffices to remark that a Dedekind completion iI of the scattered LOS I is scattered itself (it is obvious from the counter-argument). The Stone space of the Boolean algebra $B(I)$ is homomorphic to the space of $(iI)^t$, but any subspace of the space $(iI)^t$ for a scattered LOS obviously contains isolated points and, hence, by the equivalence of (a) $\leftrightarrow$ (f) of theorem **1.5** of the algebra $B(I)$ is superatomic.

For countable superatomic Boolean algebras this result allows a refinement.

Theorem 1.6. If $\mathcal{B}$ is a countable superatomic Boolean algebra with a characteristic $< \alpha, n >$ and $\alpha > 0$, then $\mathcal{B} \cong B(\omega^\alpha \cdot n)$.

Theorem **1.6** makes it possible to refine the formulation of theorem **1.4**.. As has been remarked earlier, any countable not scattered LOS can be represented as $\sum_{i \in \beta} \alpha_i$, where α_i are scattered LOS, while β is a countable densely ordered LOS. It appears, however, that in the representation of non-superatomic countable Boolean algebras as interval ones one can do (as was the case for superatomic algebras in theorem **1.6**) do with ordinals instead of arbitrary scattered LOS.

Theorem 1.7. For any countable non-superatomic Boolean algebra $\mathcal{B}$ there exist ordinals $\alpha_i (i \in \eta)$, where η are LOS ordered by the type of rational numbers that $\mathcal{B} \cong B(\sum_{i \in \eta} \alpha_i)$.

Let us now remark that if $\alpha < \beta$, or $\alpha = \beta$ and $m < n$, then a Boolean algebra $B(\omega^\beta \cdot n)$ obviously maps homomorphically to a Boolean algebra $B(\omega^\alpha \cdot m)$, and, according to what has been remarked in the beginning of this section, $B(\omega^\alpha \cdot m)$ will, in turn, be isomorphic to a certain subalgebra of the algebra $B(\omega^\beta \cdot n)$. On the other hand, since for any countable not scattered LOS I there exist isotonic mappings of both a LOS of the ordered type η^+ on I^+ and a LOS I^+ onto a LOS of the ordered type η^+, and, by theorem **1.4** and what has been said above, any non-superatomic Boolean algebra is isomorphic to $B(I)$ for a certain countable not scattered LOS I, the following corollary is valid.

Corollary 1.2. Any set of countable non-singleton superatomic Boolean algebras is comparable by the relations of embedding and epimorphism, these relations coinciding in the class of countable non-singleton superatomic Boolean algebras. Any countable Boolean algebra is embeddable into, and is a homomorphic image of, any countable non-superatomic Boolean algebra.

By $BA, BA_{\aleph_0}, BA'_{\aleph_0}, SBA, IBA$ we will henceforth mean the families of all, respectively, not more than countable, non-singleton, superatomic and interval Boolean algebras.

For any class of algebras $\mathfrak{K}$ by $\mathfrak{J}\mathfrak{K}$ we will mean the families of the types of the isomorphism of algebras of the class $\mathfrak{K}$. Let us introduce the relations of the quasi-orders $\leq$ and $\ll$ on $\mathfrak{J}\mathfrak{K}$ in the following way: for all $a,b \in \mathfrak{J}\mathfrak{K}$ we have $a \leq b (a \ll b)$, provided that a is a type of the isomorphism of a certain subalgebra of the algebra of the type of the isomorphism of b (if a is a type of the isomorphism of a certain homomorphic image of an algebra of the type of the homomorphism of b).

Definition 1.12. A skeleton of epimorphism of the class of algebras $\mathfrak{K}$ will be called a quasi-ordered class $< \mathfrak{J}\mathfrak{K}; \ll >$. A skeleton of embedding of the class $\mathfrak{K}$ will be a quasi-ordered class $< \mathfrak{J}\mathfrak{K}; \leq >$.

For the quasi-orders $\ll, \leq$ on $\mathfrak{J}\mathfrak{K}$ let us introduce equivalence relations naturally associated with it: for $a,b \in \mathfrak{K}$ $a \equiv_{\ll} b$ $(a \equiv_{\leq} b)$ iff $a \ll b$ and $b \ll a$ (when $a \leq b$ and $b \leq a$).

A subclass B of the quasi-ordered class $< A; \leq >$ is called a semi-ideal if for any $a \in A, b \in B$ from $a \leq b$, we have $a \in B$.

By corollary **1.1** a family $\mathfrak{J}SBA$ is a semi-ideal both in the skeleton of epimorphism and in that of embedding of a variety of all Boolean algebras BA. As has been noticed above, $\mathfrak{J}IBA$ is a semi-ideal in the skeleton of epimorphism of BA but not an ideal in the skeleton of embedding of BA. Besides, the quasi-order $\leq$ is an extension of the quasi-order $\ll$ on the class $\mathfrak{J}IBA$, i.e., for any $a,b \in \mathfrak{J}IBA$ from $a \ll b$ we have $a \leq b$. Corollary **1.2** implies the existence of the following isomorphisms: $< \mathfrak{J}BA'_{\aleph_0}; \leq > \cong < \mathfrak{J}BA_{\aleph_0}; \ll > \cong \omega_1 \oplus 1^*$, where the quasi-order $\omega_1 \oplus 1^*$

is obtained from the ordinal ω_1 by adding $2^{\aleph_0}$ mutually equivalent relative $\equiv_{\ll}$ and $\equiv_{\leq}$ elements as the last ones. These $2^{\aleph_0}$ elements form the types of the isomorphism of countable non-superatomic Boolean algebras.

For uncountable superatomic, though even interval, Boolean algebras theorem **1.6** is no longer valid. One can easily see that a superatomic Boolean algebra $B(\omega + \omega_1^*)$ is isomorphic to no Boolean algebra of the type $B(\gamma)$, where γ is an ordinal. Indeed, the only real candidate to play the part of γ is obviously ω_1. But $B(\omega + \omega_1) \not\leq B(\omega_1)$, as any countable set of atoms of the algebra $B(\omega_1)$ is contained in a certain element of this algebra which belongs to an ideal $I_i(B(\omega_1))$, for $i < \omega_1$. This statement is obviously false for the element $(n,n+1]$ of the Boolean algebra $B(\omega + \omega_1^*)$, where $n \in \omega$. Nonetheless, for uncountable interval superatomic Boolean algebras there also exists, to the accuracy of mutual embedding, their representation in the form of the algebras $B(I)$ for certain LOS I in the sense of a canonical form.

$\mathfrak{K}^k$, where k is an arbitrary cardinal, will denote a family of algebras of the class $\mathfrak{K}$ of the power k. For any ordinal α, any natural m,n such that $m+n \geq 1$, $B_{\alpha,m,n}$ will denote an interval superatomic Boolean algebra $B(\omega^{\alpha} \cdot m + (\omega^{\alpha} + (\omega^{\alpha})^*) \cdot n)$. For any cardinal $k \geq \aleph_1$ N_k will denote $\{B_{\alpha,m,n} \mid |\alpha| = k;\ m,n \in \omega;\ m+n \geq 1\}$. It should be remarked that the algebras $B_{\alpha,m,n}$ have a characteristic $< \alpha, m+n >$. By *SIBA* we will mean the family of all interval superatomic Boolean algebras.

The theorem presented below describes the skeleton of embedding of the class of superatomic interval Boolean algebras of an arbitrary fixed power.

Theorem 1.8.

(1) Let $\mathcal{B}$ be an interval superatomic Boolean algebra of a power k and a characteristic $< \alpha, p >$, in which case:

(a) if $cf(\omega^{\alpha}) = \omega$, then there is a unique algebra $B_{\alpha,m,0} \in N_k$ such that $\mathcal{B} \equiv_{\leq} B_{\alpha,m,0}$, in which case $m = p$;

(b) if $cf(\omega^{\alpha}) \neq \omega$, then there is a unique algebra $B_{\alpha,m,n} \in N_k$ such that $\mathcal{B} \equiv_{\leq} B_{\alpha,m,n}$, in which case $m+n = p$.

(2) Let algebras $B_{\alpha,m,n}$, $B_{\beta,p,q}$ belong to the family N_k, in which case $B_{\alpha,m,n} \leq B_{\beta,p,q}$ iff:

(a) $\alpha < \beta$;

(b) $\alpha = \beta$, $cf(\omega^{\alpha}) = \omega$ and $m+n \leq p+q$, in which case, if $m+n = p+q$, then $B_{\alpha,m,n}$, $B_{\alpha,p,q}$, $B_{\alpha,m+n,0}$ are isomorphic, or

(c) $\alpha = \beta$, $cf(\omega^{\alpha}) > \omega$, $m+n \leq p+q$ and $m+2n \leq p+2q$.

(3) $<\mathfrak{I}SIBA^k/\equiv_{\le};\le>\cong<\mathfrak{I}N'_k;\le>$, and they are distributive lattices, each of their elements having only a finite number of elements incomparable with them. Here $N'_k=\{B_{\alpha,m,n}|\alpha\in Ord;|\alpha|=k,m,n\in\omega;m+n\ge 1$ and, if $cf(\alpha)=\omega$, then $n=0\}$.

As an example, let us present the initial interval of the lattice $<\mathfrak{I}N^{\alpha};\le>$ at $cf(\omega^{\alpha})>\omega$. In this case $<m,n>$ will denote the algebra $B_{\alpha,m,n}$, $N^{\alpha}=$ $=\{B_{\alpha,m,n}|m,n\in\omega\}$.

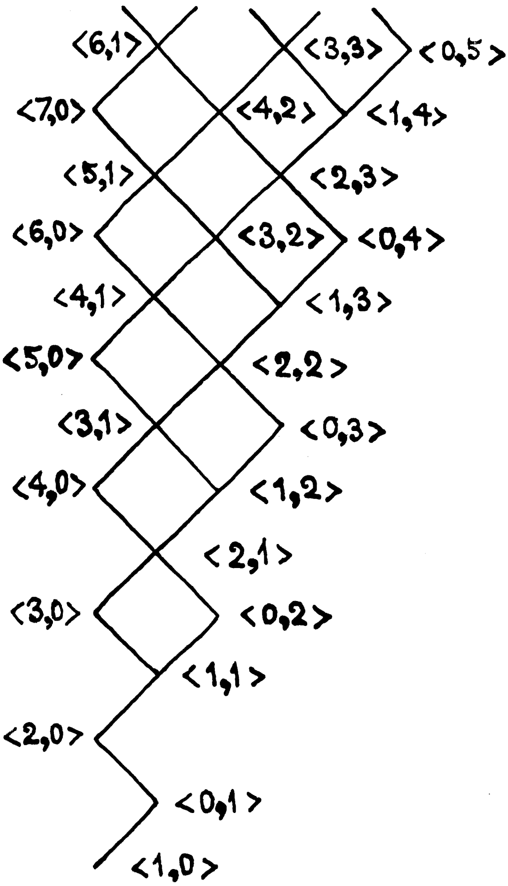

Fig. 2

It should be recalled that for countable non-singleton Boolean algebras the relations of embedding and epimorphism coincide. The problem of a complete description of the skeletons of epimorphism of the classes of superatomic interval Boolean algebras of a fixed uncountable power is still open to discussion. However, as follows from the theorem presented below, for the skeletons of epimorphism of these classes the situation is essentially different from that described in theorem **1.8**.

Theorem 1.9. For any uncountable cardinal $\aleph_i$ and for any $I \subseteq \omega_i$, there exists a superatomic interval Boolean algebra $\mathcal{B}_I$ of a power $\aleph_i$ such that for any $I, J \subseteq \omega_i$ $\mathcal{B}_I$ is embeddable into $\mathcal{B}_J$, and $\mathcal{B}_I \ll \mathcal{B}_J$ iff $I \subseteq J$.

The proof of this theorem is given in §**14** of Applications and employs theorem **1.3**..

Corollary 1.3. For any uncountable cardinal $\aleph_i$:

(a) there are $2^{\aleph_i}$ of mutually embeddable superatomic interval Boolean algebras of a power $\aleph_i$, none of which is a homomorphic image of the other;

(b) any partially ordered set of the power not greater than $\aleph_i$ is isomorphically embeddable into $< \mathfrak{Z}SIBA_{\aleph_i}; \ll>$, in such a way that the images of the elements of this set are mutually embeddable into each other.

The proof of this corollary results immediately from the statements of theorem **1.9** that for any cardinal $\aleph_i$ there exist $2^{\aleph_i}$ mutually incomparable subsets of the ordinal ω_i, and that any partially ordered set of the power not greater than $\aleph_i$ is isomorphically embeddable into the set of all subsets of the ordinal ω_i.

Further on we will also need the following statement which results from the proof of theorem **1.9**.

Corollary 1.4. There is an infinite number of mutually embeddable interval Boolean algebras $\mathfrak{C}_i$ of a power $\aleph_1$, none of which is a homomorphic image of any of these algebras, in which case for every set of algebras $\mathfrak{C}_i$ the set A forms an ultrafilter on $\mathfrak{C}_i$, where $A=\{d \in \mathfrak{C}_i | \mathfrak{C}_i | d$ contains a chain of elements of the ordinal type $\eta \cdot \omega_1\}$.

It should be recalled that a universal algebra $\mathfrak{A}_1$ is called a retract of the algebra $\mathfrak{A}_2$ provided that there is a homomorphism f of the algebra $\mathfrak{A}_2$ on $\mathfrak{A}_1$, and an embedding h of the algebra $\mathfrak{A}_1$ into $\mathfrak{A}_2$ such that fh is identical on $\mathfrak{A}_1$.

A universal algebra $\mathfrak{A}$ is called retractive if any non-singleton homomorphic image of $\mathfrak{A}$ is its retract. An equivalent definition: $\mathfrak{A}$ is retractive if for any non-unit congruence α of the algebra $\mathfrak{A}$ there is a subalgebra $\mathfrak{A}_1$ of the algebra $\mathfrak{A}$ such that for any $a \in \mathfrak{A}$ $|a/\alpha \cap \mathfrak{A}_1| = 1$.

As has been noted above, interval algebras are retractive. But in fact retractivity is inherent to a much larger class of Boolean algebras.

Theorem 1.10. Any subalgebra of an interval Boolean algebra is retractive.

Rotman [203] put forward a hypothesis that subalgebras of interval algebras exhaust the class of retractive Boolean algebras. Rubin [204] disproved this hypothesis under various set-theoretical propositions $(\Diamond_{\aleph_1}, MA, CH)$.

Theorem 1.11. $(\Diamond_{\aleph_1}, MA, CH)$. There exist retractive Boolean algebras not embeddable in any interval ones.

In a more general situation of *ZFC* the problem of the existence of a retractive Boolean algebra not embeddable into any interval algebra is still open to discussion. The question whether a subalgebra of a retractive Boolean algebra is always retractive also remains unsolved.

C. *Rigid Boolean Algebras*

A number of results of this section has been formulated under the assumption of a continuum or a generalized continuum hypothesis. A factual proof of these results often requires weaker but also less popular set-theoretical assumptions, while the formulations presented here are due to the author's unwillingness to get deep in the 'swamp' of set-theoretical assumptions in this monograph.

Definition 1.13. A LOS $<I;\leq>$ is said complete if any of its limited subsets has a least upper and a biggest lower bound in $<I;\leq>$. A LOS is said uniform if for any $a,b \in I$ such that $a < b$, the equality $|(a,b)| = |I|$ holds.

A uniform non-singleton LOS is obviously densely ordered.

Definition 1.14. A LOS $<I;\leq>$ of a regular infinite power will be said formally real if $<I;\leq>$ is complete, uniform, and there is a dense subset I_1 in

$<I;\leq>$ such that $2^{|I_1|}=|I|$, and for any interval (a,b) of the set I, there exists a monotonous embedding of (a,b) into itself without fixed points.

An example of a formally real LOS is the LOS of all real numbers. ${}^{\omega_i}2$ will denote the lexicographically ordered LOS of all the sequences of the length ω_i, consisting of 0 and 1 and such that they have no last zero. By $({}^{\omega_i}2)_0$ we will mean the subset of the LOS ${}^{\omega_i}2$ which consists of the sequences with the totality of their units limited. It should be remarked that at $i=0$, ${}^{\omega_i}2$ and $({}^{\omega_i}2)_0$ can be identified with the LOS of all real numbers and that of all rational numbers. One can easily notice that under the assumption of *GCH* ${}^{\omega_i}2$ will be a formally real LOS, while $({}^{\omega_i}2)_0$ will be a subset dense in ${}^{\omega_i}2$, with $|{}^{\omega_i}2|=2^{|({}^{\omega_i}2)_0|}$. Therefore, under the assumption of *GCH*, formally real LOSes do exist in any uncountable unlimit power. Further on we will supply the formally real LOS $<I;\leq>$ with an interval topology with a basis of open sets of the type (a,b) for $a,b\in I$. The following lemma will play a major role in constructing interval rigid Boolean algebra.

Lemma 1.1. Let $<I;\leq>$ be a formal real LOS, in which case there is a subset $P\subseteq I$ such that

(1) for any $a<b$ from I we have $|P\cap[a,b]|=|I|$,

(2) for any $P'\subseteq P$ any strictly monotonous (increasing or decreasing) mapping f from the LOS $<P';\leq>$ to $<P;\leq>$, the inequality $|\{x\in P'|f(x)\neq x\}|<|I|$ is valid,

(3) $I\setminus P$ is dense in I.

Definition 1.15.

(a) An algebra $\mathfrak{A}$ is called rigid if it has no non-trivial automorphisms;

(b) the algebra $\mathfrak{A}$ is called strictly rigid if the only embedding into it is identical;

(c) The Boolean algebra $\mathfrak{A}$ is called Bonnet-rigid if for any Boolean algebra $\mathfrak{A}_1$, any homomorphism f_1 of $\mathfrak{A}$ on $\mathfrak{A}_1$, and any embedding f_2 of $\mathfrak{A}$ into $\mathfrak{A}_1$ the equality $f_1=f_2$ holds.

__Lemma 1.2.__ A Boolean algebra $\mathfrak{A}$ is Bonnet-rigid iff for any Boolean algebra $\mathfrak{A}_2$, any embedding f_1 of $\mathfrak{A}_2$ in $\mathfrak{A}$, and any homomorphism f_2 of $\mathfrak{A}_2$ on $\mathfrak{A}$, the equality $f_1 = f_2$ is valid.

Strictly rigid and Bonnet-rigid Boolean algebras are interrelated in the following way.

__Lemma 1.3.__ If a Boolean algebra is retractive and strictly rigid, it is also Bonnet-rigid.

Before going over to the statements on the existence of rigid Boolean algebras, let us consider in detail the situation with interval Boolean algebras of the type $B(P)$, where P is a subset of formally real LOS I of a power $|I|$.

Let us assume that both P and $I \setminus P$ are dense in I. Let $\mathbb{A}$ be an arbitrary set of elements of a Boolean algebra $B(P)$ such that $|\mathbb{A}|=|I|$. Let us number the elements of $\mathbb{A}$ and let $\mathbb{A} = \{u_j | j < \omega_i\}$, where $\aleph_i = |I|$. Let us assume that $u_j = \bigcup_{k=1}^{l(j)} (a_k^j, b_k^j]$ is a canonical presentation of the elements u_j of the Boolean algebra $B(P)$. Then, since $\aleph_i > \aleph_0$ and is regular, there is a natural $m \geq 1$ and a subset $R \subseteq \omega_i$ such that $|R|=|I|$ and for any $j \in R$ $l(j) = m$. As I_1 is dense in I, for any $j \in R$ there are $r_k^j, q_k^j \in I_1$ such that $a_k^j < r_k^j < q_k^j < b_k^j$ for $k \leq l(j) = m$. Since $|I_1|<|I|$ and $|I|$ is regular, then there is a $R_0 \subseteq R$ such that $|R_0|=|I|$, and there are $r_1 < q_1 < r_2 <...< r_m < q_m \in I_1$ such that for $j \in R_0$ we have $a_k^j < r_k < q_k < b_k^j$ at $k \leq l(j) = m$.

On R_0 let us now introduce an equivalence relation θ in the following way: $<i, j> \in \theta_1$ iff $a_1^i = a_1^j$. If $|R_0 / \theta_1|=|I|$, let us choose a certain set R_1 of representatives of equivalence classes over θ_1 of the set R_0 and then $|R_1|=|I|$ and for any $i,j \in R_1$ $a_1^i \neq a_1^j$. If $|R_0 / \theta_1|<|I|$, then since $|I|$ is regular, one of the equivalence classes (let us denote it as R_1) has the power $|I|$, in which case for any $i,j \in R_1$ we have $a_1^i = a_1^j$. The equivalence θ_2 on R_1 will be defined analogously to the equivalence θ_2, stemming from elements b_1^i instead of a_1^i. Continuing this process, we finally get a set $R_{2m} \subseteq \omega_i$ such that $|R_{2m}|=|I|$, and for any $k \leq m$, or for any $i,j \in R_{2m}$ we have $b_k^i \neq b_k^j$, while for any $i,j \in R_{2m}$ we have $b_k^i = b_k^j$. An analogous property is also valid for a_k^i instead of b_k^i. It should be recalled that in the case when for a given k for any $i \neq j \in R_{2m}$ we have $b_k^i \neq b_k^j$ $(a_k^i \neq b_k^j)$, then there is no more than one $i(k,+\infty) \in R_{2m}$ such that $b_k^{i(k,+\infty)} = +\infty$ (not more than one $i(k,-\infty) \in R_{2m}$ such that $a_k^{i(k,-\infty)} = -\infty$). Let us set $R' = R_{2m} \setminus \{i(1,-\infty), \quad i(m,+\infty)\}$.

A set $R(\mathcal{A}) = \{u_i | i \in R'\}$ will be called a subtraction of the set $\mathcal{A}$ of the elements of the Boolean algebra $B(P)$, the number m will be called a general number for $R(\mathcal{A})$, while a sequence $r_1 < q_1 < r_2 < ... < r_m < q_m$ will be called a separating sequence for the elements of $R(\mathcal{A})$. Let $\rho^1_R(\mathcal{A})(\rho^2_R(\mathcal{A}))$ be a set of such $k \leq m$, that $a^i_k \neq a^j_k$ $(b^i_k \neq b^j_k)$ for various $i, j \in R'$. Let us also remark that $|R(\mathcal{A})| = |I|$, and for any $S \subseteq R(\mathcal{A})$ such that $|S| = |I|$, S is a subtraction for $\mathcal{A}$.

Lemma 1.4. Let I be a formally real LOS, a subset $P \subseteq I$ obey the conclusion of lemma **1.1**, and let $\mathcal{A}$ be a chain of elements of a Boolean algebra $B(P)$, $|\mathcal{A}| = |I|$ and $R(\mathcal{A}) = \mathcal{A}$. In this case we have either $|\rho^1_R(\mathcal{A})| = 1$ and $\rho^2_R(\mathcal{A}) = \varnothing$, or $\rho^1_R(\mathcal{A}) = \varnothing$ and $|\rho^2_R(\mathcal{A})| = 1$.

Let us denote the only $k \leq m$ belonging to $\rho^i_R(\mathcal{A}) \cup \rho^2_R(\mathcal{A})$ by $k(\mathcal{A})$ and $j(\mathcal{A}) = 1$, or by 2, depending on the fact if this k belongs to $\rho^1_R(\mathcal{A})$ or to $\rho^2_R(\mathcal{A})$. Therefore, under the conditions of lemma **1.4** for any $u_i, u_j \in \mathcal{A}$, for $k \neq k(\mathcal{A})$ we have $a^i_k = a^j_k$, $b^i_k = b^j_k$, while for $k = k(\mathcal{A})$ we have $a^i_k = a^j_k$ and $b^i_k < b^j_k$ iff $u_i \subseteq u_j$, if $j(\mathcal{A}) = 2$ and $b^i_k = b^j_k$, while $a^i_k < a^j_k$ iff $u_i \supseteq u_j$ if $j(\mathcal{A}) = 1$.

These statements result in the following lemma.

Lemma 1.5. Let $P, I, \mathcal{A}$ be such as in lemma **1.4** and let h be a strictly increasing (strictly decreasing) function from $\mathcal{A}$ in $B(P)$. Therefore, $h(\mathcal{A})$ is a chain of the power $|I|$ in $B(P)$. Let $G_1 = R(h(\mathcal{A}))$ and $\mathcal{A}_1 = h^{-1}(G_1)$, then $\mathcal{A}_1$ and G_1 are chains, and $j(\mathcal{A}_1) = j(G_1)(j(\mathcal{A}_1) \neq j(G_1))$. Let us set

$$\sum(\mathcal{A}_1, G_1) = \{u \in \mathcal{A}_1 | a^u_{k(\mathcal{A}_1)} \neq a^{h(u)}_{k(G_1)}\}$$

if $j(\mathcal{A}_1) = j(G_1) = 1$;

$$\sum(\mathcal{A}_1, G_1) = \{u \in \mathcal{A}_1 | b^u_{k(\mathcal{A}_1)} \neq b^{h(u)}_{k(G_1)}\}$$

if $j(\mathcal{A}_1) = j(G_1) = 2$;

$$\sum(\mathcal{A}_1, G_1) = \{u \in \mathcal{A}_1 | a^u_{k(\mathcal{A}_1)} \neq b^{h(u)}_{k(G_1)}\}$$

if $j(\mathcal{A}_1) = 1$, $j(G_1) = 2$, and

$$\sum(\mathcal{A}_1, G_1) = \{u \in \mathcal{A}_1 \mid b^{u}_{k(\mathcal{A}_1)} \neq a^{h(u)}_{k(G_1)}\}$$

if $j(\mathcal{A}_1) = 2,\quad j(G_1) = 1.$

In this case $|\sum(\mathcal{A}_1, G_1)| < |I|$.

Now on the basis of lemmas **1.1-1.5** the following theorem is proved.

Theorem 1.12. Let I be a formally real LOS, $P \subseteq I$ and obey the conclusions of lemma **1.1**. Let P_1 be a subset of P such that for any $a < b \in I$ $|(a,b) \cap P_1| = |I|$ and f is a strictly increasing mapping from $B(P_1)$ into $B(P)$. In this case, f is identical and, in particular, the Boolean algebra $B(P)$ is strictly rigid.

Theorem **1.12**, lemma **1.3**, theorem **1.10** and the remark on the existence of a formally real LOS after definition **1.14** together yield the following corollary.

Corollary 1.5. There are Bonnet-rigid interval Boolean algebras of a continual power. Under the conditions of *GCH* there are Bonnet-rigid interval Boolean algebras of any power.

By analogy with the proof of theorem **1.12** we can prove the following theorem.

Theorem 1.12'. Let I be a formally real LOS, $P \subseteq I$ and obey the conclusion of lemma **1.1.** Let P_1 be a subset P of a power $|I|$. Then there is a subset $\hat{P}_1 \subseteq P_1$ such that

(1) $|P_1 \setminus \hat{P}_1| < |I|$;

(2) for any $a < b \in I$ $\quad |(a,b) \cap \hat{P}_1| = |I|$ or $(a,b) \cap \hat{P}_1 = \varnothing$;

(3) if $a < b \in I$ and $(a,b) \cap \hat{P}_1 \neq \varnothing$, then $(a,b) \cap \hat{P}_1$ has no first or last element, in which case $B(\hat{P}_1)$ is strictly rigid.

Besides proving the existence of separate rigid Boolean algebras, the above construction makes it possible to construct large families of Bonnet-rigid Boolean algebras possessing some properties pertaining to the relations of embedding and epimorphism.

Theorem 1.13. If I is a certain formally real LOS, then

(a) there is a family $\{B_l^0 | l \in 2^{|2|}\}$ of mutually unembeddable Bonnet-rigid Boolean algebras of the power $|I|$;

(b) (GCH) there exists a family $\{B_l^1 | l \in 2^{|2|}\}$ of Bonnet-rigid Boolean algebras of the power $|I|$, ordered linearly by the relation of embedding ;

(c) there is a family $\{B_j^2 | j \in \omega_i\}$, $\{B_j^3 | j \in \omega_i\}$ of Bonnet-rigid Boolean algebra such that $B_{j_1}^2 \le B_{j_2}^2$ iff $j_1 \le j_2$, and $B_{j_1}^3 \le B_{j_2}^3$ iff $j_1 \ge j_2$. Here ω_i is the initial ordinal of the power $|I|$.

It should be recalled that a family of subsets G of a certain infinite set R is called almost disjunct iff for any $A, B \in G$, $|A| = |B| = |R|$ and $|A \cap B| < |R|$. Then the following statement is valid.

Lemma 1.6. Let I be a formally real LOS of a power $\aleph$, $P \subseteq I$ obey the conclusions of lemma **1.1**. Let $P', P'' \subseteq P$ $|P'| = |P''| = \aleph$ and $|P' \cap P''| < \aleph$, in which case if $\mathcal{B}$ is a Boolean algebra isomorphic to subalgebras of the algebras $B(P')$ and $B(P'')$, then $|\mathcal{B}| < \aleph$.

Using this lemma and the fact that under the assumption of GCH, as is well known, for any infinite set of a power k there is a family of its almost disjunct subsets of a power 2^k, the following theorem is proved.

Theorem 1.14 (GCH) For any unlimited cardinal $\aleph$ there is a family $G = \{B_i | i \in I\}$ of strictly rigid Boolean algebras of a power $\aleph$ such that:

(a) $|I| = 2^{\aleph}$;

(b) for any Boolean algebra $\mathcal{B}$ embeddable into any pair of different algebras of the family G, the power of $\mathcal{B}$ is less than $\aleph$;

(c) for any Boolean algebra $\mathcal{B}$ which is a homomorphic image of a pair of different algebras of the family G, the power of $\mathcal{B}$ is less than $\aleph$.

Definition 1.16. A chain B in a quasi-ordered set $<A;\leq>$ is called noncompactable iff for any $a \in A$, a belongs to the chain B if for some $c_1, c_2 \in B$ we have $c_1 \leq a \leq c_2$ and a is comparable with any element of the chain B.

It follows from clause (b) that in the skeleton of epimorphism of a variety of Boolean algebras, a family of countable Boolean algebras is an ideal and form a noncompactable chain of a scattered quasi-order $\omega_1 \oplus 1^*$. In the same way a family of Boolean algebras of the type $B(\alpha)$, where $\alpha \in Ord$, is an ideal in $<\mathfrak{Z}BA;\ll>$, and forms a noncompactable chain isomorphic to an ordered class of all ordinals. The natural question arises whether in $<\mathfrak{Z}BA;\ll>$ there exist noncompactable chains of dense order types, i.e., noncompactable chains $B \subseteq \mathfrak{Z}BA$ such that $<B/\equiv_{\ll};\ll>$ is a densely ordered set.

Let R be an ordered set of real numbers, $Q \subseteq R$ be a set of rational numbers, and let $P \subseteq R$ obey the conclusion of lemma **1.1**. For any $a \in R$ let us define P_a as $\{x \in P | x < a\}$.

Lemma 1.7.

(a) Boolean algebras $B(P_a)$ are Bonnet-rigid;

(b) for $a < b \in R$, $B(P_a) \ll B(P_b)$, and for any non-singleton Boolean algebra $\mathcal{B}$, $B(P_a) \times \mathcal{B} \not\ll B(P_a)$;

(c) for any a and any Boolean algebra $\mathcal{B}$, it follows from $\mathcal{B} \equiv_{\ll} B(P_a)$ that $\mathcal{B} \cong B(P_a)$;

(d) (CH) for $a \in R$ and any Boolean algebra $\mathcal{B}$, if for all $b \in R$ such that $b > a$, $B(P_a) \ll \mathcal{B} \ll B(P_b)$, then there exists a countable set $D \subseteq \{x \in P | x \geq a\}$ such that $\mathcal{B} \cong B(P_a \cup D)$;

(e) for $a \in R$ and any Boolean algebra $\mathcal{B}$, if for all $b \in R$ such that $b < a$, $B(P_b) \ll \mathcal{B} \ll B(P_a)$, then we have $\mathcal{B} \cong B(P_a)$.

This lemma can be used to prove the following theorem.

Theorem 1.15. (CH) In the skeleton of epimorphism of a variety of Boolean algebras there is a noncompactable chain B of a dense ordered type, i.e.,

$< B/\equiv_{\ll};\ll>$, has the order type of a set of real numbers.

Let us give an example of constructing rigid Boolean algebras. The roots of this construction originate from S.Shelah [207], who constructed large families of mutually unembeddable models of not superstable theories, which are based on constructing corresponding families of trees. It should be recalled that a tree is a partially ordered set $< C;\leq>$ which has the following property: for any set $a \in C$, $\{b \in C | b \leq a\}$ is a well-ordered subset of $< C;\leq>$.

Henceforth if ξ is a certain sequence of $< a_0,a_1,...,a_\alpha,... | \alpha < \beta >$, then $l(\xi)$ will denote its length, i.e., the ordinal β. For any $\alpha < \beta$, $\xi | \alpha$ will denote the initial segment of this sequence, of length $\alpha < a_0, a_1,...a_\gamma,... | \gamma < \alpha >$, while $\xi[\alpha]$ will denote an element a_α. For any sequences ξ_1,ξ_2, $\xi_1{}^\wedge\xi_2$ denotes a sequence obtained by putting the sequence ξ_2 in the end of ξ_1 with a corresponding reindexation of the elements of the sequence ξ_2. For any ordinals λ,μ, $\lambda^{\leq\mu}(\lambda^{<\mu})$ will denote a family of all the sequences of length not greater than μ (strictly less than μ) consisting of the elements of the ordinal λ.

Definition 1.17. Let $\mathfrak{K}$ be an arbitrary class of models of a fixed signature and let the models $\mathfrak{A}_\alpha (\alpha \in J)$ belong to the class $\mathfrak{K}$. A discrete sum of the models $\mathfrak{A}_\alpha (\alpha \in J)$ is a model $\sum_{\alpha \in J} \mathfrak{A}_\alpha$ with a basis set of the type $\coprod_{\alpha \in J} \{\alpha\} \times \mathfrak{A}_\alpha$ such that for any signature predicate $R(x_1,...,x_n)$, any $\alpha_1,...,\alpha_n \in J$ and any $a_l \in \mathfrak{A}_{\alpha_l}, l \leq n$ on $\sum_{\alpha \in J} \mathfrak{A}_\alpha$, $R(<\alpha_1,a_1>,...,<\alpha_n,a_n>)$ is true iff $\alpha_1 = \alpha_2 = ... = \alpha_n$ and $\mathfrak{A}_{\alpha_1} \models R(a_1,...a_n)$.

Let L be the language of a countable functional signature which consist of $\aleph_0$ of different n-unary functional symbols for any $n < \omega$.

Definition 1.18. For any model $\mathfrak{A} \in \mathfrak{K}$, $M(\mathfrak{A})$ will denote an algebraic system of the signature comprising the signature of the language L, the signature of the class $\mathfrak{K}$ and one more unary predicate P. In this case the reduction of $M(\mathfrak{A})$ to the signature of the language L is an absolutely free L-algebra generated by the basic set of the model $\mathfrak{A}$; the predicate P singles out the basic set of the model $\mathfrak{A}$ in $M(\mathfrak{A})$, the reduction of $M(\mathfrak{A})$ to the language of the class $\mathfrak{K}$ on the set $\mathfrak{A}$ coincides with the model $\mathfrak{A}$, and for any $a_1,...,a_n \in M(\mathfrak{A})$, any predicate $R(x_1,...,x_n)$ of the signature of the class $\mathfrak{K}$, if for a certain $i \leq n$ $a_i \in (\mathfrak{A}) \setminus \mathfrak{A}$, then $M(\mathfrak{A}) \models \neg R(a_1,...a_n)$.

Definition 1.19. By $\mathfrak{K}_{tr}^{\omega}$ we will mean the following class of the models $\mathfrak{A}$:

(a) a basic set of the model $\mathfrak{A}$ is a certain family of sequences of ordinals whisch are either finite or of a length ω such that all initial segments of this sequence enter $\mathfrak{A}$ with every such sequence belonging to $\mathfrak{A}$;

(b) the predicates $P_i(i \leq \omega), <_1, <, Eq_i(i \in \omega)$ and a constant $<>$ are defined on $\mathfrak{A}$ in the following way: $P_i = \{\xi \in \mathfrak{A} \mid l(\xi) = i\}$; $\xi < \gamma$ iff $\xi = \gamma \mid l(\xi)$; $<_1 =$ $= \{<\xi^\wedge <\alpha>, \xi^\wedge <\beta>> \mid \xi^\wedge <\alpha>, \xi^\wedge <\beta> \in \mathfrak{A}, \alpha < \beta\}$;
$Eq_i = \{<\xi, \nu> \mid \xi, \nu \in \mathfrak{A}, \xi \mid i = \nu \mid i\}$; the constant $<>$ coincides with an empty sequence.

Therefore, the models $\mathfrak{A} \in \mathfrak{K}_{tr}^{\omega}$ are refined trees: the basic set of the model with the predicate $<_1$, and the discrete sum of the $\mathfrak{K}_{tr}^{\omega}$-models (with obvious additional definitions related to the addition of an empty sequence to the discrete sum) is again a model of the class $\mathfrak{K}_{tr}^{\omega}$.

By $\psi(x_0, x_1, y_0 y_1)$ we will mean the following formula of calculus $L_{\omega_1,\omega}$ of the signature of the class $\mathfrak{K}_{tr}^{\omega}$:

$$\psi(x_0, x_1, y_0, y_1) = \bigvee_{i+1<\omega} [P_{i+1}(x_0) \& P_{i+1}(y_0) \&$$
$$P_\omega(x_1) \& Eq_i(x_0, y_0) \& x_0 \neq y_0 \& x_1 = y_1 \& x_0 <_1 y_1 \& y_0 < x_0].$$

Definition 1.20. The model $\mathfrak{A} \in \mathfrak{K}_{tr}^{\omega}$ is said ψ – unembeddable into a model $\mathfrak{C} \in \mathfrak{K}_{tr}^{\omega}$ if for any mapping f of the model $\mathfrak{A}$ into $M(\mathfrak{C})$, for any finite subset $A \subseteq \mathfrak{C}$ there can be found elements $a_0, a_1, b_0, b_1 \in \mathfrak{A}$ such that $\mathfrak{A} \models \psi(a_0, a_1, b_0, b_1)$, and for some terms $\tau_i(x_1, \ldots, x_{n_i})$ ($i = 0,1$) of the language L, for some $c_1^0, \ldots, c_{n_0}^0, c_1^1, \ldots, c_{n_1}^1, d_1^0, \ldots, d_{n_0}^0, d_1^1, \ldots, d_{n_1}^1 \in \mathfrak{C}$ at $i = 0,1$

$$f(a_i) = \tau_i(c_1^i, \ldots, c_{n_i}^i),\ f(b_i) = \tau_i(d_1^i, \ldots, d_{n_i}^i),$$

while the tuples $< c_1^0, \ldots, c_{n_0}^0, c_1^1, \ldots, c_{n_1}^1 >, < d_1^0, \ldots, d_{n_0}^1, d_1^1, \ldots, d_{n_1}^1 >$ implement the same quantifierless type over A in $\mathfrak{C}$.

Definition 1.21. A class $\mathfrak{K}_{tr}^{\omega}$ will be called $(\aleph, \lambda)$-wide (where $\aleph, \lambda$ are arbitrary cardinals), if there is a family $\mathfrak{A}_i (i < \aleph)$ of the $\mathfrak{K}_{tr}^{\omega}$-models such that $|\mathfrak{A}_i| = \lambda$, and for any $i \neq j < \aleph$, $\mathfrak{A}_i$ is ψ-unembeddable into $\mathfrak{A}_j$. The class $\mathfrak{K}_{tr}^{\omega}$ will be called $(\aleph, \lambda)$-superwide if there is a family $\mathfrak{A}_i (i < \aleph)$ of the $\mathfrak{K}_{tr}^{\omega}$-models such that $|\mathfrak{A}_i| = \lambda$, and for any $i < \aleph$, $\mathfrak{A}_i$ is ψ-unembeddable into $\sum_{j<\aleph, j\neq i} \mathfrak{A}_i$.

Lemma 1.8. If $\mathfrak{K}_{tr}^{\omega}$ is an $(\aleph,\lambda)$-superwide class, then $\mathfrak{K}_{tr}^{\omega}$ is an $(\aleph',\lambda)$-wide class, where $\aleph' = \min\{2^{\aleph}, 2^{\lambda}\}$.

Theorem 1.16.

(a) If λ is a regular cardinal, $\lambda > \aleph_0$ and $\lambda^* \geq \lambda$, then the class of models $\mathfrak{K}_{tr}^{\omega}$ is (λ,λ^*)-superwide;

(b) if λ is a singular cardinal, $2^{\aleph_0} < \lambda$ and $\lambda^{\aleph_0} = \lambda$, then $\mathfrak{K}_{tr}^{\omega}$ is a (λ,λ)-superwide class;

(c) if $\lambda^{\aleph_0} = \lambda$, then $\mathfrak{K}_{tr}^{\omega}$ is $(2^{\lambda},\lambda^{+})$-superwide.

The above theorem and lemma **1.8** give rise to the following corollary.

Corollary 1.6. For any regular uncountable λ, the class $\mathfrak{K}_{tr}^{\omega}$ is $(2^{\lambda},\lambda)$-wide.

The constructed $\mathfrak{K}_{tr}^{\omega}$-models will be now used for constructing rigid Boolean algebras and families of Boolean algebras which are not mutually interrelated with relations of embedding and epimorphism.

Definition 1.22. An ordinal tree is an arbitrary family of sequences with the relation $<$. If $\mathfrak{A}$ is a certain ordinal tree, then $B(\mathfrak{A})$ will denote a Boolean algebra freely generated by a set of elements $\{x_\eta | \eta \in \mathfrak{A}\}$ modulo the following defining relations:

(1) for $\alpha \neq \beta$ if $\eta^\wedge\langle\alpha\rangle, \eta^\wedge\langle\beta\rangle \in \mathfrak{A}$, then $x_{\eta^\wedge\langle\alpha\rangle} \cap x_{\eta^\wedge\langle\beta\rangle} = 0$;

(2) if $\eta < \nu$, then $x_\nu \leq x_\eta$;

(3) if η has a finite number of extensions of the type $\eta^\wedge\langle\alpha\rangle$ in $\mathfrak{A}$, and $\eta^\wedge\langle\alpha_l\rangle\ (l < k)$, where $k < \omega$ are all these extensions, then $x_\eta = \bigcup_{l<k} x_{\eta^\wedge\langle\alpha_l\rangle}$;

(4) if $\eta < \nu$ and for any ρ such that $\eta < \rho < \nu$ there is the only extension of

the type $\rho^\wedge<\alpha>$ in $\mathfrak{A}$, then $x_\eta = x_\nu$. One can also remark that $B(\mathfrak{A})$ will be an interval Boolean algebra.

Definition 1.23.

(a) Let $\mathfrak{A} \in \mathfrak{K}_{tr}^{\omega}$ and let $\bar{l}, \bar{h}$ be tuples formed of the elements of $M(\mathfrak{A})$. Then if $\bar{l} \approx \bar{h} (\mathrm{mod}\, M(\mathfrak{A}))$ $(l(\bar{l}) = l(\bar{h}))$, then there is a tuple $\bar{\tau}(x_1,\dots,x_n)$ of the terms of the language L, and there are tuples $\bar{c}, \bar{d}$ of the elements of the model $\mathfrak{A}$ such that $l(\bar{c}) = l(\bar{d}) = n$; $\bar{\tau}(\bar{c}) = \bar{l}, \bar{\tau}(\bar{d}) = \bar{h}$, while the tuples $\bar{c}, \bar{d}$ implement the same quantifierless type in the model $\mathfrak{A}$.

(b) An arbitrary model $\mathfrak{C}$ is representable in $M(\mathfrak{A})$ if there is a function f mapping the model $\mathfrak{C}$ in $M(\mathfrak{A})$ such that for any $a_1,\dots,a_n, b_1,\dots,b_n \in \mathfrak{C}$, it follows from $< f(a_1),\dots,f(a_n) > \approx < f(b_1),\dots,f(b_n) > (\mathrm{mod}\, M(\mathfrak{A}))$ that $< a_1,\dots,a_n >$ and $< b_1,\dots,b_n >$ implement the same quantifierless type in the model $\mathfrak{C}$.

Evidently, if for a certain $\mathfrak{C} \in \mathfrak{K}_{tr}^{\omega}$ either $\mathfrak{C}$ itself or a Boolean algebra $B(\mathfrak{C})$ is representable in $M(\mathfrak{A})$ for a certain $\mathfrak{A} \in \mathfrak{K}_{tr}^{\omega}$, then the model $\mathfrak{C}$ cannot be ψ-unembeddable into $\mathfrak{A}$.

Lemma 1.9. If for a certain ordinal tree $\mathfrak{A}$ a Boolean algebra $\mathfrak{B}_0 = B(\mathfrak{A})$ is representable in $M(\mathfrak{A}')$ for a certain $\mathfrak{A}' \in \mathfrak{K}_{tr}^{\omega}$, then for any Boolean algebra $\mathfrak{B}$ which is a homomorphic image of the algebra $\mathfrak{B}_0$, there is an ordinal tree $\mathfrak{C}$ such that $\mathfrak{B} \cong B(\mathfrak{C})$, and $\mathfrak{B}$ is also representable in $M(\mathfrak{A}')$.

Definition 1.24. Let a family $\mathfrak{A}_i (i < \lambda)$ of $\mathfrak{K}_{tr}^{\omega}$-models implement the (λ,λ)-superwideness of the class $\mathfrak{K}_{tr}^{\omega}$ for a certain cardinal λ. Let us construct an increasing (by inclusion) continuous sequence of Boolean algebras $\mathfrak{B}_i (i < \lambda)$ in the following way: $\mathfrak{B}_0$ is a two-element Boolean algebra, $\mathfrak{B}_{i+1} = \mathfrak{B}_i | \neg a_i \times B(\mathfrak{A}_i)$, where the sequence $< a_i | i < \lambda >$ is a certain sequence of all the atoms of the algebras $\mathfrak{B}_i (i < \lambda)$, in which case $a_i \in \mathfrak{B}_i$. Let us denote by R a sequence $<< \mathfrak{A}_i, a_i > | i < \lambda >$ taking part in the construction of a sequence of Boolean algebras $\mathfrak{B}_i$, and define a Boolean algebra $\mathfrak{B}_R$ as $\bigcup_{i<\lambda} \mathfrak{B}_i$.

Lemma 1.10. In the notations of definition **1.24** the following are valid:

(a) a Boolean algebra $\mathcal{B}_i$ is representable in $M(\sum_{j<i}\mathcal{U}_j)$;

(b) a Boolean algebra $\mathcal{B}_i|(1\setminus a_j)$ is representable in $M(\sum_{l<i,l\neq j}\mathcal{U}_l)$ for any $j<i$.

By choosing suitable algebras of the type $\mathcal{B}_R$, and using lemma **1.10**, we can prove the following theorem.

__Theorem 1.17.__ If the class $\mathfrak{K}_{tr}^{\omega}$ is (λ,λ)-superwide, then the following statements are valid:

(1) there is a Boolean algebra $\mathcal{B}$ of the power λ such that for any $a,b\in\mathcal{B}\setminus\{0\}$, if $a\cap b=0$, then there is no embedding of the algebra $\mathcal{B}|a$ into any homomorphic image of the Boolean algebra $\mathcal{B}|b$.

(2) there are Boolean algebras $\mathcal{B}_i(i<2^{\lambda})$ of the power λ such that for any $i\neq j<2^{\lambda}$, any $a\in\mathcal{B}_i\setminus\{0\}, b\in\mathcal{B}_j\setminus\{0\}$, there is no embedding of the algebra $\mathcal{B}_i|a$ into any homomorphic image of the algebra $\mathcal{B}_j|b$.

Theorems **1,16** and **1.17** yield, in particular, the following corollary.

__Corollary 1.7.__ For any regular cardinal λ greater than $\aleph_0$, there is a family $\mathcal{B}_i(i\in 2^{\lambda})$ of Boolean algebras of the power λ which are mutually unembeddable and are not homomorphic images of each other.

Theorems **1.16** and **1.17** yield, for example, the validity of the statement of corollary **1.7** for singular λ such that $2^{\aleph_0}<\lambda$, and $\lambda^{\aleph_0}=\lambda$.

It should be recalled that the Boolean algebra $\mathcal{B}$ obeying clause (1) of theorem **1.17** is Bonnet-rigid and, in particular, has no injective endomorphism onto itself. Indeed, let us assume that f_1 is a certain homomorphism from $\mathcal{B}$ to an arbitrary algebra $\mathcal{B}_1$, and f_2 is an embedding of $\mathcal{B}$ into $\mathcal{B}_1$ such that $f_1\neq f_2$. If f_1 is not injective, then there is an $a\in\mathcal{B}$ such that $a\neq 0$ and $f_1(a)=0$. For any $b\in\mathcal{B}$, $f_1(b\setminus a)=f_1(b)$ and, hence, $\mathcal{B}_1$ is a homomorphic image of the algebra $\mathcal{B}|(1\setminus a)$. On the other hand, $\mathcal{B}|a$ is embeddable into $\mathcal{B}_1$. The thus obtained contradiction proves the impossibility of a similar situation. If f_1 is injective, then f_1 is an isomorphism from $\mathcal{B}$ to $\mathcal{B}_1$ and, hence, $f_1^{-1}f_2:\mathcal{B}\to\mathcal{B}$ is not an identical embedding of $\mathfrak{B}$ into itself. Therefore, for a certain $a \in \mathfrak{B}$, $a \cap f_2^{-1} f_2(a) = 0$, but in this case the elements a and $b = f_1^{-1} f_2(a)$ contradict the statement (1) of theorem **1.17**. It is this fact that proves that any Boolean algebra obeying this statement is Bonnet-rigid.

It should be remarked in this context that S.Shelah [208] proved the existence of rigid Boolean algebras of any uncountable power.

D. Invariants of Countable Boolean Algebras and their Monoid

A system of invariants for countable Boolean algebras was first suggested by J.Ketonen [104]. Y.L.Ershov [57] extended this system to the class of distributive lattices with a relative complement, in which case, unlike J.Ketonen who used Stone spaces of Boolean algebras , Y.L.Ershov's proofs are purely algebraic.

Definition 1.25. A distributive lattice $< A;\cup,\cap,0>$ with the least element 0 is a lattice with relative complements, if for any of its elements a,b, the inequality $a \le b$ yields that there exists an element $c \in A$ such that $a \cup c = b$ and $a \cap c = 0$. This element c is called the complement of a relative to b. From now on, a distributive lattice with relative complements will be called a DILARC.

It should be remarked that for any DILARC $\mathfrak{A}$ and any $a \le b \in \mathfrak{A}$, the complement of a relative to b is unique. This makes it possible to introduce an operation $\setminus$ on the DILARC $\mathfrak{A}$, setting for $c,d \in \mathfrak{A}$ $c \setminus d$ equal to the complement of the element $c \cap d$ relative to c. It should be remarked that any homomorphism from the DILARC $\mathfrak{A}$ to the DILARC $\mathfrak{B}$ in the signature $< \cup,\cap,0 >$ will also be a homomorphism in the signature $< \cup,\cap,\setminus,0>$. A DILARC is a Boolean algebra iff it has a biggest element. Any ideal of a Boolean algebra is a DILARC. On the other hand, any DILARC can be represented as an ideal, and even as a maximal ideal of a Boolean algebra. If a DILARC is a Boolean algebra itself, this is obvious. Now let $\mathfrak{A} =< A;\cup,\cap,0 >$ be a DILARC with no biggest element. Let $\bar{A}$ denote a family $A \cup \{\neg a | a \in A\}$, where $\{\neg a | a \in A\}$ is a set disjunct from A which is in a one-to-one correspondence $a \rightarrow \neg a$ with A. The operations $\cup,\cap$ are naturally extended from A to $\bar{A}$, owing to the fact that the element $\neg a$ must play the role of a complement of the element $a \in A$ in a Boolean algebra $< \bar{A};\cup,\cap,\neg,0,1 = \neg 0 >$. In this case, $\mathfrak{A}$ is obviously a maximal ideal of the constructed Boolean algebra. The notions of an ideal and of a filter of a DILARC are introduced in the same way as for Boolean algebras, and any congruence of a DILARC is uniquely defined by an ideal, i.e., by a class of congruence containing 0. Let $\hat{a}$ be the principal ideal of the DILARC $\mathfrak{A}$ generated by an element a.

Definition 1.26. An embedding φ of a DILARC $\mathfrak{A}_0$ into a Boolean algebra $\mathfrak{A}$ will be called an extension of the DILARC $\mathfrak{A}_0$ using a Boolean algebra $\mathfrak{A}_1$ provided that $\varphi(\mathfrak{A}_0)$ is an ideal in $\mathfrak{A}$ and $\mathfrak{A} / \varphi(\mathfrak{A}_0) \cong \mathfrak{A}_1$, where the latter isomorphism is fixed (its composition with a natural homomorphism $\mathfrak{A} \to \mathfrak{A} / \varphi(\mathfrak{A}_0)$ will be denoted as $\hat{\varphi}$).

The next aim will be a description of all the extensions of the DILARC $\mathfrak{A}_0$ using the Boolean algebra $\mathfrak{A}_1$. Two extensions, $\varphi':\mathfrak{A}_0 \to \mathfrak{A}'$ and $\varphi'':\mathfrak{A}_0 \to \mathfrak{A}''$, will be said equivalent if there exists an isomorphism ψ of the DILARC $\mathfrak{A}'$ with the DILARC $\mathfrak{A}''$ such that $\psi\varphi' = \varphi''$ and $\hat{\varphi}''\psi = \hat{\varphi}'$. $Ext(\mathfrak{A}_0, \mathfrak{A}_1)$ will denote a family of all extensions of the DILARC $\mathfrak{A}_0$ using the Boolean algebra $\mathfrak{A}_1$ to the accuracy of the equivalence introduced.

Definition 1.27. Let $\mathfrak{A}$ be an ideal of a DILARC $\overline{\mathfrak{A}}$.Let $\overline{\mathfrak{A}}$ be an ideal complement of the DILARC $\mathfrak{A}$, if for any embedding φ of the DILARC $\mathfrak{A}$ there exists, as an ideal into any DILARC $\mathfrak{A}_1$, and the only one, a homomorphism ψ of the DILARC $\mathfrak{A}_1$ into the DILARC $\overline{\mathfrak{A}}$ such that $\psi\varphi$ is identical on $\mathfrak{A}$.

The definition of an ideal complement $\overline{\mathfrak{A}}$ for any DILARC $\mathfrak{A}$ obviously yields its uniqueness as an extension of the DILARC $\mathfrak{A}$. Let us show the existence of an ideal complement for any DILARC $\mathfrak{A}$. $J(\mathfrak{A})$ will denote a family of all ideals of an arbitrary DILARC $\mathfrak{A}$ with the lattice operations introduced in a standard way on $J(\mathfrak{A})$: for $J_1, J_2 \in J(\mathfrak{A})$, $J_1 \cap J_2$ is the intersection of J_1 and J_2 as sets, $J_1 \cup J_2$ is the least ideal of the DILARC $\mathfrak{A}$ containing J_1 and J_2. $J'(\mathfrak{A})$ will denote a subfamily of the family $J(\mathfrak{A})$, consisting of the so-called locally principal ideals; an ideal $J \in J(\mathfrak{A})$ is called locally principal if the intersection of J with any principal ideal of the lattice of $\mathfrak{A}$ is also a principal ideal. A family $J'(\mathfrak{A})$ is a sublattice of the lattice of $J(\mathfrak{A})$. Any principal ideal is locally principal, and a mapping $a \to \hat{a}$ will be an embedding of the lattice of $\mathfrak{A}$ into the lattice of $J'(\mathfrak{A})$. In this case the principal ideals of the lattice of $\mathfrak{A}$ form an ideal (henceforth it will be identified with the lattice of $\mathfrak{A}$ itself) in the lattice of locally principal ideals of the DILARC $\mathfrak{A}$. It should be also remarked that for Boolean algebras $\mathfrak{A}$ the notion of a locally principal ideal coincides with that of a principal ideal and, hence, $J'(\mathfrak{A}) \cong \mathfrak{A}$.

Lemma 1.11. The lattice of locally principal ideals of any DILARC $\mathfrak{A}$ is an ideal complement of the DILARC $\mathfrak{A}$.

For any DILARC $\mathfrak{A}$ by $\mathfrak{A}'$ we will define a factor $J'(\mathfrak{A})/\mathfrak{A}$. The following result describes all the extensions of the DILARC using Boolean algebras.

Theorem 1.18. For any DILARC $\mathfrak{A}_0$ and a Boolean algebra $\mathfrak{A}_1$ there exists a one-to-one correspondence between all the extensions $Ext(\mathfrak{A}_0,\mathfrak{A}_1)$ of the DILARC $\mathfrak{A}_0$ using $\mathfrak{A}_1$ (to the accuracy of the equivalence introduced above) and all the homomorphisms $Hom(\mathfrak{A}_1,\mathfrak{A}_0')$.

It should be remarked that either directly or using theorem **1.18** and the fact that $J'(\mathfrak{A}_0)=\mathfrak{A}_0$ for any Boolean algebra $\mathfrak{A}_0$, any extension of a Boolean algebra $\mathfrak{A}_0$ using the Boolean algebra $\mathfrak{A}_1$ has the form $\mathfrak{A}_0\times\mathfrak{A}_1$.

As is the case for Boolean algebras, we can introduce the notion of an atom for a DILARC as well: an element a of the DILARC $\mathfrak{A}$ is called an atom if for any $c\in\mathfrak{A}$, $a\cap c=0$ or $a\cap c=a$. The DILARC $\mathfrak{A}$ is called atomless if it has no atoms, or atomic if for any $c\in\mathfrak{A}$ there is an atom a such that $a\leq c$. The DILARC $\mathfrak{A}$ is called superatomic if any of its homomorphic images is atomic. By reproducing the definitions from section (b) we can introduce the notion of a Frechet ideal $I(\mathfrak{A})$ for an arbitrary DILARC $\mathfrak{A}$ and, iterating this notion, obtain a sequence of ideals $I_\alpha(\mathfrak{A})$, where α is an arbitrary ordinal. As is the case for Boolean algebras, an atomic rank $at(\mathfrak{A})$ for the DILARC $\mathfrak{A}$ will be the least ordinal i such that $I_{i+1}(\mathfrak{A})=I_i(\mathfrak{A})$. However, unlike Boolean algebras, as can be easily seen, the atomic rank of a DILARC can, generally speaking, be a limit ordinal.

Definition 1.28. The atomic type of a superatomic DILARC $\mathfrak{A}$ will be a triple $\tau(\mathfrak{A})=<\alpha^*,\alpha_1,n>$, where α^*,α_1 are ordinals, and n is a natural number such that α_1 is the least ordinal of β, for which $\mathfrak{A}/I_\beta(\mathfrak{A})$ has a biggest element (i.e., is a Boolean algebra). We, obviously, have $\alpha_1\leq at(\mathfrak{A})$. If $\alpha_1=at(\mathfrak{A})$, then $\alpha^*=at(\mathfrak{A})$ and $n=0$. If $\alpha_1<at(\mathfrak{A})$, then one can easily see that $at(\mathfrak{A})$ is not limiting, and then α^* is such that $at(\mathfrak{A})=\alpha^*+1$, and n is a number of atoms of the Boolean algebra $\mathfrak{A}/I_{\alpha^*}(\mathfrak{A})$.

It should be remarked that for any superatomic Boolean algebra $\mathfrak{A}$ of type $<\alpha,n>$, its atomic type will be $<\alpha,0,n>$. An atomic type $\tau(a)$ of an element a of a superatomic DILARC $\mathfrak{A}$ will be an atomic type of the DILARC $\hat{a}$.

Definition 1.29. For any limit ordinal α, $\mathfrak{D}(\alpha)$ will denote a DILARC (which is not a Boolean algebra) of subsets of the ordinal α generated in the signature $<\cup,\cap,\setminus,0>$ by intervals of the type $(a,b]$ for $a \le b \in \alpha$. In this case the following analog of theorem **1.6** is valid.

Lemma 1.12. For any countable ordinal α, if a DILARC $\mathfrak{A}$ is such that $\tau(\mathfrak{A}) = <\alpha,\alpha,0>$, then $\mathfrak{A} \cong \mathfrak{D}(\omega^{\alpha})$.

Theorem 1.19. For any countable ordinals $\beta \le \alpha$, natural number n, and DILARC $\mathfrak{A}$ such that $\tau(\mathfrak{A}) = <\alpha,\beta,n>$, we have:

(a) if $\beta = 0$, then $\mathfrak{A} \cong B(\omega^{\alpha} \cdot n)$;

(b) if $\beta = \alpha, n = 0$, then $\mathfrak{A} \cong \mathfrak{D}(\omega^{\alpha})$;

(c) if $\alpha \ge \beta > 0, n > 0$, then $\mathfrak{A} \cong B(\omega^{\alpha} \cdot n) \times \mathfrak{D}(\omega^{\beta})$.

This theorem immediately yields the following corollary.

Corollary 1.8. Any countable superatomic DILARCs of identical atomic types are isomorphic.

Definition 1.30. The atomic type of an arbitrary DILARC $\mathfrak{A}$ will be a triple $\tau(\mathfrak{A}) = \tau(I_{at(\mathfrak{A})}(\mathfrak{A}))$, and the atomic type of an element $a \in \mathfrak{A}$ will be a triple $\tau(a) = \tau(\hat{a} \cap I_{at(\mathfrak{A})}(\mathfrak{A}))$. $\delta(\mathfrak{A})(\delta(a))$ will denote the second component of the triple $\tau(\mathfrak{A})(\tau(a))$ which will be called a special rank of the DILARC $\mathfrak{A}$ (of the element a).

Definition 1.31. A function f of a DILARC $\mathfrak{A}$ in a certain ordinal will be called additive if f obeys the condition: for any $a,b \in \mathfrak{A}$ $f(a \cup b) = \max\{f(a), f(b)\}$ and $f(0) = 0$.

Lemma 1.13. For any DILARC $\mathfrak{A}$, the function of a special rank $\delta(a)$ is additive.

Definition 1.32. A countable superatomic DILARC $\mathfrak{A}$ will be called special if its atomic type has the form $<\delta(\mathfrak{A}),\delta(\mathfrak{A}),0>$.

Definition 1.33. For any set of DILARCs $\mathfrak{A}_i (i \in I)$, $\sum_{i\in I}\mathfrak{A}_i$ will denote a sublattice of the direct product $\prod_{i\in I}\mathfrak{A}_i$ of a DILARC $\mathfrak{A}_i$ with a basic set

$$\{f \in \prod_{i\in I}\mathfrak{A}_i \,|\, |\{j \in I \,|\, f(j) \neq 0\}| < \aleph_0\}.$$

From lemma **1.11** one can easily deduce the following corollary.

Corollary 1.9. An ideal complement of the DILARC $\sum_{i\in I}\mathfrak{A}_i$ is isomorphic to the direct product of the ideal complements of the DILARC $\mathfrak{A}_i$.

Lemma 1.14. Let $\mathfrak{A}$ be a special DILARC.

(a) if $\delta(\mathfrak{A}) = \alpha + 1$, then $\mathfrak{A}$ is representable as $\mathfrak{A} = \sum_{n\in\omega}\mathfrak{A}_n$, where $\mathfrak{A}_n$ is a superatomic Boolean algebra of the atomic type $<\alpha,0,1>$;

(b) if $\delta(\mathfrak{A})$ is limiting and $\delta(\mathfrak{A}) = \lim_{n\in\omega}\alpha_n$, where α_n is an increasing sequence of ordinals, then $\mathfrak{A}$ is representable as $\mathfrak{A} \cong \sum_{n\in\omega}\mathfrak{A}_n$, where $\mathfrak{A}_n$ are superatomic Boolean algebras of the atomic type $<\alpha_n,0,1>$.

According to corollary **1.9** and to the above, an ideal complement of a Boolean algebra is isomorphic to it, an ideal complement $J'(\mathfrak{A})$ of a special DILARC $\mathfrak{A}$ has the form $\prod_{n\in\omega}\mathfrak{A}_n$, where $\mathfrak{A}_n$ are Boolean algebras used in the statements (a) and (b) of lemma **1.14**. In $J'(\mathfrak{A}) = \prod_{n\in\omega}\mathfrak{A}_n$ let us determine a chain of ideals $\bar{I}_\beta(\beta \leq \delta(\mathfrak{A}))$, setting $\bar{I}_\beta = \{f \in \prod_{n\in\omega}\mathfrak{A}_n | f(n) \in I_\beta(\mathfrak{A})\}$. Therefore, for any locally principal ideal J of a special DILARC $\mathfrak{A} = \sum_{n\in\omega}\mathfrak{A}_n$ $J \in \bar{I}_\beta$ iff $J \subseteq I_\beta(\mathfrak{A})$, where β is an arbitrary ordinal. This fact implies, in particular, that the definition of ideals $\bar{I}_\beta$ on an ideal extension $J'(\mathfrak{A})$ of a special DILARC $\mathfrak{A}$ is independent of the choice of its decomposition $\sum_{n\in\omega}\mathfrak{A}_n$.

Definition 1.34. For a special DILARC $\mathfrak{A}$, ψ_β (β-ordinal $\leq \delta(\mathfrak{A})$) will denote the image of an ideal $\bar{I}_\beta$ under a natural homomorphism

$$J'(\mathfrak{A}) = \prod_{n\in\omega} \mathfrak{A}_n \to \mathfrak{A}' = J'(\mathfrak{A})/\mathfrak{A} = \prod_{n\in\omega}\mathfrak{A}_n \Big/ \sum_{n\in\omega}\mathfrak{A}_n,$$

where $\mathfrak{A} = \sum_{n\in\omega}\mathfrak{A}_n$ is a decomposition discussed in lemma 1.14. The function $\rho:\mathfrak{A}' \to \delta(\mathfrak{A})+1$ will be defined in the following way: for $a\in\mathfrak{A}'$ we have $\rho(a) = \min\{\beta |\, a\in\psi_\beta\}$.

Lemma 1.15. For any special DILARC $\mathfrak{A}$, any ideal $J\in J'(\mathfrak{A})$, if d is an image of J at a natural homomorphism $J'(\mathfrak{A})\to\mathfrak{A}'$, then $\rho(d) = \delta(J)$.

Lemma 1.16. For any special DILARC $\mathfrak{A}$, the mapping $\rho:\mathfrak{A}'\to\delta(\mathfrak{A})+1$ has the following properties:

(1) $\rho(d_1\cup d_2) = \max\{\rho(d_1),\rho(d_2)\}, \rho(d) = 0 \Leftrightarrow d = 0$;

(2) for any $\beta\leq\delta(\mathfrak{A})$ there is a $d\in\mathfrak{A}'$ such that $\rho(d) = \beta$;

(3) if $\beta\leq\rho(d)$, then there is a $d_0\in\mathfrak{A}'$ such that $d_0\leq d$, $\rho(d_0) = \beta$ and $\rho(d\setminus d_0) = \rho(d)$.

For the special DILARC $\mathfrak{A}$, the pair $<\mathfrak{A}',\rho>$ is, as can be seen in the following statement, universal.

Lemma 1.17. Let $\mathfrak{A}$ be a special DILARC, $\alpha = \delta(\mathfrak{A})$, $\mathfrak{A}_0$ be not more than a countable Boolean algebra, $r:\mathfrak{A}_0\to\alpha+1$ be an additive function, and $r(1_{\mathfrak{A}_0}) = \alpha$. Then there is a homomorphism $\psi:\mathfrak{A}_0\to\mathfrak{A}'$ such that $r(d) = \rho(\psi(d))$ for any $d\in\mathfrak{A}_0$.

Since all countable atomless Boolean algebras are isomorphic, and any Boolean algebra $\mathfrak{A}/I_{at(\mathfrak{A})}(\mathfrak{A})$ is atomless, any countable Boolean algebra $\mathfrak{A}$ is either superatomic or is an extension of the superatomic DILARC $I_{at(\mathfrak{A})}(\mathfrak{A})$ by a countable atomless Boolean algebra $B(\eta)$. It should be recalled that by theorem **1.6** a complete system of invariants for countable superatomic Boolean algebras will be their types.

Definition 1.35. A countable Boolean algebra $\mathfrak{A}$ is called normalized provided that $I_{at(\mathfrak{A})}(\mathfrak{A})$ is special and $\mathfrak{A} / I_{at(\mathfrak{A})}(\mathfrak{A}) \cong B(\eta)$.

Lemma 1.18. Any countable non-superatomic Boolean algebra $\mathfrak{A}$ can be represented as $\mathfrak{A}_0 \times \mathfrak{A}_1$, where:

(1) $\mathfrak{A}_0$ is a normalized Boolean algebra;

(2) $\mathfrak{A}_1$ is either a singleton or a superatomic Boolean algebra of type $< \alpha,n >$, in which case $\alpha \geq \delta(\mathfrak{A}_0)$.

Let us now consider the types of the isomorphism of normalized Boolean algebras.

Definition 1.36. Let $\mathfrak{A}_0$ be a special DILARC of a special rank α, and $\mathfrak{A}$ be an extension of $\mathfrak{A}_0$ by an atomless countable Boolean algebra $B(\eta)$ (we assume $\mathfrak{A}_0 \subseteq \mathfrak{A}$). Let $\varphi_{\mathfrak{A}}$ be a natural homomorphism from $\mathfrak{A}$ to $B(\eta)$ arising when $\mathfrak{A}$ is factorized over $\mathfrak{A}_0$. By lemma **1.13**, the function $\delta{:}\mathfrak{A} \to \alpha+1$ is additive. As for any superatomic element $d \in \mathfrak{A}$, $\delta(d)=0$, the epimorphism $\varphi_{\mathfrak{A}}$ induces a mapping $r_{\mathfrak{A}}: B(\eta) \to \alpha+1$ such that $r_{\mathfrak{A}}(\varphi_{\mathfrak{A}}(d)) = \delta(d)$ for any $d \in \mathfrak{A}$. Obviously, $r_{\mathfrak{A}}$ is an additive function on $B(\eta)$. The pair $< B(\eta); r_{\mathfrak{A}} >$ will be called an atomic approximation of the extension of $\mathfrak{A}$.

Lemma 1.19. If $\mathfrak{A}_1, \mathfrak{A}_2$ are two extensions of a special DILARC $\mathfrak{A}_0$ by a countable atomless Boolean algebra $B(\eta)$, then the Boolean algebras of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic iff the pairs $< B(\eta); r_{\mathfrak{A}_1} >$ and $< B(\eta); r_{\mathfrak{A}_2} >$ are isomorphic, i.e., when there is an automorphism μ of the algebra $B(\eta)$ such that $r_{\mathfrak{A}_1} = r_{\mathfrak{A}_2} \cdot \mu$.

Lemma 1.20. Let $\mathfrak{A}$ be a special DILARC, $\delta(\mathfrak{A}) = \alpha$. Then for any additive function $r{:}B(\eta) \to \alpha+1$ such that $r(1_{B(\eta)}) = \alpha$, there is a Boolean algebra $\mathfrak{A}_r$ that is an extension of the DILARC $\mathfrak{A}$ with a Boolean algebra $B(\eta)$, and such that the atomic approximation of $\mathfrak{A}$ $< B(\eta); r_{\mathfrak{A}} >$ is equal to $< B(\eta); r >$.

Lemma 1.21. A decomposition of countable non-superatomic Boolean algebras $\mathfrak{A}$ into a product of normalized and superatomic (or singleton) ones, discussed in lemma **1.18** is unique, i.e., if $\mathfrak{A} \cong \mathfrak{A}_0 \times \mathfrak{A}_1 \cong \mathfrak{A}_0^* \times \mathfrak{A}_1^*$ are two such representations, then $\mathfrak{A}_0 \cong \mathfrak{A}_0^*$, $\mathfrak{A}_1 \cong \mathfrak{A}_1^*$.

<u>**Definition 1.37.**</u> We will call a system of invariants for a countable Boolean algebra $\mathfrak{A}$:

(a) the type $<\alpha,n>$ of the algebra $\mathfrak{A}$ if $\mathfrak{A}$ is superatomic;

(b) the set $<<B(\eta);r_{\mathfrak{A}_0}>, \ \tau(\mathfrak{A}_1)>$, where $\mathfrak{A}_0 \times \mathfrak{A}_1$ is a representation of the algebra $\mathfrak{A}$ as a product of a normalized algebra $\mathfrak{A}_0$ by a superatomic algebra $\mathfrak{A}_1$, and where $\delta(\mathfrak{A}_0) \le \tau_1(\mathfrak{A}_1)$ (here $\tau_1(\mathfrak{A}_1)$ is the first component of the triple $\tau(\mathfrak{A}_1)$ when $\mathfrak{A}$ is not superatomic).

A system of invariants of the algebras $\mathfrak{A}, \mathfrak{A}^*$ will be called isomorphic if they coincide in the case of superatomic algebras $\mathfrak{A}$ and $\mathfrak{A}^*$, or if $<B(\eta);r_{\mathfrak{A}_0}> \cong <B(\eta);r_{\mathfrak{A}_0^*}>$, $\tau(\mathfrak{A}_1) = \tau(\mathfrak{A}_1^*)$ for the corresponding decompositions $\mathfrak{A} \cong \mathfrak{A}_0 \times \mathfrak{A}_1$, $\mathfrak{A}^* \cong \mathfrak{A}_0^* \times \mathfrak{A}_1^*$ for the case of non-superatomic algebras $\mathfrak{A}$ and $\mathfrak{A}^*$.

The statements of lemmas **1.18-1.21** and theorem **1.6** imply the following theorem.

<u>**Theorem 1.20.**</u>

(a) Countable Boolean algebras are isomorphic iff they have isomorphic systems of invariants;

(b) for any cardinal $\alpha < \omega_1$, and any natural number $n > 0$ there exists a countable superatomic algebra with a system of invariants equal to $<\alpha,n>$. For any ordinals $\alpha \le \beta < \omega_1$, any natural number $n > 0$ and any additive function $r: B(\eta) \to \alpha + 1$ such that $r(1_{B(\eta)}) = \alpha$, there is a non-superatomic countable Boolean algebra with a system of invariants $<<B(\eta),r>,<\beta,0,n>>$.

Employing the obtained system of invariants of countable Boolean algebras, J.Ketonen obtained the most important property of the so-called Cartesian skeleton of the class of countable Boolean algebras, answering a number of known problems.

<u>**Definition 1.38.**</u> For any class of algebras $\mathfrak{K}$, closed relative to finite Cartesian products (if $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{K}$, then $\mathfrak{A}_1 \times \mathfrak{A}_2 \in \mathfrak{K}$), the Cartesian (countable

Cartesian) skeleton of the class $\mathfrak{K}$ is $<\mathfrak{J}\mathfrak{K};\times>(\mathfrak{J}\mathfrak{K}_{\aleph_0};\times>)$, where $\mathfrak{K}_{\aleph_0}=\{\mathfrak{A}\in\mathfrak{K}\mid|\mathfrak{A}|\leq\aleph_0\}$, and the operation $\times$ is defined on the elements $\mathfrak{J}\mathfrak{K}(\mathfrak{J}\mathfrak{K}_{\aleph_0})$ in the following way: if $a,b,c\in\mathfrak{J}\mathfrak{K}$ and a,b,c are the types of the isomorphism of the algebras $\mathfrak{A},\mathfrak{B},\mathfrak{C}$, respectively, then $a\times b=c$ iff $\mathfrak{A}\times\mathfrak{B}\cong\mathfrak{C}$.

It is obvious that the class $\mathfrak{J}\mathfrak{K}(\mathfrak{J}\mathfrak{K}_{\aleph_0})$ modified by the operation $\times$ satisfies the axioms of a commutative semigroup, while if $\mathfrak{K}$ contains a singleton algebra, then $<\mathfrak{J}\mathfrak{K},\times>(<\mathfrak{J}\mathfrak{K}_{\aleph_0};\times>)$ obeys the axioms of a monoid, where the role of 1 is played by the singleton algebra.

The basic result obtained by J.Ketonen with a system of invariants for countable Boolean algebras in the course of a rather lengthy proof is as follows.

Theorem 1.21. Any countable commutative semigroup is isomorphically embeddable into $<\mathfrak{J}BA_{\aleph_0};\times>$, which is a countable Cartesian skeleton of a variety of Boolean algebras.

For an algebraic description of a countable Cartesian skeleton of a variety of Boolean algebras a number of definitions will be required.

Definition 1.39.

(a) A commutative monoid $<M;\cdot,1>$ is called canonical if the equality $xy=1$ yields the equalities $x=y=1$.

(b) A general refinement of the sequences $<x_i|i\leq n><y_j|j\leq m>$ of the elements of a commutative monoid with the property $\prod_{i\leq n}x_i=\prod_{j\leq m}y_j$ is a sequence $<r_{ij}|i\leq n,j\leq m>$ such that for any $i\leq n,j\leq m$

$$x_i=\prod_{j\leq m}z_{ij},\ y_j=\prod_{i\leq n}z_{ij}.$$

(c) The monoid $<M;\cdot,1>$ is called a refinement monoid if it is canonical and any sequences of the elements of M $<x_i|i\leq n>$, $<y_j|j\leq m>$ with the property $\prod_{i\leq n}x_i=\prod_{j\leq m}y_j$ have a general refinement.

(d) If $\mathfrak{M}=<M;\cdot,1>$ and $\mathfrak{N}=<N;+,1>$ are two monoids, then the relation $R\subseteq N\times M$ is called a left V-relation, provided that:

(1) $<1,y>\in R$ implies the equality $y=1$;

(2) if $<x,y>\in R$ and $x=x_1x_2$, then there are $y_1,y_2\in\mathfrak{M}$ such that $y=y_1y_2$ and $<x_1,y_1>,<x_2,y_2>\in R$.

(e) The relation $R\subseteq N\times M$ is called a right V-relation if R^{-1} is a left V-relation. If R is both a left and right V-relation, we will call R simply a V-relation; if in this case $\mathfrak{N}=\mathfrak{M}$, then R will be called a V-relation on $\mathfrak{M}$.

(f) A monoid $\mathfrak{M}$ is called a V-monoid if $\mathfrak{M}$ is a refinement monoid, and on $\mathfrak{M}$ the V-criteria is fulfilled: if R is a V-relation on $\mathfrak{M}$ and $<x,y>\in R$, then $x=y$, i.e., the only V-relation on $\mathfrak{M}$ is the equality relation.

Let us recall a known Vaught criterion of the isomorphism of countable Boolean algebras.

Theorem 1.22. Two at most countably infinite Boolean algebras $\mathfrak{A}_1,\mathfrak{A}_2$ will be isomorphic iff there is a correspondence between $\mathfrak{A}_1$ and $\mathfrak{A}_2$, i.e., when there is a set $C\subseteq\mathfrak{A}_1\times\mathfrak{A}_2$ such that:

(1) $<1,b>\in C\Leftrightarrow b=1,\quad <a,1>\in C\Leftrightarrow a=1$;

(2) $<a,b>\in C\Leftrightarrow<\neg a,\neg b>\in C$;

(3) if $<a,b>\in C$ and $c\in\mathfrak{A}_1$, then there is a $d\in\mathfrak{A}_2$ such that $<a\cap c,b\cap d>,<a\setminus c,b\setminus d>\in C$;

(4) if $<a,b>\in C, d\in\mathfrak{A}_2$, then there is a $c\in\mathfrak{A}_1$ such that $<a\cap c,b\cap d>,<a\setminus c,b\setminus d>\in C$.

A countable Cartesian skeleton of a variety of Boolean algebras $<\mathfrak{Z}BA_{\aleph_0};\times>$, refined with a singleton Boolean algebra as a constant 1 will obviously be a refinement monoid, and by virtue of the aforementioned Vaught criterion, it will also be a V-monoid.

Definition 1.40.

(a) A submonoid $\mathfrak{N}$ of a monoid $\mathfrak{M}$ is said hereditary if for any

$x \in \mathfrak{N}$, $y, z \in \mathfrak{M}$ the equality $x = yz$ yields $y, z \in \mathfrak{N}$. Obvious is the fact that any hereditary submonoid of a monoid of refinement (a V-monoid) is a refinement monoid itself (a V-monoid).

(b) The rank of summation of the monoid $\mathfrak{M}$ is the least upper boundary of the powers of the sets $\{y \in \mathfrak{M} \mid \text{there is a } z \in \mathfrak{M} \quad yz = x\}$ for $x \in \mathfrak{M}$.

(c) A V-monoid $\mathfrak{M}$ is called a universal V-monoid of the summation rank k (k is an arbitrary infinite cardinal), if $\mathfrak{M}$ has a rank of summation k, and any V-monoid $\mathfrak{N}$ of the rank of summation not greater than k is isomorphic to a certain hereditary submonoid of the monoid $\mathfrak{M}$.

Theorem 1.23.

(a) For any infinite cardinal k there is a unique universal V-monoid of the rank of summation k.

(b) If f is an isomorphism from a V-monoid $\mathfrak{N}$ to a hereditary submonoid of a V-monoid $\mathfrak{M}$, then f is uniquely defined.

The following result gives an algebraic characterization of a countable Cartesian skeleton of a variety of Boolean algebras, using the notions introduced above.

Theorem 1.24. A countable Cartesian skeleton of a variety of Boolean algebras $< \mathfrak{J}BA_{\aleph_0}; \times, 1_{BA} >$, refined with a constant which is a type of the isomorphism of a singleton Boolean algebra, is a universal V-monoid of the rank of summation $\aleph_0$.

E. Mad-Families and Boolean Algebras

Here we will consider one more way of constructing Boolean algebras, which enables one to construct a family of these algebras which have different properties and are interrelated by, in particular, relations of embedding and epimorphism. The method is based on the so-called almost disjunct families of sets. It should be remarked that the notion of almost-disjunctness defined below is other than that introduced in section *C*.

Definition 1.41.

(a) A family X of infinite subsets of a set A is said (pairwise) almost-disjunct if for any $R_1, R_2 \in X$ we have $|R_1 \cap R_2| < \aleph_0$.

(b) An almost disjunct family X of subsets of a set A is said an *ad*-family, if for any finite $X' \subseteq X$, the set $A \setminus \bigcup X'$ is infinite, and an *ad*-family X is called a *mad*-family, if it is maximal in terms of the inclusion of the subsets of the set A among *ad*-families.

(c) A family X of subsets of the set A has the property of a finite intersection (the *fip* property), provided that we have $|\bigcap X'| \geq \aleph_0$ for any finite $X' \subseteq X$.

The relation "$P \setminus R$ is finite" between subsets of the set A will be denoted by $P \subseteq_* R$, $P =_* R$ if $(P \setminus R) \cup (R \setminus P)$ is finite.

For any *ad*-family X of subsets of the set A, $F(X)$ will denote $\{B \subseteq A \,|\, |\{s \in X \,|\, s \subseteq_* B\}| = 2^{|A|}\}$.

By $P(2^{\omega})$ we will mean the following set-theoretical proposition introduced by Rothberger [202]: if $F \subseteq P(\omega)$ has the *fip* property, and $|F| < 2^{\aleph_0}$, then there is an infinite $P \subseteq \omega$ such that $P \setminus R$ is finite for any $R \in F$. It has been proved [114] that the statement $P(2^{\omega})$ follows from the Martin hypothesis and, hence, moreover from the continuum-hypothesis. Indeed, for the case when $CH, |F| = \aleph_0$ we set $F = \{A_i | i < \omega\}$, and then we construct P in an inductive way, choosing $x_i \in \omega$ in the following way: $x_0 \in A$; if we have constructed $x_0, \ldots, x_k$ for $k < \omega$, then let $x_{k+1} \in A_0 \cap \ldots \cap A_k \setminus \{x_0, \ldots x_k\}$. Setting $P = \{x_i | i < \omega\}$, we obviously get $|P \setminus R| < \aleph_0$ for any $R \in F$.

Lemma 1.22. Under the assumption $P(2^{\omega})$ for any non-principal ultrafilter P, on ω there is a *mad*-family X of subsets ω such that $F(X) = P$.

For any family T of subsets of the set A, $B(A,T)$ will denote a subalgebra of the Boolean algebra of all subsets of the set A, generated by the elements included in T and elements of the type $\{a\}$, where $a \in A$. If X is a *mad*-family of the subsets of A, then $B(A, X)$ is, as can be seen easily, a superatomic Boolean algebra of type $< 2,1 >$.

Definition 1.42. By βA we will mean the family of all ultrafilters on the set A. The Rudin-Keisler quasi-order $\leq_1$ on different ultrafilters is defined in the following way: for $p \in \beta A, q \in \beta B$, the relation $p \leq_1 q$ holds iff there is a mapping f

of a certain set $X \in q$ in A such that for any $Y \subseteq A, Y \in p$ iff for a certain $Z \in q$, $f(Z) \subseteq Y$. The finite Rudin-Keisler quasi-order $\leq$ is defined in an analogous way, but there is an additional requirement on f: for any $a \in A$, $|f^{-1}(a)| < \aleph_0$.

Theorem 1.25. There is $2^{\aleph_0}$ ($2^{2^{\aleph_0}}$ under CH) of non-principal ultrafilters on ω, which are mutually incomparable relative to the Rudin-Keisler quasi-order.

Let p be an arbitrary non-principal ultrafilter on ω, and let X_p be a *mad*-family of the subsets ω constructed in lemma **1.22** such that $F(X_p) = p$. By $\mathcal{B}_p$ we will mean a Boolean algebra $B(\omega, X_p)$. It should be remarked that the existence of the algebra $B(\omega, X_p)$ has been proved under the assumption of the set-theoretical hypothesis $P(2^\omega)$, or a stronger one, CH.

Lemma 1.23. For any non-principal ultrafilters p,q on ω, if a Boolean algebra $\mathcal{B}_p$ is isomorphically embeddable into the Boolean algebra $\mathcal{B}_q$, then $p \leq q$.

The statement of lemma **1.23** obviously remains true for any ultrafilters p,q defined on arbitrarily countable sets A and B.

From theorem **1.25** and lemma **1.23** one can deduce, as a corollary, (assuming $P(2^\omega)$) the existence of $2^{\aleph_0}$ ($2^{2^{\aleph_0}}$ under CH) of mutually unembeddable Boolean algebras of the powers $2^{\aleph_0}$. It should be recalled that in section (c) we obtained a stronger result with no additional set-theoretical assumptions: for any $\aleph > \aleph_0$ there are $2^{\aleph}$ of mutually unembeddable Boolean algebras of the power $\aleph$. The construction discussed above, however, will be used for constructing families of mutually unembeddable Boolean algebras with an additional property: they will be homomorphic images of each other, i.e., equivalent in terms of $\equiv_{<<}$.

Theorem 1.26. Under the assumption $P(2^\omega)$ (or under a stronger one, CH) for any $n \in \omega$ there are Boolean algebras $\mathcal{B}_0, \ldots \mathcal{B}_{n-1}$ such that $\mathcal{B}_i$ are mutually unembeddable, and for any $i,j < n$, we have $\mathcal{B}_i << \mathcal{B}_j$.

The latter statement, combined with theorem **1.9**, makes it possible to prove the following theorem, which is of primary importance in this section.

Theorem 1.27. Under the assumption $P(2^\omega)$, for any finite set $B = \{b_0, \ldots, b_{s-1}\}$ modified by two arbitrary quasi-orders $\leq_1, \leq_2$, there are mutually non-isomorphic Boolean algebras $\mathcal{C}_0, \ldots \mathcal{C}_{s-1}$ (of the power $2^{\aleph_0}$) such that for $i,j < s$ $\mathcal{C}_i \leq \mathcal{C}_j$ iff $b_1 \leq_1 b_j$, and $\mathcal{C}_i << \mathcal{C}_j$ iff $b_1 <<_2 b_2$.

The proof of this theorem is given in section **14** of Applications.

By way of concluding this section it should be remarked that not all the basic problems of the theory of Boolean algebras have been discussed here, my choice being governed by the applicability of the material presented in Chapters **2** and **3**. A detailed and basically complete presentation of the modern theory of Boolean algebras can be found in "Handbook of Boolean Algebras" mentioned earlier. In section **14** of Applications in this monograph one can find proofs of the theorems of the present section not to be found in "Handbook of Boolean Algebras" and in other monographs on Boolean algebras.

Priorities. Theorem **1.1** is by M.N.Stone [220], and its proof can be found in any sufficiently complete textbook on topology and theory of Boolean algebras. Theorem **1.2** belongs to G.Fodor [63], the statement of theorem **1.3** to R.M.Solavay [216], the proof of these theorems can be found, for instance, in a monograph by A.Levy [124]. Theorem **1.4** was proved by S.Mazurkiewicz and W.Sierpinski [133], theorem **1.5** by A.Mostowsli and A.Tarski, theorem **1.6** is to be found in a paper by G.Day [49]. Theorem **1.7** belongs to R.S.Pierce [159], theorem **1.8** to R.Bonnet [20]. The statement of theorem **1.9** for the case $\aleph_i = \aleph_1$ is a variation of lemma **1** from a paper by A.G.Pinus [178], and in a general form this statement can be found in a paper by R.Bonnet and H.Si-Kaddour [19] (the proof is given in section **14** of Applications of the present monograph).

It should be remarked that part of the material pertaining to interval and superatomic Boolean algebras can be found in "Handbook of Boolean Algebras" mentioned earlier and also, for instance, in monographs by Yu.L.Ershov [59] and S.S.Goncharov [78]. The proof of theorem **1.9**, theorems **1.10** and **1.11** are by M.Rubin [204].

Lemma **1.1** for the case of an ordered set of real numbers stems from a paper by W.Sierpinski [211] and has been repeatedly generalized for other LOSs by various authors. The form of lemma **1.1** presented here is closest to that presented by R.Bonnet [21]. Lemmas **1.2-1.6** and theorems **1.12-1.14** also belong to R.Bonnet [21]. Lemma **1.7** and theorem **1.15** can be found in a paper by A.G.Pinus [169], the proofs of these statements are given in section **14** of Applications.

The constructions pertaining to definitions **1.17-1.24**, as well as the statements of lemmas **1.8-1.10**, of theorems **1.16-1.17** and corollaries **1.6-1.7** are by S.Shelah [207].

As has been mentioned earlier, the system of invariants for countable Boolean algebras was borrowed from J.Ketonen [104], a purely algebraic construction of this system of invariants and a generalized result for the class of countable DILARCs was borrowed from Yu.L.Ershov [56]. The contents of section (d) of this section, up to and including theorem **1.20** is after Yu.L.Ershov [56].

Theorem **1.21** that has accounted for a number of known problems was proved by J.Ketonen [104]. Theorem **1.22** is the classical Vaught criterion of the isomorphism of countable Boolean algebras [233]. The results pertaining to the notion of V-monoids, i.e. theorems **1.23-1.24**, belong to H.Dobbertin [53].

The statement of lemma **1.22** is by M.Weese [235], who also remarks in his work that Boolean algebras $\mathfrak{B}_p, \mathfrak{B}_q$ are non-isomorphic when the ultrafilters p,q are incomparable relative to the Rudin-Keisler order. Theorem **1.25** was proved by K.Kunen [116], and its proof, as well as more details on the Rudin-Keisler order on ultrafilters, can be found in a monograph by W.W.Comfort and S.Negrepontis [41]. Lemma **1.23** and theorems **1.26** and **1.27** are to be found in a paper by A.G.Pinus [178] (their proofs are presented in section **14** in Applications).

2. Basic Notions of Universal Algebra

The purpose of this section is to recall the basic facts of the theory of universal algebras to be used later in this monograph. Some standard notations and definitions of the theory of universal algebras, which can be found in monographs by A.I.Mal'tzev [128], A.G.Pinus [161], G.Grätzer [84], P.M.Cohn [37], S.Burris, H.P.Sankappanavar [28] and others, will be made use of. Let us recall some of them.

For any class of algebras $\mathfrak{K}$, $I(\mathfrak{K})$ will denote the class of all algebras isomorphic to the algebras of the class $\mathfrak{K}$. Let $S(\mathfrak{K})$ be a class of all subalgebras of the algebras of the class $\mathfrak{K}$. $H(\mathfrak{K})$ will be a class of all homomorphic images of the algebras of the class $\mathfrak{K}$, $P(\mathfrak{K})$, $P_s(\mathfrak{K})$ the classes of all direct and subdirect products of $\mathfrak{K}$-algebras, $P_p(\mathfrak{K})$ the class of all ultraproducts of $\mathfrak{K}$-algebras, and $P_F(\mathfrak{K})$ the class of all filtered products of $\mathfrak{K}$-algebras.

The class of algebras $\mathfrak{K}$ is called a variety (a quasi-variety) if it consists of all the algebras of a given fixed signature which obeys a certain system of identities (quasi-identities). For an arbitrary class of algebras $\mathfrak{K}$, $\mathfrak{M}(\mathfrak{K})$ ($\mathfrak{Q}(\mathfrak{K})$) will denote the least variety (quasi-variety) containing the class $\mathfrak{K}$. The following statements are the corner-stones of the theory of varieties (quasi-varieties) of universal algebras (for simplicity, an at most countably infinite signature is assumed throughout).

Theorem 2.1. The class of algebras $\mathfrak{K}$ is a variety iff $S(\mathfrak{K}) \subseteq \mathfrak{K}$,

$H(\mathfrak{K}) \subseteq \mathfrak{K}$ and $P(\mathfrak{K}) \subseteq \mathfrak{K}$. For any class of algebras $\mathfrak{K}$ we have $\mathfrak{M}(\mathfrak{K}) = HSP(\mathfrak{K})$.

$\mathfrak{K}^+$ will denote an extension of the class $\mathfrak{K}$ by adding a one-element algebra to it.

Theorem 2.2. The class of algebras $\mathfrak{K}$ is a quasi-variety iff $S(\mathfrak{K}) \subseteq \mathfrak{K}$, $P_F(\mathfrak{K}) \subseteq \mathfrak{K}$ and $\mathfrak{K}^+ \subseteq \mathfrak{K}$. The class of algebras $\mathfrak{K}$ is a quasi-variety iff $S(\mathfrak{K}) \subseteq \mathfrak{K}$, $P(\mathfrak{K}) \subseteq \mathfrak{K}$, $P_p(\mathfrak{K}) \subseteq \mathfrak{K}$ and $\mathfrak{K}^+ \subseteq \mathfrak{K}$. For any class of algebras $\mathfrak{K}$ we have $\mathfrak{Q}(\mathfrak{K}) = SP_F(\mathfrak{K}^+)$.

A variety $\mathfrak{M}$ is called finitely generated if there exists a finite class $\mathfrak{K}$ of finite algebras such that $\mathfrak{M} = \mathfrak{M}(\mathfrak{K})$, or, which is equivalent, $\mathfrak{M}$ is generated by a certain finite algebra. By $\mathfrak{F}_{\mathfrak{M}}(\aleph)$ we will mean an $\aleph$-generated free $\mathfrak{M}$-algebra. For any variety $\mathfrak{M}$ $\mathfrak{M} = HSP(\mathfrak{F}_{\mathfrak{M}}(\aleph_0))$.

π_i will denote a projection of a direct product $\prod_{i \in I} \mathfrak{B}_i$ and its subalgebras on the algebra $\mathfrak{B}_i$. The algebra $\mathfrak{A}$ is called subdirectly non-decomposable if for any algebras $\mathfrak{B}_i (i \in I)$ the fact that $\mathfrak{A}$ is a subdirect product of the algebras. $\mathfrak{B}_i (i \in I)$ implies that for a certain $i_0 \in I$, π_{i_0} is an isomorphism of $\mathfrak{A}$ on $\mathfrak{B}_{i_0}$.

Theorem 2.3. Any algebra $\mathfrak{A}$ is isomorphic to a subdirect product of subdirectly non-decomposable algebras.

Since the subdirectly non-decomposable algebras in the above statement belong to any variety to which the algebra $\mathfrak{A}$ belongs, any variety is uniquely definable by its subdirectly non-decomposable algebras. In particular, for any class $\mathfrak{K}$, the equality $\mathfrak{M}(\mathfrak{K}) = HP_S(\mathfrak{K})$ [108] holds.

By $\mathfrak{K}_{SI}$ we will mean a class of subdirectly non-decomposable $\mathfrak{K}$-algebras. For any algebra $\mathfrak{A}$, $Con\mathfrak{A}$ is a lattice of congruences of the algebra $\mathfrak{A}$; $\vee, \wedge$ are the corresponding lattice operations on $Con\mathfrak{A}$, $\nabla_{\mathfrak{A}}$, $\Delta_{\mathfrak{A}}$ are the biggest and the least, or unit and zero, congruences on $\mathfrak{A}$. An element a of an arbitrary lattice L is said compact if for any subset $A \subseteq L$, $\vee A \geq a$ implies the existence of a finite $B \subseteq A$ such that $\vee B \geq a$. A lattice is called algebraic if it is complete and any of its element is the upper bound of a certain family of compact elements. For any algebra $\mathfrak{A}$ and any $a,b \in \mathfrak{A}$, $\theta_{a,b}$ will denote the least congruence on $\mathfrak{A}$ containing a pair $< a,b >$ (such congruences are said principal). $Con_p\mathfrak{A}$ stands for the family of principal congruences partially ordered in terms of the inclusion on $\mathfrak{A}$. For any algebra $\mathfrak{A}$, the compact elements of $Con\mathfrak{A}$, and only them, are finite unions of principal congruences. The lattice $Con\mathfrak{A}$ is algebraic for any algebra. The converse is also valid.

Theorem 2.4. Any algebraic lattice is isomorphic to the congruence lattice of a certain algebra. In this case, if the biggest element of the lattice is compact, then the algebra of a finite signature can be chosen.

If f is a certain homomorphism of the algebra $\mathfrak{A}$ on $\mathfrak{B}$ and $\theta \in Con\mathfrak{B}$, then $\check{f}\theta$ will denote a congruence on $\mathfrak{A}$ which is equal to $\{<a,b>\in\mathfrak{A}^2 | <f(a),f(b)>\in\theta\}$. The kernel of f, $\ker f$, is, by the definition, equal to $\check{f}\Delta_{\mathfrak{B}}$.

If $\mathfrak{A} \subseteq \prod_{i\in I}\mathfrak{A}_i$, then we will write $[\![f=g]\!]$ for $\{i\in I | f(i)=g(i)\}$ $f,g\in\mathfrak{A}$, and $[\![f\neq g]\!]$ for $I\setminus[\![f=g]\!]$. If $\theta_i\in Con\mathfrak{A}$ $(i\in I)$, and $\bigwedge_{i\in I}\theta_i=\Delta_{\mathfrak{A}}$, then $\mathfrak{A}$ is isomorphic to a subdirect product of algebras $\mathfrak{A}/\theta_i(i\in I)$. Conversely, if $\mathfrak{A}\subseteq\prod_{i\in I}\mathfrak{B}_i$ has a subdirect product (the latter will be denoted by $\mathfrak{A}\subseteq_p\prod_{i\in I}\mathfrak{B}_i$), then $\bigwedge_{i\in I}\ker\pi_i=\Delta_{\mathfrak{A}}$.

Therefore, $\mathfrak{A}$ is subdirectly non-decomposable iff there exists a least non-trivial congruence on $\mathfrak{A}$, which is called a monolith of $\mathfrak{A}$ and denoted by $\beta(\mathfrak{A})$. Since any congruence is a union of principal ones, $\beta(\mathfrak{A})$ is always principal. An element a of the lattice L will be called non-decomposable at an intersection in L if for any $b_i\in L\,(i\in I)$, the equality $a=\bigwedge_{i\in I}b_i$ implies the existence of an $i_0\in I$ such that $a=b_{i_0}$. For any $\theta\in Con\mathfrak{A}$, $\mathfrak{A}/\theta$ is a subdirectly non-decomposable algebra iff θ is non-decomposable at an intersection in $Con\mathfrak{A}$. For any $\theta\in Con\mathfrak{A}$ a natural homomorphism of $\mathfrak{A}$ on $\mathfrak{A}/\theta$ generates a natural isomorphism of the lattice $Con\mathfrak{A}/\theta$ with a sublattice $[\theta;\nabla_{\mathfrak{A}}]$ of the lattice $Con\mathfrak{A}$, where $[\alpha;\beta]=\{\gamma\in L | \alpha\le\gamma\le\beta\}$ for the lattice L and any $\alpha\le\beta\in L$.

Let φ,ψ be arbitrary binary relations on $\mathfrak{A}$, $\varphi\cdot\psi$ denote their composition, $\varphi\cdot\psi=\{<a,b>\in\mathfrak{A}^2 | \ \exists c\in\mathfrak{A}:\ <a,c>\in\varphi, <c,b>\in\psi\}$. For $\theta_1.\theta_2\in Con\mathfrak{A}$, the following conditions are equivalent:

(1) $\theta_1\cdot\theta_2=\theta_2\cdot\theta_1$;

(2) $\theta_1\cdot\theta_2\subseteq\theta_2\cdot\theta_1$;

(3) $\theta_1 \cdot \theta_2 = \theta_2 \vee \theta_1$.

In this case the congruences θ_1, θ_2 are said commutable. If $\theta_1, \theta_2 \in Con\mathfrak{A}$, $\theta_1 \wedge \theta_2 = \Delta$, $\theta_1 \vee \theta_2 = \nabla$, and θ_1, θ_2 are commutable, then $\mathfrak{A} \cong \mathfrak{A} / \theta_1 \times \mathfrak{A} / \theta_2$. The converse is also valid: if $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$, then there are $\theta_1, \theta_2 \in Con\mathfrak{A}$ such that $\mathfrak{A}_i \cong \mathfrak{A} / \theta_i$ and θ_1, θ_2 have the above properties.

A variety of algebras whose congruences are commutable, is said congruence-commutable.

Theorem 2.5. For a variety $\mathfrak{M}$, the following conditions are equivalent:

(1) $\mathfrak{M}$ is congruence-commutable;

(2) $\mathfrak{F}_{\mathfrak{M}}(3)$ is congruence-commutable;

(3) there is a term p of three variables such that on $\mathfrak{M}$ the following identities hold:

$$p(x,z,z) = x, \quad p(x,x,z) = z.$$

A lattice is called distributive if it satisfies either of the following equivalent equations:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice L is called modular if it satisfies the following equality:

$$(x \wedge y) \vee (y \wedge z) = y \wedge ((x \wedge y) \vee z),$$

or a quasi-equality equivalent to it:

$$x \wedge y = x \rightarrow x \vee (y \wedge z) = y \wedge (x \vee z).$$

Any distributive lattice is modular.

Theorem 2.6. A lattice L is non-modular iff a lattice N_5 is isomorphically imbeddable in L (Fig. 3).

Theorem 2.7. A lattice L is non-distributive iff a lattice M_3 or N_5 is isomorphically imbeddable into L (Fig. 3).

Fig. 3

A variety of algebras such that $Con\,\mathfrak{A}$ is distributive (modular) for all its algebras $\mathfrak{A}$ is called a congruence-distributive (congruence-modular) variety. A variety which is both congruence-commutable and congruence-distributive is said arithmetic.

Theorem 2.8. For a variety $\mathfrak{M}$, the following conditions are equivalent:

(1) $\mathfrak{M}$ is congruence-modular;

(2) $\mathfrak{F}_{\mathfrak{M}}(4)$ is congruence-modular;

(3) for any $\mathfrak{A} \in \mathfrak{M}$ and any $a,b,c,d \in \mathfrak{A}$,

$$<a,b> \in \theta_{b,c} \vee ((\theta_{a,d} \vee \theta_{b,c}) \wedge (\theta_{a,b} \vee \theta_{c,d}));$$

(4) for a certain natural number $n \geq 1$, there are terms $p_0, \ldots, p_n$ of four

variables such that for $i = 0,\dots,n-1$, the following equalities are valid on $\mathfrak{M}$:

$$p_0(x,y,z,u) = x,\ p_n(x,y,z,u) = u,\quad p_i(x,y,y,x) = x;$$
$$p_i(x,y,y,u) = p_{i+1}(x,y,y,u) \text{ for even } i;$$
$$p_i(x,x,u,u) = p_{i+1}(x,x,u,u) \text{ for odd } i.$$

Theorem 2.9. For a variety $\mathfrak{M}$, the following conditions are equivalent:

(1) $\mathfrak{M}$ is congruence-distributive;

(2) $\mathfrak{F}_{\mathfrak{M}}(3)$ is congruence-distributive;

(3) for any $\mathfrak{A} \in \mathfrak{M}$ and any $a,b,c \in \mathfrak{A}$,

$$< a,c > \in (\theta_{a,b} \wedge \theta_{a,c}) \vee (\theta_{b,c} \wedge \theta_{a,c});$$

(4) for a certain natural number $n \geq 1$, there are terms $p_0,\dots,p_n$ of three variables such that for $i = 0,\dots,n-1$, the following equalities are valid on $\mathfrak{M}$:

$$p_0(x,y,z) = x,\quad p_n(x,y,z) = z,\quad p_i(x,y,x) = x;$$
$$p_i(x,x,z) = p_{i+1}(x,x,z) \text{ for even } i;$$
$$p_i(x,z,z) = p_{i+1}(x,z,z) \text{ for odd } i.$$

Theorem 2.10. For a variety $\mathfrak{M}$, the following conditions are equivalent:

(1) $\mathfrak{M}$ is arithmetic;

(2) there is a term p of three variables such that $\mathfrak{M}$ satisfies the following equations:

$$p(x,y,x) = p(x,y,y) = p(y,y,x) = x.$$

It should be noted, for example, that any variety of groups, rings and quasi-groups is congruence-commutable. Either directly, or using theorems **2.5** and **2.8**, one can also observe that any congruence-commutable variety is congruence-modular, and any variety of lattices is congruence-distributive. Another source of examples of congruence-distributive varieties are the discriminator varieties to be introduced below.

The algebra $\mathfrak{A}$ is called simple if its lattice of congruences is two-element, i.e., any of its factors over a nonzero congruence is a one-element algebra. Of major importance in the theory of varieties of algebras is the following result.

Theorem 2.11. Any nontrivial variety contains a simple algebra.

In the proof of all preceding statements use has been made of the following fundamental statement about the structure of congruences generated by a given set of pairs of elements of a universal algebra.

Theorem 2.12. For any universal algebra $\mathfrak{A}$, any set T of non-ordered pairs of elements of $\mathfrak{A}$, and for any $a,b \in \mathfrak{A}$, a pair $< a,b >$ belongs to a congruence on $\mathfrak{A}$ generated by the set T iff there are terms $p_i(x, y_1,\dots, y_m)$ of the signature of the algebra $\mathfrak{A}$ $(1 \le i \le l)$, elements $e_1,\dots,e_m \in \mathfrak{A}$, and pairs $\{s_i,t_i\} \in T$ $(1 \le i \le l)$ such that:

$$a = p_1(s_1,e_1,\dots,e_m),$$

for $1 \le i < l$ $p_i(t_i,e_1,\dots,e_m) = p_{i+1}(s_{i+1},e_1,\dots,e_m)$, and

$$p_l(t_l,e_1,\dots,e_m) = b.$$

This theorem obviously entails the following corollary.

Corollary 2.1. For any subalgebra $\mathfrak{A}_1$ of an algebra $\mathfrak{A}$ and any set T of non-ordered pairs of elements of $\mathfrak{A}_1$, if a congruence $\alpha \in Con\mathfrak{A}$ is generated by the set T, then:

(a) the existence of a homomorphism f of the algebra $\mathfrak{A}$ on $\mathfrak{A}_1$ (leaving the elements of $\mathfrak{A}_1$ fixed) yields that α restricted on $\mathfrak{A}_1$ is equal to the congruence generated by the set T on the algebra $\mathfrak{A}_1$;

(b) for any $c \in \mathfrak{A}_1, d \in \mathfrak{A}$, if there is a homomorphism g of the algebra $\mathfrak{A}$ on $\mathfrak{A}_1$ fixing the elements of $\mathfrak{A}_1$ and such that $g(d) = c$, then the restriction of the congruence $\alpha \vee \theta_{c,d}$ on $\mathfrak{A}_1$ is equal to that of the congruence α on $\mathfrak{A}_1$.

$\mathfrak{K}_s$ will denote the family of all simple algebras from a class $\mathfrak{K}$. It should be noted that for any non-trivial quasi-variety of algebras, there is a simple algebra relative to this quasi-variety, i.e. such that any factor of this algebra over a congruence other than zero and unity, does not belong to the given quasi-variety (fot the proof see [79]). .

The algebra $\mathfrak{A}$ is called congruence-uniform if for any $\theta \in Con\mathfrak{A}$, all the equivalence classes over θ on $\mathfrak{A}$ are of the same power. A variety is congruence-uniform if its all algebras are congruence-uniform.

The class of algebras $\mathfrak{K}$ is called locally finite if any finitely generated $\mathfrak{K}$-algebra is finite. The variety $\mathfrak{M}$ is locally finite iff the algebras $\mathfrak{F}_{\mathfrak{M}}(n)$ are finite for any $n \in \omega$. For any finite class $\mathfrak{K}$ of finite algebras, a variety $\mathfrak{M}(\mathfrak{K})$ is locally finite.

The finite algebra $\mathfrak{A}$ is called primal if for any $n \in \omega$ and any n-ary function $f(x_1,\dots,x_n)$ defined on $\mathfrak{A}$, there is a term $t(x_1,\dots,x_n)$ of the algebra $\mathfrak{A}$ such that for any $a_1,\dots,a_n \in \mathfrak{A}$ $f(a_1,\dots,a_n) = t(a_1,\dots,a_n)$, i.e., any n-unary function on $\mathfrak{A}$ is defined by the term.

<u>Theorem 2.13.</u> For a finite algebra $\mathfrak{A}$, the following conditions are equivalent:

(1) $\mathfrak{A}$ is primal;

(2) $\mathfrak{M}(\mathfrak{A})$ is arithmetic, $\mathfrak{A}$ is simple, has no proper subalgebras and is rigid, i.e., has no non-trivial automorphisms.

A discriminator on algebra $\mathfrak{A}$ is a function $d(x,y,z)$ such that for any $a,b,c \in \mathfrak{A}$, $d(a,b,c) = a$ if $a \neq b$ and $d(a,b,c) = c$ if $a = b$. The algebra $\mathfrak{A}$ is called a discriminator algebra if the discriminator on $\mathfrak{A}$ is defined by a term of the algebra $\mathfrak{A}$. A finite discriminator algebra is called a quasi-primal algebra.

<u>Theorem 2.14.</u> For a finite algebra $\mathfrak{A}$, the following conditions are equivalent:

(1) $\mathfrak{A}$ is quasi-primal;

(2) $\mathfrak{M}(\mathfrak{A})$ is arithmetic, $\mathfrak{A}$ and its all subalgebras are simple;

(3) any function $f: A^k \to A$, where A is a basic set of the algebra $\mathfrak{A}$, preserving all subalgebras of the algebra $\mathfrak{A}$ and all isomorphisms among these subalgebras, is defined by a certain term of the algebra $\mathfrak{A}$.

A variety $\mathfrak{M}$ is a discriminator variety if there is a class $\mathfrak{K} \subseteq \mathfrak{M}$ such that $\mathfrak{M}(\mathfrak{K}) = \mathfrak{M}$, and on all $\mathfrak{A} \in \mathfrak{K}$ the discriminators are determined by a general term. In this case it appears that $\mathfrak{K} \subseteq \mathfrak{M}_{SI} = \mathfrak{M}_S$.

Theorem 2.15. For any variety $\mathfrak{M}$, the following conditions are equivalent:

(1) $\mathfrak{M}$ is a discriminator variety;

(2) there is a term p of three variables such that the following equations are satisfied on $\mathfrak{M}$:

$$p(x,z,z) = x, \quad p(x,y,x) = x, \quad p(x,x,z) = z,$$
$$p(x,p(x,y,z),y) = y,$$

and for any signature function f, the following identity holds on $\mathfrak{M}$:

$$p(x,y,f(z_1,\dots,z_k)) = p(x,y,f(p(x,y,z_1),\dots,p(x,y,z_k))).$$

A set $\mathfrak{M}$ is said to be semisimple if any of its subdirectly non-decomposable algebras is simple. As has been remarked earlier, for discriminator varieties the equality $\mathfrak{M}_S = \mathfrak{M}_{SI}$ is true, i.e., any discriminator variety is semisimple. By theorems **2.10** and **2.15**, any discriminator variety is arithmetic.

The following statement is of great importance for the study of congruence-distributive varieties of algebras.

Theorem 2.16. (Jonsson lemma). If for a certain class of algebras $\mathfrak{K}$, $\mathfrak{M}(\mathfrak{K})$ is congruence-distributive, then $\mathfrak{M}(\mathfrak{K})_{SI} \subseteq HSP_P(\mathfrak{K})$.

Let us now recall the basic notions and results of the theory of commutators of congruence-modular varieties of algebras, the theory which was a major breakthrough in the theory of varieties of algebras in recent years.

The cornerstone of this theory was laid in a monograph by J.D.H.Smith [213] on congruence-commutable varieties and later developed for congruence-modular varieties by J.Hagemann, C.Herrmann and H.P.Gumm ([89], [92], [86], [87]). A systematic presentation of the theory of commutators can be found elsewhere ([88], [72], [161]). Let henceforth $\mathfrak{M}$ stand for an arbitrary congruence-modular variety.

For any algebra $\mathfrak{A} \in \mathfrak{M}$, any congruences $\alpha, \beta \in Con\mathfrak{A}$, $\mathfrak{A}^\alpha$ will denote a subalgebra of the algebra $\mathfrak{A}$ with a basis set equal to α, and we will define the congruence Δ_α^β on $\mathfrak{A}^\alpha$ as a congruence generated by the set $\{<<x,x>,<y,y>> \mid <x,y> \in \beta\}$. A commutator of the congruences α,β will be a relation $[\alpha,\beta]$ on the algebra $\mathfrak{A}$ such that $<x,y> \in [\alpha,\beta]$ iff $<<x,x>,<x,y>> \in \Delta_\alpha^\beta$. For any $\alpha,\beta \in Con\mathfrak{A}$, $[\alpha,\beta] \in Con\mathfrak{A}$.

In the following theorem one can find an abstract definition of a commutator.

Theorem 2.17. A commutator of congruences on algebras $\mathfrak{A}$ of a congruence-modular variety $\mathfrak{M}$ is said to be the biggest binary operation $f(x,y)$ (relative to the order on $Con\mathfrak{A}$) defined on $Con\mathfrak{A}$ ($\mathfrak{A} \in \mathfrak{M}$) and having the following properties for all $\alpha,\beta,\gamma \in Con\mathfrak{A}$:

(1) $f(\alpha,\beta) \leq \alpha \wedge \beta$;

(2) $f(\alpha,\beta \vee \gamma) = f(\alpha,\beta) \vee f(\alpha,\gamma)$;

(3) $f(\alpha \vee \beta,\gamma) = f(\alpha,\beta) \vee f(\beta,\gamma)$,

(4) for any homomorphism φ of an algebra $\mathfrak{B} \in \mathfrak{M}$ on the algebra $\mathfrak{A}$ we have $\overset{\vee}{\varphi} f(\alpha,\beta) = f(\overset{\vee}{\varphi}\alpha,\overset{\vee}{\varphi}\beta) \vee \ker\varphi$.

For our further proofs we will need more properties for the commutator resulting from theorem **2.17**, i.e.,

(5) if $\mathfrak{A} = \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_n$, $\alpha_i,\beta_i \in Con\mathfrak{A}_i$, then $[\bigwedge_{i \leq n} \overset{\vee}{\pi_i}\alpha_i, \bigvee_{i \leq n} \overset{\vee}{\pi_i}\beta_i] = \bigwedge_{i \leq n} \overset{\vee}{\pi_i}[\alpha_i,\beta_i]$, where π_i are projections of $\mathfrak{A}$ on $\mathfrak{A}_i$.

The following statement is another definition of a commutator, different from that in theorem **2.17**.

Theorem 2.18. If $\alpha,\beta \in Con\mathfrak{A}$, then $[\alpha,\beta]$ is the least congruence γ on $\mathfrak{A}$ with the following properties: for any $\bar{a},\bar{b} \in \mathfrak{A}^n$, $p,q \in \mathfrak{A}$, if $<\pi_i\bar{a}, \pi_i\bar{b}> \in \alpha (i \leq n)$, $<p,q> \in \beta$, then for any term $t(x_1,\ldots,x_n,y)$ of the algebra $\mathfrak{A}$, the fact that $<t(\bar{a},p), t(\bar{a},q)> \in \gamma$ implies that $<t(\bar{b},p), t(\bar{b},q)> \in \gamma$.

The center of the algebra $\mathfrak{A}$ is said to be the biggest congruence $\alpha \in Con\mathfrak{A}$ such that $[\nabla,\alpha] = \Delta$. The congruence $\beta \in Con\mathfrak{A}$ is called Abelian, if $[\beta,\beta] = \Delta_{\mathfrak{A}}$. The algebra $\mathfrak{A}$ is Abelian if $\nabla_{\mathfrak{A}}$ is also Abelian. A variety is Abelian if all its algebras are Abelian. It should be noted that in the case when $\mathfrak{M}$ is a variety of groups, $\mathfrak{A} \in \mathfrak{M}$, $\alpha,\beta \in Con\mathfrak{A}$ and correspond to normal subgroups $\mathfrak{A}_1,\mathfrak{A}_2$ of the algebra $\mathfrak{A}$, then $[\alpha,\beta]$ corresponds to a group-theoretical commutator $[\mathfrak{A}_1,\mathfrak{A}_2]$ of the subgroups $\mathfrak{A}_1,\mathfrak{A}_2$. Any group is Abelian in the sense of the theory of commutators iff it is Abelian (i.e., commutative) in the group-theoretical sense. A ring is Abelian iff it is a ring with a zero multiplication.

Theorem 2.19. If $\mathfrak{M}$ is a congruence-modular variety, then there is a term $p(x,y,z)$ such that for any algebra $\mathfrak{A} \in \mathfrak{M}$, any Abelian $\alpha \in Con\mathfrak{A}$ and any $d \in \mathfrak{A}$ on the congruence-class d/α, the operations of an Abelian group + are definable, so that for $a,b,c \in d/\alpha$, $p(a,b,c) = a - b + c$, and for any signature operation $f(x_1,\dots x_k)$, for any $a_1,\dots a_k, b_1,\dots b_k, c_1,\dots c_k \in \mathfrak{A}$ such that for $i \le k$ a_i,b_i,c_i are α-equivalent, the following equality is valid:

$$f(a_1 - b_1 + c_1,\dots,a_k - b_k + c_k) = f(a_1,\dots,a_k) - f(b_1,\dots,b_k) + f(c_1,\dots,c_k).$$

Corollary 2.2.

(1) If $\mathfrak{A}$ is an Abelian algebra, then $\mathfrak{M}(\mathfrak{A})$ is an Abelian variety. A class of all Abelian algebras of a congruence-modular variety $\mathfrak{M}$ is a variety, the term p of theorem **2.19** determining the operation $x - y + z$ of an Abelian group on all Abelian $\mathfrak{M}$-algebras simultaneously.

(2) If $\alpha \in Con\mathfrak{A}$ is such that $\mathfrak{A}/\alpha$ is an Abelian algebra, then for any $\gamma \in Con\mathfrak{A}$, we have $\alpha \cdot \gamma = \gamma \cdot \alpha$.

It should be recalled that polynomials on the algebra $\mathfrak{A}$ are said to be functions of the type $f(x_1,\dots x_k) = t(x_1,\dots x_k, a_{k+1},\dots, a_n)$, where $t(x_1,\dots,x_n)$ is a term of $\mathfrak{A}$, $a_{k+1},\dots,a_n \in \mathfrak{A}$. Two algebras with the same basic set are polynomially equivalent, provided that the families of polynomials on them coincide. The varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of the signatures δ_1,δ_2, respectively, are polynomially equivalent if there exist:

(1) bijective mappings $\psi_1(\psi_2)$ $\mathfrak{M}_1(\mathfrak{M}_2)$ on $\mathfrak{M}_2(\mathfrak{M}_1)$;

(2) the mappings $\varphi_1(\varphi_2)$ of the functions of the signature $\delta_1(\delta_2)$ into the terms of the signature $\delta_2(\delta_1)$;

(3) the mappings $\pi_1(\pi_2)$ of the products $\mathfrak{M}_1 \times \delta_1(\mathfrak{M}_2 \times \delta_1)$ such that for $\mathfrak{A} \in \mathfrak{M}_1(\mathfrak{M}_2)$, $f \in \delta_1(\delta_2)$ $\pi_1(\mathfrak{A}, f)$ $(\pi_2(\mathfrak{A}, f))$ there is a certain tuple of elements of the algebra $\mathfrak{A}$ in which case the algebras $\mathfrak{A} \in \mathfrak{M}_1(\mathfrak{M}_2)$ and $\psi_1(\mathfrak{A}) \in \mathfrak{M}_2$ $(\psi_2(\mathfrak{A}) \in \mathfrak{M}_1)$ have the same basic set A, and for $f(x_1,\ldots,x_n) \in \delta_1(\delta_2)$ the function $f(x_1,\ldots x_n)$ and the term $\varphi_1(f)(x_1,\ldots,x_n, \pi_1(\mathfrak{A}, f))$ $(\varphi_2(f)(x_1,\ldots,x_n, \pi_2(\mathfrak{A}, f)))$ coincide on this set A.

Theorem 2.20 If $\mathfrak{M}$ is an Abelian variety of algebras, then $\mathfrak{M}$ is polynomially equivalent to a variety of the left unitary $R_{\mathfrak{M}}$-modules over a certain ring $R_{\mathfrak{M}}$ with unit.

The proof of this theorem yields the following corollary.

Corollary 2.3. Any locally finite Abelian variety is finitely generated.

Some general properties pertaining to polynomial equivalence should be pointed out here. Since a family of congruences of any algebra is uniquely determined by a set of their polynomials, the polynomial equivalence of the two algebras implies a coincidence of their congruences. Therefore, the properties of the algebra which can be formulated in the language of congruences are transferred from the algebra itself onto any algebra polynomially equivalent to it. Such properties are, for instance, congruence-commutability, congruence-distributivity, congruence-modularity, simplicity, subdirect irreducibility and others. Thus, the following theorem holds.

Theorem 2.21.

(a) If a variety $\mathfrak{M}_1$ is congruence-commutable (congruence-distributive, congruence-modular, semisimple), then the varieties polynomially equivalent to it have the same property.

(b) If the algebra $\mathfrak{A}$ is simple (subdirectly non-decomposable, directly non-decomposable), then the algebras polynomially equivalent to it have the same property.

(c) If the algebra $\mathfrak{A}$ is a direct (subdirect) product of certain algebras $\mathfrak{A}_i$, then any algebra polynomially equivalent to the algebra $\mathfrak{A}$ can be represented as a direct (subdirect) product of the algebras which are polynomially equivalent to the algebras $\mathfrak{A}_i$.

The algebra $\mathfrak{A}$ is called congruence-regular if for any $\alpha,\beta \in Con\mathfrak{A}$, a coincidence of any classes of congruence over both α and β yields the equality $\alpha = \beta$. A variety is congruence-regular if its all algebras are congruence-regular.

Since any module is congruence-commutable, congruence-regular and congruence uniform, theorems **2.20** and **2.21** yield the following corollary.

Corollary 2.4. Any Abelian variety is congruence-commutable, congruence-regular and congruence-uniform.

An algebra $\mathfrak{A}$ is called neutral if for any $\alpha,\beta \in Con\mathfrak{A}$ $[\alpha,\beta] = \alpha \wedge \beta$. A variety is said neutral if all its algebras are neutral.

Theorem 2.22.

(a) A subdirect product of two neutral algebras is neutral.

(b) A congruence-modular variety $\mathfrak{M}$ is neutral iff it is congruence-distributive.

An element a of a lattice L is said neutral if for any $b,c \in L$, a sublattice generated by the elements a,b,c in L is distributive. The element a of a modular lattice is known to be neutral iff for any $b,c \in L$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Theorem 2.23. If $\alpha \in Con\mathfrak{A}$ and $\mathfrak{A} / \alpha$ is congruence-distributive, then α is a neutral element of $Con\mathfrak{A}$.

A variety $\mathfrak{M}$ is called residually small (residually finite) if the powers of its subdirectly non-decomposable algebras are bounded (finite). The variety $\mathfrak{M}$ is k-residual for a certain cardinal k if for any subdirectly non-decomposable $\mathfrak{M}$-algebra $\mathfrak{A}$, $|\mathfrak{A}| < k$.

The following statements are known as regards residual smallness.

Theorem 2.24. If a variety $\mathfrak{M}$ is residually small, then it is $(2^{\aleph_0})^+$-residual.

Theorem 2.25. For any finite algebra $\mathfrak{A}$ such that $\mathfrak{M}(\mathfrak{A})$ is congruence-modular, the following conditions are equivalent:

(1) $\mathfrak{M}(\mathfrak{A})$ is residually small;

(2) $\mathfrak{M}(\mathfrak{A})$ is n-residual, where $n=(l+1)!\,m+1, l=m^{m^{m+1}}$ and $m=|\mathfrak{A}|$;

(3) for any $\mu,\nu \in Con\mathfrak{C}$, $\mathfrak{C} \in S(\mathfrak{A})$ the inequality $\nu \leq [\mu,\mu]$ implies the equality $\nu=[\nu,\mu]$.

Let us now dwell on some known facts of the theory of modules and rings we will need for further proofs. A variety of all modules over a ring R will be denoted by M_R. A finite ring R is called a ring with a finite type of representations iff in M_R there is only a finite number (to the accuracy of an isomorphism) of directly non-decomposable finite modules.

Theorem 2.26.

(a) Any module over a ring with a finite type of representations is isomorphic to a direct sum of finite directly non-decomposable modules.

(b) If R is a ring with a finite type of representations, and for a finite R-module $\mathcal{M}$ we have $\mathcal{M} \cong \bigoplus_{i\in I} \mathcal{M}_i$, $\mathcal{M} \cong \bigoplus_{j\in J} \mathcal{N}_j$, where $\mathcal{M}_i, \mathcal{N}_j$ are nonzero directly non-decomposable modules, then there is a bijective mapping f of a set I on a set J such that for any $i \in I$ $\mathcal{M}_i \cong \mathcal{N}_{f(i)}$.

Theorem 2.27. Any finite simple ring is isomorphic to a ring of all $n\times n$-matrices over a finite field of a certain natural number n.

By way of concluding this section, let us formulate a known theorem on Cartesian powers of finite algebras.

Theorem 2.28. If $\mathfrak{A}, \mathfrak{B}$ are finite algebras, and for a certain $n\in\omega$ $\mathfrak{A}^n \cong \mathfrak{B}^n$, then $\mathfrak{A} \cong \mathfrak{B}$.

It should be noted that the choice of definitions and theorems of the present section was prompted not by the desire to make a complete survey of the theory of universal algebras, but by the requirements of the material to be further discussed in this work. For instance, the theory of clones and many other interesting and developing fields of the theory of universal algebras have not been discussed here.

Priorities. As has been pointed out earlier in this section, the proofs of the cited results can be found in the monographs listed there. Let us again emphasize the value of the monograph by S.Burris and H.P.Sankappanavar [28] as an introduction to the modern theory of universal algebras, and that by R.Freese and R.McKenzie [71] as an introduction to the theory of commutators.

Theorem **2.1** is by G.Birkhoff [15], theorem **2.2** by A.I.Mal'tzev [130]. Theorem **2.3** can be found in a paper by G.Birkhoff [16]. The first statement in theorem **2.4** belongs to G.Grätzer and E.T.Smidt [82]. The refinement on the finiteness of a signature belongs to W.A.Lampe [120]. Theorem **2.5** is a pioneer work by A.I.Mal'tzev [131], which initiated the study of congruence-classes of varieties. Theorem **2.6** was proved by R.Dedekind [50], theorem **2.7** by G.Birkhoff. Theorem **2.8** can be found in a paper by A.Day [48], theorem **2.9** in a work by B.Jonsson [98], theorem **2.10** in that by A.F.Pixley [182]. Theorem **2.11** belongs to R.Magari [127]. One of the cornerstone results in the theory of universal algebras are theorem **2.12** and corollary **2.1** resulting from it, which were obtained by A.I.Mal'tzev [131]. Theorem **2.13** is by A.L.Foster and A.F.Pixley [65], theorem **2.15** was proved by R.McKenzie [137], theorem **2.16** by B.Jonsson [98], theorem **2.17** by C.Hermann [88], theorem **2.18** by H.P.Gumm [71]. Theorems **2.19** and **2.10**, as well as corollaries **2.2** and **2.3** belong to C.Hermann [92]. Theorems **2.22** and **2.23** can be found in a paper by J.Hagemann and C.Herrmann [89], theorem **2.24** in that by W.Taylor [226], theorem **2.25** in a paper by R.Freese and R.McKenzie [72]. The statement of theorem **2.26** is by W.Baur [11] and S.Garavaglia [75]. Theorem **2.27** belongs to J.H.M.Wedderburn, theorem **2.28** can be found in a paper by Lovàsz [126].

CHAPTER 2
BOOLEAN CONSTRUCTIONS IN UNIVERSAL ALGEBRAS

3. Boolean Powers

One of the basic ways the theory of Boolean algebras has been affecting the theory of universal algebras on the whole during the last decades, has been the introduction and wide use of the construction of Boolean powers and their various modifications in universal algebra.

By $C(X,Y)$ we will henceforth mean a set of continuous mappings of a topological space X to a space Y. If not otherwise stated, universal algebras with a discrete topology will be considered.

Definition 3.1. If $\mathfrak{A}$ is an arbitrary algebra, $\mathfrak{B}$ is a Boolean algebra, and $\mathfrak{B}^*$ is a Stone space of the Boolean algebra $\mathfrak{B}$, then a Boolean power $\mathfrak{A}^{\mathfrak{B}}$ of the algebra $\mathfrak{A}$ over the Boolean algebra $\mathfrak{B}$ is said to be a subalgebra of the direct power $\mathfrak{A}^{\mathfrak{B}^*}$ of the algebra $\mathfrak{A}$ with a basic set $C(\mathfrak{B}^*,\mathfrak{A})$.

Since the constant functions of $\mathfrak{A}^{\mathfrak{B}^*}$ are contained in $C(\mathfrak{B}^*,\mathfrak{A})$, $\mathfrak{A}^{\mathfrak{B}}$ will be a subdirect power of the algebra $\mathfrak{A}$. As $\mathfrak{B}^*$ is compact, and the topology on $\mathfrak{A}$ is discrete, the domain of values of any element f contained in $\mathfrak{A}^{\mathfrak{B}}$ will be finite, and for any element a in this domain $f^{-1}(a)$ is open-closed in $\mathfrak{B}^*$, i.e., it is identical with a certain element of the Boolean algebra $\mathfrak{B}$. The converse is also valid: for any partition 1 of the Boolean algebra $\mathfrak{B}$ into a finite number of elements $b_1,\ldots,b_n$, for any sequence of elements $a_1,\ldots a_n$ of the algebra $\mathfrak{A}$ there is an element $f\in\mathfrak{A}^{\mathfrak{B}}$ such that for every $i\leq n$, any element x of the space $\mathfrak{B}^*$, if $x\in B_i$ (i.e., to an open-closed subset of the space $\mathfrak{B}^*$ corresponding to an element b_i), then $f(x)=a_i$. Therefore, the elements of a Boolean power can be set as tuples $<b_1,\ldots,b_n;a_1,\ldots a_n>$, where $b_1,\ldots b_n$ is the partition 1 in the Boolean algebra $\mathfrak{B}$, while $a_1,\ldots a_n$ are pairwise different elements of the algebra $\mathfrak{A}$. Such setting of elements of $\mathfrak{A}^{\mathfrak{B}}$ will be termed canonical, while that without the condition of pairwise difference of $a_1,\ldots,a_n$ will be called quasi-canonical. Obviously, for any quasi-canonical setting $< b_1,\dots,b_n; a_1,\dots,a_n >$, there exists a canonical setting such that the same element of the Boolean algebra $\mathfrak{A}^{\mathcal{B}}$ corresponds to them, for which purpose it would be sufficient to join together those b_i which correspond to the same values of a_i.

The definition of a Boolean power obviously entails that any class of algebras are closed relative to direct powers and subalgebras will be closed relative to the Boolean powers as well, and, in particular, so will be arbitrary varieties and quasi-varieties of algebras.

Let us recall without proof a number of obvious properties of the operation of a Boolean degree. By $\underline{2}$ we will henceforth denote a two-element Boolean algebra, by $P(A)$ a Boolean algebra of all subsets of an arbitrary set A.

Lemma 3.1. For any algebras $\mathfrak{A}, \mathfrak{A}_1$, any Boolean algebras $\mathcal{B}_1,\dots,\mathcal{B}_n$, any set C the following statements are true:

(a) $\mathfrak{A}^{\underline{2}} \cong \mathfrak{A}$;

(b) $\mathfrak{A}^{\mathcal{B}_1\times\dots\times\mathcal{B}_n} \cong \mathfrak{A}^{\mathcal{B}_1}\times\dots\times\mathfrak{A}^{\mathcal{B}_n}$;

(c) $\underline{2}^{\mathcal{B}_1} \cong \mathcal{B}_1$;

(d) if $\mathfrak{A}$ is finite, then $\mathfrak{A}^{P(C)} \cong \mathfrak{A}^C$;

(e) if $\mathcal{B}_1$ is a subalgebra of the algebra $\mathcal{B}_2$, then $\mathfrak{A}^{\mathcal{B}_1}$ is isomorphically embeddable in $\mathfrak{A}^{\mathcal{B}_2}$;

(f) if $\mathfrak{A}_1$ is a subalgebra of the algebra $\mathfrak{A}$, then $\mathfrak{A}_1^{\mathcal{B}_1}$ is also a subalgebra of the algebra $\mathfrak{A}^{\mathcal{B}_1}$;

(g) if $\mathcal{B}_1$ is a homomorphic image of the algebra $\mathcal{B}_2$, then the algebra $\mathfrak{A}^{\mathcal{B}_1}$ is a homomorphic image of the algebra $\mathfrak{A}^{\mathcal{B}_2}$;

(h) if $\mathfrak{A}_1$ is a homomorphic image of $\mathfrak{A}$, then $\mathfrak{A}_1^{\mathcal{B}_1}$ is a homomorphic image of $\mathfrak{A}^{\mathcal{B}_1}$.

Below we will obtain, under certain conditions, the converses of classes (e) and (g).

If F is a filter on a Boolean algebra $\mathcal{B}$, then the relation θ_F on the Boolean power $\mathfrak{A}^{\mathcal{B}}$ will be defined in the following way: $<f,g>\in\theta_F$ iff $[\![f=g]\!]\in F$. One can immediately check if θ_F is a congruence on $\mathfrak{A}^{\mathcal{B}}$.

Lemma 3.2.

(a) If $\mathfrak{A}$ is not a singleton algebra, then the mapping $F\to\theta_F$ is an embedding (preserving all sup and inf) of the lattice of filters of the Boolean algebra $\mathcal{B}$ in the lattice $Con(\mathfrak{A}^{\mathcal{B}})$ of congruences of the algebra $\mathfrak{A}^{\mathcal{B}}$;

(b) $\mathfrak{A}^{\mathcal{B}}/\theta_F\cong\mathfrak{A}^{\mathcal{B}/F}$.

Proof. Statement (a) can be directly checked. f/θ_F will denote an equivalence class θ_F containing an element f of the algebra $\mathfrak{A}^{\mathcal{B}}$, and b/F an equivalence class relative to the filter F containing an element b of the Boolean algebra $\mathcal{B}$. One can easily see that the mapping $\varphi:\mathfrak{A}^{\mathcal{B}}/\theta_F\to\mathfrak{A}^{\mathcal{B}/F}$ defined as

$$\varphi(f/\theta_F)=<b_1/F,\ldots,b_n/F;a_1,\ldots,a_n>,$$

where $<b_1,\ldots,b_n;a_1,\ldots,a_n>$ is a canonical representation of the element f, is an isomorphism of the algebras $\mathfrak{A}^{\mathcal{B}}/\theta_F$ and $\mathfrak{A}^{\mathcal{B}/F}$. ■

Lemma 3.3. If $\mathfrak{A}$ is arbitrary, and $\mathcal{B}_1,\mathcal{B}_2$ are Boolean algebras, then

$$(\mathfrak{A}^{\mathcal{B}_1})^{\mathcal{B}_2}\cong\mathfrak{A}^{(\mathcal{B}_1{}^{\mathcal{B}_2})}\cong\mathfrak{A}^{\mathcal{B}_1*\mathcal{B}_2},$$

where $\mathcal{B}_1*\mathcal{B}_2$ is a free product of the Boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proof. It suffices to show that:

(1) for any Boolean algebras $\mathcal{B}_1,\mathcal{B}_2$ there is an isomorphism $\mathcal{B}_1{}^{\mathcal{B}_2}\cong\mathcal{B}_1*\mathcal{B}_2$;

(2) $(\mathfrak{A}^{\mathcal{B}_1})^{\mathcal{B}_2}\cong\mathfrak{A}^{(\mathcal{B}_1{}^{\mathcal{B}_2})}$.

To prove statement (1), it is sufficient to directly check if the mapping $\varphi(<b_1,\ldots,b_n;a_1,\ldots,a_n>)=\bigvee_{i=1}^{n}(b_i\wedge a_i)$, where $<b_1,\ldots,b_n;a_1,\ldots,a_n>$, is a canonical representation of an element of the algebra $\mathcal{B}_1^{\mathcal{B}_2}$ is an isomorphism of the algebra $\mathcal{B}_1^{\mathcal{B}_2}$ on the algebra $\mathcal{B}_1 * \mathcal{B}_2$.

Statement (2) is proved by the same direct checking. Indeed, if

$$< b_1,\dots,b_n; d_1,\dots,d_n > \in (\mathfrak{A}^{\mathcal{B}_1})^{\mathcal{B}_2},$$

in which case $d_i = < c_1^i,\dots,c_{m_i}^i; a_1^i,\dots,a_{m_i}^i > \in \mathfrak{A}^{\mathcal{B}_1}$, then we will consider the elements $r_{ij} = < b_1,\dots,b_n; \alpha_1^{ij},\dots,\alpha_n^{ij} > \in \mathcal{B}_1^{\mathcal{B}_2}$, where $i \le n; j \le m_i$, while $\alpha_k^{ij} = c_j^i$ at $k = i$, and $\alpha_k^{ij} = 0$ at $k \ne i$. Obviously, the elements $r_{ij}(i \le n, j \le m_i)$ perform the partition of unity in the Boolean algebra $\mathcal{B}_1^{\mathcal{B}_2}$, and we can easily check that the mapping

$$\varphi(< b_1,\dots,b_n; d_1,\dots,d_n >) = < r_{ij}(i \le n, j \le m_i);\ a_j^i(i \le n, j \le m_i) >$$

is an isomorphism of the algebra $(\mathfrak{A}^{\mathcal{B}_1})^{\mathcal{B}_2}$ on $\mathfrak{A}^{(\mathcal{B}_1^{\mathcal{B}_2})}$. ■

The following statement contains a sufficient condition for subdirect powers of the algebra $\mathfrak{A}$ to be isomorphic to Boolean powers of this algebra.

Lemma 3.4. Let a subalgebra $\mathfrak{C}$ of a direct power $\mathfrak{A}^I$ of the algebra $\mathfrak{A}$ have the following properties:

(1) all constant functions from $\mathfrak{A}^I$ are in $\mathfrak{C}$;

(2) the range of any function $f \in \mathfrak{C}$ is finite;

(3) for any $f_1, f_2, f_3, f_4 \in \mathfrak{C}$, if $g \in \mathfrak{A}^I$ is such that for $i \in \|[f_1 = f_2]\|$ $g(i) = f_3(i)$, and for $i \in \|[f_1 \ne f_2]\|$ $g(i) = f_4(i)$, in which case $g \in \mathfrak{C}$, then there is a Boolean algebra $\mathcal{B}$ such that $\mathfrak{C} \cong \mathfrak{A}^{\mathcal{B}}$.

Proof. Let $\bar{a}$ be a function from $\mathfrak{A}^I$ assuming a constant value $a \in \mathfrak{A}$. Let us consider a family S of subsets of the set I that consists of sets of the type $\|[f = g]\|$ where $f, g \in \mathfrak{C}$. It should be remarked that S is a subalgebra $\mathcal{B}$ of the Boolean algebra of subsets of the set I. Indeed, if, for instance, $A = \|[f = g]\|$, $B = \|[h = k]\|$ for $f, g, h, k \in \mathfrak{C}$, then, by property (3), there are functions $p, q \in \mathfrak{C}$ such that $\|[p = f]\| = \|[q = f]\| = \|[f = g]\|$ and $\|[p = h]\| = \|[f \ne g]\|$, $\|[q = k]\| = [f \ne g]$. In this case, obviously, $A \cup B = \|[p = q]\|$. In an analogous way we can prove that $A \cap B$ and $\neg A$ also belong to S.

Let $f \in \mathfrak{C}$ and $x \in \mathfrak{B}^*$. As Rf (the range of f) is finite, x is an ultrafilter on $\mathfrak{B}$, and $\bigsqcup_{a \in Rf} f^{-1}(a) = I \in x$; as $f^{-1}(a) = [\![f = \bar{a}]\!] \in \mathfrak{B}$, and $f^{-1}(a)$ $(a \in Rf)$ are disjunct, $f^{-1}(a) \in x$ to the accuracy of a single $a \in Rf$. Let us denote this a by $\sigma(f,x)$. Let us define a mapping α of the algebra $\mathfrak{C}$ into a direct power $\mathfrak{A}^{\mathfrak{B}^*}$ of the algebra $\mathfrak{A}$, assuming $\alpha(f)(x) = \sigma(f,x)$. Obviously, in this case the elements of the algebra $\mathfrak{C}$ of the type $\bar{a}$ lead to constant functions from $\mathfrak{A}^{\mathfrak{B}^*}$ and assume the same a value (the latter denoted by $\hat{a}$). For any $f_1, f_2 \in \mathfrak{C}$

$$[\![\alpha(f_1) = \alpha(f_2)]\!] = \{x \in \mathfrak{B}^* \mid [\![f_1 = f_2]\!] \in x\}.$$

Indeed, we have

$$[\![\alpha(f_1) = \alpha(f_2)]\!] = \{x \in \mathfrak{B}^* \mid \alpha(f_1)(x) = \alpha(f_2)(x)\} =$$
$$\{x \in \mathfrak{B}^* \mid \sigma(f_1,x) = \sigma(f_2,x)\} = \{x \in \mathfrak{B}^* \mid f_1^{-1}(a) \in x.\ f_2^{-1}(a) \in x$$

for some $a \in \mathfrak{A}\} = \{x \in \mathfrak{B}^* \mid [\![f_1 = f_2]\!] \in x\}$.

It is obvious that for $f_1 \neq f_2 \in \mathfrak{C}$, $\alpha(f_1) \neq \alpha(f_2)$. α is an isomorphism of $\mathfrak{C}$ on a subalgebra of the algebra $\mathfrak{A}^{\mathfrak{B}^*}$. By virtue of the equalities $\alpha(\bar{a}) = \hat{a}$ and $[\![\alpha(f) = \alpha(\bar{a})]\!] = \{x \in \mathfrak{B}^* \mid [\![f = \bar{a}]\!] \in x\}$, the set $[\![\alpha(f) = \hat{a}]\!]$ is open-closed in the space $\mathfrak{B}^*$. Moreover, by the definition of the mapping α, for any $f \in \mathfrak{C}$, the range of the function $\alpha(f)$ is equal to Rf. Therefore, if $Rf = \{a_1, \ldots, a_n\}$, then αf coincides with an element of the Boolean power $\mathfrak{A}^{\mathfrak{B}}$ with a canonical setting $< [\![\alpha(f) = \hat{a}_1]\!], \ldots, [\![\alpha(f) = \hat{a}_n]\!]; a_1, \ldots a_n >$. Hence, the algebra $\alpha(\mathfrak{C})$ is a subalgebra of the algebra $\mathfrak{A}^{\mathfrak{B}}$.

Let now $h \in \mathfrak{A}^{\mathfrak{B}}$ and $< [\![f_1 = g_1]\!], \ldots, [\![f_n = g_n]\!]; a_1, \ldots, a_n >$ be its canonical setting. Property (3) implies the existence of an element $g \in \mathfrak{C}$ such that $[\![g = \bar{a}_i]\!] = [\![f_i = g_i]\!]$ for $i \leq n$. In this case $\alpha(g) = h$ and, hence, α is an isomorphism of the algebra $\mathfrak{C}$ on the Boolean power $\mathfrak{A}^{\mathfrak{B}}$. ■

Let us recall that if $\mathfrak{A}$ is a certain finite group or a module, then an arbitrary direct sum of $\mathfrak{A}$ is isomorphic to a certain Boolean power of the algebra $\mathfrak{A}$. Indeed, let $\mathfrak{C} = \bigoplus_{i \in \aleph} \mathfrak{A}_i$, where $\aleph$ is a cardinal and $\mathfrak{A}_i = \mathfrak{A}$. Let us construct a sequence $f_i (i \in \aleph)$ of isomorphic embeddings of the algebras $\mathfrak{C}_i = \bigoplus_{j<i} \mathfrak{A}_j$ into an algebra $\mathfrak{C}' = \mathfrak{A}^{\aleph}$ such that for $i < j \in \aleph$ f_j is an extension of f_i. Let us set

$f_1(a)(i) = a$ for any $a \in \mathfrak{C}_1 = \mathfrak{A}$ and any $i \in \aleph$. For limiting i, $f_i = \bigsqcup_{j<i} f_i$, and for any $i \in \aleph$, f_{i+1} is an extension of f_i from the algebra $\mathfrak{C}_i = \bigoplus_{j<i} \mathfrak{A}_j$ on an algebra $\mathfrak{C}_{i+1} = (\bigoplus_{j<i} \mathfrak{A}_j) \oplus \mathfrak{A}$, defined by the following condition: for any $a \in \mathfrak{A}$ (the last addent) $f_{i+1}(a)(j) = 0$ if $j < i$, and $f_{i+1}(a)(j) = a$ if $j \geq i$. One can easily check the fact that such an isomorphic embedding f_{i+1} exists. One can also make a direct remark that if $f = \bigsqcup_{i \in \aleph} f_i$, then $f(\mathfrak{C}) \subseteq \mathfrak{A}^{\aleph}$ obeys the conditions of lemma 3.4. Therefore, both $f(\mathfrak{C})$ and $\mathfrak{C}$ are isomorphic to some Boolean power of the algebra $\mathfrak{A}$.

Definition 3.2. An algebra $\mathfrak{A}$ is called Boolean-separated, if for any Boolean algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$ the isomorphism of the Boolean powers $\mathfrak{A}^{\mathfrak{B}_1}$ and $\mathfrak{A}^{\mathfrak{B}_2}$ implies that of the Boolean algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

Theorem 3.1. If $\mathfrak{A}$ is a non-Abelian subdirectly non-decomposable algebra, and $\mathfrak{M}(\mathfrak{A})$ is congruence-modular, then $\mathfrak{A}$ is Boolean-separated.

Proof. Let $\mathfrak{A}$ obey the conditions of the theorem. Let us construct an $L_{\omega_1,\omega}$ interpretation of any Boolean algebra $\mathfrak{B}$ in a Boolean power $\mathfrak{A}^{\mathfrak{B}}$. It should be recalled that the notion of the center of the algebra of a congruence-modular variety of algebras, of a commutator of congruences of such a variety, as well as the basic properties of these notions are given in section **2**. Let $Z_{\mathfrak{A}}$ be the center of the algebra $\mathfrak{A}$, i.e., $Z_{\mathfrak{A}}$ is the biggest congruence γ on $\mathfrak{A}$ such that $[\nabla, \gamma] = \Delta$. Let β be a monolith of $\mathfrak{A}$. The relation $<x,y> \in \theta_{u,v}$ on $\mathfrak{A}$ is definable with the $L_{\omega_1,\omega}$ -formula (see theorem **2.12**). Let

$$\| fZg \| = \{x \in \mathfrak{B}^* \mid < f(x), g(x) > \in Z_{\mathfrak{A}}\}.$$

According to theorem **2.18**, for any algebra $\mathfrak{C} : <a,b> \in Z_{\mathfrak{A}}$ iff

$$\mathfrak{C} \models \bigwedge_t \forall \bar{x}, \bar{y}(t(\bar{x},a) = t(\bar{y},a) \leftrightarrow t(\bar{x},b) = t(\bar{y},b)),$$

where the conjunction ranges over all the terms of the algebra $\mathfrak{C}$ and $\bar{x}, \bar{y}$-tuples of the variables of the length n, if t has $n+1$ variables. Therefore, it is obvious that $\{\| fZg \| \mid f, g \in \mathfrak{A}^{\mathfrak{B}}\} = \mathfrak{B}$. Let us consider the following $L_{\omega_1,\omega}$-formulas:

(1) $red'(a,b,c,d) = \bigvee_t \exists \bar{r},\bar{s}([\{a,b\} = \{t(\bar{r},c),t(\bar{s},c)\} \wedge$
$\wedge\, t(\bar{r},d) = t(\bar{s},d)] \vee [\{a,b\} = \{t(\bar{r},d),t(\bar{s},d)\} \wedge t(\bar{r},c) = t(\bar{s},c)])$.

The disjunction is taken here over all the terms of the signature of a variety $\mathfrak{M}$ and the length of the tuples $\bar{r},\bar{s}$, a unity smaller than the number of the variables of the term t. It is obvious that for $a,b,c,d \in \mathfrak{A}$,

$$\mathfrak{A} \models red'(a,b,c,d) \,\&\, < c,d > \in Z_{\mathfrak{A}} \rightarrow a = b.$$

(2) $red(a,b,c,d) = \exists x,y(< a,b > \in \theta_{x,y} \wedge red'(x,y,c,d))$.

One can easily observe that:

$$\mathfrak{A} \models red(a,b,c,d) \,\&\, < c,d > \in Z_{\mathfrak{A}} \Rightarrow a = b,$$
$$\mathfrak{A}^{\mathcal{B}} \models red(f,g,h,k) \rightarrow \mathfrak{A} \models red(f(i),g(i),h(i),k(i))$$

for all $i \in \mathcal{B}^*$. Therefore, if $\mathfrak{A}^{\mathcal{B}} \models red(f,g,h,k)$, then $\|hZk\| \subseteq [f = g]$.

(3) $\perp(a,b,c,d) = \forall x,y(red(x,y,a,b) \wedge red(x,y,c,d) \rightarrow x = y)$.

Let us show that for $f_0,f_1,f_2,f_3 \in \mathfrak{A}^{\mathcal{B}}$, the property

$$\mathfrak{A}^{\mathcal{B}} \models \perp(f_0,f_1,f_2,f_3)$$

holds iff

(*) $\|f_0Zf_1\| \cup \|f_2Zf_3\| = \mathcal{B}^*$.

In one direction this equivalence results immediately from the property of the relation *red* mentioned before the relation $\perp$ was introduced. Let us now assume that $x_0 \in \mathcal{B}^*$, $< f_0(x_0),f_1(x_0) \notin Z_{\mathfrak{A}}$, and $< f_2(x_0),f_3(x_0) \notin Z_{\mathfrak{A}}$. Let $U = \bigcap_{i=0}^{3}(f_i^{-1}(f_i(x_0)))$, then $U \in \mathcal{B}$. By theorem **2.18**, there are terms t^0,t^1 and tuples of the elements of the algebra $\mathfrak{A}$

$$\bar{r}^0 =< r_0^0,\ldots,r_{k-1}^0 >, \quad \bar{s}^0 =< s_0^0,\ldots,s_{k-1}^0 >,$$

such that

$$t^0(\bar{r}^0, f_0(x_0)) = t^0(\bar{s}^0, f_0(x_0)), t^0(\bar{r}^0, f_1(x_0)) \neq t^0(\bar{s}^0, f_1(x_0)),$$
$$t^1(\bar{r}^1, f_2(x_0)) = t^1(\bar{s}^1, f_2(x_0)), t^1(\bar{r}^1, f_3(x_0)) \neq t^1(\bar{s}^1, f_3(x_0)).$$

Let us fix a certain element $e \in \mathfrak{A}$, and for $\varepsilon = 0,1$ we will define $\bar{r}^\varepsilon(x) = \bar{r}^\varepsilon$, if $x \in U$ and $\bar{r}^\varepsilon = < e,...,e >$, if $x \notin U$. Analogously, $\bar{s}^\varepsilon(x) = \bar{s}^\varepsilon$, if $x \in U$, and $\bar{s}^\varepsilon = < e,...,e >$ if $x \notin U$. Therefore, $\bar{r}^\varepsilon, \bar{s}^\varepsilon$ is a tuple of the elements of the algebra $\mathfrak{A}^{\mathcal{B}}$. Let

$$\gamma^0 = t^0(\bar{r}^0, f_1), \delta^0 = t^0(\bar{s}^0, f_1), \gamma^1 = t^1(\bar{r}^1, f_3), \delta^1 = (\bar{s}^1, f_3).$$

In this case $\mathfrak{A}^{\mathcal{B}} \models red'(\gamma^0, \delta^0, f_0, f_1) \wedge red'(\gamma^1, \delta^1, f_2, f_3)$. The equality $\|\gamma^0 = \delta^0\| = \|[\gamma^1 = \delta^1]\| = \mathcal{B}^* \setminus U$ is valid, and $\gamma^0, \gamma^1, \delta^0, \delta^1$ are constants on U. Let us, finally, show that $\mathfrak{A}^{\mathcal{B}} \not\models \perp(f_0, f_1, f_2, f_3)$. Indeed, let us choose $a \neq b \in \mathfrak{A}$ so that $< a,b > \in \beta$. Let $\gamma, \delta \in \mathfrak{A}^{\mathcal{B}}$, $\gamma(x) = a$ for all $x \in \mathcal{B}^*$, and $\delta(x) = a$ if $x \notin U$ and $\delta(x) = b$ if $x \in U$.

Let us show that $< \gamma, \delta > \in \theta_{\gamma^0, \delta^0} \cap \theta_{\gamma^1, \delta^1}$, in which case $\gamma \neq \delta$. This property, combined with the one cited above, i.e.,

$$\mathfrak{A}^{\mathcal{B}} \models red'(\gamma^0, \delta^0, f_0, f_1) \wedge red'(\gamma^1, \delta^1, f_2, f_3),$$

implies that on $\mathfrak{A}^{\mathcal{B}}$ the formula $red(\gamma, \delta, f_0, f_1) \wedge red(\gamma, \delta, f_2, f_3) \wedge \gamma \neq \delta$ is true, i.e., it implies the required statement, $\mathfrak{A}^{\mathcal{B}}) \not\models \perp(f_0, f_1, f_2, f_3)$.

Let $\mathfrak{A}_U^{\mathcal{B}}$ be a subalgebra of the algebra $\mathfrak{A}^{\mathcal{B}}$, consisting of elements $\mathfrak{A}^{\mathcal{B}}$ which are constants on U. Let us show that $< \gamma, \delta > \in \theta_{\gamma^0, \delta^0}$ in the subalgebra $\mathfrak{A}_U^{\mathcal{B}}$ and, moreover, $< \gamma, \delta > \in \theta_{\gamma^0, \delta^0}$ in the algebra $\mathfrak{A}^{\mathcal{B}}$. Let

$$\eta_1 = \{< \mu, \nu > \in (\mathfrak{A}_U^{\mathcal{B}})^2 \mid \mu \mid \mathcal{B}^* \setminus U = \nu \mid \mathcal{B}^* \setminus U\},$$
$$\eta = \{< \mu, \nu > \in (\mathfrak{A}_U^{\mathcal{B}})^2 \mid \mu(x_0) = \nu(x_0)\},$$
$$\theta_0 = \theta_{\gamma^0, \delta^0}$$

in the subalgebra $\mathfrak{A}_U^{\mathcal{B}}$. Obviously, $\eta \times \eta^1 = \Delta$ and $\eta^1 \geq \theta_0$, since γ^0, δ^0 coincide on $\mathcal{B}^* \setminus U$. Let $\bar{\beta} = \{< \mu, \nu > \in (\mathfrak{A}_U^{\mathcal{B}})^2 \mid < \mu(x_0), \nu(x_0) > \in \beta\}$. In this case $\mathfrak{A}_U^{\mathcal{B}} / \eta \cong \mathfrak{A}$

and, hence, $\bar{\beta} > \eta$, and $\bar{\beta}$ covers η in $Con\mathfrak{A}_U^{\mathfrak{B}}$. Therefore, $\eta \vee \theta_0 = \bar{\beta}$. Moreover, $\eta^1 \geq \theta_0$ and, hence, since $Con\mathfrak{A}_U^{\mathfrak{B}}$ is modular, we get:

$$\bar{\beta} \wedge \eta^1 \leq (\eta \vee \theta_0) \wedge \eta^1 = (\eta \wedge \eta^1) \vee \theta_0 = \theta_0.$$

$<\gamma,\delta> \in \bar{\beta} \wedge \eta^1$, theirfore, $<\gamma,\delta> \in \theta_0$, i.e., $<\gamma,\delta> \in \theta_{\gamma^0,\delta^0}$ in the algebra $\mathfrak{A}^{\mathfrak{B}}$. In an analogous way one can deduce that $<\gamma,\delta> \in \theta_{\gamma^1,\delta^1}$ in $\mathfrak{A}^{\mathfrak{B}}$. As has been noted earlier, this fact implies $\mathfrak{A}^{\mathfrak{B}} \models \perp(f_0, f_1, f_2, f_3)$, and, therefore, the equivalence in (*) is proved.

The equivalence (*) and the equality discussed above $(\{\|[fZg]\| \mid f,g \in \mathfrak{A}^{\mathfrak{B}}\} = \mathfrak{B})$ imply $L_{\omega_1,\omega}$-definability of $\mathfrak{B}$ in $\mathfrak{A}^{\mathfrak{B}}$. Elements b of the Boolean algebra $\mathfrak{A}$ are interpretable by pairs of elements $<f,g>$ of the algebra $\mathfrak{A}^{\mathfrak{B}}$ such that $\|[fZg]\| = b$, in which case $\|[fZg]\| \subseteq \|[hZk]\|$ iff

$$\mathfrak{A}^{\mathfrak{B}} \models \forall x,y(\perp(f,g,x,y) \rightarrow \perp(h,k,x,y)). \ \blacksquare$$

In the case of a congruence-distributive variety, a Boolean degree in the Boolean power for algebras of the given variety can be singled out in a more direct and algebraic way, which, in particular, enables one to transfer the results on relations of embedding and epimorphism from Boolean algebras to algebras of congruence-distributive varieties. The results of theorems **3.2** and **3.3** are essential in this respect.

Theorem 3.2. If $\mathfrak{M}$ is a congruence-distributive variety, $\mathfrak{A} \in \mathfrak{M}$, $\mathfrak{B}$ is a Boolean algebra, and $f,g,h,k \in \mathfrak{A}^{\mathfrak{B}}$, then $<f,g> \in \theta_{h,k}^{\mathfrak{A}^{\mathfrak{B}}}$ iff for any $x \in \mathfrak{B}^*$ we have $\theta_{f(x),g(x)}^{\mathfrak{A}} \subseteq \theta_{h(x),k(x)}^{\mathfrak{A}}$. In particular, it entails the equality $Con_p(\mathfrak{A}^{\mathfrak{B}}) \cong (Con_p\mathfrak{A})^{\mathfrak{B}}$.

Proof. Let U be an ultrafilter on $\mathfrak{B}$ and $\theta_1 \in Con\mathfrak{A}$, then $\theta_1(U)$ will denote a congruence on $\mathfrak{A}^{\mathfrak{B}}$ defined in the following way: for $f,g \in \mathfrak{A}^{\mathfrak{B}}$ $<f,g> \in \theta_1(U)$ iff $<f(U),g(U)> \in \theta_1$.

Let us first of all prove the following variation of the Jonsson lemma:

(*) any congruence on $\mathfrak{A}^{\mathfrak{B}}$ is an intersection of congruences of the type $\theta_1(U)$, where $\theta_1 \in Con\mathfrak{A}, U \in \mathfrak{B}^*$.

Let us note that, for any $\theta \in Con(\mathfrak{A}^{\mathcal{B}})$ that is non-decomposable at the intersection, there is a $U \in \mathcal{B}^*$ such that $\Delta(U) \leq \theta$. Let $V = \{b \in \mathcal{B} \mid \Delta_b \leq \theta\}$. Here

$$\Delta_b = \{< f, g > \in (\mathfrak{A}^{\mathcal{B}})^2 \mid \; [\![f = g]\!] \supseteq b\}.$$

Obviously, if $b_1 \subseteq b_2$ and $b_1 \in V$, then $b_2 \in V$. Assume now that $b_1 \vee b_2 \in V$. Then $\Delta_{b_1 \vee b_2} = \Delta_{b_1} \wedge \Delta_{b_2}$ and, since $\Delta_{b_1 \vee b_2} \leq \theta$,

$$\theta = \theta \vee \Delta_{b_1 \vee b_2} = \theta \vee (\Delta_{b_1} \wedge \Delta_{b_2}) = (\theta \vee \Delta_{b_1}) \wedge (\theta \vee \Delta_{b_2})$$

(as $\mathfrak{M}$ is congruence-distributive). Since θ is non-decomposable at the intersection of congruences, we have either $\theta = \theta \vee \Delta_{b_1}$ (i.e., $b_1 \in V$), or $\theta = \theta \vee \Delta_{b_2}$ (i.e., $b_2 \in V$). It should be remarked that $V = \mathcal{B}$ iff $\theta = \nabla$. As the statement (*) is obvious for $\theta = \nabla$, we will assume $V \neq \mathcal{B}$.

Let now $\mathfrak{D}$ be a maximal filter among the filters contained in V. Let us show that $\mathfrak{D}$ is an ultrafilter. Assume to the contrary that $b \in \mathcal{B}$ and $b \notin \mathfrak{D}, \neg b \notin \mathfrak{D}$. If for any $d \in \mathfrak{D}$ we had $b \cap d \in V$, then $\mathfrak{D}$ and b would generate a filter contained in V and strictly greater than $\mathfrak{D}$. Therefore, we can find a $b_0 \in \mathfrak{D}$ such that $b \cap b_0 \notin V$. Analogously, there is a $b_1 \in \mathfrak{D}$ such that $\neg b \cap b_1 \notin V$. Let $b_2 = b_0 \wedge b_1$. In this case, $b_2 \in \mathfrak{D} \subseteq V$, $b_2 \wedge b \notin V$ and $b_2 \wedge \neg b \notin V$, which contradicts the above-mentioned property of V and the fact that $b_2 = (b_2 \vee b) \wedge (b_2 \vee \neg b) \in V$. Thus, $\mathfrak{D}$ is indeed an ultrafilter, i.e., $\mathfrak{D} \in \mathcal{B}^*$.

If $f, g \in \mathfrak{A}^{\mathcal{B}}$ and $< f,g > \in \Delta(\mathfrak{D})$, then there is a $b \in \mathfrak{D}$ such that $< f,g > \in \Delta_b$ and, therefore, $< f,g > \in \theta$. Hence, $\Delta(\mathfrak{D}) \leq \theta$, i.e., indeed, for any congruence $\theta \in Con(\mathfrak{A}^{\mathcal{B}})$ non-decomposable at the intersection, there is a $U \in \mathcal{B}^*$ such that $\Delta(U) \leq \theta$.

Assume that $\theta \in Con(\mathfrak{A}^{\mathcal{B}})$, that θ is non-decomposable at the intersection, and that $U \in \mathcal{B}^*$ is such that $\Delta(U) \leq \theta$. Let $\theta_1 = \{< a,b > \in \mathfrak{A}^2 \mid$ for some $f, g \in \mathfrak{A}^{\mathcal{B}}$ $< f,g > \in \theta$ and $f(U) = a, g(U) = b\}$.

Let us show that $\theta_1(U) = \theta$. $\theta \leq \theta_1(U)$ being obvious, let us prove the converse. Assume that $< f,g > \in \theta_1(U)$, then $< f(U), g(U) > \in \theta_1$. By the definition of θ_1, there are $h,k \in \mathfrak{A}^{\mathcal{B}}$ such that $< h,k > \in \theta$ and $h(U) = f(U), k(U) = g(U)$. As $\theta \geq \Delta(U)$, $< f,h > \in \theta, < k,g > \in \theta$, and all this implies that $< f,g > \in \theta$. Therefore, $\theta_1(U) \leq \theta$ and, as a result, $\theta = \theta_1(U)$, and the statement (*) is proved.

Assume now that $f, g \in \mathfrak{A}^{\mathcal{B}}$. Since $< f,g > \in \bigcap_{U \in \mathcal{B}^*} \theta_{f(U)g(U)}(U)$,

$\theta_{f,g} \le \bigcap_{U \in \mathcal{B}^*} \theta_{f(U),g(U)}(U)$. Let us prove the converse. By the statement (*), there are $\theta_j \in Con\mathfrak{A}$ and $U_j \in \mathcal{B}^*$ ($j \in J$) such that $\theta_{f,g} = \bigcap_{j \in J} \theta_j(U_j)$. Since $< f,g > \in \theta_{f,g}$, $< f(U_j), g(U_j) > \in \theta_j$ and, hence, for any $j \in J$, $\theta_j \ge \theta_{f(U_j),g(U_j)}$. This implies the inequality $\theta_j(U_j) \ge \theta_{f(U_j),g(U_j)}(U_j)$, i.e.,

$$\theta_{f,g} = \bigcap_{j \in J} \theta_j(U_j) \ge \bigcap_{j \in J} \theta_{f(U_j),g(U_j)}(U_j) \ge \bigcap_{U \in \mathcal{B}^*} \theta_{f(U),g(U)}(U).$$

Together with the above-mentioned inequality, the latter one implies the equality $\theta_{f,g} = \bigcap_{U \in \mathcal{B}^*} \theta_{f(U),g(U)}(U)$.

One can easily notice that a similar representation of $\theta_{f,g}$ is one-to-one. As $f,g \in \mathfrak{A}^{\mathcal{B}}$, there is a finite partition of $\mathcal{B}^*$ by the elements $b_1,\dots,b_n$ of $\mathcal{B}$ such that the functions f,g on b_i are constant, $\theta_{f(U),g(U)}$ are also constant on b_i. Therefore, putting into correspondence to the congruence $\theta = \theta_{f,g}$ an element $\varphi(\theta) \in \prod_{U \in \mathcal{B}^*} Con_p\mathfrak{A}$ such that $\varphi(\theta)(U) = \theta_{f(U),g(U)}$, we get $\varphi(\theta) \in (Con_p\mathfrak{A})^{\mathcal{B}}$. It is quite obvious that the mapping φ is from $Con_p(\mathfrak{A}^{\mathcal{B}})$ to $(Con_p\mathfrak{A})^{\mathcal{B}}$ and preserves the order. Since the presentation of $\theta_{f,g}$ is injective, and, hence, so is the mapping φ, φ is an isomorphism from $Con_p(\mathfrak{A}^{\mathcal{B}})$ to $(Con_p\mathfrak{A})^{\mathcal{B}}$. ■

Corollary 3.1. If $\mathfrak{M}$ is a congruence-distributive variety and $\mathfrak{A}$ is a simple $\mathfrak{M}$-algebra, then:

(a) for any Boolean algebra $\mathcal{B}$ and for $f,g,h,k \in \mathfrak{A}^{\mathcal{B}}$, $< f,g > \in \theta_{h,k}^{\mathfrak{A}^{\mathcal{B}}}$ iff $\{x \in \mathcal{B}^* | f(x) = g(x)\} \supseteq \{x \in \mathcal{B}^* | h(x) = k(x)\}$; $Con_p(\mathfrak{A}^{\mathcal{B}}) \cong \mathcal{B}$;

(b) for any Boolean algebras $\mathcal{B}_1, \mathcal{B}_2$ the relations $\mathfrak{A}^{\mathcal{B}_1} << \mathfrak{A}^{\mathcal{B}_2}$ and $\mathcal{B}_1 << \mathcal{B}_2$ are equivalent;

(c) for any Boolean algebra $\mathcal{B}$ and $\theta \in Con\mathfrak{A}^{\mathcal{B}}$ there is a $\psi \in Con\mathcal{B}$ such that $\mathfrak{A}^{\mathcal{B}} / \theta \cong \mathfrak{A}^{\mathcal{B}/\psi}$.

Proof. Statement (a) directly follows from theorem **3.2.** In order to prove statement (b), let us recall that, by lemma **3.1** (g), $\mathcal{B}_1 << \mathcal{B}_2$ entails $\mathfrak{A}^{\mathcal{B}_1} << \mathfrak{A}^{\mathcal{B}_2}$. Assume now that $\mathfrak{A}^{\mathcal{B}_1} << \mathfrak{A}^{\mathcal{B}_2}$. Then, obviously, $\mathrm{Con}_p(\mathfrak{A}^{\mathcal{B}_1}) << \mathrm{Con}_p(\mathfrak{A}^{\mathcal{B}_2})$ but, as $\mathfrak{A}$ is simple, $Con_p\mathfrak{A} = \underline{2}$ (a two-element Boolean algebra) and, by theorem **3.2,**

$Con_p(\mathfrak{A}^{\mathcal{B}_i}) \cong \underline{2}^{\mathcal{B}_i}$. At the same time, the algebras $\underline{2}^{\mathcal{B}_i}$ are isomorphic with the Boolean algebras $\mathcal{B}_i$ (by lemma **3.1** (c)). Therefore, indeed, $\mathfrak{A}^{\mathcal{B}_1} << \mathfrak{A}^{\mathcal{B}_2}$ entails $\mathcal{B}_1 << \mathcal{B}_2$. ■

The validity of clause (c) is readily deduced from clause (a).

As to clause (a), it should be remarked that a complete description of the structure of finite Boolean-separated algebras in congruence-distributive varieties by D.Bigelow and S.Burris [13] are in full accord with the description of finite Boolean-separated groups by A.B.Apps [4].

Definition 3.3. A variety $\mathfrak{M}$ has extendable congruences if for any algebra $\mathfrak{A} \in \mathfrak{M}$, its any subalgebra $\mathfrak{A}_1$, and any congruence $\theta_1 \in Con\mathfrak{A}_1$, there is a congruence $\theta \in Con\mathfrak{A}$ (an extension of θ_1 onto the algebra $\mathfrak{A}$) such that $\theta|\mathfrak{A}_1 = \theta_1$.

Many properties of the relation << on congruence-distributive varieties can be transferred to relations of embedding $\leq$ if we require in addition that the variety should have the property of congruence extension and, in particular, that there should be a corollary analogous to **3.1** for the relation of embedding.

Theorem 3.3. Let $\mathfrak{M}$ be a congruence-distributive variety with extendable congruences and $\mathfrak{A}$ a simple $\mathfrak{M}$-algebra. Then for any non-singleton Boolean algebras $\mathcal{B}_1$ and $\mathfrak{A}_2$ the relations $\mathfrak{A}^{\mathcal{B}_1} \leq \mathfrak{A}^{\mathcal{B}_2}$ and $\mathcal{B}_1 \leq \mathcal{B}_2$ are equivalent.

Proof. The entailment $\mathcal{B}_1 \leq \mathcal{B}_2 \Rightarrow \mathfrak{A}^{\mathcal{B}_1} \leq \mathfrak{A}^{\mathcal{B}_2}$ requires no extra assumptions, as been mentioned in lemma **3.1** (f). Let us now prove the converse. Let f be an embedding of $\mathfrak{A}^{\mathcal{B}_1}$ in $\mathfrak{A}^{\mathcal{B}_2}$. If $a,c \in \mathfrak{A}$ and b is an element of the Boolean algebra $\mathfrak{A}$, then $\bar{a}, \frac{a}{c}|b$ will denote the elements of the algebra $\mathfrak{A}^{\mathcal{B}}$ with the canonical representations $< 1;a >, < b, \neg b; c,a >$, respectively. By corollary **3.1**, for any $h_1,h_2,g_1,g_2 \in \mathfrak{A}^{\mathcal{B}}$ we have $< g_1,g_2 > \in \theta^{\mathfrak{A}^{\mathcal{B}}}_{h_1,h_2}$ iff $\|[h_1 = h_2]\| \subseteq \|[g_1 = g_2]\|$.

Let us fix a pair of elements $a \neq c$ of the algebra $\mathfrak{A}$ and construct a mapping ψ of the algebra $\mathcal{B}_1$ to $\mathcal{B}_2$ in the following way: for $b \in \mathcal{B}_1$ let $\psi(b) = \|[f(\bar{a}) \neq f(\frac{a}{c}|b)]\|$. Since $\mathfrak{M}$ has extendable congruences, the mapping $\varphi_1(\theta^{\mathfrak{A}^{\mathcal{B}_1}}_{\bar{a},\frac{a}{c}|b}) = \theta^{\mathfrak{A}^{\mathcal{B}_2}}_{f(\bar{a}),f(\frac{a}{c}|b)}$ will be an injective mapping from $Con_p(\mathfrak{A}^{\mathcal{B}_1})$ to $Con_p(\mathfrak{A}^{\mathcal{B}_2})$ which preserves the order. Let us consider the mappings

$\psi_0(b) = \theta^{\mathfrak{A}^{\mathcal{B}_1}}_{\bar{a},\frac{a}{c}|b}$ (φ_0, as has been noted above, is an isomorphism between $\mathcal{B}_1$ and $Con_p(\mathfrak{A}^{\mathcal{B}_1})$) and

$$\varphi_2(\theta^{\mathfrak{A}^{\mathcal{B}_2}}_{f(\bar{a}),f(\frac{a}{c}|b)}) = \|f(\bar{a}) \neq f(\tfrac{a}{c}|b)\|$$

(φ_2 is an order-preserving injective mapping from $\varphi_1(Con_p(\mathfrak{A}^{\mathcal{B}_1}))$ to $\mathcal{B}_2$). As $\psi = \varphi_2 \cdot \varphi_1 \cdot \varphi_0$, ψ is an injective, order-preserving mapping from $\mathcal{B}_1$ to $\mathcal{B}_2$. Let us now show that ψ is an isomorphic embedding of $\mathcal{B}_1$ in a Boolean algebra $\mathcal{B}_2 | \|f(\bar{a}) \neq f(\bar{c})\|$, i.e., in a Boolean algebra with a basic set $\{b \in \mathcal{B}_2 | b \subseteq \|f(\bar{a}) \neq f(\bar{c})\|\}$ and the operations induced from $\mathcal{B}_2$. The latter fact, obviously, implies the isomorphic embedding that we have been looking for, i.e., that of the Boolean algebra $\mathcal{B}_1$ in $\mathcal{B}_2$.

In order to prove that ψ is an isomorphic embedding of $\mathcal{B}_1$ in $\mathcal{B}_2 | \|f(\bar{a}) \neq f(\bar{c})\|$, it suffices to show that $\psi(b) \cup \psi(\neg b) = \psi(1_{\mathcal{B}_1})$ and $\psi(0_{\mathcal{B}_1}) = 0_{\mathcal{B}_2}$. The latter equality is obvious:

$$\psi(0_{\mathcal{B}_1}) = \left\|f(\bar{a}) \neq f(\tfrac{a}{c}|0_{\mathcal{B}_1})\right\| = \|f(\bar{a}) \neq f(\bar{a})\| = 0_{\mathcal{B}_2}.$$

Let us prove the first equality. Since ψ preserves the order,

$$\|f(\bar{a}) \neq f(\tfrac{a}{c}|b)\| = \psi(b), \quad \|f(\bar{a}) \neq f(\tfrac{a}{c}|\neg b)\| = \psi(\neg b) \subseteq$$
$$\subseteq \left\|f(\bar{a}) \neq f(\tfrac{a}{c}|1_{\mathcal{B}_1})\right\| = \|f(\bar{a}) \neq f(\bar{c})\|.$$

Now we have to show that $\psi(1_{\mathcal{B}_1}) \subseteq \psi(b) \cup \psi(\neg b)$. Let us assume that, to the contrary, $i \in \psi(1_{\mathcal{B}_1}) \setminus (\psi(b) \cup \psi(\neg b))$. Since

$$\theta^{\mathfrak{A}^{\mathcal{B}_1}}_{\bar{a},\frac{a}{c}|b} = \theta^{\mathfrak{A}^{\mathcal{B}_1}}_{\bar{c},\frac{a}{c}|\neg b}, \quad \theta^{\mathfrak{A}^{\mathcal{B}_1}}_{\bar{a},\frac{a}{c}|\neg b} = \theta^{\mathfrak{A}^{\mathcal{B}_1}}_{\bar{c},\frac{a}{c}|b},$$

$$\theta^{\mathfrak{A}^{\mathcal{B}_2}}_{f(\bar{a}),f(\frac{a}{c}|b)} = \theta^{\mathfrak{A}^{\mathcal{B}_2}}_{f(\bar{c}),f(\frac{a}{c}|\neg b)}, \quad \theta^{\mathfrak{A}^{\mathcal{B}_2}}_{f(\bar{a}),f(\frac{a}{c}|\neg b)} = \theta^{\mathfrak{A}^{\mathcal{B}_2}}_{f(\bar{c}),f(\frac{a}{c}|b)},$$

i.e.,

$$\|f(\bar{a}) \neq f(\tfrac{a}{c}|b)\| = \|f(\bar{c}) \neq f(\tfrac{a}{c}|\neg b)\|,$$

$$\|[f(\bar{a}) \neq f(\tfrac{a}{c}|\neg b)]\| = \|[f(\bar{c}) \neq f(\tfrac{a}{c}|b)]\|$$

Therefore,

$$i \in \|[f(\bar{a}) \neq f(\bar{c})]\| \setminus (\|[f(\bar{c}) \neq f(\tfrac{a}{c}|\neg b)]\| \cup \|[f(\bar{c}) \neq f(\tfrac{a}{c}|b)]\|) =$$
$$= \|[f(\bar{a}) \neq f(\bar{c})]\| \cap \|[f(\bar{c}) = f(\tfrac{a}{c}|\neg b)]\| \cap \|[f(\bar{c}) = f(\tfrac{a}{c}|b)]\|,$$

i.e.,

$$f(\tfrac{a}{c}|\neg b)(i) = f(\tfrac{a}{c}|b)(i) \neq f(\bar{a})(i).$$

On the other hand, since

$$i \in \psi(1_{\mathcal{B}_1}) \setminus (\psi(b) \cup \psi(\neg b)) = \|[f(\bar{a}) \neq f(\bar{c})]\| \setminus .$$
$$\setminus \|[f(\bar{a}) \neq f(\tfrac{a}{c}|\neg b)]\| \cup \|[f(\bar{a}) \neq f(\tfrac{a}{c}|b)]\| =$$
$$= \|[f(\bar{a}) \neq f(\bar{c})]\| \cap \|[f(\bar{a}) = f(\tfrac{a}{c}|\neg b]\| \cap \|[f(\bar{a}) = f(\tfrac{a}{c}|b)]\|,$$

$$f(\tfrac{a}{c}|\neg b)(i) = f(\tfrac{a}{c}|b)(i) = f(\bar{a})(i).$$

The obtained contradiction proves that the set $\psi(1_{\mathcal{B}_1}) \setminus (\psi(b) \cup \psi(\neg b))$ is empty, which fact, combined with what has been proved above, proves the theorem. ■

It should be remarked that the condition of extension of the congruences in the formulation of the latter theorem is necessary. Let $Part(A)$ be a lattice of partitions of the set A. It is well-known from O.Ore [152] (see, for instance, [83]), that $Part(A)$ is a simple lattice. It is also known that any variety of lattices is congruence-distributive. Let $\mathcal{B}_4$ be a four-element Boolean algebra. Then $\mathcal{B}_4 \not\leq \underline{2}$ but, on the other hand, for any infinite set A the following obvious relations are valid:

$$Part(a)^{\mathcal{B}_4} \cong Part(A) \times Part(A) \leq Part(A \cup A) \cong Part(A) \cong Part(A)^{\underline{2}}.$$

Let us now go over to the interrelation of Boolean powers of algebras and the properties expressible in the language of the first-order predicate calculus. Let $\mathcal{B}$ be an arbitrary Boolean algebra and F be a filter on $\mathcal{B}$. Let $\mathcal{B}(*)$ be a representation of the Boolean algebra $\mathcal{B}$ by the open-closed subsets of a Stone space $\mathcal{B}^*$ with a predicate $\mathfrak{F}$ which singles out the elements of $\mathcal{B}(*)$ corresponding to the filter F in this presentation.

There is a variation of the Feferman-Vaught theorem on generalized powers, which is as follows. Let us establish a correspondence between any elementary formula $\sigma(x_1,\dots,x_n)$ with free variables $x_1,\dots x_n$ of the signature of the algebra $\mathfrak{A}$ and a tuple $T(\sigma) = <\Phi;\theta_1,\dots\theta_m>$ of the elementary formulas, where Φ are signatures of the algebra $\mathcal{B}(*)$ with free variables $X_1,\dots X_m$, θ_i are signatures of the algebra $\mathfrak{A}$ with free variables $x_1,\dots x_n$. The tuple $T(\sigma)$ is defined by induction over the construction of σ in the following way (the only logical connection of the formula is assumed to be the Sheffer sign, $|$).

(1) σ is an atomic formula of the type $p(x_1,\dots x_n) = q(x_1,\dots,x_n)$, where p,q are terms of the algebra $\mathfrak{A}$, in which case $T(\sigma) = <\mathfrak{F}(x); p = q>$;

(2) $\sigma = \sigma_1 | \sigma_2$ and $T(\sigma_i) = <\Phi_{\sigma_i}(X_1,\dots,X_{m_i}); \theta^i_1,\dots,\theta^i_{m_i}>$, then

$$T(\sigma) = <\Phi_{\sigma_1}(X^1_1,\dots,X^1_{m_1}) | \Phi_{\sigma_2}(X^2_1,\dots,X^2_{m_2});\ \theta^1_1,\dots\theta^1_{m_1}, \theta^2_1,\dots,\theta^2_{m_2}>.$$

(3) $\sigma = \exists x_k \sigma_1$ and $T(\sigma_1) = <\Phi(X_1,\dots,X_m);\theta_1,\dots,\theta_m>$. Let $m' = 2^m$ and $A_1,\dots,A_{m'}$ be all subsets of the set $\{1,\dots,m\}$, and let $S_l = \{i \le m' | l \in A_i\}$ at $l \le m$. Let

$$\Phi'(X_1,\dots,X_{m'}) = \Phi(\bigcup_{i\in S_1} X_i,\dots,\bigcup_{i\in S_m} X_i)$$

and for $i \le m'$ $\theta'_i = \underset{j\in A_i}{\&}\theta_j \ \& \underset{j\notin A_i}{\&} \neg\theta_j$.

By $Part(Y_1,\dots Y_{m'})$ we will mean

$$(\underset{i<j\le m'}{\&}(Y_i \cap Y_j = 0) \& \underset{i\le m'}{\vee} Y_i = 1).$$

In this case we set

$$T(\delta) = <\exists Y_1,\dots,Y_{m'}(Part(Y_1,\dots,Y_{m'}) \& ((\underset{i\le m'}{\&} Y_i \subseteq X_i) \&$$
$$\& \Phi'(Y_1,\dots Y_{m'});\ \exists x_k\theta'_1,\dots,\exists x_k\theta'_{m'}>.$$

Let f/F be a class of θ_F containing an element $f \in \mathfrak{A}^{\mathcal{B}}$, in which case for any formula $\sigma(x_1,\dots,x_k)$ and any $f_1,\dots,f_k \in \mathfrak{A}^{\mathcal{B}}$, the following lemma is valid.

Lemma 3.5. $\mathfrak{A}^{\mathcal{B}} / \theta_F \models \sigma(f_1 / F, \dots, f_k / F)$ iff $\mathcal{B}(*) \models \Phi(I(\theta_1), \dots, I(\theta_n))$, where

$$T(\sigma) = < \Phi; \theta_1, \dots, \theta_n > \text{ and}$$
$$I(\theta_i) = \{i \in \mathcal{B}^* \mid \mathfrak{A} \models \theta_i(f_1(i), \dots f_k(i))\}.$$

Proof. The proof will be carried out by induction over the complexity of the construction of the formula σ. The proofs of the statements corresponding to the basis of the induction, i.e., the case of an atomic formula and the induction step corresponding to the case when $\sigma = \sigma_1 | \sigma_2$, do not differ from the proofs of the corresponding cases in the proper formulation of the Feferman-Vaught theorem (see, for instance, [57], [35]). Let us consider an induction step corresponding to the case when $\sigma = \exists x_k \sigma_1$. Let us assume that

$$\mathfrak{A}^{\mathcal{B}} / \theta_F \models \exists x_k (\sigma_1 (x_k, f_1 / F, \dots, f_n / F))$$

and let $g \in \mathfrak{A}^{\mathcal{B}}$ be such that $\mathfrak{A}^{\mathcal{B}} / \theta_F \models \sigma_1 (g / F, f_1 / F, \dots, f_n / F)$. Let $T(\sigma_1) = \\ = < \Phi(X_1, \dots, X_m); \theta_1, \dots, \theta_m >$. Then, by the induction proposition, we have $\mathcal{B}(*) \models \Phi(I(\theta_1), \dots, I(\theta_m))$. It is also obvious that if for any $i \leq m'$,

$$I_i = \bigcap_{j \in A_i} I(\theta_j) \wedge \bigcap_{j \notin A_i} (I \setminus I(\theta_j)),$$

then $\{I_i \mid i \leq m'\}$ is the partition of $\mathcal{B}^*$. Moreover, $I_i \subseteq I(\exists x_k \theta_i')$. Therefore,

$$\mathcal{B}(*) \models Part\,(I_1, \dots, I_{m'}) \& (\underset{i \leq m'}{\&} I_i \subseteq I(\exists x_k \theta_i')) \& \Phi'(I_1, \dots, I_{m'}),$$

i.e., by the definition of $T(\sigma)$, $\mathcal{B}(*) \models \psi(I(\theta_1), \dots, I(\theta_m))$; where

$$T(\sigma) = < \psi(X_1, \dots, X_m); \quad \theta_1, \dots \theta_m >$$

Let us now try to prove the converse statement. Let $T(\sigma_1) = \\ < \Phi(X_1, \dots X_m); \theta_1, \dots, \theta_m >$ and $\mathcal{B}(*) \models \exists Y_1, \dots Y_{m'} [Part\,(Y_1, \dots, Y_{m'}) \& \underset{i \leq m'}{\&} Y_i \subseteq I(\exists x_k \theta_i') \& \Phi'(Y_1, \dots, Y_{m'})]$. Let $C_1, \dots, C_l$ be a partition of $\mathcal{B}^*$ by elements of the Boolean algebra $\mathcal{B}(*)$ such that the elements $f_1, \dots f_n$ of the Boolean power $\mathfrak{A}^{\mathcal{B}}$ are constant functions on $C_1, \dots, C_l$. Let $D_1, \dots, D_{m'}'$ be a partition of $\mathcal{B}^*$ by elements of the algebra $\mathcal{B}(*)$ such that $D_i \subseteq I(\exists x_k \theta_i')$ and $\mathcal{B}(*) \models \Phi'(D_1, \dots, D_{m'})$.

For any $i \le l, j \le m'$ there are elements $a_1^{ij},\dots,a_n^{ij} \in \mathfrak{A}$ such that for any $p \in D_j \cap C_i$ we have $f_1(p) = a_1^{ij},\dots,f_n(p) = a_n^{ij}$. Since $D_j \subseteq I(\exists x_k \theta_i)$, there is an element $b_{ij} \in \mathfrak{A}$ such that $\mathfrak{A} \models \theta_i'(b_{ij}, a_1^{ij},\dots,a_n^{ij})$. Let an element $g \in \mathfrak{A}^{\mathfrak{B}}$ be such that for any $i \le l, j \le m'$ and any $p \in D_j \cap C_i$, we have $g(p) = b_{ij}$. Then, obviously, we get $\mathfrak{B}(*) \models \Phi(I(\theta_1),\dots,I(\theta_m))$, where $I(\theta_j) = \{p \in \mathfrak{B}^* \mid \mathfrak{A} \models \theta_j(g(p), f_1(p),\dots,f_n(p))\}$. By the induction supposition, $\mathfrak{A}^{\mathfrak{B}} / \theta_F \models \sigma_1(g/F, f_1/F,\dots,f_n/F)$, i.e., $\mathfrak{A}^{\mathfrak{B}} / \theta_F \models \sigma(f_1/F,\dots,f_n/F)$. Therefore, the induction step corresponding to the case when $\sigma = \exists x_k \sigma_1$ is proved and, hence, the lemma is proved. ■

Considering $F = \{1_{\mathfrak{B}}\}$, we confirm the statement that the formulas $L_{\omega,\omega}$, on the Boolean powers $\mathfrak{A}^{\mathfrak{B}}$ are true.

__Corollary 3.2.__ For any algebras $\mathfrak{A}_0, \mathfrak{A}_1$ any Boolean algebras $\mathfrak{B}_0, \mathfrak{B}_1$ the following statements hold:

(a) if $\mathfrak{A}_0 \equiv \mathfrak{A}_1$ and $\mathfrak{B}_0 \equiv \mathfrak{B}_1$, then $\mathfrak{A}_0^{\mathfrak{B}_1} \equiv \mathfrak{A}_1^{\mathfrak{B}_1}$;

(b) if $\mathfrak{A}_0 \prec \mathfrak{A}_1$ and $\mathfrak{B}_0 \prec \mathfrak{B}_1$, then $\mathfrak{A}_0^{\mathfrak{B}_0} \prec \mathfrak{A}_1^{\mathfrak{B}_1}$;

(c) if $\mathfrak{K}$ is a class of Boolean algebras with a solvable elementary theory $Th(\mathfrak{K})$ and $Th(\mathfrak{A}_0)$ is also solvable, then $Th(\{\mathfrak{A}_0^{\mathfrak{B}_1} \mid \mathfrak{B} \in \mathfrak{K}\})$ is solvable;

(d) for any set I and a filter F on I the algebra $\mathfrak{A}_0^{(2^I/F)}$ is isomorphic to an elementary subalgebra of a filtered power $\mathfrak{A}_0^I / F$;

(e) any Boolean power of the algebra $\mathfrak{A}_0$ is elementary equivalent to a certain filtered power of the algebra $\mathfrak{A}_0$ and, conversely, any filtered power of the algebra $\mathfrak{A}_0$ is elementary equivalent to a certain Boolean power of this algebra.

The proof of statements (a), (b) and (c) immediately results from lemma **3.5**. The isomorphism of the algebra $\mathfrak{A}_0^{(2^I/F)}$ and of an elementary subalgebra $\mathfrak{A}_0^I / F$ in statement (d) is constructed in the following way. Assume

$$d = \langle b_1/F,\dots,b_n/F; \quad a_1,\dots,a_n \rangle \in \mathfrak{A}_0^{(2^I/F)},$$

where b_i / F are equivalence classes over the filter F, containing elements $b_i \in \underline{2}^I$.

Since $b_1/F,...b_n/F$ form the partition of unity in the Boolean algebra $\underline{2}^I/F$, $b_1,...,b_n$ can be chosen to perform the partition of the set I. Let us establish a correspondence between an element d and an element $\varphi(d)=[f]\in\mathfrak{A}_0^I/F$, where $[f]$ is the equivalence class over the filter F of the algebra $\mathfrak{A}_0^I$, which contains an element $f\in\mathfrak{A}_0^I$ such that $f(i)=a_j$ for any $i\in b_j(j\le n)$. One can easily see that the definition of the mapping φ is independent of the choice of representatives $b_1,...,b_n$ in the equivalence classes $b_1/F,...,b_n/F$, that φ is an isomorphism between $\mathfrak{A}_0^{(\underline{2}^I/F)}$ and some subalgebra of the algebra $\mathfrak{A}_0^I/F$, and this subalgebra is an elementary subalgebra of the algebra $\mathfrak{A}_0^I/F$.

To prove statement (e), a well-known result by Yu.L.Ershov [61] should be recalled: any Boolean algebra $\mathfrak{B}$ is elementary equivalent to a certain Boolean algebra of the type $\underline{2}^\omega/F$ for a suitable filter F. By statement (a) of the corollary under discussion, for any algebra $\mathfrak{A}^{\mathfrak{B}}\equiv\mathfrak{A}^{(\underline{2}^\omega/F)}$, but, by statement (d), $\mathfrak{A}^{(\underline{2}^\omega/F)}\equiv\mathfrak{A}^\omega/F$, i.e., the arbitrary Boolean power $\mathfrak{A}^{\mathfrak{B}}$ is elementary equivalent to a certain filtered power of the algebra $\mathfrak{A}$. The converse is proved in an analogous manner. ■

Theorem 3.4.

(a) A formula of the first-order predicate calculus is preserved relative to Boolean powers iff it is equivalent to a disjunction of Horn formulas.

(b) The axiomatazible class $\mathfrak{K}$ is closed relative to Boolean powers iff $\mathfrak{K}$ is axiomatizable by the formulas which are disjunct Horn formulas.

The proof of the theorem results immediately from statement (e) of corollary **3.2** and well-known facts (see, for instance, [34]):

(a) a formula of the first-order predicate calculus is preserved relative to filtered powers iff it is equivalent to a disjunction of Horn formulas;

(b) the axiomatizible class $\mathfrak{K}$ is closed relative to filtered powers iff $\mathfrak{K}$ is axiomatizable by formulas which are disjunctions of Horn formulas.

Corollary 3.3. For any algebra $\mathfrak{A}$ and any non-singleton Boolean algebras $\mathfrak{B}_0,\mathfrak{B}_1$ there exists a set I and a filter F on I such that $(\mathfrak{A}^{\mathfrak{B}_0})^I/F\cong(\mathfrak{A}^{\mathfrak{B}_1})^I/F$.

Proof. Let $\mathcal{D}$ be a filter of co-finite subsets ω. In this case, since $\mathcal{B}_0^{\omega}/\mathcal{D}$ and $\mathcal{B}_1^{\omega}/\mathcal{D}$ are atomless, $\mathcal{B}_0^{\omega}/\mathcal{D} \equiv \mathcal{B}_1^{\omega}/\mathcal{D}$. According to a well-known Keisler-Shelah theorem [204], there is a set J and an ultrafilter U such that $(\mathcal{B}_0^{\omega}/\mathcal{D})^J/U \cong (\mathcal{B}_1^{\omega}/\mathcal{D})^J/U$. But, as is well known (see, for instance, [34]), for a suitable filter E on the set $\omega \times J$ we have $(\mathcal{B}_i^{\omega}/\mathcal{D})^J/U \cong \mathcal{B}_i^{\omega\times J}/E$ and, hence, $\mathcal{B}_0^{\omega\times J}/E \cong \mathcal{B}_1^{\omega\times J}/E$. By clause (d) of corollary **3.2**, $\mathcal{B}_i^{\omega\times J}/E \equiv \mathcal{B}_i^{(2^{\omega\times J}/E)}$. At the same time, by clauses (a) and (d) of the same corollary, we get

$$\mathfrak{A}^{\mathcal{B}_i^{\omega\times J}/E} \equiv \mathfrak{A}^{(\mathcal{B}_i^{(2^{\omega\times J}/E)})} \cong (\mathfrak{A}^{\mathcal{B}_i})^{(2^{\omega\times J}/E)} \equiv (\mathfrak{A}^{\mathcal{B}_i})^{\omega\times E}/E$$

(for $i = 0,1$). This fact, combined with the isomorphism $\mathcal{B}_0^{\omega\times J}/E \cong \mathcal{B}_1^{\omega\times J}/E$ mentioned above, implies that the algebras $(\mathfrak{A}^{\mathcal{B}_0})^{\omega\times J}/E$ and $(\mathfrak{A}^{\mathcal{B}_1})^{\omega\times J}/E$ are elementary equivalent. Now, again by the Keisler-Shelah theorem, we can find a set A and an ultrafilter B on A such that $((\mathfrak{A}^{\mathcal{B}_0})^{\omega\times J}/E)^A/B \cong$ $\cong ((\mathfrak{A}^{\mathcal{B}_1})^{\omega\times J}/E)^A/B$. But, in this case, for a suitable filter F on the set $I = \omega \times J \times A$ we have

$$((\mathfrak{A}^{\mathcal{B}_i})^{\omega\times J}/E)^A/B \cong (\mathfrak{A}^{\mathcal{B}_i})^I/F,$$

and, therefore,

$$(\mathfrak{A}^{\mathcal{B}_0})^I/F \cong (\mathfrak{A}^{\mathcal{B}_1})^I/F. \blacksquare$$

By way of concluding this section, let us recall the following obvious property pertaining to theorem **2.21**: if an algebra $\mathfrak{A}$ is isomorphic to a Boolean power $\mathfrak{A}_1^{\mathcal{B}}$ of the algebra $\mathfrak{A}_1$, then any algebra $\mathfrak{A}'$ polynomially equivalent to the algebra $\mathfrak{A}$ is isomorphic to the Boolean power $(\mathfrak{A}_1')^{\mathcal{B}}$, where $\mathfrak{A}_1'$ is an algebra polynomially equivalent to the algebra $\mathfrak{A}_1$.

Priorities. Particular cases of considering the notion of a Boolean power stem from the works by M.H.Stone [218], I.M.Gelfand [76], R.F.Arens and J.Kaplansky [5] and others. The notion of a generalized Boolean power (or simply a Boolean power) was introduced in a general form by A.L.Foster [67], and the notion of a Boolean power (a boundered Boolean power) also belongs to him [66]. The first attempt to systematize the results on Boolean powers supplied with a detailed historic review is by S.Burris [30], who later modified it [24]. Theorem **3.1** was first proved by J.T.Baldwin and R.McKenzie [7]. Lemma **3.3** belongs to S.Burris [30], while the result $\mathcal{B}_1^{\mathcal{B}_2} \cong \mathcal{B}_1 * \mathcal{B}_2$ was borrowed by him from R.W.Quackenbush [194]. Lemma **3.4** stems from a paper by A.L.Foster [66] and M.Gould-G.Grätzer [80]. Theorem **3.1**, suggested by S.Burris [30], was developed and obtained in the present formulation by A.G.Pinus [178]. Corollaries **3.1** and theorem **3.3** are proved by A.G.Pinus [178]. Lemma **3.5**, corollaries **3.2**, **3.3** and theorem **3.4** were proved by S.Burris [30].

4. Other Boolean Constructions

The purpose of the present section is the definition and presentation of the basic properties of the construction of a Boolean product, a filtered and congruence-Boolean power, as well as other modifications of the Boolean power construction studied in the previous section.

Definition 4.1. For an algebra $\mathfrak{A}$, any Boolean algebra $\mathcal{B}$, the subalgebra $\mathfrak{C}$ of a Boolean power $\mathfrak{A}^{\mathcal{B}}$ is called a sub-Boolean power, provided that for any $f,g \in \mathfrak{C}$ and any open-closed subset N of a space $\mathcal{B}^*$, the element $f|N \cup g|\mathcal{B}^* \setminus N$ also belongs to $\mathfrak{C}$. Here $f|N \cup g|\mathcal{B}^* \setminus N$ denotes an element $h \in \mathfrak{A}^{\mathcal{B}}$ such that $h(i) = f(i)$ for $i \in N$, and $h(i) = g(i)$ for $i \in \mathcal{B}^* \setminus N$.

A family of all sub-Boolean powers of the class $\mathfrak{K}$ will be denoted by $P_{SB}(\mathfrak{K})$.

Definition 4.2. A subdirect product $\mathfrak{D} \subseteq \prod_{x \in \mathcal{B}^*} \mathfrak{A}_x$ of algebras $\mathfrak{A}_x$, where $\mathcal{B}^*$ is a Stone space of a Boolean algebra $\mathcal{B}$ is called a Boolean product if the following conditions are met:

(a) for any $f,g \in \mathfrak{D}$ $\|f = g\|$ is open-closed in $\mathcal{B}^*$;

(b) for any $f,g \in \mathfrak{D}$ and any open-closed $N \subseteq \mathcal{B}^*$ the element $f|N \cup g|\mathcal{B}^* \setminus N$ also belongs to $\mathfrak{D}$.

In this case, $\mathcal{B}$ will be called the degree of this Boolean product. A family of all Boolean products of algebras of the class $\mathfrak{K}$ will be denoted by $\Gamma^a(\mathfrak{K})$. For any class of the algebras $\mathfrak{K}$, the following inclusions obviously hold:

$$P_B(\mathfrak{K}) \subseteq P_{SB}(\mathfrak{K}) \subseteq \Gamma^a(S(\mathfrak{K})).$$

Let us recall the simplest properties of a Boolean product.

Lemma 4.1. Let $\mathfrak{A}$ be a Boolean product of algebras $\mathfrak{A}_i (i \in \mathfrak{B}^*)$ with a degree $\mathfrak{B}$, in which case:

(a) if N is an open-closed subset of the space $\mathfrak{B}^*$, and $\varnothing \neq N \neq \mathfrak{B}^*$, then $\mathfrak{A} \cong \mathfrak{A} | N \times \mathfrak{A} | \mathfrak{B}^* \setminus N$, where $\mathfrak{A} | M = \{f | M | f \in \mathfrak{A}\}$ for $M \subseteq \mathfrak{B}^*$. In this case, $\mathfrak{A} | N$, $\mathfrak{A} | \mathfrak{B}^* \setminus N \in \Gamma^a(\mathfrak{K})$;

(b) if $j \in \mathfrak{B}^*$, $\mathfrak{A}_j$ is a finite algebra, and $|\mathfrak{A}_j| = n$, then there are $f_1, \ldots, f_n \in \mathfrak{A}$ and an open-closed neighborhood N of a point $j \in \mathfrak{B}^*$ such that for $i \in N$ there are one-to-one mappings λ_{ji} of the algebra $\mathfrak{A}_j$ in algebras $\mathfrak{A}_i$, defined by the equalities $\lambda_{ji}(f_l(j)) = f_l(i)$;

(c) if $\mathfrak{A}_j$ is a finite algebra, and if the signature of the algebra $\mathfrak{A}$ is finite, or if there is a finite algebra $\mathfrak{C}$ and a neighborhood M of a point j such that for any $i \in M$, $\mathfrak{A}_i$ are isomorphically imbeddable in the algebra $\mathfrak{C}$, then the neighborhood N in property (b) can be chosen in such a way that the mappings $\lambda_{ji}(i \in N)$ will be isomorphic imbeddings of the algebra $\mathfrak{A}_j$ in algebras $\mathfrak{A}_i$.

The proof of the lemma results immediately from the definition of a Boolean product of algebras. ■

By $P_{fin}(\mathfrak{K})$ we will mean the family of all algebras representable as the Cartesian product of a finite number of algebras of the class $\mathfrak{K}$.

Lemma 4.2. For any class of algebras $\mathfrak{K}$:

(a) $P_{fin}(\Gamma^a(\mathfrak{K})) \subseteq \Gamma^a(\mathfrak{K})$;

(b) if an algebra $\mathfrak{A}$ is a Boolean product of $\mathfrak{K}$-algebras and it is finite, then $\mathfrak{A} \in P_{fin}(\mathfrak{K})$;

(c) for any finite algebra $\mathfrak{A}$, $I\Gamma^a(\mathfrak{A}) = IP_B(\mathfrak{A})$;

(d) if the algebras $\mathfrak{A}_1, \mathfrak{A}_2$ contain one-element subalgebras, then

$P_{SB}(\mathfrak{A}_1) \times P_{SB}(\mathfrak{A}_2) \subseteq P_{SB}(\mathfrak{A}_1 \times \mathfrak{A}_2)$.

Proof. Statement (a) which is, in a certain sense, the converse of statement (a) of lemma **4.1**, is proved directly, in which case, if $\mathfrak{A}_1,...,\mathfrak{A}_n$ are Boolean products of algebras of the class $\mathfrak{K}$ with the corresponding degrees $\mathcal{B}_1,...,\mathcal{B}_n$, then $\mathfrak{A}_1 \times ... \times \mathfrak{A}_n$ will be a Boolean product of $\mathfrak{K}$-algebras with a degree $\mathcal{B}_1 \times ... \times \mathcal{B}_n$.

Let us prove statement (b). Let $\mathfrak{A}$ be a subdirect product of algebras $\mathfrak{A}_i (i \in \mathcal{B}^*)$ which is a Boolean product with a degree equal to a Boolean algebra $\mathcal{B}$. Let $\mathfrak{A}$ be finite. Let us define an equivalence relation on $\mathcal{B}^*$: $i \sim j$ iff for any $f, g \in \mathfrak{A}$, $i \in \|f = g\|$ is equivalent to $j \in \|f = g\|$. It is obvious that for any i the class $[i]\sim$ of $\sim$-equivalence containing an element i is an open-closed subset of the space $\mathcal{B}^*$. Let Y be a fixed family of representatives of the classes of $\sim$-equivalence on $\mathcal{B}^*$. As $\mathfrak{A}$ is a finite algebra, by the definition of $\sim$-equivalence, Y is finite. Let $\mathfrak{A}' = \mathfrak{A} \restriction Y \subseteq \prod_{i \in Y} \mathfrak{A}_i$. The definition of $\sim$-equivalence implies an isomorphism between $\mathfrak{A}'$ and $\mathfrak{A}$. Property (b) of definition **4.2** immediately guarantees that $\mathfrak{A}'$ and $\prod_{i \in Y} \mathfrak{A}_i$ coincide. Therefore, $\mathfrak{A} \cong \prod_{i \in Y} \mathfrak{A}_i$ and, hence, statement (b) is proved.

Let us now prove statement (c). Assume that $\mathfrak{D} \in \Gamma^a(\mathfrak{A})$ and $\mathfrak{D} \subseteq \mathfrak{A}^{\mathcal{B}^*}$ for a Boolean algebra $\mathcal{B}$, and let $\mathfrak{D}$, as a subalgebra of the algebra $\mathfrak{A}^{\mathcal{B}^*}$, obey conditions (a) and (b) of definition **4.2** of a Boolean product. By statements (b) and (c) of lemma **4.1**, for any $i \in \mathcal{B}^*$ we choose an open-closed neighborhood N_i and elements $f_1^i,...,f_n^i \in \mathfrak{D}$ (where $n = |\mathfrak{A}|$) such that $\mathfrak{A} = \{f_1^i(i),...,f_n^i(i)\}$, and for $j \in N_j$ the mappings λ_{ij}, such that $\lambda_{ij}(f_l^i(i)) = f_l^i(j)$ are isomorphic embeddings of the algebra $\mathfrak{A}$ in $\mathfrak{A}$. As $\mathfrak{A}$ is finite, λ_{ij} are automorphisms of $\mathfrak{A}$. Since $\mathcal{B}^*$ is compact, one can find $i_1,...i_k \in \mathcal{B}^*, k < \omega$ such that $\mathcal{B}^* = \bigcup_{l \leq k} N_{i_l}$. We can, evidently, assume that $N_{i_l} (l \leq k)$ is a partition of the space $\mathcal{B}^*$. Let us define the mapping $\pi: \mathfrak{A}^{\mathcal{B}^*} \to \mathfrak{A}^{\mathcal{B}^*}$ setting $\pi(f)(j) = \lambda_{i,j}^{-1}, j(f(j))$ at $j \in N_{i_l}$. We can directly check that π boundered on $\mathfrak{D}$ is an isomorphism of $\mathfrak{D}$ on $\mathfrak{A}^{\mathcal{B}}$. Therefore, indeed, $I\Gamma^a(\mathfrak{A}) \subseteq IP_B(\mathfrak{A})$. By virtue of the validity of the converse statement, (c) is proved.

Statement (d) is obvious. ■

<u>**Definition 4.3.**</u> For any algebra $\mathfrak{A}$ and a Boolean algebra $\mathcal{B}$, a subalgebra $\mathfrak{D}$ of the Boolean power $\mathfrak{A}^{\mathcal{B}}$ is called a filtered Boolean power of the algebra $\mathfrak{A}$

if for a family $\mathfrak{A}_i (i \in I)$ of subalgebras of the algebra $\mathfrak{A}$ there is a family $X_i (i \in I)$ of closed sets of a space $\mathfrak{B}^*$ such that $\mathfrak{D} = \{f \in \mathfrak{A}^{\mathfrak{B}} |$ for any $i \in I \;\; f(X_i) \subseteq \mathfrak{A}_i\}$.

A family of all filtered Boolean powers of algebras of the class $\mathfrak{K}$ will be denoted by $P_{FB}(\mathfrak{K})$. Obviously, for any class of the algebras $\mathfrak{K}$, the following inclusions are valid: $P_B(\mathfrak{K}) \subseteq P_{FB}(\mathfrak{K}) \subseteq P_{SB}(\mathfrak{K})$.

<u>**Lemma 4.3.**</u> For any finite algebra $\mathfrak{A}$, $P_{FB}(\mathfrak{K}) = P_{SB}(\mathfrak{A})$.

Proof. Let $\mathfrak{A}_1,...\mathfrak{A}_k$ be all subalgebras of a finite algebra $\mathfrak{A}$, and let $\mathfrak{D} \subseteq \mathfrak{A}^{\mathfrak{B}}$ be a sub-Boolean power of the algebra $\mathfrak{A}$. For $i \in \mathfrak{B}^*$, $\mathfrak{D}_i$ will denote the i-th projection of the algebra $\mathfrak{D}$, i.e., $\mathfrak{D}_i = \{f(i) | f \in \mathfrak{D}\}$. $X_j (j \le k)$ will denote $\{i \in \mathfrak{B}^* | \mathfrak{D}_i \subseteq \mathfrak{A}_j\}$. Since $\mathfrak{A}$ is finite, and for any $f \in \mathfrak{D}$ and any $a \in \mathfrak{A}$, the set $\{i \in \mathfrak{B}^* | f(i) = a\}$ is open-closed, any set $U_j = \{i \in \mathfrak{B}^* | \mathfrak{A}_j \subseteq \mathfrak{D}_i\}$ is an open subset of the space $\mathfrak{B}^*$. Then for $j \le k$, $X_j = \bigcap_{\mathfrak{A}_l \not\subseteq \mathfrak{A}_j} (\mathfrak{B}^* \setminus U_l)$ will be closed in $\mathfrak{B}^*$. Then it is obvious that for $f \in \mathfrak{D}$ and $j \le k$, $f(X_j) \subseteq \mathfrak{A}_j$. Assume now that $f \in \mathfrak{A}^{\mathfrak{B}}$, and for any $j \le k$ we have $f(X_j) \subseteq \mathfrak{A}_j$. Then for any $i \in \mathfrak{B}^*$, $f(i) \in \mathfrak{D}_i$ and, hence, we can find a $f_i \in \mathfrak{D}$ such that $f_i(i) = f(i)$. Therefore, $\bigcup_{i \in \mathfrak{B}^*} \|[f = f_i]\| = \mathfrak{B}^*$. Since $\mathfrak{B}^*$ is compact, one can find $f^1,...f^n \in \mathfrak{D}$ such that $\mathfrak{B}^* = \bigcup_{i \le n} \|[f = f^J]\|$. One can consider $N_j = [f = f^j]$ to be a partition of the space $\mathfrak{B}^*$, and $f = f^1 | N_1 \cup ... \cup f^n | N_n$. As $f^j \in \mathfrak{D}$, by the definition of a sub-Boolean power, $f \in \mathfrak{D}$. Therefore, $\mathfrak{D} = \{f \in \mathfrak{A}^{\mathfrak{B}} | f(X_j) \subseteq \mathfrak{A}_j$ for $j \le k\}$, i.e., any algebra $\mathfrak{D} \in P_{SB}(\mathfrak{A})$ is a filtered Boolean power of the algebra $\mathfrak{A}$. ■

<u>**Definition 4.4.**</u> A congruence α on the algebra $\mathfrak{A}$ is said complementable if there exists a congruence $\beta(= \neg\alpha)$ on the algebra $\mathfrak{A}$ such that $\alpha \wedge \beta = \Delta_{\mathfrak{A}}$, $\alpha \vee \beta = \nabla_{\mathfrak{A}}$, and α, β are commutable, i.e., if $\mathfrak{A} \cong \mathfrak{A} / \alpha \times \mathfrak{A} / \beta$.

<u>**Theorem 4.1.**</u> If complementable congruences on an algebra $\mathfrak{A}$ form a Boolean algebra $\mathfrak{B} \subseteq Con\mathfrak{A}$, and for any $a,b \in \mathfrak{A}$ there is an $\inf\{\gamma \in \mathfrak{B} | \theta^{\mathfrak{A}}_{a,b} \subseteq \gamma\}$, then the algebra $\mathfrak{A}$ is isomorphic to a Boolean product of some direct non-decomposable algebras of degree $\mathfrak{B}$.

Proof. For $\alpha \in \mathcal{B}^*$ α' will denote the maximal ideal $\{\beta \in \mathcal{B} \mid \neg\beta \in \alpha\}$ in the Boolean algebra $\mathcal{B}$. By $f(\alpha)$ we will mean $\bigcup_{\beta \in \alpha'} \beta$. For any congruence $\beta \in \mathcal{B}$, $\overline{\beta}$ will denote an open-closed subset of the space $\mathcal{B}^*$ which corresponds to an element $\overline{\beta} = \{\alpha \in \mathcal{B}^* \mid \beta \in \alpha\}$. The equality $\bigcap_{\alpha \in \mathcal{B}^*} f(\alpha) = \Delta$ obviously holds and, therefore, a homomorphism φ from the algebra $\mathfrak{A}$ to the product $\prod_{\alpha \in \mathcal{B}^*} \mathfrak{A} / f(\alpha)$, defined as $\varphi(a)(\alpha) = a / f(\alpha)$, is an isomorphic embedding, i.e., the algebra $\mathfrak{A}$ is isomorphic to a subdirect product $\varphi(\mathfrak{A})$ of the algebras $\mathfrak{A} / f(\alpha)$ $(\alpha \in \mathcal{B}^*)$. In order to prove the theorem, we have to check if conditions (a) and (b) of definition **4.2** hold for $\varphi(\mathfrak{A})$. If $a,b \in \mathfrak{A}$, and for some $\alpha \in \mathcal{B}^*$ $\varphi(a)(\alpha) = \varphi(b)(\alpha)$ then, by the definition of φ, we get $a / f(\alpha) = b / f(\alpha)$ and, hence, $\theta^{\mathfrak{A}}_{a,b} \subseteq f(\alpha) = \bigcup_{\beta \in \alpha'} \beta$. Since the congruence $\theta^{\mathfrak{A}}_{a,b}$ is compact, there is a finite set $\beta_1, \ldots, \beta_n \in \alpha'$ such that $\theta^{\mathfrak{A}}_{a,b} \subseteq \beta_1 \vee \ldots \vee \beta_n$. Since $\beta_i \in \mathcal{B}$, $\beta_1 \vee \ldots \vee \beta_n \in \mathcal{B}$. Let us write β for $\beta_1 \vee \ldots \vee \beta_n$. Thus, $a / \beta = b / \beta$ and $\beta \in \alpha'$. Therefore, $a / f(\gamma) = b / f(\gamma)$ for any $\gamma \in \mathcal{B}^*$ such that $\beta \in \gamma'$, i.e., such that $\gamma \in \overline{\neg\beta}$. Hence, for any $\alpha \in [\![\varphi(a) = \varphi(b)]\!]$ there is a neighborhood $\overline{\neg\beta}$ of a point α which entirely belongs to the set $[\![\varphi(a) = \varphi(b)]\!]$, i.e., a set of the type $[\![\varphi(a) = \varphi(b)]\!]$ for any $a,b \in \mathfrak{A}$ is open in $\mathcal{B}^*$. Moreover, this means that

$$[\![\varphi(a) = \varphi(b)]\!] = \bigcup\{\overline{\neg\beta} \mid \beta \in \mathcal{B},\ a / \beta = b / \beta\} =$$
$$= \bigcup\{\overline{\neg\beta} \mid \beta \in \mathcal{B},\ \theta^{\mathfrak{A}}_{a,b} \subseteq \beta\} = \overline{\neg\inf\{\beta \in \mathcal{B} \mid \theta^{\mathfrak{A}}_{a,b} \subseteq \beta\}}$$

By the condition of the theorem, $\inf\{\beta \in \mathcal{B} \mid \theta^{\mathfrak{A}}_{a,b} \subseteq \beta\} = \beta_1$ for some $\beta_1 \in \mathcal{B}$. Therefore, $[\![\varphi(a) = \varphi(b)]\!] = \overline{\neg\beta_1}$, i.e., $[\![\varphi(a) = \varphi(b)]\!]$ is an open-closed subset of the space $\mathcal{B}^*$, and condition (a) of definition **4.2** for the algebra $\varphi(\mathfrak{A}) \subseteq \prod_{\alpha \in \mathcal{B}^*} \mathfrak{A} / f(\alpha)$ is checked.

Assume now $a,b \in \mathfrak{A}$, $\alpha \in \mathcal{B}$. Since $\mathfrak{A} \cong \mathfrak{A} / \alpha \times \mathfrak{A} / \neg\alpha$, there is an element $c \in \mathfrak{A}$ such that $c / \alpha = a / \alpha$ and $c / \neg\alpha = b / \neg\alpha$. One can also directly check that $\varphi(c) = \varphi(a) \mid \overline{\alpha} \vee \varphi(b) \mid \mathcal{B}^* \setminus \overline{\alpha}$, i.e., condition (b) of definition **4.2** for the algebra $\varphi(\mathfrak{A})$ also holds. Therefore, the algebra $\mathfrak{A}$ is indeed isomorphic to the Boolean product $\varphi(\mathfrak{A})$ of the algebras $\mathfrak{A} / f(\alpha)$, which are evidently directly non-decomposable. ■

For congruence-distributive varieties a simple and transparent description of principal congruences on Boolean product can be given. Namely, by repeating the proof of theorem **3.2** word per word, we get the following statement.

Theorem 4.2. If $\mathfrak{M}$ is a congruence-distributive variety, then for any Boolean product $\mathfrak{C} \subseteq \prod_{i \in \mathfrak{B}^*} \mathfrak{A}_i$ of $\mathfrak{M}$-algebras relative to a Boolean algebra $\mathfrak{B}$, for any $f,g,h,k \in \mathfrak{C}$ $<f,g> \in \theta^{\mathfrak{C}}_{h,k}$ iff $<f(i), g(i)> \in \theta^{\mathfrak{A}_i}_{h(i)k(i)}$ for any $i \in \mathfrak{B}^*$.

The statement of theorem **4.2** immediately yields, in particular, the following statement:

if $\mathfrak{A}_1, \mathfrak{A}_2$ are algebras from a congruence-distributive variety, then for any $\theta \in Con(\mathfrak{A}_1 \times \mathfrak{A}_2)$ there are $\theta_i \in Con\mathfrak{A}_i$ such that $\theta = \theta_1 \times \theta_2$, where for $f,g \in \mathfrak{A}_1 \times \mathfrak{A}_2$ $<f,g> \in \theta_1 \times \theta_2$ iff $<f(i),g(i)> \theta_i$ $(i=1,2)$.

Definition 4.5. For any algebra $\mathfrak{A}$, any congruence α on $\mathfrak{A}$, any Boolean algebra $\mathfrak{B}$ and its subalgebra $\mathfrak{B}_1$, a double Boolean power of the algebra $\mathfrak{A}$ with respect to a pair $<\mathfrak{B},\mathfrak{B}_1>$ and congruences α will be said to be a subalgebra $\mathfrak{A}^{<\mathfrak{B},\mathfrak{B}_1>}(\alpha)$ of the algebra $\mathfrak{A}^{\mathfrak{B}}$ such that $\mathfrak{A}^{<\mathfrak{B},\mathfrak{B}_1>}(\alpha) = \{f \in \mathfrak{A}^{\mathfrak{B}} |$ for any $a \in \mathfrak{A}$ $f^{-1}(a/\alpha) \in \mathfrak{B}_1\}$. Here a/α is a class of α-congruence containing an element a. When $\mathfrak{B}_1 = \{0,1\}$ the double Boolean power $\mathfrak{A}^{<\mathfrak{B},\mathfrak{B}_1>}(\alpha)$ will be called a congruence-Boolean power and denoted by $\mathfrak{A}(\alpha)^{\mathfrak{B}}$.

Therefore, $\mathfrak{A}(\alpha)^{\mathfrak{B}} = \{f \in \mathfrak{A}^{\mathfrak{B}} |$ for any $i,j \in \mathfrak{B}^*$ $<f(i), f(j)> \in \alpha\}$.

Theorem 4.3. If $\mathfrak{M}$ is a congruence-distributive variety, $\mathfrak{A} \in \mathfrak{M}$, $\alpha \in Con\mathfrak{A}$ and $\mathfrak{B}$ is a Boolean algebra, then on $\mathfrak{A}(\alpha)^{\mathfrak{B}}$ there is a congruence θ such that

$$Con_p(\mathfrak{A}(\alpha)^{\mathfrak{B}})| \le \theta \cong (Con_p\mathfrak{A} | \le \alpha)^{\mathfrak{B}}, \text{ and}$$
$$Con_p(\mathfrak{A}(\alpha)^{\mathfrak{B}})| > \theta \cong Con_p\mathfrak{A} | > \alpha.$$

Proof. By $\alpha^{\mathfrak{B}}$ we will denote the following congruence on $\mathfrak{A}(\alpha)^{\mathfrak{B}}$: $\alpha^{\mathfrak{B}} = \{<f,g> | f,g \in \mathfrak{A}(\alpha)^{\mathfrak{B}}$, and for any $i \in \mathfrak{B}^* <f(i), g(i)> \in \alpha\}$. It is this congruence, $\alpha^{\mathfrak{B}}$, that will play the role of θ in the statement of the theorem. Repeating nearly word per word the proof of theorem **3.2**, we should note that for $f,g \in \mathfrak{A}(\alpha)^{\mathfrak{B}}$, if the principal congruence $\theta_{f,g}$ generated by the pair $<f,g>$ on the algebra $\mathfrak{A}(\alpha)^{\mathfrak{B}}$ is contained in $\alpha^{\mathfrak{B}}$, then

$$\theta_{f,g} = \{< h,g > \mid h,g \in \mathfrak{A}(\alpha)^{\mathcal{B}}, \text{ and for all } i \in \mathcal{B}^*,\ \theta^{\mathfrak{A}}_{h(i),g(i)} \subseteq \theta^{\mathfrak{A}}_{f(i),g(i)}\}.$$

Therefore, indeed,

$$Con_p(\mathfrak{A}(\alpha)^{\mathcal{B}})| \leq \alpha^{\mathcal{B}} \cong (Con_p\mathfrak{A}| \leq \alpha)^{\mathcal{B}}.$$

One can also easily see that $\mathfrak{A}(\alpha)^{\mathcal{B}} / \alpha^{\mathcal{B}} \cong \mathfrak{A} / \alpha$, and this isomorphism implies the isomorphisms $Con_p(\mathfrak{A}(\alpha)^{\mathcal{B}}| > \alpha^{\mathcal{B}}$ and $Con_p\mathfrak{A}| > \alpha$. ■

__Definition 4.6.__ A congruence α on the algebra $\mathfrak{A}$ will be called overlapping if $Con\mathfrak{A} = Con\mathfrak{A}| \leq \alpha \oplus Con\mathfrak{A}| > \alpha$, where $\oplus$ is the lexicographical addition of ordered sets.

__Corollary 4.1.__ If α is an overlapping congruence on the algebra $\mathfrak{A}$ belonging to a congruence-distributive variety, then

$$Con_p(\mathfrak{A}(\alpha)^{\mathcal{B}}) \cong (Con_p\mathfrak{A}| \leq \alpha)^{\mathcal{B}} \oplus Con_p\mathfrak{A}| > \alpha.$$

In particular, if $\mathfrak{A}$ is subdirectly non-decomposable and α is a monolith of $\mathfrak{A}$, then

$$Con_p(\mathfrak{A}(\alpha)^{\mathcal{B}}) \cong \mathcal{B} \oplus Con_p\mathfrak{A}| > \alpha.$$

The statement of the corollary results from theorem **4.3** and from the fact that, by the definition of $\mathfrak{A}(\alpha)^{\mathcal{B}}$, if for $f,g \in \mathfrak{A}(\alpha)^{\mathcal{B}}$ and a certain $i \in \mathcal{B}^*$, we have $\theta^{\mathfrak{A}}_{f(i),g(i)} \leq \alpha$, then for any $j \in \mathcal{B}^*$, we have $\theta^{\mathfrak{A}}_{f(j),g(j)} \leq \alpha$. ■

__Lemma 4.4.__ For any Boolean algebras $\mathcal{B}_1, \mathcal{B}_2$ and partially ordered sets $< A;\leq >$, if there is a mapping f from $\mathcal{B}_2 \oplus < A;\leq >$ to $\mathcal{B}_1 \oplus < A;\leq >$ that preservs finite inf and sup and is such that $f(\mathcal{B}_2) \neq \{0_{\mathcal{B}_1}\}$, then there is a homomorphism of the Boolean algebra $\mathcal{B}_2$ on the algebra $\mathcal{B}_1$.

Proof. Let $1_{\mathcal{B}}$ be the unit element of a Boolean algebra $\mathcal{B}$. Then either $f(1_{\mathcal{B}_2}) = 1_{\mathcal{B}_1}$, or $0_{\mathcal{B}_1} < f(1_{\mathcal{B}_2}) < 1_{\mathcal{B}_1}$, or $f(1_{\mathcal{B}_2}) > 1_{\mathcal{B}_1}$. In the first case f is a mapping from $\mathcal{B}_2$ to $\mathcal{B}_1$ preserving finite inf and sup and is, obviously, a homomorphism of the Boolean algebra $\mathcal{B}_2$ on the algebra $\mathcal{B}_1$. If $0_{\mathcal{B}_1} < f(1_{\mathcal{B}_2}) < 1_{\mathcal{B}_1}$, then let a be a complement of $f(1_{\mathcal{B}_2})$ in $\mathcal{B}_1$. Then for any $c \in f^{-1}(a)$, we have $c > 1_{\mathcal{B}_2}$, which contradicts the fact that f is isotonic, and the fact that $f(c) = a$ and $f(1_{\mathcal{B}_2})$ are incomparable. Therefore, the second case is impossible. Let now $f(1_{\mathcal{B}_2}) > 1_{\mathcal{B}_1}$. Setting $h(a) = f(a)$ for $a \in \mathcal{B}_2$ if $f(a) \in \mathcal{B}_1$, and $h(a) = 1_{\mathcal{B}_1}$ if $f(a) \notin \mathcal{B}_1$, we get a mapping of $\mathcal{B}_2$ on $\mathcal{B}_1$ preserving finite inf and sup and, therefore, a homomorphism of the Boolean algebra $\mathcal{B}_2$ on the algebra $\mathcal{B}_1$. ■

<u>**Corollary 4.2.**</u> If $\mathfrak{A}$ is a subdirectly non-decomposable algebra of a congruence-distributive variety, and θ is its monolith, then:

(a) for any Boolean algebras $\mathcal{B}_1, \mathcal{B}_2$, $\mathfrak{A}(\theta)^{\mathcal{B}_1} << \mathfrak{A}(\theta)^{\mathcal{B}_2}$ iff $\mathcal{B}_1 << \mathcal{B}_2$ or $\mathfrak{A}(\theta)^{\mathcal{B}_1} << \mathfrak{A}/\theta$. In particular, if $|\mathfrak{A}|^2 < |\mathcal{B}_1|$, then $\mathfrak{A}(\theta)^{\mathcal{B}_1} << \mathfrak{A}(\theta)^{\mathcal{B}_2}$ iff $\mathcal{B}_1 << \mathcal{B}_2$.

(b) For any Boolean algebra $\mathcal{B}$ and $\sigma \in Con\,\mathfrak{A}(\theta)^{\mathcal{B}}$ such that $\sigma < \theta^{\mathcal{B}}$, there is a $\gamma \in Con\mathcal{B}$ such that $\mathfrak{A}(\theta)^{\mathcal{B}} / \sigma \cong \mathfrak{A}(\theta)^{\mathcal{B}/\gamma}$.

P r o o f . If $\mathcal{B}_1 << \mathcal{B}_2$ or $\mathfrak{A}(\theta)^{\mathcal{B}_1} << \mathfrak{A}/\theta$ then, obviously, $\mathfrak{A}(\theta)^{\mathcal{B}_1} << \mathfrak{A}(\theta)^{\mathcal{B}_2}$. Let us now assume that g is a homomorphism of $\mathfrak{A}(\theta)^{\mathcal{B}_2}$ on $\mathfrak{A}(\theta)^{\mathcal{B}_1}$. Then the mapping f:

$$Con_p\mathfrak{A}(\theta)^{\mathcal{B}_2} \cong \mathcal{B}_2 \oplus Con_p\mathfrak{A} \mid\ > \theta$$

on

$$Con_p\mathfrak{A}(\theta)^{\mathcal{B}_1} \cong \mathcal{B}_1 \oplus Con_p\mathfrak{A} \mid\ > \theta$$

will be defined in the following way:

$$f(\theta_{a,b}^{\mathfrak{A}(\theta)^{\mathcal{B}_2}}) = \theta_{g(a)g(b)}^{\mathfrak{A}(\theta)^{\mathcal{B}_1}}.$$

The mapping f preserves the finite inf and sup, since there is an isomorphism of h $Con_p\mathfrak{A}(\theta)^{\mathcal{B}_1}$ on $Con_p\mathfrak{A}(\theta)^{\mathcal{B}_2}|\geq\alpha$, where α is the kernel of the homomorphisms g:

$$h(\theta^{\mathfrak{A}(\theta)^{\mathcal{B}_1}}_{g(a)g(b)}=\theta^{\mathfrak{A}(\theta)^{\mathcal{B}_2}}_{a,b}\vee\alpha$$

and in this case the variety containing $\mathfrak{A}$ is congruence-distributive. Therefore, by lemma **4.5**, we have either $f(\mathcal{B}_2)=\Delta_{\mathfrak{A}(\theta)^{\mathcal{B}_1}}$, i.e., $\mathfrak{A}(\theta)^{\mathcal{B}_1}<<\mathfrak{A}/\theta$, or $\mathcal{B}_1<<\mathcal{B}_2$. For the case when $|\mathfrak{A}|<|\mathcal{B}_1|$, since $|Con_p\mathfrak{A}|>\theta|\leq|\mathfrak{A}|^2<|\mathcal{B}_1|$ and f is a mapping "on", we get $f(\mathcal{B}_2)\neq\Delta_{\mathfrak{A}(\theta)^{\mathcal{B}_1}}$, and, therefore, $\mathcal{B}_1<<\mathcal{B}_2$.

Statement (b) is proved in an analogous way (see also corollary **3.1**).

<u>**Corollary 4.3.**</u> If directly non-decomposable algebras of a congruence-distributive variety $\mathfrak{M}$ are limited in power, then $\mathfrak{M}$ is semi-simple.

Proof. Let us assume the converse to be valid, and let $\mathfrak{A}$ be a subdirectly non-decomposable not simple $\mathfrak{M}$-algebra and θ be a monolith of $\mathfrak{A}$, then $Con\mathfrak{A}|>\theta\neq\varnothing$. By corollary **4.2**, $Con\mathfrak{A}(\theta)^{\mathcal{B}}\cong Con\mathcal{B}\oplus Con\mathfrak{A}|>\theta$ for any Boolean algebra $\mathcal{B}$. This formula entails that for any $\alpha,\beta\in Con\mathfrak{A}(\theta)^{\mathcal{B}}$, the equalities $\alpha\wedge\beta=\Delta$, $\alpha\vee\beta=\nabla$ imply either $\alpha=\nabla$, $\beta=\Delta$ or $\alpha=\Delta$, $\beta=\nabla$. Therefore, the algebras $\mathfrak{A}(\theta)^{\mathcal{B}}$ are directly non-decomposable for any Boolean algebra $\mathcal{B}$, and the statement of the corollary results from the fact that $|\mathfrak{A}(\theta)^{\mathcal{B}}|\geq|\mathcal{B}|$. ■

In §**3**, a variant of the Feferman-Vaught theorem pertaining to the elementary properties of Boolean algebras was proved. An analogous statement is valid for a more general construction, i.e., filtered Boolean powers. Let $\mathfrak{A}$ be an arbitrary algebra, and let $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$ be a finite family of its subalgebras. By $\overline{\mathfrak{A}}$ we will denote an extension of the algebra $\mathfrak{A}$ with new unary predicates which select subalgebras $\mathfrak{A}_i(i=1,\ldots,n)$ in the algebra $\mathfrak{A}$. Let $\mathfrak{A}'$ be a filtered Boolean power of the algebra $\mathfrak{A}$, in which case $\mathfrak{A}'=\{f\in\mathfrak{A}^{\mathcal{B}}|$ for $i=1,\ldots,n$ $f(X_i)\subseteq\mathfrak{A}_i\}$ for a certain Boolean algebra $\mathcal{B}$ and closed subsets X_i of a Stone space $\mathcal{B}^*$. F_i will denote filters $\{N\in\mathcal{B}|\ X_i\subseteq N\}$ of the Boolean algebra $\mathcal{B}$, and $\overline{\mathcal{B}}$ an extension of $\mathcal{B}$ with new unary predicates which select filters $F_i(i=1,\ldots n)$ in the Boolean algebra $\mathcal{B}$. As was the case in lemma **3.5**, for any elementary formula φ of the signature of the algebra $\mathfrak{A}$ we can define a tuple $T(\varphi) = <\phi;\theta_1,...,\theta_m>$ in a natural way, where ϕ is the formula of the signature of the algebra $\overline{\mathfrak{B}}$ and θ_i are the formulas of the signature of the algebra $\overline{\mathfrak{A}}$. In this case, the following lemma holds.

Lemma 4.5. $\mathfrak{A}' \models \varphi(f_1,...,f_n)$ iff $\overline{\mathfrak{B}} \models \phi(B_1,...,B_m)$, where for $j = 1,...,m$, $B_j = \{i \in \mathfrak{B}^* \mid \overline{\mathfrak{A}} \models \theta_j(f_1(i),...,f_n(i))\}$.

The proof of this statement is absolutely analogous to that of lemma **3.5** and is left to the reader as an exercise.

By way of concluding the list of principal Boolean constructions in universal algebra, let us briefly dwell on the following. Let $\mathfrak{A}$ be an arbitrary algebra and $\mathfrak{B}$ an arbitrary Boolean algebra. Let G be an arbitrary finite group, φ a certain homomorphism of the group G into a group of all automorphisms of the algebra $\mathfrak{A}$ and ψ a homomorphism of the group G into the group of all automorphisms of the Boolean algebra $\mathfrak{B}$. For any $g \in G$, the automorphism $\psi(g)$ of the algebra $\mathfrak{B}$ naturally induces, using the Stone duality, a homomorphism $\psi^*(g)$ of a Stone space $\mathfrak{B}^*$. $\mathfrak{A}_G^{\mathfrak{B}}$ will denote a subalgebra of the Boolean power $\mathfrak{A}^{\mathfrak{B}}$ with a basic set

$$\{f \in \mathfrak{A}^{\mathfrak{B}} \mid f(\psi^*(g)(i)) = \varphi(g)(f(i)) \text{ for any } i \in \mathfrak{B}^* \text{ and any } g \in G\}.$$

The algebra $\mathfrak{A}_G^{\mathfrak{B}}$ will be called a G-power of the algebra $\mathfrak{A}$, with the family of all Boolean G-powers of the algebra $\mathfrak{A}$ under a fixed action φ of the group G on $\mathfrak{A}$ denoted as $P_G(\mathfrak{A})$. Any algebra of the class $P_G(\mathfrak{A})$ has been proved [25] to be elementary equivalent to a certain filtered Boolean power of the algebra $\mathfrak{A}$ if G is Abelian, or if the restriction of any G-automorphism of the algebra $\mathfrak{A}$ on any subalgebra of the algebra $\mathfrak{A}$ is an automorphism of this subalgebra. The same authors have shown any filtered Boolean power of the algebra $\mathfrak{A}$ to be isomorphic to some algebra of the class $P_G(\mathfrak{A})$ for a suitable group G, if the subalgebras $\mathfrak{A}_i$ participating in the definition of a filtered power have the form $\{a \in \mathfrak{A} \mid f(a) = a$ for $f \in H_i\}$, where H_i are some subgroups of the group $Aut\,\mathfrak{A}$.

To conclude this section, let us consider the notion of the direct product of varieties of algebras.

Definition 4.7. Subvarieties $\mathfrak{M}_1, \mathfrak{M}_2$ of a variety of algebras $\mathfrak{M}$ are called independent if there is a term $f(x,y)$ such that $\mathfrak{M}_1 \models f(x,y) = x$, $\mathfrak{M}_2 \models f(x,y) = y$.

The intersection of independent varieties obviously contains a one-element algebra only.

Lemma 4.6. If $\mathfrak{M}_1, \mathfrak{M}_2$ are independent subvarieties of a variety $\mathfrak{M}$, then for any $\mathfrak{A} \in \mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2)$ there are $\mathfrak{A}_1 \in \mathfrak{M}_1, \mathfrak{A}_2 \in \mathfrak{M}_2$ such that $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$.

Proof. Since $\mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2) = HSP(\mathfrak{M}_1 \cup \mathfrak{M}_2)$, then it suffices to show that for any $\mathfrak{A}_i \in \mathfrak{M}_i$ the following statements are valid:

(1) if $\mathfrak{B} \subseteq \mathfrak{A}_1 \times \mathfrak{A}_2$, then $\mathfrak{B} = \pi_1(\mathfrak{B}) \times \pi_2(\mathfrak{B})$;

(2) if φ is a homomorphism of $\mathfrak{A}_1 \times \mathfrak{A}_2$ on the algebra $\mathfrak{B}$, then $\mathfrak{B} \cong \varphi_1(\mathfrak{A}_1) \times \varphi_2(\mathfrak{A}_2)$, where φ_i are homomorphisms defined on $\mathfrak{A}_i$.

Let us assume $\mathfrak{B} \subseteq \mathfrak{A}_1 \times \mathfrak{A}_2$ and that η_i are kernels of π_i projections on the algebra $\mathfrak{B}$. We have, obviously, $\eta_1 \wedge \eta_2 = \Delta$ and $\eta_1 \vee \eta_2 = \nabla$. Therefore, in order to prove that $\mathfrak{B} = \pi_1(\mathfrak{B}) \times \pi_2(\mathfrak{B})$, it suffices to prove that η_1 and η_2 are permutable on $\mathfrak{B}$. Assume that $< a,b > \in \eta_1 \circ \eta_2$, i.e., there is a $c \in \mathfrak{B}$ such that $< a,c > \in \eta_1$ and $< c,b > \in \eta_2$, and assume that $a_i, b_i, c_i \in \mathfrak{A}_i$ $(i = 1,2)$ such that $a = < a_1, a_2 >$, $b = < b_1, b_2 >$, $c = < c_1, c_2 >$, in which case $a_1 = c_1$, $c_2 = b_2$. Let $f(x,y)$ be a term occurring in the definition of the independence of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then $f(< b_1,b_2 >, < a_1,a_2 >) = \; f(b_1,a_1), f(b_2,a_2) >=< b_1,a_2 >$, i.e., $< b_1,a_2 > \in \mathfrak{B}$. But we have

$$< a, < b_1,a_2 >> \in \eta_2, << b_1,a_2 >, b > \eta_1$$

and, hence, $< a,b > \in \eta_2 \circ \eta_1$. Therefore, $\eta_2 \circ \eta_1 \leq \eta_2 \circ \eta_1$. We can prove the converse and, thus, the permutability of η_1 and η_2, in an analogous way.

To prove property (2), let us assume that φ is a homomorphism of the algebra $\mathfrak{A}_1 \times \mathfrak{A}_2$ on $\mathfrak{B}$, and α is the kernel of φ. Let us define congruences ψ_i on $\mathfrak{A}_1 \times \mathfrak{A}_2$ in the following way:

$$<< a_1,a_2 >, < b_1,b_2 >> \in \psi_1$$

iff there exist $a_3, b_3 \in \mathfrak{A}_2$ such that $<< a_1,a_3 >, < b_1,b_3 >> \in \alpha$;

$$<< a_1,a_2 >, < b_1,b_2 >> \in \psi_2$$

iff there are $a_4, b_4 \in \mathfrak{A}_1$ such that $\ll a_4, a_2 >, < b_4, b_2 \gg \in \alpha$.

Let us prove that $\psi_1 \vee \psi_2 = \nabla$, $\psi_1 \wedge \psi_2 = \alpha$, $\psi_i \geq \eta_i$ and $\psi_1 \circ \psi_2 = \psi_2 \circ \psi_1$. In this case,

$$\mathfrak{B} \cong \mathfrak{A}_1 \times \mathfrak{A}_2 / \alpha \cong (\mathfrak{A}_1 \times \mathfrak{A}_2 / \psi_1) \times (\mathfrak{A}_1 \times \mathfrak{A}_2 / \psi_2) \cong$$
$$(\mathfrak{A}_1 \times \mathfrak{A}_2 / \eta_1 / (\psi_1 / \eta_1) \times (\mathfrak{A}_1 \times \mathfrak{A}_2 / \eta_2) / (\psi_2 / \eta_2) \cong$$
$$\mathfrak{A}_1 / (\psi_1 / \eta_1) \times \mathfrak{A}_2 / (\psi_2 / \eta_2),$$

where for $\theta_1 \subseteq \theta_2 \in Con\mathfrak{A}$, θ_2 / θ_1 stands for a congruence corresponding to that of θ_2 on the algebra $\mathfrak{A} / \theta_1$ under a canonical homomorphism of $\mathfrak{A}$ on $\mathfrak{A} / \theta_1$. It is the homomorphism $\mathfrak{B} \cong \mathfrak{A}_1 / (\psi_1 / \eta_1) \times \mathfrak{A}_2 / (\psi_2 / \eta_2)$ that completely proves, as has been noted above, the statement of the lemma.

The inequalities $\psi_i \geq \eta_i$ follow from the definition of ψ_i, and, since $\eta_1 \vee \eta_2 = \nabla$, $\psi_1 \vee \psi_2 = \nabla$. Moreover, by virtue of permutability of η_1 and η_2 we have $\eta_1 \circ \eta_2 = \eta_1 \vee \eta_2$. Therefore, $\psi_1 \circ \psi_2 = \psi_2 \circ \psi_1 = \nabla$ and ψ_1, ψ_2 are permutable. By the definition, $\psi_1 \geq \alpha$. Now we have to show that $\psi_1 \wedge \psi_2 \leq \alpha$. Let $\ll a_1 a_2 >, < b_1, b_2 \gg \in \psi_1 \wedge \psi_2$, i.e., there are $a_3, b_3 = \mathfrak{A}_2$, $a_4, b_4 \in \mathfrak{A}_1$ such that

$$\ll a_1, a_3 >, < b_1, b_3 \gg \in \alpha, \ll a_4, a_2 >, < b_4, b_2 \gg \in \alpha.$$

In this case,

$$< f(< a_1, a_3 >, < a_4, a_2 >), \quad f(b_1, b_3), f(b_4, b_2) > \in \alpha$$

and

$$< f(< a_1, a_3 >, < a_4, a_2 >) = < f(a_1, a_4), f(a_3, a_2) > = < a_1, a_2 >$$

Analogously, $f(< b_1, b_3 >, < b_4, b_2 >) = < b_1, b_2 >$, i.e., $\ll a_1, a_2 >, < b_1, b_2 \gg \in \alpha$, which was to be proven. ■

In a congruence-modular case lemma **4.6** assumes an inversion.

__Lemma 4.7.__ If $\mathfrak{M}_1 \cap \mathfrak{M}_2$ contains a one-element algebra only, $\mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2)$ is congruence-modular, and for any $\mathfrak{A} \in \mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2)$ there are $\mathfrak{A}_1 \in \mathfrak{M}_1$, $\mathfrak{A}_2 \in \mathfrak{M}_2$ such that $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are independent.

Proof. Let $\mathfrak{A} \in \mathfrak{M}_1$, $\mathfrak{B} \in \mathfrak{M}_2$ such that $\mathfrak{F}_{\mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2)}(2) = \mathfrak{F} \cong \mathfrak{A} \times \mathfrak{B}$. Let x, y be free generators of $\mathfrak{F}$, and let θ_1, θ_2 be the least congruences on $\mathfrak{F}$ such that $\mathfrak{F} / \theta_1 \in \mathfrak{M}_1$, $\mathfrak{F} / \theta_2 \in \mathfrak{M}_2$. In this case, $\theta_i \leq \eta_i$, where η_1, η_2 are the kernels of $\mathfrak{F}$ projections on $\mathfrak{A}$ and $\mathfrak{B}$, respectively. As $\mathfrak{F} / \theta_1 \vee \theta_2 \in \mathfrak{M}_1 \cap \mathfrak{M}_2$, and $\mathfrak{M}_1 \cap \mathfrak{M}_2$ contains a one-element algebra only, then $\theta_1 \vee \theta_2 = \nabla$. By the choice of θ_i, any mapping from x,y to the generators of the algebra $\mathfrak{A}_1 \in \mathfrak{M}_1 (\mathfrak{B}_1 \in \mathfrak{M}_2)$ induces the homomorphism $\mathfrak{F} / \theta_1 \wedge \theta_2$ in $\mathfrak{A}_1(\mathfrak{B}_1)$ and, hence, in $\mathfrak{A}_1 \times \mathfrak{B}_1$ as well. As, be the lemma condition, any $\mathfrak{M}(\mathfrak{M}_1 \vee \mathfrak{M}_2)$-algebra has the form $\mathfrak{A}_1 \times \mathfrak{B}_1 (\mathfrak{A}_1 \in \mathfrak{M}_1, \mathfrak{B}_1 \in \mathfrak{M}_2)$, then $\mathfrak{F} / \theta_1 \wedge \theta_2$ will be a free two-generated algebra in $\mathfrak{M}(\mathfrak{M}_1 \vee \mathfrak{M}_2)$, i.e., $\mathfrak{F} / \theta_1 \wedge \theta_2 \cong \mathfrak{F}$. Therefore, we can assume $\theta_1 \wedge \theta_2 = \Delta$. Since $Con\mathfrak{F}$ is modular, the equalities $\theta_1 \vee \theta_2 = \nabla$, $\theta_1 \wedge \theta_2 = \nabla$, $\eta_1 \vee \eta_2 = \nabla$, $\eta_1 \wedge \eta_2 = \Delta$ and $\eta_i \geq \theta_i$ imply the equalities $\theta_i = \eta_i$. Thus, $\mathfrak{F} \cong \mathfrak{F} / \theta_1 \times \theta_2$, and $\mathfrak{F} / \theta_1$ is a free two-generated $\mathfrak{M}_i$-algebra, and let its generators be x^i, y^i.

In this case, the discussed isomorphism of $\mathfrak{F}$ and $\mathfrak{F} / \theta_1 \times \mathfrak{F} / \theta_2$ transforms x,y in $< x^1, x^2 >$, $< y^1, y^2 >$, respectively. Since $\mathfrak{F} / \theta_1 \times \mathfrak{F} / \theta_2$ is generated be the elements $< x^1, x^2 >$, $< y^1, y^2 >$, there is a term f such that $f < x^1, x^2 >$, $< y^1 y^2 > = < x^1, y^2 >$. Thus,

$$\mathfrak{F} / \theta_1 | = f(x^1, y^1) = x^1, \quad \mathfrak{F} / \theta_2 | = f(x^2, y^2) = y^2.$$

Since $\mathfrak{F} / \theta_i$ are free in $\mathfrak{M}_i$, the identities $f(x,y) = x$, $f(x,y) = y$ will be fulfilled on $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively, which implies that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are independent. ■

Theorem 4.4. If $\mathfrak{M}_1$ is an Abelian and $\mathfrak{M}_2$ a congruence-distributive subvariety of a congruence-modular variety $\mathfrak{M}$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are independent.

Proof. It should be remarked that $\mathfrak{M}_1 \cap \mathfrak{M}_2$ consists of a one-element algebra only, as $\mathfrak{M}_1 | = [\nabla, \nabla] = \Delta$, and $\mathfrak{M}_2 | = [\nabla, \nabla] = \nabla$. Therefore, by lemma **4.7**, it suffices to show that any $\mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2)$-algebra is presentable as a direct product of algebras from $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

If $\mathfrak{D} \in \mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2)$, then there is a $\mathfrak{C}$ which is a subdirect product of the algebras $\mathfrak{A} \in \mathfrak{M}_1$ and $\mathfrak{B} \in \mathfrak{M}_2$, and there are $\theta \in Con\mathfrak{C}$ such that $\mathfrak{D} \cong \mathfrak{C} / \theta$. Let η_1, η_2 be the kernels of the $\mathfrak{C}$ projections on $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then $\mathfrak{C} / \eta_1 \vee \eta_2 \in \mathfrak{M}_1 \cap \mathfrak{M}_2$ and, hence, $\eta_1 \vee \eta_2 = \nabla_{\mathfrak{C}}$. By corollary 2.2, η_1 and η_2 are permutable and, therefore, $\mathfrak{C} = \mathfrak{A} \times \mathfrak{B}$.

By theorem **2.23**, η_2 is a neutral element of $Con\mathfrak{C}$ and, hence, $\theta = \theta \vee (\eta_1 \wedge \eta_2) = (\theta \vee \eta_1) \wedge (\theta \vee \eta_2)$, i.e., $\mathfrak{D} = \mathfrak{C} / \theta$ is a subdirect product of the algebras $\mathfrak{D}_1, \mathfrak{D}_2$ from the varieties $\mathfrak{M}_1, \mathfrak{M}_2$, respectively. As was the case in the preceding section, the algebra $\mathfrak{D}$ must be a direct product of these algebras $\mathfrak{D}_1, \mathfrak{D}_2$. ■

Definition 4.8. If $\mathfrak{M}_1, \mathfrak{M}_2$ are independent varieties of the same signature, then the variety $\mathfrak{M}(\mathfrak{M}_1 \cup \mathfrak{M}_2)$ is called a direct product of the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and is denoted by $\mathfrak{M}_1 \otimes \mathfrak{M}_2$.

Therefore, by lemma **4.6**, any algebra $\mathfrak{A} \in \mathfrak{M}_1 \otimes \mathfrak{M}_2$ can be represented as $\mathfrak{A}_1 \times \mathfrak{A}_2$, where $\mathfrak{A}_i \in \mathfrak{M}_i$ and, by theorem **4.4**, the union of an Abelian and a congruence-distributive varieties of the same signature is their direct product.

Lemma 4.8. Let the equality $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$ hold for the varieties of algebras $\mathfrak{M}$, $\mathfrak{M}_1, \mathfrak{M}_2$ and let the algebra $\mathfrak{D}$ and the class of algebras $\mathfrak{K}$ belong to $\mathfrak{M}$. In this case,

(a) if $\mathfrak{D} = \mathfrak{A}^{\mathfrak{B}}$ and $\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$, $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$, where $\mathfrak{D}_i, \mathfrak{A}_i \in \mathfrak{M}_i$, then $\mathfrak{D}_i = \mathfrak{A}_i^{\mathfrak{B}}$;

(b) if $\mathfrak{D}$ is a sub-Boolean power of the algebra $\mathfrak{A}$ with a degree $\mathfrak{B}$, $\mathfrak{D} = \mathfrak{D}_1 \times \mathfrak{D}_2$, $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$, where $\mathfrak{A}_i, \mathfrak{D}_i \in \mathfrak{M}_i$, then $\mathfrak{D}_i$ are sub-Boolean powers of the algebras $\mathfrak{A}_i$ of degree $\mathfrak{B}$;

(c) if $\mathfrak{D}$ is a Boolean product of the algebras $\mathfrak{A}_i (i \in \mathfrak{B}^*)$ of degree $\mathfrak{B}$, $\mathfrak{D}_1 \times \mathfrak{D}_2$ and $\mathfrak{A}_i = \mathfrak{A}_i^1 \times \mathfrak{A}_i^2$, where $\mathfrak{D}_j, \mathfrak{A}_i^j \in \mathfrak{M}_j$, then $\mathfrak{D}_j$ are Boolean products of the algebras $\mathfrak{A}_i^j (i \in \mathfrak{B}^*)$ with a power $\mathfrak{B}$;

(d) $\mathfrak{M} = I\Gamma^a(\mathfrak{K})$ iff $\mathfrak{M}_i = I\Gamma^a(\mathfrak{K} \cap \mathfrak{M}_i)$ for $i = 1,2$.

Statements (a) - (c) of the lemma under discussion can be directly checked. Statement (d) follows from statement (a) of lemma **4.2**, and statement (c) of the present lemma. ■

As was the case in the end of section **3**, one can easily observe that if the algebra $\mathfrak{A}$ can be represented as a Boolean product $\mathfrak{A} \subseteq \prod_{i \in \mathfrak{B}^*} \mathfrak{A}_i$ of the algebras $\mathfrak{A}_i$ (as a congruence-Boolean power $\mathfrak{A}_1(\alpha)^{\mathfrak{B}}$ of a certain algebra $\mathfrak{A}_1$), then any algebra $\mathfrak{A}'$ polynomially equivalent to $\mathfrak{A}$ can be represented as a Boolean product $\mathfrak{A}' \subseteq \prod_{i \in \mathfrak{B}^*} \mathfrak{A}_i'$ of algebras $\mathfrak{A}_i'$ polynomially equivalent to the algebras $\mathfrak{A}_i$ (as a congruence-Boolean power $\mathfrak{A}_1'(\alpha)^{\mathcal{B}}$ of some algebra $\mathfrak{A}_1'$ polynomially equivalent to the algebra $\mathfrak{A}_1$).

And, finally, let us briefly dwell on the relation between the notions of a Boolean product and an algebra of global sections of a sheaf of algebras. It should be recalled that a triple $< S,X,\pi >$ is called a sheaf of algebras if:

(1) S and X are topological spaces,

(2) $\pi:S \to X$ is a local homomorphism of S on X,

(3) $S_x = \pi^{-1}(x)$ are algebras of the same signature σ for all $x \in X$,

(4) if f is a functional symbol of the signature σ, and S_n is a subspace $(\prod_{x\in X} S_x)^n$ of the space $(S^X)^n$, then under a natural definition of the mapping $f: S_n \to S^X$, f is continuous.

A global section of the sheaf $< S,X,\pi >$ is any continuous mapping $g:X \to S$ such that $\pi \cdot g$ is identical on X. The algebra of global sections of the sheaf $< S,X,\pi >$ is called a subalgebra of the direct product $\prod_{x\in X} S_x$, the basic set of which consists of global sections of the sheaf $< S,X,\pi >$. Let us denote this algebra with $\gamma(S,X,\pi)$. The sheaf $< S,X,\pi >$ is called a Hausdorff sheaf, if the space S is a Hausdorff space.

One can also directly check the following statement.

Theorem 4.5.

(a) If $< S,\mathcal{B}^*,\pi >$ is a Hausdorff sheaf and $\mathcal{B}^*$ is a Stone space of a Boolean algebra $\mathcal{B}$, then $\gamma(S,\mathcal{B}^*,\pi)$ is a Boolean product of the algebras S_x of degree $\mathcal{B}$.

(b) Let $\mathfrak{A}$ be a Boolean product of algebras $\mathfrak{A}_x$ of degree X. Let $S = \bigcup_{x\in X} \{x\}\times\mathfrak{A}_x$, and let us define a topology on S with a basis of open neighborhoods of the type $\{< x,f(x) > \mid x \in N\}$, where $f \in \mathfrak{A}$ and N is an open-closed subset of X. Let us define the mapping $\pi:S \to X$ with the equalities $\pi(< x,a >) = x$. Then $< S,X,\pi >$ is a Hausdorff sheaf of algebras, and $\gamma(S,X,\pi)$ is isomorphic to $\mathfrak{A}$ under a homomorphism α defined by the equalities $\alpha(g)(x) = a$ if $g(x) =< x,a >$.

Priorities. The notion of a Boolean product was introduced by S.Burris and H.Werner [29] as a reformulation of the construction of a Boolean sheaf, which has been studied in detail starting from a paper by J.Dauns and K.H.Hofman [45]. The construction of a filtered Boolean power for a variety of rings stems from a work by R.F.Arens and J.Kaplansky [5]. The statements of lemma **4.1** belong to S.D.Comer [39]. Lemmas **4.2**, **4.3** and **4.7** are by S.B.Burris and R.McKenzie [27], theorem **4.1** is by S.D.Comer [40]. The construction of a double Boolean power belongs to S.Burris and has been effectively used in [27]. The construction of a congruence-Boolean power was introduced by A.G.Pinus [168] and used by him to study skeletons of epimorphism of congruence-distributive varieties [168, 169].

In its implicit form theorem **4.2** can be found in a number of papers. It should be remarked that its simplest variant for the case of congruences on a Cartesian product of two algebras from a congruence-distributive variety is, in essence, the statement of a known Fraser-Horn theorem [70]. Theorem **4.3** and corollary **4.1** are by A.G.Pinus [168], lemma **4.4** and corollary **4.2** also belong to him [169]. Lemma **4.5** was proved by S.Burris and D.Clark [25]. The definitions of independence and of a direct product of varieties, as well as lemmas **4.6** and **4.7** stem from a paper by G.Grätzer, H.Lakser and J.Plonka [81]. Theorem **4.4** belongs to C.Herrmann [92].As regards theorem **4.5**, see [29]. Some details on algebras of global sections can be found, for instance, in [112].

5. Discriminator Varieties And Their Specific Algebras

In many cases studies of varieties representable by Boolean constructions is reduced to those of Abelian and discriminator varieties. By theorem **2.20**, the former are polynomially equivalent varieties of unitary modules over some ring with unity which have been quite thoroughly studied in the literature [see, for instance, [97], [191], [239]). The present section will be devoted to the description of the structure of discriminator varieties using constructions of a Boolean product, followed by the demonstration of the resulting possibilities of reducing descriptions of various special algebras in discriminator varieties to considering the corresponding Boolean algebras. Namely, we will describe the construction of injective, equationally compact, topologically compact, algebraically closed and other algebras of discriminator varieties, starting with a number of examples of discriminator varieties and their various characterizations.

Here are some examples of discriminator varieties:

(1) +-Heyting algebras. These are varieties consisting of algebras $\mathfrak{H} = < H; \wedge, \vee, \rightarrow, +, 0, 1 >$ such that:

(a) $<H;\wedge,\vee,0,1>$ is a bounded distributive lattice;

(b) $x \wedge y \to y = 1$; (the operation $\to$ is here a relative pseudocomplement)

(c) $x \wedge (x \to y) = x \wedge y$;

(d) $x \wedge (x \wedge y \to z) = x \wedge (y \to z)$;

(e) $1^{+} = 0$; (the operation $+$ is here a dual pseudocomplement)

(f) $x \vee x^{+} = 1$;

(g) $x \vee (x \vee y)^{+} = x \vee y^{+}$;

(h) $x^{+} \wedge x^{++} = 0$.

Subdirectly non-decomposable algebras of this variety are $<H;\wedge,\vee,\to,+,0,1>$, where $<H;\wedge,\vee,0,1>$ is an arbitrary bounded distributive lattice with $\vee$-non-decomposable unity, and

$$x^{+} = \begin{cases} 1, & if \quad x \neq 1 \\ 0, & if \quad x = 1 \end{cases}.$$

The discriminator for such algebras on H is the term $t(x,y,z) = [z \wedge ((x \to y) \wedge (y \to x))^{++}] \vee [x \wedge (((x \to y) \wedge (y \to x))^{++} \to 0)]$.

(2) **Boolean algebra**. The only subdirectly non-decomposable algebra of this variety is the two-element Boolean algebra $<\{0,1\};\wedge,\vee,\neg,0,1>$. The discriminator on it is defined by the term $t(x,y,z) = ((x \wedge z) \vee \neg y) \wedge (x \vee z)$.

(3) **Lukasiewicz algebras of the order** n. The variety consists of algebras of the form $\mathcal{B} = <L;\wedge,\vee,-,D_1,\dots,D_{n-1},0,1>$ such that:

(a) $<L;\wedge,\vee,0,1>$ is a bounded distributive lattice;

(b) $\overline{\overline{x}} = x$, $\overline{x \wedge y} = \overline{x} \vee \overline{y}$;

(c) $D_i(x \wedge y) = D_i(x) \wedge D_i(y)$, $1 \leq i < n$,

$D_i(x \vee y) = D_i(x) \vee D_i(y), \quad 1 \le i < n;$

(d) $D_i(x) \ge D_j, \quad 1 \le i \le j < n;$

(e) $D_i(x) \vee \overline{D_i(x)} = 1, \quad 1 \le i < n,$
$D_i(x) \wedge \overline{D_i(x)} = 0, \quad 1 \le i < n;$

(f) $D_i(\bar{x}) \wedge \overline{D_{n-i}(x)}, \quad 1 \le i < n;$

(g) $D_i(D_j(x)) = D_j(x), \quad 1 \le i, j < n;$

(h) $D_{n-1}(x) \le x < D_1;$

(i) $\bar{x} \wedge D_{n-1}(x) = 0, \quad \bar{x} \vee D_1(x) = 1;$

(j) $D_i(0) = 0, \quad D_i(1) = 1, \quad 1 \le i < n;$

(k) $y \le x \vee \overline{D_i(x)} \vee D_{i+1}(y), \quad 1 \le i \le n-2.$

The operation $x \to y = y \vee \bigwedge_{1 \le i < n} \overline{(D_i(x)} \vee D_i(y))$ has been shown [236] to be a relative pseudocomplement in the lattice $< L; \wedge, \vee, 0, 1 >$. The operation $x^+ = D_1(\bar{x})$ is a dual pseudocomplement on $< L; \wedge, \vee, 0, 1 >$. Therefore, $< L; \wedge, \vee, \to, +, 0, 1 >$ is a $^+$-Heyting algebra, and a variety of Lukasiewicz algebra of the order n is discriminatory. Subdirectly non-decomposable algebras of this variety are subalgebras of an algebra of the type $< \{0,\dots,n-1\}; \wedge, \vee, -, D_1, \dots, D_{n-1}, 0, n-1 >$, where $\bar{x} = n-1-x$, $D_i(x) = n-1$, if $i \le x$ and is equal to 0, if $i > x$, $x \wedge y = \min\{x,y\}$, $x \vee y = \max\{x,y\}$. The discriminator on them is defined by the term which determines a discriminator on subdirectly non-decomposable $^+$-Heyting algebras under the above-mentioned representation of the functions $\to, +$ in the signature of Lukaciewicz algebras.

(4) Cylindric algebras of dimension n. The variety consists of algebras of the type $\mathfrak{C} = < C; \wedge, \vee, \neg, 0, 1, c_1, \dots, c_n >$ such that:

(a) $< C; \wedge, \vee, \neg, 0, 1 >$ is a Boolean algebra;

$$\left.\begin{array}{l} (b)\ c_k(0) = 0 \\ (c)\ x \le c_k(x) \\ (d)\ c_k(x \wedge c_k(y)) = c_k(x) \wedge c_k(y) \\ (e)\ c_k c_j(x) = c_j c_k(x) \end{array}\right\} \quad 1 \le k, j \le n.$$

Subdirectly non-decomposable algebras of this variety are algebras of the type $\mathfrak{C} = < C; \wedge, \vee, \neg, 0, 1, c_1, \ldots, c_n >$, where $< C; \wedge, \vee, \neg, 0, 1 >$ is an arbitrary Boolean algebra, and $c_1, c_2, \ldots c_n(x) = 1$ if $x = 1$ and is equal to zero if $x \neq 1$. The discriminator on this algebras is defined by the same term as that on $^+$-Heyting algebras if we assume $x \rightarrow y = \neg x \vee y$, $x^{++} = c_1 c_2, \ldots, c_n(x)$.

(5) **Relation algebras.** The variety consists of algebras of the type $\mathfrak{R} = < R; \wedge, \vee, \neg, 0, 1, ;, +, \Delta >$ such that:

(a) $< R; \wedge, \vee, \neg, 0, 1 >$ is a Boolean algebra;

(b) $x \cdot (y \cdot z) = (x \cdot y) \cdot z, x \cdot \Delta$;

(c) $x^{++} = x$, $(x \cdot y)^+ = y^+ \cdot x^+$, $(x \vee y)^+ = x^+ \vee y^+$;

(d) $(x \vee y) \cdot z = (x \vee z) \cdot (y \vee z)$;

(e) $x \cdot \neg (x^+ \cdot y) \leq \neg y$.

Subdirectly non-decomposable algebras of this variety are algebras of the type $\mathfrak{R} = < R; \wedge, \vee, \neg, 0, 1, +, \Delta >$ such that for any $x \in R$ $1 \cdot x \cdot 1 = 1$ if $x \neq 0$ and is zero if $x = 0$. The discriminator on these algebras is defined by the same term as that on $^+$-Heyting algebras, if we set $x \rightarrow y = \neg x \vee y$, $x^{++} = \neg(1 \cdot x \cdot 1)$.

(6) Rings. Since the lattice of the congruence of any ring is isomorphic with that of its ideals, and the lattice of the congruence of an algebra of a discriminator variety is distributive, then any discriminator variety of rings must consist of rings with a distributive lattice of ideals, i.e., of arithmetic rings.

Theorem 5.1. For an arbitrary variety $\mathfrak{M}$, the following conditions are equivalent:

(a) $\mathfrak{M}$ is generated by a finite set of finite fields;

(b) $\mathfrak{M}$ is a discriminator variety;

c) all $\mathfrak{M}$-rings are arithmetic;

(d) there is a polynomial $t(x)$ with integer coefficients and with no free term such that on any ring $\mathfrak{R} \in \mathfrak{M}$, the identity $x \cdot t(x) = x$ holds;

(e) there is an integer $n \geq 2$ such that on any ring $\mathfrak{R} \in \mathfrak{M}$, the identity $x^n = x$ is true.

Proof. The implication (1) → (2) is fulfilled because for any field $GF(q)$ for any $n \in \omega$ such that $n - 1 \equiv 0 \pmod{q-1}$, the discriminator $t(x,y,z)$ on $GF(q)$ is set by the term $z + (x-z)(y-x)^n$.

The implication (2) → (3) has been discussed earlier, the implications (5) → (4) and (1) → (5) are obvious. Let us prove that the implication (3) → (4) holds. If the variety of rings $\mathfrak{M}$ is arithmetic then, by theorem **2.10**, there is a term $p(x,y,z)$ (i.e., a polynomial with integer coefficients) such that the following identities are true on $\mathfrak{M}$: $p(x,x,y) = x = p(x,y,x) = p(y,x,x)$. These identities imply that the polynomial

$$s(x,y,z) = p(x,y,z) - p(0,0,z) - p(0,y,0) - \\ p(x,0,0) + 2p(0,0,0)$$

has integer coefficients. Besides, the free term of the polynomial $s(x,y,z)$ and its coefficients at x, y, z are zero. Direct checking shows that for $s(x,y,z)$ the same identities are valid as those mentioned above for $p(x,y,z)$. Putting x beyond the brackets in the polynomial $s(x,x,x)$ we get a polynomial $t(x)$ such that $x \cdot t(x) = s(x,x,x)$. But $s(x,x,x) = x$ is an identity on $\mathfrak{M}$ and, hence, the implication (3) → (4) is true.

Concluding the proof of this theorem, let us prove the implication (4) → (1). First of all, since $t(x)$ is a polynomial with no free term, i.e., $t(x) = x \cdot q(x)$ for a certain polynomial $q(x)$, the equality $x^2 = 0$ implies that $x \cdot t(x) = 0$ as well and, hence, according to (4), $x = 0$. It should be noticed now that if e is an idempotent of the ring $\mathfrak{R}$, then e is central, i.e., for any elements $e, x \in \mathfrak{R}$ $e^2 = e$ imply $ex = xe$. Indeed,

$$(ex - exe)^2 = exex - exexe - exeex + exeexe = 0$$

and, due to the above, $ex = exe$. In an analogous way it should be observed that $xe = exe$, i.e., $ex = xe$. Since $x \cdot t(x) = x$ and $t(x) = x \cdot q(x) = q(x) \cdot x$, $t(x) \cdot t(x) = t(x)$. The same equality $x \cdot t(x) = x$ implies that if $t(a) = 0$, then $a = 0$ for any element $a \in \mathfrak{R}$. If $\mathfrak{R}$ is a directly non-decomposable ring, then the only non-zero central idempotent of the ring $\mathfrak{R}$ is unity. Therefore, a directly non-decomposable $\mathfrak{R}$ must have unity, and for any $a \in \mathfrak{R}$ we have $t(a) = 1$ if $a \neq 0$, and $t(a) = 0$ if $a = 0$. As $t(x) = x \cdot q(x)$, then for any $a \neq 0$ $a \cdot q(a) = 1$, i.e., $\mathfrak{R}$ is a sfield obeying the identity $x \cdot t(x) - x = 0$. Hence, $\mathfrak{R}$ has not more elements than the power of the polynomial $x \cdot t(x) - x$. By the Wedderburn theorem (see, for instance, [96]), every finite sfield is a field. Thus, $\mathfrak{R}$ will be a finite field with a bounded number (the power of the polynomial $x \cdot t(x) - x$) of elements, i.e., (1) follows from (4). ■

(7). Rings with operators. Let us consider a variety of $\mathfrak{R}_d$-algebras obtained by adding one unary operation $g(x)$ into the ring signature and consisting of algebras $\mathfrak{R} = \langle R;+,-,0,\cdot,g \rangle$ such that:

(a) $\langle R;+,-,0,\cdot \rangle$ is an associative ring;

(b) $g(x) \cdot y = y \cdot g(x)$, $g(x) \cdot g(x) = g(x)$;

(c) $g(x) \cdot x = x$;

(d) $g(x \cdot g(y)) = g(x) \cdot g(y)$,
$g(x - x \cdot g(y)) = g(x) - g(x) \cdot g(y)$.

The subdirectly non-decomposable algebras of this variety are algebras of the type $\langle R;+,-,0,;g \rangle$, where $g(x) = 1$ if $x \neq 0$, and $g(x) = 0$ if $x = 0$. The discriminator on these algebras is defined by the term $t(x,y,z) = z + (x - z)g(y - x)$.

The ring $\mathfrak{R}$ is biregular if for any $x \in \mathfrak{R}$ there is a central idempotent x^+ which generates the same principal ideal in $\mathfrak{R}$ that x does. It is evident that x^+ is uniquely defined by x and, by modifying the biregular ring with the operation $g(x) = x^+$, we see that every biregular ring modified in this way is an algebra from the variety $\mathfrak{R}_d$.

A ring is called a Baer*-ring if for any $x \in \mathfrak{R}$, the ideal of annihilators of x is generated by a certain central idempotent x^* (x^* is uniquely obtained by x). The ring $\mathfrak{R}$ is called strictly regular if for any $x \in \mathfrak{R}$ there is an element $x^{-1} \in \mathfrak{R}$ such that $x^2 \cdot x^{-1} = x$ (x^{-1} is uniquely defined except for the case $x = 0$ when we set $0^{-1} = 0$). Extension of Baer*- and strictly regular rings, respectively, with unary operations $x \to x^*, x \to x^{-1}$ converts these families of rings into discriminator varieties.

By way of concluding this series of examples of discriminator varieties, it should be remarked that despite the fact that the above-mentioned varieties are not of prime importance in modern algebra, their investigation is worth undertaking not only in view of the examples listed above (the list, incidentally, can be extended), but in view of the following result as well. Let us first recall the definition of quasi-primarity discussed in Section **2**.

Definition 5.1. A finite algebra $\mathfrak{A}$ is called quasi-primal if it is subdirectly non-decomposable, and $\mathfrak{M}(\mathfrak{A})$ is a discriminator variety. In other words, $\mathfrak{A}$ is quasi-primal if the discriminator on it is defined by some term.

Theorem 5.2. If $G(n)$ is a number of mutually non-isomorphic groupoids of the power n, and $Q(n)$ is a number of mutually non-isomorphic quasi-primal groupoids of the power n, then $\lim_{n\to\infty}\frac{Q(n)}{G(n)}=1$, i.e., 'nearly all' finite groupoids of greater power are quasi-primal.

The following characteristics of quasi-primal algebras are known.

Theorem 5.3. A finite algebra $\mathfrak{A}$ is quasi-primal iff $\mathfrak{A}$ and its all subalgebras are simple, and $\mathfrak{M}(\mathfrak{A})$ is arithmetic.

Theorem 5.4. A finite algebra $\mathfrak{A}$ is quasi-primal iff for any n-ary function f set on $\mathfrak{A}$ and such that any subalgebra of the algebra $\mathfrak{A}^2$ which is a graph of the isomorphism between any subalgebras of the algebra $\mathfrak{A}$ is closed relative to f, then for f there is a term of the algebra $\mathfrak{A}$ defining f on $\mathfrak{A}$.

The class of discriminator varieties among the class of all arithmetic ones is singled out with the help of a certain property of congruence determinability.

Definition 5.2. The principal congruences on a variety $\mathfrak{M}$ are definable by bounded identities if there is a finite set of pairs of terms of four variables $<t_1,q_1>,\dots,<t_n,q_n>$ such that for any algebra $\mathfrak{A}\in\mathfrak{M}$ and any $a,b,c,d\in\mathfrak{A}$ $<a,b>\in\theta_{c,d}$ iff

$$\mathfrak{A}\models \mathop{\&}_{i=1}^{n} t_i(a,b,c,d)=q_i(a,b,c,d).$$

Theorem 5.5. A variety $\mathfrak{M}$ is a discriminator variety iff $\mathfrak{M}$ is arithmetic and the principal congruences on $\mathfrak{M}$ are definable by bounded identities.

This, in particular, entails the properties of extensibility of congruences for discriminator varieties.

Without proof let us recall the following description of principal congruences on subalgebras of direct products of simple algebras of discriminator varieties to be used below.

Theorem 5.6. If $\mathfrak{A}_i$ are simple algebras belonging to a discriminator variety (congruence-distributive, with extensible congruences, and such that $\mathfrak{M}_{SI}$ is approximated and the principal congruences on $\mathfrak{M}$ are elementary definable), and $\mathfrak{C}$ is a subalgebra of a direct product $\prod_{i\in I}\mathfrak{A}_i$, then for any $f,g,h,k\in\mathfrak{C}$,

$$<f,g>\in\theta^{\mathfrak{C}}_{h,k} \text{ iff } \{i\in I \mid f(i)=g(i)\}\supseteq\{i\in I \mid h(i)=k(i)\}.$$

The proof of this theorem can be compared with the description of principal congruences of Boolean powers of simple algebras in congruence-distributive varieties given in theorem **3.2** (see also theorem **4.2**).

As has been mentioned in the beginning of this section, discriminator varieties allow a simple description using the construction of a Boolean product. By $\mathfrak{M}^+_{SI}(\mathfrak{M}^+_S)$ we will mean a class $\mathfrak{M}_{SI}(\mathfrak{M}_S)$ with a singleton algebra added to it.

Theorem 5.7. If $\mathfrak{M}$ is a discriminator variety, then $\mathfrak{M}=I\Gamma^a(\mathfrak{M}^+_{SI})$.

Proof. Let a term $t(x,y,z)$ define the function of a discriminator on subdirectly non-decomposable $\mathfrak{M}$-algebras. $S_p\mathfrak{A}$ will denote a set $\{\theta\in Con\mathfrak{A} \mid$ for any $\alpha\in Con\mathfrak{A}$, $\theta\subseteq\alpha$ implies either $\theta=\alpha$ or $\alpha=\nabla\}$ for any algebra $\mathfrak{A}$ of $\mathfrak{M}$, i.e., $S_p\mathfrak{A}$ is a family of all maximal congruences on $\mathfrak{A}$ plus the congruence ∇. For any elements $x,y\in\mathfrak{A}$, $E(x,y)$ will denote the set $\{\theta\in S_p\mathfrak{A} \mid <x,y>\in\theta\}$, while $D(x,y)$ will denote the set $\{\theta\in S_p\mathfrak{A} \mid <x,y>\notin\theta\}$. Let us define on $S_p\mathfrak{A}$ the topology the subbasis of open sets of which is a family $\mathfrak{T}$ of all subsets $S_p\mathfrak{A}$ of the type $E(x,y)$, $D(x,y)$ for any $x,y\in\mathfrak{A}$. Let us first of all note that this family forms a Boolean algebra and is the basis of the given topology, which consists of open-closed subsets of the space $S_p\mathfrak{A}$. Since $D(x,y)=S_p\mathfrak{A}\setminus E(x,y)$, it suffices to show that the family $\{E(x,y),D(x,y) \mid x,y\in\mathfrak{A}\}$ is closed relative to intersections. Let θ be a maximal congruence on $\mathfrak{A}$, i.e., $\theta\in S_p\mathfrak{A}\setminus\{\nabla\}$, in which case for any $x,y,z\in\mathfrak{A}$ we have, by the definition of $t(x,y,z)$:

$$<x,y>\in\theta \Rightarrow <t(x,y,z),z>\in\theta$$

and

$$<x,y>\notin\theta \Rightarrow <t(x,y,z),x>\in\theta.$$

Therefore, for any $r,s,u,v\in\mathfrak{A}$,

$$<r,s>\in\theta \text{ and } <u,v>\in\theta \Leftrightarrow <t(r,s,u),t(s,r,v)>\in\theta,$$
$$<r,s>\in\theta \text{ and } <u,v>\notin\theta \Leftrightarrow <t(r,s,u),t(s,r,v)>\notin\theta,$$
$$<r,s>\in\theta \text{ and } <u,v>\notin\theta \Leftrightarrow <t(r,t(r,s,u),u),\ t(r,\ t\ t(s,r,v)v>\notin\theta.$$

The above equivalences together with the fact that $\nabla\in E(x,y)$, $\nabla\notin D(x,y)$ for any $x,y\in\mathfrak{A}$ imply the required equalities:

$$E(r,s) \cap E(u,v) = E(t(r,s,u), t(s,r,v)),$$
$$E(r,s) \cap D(u,v) = D(t(r,s,u), t(r,s,v)),$$
$$E(r,s) \cap D(u,v) = D(t(r,t(r,s,u),t(r,s,v)) \quad t(r,t(r,s,v),v)).$$

Therefore, the family $\mathfrak{C}$ of subsets of the space $S_p\mathfrak{A}$ indeed forms a Boolean algebra, consists of open-closed subsets of this space and forms its basis.

Let now C be a closed subset of the space $S_p\mathfrak{A}$ and $\nabla \in C$. Then, since $\mathfrak{C}$ is the basis of the space $S_p\mathfrak{A}$ consisting of open-closed sets, $C = \cap\{A \in \mathfrak{C} \mid C \subseteq A\}$. Since for any $r,s \in \mathfrak{A}$ $\nabla \notin D(r,s)$, $C = \cap\{E(r,s) \in \mathfrak{C} \mid C \subseteq E(r,s)\}$.

Turning to the complement of C, we get the following statement: for any open subset C_1 of the space $S_p\mathfrak{A}$ such that $C_1 \not\ni \nabla$, the equality $C_1 = \cup\{D(r,s) \in \mathfrak{C} \mid C_1 \not\subseteq E(r,s)\}$ holds. Let now α be an arbitrary congruence on the algebra $\mathfrak{A}$, and let us establish a correspondence between α and the set $U(\alpha) = \{\theta \in S_p\mathfrak{A} \mid \alpha \not\subseteq \theta\}$ of the space $S_p\mathfrak{A}$. Evidently, $U(\alpha)$ is an open subset, and $\nabla \notin U(\alpha)$. Using the above-mentioned representation of similar open sets C_1, one can directly check that U is an isomorphism between the lattice $Con\mathfrak{A}$ and the lattice of open subsets of the space $S_p\mathfrak{A}$ containing no element ∇.

Let us now show that $S_p\mathfrak{A}$ is a Boolean space, and a Boolean algebra of all open-closed subsets of the space $S_p\mathfrak{A}$ coincides with the family $\mathfrak{C}$. Since $\mathfrak{C}$ is a basis of $S_p\mathfrak{A}$ consisting of open-closed subsets of the space $S_p\mathfrak{A}$, $S_p\mathfrak{A}$ is a 0-dimensional space, and in order to prove that $S_p\mathfrak{A}$ is Boolean, one has to show that it is compact and is a Hausdorff space. If $\theta,\alpha \in S_p\mathfrak{A}$ and $\theta \not\subseteq \alpha$, then there are $x,y \in \mathfrak{A}$ such that $<x,y> \in \theta, <x,y> \notin \alpha$ and, hence, $E(x'y), D(x,y)$ are disjunct neighborhoods of the points θ,α, which proves that $S_p\mathfrak{A}$ is a Hausdorff space. Let now $\{U_i \mid i \in I\}$ be an open covering of the space $S_p\mathfrak{A}$. There is an $i_0 \in I$ such that $\nabla \in U_{i_0}$ and, therefore, since $\mathfrak{C}$ is a basis and $\nabla \notin D(x,y)$, there are $r,s \in \mathfrak{A}$ such that $\nabla \in E(r,s) \subseteq U_{i_0}$. Let $V_i = U_i \cap D(r,s)$ for $i \in I$, then $\{V_i \mid i \in I\}$ is an open covering of the set $D(r,s)$. Let us choose an $\alpha_i \in Con\mathfrak{A}$ such that $U(\alpha_i) = V_i$, where U is the above constructed isomorphism $Con\mathfrak{A}$ and the lattice of open subsets of $S_p\mathfrak{A}$ containing no ∇. Then

$$U(\theta_{r,s}) = D(r,s) = \bigcup_{i \in I} V_i = \bigcup_{i \in I} U(\alpha_i) = U(\bigcup_{i \in I} \alpha_i).$$

Therefore, as U is an isomorphism, $\theta_{r,s} = \bigcup_{i \in I} \alpha_i$ and, since $\theta_{r,s}$ are compact, there are $i_1,\dots,i_n \in I$ such that $\theta_{r,s} = \alpha_{i_1} \vee \dots \vee \alpha_{i_n}$. But in this case

$$U(\theta_{r,s}) = D(r,s) = U(\alpha_{i_1}) \cup ... \cup U(\alpha_{i_n}) = V_{i_1} \cup ... \cup V_{i_n}$$

and, hence, $S_p\mathfrak{A} = U_{i_0} \cup U_{i_1} \cup ... \cup U_{i_n}$. This is the proof that the space $S_p\mathfrak{A}$ is compact and, at the same time, Boolean. The family $\mathfrak{T}$ is the basis of $S_p\mathfrak{A}$ which consists of open-closed subsets of $S_p\mathfrak{A}$ and is a Boolean algebra, but then, as is known, $\mathfrak{T}$ is the family of all open-closed subsets of $S_p\mathfrak{A}$.

Let π_θ ($\theta \in S_p\mathfrak{A}$) be a canonical homomorphism from the algebra $\mathfrak{A}$ to $\mathfrak{A}/\theta$ and f a mapping from $\mathfrak{A}$ to $\prod_{\theta \in S_p\mathfrak{A}} \mathfrak{A}/\theta$ such that for $x \in \mathfrak{A}$, $\theta \in S_p\mathfrak{A}$ we get $f(x)(\theta) = \pi_\theta(x)$.

It should be remarked that for $\theta \in S_p\mathfrak{A}$, $\mathfrak{A}/\theta$ is either simple or singleton, i.e., in particular, f is a mapping of $\mathfrak{A}$ on a subdirect product of $\mathfrak{M}^+_{SI}$-algebras. If $x \neq y \in \mathfrak{A}$, then there is a $\theta \in S_p\mathfrak{A}$ such that $\pi_\theta(x) \neq \pi_\theta(y)$. Indeed, let $\alpha = \cup\{\gamma \in Con\mathfrak{A} \mid <x,y> \notin \gamma\}$. Then $\mathfrak{A}/\alpha \in \mathfrak{M}_{SI}$ and, since $\mathfrak{M}$ is a discriminator variety, $\mathfrak{M}_{SI} = \mathfrak{M}_S$ (see section **2**) and, hence, $\alpha \in S_p(\mathfrak{A})$ and $\pi_\alpha(x) \neq \pi_\alpha(y)$. Therefore, $f(x) \neq f(y)$, i.e., f is an isomorphism.

Let us prove that $f(\mathfrak{A})$ is a Boolean product of the algebras $\mathfrak{A}/\theta$, $\theta \in S_p\mathfrak{A}$, which will require the validity of the following conditions:

(a) for any $x, y \in \mathfrak{A}$ $[\![f(x) = f(y)]\!]$ is open-closed in $S_p\mathfrak{A}$;

(b) for any $x, y \in \mathfrak{A}$ and any open-closed subset $A \subseteq S_p\mathfrak{A}$ there is a $z \in \mathfrak{A}$ such that $f(z) = f(x) \mid A \cup f(y) \mid S_p\mathfrak{A} \setminus A$.

Condition (a) is obviously fulfilled, since by the definition of f and the set $E(x,y)$ we have $[\![f(x) = f(y)]\!] = E(x,y)$.

Assume now $x, y \in \mathfrak{A}$ and $A \in \mathfrak{T}$. For the sake of definiteness, let us assume that $A = E(r,s)$, in which case $S_p\mathfrak{A} \setminus A = D(r,s)$. One can directly check that $z = t(t(r,s,x), t(r,s,y), y)$ indeed has the properties of condition (b). Therefore, indeed, $f(\mathfrak{A}) \in \Gamma^a(\mathfrak{M}^+_{SI})$, i.e., the arbitrary algebra $\mathfrak{A} \in \mathfrak{M}$ is isomorphic with a Boolean product of $\mathfrak{M}^+_{SI}$-algebras. ■

The proof of theorem **5.7** makes it also possible to establish a relationship between algebras of an arbitrary discriminator variety and a variety of distributive lattices with relative complements. Let $\mathfrak{D}(\mathfrak{A})$ denote the lattice of open-closed subsets of the space $S_p(\mathfrak{A})$ containing no ∇ for an algebra $\mathfrak{A}$ which belongs to a discriminator variety $\mathfrak{M}$. $\mathfrak{D}(\mathfrak{A})$ is a distributive lattice with relative complements. Let us fix an arbitrary element a from $\mathfrak{A}$, and define the operations $x \wedge y, x \vee y, x \setminus y$

on $\mathfrak{A}$ in the following way:

$$x \wedge y = t(a,t(a,x,y),y),\ x \vee y = t(x,a,y),\ x \setminus y = t(a,y,x).$$

One can also directly check that a mapping $d:\mathfrak{A} \to \mathfrak{D}(\mathfrak{A})$, defined as $d(r) = D(a,r)$ for $r \in \mathfrak{A}$, is a homomorphism from the basic set of the algebra $\mathfrak{A}$ with polynomial operations $\vee, \wedge, \setminus$ to the distributive lattice with relative complements $\mathfrak{D}(\mathfrak{A})$.

The proofs of theorem **5.7** and the statement of theorem **5.6** obviously yield the following statements.

Corollary 5.1. For any algebra $\mathfrak{A}$ from a discriminator variety $\mathfrak{M}$, $Con_p\mathfrak{A}$ is a Boolean algebra if $\nabla_{\mathfrak{A}} \in Con_p\mathfrak{A}$.

Corollary 5.2. For any quasi-primal algebra $\mathfrak{A}$, for any Boolean algebra $\mathfrak{B}$ and subalgebra $\mathfrak{A}_1$ of the algebra $\mathfrak{A}^{\mathfrak{B}}$, if $\mathfrak{A}_1$ contains all constant elements of the algebra $\mathfrak{A}^{\mathfrak{B}}$, then there is a subalgebra $\mathfrak{B}_1$ of the Boolean algebra $\mathfrak{B}$ such that $\mathfrak{A}_1 \cong \mathfrak{A}^{\mathfrak{B}_1}$. Indeed, $\mathfrak{B}_1 = \{[\![f = g]\!] \mid f,g \in \mathfrak{A}_1\}$.

Let us now turn to the description of the construction of various special algebras in discriminator varieties. Let $\mathfrak{M}$ be a discriminator variety, and let $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}$. In this case, by theorem **5.7**, the algebras $\mathfrak{A}_i$ are representable as Boolean products of simple and singleton $\mathfrak{M}$-algebras. Let us use the notations of the proof of this theorem, while the algebras $\mathfrak{A}_i$ proper will be identified with the corresponding subalgebras of the algebras $\prod_{\theta \in S_p\mathfrak{A}_i} \mathfrak{A}_i/\theta$. Let g be a homomorphism from the algebra $\mathfrak{A}_1$ to the algebra $\mathfrak{A}_2$. Since subalgebras of simple $\mathfrak{M}$-algebras are simple, for any $\theta \in S_p\mathfrak{A}_2$, when π_θ is a canonical projection of $\mathfrak{A}_2$ on $\mathfrak{A}_2/\theta$, we get $\ker(\pi_\theta \cdot g) \in S_p\mathfrak{A}_1$. Let us refer to $\ker(\pi_\theta \cdot g)$ as $g^+(\theta)$ and we can easily check that g^+ is a continuous mapping from the space $S_p\mathfrak{A}_2$ to the space $S_p\mathfrak{A}_1$. As $S_p\mathfrak{A}_i$ are Stone spaces of Boolean algebras

$$\mathfrak{C}_{\mathfrak{A}_i} = \{D_{\mathfrak{A}_i}(x,y), E_{\mathfrak{A}_i}(x,y) \mid x,y \in \mathfrak{A}_i\},$$

a mapping g^* from the Boolean algebra $\mathfrak{C}_{\mathfrak{A}_1}$ to the Boolean algebra $\mathfrak{C}_{\mathfrak{A}_2}$ (a dual of g^+) will be a homomorphism, in which case for any $x,y \in \mathfrak{A}_1$ we get $g^*(D_{\mathfrak{A}_1}(x,y)) = D_{\mathfrak{A}_2}(g(x),g(y))$. Let $S_{p_0}\mathfrak{A}_i = S_p\mathfrak{A}_i \setminus \{\nabla_{\mathfrak{A}_i}\}$, and let us notice that the mapping g^+ preserves the points of $\nabla_{\mathfrak{A}_i}$, i.e., $g^+(\nabla_{\mathfrak{A}_2}) = \nabla_{\mathfrak{A}_1}$.Let $\varphi_{\mathfrak{A}_i}$ be an ultrafilter on $\mathfrak{T}_{\mathfrak{A}_i}$ equal to $\{E(x,y) | x,y \in \mathfrak{A}_i\}$, in which case g^* preserves this filter, i.e., $(g^*)^{-1}(\varphi_{\mathfrak{A}_2}) = \varphi_{\mathfrak{A}_1}$. The ideal $J_{\mathfrak{A}_i}$ of the Boolean algebra $\mathfrak{T}_{\mathfrak{A}_i}$ is also the dual of the subspace $S_{p_0}\mathfrak{A}_i$: $J_{\mathfrak{A}_i} = T_{\mathfrak{A}_i} \setminus \varphi_{\mathfrak{A}_i} = \{D(x,y) | x,y \in \mathfrak{A}_i\}$ and, although g^+ may not be a mapping from $S_{p_0}\mathfrak{A}_2$ to $S_{p_0}\mathfrak{A}_1$, g^* is a homomorphism from a lattice with relative complements of $J_{\mathfrak{A}_1}$ to $J_{\mathfrak{A}_2}$.

Before we start studying the structure of special embeddings and special algebras in discriminator varieties, let us cite some necessary results related to the truth of elementary formulas about the algebras of similar varieties.

Lemma 5.1.

(a) For any algebra $\mathfrak{A} \in \mathfrak{M}_S$, any $a_1 \neq a_2 \in \mathfrak{A}$, for any formula φ of the type $p(\bar{a}) = q(\bar{a})$ where p,q are terms and $\bar{a}$ is a tuple of elements of the algebra $\mathfrak{A}$, $\mathfrak{A} \models p(\bar{a}) \neq q(\bar{a})$ iff

$$\mathfrak{A} \models t(p(\bar{a}), q(\bar{a}), a_1) = t(p(\bar{a}), q(\bar{a}), a_2).$$

(b) For any algebra $\mathfrak{A} \in \mathfrak{M}$, any formula φ of the type $\forall \bar{x} \exists \bar{y}$ $(\mathop{\&}_{i=1}^{n} P_i(\bar{x},\bar{y}) = q_i(\bar{x},\bar{y}))$, where p_i,q_i are terms and $\bar{x},\bar{y}$ are tuples of variables, $\mathfrak{A} \models \varphi$ iff for any $\Phi \in S_{p_0}\mathfrak{A}$ we have $\mathfrak{A} / \Phi \models \varphi$.

Proof. Statement (a) can be checked directly, owing to the fact that the term $t(x,y,z)$ defines the discriminators on $\mathfrak{M}_S$ algebras. In order to prove statement (b), one should notice that, since $\mathfrak{A} \models \varphi$, φ is true on $\mathfrak{A} / \Phi$ for any $\phi \in Sp\mathfrak{A}$ because the formula of φ is positive. Let us prove the converse case, so let $\mathfrak{A} / \Phi \models \varphi$ for any $\Phi \in S_{p_0}\mathfrak{A}$ and, therefore, $\mathfrak{A} / \Phi \models \varphi$ for any $\Phi \in S_p\mathfrak{A}$. Assume that $\bar{a} \in \mathfrak{A}$, and let $\pi_\Phi(\bar{a})$ ($\Phi \in S_p\mathfrak{A}$) stand for a tuple consisting of π_Φ-images of the elements of the tuple $\bar{a}$. Then for a certain tuple $\bar{b}_\Phi \in \mathfrak{A} / \Phi$ we have

$$\mathfrak{A} / \Phi \models \mathop{\&}_{i=1}^{n} p_i(\pi_\Phi(\bar{a}), \bar{b}_\Phi) = q_i(\pi_\Phi(\bar{a}), \bar{b}_\Phi).$$

Assume that $\bar{b}^\Phi \in \mathfrak{A}$ so that $\pi_\Phi(\bar{b}^\Phi) = \bar{b}_\Phi$. Then $A_\Phi = \{\psi \in S_p\mathfrak{A} \mid \mathfrak{A} / \psi \models \mathop{\&}_{i=1}^{n} p_i(\pi_\psi(\bar{a}), \pi_y(\bar{b}^\Phi)) = q_i(\pi_y(\bar{a}), \pi_y(\bar{b}^\Phi))\}$ is open-closed in $S_p\mathfrak{A}$, and contains a point Φ of the space $S_p\mathfrak{A}$. Choosing a finite subcovering $A_{\Phi_1},\ldots,A_{\Phi_n}$ from the covering $\{A_\Phi \mid \Phi \in S_p\mathfrak{A}\}$ of the space $S_p\mathfrak{A}$, one can consider the latter a subdivision of the space $S_p\mathfrak{A}$. Since $\mathfrak{A}$ is a Boolean product of the algebras $\mathfrak{A}/\Phi$ for $\Phi \in S_p\mathfrak{A}$, there is a $\bar{b} \in \mathfrak{A}$ such that for $i \le n$, any $\psi \in A_{\Phi_i}$ we have $\pi_\psi(\bar{b}) = \pi_\psi(\bar{b}^{\Phi_i})$. Therefore, for any $\Phi \in S_p\mathfrak{A}$,

$$\mathfrak{A}/\Phi \models \underset{i=1}{\overset{n}{\&}} p_i(\pi_\Phi(\bar{a}), \pi_\Phi(\bar{b})) = q_i(\pi_\Phi(\bar{a}), \pi_\Phi(\bar{b}))$$

and, hence, $\mathfrak{A} \models \underset{i=1}{\overset{n}{\&}} p_i(\bar{a},\bar{b}) = q_i(\bar{a},\bar{b})$, which means that there is an $\mathfrak{A} \models \varphi$. ■

Lemma 5.2. For any set of $\forall\exists$-formulas Σ there exists a set Σ' of formulas of the kind $\forall\bar{x}\exists\bar{y}(p(\bar{x},\bar{y}) = x_1)$, where p is a term and $\bar{x},\bar{y}$ are tuples of variables such that for any algebra $\mathfrak{A} \in \mathfrak{M}$ $\mathfrak{A} \models \Sigma'$ iff for any $\psi \in Sp_0\mathfrak{A}$ $\mathfrak{A}/\psi \models \Sigma$.

Proof. Let us assume that all the negations occurring in the Σ-formulas refer only to atomic subformulas, i.e., occur as inequalities $p(\bar{x}) \ne q(\bar{x})$. Let us replace every occurrence of a similar inequality σ in the Σ-formula with a corresponding positive $\forall$-formula σ' of the type $\forall x,y(t(p(\bar{x}),q(\bar{x}),x) = t(p(\bar{x}),q(\bar{x}),y)$. Then for the algebras $\mathfrak{B} \in \mathfrak{M}_S$, $\mathfrak{B} \models \sigma$ iff $\mathfrak{B} \models \sigma'$. Let Σ'' be a result of the substitution of the inequalities σ in Σ-formulas with subformulas of the type σ'. Σ'' is a family of positive $\forall\exists$-formulas and, obviously, for $\mathfrak{A} \in \mathfrak{M}_S$, $\mathfrak{A} \in \Sigma'' \Leftrightarrow \mathfrak{A} \models \Sigma$.

Let us now consider a certain standard transformation of positive quantifierless formulas in discriminator varieties. Let us, first of all, take into consideration the following terms:

$$n(x,y,z,u) = t(t(x,y,z),t(x,y,u),u) \text{ and}$$
$$s(x,y,z) = n(x,y,z,t(x,z,y)).$$

One can observe directly that for $\mathfrak{A} \in \mathfrak{M}_S$, for any $a,b,c,d \in \mathfrak{A}$, $n(a,b,c,d) = c$ if $a = b$, is equal to d if $a \ne b$, $s(a,b,c) = a$, if $a \ne b,c$, is equal to b if $a = c$, is equal to c if $a = b$. By way of induction, let us establish a correspondence between any positive quantifierless formula φ with variables $x_1,\ldots,x_n$ and a certain term $T(\varphi)$ according to the following rules:

(a) if φ is an equality of terms $p(\bar{x}) = q(\bar{x})$, then $T(\varphi) = s(p(\bar{x}), q(\bar{x}), x_1)$;

(b) if $\varphi = \alpha \vee \beta$, then $T(\alpha \vee \beta) = n(T(\alpha), x_1, x_1, T(\beta))$;

(c) if $\varphi = \alpha \wedge \beta$, then $T(\alpha \wedge \beta) = t(T(\alpha), x_1, T(\beta))$.

One can also directly check that for any algebra $\mathfrak{A} \in \mathfrak{M}_S$, and $a_1,\dots,a_n \in \mathfrak{A}$, $\mathfrak{A} \models \varphi(a_1,\dots,a_n) \Leftrightarrow \mathfrak{A} \models T(\varphi)(a_1,\dots,a_n) = a_1$.

For any formula $\psi = \forall\bar{x}\exists\bar{y}\,\varphi(\bar{x},\bar{y}) \in \Sigma''$, we will construct a formula $\psi^* = \forall\bar{x}\exists\bar{y}\,(T(\varphi)(\bar{x},\bar{y}) = x_1)$ and let $\Sigma' = \{\psi^* \mid \psi \in \Sigma''\}$. By lemma **5.1** we see that for any algebra $\mathfrak{A} \in \mathfrak{M}$, $\mathfrak{A} \models \Sigma'$ iff for any $\psi \in Sp_0\mathfrak{A}$, $\mathfrak{A} / \psi \models \Sigma'$. Since for any $\mathfrak{A} \in \mathfrak{M}_S$,

$$\mathfrak{A} \models \Sigma' \Leftrightarrow \mathfrak{A} \models \Sigma'' \Leftrightarrow \mathfrak{A} \models \Sigma,$$

the family of the Σ'-formulas is the one we were loking for. ■

Definition 5.3.

(a) A subalgebra $\mathfrak{A}_1$ is called an essential subalgebra of the algebra $\mathfrak{A}$ if for any homomorphism f from the algebra $\mathfrak{A}$ to the algebra $\mathfrak{B}$, f is an isomorphism iff $f|\mathfrak{A}_1$ is an isomorphic embedding of $\mathfrak{A}_1$ in $\mathfrak{B}$. An embedding g of the algebra $\mathfrak{A}_2$ to the algebra $\mathfrak{A}$ is said essential if the algebra $g(\mathfrak{A}_2)$ is an essential subalgebra of the algebra $\mathfrak{A}$.

(b) The algebra $\mathfrak{A}$ is called a pure extension of its subalgebra $\mathfrak{A}_1$ if for any finite set of equalities $\{t_i^1(\bar{x},\bar{a}) = t_i^2(\bar{x},\bar{a}) \mid i \in I\}$, where t_i^1, t_i^2 are terms, $\bar{a}$ is a tuple of elements of $\mathfrak{A}_1$ and $\bar{x} = x_1,\dots,x_n$, the fulfilment of the conjunction $\underset{i \in I}{\&} t_i^1(\bar{x},\bar{a}) = t_i^2(\bar{x},\bar{a})$ in $\mathfrak{A}$ implies the fulfilment of this conjunction in $\mathfrak{A}_1$. The embedding g of the algebra $\mathfrak{A}_2$ in the algebra $\mathfrak{A}$ is called pure if $\mathfrak{A}$ is a pure extension of the algebra $g(\mathfrak{A}_2)$.

(c) The algebra $\mathfrak{A}$ is called an existential extension of its subalgebra $\mathfrak{A}_1$, if for any finite set of equalities and inequalities

$$\{t_i^1(\bar{x},\bar{a}) = t_i^2(\bar{x},\bar{a}), t_i^3(\bar{x},\bar{b}) \neq t_j^4(\bar{x},\bar{b}) \mid i \in I, j \in J\},$$

where $t_i^1, t_i^2, t_j^3, t_j^4$ are terms, $\bar{a}, \bar{b}$ are tuples of the elements from $\mathfrak{A}_1$ and $\bar{x} = x_1,\dots,x_n$, the fulfilment of the conjunction

$$\underset{i \in I}{\&} t_i^1(\bar{x},\bar{a}) = t_i^2(\bar{x},\bar{a}) \,\&\, \underset{j \in J}{\&} t_j^3(\bar{x},\bar{b}) \neq t_j^4(\bar{x},\bar{b})$$

in $\mathfrak{A}$ implies that in $\mathfrak{A}_1$. The embedding g of the algebra $\mathfrak{A}_2$ in the algebra $\mathfrak{A}$ is called existential if $\mathfrak{A}$ is an existential extension of the algebra $g(\mathfrak{A}_2)$.

Lemma 5.3. If $X \subseteq Sp_0\mathfrak{A}$, then the following conditions are equivalent:

(a) a family of canonical projections $\{\pi_\theta \mid \theta \in X\}$ implies a subdirect decomposition of the algebra $\mathfrak{A}$;

(b) $\cap\{\theta \mid \theta \in X\} = \Delta$;

(c) the set $X \cup \{\nabla\}$ is dense in the space $Sp\mathfrak{A}$.

Proof. $\cap\{\theta \mid \theta \in X\} = \Delta$ iff for any $x, y \in \mathfrak{A}$, when $x \neq y$, there is a $\theta \in X$ such that $< x, y > \notin \theta$. The latter statement is equivalent to the fact that for any $x, y \in \mathfrak{A}$, $x = y$ if for all $\theta \in X$ $< x, y > \in \theta$, which is, in turn, equivalent to the fact that for any $x, y \in \mathfrak{A}$, the inclusion $X \subseteq E(x,y)$ implies the equality $x = y$. ■

Lemma 5.4. The following conditions are equivalent for the embedding $f: \mathfrak{A} \to \mathfrak{C}$:

(1) f is essential;

(2) for any $C \subseteq Sp\mathfrak{C}$ such that $\nabla \in C$, C is dense in $Sp\mathfrak{C}$ iff $f^+(C)$ is dense in $Sp\mathfrak{A}$;

(3) if C is a closed subset in $Sp\mathfrak{C}$, then $Sp_0\mathfrak{A} \subseteq f^+(C)$ iff $Sp_0\mathfrak{C} \subseteq C$;

(4) the embedding $f^*: J_\mathfrak{A} \to J_\mathfrak{C}$ is essential in the class of distributive lattices with relative complements.

Proof. In the definition of the essentiality of an embedding of the algebra $\mathfrak{A}$ in the algebra $\mathfrak{C}$ it is obviously sufficient to consider only the principal congruences of the algebra $\mathfrak{C}$ and, in this case, the implication (1) → (2) directly results from the criterion (b) of density of the subset of the space $Sp(\mathfrak{C})(Sp(\mathfrak{A}))$ of lemma 5.3. The implication (2) → (3) is obvious. The implication (3) → (1) can be also directly proved using lemma **5.3**. At the same time, the equivalence of conditions (3) and (4) results from the fact that any congruence on the lattice of $J_\mathfrak{A}$ is uniquely determined by a certain ideal of this lattice, while the latter is uniquely determined by a certain open subset of the space $Sp\mathfrak{A}$ containing no element ∇. ■

Lemma 5.5. If f is an embedding of the algebra $\mathfrak{A}$ in the algebra $\mathfrak{C}$, and there is an $X \subseteq Sp\mathfrak{C}$ such that $f^+(X) = Sp_0\mathfrak{A}$, and for any $\Phi \in X$ the induced embedding $f_\Phi : \mathfrak{A} / f^+(\Phi) \to \mathfrak{C} / \Phi$ is pure, then the embedding f is pure itself.

Proof. Let

$$\varphi(y_1,\dots,\varphi_s) = \exists x_1,\dots,x_n(\mathop{\&}_{i=1}^{m} t_i^1(\bar{x},\bar{y}) = t_i^2(\bar{x},\bar{y}))$$

and $a_1,\dots,a_s \in \mathfrak{A}$. In this case, if $\mathfrak{C} \models \varphi(f(a_1),\dots,f(a_s))$, then, since φ is a positive formula, for any $\Phi \in Sp\mathfrak{C}$,

$$\mathfrak{C} / \Phi \models \varphi(f(a_1)/\Phi,\dots,f(a_s)/\Phi).$$

As f_Φ are pure embeddings of the algebras $\mathfrak{A} / f^+(\Phi)$ in $\mathfrak{C} / \Phi$, for any $\Phi \in X$,

$$\mathfrak{A} / f^+(\Phi) \models \varphi(a_1 / f^+(\Phi),\dots,a_s / f^+(\Phi)).$$

And since $f^+(X) = Sp\mathfrak{A} \setminus \{\nabla\}$, for any $\psi \in Sp_0\mathfrak{A}$ we have $\mathfrak{A} / \psi \models \varphi(a_1 / \psi,\dots,a_s / \psi)$.
By lemma **5.1** (b), we get from here that $\mathfrak{A} \models \varphi(a_1,\dots a_s)$. ■

Lemma 5.6. Let f be an embedding of the algebra $\mathfrak{A}$ in the algebra $\mathfrak{C}$, and let the space $Sp\mathfrak{A}$ have no other isolated points but ∇. If there is a subset X dense in $Sp_0\mathfrak{C}$ such that $f^+(X) = Sp_0\mathfrak{A}$, and for any $\Phi \in X$ the induced embedding $f_\Phi : \mathfrak{A} / f^+(\Phi) \to \mathfrak{C} / \Phi$ is existential, then the embedding f is existential itself.

Proof. Let the formula

$$\varphi(y_1,\dots,y_l) = \exists x_1,\dots x_n(\mathop{\&}_{i=1}^{m} t_i^1(\bar{x},\bar{y}) =$$
$$= t_i^2(\bar{x},\bar{y}) \& \mathop{\&}_{j=1}^{k} t_j^3(\bar{x},\bar{y}) \neq t_j^4(\bar{x},\bar{y})).$$

φ_j will denote the formulas

$$\exists x_1,\dots,x_n(\mathop{\&}_{i=1}^{m} t_i^1(\bar{x},\bar{y}) = t_i^2(\bar{x},\bar{y}) \& t_j^3(\bar{x},\bar{y}) \neq t_j^4(\bar{x},\bar{y})),$$

for $j \le k$, and φ' the formula

$$\exists x_1,\ldots,x_n(\mathop{\&}_{i=1}^{m} t_i^1(\bar{x},\bar{y}) = t_i^2(\bar{x},\bar{y})).$$

Let $a_1,\ldots,a_l \in \mathfrak{A}$ and $\mathfrak{C} \models \varphi(f(a_1),\ldots,f(a_l))$. The task is to show that $\mathfrak{A} \models \varphi(a_1,\ldots,a_l)$.

Since $\mathfrak{C}$ is a subdirect product of algebras $\mathfrak{C}/\Phi$, where $\Phi \in Sp\mathfrak{C}$, it is obvious that for any $\Phi \in Sp\mathfrak{C}$,

$$\mathfrak{C}/\Phi \models \varphi'(f(a_1)/\Phi,\ldots,f(a_l)/\Phi)$$

and there are $\Phi_1,\ldots,\Phi_k \in Sp\mathfrak{C}$ such that

$$\mathfrak{C}/\Phi_j \models \varphi_j(f(a_1)/\Phi_j,\ldots,f(a_l)/\Phi_j$$

for all $j \le k$.

It is obvious that, since $\mathfrak{C}/\nabla$ is singleton, $\Phi_1,\ldots,\Phi_k$ are different from ∇.

It is also obvious (see proof of lemma **5.1** (b)) that for any existential formula, the definition of a Boolean product entails that the truth of this formula for a certain position of the projection π_i of this Boolean product implies that of this formula for any projection occurring in a certain neighborhood of the projection π_i. By virtue of this remark and since X is dense in $Sp_0\mathfrak{C}$, there are $\Phi_1',\ldots,\Phi_k' \in X$ such that

$$\mathfrak{C}/\Phi_j' \models \varphi_j(f(a_1)/\Phi_j',\ldots,f(a_l)/\Phi_j') \text{ for } j \le k.$$

Since the embeddings f_Φ of the algebras $\mathfrak{A}/f^+(\Phi)$ in the algebras $\mathfrak{C}/\Phi$ are existential for any $\Phi \in X$,

$$\mathfrak{A}/f^+(\Phi_j') \models \varphi_j(a_1/f^+(\Phi_j'),\ldots,a_l/f^+(\Phi_j'))$$

for any $j \le k$, and for any $\Phi \in X$ $\mathfrak{A}/f^+(\Phi) \models \varphi'(a_1/\Phi,\ldots,a_l/\Phi)$. But $f^+(X) = Sp_0\mathfrak{A}$ and, therefore, $\mathfrak{A}/\Phi \models \varphi'(a_1/\Phi,\ldots,a_l/\Phi)$ for any $\Phi \in Sp\mathfrak{A}$. As in the case considered above, the validity of $\varphi_j(a_1/f^+(\Phi_j'),\ldots,a_l/f^+(\Phi_j'))$ for the algebra $\mathfrak{A}/f^+(\Phi_j')$ implies that of the formula $\varphi_j(a_1/\Phi,\ldots,a_l/\Phi)$ for the algebra $\mathfrak{A}/\Phi$ for any Φ from a certain neighborhood S_j of the point $f^+(\Phi_j')$ of the space $Sp\mathfrak{A}$. But $Sp\mathfrak{A}$ has no other isolated points but ∇ and, obviously, $f^+(\Phi_j') \ne \nabla$. This enables us to choose points $\Phi_j'' \in S_j$ in such a way that they are mutually different at $j \leq k$ and, therefore, for pairwise different Φ_j'' (at $j \leq k$), we have

$$\mathfrak{A} / \Phi_j'' \models \varphi_j(a_1 / \Phi_j'', \ldots, a_l / \Phi_j'').$$

From this fact, and since $\mathfrak{A} / \Phi \models \varphi'(a_1 / \Phi, \ldots, a_l / \Phi)$ for any $\Phi \in Sp\mathfrak{A}$, one can deduce in a standard way (see, for instance, the proof of lemma **5.1.**(b)) the truth of the formula $\varphi(a_1, \ldots, a_l)$ for the algebra $\mathfrak{A}$, which is a Boolean product of algebras $\mathfrak{A} / \Phi (\Phi \in Sp\mathfrak{A})$ over the Boolean space $Sp\mathfrak{A}$. ■

Definition 5.4.

(a) The algebra $\mathfrak{A}$ is called injective in the class $\mathfrak{K}$ iff for any embedding $h: \mathfrak{B} \to \mathfrak{C}$ of the algebras $\mathfrak{B}, \mathfrak{C} \in \mathfrak{K}$ and any homomorphism f of the algebra $\mathfrak{B}$ in the algebra $\mathfrak{A}$, there is a homomorphism g from the algebra $\mathfrak{C}$ to the algebra $\mathfrak{A}$ such that $f = g \circ h$.

(b) The algebra $\mathfrak{A}$ is weakly injective in the class $\mathfrak{K}$ iff for any embedding $h: \mathfrak{B} \to \mathfrak{C}$ of algebras $\mathfrak{B}, \mathfrak{C} \in \mathfrak{K}$, and for any homomorphism f from the algebra $\mathfrak{B}$ to the algebra $\mathfrak{A}$, there is a homomorphism g from the algebra $\mathfrak{C}$ to the algebra $\mathfrak{A}$ such that $f = g \circ h$.

(c) The algebra $\mathfrak{A}$ is an absolute subretract in $\mathfrak{K}$ if for any embedding $h: \mathfrak{A} \to \mathfrak{C} \in \mathfrak{K}$ there is a homomorphism g from the algebra $\mathfrak{C}$ to $\mathfrak{A}$ such that $g \cdot h$ is identical on $\mathfrak{A}$.

(d) The algebra $\mathfrak{A}$ has no proper essential extension in the class $\mathfrak{K}$ if any essential embedding $h: \mathfrak{A} \to \mathfrak{C} \in \mathfrak{K}$ is an isomorphism.

(e) The algebra $\mathfrak{A}$ is algebraically closed in the class $\mathfrak{K}$ if any embedding $h: \mathfrak{A} \to \mathfrak{C} \in \mathfrak{K}$ is pure.

(f) The algebra $\mathfrak{A}$ is existentially closed in the class $\mathfrak{K}$ if any embedding $h: \mathfrak{A} \to \mathfrak{C} \in \mathfrak{K}$ is existential.

(g) The algebra $\mathfrak{A}$ is called equationally compact if any set $\{t_i^1(\bar{x}_i, \bar{a}_i) = t_i^2(\bar{x}_i, \bar{a}_i) \mid i \in I\}$, where t_i^1, t_i^2 are terms, $\bar{x}_i, \bar{a}_i$ are tuples of variables and $\mathfrak{A}$-elements, respectively, each of the finite subsets of which is fulfilled in $\mathfrak{A}$, is self-fulfilled in $\mathfrak{A}$.

One can now easily prove the following lemma.

Lemma 5.7. If a variety $\mathfrak{M}$ has the property of extensibility of congruences, then the following conditions on the algebra $\mathfrak{A} \in \mathfrak{M}$ are equivalent:

(a) $\mathfrak{A}$ is weakly injective in $\mathfrak{M}$;

(b) $\mathfrak{A}$ is an absolute subretract in $\mathfrak{M}$;

(c) $\mathfrak{A}$ has no proper essential extension in $\mathfrak{M}$.

Let then $\mathfrak{M}$ be a discriminator variety. $\mathfrak{M}_S^+$, as earlier, will stand for the class of algebras $\mathfrak{M}_{SI} = \mathfrak{M}_S$ modified with a one-element algebra.

Lemma 5.8.

(a) For any $\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}_S^+$, any embedding $f\colon \mathfrak{A} \to \mathfrak{B}$

(1) is essential but for the case when $|\mathfrak{A}| = 1, |\mathfrak{B}| > 1$;
(2) if $\mathfrak{A}$ is finite, then f is pure iff f is an isomorphism, or $|\mathfrak{A}| = 1$;
(3) if $\mathfrak{A}$ is finite, then f is existential iff f is an isomorphism.

(b) For $\mathfrak{B} \in \mathfrak{M}_S^+$, the following conditions are equivalent:

(1) $\mathfrak{B}$ is injective in $\mathfrak{M}_S^+$;

(2) $\mathfrak{B}$ is finite, and any isomorphism between non-singleton subalgebras of the algebra $\mathfrak{B}$ is extendable up to the automorphism of the algebra $\mathfrak{B}$, and for any $\mathfrak{A} \in \mathfrak{M}_S^+$, either $\mathfrak{A}$ is imbeddable in $\mathfrak{B}$, or $\mathfrak{B}$ and $\mathfrak{A}$ have no non-singleton isomorphic subalgebras;

(3) $\mathfrak{B}$ is injective in $\mathfrak{M}$.

(c) For a non-singleton $\mathfrak{B} \in \mathfrak{M}_S^+$, the following conditions are equivalent:

(1) $\mathfrak{B}$ is weakly injective in $\mathfrak{M}_S^+$;
(2) $\mathfrak{B}$ has no proper extensions in $\mathfrak{M}_S^+$;
(3) $\mathfrak{B}$ is finite and algebraically closed in $\mathfrak{M}_S^+$;

(4) $\mathfrak{B}$ is finite and existentially closed in $\mathfrak{M}_S^+$;
(5) $\mathfrak{B}$ is weakly injective in $\mathfrak{M}$.

(d) $\mathfrak{B} \in \mathfrak{M}_S^+$ is equationally compact iff $\mathfrak{B}$ is finite.

Proof. Statement (a)(1) is obvious.

Statements (a)(2), (a)(3) are obvious for the case when $|\mathfrak{A}|=1$. Let $|\mathfrak{A}|>1$, $\mathfrak{A}=\{a_1,\dots,a_n\}$ and $a_1 \neq a_2$. For any algebra $\mathfrak{C}$ such that $f(\mathfrak{A}) \subseteq \mathfrak{C} \subseteq \mathfrak{B}$, the formula $\underset{i=1}{\overset{n}{\&}} t(a_i,x,a_1)=t(a_i,x,a_2)$ is fulfilled on an element $b \in \mathfrak{C}$ iff $b \notin f(\mathfrak{A})$. Therefore, if f is pure and existential, then f is certainly an isomorphism between the algebras $\mathfrak{A}$ and $\mathfrak{B}$. Hence, statements (a)(2) and (a)(3) are completely proved.

Let us show that (b)(1) and (b)(2) are equivalent. Let us first of all prove that an algebra $\mathfrak{A}$ that is weakly injective in the class $\mathfrak{M}_S^+$ must be finite. Let us assume that the opposite holds, i.e., let $\mathfrak{A}$ be infinite, $\mathfrak{A}=\{a_i \mid I \in I\}$, and let an element $b \notin \mathfrak{A}$, then the family of the statements $\{a_i \neq b \mid i \in I\} \cup \{t(x,y,z)$-discriminator$\} \cup \{$diagram of the algebra $\mathfrak{A}\}$ is locally compactable, and, hence, according to the compactness theorem, it is compactable. The model of $\mathfrak{B}$ of this family of statements will be a proper extension of the algebra $\mathfrak{A}$ in the class $\mathfrak{M}_S^+$, i.e., in particular, a simple algebra. And again, since the algebra $\mathfrak{A}$ is weakly injective, the algebra $\mathfrak{B}$ must homomorphically map on $\mathfrak{A}$. The obtained contradiction proves the finiteness of any weakly injective in $\mathfrak{M}_S^+$ algebra.

Let now h be an embedding of the algebra $\mathfrak{C}$ in the algebra $\mathfrak{D}$, and $\mathfrak{C},\mathfrak{D} \in \mathfrak{M}_S^+$. Let f be a homomorphism from $\mathfrak{C}$ to the algebra $\mathfrak{B}$, in which case (b)(1) states the existence of a homomorphism g from the algebra $\mathfrak{D}$ to the algebra $\mathfrak{B}$ such that $f=h{:}g$, while (b)(2) states the existence of such a homomorphism g only in the case when $|f(\mathfrak{C})|>1$. In the case when $|f(\mathfrak{C})|=1$, it suffices to choose g in such a way that $g(\mathfrak{D})=f(\mathfrak{C})$.

Let us show that (b)(1) $\to$ (b)(3). Let h be an embedding $\mathfrak{A} \to \mathfrak{C}$, where $\mathfrak{A},\mathfrak{C} \in \mathfrak{M}$, and f be a homomorphism from the algebra $\mathfrak{A}$ to the algebra $\mathfrak{B}$. Since $f(\mathfrak{A})$ is a simple subalgebra of the algebra $\mathfrak{B}$ and since the congruences on $\mathfrak{M}$ are extendable, there exists a congruence ψ maximal in $Con\mathfrak{C}$ and a congruence ϕ maximal in $Con\mathfrak{A}$ such that the following diagram is commutative:

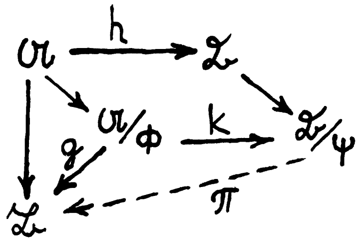

Fig. 4

The injectivity of the algebra $\mathfrak{B}$ in the class $\mathfrak{M}_S^+$ implies the existence of a homomorphism $\pi : \mathfrak{C} / \psi \rightarrow \mathfrak{B}$ such that $g = \pi \cdot k$. But in this case, $\pi \cdot \varphi$ is a homomorphism from the algebra $\mathfrak{C}$ to the algebra $\mathfrak{B}$ and, obviously, is the one to prove the injectivity of the algebra $\mathfrak{B}$ in the variety $\mathfrak{M}$.

Statement (b)(3) $\rightarrow$ (b)(1) is obvious.

Statements (c)(5) $\rightarrow$ (c)(1), (c)(1) $\rightarrow$ (c)(2), (c)(4) $\rightarrow$ (c)(3) are also obvious. Implication (c)(3) $\rightarrow$ (c)(2) results from statement (a)(2). The proof of statement (c)(2) $\rightarrow$ (c)(4) results from the considerations of the proof of statements (b)(1) $\leftrightarrow$ (b)(2) and (a)(2). The statement (c)(2) $\rightarrow$ (c)(5) is proved analogously to the implication (b)(1) $\rightarrow$ (b)(3).

Statement (d) results from the fact that for an infinite algebra $\mathfrak{A} \in \mathfrak{M}_S^+$, if $\mathfrak{F}$ is not the principal ultrafilter on ω, then $\mathfrak{A}^{\omega} / \mathfrak{F}$ is a proper extension of $\mathfrak{A}$, and since the term $t(x,y,z)$ will define the discriminator on $\mathfrak{A}^{\omega} / \mathfrak{F}$, as well as on $\mathfrak{A}$, $\mathfrak{A}^{\omega} / \mathfrak{F}$ is simple. On the other hand, taking a positive diagram of $\mathfrak{A}^{\omega} / \mathfrak{F}$ in a standard way and replacing the constant elements in $\mathfrak{A}^{\omega} / \mathfrak{F}$ with the corresponding constants from $\mathfrak{A}$, and the rest of the elements with various variables, we obtain a system of equations finitely fulfillable in $\mathfrak{A}$, with its fulfillability in $\mathfrak{A}$ implying that

$\mathfrak{A}$ is a retract of $\mathfrak{A}^{\omega}/\mathfrak{F}$. The contradiction obtained proves statement (d). ■

Lemma 5.9. Any non-singleton algebra $\mathfrak{A}$ algebraically closed in $\mathfrak{M}_S^+$ is existentially closed in $\mathfrak{M}_S^+$.

Proof. It suffices to notice that for any $\mathfrak{M}_S^+$-algebra $\mathfrak{A}$ containing elements $a_1 \neq a_2$, the solvability of any inequality $f(\bar{x},\bar{a}) \neq g(\bar{x},\bar{a})$ in $\mathfrak{A}$, where f,g are terms, $\bar{x}$ is a tuple of variables and $\bar{a}$ is a tuple of the elements of the algebra $\mathfrak{A}$, is equivalent to the solvability of the equation

$$t(f(\bar{x},\bar{a}),g(\bar{x},\bar{a}),a_1) = t(f(\bar{x},\bar{a}),g(\bar{x},\bar{a}),a_2),$$

where t is a term determining the discriminator on $\mathfrak{A}$. ■

Theorem 5.8. Let $\mathfrak{M}$ be a discriminator variety and $\mathfrak{A} \in \mathfrak{M}$ such that for any $\phi \in Sp_0\mathfrak{A}$, the algebra $\mathfrak{A}/\phi$ is algebraically closed in $\mathfrak{M}_S^+$. Then:

(a) $\mathfrak{A}$ is algebraically closed in $\mathfrak{M}$;

(b) $\mathfrak{A}$ is existentially closed in $\mathfrak{M}$ provided that one of the following conditions are met:

(1) the non-singleton $\mathfrak{M}_S^+$-algebra has no one-element subalgebras, and the space $Sp\mathfrak{A}$ has no isolated points other than, possibly, ∇;

(2) a certain non-singleton $\mathfrak{M}_S^+$-algebra has a one-element subalgebra, while the space $Sp\mathfrak{A}$ has no isolated points.

Proof.

(a) Let f be an embedding of the algebra $\mathfrak{A}$ in the algebra $\mathfrak{B} \in \mathfrak{M}$. As $\mathfrak{A}/\phi$ are algebraically closed in $\mathfrak{M}_S^+$ for any $\phi \in Sp_0\mathfrak{A}$, for any $\psi \in Sp_0\mathfrak{C}$ such that $f^+(\psi) \in Sp_0\mathfrak{A}$, the embedding f_ψ of the algebra $\mathfrak{A}/f^+(\psi)$ in the algebra $\mathfrak{C}/\psi$ is pure. Therefore, by lemma **5.5**, the embedding f is pure and statement (a) is proved.

b1). It should be first of all noticed that, since non-singleton $\mathfrak{M}_S^+$-algebras contain no singleton subalgebras, for any embedding $f:\mathfrak{A} \to \mathfrak{C}$, where $\mathfrak{C} \in \mathfrak{M}$, for $\psi \in Sp\mathfrak{C}$, the equality $f^+(\psi) = \nabla$ yields the equality $\psi = \nabla$. Therefore, for any

$\psi \in Sp_0\mathfrak{C}$, the algebra $\mathfrak{A}/_{f^+(\psi)}$ is a non-singleton algebra algebraically closed in $\mathfrak{M}_S^+$. According to lemma **5.9**, the algebra $\mathfrak{A}/_{f^+(\psi)}$ is existentially closed and, hence, the embedding $f_\psi : \mathfrak{A}/_{f^+(\psi)}$ into the algebra $\mathfrak{C}/_\psi$ is existential for any $\psi \in Sp_0\mathfrak{C}$. The embedding f is, hence, existential by lemma **5.6**.

b2). If ∇ is not an isolated point in $Sp\mathfrak{A}$, then for any embedding $f:\mathfrak{A} \to \mathfrak{C}$, the set $\{\psi \in Sp\mathfrak{C} \mid f^+(\psi) \neq \nabla\}$ is dense in the subspace $Sp_0\mathfrak{C}$, since the mapping f^+ is a continuous mapping from the space $Sp\mathfrak{C}$ to the space $Sp\mathfrak{A}$. Now the statement b(2) follows, like the statement b(1), from lemma **5.6.** ■

Let us now assume that $\mathfrak{M}$ is a finitely generated discriminator variety. By the Jonsson lemma and by the fact that $\mathfrak{M}_{SI} = \mathfrak{M}_S$, the class of $\mathfrak{M}_{SI}$-algebras is finite, as all $\mathfrak{M}_{SI}$-algebras are. By $\mathfrak{M}_{\max}$ we will mean the $\mathfrak{M}_{SI}$-algebras which are maximal in $\mathfrak{M}_{SI}$ relative to the embedding. As $\mathfrak{M}$ is congruence-distributive and, by the Baker theorem [6], ([161]) $\mathfrak{M}$ is finitely generated, (in fact, by the Padmanabhan-Quackenbush theorem [153], ([161]), $\mathfrak{M}$ is even one-based). Let $\varepsilon_1(x_1,\dots,x_n),\dots,\varepsilon_m(x_1,\dots,x_n)$ be the basis of identities for $\mathfrak{M}$. In this case, the class of $\mathfrak{M}_{SI}$-algebras is axiomatizable by the universal positive formula

$$\alpha = \forall x_1,\dots,x_n(\underset{i=1}{\overset{m}{\&}}\varepsilon_i(x_1,\dots,x_n)\&(x_1 = x_2 \vee t(x_1,x_2,x_3) = x_1)\&t(x_1,x_1,x_3) = x_3),$$

where the term $t(x,y,z)$ defines the determinators on $\mathfrak{M}_{SI}$-algebras.

For any algebra $\mathfrak{A} \in \mathfrak{M}_{SI}$ let $\varphi_{\mathfrak{A}}(x_1,\dots,x_m)$ be a positive diagram of the algebra $\mathfrak{A}$. If $\mathfrak{A} = \{a_1,\dots,a_m\}$, then the positive $\forall\exists$-formula

$$\delta_{\mathfrak{A}} = \forall x,y\exists x_1,\dots,x_m(\varphi_{\mathfrak{A}}(x_1,\dots,x_m)\&\underset{i<j\leq m}{\&} t(x_i,x_j,x) = t(x_i,x_j,y)$$

is true only for a one-element algebra or for those $\mathfrak{M}_{SI}$-algebras which contain subalgebras isomorphic to $\mathfrak{A}$. Let $\{\mathfrak{A}_1,\dots,\mathfrak{A}_k\} = \mathfrak{M}_{\max}$, then a positive $\forall\exists$-formula $\varphi = \alpha\&\underset{i\leq k}{\vee}\delta_{\mathfrak{A}_i}$ is a system of axioms for a class which consists of $\mathfrak{M}_{\max}$ and a one-element algebra. Now we evidently can, depending on whether $\mathfrak{M}_{\max}$ contains a one-element algebra or not, write a $\forall\exists$-formula φ_1 as a system of axioms for $\mathfrak{M}_{\max}$.

Let us now take into consideration the following $\forall\exists$-formulas:

$$\xi_0 = \forall x,y\exists z(x \neq y) \to t(x,y,z) = x\&t(x,z,y) \neq x\& x \neq z),$$

$$\xi = \forall x, y \exists u, v((x \neq y \to t(x,y,u) = x \& t(x,u,y) \neq$$
$$\neq x \& x \neq u) \& (t(x,v,y) = x \& t(x,y,v) \neq x)).$$

Since any $\mathfrak{M}$ -algebra is representable as a Boolean product of $\mathfrak{M}_S^+$-algebras, i.e. such that the term $t(x,y,z)$ defines the discriminator on it, and, as for the elements f,g,h from this Boolean representation of the $\mathfrak{M}$ -algebra $\theta_{f,g} \subseteq \theta_{f,h}$ iff $\|f \neq h\| \supseteq \|f \neq g\|$, then the inclusion $\theta_{x,z} \subseteq \theta_{x,y}$ for any elements x,y,z of an arbitrary $\mathfrak{M}$ -algebra is equivalent to the fact that the formula $t(x,y,z) = x$ is true on this algebra. Therefore, one can easily conclude that for an arbitrary algebra, $\mathfrak{A} \subseteq \mathfrak{M} : \mathfrak{A} \models \xi$ iff $Sp\mathfrak{A}$ has no other isolated point than, possibly, ∇.

Lemma 5.10.. Let $\mathfrak{M}$ be a finitely-generated discriminator variety. The formulas φ_1, ξ_0, ξ have been constructed above and are such that $\{\mathfrak{A} \in \mathfrak{M} \,|\mathfrak{A} \models \varphi_1\} = \mathfrak{M}_{\max,}$ $\{\mathfrak{A} \in \mathfrak{M} \,|\mathfrak{A} \models \xi_0\} = \mathfrak{A} \in \mathfrak{M} \,|sp\mathfrak{A}$ has no isolated points besides, possibly, the point $\nabla\}$, $\{\mathfrak{A} \in \mathfrak{M} \,|\mathfrak{A} \models \xi\} = \{\mathfrak{A} \in \mathfrak{M} \,|sp\mathfrak{A}$ has no isolated points$\}$.

Let φ_1' be the formula constructed by φ_1 according to lemma **5.2.**. In this case, we have:

(a) the $\mathfrak{M}$ -algebra $\mathfrak{A}$ is algebraically closed iff $\mathfrak{A} \models \varphi_1'$;

(b) when non-singleton $\mathfrak{M}_S$-algebras have one-element subalgebras, the $\mathfrak{M}$ -algebra $\mathfrak{A}$ is existentially closed iff $\mathfrak{A} \models \varphi_1' \& \xi_0$;

(c) if a certain $\mathfrak{M}_S$-algebra has a one-element subalgebra, then the $\mathfrak{M}$ -algebra $\mathfrak{A}$ is existentially closed iff $\mathfrak{A} \models \varphi_1' \& \xi$.

Proof. Since any $\mathfrak{M}_S$-algebra is embeddable into an $\mathfrak{M}_{\max}$-algebra, an arbitrary $\mathfrak{M}$ -algebra $\mathfrak{A}$ can be embedded into an $\mathfrak{M}$ -algebra $\mathfrak{A}'$ so that for any $\psi \in Sp\mathfrak{A}'$, we have $\mathfrak{A}'/\psi \in \mathfrak{M}_{\max}$, i.e. so that $\mathfrak{A}' \models \varphi_1'$.In this case, depending on whether the non-singleton $\mathfrak{M}_{\max}$-algebra with a one-element subalgebra exists or not, $\mathfrak{A}'$ can be chosen in such a way that either the formula ξ or the formula ξ_0, respectively, is valid on $\mathfrak{A}'$. The latter requirements are, in other words, reduced to the fact that the Boolean algebra corresponding to the space $Sp\mathfrak{A}$ be atomless (this can be achieved as any Boolean algebra is embeddable into an atomless one), as well as to the fact that for the case when a one-element algebra is contained as a subalgebra in a certain non-singleton $\mathfrak{M}_{\max}$-algebra, the congruence ∇ on $\mathfrak{A}'$ be not principal. The latter can be achieved, for instance, in the following way: if

$\mathfrak{A}_1 \in \mathfrak{M}_{\max}$, $a \in \mathfrak{A}_1$, and $\{a\}$ is a subalgebra of $\mathfrak{A}_1$, then assume that $\mathfrak{A}_2 \subseteq \mathfrak{A}' \times \mathfrak{A}_1^{\omega}$ and $\mathfrak{A}_2 = \left\{ <b, f> \middle| b \in \mathfrak{A}', f \in \mathfrak{A}_1^{\omega}\right.$, and there is a $k \in \omega$ such that for all $n \geq k$, $f(n) = a\}$. Then $\mathfrak{A}' \subseteq \mathfrak{A}_2$, and $\mathfrak{A}' \models \xi$.

Thus, let $\mathfrak{A}'$ obey the requirements formulated above, and let $\mathfrak{A} \subseteq \mathfrak{A}'$. If $\mathfrak{A}$ is algebraically closed, then the embedding of $\mathfrak{A}$ into $\mathfrak{A}'$ is pure and, since the formula φ_1' has the form $\forall \bar{x} \exists \bar{y} (p(\bar{x}, \bar{y}) = x_1)$, the truth of φ_1' for $\mathfrak{A}'$ results in that of φ_1' for $\mathfrak{A}$. Conversely, if for a certain algebra $\mathfrak{B} \in \mathfrak{M}$ the formula φ_1' holds, then $\mathfrak{B}$ is algebraically closed in $\mathfrak{M}$ by theorem **5.8**. The statements of the lemma related to existentially closed algebras are proved in an analogous way. ■

Lemma **5.10** results in the following statement.

__**Theorem 5.9.**__. Let $\mathfrak{M}$ be a finitely-generated discriminator variety. In this case, we have:

(a). an arbitrary $\mathfrak{M}$-algebra $\mathfrak{A}$ is algebraically closed iff its any non-singleton simple factor-algebra lies in $\mathfrak{M}_{\max}$;

b) an arbitrary $\mathfrak{M}$-algebra $\mathfrak{A}$ is existentially closed iff any of its non-singleton simple factor-algebras lies in $\mathfrak{M}_{\max}$, the Boolean algebra $Con_p\mathfrak{A}$ is atomless, and $\nabla \notin Con_p\mathfrak{A}$ in the case when a certain $\mathfrak{M}_{\max}$-algebra contains a one-element subalgebra.

Before we describe the construction of injective, weakly injective, equationally compact and topologically compact algebras in finitely-generated discriminator varieties, we will require another subsidiary result.

__**Definition 5.5.**__ The algebra $\mathfrak{A}$ is called a subdirect retract of the algebras $\mathfrak{A}_i (i \in I)$ if there exists an embedding f of the algebra $\mathfrak{A}$ into $\prod_{i \in I} \mathfrak{A}_i$, and a homomorphism g from the algebra $\prod_{i \in I} \mathfrak{A}_i$ to $\mathfrak{A}$ such that $g \cdot f$ is identical on $\mathfrak{A}$, and for any $i \in I$, $\pi_i \cdot f$ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_i$.

__**Lemma 5.11.**__ Let $\mathfrak{A}$ be a subdirect retract of the algebras $\mathfrak{A}_i \in \mathfrak{M}_S$ $(i \in I)$ and for any $i \in I |\mathfrak{A}_i| \neq 1$, in which case we have:

(a) if $\mathfrak{K}$ is a certain class of simple $\mathfrak{M}$-algebras elementary in $\mathfrak{M}$, then

$\left\{\varphi \in Sp\mathfrak{A} \mid \mathfrak{A}/_{\varphi} \in \mathfrak{K}\right\}$ is open-closed in $Sp\mathfrak{A}$;

(b) $\mathfrak{T}_0(\mathfrak{A})$ the Boolean algebra of open-closed subsets of the space $Sp_0\mathfrak{A}$ is complete;

(c) if all $\mathfrak{A}/_{\varphi}$ are isomorphic with a certain finite algebra $\mathfrak{A}_1$, then $\mathfrak{A} \cong \mathfrak{A}_1^{\mathfrak{T}_0(\mathfrak{A})}$.

Proof.

(a) Let the mappings f and g be the same as in Definition **5.5**, and let $\mathfrak{C} = \prod_{i \in I} \mathfrak{A}_i$. It should be remarked that $Sp_0\mathfrak{C}$ is exactly the space of all the ultrafilters over the set I, and the Boolean algebra of open-closed subsets of the space $Sp_0\mathfrak{C}$ is the Boolean algebra 2^I of all the subsets of the set I. It should be also remarked that as $\nabla \in Con_p\mathfrak{C}$, ∇ is an isolated point of the space $Sp\mathfrak{C}$.

Let us consider dual mappings $f^+ : Sp\mathfrak{C} \to Sp\mathfrak{A}$ and $g^+ : Sp\mathfrak{A} \to Sp\mathfrak{C}$. In this case, $f^+ \cdot g^+ = (g \cdot f)^+ = (id_{\mathfrak{A}})^+ = id_{Sp\mathfrak{A}}$, $id_{\mathfrak{C}}$ being here an identical mapping on the set C. In particular, f^+ is a mapping "onto", and g^+ is an embedding. Since g is the homomorphism "onto", for any $\Phi \in Sp\mathfrak{A}$ in the following commutative diagram

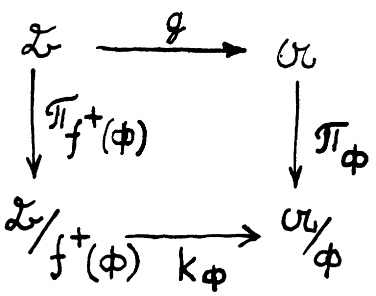

Fig. 5

the canonical homomorphism k_{Φ} will be a mapping "onto", and, since $\mathfrak{C}/_{f^+(\Phi)}$ is simple, k_{Φ} is an isomorphism for $\Phi \in Sp_0\mathfrak{A}$. Therefore,

$$(g^+)^{-1}\left(\left\{\Phi \in Sp\mathfrak{C} \mid \mathfrak{C}/_{\Phi} \in \mathfrak{K}\right\}\right) = \left\{\Phi \in Sp\mathfrak{A} \mid \mathfrak{A}/_{\Phi} \in \mathfrak{K}\right\}.$$

But, according to the above remark on the space $Sp_0\mathfrak{C}$, for any $\Phi \in Sp_0\mathfrak{C}$ we have

$$\mathfrak{C}/_{\Phi} \cong \prod_{i \in I} \mathfrak{A}_i/_{\Phi^*},$$

where Φ^* is an ultrafilter on I corresponding to Φ. Since $\mathfrak{K}$ is an elementary, i.e., a finitely axiomatizable class, by the Los's theorem on ultraproducts [35],

$$\mathfrak{C}/_{\Phi} \cong \prod_{i \in I} \mathfrak{A}_i/_{\Phi^*} \in \mathfrak{K} \text{ iff } \left\{i \in I \mid \mathfrak{A}_i \in \mathfrak{K}\right\} \in \Phi^*.$$

Therefore, $\{\Phi \in Sp_0\mathfrak{C} \mid \mathfrak{C}/_{\Phi} \in \mathfrak{K}\}$ is open-closed in $Sp_0\mathfrak{C}$. As has been remarked above, ∇ is an isolated point of the space $Sp\mathfrak{C}$ and, hence, the set $\{\Phi \in Sp\mathfrak{C} \mid \mathfrak{C}/_{\Phi} \in \mathfrak{K}\}$ is also open-closed in $Sp\mathfrak{C}$. But in this case, $\{\Phi \in Sp\mathfrak{A} \mid \mathfrak{A}/_{\Phi} \in \mathfrak{K}\}$, as a preimage of the latter under a continuous mapping g^+, will also be open-closed in the space $Sp\mathfrak{A}$, which is the proof of the statement (a).

(b) From the diagram in the proof of the statement (a) given above, and from the fact that the mapping k_{Φ} is an isomorphism, one can readily deduce that $(g^+)^{-1}(\nabla_{\mathfrak{C}}) = \nabla_{\mathfrak{A}}$. Since $id_{Sp\mathfrak{A}} = f^+ \cdot g^+$ for the continuous mappings $f^+ : Sp\mathfrak{C} \to Sp\mathfrak{A}$ and $g^+ : Sp\mathfrak{A} \to Sp\mathfrak{C}$, all the open-closed subsets of the space $Sp\mathfrak{A}$ have the form $(g^+)^{-1}(\mathfrak{N})$, where $\mathfrak{N}$ is open-closed in $Sp\mathfrak{C}$. If we take into account that $\nabla_{\mathfrak{C}}, \nabla_{\mathfrak{A}}$ are isolated points in $Sp\mathfrak{C}$, $Sp\mathfrak{A}$, respectively, as well as the above-mentioned equality $(g^+)^{-1}(\nabla_{\mathfrak{C}}) = \nabla_{\mathfrak{A}}$ for the Boolean algebras $\mathfrak{T}_0(\mathfrak{A}), \mathfrak{T}_0(\mathfrak{C})$, which are open-closed subsets of the spaces $Sp_0\mathfrak{A}, Sp_0\mathfrak{C}$, then for the mappings $f^* = (f^+)^{-1}, g^* = (g^+)^{-1}$, f^* maps $\mathfrak{T}_0(\mathfrak{A})$ into $\mathfrak{T}_0(\mathfrak{C})$, while g^* maps $\mathfrak{T}_0(\mathfrak{C})$ into $\mathfrak{T}_0(\mathfrak{A})$. In this case, $\mathfrak{T}_0(\mathfrak{A})$ proves to be a retract of the Boolean algebra $\mathfrak{T}_0(\mathfrak{C})$ as $f^+ \cdot g^+ = id_{Sp\mathfrak{A}}$. As we have noted in the proof of the statement (a) above, $\mathfrak{T}_0(\mathfrak{C})$ is isomorphic to the Boolean algebra of all the subsets of the set I which, in particular, implies that it is complete. In this case, $\mathfrak{T}_0(\mathfrak{A})$ is, as a retract of a complete Boolean algebra, also complete.

(c) Let $\mathfrak{A}_i \cong \mathfrak{A}_1$ for any $i \in I$, where $\mathfrak{A}_1$ is a certain finite algebra. In this case, one can assume $\mathfrak{C} = \mathfrak{A}_1^I$. Since for any $\Phi \in Sp\mathfrak{C}$

$$\mathfrak{C}/_{\Phi} \cong \mathfrak{A}_1^I/_{\Phi^*},$$

where Φ^* is an ultrafilter on I corresponding to Φ, and $\mathfrak{A}_1^I/_{\Phi^*} \cong \mathfrak{A}_1$, for any $\Phi \in Sp_0\mathfrak{C}$, we have $\mathfrak{C}/_{\Phi} \cong \mathfrak{A}_1$. The proof of the statement (a) entails that for any $\Phi \in Sp\mathfrak{A}$, we have $\mathfrak{A}/_{\Phi} \cong \mathfrak{A}_1$.

The algebra $\mathfrak{A}$ is isomorphic to the Boolean product of algebras $\{\mathfrak{A}_{\Phi} | \Phi \in Sp\mathfrak{A}\} = \{\mathfrak{A}_1$, a one-element algebra$\}$ by the Boolean space $Sp\mathfrak{A}$. Besides, $\mathfrak{A}/_{\Phi}$ is singleton iff $\Phi = \nabla$, in which case ∇ is an isolated point in $Sp_0\mathfrak{A}$. Therefore, the algebra $\mathfrak{A}$ is the Boolean product of the algebra $\mathfrak{A}_1$ by the Boolean space $Sp\mathfrak{A}$. One can easily check that the conditions of lemma **3.4** are satisfied and, hence, we have $\mathfrak{A} \cong \mathfrak{A}_1^{\mathfrak{B}}$ for some algebra $\mathfrak{B}$. The fact that $\mathfrak{B} = \mathfrak{C}_0(\mathfrak{A})$ can also be checked easily. ■

Theorem 5.10. Let $\mathfrak{M}$ be a finitely generated discriminator variety. In this case,

(a) $\mathfrak{A} \in \mathfrak{M}$ is equationally compact iff $\mathfrak{A} \cong \mathfrak{A}_1^{\mathfrak{B}_1} \times \ldots \times \mathfrak{A}_n^{\mathfrak{B}_n}$, where $\mathfrak{A}_i \in \mathfrak{M}_S$ and $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ are complete Boolean algebras;

(b) $\mathfrak{A} \in \mathfrak{M}$ is weakly injective in $\mathfrak{M}$ iff $\mathfrak{A} \in \mathfrak{A}_1^{\mathfrak{B}_1} \times \ldots \times \mathfrak{A}_n^{\mathfrak{B}_n}$, where $\mathfrak{A}_i \in \mathfrak{M}_{\max}$ and $\mathfrak{B}_1, \ldots \mathfrak{B}_n$ are complete Boolean algebras;

(c) $\mathfrak{A} \in \mathfrak{M}$ is topologically compact iff we have $\mathfrak{A} \cong \mathfrak{A}_1^{I_1} \times \ldots \times \mathfrak{A}_n^{I_n}$ for some sets $I_1, \ldots, I_n$ and some algebras $\mathfrak{A}_i \in \mathfrak{M}_S$;

(d) $\mathfrak{A} \in \mathfrak{M}$ is injective in $\mathfrak{M}$ iff $\mathfrak{A} \in \mathfrak{A}_1^{\mathfrak{B}_1} \times \ldots \times \mathfrak{A}_n^{\mathfrak{B}_n}$, where $\mathfrak{A}_i \in \mathfrak{M}_{\max}$, any isomorphism between non-singleton subalgebras of the algebras $\mathfrak{A}_i$ can be extended to the automorphisms of $\mathfrak{A}_i$, any of the algebras $\mathfrak{A}_i$ has no non-singleton subalgebras isomorphic to the subalgebras of any $\mathfrak{M}_{\max}$-algebras other than $\mathfrak{A}_i$ and the Boolean algebras $\mathfrak{B}_1, \ldots \mathfrak{B}_n$ are complete.

Proof.

(a) Let $\mathfrak{A}$ be equationally compact. By lemma **5.5**, the embedding of the algebra $\mathfrak{A}$ into the algebra $\prod_{\Phi \in Sp\mathfrak{A}} \mathfrak{A}/_{\Phi}$ induced by the representation of the algebra $\mathfrak{A}$

as a Boolean product of the algebras $\mathfrak{A}/_{\Phi}(\Phi \in Sp\mathfrak{A})$ is pure. Therefore, the algebra $\mathfrak{A}$ is a retract of the algebra

$$\prod_{\Phi \in Sp\mathfrak{A}} \mathfrak{A}/_{\Phi} \cong \prod_{\Phi \in Sp_0\mathfrak{A}} \mathfrak{A}/_{\Phi}.$$

Indeed, the algebra $\mathfrak{A}$ is a subdirect retract of the algebras $\mathfrak{A}/_{\Phi}(\Phi \in Sp_0\mathfrak{A})$. In this case, for any algebra $\mathfrak{A}_i \in \mathfrak{M}_S$, since $\mathfrak{A}_i$ is finite, $\{\mathfrak{A}_i\}$ is an elementary class, and, by lemma **5.11(a)**, $\{\Phi \in Sp\mathfrak{A} \mid \mathfrak{A}/_{\Phi} \cong \mathfrak{A}_1\}$ is open-closed in $Sp\mathfrak{A}$. Since $\mathfrak{M}_S$ is a finite set, there can be found various $\mathfrak{A}_1,\dots,\mathfrak{A}_n \in \mathfrak{M}_S$ such that the family $\{\Phi \in Sp\mathfrak{A} \mid \mathfrak{A}/_{\Phi} \cong \mathfrak{A}_i\}$ $(i \le n)$ is an open-closed division of the space $Sp\mathfrak{A}$. Therefore, since $\mathfrak{A}$ is isomorphic to the Boolean product of the algebras $\mathfrak{A}/_{\Phi}$ by the Boolean space $Sp\mathfrak{A}$, we observe that $\mathfrak{A} \cong \prod_{i \le n} \mathfrak{C}_i$, where $\mathfrak{C}_i$ is the Boolean product of the algebras $\mathfrak{A}/_{\Phi}$ by the Boolean space $T_i = \{\Phi \in Sp\mathfrak{A} \mid \mathfrak{A}/_{\Phi} \cong \mathfrak{A}_i\}$. By lemma **5.11(c)**, $\mathfrak{C}_i \cong \mathfrak{A}_i^{\mathfrak{C}_0(\mathfrak{C}_i)}$. Besides, since by lemma **5.11(b)**, $\mathfrak{C}_0(\mathfrak{A})$ is a complete Boolean algebra, $\mathfrak{C}_0(T_i)$ will also be complete. Thus, $\mathfrak{A} \cong \mathfrak{A}_1^{\mathfrak{C}_0(T_1)} \times \dots \times \mathfrak{A}_n^{\mathfrak{C}_0(T_n)}$, and the Boolean algebras $\mathfrak{C}_0(T_i)$ are complete. Therefore, the statement (a) has been proved in one direction, while in the other direction it can be checked directly.

(b) Let $\mathfrak{A}$ be weakly-injective. As $\mathfrak{A}$ is equationally compact in this case, it follows that, according to (a), $\mathfrak{A} \cong \mathfrak{A}_1^{\mathfrak{B}_1} \times \dots \times \mathfrak{A}_n^{\mathfrak{B}_n}$ for certain $\mathfrak{A}_i \in \mathfrak{M}_S$ and complete Boolean algebras $\mathfrak{B}_i$. Let us assume that, for instance, $\mathfrak{A}_1 \notin \mathfrak{M}_{\max}$ and $\mathfrak{A}_0 \in \mathfrak{M}_{\max}$, $\mathfrak{A}_0 \supset \mathfrak{A}_1$. In this case, one can easily see that the algebra $\mathfrak{A}$ is not a retract of the algebra $\mathfrak{A}_0^{\mathfrak{B}_1} \times \mathfrak{A}_2^{\mathfrak{B}_2} \times \dots \times \mathfrak{A}_n^{\mathfrak{B}_n}$, which contradicts the weak injectivity of $\mathfrak{A}$. Therefore, all the algebras $\mathfrak{A}_i$ are to lie in $\mathfrak{M}_{\max}$. One can directly check the converse statement, i.e., the one that if $\mathfrak{A} = \mathfrak{A}_1^{\mathfrak{B}_1} \times \dots \times \mathfrak{A}_n^{\mathfrak{B}_n}$, $\mathfrak{A}_i \in \mathfrak{M}_{\max}$, and $\mathfrak{B}_i$ are complete Boolean algebras, then $\mathfrak{A}$ is weakly-injective in $\mathfrak{M}$.

(c) Since a homomorphic image of an algebra injective in $\mathfrak{M}$ is injective, it follows that, by lemma **5.8**(b), for any $\Phi \in Sp\mathfrak{A}$ where $\mathfrak{A}$ is an algebra injective in $\mathfrak{M}$, $\mathfrak{A}/_{\Phi}$ satisfies the condition b(2) of the same lemma, i.e., $\mathfrak{A}/_{\Phi} \in \mathfrak{M}_{\max}$, any isomorphism between non-singleton subalgebras of the algebra $\mathfrak{A}/_{\Phi}$ can be extended to the automorphism of the algebra $\mathfrak{A}/_{\Phi}$, and $\mathfrak{A}/_{\Phi}$ has no non-singleton subalgebras isomorphic with subalgebras of some $\mathfrak{M}_{\max}$-algebras other than $\mathfrak{A}/_{\Phi}$. On the other hand, since $\mathfrak{A}$ is weakly injective in $\mathfrak{M}$, it follws that, according to the statement (b), $\mathfrak{A} \cong \mathfrak{A}_1^{\mathfrak{B}_1} \times \dots \times \mathfrak{A}_n^{\mathfrak{B}_n}$, where $\mathfrak{A}_i = \mathfrak{A}/_{\Phi_i}$ for some $\Phi_i \in Sp_0\mathfrak{A}$, and $\mathfrak{B}_i$ are complete Boolean algebras. As with the above proof related to the algebras $\mathfrak{A}/_{\Phi_i}$, the statement (d) is proved in one direction, and its proof in the other direction is easily obtained.

(c) Let the algebra $\mathfrak{A} \in \mathfrak{M}$ possess a compact Hausdorff topology which agrees with the algebraic structure of $\mathfrak{A}$, i.e., let $\mathfrak{A}$ be topologically compact. Since, obviously, $\mathfrak{A}$ will also be existentially compact in this case, it follws that, according to the statement (a), $\mathfrak{A} \cong \mathfrak{A}_1^{\mathcal{B}_1} \times \ldots \times \mathfrak{A}_n^{\mathcal{B}_n}$, where $\mathfrak{A}_i \in \mathfrak{M}_S$ and $\mathcal{B}_i$ are complete Boolean algebras. The topological compactness of $\mathfrak{A}$ implies that of each of the algebras of the type $\mathfrak{A}_i^{\mathcal{B}_i}$. Let us fix a certain element h in the algebra $\mathfrak{A}_i^{\mathcal{B}_i}$, and let us define the following arbitrary operations on $\mathfrak{A}_i^{\mathcal{B}_i}$ for any $f, g \in \mathfrak{A}_i^{\mathcal{B}_i}$:

$$f \wedge g = t(h, t(h, f, g), g),$$
$$f \vee g = t(f, h, g)$$

and

$$f \setminus g = t(h, g, t).$$

Let $(\mathfrak{A}_i^{\mathcal{B}_i})^*$ be an algebra given on the basic set of the algebra $\mathfrak{A}_i^{\mathcal{B}_i}$, its signature consisting of the functions $\wedge, \vee, \setminus$ defined in a similar way. One can easily check that the mapping $d: (\mathfrak{A}_i^{\mathcal{B}_i})^* \to \mathfrak{C}_0(\mathfrak{A})$, defined as $d(f) = \{\Phi \in Sp_0\mathfrak{A} \mid < f, h > \notin \Phi\}$ is a homomorphism from the algebra $(\mathfrak{A}_i^{\mathcal{B}_i})^*$ to the Boolean algebra $\mathfrak{C}_0(\mathfrak{A})$. The topological compactness of the algebra $\mathfrak{A}_i^{\mathcal{B}_i}$ implies that of the algebra $(\mathfrak{A}_i^{\mathcal{B}_i})^*$ and, hence, that of the homomorphic image of the algebra $\mathfrak{C}_0(\mathfrak{A}) \cong \mathcal{B}_i$. But, as is well known (see, for instance, [39]), the only topologically compact Boolean algebras are those of the type 2^I of all the subsets of a set. Therefore, for any $i \leq n$ there is a set I_i such that $\mathcal{B}_i \cong 2^{I_i}$, in which case $\mathfrak{A}_i^{\mathcal{B}_i} \cong \mathfrak{A}_i^{I_i}$ (as $\mathfrak{A}_i$ is finite). Thus, $\mathfrak{A} \cong \mathfrak{A}_1^{I_1} \times \ldots \times \mathfrak{A}_n^{I_n}$, and the statement (c) is proved in one direction. The converse statement is obvious: it suffices to choose a Tikhonov's topology of the product on the algebra $\mathfrak{A}_1^{I_1} \times \ldots \times \mathfrak{A}_n^{I_n}$ when choosing a discrete topology on every $\mathfrak{A}_i \in \mathfrak{M}_S$. ■

By way of concluding this section, let us obtain one more characteristic feature of discriminator varieties in addition to theorems **2.15** and **5.5** which will be required below in section **7**. But first let us prove a lemma.

<u>**Lemma 5.12.**</u> For any algebra $\mathfrak{A}$ of a congruence-distributive variety which is a Boolean product of simple algebras, for any principal congruence $\theta^{\mathfrak{A}}_{a,b}$ there is

a complement to $\theta^{\mathfrak{A}}_{a,b}$ in the lattice $Con\mathfrak{A}$.

Proof. Assume that $\mathfrak{A} \subseteq \prod_{i \in \mathfrak{B}^*} \mathfrak{A}_i$ is a Boolean product of simple algebras $\mathfrak{A}_i$ with a Boolean power $\mathfrak{B}$ .Let $a,b \in \mathfrak{A}$ and $i \in \lfloor a=b \rfloor\!\rfloor$. Using the conditions of defining a Boolean product in a standard way, one can find an element $c_i \in \mathfrak{A}$ such that $\lfloor a=c_i \rfloor\!\rfloor \supseteq [\![a \neq b]\!]$ and $i \in \lfloor a \neq c_i \rfloor\!\rfloor$. But in this case, by virtue of the description of the principal congruences on Boolean products in congruence-distributive varieties from theorem **4.2**, the congruence $\bigvee_{i \in [\![a=b\rfloor} \theta^{\mathfrak{A}}_{a,c_i}$ will be an addition to the congruence $\theta^{\mathfrak{A}}_{a,b}$ in the lattice $Con\mathfrak{A}$. ∎

<u>**Theorem 5.11.**</u> The variety $\mathfrak{M}$ is a discriminator variety iff it is arithmetic, and for any $\mathfrak{M}$ -algebra $\mathfrak{A}$ any principal congruence $\theta^{\mathfrak{A}}_{a,b}$ on the algebra $\mathfrak{A}$ has a complement in $Con\mathfrak{A}$.

Proof. The fact that a discriminator variety is arithmetic has been noted in section **2**, and it can also be proved easily using theorem **2.10.**. The existence of complements of the principal congruences of discriminator varieties results from lemma **5.12.**. Therefore, the theorem is proved in one direction.

Let now $\mathfrak{M}$ be arithmetic, and let the principal congruences have complements in $Con\mathfrak{A}$ for any $\mathfrak{A} \in \mathfrak{M}$. Let us prove that $\mathfrak{M}$ is a discriminator variety. Let X be an arbitrary infinite set, and x,y,z,u be various elements not incorporated into X. Let us set $X_1 = X \cup \{x,y,z,u\}$. As, by the condition, $\mathfrak{F}_{\mathfrak{M}}(X_1)$ is arithmetic, we get:

$$\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} = \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge (\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y} \vee \gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}) = (\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}),$$
$$(\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}) = (\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}) \cdot (\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}),$$

where $\gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}$ is the complement of $\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}$ in $Con\mathfrak{F}_{\mathfrak{M}}(X_1)$. As $< z,u > \in \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u}$, it follows that, according to the equality given above, there is a term $g(x,y,z,u,x_1,\dots x_m)$ such that

$$< z, g(x,y,z,u,x_1,\dots x_m) > \in \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}$$

and

$$< g(x,y,z,u,x_1,\dots x_m), u > \in \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}.$$

Let $g(x,y,z,u) = g(x,y,z,u,x,\dots,x)$, i.e., $g(x,y,z,u)$ is a homomorphic image of the element $g(x,y,z,u,x_1,\dots,x_m) \in \mathfrak{F}_{\mathfrak{M}}(X_1)$ under the homomorphism $\mathfrak{F}_{\mathfrak{M}}(X_1)$ on $\mathfrak{F}_{\mathfrak{M}}(x,y,z,u) = \mathfrak{A}$ which is an induced mapping of X in x and identical on $\{x,y,z,u\}$. Then, by corollary **2.1.**,

$$< z,g(x,y,z,u) > \theta^{\mathfrak{A}}_{z,u} \wedge \theta^{\mathfrak{A}}_{x,y}, < g(x,y,z,u),u > \in \theta^{\mathfrak{A}}_{z,u} \wedge \gamma^{\mathfrak{A}}_{x,y}.$$

Considering the homomorphism $\mathfrak{F}_{\mathfrak{M}}(x,y,z,u) = \mathfrak{A}$ on $\mathfrak{F}_{\mathfrak{M}}(x,y,u)$ induced by the mapping $\{x,y\}$ in x and identical on $\{z,u\}$, we again get, by corollary **2.1.**, the equality $z = g(x,x,z,u)$. Therefore, the identity $z = g(x,x,z,u)$ is true on $\mathfrak{M}$.

Let us now show that for any $\mathfrak{M}_{SI}$-algebra $\mathfrak{C}$ and $a,b,c,d \in \mathfrak{C}$, $g(a,b,c,d) = d$ if $a \neq b$. Let us, first of all, notice that the condition of the existence of a complement for any principal congruence in the lattices of the congruences of the $\mathfrak{M}$-algebras evidently implies the simplicity of any subdirectly non-decomposable algebra in $\mathfrak{M}$. If $|X| \geq \mathfrak{C}$, then let φ be a homomorphism of $\mathfrak{F}_{\mathfrak{M}}(X_1)$ induced by the mapping $X \cup \{x\}$ in a, y in b, z in c, u in d, and $\psi = ket\varphi$. In this case, since $\mathfrak{C}$ is simple, ψ is a maximal congruence in $Con\mathfrak{F}_{\mathfrak{M}}(X_1)$. As $\mathfrak{M}$ is arithmetic, we have the equality

$$\psi = \psi \vee (\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y} \wedge \gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y} = (\psi \vee \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y} \wedge (\psi \vee \gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}).$$

Since $a \neq b$, $\theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y} \nleq \psi$, i.e., $\psi \vee \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y} = \nabla_{\mathfrak{F}_{\mathfrak{M}}(X_1)}$ and, hence, $\gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y} \leq \psi$. But $< g(x,y,z,u),u > \in \gamma^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}$ and, hence, $< g(x,y,z,u),u > \in \gamma$, i.e., $g(a,b,c,d) = d$.

Assume now that

$$F_X = \{h(x,y,z,u,x_1,\dots,x_{m(h)}) \in \mathfrak{F}_{\mathfrak{M}}(X_1) |$$
$$< z,h(x,y,z,u,x_1,\dots,x_{m(h)} > \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}$$

and

$$< h(x,y,z,u,x_1,\dots,x_{m(h)}),u > \in \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{z,u} \wedge \theta^{\mathfrak{F}_{\mathfrak{M}}(X_1)}_{x,y}\}.$$

It is obvious that the inclusion $F_y \subseteq F_X$ is valid for $X \subseteq Y$ and, since $F_X \neq \emptyset$ for any X, there is a $g(x,y,z,u,x_1,\dots x_m) \in F_X$ for arbitrarily large sets X. As has been proved above, the identity $z = g(x,x,z,u)$ is true for such a $g(x,y,z,u)$ on $\mathfrak{M}$, and for any $\mathfrak{M}_{SI}$-algebra of $\mathfrak{C}$ and $a,b,c,d \in \mathfrak{C}$, if $a \neq b$, then $g(a,b,c,d) = d$. Therefore, the term $h(x,y,z) = g(x,y,z,z)$ is a discriminator on $\mathfrak{M}_{SI}$-algebras, i.e.,

$\mathfrak{M}$ is a discriminator variety. ■

Priorities. Sufficient information on discriminator varieties can be found in the H.Werner's monograph specially devoted to this class of varieties [236], which also considers some examples of discriminator varieties, both the ones cited in the present section and a number of others. Theorem **5.1.** is by G. Michler and R.Wille [139], theorem **15.2** belongs to G.L.Mursky [144]. Theorems **5.3** and **5.4** are contained in the work by A.F.Pixley[188] (see also [161], [238]). Theorem **5.5** can be found in a work by W.J.Blok and D.Pigozzi [18], theorem **5.6** in a paper by E.Fried, G.Grätzer and R.Quackenbush [73]. Theorem **5.7** for finitely generated varieties has been proved by K.Keimel and H.Werner [102], while in the total completion it was achieved by S. Bulman-Fleming and by H. Werner [22]. The statement of corollary **5.2** belongs to R.W. Quackernbush [196]. The remaining material of the section, except lemma **5.12** and theorem **5.11** is from the material of the H. Werner's monograph [236]. Theorem **5.11** is by E. Fried and E.W.Kiss [74].

6. Direct Presentation of a Variety and Algebras with a Minimal Spectrum

The present section is devoted to the description of the so-called directly presentable varieties and algebras with a minimal spectrum. Though the description of these varieties has interest of its own, in the context of the present chapter these results play an auxiliary role for the description of varieties representable with Boolean constructions.

Definition 6.1. A variety is termed directly presentable iff it is finitely generated and contains (to the accuracy of isomorphism) only a finite set of finite directly non-decomposable algebras.

Therefore, according to the definitions given in the end of section **2**, a finite ring R has a finite type of representations iff the variety of R-modules is directly presentable. On the other hand, a finitely generated variety $\mathfrak{M}$ is directly presentable iff any finite $\mathfrak{M}$-algebra is representable as a Boolean product (see lemma **4.2**) of the algebras of a certain fixed finite set of directly non-decomposable $\mathfrak{M}$-algebras. Theorem **6.3** given below describes directly presentable congruence-modular varieties relative to the module of directly representable Abelian varieties, i.e., by theorem

2.20, relative to the module of the varieties which are polynomially equivalent to those of R-modules, where R is a finite ring with a finite type of representations.

Let us consider some auxiliary lemmas and theorems before we start proving this theorem.

Definition 6.2. If $\mathfrak{K}$ is a certain class of algebras, then $\Pr(\mathfrak{K})$ will denote a set of simple divisors of the powers of finite $\mathfrak{K}$-algebras. The class $\mathfrak{K}$ is called narrow if $\Pr(\mathfrak{K})$ is finite.

Lemma 6.1. If for a certain finite algebra $\mathfrak{A}$ the class $SP(\mathfrak{A})$ is narrow, then $\mathfrak{A}$ is congruence-homogenous.

Proof. Let $\theta \in Con\mathfrak{A}$ and $a_1,\ldots,a_k$ be the powers of all θ-classes on $\mathfrak{A}$. Let us prove that $a_1=\ldots=a_k$. For any $n\geq 1$ let $s_n(\bar{a})=a_1^n+\ldots+a_k^n$, where $\bar{a}=<a_1,\ldots,a_k>$. Let

$$\mathfrak{A}_n=\mathfrak{A}^n(\theta)=\{f\in\mathfrak{A}^n |\text{ for all } i,j \ <f(i),f(j)>\in\theta\}.$$

Then we have $\mathfrak{A}_n\in SP(\mathfrak{A})$, and it is obvious that $|\mathfrak{A}_n|=s_n(\bar{a})$. Therefore, any simple divisor of the number $s_n(\bar{a})$ belongs to $\Pr(SP(\mathfrak{A}))$. To complete the proof it suffices to demonstrate that if $\Pr(\{\mathfrak{A}_n | n\in\omega\})$ is finite for any positive natural numbers $a_1,\ldots,a_k$, then $a_1=\ldots=a_k$. This theoretico-numerical result belongs to Polya, and for the sake of completeness of representation let us give its proof here.

Let $\Pr(\{\mathfrak{A}_n | n\in\omega\})=\{p_1,\ldots,p_t\}$. Let us divide all a_i by $GCD(a_i | i\leq k)$ (GCD stands for greatest common divisor), and one can obviously assume, with the generality preserved, that $GCD(a_i | i\leq k)=1$. Let $m=(p-1)p^k$ and $b_i=a_i^m$ for any simple p. Then p^{k+1} divides none of $s_n(\bar{b})$ at $n\geq 1$. Indeed, by the Fermat theorem $a_i^{p-1}\equiv 0$, or $\equiv 1(\bmod p)$, so $a_l^m\equiv 0$, or $\equiv 1(\bmod p^{k+1})$, respectively,. Let α be a number of different $i\leq k$ such that p does not divide α_i. In this case, $1\leq\alpha\leq k$, since $GCD(a_i | i\leq k)=1$. For $n\geq 1$ we have

$$s_n(\bar{b})=\sum_{i=1}^{k} a_i^{mn}\equiv\alpha(\bmod p^{k+1}),$$

and, since $1\leq\alpha\leq k<p^{k+1}$, indeed, p^{k+1} does not divide $s_n(\bar{b})$.

Let now $m=\prod_{i=1}^{t}(p_i-1)p_i^k$. For any $n\geq 1$, p_i^{k+1} does not divide $s_{mn}(\bar{a})$.

Therefore, since $\{p_1,...,p_t\}$ are all simple divisors of the numbers of the type $s_l(\overline{a})$, for all $n \geq 1$ we have $s_{mn}(\overline{a}) \leq \left(\prod_{i=1}^{t} p_i\right)^k$. Hence, the sequence $< s_{mn}(\overline{a}) | n \in \omega >$ is bounded, which is possible only when $a_1,...,a_k = 1$. ■

Lemma 6.2. If $S(\mathfrak{A}^2)$ is a class of congruence-uniform algebras, then $\mathfrak{A}$ is a congruence-permutable algebra.

Proof. Let $\gamma, \theta \in Con\mathfrak{A}$. It suffices to demonstrate that $\theta \cdot \gamma \subseteq \gamma \cdot \theta$. The relation $\gamma \cdot \theta$ can be obviously viewed as a certain subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}^2$. Let us define the congruence θ' on $\mathfrak{B}$ in the following way: $<< x,y >,< u,v >> \in \theta'$ iff $< x,u > \in \theta$ and $< y,v > \in \theta$. Since, as a diagonal algebra, $\mathfrak{A} \in IS(\mathfrak{A}^2)$, then both θ on $\mathfrak{A}$ and θ' on $\mathfrak{B}$ are uniform. Let r be the power of the θ-classes on $\mathfrak{A}$, and s be that of the θ'-classes on $\mathfrak{B}$. For any $a \in \mathfrak{A}$ we have $\left(a/_\theta\right) \times \left(a/_\theta\right) \subseteq \mathfrak{B}$, and, hence, $s = \left| < a,a > /_{\theta'} \right| = r^2$. Therefore, for all $< a,b > \in \mathfrak{B}$ we have $\left(a/_\theta\right) \times \left(b/_\theta\right) \subseteq \mathfrak{B}$, or, otherwise, $\left| < a,b > /_{\theta'} \right| < r^2$. The inclusion $\left(a/_\theta\right) \times \left(b/_\theta\right) \subseteq \mathfrak{B}$ implies the inclusion $\theta \cdot \gamma \cdot \theta \cdot \theta \leq \gamma \cdot \theta$, or $\theta \cdot \gamma \leq \gamma \cdot \theta$, which is the required proof. ■

Theorem 6.1. If $\mathfrak{K}$ is a finite set of finite algebras and $\mathfrak{M}(\mathfrak{K})$ is a directly representable variety, $\mathfrak{M}(\mathfrak{K})$ is congruence-permutable.

Proof. Being directly representable, the variety $\mathfrak{M}(\mathfrak{K})$ is narrow. Besides, since for any finite algebra $\mathfrak{A} \in SP(\mathfrak{K})$ we have $SP(\mathfrak{A}) \subseteq SP(\mathfrak{K})$, the class $SP(\mathfrak{A})$ will also be narrow for any finite $\mathfrak{A} \in SP(\mathfrak{K})$. By lemma **6.1**, any finite $\mathfrak{A} \in SP(\mathfrak{K})$ is congruence-uniform. As for any finite algebra $\mathfrak{B} \in \mathfrak{M}(\mathfrak{K})$ there exists a finite algebra $\mathfrak{A} \in SP(\mathfrak{K})$ such that $\mathfrak{B} \in H(\mathfrak{A})$, and since the property of congruence-uniformity is preserved under homomorphisms, any finite $\mathfrak{M}(\mathfrak{K})$-algebra is congruence-uniform. Therefore, for any finite $\mathfrak{A} \in \mathfrak{M}(\mathfrak{K})$ any algebra $\mathfrak{B} \in S(\mathfrak{A}^2)$ is congruence-uniform. By lemma **6.2**, any finite $\mathfrak{A} \in \mathfrak{M}(\mathfrak{K})$ is congruence-permutable. As $\mathfrak{M}(\mathfrak{K})$ is finitely generated, $\mathfrak{F}_{\mathfrak{M}(\mathfrak{K})}(3)$ is a finite algebra and hence, in particular, $\mathfrak{F}_{\mathfrak{M}(\mathfrak{K})}(3)$ is congruence-permutable. By theorem **2.5**, this means that the whole variety $\mathfrak{M}(\mathfrak{K})$ is congruence-permutable. ■

Lemma 6.3. If $\mathfrak{B}$ is a finite algebra and if any finitely generated subalgebra of a certain algebra $\mathfrak{C}$ is contained in the class $P_S HS(\mathfrak{B})$, then $\mathfrak{C}$ itself is contained in this class.

Proof. Since any finitely generated subalgebra of the algebra $\mathfrak{C}$ is embeddable into a locally finite class $P_S HS(\mathfrak{B})$, $\mathfrak{C}$ itself is locally finite. Let $\overline{\mathfrak{D}} = \{\mathfrak{D} \mid \mathfrak{D}$ is a finite subalgebra of the algebra $\mathfrak{C}\}$, then $<\overline{\mathfrak{D}};\subseteq>$ is a directed set. By the condition of the lemma, for any $\mathfrak{D} \in \overline{\mathfrak{D}}$ there is a $\mathfrak{D}' \in P_S HS(\mathfrak{B})$ such that $\mathfrak{D} \cong \mathfrak{D}'$. Let $\mathfrak{D}' \subseteq \prod_{i \in I_{\mathfrak{D}}} \mathfrak{U}_i^{\mathfrak{D}}$, where $\mathfrak{U}_i^{\mathfrak{D}} \in HS(\mathfrak{B})$. As $HS(\mathfrak{B})$ is finite, and $<\overline{\mathfrak{D}};\subseteq>$ is directed, choosing in the opposite case the co-final subset $<\overline{\mathfrak{D}};\subseteq>$, one can assume that $\left\{\mathfrak{U}_i^{\mathfrak{D}} \mid i \in I_{\mathfrak{D}}\right\}$ is the same for all $\mathfrak{D} \in \overline{\mathfrak{D}}$, and let it be equal to $\left\{\mathfrak{U}_1,\dots,\mathfrak{U}_n\right\}$. It is also obvious that we can assume that $I_{\mathfrak{D}} = I$ for all $\mathfrak{D} \in \overline{\mathfrak{D}}$, and that for any $j \le n$ $\{i \mid i \in I_{\mathfrak{D}}, \mathfrak{U}_i^{\mathfrak{D}} = \mathfrak{U}_j\}$ is not changeable for all $\mathfrak{D} \in \overline{\mathfrak{D}}$.

Let $\{i \mid i \in I_{\mathfrak{D}}, \mathfrak{U}_i^{\mathfrak{D}} = \mathfrak{U}_j\} = I_j$, and k_j be a certain fixed element of I_j. $T(\mathfrak{D})$ will denote the family of all the isomorphisms from the algebra $\mathfrak{D}$ to $\mathfrak{D}'$. The set $T(\mathfrak{D})$ is finite. For $\mathfrak{D}_1,\mathfrak{D}_2 \in \mathfrak{D}'$ and $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$, $\psi_{\mathfrak{D}_1}^{\mathfrak{D}_2}$ will denote the mapping from $T(\mathfrak{D}_2)$ to $T(\mathfrak{D}_1)$ such that for $f \in T(\mathfrak{D}_2)$, $h_{\mathfrak{D}_1}^{\mathfrak{D}_2}\left(\psi_{\mathfrak{D}_1}^{\mathfrak{D}_2}(f)\right)$ is the restriction of f to the algebra $\mathfrak{D}_1$ (here $h_{\mathfrak{D}_1}^{\mathfrak{D}_2}$ is an embedding of $\mathfrak{D}_1'$ into $\mathfrak{D}_2'$ fixed for the pair $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$.

Let us consider the inverse spectrum φ of the finite sets $\{T(\mathfrak{D}) \mid \mathfrak{D} \in \overline{\mathfrak{D}}\}$ and the mappings between them $\{\psi_{\mathfrak{D}_1}^{\mathfrak{D}_2} \mid \mathfrak{D}_1,\mathfrak{D}_2 \in \overline{\mathfrak{D}}, \mathfrak{D}_1 \subseteq \mathfrak{D}_2\}$. By a well-known theorem (see, for instance, [59]), there is a non-empty inverse limit $\varprojlim \varphi$ of this spectrum. Let $g \in \varprojlim \varphi$, i.e., $g \in \prod_{\mathfrak{D} \in \mathfrak{D}} T(\mathfrak{D})$, in which case for $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$ we have $\varphi_{\mathfrak{D}_1}^{\mathfrak{D}_2}(g(\mathfrak{D}_2)) = g(\mathfrak{D}_1)$. Let us define the mapping h of the algebra $\mathfrak{C}$ in $\prod_{\mathfrak{D} \in \mathfrak{D}} \mathfrak{D}' \subseteq \prod_{\mathfrak{D} \in \mathfrak{D}} (\prod_{i \in I} \mathfrak{U}_i^{\mathfrak{D}})$, where $\mathfrak{U}_i^{\mathfrak{D}} \in HS(\mathfrak{B})$, in the following way. For any $c \in \mathfrak{C}$ and any $\mathfrak{D} \in \overline{\mathfrak{D}}$, if $c \in \mathfrak{D}$, then $h(c)(\mathfrak{D}) = g(\mathfrak{D})(c)$. Since $\left\{h(c)(\mathfrak{D})(k_j) \mid \mathfrak{D} \in \overline{\mathfrak{D}}, c \in \mathfrak{D}\right\}$ is finite for $j \le n$, let $b_j(c)$ be such that for the co-final subset A_j of the set $<\overline{\mathfrak{D}};\subseteq>$, we have $h(c)(\mathfrak{D})(k_j) = b_j(c)$ for $\mathfrak{D} \in A_j, c \in \mathfrak{D}$. In the case when $c \notin \mathfrak{D}, \mathfrak{D} \in \overline{\mathfrak{D}}$, let us set $h(c)(\mathfrak{D})(i) = b_j(c)$ for any $i \in I_j$. By the definition of h we can directly check that h is an isomorphic embedding of $\mathfrak{C}$ into $\prod_{\mathfrak{D} \in \mathfrak{D}} (\prod_{i \in I} \mathfrak{U}_i^{\mathfrak{D}})$, and that all the projections $h(\mathfrak{C})$ on the co-factors of the type $\mathfrak{A}_i^{\mathfrak{D}}$ coincide with $\mathfrak{A}_i^{\mathfrak{D}}$. Therefore, we get $\mathfrak{C} \in P_S HS(\mathfrak{B})$. ■

Theorem 6.2. If for a certain algebra $\mathfrak{A}$ the variety $\mathfrak{M}(\mathfrak{K})$ is locally finite, and in $\mathfrak{M}(\mathfrak{K})$ there exists (to the accuracy of isomorphism) only a finite number of finite subdirectly non-decomposable algebras, then $\mathfrak{M}(\mathfrak{K})$ is n-residual for a certain $n < \omega$.

Proof. Let $\mathfrak{A}_1,\dots,\mathfrak{A}_n$ be all (to the accuracy of isomorphism) subdirectly non-decomposable finite algebras of the variety $\mathfrak{M}(\mathfrak{A})$. Then for any finite algebra $\mathfrak{B} \in \mathfrak{M}(\mathfrak{A})$, $\mathfrak{B} \in P_S(\mathfrak{A}_1,\dots,\mathfrak{A}_n) \subseteq P_S HS(\mathfrak{A}_1 \times \dots \times \mathfrak{A}_n)$.

Since $\mathfrak{M}$ is locally finite, for any algebra $\mathfrak{B} \in \mathfrak{M}(\mathfrak{A})$ we have $\mathfrak{B} \in P_S HS(\mathfrak{A}_1,\dots,\mathfrak{A}_n)$, i.e., $\mathfrak{M}(\mathfrak{A}) = P_S HS(\mathfrak{A}_1,\dots,\mathfrak{A}_n)$, by lemma **6.3**. But we have $HS(\mathfrak{A}_1 \times \dots \times \mathfrak{A}_n) \subseteq P_S(\mathfrak{A}_1,\dots,\mathfrak{A}_n)$, then $\mathfrak{M}(\mathfrak{A}) = P_S(\mathfrak{A}_1,\dots,\mathfrak{A}_n)$ and, hence, $\mathfrak{A}_1,\dots,\mathfrak{A}_n$ are all subdirectly non-decomposable algebras of the variety $\mathfrak{M}$. ■

Lemma 6.4. For a congruence-modular variety $\mathfrak{M}$ and an algebra $\mathfrak{A} \in \mathfrak{M}$, the following conditions are equivalent:

(1) for any $\mu,\nu \in Con\mathfrak{A}$ we have $\lfloor \mu,\mu \rfloor \wedge \nu \le \lfloor \mu,\nu \rfloor$;

(2) for any $\mu,\nu \in Con\mathfrak{A}$ the inequality $\nu \le \lfloor \mu,\mu \rfloor$ results in the equality $\lfloor \nu,\mu \rfloor = \nu$.

Proof. Let condition (2) be valid. Then, since for any $\mu,\nu \in Con\mathfrak{A}$ we have $\lfloor \mu,\mu \rfloor \wedge \nu \le \lfloor \mu,\mu \rfloor$, the equality $\lfloor \mu,\mu \rfloor \wedge \nu = \lfloor \lfloor \mu,\mu \rfloor \wedge \nu,\mu \rfloor$ is true. As $\lfloor \mu,\mu \rfloor \wedge \nu \le \nu$, it follows that, since the commutator is monotonous, $\lfloor \lfloor \mu,\mu \rfloor \wedge \nu,\mu \rfloor \le \lfloor \nu,\mu \rfloor = \lfloor \mu,\nu \rfloor$. Therefore, we get $\lfloor \mu,\mu \rfloor \wedge \nu \le \lfloor \mu,\nu \rfloor$ and, hence, statement (1) is valid.

Let us now suppose that statement (1) is valid. Let us show that if θ is non-decomposable in the intersection in $Con\mathfrak{A}$, and ψ is the only covering of θ from $Con\mathfrak{A}$, if $\mu \in Con\mathfrak{A}$ and $\mu \ge \theta, \lfloor \mu,\psi \rfloor \le \theta$, then $\lfloor \mu,\mu \rfloor \le \theta$. Indeed, by virtue of (1), we have $\lfloor \mu,\mu \rfloor \wedge \psi \le \lfloor \mu,\psi \rfloor \le \theta$. Therefore, $\theta = \theta \vee (\lfloor \mu,\mu \rfloor \wedge \psi) = \psi \wedge (\theta \vee \lfloor \mu,\mu \rfloor)$ (by modularity). As θ is non-decomposable in the intersection, $\theta = \theta \vee \lfloor \mu,\mu \rfloor$, i.e., indeed, $\lfloor \mu,\mu \rfloor \le \theta$.

Let us now prove that statement (2) is valid. Assume to the contrary that $\mu,\nu \in Con\mathfrak{A}$ and $\nu \le \lfloor \mu,\mu \rfloor, \lfloor \mu,\nu \rfloor < \nu$. The latter inequality implies the existence of a $\theta \in Con\mathfrak{A}$, non-decomposable in the intersection and such that $\theta \ge \lfloor \mu,\nu \rfloor$, but $\theta \not\ge \nu$.

Let ψ be the only congruence covering θ in $Con\mathfrak{A}$. Since $\psi \leq \theta \vee \nu$, $\lfloor \theta \vee \mu, \psi \rfloor \leq \lfloor \theta \vee \mu, \theta \vee \nu \rfloor \leq \lfloor \mu, \nu \rfloor \vee \theta \leq \theta$. But then, by virtue of what proved above, we get $\lfloor \theta \vee \mu, \theta \vee \mu \rfloor \leq \theta$. The latter fact implies $\nu \leq \lfloor \mu, \mu \rfloor \leq \lfloor \theta \vee \mu, \theta \vee \mu \rfloor \leq \theta$, which contradicts the choice of θ. Therefore, the assumption that $\lfloor \mu, \nu \rfloor < \nu$ is contradictory, i.e., $\nu \leq [\mu, \mu]$ results in the equality $\lfloor \nu, \mu \rfloor = \nu$, and statement (1) results from statement (2). ■

__Lemma 6.5.__ Any subdirectly non-decomposable algebra of a directly representable variety $\mathfrak{M}$ is either simple or Abelian.

Proof. Let $\mathfrak{A} \in \mathfrak{M}$, β is the monolith of $\mathfrak{A}$, $\mathfrak{M}$ is directly representable, and $\lfloor \nabla, \nabla \rfloor \geq \beta$, which means that $\mathfrak{A}$ is not Abelian. Let us assume that $\mathfrak{A}$ here is not simple, i.e., $\beta < \nabla$. By theorem **6.2**, directly presentable $\mathfrak{M}$ should be residually small. Since $\mathfrak{M}$ is, by theorem **6.1**, congruence-permutable and, hence, also congruence-modular, for any algebra $\mathfrak{B} \in \mathfrak{M}$ the identity $x \wedge \lfloor y, y \rfloor \leq \lfloor x, y \rfloor$ is valid on $Con\mathfrak{B}$ by theorem **2.25** and lemma **6.4**. Therefore, the equality $\lfloor \nabla, \beta \rfloor = \beta$ holds on $Con\mathfrak{A}$. For any $n > 1$ let us define the subalgebra $\mathfrak{B}_n$ of the algebra $\mathfrak{A}^n$ in such a way that

$$\mathfrak{B}_n = \mathfrak{A}^n(\beta) = \{f \in \mathfrak{A}^n \text{ for } \mid i,j < n < f(i), f(j) > \in \beta\}.$$

Let us prove that $\mathfrak{B}_n$ are directly non-decomposable, and, since the powers of such $\mathfrak{B}_n$ strictly increase with n growing, we arrive at a contradiction to the direct presentability of $\mathfrak{M}$. It is this contradiction that proves the simplicity of any non-Abelian subdirectly non-decomposable $\mathfrak{M}$-algebra.

For $\theta \in Con\mathfrak{A}$ and $i < n$ let us define $\theta_i \in Con\mathfrak{B}_n$ as $\{< f,g > \mid < f(i), g(i) > \in \theta\}$. It is obvious that $\beta_0 = \beta_1 = \ldots = \beta_{n-1}$. Let us denote this congruence on $\mathfrak{B}_n$ by $\overline{\beta}$. The congruences Δ_i are kernels of i-the projections of $\mathfrak{B}_n$ on $\mathfrak{A}$. Let us define $\Delta_i' = \bigwedge_{j \neq i} \Delta_j$. Obviously, the following equalities hold:

$$\Delta_i \vee \Delta_i' = \Delta_i \bigvee_{j \neq i} \Delta_j = \bigvee_{i<n} \Delta_i' = \overline{\beta}.$$

Let us now assume that $\mathfrak{B}_n$ is directly decomposable, to arrive at a contradiction. As $\mathfrak{B}_n$ is directly decomposable, then there are $\varphi, \psi \in Con\mathfrak{B}_n$ such that $\psi \vee \varphi = \Delta$, $\psi \wedge \varphi = \Delta$, $\varphi > \Delta$ and $\psi > \Delta$.

For any $i < n$ we have either

$$(\lfloor\Delta_i',\varphi\rfloor = \Delta_i' \text{ and } \lfloor\Delta_i',\psi\rfloor = \Delta)$$

or

$$(\lfloor\Delta_i',\psi\rfloor = \Delta_i' \text{ and } \lfloor\Delta_i',\varphi\rfloor = \Delta).$$

Indeed, according to the commutator properties (see theorem **2.17**), since $\lfloor\beta,\nabla\rfloor = \beta$ and $\overline{\beta} = \underset{\pi_i}{\vee}(\beta)$, we get

$$\overline{\beta} = \lfloor\overline{\beta},\nabla\rfloor \vee \ker\pi_i = \Delta_i \vee \lfloor\Delta_i \vee \Delta_i',\psi \vee \varphi\rfloor = \Delta_i \vee \lfloor\Delta_i',\psi\rfloor \vee \lfloor\Delta_i',\varphi\rfloor.$$

As β covers Δ in $Con\mathfrak{A}$, $\overline{\beta} = \beta_i$ covers Δ_i in $Con\mathcal{B}_n$ (the interval $\lfloor\overline{\beta},\Delta_i\rfloor$ of the lattice $Con\mathcal{B}_n$ is mapped to the interval $\lfloor\beta,\Delta\rfloor$ of the lattice $Con\mathfrak{A}$ while π_i is being projected. By virtue of the inequalities

$$\lfloor\Delta_i',\psi\rfloor \le \Delta_i' \wedge \psi, \quad \lfloor\Delta_i',\varphi\rfloor \le \Delta_i' \wedge \varphi,$$

and the fact that $\varphi \wedge \psi = \Delta$, we come to the conclusion that $\lfloor\Delta',\psi\rfloor$ and $\lfloor\Delta',\varphi\rfloor$ are disjunct. As we have already noticed, $\overline{\beta} = \Delta_i \vee \lfloor\Delta_i',\psi\rfloor \vee \lfloor\Delta_i',\varphi\rfloor$ and, since $\overline{\beta} \ne \Delta_i$, $\lfloor\Delta_i',\psi\rfloor$ and $\lfloor\Delta_i',\varphi\rfloor$ could not be simultaneously equal to Δ. But $\overline{\beta}$ covers $\Delta_i, \overline{\beta} = \Delta_i \vee \Delta_i'$, while Δ_i' covers Δ, and hence we get the required statement, i.e., for any $i < n$ we have either

$$(\lfloor\Delta_i',\varphi\rfloor = \Delta_i' \text{ and } \lfloor\Delta_i',\psi\rfloor = \Delta)$$

or

$$(\lfloor\Delta_i',\psi\rfloor = \Delta_i' \text{ and } \lfloor\Delta_i',\varphi\rfloor = \Delta).$$

Let now

$$S_\varphi = \{i \mid [\Delta_i',\varphi] = \Delta_i'\}, \quad S_\psi = \{i \mid [\Delta_i',\psi] = \Delta_i'\}.$$

According to what has been pproved above, we have $S_\varphi \cup S_\psi = \{0,1,\ldots,n-1\}$, and $S_\varphi \cap S_\psi = \varnothing$. Let us show that if $S_\psi \ne \varnothing$, then $\varphi \le \overline{\beta}$, and if $S_\varphi \ne \varnothing$, then $\psi \le \overline{\beta}$. Indeed, if $i \in S_\psi$ and $\varphi \not\le \overline{\beta}$, then $\varphi \not\le \Delta_i'$ and, hence, $\Delta_i \vee \varphi \ge \overline{\beta} = \beta_i$. As $\underset{j\in S_\varphi}{\vee} \Delta_j' \le \varphi$, $Con\mathcal{B}_n$ is modular, and $\underset{j\in S_\psi}{\vee} \Delta_j' \le \psi$ entails $(\underset{j\in S_\psi}{\vee} \Delta_j') \wedge \varphi = \Delta$, we get

$$\overline{\beta} = \overline{\beta} \wedge (\Delta_i \vee \varphi) = \Delta_i \vee (\overline{\beta} \wedge \varphi) = \Delta_i \vee ((\underset{j\in S_\psi}{\vee} \Delta_j' \vee \underset{k\in S_\varphi}{\vee} \Delta_k') \wedge \varphi) =$$

$$\Delta_l \vee ((\bigvee_{j\in S_\psi} \Delta'_j) \wedge \varphi) \vee \bigvee_{k\in S_\varphi} \Delta'_k = \Delta_i \vee \bigvee_{k\in S_\varphi} \Delta'_k = \Delta_i .$$

The obtained contradiction is the required proof, i.e., if $S_\psi \neq \varnothing$, then $\varphi \leq \overline{\beta}$, and if $S_\varphi \neq \varnothing$, then $\psi \leq \overline{\beta}$. By the supposition, $\overline{\beta} < \nabla$ and, since $\varphi \vee \psi = \nabla$, we have either $S_\varphi = \varnothing$ and $S_\psi = \{0,1,\ldots,n-1\}$, or $S_\psi = \varnothing$ and $S_\varphi = \{0,1,\ldots,n-1\}$. If we assume the former, then, as has been noted above, $\varphi \leq \overline{\beta}$ but, by the definition of S_ψ, we have $\Delta'_i = \lfloor \Delta'_i, \psi \rfloor \leq \psi$ for all $i < n$. Therefore, $\overline{\beta} = \bigvee_{i<n} \Delta'_n \leq \psi$, i.e., $\varphi \leq \psi$, which contradicts the choice of φ, ψ. Hence, $\varphi, \psi \in Con\,\mathcal{B}_n$ such that $\varphi \vee \psi = \nabla, \varphi \wedge \psi = \Delta, \varphi > \Delta, \psi > \Delta$ does not exist. According to what has been proved above, this is the proof of the lemma. ■

<u>**Lemma 6.6.**</u> If $\mathfrak{M}$ is a directly presentable variety and $\mathfrak{A} \in \mathfrak{M}$, then the identity $\lfloor x,y \rfloor = x \wedge y \wedge \lfloor \nabla,\nabla \rfloor$ is valid on $Con\,\mathfrak{A}$.

Proof. Let us assume, to the contrary, that $\mathfrak{A} \in \mathfrak{M}$, $\theta,\psi \in Con\,\mathfrak{A}$, and $\lfloor \theta,\psi \rfloor < \theta \wedge \psi \wedge \lfloor \nabla,\nabla \rfloor$. Let us choose an $\alpha \in Con\,\mathfrak{A}$ such that $\mathfrak{A}/\alpha$ is subdirectly non-decomposable, $\lfloor \theta,\psi \rfloor \leq \alpha$, and $\theta \wedge \psi \wedge \lfloor \nabla,\nabla \rfloor \not\leq \alpha$. By lemma **6.5**, $\mathfrak{A}/\alpha$ is either Abelian, or simple. If $\mathfrak{A}/\alpha$ is Abelian then, by theorem **2.17** (4),

$$\check{\varphi}([\nabla_{\mathfrak{A}/\alpha}, \nabla_{\mathfrak{A}/\alpha}]) = [\check{\varphi}(\nabla_{\mathfrak{A}/\alpha}), \check{\varphi}(\nabla_{\mathfrak{A}/\alpha})] \vee \alpha ,$$

where φ is a canonical homomorphism of $\mathfrak{A}$ on $\mathfrak{A}/\alpha$. However,

$$\check{\varphi}(\nabla_{\mathfrak{A}/\alpha}) = \nabla, \quad [\nabla_{\mathfrak{A}/\alpha}, \nabla_{\mathfrak{A}/\alpha}] = \Delta_{\mathfrak{A}/\alpha},$$

(as $\mathfrak{A}/\alpha$ is Abelian), and $\check{\varphi}(\Delta_{\mathfrak{A}/\alpha}) = \alpha$. Therefore, $\alpha = \lfloor \nabla,\nabla \rfloor \vee \alpha$, i.e., $\lfloor \nabla,\nabla \rfloor \leq \alpha$, which contradicts the choice of α. Thus, $\mathfrak{A}/\alpha$ must be a simple algebra, i.e., ∇ must cover α in $Con\,\mathfrak{A}$. Since,by the choice of α, $\psi,\theta \not\leq \alpha$, $\nabla = \theta \vee \alpha = \psi \vee \alpha$. Hence, we get

$$\lfloor \nabla,\nabla \rfloor = \lfloor \theta \vee \alpha, \psi \vee \alpha \rfloor = \lfloor \theta,\psi \rfloor \vee \lfloor \theta,\alpha \rfloor \vee \lfloor \alpha, \psi \rfloor \vee \lfloor \alpha,\alpha \rfloor \leq \alpha ,$$

which again contradicts the choice of α. It is the obtained contradiction that proves that the inequality $\lfloor\theta,\psi\rfloor < \theta \wedge \psi \wedge \lfloor\nabla,\nabla\rfloor$ is not possible. ■

Lemma 6.7. If on a modular $Con\mathfrak{A}$ the identity $x \wedge \lfloor y,y\rfloor \leq \lfloor x,y\rfloor$ is valid, then $\lfloor\nabla,\nabla\rfloor$ is a neutral element of $Con\mathfrak{A}$, and for any $\theta \in Con\mathfrak{A}$ we have $\lfloor\nabla,\theta\rfloor = \lfloor\nabla,\nabla\rfloor \wedge \theta$.

Proof. According to the identity given in the formulation, for any $\theta \in Con\mathfrak{A}$ we have

$$\lfloor\nabla,\nabla\rfloor \wedge \theta \leq \lfloor\nabla,\theta\rfloor \leq \lfloor\nabla,\nabla\rfloor \wedge \theta.$$

Therefore, we get $\lfloor\nabla,\theta\rfloor = \lfloor\nabla,\nabla\rfloor \wedge \theta$. To prove that $\lfloor\nabla,\nabla\rfloor$ is neutral in a modular $Con\mathfrak{A}$, it suffices, as has been shown in section 2, to show that for any $\theta,\psi \in Con\mathfrak{A}$, the following equality holds:

$$\lfloor\nabla,\nabla\rfloor \wedge (\theta \vee \psi)(\lfloor\nabla,\nabla\rfloor \wedge \theta) \vee (\lfloor\nabla,\nabla\rfloor \wedge \psi).$$

But the left-hand side of this equality is as follows:

$$\lfloor\nabla,\theta \vee \psi\rfloor = \_\nabla,\theta\rfloor \vee \lfloor\nabla,\psi\rfloor = (\lfloor\nabla,\nabla\rfloor \wedge \theta) \vee (\lfloor\nabla,\nabla\rfloor \wedge \psi). \blacksquare$$

Lemma 6.8. Let $\mathfrak{M}$ be a directly representable variety, $\mathfrak{A} \in \mathfrak{M}$, and $\mathfrak{A}$ finite, then:

(a) $\lfloor\nabla,\nabla\rfloor$ and $Z_{\mathfrak{A}}$ are neutral elements of $Con\mathfrak{A}$ complementary to each other. Therefore, $\mathfrak{A} \cong \mathfrak{A}/\lfloor\nabla,\nabla\rfloor \times \mathfrak{A}/Z_{\mathfrak{A}}$, this isomorphism inducing that of the lattice $Con\mathfrak{A}$ and the lattice $\lfloor\nabla;\lfloor\nabla,\nabla\rfloor\rfloor \times \lfloor\nabla;Z_{\mathfrak{A}}\rfloor$ are isomorphic;

(b) $\lfloor\nabla,\nabla\rfloor, Z_{\mathfrak{A}}$ is the only pair of the elements $<\alpha,\beta>$ from $Con\mathfrak{A}$ such that α,β are complements to each other, and

$$\mathfrak{A}/\alpha \models \lfloor\nabla,\nabla\rfloor = \Delta, \quad \mathfrak{A}/\beta \models \lfloor\nabla,\nabla\rfloor = \nabla;$$

(c) $\mathfrak{A}/Z_{\mathfrak{A}} \cong \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, where $\mathcal{B}_i$ are simple, non-Abelian algebras, and this decomposition is the only one.

Proof. Let $\mathfrak{A}$ obey the conditions of the lemma, $\Delta = \bigwedge_{i=1}^{m} \theta_i$, where $\theta_i \in Con\mathfrak{A}$, and such that $\mathfrak{A}/\theta_i$ are subdirectly non-decomposable, and for any $S \subset \{1,...,m\}$ we have $\Delta \neq \bigwedge_{j \in S} \theta_j$. By lemma **6.5**, one can assume that for a certain $\theta \leq k \leq m$ we have:

(1) for $1 \leq i \leq k$ we have $\theta_i \vee \lfloor \nabla,\nabla \rfloor = \nabla$, and ∇ covers θ_i in $Con\mathfrak{A}$ (in other words, $\mathfrak{A}/\theta_i\,(1 \leq i \leq k)$ are simple non-Abelian algebras);

(2) for $k+1 \leq i \leq m$ we have $\lfloor \nabla,\nabla \rfloor \leq \theta_i$ (i.e., $\mathfrak{A}/\theta_i\,(k+1 \leq i \leq m)$ are Abelian).

Let $\mu = \bigwedge_{i=1}^{k} \theta_i$, $\lambda = \bigwedge_{i=k+1}^{m} \theta_i$. Since θ_i is maximal in $Con\mathfrak{A}$ and the congruences are permutable on $\mathfrak{A}$ (by theorem **6.1**), $\mathfrak{A}/\mu$ is isomorphic to the product $\prod_{i=1}^{k} \mathfrak{A}/\theta_i$. The conditions $\theta_i \vee \lfloor \nabla,\nabla \rfloor = \nabla$ (for $1 \leq i \leq k$), as well as the properties of commutators (theorem **2.17**) imply $\mathfrak{A}/\mu \models \lfloor \nabla,\nabla \rfloor = \nabla$. And again, by property (4) of theorem **2.17** we get $\mu \vee \lfloor \nabla,\nabla \rfloor = \nabla$ in $Con\mathfrak{A}$. The definition of λ entails $\lfloor \nabla,\nabla \rfloor \leq \lambda$, $\lambda \wedge \mu = \Delta$. Hence, we get $\lfloor \nabla,\nabla \rfloor \wedge \mu = \Delta$. As $Con\mathfrak{A}$ is modular, we have

$$\lambda = \lambda \wedge \nabla = \lambda \wedge (\lfloor \nabla,\nabla \rfloor \vee \mu) = \lfloor \nabla,\nabla \rfloor \vee (\lambda \wedge \mu) = \lfloor \nabla,\nabla \rfloor.$$

To sum up, we see that $\lfloor \nabla,\nabla \rfloor \wedge \mu = \Delta$, $\lfloor \nabla,\nabla \rfloor \vee \mu = \nabla$, and $\lfloor \nabla,\nabla \rfloor, \mu$ are permutable, i.e.,

$$\mathfrak{A} \cong \mathfrak{A}/\lfloor \nabla,\nabla \rfloor \times \mathfrak{A}/\theta_1 \times ... \times \mathfrak{A}/\theta_k .$$

Let us now recall that $\mu = Z_{\mathfrak{A}}$ (the center of $\mathfrak{A}$). Indeed, by virtue of lemma **6.6**, we have $\lfloor \nabla,\mu \rfloor = \lfloor \nabla,\nabla \rfloor \wedge \mu = \Delta$. Let us prove that μ is a maximal congruence on $\mathfrak{A}$ possessing this very property. Indeed, if $\gamma > \mu$, since $Con\mathfrak{A}$ is modular, we get:

$$\gamma = \gamma \wedge (\lfloor \nabla,\nabla \rfloor \vee \mu) = (\gamma \wedge \lfloor \nabla,\nabla \rfloor \vee \mu),$$

i.e., $\gamma \wedge \lfloor\nabla,\nabla\rfloor \neq \Delta$, and, since by lemma **6.6**, $\gamma \wedge [\nabla,\nabla] = [\gamma,\nabla]$, $\lfloor\gamma,\nabla\rfloor > \Delta$. Therefore, $\lfloor\nabla,\mu\rfloor = \Delta$, and γ is maximal in $Con\mathfrak{A}$ relative to this property, i.e., by the definition, $\mu = Z_{\mathfrak{A}}$.

By lemma **6.7**, $\lfloor\nabla,\nabla\rfloor$ is a neutral element of $Con\mathfrak{A}$ and, therefore, its complement in $Con\mathfrak{A}$ is unique and neutral. One can easily see that $< \lfloor\nabla,\nabla\rfloor, Z_{\mathfrak{A}} >$ is the only pair $< \alpha,\beta >$ of elements in $Con\mathfrak{A}$ such that $\lfloor\nabla,\nabla\rfloor \leq \alpha$, $\lfloor\nabla,\nabla\rfloor \vee \beta = \nabla$, $\alpha \wedge \beta = \Delta$, i.e., statement (b) is valid.

To prove statement (c) it suffices to note that $\left|Z_{\mathfrak{A}};\nabla\right| = 2^k$. The latter results from the isomorphism of lattices $\left|Z_{\mathfrak{A}};\nabla\right|$ and $\left|\Delta;\lfloor\nabla,\nabla\rfloor\right|$, and from the fact that $\left|\Delta;\lfloor\nabla,\nabla\rfloor\right|$ is a distributive lattice according to the identity $[x,y] = x \wedge y \wedge [\nabla,\nabla]$, which is true on $Con\mathfrak{A}$ by lemma **6.6.** ∎

Lemma 6.9. Let $Con\mathfrak{A}$ be modular, and $\mathfrak{A} \cong \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, where $\mathcal{B}_i$ are simple non-Abelian algebras. In this case, the following statements are valid:

(a) for any $\varphi,\psi \in Con\mathfrak{A}$, $[\varphi,\psi] = \varphi \wedge \psi$;

(b) $Con\mathfrak{A} \cong 2^n$.

Proof. Since in every $\mathcal{B}_i$, true $\lfloor\nabla,\nabla\rfloor = \nabla$ is true, by theorem **2.17** (5), for the congruences φ,ψ on $\mathfrak{A}$ representable as intersections of the kernels Δ_i of the projections π_i of the algebra $\mathfrak{A}$ onto algebras $\mathcal{B}_i$, the equality $\lfloor\varphi,\psi\rfloor = \varphi \wedge \psi$ is valid. In particular, $\lfloor\nabla,\nabla\rfloor = \nabla$ on $Con\mathfrak{A}$. To complete the proof of the lemma, let us show that for any $\theta \in Con\mathfrak{A}$ we have $\theta = \Delta_J$, where $J = \left\{j \mid \theta \leq \Delta_j\right\}$, and for $K \subseteq \{1,\ldots,n\}$ $\Delta_K = \bigwedge_{j \in K} \Delta_j$. Let us notice that for α,β such that $\theta \vee \alpha = \theta \vee \beta = \nabla$, we have $\theta \vee (\alpha \wedge \beta) = \nabla$. Indeed:

$$\nabla = \lfloor\nabla,\nabla\rfloor = \lfloor\theta \vee \alpha, \theta \vee \beta\rfloor \leq \theta \vee \lfloor\alpha,\beta\rfloor \leq \theta \vee (\alpha \wedge \beta).$$

Therefore, since all Δ_i are maximal, we have $\theta \vee \Delta_{\{1,\ldots,n\}\setminus J} = \nabla$. As $Con\mathfrak{A}$ is modular, we get:

$$\Delta_J = \Delta_J \wedge \left(\theta \vee \Delta_{\{1,\ldots,n\}\setminus J}\right) = \theta \vee \left(\Delta_J \wedge \Delta_{\{1,\ldots,n\}\setminus J}\right) = \theta. \ ∎$$

Let us cite without proof a well-known and easily provable statement contained in the following lemma.

Lemma 6.10. If $\mathfrak{A}_1,...,\mathfrak{A}_n$ are simple algebras and $\mathfrak{M}(\mathfrak{A}_1,...,\mathfrak{A}_n)$ is congruence-permutable, then for any algebra $\mathfrak{D}$ such that $\mathfrak{D} \subseteq \mathfrak{A}_1 \times ... \times \mathfrak{A}_n$ as a subdirect product of these algebras, there exists an $I \subseteq \{1,...,n\}$ such that $\mathfrak{D} \cong \prod_{i \in I} \mathfrak{A}_1$, and, at the same time, this isomorphism is a projection of $\mathfrak{D}$ over the subset I.

Theorem 6.3. Let $\mathfrak{K}$ be an arbitrary finite set of finite algebras. The conglomeration of the following conditions is necessary and sufficient for $\mathfrak{M}(\mathfrak{K})$ to be directly representable:

(1) $\mathfrak{M}(\mathfrak{K})$ is congruence-permutable;

(2) any algebra from $S\mathfrak{K}$ is isomorphic to a direct product of simple algebras and Abelian algebras;

(3) a variety generated by Abelian direct cofactors from direct decompositions of $S\mathfrak{K}$ -algebras is directly representable.

Proof. Let $\mathfrak{K}$ be an arbitrary set of finite algebras, and $\mathfrak{M}(\mathfrak{K})$ be a directly representable variety. Then, by theorem **6.1**, $\mathfrak{M}(\mathfrak{K})$ is congruence-permutable, and by lemma **6.5**, any finite $\mathfrak{M}(\mathfrak{K})$-algebra is isomorphic to a direct product of simple algebras and Abelian algebras.
It is also obvious that a variety generated by a set of Abelian direct cofactors in direct decompositions of $S(\mathfrak{K})$-algebras is directly representable.

Let us now prove the opposite case. Let conditions (1), (2) and (3) from the formulation of the theorem be valid for the variety $\mathfrak{M}(\mathfrak{K})$. Let $\mathfrak{S}$ be a class of simple non-Abelian cofactors in direct decompositions of $S(\mathfrak{K})$-algebras, and $\mathfrak{A}$ be a variety generated by Abelian cofactors in direct decompositions of $S(\mathfrak{K})$-algebras. The class $\mathfrak{S}$ is finite to the accuracy of isomorphism, the variety $\mathfrak{A}$ is directly representable by condition (3). To prove that $\mathfrak{M}(\mathfrak{K})$ is directly representable, it suffices to show that any finite $\mathfrak{M}(\mathfrak{K})$-algebra lies in $P(\mathfrak{S} \cup \mathfrak{A})$.

Let us first consider a finite algebra $\mathfrak{F}$ from $SP(\mathfrak{K})$. In line with condition (2), $\mathfrak{F}$ is isomorphic to a subdirect product $\mathfrak{D} \subseteq \mathfrak{A} \times \mathfrak{B}_1 \times ... \times \mathfrak{B}_n$, where $\mathfrak{A} \in \mathfrak{A}$, $\mathfrak{B}_i \in \mathfrak{S}$. Let λ be the kernel of $\mathfrak{D}$ projection on $\mathfrak{A}$, and $\mu = \bigwedge_{i \leq k} \theta_i$, where θ_i are the kernels of $\mathfrak{D}$ projection on $\mathfrak{B}_i$. In this case, $\mathfrak{D}/\mu \subseteq_p \mathfrak{B}_1 \times ... \times \mathfrak{B}_k$

and, since $\mathcal{B}_i$ are simple and $\mathcal{D}/\mu$ is congruence-permutable, $\mathcal{D}/\mu \cong \mathcal{B}_1 \times \ldots \times \mathcal{B}_k$, by lemma **6.10**. As $\mathcal{U}/\theta_i$ are simple Abelian algebras at $i \leq k$, $\theta_i \vee [\nabla,\nabla] = \nabla$. By virtue of the properties of a commutator (theorem **2.17**), we get $\mathcal{U}/\mu \models [\nabla,\nabla] = \nabla$, and, again by theorem **2.17**, $\mu \vee [\nabla,\nabla] = \nabla$. The definition of λ results in the inequality $[\nabla,\nabla] \leq \lambda$. Since $Con\mathcal{U}$ is modular, we have $\lambda = \lambda \wedge ([\nabla,\nabla] \vee \mu) = [\nabla,\nabla] \vee (\mu \wedge \lambda)$ and, as $\lambda \wedge \mu = \Delta$, $\lambda = [\nabla,\nabla]$. All these facts imply $\mathcal{D} \cong \mathcal{U} \times \mathcal{B}_1 \times \ldots \times \mathcal{B}_k$, i.e., $S(P(\mathfrak{K})) \subseteq P(\mathfrak{S} \cup \mathfrak{A}))$.

Let now $\mathcal{C} \in \mathfrak{M}(\mathfrak{K}) = HSP(\mathfrak{K})$ and $\mathcal{C}$ be finite. Then for a certain finite algebra $\mathcal{D} \in SP(\mathfrak{K})$ and a certain $\theta \in Con\mathcal{D}$ we have $\mathcal{C} \cong \mathcal{D}/\theta$. Let us assume that $\mathcal{D} \cong \mathcal{U} \times \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$, where $\mathcal{U} \in \mathfrak{A}$, $\mathcal{B}_i \in \mathfrak{S}$. Let μ, as above, be equal to $\bigwedge_{i \leq n} \theta_i$. By lemma **6.9** (a) we get

$$[\nabla/\mu, {}^{\mu \vee \theta}/\mu] = \nabla/\mu \wedge ({}^{\mu \vee \theta}/\mu) = {}^{\mu \vee \theta}/\mu .$$

By theorem **1.17** (g), this formula yields $[\nabla,\mu \vee \theta] \vee \mu = (\mu \vee \theta) \vee \mu = \mu \vee \theta$, but

$$[\nabla,\mu \vee \theta] \vee \mu = [\nabla,\mu] \vee [\nabla,\theta] \vee \mu = [\nabla,\theta] \vee \mu ,$$

i.e., $\mu \vee \theta = [\nabla,\theta] \vee \mu$. Hence, because of modularity, we get:

$$[\nabla,\nabla] \wedge (\mu \vee \theta) = [\nabla,\nabla] \wedge ([\nabla,\theta] \vee \mu) = [\nabla,\theta] \vee ([\nabla,\nabla] \wedge \mu) = [\nabla,\theta],$$

as $[\nabla,\nabla] \wedge \mu = \Delta$. Therefore,

$$\theta = \theta \vee [\nabla,\theta] = \theta \vee ([\nabla,\nabla] \wedge (\mu \vee \theta)) = ([\nabla,\nabla] \vee \theta) \wedge (\mu \vee \theta).$$

The latter equality, combined with the equality $[\nabla,\nabla] \vee \mu = \nabla$ and permutability of the congruences on $\mathcal{D}$, demonstrates that $\mathcal{D}/\theta \cong \mathcal{D}/[\nabla,\nabla] \vee \theta \times \mathcal{D}/\mu \vee \theta$. But $\mathcal{D}/[\nabla,\nabla] \vee \theta \in H(\mathcal{D}/[\nabla,\nabla])$, i.e., $\mathcal{D}/[\nabla,\nabla] \vee \theta \in \mathfrak{A}$ and $\mathcal{D}/\mu \vee \theta \in H(\mathcal{B}_1 \times \ldots \times \mathcal{B}_n)$. By lemma **6.9** (b), $\mathcal{D}/\mu \vee \theta \cong \prod_{i \in I} \mathcal{B}_i$, where $I \subseteq \{1,\ldots,n\}$ and, therefore, as $\mathfrak{A}$ is directly representable, we have proved the decomposition of $\mathcal{C} = \mathcal{D}/\theta$ into a direct product of algebras from $\mathfrak{S}$ and $\mathfrak{A}$. Hence, the finite $\mathfrak{M}(\mathfrak{K})$-algebras lie in $P(\mathfrak{A} \cup \mathfrak{S})$. ■

Definition 6.3.

(a) The spectrum of an arbitrary class of algebras $\mathfrak{K}$ is a family of the powers of $\mathfrak{K}$-algebras;

(b) the finite spectrum of an arbitrary class of $\mathfrak{K}$-algebras is a set of the powers of finite $\mathfrak{K}$-algebras.

The compactness theorem for the calculus of first-order predicates yields, as is well-known, that for any variety of $\mathfrak{M}$-algebras the finite spectrum of the variety $\mathfrak{M}$ is either finite, or it is infinite, in which case the spectrum of $\mathfrak{M}$ consists of the finite spectrum of $\mathfrak{M}$ and all infinite cardinals. $Spec\,\mathfrak{K}$ and $FSpec\,\mathfrak{K}$ will denote the spectrum and the finite spectrum of the class $\mathfrak{K}$, respectively.

Definition 6.4. The finite algebra $\mathfrak{A}$ has a minimal spectrum provided that $FSpec\,\mathfrak{M}(\mathfrak{A}) = \{|\mathfrak{A}|^n \mid n \in \omega\}$.

The remaining part of the present section will be devoted to algebras with a minimal spectrum.

Lemma 6.11. Let a finite algebra $\mathfrak{A}$ be simple, and $\mathfrak{M}(\mathfrak{A})$ be congruence-permutable, $\theta \in Con\,\mathfrak{A}^n$ for a certain $n \in \omega$ and $\Delta < \theta < \nabla$. In this case, the algebra $(\mathfrak{A}^n)^2(\theta)$ (the congruence-Boolean power of the algebra $\mathfrak{A}^n$) is isomorphic to the algebra $\mathfrak{A}^{n+k}$ for a certain k such that $1 \le k \le n-1$, the congruence θ having in this case $|\mathfrak{A}^{n-k}|$ classes, each of which consists of $|\mathfrak{A}^k|$ elements.

Proof. As $\mathfrak{M}(\mathfrak{A})$ is congruence-permutable, $(\mathfrak{A}^n)^2(\theta)$, as a subdirect product of algebras isomorphic to the algebra $\mathfrak{A}$, will, by lemma **6.10**, be isomorphic to a direct product of algebras isomorphic to the algebra $\mathfrak{A}$ in a number smaller than $2n$, this isomorphism being implemented by way of projecting the algebra $(\mathfrak{A}^n)^2(\theta)$ relative to a certain subset $S \subseteq \{0,1,\dots,n-1,0',1',\dots,(n-1)'\}$ (here i' denotes the index of the i-th cofactor in the second co-ordinate of the elements of the algebra $(\mathfrak{A}^n)^2(\theta)$ when presenting these elements as pairs $<a,b>$, where $a,b \in \mathfrak{A}^n$). For any $a_1,\dots,a_n \in \mathfrak{A}$ we have

$$<< a_1,\dots,a_n >, < a_1,\dots,a_n >> \in \left(\mathfrak{A}^n\right)^2(\theta).$$

On the other hand, the choice of co-ordinates of this element in correspondence with the indices contained in S determines all the other co-ordinates of this element uniquely. But, obviously, if the co-ordinates with the indices i and i' are fixed for $1 \le i < n$, then the co-ordinates with the indices 0 and $0'$ will not be uniquely determined by the choice of fixed co-ordinates, i.e., $S \not\subseteq \{1,\dots,n-1,1',\dots,(n-1)'\}$. Analogously, for any other $i < n$ the set S contains either i or i'. Therefore, $\left(\mathfrak{A}^n\right)^2(\theta) \cong \mathfrak{A}^l$, where $n \le l < 2n$. Moreover, $l > n$, as $\theta > \Delta$. By retaining the indices we can assume $S = \{0,1,\dots,n-1, i_1',\dots,i_{l-n}'\}$.

Let $C_1,\dots,C_t$ be all equivalence classes of the algebra $\mathfrak{A}^n$ over θ, and $|C_j| = k_j$ at $j \le t$. Let us show that $k_j \le |\mathfrak{A}|^{l-n}$. Indeed, let

$$b =<< b_0,\dots,b_{n-1} >, < d_0,\dots d_{n-1} >> \in \left(\mathfrak{A}^n\right)^2(\theta)$$

also for such is that $i,i' \in S$ $b_i = d_i$ (note that there are $(l-n)$ of such is). The element

$$b' =<< b_0,\dots,b_{n-1} >, < b_0,\dots b_{n-1} >>$$

also belongs to the algebra $\left(\mathfrak{A}^n\right)^2(\theta)$, but the elements b and b' have the same co-ordinates for the indices incorporated in the set S. Hence, $b = b'$ and $b_i = d_i$ for all $i < n$.

Let now $< a_0,\dots,a_{n-1} > \in C_j$. It should be noticed that $k_j = |C_j'|$, where

$$C_j' = \left\{<< a_0,\dots,a_{n-1} >, < a_0',\dots,a_{n-1}' >> \in \left(\mathfrak{A}^n\right)(\theta)\right\}.$$

Let us consider the mapping $\pi : C_j' \to \mathfrak{A}^{l-n}$ defined as follows:

$$\pi\left(<< a_0,\dots,a_{n-1} >, < a_0',\dots,a_{n-1}' >>\right) = < a_{i_1}',\dots,a_{i_{l-n}}' >.$$

Notice that π is an embedding. Let us assume that

$$<< a_0,\dots,a_{n-1} >, < b_0,\dots,b_{n-1} >>, << a_0,\dots,a_{n-1}, < d_0,\dots d_{n-1} >> \in C_j'$$

so that their π-images coincide, i.e., $d_{i_1} = b_{i_1},\dots,d_{i_{l-n}} = b_{i_{l-n}}$. In this case, since θ is a congruence, $\langle\langle b_0,\dots,b_{n-1}\rangle,\langle d_0,\dots,d_{n-1}\rangle\rangle \in (\mathfrak{A}^n)^2(\theta)$, but, as has been noted above, this implies the equality $b_i = d_i$ for all $i < n$. Therefore, indeed, π is an embedding and, hence, $k_j \le |\mathfrak{A}|^{l-n}$. And, finally, let us note that $\sum_{j=1}^{t} k_j = |\mathfrak{A}|^n$ and $\sum_{j=1}^{t} k_i^2 = |(\mathfrak{A}^n)^2(\theta)| = |\mathfrak{A}|^l$. As $k_j \le |\mathfrak{A}|^{l-n}$, these equalities are satisfied iff $|C_j| = |\mathfrak{A}|^{l-n}$ for all $j < t$. ■

Theorem 6.4. A finite non-singleton algebra $\mathfrak{A}$ has a minimal spectrum iff it is simple, has no non-singleton proper subalgebras, and $\mathfrak{M}(\mathfrak{A})$ is congruence-permutable.

Proof. Let $\mathfrak{A}$ have a minimal spectrum, in which case $\mathfrak{A}$ is obviously simple and has no non-singleton proper subalgebras. Since the spectrum of the algebra $\mathfrak{A}$ is minimal, $SP(\mathfrak{C})$ is a narrow class of algebras, in the sense of the definition **6.2**, for any finite $\mathfrak{C} \in \mathfrak{M}(\mathfrak{A})$. By lemmas **6.1** and **6.2**, any finite algebra $\mathfrak{C} \in \mathfrak{M}(\mathfrak{A})$ will be congruence-permutable. The variety $\mathfrak{M}(\mathfrak{A})$ is finitely generated, hence, locally finite and, therefore, according to what has been proved above, a finite algebra $\mathfrak{F}_{\mathfrak{M}(\mathfrak{A})}(3)$ is congruence-permutable, which implies congruence-permutability of the whole variety $\mathfrak{M}(\mathfrak{A})$.

Let us now assume that the opposite case is true, i.e., $\mathfrak{A}$ is a simple algebra with no non-singleton proper subalgebras, and $\mathfrak{M}(\mathfrak{A})$ is congruence-permutable. By these conditions and lemma **6.10**, any finite algebra $\mathfrak{C} \in \mathfrak{M}(\mathfrak{A})$ has the form $\mathfrak{A}^n/\theta$ for certain $n \in \omega$ and $\theta \in Con\mathfrak{A}^n$. By lemma **6.11**, $|\mathfrak{A}^n/\theta| = |\mathfrak{A}|^k$ for some $k \le n$, i.e., $\mathfrak{A}$ has a minimal spectrum. ■

Before we formulate the next theorem, let us recall some facts related to modular lattices which can be found in a book by G. Birkhoff [14]. In a modular lattice of a finite length, all maximal chains are of the same length. A modular lattice is termed geometric if it is of a finite length and is a lattice with complements or, which is equivalent, if it is of a finite length and a unit is a family of atoms in it. Modular geometric lattices are self-dual. Each modular geometric lattice is representable as a product of Boolean lattices and projective geometries, i.e., lattices of the subspaces of finite-dimensional vector spaces. The filters of modular geometric lattices are modular geometric lattices themselves.

Theorem 6.5. If a finite algebra $\mathfrak{A}$ has a minimal spectrum, then either of the following two cases is possible:

(a) $\mathfrak{A}$ is quasi-primal, has no non-singleton subalgebras and in this case any finite algebra $\mathfrak{C} \in \mathfrak{M}(\mathfrak{A})$ has the form $\mathfrak{A}^n$ for a certain $n \in \omega$;

(b) $\mathfrak{A}$ is a simple Abelian algebra with no non-singleton proper subalgebras. In this case, if $\mathfrak{A}$ has a one-element subalgebra, then any finite algebra $\mathfrak{C} \in \mathfrak{M}(\mathfrak{A})$ has the form $\mathfrak{A}^n$ for some $n \in \omega$, while if $\mathfrak{A}$ has no proper subalgebras, then there is an algebra $\mathfrak{A}^\nabla \in \mathfrak{M}(\mathfrak{A})$ such that $\mathfrak{A}^\nabla \not\cong \mathfrak{A}, |\mathfrak{A}^\nabla| = |\mathfrak{A}|$, $\mathfrak{A}^\nabla$ has a one-element subalgebra, and any finite $\mathfrak{M}(\mathfrak{A})$-algebra has either the form $\mathfrak{A}^n$ or the form $(\mathfrak{A}^\nabla)^n$ for some $n \in \omega$.

Proof. As the $\mathfrak{A}$ spectrum is minimal, the algebra $\mathfrak{A}^2$ is congruence-uniform by lemma **6.1**. Let us consider a congruence on $\mathfrak{A}^2$ generated by a set of pairs $B = \left\{ << x,x >,< y,y >> | < x,y > \in \mathfrak{A}^2 \right\}$, i.e., in terms of the commutator theory, the congruence Δ_∇^∇ (see section **2**).

Since $\mathfrak{A}^2$ is congruence-uniform, all the classes of the congruence Δ_∇^∇ on the algebra $\mathfrak{A}^2$ are of the same power, and, since one of these classes contains the set $B_1 = \left\{ < x,x > | x \in \mathfrak{A} \right\}$, $\left| \mathfrak{A}^2 / \Delta_\nabla^\nabla \right| \le |\mathfrak{A}|$. As the spectrum of the algebra is minimal, either we have $\Delta_\nabla^\nabla = \nabla$ or the Δ_∇^∇- class containing the set B_1 coincides with this set. By the definition of a commutator, we have $\lfloor \nabla,\nabla \rfloor = \left\{ < x,y > | << x,x >,< x,y >> \in \Delta_\nabla^\nabla \right\}$. Therefore, in the former case the equality $\Delta_\nabla^\nabla = \nabla$ implies the equality $\lfloor \nabla,\nabla \rfloor = \nabla$, while in the latter case, i.e., when B_1 is a Δ_∇^∇-class on $\mathfrak{A}^2$, we obtain the equality $\lfloor \nabla,\nabla \rfloor = \Delta$, i.e., that the algebra A is Abelian.

Let us now prove that in the former case the variety $\mathfrak{M}(\mathfrak{A})$ is congruence-distributive. Indeed, for the simple algebra $\mathfrak{A}$ the equality $\lfloor \nabla,\nabla \rfloor = \nabla$ implies its neutrality. By theorem **2.22**, the neutrality of $\mathfrak{A}$ implies that of any algebra of the type $\mathfrak{A}^n$, where $n \in \omega$. The neutrality of an algebra evidently implies (see, for instance, theorem **2.17**), its congruence-distributivity. As has been remarked in the proofs of theorem **2.4**, for certain $n \in \omega$ and $\theta \in Con\mathfrak{A}^n$ we have $\mathfrak{F}_{\mathfrak{M}(\mathfrak{A})}(3) \cong \mathfrak{A}^n / \theta$ and, therefore, $\mathfrak{A}^n$ as well as $\mathfrak{F}_{\mathfrak{M}(\mathfrak{A})}(3)$ prove to be congruence-distributive algebras. By theorem **2.9**, the whole variety $\mathfrak{M}(\mathfrak{A})$ will also be congruence-distributive. Therefore, $\mathfrak{A}$ is a simple finite algebra, having no non-singleton proper subalgebras, generating a congruence-distributive and congruence-permutable variety $\mathfrak{M}(\mathfrak{A})$. By theorem **5.3**, $\mathfrak{A}$ is quasi-primal.

Let us now consider the latter case, when $\mathfrak{A}$ is Abelian. Since $\mathfrak{B}_1$ is a subalgebra of the algebra $\mathfrak{A}^2$ and a Δ_{∇}^{∇}-class in the case under consideration, the algebra $\mathfrak{A}^2/\Delta_{\nabla}^{\nabla}$ has a one-element subalgebra. Let us refer to $\mathfrak{A}^2/\Delta_{\nabla}^{\nabla}$ as $\mathfrak{A}^{\nabla}$. Since the spectrum of the algebra $\mathfrak{A}$ is minimal, we get the equality $|\mathfrak{A}^{\nabla}| = |\mathfrak{A}|$.

As has been noted earlier, any finite $\mathfrak{M}(\mathfrak{A})$-algebra $\mathfrak{C}$ is isomorphic to an algebra of the type $\mathfrak{A}^m/\alpha$ for certain $m \in \omega$ and $\alpha \in Con\mathfrak{A}^m$. If $\theta_i (i < m)$ are kernels of the corresponding projections π_i of the algebra $\mathfrak{A}^m$, $\Delta = \bigwedge_{i<m} \theta_i$ and, since $Con\mathfrak{A}^m$ is a modular lattice ($\mathfrak{M}(\mathfrak{A})$ is congruence-permutable and, hence, congruence-modular), then $Con\mathfrak{A}^m$ is a modular geometric lattice. As a filter of the modular geometric lattice $Con\mathfrak{A}^m$, $Con\mathfrak{C}$ will also be, according to the the facts remarked before proving of this theorem, a modular geometric lattice. In particular, a non-zero element in $Con\mathfrak{C}$ will be the intersection of a finite number of co-atoms: $\Delta = \bigwedge_{i<k} \theta_i$.. It means that the algebra $\mathfrak{C}$ can be represented as a subdirect product of simple algebras $\mathfrak{C}/\theta_i$. By lemma **6.10**, we get an isomorphism $\mathfrak{C} \cong \prod_{i \in I} \mathfrak{C}/\theta_i$, where I is a certain subset of the set $\{0,1,\ldots k-1\}$. Therefore, any finite $\mathfrak{M}(\mathfrak{A})$-algebra is representable as a direct product of simple algebras.

Let $\mathfrak{B}$ be a simple $\mathfrak{M}(\mathfrak{A})$-algebra , and for some $n \in \omega$ and $\theta \in Con\mathfrak{A}^n$ we have $\mathfrak{B} = \mathfrak{A}^n/\theta$. Let us first of all note that if $n \geq 3$, then there is a $\psi \in Con\mathfrak{A}^2$ such that $\mathfrak{A}^n/\theta \cong \mathfrak{A}^2/\psi$. Indeed, let $n \geq 3$, and let us consider subalgebras of the algebra $\mathfrak{A}^n$:

$$\mathfrak{A}_1 = \{< a_1, a_2, \ldots, a_{n-1}, a_{n-1} > \mid a_i \in \mathfrak{A}\}$$

and

$$\mathfrak{A}_2 = \{< a_2, a_2, a_3 \ldots, a_n > \mid a_i \in \mathfrak{A}\}.$$

In this case $\mathfrak{A}_1 \cap \mathfrak{A}_2 \neq \varnothing$, and $\mathfrak{A}_1 \cup \mathfrak{A}_2$ generates $\mathfrak{A}^n$: since $|\mathfrak{A}_1 \cup \mathfrak{A}_2| = 2 \cdot |\mathfrak{A}|^{n-1} - |\mathfrak{A}|^{n-2}$, and the $\mathfrak{A}$ spectrum is minimal, the subalgebra generated by the set $\mathfrak{A}_1 \cup \mathfrak{A}_2$ in $\mathfrak{A}^n$ must have the power $|\mathfrak{A}|^n$, i.e., coincide with

$\mathfrak{A}^n$. If we had $\left|{}^{\mathfrak{A}_1}\!/_{\theta}\right| = \left|{}^{\mathfrak{A}_2}\!/_{\theta}\right| = 1$, then $\left|{}^{\mathfrak{A}^n}\!/_{\theta}\right| = 1$, but θ is a co-atom in $Con\mathfrak{A}^n$. Let $\left|{}^{\mathfrak{A}_1}\!/_{\theta}\right| > 1$. As $\mathfrak{A}$ has a minimal spectrum, we get $|{}^{\mathfrak{A}_1}\!/_{\theta}| = |\mathfrak{A}|^m$ for some $m < n$. Since θ is a co-atom in $Con\mathfrak{A}^n$, $|{}^{\mathfrak{A}^n}\!/_{\theta}| = |\mathfrak{A}|$. However, ${}^{\mathfrak{A}_1}\!/_{\theta} \subseteq {}^{\mathfrak{A}^n}\!/_{\theta}$ and, hence, ${}^{\mathfrak{A}_1}\!/_{\theta} = {}^{\mathfrak{A}^n}\!/_{\theta}$. But $\mathfrak{A}_1 \cong \mathfrak{A}^{n-1}$. If we continue our considerations by induction, we can find a $\psi \in Con\mathfrak{A}^2$ such that ${}^{\mathfrak{A}^n}\!/_{\theta} \cong {}^{\mathfrak{A}^2}\!/_{\psi}$. Therefore, any simple finite $\mathfrak{M}(\mathfrak{A})$-algebra has the form ${}^{\mathfrak{A}^2}\!/_{\psi}$ for a certain $\psi \in Con\mathfrak{A}^2$, $\Delta < \psi < \nabla$.

It should be recalled that B_1 is a diagonal subalgebra of the algebra $\mathfrak{A}^2 : B_1 = \{< a,a > | a \in \mathfrak{A}\}$. Since the spectrum of the algebra $\mathfrak{A}$ is minimal, $|{}^{\mathfrak{A}^2}\!/_{\psi}| = |\mathfrak{A}|$ and we have either $|{}^{B_1}\!/_{\psi}| = |\mathfrak{A}|$, or $|{}^{B_1}\!/_{\psi}| = 1$. As $|{}^{B_1}\!/_{\psi}| \subseteq {}^{\mathfrak{A}^2}\!/_{\psi}$ and $B_1 \cong \mathfrak{A}$, we have ${}^{\mathfrak{A}^2}\!/_{\psi} = {}^{B_1}\!/_{\psi} \cong \mathfrak{A}$ for the case when $|{}^{B_1}\!/_{\psi}| = |\mathfrak{A}|$ and B_1 is not a ψ-class. If ${}^{\mathfrak{A}^2}\!/_{\psi} \not\cong \mathfrak{A}$, then $|{}^{B_1}\!/_{\psi}| = 1$ and, since $\mathfrak{A}$ is congruence-uniform and $|{}^{\mathfrak{A}^2}\!/_{\psi}| = |\mathfrak{A}|$, B_1 is a ψ-class.

Let φ, ψ be two co-atoms in $Con\mathfrak{A}^2$ such that ${}^{\mathfrak{A}^2}\!/_{\varphi}$ and ${}^{\mathfrak{A}^2}\!/_{\psi}$ are not isomorphic to the algebra $\mathfrak{A}$. As has been just noted, the set B_1 will be simultaneously both a φ- and a ψ-class. By corollary **2.4**, $\mathfrak{M}(\mathfrak{A})$ is congruence-regular and, hence, the coincidence of two φ- and ψ-classes on $\mathfrak{A}^2$ implies that of the congruences φ and ψ. Therefore, for any φ-co-atom of $Con\mathfrak{A}^2$ such that ${}^{\mathfrak{A}^2}\!/_{\varphi} \cong \mathfrak{A}$ we get:

$$ {}^{\mathfrak{A}^2}\!/_{\varphi} \cong {}^{\mathfrak{A}^2}\!/_{\Delta_\nabla^\nabla} = \mathfrak{A}^\nabla . $$

This means that the only simple algebras of the considered variety $\mathfrak{M}(\mathfrak{A})$ are the algebras $\mathfrak{A}$ and $\mathfrak{A}^\nabla$ and, since any finite $\mathfrak{M}(\mathfrak{A})$-algebra is a direct product of simple algebras, in order to complete proof of statement (b) it remains to show that for any nonzero numbers $l,n \in \omega$ we have $\mathfrak{A}^l \times \left(\mathfrak{A}^\nabla\right)^n \cong \mathfrak{A}^{l+n}$. To this end it suffices to show that the algebras $\mathfrak{A} \times \mathfrak{A}^\nabla$ and $\mathfrak{A}^2$ are isomorphic.

Let η_1 be the kernel of a projection of the algebra $\mathfrak{A}^2$ on the first cofactor, in which case $\eta_1 \wedge \Delta_\nabla^\nabla = \Delta$ (since $\mathfrak{A}^2$ is congruence-regular and B_1 is a Δ_∇^∇-class). Therefore, since $\eta_1 \wedge \Delta_\nabla^\nabla = \Delta$ and $\mathfrak{M}(\mathfrak{A})$ is congruence-permutable, we get:

$$\mathfrak{A}^2 \cong \mathfrak{A}^2/_{\eta_1 \wedge \Delta^\nabla_\nabla} \cong \mathfrak{A}^2/_{\eta_1} \times \mathfrak{A}^2/_{\Delta^\nabla_\nabla} \cong \mathfrak{A} \times \mathfrak{A}^\nabla. \blacksquare$$

Priorities. The material presented in this section and devoted to the direct presentability of varieties mostly originates from R. McKenzie [136] (lemmas **6.1**, **6.2**, **6.4**-**6.9**, theorems **6.1**-**6.3**. Lemma **6.3** is by A.F.Pixley [188]. The information on modular geometric lattices, as has been remarked in the text, can be found in a book by G.Birkhoff [14]. The data on algebras with a minimal spectrum are from R.W.Quackenbush [194] (lemma **6.11**, theorems **6.4** and **6.5**). Lemma **6.10** is by A.L.Foster and A.F.Pixley [65]. All the results presented in this section can be found in a monograph [161].

7. Representation of Varieties with Boolean Constructions

The basic aim of the present section is to clarifify of the possibilities of generation of varieties using Boolean products, Boolean powers and other Boolean constructions. In this respect, the result of theorem **7.2**, which limits the class of similar finitely generated varieties by direct products of Abelian and discriminator varieties is essential. We have already shown above (section **5**) that any algebra of a discriminator variety $\mathfrak{M}$ is isomorphic to a Boolean product of $\mathfrak{M}^+_{SI}$-algebras. For the class of congruence-distributive varieties this statement allows an inversion, i.e., the following theorem is valid.

<u>**Theorem 7.1.**</u> If $\mathfrak{M}$ is a congruence-distributive variety, then $\mathfrak{M} = I\Gamma^a(\mathfrak{M}^+_{SI})$ iff $\mathfrak{M}$ is discriminator.

Proof. By theorem **5.7**, it suffices to show that the equality $\mathfrak{M} = I\Gamma^a(\mathfrak{M}^+_{SI})$ implies that if $\mathfrak{M}$ is congruence-distributive, then it is also a discriminator variety. Let us first of all demonstrate that $\mathfrak{M}$ is semi-simple, i.e., any subdirectly non-decomposable algebra is simple. Let us assume the opposite and let $\mathfrak{A}$, thus, be a subdirectly non-decomposable algebra with a monolith β, and not simple. Let us consider a congruence-Boolean power $\mathfrak{A}^{\mathfrak{B}}(\beta)$, where $\mathfrak{B}$ is a four-element Boolean algebra. By corollary **4.1**, there is a congruence on the algebra $\mathfrak{A}^{\mathfrak{B}}(\beta)$ ($\beta^{\mathfrak{B}}$ in the notation of the proof of theorem **4.2**) other than ∇ and Δ, and comparable with any other congruence on $\mathfrak{A}^{\mathfrak{B}}(\beta)$. On the other hand, the same corollary **4.2** yields that $\mathfrak{A}^{\mathcal{B}}(\beta)$ will not be subdirectly non-decomposable: $Con_p\mathfrak{A}^{\mathcal{B}}(\beta) = \mathcal{B} \oplus Con_p\mathfrak{A}^{\mathcal{B}}(\beta)| > \beta^{\mathcal{B}}$. According to the formula $\mathfrak{M} = I\Gamma^a(\mathfrak{M}^+_{SI})$, we have $\mathfrak{A}^{\mathcal{B}}(\beta) \subseteq \prod_{i \in \mathcal{B}_1^*} \mathfrak{A}_i$, where $\mathcal{B}_1$ is a more than two-element Boolean algebra, and $\mathfrak{A}^{\mathcal{B}}(\beta)$ is a Boolean product of $\mathfrak{M}^+_{SI}$-algebras $\mathfrak{A}_i$ with respect to the Boolean algebra $\mathcal{B}_1$.

Let $b \in \mathcal{B}_1$, $b \neq 0,1$ and α_1,α_2 the kernels of the projections π_b, $\pi_{\neg b}$ of the algebra $\mathfrak{A}^{\mathcal{B}}(\beta)$ relative the open-closed subsets b, $\neg b \subseteq \mathcal{B}_1^*$. One can assume $\alpha_1,\alpha_2 \neq \Delta$ in which case, by lemma **4.1**, α_1,α_2 obey the following conditions: $\alpha_1,\alpha_2 \neq \nabla, \Delta$; $\alpha_1 \wedge \alpha_2 = \Delta$; $\alpha_1 \vee \alpha_2 = \nabla$. But these conditions contradict the fact that the congruence $\beta^{\mathcal{B}}$, which is different from Δ,∇, is comparable with α_1,α_2. It is this contradiction that proves that any subdirectly non-decomposable $\mathfrak{M}$-algebra is simple, i.e., $\mathfrak{M}$ is semi-simple, and, hence, we get $\mathfrak{M} = I\Gamma^a(\mathfrak{M}^+_S)$. By theorem **4.2**, for any algebra $\mathfrak{A}$ which is included in a congruence-distributive variety, the following statement is valid: if $\mathfrak{A}$ is a Boolean product of $\mathfrak{M}^+_S$-algebras $\mathfrak{A}_i$ relative to the Boolean algebra $\mathcal{B}_1(\mathfrak{A} \subseteq \prod_{i \in \mathcal{B}_1^*} \mathfrak{A}_i)$, then for any $f,g,h,k \in \mathfrak{A}$, we have

$< h,k > \in \theta^{\mathfrak{A}}_{f,g}$ iff

$$\{i \in \mathcal{B}_1^* \mid h(i) = k(i)\} \supseteq \{i \in \mathcal{B}_1^* \mid f(i) = g(i)\}.$$

Starting from the above description of the principal congruences of the algebra $\mathfrak{A}$, one can directly prove that the principal congruences of the algebra $\mathfrak{A}$ are permutable. Since any congruence on $\mathfrak{A}$ is a union of principal congruences, the permutability of the principal congruences implies that of any congruence on an arbitrary algebra $\mathfrak{A} \in \mathfrak{M}$. By theorem **5.10**, in order to prove that the variety $\mathfrak{M}$ is a discriminator variety it suffices to show that for any congruence $\theta^{\mathfrak{A}}_{a,b}$ in the lattice $Con\mathfrak{A}$ there is a complement to $\theta^{\mathfrak{A}}_{a,b}$, but this conclusion follows from the formula $\mathfrak{M} = I\Gamma(\mathfrak{M}^+_S)$ by lemma **5.12**. ■

As for finitely generated varieties, only those presentable as Boolean products of algebras of a certain finite class can be described completely.

<u>**Lemma 7.1.**</u> If for a certain finite set $\mathfrak{K}$ of finite algebras $\mathfrak{M}(\mathfrak{K}) = I\Gamma^a(\mathfrak{K})$, $\mathfrak{M}(\mathfrak{K}) = \mathfrak{M}_{ab} \vee \mathfrak{M}_{dist}$, where $\mathfrak{M}_{ab}(\mathfrak{M}_{dist})$ is a certain Abelian (congruence-distributive) subvariety of the variety $\mathfrak{M}$.

Proof. As, by lemma **4.2**, any finite algebra which is a Boolean product of some algebras $\mathfrak{A}_i\,(i \in I)$ is a direct product of these algebras, the equality $\mathfrak{M}(\mathfrak{K}) = I\Gamma^a(\mathfrak{K})$ implies that the variety $\mathfrak{M}(\mathfrak{K})$ is directly presentable for a certain finite set $\mathfrak{K}$ of finite algebras. Since $\mathfrak{M}(\mathfrak{K})$ is finitely generated, we get $\mathfrak{M}(\mathfrak{K}) = \mathfrak{M}(\mathfrak{F}_{\mathfrak{M}(\mathfrak{K})}(s))$ for a certain $s \in \omega$. As, by lemma **6.8**, $\mathfrak{M}(\mathfrak{K})$ is directly presentable, there is an Abelian algebra $\mathfrak{A} \in \mathfrak{M}(\mathfrak{K})$ and simple non-Abelian $\mathfrak{M}(\mathfrak{K})$-algebras $\mathfrak{B}_1,\ldots,\mathfrak{B}_n$ such that $\mathfrak{F}_{\mathfrak{M}(\mathfrak{K})}(s) \cong \mathfrak{A} \times \mathfrak{B}_1 \times \ldots \times \mathfrak{B}_n$. Let $\mathfrak{M}_{ab} = \mathfrak{M}(\mathfrak{A})$, $\mathfrak{M}_{dist} = \mathfrak{M}(\mathfrak{B}_1 \times \ldots \times \mathfrak{B}_n)$, in which case we have $\mathfrak{M}(\mathfrak{K}) = \mathfrak{M}_{ab} \vee \mathfrak{M}_{dist}$. By corollary **2.2**, the variety $\mathfrak{M}_{ab}$ generated by an Abelian algebra is Abelian itself, and for the lemma to be proved it would suffice to prove that the variety $\mathfrak{M}_{dist}$ is congruence-distributive.

Let us first show that any non-singleton subalgebra of the algebra $\mathfrak{B}_i (i \leq n)$, as well as $\mathfrak{B}_i$ itself, is a simple non-Abelian algebra.

Let us introduce the following notations: a subalgebra $\mathfrak{A}$ of the algebra $\mathfrak{B}$ will be called well-skew in $\mathfrak{B}$ provided that for any direct decomposition $\mathfrak{B} \cong \mathfrak{D}_1 \times \mathfrak{D}_2$ of the algebra $\mathfrak{B}$, i.e., for any permutable congruences $\theta,\psi \in Con\,\mathfrak{B}$ such that $\theta \cdot \psi = \nabla_{\mathfrak{B}}$, $\theta \wedge \psi = \Delta_{\mathfrak{B}}$, the inequality $(\theta|\mathfrak{A}) \cdot (\psi|\mathfrak{A}) < \nabla_{\mathfrak{A}}$ holds.

It should be recalled that, by theorem **6.1**, the directly presentable variety $\mathfrak{M}(\mathfrak{K})$ is congruence-permutable.

Let us assume that a certain non-singleton subalgebra $\mathfrak{A}_0$ of the algebra $\mathfrak{B}_0$, one of $\mathfrak{B}_1,\ldots,\mathfrak{B}_n$, is either not simple or Abelian. Let us prove that in this case there are subalgebras $\mathfrak{A}_m$ well-skew in $\mathfrak{B}_0^{2m}$ of an arbitrary large finite power in algebras of the type $\mathfrak{B}_0^{2m} (m \in \omega)$. Indeed, if $\mathfrak{A}_0 \subseteq \mathfrak{B}_0$ is not simple, and $\varphi \in Con\,\mathfrak{A}_0$, $\varphi \neq \nabla_{\mathfrak{A}_0}, \Delta_{\mathfrak{A}_0}$, let $\mathfrak{A}_m = \{ f \in \mathfrak{A}_0^m \mid \text{for any } i,j < m \quad < f(i), f(j) > \in \varphi \}$. As $\varphi \neq \Delta_{\mathfrak{A}_0}$, $|\mathfrak{A}_m| \geq 2^m$, and since $\varphi \neq \nabla_{\mathfrak{A}_0}$, and all the congruences of the algebra $\mathfrak{B}_0^m$ are, by lemma **6.9**, projections relative to the subsets of the set $\{0,\ldots,m-1\}$, one can easily observe that $\mathfrak{A}_m$ is well-skew in $\mathfrak{B}_0^m$. In the case when $\mathfrak{A}_0$ is not a one-element Abelian algebra, let $\mathfrak{A}_{2m} \subseteq \mathfrak{B}_0^{2m}$ and

$$\mathfrak{A}_{2m} = \left\{ f \in \mathfrak{A}_0^{2m} \mid f(0) + \ldots + f(m-1) = f(m) + \ldots + f(2m-1) \right\},$$

where $+$ is a certain fixed operation of addition of an Abelian group polynomially determinable in the Abelian algebra $\mathfrak{A}_0$ and correlated with all the operations of the algebra $\mathfrak{A}_0$ according to the theorem **2.20**. Let, in particular, $\mathfrak{A}_{2m}$ be a subalgebra of the algebra $\mathfrak{B}_0^{2m}$. In this case we get $|\mathfrak{A}_{2m}| = |\mathfrak{A}_0|^{2m-1}$, and again, as all the congruences of the algebra $\mathfrak{B}_0^{2m}$ are exhausted by the projections relative to the subsets of the set $\{0,...,m-1\}$,, considering pairs of the kernels of these projections $\theta,\psi \in Con\mathcal{B}_0^{2m}$ such that $\theta\cdot\psi = \nabla_{\mathcal{B}_0^{2m}}$, $\theta\wedge\psi = \Delta_{\mathcal{B}_0^{2m}}$, we can easily see that $(\theta|\mathfrak{A}_{2m})\cdot(\psi|\mathfrak{A}_{2m}) < \nabla_{\mathfrak{A}_{2m}}$, i.e., $\mathfrak{A}_{2m}$ is well-skew in $\mathcal{B}_0^{2m}$ for any $m\in\omega$. Therefore, indeed, if $\mathfrak{A}_0\subseteq\mathcal{B}_0$ is either a non-singleton Abelian or non-simple algebra, in algebras of the type $\mathcal{B}_0^{2m}$ there exist well-skew subalgebras of an arbitrary large finite power.

Let now $\mathfrak{C}$ be a certain subdirect product of simple non-Abelian algebras $\mathfrak{C}_i (i\in I)$ from the variety under consideration ($\mathfrak{M}(\mathfrak{R})$), and let $\theta,\psi\in Con\mathfrak{C}$ be such that $\theta\cdot\psi=\nabla_{\mathfrak{C}}$, $\theta\wedge\psi=\Delta_{\mathfrak{C}}$, in which case $\theta(\psi)$ is the kernel of projecting the algebra $\mathfrak{C}$ relative to the subset $A(I\setminus A)$ of the set I for a certain $A\subseteq I$. Indeed, let $\varphi_i=\ker\pi_i$ for $i\in I$. For any $i\in I$ we have either $\varphi_i\ge\theta$ or $\varphi_i\ge\psi$. In the opposite case, as $\mathfrak{C}_i$ is simple, i.e., φ_i is maximal in $Con\mathfrak{C}$, the equalities $\theta\vee\varphi_i=\psi\vee\psi_i=\nabla_{\mathfrak{C}}$ hold.

By lemma **2.17**, the equality $[\nabla,\nabla]=\nabla$ is also true on the algebra $\mathfrak{C}$, like the equalities $\lfloor\nabla,\nabla\rfloor=\nabla$ on simple non-Abelian algebras $\mathfrak{C}_i (i\in I)$. In this case we get

$$\nabla_{\mathfrak{C}}=\lfloor\nabla_{\mathfrak{C}},\nabla_{\mathfrak{C}}\rfloor=\lfloor\theta\vee\varphi_i,\psi\vee\varphi_i\rfloor\le\varphi_i\vee\lfloor\theta,\psi\rfloor\le\varphi_i\vee(\theta\wedge\psi)=\varphi_i..$$

It is the obtained contradiction that shows that for any $i\in I$ we have either $\varphi_i\ge\theta$ or $\varphi_i\ge\psi$. Let now $A=\{i\in I|\varphi_i\ge\theta\}$. Obviously, we have $\{i\in I|\varphi_i\ge\psi\}=A\setminus I$. θ_1,ψ_1 will denote $\bigwedge_{i\in A}\varphi_i$, $\bigwedge_{i\in I\setminus A}\varphi_i$, respectively. The inequalities $\theta_1\ge\theta$, $\psi_1\ge\psi$ are also obvious and, in addition, $\theta_1\wedge\psi_1=\Delta$. As has been noted above, $\mathfrak{M}$ is directly presentable and, hence, congruence-modular. Since $Con\mathfrak{C}$ is modular and we have $\theta_1\ge\theta$, $\psi_1\ge\psi$, $\theta_1\wedge\psi_1=\Delta$ and $\theta\vee\psi=\nabla$, the equalities $\theta=\theta_1,\psi=\psi_1$ hold, i.e., the mutually complementary congruences on the subdirect products of simple non-Abelian algebras of the variety under consideration have the form of projections relative to the mutually complementary sets of indices of cofactors.

Let us consider the following $\mathfrak{M}(\mathfrak{R})$-algebra: $\mathfrak{C}_m=\{f\in(\mathcal{B}_0^{2m})^{\omega}|$ for some $a\in\mathfrak{A}_{2m}|\{i\in\omega|f(i)\ne a\}|<\aleph_0\}$, where m is an arbitrary number. For any $f\in\mathfrak{C}_m$, $\hat{f}$ will denote $a\in\mathfrak{A}_{2m}$ such that $|\{i\in\omega|f(i)\ne a\}|<\aleph_0$, while for $a\in\mathfrak{A}_{2m}$, let $\bar{a}\in\mathfrak{C}_m$ such that for any $i\in\omega$ we have $\bar{a}(i)=a$.

Let us first notice that if $\mathfrak{C}_m\cong\mathfrak{D}_1\times\mathfrak{D}_2$, either $\mathfrak{D}_1$ or $\mathfrak{D}_2$ is finite, and if $a_1\ne a_2\in\mathfrak{A}_{2m}$, the images of the elements $\bar{a}_1,\bar{a}_2$ are different when projecting $\mathfrak{C}_m$ into the infinite cofactor from $\mathfrak{D}_1,\mathfrak{D}_2$. Indeed, by the definition, $\mathfrak{C}_m$ is a subdirect power of a simple non-Abelian algebra $\mathcal{B}_0$: $\mathfrak{C}_m\subseteq_p\prod_{l\in 2m\times\omega}\mathfrak{E}_l$, and $\mathfrak{E}_l=\mathfrak{E}_0$ for any

$l \in 2m \times \omega$. Let $\mathfrak{C}_m \cong \mathfrak{D}_1 \times \mathfrak{D}_2$, and $\theta, \psi \in Con\mathfrak{C}_m$ correspond to this direct decomposition of the algebra $\mathfrak{C}_m$. As has been noted earlier, we can find an $A \subseteq 2m \times \omega$ such that θ, ψ are the kernels of projection of the algebra $\mathfrak{C}_m \subseteq \prod_{l \in 2m \times \omega} \mathfrak{E}_l$ relative to the subsets $A, 2m \times \omega \setminus A$ of the set $2m \times \omega$, respectively. In order to prove that one of the algebras $\mathfrak{D}_1$, $\mathfrak{D}_2$ is finite, it suffices to prove that either one of the sets A or $2m \times \omega \setminus A$ is finite. For any $i \in \omega$ of the projections of the algebra $\mathfrak{B}_0^{2m}$ relative to the subsets $A_i = \{j \in 2m | < j,i > \in A\}$ and $B_i = \{j \in 2m | < j,i > \in 2m \times \omega \setminus A\}$ of the set $\{0,\ldots,2m-1\}$ result in a direct decomposition $\mathfrak{B}_0^{2m} \cong \mathfrak{D}_{1;n} \times \mathfrak{D}_{2,n}$. It should be remarked that these direct decompositions of the algebra $\mathfrak{B}_0^{2m}$ are trivial finite numbers $i \in \omega$, i.e., we get either $|\mathfrak{D}_{1,n}| = 1$ or $|\mathfrak{D}_{2,n}| = 1$. Indeed, otherwise there exists an infinite $I \subseteq \omega$ such that for any $i,j \in I$ we have $A_i = A_j$, $B_i = B_j$ and $A_i, B_i \neq \varnothing$.

Let θ', ψ' be the kernels of projections of the algebra $\mathfrak{B}_0^{2m}$ relative to the subsets $A' = A_i (i \in I)$ and $B' = B_i (i \in I)$ of the set $\{0,\ldots,2m-1\}$, respectively. It is obvious that $\theta' \cdot \psi' = \nabla_{\mathfrak{B}_0^{2m}}$, $\theta' \wedge \psi' = \Delta_{\mathfrak{B}_0^{2m}}$ and, as the subalgebra $\mathfrak{A}_{2m}$ is well-skew in $\mathfrak{B}_0^{2m}$, there are elements $a_1, a_2 \in \mathfrak{A}_{2m}$ such that $< a_1, a_2 > \notin (\theta' | \mathfrak{A}_{2m}) \cdot (\psi' | \mathfrak{A}_{2m})$. On the other hand, since $< \bar{a}_1, \bar{a}_2 > \in \nabla_{\mathfrak{C}_m} = \theta \cdot \psi$, there is an element $u \in \mathfrak{C}_m$ such that $< \bar{a}_1, u > \in \theta$, $< u, \bar{a}_2 > \in \psi$. As I is infinite, we get $< a_1, \hat{u} > \in \theta''$, $< \hat{u}, a_2 > \in \psi'$. As far as $\hat{u} \in \mathfrak{A}_{2m}$, we have come to a contradiction which proves that for all but a certain finite number of elements $i \in \omega$ we have either $|\mathfrak{D}_{1,i}| = 1$ or $|\mathfrak{D}_{2,i}| = 1$, in other words, either $A_i = \varnothing$ or $B_i = \varnothing$.

Let us now assume that both the equalities $A_i = \varnothing$ and $B_i = \varnothing$ are valid for an infinite set of elements $i \in \omega$. Let $a_1 \neq a_2$ be elements of $\mathfrak{A}_{2m}$. As $\theta \cdot \psi = \nabla_{\mathfrak{C}_m}$, there is a $u \in \mathfrak{C}_m$ such that $< \bar{a}_1, u > \in \theta$, $< u, \bar{a}_2 > \in \psi$. But in this case at a certain $k \in \omega$ for all $p > k$ we have $u(p) = \hat{u} \in \mathfrak{A}_{2m}$. In this case if $A_{p_1} = \varnothing$, $A_{p_2} = \varnothing$ and $p_1, p_2 > k$, $a_1 = \bar{a}_1(p_1) = u(p_1) = \hat{u}$ and $a_2 = \bar{a}_2(p_2) = u(p_2) = \hat{u}$, i.e., $a_1 = a_2$ contradicts the choice of a_1, a_2. Therefore, either $A_i = \{0,\ldots,2m-1\}$ is valid for all but the finite numbers $i \in \omega$, or this statement is valid for B_i. Thus, either A or B is finite, i.e., one of the algebras in the decomposition $\mathfrak{C}_m \cong \mathfrak{D} \times \mathfrak{D}_2$ must be finite. In this case it is obvious that for different $a_1, a_2 \in \mathfrak{A}_{2m}$, the projections of the elements $\bar{a}_1, \bar{a}_2$ onto the infinite cofactor from $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are also different.

Since $\mathfrak{K}$ is a finite set of finite algebras, let $s \in \omega$ be such that $s > |\mathfrak{D}|$ for any $\mathfrak{D} \in \mathfrak{K}$. According to the equality $\mathfrak{M}(\mathfrak{K}) = I\Gamma^a(\mathfrak{K})$, the algebra $\mathfrak{C}_s$ is representable as a Boolean product of $\mathfrak{K}$-algebras: $\mathfrak{C}_s \subseteq_{bp} \prod_{i \in \mathfrak{B}^*} \mathfrak{R}_i$, where $\mathfrak{R}_i \in \mathfrak{K}$, and $\mathfrak{B}$ is a Boolean algebra. For $i \in \mathfrak{B}^*$ the inequalities $|\mathfrak{A}_{2s}| > |\mathfrak{R}_i|$ imply covering of the space $\mathfrak{B}^*$ with open-closed sets $\lfloor|\bar{a}_1 = \bar{a}_2|\rfloor$, where $a_1 \neq a_2$ are elements of $\mathfrak{A}_{2s}$. Since $\mathfrak{B}^*$ is infinite, at least one of the sets $\lfloor|\bar{a}_1 = \bar{a}_2|\rfloor$ $(a_1 \neq a_2 \in \mathfrak{A}_{2s})$ will be infinite. But in this case $\mathfrak{C}_s = \mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_1 = \mathfrak{C}_s|\lfloor|\bar{a}_1 \neq \bar{a}_2|\rfloor$, $\mathfrak{D}_2 = \mathfrak{C}_s|\lfloor|\bar{a}_1 = \bar{a}_2|\rfloor$, in which case $\mathfrak{D}_2 = \mathfrak{C}_s|\lfloor|\bar{a}_1 = \bar{a}_2|\rfloor$ is infinite, and the images of the elements $\bar{a}_1, \bar{a}_2$ coincide when projecting $\mathfrak{C}_s$ on $\mathfrak{D}_2 = \mathfrak{C}_S|\lfloor|\bar{a}_1 = \bar{a}_2|\rfloor$, which contradicts the statement proved above. It is this contradiction that shows that all non-singleton subalgebras of the algebras $\mathfrak{B}1,\ldots,\mathfrak{B}_n$ are indeed simple and non-Abelian.

As long as $\mathfrak{M}_{dist}$ is generated by a finite set $\{\mathfrak{B}_1,\ldots,\mathfrak{B}_n\}$ of finite algebras, $\mathfrak{F}_{\mathfrak{M}_{dist}}(3)$ is finite and, hence, $\mathfrak{F}_{\mathfrak{M}_{dist}}(3) \in HSP_{fin}(\mathfrak{B}_1,\ldots\mathfrak{B}_n)$. By theorem **2.9**, in order to prove that the variety $\mathfrak{M}_{dist}$ is congruence-distributive, it suffices to show that $Con\mathfrak{F}_{\mathfrak{M}_{dist}}(3)$ is distributive, while by the inclusion $\mathfrak{F}_{\mathfrak{M}_{dist}}(3) \in HSP_{fin}(\mathfrak{B}_1,\ldots,\mathfrak{B}_n)$, it is enough to show that the lattice $Con\mathfrak{A}$ is distributive for any algebra $\mathfrak{A} \in SP_{fin}(\mathfrak{B}_1,\ldots,\mathfrak{B}_n)$. But if $\mathfrak{A} \in SP_{fin}(\mathfrak{B}_1,\ldots,\mathfrak{B}_n)$, $\mathfrak{A}$ is a subdirect product of some finite family of the subalgebras of the algebras $\mathfrak{B}_1,\ldots,\mathfrak{B}_n$. As has been shown above, these subalgebras are simple and, hence, as $\mathfrak{M}_{dist}$ is congruence-permutable, $\mathfrak{A}$ will be isomorphic to some finite direct product of these subalgebras by lemma **6.10**, i.e., $\mathfrak{A}$ is a direct product of simple non-Abelian algebras from the variety $\mathfrak{M}$. By lemma **6.9**, $Con\mathfrak{A}$ is distributive, which fact implies, as has been mentioned above, that $\mathfrak{M}_{dist}$ is congruence-distributive. Therefore, indeed, under the conditions of the lemma under discussion we have $\mathfrak{M} = \mathfrak{M}_{ab} \vee \mathfrak{M}_{dist}$, where $\mathfrak{M}_{ab}$ is Abelian, and $\mathfrak{M}_{dist}$ is congruence-distributive. ■

It should be recalled that, in line with theorem **2.20**, any Abelian variety $\mathfrak{M}$ is polynomially equivalent to the variety of left unitary R-modules over some ring R with unity. Let us denote this ring R by $R(\mathfrak{M})$. It should be also noted that for a finitely generated $\mathfrak{M}$, the ring $R(\mathfrak{M})$ is finite (for more details on the structure of the ring $R(\mathfrak{M})$ see, for instance [32], [166]).

Theorem 7.2. For any variety $\mathfrak{M}$, the following conditions are equivalent:

(a) for a certain finite set $\mathfrak{K}$ of finite $\mathfrak{M}$-algebras, the equality $\mathfrak{M} = I\Gamma^a(\mathfrak{K})$ holds;

(b) $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$, where $\mathfrak{M}_1$ is a finitely generated discriminator variety, $\mathfrak{M}_2$ is a finitely generated Abelian variety, and the ring $R(\mathfrak{M}_2)$ has a finite type of presentations.

Proof. Let us show that (b) follows from (a).

Let $\mathfrak{M} = I\Gamma^a(\mathfrak{K})$ for a certain finite set $\mathfrak{K}$ of finite algebras. According to lemma **7.1**, for some Abelian $\mathfrak{M}_{ab}$ and congruence-distributive $\mathfrak{M}_{dist}$ subvarieties of the variety $\mathfrak{M}$, we have $\mathfrak{M} = \mathfrak{M}_{ab} \vee \mathfrak{M}_{dist}$. By theorem **4.4**, $\mathfrak{M}(\mathfrak{K}) = \mathfrak{M}_{ab} \otimes \mathfrak{M}_{dist}$, while by lemma **4.8**, for the finite sets $\mathfrak{K}_1 = \mathfrak{K} \cap \mathfrak{M}_{ab}$, $\mathfrak{K}_2 = \mathfrak{K} \cap \mathfrak{M}_{dist}$ of finite algebras we have $\mathfrak{M}_{ab} = I\Gamma^a(\mathfrak{K}_1)$, $\mathfrak{M}_{dist} = I\Gamma^a(\mathfrak{K}_2)$. As has been remarked in the conclusion of section **4**, if the Abelian algebra $\mathfrak{A} \in \mathfrak{M}_{ab}$ is representable as a Boolean product of the algebras $\mathfrak{A}_i \in \mathfrak{K}_1$, the module polynomially equivalent to the algebra $\mathfrak{A}$ will be representable as a Boolean product of the modules polynomially equivalent to the algebras $\mathfrak{A}_i$. In particular, any finite $R(\mathfrak{M}_{ab})$-module will be isomorphic to a Boolean product (to a direct product, i.e., to a direct sum, by lemma **4.2**) of the modules of polynomially equivalent to the $\mathfrak{K}_1$-algebras. Therefore, $R(\mathfrak{M}_{ab})$ has a finite type of presentations.

Let us now demonstrate that $\mathfrak{M}_{dist}$ is a discriminator variety. Let us first notice that the formula $\mathfrak{M}_{dist} = I\Gamma^a(\mathfrak{K}_2)$ implies the direct presentability and, hence, the congruence-permutability of $\mathfrak{M}_{dist}$. Let us also notice, exactly as we did in the beginning of the proof of theorem **7.1**, that the formula $\mathfrak{M}_{dist} = I\Gamma^a(\mathfrak{K}_2)$ implies the semi-simplicity of the variety $\mathfrak{M}_{dist}$. Therefore, any $\mathfrak{K}_2$-algebra, since it is finite, is representable as a subdirect product of simple $\mathfrak{M}_{dist}$-algebras. Lemma **6.10** makes it possible to state, as $\mathfrak{M}_{dist}$ is permutable, that any $\mathfrak{K}_2$-algebra is isomorphic to a direct product of $(\mathfrak{M}_{dist})_s$-algebras.

Let now $\mathfrak{K}_2 = \{\mathfrak{C}_1,\ldots,\mathfrak{C}_n\}$, and $\mathfrak{A}$ an arbitrary $\mathfrak{M}_{dist}$-algebra. Let also $\mathfrak{A} \subseteq \prod_{i\in\mathcal{B}^*}\mathfrak{A}_i$ be a representation of $\mathfrak{A}$ as a Boolean product of $\mathfrak{K}_2$-algebras, i.e., in particular, $\mathfrak{A}_i \in \{\mathfrak{C}_1,\ldots,\mathfrak{C}_n\}$ for any $i \in \mathcal{B}^*$. By lemma **4.1**, for any $i \in \mathcal{B}^*$ there is an open-closed neighborhood N_i of the point i such that for any $j \in N_i$ the algebra $\mathfrak{A}_i$ is isomorphically embeddable into the algebra $\mathfrak{A}_j$. Taking into account this fact as the well compactness of $\mathcal{B}^*$, one can obtain an open-closed partition $N_1',\ldots,N_n'$ of the space $\mathcal{B}^*$ such that for $j \in N_l'$ we get $\mathfrak{A}_j \cong \mathfrak{C}_l$ (at $l \leq n$). Let, then, $\mathfrak{A}(l)$ denote the projection of the algebra $\mathfrak{A}$ relative to the subset N_l'; then $\mathfrak{A}(l)$ is the Boolean product of the algebras $\mathfrak{C}_l$, and $\mathfrak{A} \cong \prod_{l\leq n}\mathfrak{A}(l)$. Let $\mathfrak{C}_l = \prod_{k\leq m_l}\mathfrak{C}_l^k$ $(l \leq n)$ be the above-mentioned representation of the algebras $\mathfrak{C}_l$ as a direct product of $(\mathfrak{M}_{dist})_s$-algebras. If

$$\mathfrak{A}(l) \subseteq (\mathfrak{C}_l)^{\mathcal{B}_l^*} = (\prod_{k \le m_l} \mathfrak{C}_l^k)^{\mathcal{B}^*}$$

is the representation of the algebra $\mathfrak{A}(l)$ as a Boolean product of the algebras isomorphic to the algebra $\mathfrak{C}_l$, let $\rho_k(\mathfrak{A}(l))$ be a natural projection of the algebra $\mathfrak{A}(l)$ in this representation onto the algebra $\left(\mathfrak{C}_l^k\right)^{\mathcal{B}_l^*}$. The algebras $\rho_k(\mathfrak{A}(l))$ are obviously Boolean products of the algebras $\mathfrak{C}_l^k$ with the degrees $\mathcal{B}_l^*$. It is also obvious that $\mathfrak{A}(l)$ is isomorphic to a subdirect product of the algebras $\rho_1(\mathfrak{A}(l)),\dots,\rho_{m_l}(\mathfrak{A}(l))$. As the congruences on $\mathfrak{M}_{dist}$ are permutable, $\mathfrak{A}(l)$ will be isomorphic to the direct product $\prod_{k \le m_l} \rho_k(\mathfrak{A}(l))$. Therefore, the initial algebra $\mathfrak{A}$ is representable as a direct product of a finite number of Boolean products of $(\mathfrak{M}_{dist})_s$-algebras, and, hence, $\mathfrak{A}$ itself is isomorphic to a Boolean product of $(\mathfrak{M}_{dist})_s$-algebras. By theorem **7.1**, $\mathfrak{M}_{dist}$ is a discriminator variety, the implication (a) $\to$ (b) of the theorem is thus completely proved.

Let us now show that (a) follows from (b). Let the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ obey statement (b) of the theorem. According to theorem **2.26** (a), any $R(\mathfrak{M}_2)$-module is representable as a direct sum of finite directly non-decomposable modules, the number of the latter being finite (to the accuracy of isomorphism). Let $\mathfrak{D}$ be a finite family of finite directly non-decomposable $R(\mathfrak{M}_2)$-modules. Therefore, for $R(\mathfrak{M}_2)$-module, $M \cong \bigotimes_{i \in I} M_i$, where $M_i \in \mathfrak{D}$ for $i \in I$. One can assume that we have $|M_{i_0}| = 1$ for some $i_0 \in I$. Let us convert the set I into a Boolean space by considering the Boolean algebra of finite and co-finite subsets of the set $I \setminus \{i_0\}$, and by identifying i_0 with the Frechet filter of this Boolean algebra. We obviously get an isomorphism from the module M with the Boolean product of the modules M_i with respect to the Boolean Frechet algebra over the set $I \setminus \{i_0\}$. As has been remarked in the conclusion of section **4**, the $\mathfrak{M}_2$-algebra polynomially equivalent to the module M will be in this case representable as a Boolean product of $\mathfrak{M}_2$-algebras polynomially equivalent to the modules $M_i \in \mathfrak{D}^+$. Therefore, in the case under discussion, any $\mathfrak{M}_2$-algebra is isomorphic to a Boolean product of a finite number of finite $\mathfrak{M}_2$-algebras, i.e., $\mathfrak{M}_2 = I\Gamma^a(\mathfrak{K}_2)$, where $\mathfrak{K}_2$ is finite, and $\mathfrak{K}_2$-algebras are finite. By theorem **2.16**, for a finitely generated discriminator variety $\mathfrak{M}_1$, the set $(\mathfrak{M}_1)_{SI}^+$ is a finite set of finite algebras, while by theorem **7.1** we have $\mathfrak{M}_1 = I\Gamma^a((\mathfrak{M}_1)_{SI}^+)$.

Therefore, any $\mathfrak{M}$-algebra is a Cartesian product of two Boolean products of

$\mathfrak{K}_2$- and $(\mathfrak{M}_1)^+_{SI}$-algebras, respectively. By lemma **4.2**, this Cartesian product will be isomorphic to a Boolean product of $(\mathfrak{K}_2 \cup (\mathfrak{M}_1)^+_{SI})$-algebras, i.e., $\mathfrak{M} = I\Gamma^a(\mathfrak{K}_2 \cup (\mathfrak{M}_1)^+_{SI})$. ■

Corollary 7.1.

(a) For the variety of rings $\mathfrak{M}$, the equality $\mathfrak{M} = I\Gamma^a(\mathfrak{K})$ holds for a certain final family $\mathfrak{K}$ of finite rings iff $\mathfrak{M}$ is generated by some set of finite rings with zero multiplication and a set of finite fields.

(b) For the variety of groups $\mathfrak{M}$, the equality $\mathfrak{M} = I\Gamma^a(\mathfrak{K})$ holds for a certain finite family $\mathfrak{K}$ of finite groups iff $\mathfrak{M}$ is a variety of Abelian groups of a finite exponent.

Proof. (a) Let $\mathfrak{M}$ be a variety of rings, and let $\mathfrak{M} = I\Gamma^a(\mathfrak{K})$ for a certain finite family $\mathfrak{K}$ of finite rings. Then, by theorem **7.2**, $\mathfrak{M} = \mathfrak{M}(\mathfrak{K}_1) \otimes \mathfrak{M}(\mathfrak{K}_2)$, where $\mathfrak{M}(\mathfrak{K}_1)$ is Abelian, while $\mathfrak{M}(\mathfrak{K}_2)$ are discriminator varieties of rings generated by the finite sets of finite rings $\mathfrak{K}_1, \mathfrak{K}_2$, respectively. As has been observed before the proof of theorem **2.19**, $\mathfrak{M}(\mathfrak{K}_1)$ will be a variety of rings with zero multiplication, while according to theorem **5.1**, $\mathfrak{M}(\mathfrak{K}_2)$ is generated by a finite set of finite fields. Therefore, the statement (a) has been proved in one direction. The proof of this statement in the opposite direction is reduced to the following:

(1) by theorem **5.1**, the variety $\mathfrak{M}_1$ generated by a finite set of finite fields is a discriminator variety and, hence, by theorem **5.7**, $\mathfrak{M} = I\Gamma^a(\mathfrak{K})$, and it is the set $(\mathfrak{M}_1)^+_{SI}$ that is a finite family of finite fields generating $\mathfrak{M}_1$;

(2) by the Prüfer, theorem a variety generated by a finite family of finite Abelian groups (finite rings with zero multiplication) consists of direct sums (this being, obviously, a particular case for Boolean products) of finite cyclic groups (finite cyclic rings with zero multiplication);

(3) a Cartesian product of two algebras isomorphic to certain Boolean products of $\mathfrak{K}$-algebras will be, by lemma **4.2**, itself isomorphic to a Boolean product of $\mathfrak{K}$-algebras for any class of $\mathfrak{K}$.

The statement (b) results from the remark before the proof of theorem **2.19** concerning the equality of the notions of an Abelian group in the group-theoretical sense and in that of commutator theory, as well as from the fact that a non-singleton group cannot be a discriminator algebra. ■

Making use of theorem **7.2** one can obtain a complete description of finite algebras $\mathfrak{A}$ generating varieties which consist only of Boolean powers of the algebra $\mathfrak{A}$ and isomorphic to these powers of algebras, i.e., the varieties having the same, to a certain extent, structure as those of Boolean algebras.

Theorem 7.3. For an arbitrary finite algebra $\mathfrak{A}$, the equality $\mathfrak{M}(\mathfrak{A}) = IP_B(\mathfrak{A})$ is valid iff either $\mathfrak{A}$ is a quasi-primal algebra without proper subalgebras, or $\mathfrak{A}$ is a simple Abelian algebra having no non-singleton proper subalgebras but having a one-element subalgebra.

Proof. Let for a finite algebra $\mathfrak{A}$ the equality $\mathfrak{M}(\mathfrak{A}) = IP_B(\mathfrak{A})$ hold. This equality yields the following equality: $\mathfrak{M}(\mathfrak{A}) = I\Gamma^a(\mathfrak{A})$. Therefore, by theorem **7.2**, we get $\mathfrak{M}(\mathfrak{A}) = \mathfrak{M}_1 \otimes \mathfrak{M}_2$, where $\mathfrak{M}_1$ is a discriminator variety and $\mathfrak{M}_2$ is an Abelian one. Since $\mathfrak{A}$ is the only subdirectly non-decomposable algebra of the variety $\mathfrak{M}(\mathfrak{A})$, we have either $\mathfrak{M}(\mathfrak{A}) = \mathfrak{M}_1$, or $\mathfrak{M}(\mathfrak{A}) = \mathfrak{M}_2$.

In the former case, as $\mathfrak{A}$ is a finite subdirectly non-decomposable algebra generating discriminator varieties, it is quasi-primal. Since any non-singleton $\mathfrak{M}(\mathfrak{A})$-algebra has the form $\mathfrak{A}^{\mathfrak{B}}$, i.e., the power cannot be less than $|\mathfrak{A}|$, $\mathfrak{A}$ has no non-singleton subalgebras. Let us show that $\mathfrak{A}$ can have no one-element subalgebras, either. Let us assume, to the contrary, that $a \in \mathfrak{A}$ and $\{a\}$ is a subalgebra of the algebra $\mathfrak{A}$. Let us consider a subalgebra $\mathfrak{C}$ of the algebra $\mathfrak{A}^{\omega}$ such that

$$\mathfrak{C} = \left\{ f \in \mathfrak{A}^{\omega} \,|\, |\{ i \in \omega \,|\, f(i) \neq a \}| < \aleph_0 \right\}.$$

Since the equality $\mathfrak{M}(\mathfrak{A}) = IP_B(\mathfrak{A})$ holds, we have the isomorphism $\mathfrak{C} \cong \mathfrak{A}^{\mathfrak{B}}$ for some Boolean algebra $\mathfrak{B}$. By theorem **5.6**, unit congruences on Boolean powers of simple algebras in discriminator varieties are principal, i.e., according to the isomorphism $\mathfrak{C} \cong \mathfrak{A}^{\mathfrak{B}}$ we get $\nabla_{\mathfrak{C}} \in Con_p\mathfrak{C}$. Let $\nabla_{\mathfrak{C}} = \theta^{\mathfrak{C}}_{f,g}$ for some $f, g \in \mathfrak{C}$, and let $n \in \omega$ be such that for $i \geq n$ we have $f(i) = g(i) = a$. Let α be the kernel of $\mathfrak{C}$ projecting relative to the first n co-ordinates. By theorem **5.6**, we get $\theta^{\mathfrak{C}}_{f,g} \subseteq \alpha \neq \nabla_{\mathfrak{C}}$. It is the contradiction obtained that proves that $\mathfrak{A}$ cannot have one-element subalgebras, i.e., that in the case under discussion $\mathfrak{A}$ is quasi-primal with no proper subalgebras.

In the latter case $\mathfrak{A}$ is a subdirectly non-decomposable Abelian algebra. Moreover, it is an simple Abelian algebra, since by the Magari theorem the variety $\mathfrak{M}(\mathfrak{A})$ must have a simple algebra, and it is only the algebra $\mathfrak{A}$ that can be such according to the equality $\mathfrak{M}(\mathfrak{A}) = I\Gamma^a(\mathfrak{A})$. The same equality implies that $\mathfrak{A}$ has no non-singleton subalgebras, as well as that there is an isomorphism $\mathfrak{A}^{\nabla} \cong \mathfrak{A}$, where $\mathfrak{A}^{\nabla}$ is the algebra described theorem **6.5** and has a one-element subalgebra. Thus, we have proved the statement of the theorem in one direction.

Let us now prove the converse statement considering again both cases separately. Let $\mathfrak{A}$ be a quasi-primal algebra with no proper subalgebras. As $\mathfrak{A}$ contains no proper subalgebras, $\mathfrak{A}$ is the only subdirectly non-decomposable $\mathfrak{M}(\mathfrak{A})$-algebra by theorem **2.16**. Therefore, by theorem **5.7**, any $\mathfrak{M}(\mathfrak{A})$-algebra is representable as a Boolean product of a one-element algebra and the algebra $\mathfrak{A}$. In line with lemma **4.2**, any finite algebra will in this case have the form $\mathfrak{A}^n$ for some $n \in \omega$ and, as $\mathfrak{M}(\mathfrak{A})$ is locally finite, any finitely generated algebra will have the same form. This, in particular, implies that any non-singleton $\mathfrak{M}(\mathfrak{A})$-algebra contains a subalgebra isomorphic to the algebra $\mathfrak{A}$.

Let $\mathfrak{C} \in \mathfrak{M}(\mathfrak{A})$ and $|\mathfrak{C}| > 1$. As we have already noted, $\mathfrak{C} \cong \mathfrak{C}_1 \subseteq \prod_{i \in \mathfrak{B}^*} \mathfrak{A}_i$, where $\mathfrak{C}_1$ is the Boolean product of the algebras $\mathfrak{A}_i$ with the index $\mathfrak{B}$, and for $i \in \mathfrak{B}^*$ we have either $\mathfrak{A}_i \cong \mathfrak{A}$ or $|\mathfrak{A}_i| = 1$. Let $\mathfrak{A}_0$ be a subalgebra of the algebra $\mathfrak{C}_1$ isomorphic to the algebra $\mathfrak{A}$. It should be remarked that as $\mathfrak{A}$ has no proper subalgebras, we have $\pi_i(\mathfrak{A}_0) = \mathfrak{A}$ for any $i \in \mathfrak{B}^*$ provided that $\pi_i(\mathfrak{C}_1) = \mathfrak{A}$. Let $f \neq g$ be elements of the algebra $\mathfrak{A}_0$. Since $\mathfrak{A}_0$ is simple, $\theta^{\mathfrak{A}_0}_{f,g} = \nabla_{\mathfrak{A}_0}$. By theorem **5.6** we get

$$\{i \in \mathfrak{B}^* \mid f(i) \neq g(i)\} = \{i \in \mathfrak{B}^* \mid |\pi_i(\mathfrak{A}_0)| \neq 1\}.$$

Since, as has been noticed earlier, we have

$$\{i \in \mathfrak{B}^* \mid |\pi_i(\mathfrak{A}_0)| \neq 1\} = \{i \in \mathfrak{B}^* \mid |\pi_i(\mathfrak{C}_1)| \neq 1\},$$

we get the following equality:

$$\{i \in \mathfrak{B}^* \mid f(i) \neq g(i)\} = \{i \in \mathfrak{B}^* \mid |\pi_i(\mathfrak{C}_1)| \neq 1\}.$$

As $\mathfrak{C}_1$ is a Boolean product, $\{i \in \mathfrak{B}^* \mid f(i) \neq g(i)\}$ is open-closed in $\mathfrak{B}^*$; let this set correspond to an element b of the algebra $\mathcal{B}$. According to the equality $\{i\in\mathcal{B}^*|f(i)\neq g(i)\}=\{i\in\mathcal{B}^*|\pi_i(\mathfrak{C}_1)|\neq 1\}$, the projection of the algebra $\mathfrak{C}_1$ relative to the element $b\in\mathcal{B}$ is isomorphic to the algebra $\mathfrak{C}_1$ and, moreover, this projection is isomorphic to the Boolean product of the algebra $\mathfrak{A}$ relative to the Boolean algebra $\mathcal{B}|b$. Any non-singleton algebra $\mathfrak{C}_1\in\mathfrak{M}(\mathfrak{A})$ can therefore be considered a Boolean product of the algebra $\mathfrak{A}$ with respect to a certain algebra $\mathcal{B}$, while $\mathfrak{C}_1$ can be considered to contain a subalgebra $\mathfrak{A}_0\cong\mathfrak{A}$ such that for any $i\in\mathcal{B}^*$ we have $\pi_i(\mathfrak{A}_0)=\mathfrak{A}$.

Let us fix an $i_0\in\mathcal{B}$ and let f_a stand for an element of the algebra $\mathfrak{A}_0$ such that $f_a(i_0)=a$ for any $a\in\mathfrak{A}$. Let g_i $(i\in\mathcal{B}^*\setminus\{i_0\})$we will denote the mapping of the algebra $\mathfrak{A}$ to the algebra $\mathfrak{A}$ such that $g_i(a)=f_a(i)$ for any $a\in\mathfrak{A}$. It is obvious that g_i is a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}$, and, since $\mathfrak{A}$ is finite and simple, g_i is an automorphism of $\mathfrak{A}$ for any $i\in\mathcal{B}^*\setminus\{i_0\}$. Let us determine the mapping h from the algebra $\mathfrak{C}_1$ to the algebra $\mathfrak{A}^{\mathcal{B}^*}$ setting, for any $i\in\mathcal{B}^*\setminus\{i_0\}$ and any $f\in\mathfrak{C}_1$, $h(f)(i)=g_i^{-1}(f(i_0))$ and $h(f)(i_0)=f(i_0)$. By the definition of the mappings g_i, $h(f(a))$ is a constant function on $\mathcal{B}^*$ taking the value a for any $a\in\mathfrak{A}$. Since $\mathfrak{C}_1$ is a Boolean product of the algebras $\pi_i(\mathfrak{C}_1)=\mathfrak{A}$ with the index $\mathcal{B}$, in line with properties **1, 2** of the definition **4.1** of a Boolean product, we see that for $\varphi\in h(\mathfrak{C}_1)$ $\varphi^{-1}(a)$ is open-closed in $\mathcal{B}^*$ for any $a\in\mathfrak{A}$. Therefore, $h(\mathfrak{C}_1)$ is contained in the Boolean power $\mathfrak{A}^{\mathcal{B}}$ of the algebra $\mathfrak{A}$. It is directly obvious from the definition of h that it is an isomorphism between $\mathfrak{C}_1$ and $\mathfrak{A}^{\mathcal{B}}$. Any non-singleton $\mathfrak{M}(\mathfrak{A})$-algebra is thus isomorphic to an algebra of the type $\mathfrak{A}^{\mathcal{B}}$, i.e., $\mathfrak{M}(\mathfrak{A})=IP_B(\mathfrak{A})$ in the case when $\mathfrak{A}$ is quasi-primal and has no proper subalgebras.

Let now $\mathfrak{A}$ be a simple finite Abelian algebra with no non-singleton proper subalgebras, but containing a one-element subalgebra. By theorem **6.15**, any finite $\mathfrak{M}(\mathfrak{A})$-algebra will have the form $\mathfrak{A}^n(n\in\omega)$. By theorem **2.21**, the only directly non-decomposable module in the variety $M_{R_{\mathfrak{M}(\mathfrak{A})}}$ of $R_{\mathfrak{M}(\mathfrak{A})}$-modules polynomially equivalent to the Abelian variety $\mathfrak{M}(\mathfrak{A})$ will be the module $M_{\mathfrak{A}}$, which is polynomially equivalent to the algebra $\mathfrak{A}$. Therefore, the ring $R_{\mathfrak{M}(\mathfrak{A})}$ will be a ring with a finite type of representations and, by theorem **2.26**, any $R_{\mathfrak{M}(\mathfrak{A})}$-module will be isomorphic to a direct sum of modules $M_{\mathfrak{A}}$. As has already been noted (see section **3**), the direct sum $\bigoplus_{i\in I}(M_{\mathfrak{A}})_i$, where $(M_{\mathfrak{A}})_i=M_{\mathfrak{A}}$, is isomorphic to some Boolean degree $M_{\mathfrak{A}}^{\mathcal{B}}$. By theorem **2.21**, this implies an isomorphism from an arbitrary $\mathfrak{M}(\mathfrak{A})$-algebra with a certain Boolean power of the algebra $\mathfrak{A}$, i.e., in the case under consideration, the equality $\mathfrak{M}(\mathfrak{A}) = IP_B(\mathfrak{A})$ again holds. ■

By way of concluding this section, let us consider the problem of the representation of a variety generated by a given finite algebra using subBoolean powers of this algebra.

Theorem 7.4. For a finite algebra $\mathfrak{A}$, the equality $\mathfrak{M}(\mathfrak{A}_2) = IP_{SB}(\mathfrak{A})$ is equivalent to the validity of the following conditions:

(1) $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$, where $\mathfrak{A}_1$ is an Abelian algebra and $\mathfrak{M}(\mathfrak{A}_2)$ is a discriminator variety;

(2) if both algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are not singleton, they both contain singleton subalgebras;

(3) $\mathfrak{M}(\mathfrak{A}_1) = IP_{SB}(\mathfrak{A}_1)$ and $\mathfrak{M}(\mathfrak{A}_2) = IP_{SB}(\mathfrak{A}_2)$.

Proof. As the equality $\mathfrak{M}(\mathfrak{A}) = IP_{SB}(\mathfrak{A})$ yields the equality $\mathfrak{M}(\mathfrak{A}) = I\Gamma^a(S(\mathfrak{A}))$, according to theorem **7.2**, we have $\mathfrak{M}(\mathfrak{A}) = \mathfrak{M}_1 \otimes \mathfrak{M}_2$, where $\mathfrak{M}_1$ is an Abelian variety, while $\mathfrak{M}_2$ is a discriminator variety. Therefore, there are algebras $\mathfrak{A}_i \in \mathfrak{M}_i$ such that $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$, in which case $\mathfrak{A}_1$ is an Abelian algebra, while the variety $\mathfrak{M}(\mathfrak{A}_2)$, as it is contained in $\mathfrak{M}_2$, is a discriminator variety. By lemma **4.8**, any sub-Boolean power $\mathfrak{D}$ of the algebra $\mathfrak{A}$ with the index $\mathfrak{B}$ is presentable as a direct product $\mathfrak{D}_1 \times \mathfrak{D}_2$ of the sub-Boolean powers $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with the same index $\mathfrak{B}$. Therefore, as $\mathfrak{M}(\mathfrak{A}_i) \subseteq \mathfrak{M}_i$ and, hence, $\mathfrak{M}(\mathfrak{A}_i) \cap \mathfrak{M}(\mathfrak{A}_2)$ consists of only a one-element algebra, we get $\mathfrak{M}(\mathfrak{A}_1) = IP_{SB}(\mathfrak{A}_1)$ and $\mathfrak{M}(\mathfrak{A}_2) = IP_{SB}(\mathfrak{A}_2)$.

Let us now assume that both algebras, $\mathfrak{A}_1$ and $\mathfrak{A}_2$, are non-singleton, in which case $\mathfrak{A}_1$ is isomorphic to a certain sub-Boolean power of the algebra $\mathfrak{A}$ with a non-singleton index $\mathfrak{B}$. As has been noted earlier, $\mathfrak{A}_1$ will be isomorphic to the direct product $\mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_i$ are certain sub-Boolean powers of the algebras $\mathfrak{A}_i$ with the same index $\mathfrak{B}$. As $\mathfrak{M}(\mathfrak{A}_1) \cap \mathfrak{M}(\mathfrak{A}_2)$ consists of a single one-element algebra, $\mathfrak{D}_2$ must also be non-singleton, but since $\mathfrak{D}_2$ is a sub-Boolean power of the algebra $\mathfrak{A}_2$ with the one-element index $\mathfrak{B}$, this is possible only when $\mathfrak{A}_2$ contains a one-element subalgebra. The existence of a one-element subalgebra in the algebra $\mathfrak{A}_1$ is proved in an analogous way.

Let us now prove that the conditions (1) → (3) of the theorem result in the equality $\mathfrak{M}(\mathfrak{A}) = IP_{SB}(\mathfrak{A})$. The conditions (2)-(3) obviously yield the following equality:

$$I\{\mathfrak{D}_1 \times \mathfrak{D}_2 | \mathfrak{D}_i \in \mathfrak{M}(\mathfrak{A}_i)\} = IP_{SB}(\mathfrak{A}_1 \times \mathfrak{A}_2),$$

i.e., taking into account the condition (1), $I\{\mathfrak{D}_1 \times \mathfrak{D}_2 | \mathfrak{D}_i \in \mathfrak{M}(\mathfrak{A}_i)\} = IP_{SB}(\mathfrak{A})$

It also follows from the condition (1) that $\mathfrak{M}(\mathfrak{A}) = \mathfrak{M}(\mathfrak{M}(\mathfrak{A}_1) \cup \mathfrak{M}(\mathfrak{A}_2))$, but, by theorem **4.4**, $\mathfrak{M}(\mathfrak{A}_1)$ and $\mathfrak{M}(\mathfrak{A}_2)$ are independent and, hence, $\mathfrak{M}(\mathfrak{A}) = \mathfrak{M}(\mathfrak{A}_1) \otimes \mathfrak{M}(\mathfrak{A}_2)$, i.e., $\mathfrak{M}(\mathfrak{A}) = IP_{SB}(\mathfrak{A})$. ■

By virtue of the theorem just proved, the problem of the description of finite algebras $\mathfrak{A}$ for which the equality $\mathfrak{M}(\mathfrak{A}) = IP_{SB}(\mathfrak{A})$ is valid falls into the problems of describing similar Abelian algebras and those generating discriminator varieties with this property. For Abelian algebras $\mathfrak{A}$, this problem can be formulated in terms of the language of modules over the rings with a finite type of representation which are polynomially equivalent to the variety $\mathfrak{M}(\mathfrak{A})$, while for quasi-primal $\mathfrak{A}$ it is possible to obtain a complete description of such $\mathfrak{A}$ with the property $\mathfrak{M}(\mathfrak{A}) = IP_{SB}(\mathfrak{A})$. Let us first prove a number of auxiliary lemmas.

Lemma 7.2. Let $\mathfrak{A}$ be a quasi-primal algebra, in which case any $\mathfrak{M}(\mathfrak{A})$-algebra is isomorphically embeddable into some Boolean power of the algebra $\mathfrak{A}$.

Proof. As $\mathfrak{A}$ is quasi-primal, the class of subdirectly non-decomposable $\mathfrak{M}(\mathfrak{A})$-algebras coincides with the class $S(\mathfrak{A})$ by theorem **2.16**. Therefore, any $\mathfrak{M}(\mathfrak{A})$-algebra is isomorphically embeddable into some direct power $\mathfrak{A}^X$ of the algebra $\mathfrak{A}$. But since the algebra $\mathfrak{A}$ is finite, the algebra $\mathfrak{A}^X$ is obviously isomorphic to a Boolean power $\mathfrak{A}^{\mathfrak{B}}$, where $\mathfrak{B}$ is the Boolean algebra of all the subsets of the set X. Therefore, $\mathfrak{C}$ is embeddable into $\mathfrak{A}^{\mathfrak{B}}$, thus the lemma is proved. ■

Let $\mathfrak{C}$ be a subalgebra of the algebra $\mathfrak{A}^X$, a direct power of the quasi-primal algebra $\mathfrak{A}$. Let us introduce the following notations:

$$E(\mathfrak{C}) = \{[\![f = g]\!] \mid f, g \in \mathfrak{C}\},$$
$$\mathfrak{D}(\mathfrak{C}) = \{[\![f \neq g]\!] \mid f, g \in \mathfrak{C}\}$$

and

$$N(\mathfrak{D}) = E(\mathfrak{C}) \cup \mathfrak{D}(\mathfrak{C}).$$

By analogy with the proof of the theorem **5.7**, one can easily prove the validity of the following statement.

Lemma 7.3. If $\mathfrak{C}$ is a subalgebra of the Boolean power $\mathfrak{A}^{\mathfrak{B}}$ of the quasi-primal algebra $\mathfrak{A}$,:

(a) $E(\mathfrak{C})$, $\mathfrak{D}(\mathfrak{C})$ are closed relative to families and intersections, and $N(\mathfrak{C})$ is a subfield of the open-closed subsets of the space $\mathfrak{B}^*$;

(b) for $f,g \in \mathfrak{C}$ and $N \in N(\mathfrak{C})$, the element $f|N \cup g|\mathfrak{B}^* \setminus N$ also belongs to $\mathfrak{C}$;

(c) for $N \in N(\mathfrak{C})$ such that $N \neq \varnothing, \mathfrak{B}^*$, there is an isomorphism $\mathfrak{C} \cong \mathfrak{C}|N \times \mathfrak{C}|\mathfrak{B}^* \setminus N$, where $\mathfrak{C}|X$ is the projection of the algebra $\mathfrak{C}$ relative to the set $X \subseteq \mathfrak{B}^*$. Moreover, if $\mathfrak{C}$ is a sub-Boolean power of the algebra $\mathfrak{A}$, $\mathfrak{C}|N$ is also a sub-Boolean power of the algebra $\mathfrak{A}$.

For any algebra $\mathfrak{C} \subseteq \mathfrak{A}^{\mathfrak{B}}$, let $\hat{\mathfrak{C}}$ stand for the subalgebra of the algebra $\mathfrak{A}^{\mathfrak{B}}$ generated by the algebra $\mathfrak{C}$ and all constant functions from $\mathfrak{A}^{\mathfrak{B}}$. Let us introduce the following equivalence on the space $\mathfrak{B}^*$: $i \sim_{\mathfrak{C}} j$ iff the kernels of projections π_i and π_j on the algebra $\mathfrak{C}$ coincide. $N_1(\mathfrak{C})$ will denote a Boolean algebra of the subsets of the space $\mathfrak{B}^*$ generated by the set $N(\mathfrak{C})$. It should be noticed that there is a natural one-to-one correspondence of the set $\mathfrak{B}^*/\sim_{\mathfrak{C}}$ and the Stone space $(N_1(\mathfrak{C}))^*$, which for any $i \in \mathfrak{B}^*$ puts the element $\{N \in N_1(\mathfrak{C}) | i \in N\}$ of the Stone space $(N_1(\mathfrak{C}))^*$ in correspondence with the equivalence class $i/\sim_{\mathfrak{C}}$. Henceforth, we will identify $\mathfrak{B}^*/\sim_{\mathfrak{C}}$ with $(N_1(\mathfrak{C}))^*$. At the same time, the inclusions $N_1(\mathfrak{C}) \subseteq N_1(\hat{\mathfrak{C}}) \subseteq \mathfrak{B}$ imply, due to Stone duality, the existence of canonical continuous mappings of the space $\mathfrak{B}^*$ onto $(N_1(\hat{\mathfrak{C}}))^*$ and $(N_1(\hat{\mathfrak{C}}))^*$ onto $(N_1(\mathfrak{C}))^*$.

Lemma 7.4.. If $\mathfrak{C}$ is a subalgebra of the Boolean power $\mathfrak{A}^{\mathfrak{B}}$ of a quasi-primal algebra $\mathfrak{A}$, the following statements are valid:

(a) $\cap E(\mathfrak{C}) = \{i \in \mathfrak{B}^* \mid |\pi_i(\mathfrak{C})| = 1\}$ is either empty or is a $\sim_{\mathfrak{C}}$-equivalence class on $\mathfrak{B}^*$;

(b) all maximal congruences on $\mathfrak{C}$ have the form of the kernels of projections π_i for $i \notin \cap E(\mathfrak{C})$;

(c) $\mathfrak{C}$ is a sub-Boolean power of the algebra $\mathfrak{A}$ iff any $\sim_{\mathfrak{C}}$-equivalence class on $\mathfrak{B}^*$, except for, possibly, class $\cap E(\mathfrak{C})$, is one-element;

(d) $\nabla_{\mathfrak{C}} \in Con_p \mathfrak{C}$ iff $\cap E(\mathfrak{C})$ is open-closed in $\mathfrak{B}^*$.

Proof. The statement (a) is obvious. The statement (b) follows from that of theorem **5.6**. Let now $\mathfrak{C}$ be a sub-Boolean power of the algebra $\mathfrak{A}$, $i \neq j$, and $|\pi_i(\mathfrak{C})| > 1$. Let us choose elements $f, g \in \mathfrak{C}$ such that $f(i) \neq g(i)$, and an open-closed subset N of the space $\mathfrak{B}^*$ such that $i \in N$ and $j \notin N$. In this case we get $h = f|N \cup g|\mathfrak{B}^* \setminus N \in \mathfrak{C}$, $h(i) \neq g(i)$ and $h(j) = g(j)$, i.e., $i \not\sim_{\mathfrak{C}} j$.

Let us now prove the statement (b) in the other direction. Let $f, g \in \mathfrak{C}$ and N an open-closed subset of $\mathfrak{B}^*$. Let the relation $\sim_{\mathfrak{C}}$ coincide with the equality on the set $\mathfrak{B}^* \setminus \cap E(\mathfrak{C})$. In order to prove the fact that $\mathfrak{C}$ is a sub-Boolean power of the algebra $\mathfrak{A}$, we have to show that the element $f|N \cup g|\mathfrak{B}^* \setminus N$ belongs to $\mathfrak{C}$. With the generality preserved, we can assume $N \subseteq \lfloor |f \neq g| \rfloor$. In this case for any $i \in N$ and $j \notin N$ there is a set $N_{ij} \in N(\mathfrak{C})$ such that $i \in N_{ij}, j \notin N_{ij}$, (as $i \not\sim_{\mathfrak{C}} j$). Since $\mathfrak{B}^*$ is compact, N is equal to a finite family of sets from $N(\mathfrak{C})$ and, hence, by lemma **7.3** (a), N also belongs to $N(\mathfrak{C})$. By virtue of the statement (b) of the same lemma, this means that the element $f|N \cup g|\mathfrak{B}^* \setminus N$ belongs to $\mathfrak{C}$, which completes the proof of the statement (b).

The statement (d) is reduced, using theorem **5.6**, to the proof of the following equivalence: $\cap E(\mathfrak{C})$ is open-closed in $\mathfrak{B}^*$ iff there are $f, g \in \mathfrak{C}$ such that $\lfloor |f \neq g| \rfloor = \mathfrak{B}^* \setminus \cap E(\mathfrak{C})$. In one direction this statement is obvious. To prove it in the another direction, let us choose, making use of lemma **7.3** (a) and the compactness of the space $\mathfrak{B}$, a finite decomposition $N_1, \dots, N_k$ of the set $\mathfrak{B}^* \setminus \cap E(\mathfrak{C})$ such that $N_i = \lfloor |f_i \neq g_i| \rfloor$ for $i \leq k$ and some $f_i, g_i \in \mathfrak{C}$. Let us also construct, using lemma **7.3** (b), elements $f, g \in \mathfrak{C}$ coinciding with f_i and g_i, respectively, on N_i. ■

Let now $\mathfrak{C}$ be an $\mathfrak{M}(\mathfrak{A})$-algebra, where $\mathfrak{A}$ is quasi-primal. By lemma **7.2**, one can assume that $\mathfrak{C} \subseteq \mathfrak{A}^{\mathfrak{B}}$ for a certain Boolean algebra $\mathfrak{B}$. In this case, the mapping $\mathfrak{B}^*$ on $(N_1(\mathfrak{C}))^*$, following the statement of lemma **7.3**, yields an isomorphic image $\mathfrak{C}'$ of the algebra $\mathfrak{C}$ such that $\mathfrak{C}'$ is a subalgebra of the direct power $\mathfrak{A}^{(N_1(\mathfrak{C}))^*}$ which obeys the property proved in the statement (c) of lemma **7.4**, i.e., the algebra $\mathfrak{C}'$ is such that any class of $\sim_{\mathfrak{C}'}$-equivalence on $(N_1(\mathfrak{C}))^* \setminus \cap E(\mathfrak{C}')$ is non-element. If in this case $\mathfrak{C}'$ is a subalgebra of the Boolean power $\mathfrak{A}^{N_1(\mathfrak{C})}$, $\mathfrak{C}'$ proves to be a sub-Boolean power of the algebra $\mathfrak{A}$. But no one can guarantee that $\mathfrak{C}'$ will be contained in $\mathfrak{A}^{N_1(\mathfrak{C})}$, i.e., that the elements of $\mathfrak{C}'$ will be continuous functions from $(N_1(\mathfrak{C}))^*$ into a discrete $\mathfrak{A}$. Therefore, more sophisticated constructions are required for the desired representation of the algebra $\mathfrak{C}$ as a sub-Boolean power of the algebra $\mathfrak{A}$.

The following statement is obvious.

Lemma 7.5. If $\mathfrak{C} \subseteq \mathfrak{A}^{\mathcal{B}}$, the following statements are valid:

(a) for $i,j \in \mathcal{B}^*$, $i \sim_{\hat{\mathfrak{C}}} j$ iff $f(i) = f(j)$ for any $f \in \mathfrak{C}$;

(b) every class of $\sim_{\mathfrak{C}}$-equivalence is a family of a finite number (the greatest possible $|\mathfrak{A}|!$) $\sim_{\hat{\mathfrak{C}}}$-equivalence classes.

For any algebra $\mathfrak{C} \subseteq \mathfrak{A}^{\mathcal{B}}$, γ will denote a mapping from the algebra $\hat{\mathfrak{C}}$ to the algebra $\mathfrak{A}^{(N_1(\hat{\mathfrak{C}}))^*}$ such that for any $i \in \mathcal{B}^*$ we have $\gamma(f)(i/\sim_{\hat{\mathfrak{C}}}) = f(i)$. In this case the algebra $\gamma(\mathfrak{C})$ will be called a collapse of $\mathfrak{C}$.

Lemma 7.6. If $\mathfrak{C} \subseteq \mathfrak{A}^{\mathcal{B}}$, the following statements are true:

(a) γ is an isomorphism from $\hat{\mathfrak{C}}$ to $\mathfrak{A}^{(N_1(\hat{\mathfrak{C}}))^*}$ and $\gamma(\hat{\mathfrak{C}}) = \mathfrak{A}^{(N_1(\hat{\mathfrak{C}}))^*}$ and, hence, $\sim_{\gamma(\hat{\mathfrak{C}})}$ is the relation of the equality on $(N_1(\hat{\mathfrak{C}}))^*$;

(b) every $\sim_{\gamma(\mathfrak{C})}$-equivalence class is finite;

(c) if $\mathfrak{C}$ is a sub-Boolean power of $\mathfrak{A}$, the collapse of $\mathfrak{C}$ is also a sub-Boolean power of $\mathfrak{A}$.

Proof. The statement (a) of the preceding lemma obviously implies that γ is an embedding of the algebra $\hat{\mathfrak{C}}$ into the algebra $\mathfrak{A}^{N_1(\hat{\mathfrak{C}}))^*}$. Let λ be the factorization of the space $\mathcal{B}^*$ relative to $\sim_{\hat{\mathfrak{C}}}$, taking into account that $\mathcal{B}^*/\sim_{\hat{\mathfrak{C}}}$ is identified with the space $(N_1(\hat{\mathfrak{C}}))^*$. For any $f\in\hat{\mathfrak{C}}$ and $a\in\mathfrak{A}$ we have $\lambda^{-1}(\gamma(f)^{-1}(a)) = \lfloor\| f=\bar{a}\|\rfloor$, where $\bar{a}$ is a constant function taking the value a. Therefore, we get $\lambda^{-1}(\gamma(f)^{-1}(a))\in N(\mathfrak{C})$ and, hence, $\gamma(f)^{-1}(a)$ is an open-closed subset of $(N_1(\hat{\mathfrak{C}}))^*$. Hence, $\gamma(f)$ is a continuous mapping from the space $(N_1(\hat{\mathfrak{C}}))^*$ to a discrete $\mathfrak{A}$, i.e., $\gamma(f)\in\mathfrak{A}^{N_1(\hat{\mathfrak{C}})}$. It is obvious that $\gamma(\hat{\mathfrak{C}})=\gamma(\hat{\mathfrak{C}})$ contains all constant functions belonging to $\mathfrak{A}^{N_1(\hat{\mathfrak{C}})}$. As by lemma **7.3** (b), for any $f,g\in\hat{\mathfrak{C}}$ and $N\in N(\hat{\mathfrak{C}})$ we have

$$f|N\cup g|\mathfrak{B}^*\setminus N\in\hat{\mathfrak{C}},$$

then for any $h,k\in\gamma(\hat{\mathfrak{C}})$ and $N\in N_1(\hat{\mathfrak{C}})$ we get

$$h|N\cup g|(N_1(\mathfrak{C}))^*\setminus N\in\gamma(\hat{\mathfrak{C}})$$

Owing to the fact that any element of the algebra $\mathfrak{A}^{N_1(\mathfrak{C})}$ is obtained by a finite number of such constructions from constant functions included in $\mathfrak{A}^{N_1(\hat{\mathfrak{C}})}$ and that, according to the inclusion discussed above, $\gamma(\hat{\mathfrak{C}})\subseteq\mathfrak{A}^{N_1(\hat{\mathfrak{C}})}$, we get $\gamma(\hat{\mathfrak{C}})=\mathfrak{A}^{N_1(\hat{\mathfrak{C}})}$, i.e., γ is an isomorphism from $\hat{\mathfrak{C}}$ to $\mathfrak{A}^{N_1(\hat{\mathfrak{C}})}$.

The statement (b) of the lemma under consideration follows from the statement (b) of lemma **7.5**, while the statement (c) is obviously obtained from the statement (c) of lemma **7.4**. ■

The subalgebra $\mathfrak{C}$ of the algebra $\mathfrak{A}^{\mathfrak{B}}$ will be called reduced provided that:

(1) $\hat{\mathfrak{C}}=\mathfrak{A}^{\mathfrak{B}}$;

(2) for $i,j\in\cap E(\mathfrak{C})$ and $i\neq j$ there is no automorphism α of the algebra $\mathfrak{A}$ such that $\alpha(\pi_i(\mathfrak{C}))=\pi_j(\mathfrak{C})$.

For any open-close subset N of the space $\mathfrak{B}^*$, and for any automorphism α of the algebra $\mathfrak{A}$, $\rho_{N,\alpha}$ will denote the following automorphism of the algebra $\mathfrak{A}^{\mathfrak{B}}$: for $f\in\mathfrak{A}^{\mathfrak{B}}, i\in\mathfrak{B}^*$ we have $\rho_{N,\alpha}(f)(i)=f(i)$ if $i\notin N$, and $\rho_{N,\alpha}(f)(i)=\alpha(f(i))$ if $i\in N$. The following equalities are obvious:

$E(\mathfrak{C}) = E(\rho_{N,\alpha}(\mathfrak{C}))$,
$N(\mathfrak{C}) = N(\rho_{N,\alpha}(\mathfrak{C}))$,
$\sim_{\mathfrak{C}} = \sim_{\rho_{N,\alpha}}(\mathfrak{C})$.

In line with the statement (c) of lemma **7.4**, the last equality implies that $\rho_{N,\alpha}(\mathfrak{C})$ is a sub-Boolean power of the algebra $\mathfrak{A}$ iff the algebra $\mathfrak{C}$ itself is a sub-Boolean power of $\mathfrak{A}$.

Lemma 7.7. If $\mathfrak{A}$ is a quasi-primal algebra, there is a reduced representation of the algebra $\mathfrak{C}$, and if in this case $\mathfrak{C}$ is a sub-Boolean power of the algebra $\mathfrak{A}$, the reduced presentation of $\mathfrak{C}$ will also be a sub-Boolean power of $\mathfrak{A}$.

Proof. By virtue of the preceding lemmas, one can assume that $\mathfrak{C} \subseteq \mathfrak{A}^{\mathcal{B}}, \sim_{\hat{\mathfrak{C}}}$ is an equality on $\mathcal{B}^*$ (i.e., that $\hat{\mathfrak{C}} = \mathfrak{A}^{\mathcal{B}}$). Besides, $\cap E(\mathfrak{C})$ is, as a $\sim_{\mathfrak{C}}$-equivalence class on $\mathcal{B}^*$, a finite set. If for $i,j \in E(\mathfrak{C}), i \neq j$ there exists an automorphism α of the algebra $\mathfrak{A}$ such that $\alpha(\pi_i(\mathfrak{C})) = \pi_j(\mathfrak{C})$, let us choose an open-close neighborhood N of the point i such that $j \notin N$, and let $\rho = \rho_{N,\alpha}$. In this case $i \sim_{\widehat{\rho(\mathfrak{C})}} j$ and, hence, for the collapse $\gamma(\rho(\mathfrak{C})) = \mathfrak{C}'$, the set $\cap E(\mathfrak{C}')$ contains less elements then the set $E(\mathfrak{C})$, $\mathfrak{C}'$ still obeying the condition (1) of the definition of a reduced algebra. Therefore, we get a reduced representation of the algebra $\mathfrak{C}$ through a finite number of steps. ■

Let us now consider two subalgebras of different Boolean powers of the algebra $\mathfrak{A}$: $\mathfrak{C}_1 \subseteq \mathfrak{A}^{\mathcal{B}_1}, \mathfrak{C}_2 \subseteq \mathfrak{A}^{\mathcal{B}_2}$, and let λ be an isomorphism between $\mathfrak{C}_1$ and $\mathfrak{C}_2$. Let us determine the relation $R_\lambda \subseteq \mathcal{B}_1^* \times \mathcal{B}_2^*$ in the following way: $<i,j> \in R_\lambda$ iff at the isomorphism λ the kernel of projections π_i of the algebra $\mathfrak{C}_1$ corresponds to the kernel of projections π_j of the algebra $\mathfrak{C}_2$. If $<i,j> \in R_\lambda$, let λ_{ij} be a canonical isomorphism from the algebra $\pi_i(\mathfrak{C}_1)$ to the algebra $\pi_j(\mathfrak{C}_2)$ corresponding to the isomorphism λ; then we get $\lambda_{ij}(f(i)) = \lambda(f)(j)$ for $f \in \mathfrak{C}_1$. For the case when $\mathfrak{C}_1 = \mathfrak{C}_2$ and λ-identical, we get $R_\lambda = \sim_\lambda$, and λ_{ij} is the isomorphism from $\pi_i(\mathfrak{C}_1)$ to $\pi_j(\mathfrak{C}_1)$ at $i \sim_{\mathfrak{C}_1} j$.

Lemma 7.8. Let $\mathfrak{C}_1, \mathfrak{C}_2$ and λ be such as indicated above. Then the following statements are valid:

(a) if $i \in \mathcal{B}_1^*$ and $|\pi_i(\mathfrak{C}_1)| > 1$, there is an $i \in \mathcal{B}_2^*$ such that $<i,j> \in R_\lambda$;

(b) if we have either $\cap E(\mathfrak{C}_1) = \varnothing$ or $\cap E(\mathfrak{C}_2) \neq \varnothing$, for any $f, g \in \mathfrak{C}_1$ we get

$$\lfloor f = g \rfloor = R_\lambda^{-1}(\lfloor \lambda(f) = \lambda(g) \rfloor)$$

and

$$\lfloor f \neq g \rfloor = R_\lambda^{-1}(\lfloor \lambda(f) \neq \lambda(g) \rfloor).$$

Proof. Let $i \in \mathfrak{B}_1^*$ and $|\pi_i(\mathfrak{C}_1)| > 1$. Since $\pi_i(\mathfrak{C}_1)$ is, as the subalgebra of a quasi-primal algebra, simple, the kernel of projections π_i is a maximal congruence on the algebra $\mathfrak{C}_1$, and the λ-image of this kernel is a maximal congruence on the algebra $\mathfrak{C}_2$. By lemma **7.4** (b), this λ-image has the form of a kernel of a certain projection π_j of the algebra $\mathfrak{C}_2$ and, hence, we get $\langle i, j \rangle \in R_\lambda$.

Let now $f, g \in \mathfrak{C}_1$. Then if $\langle i, j \rangle \in R_\lambda$, $i \in \lfloor f = g \rfloor$ iff $\langle f(i), g(i) \rangle$ belongs to the projection kernel π_i, and this is the case iff $\langle \lambda(f)(j), \lambda(g)(j) \rangle$ belongs to the λ-image of the kernel π_i, i.e., iff $j \in \lfloor \lambda(f) = \lambda(g) \rfloor$. Therefore, now the statement (b) obviously follows from the statement (a) proved above.

__Lemma 7.9.__ Let $\mathfrak{A}$ be a quasi-primal algebra, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ its non-singleton isomorphic subalgebras such that a certain isomorphism λ of the algebra $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ has no extensions up to isomorphism from the algebra $\mathfrak{A}$. In this case there is a countable $\mathfrak{M}(\mathfrak{A})$-algebra isomorphic to no sub-Boolean power of the algebra $\mathfrak{A}$.

Proof. Let $\mathfrak{B}$ be a countable atomless Boolean algebra. Let us fix some $i_1 \neq i_2 \in \mathfrak{B}^*$. Let also $\mathfrak{C} = \left\{ f \in \mathfrak{A}^{\mathfrak{B}} \mid f(i_1) \in \mathfrak{A}_1, f(i_2) = \alpha(f(i_1)) \right\}$. It is evident that $\mathfrak{C}$ is a subalgebra of a Boolean power $\mathfrak{A}^{\mathfrak{B}}$, $\pi_{i_1}(\mathfrak{C}) = \mathfrak{A}_1, \pi_{i_2}(\mathfrak{C}) = \mathfrak{A}_2$ for $j \in \mathfrak{B}^*$ such that $j \neq i_1, i_2$, $\pi_j(\mathfrak{C}) = \mathfrak{A}$, and for $k, l \in \mathfrak{B}^*$ we have $k \sim_{\mathfrak{C}} l$ iff either $k = l$, or $k, l \in \{i_1, i_2\}$. By lemma **7.4** (d), we get $\nabla_{\mathfrak{C}} \in Con_p \mathfrak{C}$.

Let us now assume that λ is an isomorphism from the algebra $\mathfrak{C}$ to the algebra $\mathfrak{C}_1 \subseteq \mathfrak{A}^{\mathfrak{B}_1}$, which is a sub-Boolean power of the algebra $\mathfrak{A}$. Then by lemma **7.4** (d), $N = \left\{ i \in \mathfrak{B}_1^* \mid |\pi_i(\mathfrak{C}_1)| = 1 \right\}$ is open-closed in $\mathfrak{B}_1^*$. By lemma **7.3** (c), $\mathfrak{C}_1 \cong \mathfrak{C}_1 | N \times \mathfrak{C}_1 | \mathfrak{B}^* \setminus N$ and, hence, $\mathfrak{C} \cong \mathfrak{C}_1 | \mathfrak{B}^* \setminus N$, and the latter, as $\mathfrak{B}^* \setminus N$ is open-closed, is a sub-Boolean power of the algebra $\mathfrak{A}$. Therefore, without violating generality, one can assume that for any $i \in \mathfrak{B}_1^*$ we have $|\pi_i(\mathfrak{C})| > 1$. By lemmas **7.4** (c) and **7.8** (a), there exists a unique $i_0 \in \mathfrak{B}_1^*$ such that $\langle i_1, i_0 \rangle, \langle i_2, i_0 \rangle \in R_\lambda$, in which case we get $\mathfrak{A}_1 \cong \mathfrak{A}_2 \cong \pi_{i_0}(\mathfrak{C})$. On the other hand, since all the factors of the algebra $\mathfrak{C}$ relative to the maximal congruences are, by lemma **7.4** (b), isomorphic to either the algebra $\mathfrak{A}_1$ or the algebra $\mathfrak{A}$, for any $i \in \mathcal{B}_1^*$ such that $i \neq i_0$, we have $\pi_i(\mathfrak{C}_1) = \mathfrak{A}$ by the same lemma.

Let us choose a finite set of elements $\{f_1,...,f_n\}$ of the algebra $\mathfrak{C}$ such that

$$\pi_{i_0}(\mathfrak{C}_1) = \{\lambda(f_1)(i_0),...,\lambda(f_n)(i_0)\},$$

and let N be an open-closed neighborhood of the point i_0 such that all the functions $\lambda(f_i)(i \leq n)$ are constant on N. Using lemma **7.3**, one can in a standard way choose such $f, g \in \mathfrak{C}$, that $N = \lfloor \| \lambda(f) \neq \lambda(g) \| \rfloor$. Then, by lemma **7.8** (b), $M = R_\lambda^{-1}(N) \in N(\mathfrak{C})$ and $i_1, i_2 \in M$. Let us choose open-closed subsets N_i of the set M such that $i_1 \in N_1, i_2 \in N_2$, and each of the functions $f_1,...,f_n$ is constant on N_1 and N_2. Let $j_1 \in N_1 \setminus \{i_1\}, j_2 \in N_2 \setminus \{i_2\}$ and $k_1, k_2 \in \mathcal{B}_1^*$ be such that $<j_1,k_1>, <j_2,k_2> \in R_\lambda$. Since $\sim_{\mathfrak{C}_1}$ coincides with the equality on $\mathcal{B}_1^*$ and $k_i \in R_\lambda \cdot R_\lambda^{-1}(N)$ then $k_i \in N$. Therefore, obviously, $\beta = \lambda_{j_2k_2} \cdot \lambda_{j_1k_1}$ is an automorphism of the algebra $\mathfrak{A}$ extending α in contradiction to the choice of the isomorphism α. The algebra $\mathfrak{C}$ is thus isomorphic to no sub-Boolean power of the algebra $\mathfrak{A}$. ■

<u>**Lemma 7.10.**</u> Let $\mathfrak{A}$ be a quasi-algebra having a non-identical isomorphism with fixed points. In this case, there is an $\mathfrak{M}(\mathfrak{A})$-algebra of the power $\aleph_1$ isomorphic to no sub-Boolean power of the algebra $\mathfrak{A}$.

Proof. By the condition of the lemma, one can easily choose a proper subalgebra $\mathfrak{A}_0$ of the algebra $\mathfrak{A}$, and a subgroup G of all the automorphisms of the algebra $\mathfrak{A}$ such that $\mathfrak{A}_0$ is a family of points fixed relative to G. Let $\gamma: G \to Sym(n)$ be a canonical embedding of G into the group of permutations of the set $\{0,1,...,n-1\}$, where $n = |G|$.

For any limiting ordinal μ let $F_\mu = (\mu \times n) \cup \{\mu\}$, and let us introduce a partial order on F_μ defined in the following way: $<\nu,i> << \nu',j>$ iff $i = j$ and $\nu < \nu'$, and the element μ is greater than all the elements of the type $<\nu,i>$, where $\nu < \mu, i < n$. Let us consider a topology on F_μ with the basis of open sets of the type $\{x \in F_\mu | <\nu,i> << x << \nu',i>\}$, $\{x \in F_\mu | x << \nu,i>\}$, and $\{x \in F_\mu |$ for some $i < n$, $<\nu_i,i> << x\}$ at various $<\nu,i>, <\nu',i>, <\nu_0,o>,...,<\nu_{n-1},n-1> \in F_\mu$.

It is obvious that F_μ is a Boolean space, i.e., that $F_\mu = \mathcal{B}_\mu^*$, where $\mathcal{B}_\mu$ is a family of open-closed subsets of the space F_μ. Let us define the action of the group G on the space F_μ in the following way: for $g \in G$ we have

$g(<\nu,i>)=<\nu,\gamma(g)(i)>$ and $g(\mu)=\mu$. Let now $X=F_\omega \times F_{\omega_1}$ with a common Tikhonov topology of the product, and let us define the action of G on X, setting $g(<x',x''>)=<g(x'),g(x'')>$. Let $\mathcal{B}$ be a family of open-closed subsets of the space X, in which case $X=\mathcal{B}^*$.

Let us show that X has no closed subsets Z such that Z contains exactly one point of each orbit of G on X, i.e., Z contains one point of each set of the type $\{g(x)|g(\in G\}$, for all $x\in X$. Let $\nu<\omega_1$, and let $i_\nu<n$ be the only i such that $<\omega,<\nu,i>>\in Z$. For any $i\neq i_\nu$ there is an $m(i)<\omega$ such that for all $m\geq m(i)$ and any $j<n$, we have $<<m,j>,<\nu,i>>\in X\setminus Z$ since, in the opposite case, as Z is closed, we get $<\omega,<\nu,i>>\in Z$. Therefore, there is an $m_\nu<\omega$ such that for all $m\geq m_\nu$, $j<n$ and $i<n, i\neq i_\nu$ we get $<<m,j>,<\nu,i>>\in X\setminus Z$. Hence, at any fixed $m\geq m_\nu$ and $j<n$, the element $<<m,j>,<\nu,i_\nu>>$ will be the only one in its G-orbit belonging to the set Z. Let us choose an $\breve{m}<\omega$ such that $\breve{m}=m_\nu$ for an uncountable number of elements $\nu<\omega_1$. Since for any $j<n$ any neighborhood of the point $<<\breve{m},j>,\omega_1>$ of the space X contains a certain element $<<\breve{m},j>,<\nu,i_\nu>>$ such that $\breve{m}=m_\nu$ and Z is closed, we get $<<\breve{m},j>,\omega_1>\in Z$. But this implies that Z wholly includes the G-orbit of the kind $\{<<\breve{m},j>,\omega_1>|j<n\}$. It is the contradiction to the choice of Z that proves the improbability of the existence of closed $Z\subseteq X$ with the above-discussed condition relative to G-orbits.

Let us consider an algebra $\mathfrak{C}=\{f\in\mathfrak{A}^{\mathcal{B}}|g(f(i))=f(g(i))$ for all $g\in G\}$. It should be noticed that the constant functions $\bar{a}\in\mathfrak{A}^{\mathcal{B}}$ belong to $\mathfrak{C}$ iff $a\in\mathfrak{A}_0$, and that $\pi_{<\omega,\omega_1>}(\mathfrak{C})=\mathfrak{A}_0$ and $\pi_i(\mathfrak{C})=\mathfrak{A}$ for all other $i\in\mathcal{B}^*=X$. Moreover, $i\sim_{\mathfrak{C}} j$ for $i,j\in X$ iff both i and j belong to the same G-orbit.

Let us show that $\mathfrak{C}$ can be isomorphic to no sub-Boolean power of the algebra $\mathfrak{A}$. Let us assume that the opposite is the case, and let λ be an isomorphism from $\mathfrak{C}$ to $\mathfrak{C}_1\subseteq\mathfrak{A}^{\mathcal{B}_1}$, a sub-Boolean power of the algebra $\mathfrak{A}$. By lemma **7.7**, $\mathfrak{C}_1$ can be assumed to be reduced. Let us refer to an element $<\omega,\omega_1>$ of the space X by i_0, and consider two cases.

Case 1: $\mathfrak{A}_0$ is non-singleton, and, hence, $\nabla_{\mathfrak{C}}\in Con_p\mathfrak{C}$. In this case, as in the proof of the previous lemma, one can assume $|\pi_i(\mathfrak{C}_1)|>1$ for all $i\in\mathcal{B}_1^*$. And again, one can find the only $j_0\in\mathcal{B}_1^*$ such that $<i_0,j_0>\in R_\lambda$. Therefore, in particular, we get $\pi_{j_0}(\mathfrak{C}_1)=\mathfrak{A}_0$, and for any $j\in\mathcal{B}_1^*$ other than j_0 we get $\pi_j(\mathfrak{C}_1)=\mathfrak{A}$. Let us choose a finite set of constant functions $\{f_1,\dots,f_k\}$ from $\mathfrak{C}$ so that $\mathfrak{A}_0=\{f_1(i_0),\dots,f_k(i_0)\}$. Let us choose an open-closed neighborhood N of the element j_0 so that all the functions $\lambda(f_1),\dots,\lambda(f_k)$ are constant on N. In a standard way one can choose N in the form $N = [\![\lambda(g) = \lambda(h)]\!]$ for some $g,h \in \mathfrak{C}$, in which case, by lemma **7.8** (b), we get $R_\lambda^{-1}(N) \in N(\mathfrak{C})$. As i_0 is not an isolated point in $\mathcal{B}^*$, there can be found an $i' \in R_\lambda^{-1}(N) \setminus \{i_0\}$. Let $j' \in N$ be such that $< i', j' > \in R_\lambda$. Therefore, we get an automorphism $\beta = \lambda_{i'j'}$ of the algebra $\mathfrak{A}$ which extends the automorphism $\lambda_{i_0 j_0}$ of the algebra $\mathfrak{A}_0$. Without violating generality, one can assume $\lambda_{i_0 j_0}$ to be an identical mapping on $\mathfrak{A}_0$ (in the opposite case considering $\rho_{\mathcal{B}_1^*, \beta^{-1}}(\mathfrak{C}_1)$ instead of $\mathfrak{C}_1$).

Let us now choose an open-closed neighborhood M of the point i_0 such that $M \subseteq R_\lambda^{-1}(N)$, and such that M together with the action of the group G on it is homomorphic to the space $\mathcal{B}^*$ together with the action of the group G on it. For $i \in M$ let O_i be the G-orbit of the point i, and let $j_i \in \mathcal{B}_1^*$ be such that $< i, j_i > \in R_\lambda$. Then $< i', j_i > \in R_\lambda$ iff $i' \in O_i$ since, as has been already noted, $O_i = i/\sim_{\mathfrak{C}}$. For $i \in M \setminus \{i_0\}$ the automorphism λ_{ij_i} leaves all the elements of the algebra $\mathfrak{A}_0$ fixed and, hence, $\lambda_{ij_i} \in G$. For $i,i' \in M$ such that $O_i = O_{i'}$, the equality $\lambda_{ij_i} = \lambda_{i'j_i}$ holds iff $i = i'$ (indeed, if $i' = g(i) \neq i, g \in G$ then, choosing $f \in \mathfrak{C}$ in such a way that $f(i) = a$ where $g(a) \neq a$, we get $g(a) = f(i') = \lambda_{i'j_i}^{-1} \cdot \lambda_{ij_i}(a)$, i.e., $\lambda_{ij_i}(a) \neq \lambda_{i'j_i}(a)$). Therefore, for $i \in M \setminus \{i_0\}$ we get $G = \{\lambda_{i'j_i} | i' \in O_i\}$ and, in particular, there exists a unique $i \in O_i$ such that $\lambda_{i^* j_i^*}$ is identical on $\mathfrak{A}$. The same, obviously, is also true for $i = i_0$. Let now $Z = \{i^* | i \in M\}$. One can directly prove that Z is closed in M. Hence, in the subspace M homomorphic to $\mathcal{B}^*$ we have found a closed subset Z intersecting exactly one element in any G-orbit lying in M. We have already proved the improbability of the existence of such a Z. The contradiction obtained here proves that case 1 is also impossible.

Case 2: $\mathfrak{A}_0$ is a one-element algebra. In this case, since $\nabla_{\mathfrak{C}} \notin Con_p \mathfrak{C}$, $\cap E(\mathfrak{C}_1)$ is not empty, and as $\mathfrak{C}_1$ is reduced, $\cap E(\mathfrak{C}_1)$ is finite. Let $\cap E(\mathfrak{C}_1) = \{y_1, \ldots, y_r\}$. Let f be the only constant function from $\mathfrak{C}$. Let us choose pairwise non-intersecting open-closed sets $N_1, \ldots, N_r \subseteq \mathcal{B}_1^*$ containing the points $y_1, \ldots, y_r$, respectively, and such that $\lambda(f)$ is constant on each of them. In a standard way one can find $g,h \in \mathfrak{C}$ such that $[\![\lambda(g) = \lambda(h)]\!] = N_1 \cup \ldots \cup N_r$. Preserving the generality of the considerations, one can assume that none of y_i is an isolated point of the space $\mathcal{B}_1^*$ (in the opposite case, $\mathcal{B}_1^* \setminus \{y_i\}$ is considered instead of $\mathcal{B}_1^*$). Let $y_i' \in N_i \setminus \{y_i\}$. Then for some $x_i' \in \mathcal{B}^*$ we have $< x_i', y_i' > \in R_\lambda$ and, hence, $\lambda_{x' y_i'}$ is an automorphism of the algebra $\mathfrak{A}$ mapping $\mathfrak{A}_0$ onto $\pi_{y_i}(\mathfrak{C}_1)$. Since $\mathfrak{C}_1$ is reduced, the existence of such automorphisms implies the equality $r = 1$, in which case the arguments in the proof of case 1 come into play. ■

Theorem 7.5. Let $\mathfrak{A}$ be a quasi-primal algebra. Then $\mathfrak{M}(\mathfrak{A}) = IP_{SB}(\mathfrak{A})$ iff any isomorphism between non-singleton subalgebras of the algebra $\mathfrak{A}$ has a unique extension to an automorphism of the algebra $\mathfrak{A}$, the only automorphism of the algebra $\mathfrak{A}$ with fixed points being the identical one.

Proof. The necessity of the conditions on $\mathfrak{A}$ given in the formulation of the theorem follows from lemmas **7.9** and **7.10**. Let us prove that these conditions are sufficient for the equality $\mathfrak{M}(\mathfrak{A}) = IP_{SB}(\mathfrak{A})$ to hold. Let $\mathfrak{C}$ be an arbitrary algebra from $\mathfrak{M}(\mathfrak{A})$. By lemmas **7.2** and **7.7**, $\mathfrak{C} \subseteq \mathfrak{A}^{\mathfrak{B}}$, and $\mathfrak{C}$ is reduced. Let us choose an arbitrary $f \in \mathfrak{C}$, and let $M_1,\dots,M_l$ be a decomposition of $\mathfrak{B}^*$ with open-closed subsets of the space $\mathfrak{B}^*$ of the type $f^{-1}(a)$, where $a \in \mathfrak{A}$. Let $i \equiv j$ for $i,j \leq l$ iff there is an isomorphism α_{ij} from the algebra $\mathfrak{A}$ such that $\alpha_{ij}(f(M_i)) = f(M_j)$. Let us choose an automorphism ρ of the algebra $\mathfrak{A}^{\mathfrak{B}}$, which is a product of automorphisms of the type $\rho_{M_i,\alpha_{ij}}$ at $i \equiv j$ and such that for any $i \equiv j$, the function $\rho(f)$ is constant on $M_i \cup M_j$.

Let us first of all prove that the equivalence $\sim_{\rho(\mathfrak{C})}$ limited relative to the set $\mathfrak{B}^* \setminus \cap E(\rho(\mathfrak{C}))$ is contained in the equivalence $\sim_{\hat{\rho(\mathfrak{C})}}$. Let $i,j \in \mathfrak{B}^* \setminus \cap E(\rho(\mathfrak{C})) = \mathfrak{B}^* \setminus \cap E(\mathfrak{C})$ and $i \sim_{\hat{\rho(\mathfrak{C})}} j$. As has been noted after lemma **7.6**, this implies $i \sim_{\mathfrak{C}} j$. Therefore, the subalgebras $\pi_i(\mathfrak{C})$ and $\pi_j(\mathfrak{C})$ of the algebra $\mathfrak{A}$ are isomorphic, this isomorphism being, by the condition of the theorem, extendable to the automorphism α of the algebra $\mathfrak{A}$ (since $|\pi_i(\mathfrak{C})| > 1$). The uniqueness of this extension implies the equality $\alpha(f(i)) = f(j)$ and, hence, if $i \in M_k$, $j \in M_n$, $k = n$. Therefore, $\rho(f)(i) = \rho(f)(j)$. In this case the algebras $\pi_i(\rho(\mathfrak{C}))$ and $\pi_j(\rho(\mathfrak{C}))$, which are isomorphic to the algebras $\pi_i(\mathfrak{C})$ and $\pi_j(\mathfrak{C})$, respectively, are pairwise isomorphic, the element $\rho(f)(i)$ remaining fixed at this isomorphism. By the conditions of the theorem, φ is extendable to the automorphism of $\mathfrak{A}$, and, since this extension will have a fixed point, the extension itself and φ will be identical mappings. The latter consideration, obviously, implies the equivalence $i \sim_{\hat{\rho(\mathfrak{C})}} j$. Therefore, indeed, the limitation $\sim_{\rho(\mathfrak{C})}$ on $\mathfrak{B}^* \setminus \cap E(\rho(\mathfrak{C}))$ is contained in $\sim_{\hat{\rho(\mathfrak{C})}}$. But in this case the collapse $\mathfrak{C}'$ of the algebra $\rho(\mathfrak{C})$ obviously meets the condition that the classes of $\sim_{\mathfrak{C}'}$-equivalence (other than $\cap E(\mathfrak{C}')$) are one-element and, hence, by lemma **7.4**, the collapse $\mathfrak{C}'$ is a sub-Boolean power of the algebra $\mathfrak{A}$

isomorphic to the algebra $\mathfrak{C}$. ■

The theorem considered describes those quasi-primal algebras $\mathfrak{A}$ which generate varieties representable by sub-Boolean powers of the algebra $\mathfrak{A}$. With some minor modifications we can describe the quasi-primal algebras $\mathfrak{A}$ generating varieties the countable algebras of which are isomorphic to sub-Boolean powers of the algebra $\mathfrak{A}$.

<u>**Lemma 7.11.**</u> Let any isomorphism between non-singleton subalgebras of a quasi-primal algebra $\mathfrak{A}$ be extendable to an automorphism of $\mathfrak{A}$. Then any countable $\mathfrak{M}(\mathfrak{A})$-algebra is isomorphic to a Boolean product $\mathfrak{C}$ of $S(\mathfrak{A})^+$-algebras with the index $\mathcal{B}^*$, so that the following statements are true:

(a) there is no more than one $i \in \mathcal{B}^*$ with the property $|\pi_i(\mathfrak{C})| = 1$;

(b) there is no more than a countably infinite family $\{N_l \| l \in I\}$ of pairwise disjunct open-closed subsets of the space $\mathcal{B}^*$ such that $\{i \in \mathcal{B}^* \| \, |\pi_i(\mathfrak{C})| > 1\} = \bigcup_{l \in I} N_l$, and $\pi_{N_l}(\mathfrak{C})$ are sub-Boolean powers of the algebra $\mathfrak{A}$ for all $l \in I$.

Proof. Let $\mathfrak{C}_1$ be a $\mathfrak{M}(\mathfrak{A})$-algebra. By theorem **7.7**, $\mathfrak{C}_1$ is isomorphic to some Boolean product of $\mathfrak{M}(\mathfrak{A})^+_{SI}$-algebras. Since $\mathfrak{A}$ is quasi-primal, $\mathfrak{M}(\mathfrak{A})^+_{SI} = S(\mathfrak{A})^+$ and, therefore, $\mathfrak{C}_1$ is a Boolean product of $S(\mathfrak{A})^+$-algebras with some Boolean index $\mathcal{B}_1$. As $\mathfrak{C}_1$ is countable, $\mathcal{B}_1$ can also be considered countable. The set $A = \{i \in \mathcal{B}_1^* \| \, |\pi_1(\mathfrak{C}_1)| = 1\}$ is closed in the space $\mathcal{B}_1^*$. If $A = \varnothing$, the statement (a) is fulfilled for the algebra $\mathfrak{C}_1$. If $A \neq \varnothing$, let i_0 be a fixed point from A. The subspace $(\mathcal{B}_1^* \setminus A) \cup \{i_0\}$ of the space $\mathcal{B}_1^*$ is, being a continuous image of the Boolean space $\mathcal{B}_1^*$, Boolean itself. Let us identify $(\mathcal{B}_1^* \setminus A) \cup \{i_0\}$ with the space $\mathcal{B}_1^*$ for a certain Boolean algebra $\mathcal{B}_2$. Then the algebra $\mathfrak{C}_2 = \mathfrak{C}_1 | (\mathcal{B}_1^* \setminus A) \cup \{i_0\} = \mathfrak{C}_1 | \mathcal{B}_2^* \cong \mathfrak{C}_1$ is a Boolean product of $S(\mathfrak{A})^+$-algebras with the index $\mathcal{B}_2^*$, in which case the statement (a) is valid for $\mathfrak{C}_2$, i.e., $\{i \in \mathcal{B}_2^* \| \, |\pi_i(\mathfrak{C}_2)| = 1\} = 1$. We thus can consider the statement (a) to be fulfilled for $\mathfrak{C}_1$ in any case.

Let us denote $\{i \in \mathcal{B}_1^* \| \, |\pi_i(\mathfrak{C}_1)| > 1\}$ with A_1. Let us show that

(*) for any $i \in A_1$ there is an open-closed neighborhood N_i of the point i and an isomorphism ψ_i of the algebra $\pi_{N_i}(\mathfrak{C}_1)$ on a certain sub-Boolean power of the algebra $\mathfrak{A}$ with a Boolean index N_i.

The proof will be carried out by induction over the power $\mathfrak{A} \setminus \pi_i(\mathfrak{C}_1)$. Let $|\mathfrak{A} \setminus \pi_i(\mathfrak{C}_1)| = 0$, and let in this case $|\mathfrak{A}| = n$ and $f_1,\dots,f_n \in \mathfrak{C}_1$ be such that $\{f_1(i),\dots,f_n(i)\} = \pi_i(\mathfrak{C}_1)$. Since $\mathfrak{C}_1$ is a Boolean product, there is an open-closed neighborhood N_i of the point i such that at $j \in N_i$ for any signature function φ of the algebra $\mathfrak{A}$ we have $\varphi(f_{k_1}(i),\dots,f_{k_m}(i)) = f_l(i)$ iff $\varphi(f_{k_1}(j),\dots,f_{k_m}(j)) = f_l(j)$, and $f_k(i) = f_l(i)$ iff $f_k(j) = f_l(j)$, i.e., the subalgebras F_j of $\mathfrak{A}$ with the basic set $\{f_1(j),\dots,f_n(j)\}$ are isomorphic to the algebra $\pi_i(\mathfrak{C}_1) = \mathfrak{A}$, the isomorphism λ_{ij} being defined by the equality $\lambda_{ij}(f_l(i)) = f_l(j)$. Let us define the mapping ψ_i from the algebra $\pi_{N_i}(\mathfrak{C}_1)$ to the algebra $\mathfrak{A}^{N_i}$ in the following way: for any $j \in N_i$ we get $\psi_i(f(j)) = \lambda_{ij}^{-1}(f(j))$. The ψ_i-images of the elements f_l are in this case obviously constant elements of the algebra $\mathfrak{A}^{N_i}$ and, as $\pi_{N_i}(\mathfrak{C}_1)$ is a Boolean product, $\psi_i(\pi_{N_i}(\mathfrak{C}_1))$ is a sub-Boolean power of the algebra $\mathfrak{A}$, the basis of induction is thus proved.

Let $i \in \mathcal{B}_1^*$ and let the statement (*) be valid for all $j \in \mathcal{B}_1^*$ such that $|\mathfrak{A} \setminus \pi_j(\mathfrak{C}_1)| < |\mathfrak{A} \setminus \pi_i(\mathfrak{C}_1)|$. Let also $f_1,\dots,f_m \in \mathfrak{C}_1$, $m = |\pi_i(\mathfrak{C}_1)|$ and $\{f_1(i),\dots,f_m(i)\} = \pi_i(\mathfrak{C}_1)$. By analogy with the case considered above for $\pi_i(\mathfrak{C}_1) = \mathfrak{A}$, let N_i be an open-closed neighborhood of the point i such that for $j \in N_i$ the mapping from the algebra $\pi_i(\mathfrak{C}_1)$ to the algebra $\pi_j(\mathfrak{C}_1)$, defined by the equality $\lambda_{ij}(f_l(i)) = f_l(j)$, is an isomorphic embedding of $\pi_i(\mathfrak{C}_1)$ into $\pi_j(\mathfrak{C}_1) \subseteq \mathfrak{A}$. It is obvious that, if $j \in N_i$ and λ_{ij} is not a mapping “onto”, there is an open neighborhood T_j of the point j such that, for all $q \in T_j$ λ_{iq} is not a mapping “on”. Therefore, $Y = \{j \in N_i | \lambda_{ij}$ is a mapping “onto”$\}$ is closed. For any $j \in N_i \setminus Y$ we have $|\mathfrak{A} \setminus \pi_j(\mathfrak{C}_1)| < |\mathfrak{A} \setminus \pi_i(\mathfrak{C}_1)|$ and, hence, by the induction supposition, there is an open-closed neighborhood $N_j \subseteq N_i$ of the point j and an isomorphism ψ_j of the algebra $\pi_{N_j}(\mathfrak{C}_1)$ on the sub-Boolean power of the algebra $\mathfrak{A}$ with the index N_j. Moreover, since $\mathcal{B}_1$ is countable, the number of open-closed subsets of the space $\mathcal{B}_1^*$ is also countable, and one can consider the set $\{N_j | j \in N_i \setminus Y\}$ to be countable and disjunct. One can also obviously assume that ψ_{j_1} coincides with ψ_{j_2} for j_1, j_2 belonging to the same set of the type $N_j (j \in N_i \setminus Y)$, and that elements of the type $\psi_j(f_l)$ $(l \le m)$ are constant elements of the sub-Boolean power $\mathfrak{A}^{N_j}$.

Let η_{ij} be a certain automorphism of the algebra $\mathfrak{A}$ extending the isomorphic embedding ξ_{ij} from the algebra $\pi_i(\mathfrak{C}_1)$ to $\pi_j(\psi_j(\pi_{N_j}(\mathfrak{C}_1)))$, where $\xi_{ij}(f_l(i)) = \psi_j(f_l)(j)$. For $j \in Y$ let ρ_{ij} be a certain automorphism of the algebra $\mathfrak{A}$ extending the automorphism λ_{ij}. Let us then define the mapping φ_i from the algebra

$\pi_{N_i}(\mathfrak{C}_1)$ to the algebra $\mathfrak{A}^{N_i}$ in the following way: for $f \in \mathfrak{C}_1$ and $j \in Y$, we have $\varphi_i(\pi_{N_i}(f))(j) = \rho_{ij}^{-1}(f(j))$, and for $j \in N_j$ $(j \in N_i \setminus Y)$ we have $\varphi_i(\pi_{N_i}(f))(j) = = \eta_{ij}^{-1}(f(j))$. The elements of the type $\varphi_i(\pi_{N_i}(f_l))$ are obviously constant elements of the algebra $\mathfrak{A}^{N_i}$ for $l \leq m$. One can easily check that $\varphi_i(\pi_{N_i}(\mathfrak{C}_1))$ is a sub-Boolean power of the algebra $\mathfrak{A}$ with an index equal to $\mathfrak{B}$ such that $\mathfrak{B}^* = N_i$. Therefore, the induction step in the proof of the statement (*) has been made, and this statement is completely proved.

Now, in order to prove statement **(b)** of the lemma, one can use a countable family of open-closed subsets $N_i (i \in A_1)$ covering A_1, and construct a family of pairwise disjunct similar subsets in an obvious way, which proves the statement in **(b)**. ■

Theorem 7.6. Let $\mathfrak{A}$ be a quasi-primal algebra. Then the condition that every countable $\mathfrak{M}(\mathfrak{A})$-algebra is isomorphic to a sub-Boolean power of the algebra $\mathfrak{A}$, i.e., $\mathfrak{M}(\mathfrak{A})_{\aleph_0} \subseteq IP_{SB}(\mathfrak{A})$, is equivalent to the condition that any isomorphism between non-singleton subalgebras of the algebra $\mathfrak{A}$ is extendable to the automorphism of $\mathfrak{A}$.

Proof. The necessity of the above condition on the subalgebras of the algebra $\mathfrak{A}$ results from lemma **7.9**. Let us now prove the sufficiency of this condition. Let $\mathfrak{A}$ be quasi-primal and let any isomorphism between its non-singleton subalgebras be extendable to the automorphism of $\mathfrak{A}$. Let also $\mathfrak{C} \in \mathfrak{M}(\mathfrak{A})_{\aleph_0}$, and let $\mathfrak{C}$ obey the conclusion of lemma **7.11**. Let us now use the notations of the formulation of that lemma, and let in this case $i_0 \in \mathfrak{B}^*$ be the only element of i (provided that it exists) such that $|\pi_i(\mathfrak{C})| = 1$. If such an i_0 either does not exist or is an isolated point in $\mathfrak{B}^*$ then, by its compactness, the set I is finite, i.e., there exists a finite number of open-closed subsets $N_1, \ldots, N_k \subseteq \mathfrak{B}^*$ such that $\pi_{N_l}(\mathfrak{C}) \in P_{SB}(\mathfrak{A})$ for $l \leq k$, and $\mathfrak{C} \cong \mathfrak{C} | (N_1 \dot\cup \ldots \dot\cup N_k)$. Since the Cartesian product of $P_{SB}(\mathfrak{A})$-algebras is also an $IP_{SB}(\mathfrak{A})$-algebra, the inclusion $\mathfrak{M}(\mathfrak{A})_{\aleph_0} \subseteq IP_{SB}(\mathfrak{A})$ is proved.

Let us now consider i_0 to be a limiting point in $\mathfrak{B}^*$. Let us choose a certain $f_0 \in \mathfrak{C}$, and let $M = \{i \in \mathfrak{B}^* | \{\pi_i(f_0)\}$ is a subalgebra of $\mathfrak{A}\}$. Obviously, M is open-closed in $\mathfrak{B}^*$. As $\mathfrak{C}$ is a Boolean product with the index $\mathfrak{B}^*$, we get $\mathfrak{C} \cong \mathfrak{C} | M \times \mathfrak{C} | \mathfrak{B}^* \setminus M$. Since $i_0 \notin \mathfrak{B}^* \setminus M$, one can assume $\mathfrak{C} | \mathfrak{B}^* \setminus M \in P_{SB}(\mathfrak{A})$ (representing $\mathfrak{B}^* \setminus M$ as a finite union of sets of the type $N_l \cap (\mathfrak{B}^* \setminus M)$ and again recalling that the Cartesian product of $P_{SB}(\mathfrak{A})$-algebras is also an $IP_{SB}(\mathfrak{A})$-algebra). Therefore, it is sufficient to show that $\mathfrak{C} | M \in P_{SB}(\mathfrak{A})$, i.e., henceforth one can consider $\mathcal{B}^* = M$. Choosing, if required, a certain extension of the decomposition $\{i_0\}, N_l(l \in I)$ of the space $\mathcal{B}^*$ one can view f_0 as constant on every $N_l(l \in I)$.

Let $a_1,\dots,a_s$ be values of the function f on $\mathcal{B}^* \setminus \{i_0\}$. Let us set $N^{(j)} = \{N_l | l \in I, f_l(N_l) = \{a_j\}\}$. Let us define the space $\mathcal{B}_j^*$ as $\mathcal{B}^*$ with the element i_0 substituted with a new element y_j. Let now $Y_j = \cup N^{(j)} \cup \{y_j\}$ at $j \leq s$.

It should be noticed that Y_j is a closed subset in $\mathcal{B}_j^*$, and that at $j \neq k$ we have $Y_j \cap Y_k = \varnothing$. Let $Y = \bigcup_{j=1}^{s} Y_j$ be a discrete family of the spaces Y_j. Obviously, Y is a Boolean space, i.e., $Y = \mathcal{B}_1^*$ for a certain Boolean algebra $\mathcal{B}_1$, in which case $\mathcal{B}^*$ is obtained from Y by identifying the points $y_1,\dots y_s$. Let us define the embedding α of the algebra $\mathfrak{C}$ into the Cartesian degree $\mathfrak{A}^Y$: for all $f \in \mathfrak{C}$ let $\alpha(f) | Y \setminus \{y_1,\dots,y_s\} = f | Y \setminus \{y_1,\dots,y_s\}$, and for $1 \leq j \leq s$ let $\alpha(f)(y_j) = a_j$. It is obvious that $\alpha(\mathfrak{C})$ obeys the following condition: for any $f, g \in \alpha(\mathfrak{C})$, any open-closed $N \subseteq Y$ we have $f | N \cup g | Y \setminus N \in \alpha(\mathfrak{C})$ (as $\mathfrak{C} \in \Gamma^a S(\mathfrak{A})$).

Therefore, in order to prove the inclusion $\alpha(\mathfrak{C}) \in P_{SB}(\mathfrak{A})$, it suffices to show that $\alpha(\mathfrak{C})$ is a subalgebra of a Boolean power $\mathfrak{A}^{\mathcal{B}_1}$, i.e., that for any $f \in \mathfrak{C}$ the function $\alpha(f)$ is a continuous mapping from the space Y to a discrete $\mathfrak{A}$. Let $y \in Y \setminus \{y_1,\dots,y_s\}$. Then we get $y \in N_i$ for a certain N_i which is, in particular, open-closed in Y, and $\alpha(f) | N_i = f | N_i$. The function $f | N_i$ is continuous on N_i, since $\mathfrak{C} | N_i \in P_{SB}(\mathfrak{A})$. Therefore, f is continuous on a certain neighborhood of any point $y \in Y \setminus \{y_1,\dots,y_s\}$. If for a certain $1 \leq j \leq s$ we have $y = y_j$, $\alpha(f)$ is constant and, hence, continuous on the open-closed neighborhood $([\![f = f_0]\!] \cap \cup N^{(j)} \cup \{y_j\})$ of the point y_j in the space Y. Therefore, for any $f \in \mathfrak{C}$ we get $\alpha(f) \in \mathfrak{A}^{\mathcal{B}_1}$, which is the required proof. ■

It should be remarked in connection with theorems **7.4**, **7.5** and **7.6** that, taking into account the results of lemma **4.3** as well as the fact that the algebra $\mathfrak{A}$ is finite, the operation P_{SB} in the formulation of the theorems can be substituted with the operation P_{FB}, as the conditions of these theorems describe algebras $\mathfrak{A}$ such that any (countable) $\mathfrak{M}(\mathfrak{A})$-algebras are presentable by filtered Boolean powers of the algebra $\mathfrak{A}$. It is also of interest that countable algebras of a finitely generated discriminator variety $\mathfrak{M}$ are always representable by filtered Boolean powers of a certain finite algebra which is not, in fact, necessarily in $\mathfrak{M}$.

Theorem 7.7. If $\mathfrak{M}$ is a finitely generated discriminator variety, there is a finite algebra $\mathfrak{H}$ (not necessarily belonging to $\mathfrak{M}$) such that any at most countably infinite $\mathfrak{M}$-algebra is isomorphic to a certain filtered Boolean power of the algebra $\mathfrak{H}$.

Proof. As $\mathfrak{M}$ is a finitely generated discriminator variety, $\mathfrak{M}_{SI}^{+}$ consists of a finite number of finite algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$, and any $\mathfrak{M}$-algebra is isomorphic to a certain Boolean product of the algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$. Let σ be a signature of the variety $\mathfrak{M}$, and let $\mathfrak{F}_\sigma$ be a class of all finite algebras of the signature σ. Since $\mathfrak{F}_\sigma$ has the properties of mutual embedding and amalgamation, there are $\mathfrak{F}_\sigma$-algebras $\mathfrak{P}_1,\ldots,\mathfrak{P}_n$ and $\mathfrak{H}$ such that:

(1) $\mathfrak{A}_i$ is a subalgebra of the algebra $\mathfrak{P}_i$ at $1 \le i \le n$;

(2) if $\mathfrak{A}_j$ is isomorphically embeddable into $\mathfrak{A}_i$, and α is an embedding of $\mathfrak{A}_j$ into $\mathfrak{P}_i$, there is an embedding β of the algebra $\mathfrak{P}_i$ into the algebra $\mathfrak{P}_j$ such that $\beta \cdot \alpha$ is identical on $\mathfrak{A}_j$;

(3) every algebra $\mathfrak{P}_i$ is embeddable into $\mathfrak{H}$; this embedding will be referred to as φ_i.

Let $\mathfrak{A}$ be an arbitrary at most countably infinite $\mathfrak{M}$-algebra, in which case one can assume that $\mathfrak{A}$ is a Boolean product of the algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$: $\mathfrak{A} \subseteq \prod_{x \in \mathfrak{B}^*} \mathfrak{A}_x$, where $\mathfrak{A}_x \in \{\mathfrak{A}_1,\ldots,\mathfrak{A}_n\}$, and $\mathfrak{B}$ is a certain at most countably infinite Boolean algebra. Let us prove that:

(*) for any $x \in \mathfrak{B}$ there exists an open-closed neighborhood N of the point x and embeddings η_y of the algebras $\mathfrak{A}_y$ into $\mathfrak{H}$ at $y \in N$ such that for $f \in \mathfrak{A}$, $h \in \mathfrak{H}$ the set $\{y \in N \mid \eta_y(f(y)) = h\}$ is open-closed in $\mathfrak{B}^*$.

The proof of the statement (*) is similar to that of lemma **7.11**. By induction over $|\mathfrak{H} \setminus \varphi_i(\mathfrak{A}_i)|$, where $i \le n$ and $\mathfrak{A}_i = \mathfrak{A}_x$, let us prove the statement obtained from (*) by replacing the algebra $\mathfrak{H}$ with the algebra $\mathfrak{P}_i$, and the embedding η_y with σ_y. Let x be such that $|\mathfrak{H} \setminus \varphi_i(\mathfrak{A}_i)|$ is minimal among the numbers $|\mathfrak{H} \setminus \varphi_j(\mathfrak{A}_j)|$ where $j \le n$. Let us choose $f_1,\ldots,f_m \in \mathfrak{A}$ such that $\{f_1(x),\ldots,f_m(x)\} = \mathfrak{A}_x$ and $m = |\mathfrak{A}_x|$. By analogy with what has been done for the induction basis in the proof of lemma **7.11**, we find an open-closed neighborhood N of the point x such that for $y \in N$ there are isomorphisms $\lambda_{x,y}$ from the algebra $\mathfrak{A}_x$ to the algebra $\mathfrak{A}_y$ defined by the equalities $\lambda_{x,y}(f_j(x)) = f_j(y)$, where $1 \le j \le m$. In particular, $\lambda_{x,y}^{-1}$ is an embedding of the algebra $\mathfrak{A}_y$ into the algebra $\mathfrak{P}_i$, where i is such that $\mathfrak{A}_x = \mathfrak{A}_i$. Let us set $\sigma_y = \lambda_{x,y}^{-1}$. Let $h \in \mathfrak{P}_i$ and $f \in \mathfrak{A}$, we get

$$\{y \in N | \lambda_{x,y}^{-1}(f(y)) = h\} = N \cap [|f = f_i|],$$

i.e., this set is open-closed in $\mathfrak{B}^*$ provided $h = f_j(x)$. Therefore, the required statement is proved for $x \in \mathfrak{B}^*$ such that $|\mathfrak{H} \setminus \varphi_i(\mathfrak{A}_x)|$ is minimal.

Let us assume that the required statement has been proved for all $y \in \mathfrak{B}^*$ such that $|\mathfrak{H} \setminus \varphi_j(\mathfrak{A}_y)| < |\mathfrak{H} \setminus \varphi_i(\mathfrak{A}_x)|$, where i,j are such that $\mathfrak{A}_x = \mathfrak{A}_i, \mathfrak{A}_y = \mathfrak{A}_j$. And again, as has been the case in the induction step when proving lemma **7.11**, we find functions $f_1, \ldots, f_m \in \mathfrak{A}$ such that $\{f_1(x), \ldots, f_m(x)\} = \mathfrak{A}_x$, where $m = |\mathfrak{A}_x|$, we find an open-closed neighborhood N of the point x of the space $\mathfrak{B}^*$, we find embeddings $\lambda_{x,y}$ of the algebra $\mathfrak{A}_x$ in the algebras $\mathfrak{A}_y$ at $y \in N$, and we also find a closed subset $Y \subseteq N$ such that $Y = \{x \in N | \lambda_{x,y}$ is an isomorphism from $\mathfrak{A}_x$ to $\mathfrak{A}_y\}$. Since $N \setminus Y$ is an open subset of the space $\mathfrak{B}^*$ and $\mathfrak{B}$ is at most countably infinite, by the induction supposition, there is at most countably infinite set $\mathfrak{I}$ and open-closed pairwise-disjunct subsets $N_k (k \in I)$ such that $\bigcup_{k \in I} N_k = N \setminus Y$. At the same time, for every $k \in I$, one can find embeddings σ_y^k $(y \in N_k)$ such that the statements (*) hold for N_k, σ_y^k with the corresponding substitution of the algebra $\mathfrak{H}$ by algebras $\mathfrak{P}_{n_k}$, where $n_k \leq n$ and $\mathfrak{A}_i = \mathfrak{A}_x$ is isomorphically embeddable into $\mathfrak{A}_{n_k}$ for all $k \in I$. $\varepsilon_i(id)$ will denote an identical embedding of the algebra $\mathfrak{A}_i$ in $\mathfrak{P}_i(\mathfrak{A}_i)$.

Let us choose embeddings μ_y $(y \in N_k)$ (which do exist by the property (2) of the algebras $\mathfrak{A}_i, \mathfrak{P}_j, \mathfrak{H}$ noted in the beginning of the proof) such that the following diagram is commutative:

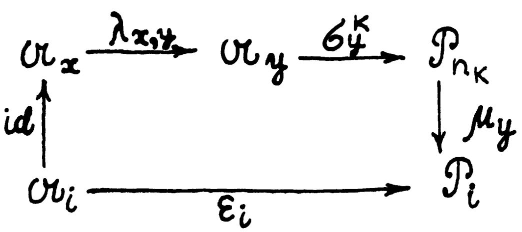

Figure 6

Let us define the embeddings σ_y of the algebra $\mathfrak{A}_y$ into $\mathfrak{P}_i$ as $\mu_y \cdot \sigma_y^k$ for $y \in N_k (k \in I)$ and for all $y \in Y$ let us set $\sigma_y = \lambda_{x,y}^{-1}$. Therefore, σ_y are defined for all $y \in Y$ and for $y \in N$, and for $j \leq m$ we have $\sigma_y(f_j(y)) = \sigma_x(f_j(x))$.

Let $f \in \mathfrak{A}$ and $h \in \mathfrak{P}_i$. If $h \in \mathfrak{A}_i$, we get

$$\{y \in N | \sigma_y(f(y)) = h\} = [\![f = f_j]\!] \cap N,$$

where f_j is such that $f_j(x) = h$. If $h \notin \mathfrak{A}_i$ then

$$\{y \in N | \sigma_y(f(y)) = h\} =$$
$$= \bigcup_{k \in I} \{y \in N_k | \mu_k \sigma_y^k(f(y)) = h\} = \bigcup_{k \in I} \{y \in N_k | \sigma_y^k(f(y)) = \mu_y^{-1}(h)\}$$

is an open set.

Therefore, for $h \in \mathfrak{P}_i$ the sets $\{y \in N | \sigma_y(f(y)) = h\}$ form an open finite disjunct decomposition of the open-closed N and, hence, indeed, for any $h \in \mathfrak{P}_i$ $\{y \in N | \sigma_y(f(y)) = h\}$ is open-closed.

Thus, the statement (*) with the algebra $\mathfrak{P}_i$ substituted for the algebra $\mathfrak{H}$ has been proved with induction. To prove the statement (*) itself it now suffices to replace the embeddings $\sigma_y : \mathfrak{A}_y \to \mathfrak{P}_i$ with the embeddings $\eta_y = \varphi_i \cdot \sigma_y$ of the algebra $\mathfrak{A}_y$ into $\mathfrak{H}$, where φ_i is an embedding of $\mathfrak{P}_i$ into $\mathfrak{H}$ mentioned in property (3) in the beginning of the proof of the theorem.

The validity of the statement (*) for the algebra $\mathfrak{A}$ makes it possible, as $\mathfrak{B}^*$ is compact, to single out a finite number of pairwise-disjunct sets $N_1, \ldots, N_k$ covering $\mathfrak{B}^*$, as well as a system of embeddings η_y of the algebras $\mathfrak{A}_y$ into $\mathfrak{H}$ obeying the statement (*). Let us define the embedding $\eta : \mathfrak{A} \to \mathfrak{H}^{\mathfrak{B}^*}$ in such a way that for any $x \in \mathfrak{B}^*$ we have $\eta(f)(x) = \eta_x(f(x))$. Let

$$\{\eta(\mathfrak{A})(x) | x \in \mathfrak{B}^*\} = \{\mathfrak{H}_1, \ldots, \mathfrak{H}_n\}.$$

Since for any $f, g \in \mathfrak{A}$ and open-closed subset $N \subseteq \mathfrak{B}^*$, the element $f | N \cup g | \mathfrak{B}^* \setminus N$ also belongs to $\mathfrak{A}$, an analogous property is also valid for the algebra $\eta(\mathfrak{A}) \subseteq \mathfrak{H}^{\mathfrak{B}^*}$. For $f \in \mathfrak{A}$ and $h \in \mathfrak{H}$ we have

$$\{y \in \mathfrak{B}^* | \eta(f)(y) = h\} = \bigcup_{1 \leq j \leq k} \{y \in N_k | \eta_k(f(y)) = h\}$$

and, hence, by the statement (*), this set is open-closed in $\mathcal{B}^*$. Thus, we get $\eta(\mathfrak{A}) \subseteq \mathfrak{H}^{\mathcal{B}}$. Let $X_j = \{x \in \mathcal{B}^* \mid \eta(\mathfrak{A})(x) \subseteq \mathfrak{H}_j\}$ for $1 \leq j \leq n$.

The sets X_j are obviously closed in $\mathcal{B}^*$. To conclude the proof of the theorem, we now have to notice that $\eta(\mathfrak{A}) = \mathfrak{H}^{\mathcal{B}}(X_1,\dots,X_n;\mathfrak{H}_1,\dots,\mathfrak{H}_n)$. In one direction the inclusion is valid by the definition of the sets X_j. Let now $f \in \mathfrak{H}^{\mathcal{B}}(X_1,\dots,X_n;\mathfrak{H}_1,\dots,\mathfrak{H}_n)$. Standard considerations implying that for any $h \in \mathfrak{H}$, $[\![f(x) = h]\!]$ is open-closed in $\mathcal{B}^*$, $\{\eta(\mathfrak{A})(x) \mid x \in \mathcal{B}^*\} = \{\mathfrak{H}_1,\dots,\mathfrak{H}_n\}$ and $\eta(\mathfrak{A})$ is closed relative to the formation of the elements $f|N \cup g|\mathcal{B}^* \setminus N$, prove that $f \in \eta(\mathfrak{A})$. Therefore, indeed, any at most countably infinite $\mathfrak{M}$-algebra proves to be isomorphic to a certain filtered Boolean power of the algebra $\mathfrak{H}$. ■

And finally let us formulate without proof some results concerning the representability of varieties with Boolean G-powers.

Theorem 7.8. For a finite algebra $\mathfrak{A}$ with a group of automorphisms G, the equality $\mathfrak{M}(\mathfrak{A}) = IP_G(\mathfrak{A})$ is equivalent to the following conditions:

(1) $\mathfrak{A} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$, where $\mathfrak{A}_1$ is an Abelian algebra, $\mathfrak{M}(\mathfrak{A}_2)$ is a discriminator variety;

(2) if both algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, are non-singleton, they both contain one-element subalgebras which are families of fixed points relative to the automorphisms for each of them;

(3) $\mathfrak{M}(\mathfrak{A}_1) = IP_{G_{\mathfrak{A}_1}}(\mathfrak{A}_1), \mathfrak{M}(\mathfrak{A}_2) = IP_{G_{\mathfrak{A}_2}}(\mathfrak{A}_2)$, where $G_{\mathfrak{A}_i}$ are groups of automorphisms of the algebras $\mathfrak{A}_i$.

Theorem 7.9. Let $\mathfrak{A}$ be a quasi-primal algebra and G be its group of automorphisms. Then we get $\mathfrak{M}(\mathfrak{A}) = IP_G(\mathfrak{A})$ iff:

(1) any isomorphism between non-singleton subalgebras of the algebra $\mathfrak{A}$ is extendable to the automorphism of $\mathfrak{A}$;

(2) any subalgebra of the algebra $\mathfrak{A}$ is a family of fixed points of $\mathfrak{A}$ relative to a certain subgroup of the group G.

By way of concluding this section let us recall one more result concerning Boolean representability. Theorem **4.1** is a generalization of a result obtained by R.S.Pierce [157]: any commutative ring with unity is representable as a Boolean product of directly non-decomposable rings. In a paper by W.D.Burgess and W.Stephenson [23], the authors proved that any ring R with unity is representable as a Boolean product of directly non-decomposable rings iff any idempotent of the ring R is central.

Priorities. Among the first results concerning the representability of varieties using Boolean constructions were those obtained by R.F.Arens and J.Kaplanski [5] on varieties of algebras over a finite field, A.L.Foster [67] on varieties generated by primal algebras, J.Dauns and K.H.Hofmann [45] on biregular rings, etc.. Theorem **7.1** of the present section is by to D.M.Clarkand and P.H.Krauss [36]. Lemma **7.1** as well as theorem **7.2** resulting from it were proved by S.Burris and R.McKenzie [31], using description of finitely generated congruence-modular varieties with a solvable elementary theory. The proof of these lemmas and the theorem cited here and employing no results on solvability is by E.W.Kiss [106]. The statement of theorem **7.3** is due to R.W.Quackenbush [195]. Theorems **7.4**, **7.5**, **7.6** and lemmas **7.2**-**7.11** pertaining to them were proved by S.Burris and R.McKenzie [31], theorem **7.7** is by S.Burris and H.Werner [33]. The proof of theorems **7.8** and **7.9** can be found in a paper by S.Burris and D.Clark [29].

CHAPTER 3

VARIETIES: SPECTRA, SKELETONS, CATEGORIES

The aim of the present chapter is to apply the methods, results and constructions considered in the first two chapters to "external" studies of universal algebra varieties. "External" studies of varieties imply consideration and description of not the algebras incorporated into a given variety but of the variety as a whole, i.e., studies of the variety as a single object the elements of which are the algebras of the variety with basic algebraic relations and operations among them such as isomorphisms, epimorphisms, embeddings, Cartesian products, etc.. Studies of the "external" structure of a variety imply, first of all, those of the categories of the algebras belonging to the variety in the case when the morphisms of the category are all homomorphisms between algebras of the given variety. Indeed, the overwhelming part of the notions related to an algebra can be formulated in terms of these categories and, therefore, the varieties with "the same" categories must be "almost the same" themselves, as we will see in the first theorems proved in section **8**. Another, rougher "external" characteristic of a variety is its spectrum and its fine spectrum. We have already discussed in section **6** some results for algebras with a minimal spectrum, these impose very rigid limitations and allow only three variants for the varieties generated by such algebras. Below, in section **8**, we will present a result describing to the accuracy of "the same category" all the varieties with a minimal fine spectrum of a certain quite definite type, as well as a number of other results on spectra and fine spectra. Well-known descriptions of category transformations also pertain to the results characterizing varieties with a fine spectrum. On the other hand, in the case when the fine spectrum of a variety is big, i.e., when the number of the types of the isomorphisms of the algebras of a given variety is big, it is interesting to study various relations and operations between the types of the isomorphism induced by algebraically important relations and operations between the algebras of the variety themselves. This results in the definition of the notion of the skeleton of a variety, and the greater part of the present chapter is devoted to studying skeletons of congruence-distributive varieties for which the application of Boolean constructions is most efficient. In particular, a number of results on countable skeletons of congruence-distributive varieties make it possible, using the language of "external" description of varieties, to express such facts as the degeneration of a variety with a quasi-primal algebra with no proper subalgebras, finite generation of discriminator varieties, etc.

8. Spectra and Categories

The present section is devoted to the problems of characterizing varieties by categories and spectra associated with them. The basic notions of the theory of categories can be found elsewhere [138].

Definition 8.1.

(a) For any variety of the algebras $\mathfrak{M}$, $\breve{\mathfrak{M}}$ will denote the category the objects of which are all $\mathfrak{M}$-algebras, and the morphisms of which are homomorphisms between $\mathfrak{M}$-algebras.

(b) The categories $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are isomorphic if they are isomorphic as partial semigroups or, which is equivalent, if there is a bijective mapping φ from the set of objects $Ob(\mathfrak{K}_1)$ of the category $\mathfrak{K}_1$ to $Ob(\mathfrak{K}_2)$, and if for any $a,b \in Ob(\mathfrak{K}_1)$, the bijective mapping $\varphi_{a,b}$ from the set $Hom(a,b)$ of $\mathfrak{K}_1$-morphisms from a to b to the set $Hom(\varphi(a),\varphi(b))$, for any $a,b,c \in Ob(\mathfrak{K}_1)$ and $\alpha \in Hom(a,b), \beta \in Hom(b,c)$, we have $\varphi_{a,c}(\alpha\beta) = \varphi_{a,b}(\alpha)\cdot\varphi_{b,c}(\beta)$.

In the case of an isomorphism of the categories $\breve{\mathfrak{M}}_1$ and $\breve{\mathfrak{M}}_2$ for some algebra varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$, we will speak about a weak equivalence of the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ denoted by $\mathfrak{M}_1 \simeq \mathfrak{M}_2$.

(c) By the erasing functor $S_{\mathfrak{M}}$ from the category of $\breve{\mathfrak{M}}$-algebras of the variety $\mathfrak{M}$ to the category of the sets *Set* we will mean a correlation between any $\mathfrak{M}$-algebra and its basic set, any homomorphism between $\mathfrak{M}$-algebras being considered in this case as a mapping between the basic sets of the algebras. The varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are called equivalent provided that there is an isomorphism F of the categories $\breve{\mathfrak{M}}_1$ and $\breve{\mathfrak{M}}_2$ such that $S_{\mathfrak{M}_2}F = S_{\mathfrak{M}_1}$; the equivalence of the varieties will be denoted by $\mathfrak{M}_1 \cong \mathfrak{M}_2$.

(d) The varieties of the algebras $\mathfrak{M}_1$ of the signature σ_1 and those of the algebras $\mathfrak{M}_2$ of the signature σ_2 are rationally equivalent, provided that there are mappings $F_1(F_2)$ from the operations of the signature $\sigma_2(\sigma_1)$ into the set of terms of the signature $\sigma_1(\sigma_2)$ such that an n-ary operation transforms into an n-ary term, in which case:

(1) for any $\mathfrak{M}_1$-algebra $\mathfrak{A} = < A;\sigma_1 >$ we have $F_1(\mathfrak{A}) = < A;\sigma_2 > \in \mathfrak{M}_2$, where the σ_2-operations on the algebra $F_1(\mathfrak{A})$ are defined using $F_1(\sigma_2)$ terms of the algebra $\mathfrak{A}$;

(2) for any $\mathfrak{M}_2$-algebra $\mathfrak{A} = < A;\sigma_2 >$ we have $F_2(\mathfrak{A}) = < A;\sigma_1 > \in \mathfrak{M}_1$;

(3) for any $\mathfrak{M}_2$-algebra $\mathfrak{A} = < A;\sigma_2 >$ we have $F_1(F_2(\mathfrak{A})) = \mathfrak{A}_1$;

(4) for any $\mathfrak{M}_1$-algebra $\mathfrak{A} = < A;\sigma_1 >$ we have $F_2(F_1(\mathfrak{A})) = \mathfrak{A}$.

The rational equivalence of the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is denoted by $\mathfrak{M}_1 \equiv_t \mathfrak{M}_2$.

(e) Let us say that the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of the signatures σ_1 and σ_2, respectively, are rationally equivalent provided that the conditions given in the definition (d) for the varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are valid for these algebras. When the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are rationally equivalent, we will write $\mathfrak{A}_1 \equiv_t \mathfrak{A}_2$.

Definition 8.2. A clone of the variety $\mathfrak{M}$ is a multi-basis algebra $< A_1, A_2, \ldots, A_n, \ldots; e_i^n, c_m^n | 1 \le i \le n < \omega, m < \omega >$, where each A_n is a family of terms of the signature of the variety $\mathfrak{M}$ from the variables $x_1, \ldots, x_n$ having been factorized with respect to the module of their equivalence on the variety $\mathfrak{M}$; e_i^n are the constants incorporated into A_n, i.e., they are the term-projections correlating the variable x_i to the set of variables $x_1, \ldots, x_n$; and, finally, c_m^n are operations such that the terms $t \in A_n, t_1, \ldots, t_n \in A_m$ are correlated to the term $c_m^n(t, t_1, \ldots, t_n) = t(t_1, \ldots, t_n)$. Such a clone of the variety will henceforth be denoted by $Clon\,\mathfrak{M}$.

The following theorem will reveal the interrelations among the notions introduced above.

Theorem 8.1. For any varieties $\mathfrak{M}_1$ and $\mathfrak{M}_2$ the following conditions are equivalent:

(a) $\breve{\mathfrak{M}}_1 \cong \breve{\mathfrak{M}}_2$;

(b) $\mathfrak{F}_{\mathfrak{M}_1}(\aleph_0) \equiv_t \mathfrak{F}_{\mathfrak{M}_2}(\aleph_0)$;

(c) there are algebras $\mathfrak{A}_i \in \mathfrak{M}_i$ such that $\mathfrak{M}_i = \mathfrak{M}(\mathfrak{A}_i)$ and $\mathfrak{A}_1 \equiv_t \mathfrak{A}_2$;

(d) $\mathfrak{M}_1 \equiv_t \mathfrak{M}_2$;

(e) $Clon\mathfrak{M}_1 \cong Clon\mathfrak{M}_2$.

Proof. The implication (b) $\rightarrow$ (c) is obvious. Let now $\mathfrak{M}_i = \mathfrak{M}(\mathfrak{A}_i)$ and $\mathfrak{A}_1 \equiv_t \mathfrak{A}_2$. Since $\mathfrak{M}_i = HSP(\mathfrak{A}_i)$, there is a set I_i, a subalgebra $\mathfrak{C}_i \subseteq \mathfrak{A}_i^{I_i}$ and a homomorphism h_i from the algebra $\mathfrak{C}_i$ to the algebra $\mathfrak{B}_i$. for any algebra $\mathfrak{B}_i \in \mathfrak{M}_i$. The equivalence $\mathfrak{A}_1 \equiv_t \mathfrak{A}_2$ implies the existence of mappings F_i from the operations of the signature $\sigma_j (j \neq i)$ of the algebra $\mathfrak{A}_j$ to the terms of the signature σ_i such that $F_i(\mathfrak{A}_i) \cong \mathfrak{A}_j (j \neq i)$ provided that the operations of the signature σ_j are defined on the basic set of the algebra $\mathfrak{A}_i$ using the F_i-corresponding terms of the signature σ_i. Obviously, a similar construction of F_i-algebras can be extended to direct powers of the algebra $\mathfrak{A}_i$, subalgebras and homomorphic images of these direct powers, with all the conditions (1) - (4) of the definition **8.1** (d) fulfilled. But at the same time this implies that $\mathfrak{M}_1 \equiv_t \mathfrak{M}_2$. Therefore, the statement (c) of the present theorem yields the statement (d).

Let now $\mathfrak{M}_1 \equiv_t \mathfrak{M}_2$ and F_1, F_2 be the mappings from the definition **8.1** (d). In this case, extending $F_1(F_2)$ in a natural way to a family of all $\sigma_2(\sigma_1)$ terms, we get an isomorphism of the multi-basis algebras $Clon\mathfrak{M}_1$ and $Clon\mathfrak{M}_2$. Therefore, (d) yields (e).

Let $\breve{\mathfrak{M}}_1 \cong \breve{\mathfrak{M}}_2$ and $f(x_1,\dots,x_m)$ one of the operations of the signature σ_1. Let us consider $\mathfrak{F}_{\mathfrak{M}_1}(m) \subseteq \mathfrak{F}_{\mathfrak{M}_1}(\aleph_0)$. Since free algebras of varieties are defined in category terms, the isomorphism $\breve{\mathfrak{M}}_1 \cong \breve{\mathfrak{M}}_2$ can be implemented by a certain functor F such that $S_{\mathfrak{M}_2}F = S_{\mathfrak{M}_1}$ implies the equalities

$$F(\mathfrak{F}_{\mathfrak{M}_1}(m)) = \mathfrak{F}_{\mathfrak{M}_2}(m), \quad F(\mathfrak{F}_{\mathfrak{M}_1}(\aleph_0)) = \mathfrak{F}_{\mathfrak{M}_2}(\aleph_0).$$

Therefore, the element $f(x_1,\dots,x_m)$ of the algebra $\mathfrak{F}_{\mathfrak{M}_1}(m)$ is equal to a certain element $\varphi(x_1,\dots,x_m)$ of the algebra $\mathfrak{F}_{\mathfrak{M}_2}(m)$, where φ is a term of the signature σ_2 of the variety $\mathfrak{M}_2$. For any $a_1,\dots,a_m \in S_{\mathfrak{M}_1}(\mathfrak{F}_{\mathfrak{M}_1}(\aleph_0)) = S_{\mathfrak{M}_2}(\mathfrak{F}_{\mathfrak{M}_2}(\aleph_0))$, by defining the mappings $h_i(x_i) = a_j$ $(i = 1,2$ and $j \leq m)$, we can find, bearing in mind that the algebras $\mathfrak{F}_{\mathfrak{M}_1}(m)$ and $\mathfrak{F}_{\mathfrak{M}_2}(m)$ are free and unique, pairwise coinciding extensions $\breve{h}_i$ of the mappings h_i to all the algebras $\mathfrak{F}_{\mathfrak{M}_i}(m)$, which are homomorphisms from the algebras $\mathfrak{F}_{\mathfrak{M}_i}(m)$ to the algebras $\mathfrak{F}_{\mathfrak{M}_i}(\aleph_0)$, respectively. In this case, as $f(x_1,\dots,x_m) = \varphi(x_1,\dots,x_m)$, we get

$$f(a_1,\dots,a_m) = \breve{h}_1(f(x_1,\dots,x_m)) = \breve{h}_2(\varphi(x_1,\dots,x_m) = \varphi(a_1,\dots,a_m).$$

Setting $F_2(f) = \varphi(x_1,\dots,x_m)$, we get a certain mapping F_2 from the operations of the signature σ_1 to σ_2-terms. The mapping F_1 from the operations of the signature σ_2 to σ_1-terms is defined analogously. The above properties of these mappings obviously imply the fulfillment of requirements (1) - (4) in definition **8.1** (e). Therefore, we get $\mathfrak{F}_{\mathfrak{M}_1}(\aleph_0) \equiv_t \mathfrak{F}_{\mathfrak{M}_2}(\aleph_0)$, i.e., condition (a) of the theorem under consideration implies condition (b).

To complete the proof of the theorem, we now have to prove that the isomorphism of multi-basis algebras $Clon\mathfrak{M}_1 \cong Clon\mathfrak{M}_2$ implies the isomorphism F of the categories $\breve{\mathfrak{M}}_1$ and $\breve{\mathfrak{M}}_2$ such that $S_{\mathfrak{M}_2}F = S_{\mathfrak{M}_1}$. So, let $Clon\mathfrak{M}_1 \cong Clon\mathfrak{M}_2$. By the definitions, this obviously implies the rational equivalence of the algebras $\mathfrak{F}_{\mathfrak{M}_1}(n)$ and $\mathfrak{F}_{\mathfrak{M}_2}(n)$ which is, at $n < \omega$, implemented by the same mappings $F_2(F_1)$ of $\sigma_1(\sigma_2)$-operations into $\sigma_2(\sigma_1)$-terms for any n. In this case the basic sets of the algebras $\mathfrak{F}_{\mathfrak{M}_1}(n)$ and $\mathfrak{F}_{\mathfrak{M}_2}(n)$, respectively, can be identified. The rational equivalence of $\mathfrak{F}_{\mathfrak{M}_1}(n)$ and $\mathfrak{F}_{\mathfrak{M}_2}(n)$ implies the coincidence of the lattices $Con\mathfrak{F}_{\mathfrak{M}_1}(n)$ and $Con\mathfrak{F}_{\mathfrak{M}_2}(n)$. The latter condition allows us to construct a functor F mapping finitely generated $\mathfrak{M}_1$-algebras to finitely generated $\mathfrak{M}_2$-algebras, a set of morphisms from $\mathfrak{F}_{\mathfrak{M}_1}(n)$ to an n-generated $\mathfrak{M}_1$-algebra $\mathfrak{A}_1$ to a set of morphisms from $\mathfrak{F}_{\mathfrak{M}_2}(n)$ to an n-generated algebra $F(\mathfrak{A}_1)$ so that $S_{\mathfrak{M}_2}F = S_{\mathfrak{M}_1}$.

Any morphism of an n-generated $\mathfrak{M}_1$-algebra $\mathfrak{A}_1$ on an n-generated $\mathfrak{M}_1$-algebra $\mathfrak{B}_1$ is uniquely defined by certain morphisms from the algebra $\mathfrak{F}_{\mathfrak{M}_1}(n)$ in $\mathfrak{A}_1$ and $\mathfrak{B}_1$. This makes it possible to extend the functor F, with the required condition on full subcategories of the categories $\breve{\mathfrak{M}}_i$ generated by families of finitely generated $\mathfrak{M}_i$-algebras fulfilled. On the other hand, any homomorphism from an arbitrary $\mathfrak{M}_1$-algebra $\mathfrak{A}$ to an algebra $\mathfrak{B}$ is uniquely defined by its limitations up to finitely generated subalgebras of the algebra $\mathfrak{A}$. This enables one to extend the functor F to the isomorphism of F of the whole category $\breve{\mathfrak{M}}_1$ on the category $\breve{\mathfrak{M}}_2$ such that $S_{\mathfrak{M}_2}F = S_{\mathfrak{M}_1}$, which is the required proof for the implication (e) $\rightarrow$ (a). ■

The condition on weak equivalence of varieties is essentially weaker than that on the equivalence of varieties, as will be shown below in the description of all varieties weakly equivalent to the variety of Boolean algebras.

Theorem 8.2. A variety of algebras $\mathfrak{M}$ is weakly equivalent to a variety of the Boolean algebras BA iff $\mathfrak{M}$ is generated by a primal algebra.

Proof. Let $\mathfrak{A}$ be a primal algebra, and let us show that $\breve{\mathfrak{M}}(\mathfrak{A}) \simeq \check{BA}$. Indeed, by theorem **7.3**, $\mathfrak{M}(\mathfrak{A}) = IP_B(\mathfrak{A})$ and, therefore, for any $\mathfrak{M}(\mathfrak{A})$-algebra $\mathfrak{A}_1$ there is a Boolean algebra $\mathcal{B}$ such that $\mathfrak{A}_1 \cong \mathfrak{A}^{\mathcal{B}}$. By virtue of congruence-distributivity of $\mathfrak{M}(\mathfrak{A})$ and theorem **3.2**, we get $Con_p\mathfrak{A}_1 \cong Con_p\mathfrak{A}^{\mathcal{B}} \cong \mathcal{B}$, i.e., $Con_p\mathfrak{A}_1 \in BA$. Let us define the mapping

$$\varphi: Ob(\breve{\mathfrak{M}}(\mathfrak{A})) = \mathfrak{M}(\mathfrak{A}) \to Ob(\check{BA}) = BA$$

as $\varphi(\mathfrak{A}_1) = Con_p\mathfrak{A}_1$.

For any $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{M}(\mathfrak{A})$ and $h \in Hom(\mathfrak{A}_1, \mathfrak{A}_2)$, let us define

$$\varphi_{\mathfrak{A}_1,\mathfrak{A}_2}(h) \in Hom(Con_p\mathfrak{A}_1, Con_p\mathfrak{A}_2)$$

as $\varphi_{\mathfrak{A}_1,\mathfrak{A}_2}(h)(\theta^{\mathfrak{A}_1}_{a,b}) = \theta^{\mathfrak{A}_2}_{h(a),h(b)}$.

To prove that $\mathfrak{M}(\mathfrak{A})$ and BA are weakly equivalent, it now suffices to notice that any homomorphism from the algebra $\mathfrak{A}^{\mathcal{B}_1}$ to the algebra $\mathfrak{A}^{\mathcal{B}_2}$ is uniquely defined by a certain homomorphism $Con_p\mathfrak{A}^{\mathcal{B}_1}$ into $Con_p\mathfrak{A}^{\mathcal{B}_2}$. To this end, let us note that at any homomorphism h from $\mathfrak{A}^{\mathcal{B}_1}$ to $\mathfrak{A}^{\mathcal{B}_2}$, a subalgebra $\mathfrak{A}_0 \subseteq \mathfrak{A}^{\mathcal{B}_1}$ such that $\mathfrak{A}_0 = \{\bar{a}^1 \mid a \in \mathfrak{A}$ and for any $i \in \mathcal{B}_1^*$ $\bar{a}^1(i) = a\}$ obeys the condition $h(\bar{a}^1) = \bar{a}^2$, where $\bar{a}^2 \in \mathfrak{A}^{\mathcal{B}_2}$, and for any $i \in \mathcal{B}_2^*$, $\bar{a}^2(i) = a$. Indeed, this directly results from the fact that $\mathfrak{A}$ is simple, has no subalgebras and its only automorphism is identical. Hence, the h-image of any element $d \in \mathfrak{A}^{\mathcal{B}_1}$ is uniquely defined by the rule

$$h(< a_1,\ldots,a_n, \neg\theta^{\mathfrak{A}_1}_{d,a_1},\ldots,\neg\theta^{\mathfrak{A}_1}_{d,a_n} >) =$$
$$= < a_1,\ldots,a_n, \varphi_{\mathfrak{A}_1,\mathfrak{A}_2}(\neg\theta^{\mathfrak{A}_1}_{d,a_1}),\ldots,\varphi_{\mathfrak{A}_1,\mathfrak{A}_2}(\neg\theta^{\mathfrak{A}_1}_{d,a_n}) >,$$

where $\{a_1,...a_n\} = \mathfrak{A}$, and $< a_1,...a_n, \neg\theta^{\mathfrak{A}_1}_{d,a_1},...,\theta^{\mathfrak{A}_1}_{d,a_n} >$ is a quasi-canonical setting of the element d of the Boolean power $\mathfrak{A}^{B_1}$ defined in section **3**.

Let us now assume that the variety $\mathfrak{M}$ is such that $\breve{\mathfrak{M}} \simeq \check{BA}$, and the isomorphism of the categories $\breve{\mathfrak{M}}$ and $\check{BA}$ is implemented with the mappings

$$\varphi: Ob(\breve{\mathfrak{M}}) = \mathfrak{M} \to Ob(\check{BA}) = BA$$

and

$$\varphi_{\mathfrak{A}_1,\mathfrak{A}_2}: Hom(\mathfrak{A}_1,\mathfrak{A}_2) \to Hom(\varphi(\mathfrak{A}_1),\varphi(\mathfrak{A}_2)).$$

Let $\mathfrak{A} \in \mathfrak{M}$ be such that $\varphi(\mathfrak{A}) = 2$ is a two-element Boolean algebra, and let us show that $\mathfrak{A}$ is primal and $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$. Let us first remark that since $Hom(1,2) = \varnothing$, where 1 is a one-element Boolean algebra, $Hom(\varphi^{-1}(1),\mathfrak{A}) = \varnothing$ and, hence, $\mathfrak{A}$ is not a one-element $\mathfrak{M}$-algebra.

Let us show that for any algebra $\mathfrak{A}_1 \in \mathfrak{M}$, the lattices of congruences of $\mathfrak{A}_1$ and $\varphi(\mathfrak{A}_1)$ are isomorphic. Let $f: \mathfrak{A}_1 \to \mathfrak{A}_2$ be a certain epimorphism from the algebra $\mathfrak{A}_1$ to $\mathfrak{A}_2$ in the sense of the theory of categories. Let us define a surjective homeomorphism f_1 from $\mathfrak{A}_1$ to $f(\mathfrak{A}_1)$ by the equalities $f_1(a) = f(a)$ for all $a \in \mathfrak{A}_1$, and let us consider the embedding i of the algebra $f(\mathfrak{A}_1)$ into the algebra $\mathfrak{A}_2$ to be identical. The equality $f = i \cdot f_1$ shows that i is an epimorphism of the category $\breve{\mathfrak{M}}$, and, since i is an embedding, i is also a monomorphism in $\breve{\mathfrak{M}}$. Therefore, $\varphi_{\varphi(\mathfrak{A}_1),\varphi(\mathfrak{A}_2)}(i)$ also is both an epi- and a monomorphism in the category of Boolean algebras but, as is well-known and can be readily checked those morphisms in $\check{BA}$ are isomorphisms. As a result, i is also an isomorphism in $\breve{\mathfrak{M}}$, i.e., any epimorphism of the category $\breve{\mathfrak{M}}$ is a surjective homomorphism of $\mathfrak{M}$-algebras. Hence, the lattice of the congruences of the algebra $\mathfrak{A}_1$ is isomorphic to the natural lattice $\phi(\mathfrak{A})$ of the epimorphisms (in the category $\breve{\mathfrak{M}}$) of the object $\mathfrak{A}_1 \in Ob(\breve{\mathfrak{M}})$ arising when identifying such epimorphisms f_1, f_2 for which there is an isomorphism $i \in Hom(f_1(\mathfrak{A}_1)), f_2(\mathfrak{A}_1)$ such that $if_1 = f_2$. But in this case the isomorphism of the categories $\breve{\mathfrak{M}}$ and $\check{BA}$ implies that of the lattices $\phi(\mathfrak{A}_1)$, and $Con\varphi(\mathfrak{A}_1) \cong \phi(\varphi(\mathfrak{A}_1))$. Therefore, indeed, for any algebra $\mathfrak{A}_1 \in \mathfrak{M}$, we get $Con\mathfrak{A}_1 \cong Con\varphi(\mathfrak{A}_1)$.

Let us now prove that the algebra $\mathfrak{A}$ is finite. Let us assume the opposite, and let $a_1,...,a_n,...$ be pairwise different elements of the algebra $\mathfrak{A}$. A free two-generated Boolean algebra $\mathfrak{F}_{BA}(2)$ is equal to 2^4 and, therefore, it has a finite lattice of congruences. Obviously, an isomorphism of the categories $\breve{\mathfrak{M}}$ and $\check{BA}$ implies that an $\mathfrak{M}$ -algebra $\varphi^{-1}(\mathfrak{F}_{BA}(2))$ is also a free two-generated $\mathfrak{M}$ -algebra and, hence, according to the isomorphism of $Con\varphi^{-1}(\mathfrak{F}_{BA}(2))$ and $Con\mathfrak{F}_{BA}(2)$ proved above, the congruence lattice of the algebra $\mathfrak{F}_{\mathfrak{M}}(2)$ is finite. Therefore, any two-generated $\mathfrak{M}$ -algebra has a finite lattice of congruences.

Let us consider a subalgebra $\mathfrak{A}_1$ of the algebra $\mathfrak{A}^{\omega}$ generated by the elements f_1 and h such that for any $n \in \omega$ we have $f_1(n) = a_1$ and $h(n) = a_n$. Since the algebra $\mathfrak{A}$, as well as the algebra $2 \in BA$, has no proper subalgebras (by virtue of the isomorphism of $\breve{\mathfrak{M}}$ and $\check{BA}$), $\mathfrak{A}_1$ contains all the elements f_k of the algebra $\mathfrak{A}^{\omega}$ such that for any $n \in \omega$, $f_k(n) = a_k$. Therefore, the kernels of projections $\pi_i(i \in \omega)$ are pairwise different congruences of the algebra $\mathfrak{A}_1$. The contradiction obtained proves that $\mathfrak{A}$ is finite.

Let now A be the basic set of the algebra $\mathfrak{A}$, and let f be a mapping from the set A^n to $\mathfrak{A}$ for some $n \in \omega$. As earlier, π_i will denote the projection of the set A^n relative to the i-th coordinate. Let us consider $\pi_i(i = 0,1,\ldots,n-1)$ as elements of $\mathfrak{A}^{A^n}$, and let $\mathfrak{A}_1$ be a subalgebra of the algebra $\mathfrak{A}^{A^n}$ generated by the set $\{\pi_0,\ldots,\pi_{n-1}\}$. For any $s \in A^n$ there is a homomorphism h_s from the algebra $\mathfrak{A}_1$ to the algebra $\mathfrak{A}$ defined by the equalities $h_s(\pi_i) = \pi_i(s)$. Let us define the homomorphism h from the algebra $\mathfrak{A}_1$ to the algebra $\mathfrak{A}^{A^n}$ with the following condition: $h(a) = \langle h_s(a) | s \in A^n \rangle$ for any $a \in \mathfrak{A}_1$. Let us prove that h is an isomorphism.

From the definition one can directly notice that h is injective, i.e., h is a monomorphism in the category $\breve{\mathfrak{M}}$. The isomorphism of the categories $\breve{\mathfrak{M}}$ and $\check{BA}$ implies that $\varphi(h)$ is also a monomorphism in the category $\check{BA}$. Since monomorphisms in $\check{BA}$ are injective homomorphisms of Boolean algebras, $|\varphi(\mathfrak{A}^{A^n})| \geq |\varphi(\mathfrak{A}_1)|$. On the other hand, $\varphi(\mathfrak{A}^{A^n})$ must be a direct product of $|A^n|$ copies of the Boolean algebras $2 = \varphi(\mathfrak{A})$ (as $\mathfrak{A}^{A^n}$ is a direct product of $|A^n|$ copies of the algebra $\mathfrak{A}$ in $\breve{\mathfrak{M}}$). Therefore, $|\varphi(\mathfrak{A}^{A^n})| = 2^{|A^n|}$, and hence $|\varphi(\mathfrak{A}_1)| \leq 2^{|A^n|}$. Moreover, $\varphi_{\mathfrak{A}_1,\mathfrak{A}}$ maps $Hom(\mathfrak{A}_1,\mathfrak{A})$ to $Hom(\varphi(\mathfrak{A}_1),2)$ bijectively. Since all homomorphisms $h_s(s \in A^n)$ from the algebra $\mathfrak{A}_1$ to $\mathfrak{A}$ are different, $|Hom(\varphi(\mathfrak{A}_1),2)| \geq |A^n|$. A finite Boolean algebra of the power 2^m has exactly m of its various homomorphisms on the algebra 2 and, hence, by virtue of the inequalities mentioned above, the Boolean algebras $\varphi(\mathfrak{A}_1)$ and $\varphi(\mathfrak{A}^{A^n})$ are isomorphic. This implies the existence of an isomorphism h of the $\mathfrak{M}$-algebras $\mathfrak{A}_1$ and $\mathfrak{A}^{A^n}$. The mapping $f \in \mathfrak{A}^{A^n}$ and, hence, $h^{-1}(f) \in \mathfrak{A}_1$, i.e., there is a term $t(x_0,\dots,x_{n-1})$ of the signature of the variety $\mathfrak{M}$ such that $h(t(\pi_0,\dots,\pi_{n-1})) = f$, the latter fact implying that the mapping f is defined on A^n with the help of the term t. The algebra $\mathfrak{A}$ is, therefore, primal.

Let us, finally, show that $\mathfrak{M}(\mathfrak{A}) = \mathfrak{M}$. Let us assume the opposite, and set $\mathfrak{A}_1 \in \mathfrak{M} \setminus \mathfrak{M}(\mathfrak{A})$. As $\mathfrak{A}$ is primal, the categories $\breve{\mathfrak{M}}(\mathfrak{A})$ and $\check{B}A$ are isomorphic, as has been demonstrated in the beginning of the proof of the theorem. Let $\varphi' : \mathfrak{M}(\mathfrak{A}) \to BA$ be the mapping of the objects of these categories implementing their isomorphism. Therefore, there is an algebra $\mathfrak{A}_2 \in BA$ such that $\varphi'(\mathfrak{A}_2)$ is isomorphic to the Boolean algebra $\varphi(\mathfrak{A}_1)$. Since $Con\mathfrak{A}_1 \cong Con\varphi(\mathfrak{A}_1)$ and $Con\mathfrak{A}_2 \cong Con\varphi'(\mathfrak{A}_2)$, $Con\mathfrak{A}_1 \cong Con\mathfrak{A}_2$. But in this case we get

$$Con\varphi(\mathfrak{A}_1) \cong Con\mathfrak{A}_1 \cong Con\mathfrak{A}_2 \cong Con\varphi(\mathfrak{A}_2).$$

It is a well-known fact that if congruences of Boolean algebras are isomorphic, the algebras are isomorphic as well. Therefore, there is an isomorphism i of the Boolean algebras $\varphi(\mathfrak{A}_1)$ and $\varphi(\mathfrak{A}_2)$ and, hence, $\varphi^{-1}_{\varphi(\mathfrak{A}_1),\varphi(\mathfrak{A}_2)}(i)$ will be an isomorphism of the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Hence, $\mathfrak{A}_1 \in \mathfrak{M}(\mathfrak{A})$, i.e., $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$. ■

As a corollary to this theorem one should remark that weak equivalence of varieties does not imply their equivalence. Indeed, if $\mathfrak{A}$ is an arbitrary primal algebra of the power more than 2, according to theorem just proved, we have $\check{\mathfrak{M}}(\mathfrak{A}) \simeq \check{B}A$ but, since all $\mathfrak{M}(\mathfrak{A})$-algebras have the form $\mathfrak{A}^{\mathfrak{B}}$, there is no two-element algebra in $\mathfrak{M}(\mathfrak{A})$ and, hence, the varieties $\mathfrak{M}(\mathfrak{A})$ and BA cannot be equivalent.

In relation with theorem **8.2** the problem of describing the categories isomorphic to the categories $\check{B}A$ arises. A number of various descriptions of the kind can be found in a paper by K.Sokolnicki. Here we will dwell on one of them in detail.

Definition 8.3. A category $\mathfrak{K}$ is called algebraic provided that there is a variety of the algebras $\mathfrak{M}$ such that $\mathfrak{K}$ and $\breve{\mathfrak{M}}$ are isomorphic.

All the category notions given below without definitions can, for instance, be found in a monograph by S.McLane [138]. A Lauvere theorem describing algebraic categories is also known.

Theorem 8.3. The category $\mathfrak{K}$ is algebraic iff

(1) $\mathfrak{K}$ contains co-equalizers and finite limits, and there is a separate object G in $\mathfrak{K}$ such that:

(2) for any set Z in $\mathfrak{K}$ there is a co-product $(G \xrightarrow{u_z} \coprod_{z \in Z} z)$ and, in particular, there is an initial object $I = \Pi_{\varnothing}$ in $\mathfrak{K}$.

(3) A morphism f is regular in $\mathfrak{K}$ iff $\mathfrak{K}(G, f)$ is a surjection.

(4) For any parallel pair f, g of morphisms from $\mathfrak{K}$, if $\mathfrak{K}(G, f), \mathfrak{K}(G, g)$ is a kernel pair in Set, f, g is a kernel pair in $\mathfrak{K}$.

(5) For any set $Z \subseteq \mathfrak{K}$ and any morphism $h: G \to \coprod_{z \in Z} z$, there is a finite subset $Y \subseteq Z$ and a morphism $k: G \to \coprod_{y \in Y} y$ such that $h = l \cdot k$, where $l: \coprod_{y \in Y} y \to \coprod_{z \in Z} z$ is a canonical "inclusion" such that $l \cdot u_y = u_y$ for $y \in Y$.

Now, taking into account the statement of theorem **8.3**, let us prove the following theorem.

Theorem 8.4. The category $\mathfrak{K}$ is isomorphic to the category $\overset{\vee}{B}A$ iff $\mathfrak{K}$ is algebraic, (i.e., the conditions (1) - (5) of theorem **8.3** are met) and, in addition, iff the following conditions are valid in terms of the statement of theorem **8.3**:

(6) $|Hom(I^2, I)| = 2$.

(7) For any $\mathfrak{K}$-object P there is a cardinal λ and a monomorphism α from the object P to I^{λ}.

(8) For any $\mathfrak{K}$-object P, if $|Hom(P, I)| = n$, where $n \in \omega$, P and I^n are isomorphic in $\mathfrak{K}$.

(9) For any set S and ultrafilter $\mathfrak{D}$ we have $Co\lim_{d \in \mathfrak{D}} I^d = I$ on S.

Here I^λ for the cardinal λ and I^d for the set d denote the product in $\mathfrak{K}$ of λ and $|d|$, respectively, copies of the object I. Here is a more exact formulation of the condition (9).

Let $\pi_i^d : I^d \to I$ $(i \in d)$ be morphisms corresponding to the fact that I^d is a product of $|d|$ copies of the object I in $\mathfrak{K}$. Let $\breve{\mathfrak{D}}$ be a category corresponding to the ultrafilter $\mathfrak{D}$ viewed as partially ordered in terms of the set inclusion, and let $\breve{\mathfrak{D}}^{op}$ be the dual of category $\breve{\mathfrak{D}}$. Let $J:(\breve{\mathfrak{D}}^{op}) \to \mathfrak{K}$ be a functor defined in the following way: for $d \in Ob(\breve{\mathfrak{D}}^{op}) = \mathfrak{D}$, $J(d) = I^d$, and if $d, d' \in \mathfrak{D}$ are such that $d \subseteq d'$ and $Hom(d,d') = \{\beta\}$, $J(\beta) \in Hom(I^{d'}, I^d)$, and $\pi_i^d \cdot J(\beta) = \pi_i^{d'}$ for any $i \in d$, where $< \pi_i^d | i \in d >, < \pi_j^{d'} | j \in d' >$ are families of the morphisms defining $I^d, I^{d'}$ as products of the object I in $\mathfrak{K}$. In this case the condition (9) of the statement of theorem **8.4** claims that for any ultrafilter $\mathfrak{D}$ we have $I = Co\lim J$.

Let us now turn to the proof of theorem **8.4**. Let us first directly recall that all conditions (1) - (9) are met for the category $\check{BA}$ when choosing I as an initial object of this category, i.e., when I is a two-element Boolean algebra. Let us only notice that the condition (9) is fulfilled on $\check{BA}$ only under the following statement: if $\mathfrak{D}$ is a filter on the set $\mathfrak{S}$, $\mathfrak{A}$ is a universal algebra and $Co\lim_{d \in \mathfrak{D}} \mathfrak{A}^d$ is defined relative to the category $\breve{\mathfrak{M}}(\mathfrak{A})$ in the same way as $Co\lim_{d \in \mathfrak{D}} I^d$ has been defined in the category $\mathfrak{K}$, $Co\lim_{d \in \mathfrak{D}} \mathfrak{A}^d$ is isomorphic to a filtered power $\mathfrak{A}^S/\mathfrak{D}$ of the algebra $\mathfrak{A}$.

Let us now prove the converse statement. Let the conditions (1) - (9) hold for the category $\mathfrak{K}$. By theorem **8.3**, there is a variety of the algebras $\mathfrak{M}$ containing an $\mathfrak{A}$ and an isomorphism of the category $\phi: \mathfrak{K} \to \breve{\mathfrak{M}}$ such that $\phi(I) = \mathfrak{A}$. As the category notions of a monomorphism and a product are preserved under an isomorphism of categories, and as the monomorphisms of categories of the type $\breve{\mathfrak{M}}$ correspond to isomorphic embeddings between $\mathfrak{M}$-algebras, we get $\mathfrak{M} \subseteq ISP(\mathfrak{A})$ in line with the property (7). But $\mathfrak{M}$ is a variety containing the algebra $\mathfrak{A}$ and, hence, $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$.

Let us show that $\mathfrak{A}$ is primal. Indeed, since, by the property (6), $|Hom(\mathfrak{A}^2, \mathfrak{A})| = 2$, $\mathfrak{A}$ is non-singleton. Let S be an infinite set and $\mathfrak{D}$ be a non-principal ultrafilter on S. As is well-known, $\mathfrak{A}^S/\mathfrak{D} \cong \mathfrak{A}$ iff $\mathfrak{A}$ is a finite algebra. But, as we have already noted, $Co\lim_{d \in \mathfrak{D}} \mathfrak{A}^d \cong \mathfrak{A}^S/\mathfrak{D}$ and, by the property (9), $Co\lim_{d \in \mathfrak{D}} \mathfrak{A}^d \cong \mathfrak{A}$. Therefore, $\mathfrak{A}$ is finite.

Let now $m = |\mathfrak{A}^n|$ and $P^{(n)}(\mathfrak{A})$ be a family of all the terms of the variables

$x_1,\ldots,x_n$ of the algebra $\mathfrak{A}$, considered as a subalgebra of the Cartesian power $\mathfrak{A}^{|\mathfrak{A}|^n} = \mathfrak{A}^m$ of the algebra $\mathfrak{A}$. Obviously, $|Hom(P^{(n)}(\mathfrak{A}),\mathfrak{A})| = m$ and, if now $C \in Ob(\mathfrak{K})$ is such that $\phi(C) = P^{(n)}(\mathfrak{A})$, $|Hom(C,I)| = m$. By the property (8), C is isomorphic to an $\mathfrak{K}$-object I^m and, hence, there is an isomorphism from the algebra $P^{(n)}(\mathfrak{A})$ to the algebra $\mathfrak{A}^m$. Since $\mathfrak{A}$ is finite and $P^{(n)}(\mathfrak{A}) \subseteq \mathfrak{A}^m$, $P^{(n)}(\mathfrak{A}) = \mathfrak{A}^m$ and, thus, any function on the basic set of the algebra $\mathfrak{A}$ of n variables coincides with a certain term of this algebra. Therefore, $\mathfrak{A}$ is primal, $\mathfrak{K}$ is isomorphic to $\breve{\mathfrak{M}}(\mathfrak{A})$ and, by theorem **8.2**, $\breve{\mathfrak{M}}(\mathfrak{A}) \cong \check{B}A$. ■

In the introduction to the present chapter, we have mentioned that one of the "external" characteristics of varieties often essentially characterizing its internal structure is the notion of a spectrum and its variations.

Definition 8.4.

(a) A spectrum $Spec\,\mathfrak{M}$ of a variety $\mathfrak{M}$ is called a family of the powers of the algebras of the given variety. A finite spectrum $FSpec\,\mathfrak{M}$ of a variety $\mathfrak{M}$ is a family of the powers of the finite algebras of the given variety.

(b) A fine spectrum of a variety $\mathfrak{M}$ is a function $Spec_{\mathfrak{M}}(\aleph)$ which puts the power of a set of types of the isomorphism of $\mathfrak{M}$-algebras of the power $\aleph$ in correspondence with an arbitrary cardinal $\aleph$, i.e., $Spec_{\mathfrak{M}}(\aleph) = |J\{\mathfrak{A} \in \mathfrak{M} \,||\mathfrak{A}| = \aleph\}|$. A finite fine spectrum $FSpec_{\mathfrak{M}}(n)$ of a variety $\mathfrak{M}$ is a limitation of the function $Spec_{\mathfrak{M}}$ to ω.

Lemma 8.1.

(a) If the varieties $\mathfrak{M}$ and $\mathfrak{M}_1$ are rationally equivalent, $Spec\,\mathfrak{M} = Spec\,\mathfrak{M}_1$, $FSpec\,\mathfrak{M} = FSpec\,\mathfrak{M}_1$, and the functions $Spec_{\mathfrak{M}}, Spec_{\mathfrak{M}_1}$ and $FSpec_{\mathfrak{M}}, FSpec_{\mathfrak{M}_1}$ coincide.

(b) If $\mathfrak{M}_0 = \mathfrak{M}(\{\mathfrak{A} \in \mathfrak{M} \mid \mathfrak{A}$ is finite$\})$, $FSpec\,\mathfrak{M} = FSpec\,\mathfrak{M}_0$, and for any $n \in \omega$, $FSpec_{\mathfrak{M}}(n) = FSpec_{\mathfrak{M}_0}(n)$.

The statements of the lemma are obvious. Therefore, we get, in particular, that spectra and fine spectra can characterize varieties only to the accuracy of rational equivalence. On the other hand, when trying to characterize varieties with a finite or a finite fine spectrum, it would be natural, as is shown by the statement (b), to limit ourselves to the varieties generated by their finite algebras. Indeed, for any variety $\mathfrak{M}$, if $\mathfrak{M}_1$ is an arbitrary variety having no non-singleton finite algebras, $FSpec\,\mathfrak{M} = FSpec\,\mathfrak{M} \otimes \mathfrak{M}_1$, and for $n \in \omega$ we have $FSpec_{\mathfrak{M}}(n) = FSpec_{\mathfrak{M} \otimes \mathfrak{M}_1}(n)$. Choosing here different not rationally equivalent varieties as $\mathfrak{M}_1$, we can get families of not rationally equivalent varieties with the same fine finite spectrum.

It should be noticed that, since the varieties are closed with respect to direct products, and by virtue of the Löwenheim-Skolem theorem for elementary classes of algebras, if $Spec\,\mathfrak{M} \neq \{1\}$, for any infinite cardinal $\aleph$ we have $Spec\,\mathfrak{M} \ni \aleph$. Therefore, only the finite spectrum of a variety can impose essential limitations on $\mathfrak{M}$.

The description of the fine and, in particular, finite fine spectrum of a variety, which is a trivial problem in a number of cases (varieties of Boolean algebras, of vector spaces), can, on the other hand, be a problem of greater complexity (varieties of groups, lattices, etc.).

In a work by W.Taylor [227] one can find a number of interesting digital functions which are finite fine spectra of some varieties of algebras.

The finite spectrum of a variety is, evidently, a multiplicatively closed subset of ω, the inverse statement being also valid.

Theorem 8.5. Let f be a mapping from ω to ω such that $f(0) = 0, f(1) = 1$, and if $f(m), f(n) > 0$, $f(m \cdot n) > 0$. In this case there is a variety $\mathfrak{M}$ such that:

(a) for any $n \in \omega$ we have $f(n) \leq FSpec_{\mathfrak{M}}(n) < \aleph_0$;

(b) $f(n) = 0 \Rightarrow FSpec_{\mathfrak{M}}(n) = 0$.

Proof. For $n \geq 2, 0 \leq m < f(n)$ let us define algebras $\mathfrak{A}_{n,m} = <\{1,...,n\}; D, g, a_i >_{i \in \omega}$, where D is a discriminator on $\{1,...,n\}$, g is a certain cyclic permutation on $\{1,...,n\}$ fixed for a given n, and $a_i = \min(n, \max(1, i - m))$. Let $\mathfrak{M} = \mathfrak{M}(\mathfrak{A}_{n,m} | n \geq 2, 0 \leq m < f(n)\}$. By theorem **2.10**, $\mathfrak{M}$ is both congruence-distributive and congruence-permutable. By theorem **2.16**, any subdirectly non-decomposable $\mathfrak{M}$-algebra belongs to the class $HS(\mathfrak{A})$, where $\mathfrak{A}$ is a certain ultraproduct of algebras of the type $\mathfrak{A}_{n,m}$. But any subalgebra of a similar ultraproduct is simple (the discriminator D belongs to the signature of the algebras $\mathfrak{A}_{n,m}$ and, hence, it is also a discriminator on the ultraproduct). Therefore, all subdirectly non-decomposable $\mathfrak{M}$-algebras belong to $S(\mathfrak{A})$, where $\mathfrak{A}$ is a certain ultraproduct of algebras of the type $\mathfrak{A}_{n,m}$. But since it belongs to the signature of the functions g, any ultraproduct of algebras of the type $\mathfrak{A}_{n,m}$ contains no finite subalgebras, except for the cases when either the ultrafilter is principal or all cofactors, except a finite number of them, coincide with the same algebra, $\mathfrak{A}_{n,m}$. Therefore, all finite subdirectly non-decomposable $\mathfrak{M}$-algebras are subalgebras of some of the algebras $\mathfrak{A}_{n,m}$, and hence, a family of finite subdirectly non-decomposable $\mathfrak{M}$-algebras coincides with the set $\{\mathfrak{A}_{n,m} | n \geq 2, 0 \leq m < f(n)\}$. Hence, any finite $\mathfrak{M}$-algebra is a subdirect product of a finite number of algebras from $\{\mathfrak{A}_{n,m} | n \geq 2, 0 \leq m < f(n)\}$. In particular, any finite $\mathfrak{M}$-algebra belongs to a certain variety $\mathfrak{M}_s = \mathfrak{M}(\{\mathfrak{A}_{n,m} | 2 \leq n \leq s, 0 \leq m < f(n)\})$, where $s \in \omega$. But $\mathfrak{M}_s$ is a finitely generated, semi-simple, congruence-permutable variety and, hence, by theorem **6.3**, any finite $\mathfrak{M}_s$ and, therefore, any finite $\mathfrak{M}$-algebra is a direct product of algebras of the type $\mathfrak{A}_{n,m}$. But this fact, since the set $\{n | f(n) > 0\}$ is multiplicatively closed, implies the statement (b) of the theorem. The statement (a) follows from the definition of constants a_i on algebras of the type $\mathfrak{A}_{n,m}$. ■

Bearing in mind the statement of the theorem and the remark made before its formulation, we get the following corollary.

Corollary 8.1. A subset K of the set ω is a finite spectrum of a certain variety (which can be chosen to be adiscriminator variety) iff $K \ni 1$ and K is multiplicatively closed.

We can prove the following theorem in an analogous way.

Theorem 8.6. If P is a certain set of simple numbers, there is a $2^{|P|}$ of pairwise rationally non-equivalent varieties $\mathfrak{M}$ generated by a family of their finite algebras and such that for all $n \in \omega$ we get $FSpec_{\mathfrak{M}}(n) = 1$ if all simple divisors of the number n lie in P, and $FSpec_{\mathfrak{M}}(n) = 0$ in the opposite case.

Proof. For any $n \in \omega$ let us define algebras $\mathfrak{A}_n = <\{1,\ldots,n\}; D, g, 1>$ and $\mathfrak{B}_n = <\{1,\ldots,n\}; D, g, 1>$, where D is a discriminator on $\{1,\ldots,n\}$, g in $\mathfrak{B}_n$ is a cyclic permutation on $\{2,\ldots,n\}$ and g in $\mathfrak{A}_n$ is a cyclic permutation on $\{1,\ldots,n\}$. Let Q_1, Q_2 be a decomposition of the set P, and $\mathfrak{M}_{Q_1Q_2} = \mathfrak{M}(\mathfrak{A}_n, \mathfrak{B}_m | n \in Q_1, m \in Q_2)$.

As was the case in the proof of the previous theorem, first we show that only direct products of the algebras $\mathfrak{A}_n, \mathfrak{B}_m (n \in Q_1, m \in Q_2)$ can be finite $\mathfrak{M}_{Q_1Q_2}$-algebras. Now we have to remark that for different decompositions Q_1, Q_2 and Q_1', Q_2' of the set P of the variety, $\mathfrak{M}_{Q_1Q_2}$ and $\mathfrak{M}_{Q_1',Q_2'}$ are not rationally equivalent. Indeed, for $n \in Q_2 \setminus Q_2'$, the algebras $\mathfrak{B}_n$ and $\mathfrak{A}_n$ are the only algebras of the power n belonging to the varieties $\mathfrak{M}_{Q_1Q_2}$ and $\mathfrak{M}_{Q_1',Q_2'}$, respectively. The algebra $\mathfrak{B}_n$, however, contains the one-element subalgebra {1}, while the algebra $\mathfrak{A}_n$ has no one-element subalgebras. ■

Using direct products and families of varieties one can easily prove the following theorem.

Theorem 8.7. A class $\mathfrak{R}$ of functions which are finite fine spectra of varieties is closed under the following operations:

(a) if $f, g \in \mathfrak{R}$, also $f*g \in \mathfrak{R}$, where $f*g(n) = \sum_{d|n} f(d)\cdot g(n/d)$;

(b) if $f \in \mathfrak{R}, k \in \omega$, also $f^{[k]} \in \mathfrak{R}$, where $f^{[k]}(n) = f(m)$ if $n = m^k$ for some m, and $f^{[k]}(n) = 0$ in the opposite case;

(c) if $f, g \in \mathfrak{R}$, there is an $h \in \mathfrak{R}$ such that for any $n \in \omega$ we get

$$f(n)g(n) \le h(n) \le n!f(n)g(n),$$

(assuming here $0 \cdot \aleph_0 = 0$). In particular, $h(n) = 0$ iff either $f(n) = 0$ or $g(n) = 0$, and $h(n)$ is finite iff both $f(n)$ and $g(n)$ are finite.

It should be remarked that one cannot claim that $\mathfrak{R}$ is closed under the product of functions. Indeed, let $\mathfrak{M} = \mathfrak{M}(\mathfrak{A}_0, \mathfrak{A}_1)$, where $\mathfrak{A}_0 = <\{0,1\}; +, 0>, \mathfrak{A}_1 = <\{0,1\}; +, 1>$ and + is a binary addition on {0,1}. Then we get

$$FSpec_{\mathfrak{M}}(2) = FSpec_{\mathfrak{M}}(16) = 2, FSpec_{\mathfrak{M}}(6) = 0.$$

However, as will be shown in the theorem to follow, for any variety $\mathfrak{M}_1$ if $FSpec_{\mathfrak{M}_1}(2) = 3 + m$ and $FSpec_{\mathfrak{M}_1}(6) = 0$, $FSpec_{\mathfrak{M}_1}(2^{3+m+s}) > 3 + m$. Therefore, the function $f = FSpec^2_{\mathfrak{M}}$ cannot be a function of the finite fine spectrum for any variety.

The following properties of the functions from $\mathfrak{R}$ can be easily deduced:

(1) $f(m^k) \geq f(m)$ for any $m,k \in \omega$;

(2) for any $n_1,\ldots,n_k \in \omega$ we have

$$f(\prod_{i=1}^{k} n_i) \geq \frac{\prod_{i=1}^{k}(n_i!)}{(\prod_{i=1}^{k} n_i)!} \prod_{i=1}^{k} f(n_i).$$

More subtle properties of functions from $\mathfrak{R}$ will be discussed in the theorem to follow. Since its proof is based on the lemmas from the proof of theorem **8.9**, it will be given after them.

Theorem 8.8. If $f \in \mathfrak{R}$,

(a) if $f(2)=1$, $f(6)=0$ and $f(2^k)>1$, we get $f(2^{k+1})>1$;

(b) if $f(2)=2$, $f(6)=0$ and $f(2^k)>2$, we get $f(2^{k+1})>2$;

(c) if $f(2)=3+m$, $f(6)=0$, $f(2^{3+m+s})>3+m$

(here k,m,s are arbitrary elements from ω).

The properties of the functions of finite fine spectra of varieties considered above leave, nonetheless, a whole number of problems open for discussion.

Problem 8.1. To find any description of the class of functions $\mathfrak{R}$.

Problem 8.2. Is the set $\mathfrak{R} \cap \omega^\omega$ closed in the space ω^ω? In other words, if f is a function from ω to ω that is finitely approximated by finite fine spectra of varieties (for any $N \in \omega$ there is a variety $\mathfrak{M}_N$ such that for $n \leq N$ $f(n) = FSpec_{\mathfrak{M}_N}(n)$), will f obligatory belong to $\mathfrak{R}$ (i.e., is there a variety $\mathfrak{M}$ such that $f(n) = FSpec_{\mathfrak{M}}(n)$ for all $n \in \omega$?) ?

The following problem is of interest in connection with theorem **8.6**.

Problem 8.3. Let $S = \{< m_1,n_1,\ldots,m_k,n_k > \mid k,m_i,n_i \in \omega$, and for no variety $\mathfrak{M}$, the equalities $FSpec_{\mathfrak{M}}(m_i) = n_i$ are simultaneously valid for all $1 \leq i \leq k\}$. Is S a recursive, or a recursively enumerable set?

In section **6**, when discussing algebras with a minimal spectrum, we have shown, in particular, that the conditions imposed on the spectrum, the finite or the fine spectrum of a variety impose quite strict limitations on the variety itself. In some cases, varieties with one or another limitation on their fine spectrum can be described even to the accuracy of rational equivalence, which will be demonstrated in the theorems to follow.

From now on, in theorem **8.9** and the related lemmas, + and $\cdot$ will denote the operations of addition and multiplication by the module 2 defined on the set {0,1}. Obviously, in this case the operations $x \wedge y$ and $x \vee y$ are defined on {0,1} by the terms $x \cdot y$ and $x + y + xy$.

Let us introduce the following notations for certain two-element algebras:

$\mathfrak{A}_1 = <\{0,1\}; x + y + z>$,

$\mathfrak{A}_2 = <\{0,1\}; x + y>$,

$\mathfrak{A}_3 = <\{0,1\}; x + y + z, xy + yz + zx>$,

$\mathfrak{A}_4 = <\{0,1\}; x + 1, x + y + z, xy + yz + zx>$,

$\mathfrak{A}_5 = <\{0,1\}; x + y + z, xy>$,

$\mathfrak{A}_6 = <\{0,1\}; x + y, xy>$,

$\mathfrak{A}_7 = <\{0,1\}; 1, x + y, xy>$,

$\mathfrak{A}_8 = <\{0,1\}; x + 1, x + y + z>$,

$\mathfrak{A}_9 = <\{0,1\}; 1, x + y>$.

In the Post classification, the varieties generated by these algebras are referred to as follows:

$\mathfrak{M}(\mathfrak{A}_1) = L_4$ is a variety of Boolean 3-groups,

$\mathfrak{M}(\mathfrak{A}_2) = L_3$ is a variety of Boolean groups,

$\mathfrak{M}(\mathfrak{A}_3) = D_1$ is a variety of Boolean 3-rings,

$\mathfrak{M}(\mathfrak{A}_4) = D_3$ is a variety of Boolean 3-algebras,

$\mathfrak{M}(\mathfrak{A}_5) = C_4$ is a variety of Boolean lattices,

$\mathfrak{M}(\mathfrak{A}_6) = C_2$ is a variety of Boolean rings,

$\mathfrak{M}(\mathfrak{A}_7) = C_1$ is a variety of Boolean algebras,

$\mathfrak{M}(\mathfrak{A}_8) = L_5$ is a variety of Boolean 3-groups with supplements,

$\mathfrak{M}(\mathfrak{A}_9) = L_1$ is a variety of Boolean groups with supplements.

Theorem 8.9. Let a variety $\mathfrak{M}$ be generated by its finite algebras. Then the following statements are valid.

(a) The fine spectrum of $\mathfrak{M}$ obeys the condition $FSpec_{\mathfrak{M}}(n) = 1$ if $n = 2^k$ for some $k \in \omega$, and $FSpec_{\mathfrak{M}}(n) = 0$ for all other $n \in \omega$ iff $\mathfrak{M}$ is rationally equivalent to one of the following varieties: $L_4, L_3, D_1, D_3, C_4, C_2, C_1$.

(b) The fine spectrum of $\mathfrak{M}$ obeys the condition $FSpec_{\mathfrak{M}}(n) = 1$ if $n = 1$, $FSpec_{\mathfrak{M}}(n) = 2$ if $n = 2^k$ for some $1 \le k \le \omega$, and $FSpec_{\mathfrak{M}}(n) = 0$ for all other $n \in \omega$ iff $\mathfrak{M}$ is rationally equivalent to one of the varieties, L_5 or L_1.

Proof. Let us first remark that the algebras $\mathfrak{A}_3, \mathfrak{A}_4, \ldots, \mathfrak{A}_7$ are quasi-primal, i.e., the term $x + z + xy + yz + zx$ definable in them is a discriminator on $\{0,1\}$. It is also obvious that the algebras $\mathfrak{A}_1, \mathfrak{A}_2$ are simple Abelian algebras with one-element subalgebras, while $\mathfrak{A}_8, \mathfrak{A}_9$ are simple Abelian algebras without one-element subalgebras. Now the statements of the present theorem on fine spectra of the varieties $\mathfrak{M}(\mathfrak{A}_i)(i = 1,\ldots,9)$ result directly from the statements of theorems **6.4** and **6.5**. Therefore, it remains to be shown that the converse is valid, i.e., that any variety generated by its finite algebras and having a fine spectrum of one of the types listed in the theorem is rationally equivalent to one of the varieties $\mathfrak{M}(\mathfrak{A}_i)$, where $i = 1,\ldots,7$ or $i = 8,9$, respectively.

Let us first analyse the terms definable on the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_9$.

Lemma 8.2.

(a) Every term of the algebra $\mathfrak{A}_1$ has the form $\sum_{i\in K} x_i$, where $|K|$ is odd.

(b) Every term of the algebra $\mathfrak{A}_2$ has the form $\sum_{i\in K} x_i$, where $|K|$ is an arbitrary natural number.

(c) The functions defined by the terms of the algebra $\mathfrak{A}_3$ are exactly those among the functions $F:\{0,1\}^k \to \{0,1\}$ which obey the conditions $F(0,\ldots,0)=0$ and $F(x_1+1,\ldots,x_k+1)=F(x_1,\ldots,x_k)+1$. The equivalent condition is as follows: it must be the set of functions $F:\{0,1\}^k \to \{0,1\}$ for which $F(\alpha_1,\ldots,\alpha_k)$ is either x or y in the case when $\alpha_1,\ldots,\alpha_k$ are contained in the set $\{x,y\}$.

(d) The functions defined by the terms of the algebra $\mathfrak{A}_4$ are exactly those among the functions $F:\{0,1\}^k \to \{0,1\}$ which obey the condition $F(x_1+1,\ldots,x_k+1)=F(x_1,\ldots x_k)+1$. And this is the equivalent condition: it is the set of functions $F:\{0,1\}^k \to \{0,1\}$ for which $F(\alpha_1,\ldots,\alpha_k)$ is either x or y, or $1+x$ or $1+y$ in the case when $\alpha_1,\ldots,\alpha_k$ are contained in the set $\{x,y\}$.

(e) The functions defined by the terms of the algebra $\mathfrak{A}_5$ are exactly those among the functions $F:\{0,1\}^k \to \{0,1\}$ which obey the conditions: $F(0,\ldots,0)=0$ and $F(1,\ldots,1)=1$.

(f) The functions defined by the terms of the algebra $\mathfrak{A}_6$ are exactly those among the functions $F:\{0,1\}^k \to \{0,1\}$ which obey the condition $F(0,\ldots,0)=0$.

(g) Any function $F:\{0,1\}^k \to \{0,1\}$ defined by the terms of the algebra $\mathfrak{A}_7$.

(h) The functions defined by the terms of the algebra $\mathfrak{A}_8$ have either the form α or $\alpha+1$, where α is one of the functions defined by the terms of the algebra $\mathfrak{A}_1$.

(i) The functions defined by the terms of the algebra $\mathfrak{A}_9$ have either the form α or $\alpha+1$, where α is one of the functions defined by the terms of the algebra $\mathfrak{A}_2$

Proof. The statements (a), (b), (h) and (i) are obvious. The remaining statements result from the fact mentioned earlier that $\mathfrak{A}_3,\ldots\mathfrak{A}_7$ are quasi-primal algebras and can be directly deduced from theorem **2.14**. ■

Lemma 8.3. For any algebra $\mathfrak{A} = <\{0,1\}; f_1, f_2, \ldots>$, the coincidence of a family of ternary functions definable by the terms of this algebra with that of ternary functions definable by the terms of one of the algebras $\mathfrak{A}_1,\ldots\mathfrak{A}_9$ implies a similar coincidence for families of all functions definable by their terms.

Proof. This statement readily follows from that of lemma **8.2**. For instance, if a family of ternary functions defined by the terms of the algebra $\mathfrak{A}$ coincides with that of ternary functions defined by the terms of the algebra $\mathfrak{A}_3$, obviously, any function defined by the terms of the algebra $\mathfrak{A}_3$ is also definable by the terms of the algebra $\mathfrak{A}$. If the converse statement was invalid,, by lemma **8.2**(c), for a certain term $t(x_1,\ldots,x_k)$ of the algebra $\mathfrak{A}$ there could be found $\alpha_1,\ldots,\alpha_k \in \{x,y\}$ such that $t(\alpha_1,\ldots,\alpha_k)$ would be equal to neither x nor y on $\mathfrak{A}$. But $t(\alpha_1,\ldots,\alpha_k)$ is a binary term and, hence, by the supposition of the lemma, the function defined by them does not coincide with a single function defined by the terms of the algebra $\mathfrak{A}_3$. The latter statement contradicts that of lemma **8.2**(c). Therefore, indeed, the families of functions definable by the terms of $\mathfrak{A}$ and $\mathfrak{A}_3$ must coincide. ■

A function $f^*(x_1,\ldots,x_k)$ defined by the equality

$$f^*(x_1,\ldots,x_k) = 1 + f(x_1+1,\ldots,x_k+1)$$

will be termed dual to the function $f(x_1,\ldots,x_k)$ defined on the set $\{0,1\}$.

For any set F of functions defined on $\{0,1\}$, F^* will denote the set $\{f^* \mid f \in F\}$, where F is self-dual provided that $F^* = F$. Obviously, the mapping $\varphi(x) = x+1$ is a homomorphism of the algebras $<\{0,1\}; f_1, f_2, \ldots>$ and $<\{0,1\}; f_1^*, f_2^*, \ldots>$.

Lemma 8.4. For any algebra $\mathfrak{A} = <\{0,1\}; f_1, f_2,,,>$, a set of binary functions on $\{0,1\}$ definable by the terms of this algebra coincides with one of the following sets or their dual sets (the sign * denotes the cases of self-dual sets):

* (a) x, y;

(b) $x, y, 0, x+y$;

* (c) $x, y, x+1, y+1$;

* (d) $x, y, x \wedge y, x \vee y$;

(e) $x, y, 0, x+y, xy, x+xy, y+xy, x+y+xy$;

* (f) the family of all binary functions on $\{0,1\}$;

* (g) $x, y, 0, 1, x+y, x+1, y+1, x+y+1$;

(h) $x, y, 0$;

(i) $x, y, 1$;

(j) x, y, xy;

(k) $x, y, xy, 0$;

(l) $x, y, xy, 1$;

(m) $x, y, xy, 0, 1$;

(n) $x, y, x \wedge y, x \vee y, 0$;

* (o) $x, y, x \wedge y, x \vee y, 0, 1$;

* (p) $x, y, x+1, y+1, 0, 1$;

(q) $x, y, 0, xy, xy+x, xy+y$.

The proof of this lemma is carried out by directly checking the following statements:

(1) each of the above mentioned sets of functions is closed under superpositions;

(2) any binary function on $\{0,1\}$ generates one of the given sets;

(3) a set of any two of the sets of functions given in the formulation of the lemma is also contained in this list. ■

Lemma 8.5. For any algebra $\mathfrak{A} = <\{0,1\}; f_1, f_2, ...>$, if $SP(\mathfrak{A})$ contains no six-element algebra, the set of binary functions on {0,1} definable by the terms of this algebra coincides with either of the following sets or their dual sets (the sign *, as above, denotes the cases of self-dual sets):

* (a) x, y;

(b) $x, y, 0, x+y$;

* (c) $x, y, x+1, y+1$;

* (d) $x, y, x \wedge y, x \vee y$;

(e) $x, y, 0, x+y, xy, x+xy, y+xy, x+y+xy$;

* (f) the family of all binary functions on {0,1};

* (g) $x, y, 0, 1, x+y, x+1, y+1, x+y+1$.

The proof is reduced to the fact that $SP(\mathfrak{A})$ contains a six-element (or even a three-element) algebra in the cases (h) - (q) in lemma **8.4**. One can also notice that, obviously, free two-generated algebras of the variety $\mathfrak{M}(\mathfrak{A})$ will also be of the same kind in the cases (h), (j), (o), (p) and (c). Therefore, only the cases (i), (k), (l), (m) and (n) are left for consideration. Let us set $X \subseteq B \subseteq \{0,1\}^2$ such that $|X| \leq 2, |B| = 3$, and let B be generated by the set X with the help of the binary operations of the algebra $\mathfrak{A}$ in each of the cases (i), (k), (l), (m) and (n). Therefore, B will be a basic set of a three-element algebra in $SP(\mathfrak{A})$.

(i) $X = \{<0,1>\}, B = \{<0,1>, <0,0>, <1,1>\}$;

(k) $X = \{<0,1>, <1,0>\}, B = \{<0,1>, <1,0>, <0,0>\}$;

(l) $X = \{<0,0>, <0,1>\}, B = \{<0,0>, <0,1>, <1,1>\}$;

(m) $X = \{<0,1>\}, B = \{<0,1>, <0,0>, <1,1>\}$;

(n) $X = \{<1,1>, <0,1>\}, B = \{<1,1>, <0,1>, <0,0>\}$. ■

Lemma 8.6. For any algebra $\mathfrak{A} = <\{0,1\}; f_1, f_2, ...>$, if $SP(\mathfrak{A})$ contains no six-element algebra, either $\mathfrak{A}$ is quasi-primal, or a set of ternary functions on {0,1} definable by the terms of this algebra coincides with either one of the following sets or their dual sets (the sign * denotes the cases of self-dual sets):

* (a) $x, y, z, x+y+z$;

(b) $0, x, y, z, x+y, x+z, y+z, x+y+z$;

* (c) $x, y, z, x+1, y+1, z+1, x+y+z, x+y+z+1$;

* (d) functions of the type α and $\alpha+1$, where α are functions from (b).

Proof. By lemma **8.5**, for binary functions on {0,1} definable by the terms of the algebra $\mathfrak{A}$ one of the cases (a) - (g) considered in lemma **8.5** is valid. The sets of binary functions in the cases (b) and (g) of this lemma obviously uniquely correspond to the sets of ternary functions described in the cases (b) and (d), respectively, of the present lemma. From the set of binary functions presented in the cases (e) and (f) of lemma **8.5** one can, obviously, deduce that the discriminator is definable on {0,1} by the terms of the algebra $\mathfrak{A}$ and, hence, that $\mathfrak{A}$ is quasi-primal. Therefore, we have to consider only the situation when a family of binary functions defined on {0,1} by the terms of $\mathfrak{A}$ coincides with one of sets given in the cases (a), (c) and (d) of lemma **8.5**.

(a) The only binary functions definable on {0,1} by the terms of $\mathfrak{A}$ are x and y. Since, by the condition of the lemma, $\mathfrak{A}^2$ has no three-element subalgebras, there is a term $F(x,y,z)$ such that we get $F(<0,1>,<0,0>,<1,0>)=<1,1>$, i.e., $F(0,0,1)=F(1,0,0)=1$ in $\mathfrak{A}^2$. But $F(x,x,y)$, as well as $F(y,x,x)$ must coincide with either x or y, thus we see that the following identity is valid on $\mathfrak{A}$: $\mathfrak{A} \models F(x,x,y)=y=F(y,y,x)$. However, since one of the identities $F(x,y,x)=x$ or $F(x,y,x)=y$ is also valid on $\mathfrak{A}$, we will consider the following two subcases.

(a_1): $\mathfrak{A} \models F(x,y,x)=y$ and the identities discussed above obviously yield $F(x,y,z)=x+y+z$. Therefore, we see that, in the case under consideration, the functions $x, y, z, x+y+z$ are defined on {0,1} by the ternary terms of the algebra $\mathfrak{A}$. The alternative now is as follows: either this set exhausts all ternary functions definable on {0,1} by the terms of the algebra $\mathfrak{A}$, and then we come to the case (a) of the present lemma, or there is one more function $G(x,y,z)$ definable on {0,1} by the ternary terms of the algebra $\mathfrak{A}$ which induces only x and y as binary functions. By lemma **8.2**, since only the case (c) of the present lemma is plausible due to the latter condition, $G(x,y,z)$ must coincide with one of the functions of the type $xy+yz+zx$, $x+y+xy+yz+zx$, $x+z+xy+yz+zx$ and $y+z+xy+yz+zx$. Superposition of these functions to the function $x+y+z$ implies that the discriminator set by the function $x+z+xy+yz+zx$ on $\{0,1\}$ is definable by the terms in $\mathfrak{A}$.

(a_2) $\mathfrak{A} \models F(x,y,x) = x$. This equality, however, combined with the above-mentioned identities $F(x,x,y) = F(y,x,x) = y$, means that F is a discriminator on $\{0,1\}$, i.e., $\mathfrak{A}$ is also quasi-primal in this case.

(c) The only binary functions definable on $\{0,1\}$ by the terms of the algebra $\mathfrak{A}$ are $x, y, 1+x, 1+y$. Let $\mathfrak{B}$ be a subalgebra of the algebra $\mathfrak{A}^3$ generated by the elements $<1,0,0>$, $<0,1,0>$ and $<0,0,1>$. Then, obviously, $\mathfrak{B}$ contains the elements $<0,1,1>$, $<1,0,1>$ and $<1,1,0>$. Since $\mathfrak{B}$ cannot be six-element, $\mathfrak{B}$ contains also one of the elements $<0,0,0>$ or $<1,1,1>$ and, in addition, as $\mathfrak{B}$ is closed under the operation $x+1$, it must contain both of these elements. Therefore, there exists a ternary term $F(x,y,z)$ such that in $\mathfrak{A}^3$ we get

$$F(<1,0,0>,<0,1,0>,<0,0,1>) = <1,1,1>, \text{ i.e.,}$$

$$F(1,0,0) = F(0,1,0) = F(0,0,1) = 1.$$

Therefore, each of the functions $F(x,y,y)$, $F(y,x,y)$ and $F(y,y,x)$ must coincide with one of the functions x, $1+y$. At the same time, all three functions under discussion must simultaneously coincide with one of the functions x, $1+y$, since in the opposite case we would get the equalities $x = F(x,x,x) = 1+x$. Thus, the alternative now is as follows: either

$$F(x,y,y) = F(y,x,y) = F(y,y,x) = x,$$

or

$$F(x,y,y) = F(y,x,y) = F(y,y,x) = 1+y.$$

Let us consider the following two cases.

(c_1) $\mathfrak{A} \models F(x,y,y) = F(y,x,y) = F(y,y,x) = 1+y$. One can easily check that this is possible only when we have

$$F(x,y,z) = 1 + xy + yz + zx,$$

but in this case we get

$$1 + F(x,y+1,z) = x + z + xy + yz + zx$$

which is a discriminator on $\{0,1\}$ and, hence $\mathfrak{A}$ is quasi-primal.

(c_2) $\mathfrak{A} \models F(x,y,y) = F(y,x,y) = F(y,y,x) = x$. We can also directly notice that in this case $F(x,y,z) = x + y + z$ and, hence, the ternary functions of the algebra $\mathfrak{A}$ define all the operations definable on $\{0,1\}$ by ternary terms of the algebra $<\{0,1\}; x+1, x+y+z>$. In this case, the alternative is as follows: either there are no other ternary functions definable on $\{0,1\}$ by the terms of the algebra $\mathfrak{A}$ and, hence, the ternary functions definable on $\{0,1\}$ by the terms of the algebra $\mathfrak{A}$ coincide with those discussed in the case (d) of the present lemma, or alongside with the ternary functions definable on $\{0,1\}$ by ternary terms of the algebra $<\{0,1\}; x+1, x+y+z>$, there are ternary functions definable by the terms of the algebra $\mathfrak{A}$.

By virtue of the statement (d) of lemma **8.2**, all ternary functions definable on $\{0,1\}$ by the terms of the algebra $\mathfrak{A}$ are also definable by the terms of the algebra $\mathfrak{A}_4$. Owing to the fact that in this case there are ternary terms of the algebra $\mathfrak{A}$ not definable by the terms of the operations $x+1, x+y+z$, we can remark that the discriminator on $\mathfrak{A}$ is definable by the terms, i.e., $\mathfrak{A}$ is quasi-primal in this case as well.

Let us now consider the remaining case, (d), when the binary functions definable on $\{0,1\}$ by the terms of $\mathfrak{A}$ are $\mathfrak{A}$ $x, y, x \vee y, x \wedge y$. Since $\mathfrak{A}^2$ must not have a three-element subalgebra, there is a ternary term $F(x,y,z)$ on $\mathfrak{A}$ such that

$$F(<0,0>, <1,0>, <1,1>) = <0,1>$$

in $\mathfrak{A}^2$, i.e.,

$$F(0,0,1) = 1, F(0,1,1) = 0.$$

Let $G(x,y,z) = F(x \wedge y \wedge z, x \wedge y, x) \vee F(x \wedge y \wedge z, y \wedge z, z)$. One can directly check that $G(x,y,z) = x + z + xy + yz + zx$, i.e., it is a discriminator on $\{0,1\}$ and, hence, in this case $\mathfrak{A}$ is also quasi-primal. ■

Lemma 8.7. Any two-element quasi-primal algebra is rationally equivalent to one of the algebras $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$, $\mathfrak{A}_6$, $\mathfrak{A}_7$.

Proof. By theorem **2.14**, for any quasi-primal algebra $\mathfrak{A}$ defined on {0,1} any function preserving the subalgebras and partial isomorphisms of this algebra must be defined by the terms of the algebra $\mathfrak{A}$. Enumerating all possible candidates to be a family of basic sets of subalgebras of the algebra $\mathfrak{A}$, i.e.,

(1) {0,1},
(2) {0,1},{0},
(3) {0,1},{1},
(4) {0,1},{0},{1},

and all possible candidates to be partial isomorphisms between the subalgebras of these families, we come to the conclusion that the statement of the lemma is true. ■

Let us now return to the proof of the theorem. If a variety $\mathfrak{M}$ has the spectrum given in the statement (a) of the theorem, there is a two-element $\mathfrak{M}$-algebra $\mathfrak{A}$ and, since $\mathfrak{M}$ is categoric at any power of the type $2^n (n \in \omega)$, all finite $\mathfrak{M}$-algebras are exhausted by algebras $\mathfrak{A}^n$. As $\mathfrak{M}$ is generated by its finite algebras, the equality $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$ is valid. By lemma **8.6**, either one of (a), (b), (c) or (d) considered in its formulation must be the place, or $\mathfrak{A}$ must be quasi-primal. Since in the cases (c) and (d) $\mathfrak{A}$ is a simple Abelian algebra without one-element subalgebras,, according to theorem **6.5**, $\mathfrak{M}$ has algebras of any power 2^n which are not isomorphic to $\mathfrak{A}^n$, where $1 \le n < \omega$. For the cases (a) and (b), or when $\mathfrak{A}$ is quasi-primal, statement (a) of the theorem results directly from lemmas **8.7** and **8.3**.

To prove statement (b) of the theorem, let us remark that, by virtue of lemma **8.6** and theorem **6.5**, the fact that a finite $\mathfrak{M}$ spectrum has the form $\{2^n | n \in \omega\}$ and $\mathfrak{M}$ is not categoric in finite powers implies that only the cases (c) and (d) out of those enumerated in the conclusion of lemma **8.6** are possible, but then, by virtue of lemmas **8.3** and **8.2**, the two-element algebra $\mathfrak{A}$ generating $\mathfrak{M}$ is rationally equivalent to either algebra $\mathfrak{A}_8$ or algebra $\mathfrak{A}_9$. ■

It should be remarked that all the varieties given in the formulation of theorem **8.9** are pairwise non-equivalent, which fact can be deduced from the equalities following directly from lemma **8.2** for $1 \le n < \omega$:

$$|\mathfrak{F}_{L_4}(n)| = 2^{n-1}, \quad |\mathfrak{F}_{L_3}(n)| = 2^n, \quad |\mathfrak{F}_{D_1}(n)| = 2^{2^{n-1}-1},$$

$$|\mathfrak{F}_{D_3}(n)| = 2^{2^{n-1}},\ |\mathfrak{F}_{C_4}(n)| = 2^{2^n-2},\ |\mathfrak{F}_{C_2}(n)| = 2^{2^n-1},$$

$$|\mathfrak{F}_{C_1}(n)| = 2^{2^n},\ |\mathfrak{F}_{L_5}(n)| = 2^n,\ |\mathfrak{F}_{L_1}(n)| = 2^{n+1}.$$

Now, let us return to the proof of theorem **8.8** employing the lemmas just proved.

Proof of theorem **8.8.**

(a) Let $f \in \mathfrak{R}$, i.e., $f = FSpec_{\mathfrak{M}}$ for a certain variety $\mathfrak{M}$, and in this case we have $f(2) = 1$, $f(6) = 0$. It suffices to show that the equality $f(2^{k+1}) = 1$ yields the equality $f(2^k) = 1$. Let $\mathfrak{A}$ be a two-element $\mathfrak{M}$-algebra, and let $\mathfrak{A}' \in \mathfrak{M}, |\mathfrak{A}'| = 2^k$. Then, in accordance with the equality $f(2^{k+1}) = 1$, the algebra $\mathfrak{A} \times \mathfrak{A}'$ is isomorphic to the algebra $\mathfrak{A}^{k+1}$ and, hence, $\mathfrak{A}' \in HP(\mathfrak{A})$. By lemma **8.6**, $\mathfrak{A}$ is either quasi-primal or one of the algebras presented in the cases (a) - (d) of the lemma under consideration. On the other hand, the condition $f(2) = 1$ excludes the cases (c) and (d). Therefore, $\mathfrak{A}$ is either quasi-primal or has the form described in the cases (a) and (b) of the present lemma. But then, by lemmas **8.7** and **8.3**, $\mathfrak{A}$ is rationally equivalent to one of the algebras $\mathfrak{A}_1 - \mathfrak{A}_7$, and the equality $f(2^k) = 1$ results now from the statement (a) of theorem **8.9**.

(b) Let $f \in \mathfrak{R}$, $f = FSpec_{\mathfrak{M}}$ and $f(6) = 0$, $f(2) = f(2^{k+1}) = 2$. Let us prove that in this case we get $f(2^k) = 2$. Let $\mathfrak{C} \in \mathfrak{M}$ and $|\mathfrak{C}| = 2^k$. $\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}$, $|\mathfrak{A}| = |\mathfrak{B}| = 2$, and $\mathfrak{A} \not\cong \mathfrak{B}$. It should be remarked that, in this case, by theorem **2.28**, we have $\mathfrak{A}^{k+1} \not\cong \mathfrak{B}^{k+1}$. Therefore, $\mathfrak{B}^k\mathfrak{A} \cong \mathfrak{A}^{k+1}$ or $\mathfrak{B}^k\mathfrak{A} \cong \mathfrak{B}^{k+1}$. Let us assume that the latter is valid, $\mathfrak{A} \in HP(\mathfrak{B})$. And again, either $\mathfrak{C}\mathfrak{B} \cong \mathfrak{B}^{k+1}$ or $\mathfrak{C}\mathfrak{B} \cong \mathfrak{A}^{k+1}$ and, hence, either $\mathfrak{C} \in HP(\mathfrak{B})$ or $\mathfrak{C} \in HP(\mathfrak{A}) \subseteq HP(\mathfrak{B})$. By lemma **8.6** applied to the algebra $\mathfrak{B}$ and theorem **6.5**, the algebra $\mathfrak{B}$ must obey one of the conclusions of lemma **8.6**, (c) or (d) and, hence, according to lemma **8.3**, $\mathfrak{B}$ is rationally equivalent either to the algebra $\mathfrak{A}_8$ or to the algebra $\mathfrak{A}_9$, and the statement (b) of the lemma under discussion now results from the statement (b) of theorem **8.9**.

(c) Let now $f = FSpec_{\mathfrak{M}}$, $f(6) = 0$ and $f(2) = f(2^{3+m+s}) = 3 + m$, and thus we come to a contradiction. Let $\mathfrak{C}_1, \ldots, \mathfrak{C}_{3+m}$ be non-isomorphic $\mathfrak{M}$-algebras of the power 2. Then the algebra $\mathfrak{C}_1^{s+1}\times\mathfrak{C}_2\times\ldots\times\mathfrak{C}_{3+m}$ is isomorphic to one of the algebras $\mathfrak{C}_j^{3+m+s}(1\le j\le 3+m)$ which are pairwise non-isomorphic, by theorem **2.28.** Let us assume $\mathfrak{C}_1^{3+m+s}\times\mathfrak{C}_2\times\ldots\times\mathfrak{C}_{3+m}\cong\mathfrak{C}_1^{3+m+s}$ and, hence, $\mathfrak{C}_1,\ldots,\mathfrak{C}_{3+m}\in HP(\mathfrak{C}_1)$. The variety $\mathfrak{M}(\mathfrak{C}_1)$, however, cannot have more than two non-isomorphic algebras of the power 2 by lemmas **8.6**, **8.3** and theorem **8.9.** ■

So far we have been limiting ourselves with to fine spectra solely. Turning to values of the function $Spec_{\mathfrak{M}}(x)$ on infinite cardinals, let us first of all recall the equality $Spec_{\mathfrak{M}}(\aleph)=2^{\aleph}$ for any infinite cardinal $\aleph$ for the case when the variety $\mathfrak{M}$ is an arbitrary non-Abelian and congruence-modular variety. Indeed, by corollary **2.2,** a non-Abelian variety $\mathfrak{M}$ must contain a non-Abelian subdirectly non-decomposable algebra $\mathfrak{A}$ which will be, according to theorem **3.1**, Boolean-separated, i.e., for any non-isomorphic Boolean algebras $\mathcal{B}_1,\mathcal{B}_2$, the algebras $\mathfrak{A}^{\mathcal{B}_1}$ and $\mathfrak{A}^{\mathcal{B}_2}$ are also non-isomorphic. Therefore, the equality $Spec_{\mathfrak{M}}(\aleph)=2^{\aleph}$ will result from the equality $Spec_{BA}(\aleph)=2^{\aleph}$ known for infinite cardinals. For the case of congruence-modular varieties, it is possible to completely describe, by the module $\aleph_0$, the values of the functions of a fine spectrum on infinite $\aleph$. Since we have considered the non-Abelian case, it suffices to analyse the functions $Spec_{\mathfrak{M}}(\aleph)$ for Abelian varieties.

Theorem 8.10. If $\mathfrak{M}$ is a congruence-modular variety and the function $n_{\mathfrak{M}}(\aleph)=Spec_{\mathfrak{M}}(\aleph)+\aleph_0$, on infinite cardinals $n_{\mathfrak{M}}(\aleph)$ coincides with one of the following functions:

(1) $n_1(\aleph_\alpha)=2^{\aleph_\alpha}$;

(2) $n_2(\aleph_\alpha)=(|\alpha|+\aleph_0)^{\aleph_0}$;

(3) $n_3(\aleph_\alpha)=|\alpha|+2^{\aleph_0}$;

(4) $n_4(\aleph_\alpha)=|\alpha|+\aleph_0$;

(5) $n_5(\aleph_\alpha)=2^{\aleph_0}$;

(6) $n_6(\aleph_\alpha)=\aleph_0$.

The proof of this theorem is, by theorem **2.20**, reduced to the analysis of the functions $n_{\mathfrak{M}}(\aleph)$ for varieties of left unitary modules over the rings with unities and will thus be omitted. A full description of fine spectra of varieties has been given by

E.A.Paljutin [153].

The case of category varieties also pertains to the cases when a variety is described to the accuracy of rational equivalence employing fine spectra. Let us not prove the corresponding results, which are of a specific model-theoretical character, and give instead their formulation with some required definitions.

Definition 8.5.

(a) A variety $\mathfrak{M}$ is called categoric if for any cardinal $\aleph$ $Spec_{\mathfrak{M}}(\aleph) \leq 1$ and $\mathfrak{M}$ is not trivial (has non-singleton algebra).

(b) A variety $\mathfrak{M}$ is categoric in the power $\aleph$ if $Spec_{\mathfrak{M}}(\aleph) \leq 1$.

The following limitations are also valid on the function of a fine spectrum of a variety pertaining to the notion of category.

Theorem 8.11.

(a) The equality $Spec_{\mathfrak{M}}(\aleph_0) = 1$ implies that the variety $\mathfrak{M}$ is categoric.

(b) The equalities $Spec_{\mathfrak{M}}(\aleph_1) = 1$ and $Spec_{\mathfrak{M}}(n) > 0$ for any n obeying the inequalities $2 \leq n < \aleph_0$ imply that the variety $\mathfrak{M}$ is categoric.

Definition 8.6. For any algebra $\mathfrak{A}$ and any $n \geq 1$, let us define the operations $d_n^{\mathfrak{A}}$ and $p_{ni}^{\mathfrak{A}}$ on the algebra $\mathfrak{A}^n$ in such a way that

$$d_n^{\mathfrak{A}}(< x_{11},\ldots,x_{1n} >,\ldots,< x_{n1},\ldots,x_{nn} >) =< x_{11},\ldots,x_{nn} > ,$$

$$p_{ni}^{\mathfrak{A}}(< x_1,\ldots,x_n >) =< x_i,\ldots,x_i > .$$

For any variety $\mathfrak{M}$, $\mathfrak{M}_n$ will denote $\{\mathfrak{A}_n | \mathfrak{A} \in \mathfrak{M}\}$, where $\mathfrak{A}_n$ is an enrichment of the algebra $\mathfrak{A}^n$ with the operations $d_n^{\mathfrak{A}}$ and $p_{n,i}^{\mathfrak{A}}$ $(1 \leq i \leq n)$. If $t(x)$ is the term of the signature of the variety $\mathfrak{M}_n$, by $T(t)$ we will denote the set $\{r(x_1,\ldots x_m) | r(x_1,\ldots,x_m)$ is the term of the signature of the variety $\mathfrak{M}_n$, and $\mathfrak{M}_n \models \forall x r(t(x),\ldots,t(x)) = t(x)\}$. $\mathfrak{M}_n(t)$ will denote a class of algebras obtained on the basic sets of $\mathfrak{M}_n$-algebras by including the functions definable on them by the terms of the set $T(t)$, as well as the functions $d_n^{\mathfrak{A}}$ and $p_{n,i}^{\mathfrak{A}}$, into the signature.

Theorem 8.12. A variety $\mathfrak{M}$ is categoric iff it is rationally equivalent to a variety of one of the following types:

(a) $\mathfrak{V}_n(t)$, where $\mathfrak{V}$ is the variety of vector spaces over a certain sfield $\mathfrak{D}$, $n \geq 1$, and $t(x)$ is a certain term of the signature of the variety $\mathfrak{V}_n$;

(b) $\mathfrak{S}_n$, where $\mathfrak{S}$ is the variety of all sets (of an empty signature), and $n \geq 1$;

(c) $\mathfrak{C}_n$, where $\mathfrak{C}$ is the variety of all sets with an element singled out (the signature consists of one constant), and $n \geq 1$.

Priorities. Theorem **8.1** comprises the equivalences of various statements proved by different authors at different times. In particular, the equivalence of the conditions (a) and (d) was proved by A.I.Malzev, that of the conditions (d) and (e) was proved by W.Taylor [225]. Theorem **8.2** is by T.K.Hu [94], theorem **8.4** by K.Sokolnicki [214]. Theorems **8.5**, **8.7**, **8.8** and **8.9** and the related proof of the lemmas are by W.Taylor [227]. The description of categoric varieties was obtained by E.A.Paljutin and S.Givant [77] independently. The formulation of theorem **8.12** used here belongs to S.Givant. Theorem **8.11** was proved by E.A.Paljutin for quasi-varieties [157] (see also [156]) when describing categoric quasi-varieties, the description of categoric positive Horn theories are also by him. Theorem **8.10** is by Y.T.Baldwin and R.McKenzie [7], the statement of corollary **8.1** is by G,Grätzer [85]. Theorem **8.6** was proved by A.Ehrenfeucht and can be found in [227]. Theorem **8.3**, as has been pointed out in the text, belongs to F.W.Lauvere [123].

9. Epimorphism Skeletons, Minimal Elements, the Problem of Cover, Universality

Let $\mathfrak{K}$ be an arbitrary class of universal algebras, and $\mathfrak{I}\mathfrak{K}$ denote a family of the types of the isomorphism of $\mathfrak{K}$-algebras. Studies of the relation of epimorphism between $\mathfrak{K}$-algebras result in the following notion: for $a,b \in \mathfrak{I}\mathfrak{K}$, the relation $a << b$ is valid iff there is a homomorphism from an algebra of the type of the isomorphism b on an algebra of the type of the isomorphism a. The relation $<<$ will be used in an analogous sense between the algebras from $\mathfrak{K}$ as well.

Obviously, << is a quasi-order relation on $\mathfrak{J}\mathfrak{K}$.

Definition 9.1. The skeleton of epimorphism of the class of algebras $\mathfrak{K}$ will be called a quasi-ordered class $<\mathfrak{J}\mathfrak{K};<<>$.

The present section, as well as the following ones, will be devoted to studying epimorphism skeletons of varieties of universal algebras (basically congruence-distributive), as well as to studying the algebras the types of isomorphisms of which occupy extreme positions in the epimorphism skeletons of the corresponding varieties themselves.

$\mathfrak{J}\mathfrak{A}$ will denote the isomorphism type of the algebra $\mathfrak{A}$. Let us first investigate relations between the epimorphism skeletons of varieties and such traditional objects of universal algebra as lattices of subvarieties and those of congruences. It should be recalled that for an arbitrary cardinal $\aleph$, $\mathfrak{M}_{\aleph}$ denotes a family of $\mathfrak{M}$-algebras whose power is not greater than $\aleph$. Bounded epimorphism skeletons of the variety $\mathfrak{M}$ will be called quasi-ordered sets $<\mathfrak{J}\mathfrak{M}_{\aleph};<<>$, a countable epimorphism skeleton of $\mathfrak{M}$ will be termed a quasi-ordered set $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$. Obviously, the epimorphism skeleton of any variety contains a least element (let us denote it with $0_{\mathfrak{M}}$) which is the isomorphism type of a one-element algebra. Any bounded epimorphism skeletons of the variety $\mathfrak{M}$, $<\mathfrak{J}\mathfrak{M}_{\aleph};<<>$ $(\aleph \geq \aleph_0)$ contains a greatest element which is the isomorphism type of a free algebra $\mathfrak{F}_{\mathfrak{M}}(\aleph)$.

For any quasi-order $<A;\leq>$, let $\equiv_{\leq}$ denote the equivalence relation naturally associated with this quasi-order: $a \equiv_{\leq} b$ iff $a \leq b$ and $b \leq a$.

If $\sim$ is a certain equivalence relation on the quasi-ordered set $<A;\leq>$ with convex classes of equivalence, $<A/\sim;\leq>$ will denote the quasi-order defined on the factor-set $A/\sim$ in the following way: $[a]_{\sim} \leq [b]_{\sim}$ iff for some $c \in [a]_{\sim}$, $d \in [b]_{\sim}$ we have $c \leq d$ (here $[a]_{\sim}$ is a class of $\sim$-equivalence containing the element a). A subset B of the quasi-ordered set $<A;\leq>$ is called a semi-ideal if for any $a \in A$, $b \in B$ it follows from $a \leq b$ that $a \in B$. Henceforth we will often consider lattices as partially ordered sets without pointing it out especially, if it is clear from the context.

Let us define the relation $\leq_{\mathfrak{J}}$ on $Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$ in the following way: $\psi \leq_{\mathfrak{J}} \theta$ iff $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta << \mathfrak{F}_{\mathfrak{M}}(\aleph)/\psi$. Obviously, $\psi \leq \theta$ yields $\psi \leq_{\mathfrak{J}} \theta$, the latter implying that there is a $\theta'' \in Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$ such that $\psi \leq \theta'$ and $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta'$. For $\theta, \theta' \in Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$, the existence of the isomorphism $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta'$ will be expressed by $\theta \cong \theta'$. Let us introduce one more equivalence relation on $Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$: $\theta \equiv \theta'$ if $\theta \leq_{\mathfrak{J}} \theta'$ and $\theta' \leq_{\mathfrak{J}} \theta$. Let us also define the relations $\leq_{\mathfrak{J}c}, \cong_c, \equiv_c$ on the lattice $Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$:

$\psi \leq_{\mathfrak{J}c} \theta$ iff there is a $\phi \geq \psi$ such that

$$< Con\mathfrak{F}_{\mathfrak{M}}(\aleph)|\geq \theta; \vee,\wedge > \cong < Con\mathfrak{F}_{\mathfrak{M}}(\aleph)|\geq \phi; \vee,\wedge >;$$

$\theta \equiv_c \theta'$ iff $\theta \leq_{\mathfrak{J}c} \theta'$ and $\theta' \leq_{\mathfrak{J}c} \theta$;

$\theta \cong_c \theta'$ iff

$$< Con\mathfrak{F}_{\mathfrak{M}}(\aleph)|\geq \theta; \vee,\wedge > \cong < Con\mathfrak{F}_{\mathfrak{M}}(\aleph)|\geq \theta'; \vee,\wedge >.$$

The following relations between the equivalence relations $\cong, \cong_c, \equiv, \equiv_c$ introduced on $Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$ and the quasi-orders $\leq_{\mathfrak{J}}, \leq_{\mathfrak{J}c}$ are also obvious:

$$\begin{array}{ccc} \cong_c & \subseteq & \equiv_c \\ \cup\!| & & \cup\!| \\ \cong & \subseteq & \equiv \end{array} \qquad \leq \subseteq \leq_{\mathfrak{J}} \subseteq \leq_{\mathfrak{J}c}.$$

It should be remarked the the equivalence classes on $Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$ will be convex subsets relative to $\equiv$ and $\equiv_c$. One can also easily note, by correlating the algebras $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta$ to congruences $\theta \in Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$, that $< \mathfrak{J}\mathfrak{M}_{\aleph}; << >$ is an antimonotonic image of a partial order $< Con\mathfrak{F}_{\mathfrak{M}}(\aleph); \leq >$ at $\aleph \geq \aleph_0$. Moreover, $< \mathfrak{J}\mathfrak{M}_{\aleph}/\equiv_{<<}; << >$ is antiisomorphic to $< Con\mathfrak{F}_{\mathfrak{M}}(\aleph)/\equiv; \leq_{\mathfrak{J}} >$.

In its turn, $< Con\mathfrak{F}_{\mathfrak{M}}(\aleph)/\equiv_c; \leq_{\mathfrak{J}c} >$ is an antimonotonic image of $< \mathfrak{J}\mathfrak{M}_{\aleph}; << >$ (by correlating the algebra $\mathfrak{A} \in \mathfrak{M}$ and congruence $\theta \in \mathfrak{F}_{\mathfrak{M}}(\aleph)$ so that $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta \cong \mathfrak{A}$).

The lattice of subvarieties $L_{\mathfrak{M}}$ of the variety $\mathfrak{M}$ considered as a partially ordered set is also a monotonic image of any limited epimorphism skeleton $< \mathfrak{J}\mathfrak{M}_{\aleph}; << >$ of the variety $\mathfrak{M}$ at $\aleph \geq \aleph_0$: it suffices to correlate the variety $\mathfrak{M}(\mathfrak{A}) \subseteq \mathfrak{M}$ to the algebra $\mathfrak{A} \in \mathfrak{M}$. The discussed relations $< \mathfrak{J}\mathfrak{M}_{\aleph}; << >$, $Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$ and $L_{\mathfrak{M}}$ are concluded in the following statement.

Statement 9.1. For any algebra variety $\mathfrak{M}$ and any infinite cardinal $\aleph$, the following antimonotonic, f, g, and monotonic, h, mappings exist, in which case f, g, h are surjections:

$$< Con\mathfrak{F}_{\mathfrak{M}}(\aleph);\leq> \xrightarrow{f} < \mathfrak{I}\mathfrak{M}_{\aleph};<<> \begin{cases} \xrightarrow{g} <Con\mathfrak{F}_{\mathfrak{M}}(\aleph)/\equiv_c;\leq_{\mathfrak{I}_c}> \\ \xrightarrow{h} <L_{\mathfrak{M}};\subseteq> \end{cases}$$

Let us recall some more simple facts pertaining only to the epimorphism skeletons of congruence-distributive varieties. As was the case in the preceding discussion, let BA be a variety of Boolean algebras, IBA be a subclass in BA consisting of only interval Boolean algebras. It should be recalled that $BA_{\aleph_0} \subseteq IBA$, and, obviously, $\mathfrak{I}IBA$ is a semi-ideal in $< \mathfrak{I}BA;<<>$.

Let L be the family of all linearly ordered sets. Correlating the linearly ordered set $< A;\leq>$ and a corresponding interval Boolean algebra , we get a mapping f: $\mathfrak{I}L$ on $\mathfrak{I}IBA$. Obviously, f preserves the relation $<<$ and, moreover, if $f(< A;\leq>) << f(< B;\leq>)$, there is a $< C;\leq>\in L$ such that $f(< A;\leq>) = f(< C;\leq>)$ and $< C;\leq> << < B;\leq>$. In other words, there is a strong homomorphism from the epimorphism skeleton of the class of linearly ordered sets on the epimorphism skeleton of interval Boolean algebras.

Let now $\mathfrak{M}$ be an arbitrary nontrivial congruence-distributive variety and let $\mathfrak{A}$ be a simple $\mathfrak{M}$-algebra existing by theorem **2.11**. Then, for any Boolean algebra $\mathfrak{B}$, any homomorphic image of the algebra $\mathfrak{A}^{\mathfrak{B}}$ has, by corollary **3.1**, the form $\mathfrak{A}^{\mathfrak{B}_1}$, where $\mathfrak{B}_1$ is a homomorphic image of the algebra $\mathfrak{B}$. Besides, by the same corollary **3.1**, for any Boolean algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$ the relation $\mathfrak{A}^{\mathfrak{B}_1} << \mathfrak{A}^{\mathfrak{B}_2}$ is equivalent to the relation $\mathfrak{B}_1 << \mathfrak{B}_2$, while $\mathfrak{A}^{\mathfrak{B}_1}$ and the algebra $\mathfrak{A}^{\mathfrak{B}_2}$ are isomorphic iff $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are isomorphic. Hence, the correlation of the $\mathfrak{M}$-algebra $\mathfrak{A}^{\mathfrak{B}}$ to the Boolean algebra $\mathfrak{B}$ is an isomorphic mapping from the epimorphism skeleton $< \mathfrak{I}BA;<<>$of a variety of Boolean algebras on a certain semi-ideal of the epimorphic skeleton $< \mathfrak{I}\mathfrak{M};<<>$ of the variety $\mathfrak{M}$. Taking all the facts just discussed into account, we arrive at the following statement.

Statement 9.2. If $\mathfrak{M}$ is a nontrivial congruence-distributive variety, then there is a semi-ideal isomorphic to $< \mathfrak{I}BA;<<>$ in $< \mathfrak{I}\mathfrak{M};<<>$, and there is a certain strong homomorphism from the epimorphism skeleton of the class of linearly ordered sets to a certain semi-ideal of $\mathfrak{M}$.

Definition 9.2. An element a of the epimorphism skeleton of the variety $\mathfrak{M}$ will be called minimal if $a \neq 0_{\mathfrak{M}}$, and for any $b \in \mathfrak{I}\mathfrak{M}$, $0_{\mathfrak{M}} << b << a$ yields either $b = 0_{\mathfrak{M}}$ or $b \equiv_{<<} a$.

The type of isomorphism of any simple $\mathfrak{M}$ -algebra is obviously minimal in the epimorphism skeleton of $\mathfrak{M}$ and, hence, by theorem **2.11**, there is at least one minimal element in the epimorphism skeleton of any nontrivial variety.

Definition 9.3. A non-singleton algebra is called pseudo-simple if any of its homomorphic images is either one-element or isomorphic to it.

Simple algebras are often particular cases of pseudo-simple ones. Unlike simple algebras the congruence lattices of which are two-element, the congruence lattices of pseudo-simple algebras can be arbitrarily large.

Theorem 9.1. The congruence lattice of any pseudo-simple algebra is well-ordered and has the form of a non-decomposable ordinal $\omega^{\beta}+1$, where β is a certain ordinal. For any ordinal β there is a pseudo-simple algebra with its congruence lattice isomorphic to $\omega^{\beta}+1$.

Proof. Let $\mathfrak{A}$ be a pseudo-simple algebra. Let us first remark that $\mathfrak{A}$ has a monolith. Indeed, let us choose $a \neq b \in \mathfrak{A}$, and let θ be a maximal congruence on $\mathfrak{A}$ such that $<a,b> \notin \theta$. Then, obviously, $\theta^{\mathfrak{A}/\theta}_{a/\theta, b/\theta}$ is the least congruence on $\mathfrak{A}/\theta$ other than $\Delta_{\mathfrak{A}/\theta}$, but $\mathfrak{A}/\theta \cong \mathfrak{A}$. Let us now show that $Con\mathfrak{A}$ is linearly ordered. Indeed, if the opposite is the case, if $\theta_1, \theta_2 \in Con\mathfrak{A}$ and are incomparable, $\mathfrak{A}/\theta$ isomorphic to $\mathfrak{A}$ has no monolith, since

$$(\theta_1/\theta_1 \wedge \theta_2) \wedge (\theta_2/\theta_1 \wedge \theta_2) = \Delta_{\mathfrak{A}}/\theta_1 \wedge \theta_2.$$

Let us show that $Con\mathfrak{A}$ is well-ordered: choosing an infinitely descending chain of congruences $\theta_1 > \theta_2 > \ldots > \theta_n > \ldots$ on $\mathfrak{A}$, we arrive at a contradiction with the fact that $\mathfrak{A}\Big/\bigwedge_{i\in\omega}\theta_i$ must have a monolith. Therefore, $Con\mathfrak{A} \cong \gamma$, where γ is an ordinal. But, on the other hand, since for any $\theta \neq \nabla_{\mathfrak{A}}$ we have $\mathfrak{A}/\theta \cong \mathfrak{A}$,

$$Con\mathfrak{A}/\theta \cong <\{\theta_1 \in Con\mathfrak{A} \mid \theta_1 \geq \theta\}; \leq> \cong Con\mathfrak{A},$$

i.e., for any $\alpha \in \gamma$ we get $<\{\delta \in \gamma \mid \delta \geq \alpha\}; \leq> \cong \gamma$. The fact that such not limit ordinals have the form $\omega^{\beta}+1$ for a certain ordinal β is well-known. Thus, the theorem is thus proved in one direction.

Let us now construct a pseudo-simple algebra $\mathfrak{A}$ such that $Con\mathfrak{A} \cong \omega^{\beta}+1$ for an arbitrary ordinal $\omega^{\beta}+1$. As the basic set of $\mathfrak{A}$ let us choose an ordinal ω^{β}, and let us define the function f on ω^β in the following way: $f(a,b,c)=0$ if $a>b,c$, and $f(a,b,c)=c$ otherwise.

We can easily notice that for any congruence θ on $\mathfrak{A} =<\omega^\beta;f>$, one of the equivalence classes relative to θ has the form of the initial interval in the ordinal ω^β, while all the other classes are one-element and conversely, any similar equivalence on ω^β is a congruence of the algebra $\mathfrak{A}$. Therefore, indeed, $Con\mathfrak{A} \cong \omega^\beta +1$. It is also obvious that $\mathfrak{A}$ is pseudo-simple. ■

Thus, the types of isomorphisms of pseudo-simple algebras, as well as those of simple algebras, are obviously minimal in the epimorphism skeletons of the varieties containing these algebras. In search of describing all algebras the types of isomorphisms of which are minimal in the epimorphism skeletons of the varieties containing them, we come to the following definition.

Definition 9.4. A one-element algebra $\mathfrak{A}$ is called quasi-simple if for any congruence α on $\mathfrak{A}$ other than the greatest, there is a congruence β on $\mathfrak{A}$ such that $\beta \geq \alpha$ and $\mathfrak{A}/\beta$ is isomorphic to $\mathfrak{A}$.

In particular, simple and pseudo-simple algebras are quasi-simple as well.

Obviously, for any variety $\mathfrak{M}$, an element $a \in \mathfrak{J}\mathfrak{M}$ is minimal in $<\mathfrak{J}\mathfrak{M};<<>$ iff a is an isomorphism type of a quasi-simple algebra. In this case, a will be an isomorphism type of either a simple or a pseudo-simple algebra iff the equivalence class $\equiv_{<<}$ on $\mathfrak{J}\mathfrak{M}$ containing the element a (let us refer to this class as $[a]_{\equiv_{<<}}$) is one-element. Therefore, the power of $[a]_{\equiv_{<<}}$ can serve as a measure of distinction of a not simple and not pseudo-simple algebra from the latter.

It should be noticed that, since any algebra is decomposable in a subdirect product of subdirectly non-decomposable algebras, for any quasi-simple algebra there is a subdirectly non-decomposable algebra $\equiv_{<<}$-equivalent to it. Therefore, the number of quasi-simple algebras of the variety $\mathfrak{M}$ which are pairwise non-equivalent in terms of $\equiv_{<<}$ (the number of minimal elements which are pairwise non-equivalent in terms of $\equiv_{<<}$ in the skeleton of $<\mathfrak{J}\mathfrak{M};<<>$) is not greater than the number of non-isomorphic subdirectly non-decomposable algebras.

The following statement shows that well-ordered congruence lattices are not characteristic of pseudo-simple algebras even in the class of quasi-simple ones and, moreover, there are quasi-simple algebras with well-ordered congruence lattices and any measure of distinction from pseudo-simple algebras. Let $\mathfrak{J}\mathfrak{A}$ denote this isomorphism type of the algebra $\mathfrak{A}$.

Theorem 9.2. For any ordinal $\beta > 0$ and cardinal $\aleph$ not greater than $|\omega^\beta|$ (strictly less than $\aleph_0$ for the case when $\beta = 1$) there is a quasi-simple algebra $\mathfrak{A}$ such that $Con\mathfrak{A} \cong \omega^\beta + 1$ and $|[\mathfrak{J}\mathfrak{A}]_{\equiv_{<<}}| = \aleph$.

Proof. Let $\mathfrak{A}_\beta = <\omega^\beta; f>$ be a pseudo-simple algebra constructed in the proof of theorem **9.1**, i.e., $Con\mathfrak{A}_\beta \cong \omega^\beta + 1$, and let any congruence $\theta \in Con\mathfrak{A}_\beta$ be such that the only non-singleton equivalence class relative to θ has the form of the initial interval on ω^β, and let any initial interval on ω^β be the equivalence class for a certain congruence on $\mathfrak{A}_\beta$.

Let g be a certain unary function on ω^β such that $g(x) \le x$ for all $x \in \omega^\beta$ and, therefore, any congruence $\theta \in Con\mathfrak{A}_\beta$ remains a congruence in the extension $\mathfrak{A}'_\beta = <\mathfrak{A}_\beta, g>$ of the algebra $\mathfrak{A}_\beta$ by adding the function g in the signature.

Let us consider the two cases:(a) when $\aleph$ is infinite, and (b) when $\aleph$ is finite.

In the case (a), let $\aleph = \omega_s$. Let us define the function $g(x)$ on ω_s in the following way: $g(n) = 0$ for all $n < \omega$. If $g(x)$ is already defined on all $i < \omega^j (j < \omega_s)$, then let us set $g(\gamma) = \gamma$ for γ $(\omega^j \le \gamma < \omega^j + \omega^j)$, and let us set $g(\gamma) = g(\delta)$ for any $\gamma = (\omega^j + \omega^j) \cdot n + \delta$ $(0 < n < \omega$ and $\delta < \omega^j + \omega^j)$. These conditions define g on the ordinals less than ω^{j+1}. Therefore, the function g is defined on all $\gamma < \omega_s$. If $\omega^\beta = \omega_s$, g is defined on the entire $\mathfrak{A}_\beta$, while if $\omega_s < \omega^\beta$, g is periodically defined furthermore on $\mathfrak{A}_\beta = \omega^\beta$ with the period ω_s.

Let now $\theta_\gamma (\gamma < \omega^\beta)$ be a congruence on $\mathfrak{A}'_\beta$ uniquely defined by the following condition: there is an equivalence class over θ_γ equal to $\{\delta | \delta < \gamma\}$. Obviously, at $j_1 < j_2 < \omega_s$, we have $\mathfrak{A}'_\beta / \theta_{\omega^{j_1}} \not\equiv \mathfrak{A}'_\beta / \theta_{\omega^{j_2}}$. Indeed, the identity $g(x) = x$ is fulfilled in $\mathfrak{A}'_\beta / \theta_{\omega^{j_1}}$ on the first ω^{j_1} elements, while in $\mathfrak{A}'_\beta / \theta_{\omega^{j_2}}$ in the first ω^{j_2} elements (the natural order on the ordinal-elements $\mathfrak{A}'_\beta$ is defined with the function f obviously). Therefore, we get $|[\mathfrak{J}\mathfrak{A}'_\beta]_{\equiv_{<<}}| = \aleph$.

For any $i < \omega^\beta$ there is a $\gamma < \omega^\beta$ and a finite set $\omega^{j_1} \ge \ldots \ge \omega^{j_k}$ of ordinals less than ω_s such that $i = \omega_s \cdot \gamma + \omega^{j_1} + \ldots + \omega^{j_k}$. In this case, by the definition of g and $\mathfrak{A}_\beta$, it is obvious for $i_1 = \omega_s \cdot \gamma + \omega^{j_1+1} + \omega^{j_1+1}$ that the mapping $h(j) = \omega_s \cdot \gamma + \omega^{j_1+1} + \omega^{j_1+1} + j$ is an isomorphism of $\mathfrak{A}'_\beta$ and $\mathfrak{A}'_\beta / \theta_{i_1}$. Therefore, $\mathfrak{A}'_\beta$ is quasi-simple and the statement of the theorem is proved for the case (a). The construction of the function f for the case (b) can be carried out analogously. ■

Definition 9.5. A partially ordered set (lattice) L will be termed upper-non-decomposable if for any not greatest element a from L there is a $b \in L$ such that $b \geq a$ and L is isomorphic to $L_b = \{c \in L | c \geq b\}$.

According to the theorem on homomorphisms, the congruence lattice of a quasi-simple algebra is algebraic and upper-non-decomposable.

Let us show that, under certain additional conditions, for an algebraic upper-non-decomposable lattice L, there is a quasi-simple algebra $\mathfrak{A}$ such that $Con\,\mathfrak{A} \cong L$. This, in particular, implies the statement on the existence of a greater number of quasi-simple algebras of any infinite power.

$\mathfrak{C}(L)$ will denote the upper semi-lattice of compact elements of the algebraic lattice L, $\mathfrak{R}(L)$ a partially ordered set of $\vee$-non-decomposable elements of the semi-lattice $\mathfrak{C}(L)$ (an element a is $\vee$-non-decomposable if for any b,c, from $a = b \vee c$ we have either $a = b$ or $a = c$). A complete lattice L is called well-distributive if for any sets $I, J_i (i \in I)$ and any families of the elements c_j $(j \in \bigcup_{i \in I} J_i)$ of the lattice L the following equality is valid:

$$\bigwedge_{i \in I}(\bigvee_{j \in J} c_j) = \bigvee_{f \in \prod_{j \in I} J_i}(\bigwedge_{i \in I} c_{f(i)}).$$

The following characterization of algebraic well-distributive lattices is known [44]: a lattice L is algebraic and well-distributive iff it is isomorphic to the lattice $I(< A;\leq >)$ of non-empty semi-ideals of a certain partially ordered set $< A;\leq >$ with a least element. In fact, one can choose the set $\mathfrak{R}(L)$ as $< A;\leq >$.

Definition 9.6. The algebraic lattice L is called strongly upper-non-decomposable if the partially ordered set $\mathfrak{R}(L)$ is upper-non-decomposable, and the biggest element of L is not compact.

By virtue of the characterization of algebraic well-distributive lattices given above, it is obvious that if such lattices are strongly upper-non-decomposable, they are upper-non-decomposable.

Theorem 9.3. For any algebraic well-distributive strongly upper-non-decomposable lattice L there is a quasi-simple algebra $\mathfrak{A}$ such that $Con\,\mathfrak{A} \cong L$.

Proof. Let the lattice L obey the conditions of the theorem. As has been noted earlier, $L \cong I(\mathfrak{R}(L))$. The functions f_1, f_2 are defined on $\mathfrak{R}(L)$ in the following way: $f_1(a,b) = 0$ if $a \neq b$, $f_1(a,b) = a$ if $a = b$ (here 0 is a least element of L), $f_2(a,b)=0$ if $a<b$, $f_2(a,b)=a$ otherwise.

Let $\mathfrak{A}_L=<R(L);f_1,f_2>$, where $R(L)$ is the basic set of the partially ordered set $\mathfrak{R}(L)$. Let us prove that $Con\mathfrak{A}_L\cong L$.

For any $\theta\in Con\mathfrak{A}_L$ there is a non-empty semi-ideal I_θ of the set $\mathfrak{R}(L)$ such that all the equivalence classes in terms of θ are $I_\theta,\{a\}$, where $a\in R(L)\setminus I_\theta$. Indeed, let I_θ be a non-singleton equivalence class in terms of θ, $a,b\in I_\theta$ and $a\neq b$. Then $a=f_1(a,a)\equiv_\theta f_1(a,b)=0$, i.e., $0\in I_\theta$. If $c\in I_\theta$ and $d<c$, $d\neq 0$, $0=f_2(d,c)\equiv_\theta f_2(d,0)=d$, i.e., any non-singleton equivalence class in terms of θ is a semi-ideal in $\mathfrak{R}(L)$ and, since $\mathfrak{R}(L)$ contains the least element 0, this class being unique.

The validity of the converse statement can also be checked directly: for any non-empty semi-ideal I in $\mathfrak{R}(L)$, the equivalence with the classes $I,\{a\}$, where $a\in R(L)\setminus I$, is a congruence on $\mathfrak{A}_L$. Therefore, $Con\mathfrak{A}_L\cong I(R(L))\cong L$.

To prove the theorem, one now has to notice that $\mathfrak{A}_L$ is quasi-simple. Indeed, for any $\theta\in Con\mathfrak{A}_L$ if $\theta\neq\nabla_{\mathfrak{A}_L},\Delta_{\mathfrak{A}_L}$, there is a semi-ideal $I_\theta\neq R(L)$ which is the only non-singleton equivalence class in terms of θ. Let $c\in R(L)\setminus I_\theta$. As L is strongly upper-non-decomposable in $R(L)$, there is a $d\geq c$ such that $\mathfrak{R}(L)$ is isomorphic to an interval $\{b\in R(L)| b\geq d\}$. Let J be a semi-ideal in $\mathfrak{R}(L)$ equal to $R(L)\setminus\{b\in R(L)| b\geq d\}$. Obviously, $J\supseteq I_\theta$ and, hence, $\theta_J\geq\theta_{I_\theta}=\theta$ (θ_J is a congruence on $\mathfrak{A}_L$ with the classes $J,\{a\}$, where $a\in\mathfrak{R}(L)\setminus J$). Besides, the isomorphism $\mathfrak{R}(L)$ and $\{b\in R(L)| b\geq d\}$, as well as the definition of $\mathfrak{A}_L$ yield the isomorphism $\mathfrak{A}_L/\theta_J$ and $\mathfrak{A}_L$. Therefore, indeed, $\mathfrak{A}_L$ is quasi-simple. ■

The condition of well-distributivity is not necessary for congruence lattices of quasi-simple algebras. Moreover, as can be seen from the next theorem, the limitations on congruence lattices of quasi-simple algebras are not local (interval) limitations, but algebraic ones.

Theorem 9.4. For any algebraic lattice L there is a quasi-simple algebra $\mathfrak{A}$ of a finite signature such that L is isomorphic to the ideal of the lattice $Con\mathfrak{A}$.

Proof. Let L be an arbitrary algebraic lattice and $L_1=L\oplus 1$, i.e., L_1 is obtained from L by adding a new greatest element. Let $\mathfrak{A}'=<A';\sigma>$ be a certain algebra (existing by theorem **2.4**) such that $Con\mathfrak{A}\cong L_1$. By the same theorem, as $\nabla_{\mathfrak{A}'}$ is compact, $\mathfrak{A}'$ can be chosen of a finite signature. Since $\nabla_{\mathfrak{A}'}$ is $\vee$-non-decomposable, let it be principal in $\mathfrak{A}'$. Let $g_1,g_2\in\mathfrak{A}'$ be such that $\nabla_{\mathfrak{A}'}=\theta^{\mathfrak{A}'}_{g_1,g_2}$. Let also $e\neq\mathfrak{A}'$, and let us define the algebra $\mathfrak{A}''=<A'\cup\{e\};\sigma>$ in such a way that $\mathfrak{A}'' \supset \mathfrak{A}'$, and for any $f \in \sigma$, any chain $\bar{a}$ of the elements of $\mathfrak{A}''$ we get $f(\bar{a}) = e$ if at least one of the elements in $\bar{a}$ is e. Obviously, L_1 is isomorphic to the ideal of the lattice $Con\mathfrak{A}''$ generated by a congruence on $\mathfrak{A}''$ with the equivalence classes $A', \{e\}$. For $i \in \omega$ let $\mathfrak{A}_i = \langle A_i; \sigma \rangle$ be pairwise disjunct algebras isomorphic to $\mathfrak{A}''$, the elements g_1^i, g_2^i, e^i of the algebra $\mathfrak{A}_i$ corresponding to the elements g_1, g_2, e of the algebra $\mathfrak{A}''$ under these isomorphisms. Let us define the algebra $\mathfrak{A} = \langle A; \sigma, g, h \rangle$ in the following way: $A = \bigcup_{i \in \omega} A_i$, $\mathfrak{A} \supseteq \mathfrak{A}_i$ for all $i \in \omega$, and $f(\bar{a}) = e_0$ for $f \in \sigma$ provided that the chain $\bar{a}$ belongs to no A_i. The ternary function g will be defined on A in the following way:

$$g(a,b,c) = g_1^{i_0} \text{ if } a = g_1^{i_0}, b \neq c, b \in A_{i_1}, c \in A_{i_2} \text{ and } i_0 < \max\{i_1, i_2\},$$

$$g(a,b,c) = g_2^{i_0} \text{ if } a = g_2^{i_0}, \text{ the conditions on } b,c \text{ being the same,}$$

$$g(a,b,c) = e^{i_0} \text{ if } a = e^{i_0}, \text{ the conditions on } b,c \text{ being the same,}$$

$$g(a,b,c) = e^{i_1} \text{ if } b = c \in A_{i_1},$$

$$g(a,b,c) = e^0 \text{ otherwise.}$$

The unary function h will be defined by the following condition:

$$h(a) = e^i \text{ if } a = g_1^i,$$

$$h(a) = e^{i+1} \text{ if } a = g_2^i,$$

$$h(a) = e^0 \text{ otherwise.}$$

Let $\psi \in Con\mathfrak{A}$. It is clear that, if for some $b \neq c, b \in A_{i_1}, c \in A_{i_2}, i_1 \geq i_2$ and $\langle b,c \rangle \in \psi$, $\bigcup_{i < i_1} A_i \cup \{e^{i_1}\}$ is contained in one and the same equivalence class in terms of ψ. Indeed, for any $i < i_1$ we get:

$$g_k^i = g(g_k^i, b, c) \equiv_\psi e^{i_1} = g(g_k^i, b, b),$$

$$e^i = g(e^i, b, c) \equiv_\psi e^{i_1} = g(e^i, b, b)$$

and, since $\theta^{\mathfrak{A}_i}_{g_1^i, g_2^i} = \nabla_{\mathfrak{A}_i}$, all the elements of the set $\bigcup_{i<i_1} A_i \cup \{e^{i_1}\}$ belong to one and the same equivalence class in terms of ψ.

In an analogous way, using h, we see that if $<g_1^i, g_2^i> \in \psi$, $<e^i, e^{i+1}> \in \psi$. On the other hand, one can directly check that for any $\theta \in Con\mathfrak{A}_i$ such that $\theta \neq \nabla_{\mathfrak{A}_1}$, the equivalence $\psi(\theta, i)$ with the equivalence classes: $\bigcup_{j>i} A_j \cup [e^i]_\theta$, θ-classes on $\mathfrak{A}_i$ containing no e^i and, finally, one-element subsets of the set $\bigcup_{j>i} A_j$, will be a congruence on $\mathfrak{A}$. Therefore, we get

$$Con\mathfrak{A} = Con\mathfrak{A}_0 \oplus Con\mathfrak{A}_1 \oplus ... \oplus Con\mathfrak{A}_i \oplus ... \oplus 1,$$

where $i \in \omega$. Now we have to prove that $\mathfrak{A}$ is quasi-simple. Indeed, if $\psi \in Con\mathfrak{A}$ and $\psi \neq \nabla_{\mathfrak{A}}$, there are $i \in \omega$ and $\theta \in Con\mathfrak{A}_i$ such that $\psi = \psi(\theta, i)$. Then $\psi_1 = \psi(\nabla_{\mathfrak{A}_i}, i) \geq \psi$. From the definition of $\mathfrak{A}$ and $\psi(\nabla_{\mathfrak{A}_i}, i)$ one can directly see that $\mathfrak{A}/\psi_1 \cong \mathfrak{A}$, i.e., $\mathfrak{A}$ is quasi-simple. ■

Now the following problem is open for discussion.

Problem 9.1. Is there a quasi-simple algebra $\mathfrak{A}$ such that $Con\mathfrak{A} \cong L$ for any algebraic upper-non-decomposable lattice L?

By way of concluding the discussion of quasi-simple algebras let us recall some more of its elementary properties.

Theorem 9.5. If $\mathfrak{A}$ is a quasi-simple algebra which is not simple,:

(1) $\nabla_{\mathfrak{A}}$ is not compact;

(2) if the $\mathfrak{A}$ signature is finite, $\mathfrak{A}$ contains a one-element subalgebra;

(3) if the $\mathfrak{A}$ signature is finite, $\mathfrak{A}$ is not finitely generated.

Proof.

(1) Let R be a certain chain in $Con\mathfrak{A} \setminus \{\nabla_{\mathfrak{A}}\}$ maximal in terms of inclusion. In this case, R has no greatest element, since if θ were the greatest element in R

then, as $\mathfrak{A}$ is quasi-simple, there would be a congruence $\theta' \geq \theta$ such that $\mathfrak{A}/\theta' \cong \mathfrak{A}$. Let f be an isomorphism of $\mathfrak{A}/\theta'$ on $\mathfrak{A}$. Since $\mathfrak{A}$ is not simple, there is a $\phi \in Con\mathfrak{A}$ such that $\Delta_{\mathfrak{A}} < \phi < \nabla_{\mathfrak{A}}$, but in this case $\theta < \breve{f}(\phi) < \nabla_{\mathfrak{A}}$ which contradicts the maximality of the chain R. Hence, R has no greatest element. On the other hand, it is obvious that, since R is maximal, $\bigvee_{\theta \in R} \theta = \nabla_{\mathfrak{A}}$. As $R \subseteq Con\mathfrak{A} \setminus \{\nabla_{\mathfrak{A}}\}$, and R is the chain without a greatest element, $\nabla_{\mathfrak{A}}$ cannot be a finite family of elements from R, which proves the fact that $\nabla_{\mathfrak{A}}$ is not compact.

(2) If $\sigma = < f_1^{n_1},\ldots,f_k^{n_k} >$ is the signature of $\mathfrak{A}$, let a be an arbitrary element of $\mathfrak{A}$. Let $b_i = f_i^{n_i}(a,\ldots,a), i \leq k$ and $\theta = \bigvee_{i=1}^{k} \theta_{a,b_i}^{\mathfrak{A}}$. Since, according to (1), $\nabla_{\mathfrak{A}}$ is not compact, $\theta < \nabla_{\mathfrak{A}}$ and, as $\mathfrak{A}$ is quasi-simple, there is a $\theta' > \theta$ such that $\mathfrak{A}/\theta' \cong \mathfrak{A}$. However,

$$\mathfrak{A}\Big/\theta' \models \mathop{\&}_{i=1}^{k} f_i([a]_{\theta'},\ldots,[a]_{\theta'}) = [a]_{\theta'},$$

i.e., $\mathfrak{A}/\theta'$ and, hence, $\mathfrak{A}$ as well, have a one-element subalgebra. Obviously, we can claim even more: for any positive formula $\varphi(x_1,\ldots,x_n)$ fulfilled on $\mathfrak{A}$, there is a one-element subalgebra $\{e\}$ such that $\mathfrak{A} \models \varphi(e,\ldots,e)$.

(3) If $\mathfrak{A}$ was generated by a finite set of its elements $a_1,\ldots,a_n$, and $\{e\}$ was a one-element subalgebra of the algebra $\mathfrak{A}$ ($\{e\}$ exists by virtue of (2)), $\nabla_{\mathfrak{A}} = \bigvee_{i=1}^{m} \theta_{a_i,e}$ was compact, which contradicts the statement (1). ■

Corollary 9.1. In the epimorphism skeleton of any variety $\mathfrak{M}$ of a finite signature only one-elements $\equiv_{<<}$-equivalence classes have minimal elements less than $\mathfrak{F}_{\mathfrak{M}}(n), n < \omega$.

The proof of the corollary results immediately from the statement (3) of the previous theorem.

It should be noticed that, in line with what has been proved by G.Tardos [22], for varieties of in infinite signature the corresponding statements are no longer valid: there are finitely generated pseudo-simple not simple algebras of an infinite signature.

Definition 9.7. An element a is called the cover of an element b in a quasi-ordered set $< A;\leq >$ if $a \leq b, [a]_{\equiv_{<<}} \neq [b]_{\equiv_{<<}}$ and, for any $c \in A$ such that

$a \leq c \leq b$, we have either $c \leq a$ or $b \leq c$. An algebra $\mathfrak{A}$ is called the cover of the algebra $\mathfrak{A}'$ in the variety $\mathfrak{M}$ provided that $\mathfrak{Z}\mathfrak{A}$ is the cover of $\mathfrak{Z}\mathfrak{A}'$ in the epimorphism skeleton of $\mathfrak{M}$.

Therefore, minimal elements of the epimorphism skeleton of any variety are the covers of the least element (of a one-element algebra) and, as has been noted earlier, there exists at least one cover of the least element. The problem arises whether there exists a cover for any element in the epimorphism skeleton of an arbitrary variety. In the general case, the answer remains obscure. In the considerations to follow, a number of sufficient conditions for the existence of covers of various algebras will be given, basically for congruence-distributive varieties.

Let $k(\mathfrak{A})$ stand for the least number of generating elements for $\mathfrak{A}$ for any finitely generated algebra $\mathfrak{A}$.

Definition 9.8. A variety $\mathfrak{M}$ has the basis property if for any finitely generated algebra $\mathfrak{A} \in \mathfrak{M}$ and finitely generated proper subalgebra $\mathfrak{A}_1$ of the algebra $\mathfrak{A}$ the equality $k(\mathfrak{A}_1) < k(\mathfrak{A})$ is valid.

Examples of varieties with the basis property are, obviously, the varieties of vector spaces over any fixed field, and, generally speaking, the varieties of algebras in which the sets of independently generating algebras have the same number of elements, and any set of independent elements is extendable to the set of independently generating algebras.

Theorem 9.6. If a variety $\mathfrak{M}$ has the basis property, any finitely generated $\mathfrak{M}$-algebra has a cover in the epimorphism skeleton of $\mathfrak{M}$.

Proof. The proof of this theorem basically follows the ideas used in proving theorem **2.11**. Let $\mathfrak{M}$ be a variety with the basis property, $\mathfrak{A}$ be a finitely generated $\mathfrak{M}$-algebra and $k = k(\mathfrak{A})$. By theorem **2.11**, $\mathfrak{A}$ can be considered non-singleton. Let $\mathfrak{F}_1 = \mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_{k+1}), \mathfrak{F} = \mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_k)$ and let $\theta \in Con\,\mathfrak{F}$ be such that $\mathfrak{A} = \mathfrak{F}/\theta$. Let θ_1 denote a congruence on $\mathfrak{F}_1$ generated by the pairs $< a,b > \in \theta$. Considering the homomorphism from the algebra $\mathfrak{F}_1$ to $\mathfrak{F}$ defined by the conditions $f(x_i) = x_i$ for $i \leq k$ and $f(x_{k+1}) = x_k$ we see, by corollary **2.1**, that the bound θ_1 on $\mathfrak{F}$ $(\theta_1|\mathfrak{F})$ is θ.

It should be noticed that $\theta_1 \not\geq \theta_{x_1,x_{k+1}}$. Assume, conversely, that $\theta_1 \geq \theta_{x_1,x_{k+1}}$. Let us choose an $h(x_1,\ldots,x_k) \in \mathfrak{F}_1$ such that $< h(x_1,\ldots,x_k), x_1 > \notin \theta_1$. Such an h does exist since $\theta_1|\mathfrak{F} = \theta$ and $\mathfrak{F}/\theta$ is non-singleton. In this case, however, we get $\theta_1 \vee \theta_{x_{k+1},h} \geq \theta_{x_1,x_{k+1}} \vee \theta_{x_{k+1},h} \geq \theta_{x_1,h}$. On the other hand, by corollary **2.11**, we have the equality $(\theta_1 \vee \theta_{x_{k+1},h})|\mathfrak{F} = \theta$ (it suffices to consider the homomorphism f_1 from the algebra $\mathfrak{F}_1$ to $\mathfrak{F}$ defined by the conditions: $f(x_i) = x_i$ for $i \le k$ and $f(x_{k+1}) = h$). Therefore, we see that $< x_1,h > \in \theta$ contradicts the choice of h and, hence, indeed, $\theta_1 \not\le \theta_{x_1,x_{k+1}}$.

Let $R = \{\alpha \in Con\mathfrak{F} | \alpha \le \theta_1 \vee \theta_{x_1,x_{k+1}}$ and $\alpha \not\le \theta_{x_1,x_{k+1}}\}$. One can easily see that R is inductive by inclusion and $\theta_1 \in R$. Therefore, there is a $\theta^* \in R$, maximal in R and such that $\theta_1 \le \theta^*$. Let $\mathfrak{A}_1 = \mathfrak{F}_1/\theta^*$. It should be noticed that, by corollary **2.1**, the equality $\theta_1 \vee \theta_{x_1,x_{k+1}}\mathfrak{F} = \theta$ yields the isomorphism $\mathfrak{F}_1/\theta_1 \vee \theta_{x_1,x_{k+1}} \cong \mathfrak{A}$.

In accordance with the choice of θ^*, for any $\alpha \in Con\mathfrak{F}_1$ from $\theta^* \le \alpha \le \theta_1 \vee \theta_{x_1,x_{k+1}}$ we get either $\alpha \le \theta^*$ or $\theta_{x_1,x_{k+1}} \le \alpha$. Therefore, for any algebra $\mathfrak{C}$ from $\mathfrak{A} << \mathfrak{C} << \mathfrak{A}_1$, we get either $\mathfrak{C} << \mathfrak{A}$ or $\mathfrak{A}_1 << \mathfrak{C}$. In order to prove that $\mathfrak{A}_1$ is the cover of $\mathfrak{A}$, we now have to show that $\mathfrak{A}_1 \not<\not< \mathfrak{A}$.

Let us first remark that the class of θ^*-equivalence containing x_{k+1} does not intersect with $\mathfrak{F}$. Conversely, if we had $< h(x_1,\dots,x_k), x_{k+1} > \in \theta^*$ for a certain $h(x_1,\dots,x_k)$, then, by the inequality $\theta^* \le \theta_1 \vee \theta_{x_1,x_{k+1}}$, we would have the inclusion

$$< h(x_1,\dots,x_k), x_{k+1} > \in \theta_1 \vee \theta_{x_1,x_{k+1}},$$

i.e.,

$$< x_1,h(x_1,\dots,x_k) > \in \theta_1 \vee \theta_{x_1,x_{k+1}}|\mathfrak{F}.$$

However, as has been noted earlier, $\theta_1 \vee \theta_{x_1,x_{k+1}}|\mathfrak{F} = \theta$ and, since $\theta \le \theta^*$, $< x_1,h(x_1,\dots,x_k) > \in \theta^*$. Thus, $< x_1,x_{k+1} > \in \theta^*$ contradicts the definition of θ^*. Hence, if φ is a natural homomorphism from $\mathfrak{F}_1$ to $\mathfrak{A}$, $\varphi(x_{k+1})$ does not belong to the subalgebra generated by the elements $\varphi(x_1),\dots,\varphi(x_k)$. Since the minimum number of elements generating the algebra $\mathfrak{A} \cong \varphi(\mathfrak{F})$ is k, the fact that the basis property is valid for $\mathfrak{M}$ yields that the minimum number of elements generating the algebra $\mathfrak{A}_1$ is $k+1$. Therefore, $\mathfrak{A}_1 \not<\not< \mathfrak{A}$, i.e., $\mathfrak{A}_1$ is the $\mathfrak{A}$ cover. ■

Let us now consider the problem of the existence of covers for congruence-distributive varieties. Let I be a certain semi-ideal in $< \mathfrak{J}\mathfrak{M}; << >$. The algebra $\mathfrak{A}_1$ will be called I-extendable in the algebra $\mathfrak{A}_2$ if $\mathfrak{A}_1 << \mathfrak{A}_2$ and there is an $\mathfrak{A}_3 \in I$ such that $\mathfrak{A}_1 \times \mathfrak{A}_3 \equiv_{<<} \mathfrak{A}_2$.

<u>**Lemma 9.1.**</u> If $\mathfrak{M}$ is a congruence-distributive variety, I is a semi-ideal in $< \mathfrak{J}\mathfrak{M}; << >$ closed under direct products of a finite number of algebras, for any $\mathfrak{M}$-algebra $\mathfrak{A}$ there is no more than one algebra (to the accuracy of $\equiv_{<<}$) which is

minimal relative to the quasi-order << among algebras I-extendable to $\mathfrak{A}$.

Proof. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be minimal among algebras I-extendable to $\mathfrak{A}$, i.e., there are algebras $\mathfrak{A}'$, $\mathfrak{A}'' \in I$ such that $\mathfrak{A} \equiv_{<<} \mathfrak{A}_1 \times \mathfrak{A}', \mathfrak{A} \equiv_{<<} \mathfrak{A}_2 \times \mathfrak{A}''$, and for any $\mathfrak{A}''', \mathfrak{A}'''' \in I$ and any algebras $\mathfrak{A}_3, \mathfrak{A}_4$ we get $\mathfrak{A}_1 << \mathfrak{A}_3$ from $\mathfrak{A}_3 << \mathfrak{A}_1$ if $\mathfrak{A} \equiv_{<<} \mathfrak{A}_3 \times \mathfrak{A}'''$, and we get $\mathfrak{A}_2 << \mathfrak{A}_4$ from $\mathfrak{A}_4 << \mathfrak{A}_2$ if $\mathfrak{A} \equiv_{<<} \mathfrak{A}_4 \times \mathfrak{A}''''$. Let us show that in this case we have $\mathfrak{A}_1 << \mathfrak{A}_2$ (by symmetrical consideration we then get $\mathfrak{A}_2 << \mathfrak{A}_1$, i.e., $\mathfrak{A}_1 \equiv_{<<} \mathfrak{A}_2$, and the lemma is thus proved). By theorem **4.2**, as has been noted in section **4**, in a congruence-distributive variety $\mathfrak{M}$ for any algebras $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathfrak{M}$ and any $\theta \in Con(\mathfrak{C}_1 \times \mathfrak{C}_2)$ there are $\theta_i \in Con\,\mathfrak{C}_i (i = 1,2)$ such that $\mathfrak{C}_1 \times \mathfrak{C}_2/\theta \cong \mathfrak{C}_1/\theta_1 \times \mathfrak{C}_2/\theta_2$. Since $\mathfrak{A}_2 \times \mathfrak{A}'' << \mathfrak{A}_1 \times \mathfrak{A}'$, $\mathfrak{A}_2 << \mathfrak{A}_1 \times \mathfrak{A}'$ and, according to what we have just discussed, there are $\mathfrak{A}_5 \mathfrak{A}^{v}$ such that $\mathfrak{A}_5 << \mathfrak{A}_1$, $\mathfrak{A}^{v} << \mathfrak{A}'$ and $\mathfrak{A}_2 \cong \mathfrak{A}_5 \times \mathfrak{A}^{v}$. As I is a semi-ideal closed under finite direct products, we get $\mathfrak{A}^{v}, \mathfrak{A}^{v} \times \mathfrak{A}'' \in I$. Therefore, $\mathfrak{A} \equiv_{<<} \mathfrak{A}_5 \times (\mathfrak{A}^{v} \times \mathfrak{A}''), \mathfrak{A}_5 << \mathfrak{A}_1$ and, since $\mathfrak{A}_1$ is minimal among the algebras I-extendable to $\mathfrak{A}$, we get $\mathfrak{A}_1 << \mathfrak{A}_5$. However, $\mathfrak{A}_5 << \mathfrak{A}_2$ and, hence, indeed, $\mathfrak{A}_1 << \mathfrak{A}_2$. ■

Lemma 9.2. If $\mathfrak{M}$ is a non-trivial congruence-distributive variety, then there is a well-ordered semi-ideal in $< \mathfrak{I}\mathfrak{M}; << >$ closed under direct products of a finite number of algebras and not a set itself.

Proof. Let *Ord* be the family of all ordinals. It should be recalled that for any linear order $< A; \leq >$ by $B < A; \leq >$ we mean an interval Boolean algebra constructed on the order $< A; \leq >$ (see section **1**). One can easily see that the family $\{B(\omega^{\alpha} \cdot n) | \alpha \in Ord, n \in \omega\}$ forms in $< \mathfrak{I}BA; << >$ a semi-ideal of the type presented in the formulation of the lemma. Let $\mathfrak{A}$ be a simple $\mathfrak{M}$-algebra. Then, by corollary **3.1**, we obviously get that $\{\mathfrak{A}^{B(\omega^{\alpha} \cdot n)} | \alpha \in Ord, n \in \omega\}$ is the semi-ideal in $< \mathfrak{I}\mathfrak{M}; << >$ with the properties required. ■

The semi-ideals with the properties presented in lemma **9.2** will be called *Ord*-chains. The same notation will be also used for the semi-ideals in $< \mathfrak{I}\mathfrak{M}; << >$ obeying the conditions of lemma **9.2** with the requirement of well-ordering replaced with that of the factor of the semi-ideal relative to $\equiv_{<<}$. It should be noticed that, by choosing quasi-simple $\mathfrak{M}$-algebras incomparable relative to << as $\mathfrak{A}$ in the proof of lemma **9.2**, we get different *Ord*-chains in $< \mathfrak{I}\mathfrak{M}; << >$ with the only common element $0_{\mathfrak{M}}$. Therefore, the number of different *Ord*-chains in $< \mathfrak{I}\mathfrak{M}; << >$ is not less than the number of pairwise $\equiv_{<<}$-non-equivalent minimal elements in $< \mathfrak{I}\mathfrak{M}; << >$.

Definition 9.9. The power of the maximal set of pairwise $\equiv_{<<}$-non-equivalent minimal elements in $<\mathfrak{J}\mathfrak{M};<<>$ will be called the initial width of the epimorphism skeleton of a variety $\mathfrak{M}$.

Theorem 9.7. If $\mathfrak{M}$ is a congruence-distributive variety, $\mathfrak{A}\in\mathfrak{M}$, and the principal semi-ideal generated by an element $\mathfrak{J}\mathfrak{A}$ in $<\mathfrak{J}\mathfrak{M}/\equiv_{<<};<<>$ is well-founded, $\mathfrak{A}$ has a cover in $<\mathfrak{J}\mathfrak{M};<<>$, and the number of such $\equiv_{<<}$-equivalent covers is not less then the initial width of $<\mathfrak{J}\mathfrak{M};<<>$.

Proof. Let I be any of *Ord*-chains in $<\mathfrak{J}\mathfrak{M};<<>$ existing by lemma **9.2**. Let $\mathfrak{A}_1$ be minimal among the algebras I-extendable to $\mathfrak{A}$. The algebra $\mathfrak{A}_1$ does exist since the principal semi-ideal generated by the element $\mathfrak{J}\mathfrak{A}$ in $<\mathfrak{J}\mathfrak{M}/\equiv_{<<};<<>$ is well-founded. Let $\mathfrak{A}=\mathfrak{A}_1\times\mathfrak{A}',\mathfrak{A}'\in I$. Let $\mathfrak{A}''$ be a minimal algebra in terms of the order $<<$ in the *Ord*-chain among the $\mathfrak{A}'''\in I$ such that $\mathfrak{A}\not\gg\mathfrak{A}_1\times\mathfrak{A}'''$. The existence of such $\mathfrak{A}'''$ is prompted by power considerations (I is a proper class, not a set), while the existence of a minimal $\mathfrak{A}''$ follows from the fact that I is well-ordered. Let us show that $\mathfrak{A}_1\times\mathfrak{A}''$ is the cover of $\mathfrak{A}$. Indeed, $\mathfrak{A}'<<\mathfrak{A}''$ and, hence, $\mathfrak{A}\cong\mathfrak{A}_1\times\mathfrak{A}'<<\mathfrak{A}_1\times\mathfrak{A}''$. By definition, $\mathfrak{A}_1\times\mathfrak{A}''\not<\not<\mathfrak{A}$. Let now $\mathfrak{C}\in\mathfrak{M}$ be such that $\mathfrak{A}<<\mathfrak{C}<<\mathfrak{A}_1\times\mathfrak{A}''$. By theorem **5.2**, for some $\mathfrak{A}_2<<\mathfrak{A}_1$ and $\mathfrak{A}''''<<\mathfrak{A}''$ we have $\mathfrak{C}\cong\mathfrak{A}_2\times\mathfrak{A}''''$. Since $\mathfrak{A}''''\in I$, and $\mathfrak{A}_1$ has been chosen minimal among I-extendable algebras, we get $\mathfrak{A}_2\equiv_{<<}\mathfrak{A}_1$. If $\mathfrak{A}''''$ is strictly $<<$-less than $\mathfrak{A}''$, $\mathfrak{C}<<\mathfrak{A}$, while in the opposite case $\mathfrak{A}''''\equiv_{<<}\mathfrak{A}''$ and, hence, $\mathfrak{C}>>\mathfrak{A}_1\times\mathfrak{A}''$. Therefore, $\mathfrak{A}_1\times\mathfrak{A}''$ is, indeed, the cover of $\mathfrak{A}$. Obviously, choosing different *Ord*-chains we get different covers of $\mathfrak{A}$. The statement of the theorem on the number of covers follows now from the remark on the number of different *Ord*-chains made after the proof of lemma **9.2**. ■

Corollary 9.2. If $\mathfrak{M}$ is a congruence-distributive variety, $\mathfrak{A}\in\mathfrak{M}$ and $(Con\,\mathfrak{A})^*$ is well-founded, $\mathfrak{A}$ has a cover in $<\mathfrak{J}\mathfrak{M};<<>$, and the number of such $\equiv_{<<}$-non-equivalent covers is not less than the initial width of $<\mathfrak{J}\mathfrak{M};<<>$. Here $(Con\,\mathfrak{A})^*$ is a lattice dual to the lattice $Con\,\mathfrak{A}$.

The proof results from the statement of theorem **9.7** and the fact that the principal semi-ideal generated by the element $\mathfrak{J}\mathfrak{A}$ in $<\mathfrak{J}\mathfrak{M};<<>$ will be, as a monotonous image (see **9.1**) of a well-founded lattice $(Con\,\mathfrak{A})^*$, well-founded itself.

Theorem 9.8. If $\mathfrak{M}$ is a congruence-distributive variety and $\mathfrak{A}$ is a subdirectly non-decomposable $\mathfrak{M}$-algebra, $\mathfrak{A}$ has a cover in $<\mathfrak{J}\mathfrak{M};<<>$.

Proof. A family of Boolean algebras of the type $B(\omega^{\alpha}\cdot n)$ forms, as has been noted earlier, an *Ord*-chain in $<\mathfrak{J}BA;<<>$. Let β be a monolith of the algebra $\mathfrak{A}$ and $B(\omega^{\alpha_1}\cdot n_1)$ be the least in the given *Ord*-chain among such $B(\omega^{\alpha}\cdot n)$ that $\mathfrak{A}(\beta)^{B(\omega^{\alpha}\cdot n)} \not\ll \mathfrak{A}$. The Boolean algebra $B(\omega^{\alpha_1}\cdot n_1)$ does exist since the *Ord*-chain is well-ordered and by virtue of power considerations. Let us show that $\mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}$ covers $\mathfrak{A}$. Indeed, it is obvious that $\mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)} >> \mathfrak{A}$ and, by the choice of $B(\omega^{\alpha_1}\cdot n_1)$, $\mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)} \not\ll \mathfrak{A}$. Let now $\mathfrak{C}$ be such that $\mathfrak{A} << \mathfrak{C} << \mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}$ and let $\theta \in Con(\mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)})$ be such that $\mathfrak{C} \cong \mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}/\theta$.

By corollary **4.1**, we get either $\theta < \beta^{B(\omega^{\alpha_1}\cdot n_1)}$ or $\theta \geq \beta^{B(\omega^{\alpha_1}\cdot n_1)}$. In the former case, by corollary **4.2**, there is a $\gamma \in B(\omega^{\alpha_1}\cdot n_1)$ such that

$$\mathfrak{C} \cong \mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}/\theta \cong \mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)/\gamma}.$$

However, $B(\omega^{\alpha_1}\cdot n_1)/\gamma$ is either isomorphic to $B(\omega^{\alpha_1}\cdot n_1)$ and then $\mathfrak{C} >> \mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}$, or $B(\omega^{\alpha_1}\cdot n_1)/\gamma$ is strictly $<<$-less than $B(\omega^{\alpha_1}\cdot n_1)$ and then, by the choice of $B(\omega^{\alpha_1}\cdot n_1)$, we get $\mathfrak{C} \cong \mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)} << \mathfrak{A}$. In the latter case we get $\theta >> \beta^{B(\omega^{\alpha_1}\cdot n_1)}$ and, since

$$\mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}/\beta^{B(\omega^{\alpha_1}\cdot n_1)} \cong \mathfrak{A}/\beta,$$

$$\mathfrak{C} \cong \mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}/\theta \cong \mathfrak{A}/\delta$$

for some $\delta \in Con\mathfrak{A}$, i.e., $\mathfrak{C} << \mathfrak{A}$. Hence, $\mathfrak{A}(\beta)^{B(\omega^{\alpha_1}\cdot n_1)}$ indeed covers $\mathfrak{A}$. ■

Let us consider one more algebra type having a cover in the epimorphism skeletons of congruence-distributive varieties.

Lemma 9.3. If $\mathfrak{M}$ is a congruence-distributive variety, $\aleph$ is an infinite cardinal, the algebra $\mathfrak{F}_{\mathfrak{M}}(\aleph)$ has a one-element algebra, $\mathfrak{F}_{\mathfrak{M}}(\aleph) \cong \mathfrak{A}_1 \times \mathfrak{A}_2$ and $\nabla_{\mathfrak{A}_2}$ is compact then $\mathfrak{A}_1 >> \mathfrak{F}_{\mathfrak{M}}(\aleph)$.

Proof. Let us assume that $\mathfrak{F}_{\mathfrak{M}}(\aleph) \cong \mathfrak{A}_1 \times \mathfrak{A}_2$. Let $\{e_\alpha | \alpha < \aleph\}$ be free-generating elements of $\mathfrak{F}_{\mathfrak{M}}(\aleph)$. By theorem **4.2**, any congruence on $\mathfrak{A}_1 \times \mathfrak{A}_2$ has the form $\theta_1 \times \theta_2$ where $\theta_i \in Con\mathfrak{A}_i$. This fact combined with the compactness of $\nabla_{\mathfrak{A}_2}$ in $Con\mathfrak{A}_2$ results in the compactness of the congruence $\Delta_{\mathfrak{A}_1} \times \nabla_{\mathfrak{A}_2}$ in $Con\mathfrak{F}_{\mathfrak{M}}(\aleph)$. Let $a \in \mathfrak{F}_{\mathfrak{M}}(\aleph)$ be such that $\{a\}$ is a subalgebra of $\mathfrak{F}_{\mathfrak{M}}(\aleph)$. Then, since

$$\nabla_{\mathfrak{F}_{\mathfrak{M}}(\aleph)} = \bigvee_{\alpha \in \aleph} \theta_{e_\alpha, a} \geq \Delta_{\mathfrak{A}_1} \times \nabla_{\mathfrak{A}_2},$$

there is a finite set $A = \{\alpha_1, \dots, \alpha_n\} \subseteq \aleph$ such that

$$\theta' = \bigvee_{\alpha \in A} \theta_{e_\alpha, a} \geq \Delta_{\mathfrak{A}_1} \times \nabla_{\mathfrak{A}_2}.$$

Let $\{a_1\}$ be a one-element subalgebra of the algebra generated by the elements $\{e_\alpha | \alpha < \aleph \setminus A\}$ in $\mathfrak{F}_{\mathfrak{M}}(\aleph)$. Then we get

$$\theta'' = \bigvee_{\alpha \in A} \theta_{e_\alpha, a_1} \vee \theta_{a, a_1} \geq \Delta_{\mathfrak{A}_1} \times \nabla_{\mathfrak{A}_2}.$$

As has been noticed earlier, there are θ_1, θ_2 belonging to $Con\mathfrak{A}_1, Con\mathfrak{A}_2$, respectively such that $\theta'' = \theta_1 \times \theta_2$, in which case $\theta_2 \geq \nabla_{\mathfrak{A}_2}$. Then we get

$$\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta'' = \mathfrak{A}_1 \times \mathfrak{A}_2 / \theta_1 \times \theta_2 \cong \mathfrak{A}_1/\theta_1.$$

Since $\aleph$ is infinite, $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta'' \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)$, i.e., $\mathfrak{A}_1/\theta_1 \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)$ and, hence, $\mathfrak{A}_1 >> \mathfrak{F}_{\mathfrak{M}}(\aleph)$. ■

Theorem 9.9. If $\mathfrak{M}$ is a congruence-distributive variety with all its algebras having one-element subalgebras, any $\mathfrak{M}$-free algebra with an infinite number of generating elements has a cover in $< \mathfrak{I}\mathfrak{M}; << >$, and the number of such $\equiv_{<<}$-non-equivalent covers is not less than the initial width of $< \mathfrak{I}\mathfrak{M}; << >$.

Proof. Let $\mathfrak{A}$ be a simple $\mathfrak{M}$-algebra, $\aleph$ an infinite cardinal, $\aleph^+$ the cardinal succeeding $\aleph$, and $\mathcal{B}_{\aleph^+}$ a Boolean Frechet algebra over a family of atoms of the power $\aleph^+$, i.e., a Boolean algebra of subsets of a set of the power $\aleph^+$ generated by one-element subsets. Let us show that $\mathfrak{F}_{\mathfrak{M}}(\aleph) \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}}$ covers $\mathfrak{F}_{\mathfrak{M}}(\aleph)$. Indeed, owing to the power considerations, we get

$$\mathfrak{F}_{\mathfrak{M}}(\aleph) << \mathfrak{F}_{\mathfrak{M}}(\aleph) \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}},$$

$$\mathfrak{F}_{\mathfrak{M}}(\aleph) \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}} \not<\not< \mathfrak{F}_{\mathfrak{M}}(\aleph).$$

Let now $\mathfrak{C}$ be such that

$$\mathfrak{F}_{\mathfrak{M}}(\aleph) << \mathfrak{C} << \mathfrak{F}_{\mathfrak{M}}(\aleph) \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}}.$$

As has been repeatedly noted in the proofs of the previous theorems, there are $\theta_1 \in Con\,\mathfrak{F}_{\mathfrak{M}}(\aleph)$, $\theta_2 \in Con\,\mathfrak{A}^{\mathcal{B}_{\aleph^+}}$ such that

$$\mathfrak{C} \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1 \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}}/\theta_2.$$

For any $\gamma \in Con\,\mathcal{B}_{\aleph^+}$ we get either $\mathcal{B}_{\aleph^+}/\gamma \cong \mathcal{B}_{\aleph^+}$ or $|\mathcal{B}_{\aleph^+}/\gamma| \le \aleph$. By corollary **3.1**, there is a $\delta \in Con\,\mathcal{B}_{\aleph^+}$ such that $\mathfrak{A}^{\mathcal{B}_{\aleph^+}}/\theta_2 \cong \mathfrak{A}^{\mathcal{B}_{\aleph^+}/\delta}$. If $|\mathcal{B}_{\aleph^+}/\gamma| \le \aleph$,

$$|\mathfrak{C}| = |\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1 \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}}/\theta_2| \le \aleph$$

and, hence, $\mathfrak{C} << \mathfrak{F}_{\mathfrak{M}}(\aleph)$. Therefore, the case $\mathfrak{C} \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1 \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}} >> \mathfrak{F}_{\mathfrak{M}}(\aleph)$ remains to be considered. And again, as above, there are $\theta_3 \in Con\,\mathfrak{F}_{\mathfrak{M}}(\aleph)$, $\theta_3 \ge \theta_1$ and $\theta_4 \in Con\,\mathfrak{A}^{\mathcal{B}_{\aleph^+}}$ such that

$$\mathfrak{F}_{\mathfrak{M}}(\aleph) \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_3 \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}}/\theta_4.$$

However, $\nabla_{\mathfrak{A}^{\mathcal{B}_{\aleph^+}}}$ is compact and, hence, compact is $\nabla_{\mathfrak{A}^{\mathcal{B}_{\aleph^+}}/\theta_4}$.

Therefore, by lemma **9.3**, $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_3 >> \mathfrak{F}_{\mathfrak{M}}(\aleph)$, Since $\theta_1 \le \theta_3$, $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1 >> \mathfrak{F}_{\mathfrak{M}}(\aleph)$ and, hence,

$$\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1 \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}} >> \mathfrak{F}_{\mathfrak{M}}(\aleph) \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}},$$

which completes the proof that $\mathfrak{F}_{\mathfrak{M}}(\aleph) \times \mathfrak{A}^{\mathcal{B}_{\aleph^+}}$ is a cover of $\mathfrak{F}_{\mathfrak{M}}(\aleph)$. By choosing

$\mathfrak{A}$ to be various pairwise $\equiv_{<<}$-non-equivalent minimal elements from $< \mathfrak{J}\mathfrak{M}; << >$, obviously, we get pairwise $\equiv_{<<}$-non-equivalent covers of the algebra $\mathfrak{F}_{\mathfrak{M}}(\aleph)$. ■

Let us recall an obvious result establishing a relationship between covers in epimorphism skeletons of congruence-distributive varieties and quasi-simple algebras: if $\mathfrak{M}$ is congruence-distributive, and there are quasi-simple algebras of any great power in $\mathfrak{M}$, then any $\mathfrak{M}$-algebra $\mathfrak{A}$ has a cover in the epimorphism skeleton of $\mathfrak{M}$. Indeed, it suffices to choose an algebra $\mathfrak{A} \times \mathfrak{A}_1$ as such a cover where $\mathfrak{A}_1$ is a quasi-simple $\mathfrak{M}$-algebra of a power greater than $|\mathfrak{A}|$. This, in particular, entails that in any variety $\mathfrak{M}$ of lattice-ordered groups any $\mathfrak{M}$-group has a cover in $< \mathfrak{J}\mathfrak{M}; << >$. Indeed, by bringing to a linear antilexicographic order any group of the type $\prod^*_{i \in \omega^\alpha} G_i$ where $\prod^*_{i \in \omega^\alpha} G_i$ is a direct sum of the groups G_i, and G_i are equal to $Z \times R$ for any i (here $Z \times R$ is a lexicographic product of linearly ordered groups of integer and real numbers), we get quasi-simple linearly ordered groups lying in any variety of lattice-ordered groups.

The following problem is now open for discussion.

Problem 9.2. Does any algebra of a congruence-distributive variety have a cover in the epimorphism skeleton of this variety ?

The results discussed above make it possible to expect a positive answer. If, however, we consider relative covers, i.e., those not in the whole skeleton of a variety but in a prefixed interval within this skeleton such covers might not exist. Before we prove this to be the case, let us obtain a number of statements on the structure of epimorphism skeletons of arbitrary non-trivial congruence-distributive varieties which are proved using the constructions of Boolean powers and congruence-Boolean powers, and are analogous to the statements for Boolean algebras in Chapter 1.

It should be recalled that the relation $\mathfrak{A}_1 \leq \mathfrak{A}$ between algebraic systems $\mathfrak{A}_1, \mathfrak{A}$ implies that $\mathfrak{A}_1$ is isomorphically embeddable into $\mathfrak{A}$. 2^A will denote the set of all subsets of the set A. For an arbitrary set of Boolean algebras $\mathfrak{B}_i (i \in I)$, $\sum_{i \in I} \mathfrak{B}_i$ will denote a subalgebra of the algebra $\prod_{i \in I} \mathfrak{B}_i$ generated by those elements f from $\prod_{i \in I} \mathfrak{B}_i$ for which $f(i) \neq 0$ only for a finite number i from I.

Lemma 9.4. If $\mathfrak{M}$ is a congruence-distributive variety, $\mathfrak{A} \in \mathfrak{M}$, and $\aleph$ is a regular cardinal, $\aleph > |\mathfrak{A}|, \aleph_0$, there is an $\mathfrak{M}$-algebra $\mathfrak{A}(\aleph)$ such that $|\mathfrak{A}(\aleph)| = \aleph$ and $\langle [\mathfrak{I}\mathfrak{A}, \mathfrak{I}\mathfrak{A}(\aleph)]/\equiv_{<<}; << \rangle \geq \langle 2^{\aleph}; \subseteq \rangle$. If $\mathfrak{A}$ is subdirectly non-decomposable, $\mathfrak{A}(\aleph)$ meets an additional condition:

$$[0_{\mathfrak{M}}, \mathfrak{I}\mathfrak{A}(\aleph)] = [0_{\mathfrak{M}}, \mathfrak{I}\mathfrak{A}) \oplus [\mathfrak{I}\mathfrak{A}, \mathfrak{I}\mathfrak{A}(\aleph)].$$

Proof. According to the statement of theorem **1.17**, there are Boolean algebras $\mathcal{B}_i$ of the power $\aleph (i \in \aleph)$ with the following properties:

(a) for any $i \in \aleph$, any $a \in \mathcal{B}_i \setminus \{0\}$ $|\{b \in \mathcal{B}_i | b \leq a\}| = \aleph$;

(b) for any $i \neq j \in \aleph$ and any $a \in \mathcal{B}_i \setminus \{0\}$, $b \in \mathcal{B}_j \setminus \{0\}$ $\mathcal{B}_j | b \not<\not< \mathcal{B}_i | a$.

By defining a Boolean algebra $\mathcal{B}_I$ $(I \subseteq \aleph)$ as $\sum_{i \in I} \mathcal{B}_I$, we come to the obvious conclusion that for $I_1, I_2 \subseteq \aleph$, the relation $\mathcal{B}_{I_1} << \mathcal{B}_{I_2}$ is valid iff $I_1 \subseteq I_2$.

Let $\mathfrak{A}$ be an arbitrary $\mathfrak{M}$-algebra, and $\mathfrak{A}_0$ an arbitrary simple $\mathfrak{M}$-algebra. $\mathfrak{A}(I)$ will denote the algebra $\mathfrak{A}_0^{\mathcal{B}_I} \times \mathfrak{A}$. For any $I_1 \subseteq I_2 \subseteq \aleph$, we obviously get $\mathfrak{A} << \mathfrak{A}(I_1)$ and $\mathfrak{A}(I_1) << \mathfrak{A}(I_2) << \mathfrak{A}(\aleph)$. Let us show that $\mathfrak{A}(I_1) << \mathfrak{A}(I_2)$ implies the inclusion $I_1 \subseteq I_2$. The relation $\mathfrak{A}(I_1) << \mathfrak{A}(I_2)$ implies the relation $\mathfrak{A}_0^{\mathcal{B}_{I_1}} << \mathfrak{A}(I_2)$, and let $\alpha \in Con\mathfrak{A}(I_2)$ be such that $\mathfrak{A}(I_2)/\alpha \cong \mathfrak{A}_0^{\mathcal{B}_{I_1}}$. Since $\mathfrak{M}$ is congruence-distributive, there can be found $\alpha_1 \in Con\mathfrak{A}_0^{\mathcal{B}_{I_2}}$, $\alpha_2 \in Con\mathfrak{A}$ such that

$$\mathfrak{A}_0^{\mathcal{B}_{I_2}}/\alpha_1 \times \mathfrak{A}/\alpha_2 \cong \mathfrak{A}_0^{\mathcal{B}_{I_1}}.$$

However, $Con_p\mathfrak{A}_0^{\mathcal{B}_{I_1}} \cong \mathcal{B}_{I_1}$, and

$$Con_p[(\mathfrak{A}_0)^{\mathcal{B}_{I_2}}/\alpha_1 \times \mathfrak{A}/\alpha_2] \cong \mathcal{B}_{I_2}/\gamma \times Con_p\mathfrak{A}/\alpha_2,$$

where γ is a congruence of the algebra $\mathcal{B}_{I_2}$. At the same time, as has been noticed earlier, for any $a \in \mathcal{B}_{I_1} \setminus \{0\}$ we have $|\{b \in \mathcal{B}_{I_1} | b \leq a\}| = \aleph$, while for $\delta \in Con_p\mathfrak{A}/\alpha_2$ we get $|\{\gamma \in Con_p\mathfrak{A}/\alpha_2 | \gamma \leq \delta\}| \leq |\mathfrak{A}| + \aleph_0 < \aleph$. Therefore, the congruence α_2 must be equal to $\nabla_{\mathfrak{A}}$. Hence, we get

$$\mathfrak{A}(I_2)/\alpha \cong \mathfrak{A}_0^{\mathcal{B}_{I_2}}/\alpha_1 \cong \mathfrak{A}_0^{\mathcal{B}_{I_1}}$$

and, since $Con_p\mathfrak{A}_0^{\mathcal{B}_{I_1}} \cong \mathcal{B}_{I_1}$, $Con_p\mathfrak{A}_0^{\mathcal{B}_{I_2}} \cong \mathcal{B}_{I_2}$, and $\mathfrak{M}$ is congruence-distributive, $\mathcal{B}_{I_1} << \mathcal{B}_{I_2}$. According to the remark we have made earlier, this implies the inclusion $I_1 \subseteq I_2$, which proves that

$$< 2^{\aleph};\subseteq> \leq < [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{A}(\aleph)] \equiv_{<<}; << >.$$

If $\mathfrak{A}$ is a subdirectly non-decomposable algebra and θ is its monolith, for $I \subseteq \aleph$ we define the algebras $\mathfrak{A}(I)$ to be equal to $\mathfrak{A}(\theta)^{\mathcal{B}_I}$. The inequality $< 2^{\aleph};\subseteq> \leq < [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{A}(\aleph)] \equiv_{<<}; << >$ follows from the fact that $I_1 \subseteq I_2 \subseteq \aleph$ iff $\mathcal{B}_{I_1} << \mathcal{B}_{I_2}$, while the latter, by corollary **4.2**, is valid iff

$$\mathfrak{A}(I_1) = \mathfrak{A}(\theta)^{\mathcal{B}_{I_1}} << \mathfrak{A}(\theta)^{\mathcal{B}_{I_2}} = \mathfrak{A}(I_2).$$

The equality $[0_{\mathfrak{M}}, \mathfrak{J}\mathfrak{A}(\aleph)] = [0_{\mathfrak{M}}, \mathfrak{J}\mathfrak{A}) \oplus [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{A}(\aleph)]$ obviously results from the fact that $[0_{\mathfrak{M}}, \mathfrak{J}\mathfrak{A}(\aleph)]$ is an antiisotonic image of the lattice $Con\mathfrak{A}(\aleph)$ which, by corollary **4.1**, equals $Con\mathfrak{A}(\aleph)|\leq \theta^{\mathcal{B}_\aleph} \oplus Con\mathfrak{A}|> \theta$. ■

Lemma **9.4** prompts that the epimorphism skeletons of non-trivial congruence-distributive varieties in the class of all quasi-ordered sets are universal.

Theorem 9.10. For any non-trivial congruence-distributive variety $\mathfrak{M}$, any regular cardinal $\aleph > \aleph_0$ and any algebra $\mathfrak{A}$ of a power less than $\aleph$, an arbitrary quasi-ordered set of a power not greater than $\aleph$ is isomorphically embeddable into $< \mathfrak{J}\mathfrak{M}_\aleph|>> \mathfrak{J}\mathfrak{A}; << >$. In particular, for a non-trivial congruence-distributive variety $\mathfrak{M}$, any quasi-ordered set is isomorphically embeddable into $< \mathfrak{J}\mathfrak{M}; << >$.

Proof. Since any partially ordered set of the power $\leq \aleph$ is isomorphically embeddable into $< 2^{\aleph};\subseteq>$, by lemma **9.4**, any such set is isomorphically embeddable into $< \mathfrak{J}\mathfrak{M}_\aleph|>> \mathfrak{J}\mathfrak{A}; << >$. For $\alpha < \aleph$ let $\mathcal{B}^\alpha = B((\omega^{\alpha+\omega} + \eta)\cdot\aleph)$ (here, as in Chapter **1**, η is the ordinal type of rational numbers). One can easily see that $\beta \neq \alpha < \aleph$ $\mathcal{B}^\alpha \not\cong \mathcal{B}^\beta$, but $\mathcal{B}^\alpha \equiv_{<<} \mathcal{B}^\beta$, and the powers of all Boolean algebras $\mathcal{B}^\alpha$ are equal to $\aleph$. One can also see that for any $I \subseteq \aleph$ we have $\mathcal{B}_I \times \mathcal{B}^\alpha \not\cong \mathcal{B}_I \times \mathcal{B}^\beta$, but in this case $\mathcal{B}_I \times \mathcal{B}^\alpha \equiv_{<<} \mathcal{B}_I \times \mathcal{B}^\beta$. Besides, for $I_1, I_2 \subseteq \aleph$, $\mathcal{B}_{I_1} \times \mathcal{B}^\alpha << \mathcal{B}_{I_1} \times \mathcal{B}^\beta$ iff $I_1 \subseteq I_2$. Indeed, if $\mathcal{B}_{I_1} \times \mathcal{B}^\alpha << \mathcal{B}_{I_2} \times \mathcal{B}^\beta$, $\mathcal{B}_{I_1} << \mathcal{B}_{I_2} \times \mathcal{B}^\beta$. If f is a homomorphism from $\mathcal{B}_{I_2} \times \mathcal{B}^\beta$ to $\mathcal{B}_{I_1}$, and π_2 is a projection of $\mathcal{B}_{I_2} \times \mathcal{B}^\beta$ to $\mathcal{B}^\beta$ then, for the case when $\ker f \supseteq \ker \pi_2$, we get the inequality $\mathcal{B}_{I_1} << \mathcal{B}_{I_2}$ which, by lemma **9.4**, yields the inclusion $I_1 \subseteq I_2$.

If $\ker f \not\supseteq \ker \pi_2$, $\mathcal{B}_{I_1} \cong \mathcal{B}_{I_2}/\delta \times \mathcal{B}^{\beta}/\varphi$ for some $\delta \in Con\mathcal{B}_{I_2}, \varphi \in Con\mathcal{B}^{\beta} \setminus \{\nabla\}$. In the algebra $\mathcal{B}^{\beta}/\varphi$ there are always elements a such that $|\mathcal{B}^{\beta}/\varphi| \leq a| \leq \aleph_0$, while in the algebra $\mathcal{B}_{I_1}$ there are no such elements, i.e., for any element $a \in \mathcal{B}_{I_1} \setminus \{0\}$, $|\{b \in \mathcal{B}_{I_1} | b \leq a\}| = \aleph$. Therefore the case $\ker f \not\supseteq \ker \pi_2$ is impossible and, indeed, the inequality $\mathcal{B}_{I_1} \times \mathcal{B}^{\alpha} << \mathcal{B}_{I_2} \times \mathcal{B}^{\beta}$ is equivalent to the inclusion $I_1 \subseteq I_2$. Using algebras of the type $\mathcal{B}_I \times \mathcal{B}^{\alpha}(I \subseteq \aleph, \alpha < \aleph)$ instead of the algebras $\mathcal{B}_I$ in the construction of lemma **9.4**, we get an embedding into $< \mathfrak{J}\mathfrak{M}_{\aleph}|>> \mathfrak{J}\mathfrak{A};<<>$ of a quasi-ordered set obtained from the partial order $< 2^{\aleph};\subseteq>$ by "smearing" every element into the class consisting of $\aleph$ elements pairwise equivalent in terms of the quasi-order. This implies not only that any partial order of the power not greater than $\aleph$ is embeddable into $< 2^{\aleph};\subseteq>$, but also that any quasi-order of the power not greater than $\aleph$ is isomorphically embeddable into $< \mathfrak{J}\mathfrak{M}_{\aleph}|>> \mathfrak{J}\mathfrak{A};<<>$. ■

In connection with lemma **9.4** the problem arises whether there are intervals $\equiv_{<<}$-equivalent to the partially ordered set $< 2^{\aleph};\subseteq>$ in the epimorphism skeletons of congruence distributive varieties $\mathfrak{M}$. Though this problem remains open to discussion in the general case, it appears possible to prove the existence of such intervals under certain circumstances.

For a finite set of the algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$ and their homomorphisms $f_1,\ldots,f_n$ to the same fixed algebra $\mathfrak{A}$, let us define an algebra $\prod_{i\leq n} < \mathfrak{A}_i, f_i >$ as the subdirect product of the algebras $\mathfrak{A}_i (i \leq n)$ with a basic set $\{g \in \prod_{i\in I} \mathfrak{A}_i |$ for $i,j \leq n$ $f_i(g(i)) = f_j(g(j))\}$. Let θ_i be kernels of the homomorphisms f_i. If $\alpha_i \in Con\mathfrak{A}_i$, then $\prod_{i\leq n} \alpha_i$ will denote the following congruence on $\prod_{i\leq n} < \mathfrak{A}_i, f_i >$: for $g,h \in \prod_{i\leq n} < \mathfrak{A}_i, f_i >$ we get $< g,h > \in \prod_{i\leq n} \mathfrak{A}_i$ iff for any $i \leq n$ $< g(i),h(i) > \in \alpha_i$. One can directly check that for any $g,h \in \prod_{i\leq n} < \mathfrak{A}_i, f_i >$ if $< g(i),h(i) > \in \theta_i$ for some $i \leq n$, then for any $j \leq n$ we get $< g(j),h(j) > \in \theta_j$ and, thus, $\prod_{i\leq n} < \mathfrak{A}_i, f_i > / \prod_{i\leq n} \theta_i \cong \mathfrak{A}$. In the case when all the algebras $\mathfrak{A}_i$ belong to a certain congruence-distributive variety, we get, analogously to the proof of theorem **4.2**, that for any congruence $\theta \in Con \prod_{i\leq n} < \alpha_i, f_i >$ less than $\prod_{i\leq n} \theta_i$, there are $\theta_i' \leq \theta_i$-congruences of the algebras $\mathfrak{A}_i$ such that $\theta = \prod_{i\leq n} \theta_i'$.

These remarks prove the following statement.

Lemma 9.5. If $\mathfrak{M}$ is a congruence-distributive variety, $\mathfrak{A}_1,\dots,\mathfrak{A}_n \in \mathfrak{M}$ and $f_i (i \le n)$ are homomorphisms from the algebras $\mathfrak{A}_i$ to a certain fixed algebra $\mathfrak{A}$,

$$Con\prod_{i\le n} < \mathfrak{A}_i, f_i > \rceil \le \prod_{i\le n} \theta_i \cong \prod_{i\le n} Con\,\mathfrak{A}_i \rceil \le \theta_i ,$$

and $\prod_{i\le n} < \mathfrak{A}_i, f_i > \Big/ \prod_{i\le n} \theta_i \cong \mathfrak{A}$. If in this case all θ_i are intercepting, we get

$$Con\prod_{i\le n} < \mathfrak{A}_i, f_i > \cong \prod_{i\le n} Con\,\mathfrak{A}_i \rceil \le \theta_i \oplus (Con\,\mathfrak{A} \setminus \{\Delta\}) .$$

If, moreover, for any $i \ne j \le n$ and $\theta' \le \theta_i, \theta'' \le \theta_j$ it follows from $\mathfrak{A}_i/\theta' \not\cong \mathfrak{A}$ and $\mathfrak{A}_j/\theta'' \not\cong \mathfrak{A}$ that $\mathfrak{A}_i/\theta' \not\cong \mathfrak{A}_j/\theta''$, then

$$< [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\prod_{i\le n} < \mathfrak{A}_i, f_i >] \Big/ \equiv_{<<};\ <<> \cong \prod_{i\le n} < [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{A}_i] \Big/ \equiv_{<<}; <<> .$$

An algebra $\mathfrak{A}_1$ that is a cover of the algebra $\mathfrak{A}$ in the epimorphism skeleton of the variety $\mathfrak{M}$ will be called a strong cover of $\mathfrak{A}$ provided that there is an intercepting congruence θ of the algebra $\mathfrak{A}_1$ such that $\mathfrak{A}_1/\theta \cong \mathfrak{A}$.

Theorem 9.11.

(a) If in the epimorphism skeleton of a variety $\mathfrak{M}$ the algebra $\mathfrak{A}$ has $k < \aleph$ pairwise $\equiv_{<<}$-non-equivalent covers, there is an algebra $\mathfrak{A}' \in \mathfrak{M}$ such that

$$< [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{A}'] / \equiv_{<<}; <<> \cong < 2^k; \subseteq > .$$

(b) If for a countable algebra $\mathfrak{A}$ in a congruence-distributive variety $\mathfrak{M}$ there are $k < \aleph_0$ pairwise $\equiv_{<<}$-non-equivalent subdirectly non-decomposable algebras $\mathfrak{C}_i (i < k)$ with monoliths θ_i such that $\mathfrak{C}_i/\theta_i \cong \mathfrak{A}$, there is an algebra $\mathfrak{A}' \in \mathfrak{M}$ such that

$$< [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{A}'] / \equiv_{<<}; <<> \cong \omega_1 \times \dots \times \omega_1 \quad (k \text{ times}).$$

The statement (a) of the lemma under consideration follows immediately from the statement of lemma **9.5**, since in this case $< [\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{A}'] / \equiv_{<<}; <<> \cong 2$. To prove the statement (b) it suffices to replace the algebras $\mathfrak{A}_i$ in lemma **9.5** with the algebras $\mathfrak{C}_i^{B(\omega_1)}(\theta_i)$, since

$$<[\mathfrak{Z}\mathfrak{A}, \mathfrak{Z}\mathfrak{C}_i^{B(\omega_1)}(\theta_i)]/\equiv_{<<};<<>\cong \omega_1. \blacksquare$$

The definition of a non-compactable chain has been given in section **1**. The following statement shows that in the epimorphism skeletons of non-trivial congruence-distributive varieties there are non-compactable chains of both discrete and dense ordinal types. It should be recalled that r denotes the ordinal type of real numbers.

Theorem 9.12. Let $\mathfrak{M}$ be a non-trivial congruence-distributive variety, $\mathfrak{A}$ be a subdirectly non-decomposable $\mathfrak{M}$ -algebra. In this case the following statements are valid:

(a) in the epimorphism skeleton of $\mathfrak{M}$ there is a non-compactable chain isomorphic to an ordered class of ordinals such that its lower bound in $<\mathfrak{Z}\mathfrak{M};<<>$ is equal to $\mathfrak{Z}\mathfrak{A}$;

(b) (CH) in the epimorphism skeleton of $\mathfrak{M}$ there is a non-compactable chain B such that $<B/\equiv_{<<};<<>$ is isomorphic to an ordered set of real numbers.

Proof. Let $\mathfrak{A}$ be a subdirectly non-decomposable $\mathfrak{M}$ -algebra. By corollaries **4.1** and **4.2**, a family of the types of isomorphism of $\mathfrak{M}$ -algebras of the type $\mathfrak{A}(\theta)^{B(\alpha)}$, where $\alpha \in Ord$ and θ is the monolith of $\mathfrak{A}$, forms a non-compactable chain isomorphic to the ordered class Ord. On the other hand, according to the same corollaries **4.1** and **4.2** and by theorem **1.15**, a family of the types of isomorphisms of algebras of the type $\mathfrak{A}_0(\theta)^{\mathfrak{B}}$, where $\mathfrak{B}$ runs the non-compactable chain in $<\mathfrak{Z}BA;<<>$ considered in theorem **1.15**, and $\mathfrak{A}_0$ is an arbitrary countable subdirectly non-decomposable algebra, forms itself a similar chain isomorphic to a set of real numbers in the epimorphism skeleton of $\mathfrak{M}$. $\blacksquare$

Let us now show that in the epimorphism skeleton of any non-trivial congruence-distributive variety there are algebras having no relative covers.

Theorem 9.13.(CH) If $\mathfrak{M}$ is a non-trivial congruence-distributive variety, there are $\mathfrak{M}$ -algebras $\mathfrak{A}_1, \mathfrak{A}_2$ such that $\mathfrak{A}_1 << \mathfrak{A}_2$, $\mathfrak{A}_2 \not<\not< \mathfrak{A}_1$, and the algebra $\mathfrak{A}_1$

has no covers lying in the interval $[\mathfrak{J}\mathfrak{A}_1, \mathfrak{J}\mathfrak{A}_2]$ of the epimorphism skeleton of $\mathfrak{M}$.

Proof. Let $\mathfrak{A}$ be a countable subdirectly non-decomposable $\mathfrak{M}$-algebra with a monolith θ. In the proof of theorem **14.2** (theorem **1.15**) a set of real numbers P and Boolean algebras $B(P_\alpha)$ are constructed for any real number α. Let us fix a certain pair of real numbers $\beta_1 < \beta_2$, and let

$$\mathcal{B}_1 = B(P_{\beta_1}) \times B(\eta), \mathcal{B}_2 = B(P_{\beta_2}) \times B(\eta).$$

As $\mathfrak{A}_1, \mathfrak{A}_2$ let us consider algebras $\mathfrak{A}(\theta)^{\mathcal{B}_1}, \mathfrak{A}(\theta)^{\mathcal{B}_2}$, respectively. From the properties of the algebras $B(P_\alpha)$ considered in the proof of theorem **1.15**, it is obvious that $\mathfrak{A}_1 << \mathfrak{A}_2$ and $\mathfrak{A}_2 \not\ll \mathfrak{A}_1$. Let us show that $\mathfrak{A}_1$ has no covers in the interval $[\mathfrak{J}\mathfrak{A}_1, \mathfrak{J}\mathfrak{A}_2]$. Indeed, let $\mathfrak{A}_3 \in \mathfrak{M}$ such that $\mathfrak{A}_1 << \mathfrak{A}_3 << \mathfrak{A}_2$, $\mathfrak{A}_3 \not\ll \mathfrak{A}_1$. By corollary **4.2**, the algebra $\mathfrak{A}_3$ has the form $\mathfrak{A}(\theta)^{\mathfrak{C}}$, where the Boolean algebra $\mathfrak{C}$ is such that $B(P_{\beta_1}) \times B(\eta) << \mathfrak{C} << B(P_{\beta_2}) \times B(\eta)$ and $\mathfrak{C} \not\ll B(P_{\beta_1}) \times B(\eta)$. As long as $\mathfrak{C} << B(P_{\beta_2}) \times B(\eta)$, as is noted in the proof of theorem **14.2**, we get $\mathfrak{C} \cong B(P_{\beta_1} \oplus 1 \oplus D)$ for some $D \subseteq P \cap (\beta_1, \beta_2) \oplus \eta$ and, hence, $\mathfrak{C} \cong B(P_{\beta_1}) \times B(D)$. The set D cannot be countable, since otherwise we would have $B(D) << B(\eta)$ and $\mathfrak{C} << \mathcal{B}_1$, which contradicts the above deductions. By assuming CH, $|D| = 2^{\aleph_0}$ and, therefore, there is a $\delta \in (\beta_1, \beta_2)$ such that $|D \cap (\beta_1, \delta| = |D \cap (\delta, \beta_2)| = 2^{\aleph_0}$. Let $\mathcal{B}'$ denote a Boolean algebra $B(P_{\beta_1} \oplus 1 \oplus (D \cap (\beta_1, \delta))) \times B(\eta)$. Obviously, $\mathcal{B}_1 << \mathcal{B}' << \mathfrak{C}$ but, on the other hand, considerations analogous to those in the proof of theorem **14.2** show that $\mathcal{B}' \not\ll \mathcal{B}_1$ and $\mathfrak{C} \not\ll \mathcal{B}'$. In this case, by corollary **4.2**, we get $\mathfrak{A}_1 << \mathfrak{A}^{\mathcal{B}'} << \mathfrak{A}_3$, $\mathfrak{A}^{\mathcal{B}'} \not\ll \mathfrak{A}_1$ and $\mathfrak{A}_3 \not\ll \mathfrak{A}^{\mathcal{B}'}$. ■

<u>**Definition 9.10.**</u> A pair of algebras $\mathfrak{A}_1, \mathfrak{A}_2$ is said totally disjunct if for any algebra $\mathfrak{A}$ such that $\mathfrak{A} << \mathfrak{A}_1, \mathfrak{A}_2$ the power of the algebra $\mathfrak{A}$ is strictly less that the powers of the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

<u>**Theorem 9.14.**</u> (GCH) Let $\mathfrak{M}$ be a non-trivial congruence-distributive variety. Then for any non-limit cardinal $\aleph$ there is a family of the power $\aleph$ consisting of totally disjunct $\mathfrak{M}$-algebras of the power $\aleph$.

Proof. In line with theorem **1.14**, there is a similar family G consisting of Boolean algebras. Let $\mathfrak{A}$ be a countable simple $\mathfrak{M}$-algebra. By corollary **3.1**, the family of $\mathfrak{M}$-algebras $\mathfrak{A}^{\mathcal{B}}$ $(\mathcal{B} \in G)$ has the required properties.

Priorities. Studies of epimorphism skeletons of various classes of algebraic systems with no special definition introduced have been carried out in a number of papers by many authors. For instance, C.Landraitis [121] studied the epimorphism skeleton of countable linearly ordered sets. The epimorphism skeleton of uncountable linearly ordered sets was investigated by A.G.Pinus [162]. A number of results pertaining to skeletons of the class of ordered sets belonging basically to the French school were presented in a monograph by R.Fräisse [69].

The notion of an epimorphism skeleton of algebras has been introduced by A.G.Pinus [178]. Pseudo-simple algebras have been studied in papers by H.Andrèka, I.Nèmeti [3], D.Monk [40], Szelpal [222]. Theorem **9.1** was, in particular, presented in a paper by D.Monk [140]. The notion of a quasi-simple algebra has been introduced in a paper by A.G.Pinus [171], where he also proved theorems **9.2**, **9.3**, **9.4** and **9.5**. Theorems **9.6** - **9.9** also belong to A.G.Pinus [168], as well as theorems **9.10** - **9.13** [169].

10. Countable Epimorphism Skeletons of Discriminator Varieties

According to theorem **9.11** of the preceding section, limited epimorphism skeletons $<\mathfrak{Z}\mathfrak{M}_{\aleph};<<>$ of any non-trivial variety $\mathfrak{M}$ are, when $\aleph$ is uncountable, universal in the class of all quasi-ordered sets of the power $\leq\aleph$. The situation changes when considering $\aleph=\aleph_0$. Indeed, as has been noted earlier in section **1**, the countable epimorphism skeleton of a variety of Boolean algebras is equal to $\omega_1\oplus 1^*$, where the order of $\omega_1\oplus 1^*$ is obtained by adding to the set ω_1 of all countable ordinals (isomorphism types of countable superatomic Boolean algebras) a continuum of elements pairwise equivalent in terms of the quasi-order as the latter (isomorphism types of countable non-superatomic Boolean algebras). The factor-order of the quasi-ordered set $\omega_1\oplus 1^*$ is linear as regards the natural relation of equivalence.

Definition 10.1. A quasi-ordered set $<A;\leq>$ is called linear-factor-ordered if the factor of this set is a linear order as regards the natural equivalence relation $\equiv_{\leq}$.

Therefore, a countable epimorphism skeleton of a variety of Boolean algebras is linear-factor-ordered. On the other hand, if $\mathfrak{M}$ is an arbitrary non-trivial congruence-distributive variety, and $\mathfrak{A}$ is a simple $\mathfrak{M}$-algebra existing by the Magari theorem,, by corollary **3.1**(b), for any Boolean algebras $\mathcal{B}_1,\mathcal{B}_2$ the relation $\mathfrak{A}^{\mathcal{B}_1}<<\mathfrak{A}^{\mathcal{B}_2}$ holds iff $\mathcal{B}_1<<\mathcal{B}_2$. Moreover, since we have earlier agreed that only varieties ofat most countably infinite signature are considered, we can assume $|\mathfrak{A}| \leq \aleph_0$ and, hence, they are at most countably infinite for at most countably infinite Boolean algebras $\mathfrak{B}$ and the algebra $\mathfrak{A}^{\mathfrak{B}}$. All these facts imply that the countable epimorphism skeleton of a variety of Boolean algebras is isomorphically embeddable into that of any non-trivial congruence-distributive variety, i.e. the countable epimorphism skeleton of a variety of Boolean algebras is minimal as regards embedding in the class of all countable epimorphism skeletons in non-trivial congruence-distributive varieties.

The following theorem gives a complete description of congruence-distributive varieties with minimal or, which proves equivalent, linear-factor-ordered countable epimorphism skeletons.

Theorem 10.1. Let $\mathfrak{M}$ be a non-trivial congruence-distributive variety, the following conditions are equivalent:

(a) $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<> \cong \omega_1 \oplus 1^*$;

(b) $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered;

(c) $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$, where $\mathfrak{A}$ is a quasi-primal algebra with no proper subalgebras.

Proof. Let us show that (c) follows from (b). Let $\mathfrak{M}$ be a non-trivial congruence-distributive variety such that $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered, and let $\mathfrak{A}$ be a simple $\mathfrak{M}_{\aleph_0}$-algebra. Since $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered, $\mathfrak{A}$ is the only simple algebra in $\mathfrak{M}_{\aleph_0}$.

Let $\mathfrak{B}_F$ be a Boolean algebra of finite and co-finite subsets of a countable set. As $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered, for any finite n we get either $\mathfrak{F}_{\mathfrak{M}}(n) << \mathfrak{A}^{\mathfrak{B}_F}$ or $\mathfrak{A}^{\mathfrak{B}_F} << \mathfrak{F}_{\mathfrak{M}}(n)$. The latter inequality, however, would entail that the algebra $\mathfrak{A}^{\mathfrak{B}_F}$ is finitely generated and, by the definition of a Boolean power, the same would be true for the algebra $\mathfrak{B}_F$ as well, which is impossible. Therefore, for any $n \in \omega$ we have $\mathfrak{F}_{\mathfrak{M}}(n) << \mathfrak{A}^{\mathfrak{B}_F}$. By corollary **3.1**(a), any principal congruences on $\mathfrak{A}^{\mathfrak{B}_F}$ are permutable and, hence, any congruences on $\mathfrak{A}^{\mathfrak{B}_F}$ and its homomorphic images-algebras $\mathfrak{F}_{\mathfrak{M}}(n)$ are permutable. Since $\mathfrak{F}_{\mathfrak{M}}(3)$ is congruence-permutable, the whole variety $\mathfrak{M}$ is also congruence-permutable by theorem **2.5**. Therefore, $\mathfrak{M}$ is arithmetic. On the other hand, $\mathfrak{M} = \mathfrak{M}(\{\mathfrak{F}_{\mathfrak{M}}(n) | n \in \omega\})$ and, hence, $\mathfrak{M} = \mathfrak{M}(\mathfrak{A}^{\mathfrak{B}_F}) = \mathfrak{M}(\mathfrak{A})$.

Let us demonstrate that $\mathfrak{A}$ contains no non-singleton proper subalgebras. Let us, assume to the contrary that $\mathfrak{A}_1$ is a proper subalgebra of the algebra $\mathfrak{A}$, and $|\mathfrak{A}_1|>1$. One can also assume that $\mathfrak{A}_1$ is finitely generated. Since $\mathfrak{A}_1 << \mathfrak{F}_{\mathfrak{M}}(n)$ for some $n\in\omega$ and, as has been shown above, $\mathfrak{F}_{\mathfrak{M}}(n) << \mathfrak{A}^{\mathcal{B}_F}$ and $\mathfrak{F}_{\mathfrak{M}}(n) \not\cong \mathfrak{A}^{\mathcal{B}_F}$, we get $\mathfrak{F}_{\mathfrak{M}}(n) \cong \mathfrak{A}^{\mathcal{B}}$ for some algebra $\mathcal{B}$ such that $\mathcal{B} << \mathcal{B}_F$ and $\mathcal{B} \not\cong \mathcal{B}_F$ by corollary **3.1**(b). All such $\mathcal{B}$s, however, have the form of finite Boolean algebras. Therefore, for some $m_n \in\omega$ we get $\mathfrak{F}_{\mathfrak{M}}(n) \cong \mathfrak{A}^{m_n}$. All homomorphic images of the algebra $\mathfrak{F}_{\mathfrak{M}}(n) \cong \mathfrak{A}^{m_n}$ also have, by corollary **3.1**(c), the form $\mathfrak{A}^k$ for some $k\in\omega$. In particular, the algebra $\mathfrak{A}_1$ is isomorphic to the algebra $\mathfrak{A}^l$ for some $l\in\omega$ and, hence, the algebra $\mathfrak{A}_1$ contains a subalgebra isomorphic to the algebra $\mathfrak{A}$. As a result, if $\mathfrak{A}$ contains a non-singleton proper subalgebra, $\mathfrak{A}$ contains a proper subalgebra isomorphic to itself. However, this implies the existence of a strictly ascending chain of algebras $\mathfrak{D}_0 \subset \mathfrak{D}_1 \subset \ldots \subset \mathfrak{D}_n \subset \ldots$, each of which is isomorphic to the algebra $\mathfrak{A}$.

Let $\mathfrak{D} = \bigcup_{i\in\omega} \mathfrak{D}_i$, $\mathfrak{D}\in\mathfrak{M}$. Since all $\mathfrak{D}_i$ are simple, the algebra $\mathfrak{A}$ will be simple as well. Indeed, for any $a,b,c,d\in\mathfrak{D}$ there is an $i\in\omega$ such that $a,b,c,d\in\mathfrak{D}_i$, and if $a\neq b$ then, as $\mathfrak{D}_i$ is simple, $<c,d>\in\theta_{a,b}^{\mathfrak{D}_i}$ and, since $\theta_{a,b}^{\mathfrak{D}_i} \subseteq \theta_{a,b}^{\mathfrak{D}}$, $<c,d>\in\theta_{a,b}^{\mathfrak{D}}$. Therefore, for any $a,b,c,d\in\mathfrak{D}$, $a\neq b$ implies $<c,d>\in\theta_{a,b}^{\mathfrak{D}}$, i.e., $\mathfrak{D}$ is a simple algebra. If we take into account the fact that $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered, $\mathfrak{A} \cong \mathfrak{D}$. Therefore, $\mathfrak{A}$ is a family of strictly ascending chain of proper subalgebras and, hence, it cannot be finitely generated. On the other hand, as $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered, we get $\mathfrak{A} << \mathfrak{F}_{\mathfrak{M}}(2)$, i.e., $\mathfrak{A}$ must be finitely generated. The contradiction obtained proves that $\mathfrak{A}$ has no non-singleton proper subalgebras.

Let us show that $\mathfrak{A}$ has no one-element subalgebras, either. Let us assume to the contrary, assume that that it has one-element subalgebras and let $a\in\mathfrak{A}$ be such that $\{a\}$ is a subalgebra of the algebra $\mathfrak{A}$. By theorem **3.2**, $Con_p(\mathfrak{A}^{\mathcal{B}_F}) \cong \mathcal{B}_F$. Let $\mathfrak{A}'$ denote a subalgebra of the algebra $\mathfrak{A}^{\mathcal{B}_F}$ such that for $f\in\mathfrak{A}^{\mathcal{B}_F}$ we get $f\in\mathfrak{A}'$ iff $\{i\in\mathcal{B}_F^*|f(i)\neq a\}$ is finite. Repeating nearly word by word the proof of theorem **3.2**, we see that $Con_p\mathfrak{A}'$ is isomorphic to the lattice $\mathcal{G}$ of all finite subsets of ω. Since $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered, we get either $\mathfrak{A}^{\mathcal{B}_F} << \mathfrak{A}'$ or $\mathfrak{A}' << \mathfrak{A}^{\mathcal{B}_F}$. But in this case it follows from the theorem on homomorphisms that if $Con_p(\mathfrak{A}^{\mathcal{B}_F}) \cong \mathcal{B}_F$ and $Con_p\mathfrak{A}' \cong \mathcal{G}$, we get either $\mathcal{B}_F << \mathcal{G}$ or $\mathcal{G} << \mathcal{B}_F$. Obviously, neither of the cases is possible and, hence, the algebra $\mathfrak{A}$ has neither one-element nor other proper subalgebras at all.

Let us now prove that $\mathfrak{A}$ is finite. Let us assume, to the contrary, that $\mathfrak{A}$ is infinite and, as has been proved above, it is simple and has no proper subalgebras. Let f be a bijective mapping of ω on $\mathfrak{A}$, g a certain mapping from ω to an element (for instance, d) from $\mathfrak{A}$, and $\mathfrak{C}$ a subalgebra of the algebra $\mathfrak{A}^{\omega}$ generated by the functions f and g. In this case, $\mathfrak{C} << \mathfrak{F}_{\mathfrak{M}}(2) \cong \mathfrak{A}^{m_2}$ and, as has been noticed earlier, there is an $l < m_2$ such that $\mathfrak{C} \cong \mathfrak{A}^{l}$. In particular, since all the congruences of the algebra $\mathfrak{A}^{l}$ are projections by corollary **3.1**, $Con\mathfrak{C} \cong Con\mathfrak{A}^{l}$ contains only l different co-atoms. It should be noticed that as $\mathfrak{A}$ is simple with no proper subalgebras, $\mathfrak{C}$ is a subdirect product of $\mathfrak{A}$ into $\mathfrak{A}^{\omega}$, and for any $i \in \omega$ the θ_i-kernel of the i-th projection of $\mathfrak{C} \subseteq \mathfrak{A}^{\omega}$ on $\mathfrak{A}$ is a co-atom in $Con\mathfrak{C}$. For any $j \in \omega$ let $r_j(x)$ be a term of the signature of the algebra $\mathfrak{A}$ such that $r_j(d) = f(j)$ (since $\mathfrak{A}$ is generated by any of its elements, for any $j \in \omega$, $r_j(x)$ does exist). Therefore, $r_j(g)(j) = f(j)$, and, since for any $k,l \in \omega$ we have $r_j(g)(k) = r_j(g)(l)$, for $n \in \omega$ and $n \neq j$ we get $r_j(g)(n) \neq f(n)$. Therefore, for any $j \neq n \in \omega$ we get $< r_j(g), f > \in \theta_j$ and $< r_j(g), f > \notin \theta_n$, i.e., all $\theta_j (j \in \omega)$ are different co-atoms of $Con\mathfrak{C}$, which contradicts the fact that the family of these co-atoms is finite. The contradiction obtained proves that $\mathfrak{A}$ is finite.

Thus, $\mathfrak{A}$ is finite, simple, has no proper subalgebras, and $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$ is arithmetic. By theorem **2.14**, $\mathfrak{A}$ is quasi-primal and, hence, the implication (b) $\rightarrow$ (c) is proved.

The implication (a) $\rightarrow$(b) is obvious, while the implication (c) $\rightarrow$ (a) follows from theorem **7.3** and the isomorphism $< \mathfrak{I}BA_{\aleph_0};<<> \cong < \mathfrak{I}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B} \in BA\};<<>$ resulting from corollary **3.1**. ■

It should be remarked that, as the epimorphism skeleton of a congruence-distributive variety is linearly ordered, the epimorphism relation << and the embedding relation ≤ coincide on non-singleton countable algebras. Let $\mathfrak{M}'$ be a family of non-singleton $\mathfrak{M}$-algebras.

<u>**Corollary 10.1.**</u> If a countable epimorphism skeleton of a congruence-distributive variety $\mathfrak{M}$ is linear-factor-ordered, the relations << and ≤ coincide on $\mathfrak{I}\mathfrak{M}'_{\aleph_0}$.

Proof. It follows from the proof of theorem **10.1** that in the case under consideration the fact that $< \mathfrak{I}\mathfrak{M}_{\aleph_0};<<>$ is linear-factor-ordered implies an isomorphism of $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ and $<\mathfrak{J}BA_{\aleph_0};<<>$ by way of putting into correspondence $\mathfrak{B}\to\mathfrak{A}^{\mathfrak{B}}$ for any at most countably infinite Boolean algebra $\mathfrak{B}$. Since in this case $\mathfrak{M}$ proves to be a discriminator variety, it has the property of extending congruences and, by theorem **3.3**, for any Boolean algebras $\mathfrak{B}_1,\mathfrak{B}_2$, the embedding relation $\mathfrak{B}_1\leq\mathfrak{B}_2$ holds iff $\mathfrak{A}^{\mathfrak{B}_1}\leq\mathfrak{A}^{\mathfrak{B}_2}$. On the other hand, as follows from section **1** for countable non-singleton Boolean algebras $\mathfrak{B}_1,\mathfrak{B}_2$, $\mathfrak{B}_1<<\mathfrak{B}_2$ iff $\mathfrak{B}_1\leq\mathfrak{B}_2$, while $\mathfrak{B}$ is one-element iff so is the algebra $\mathfrak{A}^{\mathfrak{B}}$. ■

In the case when the variety $\mathfrak{M}$ is a discriminator variety, it is possible to make a complete analysis of the construction of countable epimorphism skeletons. The definition of a better quasi-order, as well as the proof of a number of theorems on the properties of such quasi-orders are given in section **15** of the present monograph.

Theorem 10.2. If $\mathfrak{M}$ is a finitely generated discriminator variety, the countable epimorphism skeleton of $\mathfrak{M}$ is a better quasi-order. In particular, $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ contains neither infinite anti-chains nor infinite strictly descending chains.

Proof. Let $\mathfrak{M}$ be a finitely generated discriminator variety. By theorem **7.7**, there is a finite algebra $\mathfrak{A}$ and a finite set of its subalgebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$ such that any at most countably infinite $\mathfrak{M}$-algebra has the form of a filtered Boolean product $\mathfrak{A}^{\mathfrak{B}}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n;F_1,\ldots,F_n)$ for a certain at most countably infinite Boolean algebra $\mathfrak{B}$ and some closed subsets $F_1,\ldots,F_n$ of the space $\mathfrak{B}^*$. The chain $<\mathfrak{B},F_1,\ldots,F_n>$ will be denoted through $\overline{\mathfrak{B}F}$.

Let ω^ω be a family of all sequences of natural numbers with a common Tikhonov topology, let 2^ω be a subset of ω^ω consisting of sequences of zeros and ones, and let the topology on 2^ω be induced by ω^ω. Therefore, 2^ω is homomorphic and, hence, it can be identified with a Stone space $\mathfrak{B}_\eta^*$ of a countable atomless Boolean algebra $\mathfrak{B}_\eta$. Since for any at most countably infinite Boolean algebra $\mathfrak{B}$ we have $\mathfrak{B}<<\mathfrak{B}_\eta$, the space $\mathfrak{B}^*$ can be identified with a certain closed subspace of the space of 2^ω according to the Stone duality.

${}^n2$ will denote a set of sequences from 0,1 of the length $n+1$, ordered in a trivial way: $a\leq b$ iff $a=b$. Therefore, ${}^n2$ is, in particular, a better quasi-order. By $a_i (i=0,\ldots,n)$ we will mean the i-th element of the sequence a for $a\in{}^n2$. For at most countably infinite Boolean algebra $\mathcal{B}$ and any closed $F_1,\dots,F_n \subseteq \mathcal{B}^*$, $l_{\overline{\mathcal{BF}}}$ will denote the mapping from ω^ω to ${}^{n+1}2$ defined in the following way: if $l_{\overline{\mathcal{BF}}}(x) = a \in {}^{n+1}2$, $a_0 = 1$, if $x \in 2^\omega$ and $a_i = 0$ for all $i = 0,\dots,n+1$, if $x \in \omega^\omega \setminus 2^\omega$, and moreover: $a_1 = 1$ iff $x \in \mathcal{B}^*$, and $a_{i+1} = 1$ iff $x \in F_i$ at $i = 1,\dots,n$. For any $a \in {}^{n+1}2$ and any $\overline{\mathcal{BF}}$ in the Borel hierarchy of sets we obviously have $l^{-1}_{(\overline{\mathcal{BF}})}(a) \in \sum_2^0$. Let us arrange the set of all mappings $l_{\overline{\mathcal{BF}}}$ into a quasi-order in the following way: $l_{(\overline{\mathcal{BF}})_1} \leq l_{(\overline{\mathcal{BF}})_2}$ iff there is a continuous self-embedding h of ω^ω such that for any $x \in \omega^\omega$ we get $l_{(\overline{\mathcal{BF}})_1}(x) \leq l_{(\overline{\mathcal{BF}})_2}(h(x))$. By theorem **15.8**, the set of mappings $l_{\overline{\mathcal{BF}}}$ (where $\mathcal{B}$ is an arbitrary at most countably infinite Boolean algebra and $F_1,\dots,F_n$ is a chain of closed subsets of the space $\mathcal{B}^*$) will be a better quasi-order. Let us now remark that if $l_{(\overline{\mathcal{BF}})_1} \leq l_{(\overline{\mathcal{BF}})_2}$,

$$\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\dots,\mathfrak{A}_n; F_1^1,\dots,F_n^1) << \mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\dots,\mathfrak{A}_n; F_1^2,\dots,F_n^2).$$

Indeed, let $l_{(\overline{\mathcal{BF}})_1} \leq l_{(\overline{\mathcal{BF}})_2}$, and h be a continuous self-embedding of ω^ω implementing this inequality. Since for any $x \in \omega^\omega$ we have $l_{(\overline{\mathcal{BF}})_1}(x) \leq l_{(\overline{\mathcal{BF}})_2}(h(x))$, the restriction of h on $\mathcal{B}_1^* \subseteq 2^\omega$ will be a continuous embedding of the space $\mathcal{B}_1^*$ into the space $\mathcal{B}_2^*$ such that for any $i = 1,\dots,n$ we get $x \in F_i^1$ iff $h(x) \in F_i^2$. Therefore, $\mathcal{B}_1^*$ can be identified with a subspace of the space $\mathcal{B}_2^*$ such that $F_i^1 = \mathcal{B}_1^* \cap F_i^2$ for $i = 1,\dots,n$. Let

$$f \in \mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\dots,\mathfrak{A}_n; F_1^2,\dots,F_n^2),$$

i.e., f is a continuous mapping from $\mathcal{B}_2^*$ to a discrete $\mathfrak{A}$ such that $f(F_i^2) \subseteq \mathfrak{A}_i$ for $i \leq n$. In this case $f|\mathcal{B}_1^*$ is obviously a continuous mapping from $\mathcal{B}_1^* \subseteq \mathcal{B}_2^*$ to $\mathfrak{A}$, in which case $(f|\mathcal{B}_1^*)(F_i^1) \subseteq \mathfrak{A}_i$. The mapping $\varphi: f \to f|\mathcal{B}_1^*$ is also obviously a homomorphism from the algebra $\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\dots,\mathfrak{A}_n; F_1^2,\dots,F_n^2)$ to the algebra $\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\dots,\mathfrak{A}_n; F_1^1,\dots,F_n^1)$. Let us prove that φ is a mapping on $\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\dots,\mathfrak{A}_n; F_1^1,\dots,F_n^1)$. Let $g \in \mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\dots,\mathfrak{A}_n; F_1^1,\dots,F_n^1)$; our task is to construct a continuous extension g_1 of the mapping g to the space $\mathcal{B}_2^*$ meeting the condition $g_1(F_i^2) \subseteq \mathfrak{A}_i$.

As g is a continuous mapping from $\mathcal{B}_1^*$ to a discrete $\mathfrak{A}$, there is a partition $A_1,\dots,A_k$ of the space $\mathcal{B}_1^*$ by open-closed subsets such that g is constant on A_j.

Since $\mathcal{B}_1^*$ is a subspace of $\mathcal{B}_2^*$, due to the Stone duality, there is a homomorphism φ from the Boolean algebra $\mathcal{B}_2$ identified with the open-closed subsets of the space $\mathcal{B}_2^*$ to the Boolean algebra $\mathcal{B}_1$ (also viewed as a set of open-closed subsets of the space $\mathcal{B}_1^*$). Let $B_1,\dots,B_k$ be a partition of $\mathcal{B}_2^*$ with the elements of $\mathcal{B}_2$ such that $\psi(B_j)=A_j$ for $j=1,\dots,k$. In this case we get $\psi(B_j)=A_j=B_j\cap\mathcal{B}_1^*$. For any $j\le k$ let $K_j=\{i\le n\mid A_j\cap F_i^2=\varnothing\}$. Since F_i^2, A_j are closed in $\mathcal{B}_2^*$, there is an open-closed C_{ij} separating A_j from F_i^2 for $i\in K_j$.

Let $B_j'=B_j\cap\bigcap_{i\in K_j}C_{ij}$. The mapping g_1 from the space $\mathcal{B}_2^*$ to $\mathfrak{A}$ will be defined in the following way: $g_1(B_j')=g(A_j)$, and let g_1 coincide with any function from $\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\dots,\mathfrak{A}_n;F_1^2,\dots,F_n^2)$ on $\mathcal{B}_2^*\setminus\bigcup_{j\le k}B_j'$. In this case $g\in\mathfrak{A}^{\mathcal{B}_2}$ and the conditions $g_1(F_j^2)\subseteq\mathfrak{A}_j$ are obviously fulfilled, since $g_1^{-1}(\mathfrak{A}_j)\cap B_i'=\varnothing$ for $j=1,\dots,n$ iff $g^{-1}(\mathfrak{A}_j)\cap A_i=\varnothing$. Therefore, $g_1\in\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\dots,\mathfrak{A}_n;F_1^2,\dots,F_n^2)$, and, since $g=g_1|\mathcal{B}_1^*$, $\varphi(g_1)=g$, i.e., φ is indeed a homomorphism from the algebra $\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\dots,\mathfrak{A}_n;F_1^2,\dots,F_n^2)$ to the algebra $\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\dots,\mathfrak{A}_n;F_1^1,\dots,F_n^1)$. Thus, we have shown that $l_{\overline{(\mathcal{BF})_1}}<<l_{\overline{(\mathcal{BF})_2}}$ implies the relation

$$\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\dots,\mathfrak{A}_n;F_1^1,\dots,F_n^1)<<\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\dots,\mathfrak{A}_n;F_1^2,\dots,F_n^2),$$

i.e., that $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is a homomorphic image of a better quasi-order. As is noted in section **15**, $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ is thus also a better quasi-order. ■

The situation changes radically when considering countable epimorphism skeletons of discriminator varieties which are not finitely generated.

Theorem 10.3. If $\mathfrak{M}$ is a discriminator variety which is not finitely generated, $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ contains an uncountable number of pairwise incomparable elements. If, moreover, either the signature of $\mathfrak{M}$ is finite or all non-singleton $\mathfrak{M}$-algebras contain a finite simple subalgebra, any countable quasi-order is isomorphically embeddable into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$.

The proof of the theorem is reduced to that of a number of lemmas. Let us first consider the simplest case, when $\mathfrak{M}$ contains an infinite number of various finite, subdirectly non-decomposable and, hence, (as $\mathfrak{M}$ is a discriminator variety) simple algebras.

Lemma 10.1. If $\mathfrak{M}$ is a discriminator variety containing an infinite number of non-isomorphic simple finite algebras, $\mathfrak{M}$ contains no infinite finitely generated simple algebras, and either the $\mathfrak{M}$ signature is finite or all non-singleton $\mathfrak{M}$-algebras contain a finite simple subalgebra, any countable quasi-order is isomorphically embeddable into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$, and $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ contains $2^{\aleph_0}$ pairwise incomparable elements.

Proof. Let the $\mathfrak{M}$ signature be finite, and let $\mathfrak{A}_1,...,\mathfrak{A}_n,...$ be finite simple non-isomorphic $\mathfrak{M}$-algebras. As the signature is finite, one can assume $|\mathfrak{A}_1|<|\mathfrak{A}_2|<...<|\mathfrak{A}_n|<....$ Let $\mathfrak{C}=\prod_{n\in\omega}\mathfrak{A}_n$ and $\mathfrak{F}$ a non-principal ultrafilter on ω. By theorem **5.6**, $Con_p\mathfrak{C}=P(\omega)$, i.e., a set of all the subsets of ω and, hence, $\mathfrak{C}/\mathfrak{F}$ is a simple $\mathfrak{M}$-algebra. As is well-known, $\mathfrak{C}/\mathfrak{F}$ is infinite. Since $\mathfrak{M}$ is a discriminator variety, all subalgebras of the algebra $\mathfrak{C}/\mathfrak{F}$ are simple and, since all finitely generated simple $\mathfrak{M}$-algebras are finite, by the lemma condition one can construct an ascending chain of finite simple $\mathfrak{M}$-algebras ($\mathfrak{C}/\mathfrak{F}$ subalgebras): $\mathfrak{C}_0\subset\mathfrak{C}_1\subset..\subset\mathfrak{C}_n\subset....$ Let $I\subseteq\omega\setminus\{0\}$ and $\mathfrak{C}_I=\sum_{i\in I}\mathfrak{C}_i(\mathfrak{C}_1)=\{f\in\prod_{i\in I}\mathfrak{C}_i\}$. There is an $n\in I$ such that for all $l,m\geq n$ we get $f(l)=f(m)\in\mathfrak{C}_1$.

If g is a homomorphism from $\mathfrak{C}_I$ to some algebra $\mathfrak{C}_j$,, since $\mathfrak{C}_j$ is finite, there is a finitely generated subalgebra $\mathfrak{D}$ of the algebra $\mathfrak{C}_I$ such that g maps $\mathfrak{D}$ on $\mathfrak{C}_j$. One can easily note that $\mathfrak{C}_I$ is locally finite and, hence $\mathfrak{D}$ is finite. Any finite subalgebra of the algebra $\mathfrak{C}_I$ lies in a subalgebra of the algebra $\mathfrak{C}_I$, which is isomorphic to an algebra of the type $\prod_{i\in I_1}\mathfrak{C}_i$ for a certain finite subset $I_1\subseteq I$. Therefore, we get $\mathfrak{C}_j<<\prod_{i\in I_1}\mathfrak{C}_i$. By theorem **6.6**, since all $\mathfrak{C}_i$ are simple, all simple factors of the algebra $\prod_{i\in I_1}\mathfrak{C}_i$ have the form $\mathfrak{C}_i$ for some $i\in I_1$. Thus, if $\mathfrak{C}_j<<\mathfrak{C}_I$, $j\in I$. Hence, for any $I_1,I_2\subseteq\omega\setminus\{0\}$ we have $\mathfrak{C}_{I_1}<<\mathfrak{C}_{I_2}$ iff $I_1\subseteq I_2$, which implies an isomorphic embedding of $P(\omega\setminus\{0\})$ to $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$. Since any countable partial order is embeddable into $P(\omega\setminus\{0\})$, any countable partial order is isomorphically embeddable into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ according to the lemma conditions.

Let η be, as was the case earlier, an ordered type of rational numbers. As has been noted in section **1**, for any $l\neq m\in\omega$ we have $B(\omega^l\cdot\eta)\equiv_{<<}B(\omega^m\cdot\eta)$ and $B(\omega^l\cdot\eta)\not\cong B(\omega^m\cdot\eta)$. Since $\mathfrak{C}_0$ is simple, for any Boolean algebras $\mathfrak{B}_1,\mathfrak{B}_2$, $\mathfrak{C}_0^{\mathfrak{B}_1}<<\mathfrak{C}_0^{\mathfrak{B}_2}$ ($\mathfrak{C}_0^{\mathfrak{B}_1}\cong\mathfrak{C}_0^{\mathfrak{B}_2}$) is equivalent to the relations $\mathfrak{B}_1<<\mathfrak{B}_2$ ($\mathfrak{B}_1\cong\mathfrak{B}_2$) by corollary **3.1**. In addition, all the factors of the algebra $\mathfrak{C}_0^{\mathfrak{B}_1}$ have the form $\mathfrak{C}_0^{\mathfrak{B}}$

for a certain Boolean algebra $\mathcal{B} << \mathcal{B}_1$. As $\mathfrak{M}$ is congruence-distributive, any factor of the product $\mathfrak{A}' \times \mathfrak{A}''$ of the algebras $\mathfrak{A}', \mathfrak{A}''$ has the form of a factor product of the algebras $\mathfrak{A}'$ and $\mathfrak{A}''$. This fact implies that for any $I_1, I_2 \subseteq \omega \setminus \{0\}$, any Boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ we get $\mathfrak{C}_{I_1} \times \mathfrak{C}_0^{\mathcal{B}_1} << \mathfrak{C}_{I_2} \times \mathfrak{C}_0^{\mathcal{B}_2}$ iff $I_1 \subseteq I_2$ and $\mathcal{B}_1 << \mathcal{B}_2$.

Let now $< C;\subseteq>$ be an arbitrary countable quasi-order and f be a certain bijective mapping of C on ω. Let h be an embedding of a partial order $< C/\equiv_{\leq};\leq>$ into $< \mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ constructed above, in which case it is obvious that the mapping $g(c) = h([c]) \times \mathfrak{C}_o^{B(\omega^{f(c)} \cdot \eta)}$, where $[c]$ is a $\equiv_{\leq}$-class containing the element c, will be an isomorphic embedding of the quasi-order $< C;\leq>$ into $< \mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$. The statement that there are $2^{\aleph_0}$ pairwise incomparable elements in $< \mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ holds true because the number of the subsets ω which do not contain each other is the same, and since we have constructed the mapping $I \subseteq \omega \setminus \{0\} \to \mathfrak{C}_I$. ■

The case when $\mathfrak{M}$ is of an infinite signature, but the algebras $\mathfrak{A}_2,\ldots,\mathfrak{A}_n,\ldots$ contain a subalgebra isomorphic to a finite simple algebra $\mathfrak{A}_1$ is considered in the same manner with the algebras $\mathfrak{A}_i$ substituted for the algebras $\mathfrak{C}_i$. ■

__Lemma 10.2.__ If $\mathfrak{M}$ is a discriminator variety, $\mathfrak{A} \in \mathfrak{M}$, and $\mathfrak{A}$ is an infinite finitely generated simple algebra, there is an uncountable set of pairwise incomparable elements in $< \mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$ and any countable quasi-order is isomorphically embeddable into $< \mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$.

Proof. Let $\mathfrak{A} = \{a_1,\ldots,a_n,\ldots\}$, and let $a_1,\ldots,a_k$ generate $\mathfrak{A}$. f_i will denote an element $\mathfrak{A}^{\omega}$ such that $f_i(n) = a_i$ for any $n \in \omega$, and assume that $g \in \mathfrak{A}^{\omega}$ and $g(n) = a_n$ for any $n \in \omega$. Let us also set $\mathfrak{A}_0$ equal to a subalgebra of the algebra $\mathfrak{A}^{\omega}$ generated by the elements $f_1,\ldots,f_k$, and $\mathfrak{A}_1$ equal to a subalgebra generated by the elements $f_1,,, f_k, g$. By corollary **5.1**, $Con_p\mathfrak{A}_1$ is a Boolean algebra, so let us consider two plausible cases: (1) $Con_p\mathfrak{A}_1$ is not superatomic and (2) $Con_p\mathfrak{A}_1$ is superatomic.

Case (1). There is an uncountable number of various ultrafilters $\mathfrak{F}_i$. on $Con_p\mathfrak{A}_1$, each of them corresponding to pairwise different congruences α_i on $\mathfrak{A}_1$, which are co-atoms in $Con\mathfrak{A}_1$, i.e., to α_i such that $\mathfrak{A}_1/\alpha_i$ are simple. The number of pairwise non-isomorphic $\mathfrak{A}_1/\alpha_i$ cannot be countable, i.e., each of the algebras of this type is countable ($\mathfrak{A}_1$ is countable) and $\mathfrak{A}_1$ is finitely generated. Indeed, if $|\mathfrak{J}\{\mathfrak{A}_1/\alpha_i | i\in 2^{\aleph_0}\}| \le \aleph_0$, let

$$|\mathfrak{J}\{\mathfrak{A}_1/\alpha_i | i\in 2^{\aleph_0}\}| = \mathfrak{J}\{\mathfrak{C}_1,\dots,\mathfrak{C}_m,\dots\}.$$

The number of $\mathfrak{A}_1$ homomorphisms in $\mathfrak{C}_i$ (since $\mathfrak{C}_i$ are countable and $\mathfrak{A}_1$ is finitely generated) is at most countably infinite and, hence, the number of various congruences, which are kernels of $\mathfrak{A}_1$ homomorphisms on the algebras of the type $\mathfrak{C}_i$, is at most countably infinite, which contradicts the existence of $2^{\aleph_0}$ pairwise different co-atoms $\alpha_i \in Con\mathfrak{A}_1$. Therefore, there is an uncountable number of those $\alpha \in Con\mathfrak{A}_1$ for which $\mathfrak{A}_1/\alpha$ are simple and pairwise non-isomorphic. In particular, this fact implies the existence of an uncountable family of pairwise incomparable elements in $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>$.

Let us choose a countable set $\alpha_1,\dots\alpha_n,\dots$among such $\alpha\in Con\mathfrak{A}_1$. Therefore, $\mathfrak{A}_1/\alpha_n$ are simple, and we have $\mathfrak{A}_1/\alpha_n \not\cong \mathfrak{A}_1/\alpha_m$ for $n\neq m$. Let us also assume that $\mathfrak{A}_1/\alpha_n \not\cong \mathfrak{A}_0$. For any $j\in\omega$ and $n\neq m\in\omega$, we get $f_n/\alpha_j \neq f_m/\alpha_j$, since otherwise, by theorem **5.6**, α_j would be a unit congruence on $\mathfrak{A}_1$. Therefore, the algebra $\mathfrak{A}_0$ (isomorphic to $\mathfrak{A}$) is a subalgebra of all simple algebras of the type $\mathfrak{A}_1/\alpha_n$ $(n\in\omega)$. As was the case in the proof of lemma **10.1**, let us, for any $I\subseteq\omega$, define an algebra $\mathfrak{A}_I = \sum_{i\in I}\mathfrak{A}_1/\alpha_i(\mathfrak{A}_0) = \{f\in\prod_{i\in I}\mathfrak{A}_1/\alpha_i$, and for some $n\in I$, all $l,m\ge n$, $f(l)=f(m)\in\mathfrak{A}_0\}$. If g is a homomorphism from the algebra $\mathfrak{A}_I$ to the algebra $\mathfrak{A}_1/\alpha_j$, either there are two elements $h_1,h_2\in\mathfrak{A}_I$ such that for some $n\in\omega$ and all $m\ge n$ we get $h_1(m)\neq h_2(m)$, in which case $g(h_1)=g(h_2)$, or , as $\mathfrak{A}_1/\alpha_j$ is simple, for any $z_1,z_2\in\mathfrak{A}_I$ the equality $g(z_1)=g(z_2)$ holds provided that for some $l\in\omega$ and all $m\ge l$ we have $z_1(m)=z_2(m)$. In the former case, $\mathfrak{A}_1/\alpha_j << \prod_{i\in I, i\le n}\mathfrak{A}_1/\alpha_i$ and, as was the case in the proof of lemma **10.1**, we get $j\in I$. In the latter case we obtain, as can be easily seen, $\mathfrak{A}_1/\alpha_j\cong\mathfrak{A}_0$, which contradicts the choice of the algebras $\mathfrak{A}_1/\alpha_n$. Therefore, for any $j\in\omega$ and $I\subseteq\omega$ we get $\mathfrak{A}_j << \mathfrak{A}_I$ iff $j\in I$. Repeating the end of the proof of lemma **10.1**, we see that the statement of the present lemma for the case (1) is proved.

Case (2). Since $Con_p\mathfrak{A}_1$ is here superatomic, we get $Con_p\mathfrak{A}_1\cong B(\omega^\alpha\cdot s)$ for some countable ordinal α and some $s\in\omega$. Any principal congruence $\theta_{a,b}$ on the algebra $\mathfrak{A}_1$ will be, by theorem **5.6**, identified with a subset $\{i\in\omega | a(i)=b(i)\}$ of the set ω. For a $d\in Con_p\mathfrak{A}_1$ let d^* denote a corresponding open-closed subset of the Stone space $(Con_p\mathfrak{A}_1)^*$. For $\mathfrak{C}\subseteq\mathfrak{A}^\omega$ and $\mathfrak{C}\supseteq\mathfrak{A}_1$, for $d\in Con_p\mathfrak{A}_1$,

$Con_p\mathfrak{C}|\neg d$ will denote a naturally arising Boolean algebra with a basic set $\{d_1 \in Con_p\mathfrak{C} | d_1 \leq \neg d\}$, while ψ_d will stand for a natural homomorphism $Con_p\mathfrak{C}$ on $Con_p\mathfrak{C}|\neg d$. Under the same conditions $\mathfrak{C}|\neg d$ will denote a subalgebra of the algebra $\mathfrak{A}^{\neg d}$ ($\neg d$ is a subset of ω) which is the projection of algebra $\mathfrak{C}$ relative to the subset $\neg d$. In this case $Con_p(\mathfrak{C}|\neg d) = Con_p\mathfrak{C}|\neg d$.

It should be noticed that if k is a homomorphism of a subalgebra $\mathfrak{C}$ of the algebra $\mathfrak{A}^{\omega}$ containing the algebra $\mathfrak{A}_1$ on $\mathfrak{A}_1|\neg d$, where d is a certain element of $Con_p\mathfrak{A}_1$, for any $i \neq j \in \omega$ and $s \in \neg d \subseteq \omega$ we get $k(f_i)(s) \neq k(f_j)(s)$. Indeed, $\theta^{\mathfrak{C}}_{f_i,f_j} = \nabla_{\mathfrak{C}}$ and, therefore, $\theta^{\mathfrak{A}_1|\neg d}_{k(f_i)k(f_j)} = \nabla_{\mathfrak{A}_1|\neg d}$, i.e., by theorem **5.6**, for any $s \in \neg d$ we get $k(f_i)(s) \neq k(f_j)(s)$.

It should also be noticed that for any algebras $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{M}$, for any homomorphism k from the algebra $\mathfrak{B}_1$ to $\mathfrak{B}_2$, the mapping φ_k from $Con_p\mathfrak{B}_1$ to $Con_p\mathfrak{B}_2 \cong Con_p\mathfrak{B}_1|\geq \ker k$, defined as $\varphi_k(\theta^{\mathfrak{B}_1}_{f,h}) = \theta^{\mathfrak{B}_2}_{k(f),k(h)}$ equal to $\theta^{\mathfrak{B}_1}_{f,h} \vee \ker k$ when identifying $Con_p\mathfrak{B}_2$ with $Con_p\mathfrak{B}_1|\geq \ker k$, is, since $\mathfrak{M}$ is congruence-distributive, a homomorphism from the Boolean algebra $Con_p\mathfrak{B}_1$ to $Con_p\mathfrak{B}_2$.

Let now $R = \{r_1,\dots,r_s\}$ be a subset of a Stone space $(Con_p\mathfrak{A}_1)^*$ of the Boolean algebra $Con_p\mathfrak{A}_1$ composed of all the points of this space having the highest Cantor-Bendixon rank. Let $D = \{d_1,\dots,d_n,\dots\}$ be elements of $Con_p\mathfrak{A}_1$ such that $d_n^* \ni r_i$ for some $r_i \in R$.

Let us enumerate: $k_1,\dots,k_n,\dots$ are the homomorphisms of the algebra $\mathfrak{A}_1$ in algebras of the type $\mathfrak{A}_1|d$, where d is a certain element of the set D (let us refer to this d corresponding to k_n as $d(n)$), corresponding to the following conditions:

(1) for $i \neq j \in \omega$ and for $s \in d(n)$ $k_n(f_i)(s) \neq k_n(f_j)(s)$;

(2) φ_{k_n}, a homomorphism from the Boolean algebra $Con_p\mathfrak{A}_1$ to the Boolean algebra $Con_p\mathfrak{A}_1|d(n)$, is a homomorphism "onto".

The number of such ks is at most countably infinite, since D is countable and $\mathfrak{A}_1$ is finitely generated.

The homomorphisms φ_{k_n} induce dual continuous embeddings

$$(Con_p\mathfrak{A}_1|d(n))^* = (Con_p\mathfrak{A}_1)^*|d(n) = d(n)^*$$

into $(Con_p\mathfrak{A}_1)^*$. Let us refer to these embeddings as ψ_n.

Let $R_d = R \cap d^* = \{r_1^d,\dots,r_{m(d)}^d\}$. For any $n \in \omega$ we get $R_{d(n)} \neq \varnothing$. It should be also noticed that:

(a) for any $n \in \omega$, $j \leq m(d(n))$ we have $\psi_n(r_j^{d(n)}) \in R$;

(b) if t is an isolated point of the set $d(n)^*$, we get $\psi_n(t) \notin R$, as ψ_n are embeddings except, possibly, for a finite number of such ts. Hence, $K_n = \{t$ is an isolated point of $d(n)^* | \psi_n(t) \notin R\} = \{t_1^n, \ldots, t_m^n, \ldots\}$ is an infinite set. All isolated points of the space $(Con\mathfrak{A}_1|d(n))^*$ have the form $\theta_{f_j,g}^{\mathfrak{A}_1|d(n)}$ for a certain $j \in \omega$; let $0(n,m) \in \omega$ be such that $t_n^m = \theta_{f_{0(n,m)},g}^{\mathfrak{A}_1|d(n)}$.

For any $n \in \omega$ let us choose a sequence $t_{(1)}^n, \ldots, t_{(m)}^n, \ldots, K_n$ such that $\lim_{m\to\infty} t_{(m)}^n = r_1^{d(n)}$ and, hence, since ψ_n is continuous, $\lim_{m\to\infty} \psi_n(t_{(m)}^n) \in R$. Let $r(n)$ denote the latter limit, i.e., $\lim_{m\to\infty} \psi_n(t_{(m)}^n) = r(n)$.

Let η be a bijective enumeration of pairs of numbers. By induction, let us construct a sequence $b_1, \ldots, b_n, \ldots$ of disjunct elements of $Con_p\mathfrak{A}_1$, and a subsequence $t_{j_1^n}^n, \ldots, t_{j_k^n}^n, \ldots$ of the sequence $t_{(m)}^n$ such that if $\eta(m,q) = n$,

(*) $b_n^* \ni \psi_m(t_{j_q^m}^m)$ and $b_n^* \cap R = \varnothing$.

Let $b_1, \ldots, b_{n-1}$ have been constructed, and let $\eta(m,q) = n$. Since $\lim_{l\to\infty} \psi_m(t_{(l)}^m) = r(m)$ and $(b_1 \vee \ldots \vee b_{n-1})^* \not\ni r(m)$, there is a $p \in \omega$ such that

$$(b_1 \vee \ldots \vee b_{n-1})^* \not\ni \psi_m(t_{(p)}^m), \psi_m(t_{(p+1)}^m), \ldots .$$

Let b be an arbitrary element of $Con_p\mathfrak{A}_1$ containing a point $\psi_m(t_{(p)}^m)$, not containing the points of the set R, and disjunct from $b_1 \vee \ldots \vee b_{n-1}$. Let us set $t_{j_q^m}^m = t_{(p)}^m$ and $b_n = b$. The condition (*) is now met. One can assume in addition that the sequence $b_1, \ldots, b_n, \ldots$ results in a partition of the unit in $Con_p\mathfrak{A}_1$, i.e., that any isolated point of $(Con_p\mathfrak{A}_1)^*$ belongs to one of $b_n^* (n \in \omega)$. One should also remark that the condition (*) implies

(**) $(\varphi_{k_m}(b_n))^* \ni t_{j_q^m}^m$, i.e., $\varphi_{k_m}(b_n) \subseteq \theta_{f_{0(j_q^m,m)},g}^{\mathfrak{A}_1|d(n)}$.

Let $w_1,\ldots,w_n,\ldots$ be an enumeration of all the elements of the algebra $\mathfrak{A}_1$. Let $t_1(x_1,\ldots,x_k,x,y),\ldots,t_i(x_1,\ldots,x_k,x,y),\ldots$ be an enumeration of all $(k+2)$-ary terms of the signature of the variety $\mathfrak{M}$, i.e., for any $h\in\mathfrak{A}^{\omega}$ the algebra $<\mathfrak{A}_1,h>$, which is a subalgebra of the algebra $\mathfrak{A}^{\omega}$ generated by the elements of the algebra $\mathfrak{A}_1$ and the element h, has $\{t_i(f_1,\ldots,f_k,g,h)|i\in\omega\}$ as its basic set.

By induction over $n\in\omega$ we will define the value a_{p_n} of the element h on $i\in b_n$ and, simultaneously, certain infinite subsets $A_1\supseteq A_2\supseteq\ldots\supseteq A_n\supseteq\ldots$ of the set $\{a_1,\ldots,a_n,\ldots\}$ of the elements of algebra $\mathfrak{A}$.

Let A_{n-1} have been constructed and let the function h have been defined for $i\in b_1\vee\ldots\vee b_{n-1}$. Let $n=\eta(m,q)$ and $w_q(0(j_q^m,m))=a_l$. By the condition (1),

$$k_m(f_{s_1})(0(j_q^m,m))\neq k_m(f_{s_2})(0(j_q^m,m))$$

for any $s_1\neq s_2\in\omega$. Therefore, for all but possibly one values of $a_j\in A_{n-1}$ we get

$$k_m(f_j)(0(j_q^m,m))\neq w_q(0(j_q^m,m)).$$

Let us refer to the set $a_j\in A_{n-1}$ as A'_n. We see that A'_n is infinite, and if $a_j\in A'_n$,

$$k_m(f_j)(0(j_q^m,m))\neq w_q(0(j_q^m,m)).$$

Let us now construct a partition of the set A'_n into subsets $A(s)$, where s runs over the subsets of the set $\{\{i,j\}|i,j\leq n\}$. In this case, if $a_r\in A'_n$, $a_r\in A'(s)$ iff

$$\{\{i,j\}|t_i(f_1,\ldots,f_k,g,f_r)=t_j(f_1,\ldots,f_k,g,f_r),i,j\leq n\}=s.$$

For some s, $A(s)$ is infinite. Let us set A_n equal to this $A(s)$. Let a_{p_n} be a certain fixed value from A_n.

By virtue of the construction of h and A_n, the following statements are true:

(α) if $\eta(m,q)=n$, then $w_q(0(j_q^m,m))\neq k_m(f_{p_n})(0(j_q^m,m))$. In this case, $\psi_m(t^m_{j_q^m})\in b_n^*$, i.e., as has been noted in (**), $\varphi_{k_m}(b_n)\subseteq\theta^{\mathfrak{A}_1|d(n)}_{f_{0(j_q^m,m),g}}$, with the latter congruence identified with $0(j_q^m,m)$. Therefore, $\varphi_{k_m}(b_n)\ni 0(j_q^m,m)$. Moreover, for any $i\in b_n$ we get $h(i)=f_{p_n}(i)$;

(β) for any $n \in \omega$ and $i,j \le n$, $[\![t_i(f_1,\dots,f_k,g,h) = t_j(f_1,\dots,f_k,g,h)]\!]$ either contains $\omega \setminus (b_1 \vee \dots \vee b_n)$ or does not intersect with $\omega \setminus (b_1 \vee \dots \vee b_n)$.

Bearing in mind that $Con_p < \mathfrak{A}_1,h >, Con_p\mathfrak{A}_1$ are Boolean algebras, and that

$$[\![t_i(f_1,\dots,f_k,g,h) = t_j(f_1,\dots,f_k,g,h)]\!] \cap (b_1 \vee \dots \vee b_n) =$$
$$= \bigvee_{l=1}^{n} [\![t_i(f_1,\dots,f_k,g,f_{p_l}) = t_j(f_1,\dots,f_k,g,f_{p_l})]\!] \cap b_l,$$

we get from (β) that the Boolean algebras $Con_p < \mathfrak{A}_1,h >$ and $Con_p\mathfrak{A}_1$ coincide.

Let now $\mathfrak{A}_2 = < \mathfrak{A}_1,h >$, and we will show that there is no homomorphism from the algebra $\mathfrak{A}_2$ to $\mathfrak{A}_1/\neg d$ for any $d \in D$. Assume to the contrary that $d \in D$, and k is an $\mathfrak{A}_2$ homomorphism on $\mathfrak{A}_1/\neg d$. As has been noted in the beginning of the proof of case **(2)**, the homomorphism k meets condition (1) imposed on the homomorphisms of the sequence $k_n (n \in \omega)$. By virtue of the equality $Con_p\mathfrak{A}_2 = Con_p\mathfrak{A}_1$, the condition (2) for the mapping $\varphi_{k|\mathfrak{A}_1}$ is also met. Therefore, $k|\mathfrak{A}_1 = k_m$ for some $m \in \omega$. Let us assume in this case that $k(h) = w_q/\neg d = w_q|d$, and let $\eta(m,q) = n$. By the definition of h we have $\theta^{\mathfrak{A}_2}_{h,f_{p_n}} \supseteq b_n$. Hence, we get

$$\varphi_{k_m}(\theta^{\mathfrak{A}_2}_{h,f_{p_n}}) = \theta^{\mathfrak{A}_1/\neg d(m)}_{k_m(h)k_m(f_{p_n})} = \theta^{\mathfrak{A}_1|d(m)}_{w_q|d(m),k_m(f_{p_n})} \supseteq \varphi_{k_m}(b_n) \ni 0(j^m_q, m),$$

i.e., $0(j^m_q,m) \in d$ and $w_q(0(j^m_q,m)) = k_m(f_{p_n})(0(j^m_q,m))$. According to ($\alpha$), however, we have $w_q(0(j^m_q,m)) \ne k_m(f_{p_n})(0(j^m_q,m))$. The contradiction obtained proves the impossibility of the existence of the homomorphism h.

Iterating the construction of $\mathfrak{A}_2$ relative to $\mathfrak{A}_1$, let us build a sequence of finitely generated algebras $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n, \dots$ with the property $Con_p\mathfrak{A}_n = Con_p\mathfrak{A}_1$ for any $n \in \omega$ and for any $m > n$, while for any $d \in D$ we have $\mathfrak{A}_m \not\to\!\!\!\not\to \mathfrak{A}_n|d$.

It should be recalled that $Con_p\mathfrak{A}_1 = B(\omega^\alpha \cdot s)$. Let $\mathcal{B}_1, \dots, \mathcal{B}_n, \dots$ be a sequence of Boolean algebras such that $\mathcal{B}_n = B(\omega^{\alpha+n})$ for any $n \in \omega$. Let $\mathfrak{F}_l$ $(l \le s)$ be ultrafilters on $Con_p\mathfrak{A}_1$ corresponding to the points $r_1,\dots,r_s$ of the space $(\bar{C}on_p\mathfrak{A}_1)^*$, and let $\alpha^n_1,\dots,\alpha^n_s$ be the corresponding congruences of the algebra $\mathfrak{A}_n$. For any $j \le s$, $\mathfrak{A}_n/\alpha^n_j$ is a simple algebra, and let γ^n_j be a natural homomorphism of $\mathfrak{A}_n$ on $\mathfrak{A}_n/\alpha^n_j$. Let G_n be an ultrafilter of a maximal Cantor-Bendixon rank on the algebra $\mathcal{B}_n$. Let $\mathfrak{C}_{n,j} = (\mathfrak{A}_n/\alpha^n_j)^{\mathcal{B}_n}$, and let β^n_j be a natural homomorphism from the algebra $\mathfrak{C}_{n,j}$ to $\mathfrak{C}_{n,j}/\delta^n_j$, where $\delta^n_j \in Con\mathfrak{C}_{n,j}$ and δ^n_j correspond to the ultrafilter G_n. In this case, $\mathfrak{C}_{n,j}/\delta^n_j \cong \mathfrak{A}_n/\alpha^n_j$, as $\mathfrak{C}_{n,j}/\delta^n_j$ is simple and, as has been repeatedly noted, in a congruence-distributive variety any factor of a Boolean power of the simple algebra $\mathfrak{A}_n/\alpha_j^n$ has the form of a Boolean power of this algebra. Let us set the algebra $\mathfrak{D}_n$ equal to a subdirect product of the algebras $\mathfrak{A}_n, (\mathfrak{A}_n/\alpha_1^n)^{\mathcal{B}_n}, \ldots, (\mathfrak{A}_n/\alpha_s^n)^{\mathcal{B}_n}$ with the basic set $\{< a, b_1, \ldots, b_s > \in \mathfrak{A}_n \times (\mathfrak{A}_n/\alpha_1^n)^{\mathcal{B}_n} \times \ldots \times (\mathfrak{A}_n/\alpha_s^n)^{\mathcal{B}_n} \mid \text{for any } j \le s \ \gamma_j^n(a) = \beta_j^n(b_j)\}$.

Let us prove now that for any $m, n_1, \ldots, n_k \in \omega$ such that $m \notin \{n_1, \ldots, n_k\}$, we have $\mathfrak{D}_m \not\leftarrow\!\!\not\leftarrow \mathfrak{D}_{n_1} \times \ldots \times \mathfrak{D}_{n_k}$. Let us assume to the contrary that $m \notin \{n_1, \ldots, n_k\}$, and let δ be a homomorphism from the algebra $\mathfrak{D}_{n_1} \times \ldots \times \mathfrak{D}_{n_k}$ (henceforth, $\mathfrak{D}$) to $\mathfrak{D}_m$. Since the largest congruences of the algebras $\mathfrak{D}_i$ are principal ones, there is a partition of the congruence $\nabla_{\mathfrak{D}}$ with the principal congruences $\beta_1, \ldots, \beta_k$ such that the factors of the algebra $\mathfrak{D}$ relative to the congruences $\neg\beta_i (i \le k)$ are the algebras $\mathfrak{D}_{n_1}, \ldots, \mathfrak{D}_{n_k}$. If $g_i (i \le k)$ are natural homomorphisms of the algebra $\mathfrak{D}$ into factor-algebras of the algebra $\mathfrak{D}_m$ corresponding to the congruences $\mu_i = \ker\delta \vee \neg\beta_i$ then, since $\mathfrak{M}$ is congruence-distributive, g_i are homomorphisms of the algebra $\mathfrak{D}$ on some algebras $\mathfrak{D}_m^1, \ldots, \mathfrak{D}_m^k$ such that $\mathfrak{D}_m \cong \mathfrak{D}_m^1 \times \ldots \times \mathfrak{D}_m^k$ (from now on, we will identify $\mathfrak{D}_m$ with $\mathfrak{D}_m^1 \times \ldots \times \mathfrak{D}_m^k$), in which case the congruences corresponding to the $\mathfrak{D}_m$ projections on $\mathfrak{D}_m^i$ (as well as the congruences $\neg\beta_i$) are principal congruences of the algebra $\mathfrak{D}_m$, i.e., elements from $Con_p\mathfrak{D}_m$.

Let us refer to these elements of $Con_p\mathfrak{D}_m$ as $u_1, \ldots, u_k$. As $\mathfrak{A}_m << \mathfrak{D}_m$, $(Con_p\mathfrak{A}_m)^*$ is naturally identified with a subspace of the space $(Con_p\mathfrak{D}_m)^*$, and let $d_i^* = u_i^* \cap (Con_p\mathfrak{A}_m)^*$. Since $u_1^*, \ldots, u_k^*$ form a partition of $(Con_p\mathfrak{D}_m)^*$, $d_1^*, \ldots, d_k^*$ also form a partition of $(Con_p\mathfrak{A}_m)^*$ and, hence, at least one of the elements $d_1, \ldots, d_k$ belongs to the set D. Let it be a d_1. Therefore, since $\ker g_1 = \ker\delta \vee \neg\beta_1$ and $\mathfrak{D}/\neg\beta_1 = \mathfrak{D}_{n_1}$, we get a homomorphism (we will denote it by g_1') from the algebra $\mathfrak{D}_{n_1}$ to the algebra $\mathfrak{D}_m/\neg u_1$. Extending, if required, the homomorphism g_1' by projecting the algebra $\mathfrak{D}_m/\neg u_1$, one can assume that the element d_1^* contains only one point from R. Let it be a point r_1.

γ will denote a projection of the algebra $\mathfrak{D}_m/\neg u_1$ to the algebra $\mathfrak{A}_m/\neg u_1$. Since the algebra $\mathfrak{A}_m/\neg d_1$ (as well as $\mathfrak{A}_m$) is finitely generated, let $h_1, \ldots, h_p$ be its generating algebras, while $v_1, \ldots, v_p$ be arbitrary elements of the algebra $\mathfrak{D}_{n_1}$ such that $\gamma(g_1'(v_i)) = h_i$ for $i \le p$. In this case $\gamma g_1'$ is a homomorphism from a subalgebra $< v_1, \ldots, v_p >_{\mathfrak{D}_{n_1}}$ of the algebra $\mathfrak{D}_{n_1}$, generated by the elements $v_1, \ldots, v_p$, to the algebra $\mathfrak{A}_m/\neg d_1$. Let $x_1, \ldots, x_q$ be generating elements of a finitely generated algebra $\mathfrak{A}_{n_1}$ and $y_1, \ldots, y_q$ arbitrary elements of the algebra $\mathfrak{D}_{n_1}$ of the type $y_i =< x_i, b_1^i, \ldots, b_s^i >$ for some $b_j^i \in (\mathfrak{A}_{n_1}/\alpha_j^{n_1})^{\mathcal{B}_{n_1}} (j \le s)$. The subalgebra of the algebra

$\mathfrak{D}_{n_1}$ generated by the elements $v_1,\dots,v_p$, $y_1,\dots,y_q$ will be denoted by $\mathfrak{E}$ and, since $< v_1,\dots,v_p >_{\mathfrak{D}_{n_1}} \subseteq \mathfrak{E}$, the homomorphism $\gamma g_1'$ maps $\mathfrak{E}$ on the algebra $\mathfrak{A}_m/\neg d_1$. Let π be a projection of the algebra $\mathfrak{D}_{n_1}$ on the algebra

$$(\mathfrak{A}_1/\alpha_1^{n_1})^{\mathfrak{B}_{n_1}} \times \dots \times (\mathfrak{A}_{n_1}/\alpha_s^{n_1})^{\mathfrak{B}_{n_1}}.$$

As $\mathfrak{E}$ is finitely generated, the algebra $\pi(\mathfrak{E})$ is also finitely generated. But $\pi(\mathfrak{E})$ is a subalgebra of the algebra

$$(\mathfrak{A}_{n_1}/\alpha_1^{n_1})^{\mathfrak{B}_{n_1}} \times \dots \times (\mathfrak{A}_{n_1}/\alpha_s^{n_1})^{\mathfrak{B}_{n_1}}.$$

Any finitely generated subalgebra of any Boolean power $(\mathfrak{A}_{n_1}/\alpha_j^{n_1})^{\mathfrak{B}_{n_1}}$ is contained in a subalgebra of the type $(\mathfrak{A}_{n_1}/\alpha_j^{n_1})^{\mathfrak{B}(j)}$, where $\mathfrak{B}(j)$ are finite subalgebras of the algebra $\mathfrak{B}_{n_1}$. Therefore, extending $\mathfrak{E}$ in an obvious way, if required, to another finitely generated subalgebra of the algebra $\mathfrak{D}_{n_1}$, one can assume

$$\pi(\mathfrak{E}) = (\mathfrak{A}_{n_1}/\alpha_1^{n_1})^{\mathfrak{B}(1)} \times \dots \times (\mathfrak{A}_{n_1}/\alpha_s^{n_1})^{\mathfrak{B}(s)}$$

for some finite Boolean algebras $\mathfrak{B}(1),\dots,\mathfrak{B}(s)$. If $a_i (i \le s)$ is an $\mathfrak{B}(i)$ atom such that $a_i^* \in G_{n_1}$, we get

$$\mathfrak{E} \cong \mathfrak{A}_{n_1} \times (\mathfrak{A}_{n_1}/\alpha_1^{n_1})^{\mathfrak{B}(1)|\neg a_1} \times \dots \times (\mathfrak{A}_{n_1}/\alpha_s^{n_1})^{\mathfrak{B}(s)|\neg a_s}.$$

Let $\mathfrak{B}'(i)$ denote $\mathfrak{B}(i)|\neg a_i$ and $\mathfrak{H}$ the algebra

$$(\mathfrak{A}_{n_1}/\alpha_1^{n_1})^{\mathfrak{B}'(1)} \times \dots \times (\mathfrak{A}_{n_1}/\alpha_s^{n_1})^{\mathfrak{B}'(s)}.$$

There is a principal congruence $\eta \in Con_p \mathfrak{D}_{n_1}$ such that $\mathfrak{E}/\eta \cong \mathfrak{E}|\neg\eta \cong \mathfrak{A}_{n_1}$, and $\mathfrak{E}/\neg\eta \cong \mathfrak{E}|\eta \cong \mathfrak{H}$. Let $\eta_1, \neg\eta_1 \in Con_p(\mathfrak{A}_m|d_1)$ be the images of η and $\neg\eta$, respectively, at the homomorphism $\gamma g_1'$. Since $(\eta)^* = \mathfrak{B}'(1)^* \cup \dots \cup \mathfrak{B}'(s)^*$ is a finite space, $(\eta_1)^*$ contains only a finite number of the points of the space $(Con_p \mathfrak{A}_m|d_1)^*$. Therefore, $(\neg\eta_1)^* \ni r_1$ and, hence, $\neg\eta_1 \in D$. Factorizing the algebra $\mathfrak{E}$ over the congruence η and the algebra $\mathfrak{A}_m|d_1$ over the $\gamma g_1'$-image of η, i.e., over the congruence η_1, we get a homomorphism γ_1 induced by the homomorphism $\gamma g_1'$, from the algebra $\mathfrak{E}/\eta \cong \mathfrak{A}_{n_1}$ to the algebra $\mathfrak{A}_m|d_1/\eta_1 \cong \mathfrak{A}_m|\neg\eta_1$, where $\neg\eta_1 \in D$.

By the conditions that hold for the algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n,\ldots$, the existence of the homomorphism γ_1 implies the inequality $n_1 \leq m$ and, as $m \notin \{n_1,\ldots,n_k\}$ by the assumption, $n_1 < m$. Therefore, there is now a homomorphism g_1' from the algebra $\mathfrak{D}_{n_1}$ to the algebra $\mathfrak{D}_m/\neg u_1$, and $n_1 < m$. Then $\neg u_1$ is a principal congruence on $\mathfrak{D}_m$, and for $d_1^* = u_1^* \cap (Con_p\mathfrak{A}_m)^*$ we get $d_1^* \cap D = \{r_1\}$. In this case, however, it follows from the definition of $\mathfrak{D}_m$ that $\mathfrak{D}_m/\neg u_1$ is isomorphic to the algebra $\mathfrak{F}$, which is a subdirect product of the algebras $\mathfrak{A}_m|d_1$ (it should be recalled that here d_1 is viewed as a subset of ω), $(\mathfrak{A}_m/\alpha_1^m)^{\mathcal{B}_m|l_1},\ldots,(\mathfrak{A}_m/\alpha_s^m)^{\mathcal{B}_m|l_s}$, where $l_1,\ldots,l_s \in \mathcal{B}_m$ and, moreover, $l_1,\ldots,l_s \in G_m$, where G_m is, by the definition of the algebra $\mathfrak{D}_m$, an ultrafilter of a maximal Cantor-Bendixon rank on $\mathcal{B}_m$. However, for $l_1,\ldots,l_s \in G_m$ we have $\mathcal{B}_m|l_1 \cong\ldots\cong \mathcal{B}_m|l_s \cong \mathcal{B}_m$ and, therefore, if π is a projection of the algebra $\mathfrak{F}$ on $(\mathfrak{A}_m/\alpha_1^m)^{\mathcal{B}_m|l_1}$, $\pi g_1'$ is a homomorphism from the algebra $\mathfrak{D}_{n_1}$ to the algebra $(\mathfrak{A}_m/\alpha_1^m)^{\mathcal{B}_m|l_1} \cong (\mathfrak{A}_m/\alpha_1^m)^{\mathcal{B}_m}$. Therefore, $\pi g_1'$ induces a homomorphism ψ from the Boolean algebra

$$Con_p\mathfrak{D}_{n_1} \subseteq Con_p\mathfrak{A}_{n_1} \times Con_p[(\mathfrak{A}_{n_1}/\alpha_1^{n_1})^{\mathcal{B}_{n_1}} \times\ldots\times (\mathfrak{A}_{n_1}/\alpha_s^{n_1})^{\mathcal{B}_{n_1}} =$$
$$= B(\omega^{\alpha}\cdot s) \times B(\omega^{\alpha+n_1})^s \cong B(\omega^{\alpha+n_1}\cdot s)$$

to the Boolean algebra $Con_p(\mathfrak{A}_m/\alpha_1^m)^{\mathcal{B}_m} \cong \mathcal{B}_m \cong B(\omega^{\alpha+m})$ which, by virtue of the inequality $n_1 < m$, is impossible. The contradiction obtained proves that for any $m,n_1,\ldots,n_k \in \omega$, the existence of a homomorphism from the algebra $\mathfrak{D}_{n_1}\times\ldots\times\mathfrak{D}_{n_k}$ to the algebra $\mathfrak{D}_m$ must imply $m \in \{n_1,\ldots,n_k\}$.

It should be noticed that all algebras of the type $\mathfrak{D}_n$ contain subalgebras isomorphic to the algebra $\mathfrak{A}$. Indeed, for $\mathfrak{A}_n$ it is obviously a subalgebra $\{f_1,\ldots,f_k,\ldots\}$, and for any α_i^n at $k \neq l \in \omega$, we have $f_k/\alpha_i^n \neq f_l/\alpha_i^n$. Therefore, for any $k \in \omega$, we get

$$< f_k, f_k^*/\alpha_1^n,\ldots,f_k^*/\alpha_s^n > \in \mathfrak{D}_n,$$

and

$$\{< f_k, f_k^*/\alpha_1^n,\ldots,f_k^*/\alpha_s^n > | k \in \omega\} \cong \mathfrak{A}$$

(here $f_k^*/\alpha_i^n \in (\mathfrak{A}_n/\alpha_i^n)^{\mathcal{B}_n}$ is such that $f_k^*/\alpha_i^n(p) \in f_k/\alpha_i^n$ for any $p \in (\mathcal{B}_n)^*$. Let us refer to this subalgebra of the algebra $\mathfrak{D}_n$ as $\mathfrak{A}(\mathfrak{D}_n)$, and to the element $< f_k, f_k^*/\alpha_1^n,\ldots,f_k^*/\alpha_s^n >$ as $f_k(\mathfrak{D}_n)$. For any infinite $I \subseteq \omega$, $\mathfrak{D}_I$ will denote the algebra $\sum_{n\in I}\mathfrak{D}_n(\mathfrak{A}(\mathfrak{D}_n))$, which is a subalgebra of the algebra $\prod_{n\in I}\mathfrak{D}_n$ with a basic set $\{h\in\prod_{n\in I}\mathfrak{D}_n \mid$ there are $m,k\in\omega$ such that for all $p\geq m$, $h(p)=f_k(\mathfrak{D}_m)\}$.

Let us show that for any infinite $I\subseteq\omega$ and any $n\in\omega$, we get $\mathfrak{D}_n<<\mathfrak{D}_I$ iff $n\in I$. If $n\in I$ then, obviously, $\mathfrak{D}_n<<\mathfrak{D}_I$. Let now g be a homomorphism from the algebra $\mathfrak{D}_I$ to $\mathfrak{D}_n$. If $h_1,\dots,h_t$ is a finite set of generating algebras of the algebra $\mathfrak{A}_n$, let $v_1,\dots,v_t\in\mathfrak{D}_I$ such that $g(v_i)=<h_i,\dots>\in\mathfrak{D}_n$ for $i\leq t$ as well. Let $l\in\omega$ be such that for all $m\geq l$, for all $i\leq t$ we have $v_i(m)=f_{k(i)}(\mathfrak{D}_m)$ for some $k(i)\in\omega$. It is obvious that for any $n\in\omega$ there is a homomorphism from the algebra $\mathfrak{D}_n$ to the algebra $\mathfrak{A}$ and, hence, to the algebras $\mathfrak{A}(\mathfrak{D}_m)$ isomorphic to it (at any $m\in\omega$).

Let us refer to some fixed homomorphisms of the algebras $\mathfrak{D}_n$ on the algebras $\mathfrak{A}(\mathfrak{D}_m)$ as $\varphi_{n,m}$. Let $I=\{i_1<\dots<i_m<\dots\}$ and $i_q<l\leq i_{q+1}$. Let $\mathfrak{C}$ be a subalgebra of the algebra $\mathfrak{D}_I$ with a basic set $\{h\in\mathfrak{D}_I \mid$ for all $m\in I, m\geq l$, $h(m)=f_k(\mathfrak{D}_m)$, where k is such that $f_k(\mathfrak{D}_m)=\varphi_{i_q,i_{q+1}}(h(i_q))\}$. Obviously, we get $\mathfrak{C}\cong\mathfrak{D}_{i_1}\times\dots\times\mathfrak{D}_{i_q}$. Moreover, if π is a projection of the algebra $\mathfrak{D}_I$ on the algebra $\mathfrak{D}_{i_1}\times\dots\times\mathfrak{D}_{i_q}$, $\ker\pi$ is a principal congruence.

Let β be the image of $\ker\pi$ under the homomorphism g (i.e., $\beta=\theta^{\mathfrak{D}_n}_{\pi(b),\pi(b)}$ if $\ker\pi=\theta^{\mathfrak{D}_I}_{a,b}$); β is a principal congruence on $\mathfrak{D}_n$. If π_1 is a projection of the algebra $\mathfrak{D}_n$ on the algebra $\mathfrak{A}_n$ then, by the construction of $\mathfrak{C}$, which included $<v_1,\dots,v_t>_{\mathfrak{D}_I}$, a subalgebra of the algebra $\mathfrak{D}_I$ generated by elements $v_1,\dots,v_t$, we get the equality $\pi_1 g(\mathfrak{C})=\mathfrak{A}_n$. On the other hand, $\ker\pi$ limited to $\mathfrak{C}$ is trivial (zero) and, hence, β, limited to $g(\mathfrak{C})$ is also trivial. Therefore, we found a principal congruence β on the algebra $\mathfrak{D}_n$ such that for some subalgebra $g(\mathfrak{C})$ of the algebra $\mathfrak{D}_n$, the β boundedness on this subalgebra is trivial, the projection of this subalgebra on the algebra $\mathfrak{A}_n$ coinciding with the whole algebra $\mathfrak{A}_n$. By the construction of the algebra $\mathfrak{D}_n$, the factor-algebra $\mathfrak{D}_n/\beta$ has the form $\{<a,\varphi_1(b),\dots,\varphi_s(b_s)> \mid <a,b_1,\dots,b_s>\in\mathfrak{D}_n$, and φ_i are homomorphisms from the algebras $(\mathfrak{A}_n/\alpha_i^n)^{\mathfrak{B}_n}$ induced by the projections of the Boolean algebra $\mathfrak{B}_n$ to the algebras $\mathfrak{B}_n|d_i$ for some $d_i\in G_n\}$.

It should be recalled that G_n is an ultrafilter of a maximal Cantor-Bendixon rank on the algebra $\mathfrak{B}_n\cong B(\omega^{\alpha+n})$. Then $\mathfrak{B}_n|d_i\cong\mathfrak{B}_n$ and, hence, $\mathfrak{D}_n/\beta\cong\mathfrak{D}_n$. In this case, since $\mathfrak{D}_I/\ker\pi\cong\mathfrak{D}_{i_1}\times\dots\times\mathfrak{D}_{i_q}, \mathfrak{D}_n/\beta\cong\mathfrak{D}_n$ and g induces a homomorphism from $\mathfrak{D}_I/\ker\pi$ to $\mathfrak{D}_n/\beta$, $\mathfrak{D}_n<<\mathfrak{D}_{i_1}\times\dots\times\mathfrak{D}_{i_q}$. As has been proved earlier, $n\in\{i_1,\dots,i_q\}\subseteq I$. Therefore, indeed, $\mathfrak{D}_n<<\mathfrak{D}_I$ for any $n\in\omega, I\subseteq\omega$ iff $n\in I$. Hence, it is obvious that for any $I_1,I_2\subseteq\omega$ $\mathfrak{D}_{I_1}<<\mathfrak{D}_{I_2}$ iff $I_1\subseteq I_2$.

The latter conclusion implies the existence of $2^{\aleph_0}$ pairwise incomparable elements in $<\mathfrak{JM}_{\aleph_0};<<>$, and an embedding of any countable partially ordered set into $<\mathfrak{JM}_{\aleph_0};<<>$. As was the case in the proof of lemma **10.1**, the embedding of any countable partially ordered set into $<\mathfrak{JM}_{\aleph_0};<<>$ is extended to that of any countable quasi-order in it. ■

The statement of theorem **10.3** results directly from the statements of lemmas **10.1** and **10.2**. ■

Priorities. Theorems **10.1** [173], **10.2** [163] and **10.3** [172] are by A.G.Pinus.

11. Embedding and Double Skeletons

Alongside with epimorphism relations, isomorphic embedding relations are fundamental in algebras of an arbitrary variety. We will say that the relation $a \le b$ holds between the isomorphism types a,b of certain algebras iff an algebra of the isomorphism type a is isomorphically embeddable into an algebra of the isomorphism type b. The relation $\le$ will be used between algebras themselves in an analogous sence. It is obvious that the relation $\le$ is a quasi-order relation on isomorphism types.

<u>**Definition 11.1.**</u> A quasi-ordered class $<\mathfrak{J}\mathfrak{K};\le>$ will be termed an embedding skeleton of an algebra class $\mathfrak{K}$.

The present section is devoted to embedding skeletons of congruence-distributive varieties. Let me first remark that there is a relation between embedding skeletons of congruence-distributive varieties and such traditional notions of universal algebra as subvariety lattices and subalgebra lattices. It should be recalled that an algebra $\mathfrak{A}$ of a certain class of algebras $\mathfrak{K}$ is called $\aleph$-universal in $\mathfrak{K}$ if $|\mathfrak{A}| \le \aleph$, and any $\mathfrak{K}_{\aleph}$-algebra is isomorphically embeddable into $\mathfrak{A}$. The following statement is directly deduced by analogy with the statement on epimorphism skeletons proved in section **9**.

<u>**Statement 11.1.**</u> For any variety of algebras there is an isotonic mapping

from the embedding skeleton $<\mathfrak{J}\mathfrak{M};\leq>$ to the lattice of subvarieties of the variety $\mathfrak{M}$. If $\aleph$ is an infinite cardinal, and there is an algebra $\mathfrak{U}_\aleph(\mathfrak{M})$ $\aleph$-universal in $\mathfrak{M}$, there is an isotonic mapping from the lattice of subalgebras of the algebra $\mathfrak{U}_\aleph(\mathfrak{M})$ to the limited embedding skeleton of $\mathfrak{M}$, $<\mathfrak{J}\mathfrak{M}_\aleph;\leq>$.

As was the case for epimorphism skeletons for arbitrary congruence-distributive varieties, those with extendable congruences prove universal in the class of all quasi-orders.

Theorem 11.1. If $\mathfrak{M}$ is a non-trivial congruence-distributive variety with extendable congruences, for any regular cardinal $\aleph > \aleph_0$, any quasi-order of the power not greater than $\aleph$ is embeddable into $<\mathfrak{J}\mathfrak{M}_\aleph;\leq>$.

Proof. Let $\mathfrak{A}$ be a simple at most countably infinite $\mathfrak{M}$-algebra. By theorem **1.17**, there are Boolean algebras $\mathfrak{B}_i$ of the power $\aleph (i \in \aleph)$ with the following properties: for any $i \neq j \in \aleph$ and $0 \neq a \in \mathfrak{B}_i$, $0 \neq b \in \mathfrak{B}_j$ we get $\mathfrak{B}_j|b \not\leq \mathfrak{B}_i|a$. By defining a Boolean algebra $\mathfrak{B}_I$ as $\sum_{i\in I} \mathfrak{B}_i$ for $I \subseteq \aleph$ (see the definition of this algebra before lemma **9.4**), we obviously get that for $I_1, I_2 \subseteq \aleph$ $\mathfrak{B}_{I_1} \leq \mathfrak{B}_{I_2}$, iff $I_1 \subseteq I_2$. Therefore, the partially ordered set $<2^\aleph;\subseteq>$ is isomorphically embeddable into $<\mathfrak{J}\mathfrak{M}_\aleph;\leq>$.

Let now $<A;\leq_1>$ be an arbitrary quasi-ordered set of the power not greater than $\aleph$. A partially ordered set $<A/\equiv_\leq;\leq_1>$ is isomorphically embeddable into $<2^\aleph;\subseteq>$. Let us call this isomorphism f. It should also be noticed that among the Boolean algebras just constructed, no Boolean algebras of the type $B((\omega^i + \eta)\cdot\aleph)$ (where $i \in \aleph$ and η is the ordinal type of rational numbers) have been chosen as direct co-factors. Moreover, for $i \neq j \in \aleph$ we have $B((\omega^i + \eta)\cdot\aleph) \not\leq B((\omega^j + \eta)\cdot\aleph)$, but $B((\omega^i + \eta)\cdot\aleph) \equiv_\leq B((\omega^j + \eta)\cdot\aleph)$. For any $a \in A$, let $h_{[a]}$ be a certain bijective mapping from the $[a]_{\equiv_{\leq_1}}$ equivalence class to the ordinal $\aleph$. Then for any $a \in A$ let

$$\mathfrak{B}_a = \mathfrak{B}_{f([a]_{\equiv_{\leq_1}})} \times B((\omega^{h_{[a]}(a)} + \eta)\cdot\aleph).$$

From the earlier remarks it is obvious that for $a,b \in A$ $\mathfrak{B}_a \leq \mathfrak{B}_b$ iff $a \leq b$. $\mathfrak{A}_a (a \in A)$ will denote an $\mathfrak{M}$-algebra $\mathfrak{A}^{\mathfrak{B}_a}$, in which case $|\mathfrak{A}_a| \leq |\mathfrak{B}_a| = \aleph$ for any $a \in A$, while by theorem **11.3**, $\mathfrak{A}_a \leq \mathfrak{A}_b$ iff $\mathfrak{B}_a \leq \mathfrak{B}_b$, i.e., iff $a \leq_1 b$. ■

Corollary 11.1. If $\mathfrak{M}$ is a non-trivial congruence-distributive variety with extendable congruences, any quasi-ordered set is isomorphically embeddable into the

$\mathfrak{M}$ embeddable skeleton.

As follows from the proof of theorem **11.1**, $\mathfrak{M}$-algebras implementing embeddings of arbitrary quasi-orders in $<\mathfrak{J}\mathfrak{M};\leq>$ will be of uncountable power for both embedding skeletons and epimorphism skeletons. Even "small" quasi-orders may be not embeddable into countable embedding skeletons of certain non-trivial congruence-distributive varieties with extendable congruences. For instance, $<\mathfrak{J}BA_{\aleph_0};\leq>$ is obviously isomorphic to the quasi-ordered set $(\omega_1 \oplus 1^*) \cup 1$, whose quasi-order $\omega_1 \oplus 1^*$ is described in section **10**, and $(\omega_1 \oplus 1^*) \cup 1$ is obtained by adding to $\omega_1 \oplus 1^*$ an element comparable to nothing (of the type of a one-element algebra isomorphism). Since it is often the case that a singleton algebra is not a subalgebra of other algebras of a variety, using $\mathfrak{M}'$ to denote a class of non-singleton $\mathfrak{M}$-algebras, we introduce the following definition.

Definition 11.2. A quasi-ordered set $<\mathfrak{J}\mathfrak{M}'_{\aleph_0};\leq>$ will be termed a countable *embedding skeleton of a variety $\mathfrak{M}$.

Therefore, a countable *embedding skeleton of a variety of Boolean algebras is linear-factor-ordered . On the other hand, theorem **3.3** entailss that a countable *embedding skeleton of a variety of Boolean algebras is isomorphically embeddable into a countable *embedding skeleton of any non-trivial congruence-distributive variety with extendable congruences, i.e., the countable *embedding skeleton of a variety of Boolean algebras $\omega_1 \oplus 1^*$ is minimal among countable *embedding skeletons of varieties of the class under discussion. Then the following statement holds.

Theorem 11.2. If $\mathfrak{M}$ is a congruence-distributive variety with extendable congruences, in which case either $\mathfrak{M}$ is semi-simple, $\mathfrak{M}_{SI}$ is an approximatizable class and the principal congruences are elementary definable on $\mathfrak{M}$ or $\mathfrak{M}$, is locally finite, then the following conditions are equivalent:

(a) the countable *embedding skeleton of $\mathfrak{M}$ is linear-factor-ordered;

(b) the countable *embedding skeleton of $\mathfrak{M}$ is minimal (i.e., isomorphic to $\omega_1 \oplus 1^*$);

(c) $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$, where $\mathfrak{A}$ is a certain quasi-primal algebra with no non-singleton subalgebras, and for any one-element subalgebras of the algebra $\mathfrak{A}$, there are $\mathfrak{A}$ automorphisms transferring these subalgebras into one another.

Proof. Let us first prove some lemmas.

Lemma 11.1. If $\mathfrak{M}$ is a congruence-distributive variety with extendable congruences and $<\mathfrak{J}\mathfrak{M}'_{\aleph_0};\leq>$ is linear-factor-ordered, for any simple at most countably infinite $\mathfrak{M}$-algebras $\mathfrak{A}_1,\mathfrak{A}_2$ $\mathfrak{A}_1 \equiv_{\leq} \mathfrak{A}_2$.

Proof. Assume that $\mathfrak{A}_1,\mathfrak{A}_2 \in \mathfrak{M}_{\aleph_0}$ and they are simple. Since $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\leq>$ is linear-factor-ordered, one can assume $\mathfrak{A}_1 \leq \mathfrak{A}_2$. For the same reason, we have either $\mathfrak{A}_1^2 \leq \mathfrak{A}_2$ or $\mathfrak{A}_2 \leq \mathfrak{A}_1^2$. As the congruences on $\mathfrak{M}$ are extendable, any subalgebra of a simple algebra is simple itself and, therefore, the case $\mathfrak{A}_1^2 \leq \mathfrak{A}_2$ is impossible. However, since $\mathfrak{A}_2 \leq \mathfrak{A}_1^2$ is embeddable, as $\mathfrak{A}_2$ is subdirectly non-decomposable, $\mathfrak{A}_2$ is embeddable into $\mathfrak{A}_1$. Thus, $\mathfrak{A}_1 \leq \mathfrak{A}_2$ and $\mathfrak{A}_2 \leq \mathfrak{A}_1$. ■

Lemma 11.2. If $\mathfrak{M}$ is a congruence-distributive variety with extendable congruences, $\mathfrak{A}$ is an infinite simple $\mathfrak{M}$-algebra and $<\mathfrak{J}\mathfrak{M}'_{\aleph_0};\leq>$ is linear-factor-ordered, then there is an infinite simple finitely generated $\mathfrak{M}$-algebra.

Proof. Since, due to extendable congruences on $\mathfrak{M}$, subalgebras of simple algebras are simple, $\mathfrak{A}$ can be considered countable. Let a_1,a_2 be different elements of $\mathfrak{A}$, and let $\mathfrak{A}_1$ be a subalgebra generated in $\mathfrak{A}$ by the elements a_1,a_2. $\mathfrak{A}_1$ is simple and infinite since, if it were finite,, having by lemma **11.1** the relation $\mathfrak{A}_1 \equiv_{\leq} \mathfrak{A}$, we would get a finite $\mathfrak{A}$, which contradicts the lemma conditions. Therefore, $\mathfrak{A}_1$ is countable, simple, and is generated by the elements a_1,a_2. ■

Lemma 11.3. If $\mathfrak{M}$ is a congruence-distributive variety with extendable congruences having an infinite simple finitely generated algebra, $\mathfrak{M}_{SI}$ is an approximatizable class, and the principal congruences on $\mathfrak{M}$ are elementary definable, any countable partially ordered set is isomorphically embeddable into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\leq>$.

Proof. Let $\mathfrak{A}$ be an infinite, simple, and finitely generated $\mathfrak{M}$-algebra. For the sake of simplicity, $\mathfrak{A}$ is assumed to be generated by two elements, a_1,a_2. $\mathfrak{A}_0$ will denote a diagonal subalgebra of a direct power $\mathfrak{A}^{\omega}$ of the algebra $\mathfrak{A}$, i.e., a subalgebra of the algebra $\mathfrak{A}^{\omega}$ formed by constant functions.

Let $\mathfrak{A} = \{a_1,a_2,\ldots,a_n,\ldots\}$, and f_i will denote an element of the algebra $\mathfrak{A}^{\omega}$ such that for any $m \in \omega$ we get $f_i(m) = a_i$. Therefore, $\mathfrak{A}_0 = \{f_1,f_2,\ldots,f_n,\ldots\}$, and the elements f_1, f_2 generate $\mathfrak{A}_0$. g will denote an element of the algebra $\mathfrak{A}^{\omega}$ such that for any $m \in \omega$ we have $g(m) = a_m$. Let $\mathfrak{A}_1$ be a subalgebra of the algebra $\mathfrak{A}^{\omega}$ generated by the elements f_1, f_2, g. For any set $P \subseteq \omega$, k_P will denote an element of the algebra $\mathfrak{A}^{\omega}$ such that for $m \in P$ we have $k_P(m) = a_1$, and for $m \in \omega \setminus P$ we have $k_P(m) = a_2$. Let $\mathfrak{A}(P)$ be a subalgebra of the algebra $\mathfrak{A}^{\omega}$ generated by the elements f_1, f_2, g, k_P.

We now will have to use some notions associated with the hyperarithmetic hierarchy of the subsets of the set ω. All these notions can be found, for instance, in a well-known monograph by H.Rogers [200]. For any set $P \subseteq \omega$, $\overline{P}$ will denote the hyperpower of the set P, and $\leq$ the relation of order (reducibility) in the hyperarithmetic hierarchy. Let σ be the signature of algebra $\mathfrak{A}$, and V the family of the numbers of the identities of the signature $\sigma \cup < a_1, a_2 >$ true on the algebra $< \mathfrak{A}, a_1, a_2 >$ at a certain fixed numeration of the signature $\sigma \cup < a_1, a_2 >$. According to C.Spector [218] (the remark following theorem **1**), there is a family $\{P_1, P_2, \dots P_n, \dots\}$ of subsets of the set ω such that for any $i, i_1, \dots, i_n \in \omega$ we have $\overline{P_i} \geq \overline{V}$ and, if $i \notin \{i_1, \dots, i_n\}$, $\overline{P_i} \not\leq \overline{P_{i_1}} \vee \dots \vee \overline{P_{i_n}}$.

For any $I \subseteq \omega$, $\mathfrak{A}_I$ will denote a subalgebra of the algebra $\prod_{i \in I} \mathfrak{A}(P_i)$ with a basic set $\{f \in \prod_{i \in I} \mathfrak{A}(P_i)$, there are $n \in \omega$ and $a \in \mathfrak{A}_0$ such that for any $m > n$ $f(m) = a\}$. Let us prove that for any $I, J \subseteq \omega$, the algebra $\mathfrak{A}_I$ is isomorphically embeddable into the algebra $\mathfrak{A}_J$ iff $I \subseteq J$.

Let $I \subseteq J$, and let ψ be a certain fixed homomorphism from the algebra $\mathfrak{A}_I$ to the algebra $\mathfrak{A}_0 \cong \mathfrak{A}$ (its existence is obvious). Let us define a mapping φ from the algebra $\mathfrak{A}_I$ to the algebra $\mathfrak{A}_J$ in the following way: for $f \in \mathfrak{A}_I$, let $\varphi(f)$ be such that $\varphi(f)(m) = f(m)$ if $m \in I$, and $\varphi(f)(m) = \psi(f)$ if $m \in J \setminus I$. It is obvious that φ is an isomorphic embedding of $\mathfrak{A}_I$ into the algebra $\mathfrak{A}_J$.

Let us prove the opposite case. Let there be a certain embedding φ from the algebra $\mathfrak{A}_I$ to the algebra $\mathfrak{A}_J$. Let us prove that, in this case, $I \subseteq J$. Let us assume to the contrary, that $l \in I \setminus J$. ψ will denote a fixed homomorphism from the algebra $\mathfrak{A}(P_l)$ to the algebra $\mathfrak{A}_0$. $\mathfrak{A}_l$ will denote a subalgebra of the algebra $\mathfrak{A}_I$ with a basic set $\{f \in \mathfrak{A}_I |$, for any $k \in I \setminus \{l\}$ $f(k) = \psi(f(l))\}$. Obviously, $\mathfrak{A}_l \cong \mathfrak{A}(P_l)$. Since $\mathfrak{A}(P_l)$ and, hence, $\mathfrak{A}_l$, are finitely generated, by the definition of the algebra $\mathfrak{A}_J$, there is an $n \in J$ such that for any $h \in \mathfrak{A}_l$ there is an $a(h) \in \mathfrak{A}_0$ such that for $m \in J$ obeying the inequality $m > n$ we get $\varphi(h)(m) = a(h)$. The mapping $h \to a(h)$ is, obviously, a homomorphism from the algebra $\varphi(\mathfrak{A}_l)$ to the algebra $\mathfrak{A}_0$, and the projection of the algebra $\varphi(\mathfrak{A}_l) \subseteq \prod_{i \in J} \mathfrak{A}(P_i)$ relative to the set $\{I \in J | i \leq n\}$ is an isomorphism from the algebra $\varphi(\mathfrak{A}_l)$ to a certain subalgebra of the algebra $\prod_{i\leq n, i\in J}\mathfrak{A}(P_i)$. Therefore, there is an isomorphic embedding η of the algebra $\mathfrak{A}(P_l)\cong\mathfrak{A}_l$ in the algebra $\mathfrak{A}(P_{i_1})\times\ldots\times\mathfrak{A}(P_{i_k})$, where $\{i_1,\ldots,i_k\}=\{i\in J | i\leq n\}$.

Let now $t_1^i(x_1,\ldots,x_i),\ldots,t_j^i(x_1,\ldots,x_i),\ldots$ $(i\in\omega)$ be an enumeration of all the terms of the signature of the variety $\mathfrak{M}$ from the variables $x_1,\ldots,x_i$. Then, by the definition, for any $P\subseteq\omega$ we have $\mathfrak{A}(P)=\{t_j^4(f_1,f_2,g,k_P) | j<\omega\}$. For any $i\in\omega$, however, by the definition of the element k_P, we have

$$t_j^4(f_1,f_2,g,k_P)(i)=t_j^4(f_1,f_2,g,f_1)(i) \text{ if } i\in P,$$
$$t_j^4(f_1,f_2,g,k_P)(i)=t_j^4(f_1,f_2,g,f_2)(i) \text{ if } i\notin P.$$

Let the recursive functions $\alpha(x),\beta(x)$ be such that

$$t_j^4(f_1,f_2,g,f_1)=t_{\alpha(j)}^3(f_1,f_2,g) \text{ and}$$
$$t_j^4(f_1,f_2,g,f_2)=t_{\beta(j)}^3(f_1,f_2,g).$$

The chain $<t_{\alpha(j)}^3,t_{\beta(j)}^3,P>$ will be called a canonical description of the element $t_j^4(f_1,f_2,g,k_P)\in\mathfrak{A}(P)$, and we will write $t_j^4(f_1,f_2,g,k_P)=<t_{\alpha(j)}^3,t_{\beta(j)}^3,P>$.

The embedding η of the algebra $\mathfrak{A}(P_l)$ into the algebra $\mathfrak{A}(P_{i_1})\times\ldots\times\mathfrak{A}(P_{i_k})$ is uniquely defined by the images of the elements f_1,f_2,g,k_{P_l} generating the algebra $\mathfrak{A}(P_l)$. Let the following equalities be valid for some $m_1,\ldots,m_k,n_1,\ldots,n_k,r_1,\ldots,r_k,s_1,\ldots,s_k\in\omega$:

$$\eta(f_1)=<t_{m_1}^4(f_1,f_2,g,k_{P_{i_1}}),\ldots,t_{m_k}^4(f_1,f_2,g,k_{P_{i_k}})>=$$
$$=<<t_{\alpha(m_1)}^3,t_{\beta(m_1)}^3,P_{i_1}>,\ldots,<t_{\alpha(m_k)}^3,t_{\beta(m_k)}^3,P_{i_k}>>,$$

$$\eta(f_2)=<t_{n_1}^4(f_1,f_2,g,k_{P_{i_1}}),\ldots,t_{n_k}^4(f_1,f_2,g,k_{P_{i_k}})>=$$
$$=<<t_{\alpha(n_1)}^3,t_{\beta(n_1)}^3,P_{i_1}>,\ldots,<t_{\alpha(n_k)}^3,t_{\beta(n_k)}^3,P_{i_k}>>,$$

$$\eta(g)=<t_{r_1}^4(f_1,f_2,g,k_{P_{i_1}}),\ldots,t_{r_k}^4(f_1,f_2,g,k_{P_{i_k}})>=$$
$$=<<t_{\alpha(r_1)}^3,t_{\beta(r_1)}^3,P_{i_1}>,\ldots,<t_{\alpha(r_k)}^3,t_{\beta(r_k)}^3,P_{i_k}>>,$$

$$\eta(k_{P_l})=<t_{s_1}^4(f_1,f_2,g,k_{P_{i_1}}),\ldots,t_{s_k}^4(f_1,f_2,g,k_{P_{i_k}})>=$$
$$=<<t_{\alpha(s_1)}^3,t_{\beta(s_1)}^3,P_{i_1}>,\ldots,<t_{\alpha(s_k)}^3,t_{\beta(s_k)}^3,P_{i_k}>>.$$

By virtue of the definition of the elements k_P and theorem **5.6**, the following equivalence holds for any $i \in \omega$:

$$i \in P \Leftrightarrow k_P(i) = f_1(i) \Leftrightarrow \theta^{\mathfrak{A}(P)}_{k_P, f_1} \subseteq \theta^{\mathfrak{A}(P)}_{f_i, g}.$$

As η is an embedding of the algebra $\mathfrak{A}(P_l)$ into the algebra $\mathfrak{A}' = \mathfrak{A}(P_{i_1}) \times \ldots \times \mathfrak{A}(P_{i_k})$ and the congruences on $\mathfrak{A}'$ are extendable,

$$\theta^{\mathfrak{A}(P_l)}_{k_{P_l}, f_1} \subseteq \theta^{\mathfrak{A}(P_l)}_{f_i, g} \Leftrightarrow \theta^{\mathfrak{A}'}_{\eta(k_{P_l}), \eta(f_1)} \subseteq \theta^{\mathfrak{A}'}_{\eta(f_i), \eta(g)}.$$

For $h,k \in \mathfrak{A}'$, $[h = k]$ will denote $\{<i,j> | i \leq k, j \in \omega$ and $h(i)(j) = k(i)(j)$, where $h(i),k(i) \in \mathfrak{A}(P_i), h(i)(j), k(i)(j) \in \mathfrak{A}\}$.

In this case, by theorem **5.6**, $\theta^{\mathfrak{A}'}_{\eta(k_{P_l}), \eta(f_1)} \subseteq \theta^{\mathfrak{A}'}_{\eta(f_i), \eta(g)}$ is equivalent to $[\eta(k_{P_l}) = \eta(f_1)] \supseteq [\eta(f_i) = \eta(g)]$, while the latter, by virtue of the canonical descriptions of the elements $\eta(f_1)(j), \eta(f_2)(j), \eta(g)(j), \eta(k_{P_l})(j)$ introduced above, and since $f_i = t_i^2(f_1, f_2)$, is equivalent (for any $j \leq k$) to the following relation:

$$\begin{aligned}
&([\| t_i^2(t^3_{\alpha(m_j)}(f_1, f_2, g), (t^3_{\alpha(n_j)}(f_1, f_2, g)) = t^3_{\alpha(r_j)}(f_1, f_2, g) \|] \cap P_{i_j}) \cup \\
&\cup ([\| t_i^2(t^3_{\beta(m_j)}(f_1, f_2, g), t^3_{\beta(n_j)}(f_1, f_2, g)) = t^3_{\beta(r_j)}(f_1, f_2, g) \|] \cap \neg P_{i_j}) \subseteq \\
&\subseteq ([\| t^3_{\alpha(s_j)}(f_1, f_2, g) = t^3_{\alpha(m_j)}(f_1, f_2, g) \|] \cap P_{i_j}) \cup \\
&\cup ([\| t^3_{\beta(s_j)}(f_1, f_2, g) = t^3_{\beta(m_j)}(f_1, f_2, g) \|] \neg P_{i_j}).
\end{aligned}$$

Therefore, for any $i \in \omega$, $i \in P_l \Leftrightarrow$ for any $j \leq k$ and $u \in \omega$,

$$\begin{aligned}
&(u \in P_{i_j} \Rightarrow (t_i^2(t^3_{\alpha(m_j)}(a_1, a_2, t_u^2(a_1, a_2)), (t^3_{\alpha(n_j)}(a_1, a_2, t_u^2(a_1, a_2))) = \\
&= t^3_{\alpha(r_j)}(a_1, a_2, t_u^2(a_1, a_2)) \Rightarrow t^3_{\alpha(s_j)}(a_1, a_2, t_u^2(a_1, a_2)) = \\
&= t^3_{\alpha(m_j)}(a_1, a_2, t_u^2(a_1, a_2)))) \& (u \notin P_{i_j} \Rightarrow \\
&\Rightarrow t_i^2(t^3_{\beta(m_j)}(a_1, a_2, t_u^2(a_1, a_2)), t^3_{\beta(n_j)}(a_1, a_2, t_u^2(a_1, a_2))) = \\
&= t^3_{\beta(r_j)}(a_1, a_2, t_u^2(a_1, a_2)) \Rightarrow t^3_{\beta(s_j)}(a_1, a_2, t_u^2(a_1, a_2)) = \\
&= t^3_{\beta(m_j)}(a_1, a_2, t_u^2(a_1, a_2)))).
\end{aligned}$$

It should be recalled that we have already fixed a certain Gödel numeration of the terms and identities of the signature $\sigma \cup < a_1, a_2 >$, and that V is a family of the numbers of the identities of the signature $\sigma \cup < a_1, a_2 >$ true on the algebra $< \mathfrak{A}, a_1, a_2 >$. $\gamma(i,u,w,p,q)$ will denote the number of the identity

$$t_i^2(t_w^3(a_1,a_2,t_u^2(a_1,a_2)),t_p^3(a_1,a_2,t_u^2(a_1,a_2))) = t_q^3(a_1,a_2,t_u^2(a_1,a_2)),$$

while $\delta(u,w,p)$ stands for the number of the identity

$$t_w^3(a_1,a_2,t_u^2(a_1,a_2)) = t_p^3(a_1,a_2,t_u^2(a_1,a_2)).$$

It should be noticed that both $\gamma(i,u,w,p,q)$ and $\delta(u,w,p)$ are recursive functions. In this case, as has been noted earlier, for any $i \in \omega$ we get

$$i \in P_l \Leftrightarrow \forall j \le k, \forall u((u \in P_{i_j} \Rightarrow (\gamma(i,n,\alpha(m_j),\alpha(n_j),\alpha(r_j)) \in V \Rightarrow$$
$$\Rightarrow \delta(u,\alpha(s_j),\alpha(m_j)) \in V) \& (u \notin P_{i_j} \Rightarrow (\gamma(i,n,\beta(m_j),\beta(n_j),\beta(r_j)) \in V \Rightarrow$$
$$\Rightarrow \delta(u,\beta(s_j),\beta(m_j)) \in V).$$

Therefore, if the algebra $\mathfrak{A}_I$ is embeddable into the algebra $\mathfrak{A}_J$ and $l \in I \setminus J$, there is a $k \in \omega$ and numbers $i_1,\dots,i_k \in J$ such that

$$<\omega;+,\cdot,V> \models \exists m_1,\dots,m_k,n_1,\dots,n_k,r_1,\dots,r_k,s_1,\dots,s_k[\forall i(i \in P_l \Leftrightarrow$$
$$\Leftrightarrow \forall j \le k \forall u((u \in P_{i_j} \Rightarrow (\gamma(i,u,\alpha(m_j),\alpha(n_j),\alpha(r_j)) \in V \Rightarrow$$
$$\Rightarrow \delta(u,\alpha(s_j),\alpha(m_j)) \in V)) \& (u \notin P_{i_j} \Rightarrow (\gamma(i,u,\beta(m_j),\beta(n_j),\beta(r_j)) \in V \Rightarrow$$
$$\Rightarrow \delta(u,\beta(s_j),\beta(m_j)) \in V))))].$$

The latter relation implies the inequality $\overline{P_l} \le \overline{P_{i_1}} \vee \dots \vee \overline{P_{i_k}} \vee \overline{V}$ and, since $\overline{V} \le \overline{P_{i_1}}$, we get $\overline{P_l} \le \overline{P_{i_1}} \vee \dots \vee \overline{P_{i_k}}$, which contradicts the choice of the sets P_i. The obtained contradiction proves that if $\mathfrak{A}_I$ is embeddable into $\mathfrak{A}_J$, the assumption $I \not\subseteq J$ is impossible. Therefore, indeed, for any $I,J \subseteq \omega$, the algebra $\mathfrak{A}_I$ is embeddable into $\mathfrak{A}_J$ iff $I \subseteq J$, i.e., $<2^\omega;\subseteq>$ is isomorphically embeddable into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\le>$ and, since any countable partially ordered set is isomorphically embeddable into $<2^\omega;\subseteq>$, the lemma is proved. ∎

<u>**Lemma 11.4.**</u> Let $\mathfrak{A}$ be a quasi-primal algebra without non-singleton proper subalgebras. Let $\{a_1\},\{a_2\}$ be the $\mathfrak{A}$ algebra of subalgebras, and let the countable *embedding skeleton $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$ be linear-factor-ordered. In this case, there is an automorphism φ of the algebra $\mathfrak{A}$ such that $\varphi(a_1) = a_2$.

Proof. Let us choose subalgebras $\mathfrak{A}_1,\mathfrak{A}_2$ of the algebra $\mathfrak{A}^\omega$ in such a way that $\mathfrak{A}_i = \{f \in \mathfrak{A}^\omega \mid$ there is an $n \in \omega$ such that for all $j \ge n$, $f(j) = a_i\}$. As

$<\mathfrak{J}\mathfrak{M}_{\aleph_0};\leq>$ is linear-factor-ordered, we can assume $\mathfrak{A}_1 \leq \mathfrak{A}_2$, and let φ be an embedding from $\mathfrak{A}_1$ to $\mathfrak{A}_2$. By theorem **5.6**, a factor of the algebra $\mathfrak{A}_i$ over any of its principal congruences is isomorphic to $\mathfrak{A}_i$ itself. Therefore, if n is such that for $i \geq n$ we get $\varphi(a_1^0)(i) = a_2$ (here $a_i^0 \in \mathfrak{A}_i$, and for $j \in \omega$ $a_i^0(j) = a_i$) then, by factorizing $\mathfrak{A}_2$ and $\varphi(\mathfrak{A}_1)$ relative to $\underset{j \leq n}{\vee} \ker_{\mathfrak{A}_2} \pi_i$, we get an embedding ψ of the algebra $\mathfrak{A}_1$ into $\mathfrak{A}_2$ such that $\psi(a_1^0) = a_2^0$. Let now $b \in \mathfrak{A}_1$, $b(0) \neq a_1$ and $b(i) = a_1$ $(i > 0)$. Then, if $\mathfrak{A}_3$ is a subalgebra of the algebra $\mathfrak{A}_1$ generated by the elements b and a_1^0, $\mathfrak{A}_3 \cong \mathfrak{A}$. Let i be such that $b(i) \neq a_2$, in which case, since $\psi(\mathfrak{A}_3) \cong \mathfrak{A}$ is simple, $\pi_i(\psi)$ is an isomorphism from the algebra $\mathfrak{A}_3$ to $\mathfrak{A}$, and $\pi_i(\psi)(a_i^0) = a_2$. By virtue of the isomorphism $<\mathfrak{A}_3, a_1^0> \cong <\mathfrak{A}, a_1>$, we obtain the statement required. ■

Let us now return to the proof of theorem **11.2.** Since the implication (b) → (a) is obvious, we have to prove the implications (a) → (c) and (c) →(b). Let us start with the former, and let $\mathfrak{M}$ be a non-trivial semi-simple congruence-distributive variety with extendable congruences, and let $<\mathfrak{J}\mathfrak{M}'_{\aleph_0};\leq>$ be linear-factor-ordered. Let $\mathfrak{A}$ be a simple $\mathfrak{M}$-algebra. By lemmas **11.1**, **11.2** and **11.3**, $\mathfrak{A}$ is finite. By virtue of lemma **11.1**, and since $\mathfrak{M}$ is semi-simple, $\mathfrak{A}$ is the only subdirectly non-decomposable $\mathfrak{M}$-algebra. Therefore, as $<\mathfrak{J}\mathfrak{M}'_{\aleph_0};\leq>$ is linear-factor-ordered, for any $n \in \omega$ we get $\mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_n) \leq \mathfrak{A}^{\omega}$. Since $\mathfrak{A}$ is finite and $\mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_n)$ is finitely generated, there is an $m \in \omega$ such that $\mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_n) \leq \mathfrak{A}^m$. Let s_n be the least of such m, and let us identify $\mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_n)$ with a subalgebra of the algebra $\mathfrak{A}^{s_n}$ isomorphic to it.

Let us now choose a $k \in \omega$ such that $\mathfrak{F}_{\mathfrak{M}}(x_1,x_2,x_3) \subseteq \mathfrak{A}^{s_3} \subseteq \mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_k)$, and let g be a mapping from $\{x_1,\ldots,x_k\}$ to $\{x_1,x_2,x_3\}$ such that $g(x_i) = x_i$ at $i \leq 3$, and $g(x_i) = x_3$ at $i > 3$. Let us extend g to a homomorphism from $\mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_k)$ to $\mathfrak{F}_{\mathfrak{M}}(x_1,x_2,x_3)$. Then the limitation of g to $\mathfrak{A}^{s_3}$ is a homomorphism from $\mathfrak{A}^{s_3}$ to $\mathfrak{F}_{\mathfrak{M}}(x_1,x_2,x_3)$. By theorem **5.6**, however, all homomorphic images of the algebra $\mathfrak{A}^{s_3}$ have the form $\mathfrak{A}^l$ for $l \leq s_3$. Therefore, there is an $l \in \omega$ such that $\mathfrak{F}_{\mathfrak{M}}(x_1,x_2,x_3) \cong \mathfrak{A}^l$. And again, by theorem **5.6**, all congruences on $\mathfrak{A}^l$ and, hence, on $\mathfrak{F}_{\mathfrak{M}}(x_1,x_2,x_3)$, are permutable. By theorem **2.5**, this means that $\mathfrak{M}$ is congruence-permutable. By lemma **11.1**, and since $\mathfrak{A}$ is finite, $\mathfrak{A}$ has no non-singleton subalgebras. All these facts tofether imply that $\mathfrak{A}$ is quasi-primal. Moreover, as $\mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_n) \subseteq \mathfrak{A}^{s_n}$, $\mathfrak{F}_{\mathfrak{M}}(x_1,\ldots,x_n) \in \mathfrak{M}(\mathfrak{A})$ for any $n \in \omega$ and, hence, $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$, where $\mathfrak{A}$ is a quasi-primal algebra without non-singleton proper subalgebras. By lemma **11.4**, in this case, all one-element subalgebras of the algebra $\mathfrak{A}$ must be transformed into each other by an automorphism on $\mathfrak{A}$, which proves the implication (a) → (c) for the case when $\mathfrak{M}$ is semi-simple. The case when $\mathfrak{M}$ is locally finite is considered in an analogous way but without using lemma **11.3**.

Let us now prove the implication (c) → (b). Let $\mathfrak{A}$ be a quasi-primal algebra without non-singleton proper subalgebras, with any of its one-element subalgebras transformed one into another by automorphisms on $\mathfrak{A}$. Let us show that $< \mathfrak{J}\mathfrak{M}(\mathfrak{A})'_{\aleph_0}; \leq > \cong \omega_1 \oplus 1^*$. By theorem **7.6** and lemma **4.3**, any non-singleton $\mathfrak{M}(\mathfrak{A})_{\aleph_0}$-algebra is representable as $\mathfrak{A}^{\mathcal{B}}(\mathfrak{A}_1,\dots,\mathfrak{A}_n;F_1,\dots,F_n)$ for some at most countably infinite Boolean algebra $\mathcal{B}$, closed $F_1,\dots,F_n$ and some $\mathfrak{A}$-subalgebras $\mathfrak{A}_1,\dots,\mathfrak{A}_n$. Therefore, to prove that all non-singleton $\mathfrak{M}(\mathfrak{A})_{\aleph_0}$-algebras are comparable in terms of embedding, it suffices to show that under our conditions on $\mathfrak{A}$, for at most countably infinite Boolean algebras $\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3$, any closed subsets $F_1^1,\dots,F_m^1(F_1^2,\dots,F_n^2)$ of spaces $\mathcal{B}_1^*(\mathcal{B}_2^*)$ and any one-element subalgebras $\{a_1^1\},\dots,\{a_m^1\},\{a_1^2\},\dots,\{a_1^2\}$ of the algebra $\mathfrak{A}$, the algebras

$$\mathfrak{A}_1 = \mathfrak{A}^{\mathcal{B}_1}(\{a_1^1\},\dots,\{a_m^1\};F_1^1,\dots,F_m^1),$$
$$\mathfrak{A}_2 = \mathfrak{A}^{\mathcal{B}_2}(\{a_1^2\},\dots,\{a_n^2\};F_1^2,\dots,F_n^2)$$

and $\mathfrak{A}^{\mathcal{B}_3}$ are comparable in terms of embedding. Let us first remark that since we assume $a_i^k \neq a_j^k$ at $i \neq j$, $F_i^k \cap F_j^k = \varnothing$. As all closed F_i^k, F_j^k are separable in $\mathcal{B}_k^*$ by open-closed subsets, i.e., elements of the algebra $\mathcal{B}_k$, we get for appropriate $b_1,\dots,b_m \in \mathcal{B}_1$ $(c_1,\dots,c_m \in \mathcal{B}_2)$

$$\mathfrak{A}_1 \cong \mathfrak{A}^{\mathcal{B}_1|b_1}(\{a_1^1\};F_1^1)\times\dots\times\mathfrak{A}^{\mathcal{B}_1|b_m}(\{a_m^1\};F_m^1)$$

and

$$\mathfrak{A}_2 \cong \mathfrak{A}^{\mathcal{B}_2|c_1}(\{a_1^2\};F_1^2)\times\dots\times\mathfrak{A}^{\mathcal{B}_2|c_n}(\{a_n^2\};F_n^2).$$

Therefore, in order to prove the statement (b), it suffices to prove the following statements:

(1) all algebras of the type $\mathfrak{A}^{\mathcal{B}_1}(\{a_1\};F_1), \mathfrak{A}^{\mathcal{B}_2}(\{a_2\};F_2)$, $\mathfrak{A}^{\mathcal{B}_3}$, where $\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3$ are countable, are comparable in terms of embedding;

(2) for any $\mathcal{B}_1,\mathcal{B}_2,a_1,a_2,F_1,F_2$ there are $\mathcal{B}_3,a_3,F_3$ such that

$\mathfrak{A}^{\mathcal{B}_1}(\{a_1\};F_1) \times \mathfrak{A}^{\mathcal{B}_2}(\{a_2\};F_2) \equiv_{\leq} \mathfrak{A}^{\mathcal{B}_3}(\{a_3\};F_3)$.

Let us first make the following remark. Let $f_1 \in F_1$, and let us consider a space $(\mathcal{B}_1^* \setminus F_1) \cup \{f_1\}$ such that the system of open neighborhoods of the points other than f_1 in it will be the same as that of the same points of the space $\mathcal{B}_1^*$ not intersecting with F_1, while the system of open neighborhoods of the point f_1 in the new space will be the same as that of open neighborhoods of the set F_1 in the space $\mathcal{B}_1^*$, with the subset $F_1 \setminus \{f_1\}$ taken out of these neighborhoods. As can be seen easily, this space is Boolean (let us call it $\mathcal{B}_4^*$), and we get $\mathfrak{A}^{\mathcal{B}_1}(\{a_1\};F_1) \cong \mathfrak{A}^{\mathcal{B}_4}(\{a_1\};f_1)$, i.e., in the statements (1) and (2), all closed sets F_i can be considered one-element.

Since $\mathcal{B}_4$ is countable, it is either superatomic or contains an atomless subalgebra. Let us consider only the case when $\mathcal{B}_4$ and other Boolean algebras are superatomic; the other case can be considered in an analogous way. Then we have $\mathcal{B}_4 \cong B(\gamma)$, where γ is a certain ordinal, in which case for a certain $\delta \in \gamma+1$ we get $f_1 = f_\delta = \{b \in B(\gamma) | (\alpha,\delta] \subseteq b$ for some $\alpha < \delta\}$.

If δ is not a limit ordinal then, obviously, $\mathfrak{A}^{\mathcal{B}_4}(\{a_1\},f_1) \cong \mathfrak{A}^{B(\gamma \setminus \{\delta\})}$. If δ is a limit ordinal,

$$\mathfrak{A}^{\mathcal{B}_4}(\{a_1\},f_1) \cong \mathfrak{A}^{B(\delta)}(\{a_1\},f_\delta) \times \mathfrak{A}^{B((\delta,\gamma])}.$$

Let us now notice that for any ordinals δ_1,δ_2,γ the following comparisons are obvious:

if $\delta_1 \leq \delta_2$, $\mathfrak{A}^{B(\delta_1)}(\{a_1\},f_{\delta_1}) \leq \mathfrak{A}^{B(\delta_2)}(\{a_1\},f_{\delta_2})$;
if $\gamma \geq \delta_1$, $\mathfrak{A}^{B(\delta_1)}(\{a_1\},f_{\delta_1}) \leq \mathfrak{A}^{B(\gamma)}$;
if $\gamma < \delta_1$, $\mathfrak{A}^{B(\gamma)} \leq \mathfrak{A}^{B(\delta_1)}(\{a_1\},f_{\delta_1})$.

Moreover, since for any one-element subalgebras $\{a_1\},\{a_2\}$ of the algebra $\mathfrak{A}$ there is an $\mathfrak{A}$ automorphism transforming them into each other, $\mathfrak{A}^{B(\delta)}(\{a_1\},f_\delta) \cong \mathfrak{A}^{B(\delta)}(\{a_2\},f_\delta\}$. Therefore, the statement (1) is proved, and the statement (2) can be proved analogously. ∎

It follows from theorem **11.2** that all congruence-distributive varieties $\mathfrak{M}$ with extendable congruences such that $\mathfrak{M}_{SI}$ is approximatizable and the principal congruences on $\mathfrak{M}$ are elementary definable, with minimal or, which is the same, linear-factor-ordered *embedding skeletons are discriminator varieties. One can also obtain a number of further results on the structure of countable *embedding skeletons of discriminator varieties.

Theorem 11.3. If $\mathfrak{M}$ is a finitely generated discriminator variety with all its algebras containing a one-element subalgebra, $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\leq>$ is a better quasi-order and, in particular, $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\leq>$ contains neither infinite anti-chains nor infinitely descending chains.

Proof. Let us make use of the notations in the proof of theorem **10.2** and recall that $\overline{\mathcal{B}F}$ denotes a cortege $<\mathcal{B},F_1,\ldots,F_n>$, where $\mathcal{B}$ is at most countably infinite Boolean algebra and $F_1,\ldots,F_n$ are close subsets of the space $\mathcal{B}^*$. Let $\mathfrak{A}$ be a finite algebra such that any $\mathfrak{M}_{\aleph_0}$-algebra is isomorphic to a certain filtered Boolean power of the algebra $\mathfrak{A}$. As follows from the proof of theorem **18.2**, it suffices to show that the relation $I_{\overline{(\mathcal{B}F)_1}} \leq_1 I_{\overline{(\mathcal{B}F)_2}}$ implies the relation

$$\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n;F_1^1,\ldots,F_n^1)\leq \mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n;F_1^2,\ldots,F_n^2).$$

As was the case in the proof of theorem **10.2**, for $I_{\overline{(\mathcal{B}F)_1}} \leq_1 I_{\overline{(\mathcal{B}F)_2}}$ the space $\mathcal{B}_1^*$ can be identified with a certain subspace of the space $\mathcal{B}_2^*$ in such a way that $F_i^1=F_i^2\cap\mathcal{B}_1^*$. Elements of Boolean algebras will be identified with open-closed subsets of Stone spaces. Let us also assume that the families $\{F_1^1,\ldots,F_n^1\},\{F_1^2,\ldots,F_n^2\}$ are closed under intersections. Let $a_0=1,a_1,\ldots,a_n,\ldots$ be an enumeration of all the elements of the algebra $\mathcal{B}_0$. Let us define by induction over n embeddings h_n of a Boolean algebra $\mathcal{B}(n)$, which is a subalgebra of the algebra $\mathcal{B}_1$ generated by the elements $\{a_0,\ldots,a_n\}$, into a Boolean algebra $\mathcal{B}_1|d$ for some $d\in\mathcal{B}_2$.

Let $K_0=\{i\leq n|F_i^1=\varnothing\}$ and, hence, closed disjunct subsets of $\mathcal{B}_1^*$ and $\bigcup_{i\in K_0}F_i^2$ of the space $\mathcal{B}_2^*$ are separated by an open-closed set $d\in\mathcal{B}_2$ such that $\mathcal{B}_1^*\subseteq d$ and $d\cap(\bigcup_{i\in K_0}F_i^2=\varnothing$. Let us set $h_0(1_{\mathcal{B}_1})=d, h_0(0_{\mathcal{B}_1})=0_{\mathcal{B}_2}$, and assume that h_m is defined on a Boolean algebra $\mathcal{B}(m)$, and for any $c\in\mathcal{B}(m)$, $c\subseteq h_m(c)$ and $c\cap F_i^1\neq\varnothing$ iff $h_m(c)\cap F_i^2\neq\varnothing$, where $i=1,\ldots,n$. Let $c_1,\ldots,c_k$ be all atoms of the algebra $\mathcal{B}(m)$. If either $c_j\cap a_{m+1}=\varnothing$ or $c_j\cap\neg a_{m+1}=\varnothing$, the value of h_{m+1} on the elements $c_j\cap a_{m+1}, c_j\cap\neg a_{m+1}$ is assumed to be the same as the value of h_m.

Let now $c_j\cap a_{m+1}\neq\varnothing$ and $c_j\cap\neg a_{m+1}\neq\varnothing$. K_j^1 will denote the set $\{i\leq k|F_i^1\cap c_j\cap a_{m+1}=\varnothing\}$, while by K_j^2 the set $\{i\leq k|F_i^1\cap c_j\cap\neg a_{m+1}=\varnothing\}$. The sets

$c_j \cap a_{m+1}$ and $c_j \cap \neg a_{m+1}$ are closed and disjunct in $\mathcal{B}_2^*$ and, hence, there is a $b_j \in \mathcal{B}_2$ (we can assume $b_j \subseteq h_m(c_j)$) such that $b_j \supseteq c_j \cap a_{m+1}$ and $h_m(c_j) \setminus b_j \supseteq c_j \cap \neg a_{m+1}$. Analogously, there can be found elements $e_1^j, e_2^j \in \mathcal{B}_2$ such that

$$e_1^j, e_2^j \subseteq h_m(c_j), e_1^j \supseteq c_j \cap a_{m+1}, e_1^j \cap (\bigcup_{i \in K_j^1} F_i^2 \cap h_m(c_j)) = \varnothing$$

(such e_1^j can be found since $c_j \cap a_{m+1}$ and $F_i^2 \cap h_m(c_j)$ are closed and do not intersect; the latter is valid since the inequality $c_j \cap a_{m+1} \cap F_i^2 \cap h_m(c_j) \neq \varnothing$, as $h_m(c_j) \supseteq c_j$ and $F_i^2 \cap \mathcal{B}_1^* = F_i^1$, implies the inequality $c_j \cap a_{m+1} \cap F_i^1 \neq \varnothing$),

$$e_2^j \supseteq c_j \cap \neg a_{m+1},$$
$$e_2^j \cap (\bigcup_{i \in K_j^2} F_i^2 \cap h_m(c_j)) = \varnothing.$$

If $(\neg e_2^j \cap \neg b_j \cap h_m(c_j)) \cap \bigcup_{k \in K_j^1} F_k^2 = \varnothing$,

$$(\neg e_1^j \cap \neg b_j \cap h_m(c_j)) \cap \bigcup_{e \in K_j^2} F_e^2 = \varnothing,$$

then we set

$$h_{m+1}(c_j \cap a_{m+1}) = (b_j \cap e_1^j) \cup (\neg e_2^j \cap \neg b_j \cap h_m(c_j)),$$
$$h_{m+1}(c_j \cap \neg a_{m+1}) = (h_m(c_j) \cap \neg b_j \cap e_2^j) \cup (\neg e_1^j \cap b_j \cap h_m(c_j)),$$

in which case the induction condition for h_{m+1} is obviously met for elements of the type $c_j \cap a_{m+1}$ and $c_j \cap \neg a_{m+1}$. If, for instance, $\neg e_2^j \cap b_j \cap h_m(c_j) \cap F_k^2 \neq \varnothing$ for some $k \in K_j^1$, for the case when $(\neg e_2^j \cap \neg b_j \cap h_m(c_j) \cap F_k^2) \cap (\bigcup_{e \in K_j^2} F_e^2) = \varnothing$, one can "fix" e_2^j by adding to it a certain open-closed subset of the set $\neg e_2^j \cap \neg b_j \cap h_m(c_j)$ which contains all points of the set $\neg e_2^j \cap \neg b_j \cap h_m(c_j) \cap F_k^2$ and does not contain points of $\bigcup_{e \in K_j^2} F_e^2$. The case when $(\neg e_2^j \cap \neg b_j \cap h_m(c_j) \cap F_k^2) \cap (\bigcup_{e \in K_j^2} F_e^2) \neq \varnothing$ is impossible, since otherwise there would be an $e \in K_j^2$ such that $F_e^2 \cap F_k^2 \cap h_m(c_j) \neq \varnothing$, but $F_e^2 \cap F_k^2 = F_r^2$ for a certain $r \leq n$ and, by the condition on h_m, there would be a $p \in c_j$ such that $p \in F_k^1$. In this case, however, if

$p \in c_j \cap a_{m+1}$, $p \in c_j \cap a_{m+1} \cap F_k^1 \neq \varnothing$ and, hence, $k \notin K_j^1$, while if $p \in c_j \cap \neg a_{m+1}$, $c_j \cap a_{m+1} \cap F_e^1 \neq \varnothing$ and, hence, $e \notin K_j^1$. The obtained contradiction proves that the inequality

$$(\neg e_2^j \cap \neg b_j \cap h_m(c_j) \cap F_k^2) \cap (\bigcup_{e \in K_j^1} F_e^2) \neq \varnothing$$

is impossible.

By "fixing" e_1^j in an analogous way, if required, we get a definition of the mapping h_{m+1} on all elements of the type $c_j \cap a_{m+1}$, where $j \leq k$. On those elements of the algebra $\mathcal{B}(m+1)$ which are not its atoms, we define h_{m+1} by the additivity condition. Therefore, h_{m+1}, being an extension of h_m, isomorphically embeds the Boolean algebra $\mathcal{B}(m+1)$ into the Boolean algebra $\mathcal{B}_2|d$, in which case for any $i = 1,\ldots,n$ and $c \in \mathcal{B}(m+1)$, we get $c \cap F_i^1 \neq \varnothing$ iff $h_{m+1}(c) \cap F_i^2 \neq \varnothing$. Thus, $h = \bigcup_{m \in \omega} h_m$ will be an isomorphic embedding of the algebra $\mathcal{B}_1$ into the algebra $\mathcal{B}_2| d$, which obeys the same requirements on F_i^1 and F_i^2.

The embedding φ of the algebra $\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n;F_1^1,\ldots,F_n^1)$ into the algebra $\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n; F_1^2,\ldots,F_n^2)$ will be defined in the following natural way: if $f \in \mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n; F_1^1,\ldots,F_n^1)$ and is such that for a partition $c_1,\ldots,c_l$ of the space $\mathcal{B}_1^*$ by elements of the algebra $\mathcal{B}_1$, f is constant on $c_i (i = 1,\ldots,l)$ and $f(c_i) = b_i$, where $b_i \in \mathfrak{A}$, we set $\varphi(f)$ to be constant on the subsets $h(c_i)$ of the space $\mathcal{B}_2^*$. In this case $\varphi(f)(h(c_i)) = b_i (i = 1,\ldots,l)$ and $\varphi(f)(1_{\mathcal{B}_2} \setminus h(1_{\mathcal{B}_1})) = e$, where $\{e\}$ is a certain fixed one-element subalgebra of the algebra $\mathfrak{A}$. Obviously, φ is an isomorphic embedding of the algebra $\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n; F_1^1,\ldots,F_n^1)$ into the algebra $\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n; F_1^2,\ldots,F_n^2)$, which is the required proof. ■

Indeed, in the condition of theorem **11.3**, one can join together the statements of theorems **10.2** and **11.3**, i.e., it can be seen clearly from the proofs of these theorems that the relation $l_{\overline{(\mathcal{BF})_1}} \leq_1 l_{\overline{(\mathcal{BF})_2}}$ implies the existence of such homomorphism from the algebra $\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n; F_1^2,\ldots,F_n^2)$ to the algebra $\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n;F_1^1,\ldots,F_n^1)$ and an embedding of the latter in the former such that $\mathfrak{A}^{\mathcal{B}_1}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n;F_1^1,\ldots,F_n^1)$ is a retract of the algebra $\mathfrak{A}^{\mathcal{B}_2}(\mathfrak{A}_1,\ldots,\mathfrak{A}_n;F_1^2,\ldots,F_n^2)$. Therefore, the following corollary is valid.

Corollary 11.2. If $\mathfrak{M}$ is a finitely generated discriminator variety with all its algebras containing a one-element subalgebra, any infinite family of countable $\mathfrak{M}$-algebras contains an infinite sequence of different algebras $\mathfrak{A}_1,\dots,\mathfrak{A}_n,\dots$ such that for any $i<j$ $\mathfrak{A}_i$ is a retract of $\mathfrak{A}_j$.

The statement of the corollary results from the remark made before its formulation, by applying the Ramsey theorem in a standard way.

As has been shown in section **5**, a variety of rings obeying a certain identity of the type $x^m=x$ is a finitely generated and discriminator variety and, hence, by corollary **11.2**, in any infinite family of rings obeying a certain identity $x^m=x$, an infinite sequence of different rings $\mathfrak{A}_1,\dots,\mathfrak{A}_n,\dots$ can be found for which $\mathfrak{A}_i$ is a retract of $\mathfrak{A}_j$ for any $i<j$.

Problem 11.1. Is the condition of the existence of a one-element subalgebra for any $\mathfrak{M}$-algebra in the formulation of theorem **11.3** necessary ?

Any finitely generated variety is locally finite. The following theorem proves that countable embedding skeletons of not locally finite discriminator varieties are universal in the class of countable partial orders.

Theorem 11.4. If $\mathfrak{M}$ is a discriminator variety of a finite signature which is not locally finite, any countable partially ordered set is isomorphically embeddable in the countable embedding skeleton of $\mathfrak{M}$. If, moreover, $\mathfrak{M}$ contains at least two simple algebras, any countable quasi-ordered set is embeddable into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\leq>$.

Proof. Let us first show that any not locally finite discriminator variety $\mathfrak{M}$ of a finite signature contains an infinite simple finitely generated algebra. Let $\mathfrak{A}$ be a certain finitely generated $\mathfrak{M}$-algebra. For the sake of simplicity, we assume it to be generated by two elements, a and b. By theorem **5.7**, $\mathfrak{A}$ can be considered the Boolean product of some simple $\mathfrak{M}$-algebras $\mathfrak{A}_x(x\in\mathfrak{B}^*)$ over a certain Boolean algebra $\mathfrak{B}$. Since the algebras $\mathfrak{A}_x$ are generated by the elements $a(x),b(x)$, in order to prove that there is an infinite finitely generated $\mathfrak{M}$-algebra, it suffices to show that at least one of the algebras $\mathfrak{A}_x(x\in\mathfrak{B}^*)$ is infinite. Let us assume that the opposite case is true, i.e., all $\mathfrak{A}_x$ are finite. For any $x\in\mathfrak{B}^*$, T_x will denote the finite family of the terms of the signature σ of the variety $\mathfrak{M}$ over two variables, u,z so that

$$\mathfrak{A}_x=\{\{t(a(x),b(x))\mid t(u,z)\in T_x\}.$$

Let R_x be a finite system (since both T_x and the signature σ are finite) of all equalities of the type $h(t_1(u,z),\dots,t_n(u,z)) = t_j(u,z)$, where $h \in \sigma$, $t_1,\dots,t_n,t_j \in T_x$, and

$$\mathfrak{A}_x \models h(t_1(a(x),b(x)),\dots,t_n(a(x),b(x))) = t_j(a(x)),b(x)).$$

Obviously, for any algebra $\mathfrak{C}$ generated by some elements c,d, if $\mathfrak{C} \models r(c,d)$ for any $r(u,z) \in R_x$, $\mathfrak{C}$ is a homomorphic image of the algebra $\mathfrak{A}_x$ and, as $\mathfrak{A}_x$ is simple, we get either $\mathfrak{C} \cong \mathfrak{A}_x$ or $\mathfrak{C}$ is one-element. By virtue of the definition of a Boolean product, the set $C_x = \{y \in \mathfrak{B}^* \mid \mathfrak{A}_y \models r(a(y)),b(y))$ for any $r(u,z) \in R_x\}$ is an open-closed subset in $\mathfrak{B}^*$ containing the element x. Therefore, $\{C_x \mid x \in \mathfrak{B}^*\}$ is an open cover of the compact space $\mathfrak{B}^*$ and, hence, there is a finite set $\{x_1,\dots,x_s\} \subseteq \mathfrak{B}^*$ such that $\mathfrak{B}^* = C_{x_1} \cup \dots \cup C_{x_s}$. Therefore, for any $y \in \mathfrak{B}^*$, either $\mathfrak{A}_y$ is one-element, or $\mathfrak{A}_y \cong \mathfrak{A}_{x_i}$ for some $i \le s$, in which case the latter isomorphism φ is such that $\varphi(a(y)) = a(x_i), \varphi(b(y)) = b(x_i)$. Therefore, the algebra $\mathfrak{A}$ generated by the elements a,b is finite. The contradiction obtained proves that there must be an infinite algebra among the finitely generated simple algebras $\mathfrak{A}_x$. The existence of an infinite finitely generated $\mathfrak{M}$-algebra implies, by lemma **11.3**, embedding any countable partial order into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\le>$.

Let now $\mathfrak{M}$ have at least one more simple algebra $\mathfrak{A}_1$, and let $<A;\le>$ be an arbitrary countable quasi-order. Let us assume that, for any $a \in A$, $[a]_{\equiv_\le}$ is infinite. Let $\mathfrak{K}$ be an arbitrary countable family of non-superatomic countable Boolean algebras and, therefore, for any $\mathfrak{B}_1,\mathfrak{B}_2 \in \mathfrak{K}$ we get $\mathfrak{B}_1 \le \mathfrak{B}_2$. Let $h_{[a]}$ be for $[a] \in A/{\equiv_\le}$ a bijective mapping from the set $[a]_{\equiv_\le}$ to $\mathfrak{K}$. Then, obviously, if f is an isomorphic embedding of $<A/{\equiv_\le};\le>$ into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\le>$, $\varphi(a) = f([a]_{\equiv_\le}) \times \mathfrak{A}_1^{h_{[a]}(a)}$ will be the required isomorphic embedding of the quasi-order $<A;\le>$ into the countable embedding skeleton of $\mathfrak{M}$. ■

Now the following problem is open for discussion.

Problem 11.2. Is the requirement on the finiteness of the signature necessary in the formulation of theorem **11.4** ?

Moreover, the results obtained in theorems **11.3** and **11.4** leave the problem of whether $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\le>$ is well-quasi-ordered open only for locally finite discriminator varieties which are not finitely generated.

The following problem, in particular, remains unsolved.

Problem 11.3. If $<\mathfrak{J}(\mathfrak{M}_S)_{\aleph_0};\leq>$, i.e., a family of simple $\mathfrak{M}$-algebras of at most countably infinite power, is well-quasi-ordered, does this imply that the countable embedding skeleton of a locally finite discriminator variety $\mathfrak{M}$ is also well quasi-ordered ?

By way of concluding this section, let us turn to so-called double skeletons of algebra varieties, i.e., to studying the problem of interaction between epimorphism and embedding relations.

Definition 11.3. A double skeleton of a variety $\mathfrak{M}$ is a family of the isomorphism type of $\mathfrak{M}$-algebras $\mathfrak{J}\mathfrak{M}$ having the epimorphism relations $<<$ and embedding relations $\leq$, i.e., a twice-quasi-ordered family $<\mathfrak{J}\mathfrak{M};<<,\leq>$.

Definition 11.4. Epimorphism and embedding relations are called finitely independent on a variety $\mathfrak{M}$ provided that any finite set $<A;\leq_1,\leq_2>$ with two arbitrary quasi-orders is isomorphically embeddable in $<\mathfrak{J}\mathfrak{M};<<,\leq>$.

Theorem 11.5. **(CH).** If $\mathfrak{M}$ is a non-trivial congruence-distributive variety with extendable congruences, the epimorphism and embedding relations are finitely independent on $\mathfrak{M}$.

Proof. Let $\mathfrak{A}$ be a simple $\mathfrak{M}$-algebra. Theorem **1.27**, proved under an assumption weaker than the continuum-hypothesis $P(2^{\omega})$, claims finite independence of $<<$ and $\leq$ relations on a variety of Boolean algebras, i.e., for any finite set $<A;\leq_1,\leq_2>$ there are Boolean algebras $\mathfrak{B}_a (a\in A)$ such that, for $a,b\in A$, $\mathfrak{B}_a<<\mathfrak{B}_b$ iff $a\leq_1 b$, and $\mathfrak{B}_a\leq\mathfrak{B}_b$ iff $a\leq_2 b$. By corollary **3.1** and theorem **3.3**, an analogous statement is also valid for $\mathfrak{M}$-algebras $\mathfrak{A}^{\mathfrak{B}_a}(a\in A)$. ■

Problem 11.4. Is it possible to extend the result about finite independence of $<<$ and $\leq$ on a congruence-distributive variety with extendable congruences to the embedding any countable twice quasi-ordered set in $<\mathfrak{J}\mathfrak{M};<<,\leq>$?

Let us cite some other statements pertaining to double skeletons.

Theorem 11.6. If $\mathfrak{M}$ is a non-trivial congruence-distributive variety with extendable congruences, the following statements are valid:

(a) for any $\aleph > \aleph_0$, any twice-quasi-ordered set $< A;\leq_1,\leq_2 >$ of the power not greater than $\aleph$ is such that for any $a,b \in A$, $a \leq_2 b$ is isomorphically embeddable in $< \mathfrak{J}\mathfrak{M}_\aleph;<<,\leq>$;

(b) for any regular cardinal $\aleph > \aleph_0$, any twice-quasi-ordered set $< A;\leq_1,\leq_2 >$ of a power not greater than $\aleph$ and such that $\leq_1$ and $\leq_2$ coincide on A is isomorphically embeddable in $< \mathfrak{J}\mathfrak{M}_\aleph;<<,\leq>$. In $< \mathfrak{J}\mathfrak{M}_\aleph;<<,\leq>$, there also is a set of the power $2^\aleph$ of elements pairwise incomparable either by $<<$ or by $\leq$.

Proof. Let $\mathfrak{A}$ be a simple $\mathfrak{M}$-algebra, in which case consideration of Boolean powers of the algebra $\mathfrak{A}$ using corollary **3.1** and theorem **3.3** reduces the proof of this theorem to that of the corresponding statements for a variety of Boolean algebras instead of the variety $\mathfrak{M}$. The statement (a) for a variety of Boolean algebras results from the statement of theorem **1.9** for the case when the quasi-order $\leq_1$ is a partial order. For the embedding of an arbitrary twice-ordered set $< A;\leq_1,\leq_2 >$ obeying the condition of the statement (a) in $< \mathfrak{J}\mathfrak{M}_\aleph;<<,\leq>$, it suffices to "dilute" the embedding of the set $< A/\equiv_{\leq_1};\leq_1,\leq_2>$ by additionally multiplying the corresponding Boolean algebras (images of $\equiv_{\leq_1}$-classes in this embedding) by pairwise epimorphic and mutually embeddable Boolean algebras which would not distort the relation $<<$ on the images of $\equiv_{\leq_1}$-classes. As the latter, one can use Boolean algebras of the type $B((\alpha+\eta)\cdot\aleph)$, where α runs over all the ordinals less than $\aleph$.

The statement (b) for Boolean algebras when $\leq_1$ and $\leq_2$ are partial orders results directly from theorem **1.17** and corollary **1.7**. In the case when $\leq_1 = \leq_2$ is a quasi- rather than a partial order, it suffices to "dilute" the embedding of the set $< A/\equiv_{\leq_1};\leq_1,\leq_2>$ in the same way as in the proof of the statement (a). ■

And, finally, let us formulate one more set of results directly obtainable using Boolean powers from the corresponding results on Boolean algebras: theorem **1.10** on the retractivity of interval Boolean algebras and theorem **1.8** on the relation $\leq$ on superatomic interval Boolean algebras.

Theorem 11.7. If $\mathfrak{M}$ is a non-trivial congruence-distributive variety with extendable congruences then

(a) there is an initial interval in the skeleton $< \mathfrak{J}\mathfrak{M} \setminus \{0_\mathfrak{M}\};<<>$, which is a proper class (not a set), on which the relation $\leq$ is an extension of the relation $<<$;

(b) there is a proper class (not a set) of elements in the skeleton $<\mathfrak{Z}\mathfrak{M};\leq>$, which is a distributive lattice of a final width relative to the quasi-order $\leq$.

The problem of retractive Boolean algebras considered in section **1**, in particular, in terms of double skeletons of varieties results in the following problem.

Problem 11.5. For any congruence-distributive variety $\mathfrak{M}$ with extendable congruences, describe the algebras $\mathfrak{A}\in\mathfrak{M}$ such that for each algebra $\mathfrak{A}_1\in\mathfrak{M}$, the relation $\mathfrak{A}_1 << \mathfrak{A}$ implies the relation $\mathfrak{A}_1\leq\mathfrak{A}$.

Priorities. The notion of the embedding skeleton of a variety was introduced by A.G.Pinus [178]. Theorems **11.1** and **11.2** [183], lemmas **11.3** and theorem **11.4** [167], theorem **11.3** [163], as well as theorems **11.5** and **11.6** (a) [178] were proved by A.G.Pinus.

12. Cartesian Skeletons of Congruence-Distributive Varieties

Alongside with important embedding and epimorphism relations between algebras of an arbitrary variety, the operation of a Cartesian product belongs to the basic notions of the theory of universal algebra varieties. The same level of abstraction that resulted in the notions of embedding and epimorphism skeletons also results in the following definition. If $\mathfrak{M}$ is an arbitrary variety and $a,b\in\mathfrak{Z}\mathfrak{M}$, then $a\times b$ will denote the isomorphism type of a Cartesian product of algebras of the isomorphism type a and b. It is evident that $\mathfrak{Z}\mathfrak{M}$ ($\mathfrak{Z}\mathfrak{M}_{\aleph}$ for any infinite cardinal $\aleph$) is closed under the operation $\times$, the operation itself is commutative, and the $1_{\mathfrak{M}}$ isomorphism type of a one-element $\mathfrak{M}$-algebra plays the role of a unit element in $<\mathfrak{Z}\mathfrak{M};\times>$ ($<\mathfrak{Z}\mathfrak{M}_{\aleph};\times>$). Therefore, for any infinite cardinal $\aleph$, $<\mathfrak{Z}\mathfrak{M}_{\aleph};\times,1_{\mathfrak{M}}>$ is a monoid, while $<\mathfrak{Z}\mathfrak{M};\times,1_{\mathfrak{M}}>$ differs from a monoid only in having not a set but a proper class as its basis. In this case, we will still speak about a monoid.

Definition 12.1. A Cartesian skeleton (countable Cartesian skeleton) of a variety $\mathfrak{M}$ is a monoid $<\mathfrak{Z}\mathfrak{M};\times,1_{\mathfrak{M}}>$ ($<\mathfrak{Z}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>$).

The following statement is derived directly from the results obtained in section **1**.

Theorem 12.1. For any non-trivial congruence-distributive variety $\mathfrak{M}$, any countable commutative semigroup is isomorphically embeddable in $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times>$.

Proof. By theorem **1.21**, it suffices to show that for any non-trivial congruence-distributive variety $\mathfrak{M}$, the countable Cartesian skeleton of the Boolean algebra variety $<\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>$ is isomorphically embeddable into $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>$. Let $\mathfrak{A}$ be a simple $\mathfrak{M}_{\aleph_0}$-algebra. Then, by theorem **3.2**, we get $Con_p\mathfrak{A}^{\mathcal{B}} \cong \mathcal{B}$ for any Boolean algebra $\mathcal{B}$. On the other hand, for any algebras $\mathfrak{A}_1,\mathfrak{A}_2 \in \mathfrak{M}$, according to the remark after theorem **4.2**, since $\mathfrak{M}$ is congruence-distributive, we get $Con_p(\mathfrak{A}_1 \times \mathfrak{A}_2) \cong Con_p\mathfrak{A}_1 \times Con_p\mathfrak{A}_2$. Moreover, the isomorphism of $\mathfrak{A}^{\mathcal{B}}$ and $\mathfrak{A}_1 \times \mathfrak{A}_2$ implies the relation $\mathfrak{A}_1 << \mathfrak{A}^{\mathcal{B}}$ and, hence, by corollary **3.1**, it implies the existence of a Boolean algebra $\mathcal{B}_1$ such that $\mathfrak{A}_1 \cong \mathfrak{A}^{\mathcal{B}_1}$. These remarks together prove that the correspondence of the isomorphism type of at most countably infinite Boolean algebra $\mathcal{B}$ with the isomorphism type of an $\mathfrak{M}_{\aleph_0}$-algebra $\mathfrak{A}^{\mathcal{B}}$ is an isomorphic embedding of $<\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>$ in $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>$. ■

__**Problem 12.1.**__ Is an arbitrary commutative semigroup (of the power $\aleph_1$) isomorphically embeddable into a Cartesian skeleton of any non-trivial congruence-distributive variety ?

By theorem **4.1**, the countable Cartesian skeleton of a variety of Boolean algebras plays the part of a small object among countable Cartesian skeletons of non-trivial congruence-distributive varieties. One should also recall a purely algebraic characterization of the monoid $<\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>$ as a universal V-monoid of the summation rank $\aleph_0$, obtained in section **1**. All these remarks result in the problem of describing all congruence-distributive varieties the countable Cartesian skeleton of which is isomorphic to $<\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>$.

__**Definition 12.2.**__ The Cartesian (countable Cartesian) skeleton of a variety $\mathfrak{M}$ is of a Boolean type provided that

$$<\mathfrak{J}\mathfrak{M};\times,1_{\mathfrak{M}}> \cong <\mathfrak{J}BA;\times,1_{BA}>$$
$$(<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}> \cong <\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>).$$

__**Theorem 12.2.**__ If $\mathfrak{M}$ is a congruence-distributive variety, then the countable Cartesian skeleton of the variety $\mathfrak{M}$ is of a Boolean type iff $\mathfrak{M}$ is generated by a certain quasi-primal algebra with no proper subalgebras.

Proof. Let $\mathfrak{M}$ be a non-trivial congruence-distributive variety and $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>\cong<\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>$, the corresponding isomorphism denoted by f. Let $\mathfrak{A}$ be a simple $\mathfrak{M}_{\aleph_0}$-algebra. In the proof of theorem **12.1**, we established a homomorphism h between $<\mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B}\in BA_{\aleph_0}\},\times,1_{\mathfrak{M}}>$ and $<\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>$, in which case $h(\mathfrak{A}^{\mathfrak{B}})=\mathfrak{B}$. In section **1**, we defined the notion of a V-monoid, and proved $<\mathfrak{J}BA_{\aleph_0};\times,1_{BA}>$ to be a V-monoid. Therefore, $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>$ will also be, by virtue of the homomorphism f, a V-monoid. In this case, however, the mapping $h^{-1}f$ is a self-embedding of $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>$. Then, since, as has been noted in the proof of theorem **12.1**, the isomorphism $\mathfrak{A}^{\mathfrak{B}}\cong\mathfrak{A}_1\times\mathfrak{A}_2$ implies the existence of Boolean algebras $\mathfrak{B}_1,\mathfrak{B}_2$ such that $\mathfrak{A}_i\cong\mathfrak{A}^{\mathfrak{B}_i}$ and $\mathfrak{B}\cong\mathfrak{B}_1\times\mathfrak{B}_2$, a submonoid $h^{-1}(f(\mathfrak{J}\mathfrak{M}_{\aleph_0}))$ of the monoid $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>$ will be hereditary. Therefore, by theorem **1.23**, the self-embedding $h^{-1}f$ of $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times,1_{\mathfrak{M}}>$ must be identical and, in particular, $\mathfrak{J}\mathfrak{M}_{\aleph_0}=\mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B}\in BA_{\aleph_0}\}$. Hence, $\mathfrak{M}=\mathfrak{M}(\mathfrak{A})$.

Since the algebra $\mathfrak{A}^{\mathfrak{B}}$ is finitely generated, a Boolean algebra $\mathfrak{B}$ is also finitely generated, hence, finite. In this case, for any $n\in\omega$ there is a $k_n\in\omega$ such that $\mathfrak{F}_{\mathfrak{M}}(n)\cong\mathfrak{A}^{k_n}$. By corollary **3.1**, all congruences of the algebra $\mathfrak{A}^{\mathfrak{B}}$ are generated by corresponding congruences of the Boolean algebra $\mathfrak{B}$. Therefore, all congruences of the algebra $\mathfrak{A}^{k_n}$ will be projections and, in particular, $\mathfrak{A}^{k_3}\cong\mathfrak{F}_{\mathfrak{M}}(3)$ will be congruence-permutable. By virtue of theorem **2.5**, the variety $\mathfrak{M}$ will also be congruence-permutable. Let us show that $\mathfrak{A}$ has no proper subalgebras. If $\mathfrak{A}$ contained a proper subalgebra $\mathfrak{A}'$ then, assuming $\mathfrak{A}'$ is finitely generated, $\mathfrak{A}$ would contain a non-singleton homomorphic image of the algebra $\mathfrak{F}_{\mathfrak{M}}(n)\cong\mathfrak{A}^{k_n}$ for some $n\in\omega$. As has been noted earlier, by corollary **3.1**, all these homomorphic images have the form $\mathfrak{A}^{l}(l\in\omega)$ and, therefore, $\mathfrak{A}'$, as well as $\mathfrak{A}$, would contain a proper subalgebra isomorphic to $\mathfrak{A}$. Considering a strictly ascending chain $\mathfrak{A}_0\subset\mathfrak{A}_1\subset\ldots\subset\mathfrak{A}_n\subset\ldots$ of simple algebras isomorphic to $\mathfrak{A}$, we get a simple algebra $\bigcup_{n\in\omega}\mathfrak{A}_n$. As $\mathfrak{J}\mathfrak{M}_{\aleph_0}=\mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B}\in BA_{\aleph_0}\}$, $\mathfrak{J}\mathfrak{M}_{\aleph_0}$ has a unique simple algebra, the algebra $\mathfrak{A}$, i.e., $\mathfrak{A}\cong\bigcup_{n\in\omega}\mathfrak{A}_n$. However, $\bigcup_{n\in\omega}\mathfrak{A}_n$ cannot be finitely generated, in which case all non-singleton $\mathfrak{M}_{\aleph_0}$-algebras, being isomorphic to algebras of the type $\mathfrak{A}^{\mathfrak{B}}$, also cannot be finitely generated. The contradiction obtained proves the absence of non-singleton proper subalgebras in $\mathfrak{A}$.

If $\mathfrak{A}$ has a one-element proper subalgebra with a basic set $\{a\}$, then let us choose an $\mathfrak{A}_1 = \{f \in \mathfrak{A}^{\omega} |$ there is an $n \in \omega$ such that for $i \geq n$ $f(i) = a\}$. Let $f_1, f_2 \in \mathfrak{A}_1$, and let n be such that for $k > n$, $f_1(k) = f_2(k)$. In this case, however, $\theta^{\mathfrak{A}_1}_{f_1, f_2}$ is contained in some kernel other than $\nabla_{\mathfrak{A}_1}$ of the projection of the algebra $\mathfrak{A}_1$ over the set $\{m \in n | m \leq n\} \subseteq \omega$. Therefore, the family of principal congruences of the algebra $\mathfrak{A}_1$ has no greatest element and, at the same time, since $\mathfrak{A}_1 \cong \mathfrak{A}^{\mathfrak{B}}$ for a certain Boolean algebra $\mathfrak{B}$, $\nabla_{\mathfrak{A}_1}$ is principal. The contradiction obtained proves that $\mathfrak{A}$ cannot have one-element subalgebras, either.

To finish the proof of the theorem in one direction, we have to show that $\mathfrak{A}$ is finite. Let us assume to the contrary that the opposite is the case, and let f be a bijective ω mapping on $\mathfrak{A}$. Let g be a mapping from ω to one of the elements (for instance, d) of the algebra $\mathfrak{A}$. Let $\mathfrak{C}$ be a subalgebra of the algebra $\mathfrak{A}^{\omega}$ generated by the elements f and g. In this case, $\mathfrak{C}$ is a homomorphic image of the algebra $\mathfrak{F}_{\mathfrak{M}}(2) = \mathfrak{A}^{k_2}$ and, as has been remarked earlier, there is an $m \leq k_2 \in \omega$ such that $\mathfrak{C} \cong \mathfrak{A}^m$. In particular, since all $\mathfrak{A}^m$ congruences are projections, $Con\mathfrak{C}$ is finite. On the other hand, since $\mathfrak{A}$ is simple with no proper subalgebras, there is a term $t_j(x)$ for any element $f(i)$ of the algebra $\mathfrak{A}$ such that $t_j(d) = f(j)$.

Let θ_j be the kernel of the projections of the algebra $\mathfrak{C} \subseteq \mathfrak{A}^{\omega}$ over the set $\{n \in \omega) | n \neq j\} \subseteq \omega$. Then, since for any $n \neq m \in \omega$ we have $t_j(g)(n) = t_j(g)(m) = f(j)$ and $f(n) \neq f(m)$, for any $n \neq j \in \omega$ we get $< t_j(g), f > \in \theta_j$ and $< t_j(g), f > \notin \theta_n$. Therefore, $\{\theta_n | n \in \omega\}$ is an infinite family of various configurations on $\mathfrak{C}$. The contradiction obtained proves $\mathfrak{A}$ to be finite. Thus, $\mathfrak{M} = \mathfrak{M}(\mathfrak{A})$, and $\mathfrak{A}$ is a quasi-primal algebra with no proper subalgebras, and the theorem has been proved in one direction.

Let now $\mathfrak{A}$ be a quasi-primal algebra with no proper subalgebras. Then, by theorem **7.3**, any $\mathfrak{M}(\mathfrak{A})$-algebra is isomorphic to a Boolean power of the algebra $\mathfrak{A}$, and since, moreover, $\mathfrak{A}^{\mathfrak{B}_1 \times \mathfrak{B}_2} \cong \mathfrak{A}^{\mathfrak{B}_1} \times \mathfrak{A}^{\mathfrak{B}_2}$ and the isomorphism $\mathfrak{A}^{\mathfrak{B}_1} \cong \mathfrak{A}^{\mathfrak{B}_2}$ implies the isomorphism $\mathfrak{B}_1$ and $\mathfrak{B}_2$,

$$< \mathfrak{I}\mathfrak{M}_{\aleph_0}; \times, 1_{\mathfrak{M}} > \cong < \mathfrak{I}BA_{\aleph_0}; \times, 1_{BA} >. \blacksquare$$

Let $P_{\mathfrak{A}} = \mathfrak{I}\{\mathfrak{C} |$ and there is a $\mathfrak{C}_1$ such that $\mathfrak{A} \cong \mathfrak{C} \times \mathfrak{C}_1\}$.

Definition 12.3. A variety $\mathfrak{M}$ obeys the Vaught isomorphism criterion if for any $\mathfrak{A} \in \mathfrak{I}\mathfrak{M}_{\aleph_0}$, $|P_{\mathfrak{A}}| \leq \aleph_0$ and $< \mathfrak{I}\mathfrak{M}_{\aleph_0}; \times, 1_{\mathfrak{M}} >$ is a V-monoid, i.e., if $< \mathfrak{I}\mathfrak{M}_{\aleph_0}; \times, 1_{\mathfrak{M}} >$ is a refinement monoid, and for any at most countably infinite

$\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}$, we get $\mathfrak{A} \cong \mathfrak{B}$ iff there is a relation $R \subseteq P_{\mathfrak{A}} \times P_{\mathfrak{B}}$ such that:

(1) $R(\mathfrak{J}\mathfrak{A}, \mathfrak{J}\mathfrak{B})$;

(2) if $R(a, 1_{\mathfrak{M}})$, then $a = 1_{\mathfrak{M}}$; if $R(1_{\mathfrak{M}}, b)$, then $b = 1_{\mathfrak{M}}$;

(3) if $a = a_1 \times a_2$ and $R(a,b)$, then there are b_1, b_2 such that $b = b_1 \times b_2$ and $R(a_i, b_i)$; if $b = b_1 \times b_2$ and $R(a,b)$, then there are a_1, a_2 such that $a = a_1 \times a_2$ and $R(a_i, b_i)$).

Corollary 12.1. A congruence-distributive variety obeys the Vaught isomorphism criterion iff it is generated by a quasi-primal algebra with no proper subalgebras.

Proof. If $\mathfrak{M}$ obeys the Vaught isomorphism criterion then, by the definition, $< \mathfrak{J}\mathfrak{M}_{\aleph_0}; \times, 1_{\mathfrak{M}} >$ is a V-monoid of a countable summation rank, while the universal V-monoid of a countable summation rank $< \mathfrak{J}BA_{\aleph_0}; \times, 1_{BA} >$ is isomorphic to the hereditary submonoid $< \mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B} \in BA_{\aleph_0}\}; \times, 1_{\mathfrak{M}} >$ of the monoid $< \mathfrak{J}\mathfrak{M}_{\aleph_0}; \times, 1_{\mathfrak{M}} >$, where $\mathfrak{A}$ is a simple $\mathfrak{M}_{\aleph_0}$-algebra. By theorem **1.23**, this implies the coincidence of the sets $\mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B} \in BA_{\aleph_0}\} = \mathfrak{J}\mathfrak{M}_{\aleph_0}$ and, hence, the isomorphism between $< \mathfrak{J}BA_{\aleph_0}; \times, 1_{BA} >$ and $< \mathfrak{J}\mathfrak{M}_{\aleph_0}; \times, 1_{\mathfrak{M}} >$ established by the mapping $\mathfrak{B} \rightarrow \mathfrak{A}^{\mathfrak{B}}$. By theorem **12.2**, $\mathfrak{M} = \mathfrak{M}(\mathfrak{A}_1)$, where $\mathfrak{A}_1$ is a quasi-primal algebra with no proper subalgebras. The converse is also true because the Vaught isomorphism criterion is fulfilled for Boolean algebra varieties, and because of corollary **3.1** and theorem **7.3**. ■

By way of concluding this chapter, let us mention a counple of other general problems pertaining to the notions of variety skeletons.

Problem 12.2. To describe quasi-ordered classes (sets) isomorphic to epimorphism and embedding skeletons (countable skeletons) of arbitrary congruence-distributive ‘ discriminator’ varieties.

Problem 12.3. To describe monoids isomorphic to Cartesian skeletons (countable Cartesian skeletons) of arbitrary congruence-distributive ‘discriminator’ varieties.

Priorities. The notion of a Cartesian skeleton of a variety was introduced by A.G.Pinus [166, 176]. Cartesian skeletons of various concrete varieties have been studied by a number of authors. In particular, Cartesian skeletons of a Boolean algebra variety and skeletons of Boolean topological spaces dual to them have been studied by W.Hanf [90], J.Adamek, V.Koubek and V.Trnkova [1], as well as by J.Ketonen [104]. S.Koppelberg [111] studied the Cartesian skeleton of complete Boolean algebras. A.Tarski [224] and B.Jonsson [99] studied Cartesian skeletons of varieties of Abelian groups and semigroups. Theorem **12.1** was proved by A.G.Pinus [176]. The part of the section that follows this theorem also belongs to A.G.Pinus [166].

APPENDIX

13. Elementary Theories of Congruence-Distributive Variety Skeletons

In connection to the problems of the theory of congruence-distributive variety skeletons undertaken in the preceding chapter, there are problems of estimating the complexity of formalized fragments of this theory and, as prime and traditional among them, problems of solvability of universal and elementary theories of congruence-distributive variety skeletons.

Let us first notice that the results obtained in the previous sectionss yield the following theorem.

Theorem 13.1. If $\mathfrak{M}$ is an arbitrary congruence-distributive variety,:

(a) the universal theory of the $\mathfrak{M}$ epimorphism skeleton is decidable;

(b) the universal theory of the $\mathfrak{M}$ Cartesian skeleton is decidable;

(c) the universal theory of the $\mathfrak{M}$ embedding skeleton is decidable under the additional assumption that congruences on $\mathfrak{M}$ are extendable.

Proof. Indeed, if $\mathfrak{M}$ is a trivial variety, all its skeletons are one-element, and the statement of the theorem is obvious. If $\mathfrak{M}$ is non-trivial, any quasi-ordered set, any countable commutative semigroup is isomorphically embeddable in $\langle \mathfrak{I}\mathfrak{M}; \ll \rangle$ and in $\langle \mathfrak{I}\mathfrak{M}; \times \rangle$ by virtue of theorems **8.11** and **11.1**, respectively. Therefore, the universal theories of $\langle \mathfrak{I}\mathfrak{M}; \ll \rangle$ and $\langle \mathfrak{I}\mathfrak{M}; \times \rangle$ coincide with, respectively, universal theories of all quasi-ordered sets, and all commutative semigroups. The decidability of the former of these universal theories is well-known, that of the universal theory of commutative semigroups was proved by A.I.Malcev [129] (for more details see the review [57] and a monograph by Y.L.Ershov [59]). Under the assumpition on extending congruences on $\mathfrak{M}$, by corollary **10.1**, any quasi-ordered set is isomorphically embeddable in the $\mathfrak{M}$ embedding skeleton and, hence, in this case the universal theory of $\langle \mathfrak{I}\mathfrak{M}; \ll \rangle$ also coincides with that of all quasi-ordered sets. ■

Turning now to elementary theories of congruence-distributive variety skeletons, let us first prove the hereditary undecidability of elementary theories of skeletons of the variety BA of all Boolean algebras. For any class $\mathfrak{K}$ of algebraic systems, and any algebraic system $\mathfrak{A}$, $Th(\mathfrak{K})$, $Th(\mathfrak{A})$ will denote the elementary theory of the class $\mathfrak{K}$ and of the system $\mathfrak{A}$, respectively. The basic notions pertaining to elementary theories and solvability problems can be found elsewhere [59].

Lemma 13.1. The elementary theory of the Cartesian skeleton of the variety BA is hereditary undecidable.

Proof. Let us construct a relative to elementary interpretation of the elementary theory of a class $\mathfrak{K}_2$ in the elementary theory of $<\mathfrak{Z}BA;\times>$. Here $\mathfrak{K}_2$ is a class of all finite models of the type $<\{1,\dots,n\};\sim_1,\sim_2>$, where $n\in\omega$ and $\sim_1,\sim_2$ are arbitrary equivalences on the set $\{1,\dots,n\}$ such that for $i,j\le n$, $i\sim_1 j$ and $i\sim_2 j$ entail the equality $i=j$. The hereditary undecidability of the elementary theory of the class $\mathfrak{K}$ is well-known (see, for instance, [57], [59]). Therefore, the hereditary undecidability of the elementary theory of $<\mathfrak{Z}BA;\times>$ results from the relative elementary interpretation $Th(\mathfrak{K}_2)$ in $Th(<\mathfrak{Z}BA;\times>)$.

For any $\mathfrak{A}\in\mathfrak{K}_2$, let $k_i,l_i\in\omega$ such that $i\sim_1 j(i\sim_2 j)$ iff $k_i=k_j(l_i=l_j)$, and $k_i<l_j$ for all $i,j\le n$. $L_{\mathfrak{A}}$ will denote a LOS $\sum\limits_{i\le n}((\omega^{k_i}+\eta+\omega^{l_i})\cdot\omega)$. It should be recalled that η is the ordered type of the set of rational numbers. Let us consider the formula

$$\Phi(x,y)=\exists z(x=zy)\&\forall u,v(y=uv\rightarrow y=u\vee y=v)\&$$
$$\&\forall t,w,z(t=yw\&\forall uv(t=uv\rightarrow t=u\vee t=v)\&x=zt\rightarrow t=y.$$

stating the "maximality" of the non-decomposable cofactor y of the element x. It is clear that $<\mathfrak{Z}BA;\times>\models\Phi(B(L_{\mathfrak{A}}),\mathfrak{C})$ iff $\mathfrak{C}\cong B((\omega^{k_i}+\eta+\omega^{l_i})\cdot\omega)$ for some $i\le n$. Let $\mathfrak{F}_{\mathfrak{A}}=\{\mathfrak{C}\in BA\mid <\mathfrak{Z}BA;\times>\models\Phi(B(L_{\mathfrak{A}}),\mathfrak{C})\}$.

Let us now consider formulas $E(x)$, $A(x)$, $\Gamma(x)$ of the signature $<\times>$:

$$E(x)=\forall y(yx=y),$$

$$A(x)=\forall y,z(x=yz\&\neg E(y)\rightarrow\exists t,u(y=tu\&\neg E(t)\&\forall w,v$$
$$(t=wv\rightarrow t=w\&E(v)\vee t=v\&E(w)))),$$

$$\Gamma(x)=\forall y,z(x=yz\&\neg E(y)\rightarrow$$

$$\to \neg \exists t,u(y = tu \,\&\, \neg E(t) \,\&\, \forall w,v\ (t = wv \to t = w \,\&\, E(v) \vee t = v \,\&\, E(w)))) .$$

It is obvious that $< \mathfrak{J}BA;\times > \models E(\mathfrak{C})$ iff $\mathfrak{C}$ is a one-element Boolean algebra. One can also easily check that $< \mathfrak{J}BA;\times > \models A(\mathfrak{C})(\Gamma(\mathfrak{C}))$ iff $\mathfrak{C}$ is an atomic (atomless, respectively) Boolean algebra. Let us set the formula $\Psi_1(x,y)$ equal to

$$\neg A(y) \,\&\, \neg \Gamma(y) \,\&\, y \neq x \,\&\, \exists z(x = zy \,\&\, \forall u,v(y = uv \to y = u \vee y = v) .$$

In this case,

$$< \mathfrak{J}BA;\times > \models \Psi_1(B((\omega^{k_i} + \eta + \omega^{l_i}) \cdot \omega), \mathfrak{C})$$

iff $\mathfrak{C} \cong B(\omega^{k_i} + \eta)$.

Let $\Psi_2(x,y)$ be obtained from the formula $\Psi_1(x,y)$ by replacing the conjunctive term $\neg A(y) \,\&\, \neg \Gamma(y)$ with the formula

$$A(y) \,\&\, \forall t(A(t) \,\&\, \exists z(x = zt) \,\&\, \forall u,v(t = uv \to t = u \vee t = v)$$
$$\&\, \exists w(t = yw \to t = y)) .$$

In this case,

$$< \mathfrak{J}BA;\times > \models \Psi_2(B((\omega^{k_i} + \eta + \omega^{l_i}) \cdot \omega), \mathfrak{C})$$

iff $\mathfrak{C} \cong B(\omega^{j_i})$.

In $< \mathfrak{J}BA;\times >$ the model $< \mathfrak{F}_{\mathfrak{Y}} ; P_1(x,y), P_2(x,y) >$, where $P_i(x,y) = \forall z,u(\Psi_i(x,z) \,\&\, \Psi_i(y,u) \to z = u)$, is relatively elementary definable. In this case, bearing in mind the remarks made earlier concerning formulas $\Psi_1(x,y), \Psi_2(x,y)$, we obviously get $< \mathfrak{F}_{\mathfrak{Y}} ; P_1(x,y), P_2(x,y) > \cong \mathfrak{Y}$. Therefore, indeed, the formulas $\Phi(B(L_{\mathfrak{Y}}),x), P_1(x,y), P_2(x,y)$ set a relatively elementary interpretation of $Th(\mathfrak{R}_2)$ in $Th(< \mathfrak{J}BA;\times >)$. ■

It should be noticed that, since all Boolean algebras used in the proof of lemma **13.1** are countable, the proof also entails the undecidability of the elementary theory $< \mathfrak{J}BA_{\aleph_0};\times >$ of the countable Cartesian skeleton of a Boolean algebra variety.

Lemma 13.2. **(CH)** The elementary theory of the epimorphism skeleton of the variety BA is hereditary undecidable.

Proof. Let us use the same notations as in the proof of lemma **13.1**. As was the case in that proof, it is sufficient to construct a relative to elementary interpretation of the elementary theory of the class $\mathfrak{K}_2$ in $Th(<\mathfrak{J}BA;<<>)$. Let P be a subset of the set of all real numbers constructed in lemma **1.1**. It should be recalled that P has the following properties:

(1) $2^{\aleph_0}$ is dense, i.e., for any $a<b\in P$, we have $|P\cap(a,b)|=2^{\aleph_0}$;

(2) for any subset S of the set P and any isotonic or antiisotonic mapping f from the set S to P, the equality $|\{x\in S|\, f(x)\neq x\}|\leq\aleph_0$ holds.

Let $P_0,P_1,P_{l_1+2},\ldots,P_{l_n+2},P_{k_1+2},\ldots,P_{k_n+2}$ be non-intersecting intervals of the set P. Therefore, each of these sets has the above-mentioned properties (1) and (2) as well. Moreover, they have one more property, namely:

(3) For any LOS $\mathfrak{C}$, if there are isotonic or antiisotonic mappings g_1,g_2 from some chains $\mathfrak{C}_1,\mathfrak{C}_2$ consisting of elements of Boolean algebras $B(P')$ and $B(P'')$, respectively (where P',P'' are some sets from $P_0,P_1,P_{l_1+2},\ldots,P_{l_n+2},P_{k_1+2},\ldots,P_{k_n+2}$) to $\mathfrak{C}$, $|\mathfrak{C}|\leq\aleph_0$.

Indeed, let us assume to the contrary, that $\mathfrak{C},\mathfrak{C}_1,\mathfrak{C}_2,g_1,g_2,P',P''$ meet the conditions of property (3), in which case $|\mathfrak{C}|=2^{\aleph_0}$. Using g_1,g_2, one can obviously construct the isotonic and antiisotonic embeddings h_1,h_2 of the set $\mathfrak{C}$ in the chain $\mathfrak{C}_1,\mathfrak{C}_2$ of the Boolean algebras $B(P'),B(P'')$. Making now use of the notations used in section **1**, let us take a subtraction $R(h_1(\mathfrak{C}))$ of the chain $h_1(\mathfrak{C})$ of elements of the Boolean algebra $B(P')$. Choosing $i\in\rho^1_R(h_1(\mathfrak{C}))\cup\rho^2_R(h_1(\mathfrak{C}))$ and establishing a correspondence between the element $a_i^d(b_i^d)$ and the element $d\in h_1^{-1}(R(h_1(\mathfrak{C})))$, if $i\in\rho^1_R(h_1(\mathfrak{C}))$ (if $i\in\rho^2_R(h_1(\mathfrak{C}))$), where $\bigcup_{j=1}^{m}(a_i^d,b_i^d]$ is the canonical representation of an element $h_1(d)$ in the Boolean algebra $B(P')$, we get an antiisotonic (isotonic) embedding φ from the continual subset $h_1^{-1}(R(h_1(\mathfrak{C})))$ to the LOS P'.

Let us identify $h_1^{-1}(R(h_1(\mathfrak{C})))$ with its image relative to φ. Repeating the same considerations for the linearly ordered set $h_1^{-1}(R(h_1(\mathfrak{C})))$ and the Boolean algebra $B(P'')$, we find a continual subset $T\subseteq P'$ which is either isotonically or antiisotonically embeddable into P'', which contradicts property (2) for the set P. It is this contradiction that proves that the sets $P_0,P_1,P_{l_1+2},\ldots,P_{l_n+2},P_{k_1+2},\ldots,P_{k_n+2}$ have property (3).

Let $<\{1,\ldots,n\};\sim_1,\sim_2>$ be a fixed $\mathfrak{K}_2$-model, and let $\{a_1,\ldots,a_l\}=\{l_1+2,\ldots,l_n+2\}$ and $\{b_1,\ldots,b_s\}=\{k_1+2,\ldots,k_n+2\}$, where l_i and k_j are the numbers chosen in the proof of lemma **13.1**.

Let us consider the following linearly ordered sets:

$$L = \sum_{i=1}^{n}([P_0 + 1 + P_{l_i+2} + 1]\cdot \omega + [P_1 + 1 + +P_{k_i+2} + 1]\cdot \omega^*),$$
$$L_1 = (P_0 + 1 + P_{a_1} + ... + P_{a_l} + 1)\cdot \omega + (P_1 + 1 + P_{b_1} + ... + P_{b_s} + 1)\cdot \omega^*,$$
$$L_0 = (P_0 + 1)\cdot \omega + (P_1 + 1)\cdot \omega^*,$$
$$L^* = (P_0 + 1 + P_{a_1} + ... + P_{a_l} + 1)\cdot \omega + 1 + (P_0 + 1 + \eta + 1)\cdot \omega +$$
$$+(P_1 + 1 + \eta + 1)\cdot \omega^* + 1 + (P_1 + 1 + P_{b_1} + ... + P_{b_s} + 1)\cdot \omega^*.$$

Let us consider a formula $\Phi(x)$ of the signature <<<> with parameters $B(L_0), B(L_1), B(L), B(L^*)$ defined in the following way:

$$\Phi(x) = B(L_0) << x << B(L_1) \& x << B(L) \& x \not<\not< B(L^*).$$

Let $\mathcal{B}$ be a Boolean algebra such that $B(L_0) << \mathcal{B} << B(L_1)$, and let g be a homomorphism from $B(L_1)$ to $\mathcal{B}$ and h be a homomorphism from $\mathcal{B}$ to $B(L_1)$. The Boolean algebra $B(L_1)$ has a linearly ordered basis S of the ordered type L_1 and, hence, $g(S)$, which is an isotonic image of L_1, will be the ordered basis of the algebra $\mathcal{B}$.

$$g(S) = \sum_{i\in\omega}(g(P_0^1) + 1 + g(P_{a_1}^i + ... + P_{a_l}^i + 1) + 1) +$$
$$+ \sum_{i\in\omega^*}(g(P_1^i) + 1 + g(P_{b_1}^i + ... + P_{b_s}^i) + 1),$$

where $P^i_{0(1,a_j,b_k)}$ is the i-th copy of the set $P_{0(1,a_j,b_k)}$ in the representation of the linear order $S \cong L_1$ discussed above.

On the other hand, using h, one can construct in $\mathcal{B}$ $\aleph_0$ disjunct elements $d_i (i \in \omega)$ and $\aleph_0$ chains $C_i (i \in \omega)$ consisting of elements of the Boolean algebra $\mathcal{B}$, which are less than the elements d_i, and so that P_0 is an isotonic image of each of the chains C_i. Considerations similar to those used in the proof of property (3) imply now the existence of $\aleph_0$ disjunct intervals in $g(S)$ containing subsets isotonic or antiisotonic to some continual subsets of the set P_0. By property (3) such subsets are possible only within the intervals of the set $g(S)$ which contain sets $g(P_0^i)$ only under the condition that the corresponding $g(P_0^i)$ are isomorphic to P_0. The latter is insured by the Bonnet-rigidity of the Boolean algebra $B(P_0)$ (see section **1**), or can be obtained by considerations to the contrary, analogous to those used in the proof of property (3). Therefore, there is a function φ_1 mapping the final intervals of the set ω to elements of ω such that $g(\sum_{j\in\varphi_1(m)} P_0^j) \cong P_0$ for any $m\in\omega$, and $\sum_{i\in\omega}\varphi_1(i)=\omega$. By analogy, one can find a function φ_2 from ω into finite intervals of the set ω such that, for any $m\in\omega$ $g(\sum_{j\in\varphi_2(m)} P_1^j) \cong P_1$ and $\sum_{i\in\omega}\varphi_2(i)=\omega$.

Therefore, the linearly ordered basis $g(S)$ of the algebra $\mathcal{B}$ has the form:

$$g(S)=\sum_{i\in\omega}(P_0^i+U_i)+\sum_{j\in\omega^*}(P_1^j+V_j),$$

where U_i,V_j are linear orders which are isotonic images of finite sums of sets of the type $(P_0+1+P_{a_1}+\ldots+P_{a_l}+1)$ and $(P_1+1+P_{b_1}+\ldots+P_{b_s}+1)$, respectively.

Let now $\mathcal{B}$ meet an additional condition: $\mathcal{B} << B(L)$. In this case there are $k,m\le n$ such that for sufficiently large i,j, the sets U_i,V_j in the representation of the set $g(S)$ given above are isotonic images of finite sums of sets of the type $(P_0+1+P_{l_k+2}+\eta)$ and $(P_1+1+P_{k_m+2}+\eta)$, respectively. Indeed, in the opposite case (it should be recalled that any countable LOS is an isotonic image of η) there are $r_1\ne r_2\le l$ and infinite subsets $R_1,R_2\subseteq\omega$ such that, for $i\in R_1(R_2)$, U_i contain continual subsets T_i which are isotonic images of the sets $P_{a_{r_1}}$ $(P_{a_{r_2}})$. In the Boolean algebra $B(L)$ there is an element b such that the intervals of the LOS L comprising the element b contain all the subchains of the ordered basis L of the algebra $B(L)$ of the ordinal type $P_{a_{r_1}}$, while the intervals of the set L comprising the element $\neg b$ contain all the subchains of the basis L of the ordinal type $P_{a_{r_2}}$. By property (3), if φ is a $B(L)$ homomorphism on $\mathcal{B}$, the intervals of the ordered basis $g(S)$ of the algebra $\mathcal{B}$ comprising the element $\varphi(b)$ contain no continual subchains of isotonic chain images of the type $P_{a_{r_2}}$, while the intervals comprising the element $\neg\varphi(b)$ contain no continual subchains of isotonic chain images of the type $P_{a_{r_1}}$. Therefore, an element $\varphi(b)\in\mathcal{B}$ must separate the chains $T_i(i\in R_1)$ from the chains $T_j(j\in R_2)$. Since R_1 and R_2 are infinite, there is not such an element in the Boolean algebra $\mathcal{B}=B(g(S))$. The contradiction obtained shows that there are $k,m\le n$ such that, for sufficiently large $i,j\in\omega$, the sets U_i,V_j in the above presentation of $g(S)$ are isotonic images of finite sums of sets of the type $(P_0+1+P_{l_k+2}+\eta)$ and $(P_1+1+P_{k_m+2}+\eta)$, respectively.

Moreover, $k=m$. Indeed, if $k\ne m$, a set of the ordered type $(P_0+1+P_{l_k+2}+1)\omega$ is separated from a set of the ordered type $(P_1+1+P_{k_m+2}+1)\omega^*$ in the ordered basis L of the Boolean algebra $B(L)$ by an element of this algebra.

In this case, under a homomorphism from $B(L)$ to $\mathcal{B} = B(g(S))$, the subsets $1+\sum_{i\in\omega}(P_0^i+U_i)$ and $\sum_{j\in\omega^*}(P_1^j+V_j)+1$ of the ordered basis $g(S)$ of the Boolean algebra $\mathcal{B}$ would prove to be separated in $\mathcal{B}$ by an element of the algebra $\mathcal{B}$. Obviously, the representation of $g(S)$ given above makes such a separation impossible. Thus, indeed, k must equal m.

Therefore, indeed,

$$g(S) = \sum_{i\in\omega}(P_0^i+U_i)+\sum_{g\in\omega^*}(P_1^j+V_j),$$

and there are $k \le n, p\in\omega, p_1\in\omega^*$ such that at $i\in\omega, j\in\omega^*$ and $i>p, j<p_1$, U_i is an isotonic image of finite sums of sets of the type $(P_1+1+P_{l_k+2}+\eta)$, while V_j is an isotonic image of finite sums of sets of the type $(P_1+1+P_{k_k+2}+\eta)$. At $i\le p, j\ge p_1$, U_i and V_j are isotonic images of finite sums of sets of the type $(P_0+1+P_{a_1}+\ldots+P_{a_l}+1)$ and $(P_1+1+P_{b_1}+\ldots+P_{b_s}+1)$, respectively.

Let us now notice that each of the sets U_i , V_j is either continual or countable, in the latter case being, as has been noted earlier, an isotonic image of the set η. Hence, if we add the condition $\mathcal{B} \not\leftarrow\!\!\not\leftarrow B(L^*)$ to the conditions imposed on $\mathcal{B}$ earlier then, by virtue of this remark, for some infinite subsets $W_1\subseteq\omega, W_2\subseteq\omega^*$, we get $|U_i|=|V_j|=2^{\aleph_0}$ for any $i\in W_1, j\in W_2$.

Therefore, if $<\mathfrak{Z}BA;<<>| = \Phi(\mathcal{B})$, the algebra $\mathcal{B}$ has a linearly ordered basis Q of the following type:

$$Q = \sum_{i\in\omega}(P_0^i+U_i)+\sum_{j\in\omega^*}(P_1^j+V_j),$$

in which case there are $k(\mathcal{B})\le n, p(\mathcal{B})\in\omega, p_1(\mathcal{B})\in\omega^*$ such that, at $i>p(\mathcal{B}), j<p_1(\mathcal{B})$, U_i is an isotonic image of finite sums of sets of the type $(P_0+1+P_{l_k+2}+\eta)$, while V_j is an isotonic image of finite sums of sets of the type $(P_1+1+P_{k_k+2}+\eta)$. At $i\le p(\mathcal{B}), j\ge p_1(\mathcal{B})$, U_i and V_j are isotonic images of finite sums of sets of the type $(P_0+1+P_{a_1}+\ldots+P_{a_l}+1)$ and $(P_1+1+P_{b_1}+\ldots+P_{b_s}+1)$, respectively. Moreover, there are infinite $W_1(\mathcal{B})\subseteq\omega, W_2(\mathcal{B})\subseteq\omega^*$ such that for $i\in W_i(\mathcal{B}), j\in W_2(\mathcal{B})$, we get $|U_i|=|V_j|=2^{\aleph_0}$.

Let us now consider the following linearly ordered sets:

$$A = \sum_{c\in\{a_1,\ldots,a_l\}}[(P_0+1+P_c+1)\omega+1],$$

$$B = \sum_{d \in \{b_1,\dots,b_s\}} [(P_1 + 1 + P_d + 1)\omega^* + 1].$$

Let us consider the following elementary formulas of the signature <<<> with parameters $B(L), B(L_0), B(L_1), B(L^*), B(A), B(B)$:

$$x \sim_1 y = \Phi(x) \& \Phi(y) \& \forall x_1, y_1 (x_1 << x \& x_1 << B(A) \& y_1 << y \&$$
$$y << B(A) \rightarrow \exists z, z_1 (\Phi(z) \& z_1 << B(A) \& z_1 << z \& x_1 << z_1 \& y_1 << z_1)),$$

$$x \sim_2 y = \Phi(x) \& \Phi(y) \& \forall x_1, y_1 (x_1 << x \& x_1 << B(B) \& y_1 << y \&$$
$$y << B(B) \rightarrow \exists z, z_1 (\Phi(z) \& z_1 << B(B) \& z_1 << z \& x_1 << z_1 \& y_1 << z_1)).$$

Let $< \mathfrak{J}BA; <<> \models \mathcal{B}_1 \sim_1 \mathcal{B}_2$, and let Q_1, Q_2 be linearly ordered bases of the Boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$, which have the above mentioned form (the formula and condition for Q), in which case the sets U_i, V_j corresponding to a Boolean algebra $\mathcal{B}_m (m = 1,2)$ will be denoted by U_i^m, V_j^m, , respectively. Let us show that, in this case, $l_{k(\mathcal{B}_1)} = l_{k(\mathcal{B}_2)}$.

Indeed, considerations analogous to those used above prove that the inequalities $\mathcal{B}'_m << \mathcal{B}_m, \mathcal{B}'_m << B(A) (m = 1,2)$ imply, for the Boolean algebras $\mathcal{B}'_m$, the existence of linearly ordered bases of the type:

$$Q' = \sum_{i \in \omega} [P'_0 + (u_i^m)'],$$

where $P'_0, (U_i^m)'$ are isotonic images of LOSes P_0, U_i^m from the corresponding representations of Q_i. The converse statement is also valid: any Boolean algebra with an ordered basis Q'_m of such a type obeys the inequalities $x << \mathcal{B}_m$ and $x << B(A)$.

Let now $Q'_m = \sum_{i \in \omega} (P_0 + U_i^m)$. Then, according to the facts just proved, $B(Q'_m) << \mathcal{B}_m$ and $B(Q'_m) << B(A)$. Let $\mathcal{C}, \mathcal{C}_1$ be Boolean algebras playing the parts of z, z_1 when the formula $\mathcal{B}_1 \sim_1 \mathcal{B}_2$ is valid if $x_1 = B(Q'_1), y_1 = B(Q'_2)$. Since in this case $< \mathfrak{J}BA; <<> \models \Phi(\mathcal{C}), \mathcal{C}_1 << B(A)$ and $\mathcal{C}_1 << \mathcal{C}$, $\mathcal{C}_1$ has a linearly ordered basis F_1 such that $F_1 = \sum_{i \in \omega} (P'_0 + G_i)$, in which case P'_0, an isotonic image of P_0 and G_i, are isotonic images of finite sums of sets of the type $P_0 + 1 + P_{l_{k(\mathcal{B})}+2} + \eta)$ for a sufficiently large i. It should be recalled that for some infinite $W_1(\mathcal{B}_1) \subseteq \omega, W_1(\mathcal{B}_2) \subseteq \omega^*$, we get $|U_i^1| = |U_i^2| = 2^{\aleph_0}$ for $i \in W_1(\mathcal{B}), j \in W_1(\mathcal{B}_2)$. Therefore, the inequality $\mathcal{B}'_1 << \mathcal{C}_1, \mathcal{B}'_2 << \mathcal{C}_1$ results, using the same repeatedly employed considerations, in isotonic or antiisotonic embeddings of some continual subsets of ordered sets $U_i^1(i \in W(\mathcal{B}_1)), U_j^2(j \in W(\mathcal{B}_2))$ in the LOS G_l at large l and, hence, finally, in embeddings of similar continual subsets in the LOS $P_{l_{k(\mathcal{C})}+2}$ as well. However, since $U_i^1(U_j^2)$ are embeddable in $P_{l_{k(\mathcal{B}_1)}+2}(P_{l_{k(\mathcal{B}_2)}+2})$, and by virtue of property (3), this is possible for the sets $P_{a_1},...,P_{a_l}$ only when $l_{k(\mathcal{B}_1)}+2 = l_{k(\mathcal{C})}+2$ and $l_{k(\mathcal{B}_2)}+2 = l_{k(\mathcal{C})}+2$. Therefore, indeed, if $<\mathfrak{J}BA;<<>\models \mathcal{B}_1 \sim_1 \mathcal{B}_2$, then the equality $l_{k(\mathcal{B}_1)} = l_{k(\mathcal{B}_2)}$ holds. The converse statement is also obvious: if $l_{k(\mathcal{B}_1)} = l_{k(\mathcal{B}_2)}$ and $<\mathfrak{J}BA;<<>\models \Phi(\mathcal{B}_1)\&\Phi(\mathcal{B}_2)$ on $<\mathfrak{J}BA;<<>$, the formula $\mathcal{B}_1 \sim_1 \mathcal{B}_2$ is valid.

In an analogous way one can prove that for $\mathcal{B}_1, \mathcal{B}_2$ such that $<\mathfrak{J}BA;<<>\models \Phi(\mathcal{B}_1)\&\Phi(\mathcal{B}_2)$, the formula $\mathcal{B}_1 \sim_2 \mathcal{B}_2$ is true on $<\mathfrak{J}BA;<<>$ iff $k_{k(\mathcal{B}_1)} = k_{k(\mathcal{B}_2)}$.

Let us introduce a formula $x \sim y$ equal to $x \sim_1 y \& x \sim_2 y$. Bearing in mind the facts proved above, we see that a set $\mathfrak{F}$ singled out by the formula $\Phi(x)$ in $<\mathfrak{J}BA;<<>$, factorized with respect to the equivalence relation set by the formula $x \sim y$ and having two equivalence relations set by the formulas $x \sim_1 y$ and $x \sim_2 y$, respectively, is isomorphic to the initial model $<\{1,...,n\};\sim_1,\sim_2>\in \mathfrak{K}_2$. Therefore, the relative to elementary interpretation $Th(\mathfrak{K}_2)$ in $Th(<\mathfrak{J}BA;<<>)$ has been constructed, which fact implies that the latter is hereditary undecidable. ■

Lemma 13.3.(CH) The elementary theory of the embedding skeleton of a variety BA is hereditary undecidable.

Proof. Preserving the notations from lemmas **13.1** and **13.2**, it suffices, as was the case in their proofs, to construct a relative to elementary interpretation of the elementary theory of the class of models $\mathfrak{K}_2$ in $Th(<\mathfrak{J}BA;<<>)$.

Let us fix a model $<\{1,...,n\};\sim_1,\sim_2>$ from the class $\mathfrak{K}_2$, and let $l_i, k_i (i \le n), a_j (j \le l), b_r (r \le s)$ be the same as in the proofs of lemmas **13.1** and **13.2**. The sets $P, P_0, P_1, P_{a_1},...,P_{a_l}, P_{b_1},...,P_{b_s}$ are chosen in the same way as in the proof of lemma **13.2**.

Let us first notice that the following statement holds:

(*) for any continual LOS L, if $B(L)$ is isomorphically embeddable in $B(P)$, then there is a continual subset S of the ordered set L which is isotonically or antiisotonically embeddable in P.

Indeed (let us use the notations of section **1**), if φ is an embedding of $B(L)$ in $B(P)$, there is an isotonic embedding ψ of the LOS L in $B(P)$. In this case, if $\mathcal{A} = R(\psi(L))$ is a subtract of $\psi(L)$ in the Boolean algebra $B(P)$ and $i \in \rho_R^1(\psi(L))(i \in \rho_R^2(\psi(L)))$, putting the element $a_i^d(b_i^d)$ into correspondence to the element $d \in \psi^{-1}(R(\psi(L)))$, where $\bigcup_{j=1}^{m}(a_i^d, b_i^d]$ is a canonical representation of the element $\psi(d)$ in the Boolean algebra $B(P)$, is an antiisotonic (isotonic) embedding of a continual $S = \psi^{-1}(R(\Psi(L)))$ into the ordered set P.

Choosing now suitable isomorphic copies of LOSes $P_0, P_1, P_{a_1}, \ldots, P_{a_l}, P_{b_1}, \ldots, P_{b_s}$, let us assume that these LOSes are dense subsets of a set of all real numbers.

Let us show that in this case the sets $P_0, P_1, P_{a_1}, \ldots, P_{a_l}, P_{b_1}, \ldots, P_{b_s}$ have, alongside with the properties (1) - (3) of the proof of lemma **13.2**, the following property:

(4) for any $A \subseteq \{0,1,a_1,\ldots,a_l,b_1,\ldots,b_s\}$ and $c \in \{0,1,a_1,\ldots,a_l,b_1,\ldots,b_s\}$, if φ is an embedding of a Boolean algebra $B(P_c)$ in $B(\bigcup_{a \in A} P_a)$, $c \in A$ and φ is identical on $P_c \subseteq B(P_c)$.

Indeed, by virtue of statement (*), there is an $S \subseteq P_c$ of the power $2^{\aleph_0}$ which is isotonically or antiisotonically embeddable in $\bigcup_{a \in A} P_a$. It should be also noticed that if $c \notin A$, $|P_c \cap \bigcup_{a \in A} P_a| < 2^{\aleph_0}$, since otherwise we get a continual set $R \subseteq P_c \cap \bigcup_{a \in A} P_a$ isomorphic to two disjunct subsets of the set P (to a subset of the interval of the set P which is isomorphic to P_c and to a subset of the interval of the set P which is isomorphic to one of $P_a, a \in A$) which contradicts property (2) for P. Hence, if $c \notin A$, $|S \setminus (\bigcup_{a \in A} P_a)| = 2^{\aleph_0}$, and $S \setminus \bigcup_{a \in A} P_a$ is isotonically or antiisotonically embeddable in $\bigcup_{a \in A} P_a$, which, again, contradicts property (2) for P. Thus, $c \in A$. Analogously, using the $2^{\aleph_0}$-densities of P_c one can prove φ to be identical on P_c.

Let us now consider the following linearly ordered sets:

$$M = \sum_{i=1}^{n}((P_0^i \cup P_{l_i+2}) + (P_1^i + P_{k_i+2})),$$

$$M_1 = (P_0 \cup \bigcup_{i=1}^{n} P_{l_i+2}) + (P_1 \cup \bigcup_{i=1}^{n} P_{k_i+2}),$$

$$M_2 = P_0 + P_1,$$

$$M_3 = (P_0 \cup \bigcup_{i=1}^{n} P_{l_i+2}) + 1 + (P_1 \cup \bigcup_{i=1}^{n} P_{k_i+2}).$$

Here P_1^i, P_0^i denote isomorphic copies of the LOSes P_1, P_0 supplied with indices

i only for differentiating these sets from each other as subsets of the linearly ordered set M.

Let us consider the following formula of the signature <<<> with parameters $B(M), B(M_1), B(M_2), B(M_3)$:

$$\Phi(x) = B(M_2) \leq x \leq B(M_1) \& x \leq B(M) \& \forall y (B(M_2) \leq y \leq B(M_1) \& y \leq B(M) \&$$
$$\& x \leq y \rightarrow y \leq x) \& \neg \exists z (x \leq z \& \neg z \leq x \& z \leq B(M) \& z \leq B(M_3)).$$

Let us assume that $\mathfrak{C} \in BA$ and $< \mathfrak{Z}BA; \leq > \models \Phi(\mathfrak{C})$. Let φ be an embedding of the algebra $B(M_2)$ in the algebra $B(M_1)$ induced by the embeddings implementing the inequalities $B(M_2) \leq \mathfrak{C}$ and $\mathfrak{C} \leq B(M_1)$. By virtue of property (4), one can easily notice that φ is identical on a LOS $P_0 + P_1 \subseteq B(M_2)$. Therefore, $\mathfrak{C}$ may be considered a subalgebra of the algebra $B(M_1)$, the latter containing the chains P_0 and P_1 of elements of the algebra $B(M_1)$.

Let ψ be a $\mathfrak{C}$ embedding in $B(M)$ implementing the inequality $\mathfrak{C} \leq B(M)$. Property (4) implies that ψ maps the elements of the chain $P_0 + P_1 \subseteq \mathfrak{C}$ into those of the chain $\sum_{i=1}^{n} (P_0^i + P_1^i) \subseteq B(M)$, in which case if A is an interval of the chain $P_0(P_1)$, $\psi(A) \subseteq \sum_{i=1}^{n} P_0^i (\psi(A) \subseteq \sum_{i=1}^{n} P_1^i)$, and if $\psi(A) \subseteq P_0^l (\psi(A) \subseteq P_1^l)$ for some $l \leq n$, the mapping ψ is identical when identifying P_0^l with P_0 (P_1^l with P_1). Hence, the chain $P_0 + P_1 \subseteq \mathfrak{C}$ is subdivided into a finite number of intervals A_j^l (where $l \leq n, j = 0,1$) such that $A_0^l \subseteq P_0, A_1^l \subseteq P_1, \psi(A_j^l) \subseteq P_j^l$, and ψ is identical on A_j^l when identifying P_j^l with P_j.

Let $A_0^{k_1}$ be a finite interval of the chain P_0, while $A_1^{k_2}$ be the initial interval of P_1. Since $\mathfrak{C}$ is maximal as regards embedding among algebras $\mathfrak{A}$ such that $B(M_2) \leq \mathfrak{A} \leq B(M)$ and $\mathfrak{A} \leq B(M)$,, obviously, $\mathfrak{C} \cong B(\sum_{i=1}^{n} (B_0^i + B_1^i))$ where $B_0^i(B_1^i)$ is an interval of the LOS $P_0^i \cup P_{l_i+2}(P_1^i \cup P_{k_i+2})$ cofinal and coinitial to the subset $\psi(A_0^i)(\psi(A_1^i))$.

Let us now notice that $k_1 = k_2$. Indeed, if $k_1 \neq k_2$, let $\mathfrak{D}$ be a subalgebra of the algebra $B(M)$ generated by elements of the algebra $\mathfrak{C}_1$ (which is a subalgebra of the algebra $B(M)$ isomorphic to the algebra $\mathfrak{C}$ at $\mathfrak{C} \leq B(M)$) and an element of the type $[-\infty, a)$, where a is any fixed element of the set $P_1^{k_1}$. In this case, $\mathfrak{C} \leq \mathfrak{D}, \mathfrak{D} \leq B(M)$ and $\mathfrak{D} \leq B(M_3)$. And, finally, $\mathfrak{D} \not\leq \mathfrak{C}$, since the elements

$\psi(A_0^{k_1})$ and $\psi(A_1^{k_2})$ are not separated in the algebra $\mathfrak{C}$ and they are separated in the algebra $\mathfrak{D}$ and, hence, the embedding of $\mathfrak{D}$ in $\mathfrak{C}$ would imply (see the proof of property (4)) embedding (without fixed points) of a certain continual subset of the LOS $\psi(A_0^{k_1}) \cup \psi(A_1^{k_2})$ in the LOS $\sum_{i=1}^{n}(B_0^i \cup B_1^i)$. Therefore, when $k_1 \neq k_2$, the Boolean algebra $\mathfrak{D}$ plays the part of z, the absence of which is stated in the latter part of the formula $\Phi(x)$ and, hence, $<\mathfrak{Z}BA;\leq>\models \Phi(\mathfrak{C})$ entails the equality $k_1 = k_2$. Let $k(\mathfrak{C})$ denote k_1 as.

It should be noticed that, obviously, (since the sets $P_c, P_1, P_{l_1+2}, \ldots, P_{l_n+2}, P_{k_1+2}, \ldots, P_{k_n+2}$ are rigid), we get $<\mathfrak{Z}BA;\leq>\models \Phi(\mathfrak{B}(i))$ for any $i \leq n$, where $\mathfrak{B}(i) = B(P_0^i \cup P_{l_i+2} + P_1^i \cup P_{k_i+2})$. In this case, $k(\mathfrak{B}(i)) = i$.

Let $M_4 = P_0 + 1 + P_1$. Let us consider the formula

$$\Phi_1(x,z) = \Phi(x)\& \Phi(z)\&\forall t(t \leq x\& t \leq z \rightarrow \exists u(t \leq u\& \\ \& B(M_4) \leq u \leq B(M_3)\& u \leq B(M)).$$

Let us prove that $<\mathfrak{Z}BA;\leq>\models \Phi(\mathfrak{C}_1,\mathfrak{C}_2)$ iff $k(\mathfrak{C}_1) \neq k(\mathfrak{C}_2)$. Indeed, let $<\mathfrak{Z}BA;\leq>\models \Phi(\mathfrak{C}_1)\& \Phi(\mathfrak{C}_2)$. ψ_1, ψ_2 will denote embeddings of the algebra $\mathfrak{C}_1, \mathfrak{C}_2$ in the algebra $\mathfrak{B}$ implementing the inequalities $\mathfrak{C}_1 \leq B(M)$ and $\mathfrak{C}_2 \leq B(M)$, and let us use the notations introduced when considering the formula $\Phi(x)$, adding to their left up corners indices 1 or 2, depending on whether they refer to the algebra $\mathfrak{C}_1$ or $\mathfrak{C}_2$. For instance, ${}^1A_1^i$ instead of A_1^i when considering $\mathfrak{C}_1$ instead of $\mathfrak{C}$. As was the case earlier, let us assume $\mathfrak{C}_1, \mathfrak{C}_2 \leq B(M_1)$. If $k(\mathfrak{C}_1) = k(\mathfrak{C}_2)$, as the t indicated in the formula $\Phi_1(\mathfrak{C}_1,\mathfrak{C}_2)$ it suffices to consider such Boolean algebras $\mathfrak{C} \subseteq B(M_1)$ which contain a subset

$$\psi_1^{-1}({}^1B_0^{k(\mathfrak{C}_1)} + {}^1B_1^{k(\mathfrak{C}_1)}) \cap \psi_2^{-1}({}^2B_0^{k(\mathfrak{C}_2)} + {}^2B_1^{k(\mathfrak{C}_2)}).$$

This subset has the form $C + D$, where C is a finite interval of the LOS $P_0 \cup P_{l_{k(\mathfrak{C}_1)}+2}$, while D is the initial interval of the LOS $P_1 \cup P_{k_{k(\mathfrak{C}_1)}+2}$. If now $\mathfrak{D}$ was a Boolean algebra playing the part of the element u from the formula $\Phi_1(\mathfrak{C}_1,\mathfrak{C}_2)$, the inequality $B(M_4) \leq \mathfrak{D} \leq B(M_3)$ would imply in a standard way the separation of the chains C and D contained in the algebra $\mathfrak{C} \subseteq \mathfrak{D}$ by an element of the algebra $\mathfrak{D}$. This, in its turn, would contradict embedding of $\mathfrak{D}$ in the algebra $B(M)$, since this becomes possible only under the identical embedding of the chains C and D in the LOSes $P_0 \cup P_{l_{k(\mathfrak{C}_1)}+2}$, $P_1 \cup P_{k_{k(\mathfrak{C}_1)}+2}$, this contradiction arising since these LOSes are not separated by elements of the algebra $B(M)$, while C and D

are separated by an element of the algebra $\mathfrak{D}$. Therefore, the equality $k(\mathfrak{C}_1) = k(\mathfrak{C}_2)$ indeed implies that the formula $\Phi_1(\mathfrak{C}_1,\mathfrak{C}_2)$ is not true on $<\mathfrak{J}BA;\leq>$. In an analogous way one can prove that the equality $k(\mathfrak{C}_1) = k(\mathfrak{C}_2)$ holds when $\Phi_1(\mathfrak{C}_1,\mathfrak{C}_2)$ is not true on $<\mathfrak{J}BA;\leq>$.

Let us now consider the following formulas:

$$\psi_1(x,z) = \Phi(x)\&\,\Phi(z)\&\exists x_1, z_1(\Phi(x_1)\&\,\Phi(z_1)\&\neg\Phi_1(x,x_1)\& \\ \neg\Phi_1(z,z_1)\&\forall t(t \leq B(M_5) \to (t \leq x_1 \leftrightarrow t \leq z_1)))\,,$$

$$\psi_2(x,z) = \Phi(x)\&\,\Phi(z)\&\,\exists x_1,z_1(\Phi(x_1)\&\,\Phi(z_1)\&\neg\Phi_1(x,x_1)\& \\ \neg\Phi_1(z,z_1)\&\forall t(t \leq B(M_6) \to (t \leq x_1 \leftrightarrow t \leq z_1)))\,.$$

Here $M_s = \bigcup_{i=1}^{n} P_{l_i+2}, M_6 = \bigcup_{i=1}^{n} P_{k_i+2}$.

Let $<\mathfrak{J}BA;\leq>\models\Phi(\mathfrak{C})\&\Phi(\mathfrak{A})$ and $l_{k(\mathfrak{C})} = l_{k(\mathfrak{A})}$. Taking into account what has been just proved, we get

$$<\mathfrak{J}BA;\leq>\models\Phi(\mathfrak{B}(k(\mathfrak{C})))\&\Phi(\mathfrak{B}(k(\mathfrak{A})))\& \\ \neg\Phi_1(\mathfrak{C},\mathfrak{B}(k(\mathfrak{C})))\&\neg\Phi_1(\mathfrak{A},\mathfrak{B}(k(\mathfrak{A})))\quad.$$

Moreover, since

$$\mathfrak{B}(k(\mathfrak{C})) = B(P_0 \cup P_{l_{k(\mathfrak{C})}+2} + P_1 \cup P_{k_{k(\mathfrak{C})}+2})\,,$$
$$\mathfrak{B}(k(\mathfrak{A})) = B(P_0 \cup P_{l_{k(\mathfrak{A})}+2} + P_1 \cup P_{k_{k(\mathfrak{A})}+2})\,,$$

and $l_{k(\mathfrak{C})} = l_{k(\mathfrak{A})}$ then, obviously, for any algebra $\mathfrak{D}$ embeddable in $B(M_5) = B(\bigcup_{i=1}^{n} P_{l_i+2})$, the embeddings of $\mathfrak{D}$ in $\mathfrak{B}(k(\mathfrak{C}))$ and in $\mathfrak{B}(k(\mathfrak{A}))$ are equivalent. The converse statement can also be proved easily: if $<\mathfrak{J}BA;\leq>\models\Phi_1(\mathfrak{C},\mathfrak{A})$, $l_{k(\mathfrak{C})} = l_{k(\mathfrak{A})}$. Thus, $<\mathfrak{J}BA;\leq>\models\Phi_1(\mathfrak{C},\mathfrak{A})$ iff $l_{k(\mathfrak{C})} = l_{k(\mathfrak{A})}$. An analogous statement is also true for formulas ψ_2 with the numbers l_i replaced with k_i.

Therefore, the set $\mathfrak{J}\{\mathfrak{C} \in BA < \mathfrak{J}BA;\leq>\models\Phi(\mathfrak{C})\}$ factorized over the formula $\neg\Phi_1(x,z)$ and supplied with a couple of relations set by the formulas $\psi_1(x,z),\psi_2(x,z)$ is a model of the isomorphic model $<\{1,\ldots,n\};\sim_1,\sim_2>$ chosen from the class $\mathfrak{K}$ in the beginning of the proof of the lemma, i.e., the formulas $\Phi(x),\Phi_1(x,z),\psi_1(x,z),\psi_2(x,z)$ set a relative to elementary (with the parameters

$B(M), B(M_1), B(M_2), B(M_3), B(M_4), B(M_5), B(M_6))$ interpretation of the hereditary undecidable elementary theory of the class $\mathfrak{K}$ in the elementary theory of the embedding skeleton of a Boolean algebras variety, and, hence, the latter is undecidable. ■

The lemmas just proved enable us to prove the following statement.

Theorem 13.2.(CH)

(a) If $\mathfrak{M}$ is a non-trivial congruence-distributive variety, the elementary theories of the epimorphism skeleton $<\mathfrak{J}\mathfrak{M};<<>$ and the Cartesian skeleton $<\mathfrak{J}\mathfrak{M};\times>$ of the variety $\mathfrak{M}$ are undecidable.

(b) If $\mathfrak{M}$ is a variety containing a certain quasi-primal algebra without one-element subalgebras, the elementary theory of the embedding skeleton $<\mathfrak{J}\mathfrak{M};\leq>$ of the variety $\mathfrak{M}$ is undecidable.

Proof.

(a) Let $\mathfrak{M}$ be a non-trivial congruence-distributive variety, and $\mathfrak{A}$ be a simple $\mathfrak{M}$-algebra which exists by virtue of theorem **2.11**. According to corollary **3.1**, theorem **3.2** and the remark after theorem **4.2**, for any Boolean algebra $\mathfrak{B}$, any algebras $\mathfrak{A}_1, \mathfrak{A}_2$ such that $\mathfrak{A}^{\mathfrak{B}} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$, there are Boolean algebras $\mathfrak{B}_1, \mathfrak{B}_2$ such that $\mathfrak{A}_1 \cong \mathfrak{A}^{\mathfrak{B}_1}, \mathfrak{A}_2 \cong \mathfrak{A}^{\mathfrak{B}_2}$ and $\mathfrak{B} \cong \mathfrak{B}_1 \times \mathfrak{B}_2$. Together with the isomorphism $<\mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B} \in BA\};\times> \cong <\mathfrak{J}BA;\times>$, these prove that the hereditary undecidability of $Th(<\mathfrak{J}BA;\times>)$ established in lemma **13.1** implies the undecidability of $Th(<\mathfrak{J}\mathfrak{M};\times>)$.

Analogously, by theorem **13.1**, $<\mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B} \in BA\};<<> \cong <\mathfrak{J}BA;<<>$, in which case for any $\mathfrak{M}$-algebra $\mathfrak{C}$, if for some Boolean algebra $\mathfrak{C} << \mathfrak{A}^{\mathfrak{B}}$, there is a Boolean algebra $\mathfrak{B}_1 << \mathfrak{B}$ such that $\mathfrak{C} \cong \mathfrak{A}^{\mathfrak{B}_1}$. Therefore, $<\mathfrak{J}\{\mathfrak{A}^{\mathfrak{B}} \mid \mathfrak{B} \in BA\};<<>$ is the initial interval in $<\mathfrak{J}\mathfrak{M};<<>$ and, since all the formulas participating in the interpretation of $Th(\mathfrak{K}_2)$ in $Th(<\mathfrak{J}BA;<<>)$ given in lemma **13.2** have quantifiers limited relative to the quasi-order $<<$ incorporated in the formula parameters then, by choosing the same formulas with parameters $\mathfrak{A}^{\mathfrak{B}}$ instead of the corresponding parameters, i.e., Boolean algebras $\mathfrak{B}$, we obtain a relative to interpretation of $Th(\mathfrak{K}_2)$ in $Th(<\mathfrak{J}\mathfrak{M};<<>)$, which proves that $Th(<\mathfrak{J}\mathfrak{M};<<>)$ is undecidable.

(b) Let now $\mathfrak{M}$ be an arbitrary variety containing a quasi-primal algebra $\mathfrak{A}_0$ with no one-element proper subalgebras. Hence, $\mathfrak{A}_0$ is simple, and $\mathfrak{M}(\mathfrak{A}_0)$ is a discriminator variety. Let R be a linearly ordered set of all real numbers. It should be noticed that for any $\mathfrak{M}$-algebra $\mathfrak{A}$ such that $\mathfrak{A}_0 \leq \mathfrak{A} \leq \mathfrak{A}_0^{B(R)}$, the algebra $\mathfrak{A}$ has the form $\mathfrak{A}_0^{\mathfrak{B}}$ for some Boolean algebra $\mathfrak{B}$ such that $\mathfrak{B} \leq B(R)$.

Indeed, let $\mathfrak{A}_0 \leq \mathfrak{A} \leq \mathfrak{A}_0^{B(R)}$. Let us identify $\mathfrak{A}$ with a corresponding subalgebra of the algebra $\mathfrak{A}_0^{B(R)}$, and $\mathfrak{A}_0$ with a subalgebra $\mathfrak{A}'_0$ of the algebra $\mathfrak{A}$. As $\mathfrak{A}_0$ is simple and finite, and since for any $i \in B(R)^*$ we have $|\pi_i(\mathfrak{A}'_0)| \neq 1$, we get $|\pi_i(\mathfrak{A}'_0)| = \mathfrak{A}_0$. Since $\mathfrak{A}'_0$ is finite, there is a partition $b_1, \ldots, b_k (k \in \omega)$ of the unit of the Boolean algebra $B(R)$ such that elements of the algebra $\mathfrak{A}'_0$ are constant on the elements $b_j (j \leq k)$ as on subsets of the Stone space $B(R)^*$. Moreover, as $\mathfrak{A}_0$ is finite and since for any $i \in B(R)^*$ we have $|\pi_i(\mathfrak{A}'_0)| = \mathfrak{A}_0$, for any $j_1, j_2 \leq k$ there are automorphisms ψ_{j_1, j_2} of the algebra $\mathfrak{A}_0$ such that for $i_1 \in b_{j_1}, i_2 \in b_{j_2}$ and for any $a \in \mathfrak{A}'_0$ we get $\psi_{j_1, j_2}(\pi_{i_1}(a)) = \pi_{i_2}(a)$.

Using the automorphisms ψ_{j_1, j_2}, let us define the automorphism ψ of the algebra $\mathfrak{A}_0^{B(R)}$ in the following way: for $a \in \mathfrak{A}_0^{B(R)}$, for any $i \in b_j (j \leq k)$, $\pi_i(\psi(a)) = \psi_{j,0}(\pi_i(a))$, where 0 is a certain fixed element of $B(R)^*$. It is obviously the ψ-image of the algebra $\mathfrak{A}'_0$ consisting of constant elements of the algebra $\mathfrak{A}_0^{B(R)}$. Then, since $\psi(\mathfrak{A}) \supseteq \psi(\mathfrak{A}'_0)$ and $\psi(\mathfrak{A}) \subseteq \mathfrak{A}_0^{B(R)}$, by corollary **5.2**, both $\psi(\mathfrak{A})$ and the algebra $\mathfrak{A}$ are isomorphic to an algebra of the type $\mathfrak{A}_0^{\mathfrak{B}}$ for some Boolean algebra $\mathfrak{B}$. In this case, obviously, $\mathfrak{B} \leq B(R)$. Therefore, indeed, the inequalities $\mathfrak{A}_0 \leq \mathfrak{A} \leq \mathfrak{A}_0^{B(R)}$ result in representing the algebra $\mathfrak{A}$ as $\mathfrak{A}_0^{\mathfrak{B}}$ for a Boolean algebra $\mathfrak{B}$ such that $\mathfrak{B} \leq B(R)$.

Since $\mathfrak{M}(\mathfrak{A}_0)$ is a discriminator variety and, hence, congruence-distributive with extendable congruences, by theorem **3.3**, for any Boolean algebras $\mathfrak{B}_1, \mathfrak{B}_2$ the inequality $\mathfrak{A}_0^{\mathfrak{B}_1} \leq \mathfrak{A}_0^{\mathfrak{B}_2}$ is equivalent to the inequality $\mathfrak{B}_1 \leq \mathfrak{B}_2$. Hence,

$$< \mathfrak{I}\{\mathfrak{A} \in \mathfrak{M} | \mathfrak{A}_0 \leq \mathfrak{A} \leq \mathfrak{A}_0^{B(R)}; \leq\} \cong$$
$$\cong \{\mathfrak{I} \in BA | \mathfrak{B} \leq B(R), |\mathfrak{B}| \neq 1\}; \leq\}.$$

The relative to elementary interpretation of $Th(\mathfrak{K}_2)$ in $Th(< \mathfrak{I}BA; \leq >)$ constructed in the proof of lemma **13.3** is in fact limited by the skeleton $< \mathfrak{I}\{\mathfrak{B} \in BA | \mathfrak{B} \leq B(R); |\mathfrak{B}| \neq 1\}; \leq >$. Therefore, both this interpretation and the isomorphism mentioned above result in a relative to elementary interpretation of $Th(\mathfrak{K}_2)$ in $Th(< \mathfrak{I}\mathfrak{M}; \leq >)$, and, hence, the latter is undecidable. ■

According to the remark after the proof of lemma **13.1**, under the conditions of theorem **13.2** the elementary theory of the countable Cartesian skeleton $<\mathfrak{J}\mathfrak{M}_{\aleph_0};\times>$ of any nontrivial congruence-distributive variety is also undecidable. In this case, the continuum hypothesis is not required to prove either this statement or that on the Cartesian skeleton in theorem **13.2**. The undecidability of elementary theories of countable embedding and epimorphism skeletons under the conditions of theorem **13.2**, however, can prove not to take place, which fact can be traced from the repeatedly presented earlier equalities $<\mathfrak{J}\mathfrak{M}_{\aleph_0};<<>\cong\omega_1+1^*$, $<\mathfrak{J}BA'_{\aleph_0};\leq>\cong\omega_1+1^*$.

The following problem is now open for discussion.

__**Problem 13.1.**__ Is the elementary theory of the embedding skeleton of any non-trivial congruence-distributive variety with extendable congruences undecidable ?

By way of concluding this section, let us dwell on a problem pertaining to elementary theories of variety skeletons. It is natural to assume that for large cardinals k, the bounded epimorphism, embedding and Cartesian skeletons of an arbitrary variety $\mathfrak{M}$ inherit the basic properties of the skeletons $<\mathfrak{J}\mathfrak{M};<<>$, $<\mathfrak{J}\mathfrak{M};\leq>$ and $<\mathfrak{J}\mathfrak{M};\times>$ and, in particular, their elementary properties. In this case, the mere coincidence of the elementary theories of bounded and unbounded skeletons is not of the greatest interest; what really matters is the existence of a set of algebras $\mathfrak{A}'_1,\ldots,\mathfrak{A}'_n$ (for any set of $\mathfrak{M}$-algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$) from a bounded class $\mathfrak{M}_{<k}$, the elementary properties of which in terms of epimorphism, embedding and direct expansions in the class $\mathfrak{M}_{<k}$ coincide with analogous properties of the algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$ within the whole variety $\mathfrak{M}$. In other words, here we speak about the implementation of the elementary types of element corteges of unbounded skeletons in bounded ones. The class $\mathfrak{M}_{<k}$ is considered instead of the class $\mathfrak{M}_k$, since there is a biggest element in the skeleton $<\mathfrak{J}\mathfrak{M}_k;<<>$, while in the skeleton $<\mathfrak{J}\mathfrak{M};<<>$ there is no such element.

__**Definition 13.1.**__ The Löwenheim number for the epimorphism skeleton of a variety $\mathfrak{M}$ is the least cardinal k such that for any $\mathfrak{M}$-algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n (n\in\omega)$ there is a cardinal $k'<k$ and $\mathfrak{M}_{k'}$-algebras $\mathfrak{A}'_1,\ldots,\mathfrak{A}'_n$ such that the elementary theories of the models $<\mathfrak{J}\mathfrak{M};<<,\mathfrak{A}_1,\ldots,\mathfrak{A}_n>$ and $<\mathfrak{J}\mathfrak{M}_{<k'};<<,\mathfrak{A}'_1,\ldots,\mathfrak{A}'_n>$ coincide.

The notions of the Löwenheim number for embedding and Cartesian skeletons, as well as for a complete skeleton of the variety $\mathfrak{M}$ are introduced in an analogous way, the algebraic system $<\mathfrak{J}\mathfrak{M};<<,\leq,\times>$ considered as a complete skeleton.

Second-order complete logic is an extension of the first-order predicate calculus using the formulas with existential and universal quantifiers over arbitrary predicates allowed.

Let us recall the following definition.

Definition 13.2. The Löwenheim number of the second-order complete logic is the least cardinal k such that for an arbitrary algebraic system $\mathfrak{A}$ of a finite signature there is an algebraic system $\mathfrak{A}'$ of the power not greater than k such that the theories of the systems $\mathfrak{A}$ and $\mathfrak{A}'$ coincide in the second-order complete logic.

Localization of the Löwenheim number for the second-order complete logic depends on set-theoretical assumptions (see, for instance, [231]).

Theorem 13.3. The Löwenheim number for the complete skeleton of an arbitrary finitely approximatizable variety of a finite signature is not greater than that of the second-order complete logic.

Proof. Let $\mathfrak{M}$ be an arbitrary variety of a finite signature, and k the Löwenheim number fof the second-order complete logic. Let $\mathfrak{A}_1,\dots,\mathfrak{A}_n\in\mathfrak{M}$, and let us, using induction over $m\in\omega$, build a sequence of cardinals $k_0<k_1<<\dots<k_m,\dots$ in the following way: $k_0>|\mathfrak{A}_1|,\dots,|\mathfrak{A}_n|,\aleph_0$ and, if k_m has already been constructed, let us choose k_{m+1} in such a way that for an arbitrary formula of the first-order predicate calculus of the signature $<<<,\leq,\times>$, which has the form $\varphi(x_1,\dots,x_p)=\exists y_1,\dots,y_s\psi(x_1,\dots,x_p,y_1,\dots,y_s)$, for any $c_1,\dots c_p\in\mathfrak{J}\mathfrak{M}_{<k_m}$ such that $<\mathfrak{J}\mathfrak{M};<<,\leq,\times>\models\varphi(c_1,\dots,c_p)$ is true, there are elements $b_1,\dots,b_s\in\mathfrak{J}\mathfrak{M}_{k_{m+1}}$ such that $<\mathfrak{J}\mathfrak{M};<<,\leq,\times>\models\psi(c_1,\dots,c_p,b_1,\dots b_s)$.

Let $k'=\lim\limits_{m\to\infty}k_m$. Standard model-theoretical considerations show that the skeleton $<\mathfrak{J}\mathfrak{M};<<,\leq,\times,\mathfrak{A}_1,\dots,\mathfrak{A}_n>$ is elementary equivalent to the skeleton $<\mathfrak{J}\mathfrak{M}_{<k'};<<,\leq,\times,\mathfrak{A}_1,\dots,\mathfrak{A}_n>$.

For any formula φ of the first-order predicate calculus of the signature $<<<,\leq,\times>$, there is a formula φ' of the second-order complete logic of the signature of the variety $\mathfrak{M}$ such that for an arbitrary infinite cardinal $\aleph$ and any algebras $\mathfrak{C}_1,\dots,\mathfrak{C}_q\in\mathfrak{M}_{<\aleph}$, we get $<\mathfrak{J}\mathfrak{M}_{<\aleph};<<,\leq,\times>\models\varphi(\mathfrak{C}_1,\dots,\mathfrak{C}_q)$ iff $\mathfrak{F}_{\mathfrak{M}}(\aleph)|=\varphi'(\theta_1,\dots,\theta_q)$, where for $i\leq q$ we have $\theta_1\in Con\,\mathfrak{F}_{\mathfrak{M}}(\aleph)$ and $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_i\cong\mathfrak{C}_i$. Indeed, without giving a formally inductive definition of the formula φ', let us note that any element $a\in\mathfrak{J}\mathfrak{M}_{<\aleph}$ is interpreted in $F_{\mathfrak{M}}(\aleph)$ as a congruence $\theta \in Con\,\mathfrak{F}_{\mathfrak{M}}(\aleph)$ such that $|\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta| < |\mathfrak{F}_{\mathfrak{M}}(\aleph)|$; the equality of the elements $a_1, a_2 \in \mathfrak{J}\mathfrak{M}_{<\aleph}$ is interpreted as the existence of an isomorphism from the algebra $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1$ to $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_2$, where θ_i interprets the elements a_i; the relation $a_1 << a_2$ is interpreted as the existence of a homomorphism from the algebra $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_2$ to the algebra $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1$; the equality $a_1 \times a_2 = b$ is interpreted as the existence of congruences $\theta_3, \theta_4 \in Con\,\mathfrak{F}_{\mathfrak{M}}(\aleph)$ such that

$$\theta_3 \wedge \theta_4 = \theta,$$
$$\theta_3 \circ \theta_4 = \theta_4 \circ \theta_3 = \nabla_{\mathfrak{F}_{\mathfrak{M}}(\aleph)},$$
$$\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_3 \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_1,$$
$$\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_4 \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_2$$

(here θ is a congruence on $\mathfrak{F}_{\mathfrak{M}}(\aleph)$ interpreting the element b). All the properties of the algebras $\mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta$ enumerated above are, obviously, expressed by formulas of second-order logic as properties of θ relations on the algebra $\mathfrak{F}_{\mathfrak{M}}(\aleph)$. Therefore, the formula φ' does exist.

Let now $T(\mathfrak{A}_1,\ldots,\mathfrak{A}_n)$ be the elementary theory of the complete skeleton $< \mathfrak{J}\mathfrak{M}; <<, \le, \times, \mathfrak{A}_1,\ldots,\mathfrak{A}_n >$ of the variety $\mathfrak{M}$ with the types of isomorphism of the algebras $\mathfrak{A}_1,\ldots,\mathfrak{A}_n$ chosen. According to the remarks made above, we have

$$< \mathfrak{J}\mathfrak{M}_{<k'}; <<, \le, \times, \mathfrak{A}_1,\ldots,\mathfrak{A}_n > \models T(\mathfrak{A}_1,\ldots,\mathfrak{A}_n).$$

T' will denote $\{\varphi' | \varphi \in T(\mathfrak{A}_1,\ldots,\mathfrak{A}_n)\}$. By virtue of the construction of formulas φ', on $\mathfrak{F}_{\mathfrak{M}}(k')$ the formula $\varphi'(\theta_1,\ldots\theta_n)$ holds for any $\varphi \in T(\mathfrak{A}_1,\ldots,\mathfrak{A}_n)$ where $\theta_i \in Con\,\mathfrak{F}_{\mathfrak{M}}(k')$ and $\mathfrak{F}_{\mathfrak{M}}(k')/\theta_i \cong \mathfrak{A}_i (i \le n)$. In an obvious way, due to the definition of an algebra free in the variety $\mathfrak{M}$, and since the signature of $\mathfrak{M}$ is finite and $\mathfrak{M}$ is finitely approximated, we can write a formula ψ of the second-order complete logic such that for any infinite algebra $\mathfrak{A}$ of the signature of the variety $\mathfrak{M}$, we get $\mathfrak{A} \models \psi$ iff $\mathfrak{A} \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)$ for an infinite cardinal $\aleph$. Therefore, we get $< \mathfrak{F}_{\mathfrak{M}}(k'); \theta_1,\ldots,\theta_n > \models T' \cup \{\psi\}$.

By the definition of the Löwenheim number of the second-order complete logic, there is an algebra $\mathfrak{A}$ and congruences $\theta_1',\ldots,\theta_n'$ on $\mathfrak{A}$ such that $|\mathfrak{A}| \le k$ and $< \mathfrak{A}; \theta_1',\ldots,\theta_n' > \models T' \cup \{\psi\}$. In line with the remark made earlier, $\mathfrak{A} \cong \mathfrak{F}_{\mathfrak{M}}(\aleph)$, where $\aleph = |\mathfrak{A}| \le k$. At the same time, according to the definition of the formulas φ', we get

$$< \mathfrak{J}\mathfrak{M}_{<\aleph}; \ll, \le, \times, \mathfrak{A}_1',\ldots,\mathfrak{A}_n' > \models T(\mathfrak{A}_1',\ldots,\mathfrak{A}_n') >,$$

where $\mathfrak{A}_i' = \mathfrak{F}_{\mathfrak{M}}(\aleph)/\theta_i'$, i.e.,

$$<\mathfrak{J}\mathfrak{M}_{<\aleph};\ll,\leq,\times,\mathfrak{A}_1',\ldots,\mathfrak{A}_n'>\equiv<\mathfrak{J}\mathfrak{M};\ll,\leq,\times,\mathfrak{A}_1',\ldots,\mathfrak{A}_n'>,$$

which is the required proof. ■

The statement of theorem **13.3** obviously results in the following corollary.

Corollary 13.1. The Löwenheim number for the epimorphism (embedding, Cartesian skeleton) of an arbitrary finitely approximatizable variety is not greater than that of second-order complete logic.

This corollary gives the upper bound of the Löwenheim numbers for skeletons of arbitrary varieties. In particular cases, these Löwenheim numbers can be much less than the upper bound. For instance, for any uncountable categoric variety $\mathfrak{M}$, we get $<\mathfrak{J}\mathfrak{M};\ll>\cong<\mathfrak{J}\mathfrak{M}_{<\aleph_1};\ll>\oplus Ord$, where Ord is a well-ordered class of all ordinals. Tact that $<Ord;\leq>$ is elementary equivalent to an ordinal $<\omega^{\omega};\leq>$ is well-known and, hence, for any $a_1,\ldots,a_n \in Ord$, there are $c_1,\ldots,c_n \in \omega^{\omega+1}$ such that

$$<Ord;\leq,a_1,\ldots,a_n>\equiv<\omega^{\omega+1};\leq,c_1,\ldots,c_n>.$$

Therefore, for any $\mathfrak{A}_1,\ldots,\mathfrak{A}_n \in \mathfrak{M}$, there are $\mathfrak{C}_1,\ldots,\mathfrak{C}_n \in \mathfrak{M}_{<\aleph_{\omega^{\omega+1}}}$ such that

$$<\mathfrak{J}\mathfrak{M};\ll,\mathfrak{A}_1,\ldots,\mathfrak{A}_n>\equiv<\mathfrak{J}\mathfrak{M}_{<\aleph_{\omega^{\omega+1}}};\ll,\mathfrak{C}_1,\ldots,\mathfrak{C}_n>,$$

i.e., the Löwenheim number for epimorphism skeletons of uncountable categoric varieties is equal to $\aleph_{\omega^{\omega+1}}$. It can be noticed easily that the Löwenheim numbers for embedding and Cartesian skeletons of similar varieties are also equal to $\aleph_{\omega^{\omega+1}}$. The case is different for congruence-distributive varieties, for them the Löwenheim numbers of the Cartesian skeletons and the second-order complete logic coincide.

Theorem 13.4. Let $\mathfrak{M}$ be a non-trivial finitely approximatizable congruence-distributive variety of a finite signature such that all $\mathfrak{M}$-algebras contain subalgebras isomorphic to a certain fixed simple or one-element algebra. In this case, the Löwenheim number of the Cartesian skeleton of the variety coincides with that of second-order complete logic.

Proof. By corollary **13.1**, it suffices to show that the Löwenheim number for second-order complete logic is not greater than that of the Cartesian skeleton of the variety $\mathfrak{M}$. To this end, it suffices to construct an exact relative to elementary interpretation of the second-order theories of all algebraic systems of an arbitrary finite signature in the class of all bounded Cartesian skeletons of the variety $\mathfrak{M}$. For this purpose, it is sufficient to construct an exact relative to elementary interpretation in the class of all bounded skeletons $<\mathfrak{J}\mathfrak{M}_{<k};\times>$ of the class $\mathfrak{K}$ of algebraic systems of the kind $<P(\omega_i);U,f,\subseteq,a_1,\dots,a_n>$, where $P(\omega_i)$ is a set of all subsets of an arbitrary initial ordinal ω_i, U is a unary predicate on $P(\omega_i)$ selecting one-element subsets of the ordinal ω_i (henceforth identified with elements of the ordinal ω_i), $\subseteq$ is a set-theoretical relation of inclusion on $P(\omega_i)$, f is an arbitrary binary function bijectively mapping $\omega_i \times \omega_i$ to ω_i, $a_1,\dots,a_n$ are constants belonging to $P(\omega_i)$.

In this case, the exact relative to elementary interpretation of the class $\mathfrak{K}$ in the class $\{<\mathfrak{J}\mathfrak{M}_{<k};\times>| k$ is an arbitrary cardinal$\}$ denotes the existence of an elementary property $\mathfrak{F}(\bar{a})$ for some set of parameters $\bar{a}$, and the existence of a set $\bar{S}(\bar{x})$ of elementary formulas with the parameters $\bar{a}$ so that in any skeleton $<\mathfrak{J}\mathfrak{M}_{<k};\times>$, under any choice of the parameters $\bar{a}$ from $\mathfrak{J}\mathfrak{M}_{<k}$ obeying the formula $\mathfrak{F}(\bar{a})$, the given set of formulas $\bar{S}(\bar{x})$ defined a certain algebraic system from $\mathfrak{K}$, and for any system $\mathfrak{A}$ from $\mathfrak{K}$ there is such a choice of parameters $\bar{a}$, that for cardinals k such that $\bar{a} \in \mathfrak{J}\mathfrak{M}_{<k}$ we get $<\mathfrak{J}\mathfrak{M}_{<k};\text{«»}| = \mathfrak{F}(\bar{a})$, while the given set of formulas $\bar{S}(\bar{x})$ determines systems isomorphic to $\mathfrak{A}_0$ in the skeletons $<\mathfrak{J}\mathfrak{M}_{<k};\times>$. In this case the required inequality for Löwenheim numbers results from the fact that for the exact relative to elementary interpretation of an $\mathfrak{K}$-system $<P(\omega_i);U,f,\subseteq,a_1,\dots,a_n>$ in $<\mathfrak{J}\mathfrak{M}_{<k};\times>$ constructed later, the inequality $\sum_{l<k} 2^l \geq 2^{\aleph_i}$ holds.

It should be recalled that η is the ordered type of rational numbers, r is the ordered type of real numbers. Let η' be an ordinal type with neither initial nor final elements, of the power $2^{2^{\aleph_0}}$ and with the following property: for any $a,b \in \eta'$ the interval (a,b) is isomorphic to η' if $a < b$. Let i be a fixed ordinal, and f be a fixed bijective mapping of $\omega_i \times \omega_i$ on ω_i. Let $\eta_1 = \eta, \eta_2 = r, \eta_3 = \eta'$. Let $L_{i,f}$ denote a LOS

$$1 \oplus \sum_{<j,k> \in \omega_i^2} ((\omega^j \oplus \eta \oplus \omega^k \oplus \eta_2 \oplus \omega^{f(j,k)} \oplus \eta_3) \cdot \omega) \oplus 1.$$

Let $L_i = \sum_{j \in \omega_i} ((\omega^j \oplus \eta_1) \cdot \omega) \oplus 1$. A Boolean algebra $\mathfrak{B}$ is called non-decomposable provided that for any $b \in \mathfrak{B}$ we get either $\mathfrak{B} \cong \mathfrak{B} | b$ or $\mathfrak{B} \cong \mathfrak{B} | \neg b$.

In the proof of lemma **13.1**, we introduced the formula $\Phi(x,y)$ stating the "maximality" of the non-decomposable cofactor y of the element x. It can be easily noticed that $<\mathfrak{Z}BA;\times>\models\Phi(B(L_{i,f}),\mathfrak{C})$ iff

$$\mathfrak{C}\cong B(1\oplus(\omega^j\oplus\eta_1\oplus\omega^k\oplus\eta_2\oplus\omega^{f(j,k)}\oplus\eta_3)\cdot\omega\oplus 1)$$

for some $j,k\in\omega_i$.

In an analogous way, $<\mathfrak{Z}BA;\times>\models\Phi(B(L_i),\mathfrak{C})$ iff $\mathfrak{C}\cong B(1\oplus(\omega^j\oplus\eta_1)\cdot\omega\oplus 1)$. $\mathfrak{F}_i$ will denote $\{\mathfrak{C}\in BA<\mathfrak{Z}BA;\times>\models\Phi(B(L_i),\mathfrak{C})\}$.

In the proof of lemma **13.1** we introduced the formulas $E(x),A(x),\Gamma(x)$ selecting one-element, atomic and atomless Boolean algebras, respectively, in the skeleton $<\mathfrak{Z}BA;\times>$.

It should be noticed that analogous considerations are also valid for formulas $\Phi(x,y),E(x),A(x),\Gamma(x)$ not only on the skeleton $<\mathfrak{Z}BA;\times>$, but also on bounded skeletons $<\mathfrak{Z}BA_k;\times>$.

Let us consider the following formulas:

$$R(x,y,w)=\exists z(x=yz)\&\forall u,v(y=uv\rightarrow y=u\vee y=v)\&\neg A(y)\&$$
$$\neg\Gamma(y)\&\neg\exists q(y=xq)\&\exists p(y=pw),$$

$$T(x,y)=\forall z(A(z)\rightarrow(\exists u(x=zu)\leftrightarrow\exists v(y=zv))).$$

On $<\mathfrak{Z}BA;\times>$ the formula

$$R(B(1\oplus(\omega^j\oplus\eta_1\oplus\omega^k\oplus\eta_2\oplus\omega^{f(j,k)}\oplus\eta_3)\cdot\omega\oplus 1),\mathfrak{C},B(1\oplus\eta_s\oplus 1))$$

holds iff $\mathfrak{C}\cong B(1\oplus\omega^j\oplus\eta_1\oplus 1)$ for $s=1$, or $\mathfrak{C}\cong B(1\oplus\omega^k\oplus\eta_2\oplus 1)$ for $s=2$, or $\mathfrak{C}\cong B(1\oplus\omega^{f(j,k)}\oplus\eta_3\oplus 1)$ for $s=3$. Moreover, for any $s_1,s_2=1,2,3$ and any ordinals j_1,j_2 we get

$$T(B(1\oplus\omega^{j_1}\oplus\eta_{s_1}\oplus 1),B(1\oplus\omega^{j_2}\oplus\eta_{s_2}\oplus 1))$$

iff $j_1=j_2$.

The properties of the formulas constructed above remain valid when the skeleton $<\mathfrak{Z}BA;\times>$ is replaced with bounded skeletons $<\mathfrak{Z}BA_k;\times>$. The required relative elementary interpretation of the $\mathfrak{R}$-system $<P(\omega_i);U,f,\subseteq,a_1,\dots,a_n>$ in $<\mathfrak{Z}BA_k;\times>$ is constructed for $k>\omega_i$ in the following way: one-element subsets of the set ω_i, i.e., elements of the $\mathfrak{R}$-system satisfying the predicate U, are interpreted in

$<\mathfrak{J}BA_{k};\times>$ by elements satisfying the formula $\Phi(B(L_i),x)$, i.e., a family of such elements in $<\mathfrak{J}BA_{k};\times>$ is, as has been earlier noticed, equal to $\mathfrak{F}_i$. The function f from $\omega_i \times \omega_i$ to ω_i is interpreted by the following formula:

$$\psi(x,y,z) = \Phi(B(L_i),x) \& \Phi(B(L_i),y) \& \Phi(B(L_i),z) \&$$
$$\exists t,u_1,u_2,u_3(\Phi(B(L_{i,f}),t) \& \mathop{\&}_{s=1}^{3} R(t,u_s,B(1 \oplus \eta_s \oplus 1)) \&$$
$$T(x,u_1) \& T(y,u_2) \& T(z,u_3)).$$

It is obvious that

$$<\mathfrak{J}BA_{k};\times> \models \psi(B(1 \oplus (\omega^j \oplus \eta_1) \cdot \omega \oplus 1),$$
$$B(1 \oplus (\omega^l \oplus \eta_1) \cdot \omega \oplus 1),\ B(1 \oplus (\omega^r \oplus \eta_1) \cdot \omega \oplus 1))$$

iff $f(j,l) = r$. Subsets of the ordinal ω_i are interpreted as subsets of $\mathfrak{F}_i$ with arbitrary elements $\mathfrak{B} \in \mathfrak{J}BA_k$ in the following way: an element $\mathfrak{B}$ is interpreted with a subset $B \subseteq \mathfrak{F}_i$ such that

$$B = \{\mathfrak{C} \in BA \mid <\mathfrak{J}BA_{k};\times> \models \Phi(B(L_i),\mathfrak{C}) \& \Phi(\mathfrak{B},\mathfrak{C})\}.$$

In this case, in order to interpret any subset $A \subseteq \omega_i$, it suffices to choose the algebra $B(1 \oplus \sum_{j \in A} (\omega^i \oplus \eta_1) \cdot \omega \oplus 1)$ as $\mathfrak{B}$. The equality relation on the elements ω_i is interpreted by a common equality relation on $\mathfrak{F}_i$,, the equality relation on $P(\omega_i)$ (on subsets $\mathfrak{F}_i$) is interpreted by the formula

$$\Phi_{=}(x,y) = \forall z(\Phi(B(L_i),z) \rightarrow (\Phi(x,z) \leftrightarrow \Phi(y,z))),$$

while the $\subseteq$ relation is interpreted by the formula

$$\Phi_{\subseteq}(x,y) = \forall z(\Phi(B(L_i),z) \rightarrow (\Phi(x,z) \rightarrow \Phi(y,z))),$$

the interpretation of constants being obvious.

Therefore, indeed, for any cardinal $k > \omega_i$, the formulas $\Phi(B(L_i),x), \psi(x,y,z), \Phi_{=}(x,y), \Phi_{\subseteq}(x,y)$ define the $\mathfrak{R}$-system $<P(\omega_i);U,f,\subseteq,a_1,\ldots,a_n>$ in the skeleton

$$<\mathfrak{J}BA_{k};\times, b_1 = B(L_i), b_2 = B(L_{i,f}), b_3 = B(1 \oplus \eta_1 \oplus 1),$$

$$b_4 = B(1\oplus \eta_2 \oplus 1), b_s = B(1\oplus \eta_3 \oplus 1), c_1 = \mathcal{B}_{a_1}, \ldots, c_n = \mathcal{B}_{a_n} >$$

(here $\mathcal{B}_{a_i}$ are Boolean algebras from $BA_{<k}$ interpreting constants a_i).

In an obvious way one can write an elementary formula $\mathfrak{F}(x_1,\ldots,x_5,y_1,\ldots,y_n)$ such that for any cardinal k', for any $b_1,\ldots,b_5,c_1,\ldots,c_n \in \mathfrak{Z}BA_{<k'}$, if

$$< \mathfrak{Z}BA_{<k'};\times >\models \mathfrak{F}(b_1,\ldots,b_5,c_1,\ldots,c_n >,$$

then the formulas $\Phi, \psi, \Phi_{-}, \Phi_{\subseteq}$ mentioned above define an algebraic system of the type $< A;U,f,\subseteq,a_1,\ldots,a_n >$ where $A\subseteq P(U)$ in the skeleton $< \mathfrak{Z}BA_{<k'};\times >$, provided that $b_1,b_2,\ldots$ are replaced by the parameters $B(L_i),B(L_{i,f}),\ldots,$. It should be noticed that in this case A coincides with $P(U)$, while the cardinal k' is such that $\sum_{l<k'} 2^l \geq 2^{|U|}$. The latter inequality is obvious and, therefore, for any subset $S\subseteq U\subseteq BA_{<k'}$, we now have to find a Boolean algebra $\mathfrak{C}_S$ in $BA_{<k'}$ such that

$$< \mathfrak{Z}BA_{<k'};\times,b_1,\ldots,b_5,c_1,\ldots,c_n >\models \Phi(b_2,\mathfrak{C})\&\Phi(\mathcal{B}_S,\mathfrak{C})$$

iff $\mathfrak{C}\in S$.

Let $S=\{\mathcal{B}_l | l\in I\}\subseteq U$ and, therefore, all algebras $\mathcal{B}_l$ are non-decomposable. $\mathcal{B}_S$ will denote a Boolean algebra such that: for any $l\in I$ in $\mathcal{B}_S$ there is an element b_l such that $\mathcal{B}_S | b_l \cong \mathcal{B}_l$; for $l\neq j\in I$ $b_l\cap b_j=0$ and $\mathcal{B}_S$ are generated by elements of the algebras $\mathcal{B}_S | b_l$ when $l\in I$. Hence, the factor of the algebra $\mathcal{B}_S$ relative to the ideal generated by a set $\bigcup_{l\in I}\mathfrak{F}_l$ is isomorphic to a Boolean Frechet algebra over the set I. Here $\mathfrak{F}_l$ is an ultrafilter of the algebra $\mathcal{B}_S | b_l$. Since the algebras $\mathcal{B}_l\in U$, i.e.,

$$< \mathfrak{Z}BA_{<k'};\times,b_1,\ldots,b_5,c_1,\ldots,c_n >\models \Phi(b_2,\mathcal{B}_l),$$

for any $b,j\in I$ we get $\mathcal{B}_l\cong\mathcal{B}_j$ if $\mathcal{B}_l\cong\mathcal{B}_j\times\mathcal{B}$ for some Boolean algebra $\mathcal{B}$. This fact and the construction of the Boolean algebra $\mathcal{B}_S$ directly yield that on $< \mathfrak{Z}BA_{<k'};\times,b_1,\ldots,b_5,c_1,\ldots,c_n >$ the formula $\Phi(b_2,\mathfrak{C})\&\Phi(\mathcal{B}_S,\mathfrak{C})$ holds iff $\mathfrak{C}\in S$. Therefore, in $< \mathfrak{Z}BA_{<k'};\times,b_1,\ldots,b_5,c_1,\ldots,c_n >$ the set $S\subseteq U$ is really interpreted by the Boolean algebra $\mathcal{B}_S$ and, thus, an interpretation of the $\mathfrak{R}$ class in bounded Cartesian skeletons of a Boolean algebra variety has been constructed.

Let now $\mathfrak{M}$ be a variety satisfying the conditions of the theorem, and $\mathfrak{U}$ be an at most countably infinite simple $\mathfrak{M}$-algebra existing by the Magari theorem. As has been noted in the proof of theorem **13.2**, the following statements follow from the results obtained in chapter **2**:

(1) for any Boolean algebra $\mathfrak{B}$, any $\mathfrak{M}$-algebras $\mathfrak{A}_1,\mathfrak{A}_2$, $\mathfrak{A}^{\mathfrak{B}} \cong \mathfrak{A}_1 \times \mathfrak{A}_2$ iff there are Boolean algebras $\mathfrak{B}_1,\mathfrak{B}_2$ such that for $i=1,2$, $\mathfrak{B} \cong \mathfrak{B}_1 \times \mathfrak{B}_2$ and $\mathfrak{A}_i \cong \mathfrak{A}^{\mathfrak{B}_i}$;

(2) For any Boolean algebras $\mathfrak{B}_1,\mathfrak{B}_2$ $\mathfrak{A}^{\mathfrak{B}_1} \cong \mathfrak{A}^{\mathfrak{B}_2}$ iff $\mathfrak{B}_1 \cong \mathfrak{B}_2$.

It should also be noticed that the formulas of the signature $<\times>$ used to construct an interpretation of the class $\mathfrak{K}$ in a class of bounded Cartesian skeletons of a Boolean algebra variety, had quantifiers bounded by cofactors of the parameters employed in these formulas. These remarks enable us to use the formulas in question to obtain a relative to elementary interpretation of the class $\mathfrak{K}$ in a class of bounded Cartesian skeletons of the variety $\mathfrak{M}$ with parameters $\mathfrak{A}^{\mathfrak{C}}$ substituted for the parameters $\mathfrak{C} \in BA$ in these formulas. To complete the proof of the theorem, now we have to notice that for any parameters $b_1,\ldots,b_5,c_1,\ldots,c_n \in \mathfrak{J}\mathfrak{M}_{<k}$ such that the formulas $\Phi,\psi,\Phi_{\subseteq},\Phi_{-}$ define at these parameters a system of the type $< A;U,f,\subseteq,a_1,\ldots,a_n >$ in $<\mathfrak{J}\mathfrak{M}_{<k};\times>$, where $A \subseteq P(U)$, the equality $A=P(U)$ in fact holds. In other words, we have to show that for any subset $S \subseteq U \subseteq \mathfrak{J}\mathfrak{M}_{<k}$, there is an element $\mathfrak{D} \in \mathfrak{J}\mathfrak{M}_{<k}$ such that for $\mathfrak{C} \in \mathfrak{J}\mathfrak{M}_{<k}$, $<\mathfrak{J}\mathfrak{M}_{<k};\times> \models \Phi(b_2,\mathfrak{C})\&\Phi(\mathfrak{D},\mathfrak{C})$ iff $\mathfrak{C} \in S$.

Let $S=\{\mathfrak{A}_i | i \in I\}$ and $\mathfrak{A}'$ be a simple algebra contained in all algebras $\mathfrak{A}_i$. $\mathfrak{D}$ will denote a subalgebra of the algebra $\prod_{i \in I}\mathfrak{A}_i$ such that for $f \in \prod_{i \in I}\mathfrak{A}_i$, $f \in \mathfrak{D}$ holds iff for a certain $a \in \mathfrak{A}'$ and all i but the final number of elements from I, we have $f(i)=a$. The algebra $\mathfrak{D}$ is a Boolean product of the algebras $\{\mathfrak{A}_i | i \in I\} \cup \{\mathfrak{A}'\}$ relative to a Boolean Frechet algebra over the set I. According to known descriptions of congruences on Boolean products in congruence-distributive varieties (section **4**), any direct cofactor $\breve{\mathfrak{A}}$ of the algebra $\mathfrak{D}$ has either the form $\breve{\mathfrak{A}} \cong \mathfrak{A}'_{i_1}\times\ldots\times\mathfrak{A}'_{i_m}$ or the form $\breve{\mathfrak{A}} \cong \mathfrak{A}''_{i_1}\times\ldots\times\mathfrak{A}''_{i_m}\times\mathfrak{D}'$ for some $i_1,\ldots,i_m \in I$ and some direct decompositions $\mathfrak{A}'_{i_1}\times\mathfrak{A}''_{i_1},\ldots,\mathfrak{A}'_{i_m}\times\mathfrak{A}''_{i_m}$ of the algebras $\mathfrak{A}_{i_1},\ldots,\mathfrak{A}_{i_m}$, respectively, where the algebra $\mathfrak{D}'$ is constructed of algebras $\mathfrak{A}_i (i \in I \setminus \{i_1,\ldots,i_m\}$ in the same way as the algebra $\mathfrak{D}$ of algebras $\mathfrak{A}_i (i \in I)$. Owing to this fact, one can easily notice that $\mathfrak{D}$ has the property discussed above, i.e., it interprets the subset S of the set U. Therefore, the interpretation under discussion is an interpretation of the $\mathfrak{K}$-system, which completes the proof of the theorem. ■

The following problems still remain open for discussion.

Problem 13.2.

(a) Does the Löwenheim number of the epimorphism skeleton of any non-trivial congruence-distributive variety coincide with that of second-order complete logic ?

(b) The same problem concerning embedding skeletons.

The following results should be mentioned in relation to the problems just posed.

Theorem 13.5. The Löwenheim number for the skeleton $<\mathfrak{I}BA;\ll,*>$ of a Boolean algebra variety coincides with that of second-order complete logic.

Theorem 13.6. The Löwenheim number for the skeleton $<\mathfrak{I}BA;\leq,*>$ of a Boolean algebra variety coincides with that of second-order complete logic.

Here for $a,b,c \in \mathfrak{I}BA$ such that a is the isomorphism type of $\mathfrak{A} \in BA$, b is that of $\mathfrak{B} \in BA$ and c is that of $\mathfrak{C} \in BA$, and the equality $a*b=c$ implies that $\mathfrak{B}$ is isomorphic to $\mathfrak{A}*\mathfrak{B}$, which is a free product of the Boolean algebras $\mathfrak{A}$ and $\mathfrak{B}$.

Priorities. All the results obtained in this section belong to A.G.Pinus. Theorems on the decidability of the elementary theory of skeletons were published in [164], [165] and [176]. Theorems **13.3** and **13.4** can be found in [185], theorem **13.5** in [170] and theorem **13.6** in [186].

14. Some Theorems on Boolean Algebras

In this section we will prove some statements on Boolean algebras formulated in section **1** and used in proofs of a number of theorems of Chapter **3** but not available in basic monographs on Boolean algebras.

It should be recalled that a subset S of the cardinal λ is called closed and unbounded if:

(1) for any $\alpha \in \lambda$, there is a $\beta \in S$ such that $\beta \geq \alpha$;

(2) for any $S_1 \subseteq S$ such that for a certain $\alpha \in \lambda$, $S_1 \subseteq \{\beta \in \lambda | \beta \leq \alpha\}$, $\sup S_1 \in S$.

A subset D of the cardinal λ is called stationary if its intersection with any closed and unbounded subset of λ is non-empty. Theorem **1.3** enables us to prove the following result (theorem **1.9** in Chapter **1**).

Theorem 14.1. For any uncountable cardinal $\aleph_i$ and $I \subseteq \omega_i$, there is a superatomic interval Boolean algebra $\mathcal{B}_I$ of the power $\aleph_i$ such that, for any $I, J \subseteq \omega_i$, $\mathcal{B}_I$ is embeddable in $\mathcal{B}_J$, and $\mathcal{B}_I \ll \mathcal{B}_J$ iff $I \subseteq J$.

Proof.

(1) Let $\aleph_i$ be a regular uncountable cardinal. By theorem **1.3**, there is a family $D = \{S_j | j \in \omega_i\}$ of pairwise disjunct stationary subsets of the ordinal ω_i. For any S_j let $\beta_a^{S_j} = \omega_i^*$ if $a \in S_j$, and $\beta_a^{S_j} = 1$ if $a \notin S_j$.

α_{S_j} will denote $\sum_{a \in \omega_i} \beta_a^{S_j}$. Let us show that for $l \neq j$ a Boolean algebra $B(\alpha_{S_j})$ is not a homomorphic image of the algebra $B(\alpha_{S_l})$. Let us assume to the contrary that f is a homomorphism from the algebra $B(\alpha_{S_l})$ to $B(\alpha_{S_j})$. For any $a \in \omega_i$, there is an $a' \in \omega_i$ such that

$$f\Big(\sum_{c \leq a'} \beta_c^{S_l}\Big) \supseteq \sum_{c \leq a} \beta_c^{S_j}.$$

Indeed, if it was not the case, we would have that, for some $b \in \omega_j$,

$$f\Big(\sum_{c > b} \beta_c^{S_l}\Big) \subseteq \sum_{c \leq a} \beta_c^{S_j}.$$

But then we get

$$f\Big(\sum_{c \leq b} \beta_c^{S_l}\Big) \supseteq \sum_{c > a} \beta_c^{S_j},$$

i.e., in particular, there could be found a homomorphism from the Boolean algebra $B(\sum_{c \leq b} \beta_c^{S_l})$ to $B(\sum_{c > a} \beta_c^{S_j})$. It should be noticed that $|\{c \in S_l | c \leq b\}| < \aleph_i$, and $|\{c \in S_j | c > a\}| = \aleph_i$. Hence, there are $\aleph_i$ pairwise disjunct elements of the atomic rank

ω_i in the Boolean algebra $B(\sum_{c\geq a}\beta_c^{S_j})$, i.e., intervals $(d_k,d'_k]$, where $d_k\in\beta_{C_k}^{S_j}$, $d'_k\in\beta_{C'_k}^{S_j}$, $k\in\omega_i$ and $c_0<c'_0<c_1<c'_1<\ldots<c_k<c'_k<\ldots$ is a sequence of elements ω_i such that $c_k\notin S_j$, $c'_k\in S_j$. At the same time, the number of such disjunct elements in the Boolean algebra $B(\sum_{c\leq b}\beta_c^{S_l})$ is strictly less than $\aleph_i$ (each element must contain some initial interval of the ordered set $\beta_c^{S_l}$ for some $c\in S_l$). As has been remarked in section **1**, the homomorphism of an interval Boolean algebra, $B(\sum_{c\leq b}\beta_c^{S_l})$ in the case under consideration, to the Boolean algebra $B(\sum_{c>a}\beta_c^{S_j})$ implies that the latter is isomorphically embeddable in $B(\sum_{c\leq b}\beta_c^{S_l})$. The remark made on the number of pairwise disjunct elements of the atomic rank ω_i in these algebras results in a contradiction, which fact proves that the required element $a'\in\omega_i$ such that $f(\sum_{c\leq a'}\beta_c^{S_l})\supseteq\sum_{c\leq a}\beta_c^{S_j}$ indeed exists.

One can prove analogously that for any $a\in\omega_i$ there is an $a'\in\omega_i$ such that $f(\sum_{c\leq a}\beta_c^{S_l})\subseteq\sum_{c\leq a'}\beta_c^{S_j}$. Therefore, for any $a\in\omega_i$, there is a sequence $a_0=a<a_1<\ldots<a_n<\ldots$ of elements of the ordinal ω_i such that for any β we get

$$(*)\quad \sum_{c\leq a_\beta}\beta_c^{S_j}\subseteq f(\sum_{c\leq a_\beta}\beta_c^{S_l})\subseteq\sum_{c\leq a_{\beta+1}}\beta_c^{S_j}.$$

An element $d\in\omega_i$ will be said f-limiting provided that we have $d=\lim_{\beta\in\omega_l}a_\beta$ for an ascending sequence $<a_\beta^d|\beta\in\omega_l>$ with the property $(*)$. A family of f-limiting elements is, obviously, a closed unbounded subset of the set ω_i. Since S_j is stationary, there is a $b\in\omega_i$ such that $b\in S_j$ and b is f-limiting. As $b\in S_j$, by the definition of $\beta_l^{S_j}$, there is no supremum of an ascending chain of elements $<\sum_{c\leq a_\beta^d}\beta_c^{S_l}|\beta\in\omega_l>$ in the Boolean algebra $B(\alpha_{S_j})$, where $<a_\beta^d|\beta\in\omega_l>$ is a sequence converging to b in the definition of the f-limitedness of the point b. On the other hand, since $S_l\cap S_j=\varnothing$, $b\notin S_l$ and, by the definition of $\beta_a^{S_l}$, there is a supremum of an ascending chain of elements $<\sum_{c\leq a_\beta^d}\beta_c^{S_l}|\beta\in\omega_l>$ in the Boolean algebra $B(\alpha_{S_l})$. In this case, however, by virtue of the inclusion $(*)$, the following equalities hold:

$$f(\sup_{\beta\in\omega_l}\sum_{c\leq a_\beta^d}\beta_c^{S_l}) = \sup_{\beta\in\omega_l} f(\sum_{c\leq a_\beta^d}\beta_c^{S_l}) = \sup_{\beta\in\omega_l}\sum_{c\leq a_\beta^d}\beta_c^{S_j},$$

i.e., in contrast to what has been remarked above, $\sup_{n\in\omega}\sum_{c\leq a_n}\beta_c^{S_j}$ must exist in the algebra $B(\alpha_{S_j})$ at $l\neq j$.

Let now $B_j(j\in\omega_i)$ be a partition of ω_i into sets of the power $\aleph_i$. For any $A\subseteq\omega_i$, let us set $\gamma_A = \sum_{j\in\bigcup_{k\in A}B_k}\alpha_{S_j}$.

In this case, for any $A,C\subseteq\omega_i$, the relation $B(\gamma_A)\ll B(\gamma_C)$ holds iff $A\subseteq C$, while the relation $B(\gamma_A)\leq B(\gamma_C)$ holds for any non-empty $A,C\subseteq\omega_i$. Indeed, for any non-empty $A,C\subseteq\omega_i$, the LOS γ_A is isomorphically embeddable in γ_C, which implies the fact that the Boolean algebra $B(\gamma_A)$ is embeddable in $B(\gamma_C)$. At $A\subseteq C$ a homomorphism from $B(\gamma_C)$ to $B(\gamma_A)$ is obvious. Let now $A\not\subseteq C$ and $j\in A\setminus C$. Let us assume that there is a homomorphism f from $B(\gamma_C)$ to $B(\gamma_A)$. By analogy to what has been proved earlier for algebras of the type $B(\alpha_{S_k})$, we can show that there is an $l\in C$ such that α_{S_l} will have the form $f(\alpha_{S_j})$ to the accuracy of a finite number of intervals bounded in α_{S_l}. On the other hand, obviously, for any initial and final intervals δ_1 and δ_2, any ordered sets $\alpha_{S_p},\alpha_{S_q}$, respectively, the relations $B(\alpha_{S_q})\ll B(\delta_1+\delta_2)\ll B(\alpha_{S_q})$ hold. Thus, we get a homomorphism from the Boolean algebra $B(\alpha_{S_j})$ to $B(\alpha_{S_l})$ at $j\neq l$, which contradicts the property of the Boolean algebra $B(\alpha_{S_k})$ proved earlier. Therefore, indeed, $B(\gamma_A)\ll B(\gamma_C)$ iff $A\subseteq C$.

Setting $\mathcal{B}_I = B(\gamma_I)$, we get the statement of the theorem for a regular $\aleph_i$.

(2) Let $\aleph_i$ be a singular cardinal. We will give only a schematic presentation of the proof for this case.

Lemma 14.1. If $\mathcal{B}_1,\mathcal{B}_2$ are atomic interval Boolean algebras, and if there is a homomorphism from the Boolean algebra $\mathcal{B}_1$ to $\mathcal{B}_2$, there is a homomorphism g from the Boolean algebra $\mathcal{B}_1$ to $\mathcal{B}_2$ with the following property: for any atom $a\in\mathcal{B}_2$, there exists a unique atom $a'\in\mathcal{B}_1$ such that $g(a')=a$.

The proof is obvious and consists in choosing for any preimage of the atom $a\in\mathcal{B}_2$ an atom $a'\in\mathcal{B}_1$ contained in this preimage. Setting then $g(a')=a$ and $g(b)=0$ for other atoms $b\in\mathcal{B}_1$ contained in the preimage a, g is naturally extended to the homomorphism of $\mathcal{B}_1$ to $\mathcal{B}_2$. ∎

Homomorphisms obeying the property considered in this lemma will be called *-homomorphisms.

Lemma 14.2. For any singular $\aleph_i$, any regular $\aleph_j,\aleph_k$ such that $\aleph_j,\aleph_k<\aleph_i$, the homomorphism from a Boolean algebra $B(\omega_j+\omega_i^*)$ to $B(\omega_k+\omega_i^*)$ exists iff $k=j$.

As has been remarked in section **1**, the Stone spaces of the Boolean algebras $B(\omega_j+\omega_i^*)$ and $B(\omega_k+\omega_i^*)$ are homomorphic to LOSes $\omega_j+1+\omega_i^*$ and $\omega_k+1+\omega_i^*$ with a corresponding topology coinciding in the case under discussion with the ordered one. If g is a *-homomorphism from the algebra $B(\omega_j+\omega_i^*)$ to $B(\omega_k+\omega_i^*)$, the continuous embedding h dual to g of the space $\omega_k+1+\omega_i^*$ in the space $\omega_j+1+\omega_i^*$ meets an additional condition: for any isolated point x of the space $\omega_k+1+\omega_i^*$, $h(x)$ is an isolated point of the space $\omega_j+1+\omega_i^*$. Such a continuous embedding of $\omega_k+1+\omega_i^*$ in $\omega_j+1+\omega_i^*$ can be easily proved to exist only when $k=j$. ■

Let now $\aleph_i$ be a singular cardinal and $\aleph_l=cf(\aleph_i)$. Let $k_j(j\in\omega_l)$ be an ascending chain of cardinals such that $\sum_{j\in\omega_l}k_j=\aleph_i$. For any subset $I\subseteq\omega_i$, let us set $I(j)=I\cap\omega_{j+1}$ for $j\in\omega_l$. As was the case in (1), let us choose a family $D_j=\{S_l^j\mid l\in\omega_j\}$ of pairwise disjunct stationary subsets of the ordinal ω_{j+1} with an additional condition: all elements of the set S_l^j unlimiting in S_l^j have a cofinal equal to ω_j in ω_{j+1}. Let us set $\beta_a^{S_l^j}=\omega_i^*$ if $a\in S_l^j$, and equal to 1 if $a\notin S_l^j$. As was the case in (1), $\alpha_{S_l^j}$ will denote a LOS $\sum_{a\in\omega_{j+1}}\beta_a^{S_l^j}$. The LOS $\gamma_{I(j)}$ is constructed from the LOS $\alpha_{S_l^j}$ also by analogy with the procedure used in (1). Therefore, $\gamma_{I(j)}$ will be a lexicographic ω_{j+1}^2-sum of linear orders of the type 1 and ω_i^*, in which case, in accordance with the condition imposed on the elements of the set S_l^j, any interval ω_i^* in the LOS $\gamma_{I(j)}$, provided that it corresponds to $\beta_a^{S_l^j}$, where a is not a limiting point in S_l^j, is contained in an interval of the type $\omega_j+\omega_i^*$ of the same LOS $\gamma_{I(j)}$. And, again, as in (1), we prove that for any $I,J\subseteq\omega_i$ $B(\gamma_{I(j)})\ll B(\gamma_{J(j)})$ iff $I(j)\subseteq J(j)$.

Let us set $\delta_I=\sum_{j\in\omega_l}\gamma_{I(j)}$ for $I\subseteq\omega_i$. For any $I,J\subseteq\omega_i$, $B(\delta_I)$ is obviously embeddable in $B(\delta_J)$, and for the case $I\subseteq J$ we get $B(\delta_I)\ll B(\delta_J)$. To prove the converse statement, it suffices, according to the remark made earlier on the algebras $B(\gamma_{I(j)})$ and $B(\gamma_{J(j)})$, to notice that the relation $B(\delta_I)«B(\delta_J)$ implies the relation $B(\gamma_{I(j)})«B(\gamma_{J(j)})$ for any $j\in\omega_l$.

Let $B(\delta_I)«B(\delta_J)$. By lemma **14.1**, there is a *-homomorphism g from the Boolean algebra $B(\delta_J)$ to the algebra $B(\delta_I)$. In this case, by lemma **14.2**, only those elements of the algebra $B(\delta_J)$ can map to the intervals of the LOS δ_I of the ordered type $\omega_j+\overset{*}{\omega_i}(j\in\omega_l)$ which contain intervals of the same type, $\omega_j+\overset{*}{\omega_i}$, (a fixed j is meant). We have noticed earlier that the intervals of the LOS δ_J of the ordered type $\omega_j+\overset{*}{\omega_i}$ correspond to all non-limiting elements in the sets S_l^j in the representation of $\delta_{I(j)}$ as a ω_{j+1}^2-sum of the LOSes of the ordered types 1 and $\overset{*}{\omega_i}$. Therefore, the g-preimage of the interval $\gamma_{I(j)}$ (as an element of the Boolean algebra $B(\delta_I)$) coincides with the interval $\gamma_{J(j)}$ (as an element of the Boolean algebra $B(\delta_J)$ to the accuracy of the initial interval of the former. This fact, as can be seen easily, implies the relations $B(\delta_{I(j)})«B(\delta_{J(j)})$ for any $j\in\omega_l$ for the case when $B(\delta_I)«B(\delta_J)$.

Thus, we have also proved the statement of the theorem for the case of a singular $\aleph_i$. ■

Corollary 14.1. For any uncountable cardinal $\aleph_i$,

(a) there are $2^{\aleph_i}$ of mutually embeddable superatomic interval Boolean algebras of the power $\aleph_i$, none of which is a homomorphic image of another;

(b) any partially ordered set of the power not greater than $\aleph_i$ is isomorphically embeddable in $<\mathfrak{J}SIBA_{\aleph_i};«>$ in such a way that the images of the elements of this set are pairwise embeddable into one another.

The statement of the corollary results directly from that of theorem **15.2**, i.e., that for any cardinal $\aleph_i$ there are $2^{\aleph_i}$ mutually non-embeddable subsets of the ordinal ω_i, and from the fact that any partially ordered set of the power not greater than $\aleph_i$ is isomorphically embeddable into a set of all subsets of the ordinal ω_i. ■

Later on we will make use of a statement obtained by repeating word per word the considerations of theorem **14.1** (1) for $\aleph_i=\aleph_1$, with the ordinal type η substituted for ω_i when constructing $\beta_a^{S_j}$.

Corollary 14.2. There is an infinite number of pairwise embeddable interval

Boolean algebras $\mathfrak{C}_i$ of the power $\aleph_1$, none of which is a homomorphic image of the other, in which case for each of the algebras $\mathfrak{C}_i$, the set $\{d \in \mathfrak{C}_i | \mathfrak{C}_i | d$ contains a chain of elements of the ordinal type $\eta \cdot \omega_1\}$ forms an ultrafilter on $\mathfrak{C}_i$.

Let us now prove that there are noncompactable chains of the ordinal type of real numbers in the epimorphism skeleton of a Boolean algebra variety. In section **1** we have given the notion of a formally real LOS and formulated a number of statements on the existence of rigid subsets of these LOSes, and on the properties of these rigid subsets.

Let R be an ordered set of real numbers, and $P \subseteq R$ obey the conclusion of lemma **1.1**. For any $a \in R$, let us define P_a as $\{x \in P | x < a\}$.

Lemma 14.3.

(a) Boolean algebras $B(P_a)$ are Bonnet-rigid;

(b) for $a < b \in R$, $B(P_a) \ll B(P_b)$, and for any non-singleton algebra $\mathfrak{B}$ $B(P_a) \times \mathfrak{B} \not\ll B(P_a)$;

(c) for any a and any Boolean algebra $\mathfrak{B}$, from $\mathfrak{B} \equiv_{\ll} B(P_a)$ we get $\mathfrak{B} \cong B(P_a)$;

(d) (CH) for $a \in R$ and any Boolean algebra $\mathfrak{B}$, if $B(P_a) \ll \mathfrak{B} \ll B(P_b)$ for all $b \in R$ such that $b > a$, there is a countable set $D \subseteq \{x \in P | x \geq a\}$ such that $\mathfrak{B} \cong B(P_a \cup D)$;

(e) for $a \in R$ and any Boolean algebra $\mathfrak{B}$, if $B(P_b) \ll \mathfrak{B} \ll B(P_a)$ for all $b \in R$ such that $b < a$, $\mathfrak{B} \cong B(P_a)$.

Proof. Let h be an isomorphism of the LOS $\{x \in R | x < a\}$ and LOS R. Obviously, $h(P_a) \subseteq R$ obeys all the conclusions of lemma **1.1** and, hence, by lemma **1.3** and theorem **1.12**, a Boolean algebra $B(h(P_a)) \cong B(P_a)$ is Bonnet-rigid. The relation $B(P_a) \ll B(P_b)$ is obvious for $a < b$. If $\mathfrak{B} \equiv_{\ll} B(P_a)$ then, since $\mathfrak{B} \ll B(P_a)$ and $B(P_a)$ is retractive, we see that $\mathfrak{B}$ is embeddable into the algebra $B(P_a)$. This embedding and the relation $B(P_a) \ll \mathfrak{B}$ together imply, since the algebra $B(P_a)$ is Bonnet-rigid, an isomorphism of the algebras $B(P_a)$ and $\mathfrak{B}$. For any non-singleton algebra $\mathfrak{C}$, the relation $B(P_a) \times \mathfrak{C} \ll B(P_a)$ implies $B(P_a) \times \mathfrak{C} \equiv_{\ll} B(P_a)$ and, hence, an isomorphism relation of $B(P_a) \times \mathfrak{C}$ and $B(P_a)$. The latter fact implies the existence of a non-identical embedding of the algebra $B(P_a)$ into itself, which contradicts the Bonnet-rigidity of $B(P_a)$. Therefore, the statements (a), (b) and (c) of the lemma have been proved.

Let us show that the statement (d) is valid. Let us fix a $b_0 > a$. Since there is an isomorphism from the algebra $B(P_{b_0})$ to $\mathcal{B}$, as has been repeatedly remarked, $\mathcal{B} \cong B(D_1)$ for a certain $D_1 \subseteq P_{b_0}$. Let us show that $P_a \subseteq D_1$. In the opposite case, there is a $y \in P_a \setminus D_1$. Let h be a homomorphism from $B(D_1)$ oto $B(P_a)$, and let $c \in B(D_1)$ be such that $h(c) = (-\infty, y] \cap P$. From now on, we mean intervals of the set R in the proof of the lemma by intervals. There are $x_1 < x_2 \in D_1$ such that $h((x_1, x_2] \cap D_1) \supseteq (z, y] \cap P$ for a $z \in P_a$, and, since $x_2 \neq y$, we can assume $(x_1, x_2] \cap (z, y] = \emptyset$. As $|(z, y] \cap P_a| = 2^{\aleph_0}$, $|(x_1, x_2] \cap D_1| = 2^{\aleph_0}$ as well.

In an obvious way, using h, a homomorphism from the Boolean algebra $B(D_1)|(x_1, x_2] \cap D_1$ to the algebra $B(P_a)|(z, y] \cap P_a$ is constructed. Since $B(D_1)|(x_1, x_2] \cap D_1$ is retractive, we get an embedding g of the algebra $B(P_a)|(z, y] \cap P_a$ into the algebra $B(D_1)|(x_1, x_2] \cap D_1$. Considering a subtraction of the set $\{g((z,t])| t \in P_a$ and $z < t < y\}$ and using standard considerations, we get a continual subset $S \subseteq (z, y] \cap P_a$ which is either isotonically or antiisotonically mappable to the set $(x_1, x_2] \cap D_1 \subseteq P$. Since, as has been remarked earlier, the intervals $(x_1, x_2]$ and $(z, y]$ are disjunct, we obtain a contradiction to the properties of the set P. Therefore, indeed, the inclusion $P_a \subseteq D_1$ is valid.

Analogous considerations show that $D_1 \cap \{b \in R | b < a\} = P_a$. Let now $D = D_1 \setminus P_a$. It should be noticed that $|D| < 2^{\aleph_0}$. Indeed, in the opposite case there are $b_1, b_2 \in R$ greater than a such that $|(b_1, b_2) \cap D| = 2^{\aleph_0}$. But if this was the case, the existence of a homomorphism from the Boolean algebra $B(P_b)$ to the algebra $\mathcal{B}$ would result in that of an isotonic or antiisotonic mapping of a certain continual subset $P_b \subseteq P$ into a set $(b_1, b_2) \cap D \subseteq P$ disjunct from it, which contradicts the P properties. Therefore, by virtue of the continuum hypothesis, $|D| \leq \aleph_0$ and $\mathcal{B} \cong B(P_a \cup D)$.

Let us now assume that the conditions of the statement (e) are met. Since $B(P_a) \gg \mathcal{B}$, as has been remarked earlier, $\mathcal{B} \cong B(D)$ for a certain $D \subseteq P_a$. Let b be any real number less than a. Considerations from the proof of the statement (d) show that $D \supseteq P_b$. However, since $P_a \bigcup_{b<a} P_b$, $D = P_a$ and $D \cong B(P_a)$. ■

Theorem 14.2. (CH) In the epimorphism skeleton of a Boolean algebra variety, there is a noncompactable chain B of a dense order type or, more precisely, $< B/\equiv_{\ll}; \ll >$ has the order type of a set of real numbers.

Proof. Let us consider Boolean algebras of the type $B(P_a) \times B(Q)$ for $a \in R$, where P_a, R and Q are defined before formulating lemma **14.1**. Let us first show that for any $a,b \in R$, $B(P_a) \times B(Q) \ll B(P_b) \times B(Q)$ iff $a \leq b$. It suffices to notice that the relation $B(P_a) \times B(Q) \ll B(P_b) \times B(Q)$ implies the inequality $a \leq b$. Let h be a homomorphism from the algebra $B(P_b) \times B(Q)$ to the algebra $B(P_a) \times B(Q)$. In this case, since $B(Q)$ is countable and for any non-zero element of the algebra $B(P_a)$, there is a continuum of less elements of the algebra $B(P_a)$, we get $h(<0_{B(P_b)}, 1_{B(Q)}>) = <0_{B(P_a)}, c>$ for some $c \in B(Q)$, and, hence, $h(<1_{B(P_b)}, 0_{B(Q)}>) = <1_{B(P_a)}, \neg c>$. Therefore, there is a homomorphism from the algebra $B(P_b)$ to the algebra $B(P_a)$. At the same time, the assumption $b < a$, combined with a homomorphism of $B(P_b)$ on $B(P_a)$, contradicts the Bonnet-rigidity of the algebra $B(P_b)$. Thus, indeed, the relations $B(P_a) \times B(Q) \ll B(P_b) \times B(Q)$ and $a \leq b$ are equivalent.

To complete the proof of the theorem, it suffices now to show that a family of Boolean algebras $\equiv_{\ll}$-equivalent to algebras of the type $B(P_a) \times B(Q)$, where $a \in R$, forms a noncompactable chain in $<\mathfrak{J}BA; \ll>$. Since a set of real numbers is a complete linear order, it suffices to prove the following statements:

(1) for any algebra $\mathcal{B}$ and any $a \in R$, if $\mathcal{B} \ll B(P_a) \times B(Q)$, as well as for any $b \in R$ such that $b < a$, $B(P_b) \times B(Q) \ll \mathcal{B}$, $\mathcal{B} \equiv_{\ll} B(P_a) \times B(Q)$;

(2) for any algebra $\mathcal{B}$ and any $a \in R$, if $B(P_a) \times B(Q) \ll \mathcal{B}$, as well as for any $b \in R$ such that $a < b$, $\mathcal{B} \ll B(P_b) \times B(Q)$, $\mathcal{B} \equiv_{\ll} B(P_a) \times B(Q)$.

Let $\mathcal{B}$ satisfy the condition of the statement (1). Then, according to the inequality $\mathcal{B} \ll B(P_a) \times B(Q)$, there are Boolean algebras $\mathcal{B}_1 \ll B(P_a)$ and $\mathcal{B}_2 \ll B(Q)$ such that $\mathcal{B} \cong \mathcal{B}_1 \times \mathcal{B}_2$. Since in this case $\mathcal{B}_2$ is countable, as has been noticed earlier, the inequalities $B(P_b) \times B(Q) \ll \mathcal{B}_1 \times \mathcal{B}_2$ imply $B(P_b) \ll \mathcal{B}_1$ for any $b < a$. In line with the statement (e) of lemma **14.3**, this implies an isomorphism of the algebras $\mathcal{B}_1$ and $B(P_a)$. The algebra $\mathcal{B}_2$, however, cannot be superatomic, since in that case a homomorphism of $\mathcal{B} \cong B(P_a) \times \mathcal{B}_2$ on the algebra $B(P_a) \times B(Q)$ would amount to the existence of a non-identical homomorphism of the algebra $B(P_a)$ on itself. Therefore, $\mathcal{B}_2 \equiv_{\ll} B(Q)$, $\mathcal{B} \equiv_{\ll} B(P_a) \times B(Q)$, and statement (1) is proved.

Let now $\mathcal{B}$ satisfy the conditions of the statement (2). Let us fix a $b_0 > a$, and let the Boolean algebra $B(P_{b_0}) \times B(Q)$ be isomorphic to a Boolean algebra $B(P_{b_0} + 1 + Q)$ and, hence, the relation $\mathcal{B} \ll B(P_{b_0}) \times B(Q)$ implies an isomomorphism of the algebras $\mathcal{B}$ and $B(D_1)$, where D_1 is a subset of the LOS $P_{b_0} + 1 + Q$. Considerations of the statement (d) of lemma **14.3** prove in this case also that $D_1 = P_a \oplus D$, where D is a countable LOS. Since $B(P_a) \times B(Q) \ll \mathcal{B} \cong B(P_a \oplus D)$, as was the case in the proof of (1), we notice that the Boolean algebra $B(D)$ is not superatomic. Therefore, to prove the relation $\mathcal{B} \equiv_{\ll} B(P_a) \times B(Q)$, it suffices to show that D has a least element.

Let us assume to the contrary that if ψ is a homomorphism from the algebra $B(P_a \oplus D)$ to the algebra $B(P_a) \times B(Q) \cong B(P_a \oplus 1 \oplus Q)$. Then, as was the case in the proof of lemma **14.3**, it should be noticed that for $z \in P_a$ we get $\psi((-\infty, z]) = (-\infty, z]$. But in the algebra $B(P_a \cup D)$, there is no $\sup\{(-\infty,z] \mid z \in P_a\}$, while in the algebra $B(P_a) \times B(Q)$, $\sup\{(-\infty,z] \mid z \in P_a\} = \sup\{\psi((-\infty,z]) \mid z \in P_a\}$ exists, which fact contradicts the existence of ψ. Therefore, indeed, D has a least element, i.e., $D = 1 \oplus D_2$ for some not scattered LOS and, hence, since $B(D_2) \equiv_{\ll} B(Q)$,

$$B(P_a \oplus 1 \oplus D_2) \cong B(P_a) \times B(D_2) \equiv_{\ll} B(P_a) \times B(Q),$$

which completes the proof of the statement (2). ■

By way of concluding this section let us present some proofs of independence of the embedding and epimorphism relations on a Boolean algebra variety.

The definitions of almost disjunct, *ad*-, *mad*-families of subsets, as well as the formulation of the set-theoretical assumption $P(2^{\omega})$ are given in the end of section **1**. The relation "$P \setminus R$ finite" will be denoted by $P \subseteq_* R, P =_* R$, provided that $(P \setminus R) \cup (R \setminus P)$ is finite.

Lemma 14.4. Under the assumption $P(2^{\omega})$, for any non-principal ultrafilter p on ω, there is a *mad*-family X of the subsets of ω such that $F(X) = P$.

Proof. Let p be a non-principal ultrafilter on ω, and let $\{a_i \mid i < 2^{\aleph_0}\}$ be an enumeration of the elements of the ultrafilter p such that every $a \in p$ is encountered $2^{\aleph_0}$ times in this enumeration. Let $\{b_i \mid i < 2^{\aleph_0}\}$ be an enumeration of the elements of the set $\{b \subseteq \omega \mid b \notin p$ and $|b| = \aleph_0\}$. Let us set $A_k = \{a_i \mid i < k\}$, $B_k = \{b_i \mid i < k\}$ and construct an ascending sequence $\{X_i \mid i < 2^{\aleph_0}\}$ of the *ad*-families of the subsets ω, so that:

(1) $|X_i| < 2^{\aleph_0}$ and $X_i \cap P = \varnothing$;

(2) if $i = l + k$, where l is limiting, and $k < \omega$, there is a $c \in X_{l+2k+1} \setminus X_{l+2k}$ such that $c \subseteq a_i$;

(3) if $i = l + k$, where l is limiting and $k < \omega$, there is a $d \in X_{l+2k+2}$ such that $|d \cap b_i| = \aleph_0$.

The sequence $\{X_i | i < 2^{\aleph_0}\}$ is constructed by induction over i: $X_0 = X_1 = \varnothing$, and we have X_i constructed for $i < j = l + 2n + 1$, where l is limiting, $n < \omega$. Let

$$S = A_{l+n+1} \cup \{\omega \setminus b | b \in B_{l+n}\} \cup \{\omega \setminus x | x \in X_{l+2n}\}.$$

The family S has the property *fip*, as $S \subseteq P$ and $|S| < 2^{\aleph_0}$. Under the assumption $P(2^{\omega})$, there is an infinite $a \subseteq \omega$ such that $a \setminus s$ is finite for any $s \in S$. Let then $a^* \subseteq a \cap a_{l+n}$ be such that $a^* \notin p$ and $|a^*| = \aleph_0$. Let us set $X_{l+2n+1} = X_{l+2n} \cup \{a^*\}$. If now there is an $s \in X_j$ such that $|s \cap b_{l+n}| = \aleph_0$, we set $X_{j+1} = X_j$, while if there is no such an $s \in X_j$, let

$$T = A_{l+2n+1} \cup \{\omega \setminus b | b \in B_{l+n}\} \cup \{\omega \setminus x | x \in X_{l+2n+1}\}.$$

And again, since $T \subseteq p$, T has the property *fip*, $|T| < 2^{\aleph_0}$ and, according to $P(2^{\omega})$, there is an infinite $c \subseteq \omega$ such that $c \setminus s$ is finite for any $s \in T$. Let $c^* \subseteq c \cap a_{l+n}$ be such that $c^* \notin p$ and $|c^*| = \aleph_0$. Let us set

$$X_{l+2n+2} = X_{l+2n+1} \cup \{c^* \cup b_{l+n}\}.$$

For limiting l we get $X_l = \bigcup_{i<l} X_i$. The conditions (1) - (3) are obviously satisfied for the sequence $\{X_i | i < 2^{\aleph_0}\}$ by construction considerations. The same considerations show $X = \bigcup_{i<2^{\aleph_0}} X_i$ to be an almost disjunct family of subsets ω. As $X \cap p = \varnothing$ and p is a non-principal filter, a union of any finite number of elements X has an infinite complement in ω, i.e., X is an *ad*-family. At the same time, by virtue of the conditions (2) and (3) on $\{X_i | i < 2^{\aleph_0}\}$, X is a *mad*-family. Let us show that $F(X) = p$. Let $a \in p$, for the $2^{\aleph_0}$ ordinals i we get $a = a_i$ and, hence, by condition (2), there are $2^{\aleph_0}$ different elements X contained in a, i.e., $p \subseteq F(X)$.

Assume now that $a \subseteq \omega, a \notin p$, and a is infinite. Then $a = b_i$ for some i. By construction, $\{X_j | j < 2^{\aleph_0}\}$, for all $j > i + \omega$, for any $d \in X_{j+1} \setminus X_j$ $|d \cap (\omega \setminus b_i| = \aleph_0$, i.e., $a \notin F(X)$. Therefore, indeed, we have $F(X) = p$ for the *mad*-family X constructed. ■

For any family T of subsets of a set A, $B(A,T)$ will denote a subalgebra of the Boolean algebra of all subsets of the set A, generated by elements incorporated in T and by elements of the type $\{a\}$ where $a \in A$. If X is a *mad*-family of the subsets of A, $B(A,X)$ is a superatomic Boolean algebra of the characteristic $<2,1>$.

Definition 14.1. βA will denote the family of all ultrafilters on the set A. The Rudin-Keisler quasi-order $\prec$ is defined on different ultrafilters in the following way: for $p\in\beta A, q\in\beta B$ the relation $p\prec q$ is valid iff there is a mapping f of a certain set $X\in q$ in A such that, for any $Y\subseteq A$, $Y\in p$, iff for a certain $Z\in q$ the inclusion $f(Z)\subseteq Y$ holds. The Ridin-Keisler finite quasi-order $\leq$ is defined analogously provided that there is an additional requirement on f: for any $a\in A$, $|f^{-1}(a)|<\aleph_0$.

The following theorem will be given without any proof, as it would require a lengthy digression into the theory of ultrafilters (see, for instance, theorem **10.4** in [41]).

Theorem 14.3. There are $2^{\aleph_0}$ ($2^{2^{\aleph_0}}$ under **CH**) of non-principal ultrafilters on ω which are pairwise incomparable relative to the Rudin-Keisler quasi-order.

Let p be an arbitrary non-principal ultrafilter on ω, and X_p a *mad*-family of the subsets of ω constructed by lemma **14.4** such that $F(X_p)=p$. $\mathcal{B}_p$ will denote the Boolean algebra $B(\omega,X_p)$. It should be noticed that the existence of the algebra $B(\omega,X_p)$ has been proved only under the assumption of the set-theoretical hypothesis $P(2^{\aleph_0})$ or, under a stronger one, **CH**.

Lemma 14.5. For any non-principal ultrafilters p,q on ω, if the Boolean algebra $\mathcal{B}_p$ is isomorphically embeddable into a Boolean algebra $\mathcal{B}_q$, $p\leq q$.

Proof. It should be noticed that, for any $b\in\mathcal{B}_q$, there are no two infinite subsets A,B of elements of $\mathcal{B}_q$ such that all elements of $A(B)$ are pairwise disjunct, contain an infinite number of atoms each, and for any $a\in A, d\in B$ we have $a\subseteq b, d\subseteq\neg b$. From now on, the elements of $A(B)$ will be identified with singleton subsets of ω. From the remark just made, we can deduce that if f is an embedding of $\mathcal{B}_p$ in $\mathcal{B}_q$ then, for any $n\in\omega$ ($s\in X_p$), there are $s_1,\ldots,s_k\in X_q$ such that either $f(n)=_* s_1\cup\ldots\cup s_k(f(s)=_* s_1\cup\ldots\cup s_k)$ or $f(n)=_*\varnothing(f(s)=_*\varnothing)$.

Let f be a certain fixed embedding of $\mathcal{B}_p$ in $\mathcal{B}_q$, and let g map $\bigcup_{m\in\omega} f(m)$ to ω in such a way that $n\in f(g(n))$ for any $n\in\bigcup_{m\in\omega} f(m)$. f^* will denote the mapping of a set of all subsets of ω to itself such that for any $A\subseteq\omega$, $f^*(A)$ is the set-theoretical union of the sets $f(n)$, where $n\in A$. It should be noticed that, for any $A\in\mathcal{B}_p$, the inclusion $f(A)\supseteq f^*(A)$ holds, and for an A which is equal to a finite family of atoms of the algebra $\mathcal{B}_p$ (i.e., to a finite subset of ω), $f(A) = f^*(A)$. Let $\bar{g}(A) = \{g(n) | n \in A$ and g is defined on $n\}$ for $A \subseteq \omega$. Let us show that $\bar{g}(q) = p$. Assume to the contrary that $d \in q$ and $\bar{g}(d) \notin p$. Since p is an ultrafilter, $\omega \setminus \bar{g}(d) = b \in p$. Hence, $\mathfrak{S}_1 = \{s | s \in X_p, s \subseteq_* b\}$ has the power $2^{\aleph_0}$. For $s \in \mathfrak{S}_1$ let us set $\mu(s) = s \cap b$, and let $\mathfrak{S}_2 = \{\mu(s) | s \in \mathfrak{S}_1\}$. In this case, $|\mathfrak{S}_2| = 2^{\aleph_0}$. For $s \in \mathfrak{S}_2$ we will define $t(s) = f(s) \setminus f^*(s)$. Let $T = \bigcup_{s \in \mathfrak{S}_2} t(s)$.

For any $s \in \mathfrak{S}_2$, the equalities $f(s) = f^*(s) \cup t(s) =_* s_1 \cup \ldots \cup s_k$ hold for some $s_i \in X_q$. As $\omega \setminus d \notin q$, $|\{l \in X_q | l \subseteq_* \omega \setminus d\}| < 2^{\aleph_0}$ and, hence, there is a continual $\mathfrak{S}_3 \subseteq \mathfrak{S}_2$ such that $f(s) \cap d \neq \varnothing$ for $s \in \mathfrak{S}_3$. But for $s \in \mathfrak{S}_2$, the inclusion $s \subseteq b$ is valid and, hence, $f^*(s) \subseteq f^*(b) \subseteq \omega \setminus d$, and if $f(s) \cap d \neq \varnothing$, $t(s) \neq \varnothing$. Therefore, there has been found a continual $\mathfrak{S}_3$ such that $f(s) \cap T \neq \varnothing$ for $s \in \mathfrak{S}_3$. On the other hand, for $s_1 \neq s_2 \in \mathfrak{S}_3$, the set $s_1 \cap s_2 = m$ is a finite family of atoms and, thus, $f(s_1) \cap f(s_2) = f^*(m) \subseteq f^*(b)$, i.e., $f(s_1) \cap f(s_2) \cap T = \varnothing$. Therefore, we have got a continual system of non-intersecting non-empty subsets $\{f(s) \cap T = t(s) | s \in \mathfrak{S}_3\}$ of the countable set T. The contradiction obtained proves that $\bar{g}(d) \in p$ for any $d \in q$.

In an analogous way we can prove that the domain of the definition of the function g lies within q. Thus, the fact that there is a embedding of $\mathcal{B}_p$ in $\mathcal{B}_q$ entails $p \prec q$.

Let us show that, in fact, the inequality $p \leq q$ holds. Let $b \in X_p$ and $b^* = \{n \in b | |f(n)| \geq \aleph_0\}$. It should be noticed that b^* is finite. Indeed, in the opposite case, assume $A = \{f(n) | n \in b^*\} \subseteq \mathcal{B}_q$ and $b_1, \ldots, b_n, \ldots$ are pairwise disjunct and b-disjunct elements of $\mathcal{B}_q$, each containing an infinite number of atoms. For any $a \in A, d \in B = \{f(b_i) | i \in \omega\}$, the inequalities $a \subseteq f(b)$ and $d \subseteq \neg f(b)$ hold. As has been remarked in the beginning of the proof, there are no such elements $f(b)$ in the algebra $\mathcal{B}_q$, i.e., b^* must be finite.

Let $a = \{n \in \omega | |f(n)| \geq \aleph_0\}$. Let us show that a is finite, assuming that the opposite is the case. The element a cannot belong to X_p, as all X_p elements are infinite, $a^* = a$ and, hence, a would be finite. On the other hand, if we had $a \notin X_p$ and a was infinite,, as X_p is a maximal *ad*-family, there would be an $a_1 \in X_p$ such that an $a \cap a_1$ would be infinite. In this case, $a_1^* \supseteq a \cap a_1$, and a_1^* would be infinite in contrast to the earlier remark. Therefore, indeed, a is finite.

Let D be the domain of g, and let $D_1 = D \setminus f(a)$. Then $D_1 \in q$, since in the opposite case $D \cap f(a) \in q$, as $D \in q$ and, hence, there is an $i \in a$ such that $f(i) \cap D \in q$, i.e., $f(i) \in q$. As has been remarked earlier, $f(i) =_* u_1 \cup \ldots \cup u_m$, where $u_i \in X_q$ and, hence, there is an i such that $u_i \in q$. It is obvious at the same time that none of the elements X_q can belong to q. Therefore, indeed, $D_1 \in q$. In this case, however, the restriction of $\bar{g}$ to D_1 makes it possible to state that $p \le q$. ■

The statement of the theorem is, obviously, also valid for any ultrafilters p,q defined on arbitrary countable sets A, B, respectively.

As a corollary, one can deduce from theorem **14.3** and lemma **14.5** the existence of $2^{\aleph_0}$ (under the assumption $P(2^{\omega})$) or $2^{2^{\aleph_0}}$ (under **CH**) of mutually non-embeddable Boolean algebras of the power $2^{\aleph_0}$. It should be recalled that in section **1** a stronger result was presented without any additional set-theoretical assumptions: for any $\aleph > \aleph_0$, there are $2^{\aleph}$ Boolean algebras of the power $\aleph$ mutually non-embeddable into each other. The statements discussed earlier, however, will be used to construct families of mutually non-embeddable Boolean algebras with an additional property, i.e., they are homomorphic images of each other, which means that they are equivalent in terms of $\equiv_{\ll}$.

Theorem 14.4. Under the assumption $P(2^{\omega})$ (or under a stronger one, **CH**), for any $n \in \omega$ there are Boolean algebras $\mathcal{B}_0,\ldots,\mathcal{B}_{n-1}$ such that $\mathcal{B}_i$ are mutually non-embeddable, and for any $i,j < n$ we have $\mathcal{B}_i \ll \mathcal{B}_j$.

Proof. Let p be a non-principal ultrafilter on ω, and let $< A_i | i < 2^{\aleph_0} >$ be an enumeration of all elements p, in which case for any $A \in p$ A is encountered $2^{\aleph_0}$ times in the sequence $< A_i | i < 2^{\aleph_0} >$. Let X_p be a *mad*-family of subsets ω such that $F(X_p) = p$, as was the case earlier. Let us construct a partition of X_p into subfamilies $X_p^i (i \in \omega)$ such that $|X_p^i| = 2^{\aleph_0}$ and $F(X_p^i) = p$ for all $i \in \omega$.

A subfamily X_p^i will be constructed as a union of an ascending chain of *ad*-subfamilies $X_p^i(j)$ $(j < 2^{\aleph_0})$ of the family X_p. If j is limiting, we set $X_p^i(j) = \bigcup_{k<j} X_p^i(k)$. Let the subfamilies $X_p^i(k)$ have been constructed for $k \le j < 2^{\aleph_0}$. We set $R_j = \{B \in X_p | B \subseteq_* A_j\}$. Since $A_j \in p = F(X_p)$, $|R_j| = 2^{\aleph_0}$. Let us now define $X_p^i(j+1)$ (at $i \in \omega$), by adding one element (different for various $i \in \omega$) from $R_j \setminus \bigcup_{i\in\omega} X_p^i(j)$ to $X_p^i(j)$. It is obvious that $X_p^i = \bigcup_{j<2^{\aleph_0}} X_p^i(j)$ has the properties discussed earlier.

Let φ be a bijective mapping from ω to itself such that $|\omega \setminus \varphi(\omega)| = \aleph_0$. Let $p_0,\ldots,p_{n-1}$ be non-principal ultrafilters on $\omega \setminus \varphi(\omega)$ pairwise incomparable in terms of the order $\prec$, and for any $s \in \omega$, let $r(s)$ be a subtraction of the number s over the module n. For $p,q \in \omega$ let $X_{p_q}^p = \varphi^q(X_{p_{r(q)}}^p)$ and $X_{p_q} = \bigcup_{p\in\omega} X_{p_q}^p$. Let us choose

g to be a bijective mapping from the set $\bigcup_{k<n} X_{p_k}$ to the set $\bigcup_{k<n}(X_{p_k} \setminus X^0_{p_k})$ such that $g(X^s_{p_{r(i)}}) = X^{s+1}_{p_{i-1}}$ for $1 \le i \le n$ and $s \in \omega$. T will denote a maximal *ad*-subset of the family $\{A \subseteq \omega |$ for all $i<n$ $|A \cap (\varphi^l(\omega) \setminus \varphi^{l+1}(\omega))| < \aleph_0\}$, and for $l<n$, let h_l be a bijective mapping from $X_{p_l} \setminus X^0_{p_l}$ to T. Let us define the set X'_{p_q} in the following way:

$$X'_{p_q} = X^0_{p_q} \cup \{A \cup \varphi^{q+i-1}(g(\varphi^{-(q+i)}(A))) \cup \varphi^{q+i-2}(g^2(\varphi^{-(q+i)}(A)))$$
$$\cup \ldots \cup \varphi^q(g^i(\varphi^{-(q+i)}(A))) \cup h_{r(i)}(\varphi^{-(q+i)}(A)) | A \in X^0_{p_{q+i}}, i \ge 1\},$$

where $\varphi^{-n} = (\varphi^n)^{-1}$. One can directly check that X'_{p_q} is a *mad*-family of subsets of the set $\varphi^q(\omega)$, and that $\{B \cap (\varphi^q(\omega) \setminus \varphi^{q+1}(\omega)) | B \in X'_{p_q}\} = X_{p_q}$. It is also obvious that $F(X_{p_q})$ is not a principal ultrafilter on the set $\varphi^q(\omega), \varphi^q(\omega) \setminus \varphi^{q+1}(\omega) \in F(X'_{p_q})$, and for any $Y \subseteq \varphi^q(\omega)$, $Y \in F(X'_{p_q})$ is equivalent to $Y \cap (\varphi^q(\omega) \setminus \varphi^{q+1}(\omega))$ in $F(X'_{p_q}) = \varphi^q(p_{r(q)})$. Therefore, we get $F(X'_{p_q}) \prec p_{r(q)}$ and $p_{r(q)} \prec F(X'_{p_q})$.

Let now $\mathcal{B}_i (i \in \omega)$ be a Boolean algebra of the subsets of the set $\varphi^i(\omega)$ generated by one-element subsets and subsets of X'_{p_i}. For any $k,m \in \omega$ such that $k = m (\mathrm{mod}\, n)$, it follows from the construction of X'_{p_i} that $\mathcal{B}_k \cong \mathcal{B}_m$ (the mapping φ^{m-k} is defined by an isomorphism). Since $F(X'_{p_i}) \prec p_{r(i)}$ and $p_{r(i)} \prec F(X'_{p_i})$ for $i \in \omega$, and the ultrafilters $p_0, \ldots, p_{n-1}$ are pairwise incomparable in terms of the Rudin-Keisler quasi-order, we get from lemma **14.5** that the Boolean algebras $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ are mutually non-embeddable into each other. On the other hand, for any $i<j$, the homomorphism h from a Boolean algebra $\mathcal{B}_i$ to an algebra $\mathcal{B}_j$ is defined by the following condition: $h(a) = a$ for any $a \in \varphi^j(\omega)$, and $h(a) = \varnothing$ for $a \in \varphi^i(\omega) \setminus \varphi^j(\omega)$. Therefore, $\mathcal{B}_0 \cong \mathcal{B}_n \ll \mathcal{B}_{n-1} \ll \ldots \ll \mathcal{B}_1 \ll \mathcal{B}_0$. ■

Remark. The Boolean algebras $\mathcal{B}_0, \ldots, \mathcal{B}_{n-1}$ constructed in the proof of theorem **14.4** are not only mutually non-embeddable into each other, but also have a stronger property, i.e., for any $i \in \{0, \ldots, n-1\}$ and $A \subseteq \{0, \ldots, n-1\}$, we get $\mathcal{B}_i \not\le \prod_{j \in A} \mathcal{B}_j$ provided that $i \notin A$.

Indeed, let us assume to the contrary that $A = \{j_1, \ldots, j_m\}$. The embedding of $\mathcal{B}_i$ in $\prod_{j \in A} \mathcal{B}_j$ implies that there is a partition $1_{\mathcal{B}_i}$ of the elements $C_1, \ldots, C_m$ such that $\mathcal{B}_i | C_l \le \mathcal{B}_{j_l}$ for any $l \le m$. Preserving the notations of the proof of theorem **14.5**, we get $C_l \in F(X'_{p_i})$ for one of $C_1, \ldots, C_m$. This implies the inequalities $q \le p_i$ and $p_i \le q$ for the ultrafilter $q = \{B \cap C_l | B \in F(X'_{p_i})\}$ on the set C_l. Therefore, q is incomparable with the ultrafilter $F(X'_{p_{j_l}})$ relative to the Rudin-Keisler order. On the other hand, $\mathcal{B}_i' \restriction C_l = B(\varphi^i(\omega) \cap C_l; \{X \cap C_l | X \in X'_{p_i}\})$ and $F(\{X \cap C_l | X \in X'_{p_i}\}) = q$. By lemma **14.3**, $\mathcal{B}_i' \restriction C_l$ cannot then be embedded into the Boolean algebra $\mathcal{B}_{j_l}$. The contradiction obtained proves the statement.

Theorem 14.5. Under the assumption $P(2^\omega)$, for any finite set $B = \{a_0,\ldots,a_{s-1}\}$ with two quasi-orders, $\leq_1, \leq_2$, there exist pairwise non-isomorphic Boolean algebras $\mathcal{C}_0,\ldots,\mathcal{C}_{s-1}$ (of the power $2^{\aleph_0}$) such that for $i,j < s$, $\mathcal{C}_i \leq \mathcal{C}_j$ iff $a_1 \leq_1 a_j$ and $\mathcal{C}_i \ll \mathcal{C}_j$ iff $a_1 \leq_2 a_2$.

Proof. Let $A = \{a_0,\ldots,a_{n-1}\}$ be an arbitrary n-element set $(n \in \omega)$ with a partial order $\leq$. One can assume $< A; \leq >$ to be a subset of a finite Boolean algebra $\mathcal{B}$ with n atoms $d_0,\ldots,d_{n-1}$, and the order $\leq$ to coincide with the order on the elements of the Boolean algebra $\mathcal{B}$. Let $A_0,\ldots,A_{n-1}$ be a subdivision of ω into n infinite subsets. For $a \in A$, if $a = d_{i_1} \cup \ldots \cup d_{i_l}$, let us set $\mathcal{B}_a = \prod^*_{i \in A_{i_1} \cup \ldots \cup A_{i_l}} \mathcal{B}_i'$, where $\mathcal{B}_i' = \mathcal{B}_j$ at $i \in A_j$, while $\mathcal{B}_j (j < n)$ are Boolean algebras obeying the remark following theorem **14.4**. Here $\prod^*_{i \in A_{i_1} \cup \ldots \cup A_{i_l}} \mathcal{B}_i'$ is a subalgebra of a direct product $\prod_{i \in A_{i_1} \cup \ldots \cup A_{i_l}} \mathcal{B}_i'$ consisting of those of its elements the Cartesian projections of which all except a finite number of them either equal 0 or equal 1 in corresponding Boolean algebras $\mathcal{B}_i'$. Since $\mathcal{B}_i \equiv_\ll \mathcal{B}_j$ at $i,j < n$ and the sets A_i are infinite then, obviously, $\mathcal{B}_a \equiv_\ll \mathcal{B}_b$, for any $a,b \in A$.

Let us now show that $\mathcal{B}_a \leq \mathcal{B}_b$ iff $a \leq b$. It suffices to notice that $\mathcal{B}_a \leq \mathcal{B}_b$ entailss $a \leq b$. Assuming that the opposite is the case, for some $a,b \in A$, $d_i \in \mathcal{B}$ we get $a \supseteq d_i, b \not\supseteq d_i$ and $\mathcal{B}_a \leq \mathcal{B}_b$. The considerations of the construction of the Boolean algebras $\mathcal{B}_a, \mathcal{B}_b$ show that the Boolean algebra $\mathcal{B}_i$ is embeddable into a certain Cartesian product of a finite number of Boolean algebras $\mathcal{B}_{j_1},\ldots,\mathcal{B}_{j_m}$ from the family $\{\mathcal{B}_0,\ldots,\mathcal{B}_{n-1}\} \setminus \{\mathcal{B}_i\}$, which contradicts the choice of $\mathcal{B}_0,\ldots,\mathcal{B}_{n-1}$. The contradiction obtained proves that, indeed, $\mathcal{B}_a \leq \mathcal{B}_b$ iff $a \leq b$.

Let $\mathfrak{F}_a$ be an ultrafilter of the Boolean algebra $\mathcal{B}_a$ consisting of those its elements the Cartesian projections of which, except for, possibly, a finite number, are equal to 1 of the corresponding Boolean algebras $\mathcal{B}_j'$. Let $\mathcal{D}_k$ be a Boolean algebra $B(\omega^k \cdot \eta)$ for $k < \omega$, and let Φ_k be an arbitrary non-principal ultrafilter of the Boolean algebra $\mathcal{D}_k$, consisting of non-superatomic elements. Let $\mathcal{D}$ be a Boolean algebra of finite and co-finite subsets ω_2, and let ψ be an ultrafilter of the algebra

$\mathfrak{D}$ consisting of co-finite subsets ω_2. $\mathfrak{B}_{a,k}$ will denote the subalgebra of a Cartesian product $\mathfrak{B}_a \times \mathfrak{D}_k \times \mathfrak{D}$, consisting of elements $< c,d_1,d_2 >$ which obey the following condition: $c \in \mathfrak{F}_a$ iff $d_1 \in \Phi_1$ and iff $d_2 \in \psi$.

Let $R_{a,k}$ be an ultrafilter of the algebra $\mathfrak{B}_{a,k}$ consisting of elements $< c,d_1,d_2 >$ such that $c \in \mathfrak{F}_a$, $d_1 \in \Phi_k$ and $d_2 \in \psi$. Since the algebras $\mathfrak{B}_a$ are superatomic, any element d_1 of the ultrafilter $\mathfrak{F}_k$ is non-superatomic, and $\{c \in \mathfrak{D}_k | c \le d_1\}$ is countable, for any $a,b \in A, k,r \in \omega$ and any elements $c_1 \in R_{a,k}$, $c_2 \in R_{b,r}$ we get $\mathfrak{B}_{a,k} | c_1 \le \mathfrak{B}_{b,r} | c_2$ iff $a \le b$, while $\mathfrak{B}_{a,b} | c_1 \ll \mathfrak{B}_{b,r} | c_2$; $\mathfrak{B}_{a,k} \cong \mathfrak{B}_{b,r}$ iff $a = b$ and $k = r$.

Let now $B = \{b_0, \dots, b_{s-1}\}$ be an arbitrary finite set, and $\le_1, \le_2$ be two arbitrary quasi-orders on B. For any $b \in B$, $[b]_1, [b_2]$ will denote equivalence classes in terms of the quasi-orders $\le_1, \le_2$, respectively, on the set B containing the element b. Let φ be an arbitrary embedding of the equivalence class $[b]_1$ in ω. In this case, according to a remark made earlier, we see, choosing a set $< B/\equiv_{\le_1}; \le_1 >$ as the partially ordered set $< A; \le >$, that the correlation $f: b \to \mathfrak{B}_{[b]_1, \psi_{[b]_1}(b)}$ obeys the condition $f(a) \le f(b)$ iff $a_1 \le_1 b$; $f(a) \cong f(b)$ iff $a = b$; and $f(a) \ll f(b)$ for any $a,b \in B$. The first and the latter conditions are also valid for any algebras $f(a)|d_1$, $f(b)|d_2$, where $d_1 \in R_{[a]_1, \varphi_{[a]_1}(a)}$, $d_2 \in R_{[b]_1, \varphi_{[b]_1}(b)}$. It should be also noticed that for $d \in f(b)$, the inequality $|f(b)|d \ge \aleph_2$ is equivalent to the inclusion $d_2 \in R_{[b]_1, \varphi_{[b]_1}(b)}$.

For $[b]_2 \in B/\equiv_{\le_2}$, let $h([b]_2)$ denote Boolean algebras in corollary **14.2**, such that for $b_1, b_2 \in B$ $h([b_1]_2) \ll h([b_2]_2)$ iff $b_1 \le_2 b_2$, and $h([b_1]_2) \le h([b_2]_2)$ for any $b_1, b_2 \in B$. Any Boolean algebra of the type $h([b]_2)$ has the power $\aleph_1$ and, moreover, $h([b]_2)$ contains an ultrafilter $G_{[b]_2}$ such that, for $d \in h([b]_2)$, the algebra $h([b]_2)|d$ contains a chain of elements of an ordered type $\eta \cdot \omega_1$ iff $d \in G_{[b]_2}$, in which case for any $b_1, b_2 \in B$, $d_1 \in G_{[b_1]_2}$ and $d_2 \in G_{[b_2]_2}$, $h([b_1]_2| d_1 \equiv_{\le} h([b_2]_2| d_2$, while the inequality $h([b_1]_2| d_1 \ll h([b_2]_2| d_2$ is valid iff $b_1 \le_2 b_2$. It should be also remarked that none of the algebras of $f(b) (b \in B)$ contains chains of the ordered type $\eta \cdot \omega_1$.

For $b \in B$, $\varphi(b)$ will denote the Cartesian product of Boolean algebras $h([b]_2) \times f(b)$. Bearing in mind all the facts discussed above, one can note that $\varphi(b_1) \cong \varphi(b_2)$ iff $b_1 = b_2$, $\varphi(b_1) \le \varphi(b_2)$ iff $b_1 \le_1 b_2$, and $\varphi(b_1) \ll \varphi(b_2)$ iff $b_1 \le_2 b_2$. Indeed, if the inequality $b_1 \le_2 (\le_1) b_2$ holds, the inequality $\varphi(b_1) \ll (\le) \varphi(b_2)$ follows directly from the inequalities $h(b_1) \ll (\le) h(b_2)$ and $f(b_1) \ll (\le) f(b_2)$ discussed earlier.

Let now $\varphi(b_1) \ll \varphi(b_2)$, i.e., $h([b_1]_2 \times f(b_1) \ll h([b_2]_2 \times f(b_2)$. In this case, $h([b_1]_2) \ll h([b_2]_2 \times f(b_2)$, and there is an element $d \in h([b_1]_2)$ such that $h([b_1]_2)| d \ll h([b_2]_2)$, $h([b_1]_2)| \neg d \ll f(b_2)$. Either $h([b_1]_2)| d$ or $h([b_1]_2)| \neg d$ contains a chain of elements of the ordered type $\eta \cdot \omega_1$, however, since $f(b_2)$ contains no such chains, an algebra of the type $h([b_1]_2)| d$ must contain such a chain. Therefore,

$d \in G_{[b_1]_2}$, which, as has been noted earlier, implies $b_1 \leq_2 b_2$.

Let now $\varphi(b_1) \leq \varphi(b_2)$, i.e., $h([b_1]_2) \times f(b_1) \leq h([b_2]_2) \times f(b_2)$. Then $f(b_1) \leq h([b_2]_2) \times f(b_2)$, and there is an element $d \in f(b_1)$ such that $f(b_1)|d \leq h([b_2]_2)$ and $f(b_1)|\neg d \leq f(b_2)$. Either $f(b_1)|d$ or $f(b_1)|\neg d$ have the power $\aleph_2$ and, since $|h([b_2]_2)| \leq \aleph_1$, $|f(b_1)|\neg d| = \aleph_2$. As has been established earlier, $\neg d \in R_{[b_1]_1, \varphi[b_1](b_1)}$, and in this case the embedding $f(b_1)|\neg d$ in $f(b_2)$ implies the inequality $b_1 \leq_1 b_2$. Therefore, indeed, the mapping φ is an isomorphism from $< B; \leq_1, \leq_2 >$ to $< \mathfrak{J}BA; \leq, \ll >$. ■

Priorities. The statement of theorem **14.1** for the case $\aleph_i = \aleph_1$ is a variation of lemma **1** from a paper by A.G.Pinus [178], while in a general form it can be found in a work by Bonnet and Si-Kaddour [19]. Lemma **14.3** and theorem **14.2** belong to A.G.Pinus [169]. Lemma **14.4** was proved by Weese, theorem **14.3** is by Kunen [116], while its proof, as well as more detailed information on the Rudin-Keisler order on ultrafilters can be found in a monograph by Comfort and Negrepontis [41]. Lemma **14.5** and theorems **14.4** and **14.5** are from a paper by A.G.Pinus [178].

15. On Better Quasi-Orders

In the present section the basic notions of the theory of better quasi-orders are presented, the proof of the Laver theorem on the quasi-order on trees is given, followed by the Van Engelen, Miller and Steel result deduced from it, and used in sections **10** and **11** to obtain statements on countable skeletons of finitely generated discriminator varieties.

Definition 15.1. A quasi-ordered set $< A; \leq >$ is said well quasi-ordered if it contains no infinite strictly descending chains, and any family of its pairwise incomparable elements is finite.

It obviously follows from the Ramsey theorem that the requirement on the set $< A; \leq >$ to be well-ordered is equivalent to the following statement: for any infinite subset $X \subseteq A$, there is a sequence $a_1, \ldots, a_n, \ldots$ of elements of X such that $a_i \leq a_j$ for any $i < j \in \omega$.

The validity of the following statements can be noted directly:

(1) any well-ordered set is well quasi-ordered;

(2) the union of two well quasi-ordered sets is also well quasi-ordered;

(e) the Cartesian product of a pair of well quasi-ordered sets is well quasi-ordered;

(f) if $<I;\leq>$ is well quasi-ordered, and $<A_i;\leq>$ are well quasi-ordered for any $i \in I$, $\sum_{i \in <I;\leq>} <A_i;\leq>$ is well quasi-ordered.

For any quasi-ordered set $<A;\leq>$, $A^{<\omega}$ will denote a family of all finite sequences of A elements. We will introduce a quasi-order relation on $A^{<\omega}$: $<a_1,\ldots,a_m> \leq <b_1,\ldots,b_n>$ iff there are $k_1 <\ldots< k_m \leq n$ such that $a_i \leq b_{k_i}$. The following result is one of the principal ones in the theory of well quasi-ordered sets.

Theorem 15.1. If $<A;\leq>$ is well quasi-ordered, $<A^{<\omega};\leq>$ is also well quasi-ordered.

Proof. Let us assume to the contrary that a sequence $<u_i | i \in \omega>$ of $A^{<\omega}$ elements is such that for any $i < j \in \omega$ we get $u_i \not\leq u_j$. Such sequences of elements of a quasi-ordered set will be said poor. The poor sequence $<u_i | i \in \omega>$ is said strictly minimal poor if for any $i \in \omega$ and any $a \in A^{<\omega}$ such that $a < u_i$, there is no poor sequence starting with $u_0, u_1, \ldots, u_{i-1}, a$.

Any well-founded set with a poor sequence has a strictly minimal poor sequence: it suffices to choose the least among the first elements of poor sequences as u_0, the least among the second elements of poor sequences starting with u_0 as u_1, etc.

Therefore, $<u_i | i \in \omega>$ can be assumed to be a strictly minimal poor sequence of elements $<A^{<\omega};\leq>$ ($<A^{<\omega};\leq>$ obviously being well-founded). Let a_i be the first element of A in the finite sequence u_i. Since $<A;\leq>$ is well quasi-ordered, there is a sequence $<a_{h(i)} | i \in \omega>$ where $h(o) < h(1) <\ldots< h(i) <\ldots$ such that $<a_{h(i)} | i \in \omega>$ is either constant or strictly increases. Let $v_{h(i)}$ be obtained from $u_{h(i)}$ by crossing out the first element $a_{h(i)}$ and, hence, $v_{h(i)}$ is strictly less and $u_{h(i)}$ in $<A^{<\omega};\leq>$. Since $<u_i | i \in \omega>$ is strictly minimal poor, the sequence $u_0, u_1, \ldots, u_{h(0)-1}, v_{h(0)}, v_{h(1)}, \ldots$ is not poor and, hence, it contains two elements b_1, b_2 such that $b_1 \leq b_2$ in $<A^{<\omega};\leq>$ and b_1 is encountered in this sequence earlier than b_2. As $<u_i | i \in \omega>$ is a poor sequence, b_1, b_2 cannot be both encountered among $<u_0, \ldots, u_{h(0)-1}>$. They cannot be among $v_{h(0)}, v_{h(1)}, \ldots$ either, since in this case, if $b_1 = v_{h(n)}$, $b_2 = v_{h(m)}$ and $n < m$, $u_{h(n)} = < a_{h(n)}, v_{h(n)} > \leq u_{h(m)} = < a_{h(m)}, v_{h(m)} >$, which contradicts the fact that $< u_i | i \in \omega >$ is poor. Therefore, $b_1 = u_j$ for some $j < h(0)$, $b_2 = v_{h(k)} (k \in \omega)$ and $u_j \leq v_{h(k)} \leq u_{h(k)}$, where $j \leq h(k)$, which again contradicts the choice of $< u_i | i \in \omega >$. Thus, there are no poor sequences in $< A^{<\omega}; \leq >$. ■

The quasi-order relation on the set $A^{<\omega}$ of finite sequences of elements from A can be extended to a set of infinite sequences. Let A^{ω} be the family of all ω-sequences of the elements from A, and let in this case $< a_n | n \in \omega > \leq < b_n | n \in \omega >$ iff there is a monotonous embedding f from the set ω to itself such that for any $n \in \omega$, $a_n \leq b_{f(n)}$.

It seems to be natural to try to transfer the statement of theorem **15.1** from $< A^{<\omega}; \leq >$ to $< A^{\omega}; \leq >$. However, as is shown by an example belonging to Rado [198], this is impossible.

Let $A = \omega \times \omega$, and let the partial order $\leq_0$ be defined on A in the following way: $< a,b > \leq_0 < a',b' >$ iff either $a = a'$ and $b \leq b'$, or $a' \geq a + b$ and b' is arbitrary. It is obvious that $< A; \leq_0 >$ is well-founded, i.e., $< A; \leq_0 >$ contains no infinite strictly descending chains. If $< a,b > \in A$, any element $< c,d > \in A$ incomparable with $< a,b >$ must be such that $c < a + b$ and, on the other hand, any pair of incomparable elements from A must have different first coordinates. Therefore, any family of pairwise incomparable elements of A is finite, i.e., $< A; \leq_0 >$ is well quasi-ordered. On the other hand, a sequence $< u_i | i \in \omega >$ of elements of A^{ω} in the form $u_i = << i,0 >, < i,1 >, \ldots, < i,k >, \ldots | k \in \omega >$ is, obviously, by virtue of the definition of the order on A such that for any $i < j \in \omega$ we have $u_i \not\leq u_j$. Therefore, $< A^{\omega}; \leq >$ is not well quasi-ordered.

Analysis of the Rado example and an attempt to find a sufficiently wide class of quasi-orders $< A; \leq >$ such that $< A^{\omega}; \leq >$ could remain a well quasi-ordered set, resulted in the following definitions by Nash-Williams.

Definition 15.2.

(1) A family B of strictly increasing finite sequences of elements of a certain infinite set $S \subseteq \omega$ is called a barrier provided that no sequence from B is a subsequence of any other sequence from B, and if for any strictly increasing infinite sequence $< s_i | i \in \omega >$ of elements of S, there is an $n \in \omega$ such that $< s_i | i \leq n > \in B$. $0(B)$ will denote the family of all numbers incorporated in sequences from B.

(2)Let us introduce the relation $\lhd$ on B in the following way: $t \lhd u$ implies that for some natural numbers $s_0 < s_1 < \ldots < s_n$ and an r such that $0 \le r < n$, we get $t = \langle s_i | i \le r \rangle$ and $u = \langle s_j | 1 \le j \le n \rangle$.

(3) A quasi-ordered set $\langle Q; \le \rangle$ is called a better quasi-order if for any barrier B and any mapping f from the barrier B to Q there are $t,u \in B$ such that $t \lhd u$ and $f(t) \le f(u)$.

It is obvious that the Rado example discussed above, $\langle A; \le_0 \rangle$, is an example of a well-quasi-ordered set which is not a better quasi-order. Indeed, choosing as a barrier B a subset of the set A consisting of those pairs $\langle a,b \rangle$ in which $a < b$, and identically mapping B to A, we immediately get from the definition of the quasi-order $\le_0$ that for any $t,u \in B$ such that $t \lhd u$, we have $t \not\le_0 u$.

On the other hand, any better quasi-order is a well-quasi-order. Indeed, if $\langle Q; \le \rangle$ is a better quasi-order and $\langle a_i | i \in \omega \rangle$ is an arbitrary sequence of elements of Q then, choosing as a barrier B the set of all natural numbers, and as f a mapping from ω to B such that $f(i) = a_i$, we find, since $\langle Q; \le \rangle$ is a better quasi-order, $i \lhd j$ such that $a_i \le a_j$. The relation $i \lhd j$ on ω, however, is equivalent to the relation $i < j$. Therefore, indeed, any better quasi-order is a well-quasi-order.

The following combinatory statement on barriers plays an essential role in proving various properties of better quasi-orders.

Theorem 15.2. If B is a barrier, for any division B_1, B_2 of the set B, there is an infinite subset $H \subseteq O(B)$ such that if $B(H) = \{b \in B | b$ consists of the elements of $H\}$, we get either $B(H) \subseteq B_1$ or $B(H) \subseteq B_2$, in which case $B(H)$ is a barrier and $O(B(H)) = H$.

The reader interested in the details of the proof can find it in either an original work by Nash-Williams [146] or a monograph by Fräisse [69]. Let us use this theorem to prove a number of the simplest properties of better quasi-orders, having introduced some additional notation and definitions.

Definition 15.3.

(a) For any barrier B, B^2 will denote the family of all sequences of the type $\langle s_i | i \le n \rangle$, where for some $t,u \in B$, $r < n$ such that $t \lhd u$, $t = \langle s_i | i \le r \rangle$, $u = \langle s_j | 1 \le j \le n \}$. Such a sequence $\langle s_i | i \le n \rangle$ will be denoted by $t \cup u$. It is evident that B^2 is a barrier.

(b) A barrier V is called a barrier following the barrier U if $O(V) \subseteq O(U)$, and any element of V contains an initial interval belonging to U.

(c) Let $<A;\leq>$ be an arbitrary partial order and δ a mapping from A to the ordinals. Let U and V be barriers, and f and g be the mappings from U and V, respectively, to the set A. The function g is called a function following f, if V is the barrier following U, and if for any $t \in V$ and any s, which is the initial interval of the sequence t belonging to U, we get either $t = s$, in which case $g(t) = f(s)$, or $s \neq t$, in which case $g(t) < f(s)$ and $\delta(g(t)) < \delta(f(s))$.

(d) Let $<A;\leq>$, δ, U and f be the same as in (a). The mapping f is called poor if for any $t,u \in U$ such that $t \lhd u$, we have $f(t) \not\leq f(u)$. The mapping f is called minimal poor if f is poor and for any barrier V and any mapping g from the barrier V to A such that g is poor and g is the function following f, the inclusion $V \subseteq U$ holds, and g is an restriction of f to V.

Theorem 15.3.

(a) If $<Q;\leq>$ is a well-ordered set, $<Q;\leq>$ is a better quasi-order.

(b) If $<Q_1;\leq_1>$, $<Q_2;\leq_2>$ are better quasi-orders, and $Q_1 \cap Q_2 = \varnothing$, $<Q_1 \cup Q_2;\leq_1 \cup \leq_2>$ is also a better quasi-order.

(c) A Cartesian product of better quasi-orders is a better quasi-order itself.

Proof.

(a) Let $<Q;\leq>$ be well-ordered, B an arbitrary barrier, and f a mapping from B to Q. Let $<s_i | i \in \omega>$ be a sequence of B elements such that for any $i \in \omega$ we have $s_i \lhd s_{i+1}$. Since $<Q;\leq>$ is well-ordered, there is an $i \in \omega$ with the property $f(s_i) \leq f(s_{i+1})$, which fact, however, implies that $<Q;\leq>$ is a better quasi-order.

(b) Let $<Q_1;\leq_1>$, $<Q_2;\leq_2>$ obey the requirements of the statement (b), and let $\leq = (\leq_1 \cup \leq_2)$. Let B be a barrier, f be a mapping from B to $Q_1 \cup Q_2$, and $B_1 = f^{-1}(Q_1)$; $B_2 = f^{-1}(Q_2)$. By theorem **15.2**, there is a barrier B_3 such that either $B_3 \subseteq B_1$ or $B_3 \subseteq B_2$. Let us assume $B_3 \subseteq B_1$. In this case, since $<Q_1;\leq_1>$ is a better quasi-order, there are $t,u \in B_3$ such that $t \lhd u$ and $f(t) \leq_1 f(u)$, i.e., there are $t,u \in B$ such that $t \lhd u$ and $f(t) \leq f(u)$, which completes the proof of the statement (b).

(c) Let $<Q_1;\leq_1>$, $<Q_2;\leq_2>$ be better quasi-orders, and let the relation $\leq$ be defined on $Q_1 \times Q_2$ as the Cartesian product of the relations $\leq_1$ and $\leq_2$. Let B be a barrier, f be a mapping from B to $Q_1 \times Q_2$, and π_1,π_2 be projections of $Q_1 \times Q_2$ along the first and second co-ordinates, respectively. Let us define $B_1 \subseteq B^2$ as $B_1 = \{t \cup u | \pi_1(f(t)) \leq \pi_1(f(u))\}$, and $B_2 = B^2 \setminus B_1$. According to theorem **15.2**, there is a barrier C such that we get either $C \subseteq B_1$ or $C \subseteq B_2$. We can easily notice that $B' = \{t \in B |$ for some $u \in B$ such that $t \lhd u$, $t \cup u \in C\}$ is also a barrier, in which case $(B')^2 \subseteq C$. If we had $C \subseteq B_2$, for any $t,u \in B'$ such that $t \lhd u$, we would get $\pi_1(f(t)) \not\leq_1 \pi_1(f(u))$, which would contradict the assumption that $< Q_1;\leq_1>$ is a better quasi-order. Therefore, $C \subseteq B_1$ and, hence, for the barrier B', for any $t,u \in B'$ such that $t \lhd u$, the inequality $\pi_1(f(t)) \leq \pi_1 f(u))$ would hold. As $< Q_2;\leq_2>$ is a better quasi-order, there are $t_1,u_1 \in B'$ such that $t_1 \lhd u_1$ and $\pi_2(f(t_1)) \leq_2 \pi_2 f(u_1))$. Therefore, we found $t_1,u_1 \in B$ such that $t_1 \lhd u_1$ and $f(t_1) \leq f(u_1)$, which completes the proof of the statement (c). ■

It is also obvious that any extensions of better quasi-orders will be better quasi-orders themselves.

Theorem 15.4. Let $< A;\leq>$ be a certain partial order, δ a mapping from A to the ordinals and f a poor mapping from a barrier U to A. Then there is a minimal poor mapping g from a barrier V to A such that g follows f.

Proof. For any pair of barriers U,V such that V follows U and $V \not\subseteq U$, $p(U,V)$ will denote the least of the last elements of the sequences which belong to U, consist of elements $O(V)$ and are not elements of V. Let $W_{p(U,V)} = \{t \in U |$ a set of elements t is not a subset of $O(V)$, and those elements t which do not belong to $O(V)$ are not greater than $p(U,V)\}$.

We can directly check that $V' = V \cup W_{p(U,V)}$ is a barrier following U. If, moreover, $< A;\leq>$ is a partial order, δ is a mapping from A to the ordinals, f,g are poor mappings from the barriers U and V, respectively, to A, and g follows f then, by defining g' as a mapping from V' to A that coincides with f on $W_{p(U,V)}$ and with g on V, we get a poor mapping from V' to A following f. Such g' will be called a supplement of g in f, and the g following f will be called complete provided that $g' = g$.

Let now U and f meet the conditions of the theorem. Let us set $V_0 = U$, $f_0 = f$. Let us also assume that f is not minimal poor, in which case there is a barrier V_1 following U, and a poor mapping f_1, which is not a restriction of f to V_1, following f. One can assume that $p_0 = p(U,V_1)$ is the least of all possible numbers $p(U,V)$ for such V, and that f_1 is a complete mapping.

Iterating this process, for any natural i, we find a barrier V_{i+1} and a poor complete mapping f_{i+1} from the barrier V_{i+1} to A following f_i, assuming

$p_i = p(V_i, V_{i+1})$ to be the least of all possible numbers $p(V_i, V)$ for such V. The iteration process can be interrupted only when some V_i, f_i are obtained so that f_i is a minimal poor mapping, in which case the theorem is proved. Let us, therefore, assume that none of f_{i+1} is a minimal poor mapping and, in particular, none of f_{i+1} is a restriction of the mapping f_i.

It should be noticed that, for any $i \in \omega$, $p_{i+1} \geq p_i$. Indeed, in the opposite case, if $p_{i+1} < p_i$, for a barrier V_{i+2} and its mapping f_{i+2} we get $p(V_i, V_{i+2}) < p_i$, which contradicts the choice of p_i as minimal. Moreover, $\lim_{i \in \omega} p_i = \infty$, as there is only a finite number of increasing sequences with their maximum equal to p_i, and every element of this set can play its role for the equality $p_i(V_i, V_{i+1}) = p_i$ but once, when going over from V_i to V_{i+1}. Since f_{i+1} is complete, for every $i \in \omega$, $p_i \in 0(V_j)$ and $\{m \in 0(V_i) | m \leq p_i\}$ at $j \geq i$.

A sequence of sets $< 0(V_i) | i \in \omega >$ forms a chain decreasing by inclusion. Let $H = \bigcap_{i \in \omega} 0(V_i)$, in which case H is infinite, since for any $i \in \omega$, $p_i \in H$. For any infinite $X \subseteq H$ and a natural number i, there is a unique number s_i such that the initial interval X_{s_i} of the set X, consisting of numbers not greater than s_i, is a sequence belonging to V_i. As the barrier elements are pairwise incomparable, we get the inequality $s_{i+1} \geq s_i$ for any $i \in \omega$. Since f_{i+1} is a function following f_i, the strict inequality $s_{i+1} > s_i$ results in a strict equality for ordinals $\delta(f_{i+1}(X_{s_{i+1}})) < \delta(f_i(X_{s_i}))$, where X_r is a sequence of elements of the set X not greater than r. Therefore, there is an $i_X \in \omega$ such that for any $j \geq i_X$ we get $s_j = s_{i_X}$. Let s_X denote this S_{i_X} and remark that $V = \{X_{s_X} | X \subseteq N\}$ is a barrier, and that $0(V) = H$. Indeed, for any $X \subseteq H$, a certain initial interval X_{s_X} of this X belongs to V. If $u,t \in V$, by the definition of V and the numbers s_X, there is an $i \in \omega$ such that $u,t \in V_i$ as well and, since V_i is a barrier, u and t cannot be the initial intervals of each other. Hence, V is a barrier and, obviously, the one following any of the barriers V_i.

The mapping g from the barrier V to A will be defined in the following way: for $t \in V$ we choose an $i \in \omega$ such that $t \in V_i$, and we set $g(t) = f_i(t)$. Since at $j \geq i$ every f_j follows f_i, the definition is independent of the choice of i. If $u,t \in V$ then, for a certain $i \in \omega$, $u,t \in V_i$ and $g(u) = f_i(u)$, $g(t) = f_i(t)$ and, hence, when $u \lhd t$ and f_i is a poor mapping from V_i to A, $g(u) \not\leq g(t)$, i.e., g is also a poor mapping from V to A.

It is evident that g follows any of the mappings f_i and, in particular, $f_0 = f$.

Let us show that g is a minimal poor mapping following f. Let h be a poor mapping from the barrier W to A following g and such that h is not a restriction of g to W. If $W \not\subseteq V$, there are $t \in W$, and its initial interval $u \in V$ other than t. Let us choose the least $i \in \omega$ such that $p_i > \max u$ and $u \in V_i$. Since g is a poor mapping following f_i, and h is the one following g, h is a poor mapping following f_i. Since $u \in V_i \setminus W$, by choosing at the i-th step of the construction of the sequence $\ll f_{i+1}, V_{i+1} > | i \in \omega >$, the barrier W and the mapping h instead of V_{i+1}, f_{i+1}, we would get $p(V_i, W) \leq \max u < p_i$, which contradicts the choice of minimal p_i. Therefore, $W \subseteq V$ and, by the definition of the mapping following another mapping, h is a restriction of g to W. The contradiction obtained proves that g is minimal. ■

Using the statement of this theorem makes it already possible to show that for better quasi-orders $< A; \leq >$, the set $< A^{\omega}; \leq >$ is well-quasi-ordered. Let A^{Ord} be the family of all ordinal sequences the elements of A, and for $< a_\alpha | \alpha < \gamma >, < b_\beta | \beta < \delta > \in A^{Ord}$ let the inequality $< a_\alpha | \alpha < \gamma > \leq < b_\beta | \beta < \delta >$ hold iff there is a strictly increasing embedding f of the ordinal γ in the ordinal δ such that for any $\alpha < \gamma$, $a_\alpha \leq b_{f(\alpha)}$.

Theorem 15.5. If h is a poor mapping from the barrier U to $< A^{Ord}; \leq >$, there is a subbarrier $U' \subseteq U$ such that an h bounded on U' is (when one-element sequences are identified with the elements themselves) a poor mapping from U' to $< A; \leq >$. Therefore, if $< A; \leq >$ is a better quasi-order, $< A^{Ord}; \leq >$ will be a better quasi-order as well.

Proof. Let us define a function δ mapping A^{Ord} to the ordinals in the following way: $\delta(< a_\alpha | \alpha < \gamma >) = \gamma$. Let us assume that for a barrier U there is a poor mapping h from U to A^{Ord}. By theorem **15.4**, there is a minimal poor mapping following h. Let us assume h to be a minimal poor mapping from the barrier U to A^{Ord}. Let us divide the elements of U into three classes: for $t \in U$ we get $t \in U_1 (U_2, U_3)$ if $\delta(h(t)) = 1$, ($\delta(h(t))$ is limiting, $\delta(h(t)) > 1$ and not limiting). By theorem **15.2**, there is a barrier $U' \subseteq U$ such that $U' \subseteq U_1$, $U' \subseteq U_2$, or $U' \subseteq U_3$.

Let us assume that $U' \subseteq U_2$, and let $s, t \in U'$ be such that $s \lhd t$, in which case $h(s) \not\leq h(t)$. Since $\delta(h(s))$ and $\delta(h(t))$ are limiting, standard considerations show that there is a proper initial interval of the sequence $h(s)$ which is not less than $h(t)$ in $< A^{Ord}; \leq >$. Let $V = (U')^2$ and for $v \in V$ let v_1 be the initial interval of v belonging to U', and v_2 be v without the first element. Therefore, for $v \in V$ we get $v_1 \lhd v_2$. Let us define $g(v)$ as a minimal proper initial interval of the sequence $h(v_1)$ not embeddable into $h(v_2)$. By the definition of g, $\delta(g(v)) < \delta(h(v_1))$. Hence, g is a mapping following h, in which case $V \subseteq U$. It should be remarked that g is poor. Indeed, let $u, w \in V$ such that $u \lhd w$. Then $u_1 \lhd u_2 = w_1$ and, hence, $g(u) \not\leq h(w_1)$. However, $g(w)$ is the initial interval of the sequence $h(w)$ and, hence, $g(u) \not\leq g(w)$.

The existence of such a poor g following h contradicts the assumption that h is minimal and, hence, the case $U' \subseteq U_2$ is impossible.

Let us consider the case when $U' \subseteq U_3$, i.e., the lengths of all h-images of sequences belonging to the barrier U' are not limiting and greater than unity. For $t \in U'$, let $l(t)$ be the last element of the sequence $h(t)$, and $g(t)$ the sequence $h(t)$ without the last element, $l(t)$. There is a barrier $U'' \subseteq U'$ such that for any $t,u \in U''$, $t \lhd u$ entails $l(t) \le l(u)$. Indeed, if we subdivide $(U')^2$ into two subsets, C_1 and C_2, then $w \in C_1$ iff $l(w_1) \le l(w_2)$ and $C_2 = (U')^2 \setminus C_1$. By theorem **14.2**, there is an infinite $E \subseteq 0(U')^2$ such that $(U')^2(E) \subseteq (U')^2$, and we get either $(U')^2(E) \subseteq C_1$ or $(U')^2(E) \subseteq C_2$. The latter is impossible since in this case l will be a poor mapping of the barrier $(U')^2(E)$ in the better quasi-order $< A;\le >$. Therefore, indeed, we found a barrier $U'' = (U')^2(E) \subseteq (U')^2$ with the required properties. In this case, however, since h is a poor mapping on U', g must be a poor mapping on U''. Let us define a mapping $\varphi(v) = g(v_1)$ on $(U'')^2$. It should be noticed that $\delta(\varphi(v)) = \delta(g(v_1)) < \delta(h(v_1))$. Therefore, φ follows h, in which case $(U'')^2 \subseteq U'$. As g is a poor mapping, φ is a poor mapping as well.

Indeed, let $s',t' \in (U'')^2$ and $s' \lhd t'$, $(s')_1,(t')_1 \in U''$ and $(s')_1 \lhd (t')_1$. Hence, $\varphi(s') = g((s')_1) \not\le \varphi(t') = g((t')_1)$. The existence of such a φ contradicts the fact that h is minimal. Thus, the case when $U' \subseteq U_3$ is also impossible.

Hence, there is a barrier $U' \subseteq U_1$ such that the values of the restriction of h to U' are one-element sequences, i.e., the restriction of h to U' can be identified with a poor mapping from U' to $< A;\le >$.

In the case, when $< A;\le >$ is a better quasi-order, there is no poor mapping in $< A;\le >$ and, hence, in line with what has been proved above, there can be no poor mappings in $< A^{Ord};\le >$, i.e., in this case $< A^{Ord};\le >$ is a better quasi-order. ■

Let $P(A)$ be the set of all subsets of the set A, and if $\le$ is a certain quasi-order on A, the quasi-order $\le_1$ on $P(A)$ will be defined in the following way: for $B,C \in P(A)$, $B \le_1 C$ iff there is an embedding h from the set B to the set C such that for any $b \in B$ we get $b \le h(b)$. The following corollary naturally results from theorem **15.5**.

Corollary 15.1. If $< A;\le >$ is a better quasi-order, $< P(A);\le_1 >$ is also a better quasi-order.

In fact, a formulation similar to that in the former part of theorem **15.5** is also possible on the existence of a poor mapping in $< A;\le >$ corresponding to any poor mapping in $< P(A);\le_1 >$.

Let us now recall some basic notions pertaining to the theory of trees. A tree is a partially ordered set $< A;\leq>$ such that for any $a \in A$, a set $<\{b \in A | b < a\};\leq>$ is well-ordered. By A_α, where α is an ordinal, we will mean the family of $a \in A$ such that $<\{b \in A | b < a\};\leq>$ has the ordinal type of α. Therefore, $A = \bigcup_{\alpha \in Ord} A_\alpha$, where Ord is, as was the case earlier, the class of all ordinals. The height of a tree is the least α such that $A_\alpha = \varnothing$. $A_{\leq\alpha}, A_{<\alpha}$ will denote the sets $\bigcup_{\beta \leq \alpha} A_\beta$ and $\bigcup_{\beta < \alpha} A_\beta$, respectively. A subset $X \subseteq A$ is called a chain in the tree $< A;\leq>$ provided that $< X;\leq>$ is linearly ordered. If $x \in A$, $s(x)$ will denote the family of the covers of the element x in $< A;\leq>$, while if X is a chain in A, $S(X)$ is the family of minimal elements of the set $\{y \in A |$ for any $x \in X \quad x < y\}$.

$br_A(x)$ will denote a branch of the tree $< A;\leq>$ generated by the element x. Henceforth, we will consider only trees with a certain least element, which is the root of the tree such that for any chain X in a tree with no largest element we get $|S(X)| \leq 1$. A family of such trees will be denoted by $\mathfrak{T}$. If $< A;\leq> \in \mathfrak{T}$ and $a,b \in A$, the set $\{y \in A | y \leq a, y \leq b\}$ has a largest element, which is denoted by $a \wedge b$.

If Q is an arbitrary quasi-order, a Q-tree $< A;\leq,l_A >$ is a mapping l_A from the tree $< A;\leq>$ to Q. If $U \subseteq \mathfrak{T}$, then U_Q will denote those Q-trees $< A;\leq,l_A >$ for which $< A;\leq> \in U$. The quasi-order $\leq$ will be introduced on the family of Q-trees in the following way: $< A;\leq,l_A > \leq < B;\leq,l_B >$ iff there is an embedding h of the set A in the set B such that for any $a,b \in A$ we have $h(a \wedge b) = h(a) \wedge h(b)$ and $l_A(a) \leq l_B(h(a))$. The trees themselves will be identified with Q-trees, where Q is one-element, by determining the quasi-order $\leq$ on trees in a corresponding way. $2^{<\omega}$ will denote a standard dual tree of length ω, the basic set of which is a family of finite sequences of zeros and unities, ordered according to the principle "to be the initial interval".

Definition 15.4. A tree $< A;\leq>$ is called scattered if $2^{<\omega} \not\leq < A;\leq>$. Let $\mathfrak{S}$ be a class of all scattered trees. $\mathfrak{M}$ will denote a class of trees $< A;\leq>$ such that there is a sequence $< A^n | n \in \omega >$ of initial intervals of a partial order $< A;\leq>$ such that $A = \bigcup_{n \in \omega} A^n$ and $< A^n;\leq> \in \mathfrak{S}$.

The tree $< A;\leq>$ is obtained by extending the tree $< A';\leq>$ with the help of trees $< A_P;\leq>$, where P are chains in $< A;\leq>$ which either are maximal or have a maximal element, if A_P are sets which are pairwise non-intersecting and not intersecting with A', while A is obtained by adding to A' those branches of A_P whose roots are elements a_P which directly follow P in $< A;\leq>$ and do not belong to $S(P)$ in $< A';\leq>$. One can readily notice that if $< A';\leq> \in \mathfrak{S}$ and all $< A_P;\leq> \in \mathfrak{S}$, $< A;\leq> \in \mathfrak{S}$ as well.

Let $<A;\leq>\in\mathfrak{T}$ and P be a chain in $<A;\leq>, z\in P$, in which case $\overline{P}(z)=\{br(y) \mid y\in S(z), y\notin P\}$. If $<A;\leq,l_A>$ is a Q-tree,

$$\overline{P}^{l_A}(z)=\{<A';\leq,l_A> \mid A';\leq>\in\overline{P}(z)\}.$$

The following lemma describes the inductive method of constructing the class of all scattered trees.

Lemma 15.1. Let $\mathfrak{S}_0$ consist of an empty and a one-element trees. For any ordinal α, let $\mathfrak{S}_{\alpha+1}=<A;\leq>$ and there is a maximal chain P in $<A;\leq>$ such that for all $z\in P$, $\overline{P}(z)\in\mathfrak{S}_\alpha$. If α is a limiting ordinal, $\mathfrak{S}_\alpha=\bigcup_{\beta<\alpha}\mathfrak{S}_\beta$, in which case $\mathfrak{S}=\bigcup_{\alpha\in Ord}\mathfrak{S}_\alpha$.

Proof. Let $\overline{\mathfrak{S}}=\bigcup_{\alpha\in Ord}\mathfrak{S}_\alpha$. By induction on α, we can notice that, for any $\alpha\in Ord$, $\mathfrak{S}_\alpha\subseteq\mathfrak{S}$ and, hence, $\overline{\mathfrak{S}}\subseteq\mathfrak{S}$.

Let now $<A;\leq>\in\mathfrak{T}$, $x\in A$ and $br(x)\in\overline{\mathfrak{T}}$. In this case, there are incomparable y,z such that $x<y, x<z$ and $br(y),br(z)\notin\overline{\mathfrak{S}}$. Let us assume that the opposite is the case. Let us choose a chain P_0 in $br(x)$ such that $br(z)\notin\overline{\mathfrak{S}}$ for all $z\in P_0$. Let P_0 be a maximal chain with this property, i.e., in particular, $\{br(z) \mid z\in S(P_0)\}\subseteq\overline{\mathfrak{S}}$. By the assumption, for all $z\in P_0$, $\overline{P}_0(z)\subseteq\overline{\mathfrak{S}}$, and the equality $br(x)=P_0\cup\bigcup_{\dot{z}\in S(P_0)}(\cup P_0(z))$ holds. Let us choose a $z\in S(P_0)$ if $S(P_0)\neq\varnothing$, and let $P=P_0\cup P_1$, where P_1 is a maximal chain in $br(z)$, bearing in mind that, by the definition of P_0, $br(z)\in\overline{\mathfrak{S}}$. If for any $y\in P$, $\overline{P}(y)\subseteq\mathfrak{S}_\alpha$, $br(x)\in\mathfrak{S}_{\alpha+1}$, which contradicts the assumption $br(x)\notin\overline{\mathfrak{S}}$. Therefore, indeed, in the conditions under discussion there can be found incomparable y,z such that $x<y, x<z$ and $br(y),br(z)\notin\overline{\mathfrak{S}}$.

Let us now consider the inclusion $\mathfrak{S}\subseteq\overline{\mathfrak{S}}$. Let us assume to the contrary that $<A;\leq>\in\mathfrak{S}\setminus\overline{\mathfrak{S}}$. By induction on the length of the sequence, $j\in 2^{<\omega}$, let us define an embedding h of the tree $2^{<\omega}$ in $<A;\leq>$: $h(\varnothing)$ is the root of $<A;\leq>$; if $h(j)$ has been defined and is such that $br(h(j))\notin\overline{\mathfrak{S}}$ then, according to the facts just proved, there can be found incomparable y,z such that $y,z>h(j)$ and $br(y),br(z)\notin\overline{\mathfrak{S}}$. Let us set $h(j\wedge<0>)=y$ and $h(j\wedge<1>)=z$. Let us now define the function g on $2^{<\omega}$ in the following way: $g(j)=h(j\wedge<0>)\wedge h(j\wedge<1>)$. One can directly check that for any $j_1,j_2\in 2^{<\omega}$, we get $g(j_1)\wedge g(j_2)=g(j_1\wedge j_2)$ and, hence,

$2^{<\omega} \leq < A;\leq >$. The contradiction obtained proves that $\mathfrak{S} \setminus \overline{\mathfrak{S}} = \varnothing$. Alongside with the inclusion $\overline{\mathfrak{S}} \subseteq \mathfrak{S}$ mentioned in the beginning of the proof, we get the equality $\overline{\mathfrak{S}} = \mathfrak{S}$. ■

For any $< A;\leq > \in \mathfrak{S}$, the rank of $< A;\leq >$ is the least α such that $< A;\leq > \in \mathfrak{S}_\alpha$. The rank of the tree $< A;\leq >$ will be denoted by $rg < A;\leq >$. One can easily deduce, using induction on the rank, that if $< A;\leq > \leq < B;\leq >$, $rg < A;\leq > \leq rg < B;\leq >$.

__Theorem 15.6.__ If $< Q;\leq >$ is a better quasi-order, $< \mathfrak{S}_Q;\leq >$ is also a better quasi-order.

Proof. Assume to the contrary that f is a poor mapping from the barrier B to $\mathfrak{S}_Q$. By theorem **15.4**, there is a minimal poor mapping g from a certain barrier B' to $\mathfrak{S}_Q$, following f. Let for $b \in B'$, $g(b) = < A_b;\leq,l_b >$. B_1 will denote $\{b \in B' \mid |A_b| = 1\}$. By theorem **15.2**, there is a barrier C such that either $C \subseteq B_1$ or $C \subseteq B' \setminus B_1$. The former case is impossible, since in this case g, being bounded on C, would induce a poor mapping of C on Q, which would contradict the fact that $< Q;\leq >$ is a better quasi-order. Hence, we have found a barrier C and a poor mapping $g: C \to \mathfrak{S}_Q$ such that, for any $b \in C$, $|A_b| > 1$. P_b will denote a maximal chain in $< A_b;\leq >$ for $b \in C$ such that for any $x \in P_b$ and any $< D;\leq > \in \overline{P}_b(x)$, the rank of $< D;\leq >$ is strictly less then that of $< A_b;\leq >$. Such a P_b can be found by virtue of the definition of the class of trees $\overline{\mathfrak{S}}$ and the equality $\mathfrak{S} = \overline{\mathfrak{S}}$ proved in lemma **15.1**.

Let us define the mapping $J_b: P_b \to Q \times P(\mathfrak{S}_Q)$ in the following way: $J_b(x) = < l_b(x), \overline{P}_b^{l_b}(x) >$. On a set $Q \times P(\mathfrak{S}_Q)$ the quasi-order $\prec$ is a Cartesian product of the quasi-order on Q and the quasi-order $\leq_1$ defined on $P(\mathfrak{S}_Q)$ (which is, in its turn, induced on $P(\mathfrak{S}_Q)$ by the quasi-order defined on $\mathfrak{S}_Q$). Using the quasi-order $\prec$, we will define the relation $\leq$ on $\{J_b \mid b \in C\}$ in the following way: $J_c \leq J_b$ if there is an embedding h of the chain $< P_c;\leq >$ in the chain $< P_b;\leq >$ such that, for $x < y \in P_c$, $h(x) < h(y)$ and $J_c(x) \prec J_b(h(x))$. It should be remarked that if $J_c \leq J_d$, $< A_c;\leq,l_c > \leq < A_d;\leq,l_d >$.

Indeed, if h is the embedding discussed above from P_c to P_d then, by extending h to the mapping from the set $\overline{P}_c^{l_c}(x)$ (at $x \in P_c$) to $\overline{P}_d^{l_d}(h(x))$ using the order $\leq_1$ defined earlier for $P(\mathfrak{S}_Q)$, we obviously get an embedding h_1 from A_c to A_d, which yields the inequality $< A_c;\leq,l_c > \leq < A_d;\leq,l_d >$. Since the mapping g from the barrier C to $\mathfrak{S}_Q$ is poor, the mapping $J: C \to < \{J_b \mid b \in C\};\leq >$ such that

$J(b) = J_b$ must also be poor. However, $J_b \in (Q \times P(\mathfrak{S}_Q))^{Ord}$ and, hence, by theorem **15.5**, there is a barrier $D \subseteq C$ such that the values of the restriction of J to barrier D are one-element $Q \times P(\mathfrak{S}_Q)$-sequences and, therefore, the restriction of J to the barrier D can be identified with a poor mapping from D to $Q \times P(\mathfrak{S}_Q)$ following J. By theorem **15.3**, there is a barrier $E \subseteq D$ such that either $\pi_1 J$ or $\pi_2 J$ will be poor mappings from E to Q or to $P(\mathfrak{S}_Q)$, respectively. The former case is impossible, since $<Q;\leq>$ is a better quasi-order and, hence, $\pi_2 J$ is a poor mapping from the barrier $E \subseteq B'$ to $P(\mathfrak{S}_Q)$. By corollary **15.1**, there is a subbarrier $F \subseteq E$ such that the restriction of $\pi_2 J$ to F is a poor mapping from F to a family of one-element subsets of the set $\mathfrak{S}_Q$, i.e., the restriction of $\pi_2 J$ to F can be identified with a poor mapping from F to $\mathfrak{S}_Q$.

Let us define a mapping φ from the barrier F^2 to $\mathfrak{S}_Q$ in such a way that $\varphi(l_1 \cup l_2) = \pi_2 J(l_1)$ for any $l_1, l_2 \in F$ such that $l_1 \lhd l_2$. It is obvious that φ is poor. Therefore, the barrier F^2 follows the barrier B', $F^2 \not\subseteq B'$, and for $l_1, l_2 \in F$ such that $l_1 \lhd l_2$, we get $\varphi(l_1 \cup l_2) = \pi_2 J(l_1)$, in which case l_1 is the only element of the barrier B' which is the initial interval of the sequence $l_1 \cup l_2$, and, by the definition of P_{l_1}, the rank of the Q-tree $\pi_2 J(l_1)$ is strictly less than the rank of the Q-tree $g(l_1)$. Hence, the mapping φ is a mapping following the mapping g, and is not a restriction of g to any subbarrier, which contradicts the assumption that g is minimal. This contradiction proves that there can be no poor mappings on $<\mathfrak{S}_Q;\leq>$, i.e., that $<\mathfrak{S}_Q;\leq>$ is a better quasi-order. ■

Theorem 15.7. If $<Q;\leq>$ is a better quasi-order, $<\mathfrak{M}_Q;\leq>$ is also a better quasi-order.

Proof. It can be proved easily that for any tree $<A;\leq>\in\mathfrak{M}$ there is a decomposition $A = \bigcup_{n\in\omega} A^n$ such that A^n are initial intervals in $<A;\leq>$, and for any $n \in \omega$, $<A^n;\leq>\in\mathfrak{S}$, $A^0 \subseteq A^1 \subseteq \ldots \subseteq A^n \subseteq \ldots$ and, if $A^n \neq \varnothing$, $A = A^n \cup \bigcup\{br_A(x) \mid x$ is a maximal element in $A^n\}$.

$\mathfrak{F}_Q$ will denote the class of all Q-trees $<A;\leq,l_A>$ such that $<A;\leq>\in\mathfrak{M}$, and there is no infinite sequence $x_0 < x_1 < \ldots < x_n < \ldots$ of elements of A such that $<<br(x_n),l_A> \mid n \in \omega>$ which is a strictly decreasing sequence in the quasi-order $<\mathfrak{M}_Q;\leq>$. Let us show that $<\mathfrak{F}_Q;\leq>$ is a better quasi-order. Let $<A;\leq,l_A>\in\mathfrak{F}_Q$ and $x \in A$, and let us set $a_{<A;\leq,l_A>}(x) = <\{,br(y),l_A> \mid y \in S(x)$ and $<br(y),l_A>$ is strictly less than $<A;\leq,l_A>$ in $<\mathfrak{M}_Q;\leq>\}$; $|\{y \in S(x) \mid <br(y),l_A>$ is equivalent, in terms of the quasi-order on $\mathfrak{M}_Q$, to the Q-tree $<A;\leq,l_A>\}|>$.

Therefore, $a_{<A;\leq,l_A>} \in P(\mathfrak{F}_Q) \times Card$, where $Card$ is the family of cardinals. For any $n \in \omega$, let us define $T_{<A;\leq,l_A>}(n) = < A^n;\leq,\bar{l} >$, where $\bar{l}(z) = l_A(z)$ for all z which ar not maximal in A^n, while for z maximal in A^n, we get $\bar{l}(z) = < l(z), a_{<A;\leq,l_A>}(z) >$.

Hence, $T_{<A;\leq,l_A>}(n) \in \mathfrak{S}_R$, where $R = Q \cup (Q \times P(\mathfrak{F}_Q) \times Card)$. Let $T(< A;\leq,l_A >) = < T_{<A;\leq,l_A>}(n) | n \in \omega >$, i.e., T is a mapping from $\mathfrak{M}_Q$ to $(\mathfrak{S}_R)^{Ord}$.

Let us first notice that the relation $T(< A;\leq,l_A >) \leq T(< B,\leq,l_B >)$ implies the relation $< A;\leq,l_A > \leq < B,\leq,l_B >$. Indeed, let us construct the required embedding H of the Q-tree $< A;\leq,l_A >$ in the Q-tree $< B;\leq,l_B >$ using a chain of ω steps. Let H be already defined within an initial interval $Y_n \supseteq A^n$ in such a way that:

(a) if $t \in A \setminus Y_n$, there is a $y < t$ which is maximal in Y_n;

(b) if y is maximal in Y_n, there is a bijective mapping $J_y: S(y) \to S(H(y))$ such that if $z \in S(y)$ and $< br(z),l_A >$ is strictly less then $< A;\leq,l_A >$ in $\mathfrak{M}_Q$, $< br(z),l_A > \leq < br(J_y(z)),l_B >$, while if $< br(z),l_A > \equiv_{\leq} < A;\leq,l_A >$, $< br(J_y(z)),l_B > \equiv_{\leq} < B;\leq,l_B >$.

If y is a maximal element in Y_n, and $z \in S(y)$, let us now define H on $br(z)$. When $< br(z),l_A >$ is strictly less than $< A;\leq,l_A >$, let us extend H using the embedding from $br(z)$ to $br(J_y(z))$ which implements the inequality $< br(z),l_A > \leq < br(J_y(z)),l_B >$. If $< br(z),l_A > \equiv_{\leq} < A;\leq,l_A >$, $< br(J_y(z)),l_B > \equiv_{\leq} < B;\leq,l_B >$. As $T(< A;\leq,l_A) > \leq T(< B;\leq,l_B >)$, there is a number $i \in \omega$ and a mapping $k: T_{<A;\leq,l_A>}(n) \to T_{<B;\leq,l_B>}(i)$ implementing the inequality $T_{<A;\leq,l_A>}(n) \leq T_{<B;\leq,l_B>}(i)$.

Let h be an embedding from $< B;\leq,l_B >$ to $< br(J_y(z)),l_B >$. Let us extend H to $br(z) \cap A^{n+1}$, assuming H to be equal to a mapping $h \cdot k$ on $br(z) \cap A^{n+1}$. Since k is, in particular, an embedding from the Q-tree $< br(z) \cap A^{n+1},l_A >$ to the Q-tree $< B;\leq,l_B >$, $h \cdot k$ is an embedding of Q-trees. Let us assume to the contrary that y' is maximal in $br(z) \cap A^{n+1}$. In this case $a_{<A;\leq,l_A>}(y') \leq a_{<B;\leq,l_B>}(k(y'))$. This inequality and the embedding h together induce the existence of a bijective mapping $J_{y'}$ from the set $S(y')$ to the set $S(h \cdot k(y'))$ such that, for all $z' \in S(y')$, we get $< br(z'),l_A > \leq < br(J_{y'}(z'),l_B >$ if $< br(z'),l_A > << A;\leq,l_A >$, and $< br(J_{y'}(z')),l_B > \equiv_{\leq} < B;\leq,l_B >$ if $< br(z'),l_A > \equiv_{\leq} < A;\leq,l_A >$. Therefore, an induction step in the construction of H has been made, and the existence of H proves that the inequality $T(< A;\leq,l_A >) \leq T(< B;\leq,l_B >)$ results in the inequality $< A;\leq,l_A > \leq < B;\leq,l_B >$.

Let us now define the relation $<' : < B;\leq,l_B > <' < A;\leq,l_A >$ on the class $\mathfrak{F}_Q$ only for the case when, for a certain $x \in A$ $< B;\leq,l_B >$ is isomorphic to $< br(x),l_A >$, with the latter strictly less than $< A;\leq,l_A >$ in $\mathfrak{M}_Q$. According to the definition of the class $\mathfrak{F}_Q$, the relation $<'$ is well-founded in the class $\mathfrak{F}_Q$, and $r(B;\leq,l_B)$ will denote the ordinal corresponding to the biggest chain going to $< B;\leq,l_B >$ from a minimal element in $<\mathfrak{F}_Q;<'>$.

Let us now turn directly to proving that $<\mathfrak{F}_Q;\leq>$ is a better quasi-order. Let, to the contrary, g be a poor mapping of the barrier B on $\mathfrak{F}_Q$. Let g be the minimal poor mapping on $\mathfrak{F}_Q$ existing by theorem **15.4**. For $b \in B$, let $< A_b;\leq,l_b >$ denote $g(b)$, and let $\bar{g}(b) = T(< A;\leq,l_b >)$. Taking into account the facts proved above, $\bar{g}$ is a poor mapping from the barrier B to $(\mathfrak{S}_R)^{Ord}$. By theorem **15.5**, there is a barrier $C \subseteq B$ such that $\bar{g}$ bounded on C is identifiable with a poor mapping from C to $\mathfrak{S}_R$, i.e., for any $b \in C$, $\bar{g}(b)$ is a one-element sequence consisting of a $Q \cup (Q \times P(Z_b) \times Card)$-tree from $\mathfrak{S}$. In this case $Z_b = \{< br(x),l_b > | x \in A_b$ and $< br(x),l_b >$ is strictly less than $< A_b;\leq,l_b >$ in $< \mathfrak{M}_Q;\leq>\}$.

The fact that Q and $Card$ are better quasi-orders implies, by theorem **14.3**, that there is a barrier $C' \subseteq C$ such that for $b \in C'$ we get $g'(b) \in Q \times P(Z_b) \times Card$, and the mapping $\pi_2 \cdot \bar{g}$ is a poor mapping from C' to $P(Z_b)$ (here π_2 is a projection of $Q \times P(Z_b) \times Card$ to $P(Z_b)$). By corollary **15.1**, there is a subbarrier D of the barrier C' such that the restriction of $\pi_2 \cdot \bar{g}$ to D is identifiable with a poor mapping from D to Z_b (i.e., for $b \in D$ $\pi_2 \cdot \bar{g}$ are one-element subsets of the set Z_b). Let us define a poor mapping j from a barrier D^2 to Z_b in the following way: for $b_1,b_2 \in D$ such that $b_1 \lhd b_2$, we set $j(b_1 \cup b_2) = \pi_2 \cdot \bar{g}(b_1)$. Hence, if $b \in B$, $d \in D^2$ and b is the initial interval of d, $j(d) = < br(x),l_b >$ for some $x \in A_b$, in which case $< br(x),l_b >$ is strictly less than $< A_b;\leq,l_b >$ in $\mathfrak{M}_Q$. Therefore, the mapping j is poor, follows g and is not a restriction of g to a certain subbarrier of the barrier B, i.e., g is not minimal. The contradiction to the choice of g proves that there are no poor mappings in $\mathfrak{F}_Q$, i.e., that $<\mathfrak{F}_Q;\leq>$ is a better quasi-order.

In order to complete the proof of the theorem we have to show that $\mathfrak{F}_Q = \mathfrak{M}_Q$.

Let us define $b_{<A;\leq,l>}(x), R_{<A;\leq,l>}(x), R(< A;\leq,l >)$ for $< A;\leq,l > \in \mathfrak{M}_Q$ analogously to $a_{<C;\leq,m>}(x), T_{<C;\leq,m>}(n)$ and $T(< C;\leq,m >)$, defined earlier for $< C;\leq,m > \in \mathfrak{F}_Q$. If $x \in A$, then

$$b_{<A;\leq,l>}(x) = <\{< br(y),l > | \, y \in S(x), < br(y),l > \in \mathfrak{F}_Q\}, \ |\{y \in S(x) | < br(y),l > \notin \mathfrak{F}_Q\}| > .$$

Let $R_{<A;\leq,l>}(n) = < A^n;\leq,\tilde{l} >$ for $n < \omega$, where $\tilde{l}(z) = l(z)$ for all z which are not maximal in A^n, while if z is maximal in A^n, $\tilde{l}(z) = < l(z), b_{<A;\leq,l>}(n) | n \in \omega >$.

Let us, finally, set $R(< A;\leq,l >) = < R_{<A;\leq,l>}(n) | n \in \omega >$. Let us prove that

(*) if $< A;\leq,l >, < U;\leq,m > \in \mathfrak{M}_Q \setminus \mathfrak{F}_Q$ and $R(< A;\leq,l >) \leq R(< br(u),m >)$ for any $u \in U$ such that $< br(u),m > \notin \mathfrak{F}_Q$, $< A;\leq,l > \leq < U;\leq,m >$.

To this end, let us construct, by induction on $n \in \omega$, a certain embedding I of the tree $< A;\leq,l >$ in the tree $< U,\leq,m >$ in such a way that at a step n the mapping I will be defined at the initial interval Y_n of the tree $< A;\leq >$ such that $Y_n \supseteq A^n$, in which case:

(a) if $t \in A \setminus Y_n$, there is an element y maximal in Y_n such that $y < t$;

(b) if y is an element maximal in Y_n, there is an embedding J_y of the set $S(y)$ in the set $S(I(y))$ such that, for $z \in S(y)$, we get $< br(z),l > \leq < br(J_y(z)),m >$ if $< br(z),l > \in \mathfrak{F}_Q$, and $< br(J_y(z)),m > \notin \mathfrak{F}_Q$ if $< br(z),l > \notin \mathfrak{F}_Q$.

Let us assume that I has already been defined on Y_n, let y be a certain maximal element in Y_n and $z = S(y)$. If $< br(z),l > \in \mathfrak{F}_Q$, we extend I to the embedding of the Q-tree $< br(z),l >$ in the Q-tree $< br(J_y(z)),m >$, which exists by the condition (b). If $< br(z),l > \notin \mathfrak{F}_Q$ then, according to the same condition, $< br(J_y(z)),m > \notin \mathfrak{F}_Q$ and, hence, by the condition on $< A;\leq,l >, < U,\leq,m >$, $R(< A;\leq,l >) \leq R(< br(J_y(z)),m >)$. Therefore, in this case there is a number $i \in \omega$ and an embedding k from the $Q \cup (Q \times P(\mathfrak{F}_Q) \times Card)$-tree $R_{<A;\leq,l>}(n+1)$ to the $Q \cup (Q \times P(\mathfrak{F}_Q) \times Card)$-tree $R_{<br(J_y(z)),m>}(i)$. In this case, the mapping I will be extended to $br(z) \cap A^{n+1}$, setting $I \restriction br(z) \cap A^{n+1}$ equal to $k \restriction br(z) \cap A^{n+1}$. If now y' is maximal in $br(z) \cap A^{n+1}$, the relation $br_{<A;\leq,l>}(y') \leq br_{<U,\leq,m>}(I(y'))$ makes it possible to define a mapping $J_{y'}$ from the set $S(y')$ to the set $S(I(y'))$ and, therefore, the induction hypothesis is preserved. The embedding I constructed here proves that, under the conditions specified on $< A;\leq,l >$ and $< U;\leq,m >$, $< A;\leq,l > \leq < U;\leq,m >$.

Let us now directly turn to the proof of the equality $\mathfrak{F}_Q = \mathfrak{M}_Q$. Let us assume that there is a tree $< A;\leq,l > \in \mathfrak{M}_Q \setminus \mathfrak{F}_Q$. One can obviously set $(br(x))^n = br(x) \cap A^n$ for any $x \in A$ and $n \in \omega$. This entails that, for any $t,u \in A$ such that $t \leq u$, the inequality $R(< br(u),l >) \leq R(< br(t),l >)$ holds, and that the mapping implementing this inequality is identical. By the definition, for any $< C;\leq,m > \in \mathfrak{M}_Q$

$R(<C;\leq,m>)\in(\mathfrak{S}_{Q\cup(Q\times P(\mathfrak{F}_Q)\times Card)})^{Ord}$ and therefore, since $<Q;\leq>$ is, by the assumption, a better quasi-order, by virtue of theorems **15.3**, **15.6**, and since $<\mathfrak{F}_Q;\leq>$ is, as has been proved earlier, also a better quasi-order, $<\{R(<C;\leq,m>)|<C;\leq,m>\in\mathfrak{M}_Q\};\leq>$ is a better quasi-order and, in particular, $<\{R(<br(t),l>)|t\in A\};\leq>$ is well-founded. Therefore, there is a $t\in A$ such that $<br(t),l>\notin\mathfrak{F}_Q$, and for any $u\in A$ such that $t<u$ and $<br(u),l>\notin\mathfrak{F}_Q$, $R(<br(u),l>)$ is not strictly less than $R(<br(t),l>)$. Hence, taking into account the earlier remarks, we get $R(<br(u),l>)\equiv_{\leq}R(<br(t),l>)$.

According to the statement (*) proved earlier, we see that for any $u\in A$ such that $t<u$ and $<br(u),l>\notin\mathfrak{F}_Q$, the inequality $<br(t),l>\leq<br(u),l>$ holds. On the other hand, by the definition of the class of trees $\mathfrak{F}_Q$, the fact that $<br(t),l>\notin\mathfrak{F}_Q$ implies that there is a $v\in A$ such that $t<v$, $<br(v),l>$ is strictly less than $<br(t),l>$ and $<br(v),l>\notin\mathfrak{F}_Q$. The contradiction obtained proves the equality $\mathfrak{F}_Q=\mathfrak{M}_Q$. As we have proved already that $<\mathfrak{F}_Q;\leq>$ is a better quasi-order, $<\mathfrak{M}_Q;\leq>$ is a better quasi-order as well. ■

Let us now deduce from the result of the theorem just proved the statement used in sections **10**, **11** to prove the fact that countable skeletons of finitely generated discriminator varieties are better quasi-orders.

$\omega^{<\omega}$ will denote a family of finite sequences of elements of ω ordered in terms of the relation "to be an initial interval". The tree $\omega^{<\omega}$, obviously, belongs to the class $\mathfrak{M}$. Let us consider a Tikhonov topology on the family ω^{ω} of all infinite sequences of elements ω, the basis of the neighborhoods of which is set by elements from $\omega^{<\omega}$. $\sum_n^0$, $\prod_n^0$ will denote the family of the subsets of the topological space ω^{ω} belonging to the classes $\sum_n^0$, $\prod_n^0$, respectively, in the Borel hierarchy. If $<Q;\leq>$ is a certain quasi-order, $\sum_n^0(Q)$ ($\prod_n^0(Q)$) will denote the family of all mappings φ from the space ω^{ω} to Q such that, for any $q\in Q$, we get $\varphi^{-1}(q)\in\sum_n^0$ ($\varphi^{-1}(q)\in\prod_n^0$). On sets of the type $\sum_n^0(Q)$, $\prod_n^0(Q)$ the quasi-order $\leq_1$ will be defined in the following way: for $l_1,l_2\in\sum_n^0(Q)$ ($\prod_n^0(Q)$), $l_1\leq_1 l_2$ iff there is a continuous embedding $\sigma:\omega^{\omega}\to\omega^{\omega}$ such that for any $x\in\omega^{\omega}$ we have $l_1(x)\leq l_2(\sigma(x))$. Then the following statement is valid.

Theorem 15.8. If $<Q;\leq>$ is a better quasi-order, $<\sum_2^0(Q);\leq_1>$ is also a better quasi-order.

Proof. Let us first prove that $<\prod_1^0(Q);\leq_1>$ is a better quasi-order. Let $l\in\prod_1^0$, and let us set the range of $l:\{q_\beta|\beta\in\alpha\}$ well-ordered. Let us also define

$\hat{l}:\omega^{<\omega}\to Q$ in the following way: for $s\in\omega^{<\omega}$ $\hat{l}(s)=q_\beta$, where β is the least ordinal such that $l^{-1}(q_\beta)\cap\{x\in\omega^\omega | s\subseteq s\}\neq\varnothing$. Let us now notice that, since for any $q\in Q$, $l^{-1}(q)$ is closed, $l(x)=q$ iff for an infinite family of natural numbers n the equality $\hat{l}(x|n)=q$ holds, which is equivalent to the fact that for all but a finite family, $\hat{l}(x|n)=q$ $n\in\omega$.

Therefore, if $l_1,l_2\in\prod^0_1(Q)$, and there is an embedding $\sigma:\omega^\omega\to\omega^{<\omega}$ (where $\omega^{<\omega}$ is considered as trees) such that for any $s\in\omega^{<\omega}$, we get $\hat{l}_1(s)\leq\hat{l}_2(\sigma(s))$, for any $x\in\omega^\omega$, $l(x)\leq l(h_\sigma(x))$, where $h_\sigma:\omega^\omega\to\omega^\omega$ is defined in the following way: $h_\sigma(x)=\bigcup_{n\in\omega}\sigma(x|n)$. It is obvious that h_σ is a continuous embedding from ω^ω to itself. By theorem **14.7**, $<\{\hat{l}|l\in\prod^0_1\};\leq>$ is a better quasi-order, which fact, combined with the earlier remarks, implies that in this case $<\prod^0_1(Q)\};\leq_1>$ is a better quasi-order as well.

Let now $l\in\sum^0_2(Q)$. For any $q\in Q$, let $l^{-1}(q)=\bigcup_{m\in\omega}X^m_q$, where $X^m_q\in\prod^0_1$ and $\{X^m_q|m\in\omega\}$ is a family of pairwise disjunct sets. Let us define $\breve{l}(x):\omega^\omega\to Q\times\omega$ in the following way: $\breve{l}(x)=<l(x),m>$ if $x\in X^m_{l(x)}$. Let us consider a trivial order on ω in terms of which all elements of ω are equivalent. Therefore, $Q\times\omega$ is a better quasi-order. Hence, bearing in mind the fact proved earlier, $<\prod^0_1(Q\times\omega);\leq_1>$ is also a better quasi-order and, since the inequality $\breve{l}_1\leq_1\breve{l}_2$ obviously implies the inequality $l_1\leq_1 l_2$, $<\sum^0_2(Q);\leq_1>$ is a better quasi-order as well. ■

Priorities. The notion of a well-quasi-ordered set was introduced by J.Kaplansky. The first important results were obtained, with the theory of well-quasi-ordered sets employed, by A.I.Malcev and B.Neuman: if K is a field and G is a linearly ordered group, the group algebra $K(G)$ is embeddable into a skew (for the proof see G.Higman [93]). Theorem **15.1** belongs to G Higman [93]. The definition of a better quasi-order is by Nash-Williams [147]. Theorem **15.2** can be found in [146] (see also [69]). Theorem **15.3** is by Nash-Williams [146], while theorem **15.4** by R.Laver [122]. Both theorem **15.5** and corollary **15.1** can be found in a work by Nash-Williams [146]. Lemma **15.1** and theorems **15.6**, **15.7** belong to R.Laver [122], while theorem **15.8** to F.Van Engelen, A.W.Miller and J.Steel [232]. More details pertaining to the theory of better quasi-orders can be found in a monograph by R.Fräisse [69].

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