Learning Discourse

Discursive approaches to research in mathematics education

Edited by Carolyn Kieran, Ellice Forman and Anna Sfard

Kluwer Academic Publishers

LEARNING DISCOURSE

*Discursive approaches to research in mathematics education**

Edited by

CAROLYN KIERAN
ELLICE FORMAN
and
ANNA SFARD

*This book was previously published as a PME Special Issue in *Educational Studies in Mathematics,* Vol. 46 (1–3), 2001, under the title : BRIDGING THE INDIVIDUAL AND THE SOCIAL: DISCURSIVE APPROACHES TO RESEARCH IN MATHEMATICS EDUCATION

KLUWER ACADEMIC PUBLISHERS
NEW YORK, BOSTON, DORDRECHT, LONDON, MOSCOW

eBook ISBN: 0-306-48085-9
Print ISBN: 1-4020-1024-9

Created in the United States of America

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GUEST EDITORIAL
LEARNING DISCOURSE: SOCIOCULTURAL APPROACHES TO RESEARCH IN MATHEMATICS EDUCATION

While looking at the papers collected in this volume one feels that, in spite of their diverse themes, these seven studies have quite a lot in common and, as a collection, seem to be signaling the existence of a distinct, relatively new type of research in mathematics education. A comparison with, say, a fifteen-year-old issue of *Educational Studies in Mathematics* or of *Journal for Research in Mathematics Education* would reveal a long series of differences. To begin with, the present articles simply *look* different from their older counterparts: They are longer and have a highly variable format, often not even remotely reminiscent of the classical background-method-sample-findings-discussion structure that reigns in the former research reports. Long segments of conversation transcripts take the place of the once ubiquitous graphs and tables. As we start reading, we discover substantial differences in vocabulary. The language of *mental schemes, misconceptions,* and *cognitive conflicts* seems to be giving way to a discourse on *activities, patterns of interaction,* and *communication failures.* While the older texts speak of learning in terms of personal *acquisition,* the newer ones portray it as the process of becoming a *participant* in a collective doing. And last but not least, the classroom scenes that we see as we go on reading have very little in common with what we got used to in the older papers. To be sure, finding a detailed description of a learning activity in a research paper was a rare occurrence until recently, and in the majority of cases we had to rely on our own experience while trying to imagine the life of the class in which the authors conducted their study. In spite of this limitation, much can be said also about the differences in the ways of learning investigated in the two types of research: The traditional mathematical classroom featuring one blackboard, one outspoken teacher and twenty to forty silent students seems to belong to history.[1] It has been replaced by small teams of learners talking to each other, by groups of students voicing their opinions in whole class discussions, and by children and grownups grappling with mathematical problems in real-life situations.

Educational Studies in Mathematics **46:** 1–12, 2001.

All these innovative features, when taken together, seem to make a real difference and to define a distinctive research framework that, because of its obvious emphasis on the issues of language and communication, can be called *discursive* or *communicational.* To be sure, this special framework, although quite widespread and increasingly popular these days, is still under construction. And yet, considering the progress that has already been made, the time seems ripe for an intermediary summary and reflection. The aim of this special issue is to put discursive research in the limelight and to spur some thinking about the reasons for its appearance, about its nature, and about its possible advantages and pitfalls. Let us now address these issues briefly, one by one.

The first question to ask concerns the reasons for the advent of the discursive approach. To give a proper answer, one has to take a broader look at the history, and not just of mathematics education, but of research on human thinking in general. This latter research is, and always has been, torn between two complementary, but not necessarily compatible, goals. On the one hand, the intention of the researcher is to fathom the phenomenon of human thinking in all its uniqueness and with all its ramifications. On the other hand, the method employed must be rigorous enough to put this research on a par with any other scientific endeavor with respect to cogency, trustworthiness, and, above all, usefulness. These two goals create an essential tension that fuels the incessant change. While the request of scientificity (whatever this term means at a given moment) pushes toward simplicity and feeds the belief in cross-contextual invariants, the wish for an all-encompassing, true-to-life picture of human cognitive activities implies that the formidable complexity of the phenomenon should never disappear from the researcher’s sight. No wonder then, that the relatively brief history of cognitive studies is stormy and replete with dramatic turnabouts.

On the face of it, the main question that needs to be answered before the dilemma of the conflicting goals can be solved is that of the proper *method* of inquiry. And yet, not in many fields of research is the way of conceptualizing the *object* of investigation more sensitive to methodological issues than in the study of the human mind. Judging from history, the uncompromising insistence on methodological rigor, especially if gauged according to criteria borrowed from the ‘exact’ sciences, forces researchers to bend, and eventually forget, the original focus of their endeavor. This is what happened when behaviorists decided to purge psychological discourse of any reference to mental non-observables, and this is what happened again not long after the advent of computer science, when technology brought back the hope of a truly scientific insight into the workings of the human mind. Jerome Bruner, one of the founding fathers of the 'cognitive revolution' of the late 1950s admits that his and his colleagues' 'all-out effort to establish meaning as the central concept of psychology…' (Bruner, 1990, p. 2), grounded in the computer metaphor of mind, did not achieve its goal and, in consequence, failed to deliver on its promise of groundbreaking insights into the specificity of the human intellect. As the recent proliferation of critical publications makes clear, also Piaget's impressive attempt to meet the challenge of the conflicting goals by modeling the development of human thinking on Darwinian theory of evolution proved unsatisfactory in many respects (see e.g. Bruner, 1985). The insufficiency of all these approaches expressed itself, among others, in their inability to bring about a lasting betterment of the human condition, which is the ultimate goal of any scientific endeavor. Thus, for example, none of the theories produced by the different frameworks could account in a satisfactory way for such phenomena as the persistent failure of many students in mathematics or the stubborn irreproducibility of educational success.

The first notions about possible reasons for this pervasive difficulty came following a wave of cognition-oriented cross-cultural studies that began in the early 1920s. At that time, psychologists and educators from diverse scientific traditions began arriving in cultures far removed from their own, convinced that "[i]n the realm of culture, outsideness is a most powerful factor in understanding" (Bakhtin, 1986, p.7) and keen to observe what came to be known as *higher psychological processes* cast against the background of foreign traditions (for an historical survey see Cole, 1996). Mathematical thinking, considered as a paradigmatic example of such a process, and as one that is particularly liable to rigorous investigation, became the preferred object of study.[2] The guiding assumption of the early studies was that this uniquely human form of cognitive activity may be found in pre-industrial cultures in their nascent, underdeveloped form. By watching the incipient editions of these processes, psychologists hoped to learn about the cultural invariants of human cognition. And yet, as it soon became clear, venturing into unfamiliar cultural settings to look for phenomena defined according to one's own cultural heritage is an inherently problematic, ultimately misguided, endeavor. Initially, doubts were raised about the methods of study. The traditional forms of experimental design became questioned when the experimenters realized that school mathematical problems, imported directly from the researchers' own culture, would only too often turn out to be completely foreign to the respondent. This clearly created the possibility of major misinterpretations, with the investigators conferring on their findings meanings dictated by their own cultural background (Cole, 1996).

Increasingly suspicious about the experimental method, some of the researchers began supplementing their investigations with descriptive studies in which the focus shifted from laboratory problem solving to spontaneous everyday activities. A long series of research projects devoted to what came to be known as *everyday, street*, *workplace* or *supermarket mathematics* followed. The main merit of all these studies was that they obviated the need for the researcher's regulatory intervention, at least in the initial phase of the investigation that was usually carried out as an ethnographical observation. An experimental study would then often be devised so as to make it possible for the subjects to communicate with the researcher on their own terms. The change of approach proved itself when the non-interventional studies began producing results dramatically different from those one would expect on the grounds of the subjects' former performance on school tasks purported to involve 'the same' cognitive functions. It soon became clear that the superior everyday mathematical performance of people who tended to fail on school tasks is not an accidental, isolated phenomenon. What was found among Kpelle rice sellers in the mid-1960s (Cole, 1996) was observed over and over again among Vai tailors (Reed and Lave, 1979), Brazilian street vendors (Saxe, 1991; Nunes et al., 1993), dairy warehouse workers (Scribner, 1983/1997), and American weight-watchers and shoppers (Lave, 1988).

At this point the methodological doubt turned epistemological. Psychologists started questioning what until now had been taken for granted even without being explicitly spelled out. A common denominator of all the traditional approaches to thinking was the vision of mind as a 'mirror of nature' (Rorty, 1979) – a container to be filled with reflections of, or structures residing in, the external world. Whether simply received or individually constructed, it was believed that these structures – known as *knowledge, concepts,* or *mental schemes* – were regulated by universal external factors, and should thus be more or less the same for all human beings. Once acquired, each such structure should lead to similar behaviors in all the situations in which this structure could be identified. Similarly, the cognitive processes that produced and used these entities were expected to be cross-contextually invariant, that is, governed by universal rules that remain basically the same across different social, cultural, historical and situational settings. Those who were taking a closer look at cognition across cultural and situational boundaries could not help wondering about the soundness of this assumption, or at least about its testability. Sooner or later, this essential doubt would force them to question the conceptual foundations of the traditional framework. This is how the acquisition metaphor, upon which the time-honored cognitivist approach was resting, became the primary suspect.

To this very day, the acquisitionist framework, its impressive history notwithstanding, is a target of criticism coming from the somewhat eclectic group of thinkers who are often called *sociocultural.* In fact, many different names have been given to this rich and diverse cluster of approaches.[3] What sets these approaches apart as a distinct group is the fact that most of them are associated with the Vygotskian school of thought, and that they all promote the vision of human thinking as essentially social in its origins and as inextricably dependent on historical, cultural, and situational factors. It is important to stress that our historical account by no means exhausts the list of approaches that can be called sociocultural, nor does it cover all the events that led to the advent of this variegated trend.[4] In our selective and, of necessity, very brief survey we have focused on those developments that had a direct bearing on cognitive research in general, and on research in mathematical education, in particular.

The discursive approach announced in the title of this special issue can be viewed as one of many possible implementations of the sociocultural call for research that acknowledges the inherently social nature of human thought. Not all the contributors to this volume are using the name 'discursive' and some of them may eschew any explicit descriptions of the epistemological and ontological underpinnings of their research. Nevertheless, a number of theoretical assumptions can be identified that seem to be guiding all the authors. These overarching foundational motifs are what defines the discursive framework. The reader will come across the common theoretical threads while reading the papers. The articles by van Oers, by Lerman, and by Sfard, which all deal with the conceptual infrastructure of the discursive research explicitly, will help in revealing these common threads. At this point, suffice it to say that within the discursive framework, thinking is conceptualized as a special case of the activity of communication and learning mathematics means becoming fluent in a discourse that would be recognized as mathematical by expert interlocutors. As will be explained by the contributors themselves, these deceptively simple definitions turn out to have quite far reaching theoretical and practical entailments. In the remainder of this introduction, let us limit ourselves to the question of how the discursive approach helps in resolving the dilemmas that have been challenging our research and fueling its incessant change ever since its earliest beginnings.

Let us start with the question of whether the discursive approach stands a good chance of capturing what is unique in human thinking. The first thing to note is that while the more traditional frameworks conceptualize learning as intellectual acquisition, and thus as a change in the individual learner, the discursive approach focuses on the change in one's ways of communicating with others. This complicates the picture and makes it much richer. While the place of the individual is not denied, it is conceptualized in a whole new way. No longer is the individual learner viewed as the only object of change; furthermore, the change itself is no longer regarded as stand-alone and independent of that which affects the community of learners as a whole. Indeed, the vision of learning as becoming a participant in a practice must lead to the conclusion that in this process, the practice itself is bound to undergo modifications. Thus, the inclusion of the community in the picture of learning affects the scope of things that must be considered when the change in the activities of an individual learner is studied.

When regarded not as an isolated entity but as a part of a larger whole, the learner becomes but an inextricable element of a new, much broader unit of analysis, many ingredients of which must be brought into the account even if the ultimate focus of study is change in the individual learner's activities. More specifically, when learning mathematics is conceptualized as developing a discourse, probably the most natural units of analysis can be found in the discourse itself (as opposed to such formerly favored units as *concepts, mental schemes,* or *student's knowledge*).

Indeed, the focus of the studies reported in this volume is on the discourse generated by students grappling with mathematical problems. Thus, it is interesting to see how the classroom conversation develops on both the collective and individual level when the group of children in O'Connor's study responds to the teacher's challenging question "Can any fraction be turned into a decimal?" O'Connor examines the fit between the mathematical content (rational numbers and their representational forms) and a whole class position-driven discussion in an upper elementary school classroom. Position-driven discussions occur when a teacher orchestrates an argument among a group of students of one conceptually challenging central question with a limited number of options. Like O'Connor, Forman and Ansell investigate how a teacher orchestrates the discourse in her elementary school classroom. Unlike O'Connor, however, they argue that voices from the past, present, and future, and from outside as well as inside the classroom walls, animate the discussion of students' strategies for solving multi-digit word problems. These voices come from the students' families and the teacher's educational experiences; they represent the memories, attitudes, emotions, and expectations about traditional and reform educational practices in mathematics.

While the conversations in both of these articles involve the whole class and are orchestrated by the teacher, the study in the Zack and Graves article looks at the discourse of groups of students in problem-solving situations. The particular focus is the way in which the differences among the positions of the participants function and how they enable the learners to jointly construct new knowledge. Similarly, the studies in Kieran's and Sfard's articles offer a glimpse into dyadic peer interactions. Kieran explores the emergence of collective mathematical thinking and the ways in which the mathematical discourse of some individuals changes as a result of the group experience. Sfard, on the other hand, tries to fathom the nature and the reasons for the evident ineffectiveness of an interaction between two students who try to solve a mathematical problem.

It must be immediately stressed that discourse is not the only possible source of units of analysis for sociocultural research, nor even the only one considered by the contributors to this special issue. Among the most widely known alternatives are *activity,* the unit proposed by Vygotskian scholars who call themselves activity theorists (Leont'ev, 1978; Engestrom, 1987),[5] *culture*, as preferred by at least some of cultural psychologists who view learning as *enculturation* (Tomasello, 1999); and *practice*, introduced by those among sociocultural thinkers who are most strongly oriented toward sociological issues (Lave, 1988; Lave and Wenger, 1991; Wenger, 1998).

A review of these other possibilities and the explanation of their relative advantages can be found in the article by van Oers, who organizes his exposition around the fundamental question "What is really mathematical?". He provides an historical overview of research on mathematics learning in classroom settings before articulating the discursive approach. Building upon the theories of Vygotsky and Bakhtin, van Oers outlines an emerging framework for future research in which notions of activity, practice, and discourse play prominent roles. Lerman also surveys a variety of theories that have influenced mathematics education and provides his own version of cultural, discursive psychology. In this survey, he discusses discursive psychology, cultural psychology, and sociocultural research, in order to work towards a synthesis. In contrast, Sfard makes a clear choice and argues for the advantages of the framework that takes discourse and communication as its pivotal concepts.

Whether one speaks about learning in terms of discourse, activity, culture, or practice, the focus is on the change generated by interpersonal interactions, and this, as has already been mentioned, results in a picture which is more complex and closer to life than in the traditional cognitivist studies. The question that now begs to be asked is whether all this richness does not come at the expense of the scientific elegance, cogency, and trustworthiness of the research. While looking at recent publications that can count as sociocultural, one may say that, indeed, we are still paying the methodological cost of the decision to put a premium on the goal of capturing the intricacies of learning in all their specificity and uniqueness. We must realize that when it comes to tools and techniques that would match this endeavor, we have yet a long way to go. Unlike in the former, positivist era, we now have to craft our ways of analyzing data each time anew, appropriately to the questions we are asking, and in accord with the data we were able to collect.

This does not mean, however, that it is not possible to build a basic reservoir of sound methodological tools. With its well-defined, directly accessible object of study, the discursive approach seems to be on its way to becoming a fully-fledged research framework, complete with a set of reliable methods of data analysis. In the last years, many impressive methodological advances have been made within this area. In addition to the general-purpose techniques, such as those gathered under the names *conversation analysis* and *discourse analysis,* numerous new tools specially crafted to fit the particular needs of the research in mathematics education are appearing these days with an increasing frequency.

An assortment of such methods may be found in this special issue. O'Connor, who comes to mathematics education from the field of applied linguistics, has always made extensive use of the methods of discourse analysis in her studies on interaction patterns in mathematical classrooms. In the paper included in this volume, she builds her own techniques of looking while trying to capture the ways in which mathematical content evolves as a result of interaction. In her detailed account, O'Connor shows us how the whole-class discussion unfolds, helping us understand the conceptual, pedagogical, and interpersonal dilemmas that emerge during discussions of challenging mathematical content. She uses units that parse the argument into claims and counterclaims with supporting evidence. She also identifies units that illustrate the teacher's skill at managing conceptual and communicative confusion. Forman and Ansell employ a hierarchy of units in their analysis from the molar (such as a lesson) to the molecular (sequences of talk about a particular topic). Furthermore, they examine critical junctures or changes in the structure of events, which may allow one to make inferences about participants' interpretations of those events. Zack and Graves structure the extract that they present into four parts, which emerge as a function of participants' differing roles and stances during the mathematical interaction. This structuring device affords an analysis of the process whereby individual and group developmental trajectories are constructed, as well as an exploration of the relationship between discourse and knowing. Kieran uses an interactivity flowchart, adapted from an earlier study for which it was created (cf., Sfard and Kieran, 2001), to segment the discourse of participants into personal and interpersonal channels of talk. With a focus directed toward those object-level utterances that move the mathematical dimensions of the discourse forward, Kieran hypothesizes why there might be discrepancies between partners in their subsequent individual work. In her attempt to understand the nature of and reasons for the observed communication failure, Sfard applies the interactivity flowchart along with another type of analysis developed in her former study with Kieran: She follows the course of the mathematical conversation with the help of *focal analysis* – a method that aims at 'mapping the trajectory' of the object of conversation.

In this special issue, the complexity of the phenomena under study is reflected in the multi-level analyses of the discourse. All the authors are discussing the development of mathematical communication, and while doing so, they are alternating between the analysis of students' single turns and the examination of patterns to be found in sequences of thematically connected utterances. This may be compared to the study of the mechanics of water where, at some points, the researchers may be watching regularities in the movement of individual particles, and at other times may choose to investigate the geometry and periodic recurrence of waves and whirls. The macro- and micro-level pictures obtained in these ways do not resemble each other, and yet, both are needed by those who try to understand the complex phenomenon under study. In the same vein, whatever the particular focus or level of analysis in the studies presented in this volume, the phenomenon under study remains the same: All the authors are looking at classroom communication that evolves so as to become genuinely mathematical and to allow for solving problems that were intractable within other discourses.

A message similar to the one conveyed by the above comparison can be found in the 'zoom of lens' metaphor invoked in this volume by Lerman to explain the relation between the individual and social research perspectives. The much debated split between these two perspectives is referred to in the title of this special issue, 'Bridging the Individual and the Social'. This split has been worrying researchers for some time now. The seemingly incompatible perspectives are producing two incomplete types of studies, each of which is 'telling only half of the good story' (Cobb, 1996). The call for bridging the two halves follows. We turned this call into the title for the special issue, but not necessarily because we believe that bridging is what needs to be done. Rather, we used the slogan because it points to the dilemma that seems to be still at the center of researchers' attention. Our solution to this dilemma is to deconstruct the dichotomy, and not to unify the halves. Indeed, as the water-study metaphor makes clear, by defining thinking as communicating we are sidestepping the split rather then bridging a gap. The problematic dichotomy between the individual and social research perspectives is no longer an issue when one realizes that the cognitivist ('individualistic') and interactionist ('social') approaches are but two ways of looking at what is basically one and the same phenomenon: the phenomenon of communication, one that originates between people and does not exist without the collective even if it may temporarily involve only one interlocutor. The social nature of the individual is the principal message of this special issue.

NOTES

1. Even if this is not true for many mathematics classrooms around us, it certainly is true for those in which researchers nowadays choose to conduct their studies.
2. The reasons for the particular appropriateness of mathematics for uncovering factors contributing to human cognitive development are eloquently spelled out by Reed and Lave (1979) in the paper with the telling title *Arithmetic as a tool for investigating relations between culture and cognition.*
3. Several different terms have been used to characterize the school of thought that began with Vygotsky: sociocultural, cultural-historical activity theory, cultural psychology, neo-Vygotskian. Vygotsky himself used the term cultural-historical (Cole, 1995; van Oers, 1998).
4. Among the thinkers whose work had a decisive influence on this development one should mention, above all, the Austrian-British philosopher Ludwig Wittgenstein whose seminal work on language brought the issue of communication to the center of psychological research; and of the American and German social philosophers George Herbert Mead and Alfred Schutz, who stressed, each one of them in his own way, the tight relations between human thought and social interactions. (See Valsiner and van de Veer, 2000, and Cole, 1996, for historical overviews of the relevant theories.)
5. See also http://www.edu.helsinki.fi/activity/.

REFERENCES

Bakhtin, M.M.: 1986, *Speech genres and other late essays.* (translated by V. W. McGee; edited by C. Emerson and M. Holquist), University of Texas Press, Austin.

Bruner, J.: 1985, 'Vygotsky: a historical and conceptual pesrspective', in J.V. Wertsch (ed.), *Culture, Communication, and Cognition: Vygotskian Perspectives.* Cambridge University Press, Cambridge, UK, pp. 21–34.

Bruner, J.: 1990, 'The proper study of man', in J. Bruner, *Acts of Meaning,* Harvard University Press, Cambridge, MA, pp. 1–32.

Cobb, P.: 1996, 'Accounting for mathematical learning in the social context of the classroom', in C. Alsina, J.M. Alvarez, B. Hodgson, C. Laborde and A. Perez (eds.), 8^{th} *International Congress of Mathematical Education. Selected lectures,* S.A.E.M. 'THALES', Sevilla, pp. 85–99.

Cole, M.: 1995, 'Socio-cultural-historical psychology: Some general remarks and a proposal for a new kind of cultural-genetic methodology', in J.V. Wertsch, P. del Rio and A. Alvarez (eds.), *Sociocultural Studies of Mind,* Cambridge University Press, Cambridge, UK, pp. 187–215.

Cole, M.: 1996, *Cultural Psychology*, The Belknap Press of Harvard University Press, Cambridge, MA.

Engestrom, Y.: 1987, *Learning by Expanding: An Activity-Theoretical Approach to Developmental Research,* Orienta-Konsultit, Helsinki.

Lave, J.: 1988, *Cognition in Practice,* Cambridge University Press, Cambridge, UK.

Lave, J. and Wenger, E.: 1991, *Situated Learning: Legitimate Peripheral Participation,* Cambridge University Press, Cambridge, UK.

Leont'ev, A.N.: 1978, *Activity, Consciousness, and Personality,* Prentice Hall, Englewood Cliffs, NJ.

Nunes, T., Schliemann, A. and Carraher, D.: 1993, *Street Mathematics and School Mathematics,* Cambridge University Press, Cambridge, UK, pp. 1–27.

Reed, H.J. and Lave, J.: 1979, 'Arithmetic as a tool for investigating the relations between culture and cognition', *American Ethnologist* 6, 568–582.

Rorty, R.: 1979, *Philosophy and the Mirror of Nature,* Princeton University Press, Princeton, NJ.

Saxe, G.: 1991, *Culture and Cognitive Development,* Erlbaum, Hillsdale, NJ.

Scribner, S.: 1983/1997, 'Mind in action: A functional approach to thinking', in M. Cole, Y. Engestrom and O. Vasquez (eds.), *Mind, Culture, and Activity: Seminal Papers from the Laboratory of Comparative Human Cognition,* Cambridge University Press, NY, pp. 354–368.

Sfard, A. and Kieran, C.: 2001, 'Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions', *Mind, Culture, and Activity* 8(1), 42–76.

Tomasello, M.: 1999, *The Cultural Origins of Human Cognition,* Harvard University Press, Cambridge, MA.

Valsiner, J. and van der Veer, R.: 2000, *The Social Mind: Construction of the Idea,* Cambridge University Press, Cambridge, UK.

van Oers, B.: 1998, 'From context to contextualizing', *Learning and Instruction* 8, 473–488.

Wenger, E.: 1998, *Communities of Practice: Learning, Meaning, and Community,* Cambridge University Press, NY.

ANNA SFARD, ELLICE FORMAN AND CAROLYN KIERAN

ACKNOWLEDGEMENTS

We, the guest editors, are pleased to have had the opportunity to produce this Special Issue, one of the series emanating from the International Group for the Psychology of Mathematics Education (PME). However, we hope that this Special Issue will be of interest not only to PME researchers, but also to the broader scientific community concerned with issues related to mathematical discourse and communication.

The Special Issue would not have been possible without the collaboration and cooperation of several individuals. We especially wish to thank the authors of the seven main papers who share their research with readers of this volume. Their theoretical discussions and analyses touch upon crucial aspects of discursive interactions in the mathematics classroom. We are also grateful to Celia Hoyles and Falk Seeger for their contributions in the form of commentary papers. Among the issues raised for consideration, Celia emphasizes in particular the importance of tool mediation and the design of mathematical activities, while Falk argues for more long-term studies on the formation of proficient discursive classrooms.

We want to acknowledge, as well, the work done by the reviewers of the research papers. Their timely, thorough, and insightful comments were greatly appreciated by the authors. We express our gratitude to the Kluwer editorial staff for their patient and expert handling of the various stages of the development of this Special Issue. Last, but not least, we owe special thanks to Heinz Steinbring who, as shadow editor, shepherded this volume to its completion and helped us to deal with several matters along the way. A warm thank-you to all!

ANNA SFARD

THERE IS MORE TO DISCOURSE THAN MEETS THE EARS: LOOKING AT THINKING AS COMMUNICATING TO LEARN MORE ABOUT MATHEMATICAL LEARNING

ABSTRACT. Traditional approaches to research into mathematical thinking, such as the study of misconceptions and tacit models, have brought significant insight into the teaching and learning of mathematics, but have also left many important problems unresolved. In this paper, after taking a close look at two episodes that give rise to a number of difficult questions, I propose to base research on a metaphor of *thinking-as-communicating.* This conceptualization entails viewing learning mathematics as an initiation to a certain well defined *discourse.* Mathematical discourse is made special by two main factors: first, by its exceptional reliance on symbolic artifacts as its *communication-mediating* tools, and second, by the particular *meta-rules* that regulate this type of communication. The meta-rules are the observer's construct and they usually remain tacit for the participants of the discourse. In this paper I argue that by eliciting these special elements of mathematical communication, one has a better chance of accounting for at least some of the still puzzling phenomena. To show how it works, I revisit the episodes presented at the beginning of the paper, reformulate the ensuing questions in the language of thinking-as-communication, and re-address the old quandaries with the help of special analytic tools that help in combining analysis of mathematical content of classroom interaction with attention to meta-level concerns of the participants.

In the domain of mathematics education, the term *discourse* seems these days to be on everyone's lips. It features prominently in research papers, it can be heard in teacher preparation courses, and it appears time and again in a variety of programmatic documents that purport to establish instructional policies (see e.g. NCTM, 2000). All this could be interpreted as showing merely that we became as aware as ever of the importance of mathematical conversation for the success of mathematical learning. In this paper, I will try to show that there is more to discourse than meets the ears, and that putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned. Above all, I will be arguing that communication should be viewed not as a mere aid to thinking, but as almost tantamount to the thinking itself. The *communicational approach to cognition,* which is under scrutiny in this paper, is built around this basic theoretical principle.

In what follows, I present the resulting vision of learning and explain why this conceptualization can be expected to make a significant con-

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tribution to both theory and practice of mathematics education. I begin with taking a close look at two episodes that give rise to a number of difficult questions. The intricacy of the problems serves as the immediate motivation for a critical look at traditional cognitive research, based on the metaphor of learning-as-acquisition, and for the introduction of an additional conceptual framework, grounded in the metaphor of learning-as-participation. In the last part of this article, in order to show how the proposed conceptualization works, I revisit the episodes presented at the beginning of the paper, reformulate the longstanding questions in the new language, and re-address the old quandaries with the help of specially designed analytic tools.

1. QUESTIONS WE HAVE ALWAYS BEEN ASKING ABOUT MATHEMATICAL THINKING AND ARE STILL WONDERING ABOUT

In spite of its being a relatively young discipline, the study of mathematical thinking has a rich and eventful history. Since its birth in the first half of the 20th century, it has been subject to quite a number of major shifts (Kilpatrick, 1992; Sfard, 1997). These days it may well be on its way toward yet another reincarnation. What is it that makes this new field of research so prone to change? Why is it that mathematics education researchers never seem truly satisfied with their own past achievements?

There is certainly more than one reason, and I shall deal with some of them later. For now, let me give a commonsensical answer, likely to be heard from anybody concerned with mathematics education – teachers, students, parents, mathematicians, and just ordinary citizens concerned about the well-being of their children and their society. The immediate suspect, it seems, is the visible gulf between research and practice, expressing itself in the lack of significant, lasting improvement in teaching and learning that the research is supposed to bring. It seems that there is little correlation between the intensity of research and research-based development in a given country and the average level of performance of mathematics students in this country (see e.g. Macnab, 2000; Schmidt et al., 1999; Stigler and Hiebert, 1999). This, in turn, means that as researchers we may have yet a long way to go before our solutions to the most basic problems asked by frustrated mathematics teachers and by desperate students become effective in the long run. The issues we are still puzzled about vary from most general questions regarding our basic assumptions about mathematical learning, to specific everyday queries occasioned by concrete classroom situations. Let me limit myself to just two brief examples of teachers' and researchers' dilemmas.

A function g(x) is partly represented by the table below. Answer the questions in the

x	$g(x)$
0	-5
1	0
2	5
3	10
4	15
5	20

(1) What is $g(6)$? ________
(2) What is $g(10)$? ________
(3) The students in grade 7 were asked to write an expression for the function g(x).
Evan wrote $g(x) = 5(x - 1)$
Amy wrote $g(x) = 3(x - 3) + 2(x - 2)$
Stuart wrote $g(x) = 5x - 5$
Who is right? Why?

Figure 1. *Slope* episode – The activity sheet.

Example 1: Why do children succeed or fail in mathematical tasks? What is the nature and the mechanism of the success and of the failure?

Or, better still, why does mathematics seem so very difficult to learn and why is this learning so prone to failure? This is probably the most obvious among the frequently asked questions, and it can be formulated at many different levels. The example that follows provides an opportunity to observe a ‘failure in the making’ – an unsuccessful attempt at learning that looks like a rather common everyday occurrence.

Figure 2 shows an excerpt from a conversation between two twelve year old boys, Ari and Gur, grappling together with one of a long series of problems supposed to usher them into algebraic thinking and to help them in learning the notion of function.[1] The boys are dealing with the first question on the worksheet presented in Figure 1. The question requires finding the value of the function *g(x),* represented by a partial table, for the value of *x* that does not appear in the table ($g(6)$). Before proceeding, the reader is advised to take a good look at Ari and Gur’s exchange and try to answer the most natural questions that come to mind in situation like this: What can be said about the boys’ understanding from the way they go about the problem? Does the collaboration contribute in any visible way to their learning? If either of the students experiences difficulty, what is the nature of the problem? How could he be helped? What would be an effective way of overcoming – or preventing altogether – the difficulty he is facing?

While it is not too hard to answer some of these questions, some others seem surprisingly elusive. Indeed, a cursory glance at the transcript is enough to see that while Ari proceeds smoothly and effectively, Gur is unable to cope with the task. Moreover, in spite of Ari’s apparently adequate algebraic skills, the conversation that accompanies the process of solving does not seem to help Gur. We can conclude by saying that while Ari’s performance is fully satisfactory, Gur does not ‘pass the test’.

WHAT IS DONE	WHAT IS SAID
	- 25:40 -
[1] A. is trying to get the expression from the table	[1] A.: [1a] Wait, how do we find out the slope again? [1b] No, no, no, no. Slope, no, wait, [1c] intercept is negative 5. [1d]Slope
	[2] G.: What are you talking about?
	[3] A.: I'm talking about this. It's 5.
	[4] G.: It doesn't matter if it's on (mumble)
	[5] A.: 5x. Right?
[6] A. has written 5x + -5	[6] G.: What's that?
	[7] A.: It's the formula, so you can figure it out.
	[8] G.: Oh. How'd you get that formula?
[9] to do the next task: find g(6)	[9] A.: and you replace the x by 6.
	[10] G.: Oh. Ok, I
	[11] A.: [11a] Look. Cause the, um the slope, is the zero. [11b] Ah, no, the intercept is the zero.
	[12] G.: Oh, yeah, yeah, yeah. So you got your
[13] "each": A. points to both columns, indicating that you have to check both "from zero to what": he points to the x column	[13] A.: [13a] And then you see how many is in between each, [13b] like from zero to what
[14] the left counterpart of the right-column 0 is 1	[14] G.: And the slope is, so the slope is 1.
[15] "zero": he circles the zero n the x column on G.'s sheet	[15] A.: [15a] Hum? No, the slope, [15b] see you look at zero,
[16] -5 is the f(x) value when x = 0	[16] G.: [16a] Oh <u>that</u> zero, ok. [16b] So the slope is minus 5
	[17] A.: yeah. And
	[18] G.: How are you supposed to get the other ones?
[19] A. first points to x column ("going down by ones"), then the f(x) column ("by fives"), and again to f(x) column ("look here")	[19] A.: [19a] You look how many times it's going down, like we did before. So it's going down by ones. [19b] So then it's easy. This is ah by fives. See, it's going down by ones, so you just look here
	[20] G.: Oh. So it's 5
	[21] A.: yeah. 5x plus
	[22] G.: Negative 5.
	[23] A.: Do you understand?
	[24] G.: [24a] Negative 5. Yeah, yeah, ok. [24b] So what is g 6?
	[25] A.: [25a] 5 times 6 is 30, plus negative 5 is 25. [25b] So we <u>did</u> get it right.
[26] "this column": he points to x column	[26] G.: No, but it's - in this column there?
	[27] A.: yeah
	[28] G.: [28a] Oh, <u>then</u> that makes sense. [28b] (writes) It's 30. [28c] What is g 10? ... 40
	[29] A.: 20, ah 40. No, 45.
	[30] G.: No,
	[31] A.: 45
	[32] G.: because 20
	[33] A.: 10 times 5 is 50, minus
[34] he points to the two entries in the last row	[34] G.: Well, 5 is 20, so 10 must have 40
[35] A. circles the 10 in g(10) on G.'s sheet	[35] A.: times 5
	[36] G.: Oh, we do that thing. Ok, just trying to find it.
	[37] A.: yeah
[38] again he points to the last row of the table	[38] G.: Cause I was thinking cause 5 is 20,
	[39] A.: It's 45. Yeah
	[40] G.: (mumble) So it's 45.
	- 27:42 -

Figure 2. *Slope* episode – the protocol.

So far so good: The basic question about the overall effectiveness of the students' problem-solving efforts does not pose any special difficulty. Our problem begins when we attempt a move beyond this crude evaluation and venture a quest for a deeper insight into the boys' thinking. Let us try, for example, to diagnose the nature of Gur's difficulty. The first thing to say would be "Gur does not understand the concept of function" or, more precisely, "He does not understand what the formula and the table are all about, what is their relation, and how they should be used in the present context". Although certainly true, this statement has little explanatory power. What Tolstoy said about unhappiness seems to be true also about the lack of understanding: Whoever lacks understanding fails to understand in his or her own way. We do not know much if we cannot say anything specific about the unique nature of Gur's incomprehension.

In tune with a long-standing tradition, many researchers are likely to approach the problem quite differently. As Davis (1988) pointed out, rather than asking *whether* a person understands, we should ask *how* he or she understands. Indeed, "students usually *do* deal with meanings", he says, except that they often "*create their own meanings*" (p. 9, emphases in the original). Thus, we could analyze the event in terms of students' idiosyncratic conceptual constructions. We could say, for example, that unlike his partner, Gur has not, as yet, developed an adequate conception of function. One look at the transcript now, and we identify the familiar nature of the inadequacy: The sequence [28]–[34] shows that Gur holds the ill-conceived idea of linearity, according to which the values of any function should be proportional to the argument (this belief is a variant of the well known misconception according to which any function should be linear; see e.g. Markovitz et al., 1986, Vinner and Dreyfus, 1989).[2] This is important information, no doubt, but is it enough to satisfy our need for explanation? Is it enough for us to say we have understood Gur's thinking? Is it sufficient to guide us as teachers who wish to help Gur in his learning?

Although endowed with an extensive knowledge of students' typical misconceptions, we may still be in the dark about many aspects of this conversation and, more specifically, about the reasons for Gur's choices and responses. Thus, for example, what has been said so far does not give us a clue about the sources either of Gur's lasting confusion with the equation of linear function, or of his inability to follow Ari's explanations. The misconception that certainly plays a role in the last part of the exchange does not account for Gur's earlier responses to the notion of formula. These responses seem as unexpected as they are unhelpful. Moreover, although it is obvious that Gur does struggle for understanding, and although the ideas he wishes to understand do not appear to be very complex (indeed, what could be more straightforward than the need to substitute a number into the formula in order to calculate the value of the function for this number?), all his efforts prove strangely ineffective – they do not seem to take him one step closer to the understanding of the solution explained time and again by Ari, It is not easy to decide what kind of action on the part of the 'more capable peer' (Vygotsky, 1978, p. 86) could be of help.

At this point one may claim that the difficulty we are facing as interpreters stems mainly from the scarcity of data at hand. The episode we are looking at does not provide enough information for any decisive statement on Ari's and Gur's mathematical thinking, some people are likely to say. Although certainly true, this claim does not undermine the former complaint: Although it would certainly be better to have more information, the episode at hand should also be understood on its own terms. What we need in order to make sense of the things the two boys are saying in the given situation are not just additional data, but also, and above all, better developed ways of looking, organized into more penetrating theories of mathematical thinking and learning. Before we turn to the story of the current quest after such theories, let us look at another case of mathematical learning.

Example 2: What should count as 'learning with understanding'?

The notion of understanding, so central to our present deliberations, turns out to be an inexhaustible source of difficulty for both theorists and practitioners. I will now illustrate this difficulty with yet another example related, this time, to the famous call for *meaningful learning* or *learning-with-understanding* that has been guiding our instructional policies for many years. This call was a landmark in the history of educational research in that it signaled the end of the behaviorist era and the beginning of the new direction in the study of human cognition. When more than six decades ago Brownell (1935) issued the exhortation for "full recognition of the value of children's experiences" and for making "arithmetic less a challenge to pupil's memory and more a challenge to his intelligence" (p. 31), his words sounded innovative, and even defiant. Eventually, these words helped to lift the behaviorist ban on the inquiry into the 'black box' of mind. Once the permission to look 'inside human head' was given, the issue of understanding turned into one of the central topics of research.

In spite of the impressive advances of this research, most educators agree today that finding ways to make the principle of learning-with-understanding operative is an extremely difficult task. Methods of 'meaningful' teaching "are still not well known, and most mathematics teachers probably must rely on a set of intuitions about quantitative thinking that

[1]	Rada, the teacher:	Can you count to 10?
[2]	Noa:	Yes. One, two, three, four, five, six, seven, eight, nine, ten.
[3]	Teacher:	Do you know more than ten?
[4]	Noa:	Yes. One, two, 3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20.
[5]	Teacher:	What is the biggest number you can think of?
[6]	Noa:	Million.
[7]	Teacher:	What happens when we add one to million?
[8]	Noa:	Million and one.
[9]	Teacher:	Is it bigger than million?
[10]	Noa:	Yes.
[11]	Teacher:	So what is the biggest number?
[12]	Noa:	Two millions.
[13]	Teacher:	And if we add one to two millions?
[14]	Noa:	It's more than two millions.
[15]	Teacher:	So can one arrive at the biggest number?
[16]	Noa:	Yes.
[17]	Teacher:	Let's assume that *googol* is the biggest number. Can we add one to googol?
[18]	Noa:	Yes. There are numbers bigger than googol.
[19]	Teacher:	So what is the biggest number?
[20]	Noa:	There is no such number!
[21]	Teacher:	Why there is no biggest number?
[22]	Noa:	Because there is always a number which is bigger than that?

Figure 3. Conversation between a pre-service teacher and a 7 year old girl, Noa (first grade).

involves both the importance of meaning – however defined – and computation," complains Mayer (1983, p. 77). Hiebert and Carpenter echo this concern when saying that promoting learning with understanding "has been like searching for the Holy Grail." "There is a persistent belief in the merits of the goal, but designing school learning environments that successfully promote learning with understanding has been difficult," they add (Hiebert and Carpenter, 1992, p. 65). The conversation between pre-service teacher Rada and the 7 year old girl Noa about the concept of 'the biggest number' (see Figure 3) highlights a certain aspect of the difficulty.

Clearly, for Noa, this very brief conversation becomes an opportunity for learning. The girl begins the dialogue convinced that there is a number that can be called 'the biggest' and she ends emphatically stating the opposite: "There is no such number!". The question is whether this learning may be regarded as learning-with-understanding, and whether it is therefore the desirable kind of learning.

To answer this question, one has to look at the way in which the learning occurs. The seemingly most natural thing to say if one approaches the task from the traditional perspective, already mentioned in the former example, is that the teacher leads the girl to realize the contradiction in her conception of number: Noa views the number set as finite, but she also seems aware of the fact that adding one to any number leads to an even bigger number. These two facts, put together, lead to what is called in the literature ‘a cognitive conflict’ (see e.g. Tall and Schwartzenberger, 1978), and thus call for revision and modification of her number schema. This is what the girl eventually does. On the face of it, the change occurs as a result of rational considerations, and may thus count as an instance of learning with understanding.

And yet, something seems to be missing in this explanation. Why is it that Noa stays quite unimpressed by the contradiction the first time she is asked about the number obtained by adding one? Why doesn’t she modify her answer when exposed to it for the second time? Why is it that when she eventually puts together the two contradicting claims – the claim that adding one leads to a bigger number and the claim that there is such thing as *the* biggest number – her conclusion ends with a question mark rather than with a firm assertion (see [22])? Isn’t the girl aware of the logical necessity of this conclusion?

Another possibility, one I will discuss in detail later in this paper, is that Noa’s change of mind has less to do with her understanding of the concepts than with her spontaneous use of mostly involuntary cues about the appropriateness of her answers found in the teacher’s reactions. In this case, the decision to say, in the end, that “there is no biggest number” cannot be regarded as an evidence of ‘learning-with-understanding’, at least not according to how the term ‘understanding’ is usually interpreted in this context. If so, the adherents of meaningful learning are likely to criticize the teacher for the instructional strategy she used. And yet, from my numerous encounters with teachers, I do know, that for the great majority of them, the way Rada proceeded in the present example would be the natural choice. Teacher’s intuitions are not anything to be easily dismissed by the researcher. We seem to be facing yet another dilemma likely to challenge teachers and researchers.

Summary: On the learning-as-acquisition metaphor, its advantages and its shortcomings

After having had a look at a number of questions spawned by the two brief episodes, it is time now to say a few words about research in mathematics education in general. The ways researchers have been looking at the studied phenomena may be diverse and many, but all the known approaches were, until recently, unified by the same basic vision of learning. Influenced by folk models of learning implicit in our everyday ways of talking, and further encouraged by numerous scientific theories of mind that conceptualize learning as storing information in the form of mental representations, the students of mathematical thinking and problem-solving tacitly adopted the metaphor of *learning as the acquisition* of knowledge. The emphasis here is on the term *acquisition,* which underlines the individual nature of the endeavor. The acquisition may take place either by passive reception or by active construction, resulting in a personalized version of concepts and procedures. More often than not, these individual constructions have been termed *misconceptions* rather than simply conceptions. This suggestive label implies that one should expect a disparity between learners' private versions and the 'official', 'correct' edition of mathematical concepts. Terms such as *concept image* (as opposed to concept definition; Tall and Vinner, 1981) or *tacit models* (Fischbein, 1989; Fischbein et al., 1985), which began to appear in parallel to the notion of misconception may be regarded as very close in meaning, as they imply the same basic idea of discrepancy between individual and public conceptual constructions.

The theories of conceptual development to which all these notions are somehow related draw on the idea of internal representation and on the Kantian/Piagetian concept of schemes – organizing mental structures everyone supposedly constructs for oneself from the elementary building blocks called conceptions. It is through these mental schemes that our conceptions purportedly get their meaning. Cognitive psychology equated understanding with perfecting mental representations and defined learning-with-understanding as one that effectively relates new knowledge to knowledge already possessed. Within the acquisitionist framework, therefore, understanding is a mode of knowledge, whereas knowledge itself is conceptualized as a certain object which a person either possesses or not, and learning is regarded as a process of acquiring this object (cf. Sfard, 1998). Once acquired, the knowledge is carried from one situation to another and used whenever appropriate. To put it into Jean Lave's words, within this long-standing tradition,

> mind and its contents have been treated rather like a well-filled toolbox. Knowledge is conceived as a set of tools stored in memory, carried around by individuals who take the tools (e.g. 'foolproof' arithmetic algorithms) out and use them, the more often and appropriately the better, after which they are stowed away again without change at any time during the process. (Lave, 1988, p. 24)

With its many branches in the quickly developing new science of cognition, this approach had been flourishing for a few decades, spawning a massive flow of research (see e.g. Hiebert and Carpenter, 1992).

At this point, it must be emphasized that with all the above criticism, it was by no means my intention to disparage either the acquisition metaphor or the theories that grow out of it. The idea of students' idiosyncratic conceptions and the notion of learning-with-understanding have done much good to both the theory and the practice of mathematics education, and right now seem particularly useful to those who try to bridge the science of the mind with the science of the brain. My only point is that whether we act as researchers or as practitioners, the notions grounded in the acquisition metaphor may be too crude an instrument for some of our present more advanced needs. Acquisition-based theories 'distill' cognitive activities from their context and thus tell us only a restricted part of the story of learning. The elements that they leave out of sight are often indispensable for the kind of understanding that should underlie any sensible practical decision. In the former paragraphs I illustrated this claim with two examples, and in the last part of this paper I will be arguing that these missing elements may, in fact, be significant enough to change the picture in a radical way. The conclusion I am opting for is that rather than rejecting the long-standing acquisition metaphor, we should supplement it with theories grounded in alternative metaphors. The communicational approach, deeply rooted in one such metaphor, is to be regarded as complementary rather than incompatible with the more traditional outlooks. In the next section, I precede the introduction of the communicational approach with the presentation of a complementary metaphor.

2. COMMUNICATIONAL APPROACH TO COGNITION

Participationist approach to cognition

The complementary conceptualization of learning I wish to introduce in this article grows from the sociocultural tradition. As emphasized by the editors in the introduction to this volume, the central feature of this latter trend, one that renders it its unique identity and puts it quite apart from the former approaches to human cognition, is its deeply suspicious attitude toward the long-standing sweeping claims about various cognitive invariants – whether those supposed to cross cultural borders, those expected to remain unaffected by historical changes, or those that are merely believed to be transferred by an individual from one situation to another. All this said, please note that the emphasis in this last sentence is on the word *sweeping*. While sociocultural theories issue an admonition against ungrounded assumptions about universality and alert us to the conceptual difficulty inherent in the notion, they do not claim the total non-existence of cognitive invariants (see e.g. sociocultural account of the phylo- and ontogenesis of language in Bruner, 1986; see also Cole, 1996; Tomasello, 1999; Mantovani, 2000).

Disillusioned with the explanatory power of theories that speak of context-independent traits of the individual, sociocultural psychologists prefer to view learning as *becoming a participant in certain distinct activities* rather than as becoming a possessor of generalized, context-independent conceptual schemes. Representatives of different variants of the sociocultural framework speak of learning as "peripheral participation in a community of practice" (Lave and Wenger, 1991), as "an improved participation in an interactive system" (Greeno, 1997), as "initiation to a discourse" (Edwards, 1993; Harre and Gillett, 1995) or as "a reorganization of an activity" (Cobb, 1998). There is an ontological gulf between the old and the new metaphors, and because of this deep disparity the conceptions of learning engendered by these metaphors diverge along many dimensions.

Before I survey the most immediate entailments of the participation metaphor, two cautionary remarks are in order. First, no theory is built on a single metaphor. However, of those metaphors that can be identified, one is usually the most prominent and influential. Also, not all of the differences between the different approaches are necessitated by the respective metaphors. Some of the entailments are optional and sustained by a mere habit. Both types, however, deserve attention as both of them have a considerable impact on theory and on practice. Second, dichotomy between acquisition and participation should not be mistaken for any of the well-known theoretical distinctions. As was stressed above, even if the acquisition metaphor is more common in the traditional cognitivist approach than in sociocultural theorizing, it is not altogether absent from the latter. Sometimes, it may even be quite prominent. This is certainly the case when one speaks, with Vygotsky – a thinker generally recognized as one of the founders of the sociocultural trend in psychology – about "interiorization of higher mental functions" by their transmission from "interpsychological" to "intrapsychological plane" (Vygotsky, 1931/1981, p. 163). Neither is the acquisition/participation dichotomy equivalent to the distinction between individualist and social perspectives on learning. Whereas the social dimension is salient in the participation metaphor, it is not necessarily absent from the theories dominated by the acquisition metaphor. It is important to understand that the two distinctions have been made according to different criteria: while acquisition/participation division is ontological in nature and draws on two radically different answers to the fundamental question "What is this thing called learning?", the individual/social dichotomy does not imply a controversy as to the definition of learning, but rather rests on differing visions of the mechanism of learning.

As was already said, for participationists learning is first and foremost about the development of ways in which an individual participates in well-established communal activities. The participationist researcher is therefore attuned to the ongoing interactions that spur this development, rather than to those properties of the individual that can he held responsible for the constancy of this person's behavior. This vision implies that we should be less interested in explanations based on such unobservables as mental schemes, than in descriptions of the processes of learning, their patterns and mechanisms. The descriptions may be drawn with a special attention to those hitherto ignored dimensions of a learning situation that underlie the learner's increasing ability to create and sustain the "relation of mutual accountability" with other members of the community (Wenger, 1998, p. 81). In simpler words, the participationist researcher focuses on the growth of mutual understanding and coordination between the learner and the rest of the community. All this means that while acqusitionists are mainly interested in pinpointing cross-contextual invariants of learning, participationists shift the focus to the activity itself and to its changing, context-sensitive dimensions. In the case of Ari and Gur in our first example, this means analyzing the conversation with an eye to all those elements and circumstances of the boys' joint activity which make their exchange ineffective. In the case of Noa and Rada, it means asking the parallel question about the mechanisms of interaction that led to the student's alignment with the teacher. In both cases the shift of focus to the interactional aspects of learning implies attention to many factors that, so far, were deemed irrelevant to the issue of cognitive development.

Indeed, the inclusion of the community in the picture of learning affects the scope of things that must be considered when the change in the newcomer's ways of acting is studied. When regarded not as an isolated entity but as a part of a larger whole, the learner becomes but an aspect of a new, much broader unit of analysis,[3] many elements of which must be brought into the account even if the ultimate focus of the study is the change in the individual. In the two episodes above, this means that describing all that happens between the interlocutors exclusively in terms of stand-alone cognition, that is, of the actors' abilities and the contents of their minds (whatever the sense of the last two terms), means overlooking a great many aspects and factors of change. In the final account, this is bound to lead to an impoverished, if not distorted, unhelpful picture of learning.

Not only does success in problem solving prove highly sensitive to the context of the activity, say participationists, but also the ways people act would change from one situation to another.[4] Thus, abstract scholarly learning may have the theoretical advantage of a broader scope, but in reality it would often prove much less effective than apprentice-like participation in the restricted repertoire of specific activities for which the person wishes to prepare herself. Obviously, this belief has many implications for both educational practice and research. Participationists advocate 'cognitive apprenticeship' (Brown et al., 1989) as a preferred mode of learning, and as researchers they are at least as interested in the informal and workplace "legitimate peripheral participation",[5] as in institutionalized scholarly learning.

Yet another time-honored question likely to incite passionate debates between acquisitionists and participationists regards the nature and sources of human knowing. Acquisitionist interest in universal factors with which to account for those aspects of learning that seem relatively insensitive to social, cultural, historical, and situational context implies an emphasis on human-independent circumstances of learning, such as the direct encounter between the individual and the world, and a range of biological determinants, from inheritance to physiological growth and to the structure of human brain. Participationists, who view learning as entering a certain human practice, obviously shift the emphasis to the society as the setting that produces and sustains this practice. Indeed, participationists' deep skepticism about cross-cultural invariants is fueled by their view of learning as beginning and ending in society – as spurred by the need for interaction and communication and geared towards its continual growth. Since our very survival depends on our being a part of community, it is this need for communication that seems to be inscribed in humans. High sensitivity of our ways of acting to social, cultural, historical and situational contexts is an inevitable derivative of the fact that the activities themselves, rather than being dictated by an external non-human world, have their roots in our cultural heritage and are constantly shaped and re-shaped by successive generations of practitioners. This discussion between acquisitionists and participationists clearly echoes the centuries long nature-or-nurture controversy and may thus be read as its modern version.[6] In our examples, the way participationists propose to approach the dilemma suggests that, in an attempt to explain Gur or Noa's performance, much attention should be given to a variety of contextual factors before one decides to account for children's performance in terms of permanent traits, such as their 'mathematical ability' or the lack thereof.

Conceptualizing thinking as communicating

Although the participation metaphor may now appear pretty well defined, most attempts at turning its entailments into a sound basis for research and for practical decision-making are still in their initial stages. As stated by Cole:

> Nowhere are these ideas so highly developed that it is possible to refer to them as a mature scientific paradigm with generally accepted theoretical foundations,

a methodology, and a well-delineated set of prescriptions for relating theory to practice. (Cole, 1995, p. 187)

The words ‘these ideas’ in the quote refer to the principles underlying the sociocultural approach to cognition, and the statement itself, made nearly a decade ago,[7] seems to be still pretty much in force. And yet, if not the situation itself, then at least the chances for finding what is still missing do seem better, these days. In this last decade, quite a few significant attempts have been made at constructing frameworks that would meet the standards of a ‘mature research paradigm’ while respecting the basic sociocultural principles. The *communicational approach* presented in the rest of this paper is one of the currently available products of these attempts. With its roots in Vygotskian writings and with its branches in contemporary philosophical-sociological thought (e.g. Wittgenstein and French postmodern thinkers) and in recent advances in linguistics, this outlook seems to stand a good chance for turning into a full-fledged research framework fulfilling in a reasonable way the requirements specified by Cole.

The basic tenet of the communicational approach to the study of human cognition is that *thinking may be conceptualized as a case of communication,* that is, as one’s communication with oneself. Indeed, our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own response. The conceptualization of thinking as communication is an almost inescapable implication of the thesis on the inherently social origins of all human activities. Anyone who believes, as Vygotsky did, in the developmental priority of communicational public speech over inner private speech (e.g. Vygotsky, 1987) must also admit that whether phylogenesis or ontogenesis is considered, thinking arises as a modified private version of interpersonal communication. All this amounts to the claim that thinking is nothing but our communicating with ourselves, not necessarily inner, and not necessarily verbal. At this point, it is important to stress the crucial difference between this statement and the long-standing hypothesis that equates thinking with internalized speech: the word *communication* is used here in a very broad sense and is not confined to interactions mediated by language. This conceptualization of cognition, even if not stated explicitly, seems to be finding its way into today’s psychological thinking. Harre and Gillett (1995) go so far as to declare the emergence of a new kind of psychology, one that they call *discursive. Discursive psychology* has been described by these authors as one that rests on the assumption similar to the one just stated above: “Individual and private uses of symbolic systems, which ... constitute thinking, are derived from interpersonal discursive processes that are the main feature of the human environment." (p. 27). The reason why I describe the present approach with the term *communicational* rather than discursive in spite of its clear similarity to the position taken by Harre and Gillett (and possibly shared with others; see e.g. Edwards, 1997), is that the former differs from the latter in its epistemological underpinnings and this difference proves highly consequential in terms of theoretical and methodological entailments. This difference will be explained in one of the following paragraphs.

A number of immediate entailments of this conceptualization should now be pointed out. First, since communication may be defined as a person's attempt to make an interlocutor act, think or feel according to her intentions (c.f. Levinson, 1983; Sfard, 2000a,b), research that looks at cognition as a communicational activity focuses, in fact, on the phenomenon of mutual regulation and of self-regulation. It is exactly this phenomenon which was singled out by Leont'ev as the hallmark of being human: "[W]e do not meet in the animal world any special forms of action having as their sole and special end the mastery of the behavior of other individuals by attracting their attention" (Leont'ev, 1930, p. 59, quoted in Cole, 1988). Thus, when one is looking at cognition as a form of communication, an individual becomes automatically a nexus in the web of social relations – both a reason for, and the result of, these relations. This is true whether this individual is in a real-time interaction with others or acts alone. Whatever attempt at understanding human beings is made, it must now take into account that all human actions and deeds are guided, in one way or another, by forces of social cohesion, that is by the fact that, just like different organs in our body, the individual does not exist except as parts of a larger whole.

Further, from the proposed vision of cognition it follows that thinking is subordinated to, and informed by, the demand of making communication effective. When harnessing this fact to the analysis of cognitive mechanisms, the first point to remember is that the basic driving forces, and thus basic mechanisms, are likely to be almost the same whether one considers communicating with oneself or with others. Second, in this approach the dichotomy/thought communication practically disappears and speech is no longer considered as a mere 'window to the mind' – as an activity secondary to thinking and coming just to 'express' a ready-made thought. Although there is still room for the talk about thought and speech as two different things, these two 'things' are to be understood as inseparable aspects of basically one and the same phenomenon, with none of them being prior to the other.

Learning as initiation to a discourse

Within this conceptual framework, the focus of study is on discourse. In our research, the term *discourse* will be used to denote any specific instance of communicating, whether diachronic or synchronic, whether with others or with oneself, whether predominantly verbal or with the help of any other symbolic system. The particularly broad meaning of the term in the present context implies inclusion of instances that would probably be excluded from the category of discourse by everyday users of the term. For example, the production of a written or spoken text, often considered as the defining feature of discourse, is not a necessary ingredient of what will counts for us as 'discursive'. I shall use only one rule for deciding whether a given aspect of an observed situation should count as a component of the discourse or not: Since discourses are analyzed as acts of communicating, anything that goes into communication and influences its effectiveness – body movements, situational clues, interlocutors' histories, etc. – must be included in the analysis.

Learning mathematics may now be defined as an initiation to mathematical discourse, that is, initiation to a special form of communication known as mathematical. Let us look at those factors that are automatically included in the study of thinking as communicating and which dictate what must be learned if a person is to become a skillful participant of a given discourse. Two types of such factors deserve particular attention: the *mediating tools* (or simply *mediators*) that people use as the means of communication, and the *meta-discursive rules* that regulate the communicative effort. While tools are the shapers of the content, that is, of the object-level aspects of discourse (cf. Sfard, 2000b; Sfard and Kieran, 2001a), meta-discursive rules are the molders, enablers and navigators of the communicational activities (Sfard, 2000c). The more detailed description that follows explains why both mediators and meta-discursive rules can be regarded as principal carriers of cultural heritage.

Factors that render discourses their distinct identities: Mediating tools and meta-discursive rules

Let us turn first to the *mediating tools.* "Man differs from animals in that he can make and use tools", says Luria (1928, p. 493). Communication, either inter-personal or self-orientated (thinking) would not be possible without symbolic tools, with language being the most prominent among them. In my opening examples, additional symbolic tools used by the children are the numerical notation, graphs, tables, and algebraic formulas. The tightness of the relation between the ways we conceptualize and the ways we symbolize can be seen, for example, from the fact that all our verbal references to numbers (see e.g. those of Noa in Figure 3) bear distinct marks of the decimal notation, whether the decimal numerals are actually displayed or not (think, for example, about the way we perform mentally any calculation, notably multiplication by ten).

This last statement, referring to the role of symbols in thinking, is central enough to the present discussion to deserve further elaboration. Contrary to what is implied by a common understanding of a tool in general and of symbolic tools in particular, within the communicational framework one does not conceive of artifacts used in communication as mere auxiliary means that come to provide expression to pre-existing, pre-formed thought. Rather, one thinks about them as part and parcel of the act of communication and thus of cognition (for detailed argument see Sfard, 2000a). There is therefore no sense in which one could talk about thought as having an existence independent of the symbolic tools used in the process of communication. This means, among others, that we should regard as rather senseless such statements as "the same thought has been conveyed by different means" (which, however, does not mean we cannot *interpret* two expressions in the same way, with *interpretation* and *thought* being two different things). In other words, there is no 'cognitive essence' or 'pure thought' that could be extracted from one symbolic embodiment and put into another.

Let me now say a few words about *meta-discursive rules.* While tools play a central role in shaping the visible, object-level (content-related) aspects of discourse, meta-discursive rules are what guides the general course of communicational activities. It is noteworthy that meta-discursive rules are mostly invisible and act 'from behind the scene'. Because of their implicit nature, and in spite of their ubiquity, they have not been given any direct attention in the past. These days, the situation is changing quite rapidly, as the general interest in participationist framework and in discursive activities of 'mathematically-speaking' communities begins to spread (see e.g. Voigt, 1985, 1996; Bauersfeld, 1995; Lampert, 1990; Lampert and Blunk, 1998; Forman, 1996; Forman and Larreamendy-Joerns, 1998; Cobb, Wood and Yackel, 1993; Yackel and Cobb, 1996; O'Connor, 1998; Morgan, 1996; Sfard, 2000a,b,c; for a survey see Lampert and Cobb, in press).[8]

It is important to state right away that the term *meta-rules* is very broad and that, because of certain subtleties of its intended meaning, it is prone to misinterpretations. The first thing to note is that the idea is close to many other discourse-related concepts known from philosophical, sociological, and anthropological literature. Thus, for example, it is not altogether different from what Wittgenstein (1953) calls *language games* and what Bordieu

(1999) names *dispositions* (the latter, taken together, constitute *habitus*). It is also related to what Goffman (1974) refers to as interaction *frames* (see also Bateson, 1973), and what Bruner (1983) includes in the idea of *format*. The search for family resemblance must also lead, inevitably, to the fundamental work in sociology by Schutz (1967) and in ethnomethodology by Garfinkel (1967). In the domain of mathematics education, the term *socio-mathematical norms* used by some authors (e.g. Yackel and Cobb, 1996) may be viewed as describing a certain subset of meta-discursive rules, even though there is a subtle difference between the notions *rule* and *norm* (see discussion of this difference in Sfard, 2000c). This is to say that the term *meta-discursive rule* used in this article does not come as an entirely new construct but rather as an almost self-explanatory term supposed to encompass all the phenomena signaled by the notions listed above.

It is important to stress that in concert with Wittgenstein's idea of language games and with Bordieu's approach to the issue of social regulations, meta-rule should be understood as "an explanatory hypothesis constructed by the theorist in order to explain what he sees" (Bouveresse, 1999) rather than anything that is 'really there'. That is, meta-rules are usually not anything the interlocutors would be fully aware of, or would follow consciously. What a discourse analyst views as a meta-discursive rule can be compared to what a physicist considers to be a law of nature: the regularity that is seen by those who observe, but not necessarily by those who are seen as 'implementing' the rule.[9] Taking the interpretive status of the meta-discursive rule as a point of departure, I can now be a little more specific about this concept, while trying to illustrate it with a few examples (for a much more detailed treatment see Sfard, 2000c).

Within the communicational framework, meta-discursive rules should be understood as expressing themselves in regularities observed in those aspects of communicational activities that are not directly related to the particular content of the exchange (which does not mean the rules do not have an impact on the interlocutors' grasp of the content or that they do not change when the contents change). In concert with meta-discursive rules, people undertake actions that count as appropriate in a given context and refrain from behaviors that would look out of place. In the case of mathematical discourse, this category of rules includes those that underlie the uniquely mathematical ways of defining and proving. Further, it is thanks to spontaneously, non-reflectively observed meta-rules that interlocutors are able to navigate inter-personal exchange and regulate self-communication. It is within the system of meta-rules that people's culturally-specific norms, values, and beliefs are encoded. The way symbolic tools should be used in the given type of communication is yet another aspect where a distinct category of meta-rules may be identified. There are also special sets of meta-rules involved in regulating interlocutors' mutual positioning and shaping their identities.

The variety of meta-rules navigating and molding a particular discourse is obviously very broad and heterogeneous, and, along with the meta-rules specific to this particular discourse, usually contains a sizable bulk of implicit regulations related to more general aspects of communication, and probably common to a wide range of discourses (Cazden, 1988). It is important to stress that meta-discursive rules are responsible not only for the ways people communicate, but also for the very fact that they are able to do so in the first place. These rules have an enabling effect in that they eliminate an infinity of possible discursive moves and leave the interlocutors with only a manageable number of choices.

Since meta-rules are tacit, they are usually taught and learned 'on the run', with teachers and students quite unaware of this learning. Some of the meta-rules that are included in this hidden curriculum are truly indispensable, some others may enter the scene as if against the teacher's better judgement. Close analysis that aims at eliciting these tacit ingredients of learning may lead to re-appreciation of certain educational principles. As I will argue below while revisiting the opening examples, such analysis would often show that even those 'unwanted' meta-rules may be an effective, sometimes irreplaceable, means for significant learning.

On the methodological aspects of the communicational research framework

The claim that the communicational approach has a chance to grow into a fully-fledged research framework cannot look fully convincing unless we can be certain of the possibility of supplementing it with a strong methodology. Although the efforts to build such methodology are still under way, it is quite clear that the proposed conceptualization of thinking implies a wide range of data-collecting strategies and can be expected to produce a rich and greatly diversified family of analytical methods. In addition to the already existing discourse and conversation analyses, those who work within the communicational approach to cognition have yet to construct and test their own methods of handling data, tailored according to their specific needs. Such methods seem to be on their way (see e.g. Steinbring et al., 1998; Lampert and Blunk, 1998). Above all, thanks to the disappearance of cognition/communication dichotomy, the present object of study, that is discursive processes, is much more accessible than the more traditional one – the cognitive processes 'in the mind'.

Let me add a word of caution. A few decades ago Wittgenstein (1953) issued a powerful argument against mentalism, requiring that psychological discourse be purified from any reference to 'mental states' and to the inherently unobservable entities 'in the mind'. In the now developing approaches to cognition, this exhortation is being interpreted and operationalized in more than one way. While discursivist psychologists are ready to follow Wittgenstein's call all way down (Harre and Gillett, 1995; Edwards, 1997), extreme logical behaviorism is not the outlook promoted in this paper. References to such 'unobservables' as people's *intentions* are made in the definition of communication underlying the communicational approach, and will often, if not always, feature prominently in analyses carried out within this framework. More generally, the leading assumption here is that our experiences, feelings, and intentions are central to all our decisions, and thus cannot be omitted in any serious attempt at understanding human actions. And yet, in the light of Wittgenstein's well substantiated caveat, even those who agree with this assumption may still wonder how such mentalist ideas as 'human experience' can be made researchable. Let me then remind ourselves, that when Wittgenstein was warning against mentalist language, he was doing this out of a concern about the possible circularity of the resulting definitions. It can be shown, however, that the danger of circularity is obviated if one refrains from comparisons between mental states of different people. Indeed, the use of such terms as *intentions* is safe as long as it is understood that the status of any claim about other people's intentions the researcher can make is *interpretive,* and thus any comparison that is being made is between the *researcher's own interpretations* of other people's intentions (for a more complete argument see Sfard and Kieran, 2001 a).

The ultimate conclusion from these last remarks is that the only viable possibility for the researcher is to provide a *convincing interpretation* of the observed phenomena, as opposed to their definitive explanation. The interpretation should try to be as compelling, cogent, and trustworthy as possible, but it will nevertheless always remain subject to questioning and modifications. As interpreters, we should not make any claims either to exclusivity or completeness: tentativeness is the endemic property of interpretation, and the coexistence of alternative (or complementary) interpretations is part and parcel of the interpretive framework.

3. How does the communicational approach change the picture? Initial questions revisited

It is time now to demonstrate how the communicational approach, as presented in the former section, can possibly add to our understanding of the initial questions. Let me return to these questions, then, and try to look at them through the conceptual lens that equates thinking with communicating.

Why do children succeed or fail in mathematical tasks? What is the nature and the mechanism of the failure?

Let us return to the *Slope* episode, presented in Figures 1 and 2. We are now going to engage in an activity not unlike that of archeologists who use scarce remnants of an ancient vessel to reconstruct the original whole. If thinking is communicating, then a conversation between two persons is a complex combination of several tightly interrelated, partially overlapping attempts at communication, only some of which are accessible to observers, but all of which influence all the others. What is actually heard is like those available remnants of the ancient vessel and what is added through interpretation are the replacements of the missing parts. The reconstructed elements, although but a product of the archeologist's imagination, turn the scattered pieces into an integrated whole.

Within our present framework, Gur's failure is understood as a failure to communicate. In fact, within the communicational approach this failure should no longer be called 'Gur's'. Although it is true that the boy proves unable to lead an effective dialogue either with his partner or with himself, it is probably also true that this inability is not his inherent property but rather the property, and possibly the product, of the interaction between the two boys. In order to understand this point better, I will have to take a close, detailed look at the way the communication evolves. Scrutinizing the way the mathematical content enfolds will be the first thing to do, but it will not be the only one. In the preceding paragraphs I was talking about tacit factors that may have a considerable impact on the course and effectiveness of discursive interactions. In the attempt to understand the reasons for the lasting ineffectiveness of the communication I will thus have to look at these hidden factors as well. With this goal in mind, I will now use two types of analysis which complement each other, as one of them deals with object-level aspects of communication while the other aims at the meta-level factors. These two methods, called *focal* and *preoccupational* analysis respectively, join the quickly growing set of analytic tools that are being constructed these days by those who believe, like I do, that answers to many stubborn questions about human ways of being in the world can be found in the 'discursive trace' the humans leave behind them. The two specific types of analysis presented below have been developed by Carolyn Kieran and myself while we were grappling with issues such as those that have been raised in this paper.[10]

Focal analysis. Let me first probe deeper in the issue of the effectiveness of communication that comes to the fore the moment cognition is conceptualized in communicational terms. This latter notion, *effectiveness of communication,* may be presented as dependent on the degree of clarity of the *discursive focus* – the communication will not be regarded as effective unless, at any given moment, all the participants seem to know what they are talking about and feel confident that all the parties involved refer to the same things when using the same words. The word *focus* requires explanation. While trying to define this term in our Montreal project, we first thought of it as *the expression used by an interlocutor to identify the object of her or his attention.* Later, because of our awareness of the importance of communication mediating tools, we thought that it would be important to include some indication of *what and how one is attending to* – looking at, listening to, etc. – when speaking or thinking. We decided, therefore, to consider two focal ingredients, *pronounced* and *attended* (for example, in Ari's utterance "Ah, no, the intercept is the zero" ([11b]) the pronounced focus is the words 'the intercept' and the attended focus is the scanning procedure he uses to locate the intercept in the table). We knew, however, that there is more to communication than the pronounced and attended aspects. Whatever is pronounced or seen evokes a whole cluster of experiences, and relates the person to an assortment of statements he or she is now able to make on the entity identified by the pronounced focus. We decided to give the name *intended focus* to this collection of experiences and discursive potentials (in the case of Ari's utterance quoted above, the intended focus is all the statements the boy is likely to make, and all the attended foci he is likely to enact, while using 'the intercept' as a pronounced focus). We can now use these terms to say that the difficulty of human communication stems from the fact that intended focus, which seems to be the crux of the matter, is an essentially private dynamic entity that changes from one utterance to another. This difficulty, however, may often have a straightforward solution: The attended focus can be used as a public exponent of the intended focus, and thus plays a cardinal role in the success of communication.

Let me now apply the focal analysis to the *Slope* episode. It is useful to begin with a closer look at Ari's utterances so as to prepare a contrasting background for Gur's case. The flow of Ari's tripartite focus has been

Utterances	Pronounced	Attended	Intended
[1a], [1b], [11a] [1c], [11b] [1c] [11b]	"the slope" "the intercept" "negative five" "The zero"	**Table intercept**[I]: 1. Find 0 in left column of the table 2. Find the number in the right column of the table corresponding to that 0	The intercept
[3],[5]	"Slope"	**Formula slope**[II]: The coefficient of x in the formula 5x+-5	The slope
Writes: 5x+-5		**Formula intercept**[III]: The free coefficient in the formula 5x+-5	The intercept
[1d], [13],[15], [19]	"Slope"	**Table slope**[IV]: 1. go to the 0 in the left column of the table 2. check the size of the increase between successive numbers in the left column 3. If the increase is 1, then simply find the difference between a number and the one just above it in the right column	The slope

Please note: While looking at the figure, one has to keep in mind that the words *slope* and *intercept* are used in the focal analysis as referring to abstract features of abstract mathematical objects (linear functions), rather than to any kind of symbols. For example, slope is that characteristic of the linear function which finds its 'material' expressions in the coefficient of x in the formula ax+b, in the 'jump' in the y value corresponding to the jump of 1 in the x value, and in the slant of the graph of the function. We say that the slope is represented by all these symbolical means, but is not any of them in particular.

[I] "Table intercept" is an attending procedure for identifying the intercept of a function with the help of a table; It can also be described in structural terms as "The right-column counterpart of the left-column zero".
[II] "Formula slope" is an attending procedure for identifying the slope of a function with the help of a formula
[III] "Formula intercept" is an attending procedure for identifying the intercept with the help of a formula
[IV] "Table slope" is an attending procedure for identifying the slope of a function with the help of a table

Figure 4. *Slope* episode – analysis of Ari's tripartite focus.

charted in Figure 4. Probably the most salient feature of the boy's talk is its being tightly integrated by the intended focus. While the different utterances are built around different pronounced foci, and imply differing attended foci, they all seem to speak either about slope or intercept of the same linear function.

This stability of intended focus justifies comparing Ari's discourse on function *g* to what I once called "actual reality discourses" (Sfard, 2000a), the main characteristic of which is their being about material objects, and their being guided and navigated by actual or imagined pictures of these objects. Indeed, the way the boy uses the function and related notions (such as slope, intercept, specific values of the function) reminds one, in many respects, of the way people speak of, say, trees, chairs, and persons. In Ari's discourse on functions, like in discourses on material things, the object under consideration seems to preserve its identity while its image and its attended aspects are changing from one utterance to another. It is as if Ari was performing a sequence of zoom-ins and zoom-outs from this object (the function) to its particular part (the slope), then to the whole function again, and then to its other particular ingredient (e.g. the intercept). What makes this metaphor of zooming convincing is the ease and confidence with which Ari makes the transitions from one function-related element to another. Another noteworthy phenomenon is the agility with which he moves between different representations: from his well-formed attending procedures for, say, finding the slope in the table, to the one which involves

Ari			Gur		
Pronounced	Attended	Intended	Pronounced	Attended	Intended
[11] [11a] "the slope" [11b] "the intercept" "the zero[II]"	**Table intercept**	**The intercept**			
			[12] "your.."	?	?
[13] [13a] "how many... in between each" [13b]"from zero[II] to"	**Table slope**	**The slope**			
			[14] "slope", "1"	**The reverse of table intercept[I]**	?
[15] [15a] "slope" [15b] "zero[II]"	**Table intercept**	**The intercept**			
			[16] [16a] "slope" [16b] "that zero[II]" "-5"	**Table intercept**	?

[I] We called this attended focus "The reverse of table intercept" because what Gur is looking at may be described as "The left-column counterpart of the right-column zero", and since "The right-column counterpart of the left-column zero" is the table intercept.

[II] The recurring appearance of the word zero is evidently one of the sources of confusion, since there are two zeros in the table (in the first row in the left-hand column and in the second row in the right-hand column). In particular, it is not clear which of the two zeros is referred to by Ari in [15b], since any of them could be used to appreciate the increase in the y-value corresponding to the increase of 1 in the x-value which Ari is looking for. In [14], Gur is evidently looking at the right-hand column zero, and this is why he points to 1, which is this zero's left-hand counterpart.

Figure 5. *Slope* episode – analysis of focus flow.

the formula, and then back to the first one. See, for example, how in [1d], [3], [51, [13], [15], and [19] he shifts his glance back and forth between the expression and the table, while the intended focus, the slope, remains the same. Ari's exclamation in [25] "So we *did* get it right!", made after he extrapolated $g(6)$ from the table and compared it to the $g(6)$ computed with the just constructed formula justifies the claim that in his discourse the word 'function' does not signify either the table or the expression, but rather something that unifies the two. The boy evidently knows quite well what features of the table and what kinds of calculations with the formula correspond to each other. His good sense of the isomorphism between the different symbolic systems makes him able to arrive at the same goal in many different ways, just like having a good sense of a physical object makes one able to imagine many different ways of transferring this object from one position to another. Finally, it is remarkable how Ari keeps confounding the words 'slope' and 'intercept', and how in spite of that he confidently moves on, knowing what to look at and what calculations to make. Once again, this relative immunity of the discursive flow to inadvertent verbal confusions, and Ari's ability to correct himself, are yet another feature characteristic of discourses on things that can be seen or imagined.

For the sake of brevity I will say that Ari's discourse is *objectified* (see also *object-mediated* discourse in Sfard 2000a, 2000b). From now on, I will call this name all those discourses that display features similar to those of Ari's discourse, described above. One look at Gur's part in the conversation suffices to realize that the description does not apply to his discourse. Indeed, the majority of defining features of objectified discourse are missing from Gur's talk. As has been shown by the detailed analysis of focus flow (see Figure 5), the boy cannot move with ease between table and formula, and his attended focus is extremely sensitive to the change of pronounced focus. In fact, most of the time Gur does not show any initiative of his own and while apparently following Ari, he seems to be lacking the consistent intended focus that would keep different utterances together. His interpretations of Ari's statements are ad hoc and rather unrelated to each other. As a result, various symbolic tools – the table, the formula 5x + (–5), the expressions *g*(x), *g*(6), and *g*(10) – function in his talk as self-sustained independent objects, with no evidence whatsoever for a joint intended focus that would turn them, discursively speaking, into 'representations' of different aspects of one thing. The evidence for this disintegration is more than ample. First, Gur openly wonders about the reasons for Ari's attempt to find the formula before calculating *g*(6). He does it not only at the outset, when Ari sets out to construct the formula, but also much later, when he is watching Ari doing the job ([2], [6], [8]). When the formula for *g*(x) is already there, Gur still wonders what it has to do with *g*(6) ([24]). Not only is he ignorant of the connection between the just calculated 5x + (–5) and *g*(6), but he also cannot make sense of Ari's substitution (he puts down on his worksheet as an answer the intermediate result 30 instead of the final 25). The most telling evidence for the lack of object mediation is the way Gur approaches the subsequent task of calculating *g*(10). As if the formula was never there, he assumes the table is governed by a simple proportion: he decides that if '5 is 20, then 10 must be 40' ([34]). One can say that he is unable to zoom-out from *g*(6) or *g*(10) to the function *g*(x). Neither is he able to zoom-in to the intercept and the slope from the formula ([6], [8]) or from the table ([14], [16], [18]).

To sum up, Gur does not have a sustained intended focus of his own. In each of his turns, he is building a new intended focus by constructing his interpretation to what Ari just said. Since he is not guided by his own pre-existing intended focus, his interpretations are highly depended on environmental clues, such as those that expose some fit with the words used by Ari (e.g., when Ari uses the word 'zero', Gur looks at the first zero he can find in the inscription). As a result, the involuntary slips of pronounced focus, which for Ari are but easily correctable momentary lapses, for Gur have rather grave consequences. First, it is Ari who says 'slope' by mistake, but then immediately corrects himself ([11]).[11] Gur continues with the 'slope', even though he applies Ari's explanation for finding the intercept ([14]–[16]). Moreover, as a result of another misunderstanding, he does not choose the correct attended focus. This time, the apparent reason for the error is Gur's mistaken interpretation of Ari's synecdochic use of the expression 'the zero' as a pronounced focus (compare [11], [14], [15], [16]). All this is a compelling evidence of the feebleness of his intended focus and his difficulty with becoming a skillful participant in fully-fledged objectified mathematical discourse.

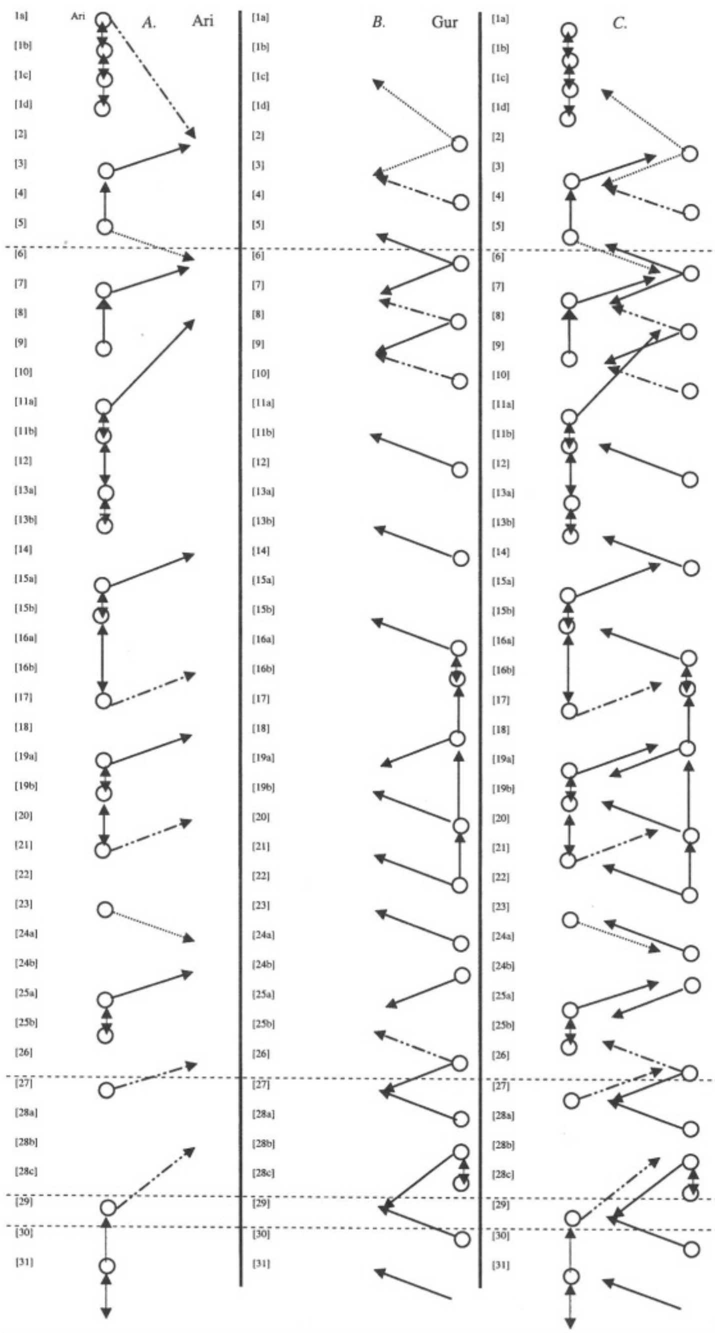
While Ari's private channel seems perfectly focused and continuous, the discourse between the two boys is incoherent. Ari does make several attempts to overcome the incoherence by explicitly pointing to his attended foci.[12] This is what he does, for example, in [13], [15], and [19]. The gesturing, however, does not work. Pointing to the attended focus, *per se,* is not enough to create an adequate intended focus. We can see two possible reasons for the ineffectiveness of Ari's intervention. First, Ari does not really try to coordinate intended and attended foci – he just points to the table or to the formula without specifying the attending procedures. Second, Ari does not probe Gur's understanding. He seems uninterested in Gur's thinking to such extent that he does not even notice Gur's slips of pronounced focus or his erroneous answers. From this point of view, the situation may be quite different for Gur who, so it seems, is keen on keeping the conversation going.

These last claims on the boys' differing attitudes toward the interaction, although plausible, are not yet grounded in a systematic analysis. Carrying out such analysis does seem a worthwhile endeavor, though. Indeed, we may be touching here upon the hidden reasons for the observed communication breaches: The boys' disparate expectations and wishes with respect to the interaction, as well as some interpersonal, mathematics-unrelated, goals and desires that may be preoccupying them as they are talking one to the other – all this is quite likely to interfere with the object-level effectiveness of the exchange. To check this conjecture, let me turn then to *preoccupation analysis.*

Preoccupation analysis. To have a better grasp of what is meant here by this last term, let us recall that interpersonal communication was defined as an attempt to make other people act or feel according to one's intentions. It is important to stress now that there are two types of intentions which may be conveyed through communicative actions. First, there are overt object-level (cognitive) intentions, related to the declared goal of a given activity. In the case of school mathematical discourse, a student may have an immediate object-level goal of solving a mathematical problem that, in turn, is embedded into the long-term goal of learning some new mathematics. In the 'Slope' episode some aspects of these object-level intentions have already been taken care of with the help of focal analysis. The other type of discursively conveyed intentions, which are usually less visible even if not less influential, is related to various aspects of the interaction, and thus has the discourse itself as its object. This latter category, which may be called meta-discursive or meta-level, is wide and multifarious, and it includes, on the one hand, interlocutors' concerns about the way the interaction is being managed and, on the other hand, the weighty, and sometimes quite charged, issues of the relationship between interlocutors. After all, every instance of communication is an occasion for re-negotiating interlocutors' mutual positioning and their respective identities. Different means are usually used by participants for communicating the object-level and meta-level intentions. While the former are best expressed in explicit ways, the latter are likely to reside in forms of utterances and in mechanisms of interaction rather than in their explicit contents. Because of the predominantly covert nature of inter-personal messages, the meta-level intentions conveyed through discourse often remain invisible even to those whom they affect (some interlocutors are more reflective and some other much less so, and thus people may be aware of their own meta-discursive intentions to varying degree; still, the concern about the meta-level is always present and we are always witnessing this coexistence of two agendas: the one related to content, and the other to the way the discourse evolves).

The two categories of discursive intentions, object- and meta-level, seem unrelated and, on the face of it, the latter could be left aside when the cognitive aspect of learning interaction is being investigated. In fact, there is a constant tension between the two types of intentions, if only because of the simple fact that they compete for being the focus. Interpersonal communication is a particularly complex phenomenon in that at any given moment each participant is simultaneously involved in a number of object-level and meta-level activities: in trying to understand the explicit contents of previous utterances and to produce new ones, in monitoring the interaction, in presenting oneself to others the way the person would like to be seen, in engineering one's position within the given group, and so on. Since all these different concerns must be attended to at the same time, it seems a miracle that people are ever up to the task of communicating at all.

Our principal tool in the preoccupational analysis is the *interactivity flowchart* that helps to evaluate the interlocutors' interest in activating different channels and in creating a real dialogue with their partners. We can look upon consecutive utterances in a discourse as endowed with invisible


A. Ari
Ari
B. Gur
C.
[1a]
[1b]
[1c]
[1d]
[2]
[3]
[4]
[5]
[6]
[7]
[8]
[9]
[10]
[11a]
[11b]
[12]
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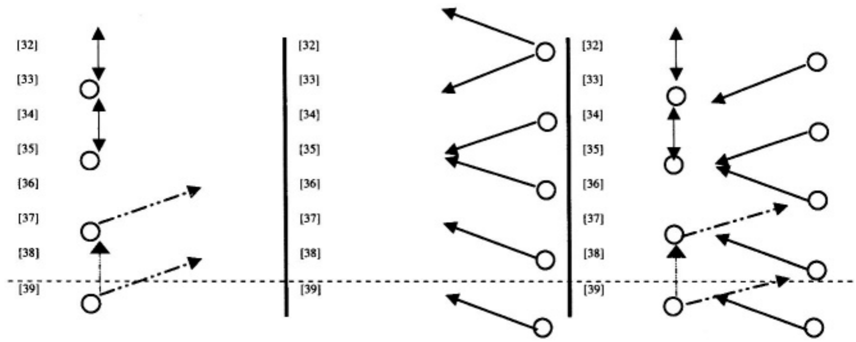

Legend. In the flowchart, the two *personal channels*, namely the respective 'parts' of the two boys, are shown in separate columns, *a* and *b*. The numbers marking the little circles correspond to the numbers of the utterances in the episode transcript.
There are two types of arrows that originate in the different utterances.

– *Reactive* arrow (an arrow which points vertically or diagonally backward/upward): this type of arrow expresses the fact that the source utterance is a reaction to the target utterance;
– *Proactive* arrow (an arrow pointing vertically or diagonally forward/downward): this type of arrow symbolizes the fact that the source utterance *invites* a response, so that the following utterance is expected to be a reaction.

Figure 6. *Slope* episode – Interactivity flowchart.

arrows that relate them to other utterances – those which have already been pronounced and those which are yet to come. These arrows express the participants' meta-discursive wishes: the wish to react to a previous contribution of a partner or the wish to evoke a response in another interlocutor (see an additional explanation in the legend of Figure 6). The conversational organization of these *reactive* and *proactive* arrows would often reveal certain regularities. The recurring forms of reactive and proactive behaviors, in their turn, may help in deciding whether interlocutors are really addressing and interpreting their partners or, in fact, are concentrating on a 'conversation with themselves'. In our Montreal study, interaction analysis has been done with the help of a diagram in which the imaginary arrows mentioned above are made visible.

The interaction flowchart of the *Slope* episode is presented and explained in Figure 6. From this graph one may learn quite a lot about Ari's and Gur's attitudes toward the interpersonal communication. A detailed analysis shows that Ari may be not genuinely interested in the interaction. He does not initiate any of the exchanges and he does not respond to many of Gur's proactive utterances. During the whole two-minute long episode he makes only two or three proactive statements (see [1], [5], [23]), all of which are meta-discursive (in comparison, Gur makes nine proactive statements, eight of them formulated as object-level questions). In fact, these utterances do not even seem to be genuinely proactive: After asking his questions Ari does not wait for an answer and makes it clear that he is eager to finish the job of explaining as quickly as possible. It is quite obvious that he never gives much thought to Gur's answers and does not really care whether his partner is sincere when he says he does understand. So much for his uninitiating attitude. Ari's unresponsiveness expresses itself in his indifference toward Gur's attempts to create an exchange. Rather than answering Gur's proactive utterances, he continues his own line of talk. All along the way he ignores Gur's questions and requests for explanations (see [4]–[6], [11]–[13] and, above all, [29]–[40], where Gur tries to explain his thinking), letting signs of Gur's incomprehension and distress go unnoticed.[13]

One look at the flowchart 6*B* reveals that Gur is still very much interested in interaction, and truly dependent on it. This is evidenced by the profusion of both reactive and proactive utterances, revealing his *initiating* and *responsive* attitude. As a result, the contrast between the two boys' discursive behaviors is now even more pronounced than before.

All this means that while Ari is keen on keeping his thinking from distractions, Gur is preoccupied with the exchange of ideas. Indeed, many of Ari's utterances take the form of a dialogue with himself (see, e.g., [1], [11], [19], [25]), whereas Gur is obviously addressing his partner. In this situation, it is not surprising that while there are long stretches of continuity along Ari's private channel (see 6*a;* in particular, notice [1]-[21] and [29]–[35]), Gur's private channel is practically non-existent (see 6*B*). It is also interesting to note that Ari's private channel has a distinct argumentative structure: Even when talking to Gur, he often sounds as if he is arguing with himself. Constant self-monitoring is one of the distinctive features of Ari's discursive actions. This is how he is able to correct his own mistakes and double-check his own solutions.[14] It is obvious that there is a *hidden part* to his discourse, in which Ari quickly performs the recursive computation mentally (see, in particular, utterance [25] which seems to sum-up such inner computation).[15]

It is also noteworthy that Ari tries to curb the discourse and, at the same time, to conceal this fact with different camouflaging techniques. Keen to protect his private channel from distractions, and aware of the fact that he is not playing according to expectations, Ari tries to soften his unsociable image by lip-service utterances. Thus, from time to time he acknowledges Gur's contributions ([15]: “Hum?”, [17], [21], [27]: “Yeah”), but it is obvious that his curt “Yes”es and “Hum?”s are only ceremonial and, in fact, do not express a genuine concern for what Gur is saying. Indeed, in all the above cases it turns out that Gur's utterances to which Ari said "Yeah" were in fact either incorrect or unfocused, and they were wrong in such a way that it should have been immediately obvious to Ari, had he really listened. Gur, in his turn, has a wide assortment of *discourse-spurring* and *face-saving* techniques. Thus, for example, he uses them to mask his misunderstandings rather than deal with them (see, e.g., his "yeah, yeah" in [12] and in [24], and his "Oh, that makes sense" in [28] when, in fact, nothing seems to make sense to him). The fact that in the first sub-episode he begins questioning Ari without even trying to solve the problem himself shows that he puts up with his partner's superiority and does not really trust his own mathematical capacities.

Let me now put together the focal and preoccupational analyses in an attempt to see what this combined outlook tells us about Ari's and Gur's learning. The first thing to say is that once thinking has been conceptualized as communicating, the dynamic, ever changing and extremely context-sensitive dimension of thinking comes to the fore. Gur's ineffective actions are no longer seen as a direct result of some stable, context independent problem-solving 'scenarios' stored in his mind and likely to repeat themselves in any other situation involving a similar task; rather, they are regarded as a chain of momentary decisions made in immediate spontaneous reaction to his partner's utterances. Since Gur does not seem to have either his own clear way of proceeding or a coherent interpretation of Ari's discursive actions, his responses are globally incoherent even if they sometimes make an impression of being locally appropriate. This uncontrolled spontaneity also accounts, at least partially, for the communication breaches that plague this conversation.

The detailed picture of the incoherent conversation provided by the focal analysis is to be understood as containing an observer's interpretive reconstruction of the participants' thinking, that is, of the 'dialogues' that take place along interlocutors' private channels. On the basis of this analysis, and in tune with what has been said above, there is a substantial difference between Ari's and Gur's thinking: While Ari is focused on self-communication and follows his own discursive line on the expense of inter-personal communication, Gur privileges his interaction with Ari to the almost total neglect of his private channel. Figuratively speaking, Gur gives up his own thinking in the attempt to interpret Ari's dialogue with himself.

If so, why does this seemingly serious effort have such an unhappy, unsatisfactory ending? A plausible answer to this question comes with the results of the preoccupational analysis. While Gur's interest in Ari's thinking is unquestionable, it is counterpoised by another, not less pervasive concern: Gur's concern about his positioning and about face-saving. It may well be that the fear of appearing as unable and unworthy prevents the boy from pursuing his wish to interpret his partner in a consistent and eventually successful way.

This two-dimensional analysis brings about a rather consequential change in our understanding of the mechanism of failure. What was seen so far as an almost direct derivative of one's personal skills and, more often than not, as an outcome of the person's given 'mathematical potential', is now regarded as a product of a collective action. The analysis has shown that when two people are engaged in communication, it takes the two to produce a failure. Ari did not contribute to Gur's predicament deliberately, but he did contribute nevertheless, if only by his presence and his insensitivity to Gur's needs. Were Gur working alone, or were he assisted by a different partner, he might have acted in a different and much more successful way. All this leads to a reasonable doubt about the soundness of research in which cognition and cognitive skills are treated as stand-alone factors that can be studied in isolation from other aspects of the situation. Not less importantly, it makes us suspicious of the common practice of trying to establish children's 'mathematical potential' on the basis of isolated, superficially evaluated, incidents of learning.

The consequences of the alternative theoretical interpretation do not end in words. First and foremost, the participationist framework that stresses change and distrusts permanent labeling brings a more hopeful picture of learning. It says, among others, that the teacher should not be too eager to project from a student's past success or failure into his or her future performance. Since permanent labeling has the quality of a self-fulfilling prophecy, the importance of this caveat can hardly be overestimated. Further, the analysis of the *Slope* episode made it abundantly clear that the beneficial effects of students' collaborative problem solving cannot be taken for granted. If students' interactions are to enhance learning, the communicative skills of the students must be taught. Careful analyses of diverse classroom episodes can be trusted to provide a good idea about what could be done in order to make mathematical communication, and thus mathematical learning, more effective. From this single episode we can already tentatively conclude that interlocutors should probably learn to make their attended foci explicit, that learning alone may sometimes be more effective than learning with others, and that one should be very careful while deciding who should be a given child's partner in collaborative learning.

What should count as 'learning with understanding'?

Let us return now to Noa's case (see Figure 3). The former attempt to interpret and explain the brief exchange raised questions about the meaning of the term *learning-with-understanding* and left us uncertain as to whether Noa's apparent change of mind with respect to the existence of 'the biggest number' was a case of meaningful learning. I will claim now that an alternative interpretation may be provided by putting the analysis of the episode in terms of discursive uses of words, and by a close inspection of the discursive mechanism that compels the girl to change this use.

Before I do this, however, let me elaborate on the idea of *objectified* discourse that appeared in the analysis of the *Slope* episode and may now prove helpful in the case of Noa and Rada's conversation on numbers. In the *Slope* episode, I described Ari's discourse on function as *objectified* because the boy was talking about functions as if these were some real, self-sustained objects. Looking at the way Ari spoke about functions, it is reasonable to say that he experienced the word function as referring to a well-defined, self-sustained entity, existing independently of the discourse itself. This property of his intended focus can be induced from the fact that all along the way Ari is making swift, smooth back-and-forth transitions from one attended focus (the table) to another (the expression 5x+(–5)) while preserving the same pronounced focus (function *g*). He is thus using the different symbols – the table, the expression, and probably also a graph, which is not presented but can be imagined – as if all these symbols were *representations* of one specific object. The special property of this objectified discourse is that it subsumes several independently created discourses, turning them into discourses 'about the same thing' and making it possible to express in the new language everything that can be said, alas in a different way, in any of the subsumed discourses. For instance, discourse about functions subsumes discourses about graphs and the discourse about algebraic expressions. In this subsuming discourse, the sentence "The intercept of this function is –5" replaces, simultaneously, the sentence "This straight line crosses the y-axis at y = 5" in the discourse on graphs, and the sentence "The free coefficient in this formula is –5" in the discourse on expressions. This subsuming effect is clearly visible in Ari's discursive actions, but can hardly be found in what Gur is saying. Being but a beginner in the discourse on functions, Gur has a visible difficulty with finding parallels between graphs and expressions. What for Ari constitutes "two different representations of the same function", for Gur remains a pair of unrelated marks on paper. As long as Gur's use of the different symbols remains unobjectified, his difficulty with following Ari's swift discursive moves is quite understandable.

Back to the *Biggest Number* episode, I will now argue that much of what is happening between Noa and Rada may be explained by the fact that unlike the teacher, the girl uses the number-related words in an unobjectified way. The term 'number' functions in Noa's discourse as an equivalent of the term 'number-word', and such words as *hundred* or *million* are things in themselves rather than mere pointers to some intangible entities. If so, Noa's initial claim that there is the biggest number is perfectly rational. Or, conversely, the claim that there is no biggest number is inconsistent with her unobjectified use of the word 'number': After all, there are only so many number-words, and one of them must therefore be the biggest, that is, must be the last one in the well ordered sequence of numbers (with the order of the sequence determining the relations 'bigger than' or 'smaller than' among its elements). Moreover, since within this type of use the expression 'million and one' cannot count as a number (but rather as a concatenation of numbers), the possibility of adding one to any number does not necessitate the non-existence of the biggest number.

Like in the case of Ari and Gur, the communication between Noa and Rada is obstructed by the fact that one of the interlocutors uses central notions in the objectified way while the other fails to do so. Unlike in the former case, however, the meta-discursive behavior of the interlocutors is now quite different, and their efforts to improve the communication are genuine enough to be ultimately quite successful. Indeed, this time, *both* interlocutors seem interested in aligning their positions. The teacher keeps repeating her question about the existence of 'the biggest number', thus issuing meta-level cue signaling that the girl's response failed to meet expectations. In order to go on, Noa tries to adjust her answers to these expectations, and she does it in spite of the fact that what she is supposed to say evidently does not fit with her use of the words *the biggest number*. The requirement of the change exposes the girl to possibilities she has not considered. Moreover, at the present stage, she does not have means to deal with the problem. Although it must be quite obvious to her that the required change has to be somehow related to the fact that one can always add one to any number, the relation between this fact and the claim about the non-existence of the biggest number cannot possibly be clear. In spite of this, the girl is evidently willing to comply with the rules of the game imposed by the more experienced interlocutor.

Thus, it looks like Noa's principal effort is to fulfill the teacher's discursive expectations. Her focus is at the communication rather than on trying to figure out for herself what is wrong with her use of numbers. One may say that she is trying to understand 'through the other' before she is going to build her own understanding. Without questioning, she looks upon the teacher's discourse as superior to her own, and as the 'correct' one. Her lack of confidence in her own discursive ways expresses itself in the last question: she already gave a satisfactory answer, now she tries to relate this answer to the other things that have been said in the encounter, thus attempting to reconstruct the teacher's reasoning.

Concerned about the issue of learning with understanding some people may say that the girl's modifications of her answers were made for all the wrong reasons: She was simply keen to please the teacher and was guessing her intentions. To attain this goal, she was playing a rational game, but her rationality was not of the kind traditional teachers would like to see. It was the rationality of guessing from meta-discursive cues rather than inferring from object-level relations. Adherents of the principle of learning-with-understanding are thus likely to join Cazden (1988) in criticizing this kind of situation as one in which established patterns of communication give but "the illusion that learning is actually occurring" (p. 48). This implies that the true learning – the one they use to call "with understanding" – should have followed a different path.

The question, however, is whether such alternative path is always possible. In Noa's case one can hardly think of any other, exclusively rational object-level route toward the eventual objectification of her discourse on numbers. To put it in a more traditional language, it is difficult to see how the child could take a more 'meaningful' path toward re-conceptualization of the notion of number. In order to change her discursive habits and dispositions, she had to undergo an experience of incomprehension – of seeing alternatives to the only possibility of which she was aware when starting the conversation. The meta-level means employed by the teacher to show her such possibilities could not be replaced with any direct object-level considerations. Indeed, Noa's discourse was perfectly coherent, and there were no contradictions between her use of number as a designated word and her claim that one of the numbers must therefore be the biggest. Thus, contrary to the traditional cognitivist analysis I have presented in the beginning, Noa's case cannot count as one of cognitive conflict stemming from holding several incompatible beliefs about number. Noa's eventual dilemma had its roots in an *inter*-discursive clash, not in *intra*-discursive contradiction. In a case like this, one has no chance to modify one's discursive habits on her own. In order to change them, one has to be led outside her own discourse by others. Only then can the conflict necessary to create the learning-engendering experience of incomprehension eventually arise.[16]

More generally, what we have seen in Noa's case may well be one of the principal forms of learning we all employ throughout our lives. It is thanks to the intricate combination of object-level and meta-level tuning to our interlocutors that we make our way toward better communication and perfect our participation in specialized discourses. Participants come to discourses with their own, possibly idiosyncratic, uses of words and their own expectations with regard to the rules of the game that is to be played. The actual shape of the exchange will be the resultant of the interaction between the expectations of all the interocutors. Of course, not all of them would influence the rules of the game to the same extent. In any specialized discourse there is usually a dominant, authoritative, voice that informs the rules more than all the others. In the classroom, the lead belongs to the teacher. Only too often, in order to learn, students have to follow the teacher before having a firm grasp of the new discourse into which they are thus led. This kind of learning is not likely to be valued by the followers of the traditionally understood principle of learning-with-understanding. And yet, this kind of learning cannot be replaced with any other. This impossibility is inherent in the claim that all our thinking is essentially social, and this is the deep meaning of Vygotsky's famous statement:

> Any function in child's cultural development appears twice, on two planes. First, it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category. (Vygotsky, 1931/1981, p. 163)

There is yet another, more general, implication of the present example. The learning that occurred in the just analyzed episode is no longer viewed as a result of *cognitive* conflict. If anything, the situation we have been witnessing can be described as one of *discursive* conflict, an occurrence quite different from that of being exposed to what looks like well-justified mind independent facts that contradict each other. Indeed, while the concept of cognitive conflict implies one's ability to *rationally* justify two colliding claims-about-the-world, the notion of discursive conflict stresses the clash of habitual uses of words, which is an inherently discursive phenomenon. In our present case, we could observe a conflict between the two interlocutors' discursive uses of the words 'number' and 'bigger number'. While aware of the fact that the teacher was applying these terms in a way quite different from her own, Noa was ignorant of the reasons for this incompatibility. In this case, therefore, the girl had to *presume* the superiority of her teacher's use in order to have any motivation at all to start thinking of rational justification for a change in her own discursive habits.

Thus, perhaps the most dramatic difference between the cognitivist and communicational interpretations of the *Biggest Number* episode lies in their respective visions of breaches-in-understanding that motivate learning. The concept of cognitive conflict assumes that the learner is in a constant quest after the truth about the world, and whatever new knowledge is acquired, it is the result of this learner's attempts at adjusting her understanding to the externally given, mind independent aggregates of facts and ideas. Clearly, this kind of endeavor could be pursued, at least in theory, without the mediation of other people. In contrast, the idea of discursive conflict stresses the need for communication as a principal drive for our cognitive actions, and points to the wish to adjust one's discursive uses of words to that of other people as one of the main motives for learning.[17]

4. AFTERWORD: CHANGING WORLDS WITH WORDS

In this article I tried to demonstrate the power of the idea of thinking-as-communicating to bring a valuable change in our vision of learning in general, and mathematical learning in particular. This change, it seems, is not just a change in words. Together with the new words come new ideas about what goes into learning and what should be done to promote this learning.

In the analyses above I did my best to show that the communicational approach, based on the learning-as-participation metaphor, does much more than add new information. What I hope to have shown is that this special outlook would often change the picture in such a way that even the 'old' parts of the image – the parts that could be seen before – acquire a new meaning. The overall transformation that occurred in our vision of the two classroom scenes as a result of communicational re-interpretation was quite remarkable. What until now was seen as a function of stable or semi-stable 'possessions' and dispositions of the individual became a dynamic property of human interactions, one that does not have an existence beyond these interactions. Teacher's decisions that, so far, were likely to be seen as somehow out of tune with the principle of learning-with-understanding, have been rehabilitated and promoted to the rank of helpful and valuable, if not outright indispensable. Above all, the hitherto ignored aspects of learning have been elicited and ascribed principal importance.

All this said, let me stress once again that the communicational approach should be seen as a complement rather than as a replacement for the more traditional outlooks. My present preference for the communicational framework and for the underlying participation metaphor does not mean rejection of the other metaphor, nor an attempt to undermine this other metaphor's valuable contribution to our understanding of learning in general, and learning mathematics in particular. In my opinion, the only reasonable conclusion from the recent criticism of the more traditional cognitivist approaches is that the manner the acquisition metaphor finds its way into scientific concepts has to be refined, and its entailments must be carefully rethought. Rather than rejecting the metaphor as such, one should cleanse the discourse on learning from its unhelpful, undesirable entailments.

This 'reconciliatory' declaration may, of course, raise some eyebrows. In the light of the rather far-reaching changes in the vision of learning entailed by the change of metaphor, how can one keep saying that the different metaphors are 'complementary' rather than incompatible? Of course, the claim about complementarity cannot be true unless the ontological and epistemological foundations of the traditional framework undergo a certain revision, and the basic notions are reconceptualized. The kind of change I am talking about is, in a sense, analogous to the one that was necessary in mathematics in order to enable the co-existence of Euclidean and non-Euclidean geometries within one consistent and surprisingly useful system; or to the change in physics, that enabled two seemingly incompatible visions of sub-atomic phenomena to be used intermittently, depending on the questions asked. In the study of human mind, like in geometry, it must be understood that the basic assumptions on which this whole framework rests are not about what the world 'really is' but only about how the world may be thought of, in certain situations. Of course, incommensurable outlooks cannot be applied to the same phenomenon at the same time, just like wave and corpuscular theories of light cannot be combined in one answer to the same question. And yet, I can think of many situations where it would be reasonable to try both these approaches in an attempt to find which one would provide a more helpful solution of the problem at hand. In this paper I was trying to show that in such conceptual 'competition', the communicational approach should be considered as a serious candidate. Our wish to model life in scientifically simplified ways with models that, nevertheless, look like life itself, is inherently insatiable; and yet, along with the numerous frameworks available to the students of the human mind these days, the one that equates thinking with communicating may have something important to offer.

NOTES

1. The episode is taken from the research project in Montreal, directed by Carolyn Kieran and myself, since 1993. The aim of the 30 session long teaching sequence produced for the sake of the study was to introduce the students to algebra while investigating their ways of constructing algebraic concepts and testing certain hypotheses about possible ways of spurring these constructions. The present episode is taken from the

21st meeting. More information on the study, as well as another outlook at the present episode, may be found in Kieran and Sfard (1999), Sfard and Kieran (2001 a, 2001b).

2. That the expectation of proportionality is a well-known phenomenon has been evidenced lately by the following episode in the popular TV series addressed to a young audience, *Friends.* A person tries to prevent an 18-year-old boy from marrying a 44-year-old woman. He says: "She is so much older than you are. And think about the future: when you are 36, she will be 88". "Yeah, I know", says the boy.
3. Quite a number of units of analysis among those proposed by representatives of different sociocultural schools seem to be good candidates for the type of study required by participationists. Among the most widely known and applied are *activity,* the unit proposed by activity theorists, *discourse* or its segments, the unit suggested by discursive psychologists, and *practice,* introduced by those among participationists who are most strongly oriented toward sociological issues.
4. It is interesting to note that this seemingly 'factual' statement is an object of fierce debate between the traditional cognitivists and the adherents of the sociocultural approach. The controversy is often framed in the language of transfer of learning: While acqusitionists' belief in the possibility of far-reaching transfer remains firm in spite of rather meager empirical evidence, participationists either deny such possibility or simply say, as Lave (1988) did, that the concept of transfer is fundamentally misconceived. Indeed, a consistent follower of a participationist framework must realize, sooner or later, that the idea of transfer, which implies 'displacement' of certain mental entities, simply does not fit with the participationist conceptualization of learning (see also the ongoing debate on transfer in *Educational Researcher,* e.g. Brown et al., 1989; Anderson et al., 1996; Greeno, 1997; Sfard, 1998; Cobb and Bowers, 1999).
5. This definition of learning was proposed by Lave and Wenger, 1991. 'Cognition at work' began to attract researchers' attention already in the 1970s and it has been turning recently into a favorite theme of study for those interested in learning (see e.g. Engstrom and Middleton, 1996; http://www.helsinki.fi/~jengestr/activity/1.htm). Many of these studies may appear as simply recording what and how people do in workplaces. However, if one accepts Wenger's (1998) definition of practices as histories of learning, then doing and learning become practically indistinguishable.
6. As noted by many, Bruner (1990) among them, the question 'nature-or-nurture' is probably ill-posed. It has its roots in what may be called 'the hardware-first fallacy', the conviction that whenever a physiological difference is found between two groups of people, this difference may be held responsible for the differences in these two groups' behavior. These days it is already clear that the uni-directional causal vision of the relation between biological and cognitive factors has little grounding. Recent findings have shown that human activities, rather than being determined by a pre-formed neural system, are partly responsible for this system's structure and functioning. It seems, therefore, that human evolving culture perpetuates itself not only by affecting human minds but also by changing their brains, with the processes of change happening on both phylogenetic and ontogenetic levels.
7. The quote is taken from a text that was first presented as a conference talk in 1992.
8. One can distinguish between two different trends in the research focusing on discourse, only one of which views communication as truly central, if not outright the same as, cognitive processes. The other, less radical trend reflects the interest in communication as an *aid* to learning rather than as an object of learning in itself.
9. To bar an undesirable entailment of the metaphor, let me immediately add that unlike unanimated physical objects, people can – and usually do – play the double role of

actors and observers. If so, meta-discursive rules often become an object of reflection and thus also of regulation.

10. For the scarcity of space, focal and preoccupational analyses will be presented here very briefly. Of necessity, the all-important discussion of their epistemological foundations will be completely omitted. For more details see Sfard and Kieran, 2001a, Sfard, 2000b.
11. Later, for a moment, Ari does show a sign of absentmindedness when he overlooks Gur's mistake and repeats 'slope' when he really means intercept – see in [15] and [17] his unjustified confirmation of what Gur has said in [14]. The question can rightly be asked, how we know that Ari's intended focus was intercept, and not slope. We can be quite sure of it for at least two reasons. First, the number both boys point to is the intercept. Second, when Gur asks Ari later ([18]) how they are "supposed to get the other ones", Ari immediately answers with a prescription for finding the slope, showing, therefore, that slope is 'the other one' – namely different from what they have just found.
12. It is noteworthy that unlike Gur, who is pointing all the time in both episodes, Ari only points for the sake of interpersonal communication, and he never makes any movements when conversing 'with himself'.
13. For example, in [24], after responding to Ari's explicit question with an assurance that he had understood his partner's former explanations, Gur asks: "So what is g(6)?". From here it is immediately obvious that he can see no connection between looking for the formula of a function and calculating its particular value, g(6) (this interpretation finds its further confirmation in the way Gur proceeds to find out g(10)). In this way, Gur makes it abundantly clear that his "yeah, yeah" in response to Ari's "Do you understand?" is but a face-saving device, and that in fact he has no inkling of what has been going on in the discourse up to this point. Ari, however, does not seem to notice his partner's predicament.
14. In [11] he 'undoes' his own slip of the tongue because he sees that what he said does not fit with what he attends to. Similarly, he corrects himself in [29]. He also verifies his own solutions, and he does it spontaneously. The most telling example, from this point of view, is his statement "So we *did* get it right" in [25], which he enthusiastically makes after computing $g(6)$. Obviously, he has criteria according to which to judge the correctness of the result. Although he doesn't say so, the only possible way for him to assess the result is to compare it to the one which may be obtained in another way; this other way can only be the recursive pattern he detected earlier in the table – the increase of the values in the right-hand column by 5 each time.
15. The interesting thing is that if he went to such trouble to find the formula, finding the value of the function with the formula must be for him the 'canonical method'. Erna Yackel and Paul Cobb (Yackel and Cobb, 1996) would say that answering the question about a concrete value of a function by using the formula is a socio-mathematical norm Ari accepted.
16. This is the way in which Noa might ever be able to overcome the learning paradox: She had to know what she was supposed to understand in order to understand this. Indeed, how could the girl understand new discursive use of the words 'number', 'bigger than', 'million and one', 'the biggest number' without being first exposed to this new use? She was not going to invent this use herself! Here is where the expecting/verifying zig-zag came into play. It is thanks to the fact that her communication with others broke down, exposing the inappropriateness of her discursive habits, that she was compelled to revise her use of the word 'number'.

17. This vision of the way in which communication breaches spur learning provides an answer to the dilemma posed by Smith, diSessa, and Rochelle (1993) who were perhaps the first writers to cast doubt on the idea of cognitive conflict. The authors wondered how resolution of a cognitive conflict could be possible at all:

> As cognitive competition, [this idea] cannot explain why expert ideas win out over misconceptions. The rational replacement of one conception with another requires criteria for judgment. As knowledge, those criteria must be constructed by the learner, and neither confrontation nor replacement explain the origins of such principles for choosing concepts, crucial data, or theories. (p. 126)

When the idea of cognitive conflict is replaced by the notion of discursive conflict, the dilemma seems to disappear. If the need for good communication is seen as a principal drive for learning and change, then student's basic readiness to yield to what they regard as expert use of words is the functional counterpart of the 'criteria for judgement' mentioned by Smith, diSessa, and Rochelle.

REFERENCES

Anderson, J.R., Reder, L.M. and Simon, H.A.: 1996, 'Situated learning and education', *Educational Researcher* 25(4), 5–11.

Bateson, G.: 1973, *Steps to an Ecology of Mind,* Frogmre, St. Albans: Paladin.

Bauersfeld, H.: 1995, 'Language games' in mathematics classroom: Their function and their effects', in P. Cobb and H. Bauersfeld (eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures,* Lawrence Erlbaum Associates, Hillsdale, NJ, pp. 271–292.

Bourdieu, P.: 1999, 'Structures, *Habitus,* practices', in A. Elliot (ed.), *The Blackwell Reader in Contemporary Social Theory,* Blackwell, Oxford, UK, pp. 107–118).

Bouveresse, J.: 1999, 'Rules, dispositions, and the *Habitus*', in R. Shusterman (ed.), *Bourdieu: A Critical Reader,* Blackwell, Oxford, UK, pp. 45–63.

Brown, J.S, Collins, A. and Duguid, P.: 1989. 'Situated cognition and the culture of learning', *Educational Researcher* 18(1), 32–42.

Brownell, W.A.: 1935, 'Psychological considerations in the learning and teaching of arithmetic', in *The Teaching of Arithmetic: Tenth Yearbook of the National Council of the Teachers of Mathematics.* Columbia University Press, New York.

Bruner, J.: 1983, 'The acquisition of pragmatic commitments', in R. Golinkoff (ed.), *The Transition from Prelinguistic to Linguistic Communication,* Lawrence Erlbaum Associates, Hillsdale, NJ, pp. 27–42.

Bruner, J.: 1986, *Actual Minds, Possible words,* Harvard University Press, Cambridge, Massachusetts.

Bruner, J.: 1990, *Acts of Meaning,* Harvard University Press, Cambridge, Mass.

Davis, R.: 1988, 'The interplay of algebra, geometry, and logic', *Journal of Mathematical Behavior* 7, 9–28.

Cazden, C.: 1988, *Classroom Discourse,* Heinemann, Portsmouth, NH.

Cobb, P.: 1998, 'Learning from distributed theories of intelligence', *Mind, Culture, and Activity* 5(3), 187–204.

Cobb, P. and Bowers, J.: 1999, ‘Cognitive and situated perspectives in theory and practice’, *Educational Researcher* 28(2), 4–15.

Cobb, P., Wood, T. and Yackel, E.: 1993, ‘Discourse, mathematical thinking, and classroom practice’, in E. Forman, N. Minick and A. Stone (eds.), *Contexts for Learning, Sociocultural Dynamics in Children’s Development,* Oxford University Press, New York, pp. 91–119.

Cole, M.: 1988, ‘Cross-cultural research in the socio-historical tradition’, *Human Development* 31, 137–151.

Cole, M.: 1995, ‘Socio-cultural-historical psychology, some general reamrks and a proposal for a new kind of cultural-genetic methodology’, in J.V. Wertsch, P. del Rio and A. Alvarez (eds.), *Sociocultural Studies of Mind,* Cambridge University Press, Cambridge, Massachusetts, pp. 187–214.

Cole, M.: 1996, *Cultural Psychology.* The Belknap Press of Harvard University Press, Cambridge, Massachusetts.

Edwards, D.: 1993, ‘But what do children really think? Discourse analysis and conceptual content in children’s talk’, *Cognition and Instruction* 11(3and4), 207–225.

Edwards, D.: 1997, *Discourse and cognition,* Sage, London.

Engestrom, Y. and Middleton, D.: 1996, *Cognition and Communication at Work,* Cambridge University Press, Cambridge, Massachusetts.

Fischbein, E.: 1989, ‘Tacit models and mathematical reasoning’, *For the learning of mathematics* 9(2), 9–14.

Fischbein, E., Deri, M., Nello, M.S. and Marino, M.S.: 1985, ‘The role of implicit models in solving verbal problems in multiplication and division’, *Journal for Research in Mathematics Education* 16, 3–17.

Forman, E.: 1996, ‘Forms of participation in classroom practice, Implications for learning mathematics’, in P. Nesher, L. Steffe, P. Cobb, G. Goldin and B. Greer (eds.), *Theories of Mathematical Learning,* Lawrence Erlbaum Associates, Hillsdale, NJ, pp. 115–130.

Forman, E. and Larreamendy-Joerns, J.: 1998, ‘Making the implicit explicit: Classroom explanations and conversational implicatures’, *Mind, Culture, and Activity* 5(2), 105–113.

Garfinkel, H.: 1967, *Studies in Ethnomethodology,* Polity Press, London.

Goffman, E.: 1974, *Frame Analysis, An Essay on Organization of Experience.* Northeastern University Press, Boston, MA.

Greeno, J.G.: 1997, ‘On claims that answer the wrong question’, *Educational Researcher* 26(1), 5–17.

Harre, R. and Gillett, G.: 1995, *The discursive mind,* Sage Publications, Thousand Oaks.

Hiebert, J. and Carpenter, T.P.: 1992, ‘Learning and teaching with understanding’, in D.A. Grouws (ed.), *The Handbook of Research on Mathematics Teaching and Learning,* Macmillan, New York, pp. 65–100.

Kieran, C. and Sfard, A.: 1999, ‘Seeing through symbols. The case of equivalent equations’, *Focus on Learning Mathematics* 21(1), 1–17.

Kilpatrick, J.: 1992, ‘A history of research in mathematics education’, in D. Grouws, (ed.), *Handbook of Research on Mathematics Teaching and Learning,* Macmillan, New York, pp. 3–38.

Krummheuer, G.: 1995, ‘The ethnography of argumentation’, in P. Cobb and H. Bauersfeld (eds.), *The Emergence of Mathematical Meaning, Interactions in Classroom Cultures*, Erlbaum, Hillsdale, NJ, pp. 229–269.

Lampert, M.: 1990, 'When the problem is not the question and the solution is not the answer, Mathematical knowing and teaching', *American Educational Research Journal* 27, 29–63.

Lampert, M. and Blunk, M.L. (eds.): 1998, *Talking Mathematics in School, Studies of Teaching and Learning,* Cambridge University Press, Cambridge, UK.

Lampert, M. and Cobb, P., in press, 'White Paper on Communication and Language for Standards 2000 Writing Group', in J. Kilpatrick, Martin, G. and Schifter, D. (eds.), *A Research Companion for NCTM Standards,* National Council for Teachers of Mathematics, Reston, VA.

Lave, J.: 1988, *Cognition in Practice,* Cambridge University Press, Cambridge.

Lave, J. and Wenger, E.: 1991, *Situated Learning, Legitimate Peripheral Participation,* Cambridge University Press, Cambridge.

Leontiev, A.N.: 1930, 'Studies in the cultural development of the child. II. The development of voluntary attention in the child', *Journal of Genetic Psychology* 37, 52–81.

Levinson, S.: 1983, *Pragmatics,* Cambridge University Press, Cambridge.

Luria, A.R.: 1928, 'The problem of the cultural development of the child', *Journal for Genetic Psychology* 35, 493–506.

Macnab, D.: 2000, 'Raising standards in mathematics education, values, vision, and TIMSS', *Educational Studies in Mathematics* 42(1), 61–80.

Mantovani, G.: 2000, *Exploring Borders, Understanding Culture and Psychology,* Routlege, London.

Markovits, Z., Eylon, B. and Bruckheimer, M.: 1986, 'Functions today and yesterday', *For the learning of mathematics* 6(2), 18–24.

Mayer, R.E.: 1983, *Thinking, problem solving, cognition,* W.H.Freeman and Company, New York.

Morgan, C.: 1996, 'The language of mathematics, Towards a critical analysis of mathematical text', *For the learning if mathematics* 16(3), 2–10.

NCTM (National Council of Teachers of Mathematics): 2000, *Principles and standards for school mathematics,* NCTM, Reston, VA.

O'Connor, M.C.: 1998, 'Language socialization in the mathematics classroom. Discourse practices and mathematical thinking', in M. Lampert and M. Blunk (eds.), *Talking Mathematics, Studies of Teaching and Learning in School,* Cambridge University Press, Cambridge, NY, pp. 17–55.

Schmidt, W.H., McKnight, C.C., Cogan, L.S., Jakwerth, P.M. and Houang, R.T.: 1999, *Facing the Consequences – Using TIMSS for a Closer Look at U.S. Mathematics,* Kluwer Academic Publishers, Dordrecht, The Netherlands.

Schutz, A.: 1967, 'The problem of social reality', in M. Natanson and H.L. van Breda (eds.), *Collected Papers,* Martinus Nijhoff, The Hague, vol. 1.

Sfard, A.: 1997, 'The many faces of mathematics, Do mathematicians and researchers in mathematics education speak about the same thing?', in A. Sierpinska and J. Kilpatrick (eds.), *Mathematics Education as a Research Domain, A Search for Identity,* Kluwer Academic Publishers, Dordrecht, Vol. 2, pp. 491–512.

Sfard, A.: 1998, 'On two metaphors for learning and on the dangers of choosing just one', *Educational Researcher* 27(2), 4–13.

Sfard, A.: 2000a, 'Symbolizing mathematical reality into being, How mathematical discourse and mathematical objects create each other', in P. Cobb, K.E. Yackel and K. McClain (eds.), *Symbolizing and Communicating, Perspectives on Mathematical Discourse, Tools, and Instructional Design,* Erlbaum, Mahwah, NJ, pp. 37–98.

Sfard, A.: 2000b, 'Steering (dis)course between metaphor and rigor: Using focal analysis to investigate the emergence of mathematical objects', *Journal for Research in Mathematics Education* 31(3), 296–327.

Sfard, A.: 2000c, 'On reform movement and the limits of mathematical discourse', *Mathematical Thinking ami Learning* 2(3), 157–189.

Sfard, A. and Kieran, C.: 2001a, 'Preparing teachers for handling students' mathematical communication, Gathering knowledge and building tools', To appear in F.L. Lin and T. Cooney (eds.), *Making Sense of Mathematics Teacher Education,* Kluwer Academic-Publishers, Dordrecht, The Netherlands.

Sfard, A. and Kieran, C.: 2001 b, 'Cognition as communication, Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions', *Mind, Culture, and Activity* 8(1), 42–76.

Smith, J.P., diSessa, A.A. and Rochelle, J.: 1993, Misconceptions reconceived, A constructivist analysis of knowledge in transition', *The Journal of the Learning Sciences* 3(2), 115–163.

Steinbring, H., Bartolini-Bussi, M.G. and Sierpinska, A. (eds.): 1998, *Language and communication in mathematics classroom*, The National Council of Teachers of Mathematics, Reston, VA.

Stigler, J. and Hiebert, J.: 1999, *The Teaching Gap, Best Ideas from the World's Teachers for Improving Education in the Classroom,* The Free Press, New York.

Tomasello, M.: 1999, *The Cultural Origins of Human Cognition,* Harvard University Press, Cambridge, Massachusetts.

Tall, D. and Schwartzenberger, R.: 1978, 'Conflicts in the learning of real numbers and limits,' *Mathematics Teaching* 82, 44–49.

Tall, D. and Vinner, S.: 1981, 'Concept image and concept definition in mathematics with particular reference to limits and continuity', *Educational Studies in Mathematics* 12, 151–169.

Vinner, S. and Dreyfus, T.: 1989, 'Images and definitions for the concept of function', *Journal for Research in Mathamtics Education* 20(4), 356–366.

Voigt, J.: 1985, 'Patterns and routines in classroom interaction', *Recherches en Didactique des Mathématiques* 6, 69–118.

Voigt, J.: 1996, 'Negotiation of mathematical meaning in classroom processes, Social interaction and learning mathematics', in L.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin and B. Greer (eds.), *Theories of Mathematical Learning,* Kluwer Academic Publishers, Mahwah, NJ, pp. 21–50.

Vygotsky, L.S.: 1931/1981, 'The genesis of higher mental functions', in J. Wertsch (ed.), *The Concept of Activity in Soviet Psychology,* Sharpe, New York.

Vygotsky, L.S.: 1978, *Mind in Society. The Development of Higher Psychological Processes,* Harvard University Press, Cambridge, MA.

Vygotsky, L.S.: 1987, 'Thinking and speech', in R.W. Rieber and A.C. Carton (eds.), *The Collected Works of L.S. Vygotsky,* Plenum Press, New York, Vol. 1, pp. 39–285.

Wenger, E.: 1998, 'Practice', in E. Wenger, *Communities of Practice: Learning, Meaning, and Community,* Cambridge University Press, New York, pp. 43–102.

Wittgenstein, L.: 1953, *Philosophical Investigations*, G.E.M. Anscombe, Trans. Oxford, UK, Blackwell (Original work published 1953).

Yackel, E. and Cobb, P.: 1996, 'Sociomathematical norms, argumentation, and autonomy in mathematics', *Journal for Research in Mathematics Education* 27, 458–477.

Faculty of Education,
The University of Haifa,
Mount Carmel, Haifa 31905,
Israel
E-mail: Sfard@netvision.net.il

BERT VAN OERS

EDUCATIONAL FORMS OF INITIATION IN MATHEMATICAL CULTURE

"Seule l'histoire peut nous débarrasser de l'histoire"
Pierre Bourdieu (1982), **Leçon sur la leçon** (p.9)

ABSTRACT. A review of literature shows that during the history of mathematics education at school the answer of what counts as 'real mathematics' varies. An argument will be given here that defines as 'real mathematics' any activity of participating in a mathematical practice. The acknowledgement of the discursive nature of school practices requires an in-depth analysis of the notion of classroom discourse. For a further analysis of this problem Bakhtin's notion of speech genre is used. The genre particularly functions as a means for the interlocutors for evaluating utterances as a legitimate part of an ongoing mathematical discourse. The notion of speech genre brings a cultural historical dimension in the discourse that is supposed to be acted out by the teacher who demonstrates the tools, rules, and norms that are passed on by a mathematical community. This has several consequences for the role of the teacher. His or her mathematical attitude acts out tendencies emerging from the history of the mathematical community (like systemacy, non-contradiction etc.) that subsequently can be imitated and appropriated by pupils in a discourse. Mathematical attitude is the link between the cultural historical dimension of mathematical practices and individual *mathematical* thinking.

KEY WORDS: activity, tool, discourse, participation, genre, attitude

1. WHAT IS REALLY MATHEMATICAL?

'Math' is widely acknowledged as an undisputed part of the school curriculum. Over the past fifty years the classroom approach to mathematics has changed radically from a drill-and-practice affair to a more insight-based problem oriented approach. Every form of mathematics education makes assumptions about what the subject matter of mathematics really is, and – consequently – how the learning individual should relate to other members of the wider culture in order to appropriate this allegedly 'real mathematics', or to put it more directly, to appropriate what is taken to be mathematics in a given community. Part of a school's responsibility is to induct students into communities of knowledge and the teaching of mathematics can be seen as a process of initiating students in the culture of the mathematical community. In fact, students are from the be-

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ginning of their life a member of a community that extensively employs embodiments of mathematical knowledge. The school focuses attention on these embodiments and their underlying insights, and by so doing draws young children into a new world of understanding, with new conventions, rules and tools. So, basically, here is a process of reacculturation in which a student is assisted to switch membership from one culture to another. Buffee's (1993) insightful analysis of this process describes reacculturation as mostly a complex and usually even painful process: "Reacculturation involves giving up, modifying, or renegotiating the language, values, knowledge, mores and so on that are constructed, established, and maintained by the community one is coming from, and becoming fluent instead in the language and so on of another community" (Buffee, 1993, p. 225).

Educational history teaches us that schools have tried to support this reacculturation process in a variety of ways. Underlying these approaches there are different assumptions concerning the nature of mathematics in the classroom, and concerning the way teachers should communicate with their pupils in the classroom. In this article I will try to apply Bakhtin's approach to the discourse in a mathematics classroom, especially focusing on the question of how the participants in this classroom are linked together and what common *background* is to be constructed in order to constitute a way of speaking and interacting that will be acknowledged as a *mathematical* discourse. The final aim is to find a way of describing some of the conditions that must be fulfilled in order to ascertain that the classroom's activity can really count as 'mathematical'. There is, however, no direct empirical way of achieving this just by observing a great number of existing classroom practices and describing the events in Bakhtinian terms. When we view the discipline of 'mathematics' as a "socially conventionalized discursive frame of understanding" (Steinbring, 1998, p. 364), we must also acknowledge – as Steinbring does – that not only factual technical mathematical operations are involved in mathematical activities in classrooms, but epistemological constraints and social conventions are also part of the process. The application of the Bakhtinian jargon requires that the hidden assumptions be brought into the open as they presumably co-determine the style and the course of the discursive process, and the authority and power relationships that are involved.

One of the values that are implicitly or explicitly applied in every mathematics classroom is an idea about what really counts as mathematical. On the basis of these notions mathematics education researchers, curriculum developers and teachers decide what is relevant or even compulsory for taking into account in the mathematics classes and courses. On the basis of their mathematical epistemology, teachers make observations of pupils' activities and select some actions as relevant or not, they value certain actions as 'good' or assess others as false or insignificant (van Oers, 2000b). Obviously, there is some normative idea at stake here about what mathematics really is, or – more modestly formulated – a norm that helps in deciding whether a particular action or utterance may count as 'mathematical' or not: one teacher focuses on number and numerals, another one on structures, while a third may stress the importance of problem solving. Introducing children in one way or another into the world of mathematics and its according speech genre probably implies teaching them the presumptions for identifying what is really mathematical and what isn't.

The idea of what mathematics really is, is of course not just an educational problem. Much of the engagement of the philosophy of mathematics is based on this very same query (see for example Rotman, 1988). Although there is probably often a relationship between the epistemological positions that can be taken with respect to mathematics as an intellectual discipline and one's view on mathematics education, I will directly focus here on the ideas about mathematics in education (school, curriculum).

As Bourdieu (1982) has already argued, education has a very important role to play in the institutionalization of a discipline through implicitly (hidden in the routines or habits of a particular community) or explicitly signaled values that create distinctions between people, and consequently mark some of them as (say) mathematicians or not, mathematically educated or not, etc. In a similar vein I shall argue here that the notion of what is mathematical and what not is developed in education, and the mastery of this value marks significantly those who will be acknowledged as mathematically educated (e.g. who may pass the exams) and who can't. Hence it is essential to find out what kind of conception of mathematics is used, and what the implications are for the relationship between teacher and pupils, as well as for the organization of the classroom discourse in mathematics. Presumably this notion of what is really mathematical in the classroom is one of the basic values that constitutes the speech genre of the mathematical classroom.

2. VIEWS ON MATHEMATICS AS SUBJECT MATTER IN SCHOOLS

There exist a number of different conceptions about what the mathematical subject matter really is. The real mathematics manifests itself with different faces in the classroom, having different implications for the relationship between teacher (as a representative of culture) and pupils, and *a fortiori,* for the conception of communicating in the mathematics classroom.

As far as mathematics education is concerned we can distinguish different views on what counts as real mathematics in the classroom.

2.1. *'Mathematics' as a school subject matter is really about arithmetical operations*

This is the classical view, which used to be very common in arithmetic education in schools in the past. Children are considered to be involved in real mathematics when they are mechanically practicing counting or sums. The focus is on mastery of arithmetical operations. This is what real mathematics is supposed to be like. This view is related to the Platonic idea of eternal mathematical truths that can be discovered with honest toil. In educational practice it is not considered useful to let all children discover mathematics for themselves. As mathematical knowledge is assumed to be constituted of fixed entities, it is also believed that the elements of mathematical knowledge can be transmitted to children. The main communicational style of this approach follows the sender-receiver model that states that direct instructive language is needed to prescribe for children what to do with numbers. This point of view inevitably implies a special authoritarian relationship of a teacher towards his pupils. The teacher (as the one who knows) transmits pieces of mathematical knowledge to pupils (who don't know yet). Public discourse on mathematics in schools still follows mostly this point of view.

2.2. *'Mathematics' as a subject matter is really about structures*

The subject matter of mathematics is here conceived as essentially dealing with abstract structures that have to be applied to concrete situations and problems. The teacher or curriculum developer who subscribes to this view believes that children are really getting involved with mathematics when they are dealing with abstract structures for the organization of practical situations or for the solution of quantitative and spatial problems. It is generally believed that the basic abstract structures can already be seen in young children's play activities (see Picard, 1970; Dienes and Golding, 1966, 1967a and b), from which these structures can be elevated and further developed into explicitly reflected mathematical structures. Both Piaget (1966) and Davydov (1972) evidently endorse such a view on mathematics in school, although their view on the essence of structures is definitely different. In their argumentation for the basic structures they both refer to the French collective of mathematicians, *Bourbaki,* who tried to write a definitive history of mathematics on the basis of a few basic mother structures that engender new, more specific embedded structures, until all mathematical knowledge can be classified as an element in one structured whole (see for example Piaget, 1969a, p. 70–71). For Piaget, however, the basic structures were a consequence of the architecture of human logical thinking; for Davydov these structures were conceived as the best historical product of human thinking for structuring the whole body of mathematical knowledge. Despite their fundamental differences, however, both Piaget and Davydov defended a view on real mathematical activity that emphasizes the importance of structures. And again, despite their theoretical differences, authors committed to this point of view all propagate active methods of learning (see Picard, 1970, p. 15; Piaget, 1969b; Davydov, 1972, 1988), in which exploration or communication may play a prominent role. The so-called 'mother structures' are taken as the real objects of mathematical teaching and communication.

From their work it is evident that no one of these educators would ever propagate a direct transmission kind of teaching. Instead, the required structures are offered in situations and problems, so that the child can step by step – with more or less help – construct the basic structures and apply these subsequently in new problem situations. The child that is constructing and applying such structures is considered to be engaged in 'real mathematical activity'.

2.3. *'Mathematics' as a subject matter is really about problem solving activity with symbolic tools*

In this view the real subject matter of mathematics in the classroom is about problem solving with the help of self-invented tools in the context of realistic situations that make sense to the pupils. The seminal work of Freudenthal is important here. In many of his books he explains his view on mathematics as a human activity of problem solving with the help of tools that are invented to organize fields of experience in a schematic way (Freudenthal, 1973, 1978, 1991). In Freudenthal's view all mathematical conceptions, structures and ideas must be conceived in relation to the phenomena for which they were created in the first place (Freudenthal, 1984, p. 9). This brings him to the position of conceiving mathematical concepts and structures always as functional and contextualized tools for the solution of problems, but they are always to be conceived in relation to the context in which they originated. Structures, then, can never be seen as eternally fixed. Structures are just temporarily stabilized ways of approaching a problem. Mathematical activity in school – in order to be realistic – should focus above all on the processes of *structuring* instead of the mastery of fixed and prescribed structures. This difference between the emphasis on structures vs the emphasis on structuring is exactly the core of Freudenthal's critique on Davydov and Piaget.

This variant of real mathematics indeed fosters active learning and communication in heterogeneous groups. Hence discussion is an important element in this approach. Freudenthal's emphasis on the real life usefulness of mathematics ("If it were not useful, mathematics would not exist", Freudenthal, 1973, p. 16) has often been interpreted as emphasizing the real-life character of the contexts from which mathematical thinking should originate. The realism of mathematics then is seen in the applicability of self-invented mathematics in a meaningful problem, and for many people this seems to mean a real-life problem. For Freudenthal this included also interactive problem solving in heterogeneous groups of pupils. The teacher follows the process from a safe distance. This view is very popular at the moment in the Netherlands, where most of the schools use a realistic maths curriculum based on the ideas of Freudenthal. Realism with regard to mathematical activity then consists in a view of constructive problem solving of an individual in the context of meaningful problems and with the help of self-invented, socially evaluated tools.

Despite the enormous innovation this view could produce in the content and activities of the mathematics classrooms, it entails a serious danger by focusing too exclusively on the real life quality of the contexts from which the mathematical thinking originates. It is inconceivable how the higher, abstract levels of mathematical thinking can be based on real life situations. How could a child ever discover that he or she is doing mathematics, let alone what mathematical argumentation, proof or systemacy implies, by just getting involved with (real life) problems? How should children ever select from their endless alternatives those actions that have mathematical relevance? Indeed, dialogues between pupils can have a selective function as to the utterances or actions that eventually may be selected as acceptable. But still, there is no basis for assuming that children in their dialogue should select *per se* the mathematically relevant propositions. Dialogues between actually present non-expert pupils lack the criteria to link their own actions to the meanings of the cultural (mathematical) practice. Such dialogues are important and necessary, but obviously not sufficient. By lack of a clear and consistent solution for this problem, teachers then tend to fall into other approaches to 'real maths' (structure-oriented or operation oriented). Of course, it is possible to stretch the meaning of the notion of 'reality-based' and let it cover every meaningful context (including personally meaningful abstract problems). Similarly, one may also accept the necessity of a teacher defining the domain of mathematics for the child and telling the child after its explorations what is mathematically acceptable or not, but this is clearly not 'realistic' in Freudenthal's sense of the word. The approach, however, does not give a clear conceptual answer to this question. Such an answer would lead us to an analysis of the problem of sense and meaning. It is unclear how these are integrated conceptually into the framework of Freudenthal's didactical phenomenology.

Broader and more liberal interpretations of Freudenthal's notion of realistic mathematics have been proposed by Gravemeijer (1994, 1997a). Individual inventions (like a method for solving multiplication problems, or geometrical problems) are seen as social products that may develop into still higher levels of abstraction and constantly feed back into the community and foster the development of the community as well. As such, the individual and the community co-develop (see for example Gravemeijer, 1997b). Gravemeijer's view justifiably draws attention to the reciprocal process of communication itself and to the ways of negotiating meanings and symbolic tools in a mathematics classroom.

3. The Discursive Approach in (Mathematics) Education

In the wake of the Vygotskian storm drifting over the world today, the notion of discursivity nowadays has acquired a great deal of pertinence in discussions about education. As the classical (Platonic) model of education and teaching, based on obedience and power, gradually turned out to fail, the more the required results of our Western education called for insight, understanding and interest. The once strong conception of knowledge as objective units of thought that can be transported from one person to another, or from one situation to another, led people – on the one hand – to conceiving education as a literal *transmission* of pieces of knowledge and abilities from a teacher to pupils, and on the other hand, to believing that instructional success was best measured in terms of *transfer* (applying elements of thought in new situations). Especially in situations where asymmetry exists between two people as to their ability and expertise (like in education), it was generally seen as unavoidable that the more knowledgeable one hands over his or her knowledge and abilities to the other.

But in practice, the transmission models of teaching mathematics turned out to be disappointing. Due to the disappointing outcomes of both the transmission model of education, and the transfer model of learning, people began scrutinizing the assumptions behind these models (Lave and Wenger, 1991; Greeno, 1997). As a result many teachers and researchers have gradually become aware of the basically reciprocal, communicative nature of human education (Bruner, 1996; Wertsch, 1985; Wells, 1999). However, although the history of the construction of this idea of the social mind is long (see Valsiner and van der Veer, 2000), we have only recently begun to envisage its compelling implications.

One of the intriguing and far-reaching questions to be raised here concerns the view of the relationship between the participants in the discourse, especially with regards to their differences in expertise. With the refutation of the transmission model and its assumptions about objective meanings, the related communication model based on a sender-receiver idea was also heavily questioned. Hence, the old idea of one person being dependent on the information given by another could not be accepted anymore as a valid description of the relationship between a person and a more knowledgeable other in an educational setting. But how to handle the asymmetry between people with respect to their expertise, without falling back into a sender-receiver transmission model? Especially in mathematics education the differences in expertise and authority between teacher and pupil were traditionally felt as a legitimization for a transmission kind of education in which the teacher demonstrates the operations and the pupils spend all their efforts in mastering these operations by intensive practicing. Developments in the last 25 years with regard to mathematics education, however, reinforced the call for a more discursive approach, taking into account the pupils' own understandings of a mathematical problem (see Cobb et al., 1993; Forman, 1996; Gravemeijer, 1994), as well as doing justice to the fact that mathematics is a cultural activity that emerges out of sociocultural practices of a community (Bishop, 1988: Saxe, 1991). Hence the study of the interrelations between the role of the community and actual communication processes for establishing common mathematical solutions is one of the major items on the future agenda of investigators of mathematics education (see Bower, 2000).

In addressing this very same problem, we will have to deal with the question of how classroom communication is turned into a *mathematical* one. Obviously, the interlocutors in a mathematical discourse must share some values or meta-rules (Sfard, 2000) in order to be able to acknowledge utterances as mathematically relevant and to discuss them at all from the given perspective. A preliminary reflection on the notion of discourse and its prerequisites is now necessary.

4. From Vygotsky to Bakhtin

Since the early 20th century the work of Vygotsky has opened a window on human functioning and development that helped scholars of human development with reconceptualizing education as a process of co-reconstruction of meanings. Essentially, for Vygotsky, this process starts with the pupil's own actions and meanings. Therefore he writes in his 'Educational Psychology' (1926/1991, p. 82/1997: 48):

> "The traditional European school system, which always reduced the process of education and instruction to a passive apprehension by the student of a teacher's lessons and outlines, was the ultimate of psychological nonsense. The educational process must be based on the student's individual activity, and the art of education should be nothing more than guiding and monitoring this activity."

Vygotsky emphasizes the importance of the student's own activity in the teaching-learning process, but he immediately hastens to add that this does not mean that the role of the teacher is minimized! The teacher should fulfill a guiding role by introducing students in significant sociocultural practices. Quite appropriately Davydov, in his introduction to a new edition of this work of Vygotsky, summarizes Vygotsky's position by saying that "the teacher may educate students in a deliberate fashion only by constantly collaborating with them, with their environment, with their desires and willingness to cooperate with the teacher" (Davydov, 1991, p. 9/cfr. 1997, p. xxiii). For Vygotsky, according to Davydov's summary, "mental functions are essentially seen as not rooted in the individual, but in the communication [obščenie] between individuals, in their relationships between each other and in their relationships with the objects created by people" (Davydov, 1991, p. 14–15/cfr. 1997, p. xxix).

Obviously, communication for Vygotsky is now more and more taken as referring to what it originally meant: sharing communalities and constructively dealing with the meanings people seem to have in common[1]. Communication is a collaborative endeavor on publicly pooled meanings.

Despite Vygotsky's undeniable merits in opening this window on human development, recent analyses of Vygotsky's ideas have also demonstrated their limitations. In his descriptions of the process of communication, Vygotsky's picture always turns out to be a neat and orderly process of meanings improving each other for the better. In-depth analyses of communication processes often demonstrate that the exchange and negotiation of meaning is a much more complicated process, pervaded by conflicts, misunderstandings, obscurities, and ambiguities. Hence, the French psychologist Clot states outspokenly about the theory of meaning that Vygotsky unfolds in his 'Thinking and Speech': "[It] is insufficiently related to the social process of intersignification that is taking place in discourses, or to the polyphony of sociodiscursive settings. Hence, it cannot improve the theory of psychological tools that remains basically a-social. The concept of 'genre' as proposed by Bakhtin, may be more helpful here as it is a tool for action that is inherently social" (Clot, 1999, p. 174).

This view on communication and its consequences for our understanding of human consciousness was deeply understood by Bakhtin (and his collaborators Voloshinov and Medvedev[2]). For Bakhtin – like Vygotsky – it was impossible to think of human consciousness as an isolated entity. Human consciousness is basically taken as a dialogical, meaning creating process and this creative activity can only emerge at the borderline of continuous interaction between individual consciousness and the outer social world, manifested in sign producing consciousnesses (see Morris, 1994, *Introduction*; Clark and Holquist, 1984). The individual and the social reflexively constitute each other in dialogue. The one can never exist without the other. This reflexive constitutive relationship is particularly manifest in human communication: every utterance is directed to an addressee, and actually anticipates the addressee's expected reactions. "Any utterance", writes Voloshinov/Bakhtin, "no matter how weighty and complete in and of itself is only a moment in the continuous process of verbal communication. But that continuous verbal communication is, in turn, itself a moment in the continuous, all-inclusive, generative process of a given collective" (Voloshinov, 1929 in Morris, 1994, p. 59). In this quotation, it is clear how Bakhtin and his group conceive of the multiple embeddedness of human 'individual' development: on the one hand human thinking is dependent on direct dialogues with social others; on the other hand this form of interacting itself is embedded in a broader cultural process of evolution of the communicating complex as a whole. What I call here 'the communicating complex' is for Bakhtin basically a historically organized institution of persons, or what he calls a "sign community" (Voloshinov, 1929 in Morris, 1994, p. 55), i.e. "[a] community which is the totality of users of the same set of signs for ideological communication". With regard to the production of signs he writes: "the forms of signs are conditioned above all by the social organization of the participants involved and also by the immediate conditions of their interaction" (Voloshinov, op.cit.). In a more modern language we would say that people's utterances in a communication process are not only regulated by the processes that occur in direct interaction, but also *by the historically developed style of communicating in that particular community of practice.* This is a very important insight of Bakhtin with regard to the question of how the individual and the social are related. Not only do communicating participants constitute each other by anticipation and mutual regulation, but their existence as a communicating unit is also deeply determined historically by others. Without this historical context this communication unit would not be possible, neither would participants be able to recognize that they have more in common (as communicators) than the incidental and ephemeral events of that actual situation. It is through this 'sign community' that people can recognize themselves as members unified in a same practice, as basically showing some shared identity and background. It is via this connection with the evolving history of a mathematical community that 'mathematics' as such can be re-invented at all.

Bakhtin applied his dialogical point of view mainly on general cultural practices like literary practices or general philosophy of the humanities. A valuable application of these ideas in the present time requires a specification of these ideas for particular areas of culture or communities of practice. In the present article I will take Bakhtin's thinking as a starting point for the further analysis of the relationship of individuals in a community of mathematical practice, especially in those cases where people have adopted an educational intention of initiating newcomers into this community of practice. Hence, I intend to focus here on mathematical education from a Bakhtinian/sociocultural point of view.

Many scholars who have been inspired by Bakhtin's work already took up the notion of speech genre as a way of analyzing the mathematical vernacular. It must be clear that for Bakhtin a genre is not just or not even primarily a thesaurus of technical terms or rules of behavior or discourse. The genre is primarily a social tool of a sign community for organizing a discourse in advance and often even unwittingly. It is a style of speaking embodied in a community's cultural inheritance, which is passed to members of that community in the same way as grammar is passed on. A genre is not so much a strict and fixed social norm, but it is a generic system of changing variants and possible utterances that fit into a community's practices; it is some kind of arena or forge where new variants of utterances are created and valued, that contribute to the essential polyphony and dissonances of meaning and discourse. Bakhtin writes:

> "Speech genres organize our speech in almost the same way as grammatical (syntactical) forms do. We learn to cast our speech in generic forms and, when hearing others' speech, we guess its genre from the very first words; we predict a certain length (that is, the approximate length of the speech whole) and a certain compositional structure; we anticipate the end; that is, from the very beginning we have a sense of the speech whole, which is only later differentiated in the course of the speech process. If speech genres did not exist and we had not mastered them, if we had to originate them during the speech process and construct each utterance at will for the very first time, speech communication would be almost impossible" (Bakhtin, 1986, pp. 271–272; see also Morris, 1994, p. 84).

Although the phenomenon of the speech genre still is not completely understood in linguistics and psychology, Bakhtin's general notion is now widely accepted as an explanation of the fact that people seem to understand each others' utterances from a wider context than is actually given in the discursive situation. According to Bakhtin, any participant always values the utterances of the discourse against a broader background of implicit, tacit, ideological knowledge. Moreover, any participant in a discourse actually expects the other participants to act in a certain way and to abide by some basic values. "Each speech genre in an area of speech communication", he writes, "has its own typical conception of the addressee, and this defines it as a genre" (Bakhtin, 1986; in Morris, 1994, p. 87). It is important to note here, that for Bakhtin the speech genre intrinsically links the interlocutors to each other, despite their possible differences in expertise (or their asymmetry in positions). The interlocutors can effectively communicate *because of* their basic alliance in the speech genre that they share. A similar position is taken by Rommetveit, when he writes: "The speaker monitors what he is saying in accordance with what he assumes to be the listener's outlook and *background information,* whereas the latter makes sense of what he is hearing by adopting what he believes to be the speakers perspective" (Rommetveit, 1985, p. 189–190, *italics added*). The speaker incorporates anticipated reactions and qualities of the listener and vice versa. Hence speaker and listener share a common background that enables them to value and interpret each other's utterances.

Thus, basically, the Bakhtinian approach to discourse focuses on the communalities of participants and on how they collaboratively fashion the heterogeneity of meanings. The asymmetry that was so evident in the sender-receiver model of communication is now made into a core element of the discursive process: heterogeneity is fundamental to the discursive process and the best result can be a consensus about the meanings that the participants are willing to take as shared. Authority, moreover, is an indispensable position in an activity for linking the actual to the historical.

In order to really value Bakhtin's contribution to the deeper understanding of mathematical processes in the classroom, a further exploration is needed that tries to apply some of the elements of Bakhtin's thinking. Bakhtin's notion of speech genre implies that utterances of the interlocutors in the discourse are not just assessed in terms of their literal meaning, but also valued from a generic background that provides meta-rules and norms which help in defining the utterances involved as mathematical or not. "No utterance can be put together without value judgment. Every utterance is above all an evaluative orientation. Therefore, each element in a living utterance not only has a meaning but also has a value" (Voloshinov, 1929, in Morris, 1994, p. 37).

5. Mathematics Education as Improvement of Participation in a Mathematical Community of Practice

When using this perspective for the analysis or description of actual mathematical practices in classrooms, it is important to first clarify the notions of activity, practice, and discourse in their mutual relationships. 'Activity' is taken here as a concept referring to any motivated and object-oriented human enterprise, having its roots in cultural history, and depending for its actual occurrence on specific goal-oriented actions. Any activity can be accomplished in a variety of ways, and it depends on the community in which the activity is carried out how much variety (or which variant) is accepted as valid. In this I follow Leont'ev's activity theory (Leont'ev, 1975; van Oers, 1987). *Mathematical activity* can then be seen as an abstract way of referring to those ways of acting that human beings have developed for dealing with the quantitative and spatial relationships of their cultural and physical environment. When we specify the activity with the values, rules and tools adopted in a specific cultural community we tend to speak of a '*mathematical practice*'. Any practice contains *performative actions* and operations that just carry out certain tasks which have mathematical meaning within that community (like performing long division). On the other hand, practices also comprise *conversational actions* that intend to communicate about the mathematical operations or even about the mathematical utterances themselves. Cobb et al. (1993) made a similar distinction between 'talk about mathematics' and 'talk about talk about mathematics'. A community committed to a particular style of accomplishing conversational actions with regard to a special category of objects can be named a community of discourse. Hence, in my view a community of practice and a community of discourse refer to slightly different concepts. A community of mathematical practice also includes people making calculations (in their own idiosyncratic ways), e.g. in the super market (see Lave, 1988), while a community of mathematical discourse mainly includes persons interested in reflectively understanding mathematical actions. This is consistent with the more general formulation of a discursive practice as "the repeated and orderly use of some sign system, where uses are intentional, that is, directed to something" (Harré and Gillett, 1994, p. 28).

'Real mathematical activity' can now be defined as the activity that is accomplished when one legitimately participates in a mathematical practice, either by acting mathematically in an acceptable way, or by discussing mathematical or discursive mathematical actions. Hence, it is not the link with meaningful problem situations as such that defines the nature of 'real' mathematics, but the observance of particular rules, the use of particular concepts and tools, the engagement with certain values that define whether one is doing mathematics or not. So the basis of the realism is the participation in mathematical *activity*. Like in Freudenthal's definition the focus is here on problem solving, tool use, and contextuality, but their relevance is rooted in the commitments to a certain type of historically rooted activity. The context of human (mathematical) action, then, is not the meaningful situation but the culturally developed activity itself (cf. van Oers, 1998). In the case of mathematical activity, certain ways of doing and talking have developed during cultural history. Real mathematics in the classroom is actually participating in this mathematical practice.

It is the function of education to initiate children in this practice, and get them involved in the mathematical speech genre. This should give them a sense of what 'real mathematics' is like. Participation in mathematical practices (like in the case of the Brazilian street vendors, see Saxe, 1991; Nunes, et al., 1993) does not automatically lead to abilities of participating in mathematical discourses. In most cases it takes (formal or informal) education to develop these discursive abilities. In the school context, doing and learning mathematics means improving one's abilities to participate in mathematical practice, both the operational part (the symbolic technology of mathematics) and the discursive part.

In the following sections of this article I shall elaborate this latter view a bit more, by focusing especially on mathematical discourse, in order to clarify how this speech genre is passed on to new generations, how pupils may get 'infected' by this view on 'real mathematics' and what is needed to strengthen their participantship in this mathematical practice.

6. THE POLYLOGICAL CHARACTER OF A COMMUNITY OF MATHEMATICAL DISCOURSE

Having explained mathematics as a historically developed practice, dealing with certain types of objects, tools and rules, it is a logical next step to reflect a bit longer on the nature of this practice and how children are enticed to become autonomous and reflective participants in this practice.

The interpretation of cultural practices in terms of activity theory raises the question of how the dynamical elements of this activity (object, motive, actions, tools) can be defined. Mathematical practice as it has been invented and developed in our culture implies an activity that is based on the construction of mental *objects* that model the numerical and spatial aspects of physical and cultural reality. As Bishop (1988) has argued, the symbolic technology (*tools*) that resulted from these constructions during cultural history has been invented and elaborated in the context of general cultural activities that had to do with cultural *key-activities* like counting, locating, measuring, designing, playing, and explaining. In the context of such activities people encountered several problems that they tried to solve (*goals*) in many different ways, but in any way it is almost certain that some kind of symbolic representation (mostly with the help of language and drawing) was invented. While struggling with these problems, people also gradually discovered the relevance of certain values to be observed. Bishop (1988) discusses several values that have played a role in the development of mathematics as a cultural practice. Those values are intrinsic to several everyday practices and as such they offer guidelines that participants of the practice are particularly supposed to obey. According to Bishop, these values are not fixed in the history of mathematics. They have changed during history and are often sensitive to circumstantial, personal, and temporal influences. In many cultural periods these values can be found in twins that have a contrary relationship (Bishop, p. 60–83): objectism vs rationalism (as the twin ideologies of mathematics), control vs progress (as the attitudinal values of mathematics), and openness vs mystery (as values that define potential ownership of mathematics). Mathematical activity, according to Bishop (1988, p. 95), accomplishes the association of a particular symbolic technology developed by the key-activities, with the values that are articulated in a certain historical period. Both the development of the technology and the reflection on the values involved is part of the responsibilities of the participants in the mathematical practice. Real mathematical activities imply both elements.

Introducing children into the culture of a mathematical practice is basically a social process, that can be described in terms of apprenticeship learning (Rogoff, 1990), or gradual progress from a legitimate peripheral participant in that practice towards a more and more extended form of participation (see Lave and Wenger, 1991, for a general description of this model of initiation in cultural practices). In the context of the present article it is important to explain how communication takes place in such a community of practice, particularly when communication aims at improvement of the participatory abilities and qualities of the participants, both with regard to the technology, and with regard to intrinsic values and norms. I will come back to that question in the next section. First it is important to clarify who should be accepted as legitimate participants in this process. In my commentary on the Freudenthal definition of realism, I already pointed out that direct dialogues between actually present pupils might not be sufficient as an explanation for the *mathematical* content. As mathematics is a historical practice, representatives of the history of mathematics always take a part in the communication within that practice. Most of the time the teacher may be considered as a representative of the cultural history of mathematics and in that quality the teacher should take part in the discourse in the classroom: not just as a guide when the process goes astray, but also as a real participant, suggesting possible solutions, strategies, concepts etc. To use a Bakhtinian terminology, one could say that the teacher represents all absent and historical voices that essentially have a say of what should be taken as 'mathematical'. Thus, instead of a dialogue among pupils, the discourse in a mathematical community is essentially a *polylogue,* a polyphonic discourse among all possible voices that helped to create the history of that community of practice (see Davydov, 1983). The implications of this point of view might look overwhelming at first for regular school practice. They probably are, but one of the first and realizable consequences is that the teacher takes a substantial (and not just a distanced guiding) role in the classroom discourse: the teacher is a serious partner in the classroom activity and discourse, suggesting serious solutions, possibilities, questions, objections. It is exactly the teacher in this role who should introduce a cultural-historical voice in the classroom discourse, a voice that can help pupils in defining 'the mathematical' in accordance with the cultural history of that practice. A similar and even more detailed analysis of this very same viewpoint is given by Sfard (2000). She convincingly argues for the notion of meta-discursive rules that regulate participation in a practice or discourse. According to Sfard, reform of mathematics education should take the appropriation of these rules more seriously in order to help children getting access to mathematical practices.

Sure enough, this requires a radical innovation in many school practices, not only in those which still practice a transmission style of frontal teaching, but also in those who have introduced forms of cooperative learning in which the core of the activity is trusted to the pupils in dialogue. Fortunately, there are already a number of experimental classrooms that have demonstrated that teachers can indeed realize parts of this ideal. The work of Cobb and his colleagues is a good illustration of how in a mathematical classroom both the technical-conceptual development and the sociomathematical norms can be put on the agenda. This is a very important starting point for getting pupils involved in the definition of their mathematical practice, taking account of the general cultural meaning of mathematical practices.

The 'Dialogue of cultures'-schools in Russia are another example that demonstrate that a discourse of 'everybody with everybody' can be practiced in an elementary school practice. The idea of the 'Dialogue of cultures' as developed by Bibler – in line with Vygotsky and Bakhtin – is that every pupil represents a multiplicity of voices, hence is a microculture in itself. The learning processes in school, according to Bibler, should be focused on developing the pupils' own cultures in dialogue with all the other cultures available (including the teacher's). Therefore, this dialogue of cultures is basically what we called previously a polylogue (Bibler, 1992). In a report on the experimental implementation of the 'Dialogue of Cultures' Berljand and Kurganov also emphasize the importance of the participation of the teacher's culture in the mathematics classroom discourse. They write about the role of the teacher in the following:

> "On the one hand, the teacher acts as one of the participants in his own right, proposing his own hypotheses and assumptions. On the other hand, the teacher directs the process in a general but very cautious way, permitting sometimes far going digressions from the original plans and intentions. (...). *Another important function for the teacher is to canalize the discussion when something new or unexpected comes up, which might not be recognized by the pupils as significant.* Sometimes a thought is unclear for a pupil or he cannot formulate it in a way that is comprehensible for the other pupils. In those cases the teacher also helps the pupils in formulating the idea" (Berljand and Kurganov, 1993, p. 37, italics added).

From a historically advanced point of view, the teacher's responsibility, according to Berljand and Kurganov, is one of introducing new cultural elements in the discourse that could never be put forward by the pupils themselves. By so doing, the teacher not only provides new unexpected information, but also demonstrates a strategy of critically and systematically evaluating and elaborating a received result with the help of new points of view. This strategy of always asking new questions, critically looking at your results from another perspective is a strategic element of a mathematical rationality that is developed through the mathematical discourse with the teacher.

What Berljand and Kurganov were describing with respect to the teacher's activity is similar to what O'Connor and Michaels (1996) called revoicing. In the act of revoicing the teacher uses his or her own background knowledge of mathematics and the values involved. The teacher's selection of concepts, and style of phrasing is colored by his or her historical knowledge. This is one legitimate way of introducing cultural history in the process. Of course this revoicing should not impose definite knowledge onto pupils. The revoiced proposition is not *a priori* better or worse than any other input in the discourse and is, consequently, open for discussion and evaluation. Revoicing, thus, is one technique for putting cultural history at work in the classroom discourse, creating a public value position from which the pupils can learn what is counted as mathematical in this community's speech genre.

7. IMPROVING PARTICIPATION IN MATHEMATICAL PRACTICES

The polylogic character of the classroom discourse articulates the heterogeneous nature of this communicative activity. It should be clear that this couldn't easily be dealt with by a sender-receiver model of communication. In this alternative Bakhtinian communication model all participants are constructors of meaning, sharing a topic that they elaborate by adding new information ('predicates' in the sense of Vygotsky, 1987, ch.7; see also Van Oers, 2000a). These predicates in fact reveal something new about the topic at hand and distinguish that topic from other topics. An example of this process can be found in the following situation:

Two 6/7 year old girls have been building a farm with blocks, and they have been playing with it for a while. The teacher starts a conversation with these girls asking about the number of blocks that the girls used for their farm. The teacher shows interest in that aspect of their work and she (implicitly) introduces a mathematical point of view by asking '*can you count them for me?*' The teacher explains that she wants to know how many blocks are needed for making such a beautiful farm in case other children at a later moment might be willing to construct something like that. She then also invites the children to fill out a graph for her so that she can immediately see how many blocks are used in this farm (she provides a big sheet of paper with a number of columns with drawings of different types of blocks at the bottom – see example in Figure 1).

Two observations are relevant here: the teacher introduces a mathematical point of view by her questions, and kind of 'defines' the situation as a counting situation. This is a first predicate that characterizes the situation in this specific way and distinguishes it from other possible perspectives on the situation (esthetic: 'how beautiful'; physical: 'how did you do it?' etc). Moreover, by providing this tool for recording their counts, the teacher structures the children's actions in a histogram-like form. This introduces a tactical element in the children's activity if not with regard to the appropriation of histograms, then possibly in a more general way regarding the fact that counts can be recorded in a structural form. So it is not purely numbers that the teacher introduces, but also more general ways of doing, either by providing specific predicates, or by providing tools (that often lead to specific predicates). Her style of acting in this case demonstrates *ritualistic* elements from the *genre* of mathematical activity.

In their activity of counting, the children encountered different practical problems (e.g. walls tumbling down), which cause them to restart their counting several times. So after repeatedly counting the blocks of the farms the counting girl suddenly shouts: "This is the table of three!" (referring to the wall with piles of three blocks). Now in fact she predicates the situation herself in a new way and makes it different from all other situations or interpretations. Her partner knows what she is talking about and starts

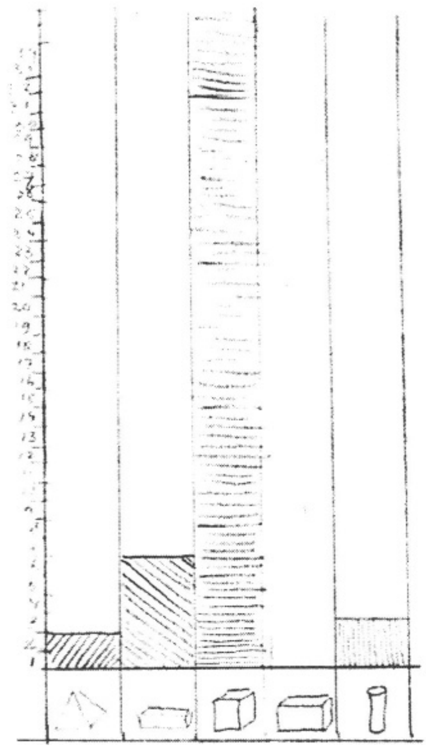

Figure 1. Histogram of 6–7 year old girls.

checking if she was right (checking systematically and answering the 'Are-you-sure?'-question is another typical element of the mathematical genre, cfr. van Oers, 1996). Actually, the other child starts evaluating this particular predicate and continues this line of reasoning by adding still other predicates (for example transforming the counting result to a *score* on the paper). Basically, this is a collaborative construction of a (mathematical) text, that is the beginning of all discursive (mathematical) thinking, and that opens the possibility of intertextual confrontation with other (historical) texts (Bibler, 1989; see also Carpay and van Oers, 1999). Finally they end up with the diagram above of the situation, which is of course a product both of the children's actions and the cultural tool provided by the teacher. It is essentially a product of a polylogic process.

Constructing meaning and negotiating meaning by constructing and evaluating new predicates is a way of talking about the processes that take place in a mathematical discourse. The diagram is one possible tool of structuring the discourse, and integrating the different (real or virtual) voices that take part in the discourse. It is clear that a multiplicity of propositions is possible all the time. The selection of propositions/predicates is a task of the community in discourse. There is, however, no universal selection principle that helps participants to decide unequivocally in advance which mathematical propositions should be used in a given situation. Within the practice, it is possible that sub-communities arise on the basis of intentional communalities among groups of participants in the practice. In general the mathematical practice comprises different groups of legitimate participants who are willing to deal with number, number relations, and spatial relations according to accepted values in the community, and above all who are willing to pursue the quest for certainty, to apply the norms of non-contradiction, systemacity, generalization, modeling etc., in short: who demonstrate *the mathematical attitude.* Hence both lay persons in the supermarket and highbrow mathematicians are to be accepted as legitimate participants in the community of mathematical practice. There is a well-known tendency to monopolize the participation in mathematical practices for the group of professional mathematicians. This is primarily an ideological struggle within our culture (and perhaps even within the community linked to mathematical practices), but the Bakhtinian theory of communication doesn't provide any principled reason why practices should be monopolized by specialist groups (experts).

What is more interesting here is the question of how participants of a mathematical practice can assist each other in order to improve their abilities for participation. There is no room here to summarize extensively the growing amount of literature that is consistent with the approach outlined here. Cobb and his colleagues have demonstrated possible ways of how pupils' mathematical understanding can be promoted through a classroom discourse. On the basis of their classroom discourse data they argue that an individual pupil's development and the development of the classroom community's understanding are reflexively related, co-existent processes. In detailed analyses they demonstrated how the development of pupils' understanding might be conceived of as a construction of a chain of signification (Cobb et al., 1997). These data provide an empirical basis for the assumption that the individual and the community are reflexively related in their discourse-based development and demonstrate what kind of processes partly constitute this development. Studies of Forman and her colleagues contributed to a further understanding of the processes of individual development in a community by analyzing the process of argumentation among participants in the discourse. It is clear from these studies that any argument always is based on common resources in the community to make up a collective argument (Forman, Larreamendy-Joerns, Stein and Brown, 1998). It is also interesting for the present argument that these authors could demonstrate the important role of the teacher in making explicit the implicit background knowledge (Forman and Larreamendy-Joerns, 1998). This probably also contributes to the emergence of the 'real' mathematical speech genre in the classroom. Both Cobb's and Forman's findings demonstrate parts of the dynamics of the development of mathematical thinking with regard to meaning development. But it is equally important to invest in building a mathematical sense in pupils. On the basis of Leont'ev's activity theory we must assume, however, that any activity always also depends on the dimension of sense, i.e. the motive-related valuation of actions and utterances. It is important to know how a person creates a chain of signification, how he or she builds arguments, but it is equally important to know *why* constructing new topic-predicate relations, chains of signification, or arguments do indeed make sense to that individual, *why* he wants to be engaged in these kind of enterprises. Basically, according to Leon'tev (1975), the development of an activity always depends on a dialectic between meaning and sense (between the 'what/how?' and the 'why?'). As 'sense' is always intrinsically related to a person's motives for acting, there is a close relationship as well with the goals that person wants to pursue. Saxe's interesting studies (see for example Saxe and Guberman, 1998) also demonstrate that the emergence of new goals (and, thus, new sources for giving sense to future actions) is dependent on collective processes. New goals emerge in a collective activity and obviously are not 'private'. The public status of newly emerging goals constitutes one of the essential elements of a shared background for communication: though individually appropriated, they provide the points for joint attention that defines part of the speech genre that may be going to be recognized by the participants in the discourse. But indeed, the mystery remains how pupils come to select the mathematically relevant goals and actions in the middle of the many possible alternatives?

All these studies, however, focus mainly on the public, goal-directed processes and qualities in a community for the development of a successful mathematical understanding in the participants in the discourse. It is becoming more and more clear that participation in a mathematical discourse presupposes the observance of a set of meta-rules (see also Sfard, 2000; Bishop, 1988) that regulate the discourse and the practice in general. These rules are culture-bound, intersubjective entities that continue to exist in the individual members of the community, that are passed on from one generation to the other, but at the same time these rules are not an authentic product of any one of them. The participation *per se* in mathematical activities with others (more mathematically advanced) covertly contributes to the development of a mathematical sense as well. It contributes to the gradual appropriation of this tacit normative background (with its included norms and meta-rules) from which students in due time start to make personal decisions about the kind of actions and goals that are assumed to be relevant in a mathematical practice. This sense cannot be instructed in a direct way. 'Sense' is formed by educative interaction (Leont'ev, 1975, p. 286).

This sense creates the personal stance that manifests itself as an attitude in a discourse. For a mathematical speech genre to arise it must be assumed now – at least theoretically – that mere mastery of mathematical meanings (knowledge and skills) is not enough. For participating autonomously in a community of mathematical discourse some conditions must be fulfilled at the personal level as well, in order to be able to value the real mathematical in the discourse. At least one of the persons involved must have the *attitude* of acting according to the meta-rules, of operating systematically, critically, non-contradictorily, and of looking for proofs and for forms of symbolization. In fact, this mathematical attitude is the *interiorized tendency of the meta personal dynamics of the mathematical speech genre* that has developed in the history of a particular community. As Billig (1986) already has extensively argued, attitudes represent *positions* taken in matters of controversy, having their roots in discursive processes. In the discourse the historical tendencies of mathematics (to be systematic, non-contradictory, to construct symbolic technology etc.) are introduced – either implicitly or explicitly – by those participants who have interiorized these historical tendencies as personal stances in matters of discourse regarding spatial and quantitative problems. As I argued elsewhere, for instance, the characteristic feature of 'abstractness', which is generally seen as a hallmark of mathematical thinking, can be interpreted as a habit of progressively focusing on imbedded relationships and assuming increasingly specific points of view (see van Oers, 2001). As I demonstrated in this latter argument 'abstraction' is also a product of discourse, intrinsically related to assuming points of view that have been shown relevant during cultural history. It is the teacher's task to help children in appropriating this habit and at the same time help them in appropriating an attitude that is generally seen as essential for mathematical thinking. There is no other way of understanding how this view on 'what mathematics really should be' finds its way into an actual discourse than by assuming that at least one of the participants convincingly demonstrates this *mathematical attitude* and regulates the discourse accordingly. The mathematical attitude is the essential link between the mathematical community's history and the development of understandings at the personal level that will be acknowledged as 'really mathematical'.

From a genetical point of view, I hypothesise that the emergence of this mathematical attitude starts out from the teacher's demonstrations of a specific type of behavior and, consequently, from her/his mathematics-related *expectations* about the pupils' activity. It seems plausible that these expectations and the pupils' ways of digesting these in actions, play a significant role in the development of mathematical sense and attitude. Further study of this theoretical hypothesis should be given top-priority on the researchers' agenda in the near future.

8. Conclusion

For the educational agenda we may conclude now that the further improvement of mathematics education requires that pupils be enticed by the teacher to take part in a mathematical practice and especially in mathematical discourse within that practice. More attention therefore should be given to the development of the mathematical *genre* (rather than just to the register clarifying the concepts, rules, tools and operations). In the interaction with the teacher, pupils can get access to the specific mathematical genre (including the meta-discursive rules, Sfard, 2000). As a result of this discourse children may interiorize the rules according to which those discourses are supposed to be regulated. This is how a mathematical sense emerges in shared practice and how a mathematical attitude can be appropriated from this. Such attitude is necessary for becoming an autonomous, critical and authentic participant in mathematical practice.

Needless to say, the provision of mathematical tools and rules is in itself not enough for developing full participation in a mathematical discourse. The tool does indeed structure the participants' actions according to implicit mathematical rules, but these rules can only be fully mastered when the participants' attention is drawn explicitly to them. So at best the tool is a starting point for discourse, and again it is the teacher who should create conditions for focusing on the hidden rules and assumptions in the tools. Recent research has provided interesting evidence in favor of such critical discourses that create the necessity for co-construction of new personalized versions of the provided tool in the pupils' community (see for example Cobb, 1999). In our own research (see van Dijk et al., 1999, 2000) we could provide evidence that the co-constructive creation of mathematical models leads to different ways of problem solving in students (as compared to transmission-based teaching). More importantly, students from a co-constructive classroom, where more exploratory and problem solving discourse took place, performed better on tasks that were relatively new for them, than students who just got ready-made models and who were involved in discourses that were primarily focused on correct application of the provided models. Hence, discursive forms of initiation lead to better performances of students on a variety of complex mathematical tasks. If these students indeed also acquired the new personal quality that we referred to as 'mathematical attitude', this is something to be investigated in the future, but the start is already there, providing a mathematical culture in the classroom with opportunities of model-based structuring, invention of symbolic tools, and creating the right atmosphere for experiencing the expectations of the mathematically more advanced partners.

A fundamental requirement for achieving this attitudinal outcome is the innovation of the teacher-pupil relationships into a form of long-lasting collaborative inquiry of mathematical actions, in the context of a shared discursive activity, in which the teacher fulfills the role of a historical resource for the pupils. It is in these conditions that they are likely to experience the historically founded, mathematics based expectations that give them a window on what it means to act mathematically.

NOTES

1. It is interesting to note that Davydov also used the word 'obščenie' that has a similar etymological root as the latin word 'communicatio' (communication), referring to what is 'common'. The translation of this word as 'intercourse' (see Davydov, 1997, p. xxix; compare my translation of this quote above) is not wrong, but hides this important connotation.
2. For more information about the somewhat enigmatic relationship between Bakhtin, Voloshinov and Medvedev see Clark and Holquist (1984). For reasons of simplicity, in my descriptions of the approach I shall take Bakhtin as the main spokesperson, even when I will also quote from sources that are officially attributed to Voloshinov or Medvedev.

REFERENCES

Bakhtin, M.M.: 1986, *Estetika slovesnogo tvorčestva* [Esthetics of verbal creativity]. Moscow: Iskusstvo.

Berljand, I.E. and Kurganov, S.Ju.: 1993, *Matemetika v škole dialoga kultur* [Mathematics in a 'Dialog of Cultures' school]. Kemerovo: Alef.

Bibler, V.S.: 1989, '*Dialog, soznanie, kultur: ideja kultura v rabotach Bachtina* [Dialogue, consciousness, culture: the idea of culture in the works of Bakhtin]. *Odissej. Čelovek v istorii. Issledovanija po social' noj istorii i istorii kulturi* [Odysseus. Man in history. Studies in social history and history of culture] (pp. 21–60) Moscow: Nauka.

Bibler, V.S.: 1992, *Škola dialoga kultur: osnovy programmy* [The Dialogue of Cultures school: fundamentals of the program]. Kemerovo: Alef.

Billig, M.: 1986, *Arguing and Thinking. A Rhetorical Approach to Social Psychology,* Cambridge University Press, Cambridge.

Bishop, A.J.: 1988, *Mathematical Enculturation. A Cultural Perspective on Mathematics Education,* Kluwer, Dordrecht.

Bourdieu, P.: 1982, *Leçon sur la leçon,* Minuit, Paris.

Bower, J.: 2000, 'Postscript: integrating themes on discourse and design', in P. Cobb, E. Yackel and K. McClain (eds.), *Symbolizing and Communicating in Mathematics Classrooms. Perspectives on Discourse, Tools, and Instructional Design,* Erlbaum, Mahwah, pp. 385–399.

Bruner, J.S.: 1996, *The Culture of Education,* Harvard University Press, Cambridge.

Buffee, K.A.: 1993, *Collaborative Learning. Higher Education, Interdependence, and the Authority of Knowledge,* John Hopkins University Press, Baltimore.

Carpay, J. and van Oers, B.: 1999, 'Didactic models and the problem of intertextuality and polyphony', in Y. Engeström, R. Miettinen, and R-L. Punamäki (eds.), *Perspectives on Activity Theory,* Cambridge University Press, Cambridge, pp. 298–313.

Clark, K. and Holquist, M.: 1984, *Mikhail Bakhtin,* Harvard University Press, Cambridge.

Clot, Y.: 1999, 'De Vygotski à Leontiev via Bakhtine', in Y. Clot (ed.), *Avec Vygotski,* La Dispute, Paris, pp. 165–185.

Cobb, P., Wood, T. and Yackel, E.: 1993, 'Discourse, mathematical thinking and classroom practice', in E.A. Forman, N. Minick and C.A. Stone (eds.), *Contexts for Learning. Sociocultural Dynamics in Children's Development,* Oxford University Press, New York, pp. 91–119.

Cobb, P., Gravemeijer, K., Yackel, E., McClain and Whitenack, J.: 1997, 'Mathematizing and symbolizing: the emergence of chains of signification in one first-grade classroom', in D. Kirshner and J.A. Whitson (eds.), *Situated Cognition. Social, Semiotic, and Psychological Perspectives,* Erlbaum, Mahwah, pp. 151–234.

Cobb, P.: 1999, 'Individual and collective mathematical learning: the case of statistical data analysis', *Mathematical Thinking and Learning* 1, 5–44.

Davydov, V.V.: 1972,Vidy *obobščenija v obučenii* [Forms of generalization in education], Moscow: Pedagogika. Translated: V.V. Davydov, Types of generalization in instruction. Reston, VA., National Council of Teachers of Mathematics, 1990.

Davydov, V.V.: 1983, 'Istoričeskie predposylki učebnoj dejatel'nosti' [Historical conditions for the learning activity], in V.V. Davydov (ed.), *Razvitie psichiki škol'nikov v processe učebnoj dejatel'nosti',* APN, Moscow.

Davydov, V.V.: 1988, 'Problems of developmental teaching', *Soviet Education* 20(8–10).

Davydov, V.V: 1991, 'L.S. Vygotskij i problemy pedagogičeskoj psichologii [L.S. Vygotsky and the problems of pedagogical pedagogy]', in: L.S. Vygotsky, *Pedagogičeskaja psichologija,* Pedagogika, Moscow, pp. 5–32, Preface. [Translation: Lev Vygotsky and educational psychology. In L.S. Vygotsky, Educational psychology. Boca Raton: Lucie Press, 1997].

Dienes, Z.P. and Golding, E.W.: 1966, *Logiques et jeux logiques. Les premiers pas en mathématiques, I,* OCDL, Paris.

Dienes, Z.P. and Golding, E.W.: 1967a, *Ensembles, nombres et puissances. Les premiers pas en mathématiques, II,* OCDL, Paris.

Dienes, Z.P. and Golding, E.W.: 1967b, *Exploration de l'espace et pratique de la mesure. Les premiers pas en mathématiques, III,* OCDL, Paris.

van Dijk, I.M.A.W., van Oers, B. and Terwel, J.: 1999, *Providing or Designing? Constructing Models as a Strategy for Working with Contextual Problems in Primary Maths Education,* Poster and paper for Earli-conferentie (Augustus 1999, Göteborg).

van Dijk, I.M.A.W., van Oers, B. and Terwel, J.: 2000, *Strategic Learning in Primary Mathematics Education: Effects of an Experimental Program in Modeling,* Paper for 9th International Congress on Mathematical Education (ICME-9), Tokyo, Japan.

Forman, E.A.: 1996, 'Learning mathematics as participation in classroom practice: implications of sociocultural theory for educational reform', in L.P. Steffe, P. Nesher, P.

Cobb, G.A Golding and B. Greer (eds.), *Theories of Mathematical Learning,* Erlbaum, Mahwah, pp. 115–130.

Forman, E.A. and Larreamendy-Joerns, J.: 1998, 'Making explicit the implicit: classroom explanations and conversational implicatures', *Mind, Culture, and Activity* 5(2), 105–113.

Forman, E.A., Larreamendy-Joerns, J., Stein, M.K. and Brown, C.A.: 1998, ' "You're going to want to find out which and prove it": collective argument in a mathematics classroom', *Learning and Instruction* 8(6), 527–548.

Freudenthal, H.F.: 1973, *Mathematics as an Educational Task,* Reidel, Dordrecht.

Freudenthal, H.F.: 1978, *Weeding and Sowing. Preface to a Science of Mathematical Education,* Reidel, Dordrecht.

Freudenthal, H.F.: 1984, *Didactische Fenomenologie van Wiskundige Structuren* [Didactical phenomenology of mathematical structures], OW&OC, Utrecht.

Freudenthal, H.F.: 1991, *Revisiting Mathematics Education. China Lectures*, Kluwer Academic Publishers, Dordrecht.

Gravemeijer, K.P.E.: 1994, *Developing Realistic Mathematics Education*, CDB press, Utrecht.

Gravemeijer, K.: 1997a, 'Mediating between the concrete and the abstract', in T. Nunes and P. Bryant (eds.), *Learning and Teaching Mathematics. An International Perspective*, Psychology Press, Hove, pp. 315–346.

Gravemeijer, K.: 1997b, 'Instructional design for reform in mathematics education', in M. Beishuizen, K.P.E. Gravemeijer and E.C.D.M. van Lieshout (eds.), *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures,* CDB press, Utrecht, pp. 13–54.

Greeno, J. G.: 1997, 'Response: On claims that answer the wrong questions', *Educational Researcher* 27(1), 5–17.

Harré, R. and Gillett, G.: 1994, *The Discursive Mind,* Sage, London.

Lave, J.: 1988, *Cognition in Practice. Mind, Mathematics, and Culture in Everyday Life,* Cambridge University Press, Cambridge.

Lave, J. and Wenger, E.: 1991, *Situated Learning. Legitimate Peripheral Participation,* Cambridge University Press, Cambridge.

Leont'ev, A.N.: 1975, *Dejatel'nost', soznanie, ličnost'* [*Activity, consciousness, personality*], Politizdat, Moscow.

Morris, P. (ed.).: 1994, *The Bakhtin Reader. Selected Writings of Bakhtin, Medvedev, Voloshinov,* Arnold, London.

Nunes, T., Schliemann, A. and Carraher, D.: 1993, *Street Mathematics and School Mathematics,* Cambridge University Press, New York.

O'Connor, M.C. and Michaels, S.: 1996, 'Shifting participant frameworks: orchestrating thinking practices in group discussion', in D. Hicks (ed.), *Discourse, Learning, and Schooling*, Cambridge University Press, Cambridge, pp. 63–104.

van Oers, B.: 1987, *Activiteit en Begrip* [Activity and concept], Free University Press, Amsterdam.

van Oers, B.: 1996, 'Are you sure? The promotion of mathematical thinking in the play activities of young children', *European Early Childhood Education Research Journal,* 4(1), 71–89.

van Oers, B.: 1998, 'From context to contextualization', *Learning and Instruction* 8(6), 473–488.

van Oers, B.: 2000a, 'The appropriation of mathematical symbols: a psychosemiotic approach to mathematics learning', in P. Cobb, E. Yackel and K. McClain (eds.), *Sym-*

bolizing and Communicating in Mathematics Classrooms. Perspectives on Discourse, Tools, and Instructional Design, Erlbaum, Mahwah, pp. 133–176.
van Oers, B.: 2000b, 'Teachers' epistemology and the monitoring of mathematical thinking in early years classrooms', Paper for the 10th conference of the European Early Childhood Education Research Association (London, August, 2000).
van Oers, B.: 2001, 'Contextualisation for abstraction', *Cognitive Sciences Quarterly*, Vol. 1, 3/4, 279–306.
Piaget, J.: 1966, 'L'initiation aux mathématiques, les mathématiques modernes et la psychologie de l'enfant', *L'enseignement mathématique* 2(12), 289–292.
Piaget, J.: 1969a, 'Education et instruction depuis 1935', in J. Piaget (ed.), *Psychologie et pédagogie,* Denoël, Paris, pp. 1–195. Originally published in 1965.
Piaget, J.: 1969b, 'Les méthodes nouvelles. Leurs bases psychologiques', in J. Piaget (ed.), *Psychologie et pédagogie,* Denoël, Paris, pp. 196–264. Originally published in 1935.
Picard, N.: 1970, *Mathematique et jeux d'enfants,* Casterman, Tournai.
Rogoff, B.: 1990, *Apprenticeship in Thinking,* Oxford University Press, New York.
Rommetveit, R.: 1985, 'Language acquisition as increasing linguistic structuring of experience and symbolic behavior control', in J.V. Wertsch (ed.), *Culture Communication and Cognition. Vygotskian Perspectives,* Cambridge University Press, Cambridge, pp. 183–204.
Rotman, B.: 1988, 'Towards a semiotics of mathematics', *Semiotica* 72(1–2), 1–35.
Saxe, G.: 1991, *Culture and Cognitive Development: Studies in Mathematical Understanding,* Erlbaum, Hillsdale, N.J.
Saxe, G. and Guberman, S.R.: 1998, 'Studying mathematics learning in collective activity', *Learning and Instruction* 8(6), 489–502.
Sfard, A.: 2000, 'On reform movement and the limits of mathematical discourse', *Mathematical Thinking and Learning* 2(3), 157–189.
Steinbring, H.: 1998, 'Mathematical understanding in classroom interaction: the interrelation of social and epistemological constraints', in: F. Seeger, J. Voigt and U. Waschescio (eds.), *The Culture of the Mathematics Classroom,* Cambridge University Press, Cambridge, pp. 344–372.
Valsiner, J. and van de Veer, R.: 2000, *The Social Mind. Construction of the Idea,* Cambridge University Press, Cambridge.
Voloshinov, V.N.: 1929/1984, 'Marxism and the philosophy of language', in P. Morris (ed.), *The Bakhtin Reader. Selected writings of Bakhtin, Medvedev, Voloshinov.,* Arnold, London.
Vygotsky, L.S.: 1987, *Thinking and Speech,* Plenum, New York.
Vygotsky, L.S.: 1926/1991, *Pedagogičeskaja psichologija,* Pedagogika, Moscow. [Translation: L.S. Vygotsky, Educational psychology. Boca Raton: Lucie Press, 1997].
Wells, G.: 1999, *Dialogic Inquiry. Toward a Sociocultural Practice and Theory of Education,* Cambridge University Press, Cambridge.
Wertsch, J.V. (ed.): 1985, *Culture, Communication and Cognition,* Cambridge University Press, Cambridge.

Department of Education and Curriculum,
Free University Amsterdam,
The Netherlands

STEPHEN LERMAN

CULTURAL, DISCURSIVE PSYCHOLOGY: A SOCIOCULTURAL APPROACH TO STUDYING THE TEACHING AND LEARNING OF MATHEMATICS

ABSTRACT. From a sociocultural perspective an object of research on mathematics teaching and learning can be seen as a particular moment in the zoom of a lens. Researchers focus on a specific part of a complex process whilst taking account of the other views that would be obtained by pulling back or zooming in. Researching teaching and learning mathematics must be seen in the same way. Thus in zooming out researchers address the practices and meanings within which students become school-mathematical actors, whilst zooming in enables a study of mediation and of individual trajectories within the classroom. In each choice of object of research the range of other settings have to be incorporated into the analysis. Such analyses aim to embrace the complexity of the teaching-learning process. This article will present a cultural, discursive psychology for mathematics education that takes language and discursive practices as central in that meanings precede us and we are constituted within language and the associated practices, in the multiple settings within which we grow up and participate.

KEY WORDS: cultural discursive psychology, learning theories, research in mathematics education, social practice, Vygotsky

INTRODUCTION

Researchers in mathematics teaching and learning draw on a range of intellectual resources for explanations, analyses and curriculum designs. The structures and meanings of mathematics (including historical and epistemological studies) and the methods and insights of psychology (especially constructivism) have provided rich theoretical fields for the mathematics education research community. They have not, however, enabled us to engage with schooling as reproduction, nor with culture or power, as they are manifest in the mathematics classroom. Sociology, anthropology and cultural studies provide intellectual resources to address these issues, and they have had their effect on psychology (e.g. Cole, 1996; Harré, 1995; Wells, 1999). In mathematics education, the last few years have seen a growing body of studies drawing on these resources (e.g. Dowling, 1998; Cooper and Dunne, 1999; de Abreu, 1998; Saxe, 1991; Nunes, Schliemann

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and Carraher, 1993; Lerman and Tsatsaroni, 1998; Evans, 2000; Adler, 2001; Lerman, 1998a) (for a more developed analysis see Lerman, 2000b).

In this article I will first describe some of the theories underpinning the move in psychology over the last decade or so to one which is fully cultural and focused on the way in which consciousness is constituted through discourse. I will argue that social practices are discursively constituted, and that people become part of practices as practices become part of them (Lerman, 2000b). Although I will return to this several times in the article, I want to emphasise here that "discourse" is to be taken to include all forms of language, including gesture, signs, artefacts, mimicking, and so on. If one focuses on learning in social practices and the manner in which the physical and cultural tools mediate learning, through all these forms of language, we can speak of 'discursive practices'.

In the second part I will focus on learning. Rather than seeing social factors as *causative* of learning, they can be seen as *constitutive* (Smith, 1993). Learning is about becoming, it is about participation in practices (Wenger, 1998). But people react differently in those practices, and perform their own trajectories through them. In arguing that people are discursively constituted the individual does not disappear; instead, the notion of individuality requires a reinterpretation. In this sense, I want to make it clear that there are a number of approaches to psychology as it relates to education. I find the perspective outlined in this article and well supported in the literature as the most persuasive and powerful, as well as fruitful for research, but other perspectives are also clearly well supported in the literature. The contrasts between sociocultural theories and individualistic ones have been well debated (e.g. Lerman, 1996; Steffe and Thompson, 2000; Lerman, 2000a) and have highlighted the contribution of each. Whilst a complementarity between some of these perspectives is sought by some (e.g. Sfard, 1998), I will take the view that many of these theories present their own world-view in terms of their understanding of human activity and consciousness and therefore notions that are familiar in one setting may need to be redefined in another. As I have argued elsewhere (Lerman, 1996, 2000a) incompatibilities lurk in incautious complementarities. I will, therefore, be advocating a particular view, that of a cultural, discursive psychology, towards which I have been working over a number of years (Lerman, 1998a, b), and not attempting to reach a complementarity with other theoretical frameworks, in particular individualistic psychologies, but I recognise that this is just one possible perspective.

Cobb and colleagues (e.g. Cobb, 2000) have developed what they see as an alternative approach, one that incorporates both psychological and sociocultural theories in a reflexive relationship. "… Each perspective constitutes the background against which mathematical activity is interpreted from the other perspective" (p. 64). The distinction is described as being about 'grain size', which has some similarities to the zoom metaphor that is employed in this article. The danger of their perspective, from my point of view, is that the social context, in the way they see it, cannot account for the forms of behaviour and activity of the individual, except in the important but superficial layer of classroom social norms (and socio-mathematical norms). 'Superficial' here is to be taken to mean the upper surface or layer of positioning in the classroom. Class, gender, ethnicity, race and other dimensions of identity seem to disappear with an appropriate social environment in the classroom. In this article I am arguing that we need an integrated account, one that brings the macro and micro together, one that enables us to examine how social forces such as a liberal-progressive position, affect the development of particular forms of mathematical thinking. I suggest that neither complementary nor emergent views can achieve this integration. In section 1.4 below I discuss a unit of analysis, from a largely Vygotskian position, that attempts to integrate the macro and the micro, and in section 2 I discuss the work of Basil Bernstein who offers an integrated sociological analysis.

In the mathematics classroom, interactions should not be seen as windows on the mind but as discursive contributions that may pull others forward into their increasing participation in mathematical speaking/thinking, in their zones of proximal development. Vygotsky's zone of proximal development is both a framework for the analysis of learning and a metaphor for the learning interaction. Elsewhere (Meira and Lerman, 2001) we have called it a symbolic space. I will outline a set of theoretical tools for the analysis of classroom interactions, drawing on this section. Readers can find initial attempts at such analyses in Lerman (2000c, 2001)

1. Discursive, Cultural Psychology

In the nineteenth century Durkheim and Marx challenged the image of the individual as the source of sense making and as the autonomous builder of her or his own subjectivity. Consciousness was to be seen as the result of social relations; in particular, relations to the means of production.

> It is not the consciousness of men that determines their being but, on the contrary, their social being that determines their consciousness. (Marx, 1859. p. 328/9)

Vygotsky's psychology was an application of Marx's theories to learning, providing a framework whereby the sociocultural roots of thought become internalised in the individual.

> Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, *between* people *(interpsychological),* and then *inside (intrapsychological)*. . .All the higher functions originate as actual relations between human individuals. (Vygotsky, 1978, p. 57)

Vygotsky was a prime inspiration in the growing interest in sociocultural roots of consciousness. Consciousness refers here not just to the culture-specific content of consciousness but also to the phenomenon of human consciousness, those features of human behaviour that distinguish the human from the animal. Vygotsky was keenly interested in studies of animal social behaviour and also of the "wolf child" precisely in order to focus on the processes whereby we become conscious in the full sense. Whilst all humans have the propensity that animals do not, the potential is not realised in the wolf child or the autistic child. I discuss this further below in relation to Tomasello's (1999) ideas.

But the move to a cultural, discursive psychology is also a response to other influences, such as Wittgenstein's later work on language, and the anti-essentialism of poststructuralism. From this perspective cultural, discursive psychology can be seen as a moment in sociocultural studies, as a particular focusing of a lens, as a gaze which is as much aware of what is not being looked at, as of what is. This image is an adaptation of Rogoff's planes of analysis, into a dynamic metaphor in which one might envisage a researcher choosing what to focus on in research through zooming in and out in a classroom, as with a video camera, and selecting a place to stop. Whilst the particular focus creates the object of research, the researcher must take into account how the object is constituted in its relations to the wider macro-situation and the micro-situations. That is to say that the task of researchers working with these theories in mathematics education is to make the links between structure and agency and between culture, history and power and students' learning of mathematics. Some authors have criticised psychology as not enabling an engagement with social life:

> Modern psychology has been incapable of making serious contributions to Third World (*sic*) development. . . it is important to point out that mainstream psychology has also failed to make significant contributions to national development and the lives of the poorest sectors in Western societies. (Harré, 1995, p. 54)

> In the process of individualizing its view of students, it (mathematics education) has lost any serious sense of the social structures and the race, gender and class relations that form these individuals. Furthermore, it is then unable to situate areas such as mathematics education in a wider, social context that includes larger programs for democratic education and a more democratic society.
> (Apple, 1995, p. 331)

I believe they are right to say that traditional psychology cannot provide such a language, when learning is seen as the individual's cognitive reorganisation, albeit *caused by* social, and physical factors (Glasersfeld, 1995, p. 66), one might add textual too, through equilibration. I argue here, however, that the move to a cultural, discursive psychology enables the link between the actions of individuals and groups in the classroom and history and culture, and that such a move is necessary for educational studies.

I have referred to the terms "sociocultural", "cultural" and "discursive" already in this article. These are familiar terms in cultural studies, in education, and increasingly in mathematics education but they are contested in their different uses of these terms. Moreover, bringing them together into a coherent account for research on mathematics teaching and learning which can meet the challenge I have offered here is the (rather ambitious) agenda of this article. There is no space to review all the uses of these terms but I will try to engage with some of the most developed work in the field in elaborating my own account. In brief, my aim is to acknowledge the insights of the emphasis on the mediating role of linguistic (and other) tools in developing human consciousness, and modern interpretations of culture, whilst also taking into account the force of discursive practices on developing identities. In education, and in particular in schooling, in which young children are encouraged (coerced?) into conforming to a collection of social practices, including that of doing school mathematics, the notion of regulation is clearly key. Describing performance in terms of discursive psychology (e.g. Edwards, 1997) and accounting for the cultural origins of human functioning, especially learning language (e.g. Tomasello, 1999), are thus essential but not sufficient for our concerns in mathematics education.

In the next three sections I will discuss discursive psychology, cultural psychology and sociocultural research, in order to make clear the orientation of this article. These terms all overlap, to some extent, and have their proponents and their disputes and it is important to locate some of those debates and find a path through them. I will conclude with a synthesis that moves the discussion on towards education and mathematics education in particular.

1.1. *Language and discursive psychology*

Culture, language and meaning precede us. We are born into a world already formed discursively. The reality or otherwise of the world or the certainty of our knowledge of it are not the issue: the issue is that we *receive* all knowledge of the world through language and other forms of communication. What things signify is learned by us as we grow into our cultures, the plurality arising from the multiple situations that constitute us: gender; class; ethnicity; colour; religion, and so on. Although we experience physical interactions in addition to social ones and we learn to use artefacts, what the physical objects and the nature of those interactions mean and what are the purposes and functions (history) of the artefacts for the individual is always mediated by culture. Physical interactions and artefacts, therefore, are also inherently social. Knowledge contents are culture specific, and consequently so too are world-views.

The idea that we *receive* knowledge of the world is intended to be a shorthand to emphasise that without the input of other humans and without the potential of an individual to benefit from that input, an individual would not develop as a human in the full sense of the word. Tomasello (1999) refers frequently to non-human primates and to autistic children to illustrate this. He argues that the key that can explain human evolution is the ability to understand conspecifics as intentional beings like the self. In the former case, non-human primates learn from their mothers and others how to function successfully, including using tools. This learning can be extended as can be seen in studies of humans teaching chimpanzees, for instance, but there are severe limitations. Chimpanzees do not become human, nor are they able to pass on what they have learned to other chimpanzees if they return to their habitats. In the latter case, autistic children cannot benefit from the input of other people because they are not able to understand others to be like themselves. These two examples illustrate Tomasello's argument for what I have referred to as the key to explain human evolution. That said, human functioning, even from a very early age, is a creative process, not a reductionist one whereby culture produces the individual. Any utterance or action can have a whole range of meanings and an individual has to try out her or his expectation of the intended meaning in her or his response. In new situations the individual has to make an informed guess at that intention, and on other occasions will initiate. These are already creative processes, to which the term *receive* does not do justice.

In addition to physical and social interactions, which I have argued become meaningful through language, there are instinctive reactions to be accounted for too but they become meaningful also through sociocultural interactions, and especially through language (though not necessarily verbal). Wittgenstein, in discussing pain, explains how the socio-cultural sense takes over from the instinctive:

> Words are connected with the primitive, the natural, expressions of the sensation and used in their place. A child has hurt himself and cries, and then adults talk to him and teach him exclamations and later sentences. They teach the children new pain behaviour. "So you are saying that the word "pain" really means crying?" – on the contrary: the verbal expression of pain replaces crying and does not describe it. (Wittgenstein. 1967, p. 89)

Vygotsky gives a very similar account of what happens for the very young child in the emergence of indicative gesture. Minick (1987) describes this process as follows:

> He [Vygotsky] argued that when the infant cries or reaches for an object, the adult attributes meaning to that behaviour. Though the infant has no communicative intent, these acts nonetheless function to communicate the infant's needs to his caretaker. Here, as in the adult's attempts to interact with the infant, the infant is included in communicative social activity before he has the capacity to use or respond adequately to communicative devices. Vygotsky argued that this provides the foundation for the transformation of the infant's behaviours into intentional indicative gestures. (Minick, 1987, p. 28)

This point is most important inasmuch as it is a view of the very foundations of all human behaviour and communication beyond the most primitive animal actions. In any case, given that communication and instruction begin from the very first moment of a child's life, discussion about basic instincts, or what is 'natural' in human behaviour, is almost exclusively rhetorical. Both Wittgenstein and Vygotsky argued, in these two quotes, that at first the child has no indicative intent; it is supplied by adults (the culture) and taken over by the child, that is, internalised (I discuss internalisation more fully below). Thus all intentional social behaviour, not solely what counts as knowledge, is constituted not at the initiative of the child but at the initiative of the adult, who supplies the meaning. As with my discussion of *receive* above, the phrase *initiative of the adult* should be taken as a shorthand with the intention of emphasising the central importance of the adult's input at all stages of the child's development, especially in the early stages. Of course beyond this early stage the child takes initiative in all aspects of social interaction whether in actions or in interpreting others' actions or in both. Edwards (1997 p. 39–41) gives other examples of studies of the very young, although he has another point to make, that being about observers' interpretations or accounts of such experiments. My point here is to argue that a discursive psychology is one that takes language and other forms of communication as critical in the possibility of an individual becoming a human being and that languages are culturally and temporally located. Thus "the subject matter of psychology has to take account of discourses, significations, subjectivities and positionings, for it is in these that psychological phenomena *actually* exist" (Harré and Gillett, 1994, p. 22).

At this point I want to address directly the apparent determinism of the account presented here. Clearly cultures change, ideas develop, and people are creative. If the challenge for individualistic psychology is to account for social life, its origins and its effects on individuals, the corresponding challenge for sociocultural theory is to account for creativity. I have already referred to the creative nature of communication and therefore learning, of negotiating meanings and intentions and initiating communication. Each person is the unique product of a range of socio-historical cultural communities and practices, of unconscious drives and desires, as well as propensities by virtue of genetic make-up and socio-cultural location. As a result, each person is positioned in any situation differently from any other person. This is the challenge for cultural, discursive psychology. "The study of the mind is a way of understanding the phenomena that arise when different sociocultural discourses are integrated within an identifiable human individual situated in relation to those discourses" (Harré and Gillett, 1994, p. 22)

The focus on language taken here opens up the possibility of analysing the role of language in two further aspects: the differential effects of social practices on different social groups; and the regulating effects of discursive practices. In education there is substantial evidence that economically disadvantaged groups do not perform as well as others,[1] and this can overlap with issues of race and ethnicity (Zevenbergen, 2000; Lubienski, 2001). Bernstein (1970) and Gee (1990), as well as others, have proposed powerful arguments that locate the source of the disadvantage in different linguistic codes of working class and middle class children. These are not deficit models, merely ones of differential opportunity being reproduced in families and communities through schooling. Poststructuralist analyses since the 1960s have indicated the role of discursive practices in regulation. Social practices are imbued with power/knowledge. This is a different view of power, which is generally seen as being held by particular people in particular situations. In this view power is a function of the discourse and its associated practices – hence discursive practices. People are positioned in practices as powerful or powerless according to the structure of the discourse and the personal histories of the participants. To take a somewhat stereotypical example, as a man I might be positioned as powerful in a male-dominated discourse on sport where a woman might be positioned as powerless: on the other hand my own history, physical attributes, or other features might position me as powerless in that discourse and a woman might be powerful. These are positionings that arise from a discourse and are located in language, in texts. I will return to these two aspects of discourse below, in relation to mathematics education.

I have discussed, albeit briefly, the fundamental role of language (verbal and other) in the development of human functioning and thus the sense in which psychology must be a discursive psychology, but the role of culture has also been referred to. I will now turn to cultural psychology.

1.2. *Cultural psychology*

In their introduction to "Sociocultural Studies of the Mind" Wertsch, del Río and Alvarez (1995) point out that there are a number of interpretations of cultural psychology and that there is not much overlap between them largely because the sources quoted by the writers differ. In attempting to draw these writers together Wertsch *et al* point to two intellectual origins, Wundt and Vygotsky, and they discuss some of the issues around the role of activity, which also divides the various writers. I will sketch some of the debates here.

Describing his approach to cultural psychology, Cole (1996, p. 349) writes: "I seek to derive its principles from activities located at the level of everyday practices and to return to those practices as a grounding for its theoretical claims." Central to his approach is the notion of artefacts (physical objects as well as language) as both material and conceptual, or ideal. "No word exists apart from its material instantiation (as a configuration of sound waves, hand movements, writing, or neuronal activity), whereas every table embodies an order imposed by thinking human beings" (ibid., p. 117). Artefacts are the product of human history and are culturally specific; hence Cole's preference for "cultural-historical psychology", or even "cultural, historical, activity theory", although he uses "cultural psychology" as the title of his book. Cole points to Wundt as the father of cultural psychology, in that he distinguished between an experimental and a cultural psychology, and also indicates that he takes inspiration from Vygotsky whose school of psychology took as its central thesis "that the structure and development of human psychological processes emerge through culturally mediated, historically developing, practical activity" (Cole, 1996, p. 108).

Shweder (1991) describes it as "the study of the way cultural traditions and social practices regulate, express, and transform the human psyche" (p. 73), and he goes on to emphasise the dialectical, dynamic nature of the dualities of subject/object, self/other, person/context, and so on. Bruner (1990) sees it as a shift away from the idea that culture is an overlay on human behaviour, the causes of which lie in biology, to the perspective that "culture and the quest for meaning within culture are the proper causes of human action" (p. 20).

There is some considerable debate concerning Vygotsky's emphasis on culture as mediating mind and consciousness and Leont'ev's emphasis on tools and objects in activity as mediating mind and consciousness (Zinchenko, 1995). I go along with Cole's goals for cultural psychology as a way of drawing together these two streams by focusing on artefacts in everyday practices, those artefacts having history in culture-specific ways.

However the discussion of the role of language in relation to consciousness in the previous section calls for a stronger emphasis on discourse. If there might be a danger of backgrounding the cultural and historical aspect of artefacts in working in a discursive psychology one can argue that there is a danger of backgrounding the discursive nature of all artefacts, material and symbolic, with an emphasis on the cultural-historical. At the same time, a cultural, discursive psychology "grounds its analysis in everyday life events" (Cole, 1996, p. 104). If one considers all the aspects of culture, including painting, making and developing physical artefacts, and so on, parts of which precede verbal communication historically, we can nevertheless see these as other forms of communication. The functions and meanings of those artefacts, paintings etc. are communicated to the young, perhaps by silent demonstration and gestures, but always through a form of language. Of course the development of verbal language was a hugely important step for the evolution of *Homo sapiens* (Tomasello, 1999) but the other forms of language continue to be essential parts of human communication.

1.3. *Sociocultural research*

> The goal of a sociocultural approach is to explicate the relationships between human action, on the one hand, and the cultural, institutional, and historical situations in which this action occurs, on the other.
>
> (Wertsch, del Río and Alvarez, 1995, p. 11)

Defined in this way, I would argue that cultural, discursive psychology lies within a sociocultural approach, as a moment of action (ibid., p. 11), rather than a separate process, although some authors attribute the label directly to Vygotsky and followers (Forman, in press), thus identifying sociocultural research as psychological.

I have argued elsewhere (Lerman, 2000a) that whilst many people refer to the work of Vygotsky and followers as social constructivism it can be very misleading. The metaphor of construction is a useful one in the context of human learning, but today constructivism is firmly associated with a school of psychology that searches for *universal* features of development. Indeed a number of writers in mathematics education who have modified their constructivist orientation by complementing it with social strands of thought have used the label social constructivism. For this reason I proposed there, in agreement with others (e.g. Forman, in press), that we use sociocultural to describe the work of Vygotsky and followers and not social constructivism.

1.4. *A synthesis*

Until the last 15 years or so, mathematics education tended to draw on mathematics itself, or psychology, as disciplines for the production of knowledge in the field (Kilpatrick, 1992). Theoretical frameworks for focusing on the social origins of knowledge and consciousness began to appear in the mathematics education literature towards the end of the 1980s. Shifts in perspectives or the development of new paradigms in academic communities are the result of a concatenation of factors within and around the community. In our own community these include: the emergence into the mainstream community of Ethnomathematics in 1984 (D'Ambrosio, 1984); a concern amongst mathematics educators in many parts of the world for a greater democratisation of the mathematics classroom through a shift in authority from the teacher to the students; a concern with increasing equity so that increased numbers of students gained certification in school mathematics; the publishing of a number of key research papers and books at around the same time (Lave, 1988; Bishop, 1988; Walkerdine, 1988; Carraher, 1988); and the special day at the Sixth International Congress on Mathematical Education devoted to Mathematics, Education and Society in 1988. The theoretical fields now drawn upon by mathematics education researchers include a range of theories that take language and social practices as constitutive of consciousness, behaviour and learning. These are: social practice theory (also called situativity, communities of practice and situated cognition); sociology; and Vygotskian theories. The elements of cultural, discursive psychology that I have gathered together here incorporate elements of each. I will end this section by discussing the issue of a unit of analysis for research.

Vygotsky searched for a suitable unit of analysis for psychology that combined all the elements of human social behaviour: affect, cognition, communication and meaning. According to Minick (1987), Vygotsky moved from "the 'instrumental act' and the 'higher mental functions' ... to the emergence of 'psychological systems' " (p. 24) and then to his third and final formulation, that "the analysis of the development of word meaning must be carried out in connection with the analysis of word in communication" (p. 26). Further on, Minick writes "In 1933 and 1934 Vygotsky began to reemphasize the central function of word meaning as a means of communication, as a critical component of social practice" (p. 26). Minick pointed out (p. 18) that there is a continuity among these three stages and that they should be seen as developments, each stage incorporating the other and extending it. In the second stage, Vygotsky and Luria had carried out their seminal study (Luria, 1976) on the effects of language development on the higher mental functions, a classic piece of research

(Brown and Dowling, 1998) and characteristic of Vygotsky's approach in that stage. What was missing was "the child's practical activity" (p. 26), and in the third stage he argued for the importance of incorporating goals and needs into the unit of analysis (p. 32). This is entirely in the spirit of the cultural, discursive psychology argued for in this paper.

The title of Vygotsky's book *Mind in Society* captures that unit, and it is also expressed by Lave and colleagues as person-in-practice and by Wertsch as person-acting with mediational means (Wertsch, 1991, p. 12). We could extend that unit further by taking account of the discussion of the regulating features of social, discursive practices. As a person steps into a new practice, in social situations, in schooling, in the workplace, or other practices, the regulating effects of that practice begin, positioning the person in that practice. Goals and needs are modified by the desire to participate, the desire not to participate, or the many other possible positions. Even if a person withdraws from a practice after a short time, she or he has been changed by that participation. We might therefore talk of practice-in-person to capture the regulative effects of participation. Combining these, we might talk of a unit of analysis of person-in-practice-in-person, or mind-in-society-in-mind (Slonimsky, 1999).

Thus, as researchers, searching for evidence of, or ways to bring about, mathematical "understanding" as a decontextualised mental process might best be abandoned. Instead, the focus for researchers would be on the developing identities of students as speakers and actors of mathematics in school classrooms, the student-in-mathematics-classroom-in-student. The elements of identity include: the ways in which the mathematical activities have been framed by the teacher, the texts, and the students' previous experiences; the ways in which the social relationships have been framed; the positions produced in the classroom; and the histories and functions of the mathematical artefacts. In the next section I will attempt to turn these elements into a procedure for research, or a language of description (Brown and Dowling, 1998).

2. Learning

From the discursive, cultural point of view, learning is an initiation into social practices and the meanings that are part of those practices. Following Lave and Wenger (1991) "participation in the cultural practice in which any knowledge exists is an epistemological principle of learning" (p. 98). Initiation and participation are not always voluntary or intentional processes. One can be misled by studies of tailoring apprenticeships in West Africa or of clerks in insurance claims offices. School classrooms are quite different kinds of practices. First, pupils do not choose to be at school. Society determines that children should attend school for a certain period of their life. The social (e.g. preparation for the world of work or fulfilment of the individual) and cultural (e.g. transmission[2] of areas of knowledge designated as desirable or essential) purpose of schooling differs across the world, based on political, economic and cultural determinants as well as, it must be said, the inertia of systems with often many decades of history. Indeed, the economic system and the very concept of modern adult working lives are built around children's absence from the home and their presence in school during the day. Second, given the age range covered by compulsory schooling, participants' identities are at their most formative, and children are particularly vulnerable to the regulating effects of social practices. Although all practices are overlapping with others, the school classroom is particularly affected by other practices since they are often of greater significance to the students than the intentions of the school and the teacher. More important to students than learning what the teacher has to offer are aspects of their peer interactions such as gender roles, ethnic stereotypes, body shape and size, abilities valued by peers, relationship to school life, and others. The ways in which individuals want to see themselves developing, perhaps as the classroom fool, perhaps as attractive to someone else in the classroom, perhaps as gaining praise and attention from the teacher or indirectly from their parents, lead to particular goals in the classroom and therefore particular ways of behaving and to different things being learned, certainly different from what the teacher may wish for the learners (Boaler, 2000). Third, the teacher intends to teach, that is the rationale for her or his function. In the work place the expert is not there to teach, but to work efficiently and be productive, the induction of a newcomer being a process to be undertaken in order to increase the efficiency and success of the enterprise. Lave's emphasis on learning, separating it from teaching, is an outcome of the focus on work-place practices. Of course learning does not always result from intentional teaching and where it does, the learning can be differentially acquired. But all learning is from others.

As researchers, the goals and desires that are associated with the multiple practices of the classroom must form a part of the analyses we carry out. We must also take into account how practices position different people in different ways. Walkerdine shows how the notion of 'child' is produced in the practices of educational psychology (1988; see also Burman, 1994), differentially positioning those who conform – white boisterous males, and those who don't – non-white people, girls, quiet boys and so on. Significations matter, they are not neutral meanings: situating meanings in practices must also take into account how those significations matter differently to different people. Practices should be seen, therefore, as discursive formations within which what counts as valid knowledge is produced and within which what constitutes successful participation is also produced. Nonconformity is consequently not just a feature of the way that an individual might react as a consequence of her or his goals in a practice or previous network of experiences, but also of the practice itself.

Walkerdine (1988; see also Klein, 1997; Evans, 2000; Walshaw, 2000) draws on post-structuralism for her account of how identities are produced in practices, in the context of mathematics classrooms. These analyses require empirical study to reveal the range of identities that will be produced. Another kind of analysis is offered by Bernstein, whose description of the links between powerful groups in society and the forms of pedagogic-practice that are legitimated was mentioned above.

To elaborate a little further, Bernstein's work over a number of decades has focused on how power and control are manifested in pedagogic relations. In particular he has looked at how the boundaries between discourses, such as those of the secondary school curriculum, are defined, what he calls the classification rules, and how control is effected within each discourse, the framing rules. As a principle, pedagogic discourse is the process of moving a practice from its original site, where it is effective in one sense, to the pedagogic site where it is used for other reasons; what he calls the principle of recontextualization. In relation to work practices he offers the example of carpentry which was transformed into woodwork (in UK schools), and now forms an element of design and technology. School woodwork is not carpentry as it is inevitably separated from all the social elements, needs, goals, and so on, which are part of the work practice of carpentry and cannot be part of the school practice of woodwork. Similarly, school physics is not physics, and school mathematics is not mathematics. Bernstein argues that recontextualization or transformation opens a space in which ideology always plays. In the transformation to schooling, values are always inherent in content selection, assessment and pedagogy. Bernstein (e.g. 1996) draws the links between dominant social groups and the practices of the classroom, which produce social positions of students and teachers. He has described a range of pedagogic styles, or modes, their origins in social ideologies and their supporting theoretical justifications, and has shown how classroom practices produce behaviours that legitimate what constitutes appropriate mathematical knowing within those modes. Dowling (1998), Cooper and Dunne (1999), Ensor (1999), Brown (1999), Lerman and Tsatsaroni (1998) and Morgan, Tsatsaroni and

Lerman (in press) are amongst those who have used Bernstein's theories in studies of mathematics teaching and learning.

The mathematical practices within a class or school, the way in which they are classified and framed, the state/community/school values which are represented and reproduced, and the teacher's own goals and motives, form the complex background to be taken into account by the research community. According to Lave, mathematics itself should be seen not as an abstract mathematical task but as something deeply bound up in socially organised activities and systems of meaning within a community. Nor, for that matter, should it be seen as a single practice. Burton (1999) has found that mathematicians identify themselves by their sub-field, as statistician, applied mathematician, mathematical modeller, or topologist. In relation to school mathematics one must be aware of the particular nature of the identities produced. Boaler (1997) has shown how different approaches to school mathematics produce different identities as school mathematicians. She suggests also that the identities produced in one of the two schools in her study, Phoenix Park school, which used a mathematics curriculum built around problem solving, overlap with students' mathematical practices outside of school. Boaler uses both Bernstein's analysis in terms of classification and framing and Lave's communities of practice as resources to explain her findings. Recently Boaler (1999) has talked of the particular practices of the two schools as offering constraints and affordances (Greeno and MMAP, 1998) as a way of interpreting the students' behaviours which resulted in them working to succeed, in the distinct terms of each school.

Summarising the points made so far, research in discursive, cultural psychology in mathematics teaching and learning includes the ideas of Vygotsky, including the zone of proximal development, intersubjectivity, internalisation and semiotic mediation; the functioning of discursive practices, including positioning and 'voice'; the social relationships in the classroom; the mathematical artefacts; and development as a process of thinking/speaking mathematics. In elaborating on each of these, I will attempt to operationalise them as tools of research, developing a toolkit for research in mathematics education from the perspective of cultural, discursive psychology. In Lerman (2000c, 2001) I have given some examples of applying this toolkit to mathematics classroom transcripts and I will give some examples here to illustrate each of the elements.

2.1. *Intersubjectivity and internalisation*

Vygotsky's thoughts about intersubjectivity are summarised in the quotation in the introduction to this article. Following on from the discussion about instruction beginning from the first day of a child's life, in that adults supply intention, meaning, and all higher functions, I would argue that intersubjectivity is prior to interactions, and forms the foundation for cultural, discursive psychology (see Lerman, 1996; Steffe and Thompson, 2000; Lerman, 2000a). Studying intersubjectivity requires examining the resources, through language, that the teacher, texts, peers and others supply as well as the ideas that emerge in joint activity.

For Vygotsky, internalisation is not the process of bringing knowledge from the external world into the pre-existing internal mental plane of the individual. It is precisely the difficulty of making sense of such an idea that was one of the motives for Piaget's genetic epistemology, since the candidate appears to be naive empiricism, or else one appears to have to rely on a form of platonism. Piaget's turn to a focus on how knowledge is acquired was his reaction to the problem, leading to the constructivist model. Instead, given the arguments above in terms of consciousness as a product of social relations, the sociocultural move is to argue that the internal mental plane is formed in the process of internalisation (Leont'ev, 1981). This formulation overcomes the mind-body duality of other notions of internal/external. Some examples may shed some light on this characterisation of the process. The first is the explanation of attribution of intention by adults overlaying the spontaneous gestures of the child, supplying meaning, or Wittgenstein's discussion of pain behaviour, as described above. Thus the process of becoming human is the forming of the mental plane in specific sociocultural situations: hence our gendered, ethnic, racial, (etc.) identities. The second example is to consider the ways in which tools transform the world for the individual. The hammer has a history of its function in human activity. Without knowledge of such a tool and its purpose, any idea of joining two objects together, hanging an object on a wall, and so on, are inconceivable. Learning about the hammer does more for the individual than give her or him something to act with; the world of possibilities for actions emerges. A more recent example from mathematics education is the 'drag' action in dynamic geometries. Internalising the tool transforms the way one can act geometrically, enabling conjectures to be generated, for example, that are unique to the dynamic geometry environment, as a result of that tool (Mariotti and Bartolini Bussi, 1998).

Evidence of internalisation can be identified in students' use of language as artefact in doing mathematics. For instance, a student turning to substitution of numbers in an algebraic expression, as a legitimate act in that context, is a micro-genetic illustration of internalisation much like the reference to and use of the drag action. Similarly, students' attempts to take over the language of the teacher, whether it is used correctly or not, are stages in that internalisation process. As Tomasello (1999) indicates in a multitude of examples, this is always an essentially creative act. The teacher's move to overlay students' language with the correct language of mathematics (Forman, in press) is another area of evidence of the process of intersubjectivity and internalisation.

2.2. *The zone of proximal development and semiotic mediation*

Vygotsky's well-known notion offers a description of the whole process of learning, whether it be from a teacher/authority, a peer, or a cultural product such as a text book, as well as a tool for studying learning. A wide range of studies have examined aspects of the zone of proximal development (zpd), some of which are included in the collections edited by Moll (1990) and by Forman, Minick and Stone (1993). In Meira and Lerman (2001) we argue, with others, that the zpd would be better conceptualized not as a physical space, in the sense of the individual's equipment (either cognitive or communicative), but as a symbolic space involving individuals, their practices and the circumstances of their activity. This view takes the zpd to be an ever-emergent phenomenon triggered, where it happens, by the participants catching each other's activity. It is often fragile and where it is sustained, a process of semiotic mediation and interaction emerges. Both teacher and student can be pulled into their zpds by a combination of the activity, the actors, and appropriate communication. That the teacher is also pulled into their zpd and therefore learns from those interactions is an important feature that is not captured in notions of scaffolding. Thus whilst the contributions of the actors in a zpd are from their individual pasts their joint activity pulls them towards their tomorrows.

From a Vygotskian point of view, the process of learning a scientific concept is a process of mediation between the subject and the object of knowledge. The interaction between the individual and the real world is regulated and transformed by the use of symbolic material, cultural tools. The ways of organising the world that are created by culture, within a social and historical context, become internalised by the person through symbolic matter, as the internal plane is formed, that is, as temporally and culturally situated consciousness develops. Our access to the real is provided through the forms of reality that are given by the symbolic systems available within the particular cultural context of learning. It is the culture that offers the individual the symbolic systems of representing reality, and through them, the universe of meanings that allow one to interpret and to organise the data collected from the real experience in the world.

In our ongoing research we have studied aspects of the functioning of the zpd and of the process of semiotic mediation. Looking for evidence of zpds involves focusing on the actors catching each other's thoughts (Vile and Lerman, 1996). It does not always happen when students or teacher and student interact (Lerman, 2001; Meira and Lerman, 2001). When it does, we claim that the emergence of a zpd enables the teacher and learner or the peer learners to become mutually orientated towards socially and culturally mediated meanings.

2.3. *Positioning and voice in classroom mathematical practices*

There is a range of positions for participants in practices, depending on the degree of their participation, perhaps best described as the development of their identity in that practice. Previous experiences, personal goals, needs and interests are key elements, and so too are the differences that are a result of the realisation of social forces in practices, as Bernstein has shown in relation to schooling. Theoretical analyses of positioning as well as empirical studies (Wenger, 1998; Evans, 2000; Morgan, 1998; Morgan and Lerman, 2000; Klein, 1997) are both required in research.

Different positions can be adopted by participants, an extreme example being resistance, especially in "coercive" practices such as schooling. Such positions can be identified in classrooms in students' behaviours. Another perspective is offered by noting that the practices of the mathematics classroom *produce* positions, as discussed at some length in this article. In the study which appears in Lerman (2001) the teacher positions one student of a collaborative pair as more able than the other and expects him to help his less able partner. My observations suggested that this judgement is not reflected in the content of the mathematical work each student produces, but is reflected in who is powerful and who is powerless in the manner of their interactions during their joint mathematical activity. The framing of the mathematical activities in the classroom, in Bernstein's (e.g. 1996) sense, positions students differently. For example, errors made by working class students to the question below (see Fig. 1), reported in Zevenbergen and Lerman (2001), were revealed as arising from the students being positioned within everyday thinking by the context of the question (Cooper and Dunne, 1999).

While many of the students were able to respond to the task correctly, particularly when the task was given to older students, it was more likely that when incorrect responses were offered, they were from students from working-class backgrounds (Zevenbergen, 1991). Typically, incorrect responses involved answering the question as if it were a task involving identification of the shape of their gardens at home. For example:

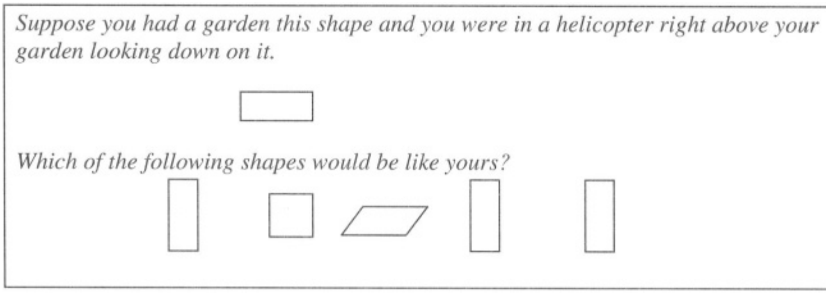

Figure 1. Shape recognition question.

R: Why did you take that shape [the square]?
Girl: Because is looks like the shape of my garden.
R: Is your garden at home like that?
Girl: Yes.

Boy: None of those.
R: Why aren't any of them the same?
Boy: My garden goes like that [draws a semi-circle in the air].

The notion of voice has at least two aspects: the expression of individuality, and the mathematical "voice" as a particular register amongst other voices (Cazden, 1993). Vygotsky's cultural psychology is often criticised as being a reductionist account of the individual, and it is then proposed that Piaget's theories can replace this missing element through a synthesis (Confrey, 1995). However, it is not the individualism of private world-views which has dominated the debate around subjectivity and voice in cultural studies in recent decades. In cultural, discursive psychology individuality is the uniqueness of each person's collection of multiple subjectivities, through the many overlapping and separate identities of gender, ethnicity, class, size, age, etc., to say nothing of the 'unknowable' elements of the unconscious.

Discourses which dominate in the classroom, and everywhere else for that matter, distribute powerlessness and powerfulness through positioning subjects (Evans, 2000). Walkerdine and the Girls and Maths Unit (1989, p. 143) report of a classroom incident in which the emergence of a sexist discourse bestows power on five-year-old boys, over their experienced teacher, dramatically illustrates the significance of a focus on discourse, not on individuals. In some research on children's interpretations of bigger and smaller, Redmond (1992) found some similar evidence of meanings being located in practices.

These two were happy to compare two objects put in front of them and tell me why they had chosen the one they had. However when I allocated the multilinks to them (the girl had 8 the boy had 5) to make a tower . . . and I asked them who had the taller one, the girl answered correctly but the boy insisted that he did. Up to this point the boy had been putting the objects together and comparing them. He would not do so on this occasion and when I asked him how we could find out whose tower was the taller he became very angry. I asked him why he thought that his tower was taller and he just replied "Because IT IS." He would go no further than this and seemed to be almost on the verge of tears. (p. 24)

Many teachers struggle to find ways to enable individual expression in the classroom, including expressing mathematical ideas, confronting the paradox of teachers giving emancipation to students from their authoritative position. But this can fruitfully be seen as a dialectic, whereby all participants in an activity manifest powerfulness and powerlessness at different times, including the teacher. When those articulations are given expression, and not denied as in some interpretations of critical pedagogy (Lerman, 1998c), shifts in relations between participants, and crucially between participants and learning, can occur (Ellsworth, 1989; Walcott, 1994).

2.4. *Social relationships*

I am using this rather catch-all term to describe the multitude of issues that matter to students in the classroom, in addition to the expectations of the teacher. Eliciting what matters will be problematic, given that the research tools, such as interviews, occur after events, and are highly contextualised, not least in that the person interviewing creates a new relationship. Inferences drawn from videos and transcripts, in addition to interviews with students and teacher can supply conjectures about goals and needs, and explanations of behaviour, but we are forced to admit that we cannot arrive at what is going on for students in our studies, only what they might choose to tell us or what we might conjecture from studying voice inflections, gestures and so on. For instance, in the Lerman (2001) study already mentioned the so-called less able student, when interviewed during video-stimulated recall, reported the incident as an occasion in which he was assisted by his friend amongst the many cases where either he helped his friend or his friend helped him. Whether this was for the benefit of his standing in conversation with the researcher or an appropriate reflection of their interactions across time we cannot know and can only surmise.

2.5. *Mathematical artefacts*

From the point of view of cultural, discursive psychology, students are provided with mathematical language, meanings, connections, strategies, artefacts such as diagrams, graphs, physical tools (rulers, calculators) and how to 'read' them, and methods, by the teacher(s), texts, peers and others. These are the tools with which students think and speak mathematically. As researchers we can examine the discourse in the classroom for the artefacts provided and for those used by the students (Adler, 1998, 2001). Clearly a key issue is the history of these artefacts. Thus, in dealing with ratios of algebraic expressions, for example, one is dealing with the sophisticated and highly abstract grammar of algebra and the generalised techniques of arithmetic. One can simplify the ratio if there are factors in common by *division* (e.g. *ab:ab*) and one can also *substitute* numerical values for the letters in order to examine, through specialising, the workings of the expression, returning to the general for the (mathematically) required expression that is the simplest (in this case 1 : 1). In Lerman (2001) the students each work with just one of these artefacts, both having been demonstrated by the teacher before the episode, but they are not able to catch each other's thoughts and share the artefacts.

2.6. *Development as a process of thinking/speaking mathematics*

I have already discussed many aspects of becoming mathematical as being able to think/speak mathematics as it is legitimised in the classroom by the teacher's framing of what are considered legitimate mathematical texts. The cultural, discursive view places practice in place of objective reality; the social practices of mathematics are constitutive of its meaning (Solomon, 1998). Thus the child is not expected to arrive at the objective reality of the structures of mathematics by herself or himself, pulling herself or himself up by the bootstraps of reflective abstraction and being pathologised if she or he cannot manage to arrive at those structures alone. Learning school mathematics is nothing more than initiation into the practices of school mathematics, hence the central role of the initiator, the teacher. The phrase "nothing more than" is not to play down the great difficulties children experience in learning mathematics, merely to emphasise that there is nothing beyond. Learning mathematics or learning to think mathematically is learning to speak mathematically. What constitutes an acceptable grammatical construction, in mathematics, is what is approved of within the discourse. Over time, studies of the development and increasing sophistication of students' language in mathematics indicate their becoming mathematical.

3. Concluding Remarks

I have attempted to construct a coherent theoretical framework within sociocultural theory, what I have called a cultural, discursive psychology, and then attempted to set out and exemplify a toolkit for realising that approach in research on mathematics teaching and learning. The specifics of that toolkit must be established in every piece of research. Thus, in previous work (Lerman, 2001) I identified features of students' interaction that would lead to generalisations about the emergence of a zpd, drawing on elements from sections 2.1 to 2.5 above. In this way the theoretical descriptions of the toolkit are translated into identifiers for analysis of data.

These are emergent thoughts, ideas and experiences, and represent an argument for one point of view in a community that draws on several different and competing perspectives. Other papers in this special issue engage with other aspects of discursive approaches. Perhaps a common feature, certainly emphasised in this article, is the notion that classroom discourse does not offer the observer a window on the mind precisely because "mind" is not static, or decontextualised, but responds to the context, the activity, and power/knowledge, and is oriented to communicate and to act. The tools for analysis, then, must view the data as student-in-mathematics-classroom-in-student in a holistic approach, taking into account both structure and agency.

Notes

1. The notion of performance should be taken as relative to middle class schooling. Studies such as Labov's (1970) demonstrate the quite different perspective of Black English, whereby it becomes evident that what is accepted as the norm is actually oppressive of other language forms, usually associated with minority ethnic groups.
2. This term should not be taken to refer to a style of teaching much opposed today, but as an inclusive term for a range of ways in which students might gain such cultural knowledge.

Acknowledgement

The author acknowledges the invaluable help given by the editors of this special issue and the reviewers.

REFERENCES

de Abreu, G.: 1998, 'Reflecting on mathematics learning in and out-of-school from a cultural psychology perspective', in A. Olivier and K. Newstead (eds.), *Proceedings of the Twenty-second Conference of the International Group for the Psychology of Mathematics Education,* Faculty of Education, The University of Stellenbosch, Stellenbosch, South Africa, Vol. 1, pp. 115–130.

Adler, J.: 1998, 'Resources as a verb: Reconceptualising resources in and for school mathematics', in A. Olivier and K. Newstead (eds.), *Proceedings of the Twenty-second Conference of the International Group for the Psychology of Mathematics Education,* Faculty of Education, The University of Stellenbosch. Stellenbosch, South Africa, Vol. 1, pp. 1–18.

Adler, J.: 2001, *Teaching mathematics in multilingual classrooms.* Kluwer Academic Publishers, Dordrecht, The Netherlands.

D'Ambrosio, U.: 1984, 'Socio-cultural bases for mathematical education', in *Proceedings of the Fifth International Congress on Mathematical Education,* Birkhäuser, Boston, pp. 1–6.

Apple, M.: 1995, 'Taking power seriously: New directions in equity in mathematics education and beyond', in W.G. Secada, E. Fennema and L.B. Adajian (eds.), *New Directions for Equity in Mathematics Education,* Cambridge University Press, Cambridge, UK, pp. 329–348.

Bernstein, B.: 1970, 'Social class differences in communication and control', in W. Brandis and D. Henderson (eds.), *Social Class, Language and Communication,* Routledge, London.

Bernstein, B.: 1996, *Pedagogy, Symbolic Control and Identity: Theory, Research, Critique,* Taylor and Francis, London.

Bishop, A.J.: 1988, *Mathematical Enculturation,* Kluwer Academic Publishers, Dordrecht, The Netherlands.

Boaler, J.: 1997, *Experiencing School Mathematics: Teaching Styles, Sex and Setting,* Open University Press, Buckingham, UK.

Boaler, J.: 1999, 'Participation, knowledge and beliefs: A community perspective on mathematics learning', *Educational Studies in Mathematics* 40, 259–281.

Boaler, J.: 2000, 'Mathematics from another world: Traditional communities and the alienation of learners', *Journal of Mathematical Behavior* 18(4), 1–19.

Brown, A.J.: 1999, *Parental participation, positioning and pedagogy: a sociological study of the IMPACT primary school mathematics project*, University of London, Unpublished PhD Thesis.

Brown, A. and Dowling, P.: 1998, *Doing research/reading research: A mode of interrogation for education,* Falmer, London.

Bruner, J.: 1990, *Acts of Meaning,* Harvard University Press, Cambridge, MA.

Burman, E.: 1994, *Deconstructing Developmental Psychology,* Routledge, London.

Burton, L.: 1999, 'The Practices of mathematicians: What do they tell us about coming to know mathematics?', *Educational Studies in Mathematics* 37, 121–143.

Carraher, T.: 1988, 'Street mathematics and school mathematics', in A. Borbás (ed.), *Proceedings of the Twelfth Conference of the International Group for the Psychology of Mathematics Education,* PME Program Committee, Veszprem, Hungary, Vol. 1, pp. 1–23.

Cazden, C.B.: 1993, 'Vygotsky, Hymes and Bakhtin: From word to utterance and voice', in E.A. Forman, N. Minick and C.A. Stone (eds.), *Contexts for Learning: Sociocultural Dynamics in Children's Development,* Oxford University Press, New York, pp. 197–212.

Cobb, P.: 2000, The importance of a situated view of learning to the design of research and instruction', in J. Boaler (ed.), *Multiple Perspectives on Mathematics Teaching and Learning,* Ablex, Westport, CT, pp. 45–82.

Cole, M.: 1996, *Cultural Psychology: A Once and Future Discipline,* Harvard University Press, Cambridge, Mass.

Confrey, J.: 1995, 'Student voice in examining "splitting" as an approach to ratio, proportions and fractions,' in L. Meira and D. Carraher (eds.), *Proceedings of the Nineteenth Conference of the International Group for the Psychology of Mathematics Education,* PME Program Committee, Recife, Brazil, Vol. 1, p. 3–29.

Cooper, B. and Dunne, M.: 1999, *Assessing Children's Mathematical Knowledge,* Open University Press, Buckingham, UK.

Dowling, P.: 1998, *The Sociology of Mathematics Education: Mathematical Myths/ Pedagogic Texts,* Falmer Press, London.

Edwards, D.: 1997, *Discourse and Cognition,* Sage, London.

Ellsworth, E.: 1989, 'Why doesn't this feel empowering? Working through the repressive myths of critical pedagogy', *Harvard Educational Review* 59(3), 297–324.

Ensor, P.: 1999, *A study of the recontextualising of pedagogic practices from a South African University preservice mathematics teacher education course by seven beginning secondary mathematics teachers,* Unpublished PhD dissertation, University of London.

Evans, J. T.: 2000, *Adults' Mathematical Thinking and Emotions: a Study of Numerate Practices,* Falmer, London.

Forman, E.: in press, 'A sociocultural approach to mathematics reform: Speaking, inscribing and doing mathematics within communities of practice', in J. Kilpatrick, G. Martin, and D. Schifter (eds.), *A Research Companion to the NCTM Standards,* National Council of Teachers of Mathematics, Reston, VA.

Forman, E.A., Minick, N. and Stone, C.A. (eds.): 1993, *Contexts for Learning: Sociocultural Dynamics in Children* 's *Development,* Oxford University Press, New York.

Gee, J.P.: 1990, *Social Linguistics and Literacies: Ideology in Discourses,* Taylor and Francis, London.

Glasersfeld, E. von: 1995, *Radical Constructivism: A Way of Knowing and Learning,* Falmer, London.

Greeno, J.G. and MMAP: 1998, 'The situativity of knowing, learning and research', *American Psychologist* 53(1), 5–26.

Harré, R.: 1995, 'But is it science? Traditional and alternative approaches to the study of social behaviour', *World Psychology* 1(4), 47–78.

Harré, R. and Gillett, G.: 1994, *The Discursive Mind,* Sage, London.

Kilpatrick, J.: 1992, 'A history of research in mathematics education', in D.A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning,* MacMillan, New York, pp. 3–38.

Klein, M.: 1997, 'Constructivist practice in preservice teacher education in mathematics: Aboriginal and Torres Strait Islander voices heard yet silenced', *Equity and Excellence in Education* 30(1), 65–71.

Labov, W.: 1970, 'The logic of non-standard English', in F. Williams (ed.), *Language and Poverty,* Markham Press, Chicago.

Lave, J.: 1988, *Cognition in Practice: Mind, Mathematics and Culture in Everyday Life,* Cambridge University Press, Cambridge, UK.

Lave, J. and Wenger, E.: 1991, *Situated Learning: Legitimate Peripheral Participation,* Cambridge University Press, New York.

Leont'ev, A.N.: 1981, 'The problem of activity in psychology', in J.V. Wertsch (ed.), *The Concept of Activity in Soviet Psychology,* Sharpe, Armonk, NY, pp. 37–71).

Lerman, S.: 1996, 'Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm?', *Journal for Research in Mathematics Education* 27, 133–150.

Lerman, S.: 1998a, 'A moment in the zoom of a lens: Towards a discursive psychology of mathematics teaching and learning', in A. Olivier and K. Newstead (eds.), *Proceedings of the Twenty-second Conference of the International Group for the Psychology of Mathematics Education,* PME Program Committee, Stellenbosch, South Africa, Vol. 1, pp. 66–81.

Lerman, S.: 1998b, 'Cultural perspectives on mathematics and mathematics teaching and learning', in F. Seeger, J. Voigt and U. Waschescio (eds.), *The Culture of the Mathematics Classroom: Analyses and Changes,* Cambridge University Press, New York, pp. 290–307.

Lerman, S.: 1998c, 'The intension/intention of teaching mathematics', in C. Kanes (ed.), *Proceedings of Mathematics Education Research Group of Australasia,* Griffith University at the Gold Coast, Australia, Vol. 1, pp. 29–44.

Lerman, S.: 2000a, 'A case of interpretations of social: A response to Steffe and Thompson', *Journal for Research in Mathematics Education* 31, 210–227.

Lerman, S.: 2000b, 'The social turn in mathematics education research', in J. Boaler (ed.), *Multiple Perspectives on Mathematics Teaching and Learning,* Ablex, Westport, CT, pp. 19–44.

Lerman, S.: 2000c, 'Some problems of socio-cultural research in mathematics teaching and learning', *NOMAD* 8(3), 55–71.

Lerman, S.: 2001, 'Accounting for accounts of learning mathematics: Reading the ZPD in videos and transcripts', in D. Clarke (ed.), *Perspectives on Meaning in Mathematics and Science Classrooms,* Kluwer Academic Publishers, Dordrecht, The Netherlands, pp. 53–74.

Lerman, S. and Tsatsaroni, A.: 1998, 'Why children fail and what mathematics education studies can do about it: The role of sociology', in P. Gates (ed.), *Proceedings of First International Conference on Mathematics, Education and Society (MEAS1),* Centre for the Study of Mathematics Education, University of Nottingham, pp. 26–33.

Lubienski, S.T.: 2001, *A second look at mathematics achievement gaps: Intersections of Race, Class and Gender in NAEP Data,* Paper presented at the annual meeting of the American Educational Research Association, Seattle.

Luria, A.R.: 1976, *Cognitive Development: Its Cultural and Social Foundations,* Harvard University Press, Cambridge, MA.

Mariotti, M.A. and Bartolini Bussi, M.G.: 1998, 'From drawing to construction: Teacher's mediation within the Cabri environment', in A. Olivier and K. Newstead (eds.), *Proceedings of the Twenty-second Conference of the International Group for the Psychology of Mathematics Education,* PME Program Committee, Stellenbosch, South Africa, Vol. 3, pp. 247–254.

Marx, K.: 1859, 'A preface to contribution to the critique of political economy', In *Marx and Engels Selected Works,* Lawrence and Wishart, London, pp. 361–365.

Meira, L. and Lerman, S.: 2001, 'The Zone of Proximal Development as a Symbolic Space', *Social Science Research Papers,* Faculty of Humanities and Social Science, South Bank University, London, UK.

Minick, N.: 1987, 'The development of Vygotsky's thought: An introduction', in R.W. Rieber and A.S. Carton (eds.), *The Collected Works of L. S. Vygotsky. Volume 1: Problems of General Psychology,* Plenum Press, New York, pp. 17–36.

Moll, L.C. (ed.): 1990, *Vygotsky and Education,* Cambridge University Press, Cambridge, UK.

Morgan, C.: 1998, *Writing Mathematically: The Discourse of Investigation,* Falmer, London.

Morgan, C. and Lerman, S.: 2000, 'Teachers' positions in assessment discourses: Including a sociological perspective on the mathematics classroom', in T. Nakahara and M. Koyama (eds.), *Proceedings of Twenty-fourth Conference of the International Group for the Psychology of Mathematics Education,* PME Program Committee, Hiroshima, Japan. Vol. 1, p. 173..

Morgan, C., Tsatsaroni, A. and Lerman, S.: in press, 'Mathematics teachers' positions and practices in discourses of assessment', *British Journal of Sociology of Education.*

Nunes, T., Schliemann, A. and Carraher, D.: 1993, *Street Mathematics and School Mathematics,* Cambridge University Press, New York.

Redmond, J.: 1992, *Are 4–7 year-old children influenced by discursive practices when asked to make comparisons using quantities?,* Unpublished manuscript, South Bank University, London, UK.

Saxe, J.: 1991, *Culture and Cognitive Development: Studies in Mathematical Understanding,* Lawrence Erlbaum Associates, Hillsdale, NJ.

Sfard, A.: 1998, 'On two metaphors for learning and the dangers of choosing just one', *Educational Researcher* 27(2), 4–13.

Shweder, R.A.: 1991, *Thinking Through Cultures,* Harvard University Press, Cambridge, MA.

Slonimsky, L.: 1999, Personal communication, September.

Smith, L.: 1993, *Necessary Knowledge: Piagetian Perspectives on Constructivism,* Lawrence Erlbaum Associates, Hove, UK.

Solomon, Y.: 1998, 'Teaching mathematics: Ritual, principle and practice', *Journal of Philosophy of Education* 32, 377–387.

Steffe, L.P. and Thompson, P.W.: 2000, 'Interaction or intersubjectivity?: A reply to Lerman', *Journal for Research in Mathematics Education* 31, 191–209.

Tomasello, M.: 1999, *The Cultural Origins of Human Cognition,* Harvard University Press, Cambridge, MA.

Vile, A. and Lerman, S.: 1996, 'Semiotics as a descriptive framework in mathematical domains', in L. Puig and A. Gutiérrez (eds.), *Proceedings of the Twentieth Conference of the International Group for the Psychology of Mathematics Education,* PME Program Committee, Valencia, Spain, Vol. 4, pp. 395–402.

Vygotsky, L.: 1978, *Mind in Society,* Harvard University Press, Cambridge, MA.

Walcott, R.: 1994, 'Pedagogical desire and the crisis of knowledge', *Discourse* 15(1), 64–74.

Walkerdine, V.: 1988, *The Mastery of Reason,* Routledge, London.

Walkerdine, V. and Girls And Maths Unit: 1989, *Counting Girls Out,* Virago, London.

Walshaw, M.: 2000, 'An unlikely alliance: Mathematics education, poststructuralism and potential affirmation', *Mathematics Teacher Education and Development* 1, 94–105.

Wells, G.: 1999, *Dialogic Inquiry: Towards a Sociocultural Practice and Theory of Education,* Cambridge University Press, Cambridge, UK.

Wenger, E.: 1998, *Communities of Practice: Learning, Meaning and Identity,* Cambridge University Press, Cambridge, UK.

Wertsch, J.V.: 1991, *Voices of the Mind: A Sociocultural Approach to Mediated Action,* Harvard University Press, Cambridge, MA.

Wertsch, J.V., del Río, P. and Alvarez, A.: 1995, 'Sociocultural studies: History, action and mediation', in J.V. Wertsch, P. del Río and A. Alvarez, (eds.), *Sociocultural Studies of the Mind,* Cambridge University Press, Cambridge, UK, pp. 1–34.

Wittgenstein, L.: 1967, *Philosophical Investigations,* Blackwell, Oxford, UK.

Zevenbergen, R.: 1991, *Children's conception of space*, Paper presented at the Annual Mathematics Education Research Group of Australasia Conference MERGA – 14, Perth.

Zevenbergen, R.: 2000, ' "Cracking the Code" of Mathematics: School success as a function of linguistic, social and cultural background,' in J. Boaler (ed.), *Multiple Perspectives on Mathematics Teaching and Learning,* Ablex, Westport, CT.

Zevenbergen, R. and Lerman, S.: 2001, 'Communicative competence in school mathematics: On being able to do school mathematics', in J. Bobis, R. Perry and M. Mitchelmore (eds.), *Proceedings of Mathematics Education Research Group of Australasia,* Mathematics Education Research Group of Australasia Inc., Sydney, Australia, Vol. 2, pp. 571–578.

Zinchenko, V.P.: 1995, 'Cultural-historical psychology and the psychological theory of activity: Retrospect and prospect', in J.V. Wertsch, P. del Río and A. Alvarez (eds.), *Sociocultural Studies of the Mind,* Cambridge University Press, Cambridge, UK, pp. 37–55.

Centre for Mathematics Education,
Faculty of Humanities and Social Science,
South Bank University,
103 Borough Road,
LONDON SE1 0AA,
United Kingdom,
Tel: +44 (0)20 7815 7440,
Fax: +44 (0)20 7815 8160,
e-mail: lermans @ sbu. ac. uk,
url: http://www.sbu.ac.uk/~lermans

ELLICE FORMAN* and ELLEN ANSELL

THE MULTIPLE VOICES OF A MATHEMATICS CLASSROOM COMMUNITY

ABSTRACT. Several mathematics educators have expressed their concern about conflicting visions of educational reform among parents and teachers, which could result in the emergence of multiple voices in discussions of achievement and instruction. The aim of this article is to examine the multiple voices of educational reform in the discourse of a third grade classroom community. In order to achieve our aim, we integrated the social and the individual as well as the past, present, and future in our analysis of the discourse in this classroom community using theoretical frameworks and methods from cultural psychology. Although our analyses focused on the classroom teacher, we employed units of analysis capable of bridging the individual and her social context. We began our analysis by focusing on a sample of whole-class discussions of students' strategies for solving multi-digit word problems. This analysis isolated two distinct voices: one that occurred during discussions of students' invented strategies and the other that emerged during talk about standard algorithms. We extended our analysis to include information about the historical, social, and institutional context of the classroom community in order to understand the origins and functions of these two voices. This additional information helped us appreciate the interconnections between the teacher's personal feelings, beliefs, recollections, and expectations; and her interpersonal transactions with her students, their parents, and other educators. We concluded with a discussion of the implications of the study for understanding one of the dilemmas of educational reform and for advancing research in classroom discourse.

KEY WORDS: classroom discourse, cultural psychology, elementary mathematics, research in mathematics education, social practice

1. INTRODUCTION

Current approaches to mathematics educational reform in North America, Europe, Japan and elsewhere in the world tend to emphasize deep conceptual understanding, complex problem solving, and communication more than procedural speed or factual accuracy (e.g., NCTM, 2000; Hatano and Inagaki, 1998). In the United States, this reform movement has had a contentious history since the 1960's due, in part, to a lack of consensus among parents and teachers about educational goals and strategies (Lehrer and Shumow, 1997; Peressini, 1998). Lehrer, Shumow, and Peressini argue that parents and teachers often employ different voices when they speak about mathematics achievement and instruction. Nevertheless, within each

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group, many voices of educational reform are also heard. For example, Lehrer and Shumow found that approximately one-third of the parents they surveyed felt that teachers should demonstrate correct procedures to their students, another third felt that indirect forms of instruction (e.g., fostering student discussions of multiple problem solving strategies) were preferable, and the remaining parents were comfortable with the indirect forms of assistance if the correct procedures were shown at some point. One source of the differing voices is each person's own experiences in mathematics as well as their expectations about their children's future experiences (Peressini, 1998). Neither of these studies, however, examined the multiple voices of educational reform in classroom discourse.

The aim of this article is to examine the emergence of multiple voices of educational reform in the discourse of a third grade classroom taught by Mrs Frances Porter[1]. We selected Mrs Porter's classroom for study because we felt that her teaching embodied many of the recommendations of the National Council of Teachers of Mathematics (NCTM) (1989, 1991, 2000). In particular, one of the standards addresses communication in mathematics classrooms (NCTM, 2000). This standard proposes that students make sense of mathematics by explaining their invented strategies for solving problems and by listening to and reflecting on the strategies of others. We found that Mrs Porter stressed a similar kind of sense-making and communication in her classroom.

Although we focused on the discourse practices of the classroom, we also collected information on the cultural context of Mrs Porter's classroom community. We included this additional information because we agree with the premise that "adults literally create different material forms of interaction based on conceptions of the world provided by their cultural experience" (Cole, 1996, p. 186). Thus, we anticipated that it might be important to integrate our discourse analysis with data about Mrs Porter's own personal experiences in mathematics education as well as with information about the institutional setting of her classroom instruction. The goal of these analyses will be to show how the individual (i.e., Mrs Porter) and the social (i.e., her classroom community) are interconnected by history and cultural artifacts (i.e., algorithms and discursive routines).

The article is organized in the following way. First, we introduce our theoretical framework in order to clarify and justify our approach to the analyses of our data. Cultural psychology is the theoretical framework employed to help us understand the process of knowledge building in communities of practice (Cole, 1996). Examples from the data that we collected in Mrs Porter's classroom are used to illustrate some of the key theoretical concepts. Second, we focus on a sample of whole-class discussions orchestrated by Mrs Porter. Third, we examine data from Mrs Porter's reports from her own educational experiences and information gathered about the institutional context of Mrs Porter's classroom in order to include these broader historical and contextual influences in our interpretation of the classroom discourse. Finally, we integrate the results of our analyses and draw conclusions about their implications for theory and research in mathematics education.

2. Theoretical framework

2.1. *Learning as a social activity of knowledge-building in communities*

The theoretical framework that guides our research comes from the field of cultural psychology (Cole, 1996). Cultural psychology requires us to view learning as a social activity involving knowledge-building in communities of practice (Lave and Wenger, 1991; Rogoff, 1990). This perspective differs from the more familiar acquisition model of learning in which abstract concepts or skills are understood as being collected, refined, and reorganized in some hypothetical mental storage device (Rogoff et al., 1995; Sfard, 1998). According to one version of the acquisition model, the teacher (expert) transfers her knowledge of a culturally valued skill (e.g., mathematics) to her students (novices). Her students, in turn, are expected to internalize the material (e.g., memorize facts) by encoding and storing information that can be retrieved later. An alternative version of the acquisition model might entail a more active construction of knowledge on the part of students. Nevertheless, the end result for students is hypothesized to be a more elaborately structured mental network of mathematical concepts and procedures.

Alternatively, the participation metaphor of learning sees knowledge building as part of the process of becoming a member of a community. Knowledge building in communities has been characterized as a system of guided participation (Rogoff, 1990) or as a practice by which newcomers become old-timers through legitimate peripheral participation (Lave and Wenger, 1991). In the participation model, classroom teachers are the old-timers whose job is to socialize their students (via guided participation) into the practice of mathematics. The participation metaphor does not entail the transfer of abstract knowledge from an expert to a novice (teacher to student) or even the active construction of abstract mental concepts and procedures by students. Instead, it focuses on old-timers and newcomers acting together to accomplish culturally valued goals, which involve the use of cultural mediating tools (e.g., mathematical symbols).

Unfortunately, community members may not always want to work together in mutually beneficial ways. One source of internal tensions among community members may occur because members participate in multiple communities. These communities (such as the family, peer group, school, local and national educational groups) may espouse conflicting norms and values. Thus, learning by participating could entail resistance as well as enthusiastic engagement due to the tensions associated with people's conflicting loyalties.

Cole adds to the participation metaphor an historical dimension: how memories of the past and anticipation of the future affect life in the present. "Only a culture-using human being can 'reach into' the cultural past, project it into the future, and then 'carry' that conceptual future 'back' into the present to create the sociocultural environment of the newcomer" (Cole, 1996, p. 186). That is, old-timers' memories of their own learning experiences and their expectations of the newcomers' possible futures influence their transactions in the present. Cole refers to the discursive mechanism by which the past and future are brought into the present as prolepsis. Prolepsis is a term from classical rhetoric that has been resurrected by psychologists, like Cole, who are interested in infusing a temporal dimension into the analysis of instructional episodes. Prolepsis means "anticipation" or "the representation of a thing as existing before it actually does or did so" (*The Concise Oxford Dictionary*, 1990). An utterance can be seen as proleptic if it asks the interpreter to fill in missing information about the speaker's intentions (Stone, 1993). We use prolepsis in this article to help us understand how a teacher, like Mrs Porter, creates a classroom community that is, in part, a product of her own experience as a learner as well as her expectations for her students.

2.2. *Learning as participating in the classroom discourse community*

Sociolinguists who have studied classroom discourse in schools in the United States have found that most whole-class discussions follow a three-part sequence known as the (I-R-E or I-R-F): *initiation* by the teacher, *response* by one or more students, *evaluation or feedback* by the teacher (Mehan, 1979; Wells, 1993). An example might be: Teacher asks, "Mary, what is the answer to this problem (15×5=)?"; Mary, answers, "75", Teacher replies, "Good." Schools elsewhere in the world may demonstrate variations of this three-part sequence. (See Inagaki, Morita, and Hatano, 1999.)

More recently, reform mathematics classrooms have been studied in which the I-R-E model has been replaced by a form of discourse that more closely resembles discussion orchestration (O'Connor and Michaels, 1993, 1996). That is, students take an active role in discourse by initiating and evaluating as well as responding, and students share with the teacher and textbooks the job of providing explanations. The teacher's role is to help students structure their talk: by organizing turn-taking, by asking students to reflect on and evaluate the explanations provided, and by orchestrating collective arguments (Forman et al., 1998; Krummheuer, 1995).

One of the distinctive features of this alternative model of classroom discourse is revoicing. That is, there is a greater tendency for students to provide the explanations in these classrooms and for the teacher to repeat, expand, recast, or translate student explanations for the speaker and the rest of the class. The teacher revoices students' contributions to the conversation so as to articulate presupposed information, emphasize particular aspects of the explanation, or disambiguate terminology (Forman et al., 1998; O'Connor and Michaels, 1993, 1996).

To further develop a version of the discursive functions of revoicing, O'Connor and Michaels draw from Goffman's writings about animation. Goffman (1974, 1981) proposes that in most conversations there are more than two people involved (which is also the case in the classroom transcripts that we analyze). Instead of a single speaker, he suggests thinking of situations that involve an animator (the person making the noise), an author (the person scripting the lines), and a principal (the person whose position is being represented). Instead of a single listener, he suggests thinking of a group of listeners, both addressed and unaddressed recipients. When a classroom teacher revoices her student's words, she is acting as the animator while the student is the principal (and perhaps the author) and the rest of the class functions as the addressed and/or unaddressed listeners. Student explanations are thus legitimated by being animated by the teacher who is the powerful, authority figure in the classroom.

We illustrate revoicing and animation with an episode from a lesson in Mrs Porter's classroom in which the students were asked to solve the following problem: "You read for 15 minutes a day. How much time will you have spent reading in one week?" Mrs Porter requested that a student explain his or her strategy by asking: "Who would like to tell us, not the answer, how they started?" Ophrah, a student in the class, replied that she had multiplied 15 times 5. After Mrs Porter asked her to elaborate ("How did you do that?"), one aspect of the ensuing conversation is depicted below[2]:

Turn	Speaker	Utterance
i	Ophrah	15 times 1 equals ...
ii	FP	15 times 1 equals ...
iii	Ophrah	15.
iv	FP	15.

v	Ophrah	15 times 2, equals 30.
vi	FP	And of course when she's saying 15 times 1 equals 15 it means one 15 is 15 and two fifteens is 30.

In turns ii and iv Mrs Porter repeats Ophrah's words. In turn vi, however, Mrs Porter expands by translating ("when she's saying ... it means") Ophrah's previous utterance. This transcript shows how Mrs Porter animates Ophrah by revoicing because Ophrah is clearly the principal of these utterances even though Mrs Porter says half of them. Another feature of the episode is Mrs Porter's use of pronouns. In the first turn, Ophrah is the addressed recipient because her name is used. Although, no names or pronouns appear in turns ii or iv, Ophrah remains the addressed recipient due to the rules of everyday conversational routines (Levinson, 1983). Mrs Porter explicitly marks her intended audience in turn vi when she refers to Ophrah as "she." This indicates that her message is directed at the rest of the class and not at Ophrah herself: Ophrah's classmates become addressed recipients while Ophrah becomes an unaddressed recipient.

In summary, we employ cultural psychology as a theoretical framework to help us understand the process of knowledge building in communities of practice. Knowledge building in communities is not free of interpersonal tensions due to participants' conflicting loyalties and personal experiences that affect their transactions in the present and their expectations of the future. These tensions are often revealed by the identification of multiple voices in a community (Wertsch, 1997). In our analyses of discourse in Mrs Porter's classroom, we focus on the multiple voices that appear when she orchestrates classroom discussion as students share the strategies they used to solve challenging mathematics problems. In the sections that follow, we provide some background information on Mrs Porter's classroom community and detail our methodology before analyzing the discourse in her classroom.

3. Methodology

3.1. *Mrs Porter's classroom community and her instructional approach*

3.1.1. *The institutional context of the classroom community*

Mrs Frances Porter's third grade classroom was located in Riverside Academy, Junior School, one of three campuses of an independent school in a medium-sized city in the northeastern part of the United States. The Junior School building was comprised of grades kindergarten through fifth. Most students enrolled at Riverside Academy came from upper middle class backgrounds and many parents were highly educated. A scholarship program enabled the school to provide financial support for the tuition payments of a small number of students, resulting in some economic diversity.

As are all independent schools in the United States, Riverside Academy was funded by tuition payments made by parents: it received no financial support from local or national governing agencies. Because of this dependence upon parental tuition payments, it was critical that parents approve of its instructional programs. Riverside Academy's mission statement included the following comment: "As a private, independent school, we answer directly to the families in our community and our Board of Trustees." The school's brochure also highlighted the importance of parental involvement in the school's activities: "Parents are important partners in our educational process. We keep in close touch through regular written reports, impromptu or planned conferences, and occasional community-wide Town Meetings."

3.1.2. *The participants in the classroom community*

Mrs Porter had been teaching at Riverside Academy (kindergarten, first, and third grade) for over 20 years when we conducted our study. There were 17 students in her classroom (seven girls, ten boys) between the ages of 8 and 9 years. About half of Mrs Porter's students were European American (three girls, seven boys), four children were African American (three girls and one boy), and three students were Asian American (one girl and two boys). Six of her students (two girls, four boys) had siblings enrolled in the school. The students in Mrs Porter's class chose to sit in a same-gender group. Each of the groups (there were five groups of three or four students) sat at a round table where they often worked on their mathematics problems individually or with a partner.

Two 90-minute blocks of time on Tuesday and Thursday mornings were set aside for mathematics instruction each week. A team of three ethnographers[3] observed these two weekly mathematics lessons from September 3 (the first day of school) until the day before winter vacation, December 17, 1998[4]. A typical mathematics instruction period entailed three sequential segments: 1) the teacher introduced one or more problems; 2) students worked individually or in pairs on the problems; 3) and the teacher led a whole-class discussion of the strategies the students had used. While students solved problems individually or in pairs, Mrs Porter often circulated around the room to answer questions and to prompt students ("Now you can't just guess. You have to have a reason."; "That's one way to start. That'll be fine."); and, during whole-class discussion, she stood in front of one of the two white boards while eliciting and recording students' strategies.

3.1.3. *Mrs Porter's instructional approach*

Mrs Porter did not use a formal curriculum in her mathematics instruction. Instead, she relied on her 30 years of teaching young children and her personal library (e.g., Burns, 1992; Kamii and Joseph, 1989). Her instructional approach centered on children's learning trajectories – a personal and informal version of more institutionalized approaches such as Cognitively Guided Instruction (Carpenter et al., 2001). She relied on tasks that can be solved by a variety of strategies. For example, she frequently asked her students to find the difference between two dates (1718 and 1607). To solve such problems, some students used strategies based on counting sequences; others used strategies that depended upon the decomposition and recomposition of quantities; and a few used standard algorithms to find the difference. She preferred tasks that she felt would appeal to her students' interests and experiences such as feeding pets or preparing for parties. She frequently employed the same tasks every year so that she could use her teaching journal to help anticipate her students' most likely strategies and common errors.

She also paid close attention to the discourse practices in her classroom. She stressed to her students that they need to listen to the strategies of their classmates while they were being explained during whole-class discussions: "This is a class where we all teach each other." In addition, she frequently repeated her student's explanations. She explained to the investigators that she revoiced her students in this way in order to put their ideas into words; to clarify them to herself and to the rest of the class; and to record them in writing to articulate and record their meaning. She felt it was important for students to listen to each other's explanations so that they could learn new strategies. In addition, by revoicing, she sought to legitimate her students' contributions to the collective discussion. Thus, Mrs Porter designed the discourse in her classroom to be closer to discussion orchestration than to the I-R-E.

Now that we have provided some contextual information about Mrs Porter's classroom, we articulate our approach to discourse analysis. In particular, we justify our choice of units of analysis and then apply those units to our data sampling strategy.

3.2. *Units of analysis in classroom discourse*

Cole's approach to cultural psychology builds on the cultural historical activity theory of Vygotsky who proposed that the unit of analysis needs to bridge the individual and the social. Because thinking is mediated by cultural artifacts such as speech, written language, and mathematical symbols, the unit of analysis must be defined so that agents and mediating tools engaged in activity can be examined as a unitary whole. (See Cole, 1996; Wertsch, 1985, 1991, 1997 for further elaboration.)

In our analysis of discussion orchestration in Mrs Porter's classroom, we defined units that include the individual (Mrs Porter) and her social context (her classroom community). We began our analysis with the need to appropriately sample major classroom events during the four-month period of classroom observation. Previous research on classroom discourse provides some guidelines for sampling data. Erickson and Shultz (1981), for example, suggest that one begin with the largest or molar units and look for critical junctures or changes in the structure of events. They argue that these junctures may index participants' interpretations of social events. Thus, following their suggestions, we began with molar units (major activities, such as lessons) before moving to increasingly molecular units (e.g. sequences of talk about a single topic).

As we transcribed the audiotapes from lessons in which discussions of invented strategies and standard algorithms were discussed, we were struck by the differences in the nature of the talk surrounding invented versus standard strategies. This was particularly salient in problems that involved subtraction or multiplication. It seemed to us that Mrs Porter devoted a great deal of time to explicating her students' invented strategies but spent very little time helping students understand the standard algorithms. When discussing standard algorithms, but not at other times, she frequently made reference to students' parents and older siblings and to her own educational experience. Also, we noticed that she mentioned and displayed different emotions in those two conversational contexts: invented strategies were associated with feelings of self-confidence and enthusiasm; standard strategies with attitudes of confusion and sadness. Changes in the structure of discourse in Mrs Porter's classroom (presence or absence of revoicing) as well as differences in lexical choices (presence or absence of references to family members; positive or negative emotions) alerted us to the possibility that these may signal discontinuities between her past experiences and future expectations and/or between parental and reform community norms and values (Erickson and Shultz, 1981).

We present, as an example, a detailed examination of one discussion of invented strategies and standard algorithms that occurred in Mrs Porter's classroom early in the academic year during a lesson when students were asked to solve a multi-digit multiplicative word problem. We also chose this example because it is unlikely that her students would have encountered similar problems in first or second grade. Thus, we may have had the chance to observe at least some of her students' first struggle to come up with strategies for solving these problems that made sense to them. Finally, we selected this example because it occurred during the second week of the academic year: a period in which the classroom norms and values were being established and so might be more explicitly marked than a few weeks later. As Erickson and Schultz (1981) recommend, times of transition, such as the beginning of a new activity, often reveal the rules guiding that activity that become tacit later after routines are more firmly established. Their recommendation was echoed by Mrs Porter who suggested that we should begin our observations as early in the academic year as possible if we wanted to watch her help her students create a distinctive classroom community.

After identifying the fall lesson during which critical junctures (i.e., multiple voices) in the structure of whole-class discourse seemed to occur, we isolated several instructional episodes: an entire exchange that occurred between Mrs Porter and her students around a single student's strategy for solving a problem. Each episode may or may not contain utterances from more than one student. The other students were assumed to be part of the audience for this exchange, as in Goffman's notion of addressed and unaddressed recipients.

The final stage of discourse analysis, according to Erickson and Shultz (1981), involves "establishing the generalizability of the single-case analysis conducted" previously (p. 157). This is done by locating multiple instances of the episodes of interest in the data corpus and finding out the degree to which they resemble each other. We identified another lesson in our corpus that resembled the September lesson in two respects: the discussion involved a comparison of invented strategies and standard algorithms; and numerous references to instruction at home by family members and to Mrs Porter's own educational history occurred. This lesson, which took place in December, will be very briefly interpreted after we present the lesson from the fall in detail.

4. THE EMERGENCE OF MULTIPLE VOICES IN TWO CLASSROOM LESSONS

During the second week of school, Mrs Porter asked her students to work independently on the reading problem described earlier: "You read for 15 minutes a day. How much time will you have spent reading in one week?" Before they began working on the problem, Mrs Porter encouraged her students to show all their work and to use a pen so that any false starts or errors would remain visible. She stressed that she was less interested in the students' accuracy than in their solution strategies. When asked if they should figure the problem out for five or seven days, Mrs Porter responded that it was up to the individual to decide. As they worked on the problem, she circulated around the room, offering encouragement and assistance. Mrs Porter also let her students know that when the class was finished with the problem, they would discuss it as a group and they would be teaching each other. After her students had completed the assigned problem, Mrs Porter called them together and stressed the importance of listening to each other. Because this lesson took place during the second week of school, Mrs Porter was aware of needing to help her students get used to providing explicit explanations of their thinking, an activity that may or may not have occurred in their previous school experience. For example, she frequently told her students during this lesson to fully explain their strategies: "I don't want an answer. I want how you figured it out."[5]

Mrs Porter called on students to explain their solution strategies and recorded them on the white board for everyone to see. She also repeated, expanded, or reformulated the explanations that she heard from her students. Eleven of the sixteen students present explained their strategies during this discussion. Two students (Raj and Pulak) employed the standard multiplication algorithm. One student's strategy (Ophrah's) was difficult to classify but it involved the creation of a 15 times table. The rest of the class used strategies that were clearly invented. Of these, one student used a counting strategy, and the other seven students used a variety of additive strategies or a combination of additive and multiplicative strategies. Almost all of the answers provided were accurate and all but one of the students in the class completed the assignment. This summary of the students' mathematical performance does not communicate the tone of the classroom discussion itself. Thus, we will sample from that discussion two instructional episodes: one focused on Lyndsey's strategy and the other focused on Pulak's strategy. This sample helps us depict the patterns of discourse that occurred when students explained their invented strategies (Lyndsey) or their use of a standard algorithm (Pulak). If, as we have claimed, multiple voices emerge in discourse in these two types of instructional episodes, then they should be apparent in these examples.

4.1. *Mrs Porter's orchestration of the discussion of an invented strategy*

Lyndsey was the third student Mrs Porter asked to explain her strategy. This instructional episode provides a typical example of Mrs Porter's transactions with her students during whole-class discussion of invented strategies. Our analyses will focus on our interpretation of the episode, with respect to the mathematics being discussed and the discursive forms and functions employed (i.e., revoicing, animation). The transcript listed below shows how this episode transpired.[6]

Turn	**Speaker**	**Utterance**
1	FP	Lyndsey, Lyndsey what did you write?
2	Lyndsey	I wrote uhm,. . . 15 and then, then plus 15 more, 30 plus 15
3	FP	What is it? . . . Did you write the one that's right first? And you said that to yourself, right?
4	Lyndsey	Ya.
5	FP	OK. She said to herself, 15 minutes, plus 15 minutes in her head, would be 30, plus 15 is 45. So, she was kind of adding them up as she went and made the row until she got to 75 minutes. And, that's another thing that people might write down as a way to try it.

Mrs Porter initiated the episode by asking Lyndsey to explain her written response ("what did you write?"). Lyndsey began her explanation by reporting what she had written and thought about. It appears from Mrs Porter's summary in turn 5 that Lyndsey had written a row of five 15's and then moved her finger along the row while successively adding up the numbers (saying 30, 45, 60, 75). By using "plus," she indicated her additive strategy. In turn 3, Mrs Porter asked Lyndsey whether she had used a verbal mediation strategy to keep track of her calculations, "you said that to yourself, right?" In this way, she acknowledged that only Lyndsey knew how she did the mental calculations. When Lyndsey agreed with Mrs Porter's characterization of her procedure, then Mrs Porter revoiced Lyndsey's explanation from turn 2 ("15 and then, then plus 15 more, 30 plus 15") and expanded it ("15 minutes, plus 15 minutes in her head, would be 30, plus 15 is 45."). The addition of "minutes" to Lyndsey's explanation articulated the presupposed information from the original problem in which the measurement unit is minutes. Mrs Porter also made explicit the mental calculation strategy that Lyndsey used ("in her head") and noted the intermediate calculation that would result ("45"). Then, she further clarified Lyndsey's strategy by mentioning it ("So, she was kind of adding them up as she went and made the row until she got to 75 minutes").

It is also important to point out that turns 1 and 3 were addressed to Lyndsey ("you") whereas turn 5 was addressed to the class ("she, her"). Thus, Mrs Porter's revoicing helped clarify Lyndsey's strategy for herself by making explicit the implicit information in Lyndsey's explanation and also articulated this strategy for her classmates. Mrs Porter ended this episode by pointing out to the class that Lyndsey's strategy (with perhaps the modification of writing the intermediate answers below the first row of numbers) would be a useful one.

Mrs Porter seemed content to accept Lyndsey's use of an additive strategy to solve a multiplication problem as she did not negatively sanction the strategy; instead, she reported it to the rest of the class in its fully articulated form. This fully articulated form required that Mrs Porter make implicit information explicit and connect her strategy with addition, "she was kind of adding them up as she went." By revoicing Lyndsey, Mrs Porter put words in Lyndsey's mouth but she also allowed Lyndsey a chance in turn 3 to object to her characterization ("right?"). Thus, Mrs Porter not only articulated Lyndsey's incomplete explanation, she also legitimated it by animating Lyndsey. Clearly, Lyndsey was the author and principal of this explanation even though Mrs Porter expanded it quite a bit. At the end of turn 5, Mrs Porter made Lyndsey's strategy more general by stating, "that's another thing that people might write down." Thus, Mrs Porter further legitimated Lyndsey's strategy by making it seem like an effective strategy that other students might want to consider.

Before interpreting the instructional episode that featured the standard multiplication algorithm, we need to briefly summarize Ophrah's multiplicative strategy that was discussed earlier in the lesson, because Mrs Porter refers to it in the instructional episode we will be interpreting next. Before Lyndsey's instructional episode, Ophrah claimed that she had written the multiplication tables for 15 on the back of her paper. After being prompted by Mrs Porter to describe how they look, Ophrah reported (with Mrs Porter's prompting): $15\times0=0$; $15\times1=15$; $15\times2=30$; $15\times3=45$; $15\times4=60$; $15\times5=75$. (An excerpt from Ophrah's instructional episode appears earlier in this article.) Mrs Porter wrote these numbers on the white board as Ophrah dictated them and they were displayed throughout the remainder of the lesson. In response to Mrs Porter's request that she explain how she knew these answers, ("Did you memorize them? Or did you figure them out?") Ophrah replied that she "figured them out." Unfortunately, Ophrah was not able to articulate how she figured them out when Mrs Porter asked her to explain further ("How did you know that?") and Mrs Porter did not inquire further. Thus, it was not clear to us whether Ophrah had used an invented strategy or memorization to arrive at her answers.

4.2. *Mrs Porter's orchestration of the discussion of the standard algorithm*

Following Lyndsey's instructional episode, two other students explained their invented strategies. Then, Mrs Porter called for a different way and Pulak was chosen to explain his approach to the problem. We have chosen to analyze the instructional episode that features Pulak because it demonstrates a different pattern of discussion than the ones presented earlier. Pulak had used the standard multiplication algorithm to solve this problem and he demonstrated his strategy, with assistance from Mrs Porter, as did Lyndsey. In this episode, you will notice that Mrs Porter does less revoicing than with Lyndsey and refers to her own educational experiences as well as to instruction by family members.

21	FP	Did anyone else do 5 times another way? Started a different way? And oh yes. OK so you did, tell us.
22	Pulak	15 times 5.
23	FP	He did 15 times 5 this way . . . Ophrah said that she did 15 times 5 too. And she did. And to figure it out, she did it this way. (Points to Ophrah's strategy that FP had written on the white board.) When Pulak figured it out, he did it a different way. And ah, Pulak show us what you did.
24	Pulak	I, I did 5 times 5 (Multiplies the numbers in the one's column to get 25.)
25	FP	Uhm-hum.
26	Pulak	And I
27	FP	And what did you write down?
28	Pulak	And I took a pen
29	FP	First, did you put something down here?
30	Pulak	Oh ya, the 5. (Records the intermediate answer, 5, in the one's column before carrying the 2 tens.)
31	FP	OK.
32	Pulak	And I put a 2 up there. (Records the 2 tens above the ten's column.)
33	FP	Which is really 20. Yes.
34	Pulak	Then I, uhm did 1 times 5 is 5 and I did 5 plus 2 is 7 (75). (Completes the answer for the ten's column by multiplying and regrouping.)
35	FP	And you got the same answer. That's the way a lot of your parents would do it. Because that's the *only* way we were allowed to do it in school. That doesn't mean it's the right way. And it's a very confusing way to a lot of people. Especially for people for whom regrouping is difficult. So, if you want to do it because you understand it and it's a good way for you, great. If not, do it a way that makes sense to you. I figured you'd know that way because your brother does it that way too. I know that from last year. OK.

This episode began with Pulak's announcement that he had used a multiplicative strategy ("15 times 5"). Looking at Pulak's paper, Mrs Porter immediately recognized that he had used the standard multiplication algorithm to solve the problem.[7] Before asking him to begin his explanation, she connected his strategy with that of another student, Ophrah, who had started the class discussion. At the beginning of turn 23, Mrs Porter did not address either Pulak or Ophrah directly but instead referred to them as "he" or "she." However, at the end of the turn, she asked Pulak to show his strategy. Therefore, most of turn 23 was devoted to making a connection between the two multiplicative strategies ("15 time 5"), yet suggesting an important difference in their approaches (without elaborating on that disparity).

Mrs Porter's transactions with Pulak contrasted with the previous instructional episode featuring Lyndsey. One dramatic difference is the lack of revoicing after turn 23 (with the possible exception of turn 33). In turn 33, Mrs Porter did translate Pulak's previous utterance ("I put a 2 up there") into ("which is really 20") but she did not elaborate on this translation so that the rest of the class could understand it. Although it is likely that Mrs Porter or Pulak pointed to the number 2 when they mentioned it, only students who understood place value well would be able to follow the connection between "2" and "really 20." Unlike her response to Lyndsey's explanations, Mrs Porter did not use her final turn with Pulak to explicate his strategy to the rest of the class. Instead, she emphasized that his strategy was accurate, that it was a strategy used by many of their parents and older siblings, she referred to her own educational experience with that strategy ("that's the *only* way we were allowed to do it in school"), and she made sure that she stressed this strategy as one of many correct strategies that could be used to solve these kinds of problems. In addition, she talked about the disadvantages of this strategy (it could be confusing or difficult) and repeated the need to use strategies that make sense.

The student who reported a strategy after Pulak used an additive invented strategy and the discussion proceeded as it had before Pulak had spoken. Later in the lesson, another student, Raj, used the standard algorithm to calculate the answer for 15×7. As she did in the case of Pulak, Mrs Porter did not explicate Raj's strategy. After each of his computational steps (which he did correctly), she merely said, "OK." When he was finished, she made the connection between his strategy and Pulak's by commenting: "So, you did it exactly the same way he (Pulak) did. You used that algorithm. OK. And you had 105 minutes. That's one way that's possible to do it. Again, (use it) if it makes sense, don't try it if it doesn't."

At the very end of the lesson, after several other students had presented their strategies, Nathan asked for further explanation of the standard algorithm. Mrs Porter did not comply with his request. Instead, she remarked that Pulak and Raj probably like the standard algorithm because it is quick but, she added, a quick strategy is not necessarily the best one to use if you do not understand how it works. She reminded him to look for alternative approaches, which may also be quick but more meaningful and to write them down so he can remember them. During the entire lesson, Mrs Porter did not explain the standard algorithm even though Nathan had asked her, indirectly, to do so. She did make sure that Nathan and the others recognized that Raj as well as Pulak appeared to understand how the algorithm works. In addition, she admitted that this strategy is fast. As we saw before, however, she stressed that the standard algorithm should only be used by students who understand it and that invented strategies were legitimate ways to solve multiplication problems such as this one.

Thus, there was a consistent difference in the discourse practices in these two different instructional contexts. During discussions of invented algorithms, Mrs Porter frequently revoiced student contributions and addressed the rest of the class as well as the focal student; whereas in discussions of the standard multiplication algorithm, she infrequently revoiced the student, mentioned that family members may teach you that way of solving problems, cautioned that this approach could be fast but also confusing and lack meaning, and referred to her own educational experiences in which the standard algorithm was taught as the only way to solve multiplication problems.

4.3. *Another occurrence of multiple voices during a second lesson*

Our dataset includes several lessons in which students explained both invented and standard strategies for solving mathematics word problems and, in one of those, another multidigit multiplication problem appeared. We examined this lesson (which occurred in December) and noticed the same kind of critical junctures in discourse as those that had emerged from our analysis of the September lesson. As Erickson and Shultz (1981) recommend, we felt it would be important to locate at least one more example of a discussion of both invented strategies and standard algorithms in order to see whether it exhibited further evidence for the existence of multiple voices. This second lesson was quite long so we will have to present it very briefly. Nevertheless, we will highlight salient similarities and differences between the two lessons.

The word problem employed in the December lesson was: "Gustave Eiffel, the man who designed the famous tower, was born this week in

1832. The Eiffel Tower is 984 feet high. How many inches would that be?" Mrs Porter's students had much more trouble completing this problem than they had experienced in September with the simpler reading problem. Only seven of the 16 students present that day (Pulak was absent) completed the problem and only two of them arrived at the correct answer. Both of the successful students (Lyndsey and Joshua) arrived at the answer by using an invented strategy of adding 984 twelve times. Eleven other students also used invented strategies (either additive or a combination of additive and multiplicative strategies) and three students used the standard multiplication algorithm (Raj, Ophrah and Karl). All of the three students who employed the standard strategy made computational errors and thus did not obtain the correct answer. None of these students appeared to understand why their procedure was incorrect.

When we examined the discursive patterns that occurred while several students (Lyndsey, Ophrah, Raj and Karl) presented their strategies during whole-class discussion, we found a partial replication of our findings from the September lesson. During both lessons, Mrs Porter seemed to speak in two different voices: one voice that was attuned to the need to tailor instruction to her students' level of understanding; a second voice that allowed standard algorithms to be demonstrated but not explicated. When she spoke in the first voice, Mrs Porter emphasized sense making, risk-taking, persistence, being logical, and finding increasingly efficient strategies. Revoicing as a strategy for orchestrating discussions was a frequent occurrence. As a result, student thinking was legitimated. In contrast, when she spoke in the second voice, Mrs Porter emphasized the confusing or complicated nature of standard algorithms. She rarely revoiced students' attempts to use the standard algorithm. She also frequently mentioned the influence of family members, especially parents and older siblings, and expressed concerns about the inflexibility of an approach that privileges the standard algorithm over more creative solutions. In addition, the emotional tone of this voice was different because Mrs Porter often mentioned her own feelings of inadequacy as a child when she was taught to memorize algorithms without understanding them.

Thus, we have documented the emergence of two distinct voices in Mrs Porter's classroom community during discussions of invented versus algorithmic strategies. In the following section, we explore several explanations for the emergence of these two discursive patterns. Our explorations took us far from the here-and-now of the classroom events that we described above. In this section, we draw connections between Mrs Porter's personal recollections, the institutional context of her classroom, and our discourse analysis.

5. Exploring the Cultural Contexts beyond the Classroom

One answer to the question of why Mrs Porter treated standard algorithms differently from invented algorithms may stem from her personal history. Cole's notion of prolepsis highlights the fact that parents and teachers recall their own past experiences, use them to imagine their children's future, and then act in the present accordingly (1996). We found Mrs Porter's differential responses to invented and standard algorithms to be proleptic. Indeed, when we looked at several statements she had written and her response to some interview questions, we heard two different versions of her educational experiences. The first version concerned her early school experiences as a learner and as a teacher. The second version involved her more recent educational experiences. We will discuss each of these versions in chronological order and will connect those experiences to the two voices we identified in her classroom.

5.1. *Mrs Porter's early school experiences*

Mrs Porter described her early experiences in mathematics in primarily negative terms. She viewed herself as a well behaved, above average, but uninspired student in general, and a confused student in mathematics. In a written statement she reported: "As a math student, I always felt weak. Math was a mystery. I never really felt that I understood how numbers worked." Mrs Porter remembered that her mathematics education entailed learning formulas in order to complete the work that earned her good grades. Unfortunately, this coping strategy left her feeling that she never understood why those formulas worked "and I just assumed that some people understood things like this and some didn't and I was one of the ones that didn't."

Mrs Porter commented to her students on her own experiences with the standard algorithm after completing the December discussion of the Eiffel Tower problem with her class:

> I am so *sad* that when I was little, people made me do it this way. Because I used to get all mixed up with multiplication and I used to think, I don't know how to do this. And when I had a list of problems and I remembered the rule, I could get the right answers. But I really didn't understand what I was doing.

Thus, it seems as if Mrs Porter's own sense of inadequacy, powerlessness. and alienation from the subject matter being taught when she was a child, and especially from mathematics, continued to color her transactions with her students as an adult. Unfortunately, this sense of alienation persisted, to some degree, during her beginning years as a teacher.

Mrs Porter's experiences with teaching did not seem very positive. She recalled, "mostly I learned how to *control* students, how to pass out and collect supplies and keep records." She taught for a few years before leaving the profession for ten years to raise her children. Her memories of those early years of teaching were vague. She remembers using workbooks to teach mathematics to her students. She also recalls that her classroom was very teacher-dominated.

When she returned to teaching in the mid 1970's, Mrs Porter found a position as a kindergarten teacher at Riverside Academy. About ten years before our study was conducted, Mrs Porter began to change her teaching practices with the encouragement of a new Headmistress. She also enrolled in a graduate program in writing and began to see herself as a teacher researcher. This is the point at which her version of her educational experiences changed from vague memories of running a teacher-dominated classroom where textbooks determined her instructional approach to distinct recollections of exciting experiences – both in her classroom and elsewhere.

5.2. *Mrs Porter's more recent experiences in education*

In Mrs Porter's graduate program, she found a new way of teaching that excited her and stimulated her thinking and self-reflection. She felt energized by the experience and used it to change her own instructional practices. This graduate program forced her to think, reflect, and to discuss her ideas with a small group of other students. In response, she rethought her approach to education. Another crucial influence on her teaching arose from a trip to Australia and New Zealand in 1990. Her trip allowed her to observe innovative approaches to both reading and mathematics instruction. After she returned from this trip to teach at Riverside Academy, she recalled that "with all of these 'new' ideas buzzing in my head, and because I was so enthusiastic and so completely interested in what the kids were able to do, we all thrived."

At this point in her career, Mrs Porter realized that she needed to extend the changes she had begun in her teaching of reading and writing to instruction in mathematics. She met a new teaching colleague, Beth Compton, at Riverside Academy with a strong background in mathematics education. Mrs Compton began leading math workshops for interested faculty (including Mrs Porter) in which tasks were introduced, tried out with their students, and discussed as a group. Mrs Compton encouraged the other teachers to take risks and pay close attention to students' thinking.

After Mrs Porter began teaching in this new way, she concluded that there had been a close connection between the way she had been taught mathematics (by memorizing formulas) and her own sense of inadequacy doing mathematics. She began to feel that if she taught children to use strategies that made sense to them, they would be able to build their confidence in their abilities. She argued that if children found their own way to solve problems and were not forced to use just one approach, then they could build on their informal knowledge of mathematics. She began keeping a teaching journal that enabled her to record and reflect on the strategies her students used to solve problems. Change also began to occur in her classroom discourse. She began to carve out a new role for herself: helping students articulate their ideas to themselves and each other. She realized that children as young as first grade often did not have the words to express their thoughts. Thus, she needed to help them explain by watching what they did and said and reiterating. One result of this change in her practice was that many of her students became increasingly able to explain their own and each other's ideas.

In summary, we found that Mrs Porter used two different voices to speak about her personal experiences in education. The voice of her earliest experiences (both as a student and a teacher) spoke of alienation, low self-confidence, passive acceptance of authority, lack of initiative, and a sense of incompetence and confusion, especially in mathematics. This first voice seemed to be connected, in part, to an instructional approach that relied upon students memorizing algorithms without understanding why they work. It also seemed linked to an approach that ignored or discounted students' own attempts to make sense of their strategies for solving problems. The voice of her more recent experiences spoke of excitement, the enjoyment of learning from others through discussion, risk-taking, a passionate commitment to fostering students' sense-making, and an intense involvement in helping her students express their ideas and reflect on them. This voice seemed associated with an instructional approach that emphasized careful attention to students' different ways of solving problems and their informal knowledge base. Both of these voices also seemed to be linked to the broader social and institutional contexts of her classroom.

5.3. *The institutional and cultural contexts of Mrs Porter's classroom community*

Unfortunately, at the time of our study, Mrs Porter found few colleagues at Riverside Academy with whom to share her interest in teaching focused on individual children's learning trajectories. Mrs Compton had moved to another city; the Headmistress who had originally encouraged her to change her teaching had also been replaced. The current administration at Riverside Academy continued to emphasize a child-centered approach to teaching, but they did not require teachers to standardize their instructional approaches. It was our impression, based on conversations with Mrs Porter, that she felt that the other teachers were not very interested in basing their mathematics instruction on a deep understanding of students' thinking and were more comfortable teaching the standard computational algorithms. In addition, she seemed to feel that her students' parents did not necessarily agree with her instructional approach and that they would have preferred to see evidence that their children were memorizing facts and learning to calculate in recognizable ways.

Mrs Porter recognized the importance of informing parents about her instructional approach. She sent weekly newsletters home with her students in which she often described the strategies they used to solve problems and the wisdom of using information about children's thinking to guide instruction. In the middle of December, she sent a newsletter to parents in which she reflected on the difficulties several of her students had experienced when they tried to use the traditional algorithm for multiplying two and three digit numbers. She explained that each of those students claimed to have been taught the algorithm at home but few of them were able to use it correctly or explain what they were doing. She argued that students need a better understanding of place value before standard algorithms can be successfully introduced.

Mrs Porter also wrote a formal statement of her teaching philosophy and sent it to the parents of her students. In her statement, she argued for the use of tasks that allow students to explore and refine their ideas about mathematics and for the need to encourage risk-taking, conjecturing, estimating, and the invention of problem-solving strategies that make sense to them. She wrote that she emphasized writing and discussion in order to promote self-reflection (activities that she had found very productive in her own graduate school experiences). As a result, she argued, her emphasis shifts from "getting the right answer to developing thinking and understanding." Her statement de-emphasized the teaching of standard algorithms.

5.4. *Some likely explanations for the multiple voices in Mrs Porter's classroom*

The source of the two discursive practices that we identified in Mrs Porter's classroom may be traced to the two different periods in Mrs Porter's educational history (before and after her graduate training). In addition, these voices may stem from an apparent lack of consensus about the instructional goals and practices among the teachers and parents of Riverside Academy. It is clear that Mrs Porter's current educational philosophy is quite different from the one she implicitly espoused as a student or young teacher. Her position comes from her own commitment to student meaning-making and at least ten years of paying close attention to her students' oral and written explanations. Mrs Porter's enthusiasm for the discovery and comparison of many different strategies seemed responsible, in part, for her students' diligent attempts to work on very difficult problems like the one about the Eiffel Tower. Also, they appeared to be intrigued by her stories about the rewards of persistence and diligence. Thus, it seemed to us that when this voice appeared in her classroom community, Mrs Porter's very positive experiences in Australia and New Zealand and in her graduate program reemerged in the present. She remembered that her own sense of alienation and confusion with mathematics ended when she began to change her approach to instruction. It makes sense that she would have similar expectations for her students' educational and occupational futures. Her encouragement and sensitivity to their needs to make sense of mathematics and to find their own ways to do mathematics are likely linked to her own experiences and anticipation of their futures. Therefore, Mrs Porter's instructional foci not only echo the voices of those involved in the educational reform movement in mathematics, they also reflect her own sense of the past and future.

In addition, Mrs Porter perceived her approach to mathematics teaching as different from her colleagues or her students' parents. She felt that many of her students' parents had different expectations about mathematics teaching than she did – especially with respect to teaching the standard algorithms. In a similar fashion, the second voice echoed Mrs Porter's early educational and occupational experiences. The influence of her own early educational experiences on this voice can be seen in her expression of sadness at the end of the Eiffel Tower lesson. At that time, she recalled her own sense of incompetence, passivity, and exclusion from the group of people who can do and understand mathematics. This sense of alienation and passivity seemed to also affect her own early occupational experiences. Thus, her cautions to her students to beware of the fake attractions of the standard algorithms seemed to arise out of her own previous negative feelings about herself as a mathematics student and her expectations for her students' own futures in school and in the workplace. In essence, she seemed to be saying to them that there are only two paths to follow in mathematics: the path towards greater autonomy, sense-making, and power or the path towards passivity, alienation, and confusion. In contrast, there seemed to be a great deal of consistency between her articulated aims and values and those of the mathematics educational reform community (at least as written in the NCTM documents from 1989, 2000).

In summary, some of the voices in Mrs Porter's classroom community seemed to have origins outside the classroom walls in the other classrooms at Riverside Academy, in her students' homes, and in the cultural environment for educational reform in the late twentieth century in North America. They began in the past (especially Mrs Porter's own educational experiences as well as those of her students' siblings and parents) and reflected expectations for the future, as in Cole's notion of prolepsis. These voices also reflected the many emotions associated with those memories and beliefs: excitement, enthusiasm, self-confidence, initiative, persistence versus passivity, confusion, conflict, and feelings of failure.

In our final section we return to some unanswered questions raised by this case study about the tensions of educational reform that are represented by the multiple voices in Mrs Porter's classroom. In addition, we want to reflect on the theory and methods employed to study those voices. What can case studies such as this one tell us about bridging the social and the individual? What are the strengths and limitations of our analytic strategies?

6. Discussion and Conclusions

We have focused on the multiple voices of educational reform that emerged in the discourse of Mrs Porter's third grade classroom. These voices, we have argued, came from the teacher's own educational experiences as well as those of her students' parents and their expectations about the children's futures. The voices also reflect the conflicting goals within the mathematics education community. Educational reformers such as Gravemeijer (1997) have identified an irreconcilable tension between the goals of the reform movement (aimed at fostering conceptual understanding and complex problem solving) and those of traditional instruction (aimed at fostering speed and accuracy in the use of algorithms). Gravemeijer argues that links cannot be built between these two approaches to instruction. Instead, he proposes that we need to choose the reform approach in order to ensure that students find mathematics meaningful.

Mrs Porter's response to an earlier version of our manuscript seemed to echo Gravemeijer's position. A sample of her written response to our analysis follows:

> I truly believe that a focus on traditional algorithms can be HARMFUL to children. I base that belief on what I've personally seen happen to a child's sense of place value when someone tries to teach them traditional algorithms and also on the writings of Constance Kamii . . . I was very purposeful in neglecting to attempt to explain the procedures when traditional algorithms were used in the lesson you

> included, particularly for the lesson in September. In September, most children really don't have much of an understanding of place value. Once I understood this, I spent lots of time early in the year trying to develop lessons that would make place value more clear. . . As I've mentioned many times, early in the third grade year, children often add to solve multiplication problems. Their understanding that multiplication is repeated addition is important as they begin to develop strategies for solving the problems that were presented. Therefore, it was important for me to keep pointing out this connection during discussions.

Thus, she defended her resistance to teaching the standard algorithms because they conflicted with her instructional philosophy, which builds on children's knowledge base. Our observations of her classroom were consistent with Mrs Porter's statements that she worked hard to create a community in which student sense making, risk-taking, communication, and diligence were valued. As in other reform classrooms (e.g., Yackel and Cobb, 1996), she fostered the norms that were consistent with those goals. Unfortunately, she seemed to feel isolated from like-minded educators and parents in her current position.

Gravemeijer (1997) cautions that choosing to base one's instruction on children's learning trajectories puts the isolated reform practitioner in a difficult position. Thus, one clear challenge for educational reform is to create the institutional communities that can support teachers like Mrs Porter and to involve parents in meaningful ways in that agenda (Lehrer and Shumow, 1997; Peressini, 1998).

We began this article by citing Lehrer, Shumow, and Peressini who mentioned the conflicting visions of educational reform among teachers and parents, which could result in the emergence of different voices in discussions of mathematics achievement and instruction. Unlike those researchers, we investigated this hypothesis by examining, in detail, transcripts of classroom instructional episodes. Our analysis of discourse employed a methodology in which units larger than individual turns at talk (e.g., instructional episodes) were examined. We adopted a top-down approach (working from the molar to the molecular), while maintaining a focus on units that contained both the social and individual. Thus, we chose not to isolate and code individual turns at talk in terms of their most obvious functions (e.g., questions, explanations, evaluations) – a procedure that depends on units too limited to bridge the individual and social. For example, we found multiple instances of animation in these instructional episodes, which indicated how the teacher was able to share her intellectual authority with her students by revoicing their strategies (O'Connor and Michaels, 1993, 1996; Goffman, 1974, 1981). Thus, authorship was distributed among the multiple participants in this community of practice.

An analysis of animation would be impossible without the use of units that link individuals and their social context.

We also used data from interviews and institutional documents to help us understand how influences outside the classroom – memories of the past, expectations of the future – can change face-to-face transactions in the present. We also found clues in the interviews to the origins of shifts in the emotional tone of the classroom discourse: from positive to negative; from active to passive; from optimistic to pessimistic. Thus, we were able to imbue our analysis of communication with a sense of the participants' beliefs about and attitudes towards learning mathematics. Nevertheless, we recognized that the empirical data itself would not be sufficient for answering the broad questions we wanted to ask about the conflicting visions or voices of educational reform. We also needed to employ interpretative concepts and strategies from cultural psychology. For example, we knew that we should look for critical junctures in discourse in order to identify participants' differing values and goals (Erickson and Shultz, 1981). In addition, we recognized that instruction is proleptic: it brings the past and future into the present (Cole, 1996). Thus, we looked at the interview data and institutional documents to try to understand the origins and functions of those junctures.

Thus, we feel that our analytic strategy and our theoretical framework have enabled us to interpret a small sample of data from classroom discourse in a single third grade classroom community in ways that may shed light on one of the current dilemmas of educational reform in mathematics. Nevertheless, the data we have been able to analyze and present is quite limited. For example, it indicated few changes in Mrs Porter's instructional approach over the three and a half months of observation. Clearly, data from additional lessons, including the other lessons in which standard algorithms were discussed would need to be analyzed to better assess the generalizability of our findings about the junctures in discourse in this classroom community. Also, further analyses of additional lessons might do a better job of revealing whether the same or different junctures in discourse practices occurred. In addition, our focus on Mrs Porter made it difficult for us to convey an articulated sense of her students' beliefs, attitudes, and mathematical practices. Data from other exemplary teachers who are also attempting to implement the recommendations of educational reformers are needed in order to provide a more complete picture of the opportunities and dilemmas of this approach to instruction.

In conclusion, cultural psychology is an emerging theory with the potential for influencing research and policy in mathematics education (e.g., Forman, in press). One advantage of cultural psychology is that it enables us to show how the individual and the social are interconnected by history and cultural artifacts. It enables us to go beyond limited analyses of interpersonal transactions in the present to their historical, social, and cultural origins and functions. This expanded analytic strategy is necessary if we want to present a richer and more adequate sense of the complexities of classroom life as well as the opportunities and dilemmas of educational reform.

ACKNOWLEDGEMENTS

This research reported herein was supported, in part, by grants to the first author from the Spencer Foundation, from the U.S. Department of Education, Office of Educational Research and Improvement, to the National Center for Improving Student Learning and Achievement in Mathematics and Science (R305A60007-98), and from the School of Education at the University of Pittsburgh. The opinions expressed do not necessarily reflect the position, policy, or endorsement of the supporting agencies. We are indebted to Deborah Dobransky-Fasiska for her thoughtful and caring attention to all aspects of the data collection and analysis process and to Mrs Porter for her honest and insightful self-reflections and her dedication to understanding and responding to children's needs. The assistance of Renee Bruckner, Elizabeth Hughes, and Jaime Munoz in data collection or analysis is gratefully acknowledged. We appreciate the extensive reviews of an earlier draft of this manuscript by Carolyn Kieran, Koeno Gravemeijer, Kristina Kumpulainen, Candia Morgan, and Elaine Olds.

NOTES

1. The teacher's and students' names are pseudonyms.
2. The turns in each lesson transcript are numbered sequentially even though they were sampled from a much longer transcript. We have kept their order of appearance consistent with the original transcript but have renumbered the turns for this discussion. "FP" refers to the teacher's turns at talk.
3. The ethnographers were: Ellice Forman, Deborah Dobransky-Fasiska, and Jaime Munoz.
4. The primary data set is comprised of field notes, audio taped classroom lessons and interviews with the classroom teacher, and classroom artifacts (photocopies of students' mathematical problem solving and answers to open-ended questions).
5. Unfortunately, we were not able to collect copies of student work for this lesson so we cannot be certain that the public discussion of each student's strategy was identical to the strategy displayed in his or her written work. Nevertheless, our comparative analyses of the transcripts of whole-class discussion from later lessons with student work completed prior to those discussions suggest that the public presentations of student strategies were quite close to the written records of students' independent work.

6. The transcription conventions are as follows: audible pauses are indicated by ...; vocal stress is indicated by an underline; overlapping speech is indicated by [ ]; additional nonverbal or contextual information is indicated by ( ).
7. Pulak's brother had been a student of Mrs Porter's during the previous academic year. In one of our conversations, she mentioned that Pulak seemed to understand the standard algorithm even better than had his older brother who had also used it frequently in her class.

REFERENCES

Burns, M.: 1992, *About Teaching Mathematics: A K-8 Resource*, Math Solutions Publications, White Plains.

Carpenter, T., Ansell, E. and Levi, L.: 2001, 'An alternative conception of teaching for understanding: Case studies of two first-grade mathematics classes', in T. Wood, B. Nelson and J. Warfield (eds.), *Beyond Classical Pedagogy: Teaching Elementary School Mathematics,* Lawrence Erlbaum, Mahwah, pp. 27–46.

Cole, M.: 1996, *Cultural Psychology: A Once and Future Discipline,* Belknap Press of Harvard University Press, Cambridge.

The Concise Oxford Dictionary: 1990, 8th ed., Oxford University Press. Oxford.

Erickson, F. and Schultz, J.: 1981, 'When is a context? Some issues and methods in the analysis of social competence,' in J.L. Green and C. Wallat (eds.), *Ethnography and Language in Educational Settings,* Ablex, Norwood, pp. 147–160.

Forman, E.A.: in press, 'A sociocultural approach to mathematics reform: Speaking, inscribing, and doing mathematics within communities of practice,' in J. Kilpatrick, G. Martin and D. Schifter (eds.), *A Research Companion to the NCTM Standards,* National Council of Teachers of Mathematics, Reston.

Forman, E.A., Larreamendy-Joerns, J., Stein, M.K. and Brown, C.A.: 1998, ' "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom', *Learning and Instruction* 8(6), 527–548.

Goffman, E.: 1974, *Frame Analysis,* Harper and Row, New York.

Goffman, E.: 1981, *Forms of Talk,* University of Pennsylvania Press, Philadelphia.

Gravemeijer, K.: 1997, 'Commentary solving word problems: A case of modelling?' *Learning and Instruction* 7(4), 389–397.

Hatano, G. and Inagaki, K.: 1998, 'Cultural contexts of schooling revisited: A review of *The Learning Gap* from a cultural psychology perspective', in S.G. Paris and H.M. Wellman (eds.), *Global Prospects for Education: Development, Culture, and Schooling,* American Psychological Association, Washington, pp. 79–104.

Inagaki, K., Morita, E. and Hatano, G.: 1999, 'Teaching-learning of evaluative criteria for mathematical arguments through classroom discourse: A cross-national study', *Mathematical Thinking and Learning* 1(2), 93–111.

Kamii, C. and Joseph, L.L.: 1989, *Young Children Continue to Reinvent Arithmetic: 2nd Grade: Implications of Piaget's Theory,* Teachers College Press, New York.

Krummheuer, G.: 1995, 'The ethnography of argumentation', in P. Cobb and H. Bauersfeld (eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures,* Erlbaum, Hillsdale, pp. 229–269.

Lave, J. and Wenger, E.: 1991, *Situated Learning: Legitimate Peripheral Participation,* Cambridge University Press, New York.

Lehrer, R. and Shumow, L.: 1997. 'Aligning the construction zones of parents and teachers for mathematics reform', *Cognition and Instruction* 15(1), 41–83.

Levinson, S.: 1983, *Pragmatics,* Cambridge University Press, Cambridge.
Mehan, H.: 1979, *Learning Lessons: Social Organization in the Classroom,* Harvard University Press, Cambridge.
NCTM: 1989, *Curriculum and Evaluation Standards for School Mathematics,* The National Council of Teachers of Mathematics, Reston.
NCTM: 1991, *Professional Standards for Teaching Mathematics,* The National Council of Teachers of Mathematics, Reston.
NCTM: 2000, *Principles and Standards for School Mathematics,* The National Council of Teachers of Mathematics, Reston.
O'Connor, M.C. and Michaels, S.: 1993, 'Aligning academic task and participation status through revoicing: Analysis of a classroom discourse strategy', *Anthropology and Education Quarterly* 24, 318–335.
O'Connor, M.C. and Michaels, S.: 1996, 'Shifting participant frameworks: Orchestrating thinking practices in group discussion' in D. Hicks (ed.), *Discourse, Learning, and Schooling,* Cambridge University Press, New York, pp. 63–103.
Peressini, D.D.: 1998, 'The portrayal of parents in the school mathematics reform literature: Locating the context for parental involvement', *Journal for Research in Mathematics Education* 29(5), 555–582.
Rogoff, B.: 1990, *Apprenticeship in Thinking: Cognitive Development in Social Context,* Oxford University Press, New York, NY.
Rogoff, B., Baker-Sennett, J., Lacasa, P. and Goldsmith, D.: 1995, 'Development through participation in sociocultural activity' in J.J. Goodnow, P.J. Miller and F. Kessel (eds.), *Cultural Practices as Contexts for Development,* Jossey-Bass, San Francisco, pp. 45–65.
Sfard, A.: 1998, 'On two metaphors for learning and the dangers of choosing just one', *Educational Researcher* 27(2), 4–13.
Stone, C.A.: 1993, 'What is missing in the metaphor of scaffolding?', in E.A. Forman, N. Minick and C.A. Stone (eds.), *Contexts for Learning: Sociocultural Dynamics in Children's Development,* Oxford University Press, New York, pp. 169–183.
Wells, G.: 1993, 'Reevaluating the IRF sequence: A proposal for the articulation of theories of activity and discourse for the analysis of teaching and learning in the classroom', *Linguistics and Education* 5, 1–37.
Wertsch, J.V.: 1985, *Vygotsky and the Social Formation of Mind,* Harvard University Press, Cambridge.
Wertsch, J.V.: 1991, *Voices of the Mind: A Sociocultural Approach to Mediated Action,* Harvard University Press, Cambridge.
Wertsch, J.V.: 1997, *Mind as Action,* Oxford University Press, New York.
Yackel, E. and Cobb, P.: 1996, 'Sociomathematical norms, argumentation, and autonomy in mathematics', *Journal for Research in Mathematics Education* 27(4), 458–477.

Department of Psychology in Education,
5C01 WWPH, University of Pittsburgh,
Pittsburgh, PA 15260, U.S.A.,
Telephone (412) 648-7022, Fax (412) 624-7231,
E-mail: ellice@pitt.edu
** Author for correspondence*

M.C. O'CONNOR

"CAN ANY FRACTION BE TURNED INTO A DECIMAL?" A CASE STUDY OF A MATHEMATICAL GROUP DISCUSSION

ABSTRACT. This case study examines two days of teacher-led large group discussion in a fifth grade about a mathematical question intended to support student exploration of relationships among fraction and decimal representations and rational numbers. The purpose of the analysis is to illuminate the teacher's work in supporting student thinking through the use of a mathematical question embedded in a position-driven discussion. The focus is an examination of the ways that the emergence of mathematical ideas is partially shaped by complex interactions among the mathematical contents of the question, the inherent properties of the discourse format and participant structure, and the available computational methods. The teacher's work is conceptualized in terms of actions and practices that coordinate these diverse tools, in constant response to students' concurrent use of them.

KEY WORDS: classroom discourse, group discussion, mathematical discourse, representation of rational number, teacher discourse

1. INTRODUCTION[1]

Despite the pervasiveness of the assumption that whole group discussion in mathematics classes may promote mathematical learning, we know little about the mechanisms that might underlie such outcomes. Nor can we answer many pragmatic questions about how student learning might best be secured through talk. Theoretical work in discourse (e.g. Clark, 1996; Vallduví and Engdahl 1996; Prince, 1992; Chafe, 1992) provides plausible accounts of how the contents of utterances are used by speakers and hearers, moment to moment, to update the information state of various entities represented in their ongoing model of the discourse and to expand the shared pool of presuppositions. This kind of research might form the basis for a schematic understanding of how ideas 'emerge' through talk. However, much of it provides only a minimal dyadic account, not an adequate model of the situation faced by a teacher trying to orchestrate a group discussion on a complex mathematical topic with 25 very different students. Empirical studies by mathematics education researchers come closer to depicting the real complexity of such settings, and are beginning to show how mathematical ideas may emerge out of the interactional ground of various forms of talk (Cobb and Bauersfeld, 1995; Cobb et al.,

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2000; Inagaki et al., 1999; Sfard, 2000a, 2000b; Lampert, 2001; O'Connor et al., 1998; Godfrey and O'Connor, 1995).

This paper aims to contribute to efforts to understand how mathematical ideas may emerge out of linguistic and discourse substance, but its primary focus is on describing the complex work of the teacher in conducting whole group discussions. It centers on a classroom discussion that took place in a fifth grade classroom over a period of several days. The discussion was framed by the teacher with a double-barrelled question: 'can any fraction be turned into a decimal, and can any decimal be turned into a fraction?'[2] The account is intended to offer a detailed view of the work involved in conducting such a discussion.

Of what use could such an account be? In order either to evaluate group discussion as an 'intervention' or to train practitioners in its use, we need a better understanding of both the general mechanisms that are at play and the specific moves and practices that may be used strategically to avoid common pitfalls and to facilitate learning. We need to understand the linkages among the mathematical content of the discussion topic, its linguistic formulations, the constraints and affordances of the activity structures occurring in particular forms of discussion, and the thinking and understanding that may result.

The specific form of this case study is a response to one of the persistent challenges of classroom discourse analysis: the competing requirements of data reduction and interpretive explicitness. To make even the most elementary claims about what action has taken place, or what has been accomplished, one must present evidence in the form of actual records of talk. Yet a complete transcript of 90 minutes of talk can rarely be made available along with the article analyzing it. Even a complete transcript, however, might still fail to yield obvious evidence for claims–interpretive work is involved, and the interpretation must be supported if the reader is to find it persuasive. Different approaches to classroom discourse analysis deal with these competing requirements in various ways.[3] For the purposes of this paper, however, only three phases of analysis are offered. First, before we can hypothesize about what the teacher may be seeking or finding in the words of her students, we must explicate the minimal knowledge involved in the mathematical domain that is the topic of discussion, and briefly review what is known about students' understanding of this domain.[4] Second, before a description of the actual conduct of the discussion, a synoptic overview of what transpired during the two days of discussion will be provided. Both the first and second phases introduce presuppositions necessary for understanding the third phase, which presents a more detailed account and makes claims about the work done by this teacher in managing the discussion.

2. Content of the Framing Question

2.1. *Rational numbers and their representation*

First let us consider what the framing question ('Can any decimal be turned into a fraction and can any fraction be turned into a decimal?') might offer in the way of openings for mathematical ideas and practices. As with most 'high level' tasks or questions in mathematics teaching (Stein et al., 2000), this one may lead to a number of profound issues, and different practitioners will have differing abilities and inclinations to follow up on these. Minimally, however, a few mathematical issues are irreducibly present. These mathematical ideas are linked with terminological choices and usages, and therefore terminology must form part of the preliminary description as well.

The question as literally posed may be puzzling to mathematicians who are not immersed in upper elementary mathematics education. The wording of the question seems to imply that 'decimals' and 'fractions' are different sorts of objects that undergo some transformation as one 'turns into' the other. A more explicit formulation might be 'Can any number that can be represented as a simple fraction be represented in decimal notation, and can any number that can be represented as a decimal also be represented as a simple fraction?' In this formulation, a simple fraction (or two integers separated by the 'fraction bar'[5] where the denominator is not zero) is a representational form that stands for a mathematical entity: a rational number. Decimal notation is a representational form that employs a decimal point and place values that denote (positive and negative) powers of ten.[6] For professional mathematicians and for teachers of more advanced mathematics classes, the point of this question would be the exploration of differences between rational and irrational numbers. Irrational numbers, numbers that cannot be expressed as a ratio of two integers, may appear as non-terminating, non-periodic decimals.

In this classroom, however, the question was not posed in hopes of introducing irrational numbers – the goal was more elementary, but as important. This fifth grade mathematics class was part of an intervention designed to increase the identification and fostering of mathematical talent in groups of students typically under-represented in higher-level mathematics classes. Suzanne Chapin, principal investigator, and Nancy Anderson, lead teacher, judged that the students in this class, with whom they had worked closely for over 18 months, had only a dim understanding that some numerals in decimal notation denote the same number as some numerals in fraction notation.[7] In their judgment, these students tended to see the written expressions[8] as things in themselves, as two different types of mathematical entities. The phrasing of the question did not directly undermine this conception. But by getting students to focus on the processes by which numbers in one format are transformed into the other format, the instructors hoped to get students to begin to see that 'decimals' and 'fractions' are alternate representational formats, not different types of numbers. Moreover, although the students had engaged in many activities foregrounding the 'quotient' subconstruct of rational or fractional numbers (what amount results when *a* is distributed among *b* portions?) (Kieren, 1975, 1993),[9] it was not clear whether they understood decimal fractions in this way; so the question was also intended to get students to encounter and puzzle over the nature of terminating decimals and non-terminating, repeating (or periodic) decimals (the latter which they had had little experience with).

These students had already spent time in fourth and fifth grade on relevant concepts and procedures. Chapin and Anderson had emphasized work on equivalence of fractions, and the text they used[10] presented situations requiring students to compare amounts represented as decimals with amounts represented as fractions. The work they had done linked the interpretation of decimals to fractions with powers of ten denominators, so they were quite able to note that 7/10 = 0.7 and 0.25 = 25/100 etc. The method of dividing numerator by denominator was also introduced but not emphasized. So although the students had had many weeks of working with fractions and with decimals, and had some grasp of their relationship, by no means did all students fully grasp that relationship. Even the students who had a strong understanding had not encountered many instances of non-terminating periodic decimals. It was hoped that in pondering the framing question, they would be led to explore the properties and constraints of each representational format and of various methods of transforming one into the other for a variety of cases, and thereby to encounter more deeply the properties of rational numbers.

2.2. *From methods to ideas and back*

The specific formulation of the mathematical question plays a role in the teacher's actions. Because the question is asking whether *all* fractions can be turned into decimals, and vice versa, the students are required to utilize various computational methods in evaluating the 'transformability' of classes of cases. Many students will know that *some* fractions can be converted: benchmark fractions like one half or one fourth have a decimal equivalent that these students will have encountered many times in previous lessons. So for these numbers at least they will believe that the quantities named by the two expressions are equivalent, and they will not have to call on a computational algorithm to verify the possibility of a transformation. These benchmark equivalences will already have the status of 'math facts.' But knowledge of these facts will not provide a complete answer to the question. The students must be able to evaluate the question for *any* fraction (or any decimal, depending on which part of the question is being answered).

Therefore, the only way this discussion can proceed is through consideration of classes of cases, in service of the question's requirement of exhaustiveness. Contributors will have to make claims about classes of fractions that can or cannot be represented as decimals, and vice versa. These claims will have to be backed up with demonstrations of methods. Discourse participants will evaluate the claims based on the evidence their colleagues give about actual transformations or on reasons why the transformations could not be done. Without computational methods and knowledge of when to apply them, the discussion cannot proceed. On the other hand, the discussion depends on more than the application of methods. Not all methods will work equally well with all numbers, and not all students are equally familiar with all methods. Finally, there are several important mathematical ideas that are not dependent on procedures and that can be discussed directly. There is thus a complex interplay among methods and ideas that I will describe in what follows.

Before considering the transcripts, which reveal the details of the teacher's integration of methods, ideas, and student contributions, I will briefly mention two methods of transforming one representation type into the other and point to some of their complex interactions with types of numbers and different states of student knowledge. The first method is not a formal algorithm, but rather a heuristic that appears in the transcripts. It allows the student to move easily from a decimal representation of a number to at least one fraction that is the equivalent of that decimal. If a student knows how to evaluate and *name* the place values of a decimal fraction such as 0.235, the simple act of naming the place values may result in a realization that this is also the name of a fraction representation. If the student says "two hundred thirty five thousandths" it yields the name of the fraction representation 235/1000. This clearly will not work with the common practice of naming the numeral as 'point two three five.' While for some students the use of this heuristic simply reflects a well-developed understanding of the fractional nature of decimal representations, for other students the transcript evidence suggests that a somewhat shaky conception is being supported by the words themselves.

This heuristic, however, works only in the case of terminating decimals (e.g. 0.235 as 'two hundred thirty five thousandths'). It does not work with non-terminating, repeating (or periodic) decimals. Repeating decimals do not provide the student any opportunity to name the corresponding fraction in which the name of the denominator is the place value of the rightmost decimal digit: one never arrives at the rightmost decimal digit (although an approximation is of course possible). So how could a student manage to transform a periodic decimal into a corresponding simple fraction representation if the periodic decimal is not one of the few well-known benchmarks (such as 0.3333. . .)? In fact, the available methods involve algebraic calculations that are not available to these fifth graders. So here we see a class of cases – periodic or repeating decimals – that can only be reached via a one-way street: these students may be able to turn simple fraction representations into repeating decimal representations, but they cannot go the other way with their available tools.

So the teacher leading the discussion of this question faces a challenge: what methods do her students have, and will those methods work in transforming the relevant cases? One might ask at this point whether these students knew the most widespread algorithm for transforming a fraction representation into a corresponding decimal: dividing the numerator by the denominator. While these students did know of the 'quotient' meaning of the fraction bar, and had been introduced to this method, only a few relied on the division of numerator by denominator as a method to generate decimal equivalents of fractions. Another method was deemed by Chapin and Anderson to be more generally productive in promoting students' computational and conceptual advancement, and had been emphasized in all the fifth grade Project Challenge classes. It requires taking a simple fraction, say 3/5, crafting an equivalent fraction with a denominator that is a power of ten, say 6/10 or 60/100, and then mapping it into decimal form: 0.60.

But the two methods ((a) numerator÷denominator and (b) building an equivalent fraction with a power of ten denominator) are not equally efficacious with respect to all classes of numbers. A student who uses method (a) (particularly with a calculator) can easily ascertain that 1/7 results in a repeating decimal: 0.1428571428571. Does method (b) allow students to encounter periodic decimals with the same ease? Imagine that a student is attempting to set up an equivalent fraction for 1/7, and starts with a denominator of 100. When the student divides this denominator by the original fraction denominator (100 divided by 7), she will see on her calculator screen that the quotient contains a decimal point and may, if the window is large enough, detect that it is repeating: 14.28571428571. . . So this method will result in an encounter with repeating decimals, but these will be encountered as the numerator of the 'power of ten' fraction: 14.28571. . ./100. When searching for an answer to the framing question, what will a student make of a fraction like 14.28571. . ./100, a numerical representation that contains both fraction and decimal notation conventions?[11]

Finally, the goal of this group discussion is not simply refinement of computational skills. Rather, it involves a development of the students' understanding of rational numbers and their representation. But ideas about rational numbers and fraction and decimal representations will not develop without computational work. Unless the students repeatedly traverse the ground between fractional and decimal notations and actually work with the various types (e.g. repeating versus terminating decimals) they are unlikely to develop a robust sense of the target domain. So the teacher must hold ideas and methods in a productive tension as the discussion proceeds.

2.3. *Setting of the case study*

At the time of the discussion analyzed here, the teacher, Mrs Anderson, was in her second year of teaching in Project Challenge, the three-year intervention mentioned above.[12] The 25 students in this episode were one of four fifth-grade classes that Mrs Anderson had worked with for about a year and a half, starting when they entered fourth grade. About 60% spoke a language other than English at home, and about 85% qualified for free or reduced prices lunches, figures reflective of the district as a whole. By the third year of the program, the mean score of this group on the California Achievement Test mathematics portion was at the 91st percentile (combined computation and concepts/applications).

In this, the second year of the program, the teacher and the PI were experimenting with a variety of approaches to language use in the classroom. A week earlier, the PI had introduced the framing question to Mrs Anderson, with the suggestion that she attempt to conduct a large-group discussion on the topic. Many of the children in the project classes (and in the district) come from communities and cultural backgrounds in which talk and argument as a means of intellectual investigation are not prominently featured as part of children's activities with parents or peers. Thus many of these students may rarely have taken part in discussions, in or out of school, in which their thoughts about a difficult non-personal question are elicited and subjected to systematic challenges. Mrs Anderson reported some trepidation about undertaking the discussion; nevertheless, the first discussion went extremely well. We suggested that she revisit the questions and allow us to videotape the class, as we had done on some other occasions. Two more days of discussion ensued, which constitute the data for this account.

3. A SKELETAL RECONSTRUCTION OF A POSITION-DRIVEN DISCUSSION

The transcripts exemplify an instance of what Michaels and Sohmer (1999) call a position-driven discussion (see Hatano and Inagaki, 1991 for discussion of similar speech activity types). A position-driven discussion involves a teacher leading a group of students in considering one central question with a fairly limited number of possible answers. Generally, the answer will not be known to any student beforehand. The point is for each student to take a position on the answer and to attempt to support that position with evidence. Challenges are encouraged, and the teacher's role is not to provide validation of correct or incorrect hypotheses or evidence, but to support and clarify the contributions of students, often through revoicing moves (O'Connor and Michaels, 1996). The teacher must also make her own contributions where necessary so that every student is following, understands the alternatives, and has a stake in the final outcome.

The teacher opens the day with one half of the *framing question*: 'Can all fractions be turned into decimals?' The students' first response is to say no. They conjecture that only fractions with denominators that are *factors of powers of ten* can be turned into decimals. (In the preceding day of discussion, not taped, they had established that decimals could be turned into fractions by the heuristic described in section 2 above and that fractions could be turned into decimals by the method of creating equivalent fractions described above.) But no student gives evidence for this claim by presenting a fraction that *cannot* be turned into a decimal, they simply assert it. The teacher *challenges the students' claim with a counterexample.* She poses a question: What about 3/6? Six is not a factor of a power of ten, but it *can* be turned into a decimal. The students respond that 3/6 is not a true counterexample, it just needs to be reduced to an equivalent fraction (e.g. 1/2) that *is* a factor of a power of ten. Then it will support the conjecture.

One student then expands the first conjecture. He informs the class that fractions with odd prime denominators become repeating decimals. In his words, repeating decimals are not 'true decimals' and if they are not true decimals, then he has identified a group of fractions that *cannot* be turned into decimals. This student's conjecture is quickly contradicted: another student presents a fraction with an odd prime denominator that *doesn't* turn into a repeating decimal: 2/5. But in turn, this counterexample is attacked: five is a factor of a power of ten, so 2/5 is covered under the very first student conjecture: all denominators that are factors of powers of ten can be turned into 'decimals'. The claim that some fractions cannot be turned into 'real' decimals still stands.

At this point the teacher goes back to attack the original conjecture with *another counterexample.* She poses a question: can't 1/8 be expressed as a decimal? Eight doesn't seem to be a factor of 10 or 100. She passes out calculators and directs students to work with partners. Eventually, the students discover that eight *is* a factor of a power of ten – 8 is a factor of 1000. Thus 1/8 has an equivalent fraction whose denominator is a factor of a power of ten (125/1000) and so 1/8 fits the students' first conjecture. As the students work with the calculators, they explore other fractions, including those with odd prime denominators. Looking at 1/7, they find that it is repeating, and reiterate the claim that repeating decimals are not 'real' decimals.

The teacher asks them to *explore a clear example.* She asks them to think about 1/3. They find with some excitement that 1/3 yields a repeating decimal, and unlike 1/7, with its confusing sequence of six digits that repeat, 1/3 consists of an infinitely repeating single digit. So now that the discussion has driven them to seek out repeating decimals, all students begin to use the method of dividing the numerator by the denominator as they use their calculators. The teacher then asks another question that *frames a sub-discussion intended to foreground the properties of repeating decimals*: is there a rule for repeating decimals? When do we get them? After some exploration with the calculators, students claim that fractions with odd denominators and their integer multiples – at least the ones that are NOT also factors of a power of ten – will result in repeating decimals (e.g. 3rds, 6ths, 9ths, 18ths, 21sts).

At this point the student who earlier made the claim about repeating decimals not being 'real' decimals reverses himself. He has thought more about it: a repeating decimal and a terminating decimal may both be transformed into a fraction representation, added together and then transformed back into another decimal, so perhaps repeating and terminating decimals are not qualitatively different. Both designate parts of a whole. So if both are 'real decimals,' then any fraction can indeed be turned into a decimal. But other students want to dispute this; they insist that repeating decimals *are* qualitatively different than terminating decimals. While they may both be parts of a whole, the repeating decimal is problematic: "you don't know how many parts there are, because the decimal never ends." And because you don't know what the smallest part is, you can't transform them back into fractions, so they are defective in some way. The second day ends with no resolution.

Schematic as it is, this synopsis reveals that some of the ideas potentially available in the question itself did come to light. By the end of the second day, the idea that decimals and fractions are the same type of entity in different guises had begun to emerge for at least a few students. The properties of the two forms of representation, fraction bar and decimal notation, had been taken up to some extent. It is also clear even in this schematic outline that the nature of the discourse event itself, the position-driven group discussion, drove some of the findings and some of the explorations. The teacher's contributions – counterexamples to claims, and examples that might provoke certain findings – are not part of the question itself, but are part of her enactment of the investigation of a conjecture. The framing question is a tool embedded within a particular discourse format (another sort of cultural tool), and she is responding to both jointly, as are the students.

This synopsis is intentionally presented as though the discussion were uniformly clear and coherent. But just as an orderly view of roads and highways seen on a map or from a mile above the earth dissolves into the complexity of one-way streets and unfinished exit ramps in a close-up, the next two sections will display a more veridical picture of the work required to conduct this discussion.

4. A CLOSER LOOK

4.1. *Challenging a robust conjecture and its proponents*

The teacher, Mrs Anderson, begins the session by asking a framing question:[13]

1 T: Can any fraction be turned into a decimal? Why don't you talk it over with your partner. See if you can recall what was brought up and if you can remember also, specifically, who said what that would be even better. Go ahead, you've got 2 minutes.

As described above, students have already had one session of discussing this question along with its companion, 'can any decimal be turned into a fraction?' – a session that went unrecorded. Mrs Anderson frames the discussion on this day with only one of the two questions. Tape-recordings of the ensuing two minute small group discussion reveal that each of the groups contains at least a few students who refer to the 'rule we made up' – that for a fraction to be turned into a decimal, the denominator of the fraction must be a factor of a power of ten, because if it isn't, 'it wouldn't work because that won't go into it evenly.' Students are using the equivalent fractions method described in Section 2. So 4/20 yields 20/100, which is 0.20.

As the large group conversation starts, several students repeat this widely (but not universally) held conjecture.

46 T: Alright, are you ready? Okay, can any fraction be turned into a decimal? Yes or no? And tell us why you think so. Mirjana?

47 M: No, because three or eight or six are not the factors of ten, hundred, thousand, ten thousand, hundred thousand, so it can't be, uh. . . so this fraction one third or one eighth or one sixth can't turn, be turned into a fraction.

48 T: So one third cannot be turned into a *decimal*? [*two turns deleted*] Why not?

51 M: Because it's not a factor of ten, hundred, thousand, ten thousand, hundred thousand, and so on [*three turns deleted*]

55 J: I think no because um if three, because threes are in the denominator so the denominator has to be a factor of ten, hundreds, thousands, and so on. And it's important because of. . . you can't change into a decimal because if you want to change into to a decimal, you have to have a factor. You have to have the denominator of the fraction to be a factor of ten, hundred, or thousand.

Notice that when the teacher introduces the fraction 1/3 Mirjana and Juana quickly dismiss it. Mrs Anderson told me that in previous lessons, months earlier, students had encountered repeating decimals, 1/3 being a parade example. Yet her prompting here is ineffective. The girls laying out this conjecture seemed quite certain they were correct. The conjecture gained support from the two methods described above, discussed on the previous day: (1) any *decimal* could be converted into a *fraction* simply by using the given digits after the decimal point as the numerator, and the place value of the rightmost digit as the denominator. (Of course all resulting fractions will have denominators that are powers of ten.) And (2), fractions can be converted to decimals, as described above, by finding an equivalent fraction with a denominator that is a power of ten. So their two methods conspire to make this conjecture seem reasonable.

In addition, the discussion has a momentum of its own. Particularly for novices in the activity of position-driven discussion, holding a position against challenges may be more compelling than the scientific search for truth. (Indeed, even adults may resist the examination of counterexamples to their claims.) So at this very early point, Mrs Anderson must juggle two obstacles to progress: the deterministic character of the students' methods and the delicate nature of challenging a novice who has made a claim.

In this regard, it is interesting to note that right in the middle of this first spate of statements about this conjecture, one student, Marco, disagrees with the conjecture and suggests that *any* fraction can be turned into a decimal through a method of 'adding zeros.' But after a lengthy attempt to understand him, Mrs Anderson bypasses Marco's method – his contribution is unintelligible, and so he is not able to alter the course of the discussion. She must turn back to the majority opinion. First she makes an important move by throwing another counterexample up against the current rule. Her example, 3/6, is a counterexample and is perhaps more accessible than 1/3.

76 T: Now, something's been on my mind since Mirjana and Juana brought up the point. Three sixths! Six is not a factor of a hundred, I agree with you. But can't we write three sixths…
77 S: As half!
78 T: Yeah. What does a half look like as a decimal?
79 J: Uh, point five.
80 T: Point five. So that kinda doesn't fit our rule. Three sixths can be written as a decimal, but six isn't a factor of a hundred. What's going on there?

Everyone in the class knows that 3/6 has a decimal equivalent. The teacher is trading on a benchmark fraction to force students to realize that their conjecture is incomplete. With this counterexample, she ensures that the dominant idea – a transparent, direct compatibility of decimal place value and fraction denominators – is challenged. But it turns out that it is not fatally challenged: students find a way to subsume this example without pushing too far towards a new understanding. What eventually follows is a revision: an extra step in the decision process. Students assert that when confronted with a fraction whose denominator is *not* a factor of a power of ten, one must first look to see if it can be reduced to a fraction whose denominator *is* a factor of a power of ten. In this case, 3/6 is obviously reducible to 1/2, whose denominator is a factor of ten. Her counterexample has been subsumed and the conjecture still holds.

4.2. *Managing student counterexamples and alternative conceptions*

At this point, Bruno speaks up with a contribution that leads eventually to one of the main strands of the discussion, a strand that had not been planned for but that eventually captures the consideration of almost every student who speaks (well over half the class). He makes a new proposal, trying to isolate exactly which fractions *cannot* be turned into decimals. His is the first contribution which directly addresses the logic of the framing question: to successfully argue that not all fractions can be turned into decimals, you need an example of a fraction that *cannot* be turned into a decimal. The example of 3/6 may have led Bruno to think about other denominators which are divisible by two: many of these will potentially be reducible to a factor of a power of ten. So what about odd numbers?

81 B: Um, I agree with Juana that, um, that the denominator has to be a factor of, of a power of ten. So, um, if the denominator is a odd prime number it cannot be a decimal. It will come out as a repeating decimal.

Bruno is the first to introduce a distinction between 'decimals' and 'repeating decimals.'[14] By the former he apparently means prototypical decimals: tractable, terminating decimals like 0.4 and 0.75. The students have encountered repeating decimals before, but never in the context of a position-driven discussion in which their properties might potentially play a role in the answer to a high-level discussion question. His words here imply that repeating decimals are not 'real' decimals.

Almost immediately, his main claim about odd prime denominators is challenged. Gina offers the counterexample of 2/5.

83 G: I disagree because five can go, uh, any fraction with five or fifths in it, can go into, can also be turned into a decimal.

84 T: Give me an example.

85 G: Um, two fifths is, is one tenth, umm, two fifths is umm is turned into...

86 T: Okay, what does two fifths look like as a decimal?

87 J: point forty.

The teacher asks Bruno to respond, and he immediately replies that fifths are covered under his original statement: fraction denominators that are factors of powers of ten (like fifths) all do allow transformation of fractions to decimals. (So his claim could have been stated more precisely, e.g.: 'any odd prime number that is not also a factor of a power of ten will result in a repeating decimal.')

At this point the class looks a bit stunned at the directness of Gina's counterexample, and the alacrity with which her challenge is returned. There is silence. Gracefully, the teacher takes this opportunity for a meta-comment that depicts the two as collaborating in an important mathematical practice:

90 T: Great, now I hope you're listening because what Gina and Bruno said was very important. Bruno made a conjecture and Gina tested

> it for him. And based on her tests he revised his conjecture because that's what a conjecture is. It means that you think that you're seeing a pattern so you're gonna come up with a statement that you think is true, but you're not convinced yet. But based on her further evidence, Bruno revised his conjecture. Then he might go back to revise it again, back to what he originally said or to something totally new. But they're doing something important. They're looking for patterns and they're trying to come up with generalizations.

The teacher has sensed that this excellent display of the mathematical practice of rinding and testing counterexamples to a claim is not without its potential risks. By portraying the exchange as positive, she is reassuring all students that the practice does not connote hostility, as it might in any informal or everyday setting. While Mrs Anderson has in the past asked students for examples that contradict one of *her* claims, rarely have the students spontaneously offered counterexamples to each other's claims. Disagreement can become personal.[15]

The conversation then returns to the question of reducing candidate fractions to equivalent fractions in lower terms. Several students attempt to express the idea that even if a fraction appears not to be a candidate for division into a power of ten, we first have to try to reduce it to lower terms, looking specifically for a denominator value that is a factor of a decimal place value.

4.3. *Posing another challenge: what about one eighth?*

Shortly thereafter, the teacher tries another fraction that students have seen before, 1/8. According to her, they had encountered its decimal notation equivalent, 0.125, some weeks before. This fraction is interesting because unlike 3/6 it cannot be reduced, and its denominator appears at first glance *not* to be a factor of a power of ten. Yet if students explore further, they will find that it is a factor of 1000. If they choose to use the 'numerator÷denominator' method, they will quickly find that the quotient of $1 \div 8$ is 0.125, a perfectly good decimal. Or the 'build an equivalent fraction' method could work as well, but only if students try dividing 8 into 1000. So this example could provide several kinds of challenges to move students further into the framing question and their inventory of methods.

4.3.1. *Interlude: Reviewing definitions and purposes*

But before Mrs Anderson lets the students start examining 1/8, she stops to clarify a term that has been widely used, 'powers of ten'. She is assessing whether the ubiquity of the term within the discussion really reflects a status of shared understanding, or whether what appears to be 'taken as shared' is really several different ideas masquerading under the same label.

123 T: Okay, so what about one eighth? Yes or no? Can you turn that into a decimal?

124 TY: I don't think so because, um, one [eighth] cannot be [xxx] uh, a fraction to a decimal [xx] uh, cause you need powers of ten.

125 T: Powers of ten. I've heard that a couple of times. Um, you Tyisha and who else used the phrase powers of ten? Very good. Who can refresh for us what that means? . . . where does that word *power* of ten come in? I don't get it.

The teacher then establishes that the powers of ten are not the same as integer multiples of ten and clarifies what they are. But this is not simply a definitional episode: she uses it also to shore up the discussion itself, which has been moving fairly quickly, by asking about the current relevance of this idea, powers of ten. 'So what do the powers of ten have to do with fractions and decimals? Who cares?' The last phrase, said with an adolescent-like shrug, is her signal to the students that she wants some reiteration of the core ideas: what does it all mean? Why should we be spending all this time talking about powers of ten in *this* discussion? How does it relate to the main purpose of the discussion? The students gather themselves to convince her of the idea's importance. Using about 30 turns in the discussion, she establishes to her satisfaction that most if not all students seem to understand this critical term in the discussion. She moves on.

At this point the teacher brings the conversation back to 1/8. No one has recalled that its decimal equivalent is 0.125. In fact, Sela, the next student to speak, seems to be using method (b), 'find an equivalent fraction that is a factor of a power of ten'. She successfully converts 2/4 into 50/100 and demonstrates her computational approach. Unfortunately, her original conversion of 1/8 into an equivalent fraction goes significantly awry.

155 S: Because you can, um, you can make it [one eighth] into an equivalent fraction that can be divided, um, that can go into a hundred, or factor a hundred, and I changed it into two fourths and, um um, I thought of something that would go into a hundred and a fourth of one hundred is twenty-five. So you need to times twenty-five by two cause you need two fourths and you get fifty, fifty hundredths.

The teacher does not correct Sela directly; she asks whether other students have a comment. Two students immediately point out the error, using good intuitive evidence of the relative sizes of the two fractions:

157 S2: Um, I disagree because, um, two fourths equals one half and one half is larger than one eighth. So one eighth can't equal two fourths.

Having gotten everyone focused on the status of one eighth, and having found that there is no equivalent simple fraction with hundredths as a denominator, the teacher then throws them back onto their own resources for a five-minute period of exploration with calculators.

160 T: Okay, well, maybe there is another way. Maybe we can turn one eighth into thousandths, or ten thousandths. Talk to your partner, I'll pass out some calculators too.

What happens next is only partially accessible, as only a few small groups were audiotaped, and their interactions are less than clear. We can discern that one group has discovered through trial and error that 8 can divide 1000 evenly, with a quotient of 125. They have discovered that it is possible for a number to be a factor of one particular power of ten but not of all powers of ten.[16] But it is also clear that they are not using method (a), numerator ÷ denominator, but instead are simply checking whether 8 can evenly divide these larger powers of ten. It is not clear whether they are doing this as a prelude to constructing a fraction that will be equivalent to a decimal or not. Most students seem to be starting with $10 \div 8$, then $100 \div 8$, then $1000 \div 8$, and so on. They do not transform this knowledge about the status of 8 as a factor of 1000 into hypotheses about the representation of the fraction 1/8 as a decimal less than one, however. Most get stuck after finding the quotient 125 or 12.5 or 1.25.

4.3.2. *Modelling confusion*

After about five minutes of intense student activity in groups of 3 or 4, the teacher begins the conversation again, asking whether 1/8 can or cannot be turned into a decimal. The first few students to answer tell her that yes, it can be turned into a decimal, but the decimals they give are incorrect. For example, one student asserts that it is 1.25. Evidence from the small group tapes suggests that the student derived this by simply dividing eight into ten. It's a state familiar to all who attempt such multi-step explorations: the students have lost sight of the goal and are bogged down in the computational details. Once they derive a numeral with a decimal point, they forget the larger goal – a decimal equivalent of 1/8.

The teacher now steps back and skillfully uses a characteristic conversational move. She models complete confusion: "alright, now I am totally confused." This allows students who are confused beyond their ability to articulate it to sit back and watch for clarification – an attempt at clarification is sure to follow. It also encourages students who do feel confident of their understanding to step up and try to help out the class. Here we have a clear example of the teacher in action, responding to the constraints and affordances of the tools she is acting with. Coordinating claims and counterclaims with searches for evidence to support the claims and counterclaims will necessarily involve moments of conceptual and communicative confusion. One of the required responses is a repertoire of moves that can reconsolidate the joint understanding of the group, bringing some unity back to the 25 separate actors by providing them with a jointly agreed upon, clarified focus.

4.3.3. *Using student insights*

It's important to note that Mrs Anderson's skillful use of this move is informed by her previous use of another tool: the five-minute peer talk discourse format. She and Ms. Brown, the classroom teacher, were able to move around the classroom and see which students were closer to methodological breakthroughs and which students had fundamental misunderstandings. So she is actually skillfully coordinating two aspects of her discursive tools: she is leveraging the knowledge she gained by listening in on peer-talk to consolidate understanding within her position-driven discussion. Sure enough, her first invitation to explain allows Bruno to bring out a methodological point that is important.

253 T: [. . .] Alright, so now I am totally confused because eight isn't a factor of one hundred, one eighth isn't equal to a fraction whose denominator is a power of one hundred, or is it? Maybe I am wrong. I know you guys have it, everybody else help me out here. Bruno.

254 B: Um, one eighth is equal to point one twenty-five because you put one eighth on the calculator, and then you press 'fraction to decimal' and that gives you zero point one two five. And to make sure to see if the fraction is equal you multiply one point two five I mean zero point one two five [xx] into eight and then that will give you one whole.

255 T: I agree with you, but Bruno, what's confusing me is the fact that it doesn't fit my rule and my rule according to Mirella, Juana, and Sela is that if my denominator is a factor of a power of a hundred then I can convert it to a decimal, but this doesn't fit my rule.

Note that in the midst of this attempt to regroup everyone around a newly fortified understanding, the teacher does not simply accept Bruno's new method and new finding. She slows him down with the obstacle of the first conjecture. She needs to have all the students understand that 8 can evenly divide a power of ten, *and* that this then yields a fraction that can be easily converted to decimal notation.

256 B: Well if it's, what we have on the board is um, with Juana, what she was saying, that fifteen thirtieths is not a factor of a hundred but it's equal to one half and one half is a factor of a power of ten. And, um, one eighth has an equivalent fraction that is a power of [ten]

257 T: It does? [*4 turns deleted*]

262 B: Um, because it has there are fractions that are beyond it, like one eighth is also equal to two sixteenths and three twenty-fourths and another fraction and another fraction until it gets to a fraction that is equal to [a power of ten].

Both methods and ideas are emerging in response to the teacher's call for clarification. Bruno not only knows where the 'fractions to decimals' button is on the calculator, he also knows that equivalent fractions can be in higher terms as well as lower terms and presents an almost visual understanding of a series of equivalent fractions. (Some of the other students have behaved as though one finds equivalent fractions only by reducing to lower terms.) What he says here at least suggests that he is constructing an understanding of decimal notation as representing fractional or rational numbers.

4.4. *Trying to leverage a new method*

Now the teacher tries to extend Bruno's reasoning (and importantly, method) beyond the example of 1/8 to a general process for all fractions. She seems to want to push them all to reconsider examples that diverge from the leading conjecture, and presumably she also wants to shore up the methodological underpinnings of the enterprise. Another student agrees with Bruno: she says that just because one eighth can't be turned into an equivalent fraction with a denominator of tenths, maybe it could if the denominator was 'millionths or ten thousandths.' The teacher then turns to the class as a whole:

265 T: Well maybe we can do that for any fraction. Maybe we can turn *any* fraction into an equivalent denominator or factor of a power of 10.

Bruno, however, objects to sending the conversation in this direction: he reminds her of the claim he made at the beginning of class, that odd prime denominators will result in repeating decimals.

266 B: No, cause, um, one of the rules that I said before that, um, asking that, like… are a odd prime number and it doesn't have a fraction equivalent to a power of ten, one hundred, [like for example] one seventh.

The teacher makes the choice to follow this up: perhaps she figures that whatever Bruno’s point is, taking up his example will get all the students to attend to the method of finding decimal equivalents with their calculators, and this will give her an entry point into what is clearly an increasingly necessary discussion about computational means and their workings.

267 T: Alright, let’s try one seventh, try that on your calculators. Do you get a decimal equivalent for one seventh? Yes or no?
[Class working, shouts of ‘Yes!’]

The visual display of a decimal point and a string of numbers will be recognized by these students as a decimal of some kind. The decimal equivalent of 1/7 is 0.142857 repeating. Bruno has at least indirectly claimed that repeating decimals are not ‘real decimals’. It is unclear from the evidence of tapes and transcripts whether the quotient that students are getting on their calculators as they try to turn 1/7 into a decimal is actually readable as a repeating decimal. On these small calculators, 0.142857 repeating appears as 0.1428571, not transparently repeating unless one is looking for it.

4.5. *Reintroducing one third, a clearer case*

As the students look at 1/7 and its decimal equivalent, the teacher suggests without discussion that they try the fraction 1/3, a fraction she first brought up in line 55, at the beginning of the lesson. At that time it was summarily dismissed by Juana as not expressible as a decimal. The teacher hasn’t mentioned the window limitations on the calculators, so it’s not clear whether she sees the problem yet.

272 T: Alright, so that works, let’s try another one. Gotta be able to find one that doesn’t work. Alright, try one third then. [*three turns deleted*]

In 272 the teacher reminds them of the logic of pursuing the conjecture, getting students to look for counterexamples or forcing them to conclude that they can’t easily find any. Students all try 1/3 and shout out their striking new decimal:

275 J: three three three three three three three three three three
276 (Several students in unison): Tens, hundreds, thousands, ten thousandths, hundred thousandths, millionths, ten millionths.
277 J3: It goes all the way up to ten millionths.

So far, only Bruno has made the claim (and quietly, at that) that fractions written as repeating decimals constitute counterexamples to the conjecture that all fractions can be turned into decimals. Now another student vacillates, starting out with the same claim. But a third student denies this when the teacher questions it in a small group discussion.

279 T: So it works. [i.e. 1/3 can be converted to a decimal]
280 S: No, because it keeps going.
281 T: So does that make it not a decimal?
282 S2: Yeah, but still because it's... but still it's a decimal.

Now the teacher tries to bring to the floor what students know about this variety of decimal.

283 T: You brought up a very good point, it does keep going. What's that – what's the term for a decimal that keeps on going?
284 X: Um, never ending
285 S: Repeating! [*one turn deleted*]
287 T: What happens when you try one third? It works but what's different about one third that hasn't been happening so far with our other examples. Yes?
288 K: You get a repeated fraction. [*two turns deleted*]
291 T: What's a repeated decimal? Do you know Mirjana? A repeating or a repeated decimal or a non-terminating decimal?
292 M: Um, it means that the number, the number, um, the number in the tens space [repeats]. It's always gonna be [repeating].
293 T: Exactly. Just like Miss Brown wrote symbolically [*on the board*]. You show it with the repeating decimal bar. This bar over the three means the three goes on forever.

4.6. *Posing a new question*

Students have now incorporated a new method, using the 'fractions to decimals' button on the calculator. At this point the teacher apparently decides to distill out of this talk and activity another framing question directed at a deeper exploration of the properties of the two varieties of decimals encountered here. Her new question does not directly reflect Bruno's tacit claim that repeating decimals are not legitimate decimals, but it does open up space for a deeper consideration of periodic vs. terminating decimals. This new question will dominate discussion in the class over the next day as well.

293 T: How come we got a repeating decimal for one third and we didn't get a repeating decimal for one fourth or one fifth or two fifths? What's the difference between one third and the other fractions like two fourths or one fourth or two fifths?

Students fumble with this question for six turns, getting nowhere. She formulates it again.

299 T: So is there a rule? When would I get a repeating decimal, when won't I? And is that the same or a different question from our original question? This is getting tough, I know it is, so I'm looking for the students who are going to stick with it.

Students begin to talk about fractions with odd denominators. Lisa finds that 2/9 yields a repeating decimal.

300 L: Well, the difference is, one third is, like, odd on the bottom and two fourths is like, even, so. Like two ninths. I tried that on my calculator and it came out like two. Like two two two two. Zero and two two two two two two, it goes on forever.

They hypothesize that odd denominators will produce repeating decimals. The classroom teacher, Ms Brown, points out that the students have forgotten about 1/5, an odd denominator that becomes a terminating decimal. Lisa responds with the point that Bruno made originally:

312 L: Well, some are like odd and they can't. Some are odd and they can. It matters if they're a power of ten.

Mrs Anderson then directs them to discuss the problem with each other, and to use their calculators. The small groups are all intensely active. One group focuses on the finding that repeating decimals are not limited to a single number repeating. Small sequences of numbers may be repeated: two digits, three digits, etc. Another group hypothesizes that fraction denominators that are multiples of three will result in repeating decimals, including 3, 6, 9, 18 and 21.

4.7. *Managing limitations of physical and discursive tools*

As the class comes back together, a problem arises. Because this is a position-driven discussion, if a student makes a claim, for example 'denominators that are multiples of 3 will result in repeating decimals,' she may appropriately be challenged by the bringing of evidence. But the relevant evidence is coming from little calculators with small windows. It now starts to become evident that the calculator is itself becoming a problem within the structure of the discussion: Juana offers the observation that 2/21 has a denominator that is a multiple of 3, but asserts that it is *not* repeating – a counterexample to the claim that odd denominators and their multiples yield repeating decimals. In fact, however, Juana is wrong: 2/21 = 0.095238095238. . ., but her calculator shows only 0.095238, so from the evidence available to her, it is non-repeating.

At this juncture the two teachers have written on the board a table with fractions that turn into terminating decimals and fractions that turn into repeating decimals. Looking at it, Mrs Anderson suggests only that 2/21 be put into the 'Don't know' column. Their calculating equipment is too limited for this case. She says that they cannot say whether it's repeating or not. Upon reading the transcript we can see that the argument may suffer, but the teacher has apparently decided not to stop and address the symbolic and physical limitations of the little blue plastic calculator. It's worth noting that at this point she has less than rive minutes left until she must move on to the next class. The next exchange, however, confronts her directly with the characteristics of the tool: they cannot be avoided.

Tyisha has decided that 1/6, another fraction relevant to the argument, is *not* repeating, because on her calculator it comes out as '0.166667.' This case is a bit different, because even with small calculators, students should be able to recognize the signs of 'rounding up.' If Mrs Anderson does not step in, faulty evidence will sink the argument. She says that she can tell them 'from experience' that 1/6 *is* a repeating decimal, and that one way to derive the decimal representation is to divide the numerator by the denominator. She moves to the board and leads them step by step through the long division, showing that no matter how many places you extend the dividend, you will have 6 as the quotient with a remainder of 4, in a never-ending sequence. She then explains that Tyisha's calculator is rounding up the last digit, so Tyisha's number '0.166667' is really an approximation to the repeating decimal. So yes, 1/6 in fact *does* fit the conjecture about denominators that are multiples of three.

The lesson ends and she congratulates them for their effort, saying that, "This is a very challenging question. So you should feel very proud of yourselves that you had so much to say about the issue. It's not straightforward at all. There's no one quick, easy response. There's a lot of things that come up. Juana, as you can see, you had a beautiful conjecture, then all of the sudden I threw a monkey wrench into it and I said 3/6, that doesn't fit. So you want to add on to your conjecture, change it a little bit . . . ."

4.8. *Day two: pushing the exploration of repeating decimals*

Two weeks later, after discussion and reflection with project PI Chapin and myself, Mrs Anderson found a day to continue this discussion. On this day she starts with two framing questions, asking students to talk together before the group begins its discussion. She reminds them that when last they carried out the discussion, they tried to answer several questions, and today she wants them to think about the issues again. She asks: "Can any fraction be turned into a decimal? Also, what is the deal with these repeating decimals?" Neither of these topics has been formally discussed during class in the intervening weeks.

Immediately students begin to talk. It is as if the previous lesson had just ended. In one small group students are still actively considering the idea that repeating decimals are not really decimals.

A1 S4: I think, I think, I think it can be turned into a – make a fraction into a decimal because, it's, even though it's a repeating number, the uh, the decimal, it's still gonna be a decimal. It's just repeating . . . in the s – s – still in the same – it's still a decimal. It's just repeating.

A2 S5: One third'll be probably point, zero point six six six, right?

A3 S4: Yeah, keep'em going.

A4 S5: That'd still be a decimal.

One of their group mates, Gina, disagrees bluntly.

A5 G: Well, I disagree with both of you. I don't think that every fraction can be turned into a decimal because I don't think a repeating decimal is a complete decimal. I don't think that a repeating decimal – that [xx] it's completely terminated. And I don't think it'll be a real decimal if it just keeps repeating.

[*eleven turns deleted*]

A17 S3: […] but it's still a decimal. I don't know how you disagree with [saying it's] gonna still be a decimal.

A18 G: Well, *hellooo,* it's not ending! So how can it be?

4.9. *Bruno's insight*

When the group comes back together after about rive minutes of this small group talk, the first question to come up is the question about what kinds of fractions turn into repeating decimals. After about six minutes, Bruno speaks up and initiates a new topic, the original framing question, whether any fraction can be turned into a decimal. He was the first student to mention, two weeks earlier, that repeating decimals and terminating decimals are 'different' in some way. Here, he seems to argue against his own initial idea, using the convertibility of fractions and decimals as part of the argument against his initial position.

54 B: Um. I would think that um any fraction can be turned into a decimal because, like, two thirds can be turned into a repeating decimal, um, point six repeating, and with the six repeating, if you wanted to add it to another decimal, like [x] point six, you could, because you could change them both into fractions and find their equivalents and add

that and you would get the answer. For point six repeating. And point six repeating is [two thirds] and um point six is six tenths. So um one of their equivalents are, two thirds has twenty thirtieths and sixth tenths has eighteen thirtieths, so if you add them together you get thirty-eight thirtieths, which is also equal to one and eight thirtieths.

I interpret his contribution as follows: repeating decimals and terminating decimals are not qualitatively different mathematical objects. If you take a repeating decimal like .6666 repeating and a terminating decimal like 0.6, you can convert them both into fractions with the same denominator. For example, .6666 repeating is 20/30, and 0.6 is 18/30. If you add the fractions together, you will get 38/30. Thus, a repeating decimal derived from a fraction like 2/3 can be turned back into another fraction, and added to non-repeating decimals or their fraction-bar equivalents. He seems to be saying that since both repeating and non-repeating decimals can be manipulated in the same way and result in the same kinds of outputs, they are not qualitatively different – they're the same kind of object.

It is a bit hard to know what to do with this contribution on the spur of the moment. Bruno's point is complex and displays a deep mathematical insight.[17] Previously, he had said that no, not any fraction could be turned into a decimal, because repeating decimals didn't really count as decimals, and we know that some fractions result in repeating decimals. Now, two weeks later, he has satisfied himself that repeating decimals are not substantively different from terminating decimals, so the answer to the original question is now yes.

Yet the issue of whether repeating decimals are or are not 'real decimals' has been only marginally addressed within the whole-group position-driven discussion. It has been playing in the background; most of the students have not directly addressed it. And as with every other turn in these discussions, the teacher does not have the luxury of a tape to play back, nor the luxury of time to ask herself just what it is that Bruno is trying to say or how it might best fit in with the current enterprise. But as she often does, Mrs Anderson quickly grasps a potential relevance of the student's contribution to the larger discussion, and harks back to the initial question, 'can any fraction be turned into a decimal?' She skillfully marks the fact that there has been a change of topic, and asks for repetition so all can follow.

61 T: Hear what he said? So if I [xx] from here take a step backwards, because when we started our class discussion, we immediately jumped into the repeating decimals discussion, which is great, but thank you Bruno. Let's just take it back a little bit and talk about the first initial

question. Can any fraction be turned into a decimal? Just to clarify that, then we'll continue going with our repeating decimal discussion. Bruno, you said yes because why? [*two turns deleted*]

64 B: Because it can be a repeating decimal.

65 T: Because you can still add it to another decimal?

66 B: Yes, because, um, 'cause they always have the – cause all decimals have an equivalent fraction to them. So you can change all the decimals into equivalent fractions and add them that way. If you wanted to [xx].

The general usefulness of Bruno's explanation is not immediately clear. He has decided that since the repeating decimals they have seen have fraction equivalents, all repeating decimals must have fraction equivalents. But what is clear is that he has now put forward a different set of related claims: repeating and non-repeating decimals are the same kind of object, not significantly different. Both are worthy of the name 'decimal'. Therefore, all fractions can be converted to decimals, because we know that, given the computational buttons available on the calculator, a simple proper fraction input will produce an output starting with a decimal point, and since there is no difference between repeating and non-repeating decimals, the answer to the original framing question from two weeks earlier must be yes.

His claim is then challenged by Gina, the student in Small Group A who argued that repeating decimals were not really 'complete' decimals.

68 G: I disagree that any frac – that, um, any fraction can change into a decimal because, um, I still think that um that repeating decimals sometimes um isn't a complete decimal. And um, I think even if you [xx] equivalents it maybe it just – it still can't. Sometimes it just doesn't end up adding, it can't turn into a decimal? Like it can't, sometimes it can't simplify any further. So I disagree and I think that only some fractions can be turned [. . .] into decimals because they terminate.

Three students then eagerly respond, all of them starting out their contribution with the words "I disagree, because..." and ending up saying in more or less articulate fashion "a repeating decimal is still a decimal" with not much in the way of supporting argumentation. The teacher intervenes about eight turns into this disagreement fest and says

75 T: Okay, let's [xxx] define what a decimal is. Because Gina brought up a good point, she said a terminating decimal is not a complete decimal. Let's just be clear on how it is that we're explaining, what a decimal is in the first place. [xxx] You all know what a decimal is. Mirjana.

The following 23 turns consist of students offering partial, fairly problematic stabs at a definition of a decimal. Several problems come up, including the fact that some students do not seem to be distinguishing tens and tenths, hundreds and hundredths. (Part of this problem may be due to the topic itself: when the focus of discussion is a fraction like 3/10, the denominator is named 'tenths' when named as part of the fraction 'three tenths', but is named 'ten' when expressed in isolation – 'the numerator is three and the denominator is ten.') Finally, Bruno returns to the conversation:

94 B: I agree with [Juana] about what she said the [x] the decimal is a level in with all the powers of ten, but also a decimal is a fractional part of a whole.
95 T: A fractional part of the whole. What does that mean?
96 B: That, um, a decimal can – that any decimal has an equivalent fraction and that they have equivalent fractions [xx we know that a] fraction is a fractional part of a whole. It's always a decimal. The whole [decimal is] a part of a whole.

Bruno's reasoning is advancing: now that he has decided that repeating and terminating decimals are the same kind of mathematical object, he is moving on to consider the relation between fractions and decimals. The turns above may be evidence that Bruno has conceptualized a unified mathematical object with two representational formats. Those representational formats have distinct properties. A decimal is 'a level in with all the powers of ten.' Presumably Bruno is referring to the fact that the places to the left of the decimal point are also powers of ten; decimals are the 'levels' smaller than one.

Mrs Anderson asks Khieu if she can put what Bruno said into her own words. She responds tentatively, and does not include the part about a decimal being a fractional part of a whole.

98 K: That, um, [xxx] because of um um the decimal [xx] turned into a fraction because, um, you say that a repeating decimal is [still] a type of decimal but not [xx]. It's still a repeating decimal.
99 T: Is that what you said?
100 B: That a decimal, even if it is a repeating decimal, it's still a fractional part of a whole.

This move, asking a student to repeat what another student said and then checking the response with the originator, is worth pointing to as another skillful adaptation of this teacher to the demands of tools with which she is working. As evidenced here, not all students will be formulating the same kinds of understandings. The teacher cannot tell if Khieu understands without hearing her talk, yet Khieu is unlikely to spontaneously respond to Bruno's contribution if she doesn't understand. Moreover, the teacher needs to make sure that all of the 24 other students have heard what Bruno has said if she wants to pursue it. So a rebroadcast is called for, both because of the sub-optimal acoustic conditions in the room and the challenging nature of the contribution itself. The move of asking a student to repeat what another student has said, the 'revoicing' move, is another tool in the hands of this skillful teacher: it is simultaneously a comprehension assessment, a rebroadcast, and a chance to gain a few more seconds to consider her own next move. When she asks Bruno to respond to the revoicing of his turn, she gets still another statement of his original point which can be attended to by the others.

Mrs Anderson then calls on Tyisha, whose turn is mostly inaudible. However, we can hear that she is repeating part of Bruno's reasoning: she mentions repeating decimals, then asserts that all decimals can be turned into fractions, so 'it' is part of a whole. Although we can't be sure, it is reasonable to assume that the antecedent of 'it' here is one of the expressions denoting a repeating decimal.

102 TY: [xxxxxxxxxxxxx] turn it into a repeating one. [xx] a decimal that has [xxx]. And um [xxxx] if it repeats, so it is and all decimals can be turned into fractions, so it is part of a whole.

Now that this claim – that repeating decimals are fractional parts of a whole – has come out of two mouths, the teacher can revoice it and bring it back to Gina. It may push her to elaborate upon why she thinks that a repeating decimal is *not* a 'complete' decimal.

103 T: You bring up a very good point. You say that a repeating decimal is still a part of a whole. Even if it goes on forever. Do you agree with that? Gina?

104 G: I don't. I don't agree with what Tyisha was saying that even if it goes on forever it's still a decimal, because I think that a decimal is actually a decimal because you don't know when it's going to stop. You don't know how many – cause I think a decimal is kind of like pieces of something of a whole. You don't know how many pieces it is that it's talking about [in the repeating decimal]. It just continues and continues, you don't know how many is in the whole, you don't know how many pieces in the whole, you just know that it just continues and it never ends. And I don't think that's really a decimal.

This turn is interesting in the way it foregrounds how Gina has heard what Bruno said about a decimal being a fractional part of a whole. She seems to agree with this. Yet she is not persuaded by the part of his argument that claims that every repeating decimal can be turned into a fraction.

It's easy to see with the benchmark 1/3, but Bruno has not presented a methodological demonstration of how it could be done for *any* repeating decimal. Gina seems to be saying that if each place value in a decimal names a portion of the whole, how can one get a complete account of that whole? How can one move to the equivalent fractional representation? And in fact it is easy to understand why Gina has this problem. These students know no easy way to take an arbitrary repeating decimal and turn it into its fraction bar counterpart. They might be able to construct various ways to guess and check, but there is no easy route as there was for changing terminating decimals into fractions.

For the next 13 turns students attempt to convince Gina that she is wrong, but they don't really present convincing arguments. Their attempts mostly take the form of "But Gina, a repeating decimal *is* a decimal." They seem to be pointing to the lexicographic plausibility of the assertion more than any mathematical evidence: the names are the same. She finally capitulates on that point, and in fact denies that she ever said it wasn't a decimal.

118 G: Okay. I do agree that it's a decimal. I agree with that. I agree with you. But I just don't admit it's a complete one. I don't think it's a complete decimal. I didn't say it was not a decimal. I just said it is, but it's not complete.

The lesson is brought to a close with no resolution. The teacher tells them that this question is complex and they will revisit it in the future.

4.10. *Summary*

At this level of analysis, closer to the moment-to-moment realities of teaching, we can see more clearly the complex interactions of discursive tools and physical tools, of methods and ideas. We have seen that the discourse format can have a powerful effect on students' thinking. The desire to find a counterexample can lead them to think of examples that will inevitably lead to further claims and counterclaims. When a counterexample is offered, it must then be scrutinized further. That scrutiny will yield its own findings. For example, a claim that repeating decimals are not 'real' decimals was made to support a larger claim that not all fractions can be turned into decimals. The subclaim will be challenged: is it true? The desire to challenge will lead to a consideration of what repeating decimals really are. Although the group never clarifies what 'real' decimals are, Gina refines the claim by insisting that they may be decimals, but they're not 'complete' decimals. And the notion of completeness is backed up by her claim about the impossibility of enumerating the parts of the whole represented by the decimal places.

Finally, the transcript reveals the vexed role played by methodological tools. Bruno's claim about repeating decimals caused the teacher to initiate a group investigation of their properties. She did not immediately take up his claim, but instead seemed to try to ensure that all students would have some awareness of the relevant properties of repeating decimals. Note that this is a requirement of the discourse format itself: if only one or two students are engaged in a position-driven discussion, and no one else can follow, the teacher's action fails. The limitations of the calculator only became apparent during this investigation, when the students began their search for evidence to respond to claims about the sources of repeating decimals. Similarly (and unsurprisingly), the course of argumentation about the original question was also limited or enhanced by the availability of computational methods of transforming fractions into decimals and vice versa.

5. Mapping further obstacles

5.1. *What is left out of the account so far?*

In the past, when I have presented papers featuring narrative accounts of classroom discussions conducted by talented and skillful teachers, I have learned to expect two types of responses, both concerned with what is not in the transcript. The more frequent type of response contains questions like these: 'Why didn't the teacher pick up on Marco's response?' 'Why did she seem to ignore the fact that this student obviously had a misconception about that idea?' 'Why didn't she just tell them about repeating decimals?' 'Why don't they know about dividing the numerator by the denominator?' To such readers, what stands out is what did not happen, mathematical ideas not pursued, misunderstandings allowed to persist, methods not presented.

Conversely, the other type of response involves bewilderment at how successful the discussion has been. Some who are very familiar with the exigencies of orchestrating classroom discussion will be suspicious about the high quality of what is depicted here, wondering how this teacher was able to get so much accomplished. They may be wondering what I have done with all the incoherent contributions, the unintelligible exchanges, the turns that go nowhere. Why is there so much that is missing from the account?

Both of these types of questions deserve answers, and in fact the same missing transcript material will provide a partial response to both types of question. One of my larger purposes in this paper is to bring the first kind of questioner to see a bit more clearly why no teacher is able to take a framing question such as the one described here and simply unfurl a discussion that hits all the major mathematical ideas – in other words, to see why simply providing teachers with a high level question like this is not sufficient to ensure student learning of the 'big ideas' it contains. On the other hand, I also want to be able to persuade teachers and researchers who see only the morass of unclarity in classroom discussion that it may in fact be possible to gain a clearer understanding of what it takes to plan, to carry out, and to review such events. If we have a better understanding of what it takes, we may better enable students to think about complex ideas in a sustained fashion.

So in this section I will delve more deeply into the less clear parts of the transcript, pointing out problems and barriers that confront the teacher in her attempts to use jointly this question and this discourse form, thus hoping to further elucidate what abilities are required to skillfully make use of these tools.

5.2. *Dealing with the unintelligible*

Let us return to the early part of day 1, where several students state their claim that not all fractions can be turned into decimals, only the ones with denominators evenly divisible into the powers of ten. One student, Marco, disagrees with this:

65 M: I disagree cause – fractions *can* be turned into decimals cause sometimes if it's a number that we can put it to a decimal. You can put it in zeroes and that would give you a repeated number.

66 T: Can you give us an example?

67 M: Like one...one seventh and on the denominator just add two zeroes like that and keep on [xxx] keep on repeating.

68 T: So turn it into one over seven hundred? [M nods] Is that what you mean? [M nods again] One seventh equals one over seven hundred?

69 M: Yeah.

70 T: So how does this help me turn it into a decimal value?

71 M: You can put, um, Seven. Zero point seven oh.

72 T: zero point seven oh? Is equal to one seventh?

73 M: It is equal to one seven hundredths.

74 T: One seven hundredths. So was what Miss Brown wrote on the board correct? [M nods, then silence] Gina?

Gina then changes the subject. Mrs Anderson, after spending ten turns on Marco's claim, has left it with no restatement, with no status as a claim in the group discussion. After this interchange, Marco did not participate again during the discussion that day. He appeared somewhat frustrated, but seemed to accept that his contribution had not made it into the mainstream of the discussion. A word with him after class revealed that he had intended to say that any fraction can be turned into a decimal using the method of dividing (although it was not utterly clear what he meant by this), and that this would overcome the problem of fractions with denominators that are not factors of powers of ten. A word with the teacher after class revealed that she had had 'no idea' what this student had been trying to say.

A researcher with months to think about an utterance can claim to have understood a student's intentions. But this teacher, in her second year of teaching, is working on-line. She cannot devote the entire hour to this student's contribution, nor even ten minutes. She has twenty-four other students whom she must keep on-line along with her. Often she must decide what response to make in less than ten seconds. The first decision point is whether she can understand what general issue in the conversational space the student is attempting to address. Sometimes it's impossible to tell.

Moreover, there are interactional consequences to pursuing the unintelligible. After a few attempts at understanding someone who is unclear (whether this is due to low competence in English or just general lack of articulateness), a sense of discomfort sets in for most teachers. Other listeners also become uncomfortable. The student him or herself may feel humiliated or attacked. Three or four attempts to clarify are often beyond the limits of tolerance, both for those seeking to understand and for those seeking to be understood.

In this classroom, where students are extremely orderly and on-task, I find at least half of the student contributions to be relatively unintelligible. Mrs Anderson is outstandingly good at eliciting intelligible, audible speech, and she is formidable in her ability to wait patiently for a faltering student to make sense. Yet sometimes even she cannot make sense of a student's contribution in the press of the moment.

This problem, which has no name that I know of, is perhaps one of the biggest determinants of the course of a classroom discussion, yet it is rarely discussed in research on this topic. Teachers talking about managing group discussion often mention it, however, as a particularly ubiquitous problem. I have heard many teachers discuss strategies intended to 'save the feelings' of a student, while at the same time saving the conversational thread they intend to pursue. For example, a teacher may pick up one tiny piece of what the student has said, affix an interpretation onto it, and run with that, attributing to the student something s/he may not have intended but which serves the purposes of the conversation. The move *is* face-saving: the student is not left with the embarrassing sense that the teacher could do absolutely nothing with his or her contribution, but the student may also be well aware that the teacher has grafted another interpretation onto what was intended, and that s/he was not truly understood after all.

Mrs Anderson does not construct such a face-saving move here, although she does sometimes resort to this. A long-term goal in the Project Challenge classrooms is to develop students' ability to stake a claim in an articulate fashion that can be understood by everyone in the discussion, to support it with evidence, and to explain why the evidence offered is good evidence for that particular claim. We have learned, as have many others, that this process is slow and laborious, and that when people (whether fifth grade students or full professors) talk about concepts that are new to them, they often sound confused, garbled, and anything but articulate. Given this long-term goal, it is not particularly productive to pretend that a student has made sense if s/he truly cannot be understood. If a student's contribution cannot be understood after three or four attempts to draw out the intended meaning with graphic or symbolic support on the board, the teacher moves on. She will often return to that student, checking in to gauge the extent of understanding, as she does with others, and offering openings to contribute again.

5.3. *Dealing with the incorrect*

Sometimes in this transcript, as in most such transcripts, a student's contribution is clear, but wrong. Some readers or visitors to the classroom may be horrified to hear a student say something incorrect, and not immediately hear a correction from the teacher. This is another topic with serious consequences for the use of group discussion in classrooms, yet it rarely appears in the literature.

In the example below, Clarence is trying to work through the process of transforming 6/8 into a decimal.

118 C: Because, um, six eighths can be reduced to three fourths and you can multiple the four to, by twenty, to get, um, to the power of ten which will be a hundred and then you multiply the three times twenty again will be sixty; that will be point sixty.

119 T: What do you think about that? [addressing Sela]

120 S: I agree because, um, because you said – so three fourths and if you time a fourth times – if you times the four times twenty you get one hundred and if you do the three times twenty you get sixty and a hundred is [xx] with powers of ten.

121 T: Well that gets back to what Juana said. If you don't have a denominator that's a factor of a hundred what should you look for, Juana?

122 J: You should look for an equivalent fraction to it that's less than that fraction.

This exchange could easily be seen as problematic. The teacher does not correct Sela's and Clarence's incorrect multiplication. Her comment "what do you think about that?" is not a recognizable correction. Nor does she comment on Juana's misuse of the expression 'equivalent fraction that's less than that fraction.' What Juana surely means is an equivalent fraction in lower terms. She has been talking about how when confronted with 4/8, one needs to look for an equivalent fraction like 1/2. Should Mrs Anderson have corrected both the terminology and the incorrect multiplication on the spot?

Many readers will be able to supply reasons why the corrections might be missing. After discussions with this teacher and others, I will say only the obvious: that it is partially a matter of conscious judgment, and partially a matter of processing load. The teacher is required to understand the gist of a student's contribution. In exploratory talk, students are maximally unclear because they themselves are under the greatest processing demands: they are trying to figure out new ideas and present them in public in coherent fashion. The teacher needs to understand them, to keep track of the sequence of contributions, and to monitor what other students are understanding, as well as to plan her own responses in a conversationally appropriate two or three seconds. The ideas the students are proposing are tenuously stated and tenuously conceived. A superb insight might be couched within a contribution that contains a hideously incorrect computation. What to focus on, and when? And how to decide?

On the one hand the teacher is less likely to notice students' mistakes when her own processing load is greatest. On the other hand, when she does notice, she has to make a judgment about whether, when and how to stop and correct the situation. Stopping to focus on precision for precision's sake, when the idea can be understood without it, is a risk in terms of everyone's concentration: the student making the contribution, she herself, and the other students listening.

But this much is obvious to anyone who has taught a complex subject using discussion as a tool. More interesting perhaps is to look at places in the lesson where Mrs Anderson *does* correct the students' unclear antecedent reference problems, their computational errors, and their opaque usage of English. These places also reveal something about the nature of the work that she is doing, and her conscious and unconscious interaction with the demands of position-driven discussion as described above.

5.4. *Stopping to solidify, refine, and make precise*

The following exchange takes place during the definitional episode described earlier. In this exchange, Mrs Anderson requires a student to supply an antecedent for a pronoun that's unclear (line 139), and corrects a misstatement about the place values in a decimal (lines 145–147).

137 T: And what do the powers of ten have to do with fractions and decimals? Who cares?
138 L2: That's the only place values you have.
139 T: The only place value who has?
140 L2: The decimal.
141 T: Do you agree with that, L? [*two turns deleted*]
144 L: I agree because, um, the place values are, um, ten, ones, tens, and hundreds and, um, all of the power of ten.
145 T: The place values of the decimals are ones, tens, hundreds, is that what you said?
146 L: Yeah.
147 T: Do you agree with that? Is there a ones place in the decimal system? [Students shake their heads]
148 T: No. What do we start with?
149 L: Um, tenths, tenths.
150 T: Exactly. And then what?

What is the difference between this exchange and others in which errors are made but left uncorrected? Here, Mrs Anderson is not in the midst of exploratory talk; she has undertaken a *review* of an idea that has been made explicit a number of times and is currently assumed by many of the students. By asking once again why these students are harping on fraction denominators being compatible with powers of ten, she is giving everyone a chance to solidify their knowledge and to practice their ability to verbally articulate what they know. She is able to pay closer attention to their mistakes because everyone's processing load is lighter. More importantly, she tightens her criteria for an acceptable answer because the speech activity she is conducting is not exploratory, it is summative. When we summarize what we know, we *must* strive for correctness and accuracy – that is when it counts, when we will be taken seriously. The teacher has appropriated this moment to take one more step towards inducting students into the practices of the field.

While many researchers and educators have noted that precise use of language is one of the hallmarks of mathematical activity, they often fail to acknowledge that socialization into this practice must proceed in fits and starts. Not all activities within mathematical classrooms in elementary and middle school will support an equally intensive focus on precise and correct language. Active exploration of mathematical ideas is difficult: ideas do not emerge fully formed, refined and gleaming. They must be shaped, revised, scrutinized, reworked, and polished. When we are in the heavy lifting and framing stages of developing new ideas, stopping to correct every flaw is disruptive to the real work. When the ideas are ready for polishing, however, correctness in every respect must be the goal.

The final example shows the same pattern. Here Mrs Anderson is getting a student to revise and refine her conjecture. The student, Juana again, has given the example of 4/8 as a fraction that according to the very first conjecture could *not* be transformed into a decimal because its denominator was not a factor of a power of ten. She updates the conjecture:

99 J: But it can be because the four is half of the eight so you can write point five. First you have to look at the fraction and if um the numerator goes into the denominator by changing it to another equivalent fraction less than that fraction.

100 T: And the equivalent fraction you change it to, what needs to be true about it?

101 J: Umm, you have to have an equivalent fraction that's less than that fraction, if it's not, it won't go, it won't change into a decimal.

In utterance 100, the teacher is indicating that Juana has left out a crucial piece of information: the fraction, when reduced to lower terms, *has to be a factor of a power of ten.* It is likely that she left this out not because she was unaware of it, but because it was so much a part of the shared conversational contents already. By this time in the transcript the phrase 'powers of ten' had been repeated dozens of times. Participants in unproblematic and informal conversations often leave out redundant information, depending instead on their co-conversationalists to know what they are taking for granted as part of the presuppositional pool. But the teacher knows that the language use requirements for the statement of a mathematical conjecture are not the same as those for participation in an informal conversation. While she allows the students a great deal of latitude in other parts of the conversation, here she gently but clearly requires the student to inspect the adequacy of her own statement. She makes a move that she uses frequently when she wants to bring students' attention to the inadequacy of their formulation: she takes their informal statement and follows it exactly, showing precisely what it leads to with a new value, and then asks for confirmation.

102 T: Okay, so two sixths. I can change the fraction, change it to the equivalent of 1/3, so you're telling me that therefore it can be written as a decimal?

104 J: No, it depends if um the fraction that you're changing to is, if the denominator is a factor of ten, a hundred or a thousand.

The teacher has ignored here Juana's infelicitous use of 'less than that fraction' mentioned above, as she concentrates on making sure that the statement of the conjecture is complete. Of course she herself knows that Juana intended to limit her statement to a search for equivalent fractions whose denominators are factors of powers of ten. But for the record, for the audience, for the community of mathematicians, it is important that Juana make everything explicit, and say it one more time. These examples suggest that the principal difference between the times that Mrs Anderson corrects students and the times she does not may be a function of what she is trying to accomplish, and of her skillful response to the demands that a particular activity places on the participants and their sometimes faltering resources. A visiting principal or other observer is likely to notice only the times when the teacher fails to correct students' unclarities or mistakes. They are liable to miss the pattern observed here, a pattern that I would argue reflects great skill. She makes corrections and insists on repairs primarily at times of maximal stability: when a widely shared (but perhaps not universally shared) fact is being reviewed, when a conjecture is being polished, in other words, when the talk is not primarily exploratory. She tightens the criterion levels for precision and correctness when the discussion provides her space for reviewing or refining, or when an idea has finally emerged and is under scrutiny during a careful attempt at formulation.[18]

6. Conclusion

This case study provides evidence that if we consider a mathematical question as a teaching tool in isolation, we will not get a complete view of how students' mathematical understanding may develop. If we consider position-driven discussion in the absence of particular mathematical content, we will similarly derive an incomplete picture. Empirical study of the fit between particular mathematical topics and particular discourse formats could profitably start as this analysis did, with a consideration of (a) the mathematical contents of a question, problem or statement considered in isolation, and (b) the likely constraints and affordances carried by the discourse format and participant structure within which the mathematical question will be pursued. After that preliminary analysis, research can then proceed with (c) a study of classrooms in which the question or problem is posed, and in which the consequences of the teacher's and students' actions within an actual discourse format can be examined. Only after a sufficient number of clearly described cases exist can we begin to draw generalizations, look at outcomes and define candidates for best practices. As Cobb has shown for concrete instructional objects like the hundreds board (Cobb, 1995), a great deal of careful work is required to make solid claims about the relationship between a child's learning and an instructional tool. This case study does not provide sufficient detail about individual children for us to be certain about how the discussion affected their thinking about the relevant mathematical ideas.

Given that caution, what can be said about the demands and potential of these particular tools, based on what is observable in this classroom? A position-driven discussion has the potential to cause a cascade of subsidiary activities: examining the logic of claims, scrutinizing the properties of evidence, returning to refine definitions in order to sharpen a claim. These activities may foster the development of students' understanding by leveraging their natural desire to affiliate, to take a position, or to reject a position, as Hatano, Inagaki, and others have previously argued. But what is demanded of the teacher who wants to ensure this kind of student activity? There are certain phases of such a discussion for which she must be prepared. After becoming familiar with the contents of the mathematical question, she should anticipate the positions that might be taken and consider the evidence that might be offered for them. Often, a particular logical path will not be followed spontaneously, and she might wish to prepare examples and counterexamples that will lead students towards the important areas of exploration. As positions are developed and confusion sets in, she must stop, summarize, clarify, and rebroadcast. Episodes in which definitions are considered and reconsidered will almost certainly emerge and these can be challenging in themselves (Godfrey and O'Connor, 1995). Finally, the tension between mathematical ideas and the computational methods required to expose them is likely to be a feature of most mathematical discussions like this one.

Yet not all discoveries or ideas or problems can be prepared for ahead of time. Every class has a different collection of knowledge and understandings about concepts, word meanings, and computational methods. Sometimes, a position-driven discussion may reveal something important that does not emerge in other venues. In this discussion, the teacher had no idea that so many students would reject repeating decimals as instances of 'real' decimals. In all of the four classes of fifth graders this same belief emerged, and in the following year, with different teachers, the same belief emerged again, much as it did here, when the framing question was repeated in the same discourse format. When the activity is as complex as this one is, it's important to realize that it may take several years before a teacher has a sense of control over the discussion and a full understanding of its potential for mathematical thinking.

We know that symbols and linguistic expressions can act as a driver – a 'pump' – for mathematical ideas that do not yet exist for the students (Sfard, 2000b), and the discourse format itself can work in the same way: a claim or a challenge introduces a new referent into the discourse model being constructed by each individual, and by the class, and this then opens up a space for exploration and elaboration.[19] This discussion is strikingly reminiscent of applications of Lakatos' model of the generation of mathematical knowledge.[20] As exemplified in Davis and Hersh (1981, p. 291–298, see also Lampert, 1990). the process consists of a series of moves starting with the posing of a conjecture, the naïve testing of this through examination of examples, reformulations resulting from counterexamples, and even shifts in the goal as reformulations present new and deeper questions. But in a fifth grade class like this the ideas will not emerge consistently or be widely shared without the kind of work done by Mrs Anderson. Bruno presents a particularly important challenge. His ideas were among the most powerful mathematically, yet Mrs Anderson skillfully addressed them without framing the entire discussion only in his terms. She managed the tension between the group's pursuit of shared mathematical understanding and individuals' conflicting contributions, and in fact she built on these divergences to create a discussion with some of the properties of real mathematical discovery.

The subtlety and complexity of this work means that outside observers and evaluators, such as principals, supervisors, parents and researchers, may not always be able to recognize the skillfulness. larger goals and success of the teacher. Teacher actions such as not correcting all mistakes and taking time to deal with unintelligible contributions may be misconstrued by outsiders. This case study indicates that we need to look more closely at how skillful teachers manage these ever-present obstacles.

Finally, as conducted by Mrs Anderson, these lessons convey strong messages that transcend the particularities of the mathematical questions explored and speak to larger issues of mathematical and scientific work in general. One message is that hard questions take lots of time and lots of thought to answer. Students find out what it means to deepen their understanding of something that they had assumed was already learned. Another message is that it's hard work for us to understand one another, but this work is important enough to warrant great effort. The cognitive and social potentials of discourse practices like the one studied here are not easy to bring to fruition. And a great deal of work remains if we are to understand, honor, and support the work of teachers and students as they struggle to coordinate their tools of symbols, computational procedures, words, and discourse actions in service of mathematical learning.

NOTES

1. Thanks are due to the Jacob K. Javits Education Foundation of the U.S. Department of Education, which funded the 3-year intervention, 'Project Challenge' (Grant #R206A980001, Principal Investigator (PI) Suzanne Chapin), during which the lessons reported on here were recorded. The support of the Spencer Foundation in the form of a major grant to the author ('Teacher Discourse in Middle School Mathematics Classrooms') is also gratefully acknowledged. The views expressed herein may not represent those of the Javits Foundation, the Dept. of Education, or the Spencer Foundation. Special thanks are due to Nancy Canavan Anderson, the teacher whose work is described here, and Suzanne Chapin, my colleague at Boston University. I have worked as a collaborator on this project for almost three years, and thank both Nancy and Suzanne for their generosity in allowing me to learn from them. Many thanks also to the editors of the journal, especially Anna Sfard and one anonymous reviewer, and to Maggie Lampert, for detailed comments. I have benefitted from discussions with Sarah Michaels, Martha Rutherford, Nicole Kent Perez, Theresa Creaney, Catherine Howell, Debra Bieler, Susan Givens. Thanks are also due to Ali Brown, the classroom teacher of the class discussed here, and to the other teachers involved in Project Challenge as well. Any errors, misconceptions, or infelicities are mine alone.
2. Suzanne Chapin, project PI, suggested this framing move to the lead teacher, Mrs Anderson. During this year of Project Challenge, Mrs Anderson spent her time teaching mathematics in four fifth-grade classrooms. The regular classroom teacher, Ms Ali Brown, assisted during all classes. During this one, she scribed student and teacher contributions on the blackboard as the discussion progressed.
3. For discussion of some aspects of this problem, see Gee, Michaels and O'Connor (1992) and Mishler (1990).
4. Because the topic of the discussion is mathematics, this problem of reduction is exacerbated. As Hiebert (1993) has pointed out, the complexity of classroom teaching and learning requires that we reduce the scope of our inquiry to the minimal possible 'slice' of mathematical knowledge necessary to explicate the topic of concern. I will follow that practice here, limiting my discussion of the mathematical content of the framing question fairly severely, as shown in the next section.
5. What I am calling here the 'fraction bar' is actually a conflation of two historically distinct symbols. One, the horizontal 'fractional line,' did not become widely used to separate the numerator and denominator of a fraction until the sixteenth century (Cajori, 1993/1928, p. 310). The typographical challenges posed by the fractional line

led De Morgan and others in the nineteenth century to propose use of the 'solidus' (Cajori, 1993/1928, p. 311), currently called a 'slash', as in 2/4.

6. Two reviewers pointed out to me that one of the factors that adds to potential confusion in a discussion of this sort is linguistic variation among different mathematical or teaching communities. In some, the term 'fractional number' is synonymous with 'rational number', and thus the term 'fraction' is ambiguous between the name of the representation that uses the 'fraction bar' and the mathematical entity it represents. Concomitantly, the term 'decimal fraction' denotes the representation of a rational number in the decimal point base-ten format. In other communities, the term 'fraction' is reserved for the representational level only, and refers to the fraction bar representation of either a rational or irrational number (e.g. 2/3 or pi/3). The term 'simple fraction' would denote numerical signifiers (fraction bar notation) that can express only rational number signifieds. In these communities the term 'decimal' refers to a number, either rational or irrational, which appears in decimal notation and the term 'decimal fraction' is not widely used. In this paper, when I refer to 'decimals' and 'fractions' I intend to refer to signifiers of rational numbers, i.e. the expressions that represent rational numbers. Irrational numbers will not be considered.
7. In this, of course, they are not alone. NAEP results indicate that only 57% of 12th graders nationwide can convert a decimal into an equivalent simple fraction in lowest terms (Wearne and Kouba, 2000, p. 172). And these results do not reveal whether the 57% who can accomplish the conversion have any understanding of the relationships among the representational means and the mathematical entities.
8. I am using the term 'written expression' as a shorthand for the linguistic-numeric representation which can be spoken, written or imagined. There are many layers of semiotic complexity here that I will resolutely ignore for the time being.
9. Kieren elaborates the construct of rational numbers into four subconstructs: quotients, measures, operators, and ratios (Kieren, 1993, p. 57). Each of these bears complex relations to physical or social applications of rational number knowledge. Together, they form a bridge between the mathematical properties of rational numbers and the intuitive actions of learners dealing with fractions and decimals. Researchers continue to explore the relationships among these ideas and actions in the world (ibid., p. 57). One might ask what the meaning of the framing question is with respect to the other subconstructs. Space limits and the complexity of these issues require that I leave this aspect untouched.
10. Connected Mathematics, 1998 (Dale Seymour). The unit relevant to this discussion was 'Bits and Pieces I'.
11. To further complicate matters, repeating decimals only reveal themselves easily to students using long division with a decimal point, or a calculator. A student may know that the fraction bar notation means 'divide numerator by denominator,' and thus that the fraction 3/7 'means' 3 divided by 7. If, however, a student works through the division of 3 by 7 using, for example, a picture to represent some interpretation of division (sharing out, repeated subtraction, etc.), he or she will come up with an answer that is simply the original fraction. (Three objects shared among seven people will yield 3/7 of an object for each person.) We do not get an answer consisting of the periodic decimal 0.428571428571 ....
12. The one hundred students identified to take part in Project Challenge in this first cohort were identified as potentially talented in mathematics using a combination of teacher recommendations, examination of student work, and an assessment designed to reflect the mathematics curriculum of the district. They constitute about 18% of the students

in their grade level, are 50% male and 50% female. Language and ethnicity closely mirror the demographics of the district as a whole. The district is a medium-sized urban district in the Northeast with some of the poorest families in the state. A large percentage of the population are immigrants.

13. Numbers to the left of turns indicate the place of the turn in the 60 minute lesson. The entire lesson from the first day of taping contains over 300 turns, and small group discussion is included in this total. 'T' indicates teacher, 'S' indicates a student whose identity is not known, and other initials indicate the name of the speaker. Students' names are all psuedonyms. Deleted turns are indicated and contain only material supportive of the interpretation, unintelligible or inaudible material, or irrelevant material (e.g. disciplinary moves). None of the interpretations presented here were contested by Chapin or Anderson after they had read the analysis. Material enclosed in square brackets is my best guess as to the contents of marginally audible utterances. Periods inside square brackets indicate deleted material inside a turn. The symbol 'x' stands for one syllable whose phonetic content is inaudible. Overlaps are rare in this classroom, and are not marked here.
14. This turn reveals something of Bruno's unusual abilities in mathematics. Like most of the other students who figure prominently in this transcript, his first language is Spanish, and his background is demographically indistinguishable from that of most students in the district. However, his use of mathematical terminology is often strikingly precise and this led us to find that he has a mathematically knowledgeable mentor in his life.
15. In fact at the end of the second day, as described in a later section, Gina is defending a claim of her own, and challenges to her position become a topic of controversy and some anger among some of the girls. For an excellent discussion of the social risks involved in mathematical argumentation within group discussion, see Lampert et al. (1996).
16. One small group recorded the following turns:

    235 S2: So it can go into everything, it can go into all thousands except ten and a hundred. [. . . .]
    238 S: But wait, we weren't using a decimal, . . . – er, um, a fraction – we were using just plain eight. [. . . .]
    240 S2: One eighth of a thousand is one twenty-five

17. Anna Sfard (p.c.) points out that Bruno's contribution is remarkable in that it reflects "probably the most basic epistemic meta-rule of mathematics: If you want to extend the meaning of a concept (in this case, 'proper' decimals – so that the category would include the repeating ones), do it in such a way that the objects you are adding preserve the properties of the original objects. Moreover, the properties. . . find their expression in the interrelations and interactions of the object with other objects in the class . . . . On the [discourse] meta-level his contribution brings forth the epistemological meta-rule that speaks of the way in which new mathematical objects are constructed and tested for legitimacy."
18. Mrs Anderson expressed the view that this account is in accord with her perceptions of her own actions and motivations. Nevertheless, a more rigorous empirical study of other such episodes is necessary to establish whether the pattern discerned in this transcript holds more generally.
19. This should not be surprising. Linguistic expressions (words and phrases) work this way in early first language acquisition, as do discursive forms (O'Connor, 1999; Nel-

son, 1991 inter alia.) The question is how to maximize their potential within the classroom.

20. Thanks to Anna Sfard for pointing this out.

REFERENCES

Cajori, F.: 1993/1928, *A History of Mathematical Notations. Volume One: Notations in Elementary Mathematics,* Dover Publications, New York. Originally published in 1928 by Open Court Publishing Company, LaSalle, Illinois.

Chafe, W.: 1996, 'Inferring identifiability and accessibility', in T. Fretheim and J.K. Gundel (eds.), *Reference and Referent Accessibility,* John Benjamins Publishing Co, Amsterdam, the Netherlands, pp. 37–46.

Clark, H.H.: 1996, *Using Language,* Cambridge University Press, Cambridge.

Cobb, P. and Bauersfeld, H. (eds.): 1995, *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures,* Erlbaum, Mahwah, N.J.

Cobb, P., Yackel, E. and McClain, K. (eds.): 2000, *Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools, and Instructional Design,* Erlbaum, Mahwah, N.J.

Davis, P. and Hersh, R.: 1981, 'The creation of new mathematics', in *The Experience of Mathematics,* Birkhauser, Boston, pp. 291–298.

Gee, J.P., Michaels, S. and O'Connor, M.C.: 1992, 'Discourse analysis', in M.D. LeCompte, W.L. Millroy and J. Preissle (eds.), *The Handbook of Qualitative Research in Education,* Academic Press, New York, pp. 227–282.

Godfrey, L. and O'Connor, M.C.: 1995, 'The vertical handspan: non-standard units, expressions, and symbols in the classroom', *Journal of Mathematical Behavior* 14(3), 327–345.

Hatano, G. and Inagaki, K.: 1991, 'Sharing cognition through collective comprehension activity', in L. Resnick, J. Levine and S. Teasley (eds.), *Perspectives on Socially Shared Cognition,* APA Press, Washington, D.C., pp. 331–348.

Hiebert, J.A.: 1993, 'Benefits and costs of research that links teaching and learning mathematics', in T.P. Carpenter, E. Fennema and T.A. Romberg (eds.), *Rational Numbers: An Integration of Research,* Lawrence Erlbaum Associates, Hillsdale, N.J., pp. 219–238.

Inagaki, K., Morita, E. and Hatano, G.: 1999, 'Teaching-learning of evaluative criteria for mathematical arguments through classroom discourse: A cross-national study', *Mathematical Thinking and Learning* 1(2), 93–111.

Kieren, T.E.: 1993, 'Rational and fractional numbers: from quotient fields to recursive understanding', in T.E. Carpenter, E. Fennema and T.A. Romberg (eds.), *Rational Numbers: An Integration of Research,* Lawrence Erlbaum Associates, Hillsdale, N.J., pp. 49–84.

Lampert, M., Rittenhouse, P. and Crumbaugh, C.: 1996, 'Agreeing to disagree: Developing sociable mathematical discourse', in D.R. Olson and N. Torrance (eds.), *The Handbook of Education and Human Development: New Models of Learning, Teaching and Schooling,* Basil Blackwell, Cambridge, MA, pp. 731–764.

Lampert, M.: 2001, 'Teaching problems: A study of classroom practice', Yale University Press.

Michaels, S. and Sohmer, R.: 1999, 'Discourses that promote academic identities', Keynote address, LERN Conference. September 1999. Alice Springs, Australia.

Mishler, E.G.: 1990, 'Validation in inquiry-guided research: The role of exemplars in narrative studies', *Harvard Educational Review* 60(4), 415–442.

Nelson, K.: 1991, 'The matter of time: Interdependencies between language and thought in development', in S.A. Gelman and J.P. Byrnes (eds.), *Perspectives on Language and Thought: Interrelations in Development,* Cambridge University Press, Cambridge, pp. 278–318.

O'Connor, M.C.; May, 1999, 'Is this square a rectangle? The linguistic sign and the contexts of classroom discussion', Paper presented at the Conference on Problems in Discourse Analysis in Mathematics and Science Classrooms, Center for Advanced Studies in the Behavioral Sciences, Stanford, CA.

O'Connor, M.C., Godfrey, L. and Moses, R.P.: 1998, 'The missing data point: negotiating purposes in classroom mathematics and science', in J. Greeno and S. Goldman (eds.), *Thinking Practices in Mathematics and Science,* Lawrence Erlbaum, Hillsdale, NJ, pp. 89–125.

O'Connor, M.C. and Michaels, S.: 1996, 'Shifting participant frameworks: orchestrating thinking practices in group discussion', in D. Hicks (ed.), *Child Discourse and Social Learning,* Cambridge University Press, Cambridge, pp. 63–102.

Prince, E.: 1992, 'The ZPG letter: Subjects, definiteness, and information-status', in William C. Mann and S.A.Thompson (eds.), *Discourse Description: Diverse Linguistic Analyses of a Fund-Raisin Text,* John Benjamins Publishing Company, Amsterdam, The Netherlands, pp. 295–325.

Sfard, A.: 2000a, 'Steering (dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects', *Journal for Research in Mathematics Education* 31(3), 296–327.

Sfard, A.: 2000b, 'Symbolizing mathematical reality into being-or how mathematical discourse and mathematical objects create each other', in P. Cobb, E. Yackel and K. McClain (eds.), *Symbolizing and Communicating in Mathematics Classrooms – Perspectives on Discourse Tools, and Instructional Design*, Erlbaum, Mahwah, NJ, pp. 37–98.

Stein, M.K., Smith, M.S., Henningsen, M.A. and Silver, E.A.: 2000, *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development,* Teachers College Press, New York.

Weame, D. and Kouba, V.L.: 2000, 'Rational numbers', in E. Silver and P.A. Kenney (eds.), *Results from the 7th Mathematics Assessment of the National Assessment of Educational Progress,* NCTM, Reston, VA, pp. 163–192.

Vallduví, E. and Engdahl, E.: 1996. 'The linguistic realization of information packaging', *Linguistics* 34(3), 459–519.

Boston University,
Contact info: 605 Commonwealth Ave.,
SED Bldg., Boston University,
Boston, MA 02215,
Telephone 617/353-3318, Fax 617/353-3924,
E-mail: mco@bu.edu

CAROLYN KIERAN

THE MATHEMATICAL DISCOURSE OF 13-YEAR-OLD PARTNERED PROBLEM SOLVING AND ITS RELATION TO THE MATHEMATICS THAT EMERGES

ABSTRACT. This paper, written within a discursive perspective, explores the co-shaping of public and private discourse, and some of the circumstances under which one occasions the other, in the evolution of mathematical thinking by pairs of 13-year-olds. The discourse of six pairs of students, engaged in interpreting and graphing problem situations involving rational functions, was analyzed by means of recently developed methodological tools. The nature of the mathematics that emerged for each pair was found to be related to several factors that included the characteristics of the interpersonal object-level utterances both before and after the solution path had been generated, the degree of activity of the personal channels of the interlocutors, and the extent to which the thoughts of participants were made explicit in the public discourse. The analysis of the discursive interactions provided evidence that adolescents within novel problem situations can experience some difficulty in making their emergent thinking available to their partners in such a way that the interaction be highly mathematically productive for both of them.

KEY WORDS: communication, discourse, graphing of rational functions, interaction, mathematical discourse, problem solving, research in mathematics education, student talk

1. INTRODUCTION

Much current research in mathematical learning has been focusing on the facilitating role played by group work in mathematical problem solving. For example, some studies (e.g., Forman, 1989; Forman and Cazden, 1985; Glachan and Light, 1982; Teasley, 1995) have found that students who engaged in discussions about strategies and explanations benefited from peer interactions. Leikin and Zaslavsky (1997) have described how small group settings led to higher engagement levels, improved attitudes, and increased mathematical communication among low-achieving ninth grade students. Hershkowitz and Schwarz (1997) have reported the ways in which ninth grade students interacted to produce and check hypotheses, and how they learned to be critical and reflect on their own and others' problem solution processes. Teasley (1995), who compared the quality of work of fourth graders in four different experimental settings involving a Logo problem-solving task – working in pairs and talking, working in pairs with no

Educational Studies in Mathematics **46:** 187–228, 2001.

talking, working alone and no talk, working alone and talking aloud – has found that talking was of significant benefit to the learning process, and that these benefits were more pronounced when that talk was directed to a partner. Kieran and Dreyfus (1998), in their research on the types of interactions engaged in by a pair of 13-year-old problem solvers and the conditions under which partner-directed talk was effective, have shown that the brief moments during which one of the interlocutors entered the "universe of thought" of the other were of critical importance (cf., Trognon, 1993).

But other research suggests that the talk engaged in by partners may not always be conducive to learning. For instance, Salomon and Globerson (1989), in a reading study of seventh graders, have reported that a particular instructional tool designed to provide explicit metacognitive guidance had more profound effects on students working individually than in pairs. In their analysis of these findings, the researchers suggested that the pairs of students were not simply individuals working side by side; they were a social system in which behaviors and cognitions became interdependent and that the communication that gradually shaped this interdependence reflected a coordination and sharing of efforts (or effort avoidance). Other related work in this area (e.g., O'Connor, 1996) has identified some of the negative social processes that underlie the surface structure of mathematical discourse and that can work against group sense-making and negotiation of meaning in sixth grade peer interactions.

In another study that focused more particularly on the nature of the mathematical talk engaged in by seventh grade students who were working in pairs during a two-month sequence on the development of algebraic thinking, Sfard and Kieran (2001) have found that mathematical communication seemed quite difficult. The students were neither precise in what they were saying nor explicit in what they were attending to. Moreover, their overall pattern of social interaction, which did not improve over the duration of the study, was clearly quite unhelpful to them. As the students did not really solve problems together – they started each problem with some private work – their conversation consisted mostly of trying to argue in favor of their own solution in the face of their partner's usually different solution. In this regard, Hershkowitz and Schwarz (1997) have pointed out that reporting one's thinking to a partner about the process of arriving at a solution is usually always a "purified" action where "details are deleted and regressions are skipped" (p. 159). But what if students do not first solve the problem and then try to explain their solution to a partner; what if the talk takes place while the students are together attempting to solve the problem? Does the same purification take place?

The studies mentioned above lead to many questions about the dynamics of students' interactive mathematical talk and the ways in which these dynamics help or hinder the mathematical growth of participants. Despite the large number of studies that have been carried out on peer interactions in the mathematics class, very few have focused on the nature of the interactive mathematical talk itself – especially for students beyond elementary-school age (see the synthesis of research by Webb, 1991) – and the role of this talk in the development of mathematical thinking for each of the interlocutors. The present paper aims to contribute to this evolving field of study. It presents research involving six pairs of 13-year-old students who, having just completed a seven-week introductory algebra sequence, worked on a set of novel problems from a family of functions not encountered in their algebra lessons. Detailed analysis of their discourse – discourse that first involved a partner and then did not – provided insights into the nature of adolescent mathematical talk in partnered, problem-solving environments and into the ways in which the characteristics of that talk can have differential effects for each interlocutor within a given pair with respect to the mathematics that emerges within the interactions.

2. A DISCURSIVE PERSPECTIVE

It was in the late 1970s and early 1980s that Vygotsky's works began to be known by the international mathematics education community (e.g., Fuson, 1980).[1] His writings provided a theoretical perspective that related directly to the instructional dimensions of the discipline of mathematics education, as well as to the wider cultural context in which mathematical learning takes place. One of the first aspects of Vygotskian theory to be appropriated was his zone of proximal development. However, more recent work by mathematics education researchers has focused on the role of language and other mediational tools in the teaching and learning of mathematics (e.g., Bartolini Bussi and Mariotti, 1999; Lerman, 1998; Noss and Hoyles, 1996).

Language was the form of mediation above all others that preoccupied Vygotsky (1981), but he also recognized a more extensive set of mediational means whenever he wrote about signs or psychological tools, a set that included various systems for counting, algebraic symbol systems, diagrams, all sorts of conventional signs. To this list of psychological tools, one must add computers and calculators, which even though they have features of physical tools, also embody sign systems.

All higher mental functions are culturally mediated – thinking, communicating with others, doing mathematics, and so on. Because the same mediational tools are common to both thinking and talking with others, the border between the individual and the social tends to disappear. In Vygotsky's view:

> Processes on both the inter-mental and the intra-mental planes are necessarily mediated by cultural artifacts. His comment that word meaning is "both [speech and thinking] at one and the same time; it is a unit of verbal thinking" (Vygotsky, 1987, p. 47) is quite telling in this connection. It is because the same basic mediational means are used on the social and individual planes that transition from the former to the latter, as well as vice versa, is possible. In fact the very boundary between social and individual, a boundary that has defined much of our thinking in psychology, comes into question in Vygotsky's writings. (Cole and Wertsch, n.d., p. 2)

Vygotskian theory may blur the boundary between the social and the individual, but a discursive perspective such as the one adopted herein attempts to deal directly with this boundary. As elaborated by Sfard (2000; see also this volume), a discursive perspective makes explicit the integration of the two in that both talking and thinking are considered examples of communication – communication with others and communication with self. In other words, the mediation that occurs on the social and individual planes is reconceptualized as two instances of communication. But the view of communication that is espoused is not one of 'exchange of meanings' where entities can be transmitted or exchanged without losing their identity. Rather, the notion of communication is one that is occasioned and shaped by the situation.

Discursive perspectives have their roots in the later writings of Wittgenstein (1953), who argued that understanding and meaning could only be discerned by looking at what people actually do with words and other sign systems. More broadly, Edwards and Potter (1992) have contended that the focus of discursive psychology is "the action orientation of talking and writing [that is, what talk and writing is being used to do]" and that "rather than seeing such discursive constructions as expressions of the speaker's underlying cognitive states, they are examined in the context of their occurrence as situated and occasioned constructions whose precise nature makes sense to participants and analysts alike in terms of the social action these descriptions accomplish" (p. 2). According to Vion (1999), there is a reciprocity between language and the social relation in that the language that is used is determined by the situation, and at the same time the situation is determined by the language.

The question for discursive psychologists, when analyzing sequences of speech-acts or utterances, is not "What does it represent?, but What is going on?" (Edwards, 1993, p. 218). Shotter and Billig (1998) have pointed out that "the kind of understanding indicated here is not of a cognitive, representational-referential kind, but is a practical, dialogical kind of understanding that is 'carried' in our ongoing languaged-activity, and is continually updated, utterance by utterance, as it unfolds" (p. 24). This continual updating presents interlocutors with a certain responsibility. According to these two authors, "[understanding in practice] is simply the practical continuation of the exchange in an intelligible manner [and] if the sharing of a 'mental-picture' or an idea is at stake, then that can only be achieved by people testing and checking each other's talk to establish whether they are in fact in agreement or not" (p. 24).

The word *discourse* as used in this paper denotes "any specific instance of communicating, whether diachronic or synchronic, whether with others or with oneself, whether predominantly verbal or with the help of any other symbolic system" (Sfard and Kieran, 2001, p. 47). Consonant with the integrating principle related to communication, as enunciated above, one of the features of this new discursive approach to psychology is that discourse might be public or private. Harré and Gillett (1994) distinguish between the two by defining public discourse as 'behavior' and private discourse as 'thought' or thinking. They also make explicit, as was suggested earlier, the connection with Vygotskian theory when they state that "individual and private uses of symbolic systems, which in this view constitute thinking, are derived from interpersonal discursive processes that are the main feature of the human environment" (p. 27).

However, to speak of thought, though it is embedded in cultural, social, and interpersonal contexts, is not to imply that the contents of thought are pictures or snatches of language in the mind. For example, a person can think about the scent of a certain perfume or a linear graph in a Cartesian plane or the notion of justice, and so on. Another assumption of this perspective is that thoughts play an important role in the explanation of public discourse because they can account for the links that people make between their present and past experiences, links that shape their activity in present experiences:

> Thinkers as concept users (competent managers of systems of signs) are active participants in experience. They select what aspects of a situation to attend to and these become a basis for the use of certain (words) [concepts]. They apply these (words)[concepts] and then can reason about the experience and link it to other experiences and more abstract thoughts. Thus the grasp of (the use of a word) [a concept] is an active discursive skill. It is selective in the face of a rich set of experiential possibilities. (Harré and Gillett, 1994, p. 48; all brackets, parentheses, and underlining in the original)

This is not to deny, according to Vygotskian theory (1981), the genetic priority of public discourse over the private. Nor does this perspective attempt to address, except in a very general way, the particular issue of how experience is 'internalized' à la Vygotsky or how interpersonal discourse is transformed in the process of being 'appropriated' for personal use (see, e.g., Wertsch, 1985).

The main point at hand is the interplay between public and private discourses. Thoughts are potentially private; they can be hidden from others. But they are also sensitive to discursive context. Harré and Gillett have argued that "the workings of each other's minds are available to us in what we jointly create conversationally, and if our private mental activity is also symbolic, using essentially the same system, then we can make it available or not, as the situation seems to require it" (p. 27).

Their commentary on private discourse leads to questions that are central to this study. Do adolescent students make the workings of their minds available to their partners when the mathematical situation seems to require it and what form does this take? What is the nature of the relationship between interlocutors' discourse in their joint mathematical problem solving and the emergence of mathematical thought for both partners? The next section presents the methodological tools of analysis used in the study and describes the ways in which answers to the above questions were sought.

3. THE STUDY

3.1. *Background of the students*

The six pairs of students involved in this study were at the end of their first year of post-elementary schooling (13 years of age) at a private high school in Montreal. They had just completed, along with their classmates, a seven-week introduction to algebra based on an object-oriented, functional approach that aimed at giving meaning to the symbols and transformations of algebra by means of graphical representations and operations with these representations (see Kieran, 1994; Kieran and Sfard, 1999). They were considered to be students of fairly high ability in mathematics, having succeeded in all of the periodic assessments during the algebra sequence with grades of over 80%. (Their mathematical proficiency in this area was confirmed by means of a pretest administered at the beginning of the study.) The content that was emphasized in the seven-week introduction to algebra centered primarily on situations involving linear functions, although some experience with quadratic and cubic functions was included. Much of the algebra sequence had involved pair-wise work using activity sheets, interspersed with classroom instruction by the teacher and whole-class discussions. Several days of the algebra sequence had been spent at the computer, where students in groups of two explored the role of parameters in graphical and symbolic representations, and their relation to the various problem situations. Thus, collaborative interaction was not new to these students, who quite often had worked together as pairs in their mathematics class during the previous several months. Most of them had also attended the same elementary school where they had had extensive experience in communicating their mathematics to their peers and in peer-wise problem solving. The six pairs agreed to participate in the research study at the conclusion of their seven-week algebra sequence.

3.2. *Methodology*

3.2.1. *What the students were asked to do*

The first part of the methodology, which dealt with what was requested of the participating students, was adapted from an approach used by Hatano and Inagaki (1994). It involved joint problem-solving work, followed by individual report writing, and then individual work on problems analogous to those worked out jointly. The six pairs of students (five male pairs and one female pair), all mathematically adept and having had the same seven-week experience in introductory algebra, were given the same problem set (see Appendix 1). It was a complex, multi-part problem that was considered by the researchers and by the classroom mathematics teacher to be fairly difficult for the students in that it was based on a type of function that they had never seen before – from the family of rational functions, that is, the independent variable was part of the denominator of the rational expression for the function. One pair of students was observed and videotaped at a time. After a couple of warm-up questions, the pair was given a single set of activity sheets. They were asked to collaborate in the solving of the given problems, to express their ideas aloud, to assist each other in understanding what they were doing, and to take as much time as they needed (it was considered that the problem set would require about 45 minutes to an hour to complete). They were also told in advance of the joint work that, upon its completion, they would be asked to write individual reports on what they had done together, and then to work on comparable problems on an individual basis. A computer was beside their work table available for their use, should they so decide (the computer was set up with the same program that they had used in their introductory algebra classes: *Math Connections: Algebra II* [Rosenberg, 1992]). Calculators were also at hand. Upon completion of the joint work, each individual was asked to write a self-report in which she/he described the reasoning they had used to solve some of the problems from the joint work, what they thought they had learned from the experience, and whether there were parts of their work about which they felt they still had only a fuzzy notion. Immediately afterward, each member of the pair was given a set of activity sheets containing a shortened problem set that was analogous to the one worked on jointly. The individual work was also videotaped.

3.2.2. *The tools for analyzing the data*

The second part of the methodology concerns the data sources and how they were analyzed. First, all of the videotapes were transcribed in such a way as to capture both what was said and what was done (see Appendix 3 for the conventions used in transcribing the videotapes). Then the two sets of activity sheets (one set consisting of the work done by each pair and the other set consisting of the follow-up work done by each individual) were analyzed. Finally, selected segments of all transcripts were subjected to a methodological tool that had been developed for an earlier study by Anna Sfard (see Sfard and Kieran, 2001) and that was adapted for the present study. Its description follows.

This tool was designed so as to synthesize from the transcripts the ways in which students interacted with each other, and to permit the researcher not only to detect at a glance the nature of the interactions but also to focus attention on those utterances that seemed to develop the mathematical content of the discourse. The tool is the interactivity flowchart. Interactivity analysis deals with the question of how the participants of a conversation move between different channels of communication (personal and interpersonal) and different levels of talk (object-level and non-object-level). Note that the personal channel of communication is not to be confused with *private discourse.* Private discourse refers to thought, while the personal channel consists of those public utterances that an interlocutor seems to be addressing more to self than to partner. The object-level of the communication concerns those discursive elements that are considered to be integral to moving the mathematical dimension of the discourse forward. The non-object-level concerns those discursive elements that simply keep the conversation going, as well as those that reflect the relationship between interlocutors.

As the conversation between the interlocutors unfolds, their attention moves between the personal and interpersonal channels, as well as between object-level and non-object-level concerns (see Appendix 4 for the representation of this flow). In constructing the interactivity flowchart, each utterance of an interlocutor is interpreted to be either *reactive* (responding to a previous utterance) or *proactive* (response inviting). If an utterance is reactive, it is represented by an arrow pointing upward, either vertically or diagonally, depending on whether the utterance is a reaction to the speaker him(her)self or his(her) partner. This type of arrow expresses the fact that the source utterance (the one in which the arrow originates) is a reaction to the target utterance (the one to which it is pointing). If an utterance is proactive, it is represented by an arrow pointing downward, either vertically or diagonally. This type of arrow expresses the fact that the source utterance invites a response. It is the presumed intentions of the speaker, and not of the partner, that are considered when deciding if an utterance is proactive or not. The vertical arrows reflect the fact that an interlocutor seems to be having a conversation more with him(her)self than with his(her) partner, although this is always a matter of interpretation (see Appendix 5 for the flowchart symbols).

The reactive/proactive arrows may be solid or dotted. Solid arrows symbolize object-level utterances while dotted arrows symbolize non-object-level utterances. Utterances are classified as being of the object-level type if they include the following: reading the problem text, rebutting some previous suggestion, offering a new suggestion regarding mathematical content, seeking information of a mathematical nature, and so on. In short, any utterance that reflects an intention to move forward with respect to the mathematical content of the interaction is considered an object-level utterance. If however the speaker merely agrees with some previously mentioned fact or observation, or repeats some already-pronounced numerical value, his(her) utterance is classified as non-object-level. The latter are viewed as having the aim of keeping the conversation going. Non-object-level utterances also include, for example, those with no mathematical content, those that might express surprise at what a partner has done (e.g., what ARE you doing?), or those concerned with the relationship between the partners.

4. The Findings

4.1. *Students' productivity*

Discourse that may be educationally productive for one of the interlocutors may be unproductive for the other. *Productive,* as discussed by Sfard and Kieran (2001), refers to discourse that can influence participants' thinking, or enhance interlocutors' ability to cope discursively with new situations, and so on. For mathematical discourse, an interaction is regarded as educationally productive if it has an impact on students' future participation in related mathematical problem-solving activity, whether that future participation involves individual or group work. According to Harré and Gillett (1994), the thoughts that emerge within interpersonal discursive activity can serve as a basis for later thoughts of an individual and private nature. If

TABLE I

Assessment of correctness of responses on the group and follow-up individual worksheets

Student pair	Group work	Work of 1st individual	Work of 2nd individual
1. Zak & Nic	75%	Zak: 79%	Nic: 71%
2. Dag & Naj	79%	Dag: 79%	Naj: 29%
3. Crs & Ali	100%	Crs: 100%	Ali: 43%
4. Jes & Jak	90%	Jes: 100%	Jak: 93%
5. Sho & Sej	85%	Sho: 100%	Sej: 57%
6. Koj & Jos	67%	Koj: 85%	Jos: 57%

such be the case for all participants in the interpersonal discursive activity, the activity is then considered to have been productive for each and every participant. If, however, the thoughts that emerge in the same interpersonal activity serve as a basis for the later thoughts of only one of the two interlocutors of a given pair, then the activity is considered to have been productive for only that participant.

Because the current study included a follow-up component involving individual work on a subset of problems analogous to those the students had just worked on with a partner, it was possible to examine the way in which each student's individual written work – a reflection of their thinking in the given mathematical situation – was based on what had been discussed and penned jointly with their partner. In other words, it was possible to identify those ideas that they might choose to draw upon from their previous partnered experience. It might be the case that, for example, if the joint work had led to a meaningful solution to a particularly difficult problem, each individual of the pair might be seen to draw in a similar way from the experience. Or they might not. In both situations, in-depth analysis of their discursive interactions could perhaps offer clues as to the characteristics of their conversations that encouraged (or not) the emergence of particular mathematical thoughts in each participant, thoughts that might be evoked immediately afterward in the follow-up individual work on an analogous problem.

An initial conjecture regarding the productivity of the six pair-wise interactions was thereby generated by analyzing the jointly-produced responses on the activity sheets of each pair. Despite the fact that participants had been presented with a rather difficult problem set involving the not-previously-experienced family of rational functions, their joint activity sheets showed that most pairs were able to answer reasonably well even some of the more difficult questions. The words they used to describe the reasoning underpinning their numerical and graphical work suggested, in most cases, a certain grasp of the mathematics of the problem situation. However, the corresponding analysis of their follow-up individual activity sheets, based on a structurally similar problem set, revealed some different explanations that were not mathematically appropriate and hinted that, for four of the six pairs, the interpersonal discourse had perhaps not been productive for both interlocutors (see Table I for my assessment of the correctness of the responses that were produced on the two sets of activity sheets). For four pairs, there was a wide discrepancy between what each of the partners of a given pair produced individually (see, in particular, the results for Sho and Sej, Crs and Ali, Dag and Naj, and Koj and Jos in Table I). For each of these four pairs, the individual work of one of the participants tended to be based on what had been written up jointly, while for the other participant it did not. For only two pairs, Jes and Jak and Zak and Nic, was the individual work of each partner a consistent reflection of what the partners had produced together. It was thus considered that, for the four former pairs, the discursive interactions were more productive for one participant than for the other, and that the lack of productivity for these latter participants was reflected in the nature of the mathematical thinking that was evoked during the individual work. In other words, for certain participants the thinking that emerged in the individual work seemed not to be based directly on the thinking that had been expressed in the public interactions of the pair just prior on similar mathematical content.

The analysis of the content of the responses on the individual activity sheets showed that the discrepancies between partners of a given pair were most marked on the question that was analogous to Question 5c from the joint work – the *Nathalie* problem. The fact that there was a much closer fit between the joint responses and the individual responses for several of the other questions – questions that admittedly were simpler than the *Nathalie* question – suggests that the interactions may have been more productive in these other instances. However, rather than analyze the interactions that took place during the solving of those other questions, I have decided to try and account for the interactions that led to discrepancies between Question 5c of the joint problem set and its analogue, the *Susan* problem, in the individual problem set (see Appendix 2 for the *Susan* problem).

The *Nathalie* problem (Question 5c) was the first question of the joint set that required the participants to plot points, and thus be explicit about a numerical procedure that would permit them to do so. Although the questions preceding 5c had been designed to encourage an awareness of the relationship underlying the situation ("*the number of hours of flying time is equal to the distance to the airport divided by the speed of the plane with no wind together with either the addition [for a tailwind] or the subtraction [for a headwind] of the wind speed*"), alternate approaches – such as reading the information off the graph – would have permitted the students to solve these earlier problems. But there was no other way to be successful with Question 5c (unless one were provided with the algebraic expression for the function); one had to be explicit about the relation among the variables and constants of the problem situation.

What had happened during the joint work on Question 5c for the four problematic pairs such that only one partner of each of these pairs could, or chose to, draw upon the partnered experience when handling a similar problem afterward? Detailed interactivity analysis of the discourse of all six pairs on this question from the joint set was then carried out in order to discern their patterns of communication and to seek for clues as to why the discursive interactions of four pairs were perhaps less productive than they might have been. Because of the lengthiness of the transcripts of the six pairs for Question 5c, they cannot all be presented and discussed in this paper. So, two pairs have been selected. The first pair, Sho and Sej, illustrate a pattern of interactive discourse that had many of the characteristics of the patterns uncovered in the four pairs where there was a large discrepancy between what each member of the pair produced in the follow-up individual work. The second pair, on the other hand – Jes and Jak – illustrate a rather different pattern of interaction and one that did not lead to the discrepancies just noted among four pairs. The sixth pair, Zak and Nic, though less successful than Jes and Jak in terms of overall correct responses on their worksheets (both pair-wise and individually), are being grouped with Jes and Jak because both Zak and Nic appeared to draw on their shared experience in a way that was more like that of Jes and Jak than that of the other four pairs. Thus, the contrast between the Sho and Sej and Jes and Jak interactions will serve as a basis for discussing and reflecting upon various patterns of interaction and the ways in which these patterns seemed to relate to the productivity of their mathematical communication. Even though the discursive interactions of the four remaining pairs are not presented in detail, the commentary that follows will, when pertinent, draw upon the analysis carried out with all the pairs.

4.2. *Interactivity analysis: The interaction of Sho and Sej*

The section of transcript from Sho and Sej's interaction that deals with Question 5c is five minutes in length (see Figure 1 for the transcript extract). In two earlier questions, 4c and 4d (see Appendix 1 for the questions

WHAT IS SAID	WHAT IS DONE
[301] Sho: Okay ! Suppose Nathalie were to	
[302] Sej: [Suppose Nathalie were.]	
[303] Sho : Fly daily to the airport one hundred and fifty kilometers away with a tailwind. Without any wind her plplplane flies at thirty kilometers per hour. Knowing that the wind can vary from day to day, sketching graph, sketch a graph that shows the various times that Nathalie could take when she flies with a tailwind. (Do your sketch in the coordinate system supplied in Figure 2).	[303] Sej laughs at the problem situation.
[304] Sej: Well, it could be anything with a tailwind. All it has to be, is going down. But ! ...Depends on the amount of wind speed.	
[305] Sho: Well of course. Ya ! But we have it by ten ten, twenty, thirty, forty. Seems it should go down the same as the other line.	[305] Sho points to where 10, 20, 30 would be along the horizontal axis; Sej marks the axis with 10, 20, 30.
[306] Sej: Well, she starts hereeee at thirty and asking for a tailwind. But, if we look back at this graph here.....you know that the line isn't aaaaaa at a constant rate.	[306] Sej shows the point (0, 5) on the graph to Sho and retrieves the Figure 1 graph of Question 1.
[307] Sho: Ya !	
[308] Sej: So, it would have to be something like that.	[308] Sej sweeps downward from left to right on the coordinate plane.
[309] Sho: Oh, why ?	[309] Sho & Sej have the Figure 1 graph of Question 1 in front of them as well.
[310] Sej: Or, well, see ! They're asking all the various types. So, it can be almost every.... Like, they're asking if.........any times with a tailwind. So, with a tailwind it should be going down.	
[311] Sho: (Three ?) That doesn't make sense. Because they're asking us to plot points that can be anything.	[311] Sho looks from time to time at the Figure 1 graph, while he questions Sej's [310] explanation.
[312] Sej: HmHm ! If they gave us a tailwind.	
[313] Sho: (It's different the two lines ?) Let's just try one of them over here. This point ? Okay ?	[313] Sho looks at the Figure 1 graph and then inserts a point (10, 4.5) on the coordinate plane of Question 5c--a kind of second reference point (but not based on any calculation).
[314] Sej: Just like that ?	
[315] Sho: Hmmm! I haven't finished.	
[316] Sej: It sayssss.. knowing that the wind can vary from day to day.	[316] Sho is silently rereading part of the problem text, which he points to with his pencil; Sej then reads it aloud.
[317] Sho: But wait ! No nooooo. Okay, I figured it out! Okay, she, she's at thirty kilometers per hour. Right ? Is that what she is ? Thirty.	
[318] Sej: Oh ! I know thirty !	
[319] Sho: [She's getting pushed forty], she's getting pushed .....forty.	[319] Sho points to the extreme right end of the graph of Figure 1 of Question 1.
[320] Sej: How do you know she's getting pushed forty?	
[321] Sho: Forty, this is what it is.	[321] Sho responds with the location of a windspeed of 40 by pointing to the horizontal axis of Figure 1 at (40, 0) and then to the point (30, 0), and then moves his pencil back and forth along the interval between (30, 0) and (40, 0).
[322] Sej: But, this isn't Nathalie.	[322] Sej refers to the Figure 1 graph of Question 1.

Figure 1. Question 5c transcript – Sho and Sej.

of the joint problem set), the boys had not made much headway. They eventually decided to skip them with concluding remarks such as “it doesn’t make much sense; at least we know that it’s not going at a constant rate” (Sho) and “there’s not really a pattern that we can find” (Sej). They moved on to Question 5a, where Sej responded “divide” after reading the question, and Sho continued with “a hundred and fifty divided by thirty" and plotted the point (0,5) (for Question 5b). At this moment, the reader is encouraged

WHAT IS SAID	WHAT IS DONE
[323] Sho : Ya ! But.... I'm saying if she's getting pushed forty or if she's getting pushed ten.	**[323]** Sho tries to communicate some relationship involving 10 and 40 while pointing to the coordinate plane of Question 5c and moving his pen point back and forth between (10, 0) and (40, 0) on the x-axis.
[324] Sej: Ya !	
[325] Sho: You have to figure out the time for that.	
[326] Sej: Okay ! Wasn't this what we did for Mr. NNN, finding the "X", finding what the value of "X" is.	**[326]** Sho picks up the calculator.
[327] Sho: Hm Aaaa !	
[328] Sej: [Because it] could be anything.	
[329] Sho: A hundred and fifty... divided by... ten. No upsss ! Sorry ! I wanted to do thirty plus ten..	**[329]** Sho calculates 150 divided by 10 on the calculator. But when he sees the result, he says "upsss." He tries another calculation in search of a more reasonable result.
[330] Sej: [ten]	
[331] Sho: Ya so ! One fifty divided by forty...	
[332] Sej: [Forty.]	
[333] Sho: Equals three point seven five.	**[333]** Sho turns the calculator towards Sej to show the answer, but Sho seems a little unsure of what he is doing.
[334] Sej: Sooooo !	**[334]** Sej waves his pen over the y-axis of the coordinate plane of Question 5c.
[335] Sho: Five. We're getting the same numbers.	**[335]** Sho suggests that the result fits with the numbers of the y-axis, to which he points.
[336] Sej: [Four]	**[336]** Sej points to the location (0, 4) on the coordinate plane, as they both try to locate 3.75 on the y-axis.
[337] Sho: But....	
[338] Sej: Ya !	
[339] Sho: So just..... this is fiveeee.	**[339]** Sho counts the numbers on the Y-axis with his pen.
[340] Sej: And that's five and that's four, that's three. So three point five, three point something.	**[340]** Sej counts the numbers off on the y-axis with his pen.
[341] Sho: Here or here?	**[341]** Sho asks Sej for his assistance with isolating the correct value on the y-axis.
[342] Sej: [Well], what's our number ? Three point seven five ?	
[343] Sho: Uhun.	**[343]** Sho remarks in agreement.
[344] Sej: Then, it's right there	**[344]** Sej points to the location on the y-axis and Sho slides along the 3.75 line to mark the point (10, 3.75).
[345] Sho: I don't want this.	**[345]** Sho crosses out the previous point at (10, 4.5).
[346] Sej: So, what, we just need to make it like that?	**[346]** Sej puts his ruler on the sheet of paper to make a straight line.
[347] Sho: No, because we still need the otherrrrr numbers.	**[347]** Sho picks up the calculator again.
[348] Sej: Oh yaaaa !	
[349] Sho: [Because] it could be different.	
[350] Sej: Because that would show the, the various times.	
[351] Sho: [One fifty divided] by fifty...equals three	

Figure 1. Continued.

to read the transcript of the interaction that occurred for Question 5c to have an initial sense of the boys' collaborative effort.

On the surface, it might appear that the boys communicated quite well. However, the interactivity analysis indicated some flaws. The left-side half of Figure 2 contains the compressed interactivity flowchart of Sho and Sej's communicative actions. (Space constraints do not permit the inclusion of a figure containing the line-by-line transcript segments along with

WHAT IS SAID	WHAT IS DONE
[352] Sej: Ten. Oh ! Just three. One, two, three.	[352] Sej begins to count up the y-axis, saying, 10; then stops himself as he sees Sho heading toward the point (10, 3.75) and moving to the right of this, toward (20, 3). Then Sej counts up the y-axis, "1,2,3" so as to br at the same location that Sho is plotting
[353] Sho: Ya ! One fifty. Well, of course one fifty.	[353] Sho enters 150 into the calculator, remarking that, "of course," he needs to starts again with 150.
[354] Sej: [Ya, one]	
[355] Sho: Okay ! And then we have thirty, soooo, sixty, Aa! One fifty divided by sixty. Two point five.	[355] At this moment, Sho points to the next main mark on the x-axis, 30, seeming a little more sure of the numerical relationship. After saying, "so 60." Sho continues with the division of 150 by 60 with the calculator. Sej looks at Sho in a somewhat questioning way.
[356] Sej: Two point five.	[356] Sej picks off 2.5 from the y-axis and begins to slide along the y = 2.5 line, while Sho at the same time plots (30, 2.5).
[357] Sho: Then, one fifty divided byyyy seventy equals two point one four two eight five seven one.	[357] Sho is now in a roll and quickly uses the calculator to compute 150 / 70.
[358] Sej: Okay ! Let's skip that one then.	[358] Sej infers that the decimal will be a challenge to locate on the y-axis.
[359] Sho: [Two point Aaaa !] Two point one.	
[360] Sej: [So, what] they are showing isssss.	[360] Sej moves his hand along the points already plotted.
[361] Sho: [No, don't] skip it !	
[362] Sej: Well, they're showing that this is the rate that it's going down.	[362] Sej again moves his pen across the plotted points.
[363] Sho: Yah, there we go.	[363] Sho plots (40, 2.1), but counts up to 2.1 from an incorrect starting point.
[364] Sej: So, let's just connect those.	[364] Sej uses his ruler to connect the different points on the plane.
[365] Sho: But, it's not constant, which means you can't use a ruler.	[365] Sho laughs.
[366] Sej: Well, it's not constant at all.	[366] Sej laughs and turns the sheet of paper to show Sho; Sej may be referring to the "misalignment" of the last point that was plotted.
[367] Sho: Okay ! Anyhow !	
[368] Sej: Okay !	[368] Sej puts aside the sheet with Question 5(c) on it and picks up the next sheet.

Figure 1. Continued.

the corresponding interactivity flowchart sections.) In this half of Figure 2, we find the arrows that represent the object- and non-object-level utterances of Sho and Sej, utterances that have been interpreted as belonging to either the personal or interpersonal channels – Sho's personal channel on the extreme left side, Sej's just to the right, and the interpersonal channel in the middle. (The numbers to the left of the circles representing the utterances of Sho and Sej refer to the corresponding lines of transcript.)

On scanning the entire column for Sho and Sej, one notes an immediate difference between the two boys' personal channels. From [313] onward, Sho's talk is tightly connected by object-level utterances, while the number of connections in Sej's personal channel is almost zero. So, we might first ask what happened prior to [313]. After the reading of the problem statement [301–303], neither boy had a clear sense of a direction to follow

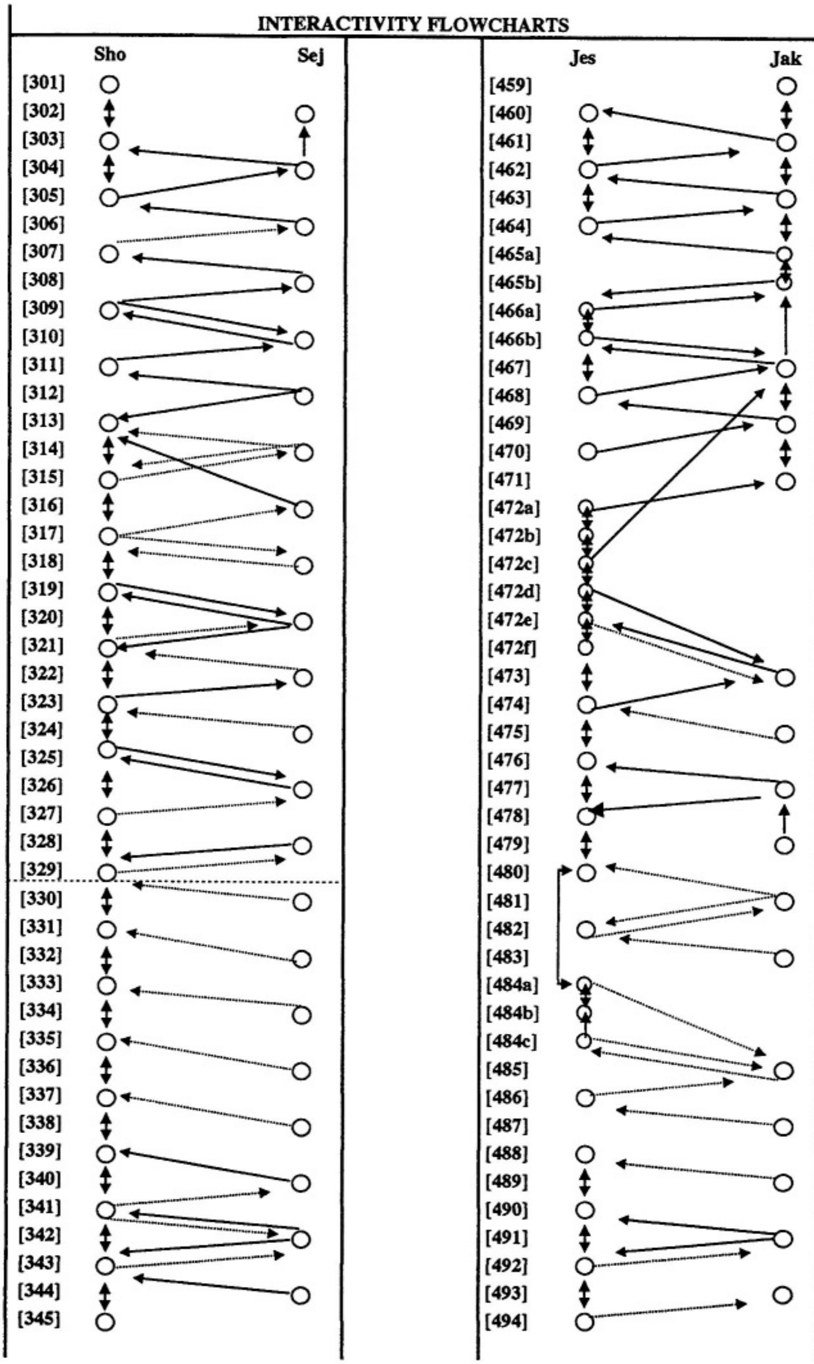


Figure 2. Interactivity flowcharts for Sho and Sej and Jes and Jak – Question 5c.

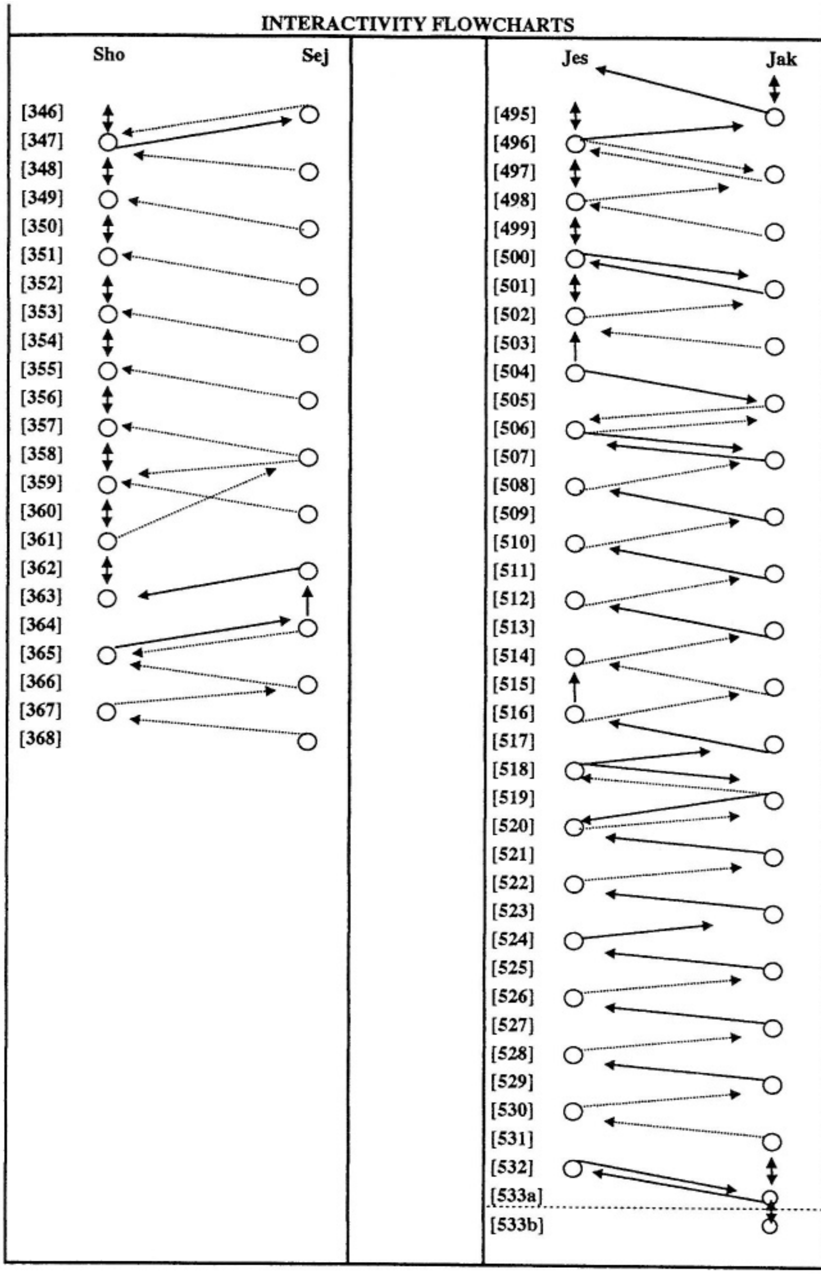


Figure 2. Continued.

for attacking the problem. Thus, the interpersonal channel from [301] to [313] consists primarily of object-level communication while both boys discussed together potential solution paths. Sej stated aloud that the graph could be anything [304], while Sho disagreed, suggesting that it should be like the ones they had already seen in figure 1 of Question 1 [305]. Sej again mentioned that they ought to be considering all the tailwind possibilities [310], while Sho countered that that would not make much sense to him because the question would not be asking them "to plot points that can be anything" [311]. Sej's next utterance, "If they gave us a tailwind" [312], signaled a shift in the nature of both the interpersonal channel and Sho's personal channel. Up to this utterance – with the exception of those related to the initial reading of the problem statement – neither boy's personal channel involved self-proaction or self-reaction. It was the interpersonal channel that was the arena of action, and it was in this arena where the ideas for a solution were being given birth.

Sej's stated wish for a specific tailwind [312] seemed to provoke in Sho a line of thought that he had not previously considered. In fact, it resulted in a focused personal channel for Sho, but not for Sej himself. It led Sho [313] to mark an uncalculated reference point on the coordinate plane of Question 5c at (10, 4.5) – as if to say that with a tailwind of 10 kilometers per hour, the time would be reduced to, say, 4.5 hours. But a few moments later, at utterance [317], Sho stated: "But wait, no, nooooo, okay, I figured it out, okay, she, she's at thirty kilometers per hour, right? is that what she is? thirty?" But what Sho "figured out" was never quite discussed in such a way that the public discourse became transparent for Sej. In [319], Sho offered, "She's getting pushed forty," to which Sej asked, "How do you know she's getting pushed forty?" [320]. Sho's response was, "Forty, this is what it is," referring possibly to either "getting pushed to forty" or "getting pushed by forty." At the same time, Sho pointed repeatedly to the points (30, 0) and (40, 0) of the figure 1 graph of Question 1. Parts of the coordinate plane seemed to be mediating Sho's thinking, although he was not being very explicit about the way in which the parts might be fitting together.

Sho then said that "they had to figure out the time" [325]. He verbalized the calculation that he was entering into the calculator: "A hundred and fifty divided by ten." His choice of "ten" may have been the result of his subtracting 30 from 40, the two values he had been pointing to on the graph. He quickly recovered on seeing the overly-large quotient displayed on the calculator screen, and said, "No, upsss, sorry, I wanted to do thirty plus ten" [329]. Even though Sho's body language suggested that he may not have been completely sure of what he was doing, he seemed to receive some positive feedback from the new quotient that was produced by the calculator – it was in the right neighborhood of values. This was confirmed by his statement to Sej: "We're getting the same numbers" [335]. After the new point (10, 3.75) was plotted and the old "placeholder" one (10, 4.5) removed, Sho continued to announce additional calculations, "one fifty divided by fifty equals three" [351], "okay, and then we have thirty, soooo, sixty, aah, one fifty divided by sixty" [355], and "one fifty divided by seventy" [357]. However, his utterances remained quite telegraphic. The communication between the two boys never really touched upon what these numbers actually signified, nor the relationship between them.

Sej participated in the interaction, by helping out with the plotting of the points. Whenever Sho announced the result of a set of calculations, Sej would run his pen up the y-axis and indicate to Sho where that value was to be found on the axis, as for example, in [344, 356]. Then Sho would depart from the y-axis point that had been identified by Sej and traverse along the line parallel to the x-axis, stopping at the x-value that corresponded to the tailwind speed that he had used in his calculations. Sej never knew where to stop his movement toward the right, along the line parallel to the x-axis, because the x-value had not been made explicit in Sho's calculation statements.

The interaction between the boys, which included some 68 utterances [301–368], can now be broken into three phases. The first phase was the relatively short pre-solution period [301–312] where the interpersonal channel showed both proactive and reactive object-level utterances on the part of the two boys. This was followed by a brief period [313–329] during which Sho's personal channel remained focused on the search for a solution path, culminating in a calculation that yielded a promising result. The object-level contributions to the interpersonal channel during this period were weak, due to Sho's attention to his personal channel. Sej posed a very explicit object-level question [320], but Sho's response seemed directed more to himself than to Sej. The rest of the interaction [330–368] served to consolidate the mathematics that had been emerging for Sho. His personal channel remained marked by a strong object-level focus. In contrast, the interpersonal channel showed very little object-level interaction. Sej's contributions were mainly of a non-object-level reactive nature.

The public discourse had included talk related to the calculations used in order to plot the points of the graph, but this public discourse on the part of Sho was more personal than interpersonal. Neither was it explicit enough to be meaningful to Sej. From [336] onward, Sej was more of a "sidekick," who assisted with the graphing, rather than a partner who contributed by questioning and testing the other's talk. The graphing tool that had mediated the thinking of Sho did in fact allow Sej to participate in the problem-solving process, but it was a somewhat limited participation – that of picking out the location of the y-values of the points to be plotted. The *Nathalie* problem, which seemed to have been productive for Sho, had not been so for Sej. It is as if, once one of the partners had found the key to the solution, the interpersonal channel became much less significant.

The pattern of discourse that characterized Sho and Sej's interactions was also seen with, for example, Crs and Ali. While Crs's personal channel was tightly focused and public, Ali's remained rather inactive. One could, of course, argue that just because an interlocutor tends to be silent does not mean that significant thinking is not taking place for that person. However, the nature of Ali's utterances in the interpersonal channel, along with his follow-up individual work, allow us to infer that the public discourse that occurred during the boys' interaction had not evoked in Ali a productive line of thought that could subsequently be called to mind on related problems. In fact, Ali's contributions to the interaction with Crs are reminiscent of Sej's in that Ali's utterances tended to be of the reactive, non-object-level type. Ali made only four object-level utterances and three of these were related to making more precise the y-coordinate location of the points being plotted – not the x-value – just as was the case with Sej. Ali too had taken on the role of graphing "sidekick." The Sho and Sej interaction pattern on Question 5c, as with that of Crs and Ali and two of the other pairs, was one where the public discourse of the problem situation seemed not to be productive for one of the two partners. The key ingredients of the solution path, which seemed to emerge for one of the partners, did not seem to similarly emerge for the other. For that, the public discourse was partially to blame. We now turn to a pair for whom the nature of the public discourse, especially the segment prior to the finding of a solution approach, was dramatically different.

4.3. *Interactivity analysis: The interaction of Jes and Jak*

The section of transcript from Jes and Jak's interaction that deals with Question 5c is ten minutes in length (see Figure 3 for the transcript extract). For two of the earlier questions, 4c and 4d, Jes and Jak, just like Sho and Sej, had made little headway and decided to skip them. They moved on to Question 5a, where Jak blurted out "one fifty divided by thirty … it's five" before Jes had even finished reading the question. When Jes was ready, Jak read the text of Question 5b and immediately answered "so five hours at thirty." Jes continued with "there's five and here's thirty," pointing to (0, 5) and (30, 0) on the y- and x-axes respectively, whereupon the two boys together marked the point (30, 5) on the coordinate plane – not seeming to

WHAT IS SAID	WHAT IS DONE
[459] Jak: Suppose Nathalie were to fly daily to the airport one hundred and fifty kilometers away with a tailwind. Without any wind her plane flies at thirty kph. Knowing that the wind can vary from day to day, sketch a graph that shows the various times that Nathalie could take when she flies with a tailwind. ... What ?	**[459]** Jak reads the question, moving his pen along as he reads, pointing to the various sections of the text.
[460] Jes: Suppose Nathalie were to fly daily to the airport one hundred and fifty kilometersss.	**[460]** Jes turns the sheet toward himself, laughs a little, and starts to read the question once again.
[461] Jak: [Away.]	**[461]** Jak & Jes both look at the question wondering what to make of all of this.
[462] Jes: Away with a tailwind.	
[463] Jak: Without any wind her plane flies at thirty kph.	
[464] Jes: [We know that]	
[465a] Jak: Knowing that the wind can vary from day to day, sketch a graph that shows the various times that Nathalie could take when she flies with a tailwind. Do your sketch in the coordinate system supplied in Figure 2.	
[465b] Jak: So, which is, it could just be ... anything ?	
[466a] Jes: No ! It has to be the sameeee.	**[466a]** Jes turns towards the computer and starts to point to it.
[466b] Jes: See ! The wind corresponding to the time has a certain value, with all headwinds. It's like, it's the same for these two. We just have to find what the wind is for both.	**[466b]** Jes points to the graph of figure 1 of Question 1 with his pen as he explains. He then points to both of the headwind graphs of figure 1.
[467] Jak: Well, the thing with this one we have to make it up.	**[467]** Jak points to the coordinate plane of Question 5c.
[468] Jes: We don't have to make it up! We have to take it from figure 1.	
[469] Jak: No ! It has nothing to do with figure 1, it's her own plane. It's the tailwind from day to day.	**[469]** Jak points to the question with his pen.
[470] Jes: [Well, we have to make], we have to make it something that could be very possible.	
[471] Jak: That's why there, that's why there's no, that's why there's numbers over there, because it varies.	**[471]** Jak points to the horizontal axis of the coordinate plane of Question 5c.
[472a] Jes: Okay look ! It takes her five hours, so she starts here. It starts here. Oh ! Hold on !	**[472a]** Jes points at (0, 5) and marks a dot there.
[472b] Jes: This is with no wind, so it's not that point ! It's over here. All right ! It starts there.	**[472b]** Jes crosses out the first point they had plotted at (30, 5) for Question 5b.
[472c] Jes: With a tailwind, it will slow her down, at the same rate as this.	**[472c]** Jes makes a downward motion on the coordinate plane and then points to the tailwind curves of the figure 1 graph of Question 1.
[472d] Jes: But, we don't know the dumb rate.	
[472e] Jes: Hummmmm ! What do we do ? ...	**[472e]** Jak & Jes keep looking at the figure 1 graphs of Question 1.
[472f] Jes: This is taking it, it's taking. The wind affects in a certain way. Like, let's say, the plane goes fifty kilometerssss per hour, it will affect it more. If it goes sixty, it will affect it less, because it's like the engine is stronger. Or what, for whatever reason.	
[473] Jak: Well, I say it can just be like, a slope that we can draw, because it doesn't say (?)	**[473]** Jak swept down the coordinate plane of Question 5c in a left-to-right direction.
[474] Jes: [All right !] It will be steeper then Matt's. Wait ! It will be less steep then Glen's.	**[474]** Jes points at the tailwind graphs of figure 1.

Figure 3. Question 5c transcript – Jes and Jak.

have noticed, for the moment, that the x-axis referred to the speed of the wind. (Observe that neither of the boys played the role of passive observer with respect to the physical plotting of the point.) They were ready to proceed to Question 5c. At this moment, the reader may wish to preview the interaction that occurred for these two boys during Question 5c, the *Nathalie* problem.

In addition to being somewhat longer than the *Nathalie* episode for Sho and Sej, the interactivity flowchart for this pair (see right-hand column of

WHAT IS SAID	WHAT IS DONE
[475] Jak: Yah !	
[476] Jes: And that's crossing at two.	**[476]** Jes points to the upper tailwind graph C of figure 1.
[477] Jak: See the first dot for her's would be right.	**[477]** Jak starts to establish where the point (0, 5) for Nathalie would be on the figure 1 graph.
[478] Jes: [And three and then four.]	**[478]** Jes continues to sweep down the C graph of figure 1.
[479] Jak: There.	**[479]** Jak points to (0, 5) of Figure 1.
[480] Jes: Yaaaa !	**[480]** Jes traces freehand a curve from (0, 5) going down to the right, slightly above the other two tailwind graphs of figure 1.
[481] Jak: So, (if it ?) goes the same. What ? Should I label these ? It would be a lot easier.	**[481]** Jak labels the horizontal axis of the coordinate system of Question 5c.
[482] Jes: This is harder then I thought Jak.	
[483] Jak: Ya !	
[484a] Jes: Aaa ! All right ! Look ! Two and three, the next one will be four. And that is for fifty. All right ?	**[484a]** Jes counts the number of grid squares that graph C traverses between one corner and the next, in its descent to the right end of the coordinate plane.
[484b] Jes: For, a plane that goes fifty kilometers per hour. Now, for sixty it is, three and then five and then seven. So, it's two. Ooooh ! I'm getting something here ! It keeps going up by two and we're, and it starts at two point five. Wait a minute !	**[484b]** Jes is looking for a visual geometric pattern to the curviness of the two given tailwind graphs of figure 1.
[484c] Jes: Go back here ! Take the mic. With one, are we doing that now? Right !	**[484c]** Jes pushes Jak to turn towards the computer.
[485] Jak: No ! (?)	
[486] Jes: Okay ! Aaaa !	
[487] Jak: We're doing this one.	**[487]** Jak picks up the question sheet from the table.
[488] Jes: I know, but we're finding out what the wind expression is. Matt issss plus three and then five. So, between those, it keeps go like two. Okay ?	**[488]** Jes points at the graph of Figure 1 with his finger and turns his head towards Jak to get a confirmation.
[489] Jak: Humhum !	
[490] Jes: So it, it's, ... Aaaa ! ... Two "X" ... Two "X". Hold it ! Two "X" two plus two.	**[490]** Jes enters the expression $2x^2+2$ into the computer.
[491] Jak: What is that ? ...	**[491]** Jak looks quizzically at the screen with its angular parabola and then at Jes.
[492] Jes: Come on (?)	
[493] Jak: (I think for this one, your supposed to ?) make up the wind.	
[494] Jes: Hold on, hold on ! Wait, wait, wait, wait, wait !	
[495] Jak: (?) [The wind is variable], it is not always telling it.	
[496] Jes: No, no, no, no ! We're not saying where it is at zero. Okay ! Look at this ! After ten, it is at, ... for Matt. What ? At fifty it is at two. At ... at forty it is at. Aaaaa ! Hold on ! We're looking at it ! Starts at two point five. Hummmm ! (We can get this ?) I'll tell you what ? Fifty, (?) at the power of two plus two. Wait, wait, wait ! (We're looking at the number ?)	**[496]** Jes eventually enters an expression into the computer, $50x^2+2.5$, which did not produce a graph of any interest to them.
[497] Jak: (There ?)	**[497]** Jak tries to help Jes find a certain key on the computer.
[498] Jes: (No ! One ?)	
[499] Jak: (?)	

Figure 3. Continued.

Figure 2) looks quite different from the previous one. The entire extract [459–533] deals with finding a solution. In contrast with the Sho and Sej interaction where a solution began to emerge for Sho quite early in the episode – from the 13th utterance – the interaction between Jes and Jak was almost all solution-seeking oriented. The interpersonal channel was also more active in terms of object-level utterances than was the case for the other pair: 42 object-level utterances, as opposed to 25 for Sho and Sej. The pattern of a lower number of object-level utterances in the inter-

WHAT IS SAID	WHAT IS DONE
[500] Jes: No it isn't ! (We're slowing down ?) It's not. Where, see, how this is curved, it's like a parabola. Where will it be level? Where will the difference be zero ?	**[500]** Jes continues to study the behavior of the two tailwind graphs of figure 1 of Question 1.
[501] Jak: At fifty, I think.	
[502] Jes: Well, we can't.	
[503] Jak: We can't see. We can't check.	
[504] Jes: [We can't check.] I think that's what he (flies. ?) But, how do we put that into the expression ? ... (Because it doesn't say that ?)	
[505] Jak: Here you go, the expression.	**[505]** Jak has looked ahead at the remaining activity sheets and has come across the expressions in Question 6(a).
[506] Jes: Where did you get that from ? Wait ! Zero point zero, zero.	**[506]** Jak looks at the expressions in Question 6(a). and enters one of the expressions into the computer $(0.005x^2+ 3)$.
[507] Jak: Five.	
[508] Jes: [Five.]	
[509] Jak: "X".	
[510] Jes: "X".	
[511] Jak: The power of two.	
[512] Jes: The power of two.	
[513] Jak: Plus three.	**[513]** Jak & Jes both look at the graph on the computer screen.
[514] Jes: Does that look	
[515] Jak: [Okay ! That's.]	
[516] Jes: Familiar to you ?	
[517] Jak: That's Graph A.	
[518] Jes: Look, look ! It's over here. No ! It isn't !	**[518]** Jes shows the Figure 1 graph of Question 1 to Jak.
[519] Jak: Well, there are three of them. The question is which one is best ? So try, zero point one.	
[520] Jes: (You're, but ?) We're not doing it right. We're like cheating Jak.	
[521] Jak: "X". No ! Not zero point zero "X".	**[521]** Jes enters another expression: 0.1x + 3
[522] Jes: Oh ! Zero point one "X".	
[523] Jak: ["X"] plus three.	
[524] Jes: I Know it's not that. That's linear.	**[524]** Jes & Jak look at the graph produced on the computer screen.
[525] Jak: Right ! One fifty divided by fifty minus "X".	**[525]** Jak dictates the last of the expressions to Jes who enters it into the computer.
[526] Jes: One fifty.	
[527] Jak: Slash.	
[528] Jes: Slash. Another fifty.	
[529] Jak: Minus "X".	
[530] Jes: Minus "X".	
[531] Jak: That's it !	**[531]** Jes & Jak look at the computer screen.
[532] Jes: That's it ! ... I get it now ! Okay ! One fifty is how long it takes. Fifty is his speed.	
[533a] Jak: Wait ! "X" represents the speed of the wind.	
[533b] Jak: So, if we want to figure out for that one, it representsss.	**[533b]** Jak is referring now to Question 4(c).

Figure 3. Continued.

personal channel was typical for those pairs whose discourse seemed not to be productive for both interlocutors.

Another difference between this interactivity flowchart and those of the four pairs, as represented by the Sho and Sej excerpt, is that both Jes's and Jak's personal channels were active in the beginning [459–474]. This period of tightly-connected personal and interpersonal activity can be broken into two segments: the first [459–465a], where each was completing the other's sentence in their shared reading of the problem statement;

and the second [465b–474], where each tried to offer arguments in favor of their initial thoughts on how to go about solving the problem. Jak, whose first publicly-uttered thoughts in this segment remind us of Sej's [304] initial reaction to the problem statement, believed that since no tailwind speed was given they could choose any they wished. Jes, on the other hand, felt certain that the relationship underlying the tailwind graphs shown in the figure for Question 1 was the same as for the *Nathalie* situation – "if only they could figure it out."

A new phase of the interaction was signaled by the non-object-level utterance of Jak in [475]: "Yah." At this moment, Jak's personal channel became less active while he began to take more the position of reactor to Jes's uttered thoughts than promoter of his own. Jes's public analysis of the tailwind graphs presented in Question 1, an analysis that involved looking for patterns in the curvature of the two graphs, continued up to [484b]. At that moment, he uttered that they needed to move to the computer, in order to "find out what the wind expression is" [488].

In the following fourth segment in the phase of searching for a solution path [488–504], the patterns of interaction shifted once more. Jes's personal channel became even more active while he tried to generate expressions that would yield graphs on the computer screen that were like the ones in Question 1. (The graphing software that was on the computer – the same software that the students had used in their prior introduction-to-algebra classes – required entry of an expression in order to produce a graph.) Once again, the interpersonal channel became alive with object-level utterances. It was the use of the computer that seemed to be provoking a change in the interpersonal channel, a channel that just a few utterances earlier had been dominated by non-object-level utterances. The computer appeared to be mediating not only Jes's personal channel but also Jak's object-level utterances in the interpersonal channel. Jak now had a more tangible point of reference with which to anchor Jes's publicly-voiced thinking. But even though the computer was facilitating the public-discourse related to the testing of conjectures, the boys were not really advancing in their search for the desired expression. They needed a breakthrough.

Although they had not yet talked about skipping the *Nathalie* question, Jak turned idly to the next page of questions in the set of activity sheets. While Jes was examining the most recent computer graph he had generated [504], Jak noticed that the following question, 6a, contained a sampling of symbolic expressions (see Appendix 1). He chose the first of the three given expressions – not, by the way, the one that corresponded with the problem situation – and then proceeded to share his discovery with Jes:

"Here you go, the expression" [505]. This marked the beginning of the last phase in their search for a problem solution.

It is not clear from the object-level talk of Jes and Jak during this episode whether either boy came to think about the graph in such a way that each point encapsulated the entire travel of a plane from take-off to landing with a given tailwind speed; in other words, that all together the graph was a summary of hypothetical options with each point on the graph representing one such option (a certain travel time for a certain speed of the wind). That such ideas about graphs are difficult for students to conceive of has been suggested by several researchers (e.g., see Leinhardt, Zaslavsky, and Stein, 1990) who have pointed out that students tend to read graphs as chronicles of one event over time. The symbolic expressions that Jak had found by thumbing through the upcoming questions were to be crucial, at least for the point-plotting activity itself, and possibly also for coming to better understand the significance of each point of the graph. Although Jes felt that they were "cheating" [520] by using them, they entered them, one at a time, into the computer. They soon concluded that the third expression, 150 / (50 – x), matched one of the graphs of the earlier Question 1, the set of graphs that they had been trying to unlock. Jes reacted with, "That's it, I get it now, Ok, one fifty is how long it takes, fifty is his speed" [532] and Jak with "Wait, x represents the speed of the wind" [533]. They now seemed to be on the right track, even if their talk with respect to the referents for each of the entities in the expression still sounded somewhat confused.

Jak, at this moment, wanted to go back to Question 4c, which they had skipped. But 4c dealt with Matt's plane, and the expression they had just graphed, 150 / (50 – x), was for Glen's. But this did not faze Jes who triumphantly announced: "Okay, I'll give you Matt's right now, watch, one fifty over sixty minus x" [546 of the non-presented part of the transcript]. And Jak continued with "Okay, so if x were fifty, then sixty minus x would be ten, so it's one fifty divided by ten." When they arrived once again at Question 5c, which had not yet been completed, they entered the expression for "Nathalie-with-tailwind" into the computer: 150 / (30 + x). In order to do their sketch of the *Nathalie* situation in the coordinate plane provided on the activity sheet, they together picked two points off the computer graph (20, 3) and (30, 2.5) and Jes transferred them to the paper sketch. He joined them to the previous point they had plotted some time ago (0, 5) [472b] with a less than elegant curve and asked Jak: "How's that?" Jak elatedly came back with "Yah, that's looks good." That the boys did in fact sort out the referents for each of the entities in the expression was illustrated in the written answer they jointly composed for the related headwind question, #8b: "We took the distance, divided by her speed minus head wind speed."

The *Nathalie* episode seemed to have ended well for Jes and Jak. But the entire interaction between these two boys, not just the episode discussed here, had had similar characteristics. Jes was very good at making his thinking public. These public utterances abounded with mathematical questions that he addressed both to himself and to his partner. In this sense the public discourse of these two boys could be said to have been more extensive, mathematically speaking, than had been the case for the other pairs whose interactions were represented by that of Sho and Sej. Jak was given the opportunity to participate in Jes's thinking in a way that was qualitatively different from what had been made feasible for Sej and several of the others. But Jak was also a more active partner than the others had been, fulfilling his part of the bargain in "checking [his partner's] talk."

Another important factor in the success of these boys' communication is conjectured to be the length of time they spent trying to find a solution. Their lengthy search yielded an interpersonal channel that was characterized by a zig-zag of object-level utterances. This is in contrast to the rather one-sided discourse that characterized Sho and Sej's interactions. For Sho and Sej, and for the three similar pairs, where there was a discrepancy between the two productions in the follow-up individual work of each member of a given pair, the interpersonal channel was characterized primarily by non-object-level reactions as soon as the solution path had been established – and rather quickly at that. In fact, after one of the partners had the solution, little mathematics seemed to emerge for the other partner during the ensuing talk.

The Jes and Jak pair provided the clearest example from among all the pairs of how "interpersonal discursive processes" can serve as the basis for "individual and private uses of symbolic systems" (Harré and Gillett, 1994, p. 27). In the *Nathalie* problem, neither boy came into the interaction with a ready-made thought as to how to proceed toward a solution. Nor did one of them have a quick insight that, as was seen with other pairs, virtually closed down the potential productivity of the interpersonal channel. No, their language and the other mediating tools they used, especially the computer and the symbolic form of the mathematical relationship underlying the *Nathalie* problem, organized their joint experience and made it one that both chose to draw upon a few minutes later. That their discursive interactions had influenced their mathematical thinking received a degree of confirmation in the analysis of their follow-up individual work. The thinking that each brought to bear on the analogous questions of the individual work reflected the interactive public discourse that they had meaningfully engaged in as a pair. The words they each used to respond to the questions were slightly different from the written responses that they had generated together, but the ideas were clearly related to that joint prior experience. Despite the fact that the joint work had led to mathematical insights for both Jes and Jak, insights that they both drew upon in their individual work afterward, it is somewhat sobering to realize that for four of the six pairs, the mathematical discourse on the *Nathalie* problem seemed not to have been equally productive for both partners.

5. DISCUSSION

To facilitate the following discussion, I will refer to the four pairs for whom the discursive interactions during the *Nathalie* problem seemed not to be equally productive for both partners as the "non-mutually-productive pairs"; the interactions of these pairs were seen through the example of Sho and Sej. The remaining two pairs, exemplified by Jes and Jak, will be referred to as the "mutually-productive pairs" in that both partners seemed to benefit to a similar extent from their discursive interactions, although no claim is being made here that the mathematical thinking that emerged in these pairs was superior to that which emerged for some of the interlocutors in the other pairs. It is the asymmetry of the productivity that is at issue here. The discussion, which will include examples drawn from the analyses carried out with the six pairs of the study, focuses primarily upon the contrasting patterns of discursive interaction seen in the two groups, and considers in particular those patterns that seemed not to be conducive to the emergence of mathematical thought for both partners in the paired work.

5.1. *A striking discursive difference between the non-mutually-productive and mutually-productive pairs*

The principal differences found in the discursive interactions of the two main groups of pairs have already been alluded to in the findings reported with respect to the pairs Sho and Sej and Jes and Jak; however, one of these differences merits further discussion. It concerns the relative absence among the non-mutually-productive pairs of focused personal channel activity of some participants, along with a concomitant weakness in the quantity and quality of the object-level utterances in the interpersonal channel. With respect to the activity of the publicly-uttered personal channel, for one of the partners in each of these four pairs, there seemed to be hardly any. Rather it was the personal channel discourse of the other partner that seemed the primary source of whatever utterances occurred of an interpersonal nature by the second participant. Wertsch (1998) has suggested that "instead of trying to 'receive' meanings that reside in speakers' utterances as envisioned by the 'conduit metaphor' (Reddy, 1979), the focus is on how an interlocutor might use texts as thinking devices and respond to them in such a way that new meanings are generated" (p. 115).[2]

Nevertheless, "trying to receive meanings" from their partner is precisely what seemed to be occurring for one of the interlocutors of these pairs. Had they been "responding to them [their partner's utterances] in such a way that new meanings were generated," there would likely have been signs of subsequent personal channel activity on their part that was linked to the object-level utterances of the interpersonal channel or sustained utterances by them in the interpersonal channel. But this did not take place. The object-level discourse of the non-mutually-productive pairs, which was centered primarily in the utterances of one interlocutor, rather than in both, was inclined to be minimalist in nature and, as it seemed more directed to self than to other, tended to have the effect of curbing meaning-making activity in the interpersonal channel.

Interactivity analysis across the six pairs disclosed that the number of interpersonal object-level utterances prior to the solution path being generated was significantly different from the non-mutually-productive pairs to the mutually-productive ones. In Figure 2, which presents the two interactivity flowcharts that have been included in this paper, a horizontal line appears in each column immediately after the occurrence of the initial utterance related to the generation of a solution path (line 329 for Sho and Sej, line 533a for Jes and Jak). The number of interpersonal object-level utterances, either proactive or reactive, from the time the problem statement was first read until the solution path was formulated can be seen to be 18 for Sho and Sej and 42 for Jes and Jak. In fact, just about all of the transcript extract for Jes and Jak deals with their efforts at trying to find the key to the underlying relationship of the problem. It is conjectured that one of the main discursive factors contributing to the mutual productivity of Jes and Jak's interactions was the greater number of interpersonal object-level utterances that occurred.

Creating mathematical meaning jointly seemed more difficult to achieve when one of the partners seemed the primary source of the utterances, and when those utterances tended more often than not to be directed to self, and when only that partner seemed to be making sense of the problem. Linn and Barbules (1993) have argued that partnered work succeeds when students are effective at communicating their ideas and able and willing to help other group members. Successful collaboration, according to these researchers, also depends on a commitment to a form of discourse that values argument and relies on explanation. Among the non-mutually-productive pairs of this study, there seemed to be little serious inclination to help one's partner. Dembo and McAuliffe (1987) have reported that perceived higher ability of some team members can lead to the higher status members giving more help than lower status ones. Assuming that the "solution finders" of the non-mutually-productive pairs of this study perceived themselves as being of higher ability, there was no evidence that this was leading them to offer help to their "lower status" partners. As soon as the "solution finders" of the *Nathalie* situation embarked on a solution path, they talked more to self than to partner, and their subsequent utterances referred mostly to the calculations they were carrying out in order to obtain additional points for the graph. No substantive explanations were put forth.

The discursive testing and interpreting carried out by Jes and Jak (of the mutually-productive pairs) in analyzing the symbolic expressions that were provided in the post-*Nathalie* problem (Question 6a) is contrasted with the interaction of one of the non-mutually-productive pairs, Crs and Ali, on the same set of expressions. When Crs and Ali read this question dealing with the three choices of expressions for the function producing one of the graphs of the earlier Question 1, Crs stated: "Okay, let's try this, one fifty divided by fifty minus, x is the speed of the wind." Ali soon afterward asked: "Why did you choose to do that one first?" to which Crs responded: "Cause it looked more reasonable." Lest anyone conjecture that simply giving one's partner the right symbolic expression would be adequate for creating a meaningful experience, he or she would be mistaken. Ali did not draw upon this exchange when tackling similar questions in the follow-up individual work.

5.2. *When the solution process was not a truly shared discursive activity*

That which occurred among the non-mutually-productive pairs could be characterized as follows: After only one of the two partners had generated a solution path, and the other partner's personal channel had not itself been active during the time that the other's had been engaged in the search, the ensuing interpersonal channel tended to be marked by mostly non-object-level utterances. No object-level utterances that were germane to the relationship underlying the calculations, which were being carried out by the more active partner, were subsequently noted in the interpersonal channel. Instead, issues of a more minor nature were raised, such as the results of the calculations and where these y-value results were to be found in the Cartesian plane. It was not the case that questioning that was related to, say, the x-values being used in the calculations, and their link to the given constants, occurred once the solution path had been established. What little there was of that which Shotter and Billig (1998) have described as the "testing and checking of each other's talk [when the sharing of a 'mental-picture' or an idea is at stake] to establish whether they are in fact in agreement or not" (p. 24) took place before the solution path had been established. This, by the way, was also the case for Jes and Jak (one of the two mutually-productive pairs) who, when their approach to a solution eventually emerged – at the end of a rather lengthy conversation – went on to apply it in a rather matter-of-fact way. In case the reader is wondering whether the partners for whom the discursive interactions were not considered mutually-productive were not being more active questioners because they were not aware that there was something in the discourse that they were not catching, there is evidence to the contrary. One of the requests made of participants in this study was the writing of personalized, individual reports at the conclusion of the joint problem-solving sequence. One of the questions of this reporting-back was the following: "Were there some questions or ideas that you worked on that are still kind of fuzzy for you?" While Crs of the Crs and Ali pair wrote, "No," Ali recorded: "Yes, I still have a little trouble finding out wich [sic] expression mached [sic] the graphs that we saw." Sej of the Sho and Sej pair described a similar awareness of his lack of understanding of what was going on regarding the expression needed for the *Nathalie* graph. So, if the public discourse was not productive for Ali and Sej and others, and they realized that it was not being productive for them, why did they not insist on a discourse that could have been more meaningful for them? Did they not want to appear unintelligent in front of their peers? Were they hoping that the sense of it all might come to them eventually? Perhaps saving face was more important to these adolescent problem solvers than was admitting to their partner that they "didn't get it." Salomon and Globerson (1989) have suspected that such phenomena are far more prevalent in real classroom teams than is reported in the educational literature.

Participants were not asked in their individual reports whether they thought that their talk had helped their partners to make sense of the mathematics in the various problems they had worked on. Nonetheless, interactivity analysis of the discourse of the four non-mutually-productive pairs suggested moments when the thinking that was conjectured to be taking place for one of the partners was not being made sufficiently available, despite hints from the other partner that the situation warranted it, for example, Sej's query to Sho, "How do you know she's getting pushed forty?" [320]. Even though Sho eventually referred publicly to the numerical procedure he had generated in order to arrive at the y-values for the points that were to be plotted, he never clarified what these numbers signified, nor why he was using certain operations. Did Sho presume that his rather cryptic answers to Sej's questions [314, 320, 326] were adequate? Sej's frequent "Ya" responses might have communicated a sense of understanding. This would be in keeping with what Grice (1968) and others have emphasized concerning the social rules governing conversation, rules which suggest that parts of an explanation that are assumed to be accepted by participants will be left unsaid rather than stated explicitly.

However, this explanation is not completely satisfactory. When Sho first entered a calculation sequence into the calculator [329], he spoke quietly, as if he did not want to lose his train of thought. And his earlier response to Sej's, "How do you know she's getting pushed forty?" [320], was accompanied by a to and fro movement between the two points of the Question 1 graph, (30, 0) and (40, 0), which suggests that he may have been distracted by the question, at the very moment that he was trying to grab hold of a newly emerging idea. It was likely the case that he was leaving out parts of an explanation, not because he assumed that Sej would be filling in the missing pieces, but because he had not yet constructed them himself – at least not up until sometime between [329] and [355]. But why not then elaborate further once the idea had become clear and had successfully yielded points that followed the sweep of the Question 1 tailwind graphs? It just didn't happen. Sho's public utterances, which were addressed mostly to himself once he had conceived of a solution path, remained fragmentary and opaque. Suffice it to say that when neither partner had yet generated a way of attacking the problem, the interpersonal channel was much more likely to maintain significant object-level utterances of both the proactive and reactive kind than it was afterward.

6. CONCLUDING REMARKS

Bridging the individual and the social has two meanings in this paper. One involves the attribution of the sociocultural origins of thought to individual thought. Because a discursive perspective is anchored in Vygotskian theory, this meaning has been taken for granted in much of the way that the findings of this study have been framed. Social in this sense refers to the cultural and social situatedness of human experience. However, it is another sense in which the individual and the social are bridged within discursive psychology that has been the main theme of this paper. In this second sense, social has to do with whether one or more than one individual participates in the immediate action. In this study, there were activities involving two persons at a time, and others involving only one.

The major thrust of the analysis was on the partnered activities and, in particular, on the interactive mathematical discourse engaged in by the participants; an accompanying analysis considered the seeming impact of this discourse on the individual's thinking in related activities afterward.

Harré and Gillett, in their book *The Discursive Mind,* stated that "the workings of each other's minds are available to us in what we jointly create conversationally, and if our private mental activity is also symbolic, using essentially the same system, then we can make it available or not, as the situation seems to require it" (1994, p. 27). They defined public-discourse as behavior and private discourse as thought (or thinking). One of the questions driving the analysis of this study was whether students do, in fact, make the workings of their minds available to partners and what form this might take. As a means of getting at the public discourse, and indirectly at the private discourse, methodological tools, which had been designed for an earlier study (Sfard and Kieran, 2001), were slightly modified so that the interactivity flowcharts of the public discourse could be used as a stand-alone representation of the locus of the (mathematical) object-level utterances of participants. The flowcharts also separated the public utterances, according to interpretations made by the researcher, into those that participants seemed to be addressing more to self (the personal channel) and those that participants seemed to be addressing to the partner (the interpersonal channel). This separation helped to make more evident some of the differences in the patterns of interaction of the various pairs during the *Nathalie* problem.

The Jes and Jak pair, representative of the two so-called mutually-productive pairs, exemplified a pattern of interaction that involved a large number of object-level utterances in the interpersonal channel during their lengthy process of seeking a solution. Jes, who tended to be the primary source of the mathematical questioning engaged in by the pair, seemed rather remarkable in the way that he made his thoughts public to his partner – thoughts that often led to dead ends, but no matter. When the two boys eventually found a solution to the problem, it seemed to be a meaningful one for both of them. This is in contrast with the patterns engaged in by the non-mutually-productive pairs. The interactions of Sho and Sej, for example, illustrated a pattern in which, once again, the personal channel of one of the partners (Sho) was more active than it was for the other partner, but in contrast to the mutually-productive pairs, these utterances seemed to be directed more to self. They also had a rather fragmentary and telegraphic aspect to them, which did not communicate well to the partner. As a consequence, the interpersonal channel lacked the intensity of the object-level interactions – except during the brief pre-solution phase – that had been characteristic of the Jes and Jak interactions. Thus, even if one were to base one's conclusions solely on the interactivity analysis of the pairs' patterns of interactions, one would be led to question the productivity of the Sho and Sej conversations, conversations in which only Sho had maintained a strong object-level focus – but this almost exclusively within the personal channel.

However, a second methodological component was included in this study – a component that was aimed at obtaining supporting evidence for what seemed to be differential productivity of the patterns of interaction disclosed by the interactivity analysis. It involved the use of a problem set that was analogous to that worked on formerly by the pairs, but this set was tackled privately this time. In this phase of the study, the same participants were alone with their thoughts as they tried to solve the *Susan* problem that was structurally identical to the *Nathalie* problem experienced just moments earlier. It was found that the individuals who successfully drew on the prior experience with their partners were either those whose patterns of interaction had been characterized by a heavy frequency of object-level utterances in the interpersonal channel (i.e., both partners of the mutually-productive pairs) or those who had maintained a strong object-level focus in their personal, but still public, channel (i.e., one partner of each of the non-mutually-productive pairs). The partners of the latter, who had not themselves manifested an equally focused personal channel and where the interpersonal channel had been very weak with respect to object-level utterances, did not (or were unable to) draw upon the solution approaches that had been promoted during the pair-wise interactions.

A discursive perspective on the mathematical problem-solving activity that was engaged in by the students of this study permitted an analysis that focused simultaneously on the individual and the pair. The patterns of interaction that were found to be most productive for both members of the pairs were those where the interpersonal channel was the site of frequent object-level utterances. Those interactions where it was the personal channel of only one of the participants that was the main site of the publicly-uttered object-level thinking – utterances that were neither complete nor ever expanded upon – seemed much less conducive to the emergence of mathematical thought for both participants. Thus, it is not just the case, as Harré and Gillett have stated, that we "make our private mental activity available to others or not." Rather, the way in which it is made available seems crucial. For partnered discourse to be productive, it would seem that thinking needs to be more than simply "made available" to partners. Otherwise, too much may be left unsaid. Earlier, the question was raised as to whether the purification that Hershkowitz and Schwarz (1997) noted when one reports one's solution approaches to a partner would also be seen in the talk that emerges between partners as they jointly attempt to solve a difficult problem. In this study, at least for the non-mutually-productive pairs, the utterances of the interlocutor who was the main site of the object-level talk tended to be abbreviated, but not in the sense of their being a trimmed-down version of some already elaborated way of thinking. Rather they were observed to be fragmentary and underelaborated – more akin to the generation of mathematical thinking than its purification. In order for that talk to be more elaborated and less fragmentary, and that it be directed as much to partner as to self, seems a tall order. The pattern of interaction engaged in by Jes and Jak, which was spread over a much longer period of time, seemed to help.

Thus, the discursive perspective, which provided a powerful theoretical lens for interpreting patterns of interaction, also allowed us to see that, for the adolescent pairs of this study, bridging the individual and the social in mathematical problem solving can be extremely difficult to put into practice, especially when it involves novel problem situations. Making one's emergent thinking available to one's partner in such a way that the interaction be highly mathematically productive for both may be more of a challenge to learners than is suggested by the current mathematics education research literature.

ACKNOWLEDGEMENTS

The research presented in this paper was made possible by a grant from the Quebec Ministry of Education – FCAR grant #97-ER-2705. The questions developed for the activities were based on a task created by Fey and Heid (1991). My appreciation is extended as well to those who contributed during various phases of the research: Anna Sierpinska, Joel Hillel, Tommy Dreyfus, Jana Trgalová, Michael Haddad, Astrid Defence, Hassane Squalli and to the students (and their teacher) who agreed to participate in the study. I also want to thank Anna Sfard with whom I collaborated on an earlier project and who was the inspiration for some of the ideas in this paper. In addition, I am very grateful to Ellice Forman and the external reviewers for their extensive reviews of and helpful comments on an earlier draft of this manuscript.

NOTES

1. However, it was not until 1985 that the research presented at PME began to include references to Vygotsky's work (see the PME Proceedings [Streefland, 1985]). The steady growth in the development of sociocultural perspectives by PME researchers during the ensuing years was reflected in, for example, the program of the 1995 PME conference where plenary and panel presentations were devoted to Vygotskian theory (Meira and Carraher, 1995).
2. *Text,* as defined by Lotman (1988, p. 37), is a "semiotic space in which languages interact, interfere, and organize themselves hierarchically."

APPENDIX 1: PROBLEM SET FOR THE JOINT WORK

HEADWINDS AND TAILWINDS

Situation: Glen and Matt own and fly experimental small airplanes. With no wind, Glen's plane can fly at 50 kilometers per hour (kph) and Matt's plane at 60 kilometers per hour. Glen and Matt are planning a trip to an airport 150 kilometers away from their home. The time for the trip will depend on the wind speed. A headwind will make the trip longer; a tailwind will make the trip shorter. The graphs in Figure 1 on the next page show the relationship between wind speed and time for the trip for each flier and for different headwind or tailwind conditions.

1.(a) If Glen is flying with a headwind of 10 kph, how many kilometers is his plane able to cover in an hour?

1.(b) If Glen is flying with a tailwind of 20 kph, how many kilometers can his plane cover in an hour?

1.(c) How long will it take Glen to get to the airport with a tailwind of 25 kph?

1.(d) State how you calculated your answer for Question Ic.

1.(e) Circle the point on the graph of Figure 1 that shows the time that Glen takes to get to the airport when he is flying with a tailwind of 25 kph?

1.(f) In Figure 1, what information does point Q give?

1.(g) In Figure 1, for Graph B what is the speed of the wind when it takes 3 hours to get to the airport?

1.(h) As you look across Graph D, going from left to right, what information do you obtain from the graph?

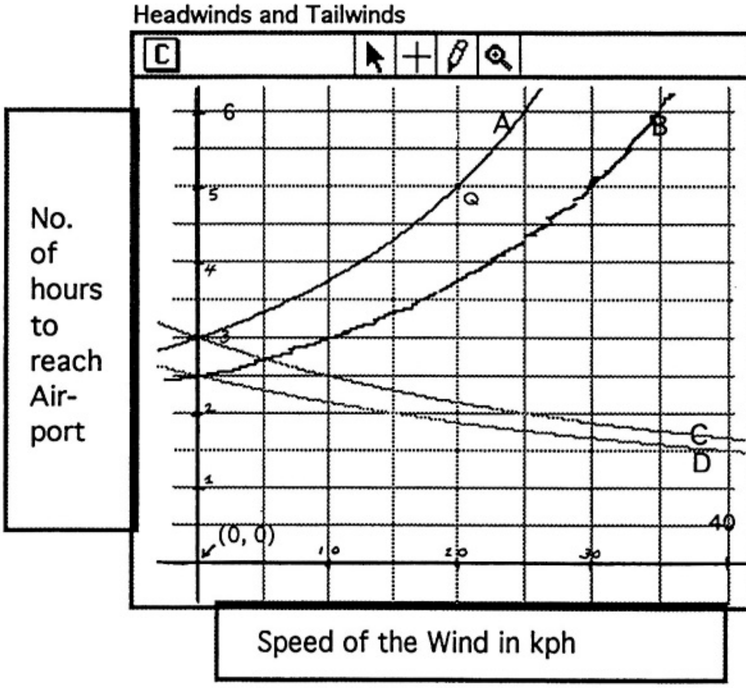


Figure 1.

IMPORTANT INFORMATION:

Speed of GLEN's plane with no wind:	50 kph
Speed of MATT's plane with no wind:	60 kph
Distance to airport:	150 km

2 Try to match each of the graphs shown in Figure 1 with the conditions under which Glen and Matt are flying, as described in the problem situation.

Glen and Tailwind	A	B	C	D	(circle one)
Glen and Headwind	A	B	C	D	(circle one)
Matt and Tailwind	A	B	C	D	(circle one)
Matt and Headwind	A	B	C	D	(circle one)

3 For which wind speed will Glen's time with a tailwind equal Matt's time with a headwind? And why?

4.(a) What explains why Graph A goes up?

4.(b) What explains why Graph C goes down?

4.(c) What distance would Matt be able to cover per hour if the speed of the headwind were 50 kph and how long would it take him to get to the airport?

4.(d) What would happen to Glen if the speed of the headwind were 50 kph?

5.(a) Nathalie also owns and flies a small plane. With no wind, her plane can fly at 30 kph. How many hours would it take her to fly to the airport that is 150 kilometers away from her home when there is no headwind or tailwind?

5.(b) Show the corresponding point in the coordinate system supplied in Figure 2.

5.(c) Suppose Nathalie were to fly daily to the airport 150 kilometers away with a tailwind. Without any wind, her plane flies at 30 kph. Knowing that the wind can vary from day to day, sketch a graph that shows the various times that Nathalie could take when she flies with a tailwind. Do your sketch in the coordinate system supplied in Figure 2.

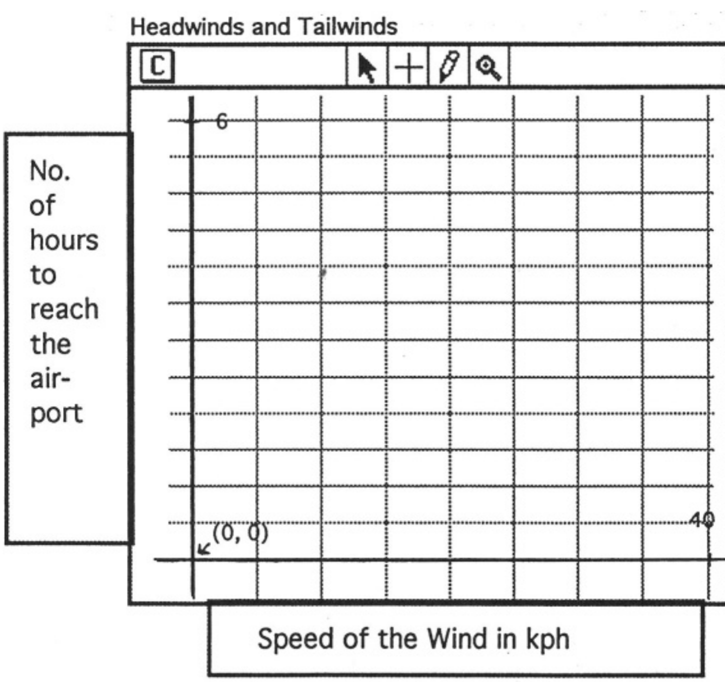


Figure 2.

6.(a) The number of hours that it takes Glen to get to the airport is a function of the speed of the wind. Here are three expressions where x represents the speed of the wind in kph. Your job is to find out which one best matches Graph A of Figure 1. Circle the one that you would choose.

$0.1x + 3$

$0.005x^2 + 3$

$\frac{150}{50 - x}$

6.(b) Give two reasons why you think it is the best one.

7.(a) Try to come up with the expression that will produce Graph B.

7.(b) Try to come up with the expression that will produce Graph D.

8.(a) Can you calculate the time for Nathalie's plane (her plane flies at a speed of 30 kph when there is no wind) flying a distance of 120 km with a headwind of 10 kph?

8.(b) Can you express what you just did with words only and no numbers?

8.(c) Write and expression for the Nathalie situation.

APPENDIX 2: THE *Susan* PROBLEM (QUESTION 3A OF THE INDIVIDUAL WORK)

Situation: Susan is training this spring for the long-distance marathon swim competition that will take place at the end of the summer. Her training consists of a daily swim up the Richelieu River, going against the current, for a distance of 20 kilometers. When there is no current, Susan swims at a speed of 4 kilometers per hour (kph). The speed of the current can vary from one day to the next. The time that Susan takes to do her swim of 20 kilometers depends on the speed of the current.

3.(a) Suppose Susan were to swim downstream the same distance of 20 kilometers, but this time going with the current. Remember that when there is no current, she swims at a speed of 4 kph. Knowing that the current can vary from day to day, sketch a graph that shows the number of hours Susan could take when she swims in the same direction as the current. Do your sketch in the coordinate system supplied in Figure 4.

APPENDIX 3: CONVENTIONS USED IN TRANSCRIBING VIDEOTAPES

[hour] Means that the word [hour] or chain of words was said by one speaker at the same time as something was said by another speaker. It always refers to the verbatim portion immediately above it in the previous verbatim line.

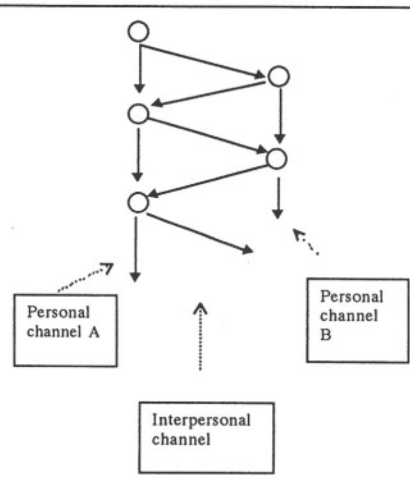
(?) Means that the transcriber could not catch a word or chain of words.

(ab ?) Means that the transcriber was not sure if the word in parentheses (ab ?) was the right word or chain of words.

eee Means that the speaker stuttered or that part of the word was accentuated.

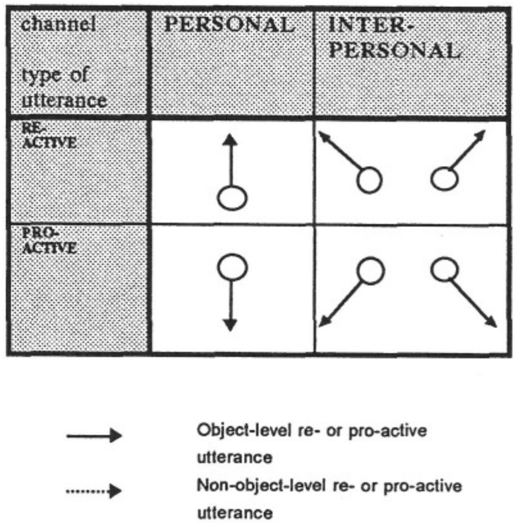
Means that a short hesitation occurred (number of dots is related to the length of the hesitation).

APPENDIX 4: REPRESENTING THE FLOW WITHIN AND BETWEEN PERSONAL AND INTERPERSONAL CHANNELS



The circles symbolize interlocutors' successive utterances. The solid arrows mark the flow of attention along both personal and interpersonal channels.

APPENDIX 5: INTERACTIVITY FLOWCHART SYMBOLS



REFERENCES

Bartolini Bussi, M.G. and Mariotti, M.A.: 1999, 'Semiotic mediation: from history to the mathematics classroom', *For the Learning of Mathematics* 19(2), 27–35.

Cole, M. and Wertsch, J.V.: n.d., *Beyond the individual-social antimony in discussions of Piaget and Vygotsky,* Web site: http://www.massey.ac.nz/~ALock/virtual/colevyg.htm.

Dembo, M.H. and McAuliffe, T.J.: 1987, 'Effects of perceived ability and grade status on social interaction and influence in cooperative groups', *Journal of Educational Psychology* 79, 415–423.

Edwards, D.: 1993, 'But what do children really think?: Discourse analysis and conceptual content in children's talk', *Cognition and Instruction* 11(3&4), 207–225.

Edwards, D. and Potter, J.: 1992, *Discursive Psychology,* Sage, London, UK.

Fey, J. and Heid, M.K.: 1991, *Computer-intensive algebra*, The University of Maryland and The Pennsylvania State University.

Forman, E.: 1989, 'The role of peer interaction in the social construction of mathematical knowledge', in N.M. Webb (ed.), *Peer Interaction, Problem-Solving, and Cognition: Multidisciplinary Perspectives,* Pergamon, Oxford, U.K., pp. 55–70.

Forman, E.A. and Cazden, C.B.: 1985, 'Exploring Vygotskian perspectives in education: The cognitive value of peer interaction', in J.V. Wertsch (ed.), *Culture, Communication, and Cognition: Vygotskian Perspectives,* Cambridge University Press, New York. pp. 323–347.

Fuson, K.C.: 1980, 'An explication of three theoretical constructs from Vygotsky', in T. Kieren (ed.), *Recent Research in Number Learning,* ERIC/SMEAC, Columbus, OH.

Glachan, M. and Light, P.: 1982, 'Peer interaction and learning: Can two wrongs make a right?', in G. Butterworth and P. Light (eds.), *Social Cognition: Studies in the Development of Understanding,* University of Chicago Press, Chicago, pp. 238–262.

Grice, H.P.: 1968, *Logic and Conversation: The William James Lectures,* Harvard University Press, Cambridge, MA.

Harré R. and Gillett, G.: 1994, *The Discursive Mind,* Sage, Thousand Oaks, CA.

Hatano, G. and Inagaki, K.: 1994, April, *A Two-Level Analysis of Collective Comprehension Activity,* Paper presented at the Symposium on Integrating the Cognitive and Social in the Construction of Mathematical and Scientific Knowledge, AERA meeting, New Orleans.

Hershkowitz, R. and Schwarz, B.: 1997, 'Unifying cognitive and sociocultural aspects in research on learning the function concept', in E. Pehkonen (ed.), *Proceedings of the 21st International Conference for the Psychology of Mathematics Education,* PME Program Committee, Lahti, Finland, Vol. 1, pp. 148–164.

Kieran, C.: 1994, 'A functional approach to the introduction of algebra: Some pros and cons', in J.P. da Ponte and J.F. Matos (eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education,* PME Program Committee, Lisbon, Portugal, Vol. 1, pp. 157–175.

Kieran, C. and Dreyfus, T.: 1998, 'Collaborative versus individual problem solving: Entering another's universe of thought', in A. Olivier and K. Newstead (eds.), *Proceedings of the 22nd International Conference for the Psychology of Mathematics Education,* PME Program Committee, Stellenbosch, South Africa, Vol. 3, pp. 112–119.

Kieran, C. and Sfard, A.: 1999, 'Seeing through symbols: The case of equivalent expressions', *Focus on Learning Mathematics* 21(1), 1–17.

Leikin, R. and Zaslavsky, O.: 1997, 'Facilitating student interactions in mathematics in a cooperative learning setting', *Journal for Research in Mathematics Education* 28, 331–354.

Leinhardt, G., Zaslavsky, O. and Stein, M.K.: 1990, 'Functions, graphs, and graphing: Tasks, learning, and teaching', *Review of Educational Research* 60, 1–64.

Lerman, S.: 1998, 'A moment in the zoom of a lens: Toward a discursive psychology of mathematics teaching and learning', in A. Olivier and K. Newstead (eds.), *Proceedings of the 22nd International Conference for the Psychology of Mathematics Education,* PME Program Committee, Stellenbosch, South Africa, Vol. 1, pp. 66–81.

Linn, M.C. and Burbules, N.C.: 1993, 'Construction of knowledge and group learning', in K. Tobin (ed.), *The Practice of Constructivism in Science Education,* American Association for the Advancement of Science, Washington, DC, pp. 91–119.

Lotman, Y.M.: 1988, 'Text within a text', *Soviet Psychology* 26(3), 32–51.

Meira, L. and Carraher, D., (eds.): 1995, *Proceedings of the 19th International Conference for the Psychology of Mathematics Education,* PME Program Committee, Recife, Brazil.

Noss, R. and Hoyles, C.: 1996, *Windows on Mathematical Meanings: Learning Cultures and Computers,* Kluwer Academic Publishers, Dordrecht, The Netherlands.

O'Connor, M.C.: 1996, 'Managing the intermental: Classroom group discussion and the social context of learning', in D.I. Slobin, J. Gerhardt, A. Kyratzis and J. Guo (eds.), *Social Interaction, Social Context, and Language,* Erlbaum, Mahwah, NJ, pp. 495–509.

Reddy, M.J.: 1979, 'The conduit metaphor: A case of frame conflict in our language about language', in A. Ortony (ed.), *Metaphor and Thought,* Cambridge University Press, Cambridge, pp. 282–324.

Rosenberg, J.: 1992, *Math Connections: Algebra II* [Computer software], WINGS for Learning, Inc., Scotts Valley, CA.

Salomon, G. and Globerson, T.: 1989, 'When teams do not function the way they ought to', in N.M. Webb (ed.), *Peer Interaction, Problem-Solving, and Cognition: Multidisciplinary Perspectives,* Pergamon Press, Oxford, UK, pp. 89–99.

Sfard, A.: 2000, 'Steering (dis)course between metaphors and rigor: Using focal analysis to investigate an emergence of mathematical objects', *Journal for Research in Mathematics Education* 31, 296–327.

Sfard, A. and Kieran, C.: 2001, 'Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions', *Mind, Culture, and Activity* 8(1), 42–76.

Shotter, J. and Billig, M.: 1998, 'A Bakhtinian psychology: From out of the heads of individuals and into the dialogues between them', in M.M. Bell and M. Gardiner (eds.), *Bakhtin and the Human Sciences,* Sage, Thousand Oaks, CA, pp. 13–29.

Streefland, L., (ed.): 1985, *Proceedings of the 9th International Conference for the Psychology of Mathematics Education,* PME Program Committee, Noordwijkerhout, The Netherlands.

Teasley, S.D.: 1995, 'The role of talk in children's peer collaborations', *Developmental Psychology* 31(2), 207–220.

Trognon, A.: 1993, 'How does the process of interaction work when two interlocutors try to resolve a logical problem?', *Cognition and Instruction* 11(3&4), 325–345.

Vion, R.: 1999, 'Linguistique et communication verbale', in M. Gilly, J.-P. Roux and A. Trognon (eds.), *Apprendre dans l'Interaction: Analyse des Médiations Sémiotiques,* Presses universitaires de Nancy, Nancy, FR, pp. 41–67.

Vygotsky, L.S.: 1981, 'The instrumental method in psychology', in J.V. Wertsch (ed.), *The Concept of Activity in Soviet Psychology,* M.E. Sharpe, Armonk, NY, pp. 134–143.

Vygotsky, L.S.: 1987, 'Thinking and speech', in R.W. Rieber and A.S. Carton (eds.), *The Collected Works of L.S. Vygotsky: Vol. 1, Problems of General Psychology* (trans. N. Minick), Plenum, NY, pp. 39–285.

Webb, N.M.: 1991, 'Task-related verbal interaction and mathematics learning in small groups', *Journal for Research in Mathematics Education* 22, 366–389.

Wertsch, J.V., (ed.): 1985, *Culture, Communication, and Cognition: Vygotskian Perspectives,* Cambridge University Press, Cambridge, UK.

Wertsch, J.V.: 1998, *Mind as Action,* Oxford University Press, New York.

Wittgenstein, L.: 1953, *Philosophical Investigations* (trans. G.E.M. Anscombe), Blackwell, Oxford, UK.

Département de Mathématiques,
Université du Québec à Montréal,
C.P. 8888, succursale Centre-Ville,
Montréal, Québec,
Canada, H3C 3P8
Telephone (office): 514-987-3000, extension 7793#
Fax: 514-987-8935
E-mail: kieran.carolyn@uqam.ca

VICKI ZACK[1] and BARBARA GRAVES[2]

MAKING MATHEMATICAL MEANING THROUGH DIALOGUE: "ONCE YOU THINK OF IT, THE Z MINUS THREE SEEMS PRETTY WEIRD"

Thinking embedded in collaborative practical activity must to a significant degree take the form of talk, gesture, use of artifacts, or some other publicly accessible instrumentality; otherwise mutual formation of ideas would be rendered impossible. Collaborative thinking opens up access to direct data on thought processes.

(Engeström, 1994, cited in John-Steiner and Meehan, 2000, pp. 44–45)

ABSTRACT. In our ongoing qualitative classroom research, we adopt a sociocultural perspective to investigate discourse, and its role in how children and teachers make meaning of mathematics in a fifth grade inquiry classroom. Our theoretical perspective draws primarily on Vygotsky (1978, 1986) and Bakhtin (1981, 1986) each of whom examines how social forms of meaning influence individual cognition. The episode described in this paper examines the process whereby individual and group developmental trajectories are constructed, and allows us to explore the relationship between discourse and knowing. We combine a longitudinal design with a case study approach to focus on the collaborative mathematical problem solving. We use video capture to help us listen to children's discussions in classroom activities and small group interactions. Our analysis of the verbal data recorded on video identifies patterns of interaction, development and change in participants' use of mathematical language and concepts, and their evolving understanding, through discussion and argument, of an algebraic expression constructed by one of the children. The findings lead us to argue for i) a more generative view of the zone of proximal development as a site of learning and of identity formation, ii) an expanded view of the role of the teacher in inquiry classrooms, and iii) an appreciation for the valuable roles difference and resistance play in knowledge building.

KEY WORDS: algebraic thinking, Bakhtin, collaborative problem solving, dialogic inquiry, Vygotsky, ZPD

1. INTRODUCTION

In our ongoing qualitative classroom research, we adopt a sociocultural perspective to investigate discourse, and its role in how children and teachers make meaning of mathematics. In this paper we focus on how three fifth-grade children construct themselves as participants in mathematical inquiry through their discourse in joint problem solving activities. Specifically we examine their adopted stances during problem solving, their

Educational Studies in Mathematics **46:** 229–271, 2001.

degree of participation in the discussion, and the ways in which that participation influenced the interaction and their roles and their identities as mathematics participants. This approach will lead us to argue for a more generative view of the zone of proximal development as a site of identity formation, an expanded view of the role of the teacher in inquiry classrooms, and an appreciation for the valuable roles difference and resistance play in knowledge building.

Our theoretical perspective draws primarily on Vygotsky (1978, 1986) and Bakhtin (1981, 1986), each of whom examines how social forms of meaning influence individual cognition. We are particularly interested in the process of meaning construction referred to as internalization understood both as mastery of cultural tools and as appropriation (Wertsch, 1998). On this latter point we are especially interested in how the ideas of others become one's own (Bakhtin, 1986; Vygotsky, 1978, 1986). Therefore, we examine the problem-solving talk of learners while they interact in joint mathematical activity in order to understand how knowledge can be created and appropriated in and through participation in the interaction.

These trends have been reflected in the reform-oriented initiatives in mathematics envisioned in the *Standards* (NCTM, 1989, 1991) in the United States, and the reform-oriented curricula in Canada such as the *Défi mathématique/Challenging Mathematics* (Lyons and Lyons, 1991, 1996) program in Quebec. The *Standards* have highlighted the role of argument and debate, justification and explanation, and have led to an increasing interest in examining the discursive practices in these learning communities. To understand the important role of language in learning, a growing number of research studies have begun to investigate the discursive practices which occur in mathematics and science classrooms (Ball, 1991; Cobb, Gravemeijer, Yackel, McClain, and Whitenack, 1997; Forman and Larreamendy-Joerns, 1995; Halliday and Martin, 1993; Lampert and Blunk, 1998; Lemke, 1991; Pimm, 1987, 1995; Roth, 1995; Walkerdine, 1990). As a result much of the data of mathematics education research are verbal data including transcripts of classroom discourse and small group discussions, students' written work, and explanations. In our own work, we combine longitudinal designs with a case study approach to focus on a particular learning community. We use video capture to help us listen to children as we study classroom activities and small group interactions. We analyze verbal data captured in video to describe patterns of interaction, development and change in students' use of mathematical language and concepts, and their evolving understanding through discussion and argument.

2. THEORETICAL FRAMEWORK

According to Vygotsky (1978, 1986), learners first construct knowledge in their interactions with people and activity contexts. From this perspective, knowledge and learning are considered to be social activities which are mediated by cultural artifacts and resources both symbolic (e.g. language, numeracy systems) and material (e.g. computers). While Vygotsky (1986) writes of mediational means including both material and symbolic resources, he focused much of his empirical research on the examination of the role of language as a central mechanism of learning. Over time and with repeated experiences with others in social interaction the learning becomes internalized by the individual and the social becomes the psychological. This is a dialectical process which "is not the transferal of an external activity to a preexisting, internal 'plane of consciousness': it is the process in which this plane is formed" (Leont'ev, 1981, p. 57).

A similar understanding can be found in the research on the socializing functions of language (Gee, 1992; Hicks, 1995; Maguire and Graves, in press; Ochs, 1993) which view social activities and related discourses as mutually constitutive phenomena that reflect and mediate one another. Also in philosophy, discussing the dialogic nature of 'personal' identity, Taylor (1991) suggests that human beings are constituted in conversation and what gets internalized in the adult is the conversation in its entirety with the inter-animation of the many participating voices (pp. 313–314). We similarly understand that the act of achieving an individual voice is accomplished by means of active dialogue with the discourses of one's social surrounding. To understand this position more fully we also draw on Bakhtin's theory of discourse and the self.

Bakhtin, analogous to Vygotsky, wrote about the individual's appropriation of language experienced in a world mediated by social texts. Bakhtin's theory, which examines how the self is formed in dialogic response to a social world, bridges the divide between the individual and the social self. Bakhtin maintained that all spoken or written language is dialogical since it is always addressed to someone. In addition, it is always delivered from a particular viewpoint. Given these properties, language constructs role-relationships between speakers (or readers) with varying conditions of privilege, formality, and authority. The dialogic self according to Bakhtin forms an active response to the discourses of an individual's social, historical and temporal worlds.

> The word in language is half someone else's. It becomes one's own only when the speaker populates it with his own intention, his own accent, when he appropriates the word adapting it to his own semantic and expressive intention. Prior to this moment of appropriation, the word does not exist in a neutral and impersonal lan-

guage . . . but rather it exists in other people's mouths, in other people's concrete contexts, serving other people's intentions: it is from there that one must take the word and make it one's own. (Bakhtin, 1981, p. 293–294)

A Bakhtinian approach suggests that the words of others carry with them their own expression, their own evaluative tones which in turn other speakers appropriate, rework and re-accentuate (1986, p. 89). Something very special occurs when one human voice addresses another which is that dialogical events always give rise to something unique.

An utterance is never just a reflection or an expression of something existing and outside it that is given and final. It always creates something that never existed before, something absolutely new and unrepeatable.
(Bakhtin, 1986, cited in Shotter and Billig, 1998, p. 13)

3. Transformative Learning in the ZPD

The zone of proximal development (ZPD) as put forth by Vygotsky has been the focus of much attention and is broadly understood as a *potential for learning* that is created in the interaction. While there has been enormous interest in this concept, there have also been some shortcomings with regard to how it is interpreted. Vygotsky's (1978) initial interest in defining the zone of proximal development was in the context of individual assessment. In Vygotsky's theory of human development, complex psychological actions such as reasoning and memory were seen to develop first in social interaction with others and only later to become part of an individual's psychology. Thus he believed that assessing a learner in interaction with a more knowledgeable other would provide more meaningful and accurate information about the learner's development. His original formulation of the ZPD was

the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.
(1978, p. 86)

By and large in the educational literature, this has led to the depiction of the ZPD as an interactional space delimited and overseen by the adult (teacher, parent). Lerman has said that the ZPD "is often described as a kind of force field which the child carries around, whose dimensions must be determined by the teacher so that activities offered are within the child's range" (Lerman, 1998, p. 71). A recent review of the extensive literature on the ZPD (Goldstein, 1999) indicates the prevalence of this view and as well the prevalence of two metaphors in the literature, namely that of the

ZPD as a ‘construction zone’ (Newman, Griffin, and Cole, 1989), and the ‘scaffolding’ metaphor (Wood, Bruner, and Ross, 1976). The limitations of the scaffolding metaphor were identified very early by Harste (Harste, Woodward, and Burke, 1984) who critiqued the ‘stimulus-response’ and ‘transmission model’ aspects. While the metaphor of the ZPD as a construction zone can be usefully applied to the classroom setting, the role of the teacher is by and large defined as very directive. Further, it is often the perception that although teachers do not emerge from their encounters unchanged, they “do not need to stretch their cognitive faculties in the same way as their students must during teaching-learning interactions” (Goldstein, 1999, p. 663).

In contrast to these views, Wells (1999, 2000) provides a detailed treatment of the transformative possibilities of the zone of proximal development based on his theoretical perspectives and classroom data from inquiry classrooms. He suggests that transformation occurs as a function of participating in meaningful activities, and that the sources of guidance do not come only from other participants but all manner of mediational means. Participation involves all aspects of the participants, their feelings and beliefs, and therefore transforms identity. Because the individual and the social worlds are mutually constitutive of each other, transformation of the learner also involves transformation of the communities and of the joint activities. While activities are situated in time and place and while there may be shared features across activities and settings, each activity is nonetheless unique since it involves the coming together of particular individuals in a particular setting with particular artifacts all of which have their own histories which in turn affect the way the activity is played out (Wells, 1999 p. 331). Consequently, outcomes of an activity cannot be completely specified in advance nor can we predict with any certainty the upper limits of knowing as this ultimately depends on the characteristics of the learner as well as on the characteristics of the interaction.

3.1. *Re-thinking the role of the teacher*

In the changing ecologies of reform-oriented inquiry classrooms such as is the case in our setting, the role of the teacher needs to be understood as highly creative, very flexible, and contingent, and as a much greater challenge than what has traditionally been espoused. We want to re-conceptualize the ZPD as an intellectual space, created in the moment as a result of the interaction of specific participants engaged with each other at that specific point in time. In this intellectual space it is not only children who learn but teachers as well, as their knowledge and roles become transformed in the process of interaction. The new reform paradigm suggests there is need for invention rather than implementation on the teacher's part and teachers who are "themselves products of the very system they now aim to change" are also revisiting and reconstructing their understandings of mathematics (Ball, 1993, pp. xi-xii). The teacher's role includes helping to develop "a social community ... that problematizes mathematics and shares in searching for solutions" (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier, and Wearne, 1996, p. 16). As well she establishes with the children both the social and sociomathematical norms (Cobb, Wood, and Yackel, 1993; Yackel and Cobb, 1996). It is vital for both teachers and peers to allow room for the children to grow, feel secure and able to explore. The climate for exploring must be safe and accepting, before the children will venture to ask for or receive help or explanations (Zack, 1993).

While the teacher's role has moved from that of provider of knowledge to a knowledgeable orchestrator, in our view this formulation does not go far enough. In fact we believe it limits the effective participation of the teacher since it does not highlight genuine inquiry as a mark of the professional teacher's involvement in the classroom (Roth, 1995; Wells, 1999). We have experienced that in open-ended project situations, in certain instances the teacher learns from the students, and learns as much as do the students (Zack, 1996). Often when we present at conferences a common response to our data is that they are often viewed simply as evidence of a talented and experienced teacher. While we are prepared to acknowledge that the successful practice we describe in her classroom is indeed very much related to Vicki's abilities as a teacher, we also want to argue that much of her 'expertise' is in her role as earnest listener (Davis, 1996) and co-learner. Writing about how the children have changed her understanding in fundamental ways, Vicki has clearly shown the children that she too is an active searcher and inquirer in mathematics (Zack, 1995, 1996, 1997). It is our contention that this type of teaching, this open-ness to inquiry, once it has been understood, can be appropriated.

3.2. *Role of difference: Disagreement, misunderstanding, and doubt*

Our work with children learning in interactional contexts has led us to consider how differences, including disagreements and misunderstandings, play an important part in generating knowledge, and we have documented how resistance and rhetorical opposition can result in mastery and appropriation of cultural tools (Graves, 1999; Zack, 1997, 1999). To theoretically frame our discussion of the role of misunderstanding and disagreement on children's knowledge in the complex dynamic of collaborative mathematical discussion, we refer readers to Wertsch's (1998) discussion of Bakhtin's idea of alterity. Bakhtin (1981) sees communication in continual tension between contrary forces, especially between forces which push towards unity, agreement and a single voice, and those forces which permit multiplicity, disagreement and multiple voices. While these two tendencies are always both present at least to some degree in social interaction, it is only when we consider the dialogic function of language which permits disagreement and multiple voices that we can begin to understand the ways in which difference may serve as a thinking device. Similarly, Dewey (1933) speaks of a state of doubt serving as the impetus to an act of searching and inquiry. Therefore in our understanding of intersubjectivity and its importance for the joint construction of knowledge we also need to include various forms of resistance and difference (Matusov, 1996; Shotter and Billig, 1998).

4. Situating the Study

4.1. *The school community and classroom setting: An ecological perspective*

In speaking about classrooms and school communities, it is becoming familiar to hear them referred to as cultures (Brilliant-Mills, 1993; Gee, 1996; Green and Dixon, 1993), communities of learners (Brown, 1997; Lave and Wenger, 1991), or ecosocial systems (Lemke, 1997). We draw on these formulations to emphasize the dynamic nature of learning communities. In any year the culture of a given classroom is unique while at the same time recognizable with respect to numerous familiar and distinguishing characteristics. In this study the school and classroom site constitute a community of learners which views learning as occurring in the process of people problem solving together in the course of shared activity.

The particular school, St. George's, is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds. The school population is pre-dominantly English-speaking, and it is likely that a number of the children would engage at home in ways of speaking which are school-like. The school was established in 1930 at which time the founders drew on the philosophy of Dewey (1900, 1933). This school has always been part of a progressive tradition in education which currently finds expression in its constructivist educational philosophy. In keeping with this constructivist orientation, the students have been tackling non-routine problems in diverse areas of the curriculum since their entry to the school, hence six years for many. As part of their participation in such a learning environment the children are expected to publicly express their thinking, and engage in intellectual practice characterized by conjecture, argument, and justification (Cobb, Wood, and Yackel, 1993). This approach pertains to many other subjects in the school curriculum such as environmental-studies/science, social studies, and literature.

The particular classroom, Vicki Zack's fifth grade classroom, is a reform-oriented classroom which has evolved over the years. There is a strong emphasis on collaborative discussion as a critical activity to promote learning in all areas. To implement collaborative discussion as a classroom reality, however, it has been necessary to socialize children into this interactive practice. While this particular dialogic activity bears some resemblance to the overall constructivist philosophy of the school, it is at the same time unique to this classroom teacher. This is most evident with respect to talking and arguing mathematics since mathematics is an institutionalized practice with its own assumptions about what it means to 'do math.' Therefore to implement a dialogic model in learning mathematics, it was important to develop and implement certain social and mathematical norms to encourage joint mathematical activity. The following norms were in place at the time of this study:

The students are expected to explain and justify their reasoning. Within that context students are expected to listen to and attempt to understand others' explanations.

Students are expected to communicate in ways which are respectful, kind, and which expand possibilities for discussion and learning.

The students are expected to credit others whose ideas they have found helpful.

This last point is particularly important as in this classroom culture the teacher explicitly develops students' awareness of and practice in acknowledging the ways in which others have contributed to their understanding of a problem. At the end of each full class discussion, students are required to complete an activity sheet citing those who have contributed to their thinking.

Mathematics classes meet daily, five times per week. Two of the sessions are 90 minutes in duration, and three are 45 minutes long. While the total class size is approximately 25 in any given year, the study of mathematics is always done in half-groups. This means that there are two mathematics sections, each consisting of 12 or 13 students of heterogeneous ability, and each of the group-of-four working teams, teacher-assigned, is heterogeneous in ability. While one group works with the teacher doing mathematics in their homeroom, the other group works elsewhere in the school with another staff member, pursuing other curriculum subjects (music, French, art, physical education). Non-routine problem-solving is at the core of the mathematics curriculum in this classroom. The students use diverse approaches. The teacher works to build upon the children's inventions, both to help them see connections between the diverse often non-standard approaches and to connect the children's ways of doing with the conventions of the culture (cf., Lampert, 1990). Throughout the week the children engage in problem-solving sessions in class which require small group discussions of cognitively challenging problems. In addition, each week the children also work on one non-routine task which requires a full written explanation; it is called the *Problem of the Week*. They first tackle this problem alone and record a description of their activity in their Math Logs. This permits the teacher to see what the child can do independently, via the child's detailed written explanation, and the child's own representations by means of drawings, tables, algorithms, arithmetic expressions, and so on. The teacher then has an indication of each child's individual understanding before engaging with other children to explore the problem further. This structure has the potential to allow the teacher to see what develops as a result of the interaction. Theoretically this corresponds with Vygotsky's view (1978) that in order to observe a child's level of potential development, we need to examine their understanding in the course of working with others. The zone is defined as the difference between a child's performance in two situations, without assistance first, then with assistance.

The Math Log serves as the initial basis of the children's group discussions which are conducted mainly by the children themselves. The teacher takes note of the individual group trajectories which emerge, and looks to see how the children appropriate ideas from others and how they use them in ways which are meaningful to them. Children first work independently, and then they discuss their work with a partner. Following their discussions in groups of two, they then meet in groups of four/five, and finally with the large group of twelve/thirteen. With this structure each problem is examined on four separate occasions and in multiple contexts. In all cases the teacher's role is to offer rich mathematical tasks, see how the trajectories unfold, and to build upon the students' ideas. During the small group discussions, the teacher observes one pair, then that pair's group of four, throughout their interaction, intervening rarely. At the culmination of the students' discussion in a group of twelve/thirteen, the teacher highlights salient ideas which have been put forth and introduces other relevant ideas appropriated from cohorts of previous years.

4.2. *The data*

We work from video capture and extensive verbal transcriptions of selected episodes. The children are videotaped throughout the school year on a rotating basis as they work in their groups; two video cameras are in use. All of the group interactions, pairs, groups of four/five and large discussion groups of twelve/thirteen are video-taped with the children seated at tables or sometimes on the floor. Each group has a camera focused on it and a flat table mike so that it is possible to hear what the members of the groups are saying. For a selected series of four tasks and their extensions, six cameras are in use so that the work of each pair and its group of four/five can be recorded. One activity in which the teacher engages during the video taping sessions, is to select one pair of children and follow their discussion as a group of two and then continue listening and taking notes as they meet with another two or three children to continue the discussion. In this way she is present to join a discussion when it seems appropriate, to make suggestions for further reflection or to request a child to clarify or re-state her thinking. In the data we present in this paper, the teacher was not monitoring the episode and so we will see no intervention on her part.

Along with the videotape records, data sources include focused observations, teacher-composed questions eliciting opinions orally and in writing, retrospective interviews, and student artifacts such as written Math Logs. Beginning with the Math Logs, we can track how much the children manage while working on the problem on their own, and from the video recordings we can examine how their thinking and knowledge then evolve in the context of the multiple group interactions. In particular, the children's written texts convey valuable information in terms of handwriting, layout, and accompanying drawings and illustrations which can be very important for interpreting the meaning of verbal text and need to be considered in any analysis of their talk. Since verbal data make sense only in relation to their multiple contexts, we also rely on specific contextual knowledge of the activity along with detailed background knowledge of the participants. The children's talk also needs to be understood in relation to the past history of their dialogues with the teacher and classmates, the group dynamics of the class, and in relation to their interpersonal skills.

4.3. *The problem-solving episode*

The mathematics episode examined in this paper focuses on a problem-solving interaction among three fifth-grade boys, Hosni, Jeff, and Micky. Since our selection and interpretation of the verbal data in part determines the content of those data, we would like to comment on how we select episodes for detailed analysis from the many hours of video capture. There are two sources which contribute to our selection of episodes: practical experience and theoretical interest. Over the years a series of challenging mathematics problems has emerged which provide rich math-

TABLE I

Frequency of speaker turns for each part of the conversational exchange (May 1994)

Speaker	Part 1-May 20 3:44 min	Part 2-May 20 7:08 min	Part 3-May 20 7:71 min	Part 4-May 24 11:22 min
Micky	36	56	83	46
Jeff	13	67	78	47
Hosni	37	24	5	absent from group

ematical content for the children to think about. Since we have been able to collect data on the responses of many groups over successive years (1990–2001), we are able to see repeating patterns among the learners as well as novel approaches. At the same time we are interested in examining how successfully various tenets of sociocultural theory explain our data. The present episode was selected because it revealed a range of mathematics behaviours, some familiar and some quite novel. We present the episode as a four-part structure which emerged as a function of the mathematical activities undertaken by the participants, and as a function of their differing roles and stances during the mathematical interaction. Table 1 shows the number of speaker turns for the participants in each of the four parts of the episode.

The mathematical activity. In our work on collaborative problem solving, it has been important to distinguish the external task which is the task assigned by the teacher, from the internal tasks which refer to problems/tasks constructed by the students within the context of the problem-solving activity (Coughlan and Duff, 1994; Graves, 1996). In the first two portions of the episode, the children focus on solving the external task, while in parts three and four they are engaged in self-directed inquiry. More specifically, in part one, to complete the assignment set by the teacher, Micky, Hosni, and Jeff check answers, confirm ways of doing, and pinpoint errors in regard to mathematical calculations. This is a very traditional interaction on the part of the children. In part two, Jeff explains his way of solving the problem and while his solution is a divergent and challenging one, the activity here is still one of solving the teacher-generated task. In part three, Micky and Jeff embark on a journey of genuine inquiry. Finally, in part four, Micky and Jeff build upon and transform their ideas, as they construct new knowledge and a deeper understanding.

Participants' roles and stances. Our representation of the episode as a four-part structure is also inferred from the roles and stances of the children. In part one, while Hosni and Micky carry much of the discussion, in that interaction Hosni assumes the role of explainer in the context of comparing answers and strategies; in part two, the roles switch and Jeff assumes the role of explainer while Micky and Hosni listen and try to follow; in part three Micky assumes the role of questioner and challenger; and finally in part four, Micky and Jeff in intense discussion adopt roles which suggest collaborative inquirers as they carry out their joint activity.

Representing conversation data

While most research accounts of analyzing video data begin with a statement informing us that 'the video data were transcribed' (the easy part) and 'then analyzed' (the difficult part), we have come to understand that even the very 'precise' transcriptions of verbal speech data into written text constitute a complex interpretive activity (Mishler, 1991). The process of transcription creates a new text whose relations to the original data are variable. Just changing from speech to writing alters our expectations and perceptions of language (Mishler, 1991; Ochs, 1979). Transcripts of actual conversation often strike the external viewer as incomplete, difficult to follow, and even incoherent. This experience certainly reflects the ways in which the written transcript is woefully inadequate as a representation of the original activity. The inadequacies stem from the fact that a written transcript is a trace only of the language, and it is a language which is disembodied from so many of the essential elements which combine to make talk coherent and meaningful.

In the case of video data, the transcribed text needs to be analyzed in conjunction with both the video images and the spoken language to permit aspects such as gesture and intonation to contribute to the interpretation of the activity. The transcripts play the important role of 'slowing down' the talk and allowing for detailed analysis of the constituent parts. However, since meaning is not made with language alone, we need to return repeatedly to the videotapes to have a more complete sense of the meanings, intentions and nuances expressed with the assistance of gesture, tone, facial expression, and body movement.

Acknowledging these constraints, we have done our best to select verbal excerpts which best illustrate our argument and adequately serve as evidence for our conclusions. The excerpts were chosen so that the reader can focus on the development of the argument itself and the evolving understanding of the students. Many excerpts contain extended stretches of dialogue and have been included to provide the reader with a sense of how ideas and positions develop over time. While we see a great deal of repetition in the talk, which to the reader may appear 'boring' or simply repetitive, in the original interactions these conversations are intense, engaging, and compelling as the children's talk invites us to eavesdrop on their thinking.

Given the many interpretive possibilities of language we rely on multiple forms of evidence to support the validity of our inferences. We try to be as explicit as possible about the process by which we make inferences from the data and select from a range of useful and principled discourse analytic techniques depending on our objectives. Our research on children's reasoning (Graves and Zack, 1996; 1997), on argument structure (Zack, 1999), and on the generative role of misunderstanding (Graves, 1999) has drawn on a range of models of discourse analysis. We have applied semantic discourse analysis (Bracewell and Breuleux, 1994; Frederiksen, 1986; Lemke, 1991) to capture subject-matter knowledge, to identify important events and themes as well as the roles of the participants and their goals. For communicative interactions we have turned to sociolinguistic approaches including conversational analysis (Gumperz and Field, 1995; Smithson and Díaz, 1996). Finally, we have applied a model of informal reasoning (Graves, 1997; Toulmin, 1995) to capture the reasoning structure of participants' explanations.

Our focus in this paper is on the discursive construction of mathematical knowing and identity in the context of mathematical activity, as well as the dialogic relationship of a speaker to his own discourse and the discourse of others. As part of our transcription notation, we use forward slashes (/ /) to indicate points of overlap in speakers' talk to draw attention to the amount of idea completion, repetition and agreement contributed by various participants. In addition we use the equal symbol (=) to identify the immediate uptake or continuation of ideas. Words enclosed in double parenthesis (( )) indicate uncertainty regarding the transcribed portion.

5. THE MATHEMATICAL DISCUSSION

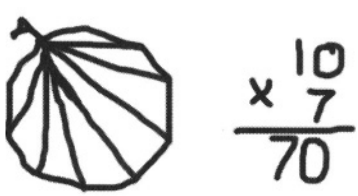
5.1. *The context of the discussion*

To situate the boys' discussion, we will first describe the larger mathematical activities of which they were a part. In the period between February and May, the children were presented with a sequence of four inter-related problems which built on each other with increasing complexity. The fourth problem, in particular, requires integrating information from the previous three in order to construct a general explanatory rule. Below is the sequence of problems:

1. **Tunnels** (a variant of the *Handshake* problem): Nine prairie dogs need to connect all their burrows to one another in order to be sure that they can evade their enemy, the ferret. How many tunnels do they need to build?
2. **Decagon Diagonals:** How many diagonal lines can be drawn inside a figure with 10 sides?
3. **25-Sided-, 52-Sided Polygons:** How many diagonals would there be in a 25-sided polygon? in a 52-sided polygon?
4. **Tunnels revisited:** Can you write a number sentence or general rule for the *Tunnels* problem?

While the focus of our discussion in this paper is on the ways in which Hosni, Jeff, and Micky engaged with problems 3, and 4, we will begin by providing some history on problem 2 as well as on how problem 4 came into being. Prior to problems 3 and 4, the children had been working on problem 2, *Decagon Diagonals,* which asks, *How many diagonal lines can be drawn inside a figure with 10 sides?* In the course of that activity, two children, each from a different mathematics section, independently identified similar key ideas which led to the construction of an algebraic expression. In one section, Cathy used a drawing and number sentence to elaborate her idea, while in the other section, Jerome expressed the same idea in words.

Cathy's way



Then 70 ÷ 2 = 35

Jerome's way

During the large group discussion of problem 2, Jerome had been looking intently at his Mathematics Log, when he said:

Oh, I just saw another pattern. Well, every time … what happens is … if you see like the five has two from each one, well from two, from two times five it's ten, but this is actually five so it's half of ten- and then over here it is seventy, but it's only 35 ... so it's always half.

In both cases the children create the same procedure to determine the number of diagonals in a decagon which is to count the number of diagonals emanating from a vertex, multiply that number by the number of sides, and divide by two. Jeff, Hosni and Micky all belong to the same mathematics cohort as Jerome and in the course of the large group discussion, Jeff expressed Jerome's words using an algebraic expression which he wrote on the chalkboard:

A * S ÷ 2 = # of diagonals (where **A** equals sides minus three, and **S** equals sides)

When others began referring to it as 'Jeff's way,' Jeff stated that it ought to be called 'Jerome-Jeff's way' since his algebraic expression was based on Jerome's idea. Indeed, Jeff credited Jerome later when he wrote, "[Jerome's] explanation helped by explaining better the pattern I saw but could not put my finger on."

Problem 4 was added to the sequence by the teacher following a discussion with Jeff in which he indicated that the *Tunnels* problem was like the *Decagon Diagonals* problem plus the addition of the sides. His ability to connect these problems in this manner impressed Vicki, and she decided to build a class activity on Jeff's connection. Consequently, she formulated an extension problem for the entire class, *Tunnels revisited,* which asks: Can you write a number sentence or rule for the *Tunnels* problem?

5.2. *PART 1: Hosni as explainer: A traditional interaction*

The mathematics episode begins with a discussion of problem 3 which asked:

How many diagonals would there be in a 25-sided polygon? in a 52-sided polygon?

Hosni opens the conversation by requesting permission from the other two to present his solution.

H: Okay, can I go first?

M: Okay.

In doing so he assumes a leadership role which he maintains throughout this part of the episode. The conversational turns are evenly distributed between Hosni and Micky, with Jeff contributing less often. Hosni assumes the role of explainer, Micky as responsive listener, while Jeff acts more as an observer and commentator.

H: Okay, well basically what I did is the same thing as Jeff

Having linked his solution to that of Jeff, Hosni goes on to explain how he made a drawing to assist him in solving the 25-sided figure. In his description of the procedure below he includes his realization that the problem can be solved without the aid of a drawing and the other boys concur.

H: Yeah, okay. So I just did them all. I did that one. I attached them with each one. Then I got twenty-two. How I realized that makes sense is you do the one that's here attaching everyone else with it, the one next to it, and the other one's side=

M: =Cannot be attached=

H: =Cannot be attached. Exactly

J: So it's always /subtract by three/

M: /Minus three/

H: /Subtract by three/

At this point the overlapping text and the similar content reveals the boys to be in agreement about this important idea. This discussion continues:

H: Yeah I figured that out after. So what I did was twenty-five minus three, and then I got twenty-two, that's how I solved here. So, well, uh what I did was did twenty-five times twenty-two.

M: /Twenty-five times twenty-two yeah/

H: /Twenty-five times twenty-two/ and then I got=

M: =You should've got uh, well-

H: Then I got five hundred and fifty, and then I did five hundred and fifty divided by two=

M: =And you got two hundred and twenty-five=

H: =Two hundred and seventy-five

M: Two twenty five

H: Two seventy five

Micky agrees with Hosni up until the point of the final calculation as Micky has made an arithmetic mistake in dividing. After some initial resistance to Hosni's repeated claims that the answer is two hundred and seventy-five, Micky recognizes this fact:

M: Oh no, it is two seventy five. Whoops I divided wrong.

The discussion of the 52-sided figure unfolds in a similar fashion, only this time it is Hosni who has missed a calculation step which is pointed out by Micky. Hosni is quick to acknowledge his mistake:

H: Oh yeah. Oh yeah, I forgot to divide by two.

What we see in part one is mainly a discussion focused on the external task which has been posed by the teacher. This almost 4-minute interaction unfolds in a way which is fairly typical of what transpires in the classroom during many small group interactions. We see an important piece of shared knowledge as the boys collectively explain the reasons why it is necessary to subtract the three. During part one there was a symmetry in the interaction between Hosni and Micky, who between them carried on most of the discussion with only a little input from Jeff. In this exchange the boys reveal themselves to be focused and competent as they share their answers and strategies for solving the 25- and 52-sided *Polygons* problems. In their solutions to the problem, Hosni and Micky had both generalized and used a number sentence. In addition, each one had miscalculated one of the answers and from the discussion each was able to see his mistake and correct it. Jeff only showed his answer after the other two had concluded their discussion. Jeff's answers concurred with the correct one for both parts of the problem.

5.3. *PART 2: Jeff as explainer: Two different responses by Micky and Hosni*

In part two Jeff presents his solution to problem 4, *Tunnels Revisited,* which asks:

Can you write a number sentence or general rule for the Tunnels problem?

At this point, neither Hosni nor Micky has attempted this part of the assignment, and Jeff appears eager to describe what he did. While it was highly unusual for Jeff to work on any assignment in advance of the due date, he may have been motivated to do so in this instance, since the assigned task was based on an observation he had made to the teacher connecting the *Decagon Diagonals* and the *Tunnels* task. Jeff provides an extended explanation in support of his thinking and uses a number of different ways to convey his meaning.

To solve problem 4, Jeff used the original expression for *Decagon Diagonals* as a core artifact, namely $(S - 3) * S \div 2$, and then added on **S** (where **S** equals sides). In his Math Log Jeff had written:

$(S - 3) * S \div 2$	+	S	=	tunnels
diagonals	+	sides	=	tunnels

Although Jeff's ideas may be clear to us as readers once the main points of his argument have been laid out, we want to present how Micky and Hosni work to make meaning of Jeff's explanation. In essence Jeff is saying that one has the **diagonals, plus the sides,** and **that's basically all the lines you can draw**. In each of his explanations Jeff attends to the first two components, that you have the diagonals plus the sides, and at times mentions the third important point.

Throughout this part of the discussion Jeff assumes the role of explainer and once again Micky is the engaged respondent. Hosni contributes less than the other two boys and participates towards the end of part two when explicitly invited by Jeff to do so. Jeff begins his explanation by highlighting the difference between *Diagonals* and *Tunnels.*

J: Cause I found that the other rule that with diagonals, in the tunnels problem it wasn't diagonals it was all sides

He repeats this idea in several ways and then goes on to review the steps to solve *Decagon Diagonals.* Both Micky and Hosni follow easily as this is work which they know, and they recognize the steps.

J: =Point **A** times sides divided by two equals diagonals=
[Note: While Jeff wrote **(S - 3) * S ÷ 2** in his Math Log, when speaking about the problem, he more often used **A * S ÷ 2** where **A** equals sides minus three.]

M: =Diagonals, and then /that's it/

J: /So/ now that's all the information I needed because=

M: =But that's basically the same as this one [indicating *Diagonals*].

J: It is /the same/

M: /It's the same/ thing=

H: =Yeah=

J: =As tunnels?=

M: =So then what's the difference? =

Micky's question, "So then what's the difference?" prompts Jeff to review which parts of *Diagonals* and *Tunnels* are exactly the same, and which are different.

J: **It's exactly the same. That was just like saying this is what I know, now how am I gonna put it into a sentence? So what I did is I did point A times sides divided by two then plus sides 'cause you get the diagonals plus the sides, and then that's all the lines you can draw.**

M: ((Hello?)) Okay, so, you're going too fast. Repeat the sentence one more time slowly.

J: Okay.

M: Okay.

J: Point **A**

M: Point **A**

J: Times sides /which is what we did/ with diagonals.

M: /You multiply by sides/.

M: Yeah.

J: And divide it by two=

M: =Divide that by two=

J: **=Which would equal diagonals, but then I did plus sides=**

M: = Plus sides=

J: **=And that would equal all the tunnels because what the /tunnels really are/=**

M: /the amount/

J: **=is diagonals plus the sides.**

Throughout this excerpt we are aware of Micky's careful monitoring. He asks Jeff to slow down, and then follows and repeats every step. We hear the two voices: "Okay," "Okay," "Point A," "Point A," and so on until the end of Jeff's explanation.

Micky then points out that he had never thought to connect the two problems, *Tunnels* and *Diagonals,* and Jeff recalls his discussion with Vicki to show why he viewed the connection was evident.

M: 'Cause I'd ne-, I would never have thought of basing it upon this rule and then just changing it around a bit. Basically, I thought it had be a whole different new rule.

J: No, I I didn't because I figured why would she mention tunnels, um=

M: =Yeah well that's true=

J: **=Well I was the one that said tunnels is almost the same as, I I said it to Vicki, tunnels is almost the same as this decagon problem only, uh, it has, it, you count the sides. And it didn't hit me then but as I was sitting down I said what did I say to Vicki? oh yeah! You just add on the sides and it works.**

M: Okay so it's basically the same thing but you just /add on the sides/

J: /add the sides/

Micky's final comment above indicates that he agrees with Jeff. He sees that the part which is the 'same' is the part which corresponds with *Diagonals,* and that the 'difference' is adding on the sides.

Jeff then asks what he did, and Micky provides his solution for the *Decagon* problems.

J: What did you do?

M: Well yeah, so like, I called mine point Z. Z minus three equals, well here wait, Z minus three equals Z minus three, times, uh, times Z, divided by two equals numbers of diagonal lines in the figure.

J: Yeah.

M: Okay. So basically it's the same thing except you just add on the sides once more.

Most of this discussion has occurred primarily between Jeff and Micky, and we can see the similarity in their solutions for the *Diagonals* problems. Micky had not solved problem 4, but after extended discussion with Jeff he appears to be convinced by Jeff's explanation. In contrast Hosni contributes less actively.While Hosni was a full and active participant in part one revealing his grasp of some important mathematical ideas which evaded other members of his class, his participation in this discussion drops off dramatically (see Table 1). Hosni seems be more involved and in tune with his partners when the discussion deals with the steps to the *Diagonals* problems which we know he understands. He becomes noticeably less vocal, however, at the point where the connection between the *Tunnels* and the *Diagonals* problems becomes key for understanding. While he does at times make himself heard, a fuller response comes when Jeff explicitly requests his input.

J: [to Hosni] Do you-, we've kinda left you out through the discussion. Do you have anything-

H: Well, there's something, a little thing I don't get.

J: What.

H: The thing that you did, the same thing, the-, the number of diagonals=

J: =Yeah=

H: =Times the sides divided by two.

J: Yeah.

H: I got that, but then you added the things for the, uh=

J: = the sides=

H: =Yeah=

Hosni has reviewed with Jeff the solution for *Diagonals,* and Hosni says, "I got that." Hosni is able to pinpoint the piece which he understands, and then to indicate the spot where he has difficulty saying, "but then you added the things for the, uh." It is Jeff who completes the utterance by supplying a key element, "the sides." Hosni answers, "Yeah."

Hosni then reviews the information and checks his understanding in the form of questions to Jeff:

H: /Oh I understand, oh okay/ You divided by two to get the diagonals=

J: =Yeah=

H: =But to go back to tunnels you have to add them again?

J: You-,¦/I didn't-, I didn't/=

H: /You just go back?/

J: =multiply it. I added sides. I added how many sides there were 'cause you see, the tunnels problem you had to do every single possible way to draw a line=

H: =Yeah=

J: =Right?

H: Right.

In the above portion, Jeff repeats his explanation with a justification pointing out that you "have to do every single possible way to draw a line." He checks to see that Hosni is following, saying "Right?" and Hosni replies, "Right."

Jeff then launches into the tenth re-iteration of his explanation. This time he supports his talk by drawing a hexagon showing three diagonals from one point. This is a simpler figure than a previous drawing already on the page in his Math Log.

J: You see, if I can use this quickly I'll make, uh, something with /not as many just like that/

H: / Oh I understand. Oh okay/

J: **You do all the diagonals, but then after you still have these to connect.**

H: Okay=

J: **=And the diagonals plus that is all of the different ways you can do it.**

H: Okay.

M: Yeah. So it's the same thing except you just add it once more.

J: Yeah (. . .) So

M: Except you add it once more without the minus three.

In response Hosni says, "Oh I understand. Oh okay." We see that on multiple occasions following Jeff's explanations, Hosni says, "Okay," and again "Okay." However, he seldom revoices Jeff's explanations in his own words as we hear Micky do, and at no time does he recast Jeff's two part explanation in his own words. While we have Hosni's claim "Oh, I understand, oh okay, you divided by two to get the diagonals," and we know that he does indeed understand how to solve the *Diagonals* task, he still seems at a loss when trying to follow what Jeff did to get the answer for *Tunnels revisited.*

Our understanding of what Hosni is thinking at this point is only conjecture. In conversational segments we often come across utterances between the children which are ambiguous and confusing for the researchers and perhaps for the children as well. What might Hosni mean when he asks, "You just go back?" What is Jeff thinking when he answers "I didn't multiply it." At the same time when the discussants actively seek clarification and persist in their search for mutual understanding, their extended talk often provides access to additional information on which to base our inferences. For example, with Micky's utterance, "So it's the same thing except you just add it once more" the referent of "it" is unclear. However, Micky continues to provide additional information when he says, "Except you add it once more without the minus three" which gives us evidence that the "it" refers to "sides" since three has only been taken away from sides in this conversation.

5.4. *PART 3: Micky as questioner: A search for meaning*

To solve the *Tunnels Revisited* problem described above, Jeff has drawn on his understanding of the relation between the *Diagonals* problem and the *Tunnels* problem and constructed a two component model which specifies that first you solve for diagonals and then you add all the sides. As reported above, Jeff in his math log represents the solution as:

$(S - 3) * S \div 2 \quad + \quad S \quad = \quad \text{tunnels}$

[diagonals + sides = tunnels]

Micky's own solution for the *Diagonals* problem was in the form

$(Z - 3) * Z \div 2 \quad = \quad \text{diagonals}$

Even though Micky had not yet tackled problem 4, we can see that his solution to the *Decagon* problem mirrors Jeff's representation. This knowledge in conjunction with Jeff's explanations of the connections between the two problems lead him to understand *Tunnels revisited* in the way that Jeff does:

M: Okay so it's basically the same thing but you just /add on the sides/.

J: /Add the sides/

M: Except you add it once more without the minus three.

While the 'minus three' has been an important and much discussed element in the boys' understanding of how to determine the number of diagonals in a polygon, it now takes on added significance as an anomaly for Micky when he says:

M: =See, but once you think of it, the Z minus three seems pretty weird.

The expression *but once you think of it* signals the type of reflection that distinguishes this part of the mathematical episode from the previous two. Micky knows that Jeff's solution works in terms of leading to the correct answer but he is puzzled, "Once you think of it, the Z minus three seems pretty weird." At this first mention, Jeff readily agrees and together the boys decide to test whether they can solve the problem by multiplying Z times Z (sides by sides) and dividing by two. This mathematical activity of multiplying Z times Z suggests that the boys have an implicit understanding that all the points must be connected. This does not give them the correct answer and they discard that line of reasoning. In the course of their discussion, however, a difference in their understanding of what happens to the 'three' emerges. As Jeff says:

J: Cause you're subtracting and then you're adding it on later

And Micky replies:

M: Well no, you're not adding it on later.
J: Yeah you are.

.
.
.

M: /when?/ you're not getting back that minus three are you?
J: Yeah you are=
M: When?=

Jeff continues to argue for the fact that "you will get the three back" while Micky simply wants to know how that happens:

J: Think of this. Twenty-three times twenty-five divided by two.
[Jeff misspeaks, and later the boys correct the first number, which should be twenty-two, not twenty-three.]
M: Yeah, will you get that three back?
J: Yes
M: When?

As the discussion continues back and forth, Micky asks again only this time Jeff switches his position:

M: When do we add back that three?
J: We don't

Now Micky who has been trying to understand how you would get the three back goes on to argue why in his view you should be getting it back:

M: We don't, but you see it has to connect with every single line in the tunnels problem. So it /would just be sides multiplied by sides/

Micky is explicit now about the idea to which he holds firm throughout: All points (burrows) must be connected, thus, if someone is subtracting three, then the three must come back or be put back at some point. Jeff counters with a reminder that they tried multiplying sides by sides divided by two and it did not work:

J: But the other way that we were trying to figure out, we just tried it out and it didn't work.

Micky continues to be puzzled, and Jeff continues to maintain it is not an issue:

M: But, I don't see when we're getting that three back from the Z.

J: We don't.

M: Why not? You have to connect every single line with every single line-burrow.=

Up until this point Micky has been arguing that on the one hand you should be getting the three back but on the other he cannot see how this occurs. Initially, Jeff agreed with him that you do get it back and then he switched to maintaining you don't get it back. Now Micky's *why not?* leads Jeff to switch again, and he replies with a question of his own:

J: Why wouldn't you get it back?

With this question Jeff engages the problem differently, and it marks the beginning of real inquiry on his part similar to Micky's *but once you think of it.* The discussion continues but there is a shift in the interaction:

M: You see, you're just taking away all this here. Well let's just say you're-, you're forgetting about three connections.

J: Yeah I know

M: When are you gonna get those back? When you're multiplying or something? You're not-, you're not gonna get them back.

J: That's a good point.

As the discussion progresses, the boys reflect together on the fact that they have an algebraic expression which allows them to successfully solve the problem but which on another level creates a puzzle for them. This then leads to the emergence of a difference in their mathematical stances:

J: It does work, but we're not exactly sure when you get the three to connect it back.

M: Well if we're not too sure about it, we can't really say it works.

Here we have two different perspectives of what it means to say "something works" in mathematics. Jeff maintains that the formula works since they have the empirical evidence in terms of getting the correct answer when they apply the algebraic expression. Micky adopts a different stance and feels if you cannot explain *why* something is the case, then you do not really understand how it works, and consequently cannot claim *that* it works.

The discussion continues with Jeff switching his position on the 'minus three' once again. To shore up his argument he uses the same strategy and draws on empirical evidence as he recalls how multiplying twenty-five times twenty-five and then dividing by two did not lead them to the correct answer.

J: No but-, no but you're not supposed to get the three back-
M: Why not? You're supposed to connect the /(( ))/
J: I know but you don't get the three back because we just tried it out. Twenty-, twenty-five times twenty-five /divided by two-/
M: /divided by two/
J: It will not work
You will never get that three back

When the question is presented again by Micky, Jeff offers yet a different response:

M: But when are we gonna get that three back? I'm still wondering. If it has to connect with every other burrow.
J: I have no idea
M: Well neither do I=

Jeff then proposes the a solution:

J: Maybe we've lost one number, then when we divide it we gain it back, or multiply it we gain it back?

And Micky in his search for meaning, asks:

M: Well how can we divide it and get it back?

Jeff tries to address Micky's concern with a number of quickly generated hypotheses which lead nowhere. He finally suggests that if they unpack every step of the process in terms of additions and subtractions, the solution to the puzzle might reveal itself:

J: At the divi-, if we did it by hand, every step by addition and then subtraction, so ma-, uh-, addition twenty-five times, the subtraction twenty-fi-, uh-, two times-
M: /I'm still not/ convinced.=
J: Well I'm convinced that it's right but … I'm not convinced where it's right.

The small group discussion on this day is interrupted as the teacher gathers the twelve children together to discuss their work on problem 3 in a large group. The boys' discussion has ended with Micky skeptical and Jeff hopeful. While we have traced the argument between Micky and Jeff, and we know that Hosni did not really contribute to building it, it is nevertheless important to examine what Hosni was doing. As Table 1 reveals Hosni contributed 5 turns to this discussion. Only two of these, however, related to the problem being discussed. In both cases he echoed Micky's words about the minus three. Since he does not appear to have understood the argument presented by Jeff in part two, we are suggesting that he aligned himself with Micky in this part of the discussion as Micky is questioning Jeff and that is something Hosni himself may have wanted to do. His contributions suggest that while he has not completely removed himself from the arena of discussion, the mathematical conversation may have moved beyond his understanding.

5.5. *PART 4: Micky and Jeff: The joint construction of new knowledge*

The discussion between Micky and Jeff resumes four days later. Hosni has opted to join another group and so is absent from this discussion. Almost as if no time has elapsed between meetings, Micky re-introduces the very same problem which has not yet been solved:

M: My only question is where did the three go? Now, that's all I'm wondering about. /I understand the rest/

His question is met this time with a very different response from Jeff, and the boys now bring new information to bear on the problem. Jeff suggests that "it's not three." Micky responds that it "should actually be minus one, cause it cannot connect with itself but in the problem it *can* connect with the others."

J: /Okay/. I have no idea where the three went. It probab-, but-, the thing is (…) why do you need the three?

'Cause it's not three. It's not three. From here [points to drawing in book] it's three but then you got this point [refers to book drawing] that's three-

M: But that should actually be minus one

cause it cannot connect with itself, but in the problem it can connect with the others.

J: Exactly.

Neither Micky nor Jeff make explicit the source of the idea of the 'minus one.' It is not an unusual idea, however, as we have seen other children solve the *Tunnels revisited* problem by going directly to S-1. In the following, Micky and Jeff proceed to explore together, and test their idea.

M: So it should be minus one, but why is it minus three (..) in your case?
J: /I don't know./Try it by subtracting one.
M: we'll see
J: I think you wouldn't have-, you could subtract by one and multiply it by sides.
M: Maybe it'll give us uh something

They proceed to try out this new understanding, (S-1) * S ÷ 2 = tunnels, on a polygon for which they know the correct answer is 10. It works.

J: =Equals ten. So you don't need to add on the sides.
M: [shakes his head] Oh cool.
J: We just found out a new rule.
M: Oh here, wait. We have to try it in like three cases.

Jeff feels that they have found a new rule, and Micky cautions that they have to "try it like in three cases."

J: Yeah, we'll try it in three cases, but let me just write it down
.
.
.
Try it, try it all you want. We've just figured out two ways to figure out tunnels.
M: But that would be the most straightforward, it'll /(( ))/
J: /That'll/ be the most straightforward because=
M: =You wouldn't have to do an extra, uh, adding on.
J: And an extra subtracting. That's where you get the two back.
It wasn't three that we were getting back. It was the /two./
M: /Two./ So this is actually better.
J: This is better than before.

The concluding discussion reveals their understanding of the principles at work in this new formulation, and their agreement concerning its superior qualities as a general expression for solving this problem. They share the view that it is more straightforward, and is a clearer representation of the underlying structure. In other words, it is a more elegant structure than its predecessor.

What drives them? In great part, it is the need to know, the need to solve the puzzle. Micky and Jeff are whole-heartedly engaged in a student-generated, self-imposed task and in the process reveal the powerful role of questioning. "The essential component of critical thinking is the ability to pose questions and evaluate their worthiness" (Borasi, 1992, p. 202). The richness of their inquiry should serve as a cautionary note to our more traditional educational practice in which we educators provide answers to questions which students have never asked. In order for students to become critical thinkers, they need the freedom to direct and focus their own inquiries as "questions … are the principle intellectual instruments available to human beings" (Postman, 1995, p. 173).

What enables them? While this type of engagement cannot be predicted by the teacher, it can indeed be fostered by implementing many of the necessary conditions which we have identified in conjunction with inquiry classrooms. Given a meaningful learning environment, and interaction, Micky and Jeff have the disposition to pursue inquiry, and the confident optimism drawn from other experiences to know that they may make headway. They are able to specify what they already know, what they do not know and what it is they need to know. Micky is also able to articulate *why* one needs to know, which reveals his understanding of the role of proving in mathematical reasoning. In a classroom where such learner initiatives are valued and the environment is one of a supportive community of learners, inquiry may follow but the characteristics are dependent on the dynamics of the interactions.

5.6. *How representative are these children and this discussion?*

The episode featured in this paper was chosen because it affords us a look at some of the ways in which dialogue leads to learning. This includes both intra-personal and inter-personal learning as the participants talk about their work, their confusion, and about how the ideas of others help or do not help them to make meaning. While the entire four-part episode is unique to these children, there remain a number of common behaviours among the participants which are representative of various aspects of this activity. For example, the discursive interaction in part one between Hosni and Micky is very typical of many of the classroom interactions, in that the students compare their answers, and talk about their strategies. If the answers are not the same, the children typically try to determine the source of the differences. They point out errors, make corrections, and generally arrive at a settled position which does not imply consensus, but rather that they have no more to say on the matter. In contrast, the Micky and Jeff exchange is less common. It is our suspicion, however, that more of this type of inquiry is going on in collaborative discussions of mathematical problems than we have been able to document. The characteristics and quality of these extended discussions often emerge only when we have time to view, re-view and transcribe episodes.

Mathematical knowledge. In terms of the mathematical knowledge, these students are very representative. In this year's class to solve problem 3, eleven of the twenty-five students were able to use the Jerome-Jeff idea, seven using an arithmetic expression, and four an algebraic expression. In the case of *Tunnels revisited,* four students used an arithmetic expression, and six constructed algebraic expressions. Other students in the class came to the same mathematical conclusions we saw Jeff and Micky reach. For example, for *Tunnels revisited,* David's solution structure matched the one the teacher had expected some of the children to construct. David explains and justifies it in talk and in writing: "It is $Z - 1 * Z \div 2$ because you can go to the sides." [David's Math Log]

We have data on similar interactions occurring in a number of small group discussions. For example, the discussion between Jerome and Michel parallels the one between Jeff and Micky vis-à-vis their conceptual structuring of the *Tunnels revisited* problem. Jerome, like Jeff, sees it as a two-part structure (diagonals plus sides = tunnels), while Michel's understanding is like Micky's. However, unlike the interaction between Jeff and Micky, there was no inquiry developed as a function of the discussion between Jerome and Michel. Michel came to the table with the same algebraic expression as we have seen stated by David above, and once Jerome heard Michel's explanation and realized that it worked, no exploration of Jerome's expression ensued. In this paper we chose to describe the inquiry on the part of Micky and Jeff working together since in this instance we are more privy to the *process* of their thinking.

Stance and disposition. In terms of children's stance and disposition in regard to what it means to be a member of this classroom problem-solving culture, the three boys in this discussion represent a range of recognizable positions. Hosni is a student who appears to understand his role as one of completing the task, listening to his partners and classmates, and trying to understand. He listens to Jeff's explanation, but does not seem as involved during the exploration of 'why minus three?' This behaviour is characteristic of many of the children who do not push themselves beyond the assigned task. As one child was heard to say: "Why do we need to do more?"

Jeff in contrast adopts a different stance and sees his role as working hard to solve the challenging tasks, using generalizations and encoding them in algebraic expressions. He is not averse to delving into *why,* but in this instance did resist the exploration at first. He also sees himself where appropriate as a liaison personality helping students connect with each other or with ideas. This is revealed when he says to Hosni, "We've kinda left you out through the discussion. Do you have anything – ?" In addition, his varied responses to Micky's persistent challenge reveal him to be an active investigator.

In contrast, as a learner Micky feels it is not enough to know *that* an algebraic expression works. He maintains it is also important to know *why,* and he states that one can be convinced only when one knows why it works. This understanding and disposition to the work leads him beyond the assigned task. In the instance of this episode, his questions led to the inquiry. While this 'going beyond' is less typical in the classroom discussions and it cannot be mandated, it can be fostered in learning environments which explicitly support it.

6. Mathematical meaning constructed in dialogue

6.1. *Otherness and own-ness*

Our understanding of the dialogic nature of learning is at the heart of our view of inquiry and of the classroom as a learning community and leads us to emphasize the importance of learners building upon each other's ideas. According to Bakhtin we individuals appropriate other people's words/ ideas by trying them out, and in that process they become transformed in accordance with our own needs.

> Our speech, that is, all our utterances (including creative works) is filled with others' words, varying degrees of otherness or varying degrees of 'our-own-ness' (Bakhtin, 1986, p. 89)

Classroom practices in this setting promote children crediting others, and using their ideas either without changing them, or by extending and building upon them. In this classroom, the students are explicitly directed to cite ideas they have found helpful, by completing in writing pertinent parts of a sheet entitled "Helpful explanations/Helpful ideas." This activity takes place at the end of each full class discussion, and has been part of the mathematical practice since 1993. The ways in which the children use other people's ideas are often revealed in their talk as well. In addition, to enhance their awareness of the dialogic nature of learning, the teacher often asks them in the course of classroom conversation to consider whose thinking has been of benefit to them. We cited one example earlier, when Jeff wrote that Jerome's explanation helped "by explaining better the pattern I saw but could not put my finger on." In relation to problem 4, *Tunnels revisited,* Michel wrote in his Math Log:

Tunels: I tried Jeff's theory but instead of subtracting 3 to get the number of diagonals for the decagon problem I subtracted 1 to get the amount of diagonals because you could conect next to each other not like the decagon problem.

In this way what the children say and do in discussing their work and ideas is filled with varying degrees of 'otherness' and of their 'own-ness'. Talk about using someone else's strategy, or pattern, and so on, by referring to it as 'Emma's way', or 'Jeff's way' is not seen as *copying*. Rather, it is the explicit implementation of Bakhtin's theory of the dialogic nature of learning and the Vygotskian view that the thoughts and practices of others become integrated in one's own.

It is not only among the children that we see this at work, but among all learning participants in the classroom including the teacher. Recall that Jeff had an idea about the connectedness of the two problems. He expressed it, and the idea was picked up and rephrased into a question by the teacher, and offered to the other children. Once offered, it became an object for thought for all the children, including Jeff himself. Jeff spoke about the process he went through in acting upon the challenge. He recalled that, "It didn't hit me then," but as he was sitting down he said to himself: "What did I say to Vicki? Oh yeah! You just add on the sides and it works." He then used his own words to jump-start his thinking again and to push it. His own words connecting the two tasks became the object of his reflection. "Tunnels is almost the same as diagonals only you connect the sides." He then knew how to proceed: "You just add on the sides and it works."

Analogical thinking also reveals aspects of own-ness and other-ness. To reason analogically in problem-solving involves a number of steps which include mentally representing the new problem, noticing a potential analogy with another problem based on some attribute, constructing an initial and often partial mapping between the two situations, and extending the mapping from the initial problem to the one under consideration to arrive at a solution to the current problem (Holyoak, 1982; see also English, 1997; Vosniadou and Ortony, 1989). We see this process in the reasoning of a number of children in the context of this mathematical activity, and it appears that the meaningfulness of the activity serves to enhance its occurrence. In particular, Jeff articulated both the similarity and difference between the two problems: "*Tunnels* is almost the same as this *Decagon* problem only, uh, it has-, it-, you count the sides." The connection Jeff made seems important to the teacher as this is what she built on to create the problem extension in *Tunnels revisited.* Her interest was to see whether any of the children might construct an arithmetic or algebraic expression for the *Tunnels* task. This action suggests that the teacher appreciated Jeff's analogy, but people take over others' ideas, and shape and transform them to suit their own personal needs and intentions. In actuality, what the teacher responded to was the fact that Jeff connected the two problems. In fact, his analogical understanding of the relationship between the two problems was substantially different from hers. Interestingly, in her first look at Jeff's work, the teacher thought his way of relating the problems was incorrect. Her comparative understanding of the problems was that they were the same except for the fact that the *Tunnels* problem unlike the *Diagonals* only required that you subtract one. While she was interested in seeing which of the children would arrive at an understanding represented algebraically or arithmetically, what she expected was that they would represent the underlying relation $(S - 1) * S \div 2$. Contrary to her expectations, the children responded to the task in diverse ways.

There is another thread in this episode with regard to Micky which is of interest because it reveals how 'otherness' and 'own-ness' may combine to constitute one's personal voice. In a post-session interview Micky revealed that he had thought of the algebraic expression $(Z - 3) * Z \div 2$ after hearing Jerome talk about the pattern while in the large circle at the chalkboard. He went on to point out, however, that he had constructed his algebraic expression independently of Jeff's work. In this way he both credited Jerome and at the same time, claimed ownership of the creation of his own algebraic-expression. Similarly, his deliberate distinction from the 'other' extended to the work he chose to record in his Math Log. Whereas Jeff valued both algebraic expressions, marking the one co-constructed with Micky as the 'best way,' Micky recorded only the final algebraic expression which fit his mental model of the problem, and he recorded it in his own notation using R rather than Z or S: $R (-1) \times R \div 2$. In these two instances we see how Micky established himself, carving out his unique space and identity.

6.2. *Reflections on the zone of proximal development*

We argued earlier for an enlarged notion of the ZPD. Our understanding builds upon definitions currently in the literature but goes beyond. We have shown previously that the four parts of this episode provide a broad range of interaction activities including checking answers, explaining how the problem was solved, and constructing new learning. All three participants have learned from and with each other. We have addressed the *what* and the *how* in relation to the ideas learned, and have been interested especially in the quality of those ideas. Lee (2000) has suggested that as the learning interaction progresses, the "student's representation of the task should evolve to a representation closer to that of the teacher" (p. 194). While this may be true, it narrowly conceives of the the ZPD as an interactional space delimited and overseen by the teacher. In regard to problem solving carried out "under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p. 86), the teacher has in mind where the children should go, and what some of them might do. This indeed is an important aspect of effective teaching. It is incomplete, however, as it does not ascribe the dynamic flexibility to the ZPD required so that it is not only children who learn but teachers as well.

The teacher. We need to be sensitive to how the teacher's zone of proximal development can be stretched in transformative and generative ways due to her interaction with her children (Confrey, 1991). In our case in this episode, using rich mathematical tasks which all children can enter, the teacher created zones of possibilities. Due to past experience with these assignments, she knew and expected some of the strategies which would be used. Part of her work entailed helping the children make connections between their inventions and ways of doing and the conventions of the culture, such as the cultural tool of algebraic expression. The work Micky and Jeff do in constructing algebraic expressions, and in deliberating together upon all and parts of their structure, attests to their mastery of this powerful cultural tool. They express their awareness of the power of algebraic expression explicitly during the large group discussion on the *25-sided Polygons.* However, and this is a big 'however,' although the teacher can appreciate the cases where transformation occurs as a result of participating in meaningful activities, both alone and with others, the teacher cannot predetermine the trajectory, or even anticipate certain parts of it. In the episode presented here, the teacher clearly did not set the upper bound to the inquiry. Indeed, she never anticipated a solution such as Jeff's, nor a peer-driven inquiry which unfolded as described.

An important part of the teacher's work included listening for these kinds of critical incidents. It involved looking closely to see how the mathematical learning was or was not accomplished interactively. Her work also included acting upon components which emerged from the interactions which might serve as catalysts for growth for all the children, and for subsequent cohorts. In this way she contributed along with the children to an ever-changing practice. Jeff and Micky's individual and collective activity, appropriated by their teacher with acknowledgements to them for their contributions, then became part of the knowledge available to the larger community.

Disagreement and misunderstanding. While much of the focus on the concept of the ZPD has been on the degree of shared understanding needed for successful communication and learning to occur, we believe that this intersubjectivity has to include a place for disagreement and misunderstanding. In such instances the catalyst which leads to learning need not be the more able person but rather may stem from a child's incorrect proposal which in turn becomes the trigger for an intensive discussion leading to a more complete understanding among the participants. We can use the Jeff-Micky interaction to explore this view.

From the episode described, we saw how the differences among the participants in this collaborative reasoning group served to affect learning and contributed to an elaborated understanding of the mathematical content on the part of Jeff and Micky. It was not just learning new information in terms of content, or being given the occasion to explain an idea, but rather it was the way the differences between the positions of the participants functioned which enabled them to jointly construct new knowledge. While Micky's persistent challenge to Jeff was a force driving the learning in this activity, the range of responses in the give and take which followed led the boys to a further articulation and clearer understanding of what they knew, of what was missing and what they still needed to figure out. As a result, their mathematical knowledge expressed as an algebraic expression became an object of reflection, and was used to support different views during the discussion including their final shared understanding. Here we see how the text as an object of discourse and more specifically as an object of reflection is present in environments which promote inquiry and understanding (Cobb, Boufi, McLain and Whitenack, 1997). This is not a question of children working in groups and helping each other or not. This type of inquiry is a practice which supports sustained engagement in which the children have constructed thinking identities and the student as intellectual participant is a role that is supported and valued. This highlights the importance of the discursive articulation of the disagreement or misunderstanding. While Hosni may disagree or not understand, his lack of expression of that misunderstanding limits its potential to contribute to the learning.

> The language of the classroom, if it is to be an invitation to reflection and culture creating, cannot be the so-called uncontaminated language of fact and 'objectivity.' It must express stance and must invite counter-stance and in the process leave place for reflection, for metacognition. It is this that permits one to reach higher ground, this process of objectifying in language or image what one has thought and then turning around on it and reconsidering it. (Bruner, 1986, p. 129)

Understanding difference as a thinking device in this way, however, also requires understanding the context and meaning of these interactions. These children have come to believe that what they think is important. This has come, in part, from the teacher's role as a teacher-researcher and from the children's participation as co-investigators into their cognitive and social activities. Taking this into account, we now see the zone as enriched by the heterogeneity of multiple perspectives, different mathematical knowledges, and differing mathematical identities.

What about Hosni? At this point we will address why a zone did not emerge for Hosni. We agree with Lerman (1998) that it is as important to examine instances in which a zone of proximal development has not emerged, as instances in which it has. It is our view that Hosni did not make meaning of Jeff's solution. Others may suggest that Hosni really did understand Jeff's way but may have chosen to be quiet, or that Hosni may have felt silenced, may not have been interested in working hard at solving the query posed by Micky, and so forth. However, we feel that he was stuck at the end of part two, and we wonder why, as he clearly understood the solution to problem 3 as evidenced from his own active role as explainer in the first part of the discussion. We will look first at the quality of Jeff's explanations, at Micky's actions, and then at the interaction between Jeff and Hosni.

In his interaction with Micky and Hosni, Jeff was very adept in conveying his ideas. His explanations were complete as he included all the essential components, his explanations were re-wordings not exact repetitions, and he used multiple representations including words, drawings, and deictic moves to refer to drawings previously done. He was enthusiastic and focused. Jeff modulated the delivery of his message, beginning with the full explanation of his solution (given three times). He then slowed down to review the part which was prior knowledge, the 'old' information, namely the *Diagonals.* Following Micky's question: "So what's the difference?" Jeff reviewed what was old and what was new. All of these behaviours suggest a high quality of explanation and teaching on Jeff's part, and we draw on Webb's characterization of the expert teacher/tutor to help us pinpoint why his explanations and his work with his partners struck us as adroit.

> [The expert teacher/tutor] relates new material to what the student already knows, uses multiple representations (mathematics symbols, numbers, pictures) to explain the concept, shows how to coordinate and translate among alternate representations so that the student can see a concept in multiple ways, provides detailed justification of each step in the problem-solving process, . . . responds to indications of misunderstanding (errors, questions, statements of confusion) with elaborated descriptions (not brief response or correction), encourages the student to freely: admit lack of understanding, disagree with others, and control the pace and content of teaching activities. (Webb, 1989, p. 35)

Thus we have the situation where an explanation can be considered exemplary but the learning does not go forward.

Micky's interaction with Jeff in the discussion revealed that he followed Jeff's reasoning carefully, tracked, made sure, went step by step, and when he needed to clarify, he asked questions like, “So what’s the difference?” and gave directives like, “Go slowly.” Micky confirmed his understanding by re-stating all the essential parts at one point in part 2: “Okay so it’s basically the same thing, but you just add on the sides.” He re-affirmed his understanding while signalling his difficulty when he said at the start of part 4, “My only question is where did the three go? Now, that’s all I’m wondering about. I understand the rest.” Micky not only demonstrated that he understood Jeff’s way, but he also showed that he had made meaning by questioning the ideas, and building upon what he understood of Jeff’s way.

What was different in Hosni’s case? We see that Hosni also asked questions. He reviewed the part of the problem which was the same, asserting that he understood and then added, “But there is one little thing I don’t get.” He identified a part of the puzzle which was missing for him. While he did pose the question, it is not clear to us that he received the answers he needed. As a result, he appeared unable to go further with additional questions. At some junctures which deal with crucial ideas in relation to the missing parts of the puzzle for him, Hosni said, “I know, okay,” and “Oh, I understand. Oh, okay.” However, we do not have any indication that Hosni had made meaning of that ‘little thing’ with which he opened his potentially fruitful exchange with Jeff. The ‘little thing’ turned out to be a ‘big thing.’ Without that missing information, Hosni was not able to participate in the discussion of *Tunnels revisited,* to the extent that he might have, had he really understood how Jeff constructed his expression for *Tunnels.* Indeed, when we look again, we see that Micky’s observation, “See, but once you think of it, the Z minus three seems pretty weird” is one which may well have confused Hosni. It is very possible that Hosni didn’t understand what troubled Micky, since the ‘Z minus three’ fit perfectly in the context which Hosni knew well, namely that of *Diagonals.* In fact, Hosni may have found it ‘weird’ that his two partners spent so much time deliberating about it. However, we can only speculate about this as Hosni never asked about why they were attending to the Z minus three.

To understand what may have prevented Hosni from engaging as fully as the other two, we offer a few conjectures based on our knowledge of his situation as a student in this school and his participation in collaborative mathematical inquiry. Hosni was a new arrival at the school that year; hence he was in the process of being enculturated into the norms of the classroom and school culture. We see him comparing answers and talking about strategies, but he is not challenging or analyzing. Not all of the children who have participated in the culture over time do analyze or think about the structure of the mathematics. With respect to Hosni, we do not know if this is the case or if he has not been part of the culture long enough. In addition, perhaps the to-and-fro interaction itself between Jeff and Micky was too rapid and connected, making it too difficult for Hosni to get a word in edgewise. He may, consequently, have felt a bit out of the loop. Also, Jeff's solution, that of adapting *Tunnels,* may be more challenging to understand than the other solution, S-1 * S ÷ 2; the latter construction has been seen far more often over the years. Had Hosni asked more questions and pursued what was not clear, might Jeff have adapted and modified his responses more in line with Hosni's needs? In order for Jeff to go beyond his part, however, Hosni would have had to be proactive. Jeff could not answer questions that Hosni had not asked. Perhaps Hosni was at a loss, and did not know what to ask. He seemed not to have made connections with Jeff's two-part idea. Thus, even when reformulated, the repetitions did not help as they were all evidence of the same line of thought.

It has been significant for us, both as teachers and as researchers, to have this example of how learning resides in the interaction. From these data we suggest that a zone of proximal development emerges only some of the time, even in cases when the conditions seem optimal. It seems clear that we cannot expect to orchestrate a desired scenario, nor can we suggest how it may be made 'right' the next time around. Thus we return again to the idea that knowledge, individual or collective, is dynamically co-constructed in the context of the activity, the participants, and the mediational tools.

7. CONCLUSION

The episode described in this paper offered us a look at the process whereby individual and group developmental trajectories are constructed, and allowed us to explore the relationship between discourse and knowing. Although Ernest (1995) has noted that conversation or *dialogue* has recently become a central metaphor for knowledge and *mind,* nevertheless *spoken words* do not equal *thoughts in the mind.* Vygotsky and Bakhtin insist upon the complexity of the process of going from thought to words, the intra-mental, and of understanding and using the ideas of others, the inter-mental or social. According to Vygotsky, thoughts do not correspond with language directly.

> In his [a speaker's] mind the whole thought is present at once, but in speech it has to be developed successively. (1962, p. 150).

Thus, the thought is re-constituted and completed as it is transformed into words, is "shaped at the point of utterance" (Britton, 1982). "Here is what I know," says Jeff in part 2 of the episode. "Now how can I put it into a sentence?" We have seen in the children's work varying degrees of 'otherness' and of 'own-ness' as they have worked to understand the ideas of others, and have at times replied to them, extended them and questioned them. Each child's ideas are made up in part of someone else's ideas and in part of their own, thus making each individual trajectory and each group trajectory unique. Hence we speak of the ZPD both as a zone of potential personal growth (Vygotsky, 1978, p. 86) and as a unique collective 'product of the moment' (Lerman, 1998). The act of achieving an individual voice is accomplished through dialogue with the others, those immediately there and those long gone whose ideas are instantiated in cultural ways of being, doing, and speaking. Dialogue bridges what others have seen as the divide between individual and social and cultural, is focal in the construction of knowing and of identity, and of individual and collective practice.

It is our conviction that we have only just begun to understand the richness of thinking possible in the discursive interaction of children around meaningful problems. While we have always been able, as teachers, to marvel at the accomplishments of our learners, it is only with the video data that we have been able to detail the intricacies and depth of their reasoning, and this has allowed us to hold it in place and think about what this talk means.

ACKNOWLEDGEMENTS

This research was supported by a Social Studies and Humanities Research Council grant to the first author from the Government of Canada #410-98-0427. We thank our external reviewers as well as Carolyn Kieran, one of the editors of this special issue, for their extensive and insightful comments on an earlier version of this paper.

REFERENCES

Bakhtin, M.M.: 1981, 'Discourse in the novel', in M. Holquist (ed.), *The Dialogic Imagination: Four Essays by M.M. Bakhtin* (C. Emerson and M. Holquist, trans.), University of Texas Press, Austin.

Bakhtin, M.M.: 1986, *Speech Genres and Other Late Essays.* C. Emerson and M. Holquist, (eds.); Y. McGee, (trans.), University of Texas Press, Austin.

Ball, D.L.: 1991, 'What's all this talk about "discourse"?', *Arithmetic Teacher* 39 (3), 44–48.

Ball, D.L.: 1993, 'Introduction', in D. Schifter and C. Twomey Fosnot (eds.), *Reconstructing Mathematics Education: Stories of Teachers Meeting the Challenge of Reform,* Teachers College, New York, pp. xi–xii.

Borasi, R.: 1992, *Learning Mathematics Through Inquiry,* Heinemann, Portsmouth.

Bracewell, R.J. and Breuleux, A.: 1994, 'Substance and romance in analyzing think-aloud protocols', in P. Smagorinsky (ed.), *Speaking about Writing: Reflections on Research Methodology,* Sage, Newbury Park, pp. 55–88.

Brilliant-Mills, H.: 1993, 'Becoming a mathematician: Building a situated definition of mathematics', *Linguistics and Education* 5(3), 359–403.

Britton, J.: 1982, 'Shaping at the point of utterance', in G. Pradl (ed.), *Prospect and Retrospect: Selected Essays of James Britton,* Heinemann, London, pp. 139–142.

Brown, A.L.: 1997, 'Transforming schools into communities of thinking and learning about serious matters', *American Psychologist* 52(40), 399–413.

Bruner, J.S.: 1986, *Actual Minds, Possible Worlds,* Harvard University Press, Cambridge.

Cobb, P., Boufi, A., McClain, K. and Whitenack, J.: 1997, 'Reflective discourse and collective reflection', *Journal for Research in Mathematics Education* 28(3), 258–277.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K. and Whitenack, J.: 1997, 'Mathematizing and symbolizing: The emergence of chains of signification in one first-grade classroom', in D. Kirshner and J.A. Whitson, (eds.), *Situated Cognition: Social, Semiotic and Psychological Perspectives,* Erlbaum, Mahwah, New Jersey, pp. 151–233.

Cobb, P., Wood, T. and Yackel, E.: 1993, 'Discourse, mathematical thinking, and classroom practice', in E. Forman, N. Minick and C.A. Stone (eds.), *Contexts for Learning: Sociocultural Dynamics in Children's Development,* Oxford University Press. New York, pp. 91–119.

Confrey, J.: 1991, 'Learning to listen: A student's understanding of powers of ten,' in E. Von Glasersfeld (ed.), *Radical Constructivism in Mathematics Education,* Kluwer, Boston, pp. 113–138.

Coughlan, P. and Duff, P.A.: 1994, 'Same task, different activities: Analysis of a SLA (Second Language) task from an activity theory perspective', in J.P. Lantolf and G. Appel (eds.), *Vygotskian Approaches to Second Language Research,* Ablex, Norwood, New Jersey, pp. 173–193.

Davis, B.: 1996, *Teaching Mathematics: Toward a Sound Alternative.* Garland, New York.

Dewey, J.: 1900, *The School and Society,* 1990, edited by P.W. Jackson, entitled *The School and Society and The Child and the Curriculum,* University of Chicago Press, Chicago.

Dewey, J.: 1933, *How We Think,* Heath, Boston.

Engeström, Y.: 1994, 'Teachers as collaborative thinkers: Activity-theoretical study of an innovative teacher team', in: I. Carlgren, G. Handal, and S. Vaage (eds.), *Teachers' Minds and Actions: Research on Teachers' Thinking and Practice,* Falmer, Bristol, pp. 43–61.

English, L.D.: (ed.), 1997, *Mathematical Reasoning: Analogies, Metaphors, and Images,* Erlbaum, Mahwah, New Jersey.

Ernest, P.: 1995, 'Criticism and the growth of knowledge', *Philosophy of Mathematics Education Newsl*etter, 8, p. 2, http://www.ex.ac.uk/~PErnest/pome/pome_8.htm

Forman, E.A. and Larreamendy-Joerns, J.: 1995, 'Learning in the context of peer collaboration: A pluralistic perspective on goals and expertise', *Cognition and Instruction* 13(4), Erlbaum, Hillsdale, New Jersey, 549–564.

Frederiksen, C.H.: 1986, 'Cognitive models and discourse analysis', in C.R. Cooper and S. Greenbaum (eds.), *Written Communication Annual vol. 1: Studying Writing: Linguistic Approaches,* Sage, Beverly Hills, pp. 227–267.

Gee, J.P.: 1992, *The Social Mind: Language, Ideology and Social Practice.* Bergin and Garvey, New York.

Gee, J.P.: 1996, *Social Linguistics and Literacies: Ideology in Discourses,* second edition, Taylor and Francis, Bristol.

Goldstein, L.S.: 1999, 'The relational zone: The role of caring relationships in the co-construction of mind', *American Educational Research Journal* 36(3), 647–673.

Graves, B.: 1996, 'Talk is not always easy: The multiple discourse tasks in peer-peer learning interactions', Paper presented at the American Educational Research Association (AERA), New York.

Graves, B.: 1997, 'Literary reasoning: The role of domain-specific and generic literary knowledge', in S. Tötösy de Zepetnek and I. Sywenky (eds.), *The Systemic and Empirical Approach to Literature and Culture as Theory and Application,* LUMIS Publications, pp. 1–11.

Graves, B.: 1999, 'Wrong answers: A catalyst for learning in children's collaborative reasoning about mathematics'. Paper presented at the American Educational Research Association (AERA), Montréal.

Graves, B. and Zack, V.: 1996, 'Discourse in an inquiry math elementary classroom and the collaborative construction of an elegant algebraic expression', in L. Puig and A. Gutiérrez (eds.), *Proceedings of the 20th International Conference, Psychology of Mathematics Education,* Vol. 3, Valencia, Spain, pp. 27–34.

Graves, B., and Zack, V.: 1997. 'Collaborative mathematical reasoning in an inquiry classroom', in E. Pehkonen (ed.), *Proceedings of the 21st International Conference, Psychology of Mathematics Education,* Vol. 3, Lahti, Finland, pp. 17–24.

Green, J. and Dixon, C.: 1993, 'Talking knowledge into being: Discursive and social practices in classrooms', *Linguistics and Education* 5(3&4), 231–239.

Gumperz, J.J. and Field, M.: 1995, 'Children's discourse and inferential practices in cooperative learning', *Discourse Processes* 19(1), 133–47.

Halliday, M.A.K. and Martin, J.R.: 1993, *Writing Science.* University of Pittsburgh Press, Pittsburgh.

Harste, J.C., Woodward, V.A. and Burke, C.: 1984, 'Examining our assumptions: A transactional view of literacy and learning', *Research in the Teaching of English* 18(1), 84–108.

Hicks, D.: 1995, 'Discourse, learning and teaching', in M.W. Apple (ed.), *Review of Research in Education* 21, American Educational Research Association, Washington, DC, pp. 49–95.

Hiebert, J., Carpenter, T.P., Fennema, E., Fuson, K., Human. P., Murray. H., Olivier, A. and Wearne, D.: 1996, 'Problem solving as a basis for reform in curriculum and instruction: The case of mathematics', *Educational Researcher* 25(4), 12–21.

Holyoak, K.J.: 1982, 'An analogical framework for literary interpretation', *Poetics* 11, 105–126.

John-Steiner, V.P. and Meehan, T.: 2000, 'Creativity and collaboration', in C.D. Lee and P. Smagorinsky (eds.), *Vygoskian Perspectives on Literacy Research: Constructing Meaning through Collaborative Inquiry,* Cambridge University Press, New York, pp. 31–48.

Lampert, M.: 1990, 'Connecting inventions with conventions', in L.P. Steffe and T. Wood (eds.), *Transforming Children's Mathematics Education: International Perspectives,* Erlbaum, Hillsdale, New Jersey, pp. 253–265.

Lampert, M. and Blunk, M.L.: (eds.), 1998, *Talking Mathematics in School: Studies of Teaching and Learning,* Cambridge University Press, New York.

Lave, J. and Wenger, E.: 1991, *Situated Learning: Legitimate Peripheral Participation,* Cambridge University Press, New York.

Lee, C.D.: 2000, 'Signifying in the zone of proximal development', in C.D. Lee and P. Smagorinsky (eds.), *Vygotskian Perspectives on Literacy Research: Constructing Meaning through Collaborative Inquiry,* Cambridge University Press, New York, pp. 191–225.

Lemke, J.L.: 1991, *Talking Science: Language, Learning and Values,* Ablex, Norwood, New Jersey.

Lemke, J.L.: 1997, 'Cognition, context, and learning: A social semiotic perspective,' in D. Kirshner and J.A. Whitson, (eds.), *Situated Cognition: Social, Semiotic and Psychological Perspectives,* Erlbaum, Mahwah, New Jersey, pp. 37–55.

Leont'ev, A.N.: 1981, 'The problem of activity in psychology', in J. Wertsch (ed.), *The Concept of Activity in Soviet Psychology,* M.E. Sharpe, Armonk, NY, pp. 37–71.

Lerman, S.: 1998, 'A moment in the zoom of a lens: Towards a discursive psychology of mathematics teaching and learning', in A. Olivier and K. Newstead (eds.), *Proceedings of the 22nd International Conference, Psychology of Mathematics Education,* Vol. 1, Stellenbosch, South Africa, pp. 66–81.

Lyons, M. and Lyons, R.: 1991, *Défi Mathématique,* Mondia éditeurs, Montréal, Québec.

Lyons, M. and Lyons, R.: 1996, *Challenging Mathematics (Grade 1–6): Teaching and Activity Guide,* V. Tétrault, (trans.), Mondia éditeurs, Montréal, Québec.

Maguire, M.H. and Graves, B.: in press, 'Speaking personalities in primary school children's second language writing', *TESOL Quarterly.*

Matusov, E.: 1996, 'Intersubjectivity without agreement' *Mind, Culture, and Activity* 3, 25–45.

Mishler, E.G.: 1991, 'Representing discourse: The rhetoric of transcription', *Journal of Narrative and Life History* 1(4), 255–280.

National Council of Teachers of Mathematics.: 1989, *Curriculum and Evaluation Standards for School Mathematics,* National Council of Teachers of Mathematics, Reston, VA.

National Council of Teachers of Mathematics.: 1991, *Professional Standards for Teaching Mathematics,* National Council of Teachers of Mathematics, Reston, VA.

Newman, D., Griffin, P. and Cole, M.: 1989, *The Construction Zone: Working for Cognitive Change in School,* Cambridge University Press, Cambridge.

Ochs, E.: 1979, 'Transcription as theory', in E. Ochs and B. Schieffelin (eds.), *Developmental Pragmatics,* Academic, New York.

Ochs, E.: 1993, 'Constructing social identity: A language socialization perspective', *Research on Language and Social Interaction* 26, 287–306.

Pimm, D.: 1987, *Speaking Mathematically: Communication in Mathematics Classrooms,* Routledge and Kegan, New York.

Pimm, D.: 1995, *Symbols and Meanings in School Mathematics,* Routledge, New York.

Postman, N.: 1995, *The End of Education: Redefining the Value of School,* Knopf, New York.

Roth, W.-M.: 1995, *Authentic School Science: Knowing and Learning in Open-Inquiry Science Laboratories,* Kluwer Academic Publishers, Norwell, MA.

Shotter, J. and Billig, M.: 1998, 'A Bakhtinian psychology: From out of the heads of individuals and into the dialogues between them', in M. Mayerfeld Bell and M. Gardiner (eds.), *Bakhtin and the Human Sciences,* Sage, London.

Smithson, J. and Díaz, F.: 1996, 'Arguing for a collective voice: Collaborative strategies in problem-oriented conversation', *Text* 16(2), 251–268.

Taylor, C.: 1991, 'The dialogical self', in D.R. Hiley, J.F. Bohman and R. Shusterman (eds.), *The Interpretive Turn: Philosphy, Science, Culture,* Cornell University Press, Ithaca, NY, pp. 304–314.

Toulmin, S.: 1995, *The Uses of Argument,* Cambridge University Press, New York.

Vosniadou, S. and Ortony, A. (eds.): 1989, *Similarity and Analogical Reasoning,* Cambridge University Press, New York.

Vygotsky, L.S.: 1962, *Thought and Language,* M.I.T. Press, Cambridge.

Vygotsky, L.S.: 1978, *Mind in Society: The Development of Higher Psychological Processes,* M. Cole, V. John-Steiner, S. Scribner and E. Souberman, (eds.), Harvard University Press, Cambridge.

Vygotsky, L.S.: 1986, *Thought and Language,* A. Kozulin, (ed.), M.I.T. Press, Cambridge.

Walkerdine, V.: 1990, 'Difference, cognition, and mathematics education', *For the Learning of Mathematics* 10, 51–56.

Webb, N.M.: 1989, 'Peer interaction and learning in small groups', *International Journal of Educational Research* 13, 21–39.

Wells, G.: 1999, *Dialogic Inquiry in Education: Towards a Sociocultural Practice and Theory of Education,* Cambridge University Press, New York.

Wells, G.: 2000, 'Dialogic inquiry in education', in C.D. Lee and P. Smagorinsky (eds.), *Vygoskian Perspectives on Literacy Research: Constructing Meaning through Collaborative Inquiry,* Cambridge University Press, New York, pp. 51–85.

Wertsch, J.V.: 1998, *Mind as Action,* Oxford University Press, New York.

Wood, D., Bruner, J.S. and Ross, G.: 1976, 'The role of tutoring in problem-solving', *Journal of Child Psychology and Child Psychiatry* 17, 89–100.

Yackel, E. and Cobb, P.: 1996, 'Sociomathematical norms, argumentation, and autonomy in mathematics', *Journal for Research in Mathematics Education* 27(4), 458–477.

Zack, V.: 1993, 'Children's perceptions of the usefulness of peer explanations', in I. Hirabayashi, N. Nohda, K. Shigematsu and F-L. Lin (eds.), *Proceedings of the 17th International Conference, Psychology of Mathematics Education,* Vol. 2, Tsukuba, Japan, pp. 286–292.

Zack, V.: 1995, 'Algebraic thinking in the upper elementary school: The role of collaboration in making meaning of "generalisation"', in D. Carraher and L. Meira (eds.), *Proceedings of the 19h International Conference, Psychology of Mathematics Education,* Vol. 2, Recife, Brazil, pp. 106–113.

Zack. V.: 1996, 'Teacher-as-learner: Reciprocal learning in an inquiry math fifth grade elementary classroom', Paper presented at the American Educational Research Association (AERA), New York.

Zack, V.: 1997, 'You have to prove us wrong: Proof at the elementary school level', in E. Pehkonen (ed.), *Proceedings of the 21st International Conference, Psychology of Mathematics Education,* Vol. 4, Lahti, Finland, pp. 291–298.

Zack, V.: 1999, '*Everyday* and *mathematical* language in children's argumentation about proof', *Educational Review* 51(2), Special issue: Culture and the Mathematics Classroom, 129–146.

[1] *St. George's Elementary School,*
3685 The Boulevard,
Montreal, Quebec, H3Y 1S9,
Canada
Telephone (514) 486–6826,
E-mail: vicki.zack@mcgill.ca

[2] *Assistant Professor,*
Faculty of Education,
University of Ottawa, K1N 6N5,
Canada
Telephone (613) 562–5800 ext. 4106,
E-mail: bgraves@uottawa.ca

CELIA HOYLES

FROM DESCRIBING TO DESIGNING MATHEMATICAL ACTIVITY: THE NEXT STEP IN DEVELOPING A SOCIAL APPROACH TO RESEARCH IN MATHEMATICS EDUCATION?

Commentary on the Special Issue of Educational Studies in Mathematics 'Bridging the Individual and the Social: Discursive Approaches to Research in Mathematics Education.'

I am delighted and honoured to have been given the opportunity to provide a commentary on the papers presented in this special issue of Educational Studies in Mathematics, edited by Carolyn Kieran, Ellice Forman and Anna Sfard. It has given me the impetus to read with care accounts of research studies that define themselves as within the socio-cultural paradigm.[1] The editors should be congratulated on bringing together a rich mix of papers that take different, but complementary, perspectives on the theme of the issue and together make a serious elaboration of the principles underlying this paradigm.

My starting point was as a learner. The papers collectively provided me with excellent summaries of a range of general theories underpinning the emerging social paradigm. I asked myself the following questions. What would the theoretical framing and methodologies of a socio-cultural approach add to the collective understandings developed in our field over the past thirty years? How can socio-cultural theory help us to understand and support students' developing *mathematical* learning? Could I propose a novel slant on some of the ideas or analyses in the papers that might offer alternative but, to me at least, fruitful interpretative frameworks? Could I identify any omissions in analytic focus that, if addressed, might usefully form part of a future research agenda?

Given restrictions in space, my commentary cannot be exhaustive nor do justice to the wealth of insights offered in this rather large corpus of work. I have chosen therefore not to engage in theoretical discussion to reconcile (or not), for example, Vygotskian or Piagetian theories (some excellent discussions appear elsewhere, see Steffe and Thompson, 2000; Lerman, 1996; Cobb, 1996). Rather I choose to discuss theoretical issues only in so far as they have illuminated an agenda of a *mathematics* education researcher or, given the aim of this issue, served to move the

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community beyond the unproductive split between individual and social research perspectives.

In seeking to specify my initial goals, I must state the obvious. My commentary will be personal, inevitably shaped by my past experience and my research in mathematics education. So let me start with a personal comment. Nobody, least of all myself, would wish to deny the influence of the social perspective on mathematics teaching and learning. It is almost a truism to argue that all learning is shaped by history, power relations and culture, and that social forces transform classrooms and the way individuals interrelate and react in them. It is important to investigate both distally and proximally social phenomena (using categories distinguished by diSessa, personal communication), but equally important to distinguish between them. How far is it legitimate to restrict attention to one category of phenomena when researching mathematics education? Is it possible to embrace both categories in any investigation in anything but a superficial way?

Before turning to the papers in the volume, I briefly discuss my own professional career in the spirit of the socio-cultural paradigm, in order to inform the reader of the background to my remarks.[2] In our book (Noss and Hoyles, 1996), Richard Noss and I commented that the community of mathematics education was little more than 25 years old, (now 30 years) but already, in this short time, there had been swings of methodologies, realignments of theoretical frameworks, and occasional paradigm shifts. We traced some of this history and noted a fundamental shift from a focus on mathematical objects and how they were understood in the school population, initially, to a concern with strategies adopted during problem solving, later to a consideration of the construction of knowledge, and eventually to an acknowledgement of the essential complementarily in activity between process and content and of the importance of analysing the totality of mathematical experience. We noted how research had shown that taking the problem situation as the arbiter of meaning was fraught with pitfalls, not least because the mapping between the mathematical and situational elements of a problem turned out to be highly ambiguous, with respect to the mathematics deemed to be relevant, the aspects of the setting considered, and the extreme sensitivity of problem-meanings to social and cultural influences.

A key insight for our theoretical work at that time, was drawn from the seminal research of Vergnaud (1982), Nunes, Carraher and Schliemann (1993) and Lave (1988), who had shown how mathematical meanings constructed within a setting were inextricably interwoven with their representations. Thus structure, context (meant more as physical rather than social setting at that time) and representation all comprised major pillars in our developing framework for understanding mathematics learning.

The next stage in my own research trajectory was a move from this largely cognitivist approach (tinged with concern about issues such as gender, and acknowledging the influence of teacher intervention), to one that included a socio-cultural perspective, in particular in investigations of the role of peer interaction with computer tools in learning mathematics. (Hoyles, Healy and Sutherland, 1991). From these studies, I argued that activity within specially-designed microworlds[3] shaped the interactions in the microworld, the tools of the microworld and the mathematical meanings developed within and from these interactions. Representations and the tools or communicative devices with which they were intimately bound, could no longer be regarded as neutral players in the process of making meaning, a position consistent with the notion of mediated action as elaborated by socio-cultural researchers, such as Wertsch (1991, 1997) and Cole (1996). But there were two dimensions central to this analysis that set it a little apart from these theorists: first, the activity in the microworlds was *designed* to foster *mathematical* meanings through construction, interaction and feedback, and second, the students could *scaffold their own thinking*[4] through communicating with the tools of the microworld and shaping them, through programming, to fit their own purposes.

Through careful design of tools and of the interactions planned to take place in activities around these tools, we noted how students together constructed and reconstructed emergent ideas, and how we, as observers of their actions and their interactions in the form of written programs, gestures and verbal communications, were able to catch sight of this construction process as it took shape – this thinking-in-change. Thus, my research agenda focussed on the design of tools and activities for learning mathematics and how these worked out in practice, by reference to the ideas expressed by small groups of children. My goal was to investigate the transformative potential of tools and the co-evolution of tools and knowledge. One outcome of this research was the elaboration of the notion of *situated abstraction,* coined by Noss and myself (Noss and Hoyles, 1992), as an attempt to capture how knowledge and symbolic technologies mutually constituted each other dialectically, through collective construction and negotiation.

While recognising the importance of the teacher in drawing attention to patterns of actions or symbols, or interesting variants and invariants in feedback, much of the research in computer-based settings could be described as cognitivist; concerned with students expressing their mathematics with the tools available. Mathematics, design and student interaction were the focal points of analysis.[5] Certainly my own work placed rather little emphasis on the wider classroom as a community of practice, where norms are negotiated and understandings taken-as-shared (see for example Yackel and Cobb, 1996), or where tools become integrated into ongoing mathematical work (see for example, Guin and Trouche, 1999).

It is notable that these analyses of tool mediation in constructionist computer-based settings have tended to be separated from the recently popular socio-cultural trend in mathematics education, and in particular that of 'discursive psychology' (Harré & Gillett, 1994): an example of the general tendency to isolate 'computer research' in a separate category from other research. Given my prior research and the limitations to which I have alluded, I am keen to ponder how bringing a discursive perspective to my research would allow a richer analysis; one which could take account of the influence of normative goals in the classroom, their interaction with students' responses and developing ideas and their orchestration by the teacher, while not sacrificing the integrity of mathematical design.

So let me turn to the contributions in this special issue. In his paper, Lerman argues that "the move to a cultural, discursive psychology enables the link between the actions of individuals and groups in the classroom and history and culture, and that such a move is *necessary* for educational studies" (my emphasis). A strong claim, and one for which Lerman provides some theoretical justification. But how will this link be theorised in empirical studies in mathematics education is less clear. In contrast, Sfard takes mathematical discourse as her starting point and argues that one of the factors that makes this discourse special is "its exceptional reliance on symbolic artefacts as its communication-mediating tools." This aspect forms a central part of her case for regarding communication, not simply as an aid to thinking, but tantamount to thinking itself. Sfard insists that the metaphor of thinking-as-communication is a way of achieving a complementarity between the cognitive research tradition based on the metaphor of learning as acquisition, and the social-cultural framework around learning as participation. In this endeavour she appears to be at odds with Lerman. Sfard does not reject the idea of a cognitive invariant. Rather she moves on from an argument about the ontological nature of learning to a presentation of "differing visions of the mechanisms of learning", visions emanating from individual or social analyses. In her detailed analysis of short extracts of student interaction, Sfard makes visible the competing influences on a child's response to mathematics: for example, his or her view of self in relation to mathematics, or the didactic contract with the teacher. Her dual analysis shows convincingly how any interpretative framework inevitably pre-judges 'findings', but, if different interpretative frameworks are used to compare and contrast and hone an argument, a researcher is better able to piece together the complex trajectory of thinking-in-change. Whose contributions are valued (or not) and why are just as important in the trajectory of learning as mathematically correct responses.

In reading Sfard's interpretations of the transcripts, I was struck by their plausibility – although I must admit occasionally to feeling that they tended to be over-judgmental (one boy was "ignorant of this" or it was "not making sense to him"). I also wanted to add to Sfard's important re-interpretation of cognitive conflict as inter- or intra-discursive contradiction, a reference to the need for the prior establishment of a meta-rule for this conflict to be experienced; namely that statements in mathematics *should* be consistent and compatible.

Sfard not only presents her theory and illustrates its principle 'in operation', but also describes in detail the new tools of analysis she has developed that have helped her to come up with her interpretations of the observed phenomena – tools that relate specifically to an analysis of the object- and meta-level aspects of discourse that she distinguishes. These methodological tools are in fact used to excellent effect in the contribution of Kieran that I shall discuss later.

But let me turn to another article, that by van Oers, who also makes explicit what he means by mathematical discourse, and makes a sustained effort to re-contextualise socio-cultural theories to study mathematical learning. Following Steinbring (1998) in describing "mathematics" as a "socially conventionalised discursive frame of understanding", van Oers acknowledges that "not only factual technical mathematical operations are involved in mathematical activities in classrooms, but epistemological constraints and social conventions are also part of the process"; and later, that "doing and learning mathematics means improving one's abilities to participate in mathematical practice, both the operational part (the symbolic technology of mathematics) and the discursive part". In mathematics classrooms, utterances, for van Oers in a similar way to Sfard, are valued according to meta-rules and norms, as well as their literal meanings (a point van Oers acknowledges is not new and discussed by, for example, Cobb and his colleagues in many papers).

Analyses of the discursive rules that regulate communication in mathematics classrooms, and which draw attention to the teacher who introduces and monitors these rules, appear as a central strand in socio-cultural research. Sfard's paper adds a further dimension, since she attempts to bring together analysis of content with that of communication. Not only does she describe, like van Oers, "the meta-discursive rules that regulate the communicative effort", but also twins this analysis with a consideration of "the mediating tools (or simply mediators) that people use as the means of communication". She argues that "tools are the shapers of the content, that is, of the object-level aspects of discourse and meta-discursive rules are the moulders, enablers and navigators of the communicational activities." It appears that object-level aspects are the bridges to a more cognitivist-oriented and individual approach, which could stand alongside and complement the social analysis, while preserving the discursive nature of both.

Returning to analyses of the regulation of interactions in classrooms, Bakhtin's notion of speech genre is used, to good effect to see, in van Oers' terms, how "people's utterances in a communication process are not only regulated by the processes that occur in direct interaction, but also *by the historically developed style of communicating in the particular community of practice*" (his emphasis). It is through interaction with a teacher, often revoicing[6] "relevant" contributions that, van Oers argues, students come to interiorise the rules that regulate the discourse of mathematics– to be systematic, consistent, symbolic, abstract. Revoicing is a distinctive methodological tool in the socio-cultural paradigm: a teacher will "repeat, expand, recast, or translate student explanations for the speaker and the rest of the class" (Forman and Ansell, this issue), and it is in this process that she/he defines what is preferred and allowable.

Abstractness, van Oers suggests, is the hallmark of mathematical thinking. Perhaps he is right, but maybe this is an idea that sits rather problematically alongside a socio-cultural approach and is certainly a term that is hotly debated (see for example Schwartz, 2001). In the Vygotskian School, for example, emphasis is placed on connections between signs, and mathematics appears as the epitome of decontextualisation, the pinnacle of abstraction. Bakhtin/Volosinov suggests that: "What interests the mathematically-minded rationalists is not the relationship of the sign to the actual reality it reflects nor the individual who is its originator, but the relationship of sign to sign within a closed system already accepted and authorised. In other words, they are interested only in the inner logic of the system of signs itself, taken, as algebra, completely independent of the ideological meanings that give the signs their content." (Volosinov, 1973, pp. 57–58).

Thus, to me, the Vygotskian tradition appears to point to mathematical discourse as a unique form, contrasting with all other sign systems. It draws attention to the ways in which meaning is produced in terms of intra-mathematical relations, in sign-sign mediation, and suggests that this is the *only* mechanism for the production of mathematical meaning: there is no effective role either for other symbol systems, or for interaction with social or physical reality (see also Confrey, 1995). If this were the case, we might at least go some way to explaining the difficulty with which so many are enculturated into mathematical discourse, but we would do so by erecting (or maintaining) a rigid barrier between social and practical activity, on the one hand, and mathematical thought, on the other. So a concern that permeated all my reading was about the place in this paradigm of new (or alternative) mathematical epistemologies, possibly brought into being by the presence of new tools. How are new meta-rules and norms and new operational procedures introduced and researched? I will return to this point later, but for the moment, trace in other papers analyses of how the culture of a mathematics classroom is developed and how the teacher enculturates students into what is allowed as mathematical and what is not.

As well as presenting a theoretical framework based on cultural psychology, Forman and Ansell's contribution defines a methodology emanating from this framework, explicitly and in detail. It involves distinguishing episodes in classroom interaction and times of transition, along with care to establish the generalisability of any single case analysed. Their research brings the personality and personal history of the teacher into the analysis of classroom interaction as another tool in the interpretation of her regulation of the classroom dialogue. Forman and Ansell again use the notion of revoicing most productively to recognise changes in the structure of a teacher's discourse, changes that may well have remained hidden in studies within another paradigm. By analysis of the discourse in a classroom community and by placing the individual teacher in her social context, the authors are able to distinguish two distinct voices: one that occurred during discussions of students' invented strategies and the other that emerged during talk about standard algorithms.

Turning to another classroom study, O'Connor set out to understand how the web of mathematical content at the focus of a position-driven discussion[7] might interact with its linguistic formulations, and the constraints and affordances of activity structures. The question under discussion was, "can any fraction be turned into a decimal?" What I took from this text is a picture of an expert teacher orchestrating discussion around this mathematical question, with all its potential meanings; she generated *mathematical dialogue* – by encouraging students to find and test counter-examples and by introducing strategic examples to open up new questions or lines of enquiry – and, at the same time, she built a *mathematical community* – by distinguishing personal disagreement from mathematical disagreement, monitoring what was 'taken-as-shared', and revoicing confusion. What was new to me too was the explicit discussion of the times when the teacher 'mis-interpreted' a student remark or was unable to make sense of it, and the repertoire of face-saving moves in the discourse that she might use. I did, however, miss any individual perspective: for example (following Sfard's analysis), I wondered if the role of counter-example was actually appreciated by the students?

O'Connor's study, like Forman and Ansell's, reveals a phenomenon that might well have remained hidden without her analytic tools: that the teacher's strategies varied, not this time in response to different student contributions, but according to phase of lesson. At times of review where ideas were widely shared, the teacher reorientated her interactions to focus on the precision and accuracy of language as a central part of the discourse of mathematics. In contrast, during exploratory discussion, criteria to evaluate student responses were deliberately loose, so students could "solidify their knowledge and practice their ability to verbally articulate what they know." I pondered this interpretation and how it fitted with principles of socio-cultural research, since it gave me the impression of 'knowledge in one's head'.

Following this thought and pursuing a more individual line of enquiry in relation to tool mediation alongside the social, I would have liked to sec-more analysis in O'Connor's study of the use of the calculator, and how this use might have mediated the meanings the students developed, alongside the dialogue with the teacher. For example, how did the physical limitations of the size of the calculator's window shape students' responses, and what was the status of one student's conjecture apparently derived from the availability of the buttons on a calculator that would allow him to convert any fraction to a decimal? I raise this point here, not to insist that the author *should* have followed up this analysis, but rather to show that while choices must be made, they can (as in this case) leave open avenues for future exploration.

In a third classroom study, again researched against a background of Vygotskian and Bakhtinian social psychology, Zack and Graves add yet another dimension to socio-cultural analysis, namely that of teacher as learner. Again we read of how an expert teacher builds a community where students are expected to conjecture, listen to each other, argue and justify their reasoning in ways that acknowledge others' contributions. The main body of the paper is an analysis of three boys engaging with two open-ended problems against a backdrop of their work in previous problems. In effect, the teacher in setting the problems was seeking to provoke the group to recognise the structural (mathematical) equivalence of 'the diagonals problem' and 'the tunnels problem', an equivalence already noticed by one of the boys. The paper describes the boys' use of mathematical language and concepts, and their evolving understanding, through discussion and argument, of an algebraic expression constructed by one of the children. The analysis focuses on the different roles the boys take, for example, to seek generalisations and encode in algebra, or to seek explanations. It also draws attention to the fact that the teacher understood the analogical relationship conjectured by one boy in a rather different way than was in fact the case, a mismatch that may have led to the impasse described but also led the teacher 'to learn'.

But could the analysis be interpreted in a different way and how could it throw light on what is for me a fascinating question; the question of 'transfer' approached from a socio-cultural perspective? A more cognitivist approach alongside the socio-cultural might have focussed on the use of algebra as a means of expression and of communication (or miscommunication) between the students. Algebra appeared to be the 'expected way' to encode the relationships perceived (a meta-discursive rule?), rather than a language for students to discuss, negotiate and manipulate. I would interpret what an individual boy had constructed interactively and externalised as the algebraic description of the number of diagonals or the number of tunnels as a *situated abstraction*: an interrelated product of constructed knowledge and algebraic expression. Written algebra framed and constrained what the boys 'saw', but at the same time, might also have served as a catalyst for 'seeing the connections', if a discursive move (by the teacher) had been made to shift attention from sign-referent connections to sign-sign connections.

Zack and Grave's research prompted me to return to the work of Balacheff (1991), who several years ago analysed (slightly older) students working on a similar diagonals problem. I wanted to compare his, constructivist and Lakatosian perspective with the socio-cultural approach. The comparison and contrasts turned out to be too numerous for me to elaborate here – it was a fascinating experience. But I simply mention a few differences: differences in research context, that is experimental and 'everyday' classroom; differences in what is produced as evidence; differences in how far the children and the teacher are given personal voices; and differences in interpretation of 'the acceptance' of a counter-example (again Sfard's analysis is a useful reference). Making the comparison also highlighted how hard it was for me to trace how the meta-theoretical tools used in Zack and Grave's study, namely those of semantic discourse analysis, sociolinguistic and conversational analysis and models of informal reasoning were actually operationalised in practice. This is a problem that must be faced by all researchers adopting this paradigm, since inevitably only illustrative data can be presented in any one article.

Methodological approach and analysis is indeed visible in the contribution of Kieran in her analysis of the mathematical discourse of 13-year-old partners solving a mathematical problem. The work consisted of joint problem-solving, followed by individual report writing and then individual work on problems analogous to those worked on jointly. My interest in this paper was more than as a commentator, as I have been involved in rather similar research with group and individual work of similar-aged children, although in my studies the computer was always used for joint problem solving, while in Kieran's research its use was optional. (See for example Hoyles, Healy and Pozzi, 1994; Healy, Pozzi and Hoyles, 1995). Kieran used what she called an interactivity flow chart, "to synthesize from the transcripts the ways in which students interacted with each other, and to permit the researcher not only to detect at a glance the nature of the interactions but also to focus attention on those utterances that seemed to develop the *mathematical* content of the discourse" (my emphasis). The analysis (following Sfard) distinguished between different channels of communication (personal and interpersonal) and different levels of talk (object-level and non-object-level). Thus Kieran focuses on a major dilemma of linking public and private discourse, by looking in detail at the interactions of children around a challenging task. The transcripts made it possible to trace how knowledge was collectively constructed and to conjecture reasons for discrepancies between partners in their subsequent individual responses – an analysis I had not seen before. Kieran's conclusions are worthy of further research, namely that; "The patterns of interaction that were found to be most productive for both members of the pairs were those where the interpersonal channel was the site of frequent object-level utterances. Those interactions where it was the personal channel of only one of the participants that was the main site of the publicly-uttered object-level thinking – utterances that were neither complete nor ever expanded upon – seemed much less conducive to the emergence of mathematical thought for both participants."

This paper, as with many others, left me pondering about where it sat in the socio-cultural paradigm, and the fruitful lines of research it opened up. For example, in the follow-up individual work, how were problems deemed to be "analogous" from this perspective to those the students worked on with a partner? Also, what were the meta-rules regulating the student work, in terms of what was valued, that is their joint products or their individual work.[8] What was the influence of a school culture where questions are presented in logical sequence, thus enabling 'copying'? What was the role of the computer in mediating the interchanges of the students? I will elaborate on this last issue. From my reading of the paper, computer use seemed to prompt, not only a change in patterns of interaction between one of the pairs, but also a change in style of problem solving, to one involving trial and evaluation, where the trial externalised the thinking of one of the pair in a public way for the other boy to build upon. Kieran, in fact, drew attention to computer mediation: and it was at this time [when the pair was actively involved with graphic software] that "the interpersonal channel became alive with object-level utterances." A complementary, individual approach might take this mediation as central and as the interpretative frame of the interactions.

So how can I summarise my reactions to the volume? The socio-cultural paradigm as represented in research reported in these papers is beginning to clarify what this theory can offer mathematics education. I was relieved to find that most authors did not seek to erase the individual perspective, and by their focus on communities of practice did not necessarily deny the integrity of an individual's reasoning. As Sfard argued: "rather than rejecting the long-standing acquisition metaphor, we should supplement it with theories grounded in alternative metaphors." Focussing on any one effort will inevitably limit analysis of others; this is the case if we simply look at the social side, as much as if we simply look at the individual. I see no argument for prioritising one over the other. However, I do insist that studies in mathematics education should involve some discussion of *mathematical* activity, however this is defined. There are invariances to our discipline that we cannot, and should not, ignore.

If we take the zooming metaphor seriously, as Lerman suggests, we must allow the researcher to zoom to interactions of individuals during mathematical activity, and while recognising its limitations *not necessarily analysing* them in the same study. I do not believe it possible or even desirable that "… the goals and desires that are associated with the multiple practices of the classroom *must* form part of the analyses we carry out" (Lerman, 1996, my emphasis). To mention social issues in largely cognitive work, all too easily leads merely to descriptive padding, not used in subsequent analysis. Despite my disagreement with Lerman on this point, he does mention an important set of potential influences on mathematics learning, each of which could usefully be the subject of research or linked to research in other paradigms: for example, class and gender (both notably absent in empirical analyses in this volume), and also tool mediation. Referring to the work of Bartolini Bussi in relation to the drag mode in dynamic geometry, Lerman mentions that "internalising the tool transforms the way one can act geometrically, enabling conjectures to be generated, for example, that are unique to the dynamic geometry environment, as a result of the tool." Research in the constructionist paradigm that I have mentioned earlier has in fact explored this idea in depth. For example, the complexities of the drag mode in use have been extensively analysed by, for example, Hölzl, 2001: the tool is not one object but is constructed differently by the learner community in different activities. If work in these different paradigms on tool mediation could build upon each other, this would be a huge step forward for our community.

Discussion of tool mediation as a unit of analysis was largely missing in this corpus of work, a remark largely referring to mediation by computer tools, but not necessarily limited to these: also absent from these analyses was reference to almost any means of interaction other than the verbal, written communication to take an example. It may be that establishing and elaborating a tool-mediation focus would help to build bridges between the individual and the social. This takes me back to a point I made earlier about new developments. Much of the research presented in this issue analysed and interpreted what was taking place in activities in classrooms. What I missed was any discussion of the *design* of the activities and the *design* or *choice* of the tools or sign systems that were introduced to foster *mathematics* learning. It is not, of course, that design will lead to outcomes in a deterministic way, but at least this focus would allow investigation of the transformative potential of tools in activities (see diSessa, Noss and Hoyles, 1995; and more recently, Cobb, 2000). Teachers not only shape the culture *in* the classroom, but also (with researchers) can play an active role in *changing* this culture – through organising the tasks and activities at an object level, as well as through interactions at a meta-discursive level. Most crucially, acknowledging design brings knowledge and epistemology back into centre stage.

NOTES

1. In the interests of clarity I have chosen to use the term socio-cultural throughout this commentary while recognising that others, including Cole, Wertsch, van Oers, and even Vygotsky himself may use different terms.
2. Much of my work has been conducted collaboratively, most notably with Richard Noss whose contribution to all these ideas I acknowledge from the outset.
3. For a description of a microworld, see Hoyles (1993) and Edwards (1995).
4. Later Richard Noss and I further developed this idea of scaffolding under user control, in our notion of webbing, see Noss and Hoyles, 1996.
5. Research papers in this paradigm can be read in the *International Journal of Computers for Mathematical Learning*.
6. A term coined by O'Connor and Michaels, 1996.
7. A discussion involving a teacher leading a group of students in exploring a central question with a limited number of answers.
8. The distinction between working 'for the group' or for one's own individual learning was found to be crucial in my research referred to earlier.

REFERENCES

Balacheff, N.: 1991, 'Treatment of refutation: Aspects of the complexity of a constructivist approach to mathematics learning', in Glasersfeld, E. (ed.). *Radical Constructivism in Mathematics Education,* Kluwer Academic Publishers, Netherlands, pp. 89–110.

Cobb, P., Wood, T. and Yackel, E.: 1993, 'Discourse, mathematical thinking and classroom practice', in Forman, E.A, Minick. N. and Stone, C.A. (eds.). *Contexts for learning. Sociocultural dynamics in children's development,* Oxford University Press, New York, pp. 91–119.

Cobb, P.: 1996, 'Constructivism and activity theory: A consideration of their similarities and differences as they relate to mathematics education', in Mansfield, H., Pateman, N. and Bednarz N. (eds.). *Mathematics for Tomorrow's Young Children: International perspectives on curriculum,* Kluwer Academic Publishers, Dordrecht, Netherlands, pp. 10–56.

Cobb, P.: 2000, 'The importance of a situated view of learning to the design of research and instruction', in Boaler, J. (ed.). *Multiple Perspectives on Mathematics Teaching and Learning,* Ablex Publishing, Westport, CT. USA, pp. 45–82.

Cole, M.: 1996, *Cultural Psychology: A Once and Future Discipline,* Belknap Press of Harvard University Press, Cambridge MA.

Confrey, J.: 1995, 'Theory of intellectual development. Part 3', *For the Learning of Mathematics* 15(2) (June), 36–48.

diSessa, A., Noss, R. and Hoyles, C. (eds.): 1995, *Computers and Exploratory Learning,* Springer-Verlag, Berlin.

Edwards, L.D.: 1995, 'Microworlds as representations', in diSessa. A., Noss, R. and Hoyles, C. (eds.). *Computers and Exploratory Learning.* Springer-Verlag, Berlin, pp. 127–194.

Guin, D. and Trouche, L.: 1999, 'The complex process of converting tools into mathematical instruments: the case of calculators', *International Journal of Computers for Mathematical Learning* 3(3), 195–227.

Harré, R. and Gillet, G.: 1994, *The discursive mind,* Sage, London.

Healy, L., Pozzi, S. and Hoyles, C.: 1995, 'Making sense of groups, computers and mathematics', *Cognition and Instruction* 13(4), 5056–523.

Hölzl, R.: 2001, 'Using dynamic geometry software to add contrast to geometric situations: A case study', *International Journal of Computers for Mathematical Learning* 6(1), 63–86.

Hoyles, C., Healy, L. and Sutherland, R.: 1991, 'Patterns of discussion between pupil pairs in computer and non-computer environments', *Journal of Computer Assisted Learning* 7, 210–228.

Hoyles, C.: 1993, 'Microworlds/schoolworlds: the transformation of an innovation', in Keitel, C. and Ruthven, K. (eds.), *Learning from Computers: Mathematics Education and Technology* NATO ASI Series, Springer-Verlag, Berlin, pp. 1–17.

Hoyles, C., Healy, L. and Pozzi, S.: 1994, 'Groupwork with computers: an overview of findings', *Journal of Computer Assisted Learning* 10, 202–215.

Hoyles, C. and Noss R.: 1996, *Windows on Mathematical Meanings: Learning Cultures and Computers,* Kluwer Academic Publishers, the Netherlands.

Lerman, S.: 1996. 'Articulating theories of mathematical learning: A challenge to the radical constructivist paradigm?', *Journal for Research in Mathematics Education* 27(2), 133–150.

Lave, J.: 1988, *Cognition in Practice,* Cambridge University Press, Cambridge.

Noss, R. and Hoyles, C.: 1992, 'Looking buck and looking forward', in Hoyles, C. and Noss, R. (eds.), *Learning Mathematics and Logo,* Massachusetts Institute of Technology. Massachusetts, pp. 431–46.

Nunes, T., Schliemann, A.D. and Carraher, D.W.: 1993, *Street Mathematics and School Mathematics,* Cambridge University Press, Cambridge.

O'Connor, M.C. and Michaels, S.: 1996, 'Shifting participants frameworks: orchestrating thinking practices in group discussion', in Hicks, D. (ed.), *Discourse, learning and schooling,* Cambridge University Press, Cambridge, pp. 63–104.

Steffe, L.P. and Thompson, P.W.: 2000, 'Interaction or Intersubjectivity? A reply to Lerman', *Journal for Research in Mathematics Education* 31(2), 191–209.

Schwartz, B. (ed.): 2001, 'Abstraction and Context'. Special Issue of *Quarterly of Cognitive Science* 1(3&4), 237–424,

Steinbring, H.: 1998, 'Mathematical understanding in classroom interaction: the interrelation of social and epistemological constraints', in Seeger, F., Voigt, J. and Waschescio. U. (eds.), *The culture of the mathematics classroom.* Cambridge University Press, Cambridge, pp. 344–372.

Vergnaud, G.: 1982, 'Cognitive and developmental psychology and research in mathematical education: some theoretical and methodological issues', *For the Learning of Mathematics* 3(23), 31–41.

Voloshinov, V.N.: 1973. Marxism and the Philosophy of Language. Seminar Press, New York.

Wertsch, J.V.: 1991, *Voices of the Mind: A Sociocultural Approach to Mediated Action,* Harvard University Press, Cambridge, MA.

Wertsch, J.V.: 1997, *Mind as Action,* Oxford University Press. New York.

Yackel, E. and Cobb, P.: 1996, 'Sociomathematical norms, argumentation, and autonomy in mathematics', *Journal for Research in Mathematics Education* 27, 458–477.

Mathematical Sciences Group,
Institute of Education,
University of London

FALK SEEGER

RESEARCH ON DISCOURSE IN THE MATHEMATICS CLASSROOM: A COMMENTARY

What's the matter with me,
I don't have much to say

(Bob Dylan, *Watching the river flow*)

My commentary on the above papers will not try to do justice to the individual papers – even though they would deserve a lot of praise, individually. They also would deserve a commentary much more founded and much more systematic than the one I have to offer. I have been hoping for a long time that 'discourse' would become more of a topic in mathematics education. Now I can witness how research on discourse flourishes and it would be appropriate to discuss as elaborately and as comprehensively as possible what we have presently at hand.

Taken together the papers present a major step in conceptualising what is going on in the mathematics classroom. If this selection of articles is in any regard representative of the current level of theoretical and empirical penetration of classroom discourse in mathematics education and of what is going on in mathematics education research, there is no need to worry about the future.

Given that situation, I did not feel the necessity to comment upon the theme of 'classroom discourse' proper. Within classroom discourse the current state of reflection and research is much advanced and in excellent condition. The present articles testify that this is not a complimentary statement. For this reason I felt more like asking questions that go beyond classroom discourse so as to situate the discussion on discourse: How representative is classroom discourse if I look at the teaching/learning processes in its entirety? How are the change processes studied that are characteristic of learning, and how are slow and fast processes combined? How far does the equation 'discourse is thinking' carry our research and conceptual work? Does it makes sense to conceive of discourse as the bridge between individual and social?

The answer to these questions is closely connected with what for me is currently a feature sticking out in the theoretical debate of mathematics education. The theoretical debates are largely what they are as a result of the appearance of constructivism in its different forms. Theor-

Educational Studies in Mathematics **46:** 287–297, 2001.

etical approaches present themselves today either as being a critique of constructivist ideas or being an elaboration, a new model so to speak. As a result of constructivist philosophy, theoretical tolerance appears to be a hallmark of current theoretical debates. Often, new ideas are developed as a result of an attempt to reconcile opposing approaches.

In my view, we find ourselves in a situation where thinking deep about the fundaments and perspectives of mathematics education has led to a mutual toleration but a basic incommensurability of perspectives.

It can be said that this statement is a result of some of Anna Sfard's papers on the current state of theorizing in mathematics education and educational psychology. She has done a lot of conceptual work in comparing and critically analysing and integrating the diverse, as she called them, research metaphors. She finds (Sfard, 1998a) the fundamental and opposing metaphors of acquisition and participation and concludes in her analysis of these two metaphors that they are complementary. But also that a more general level of theory building cannot be found, a level that would make it possible to include the two metaphors in one comprehensive theory of learning. The two metaphors are, in Sfard's analysis, divergent but mutually not amenable to critique. It may be regrettable, but the grand panorama of a theory of learning must, thus, remain a patchwork.

Reading her analytical work and papers from recent discussions, like the review on individual and social aspects of learning by Salomon and Perkins (1998) and the debate between Greeno and Anderson, Reder and Simon (see Anderson et al., 1996; Greeno, 1997), one can try the following thought experiment.

It seems like positions in these debates are moving along the dimension 'social – individual' on the one hand and 'construction – reception' on the other. Classically, a radical constructivist view would put an emphasis on the individual and constructive dimension in contradistinction to a cultural-historical view emphasising the social and receptive dimension.

In past theoretical discourses, the individual approaches have modified or adapted some of their basic, often implicit, tenets. For instance, constructivist principles have been modified as a result of a critique of an overemphasis of the individualistic, even supposedly solipsist, tendencies of radical constructivism. Similarly, the cultural-historical approach has been criticised for an overemphasis on the receptive, adult-dominated character of learning. For radical constructivism this has led to the well-known approaches of a social constructivism (see, e.g., Bauersfeld, 1993; Bauersfeld and Cobb, 1995) and in the cultural-historical approach it has led to approaches building on the communicative foundation of a cultural-historical stance.

	individual	social
construction		
acquisition		

Figure 1. The dimensions 'construction – acquisition' and 'individual – social'.

The following simple diagram (see Figure 1) is meant to represent what I am trying to say. It uses the dimensions 'construction – reception' (the figure uses 'acquisition' rather than 'reception') and 'individual – social' to arrive at a simple two-by-two grid. The two 'classic' conceptions of constructivism and activity theory would be located in the shaded fields, constructivism would enter under 'construction' and 'individual' and activity theory under 'acquisition' and 'social'. It is easy to see that the empty fields can be filled with entries that make a lot of sense. Under 'construction' and 'social' one would find social constructivism – I have already mentioned Bauersfeld whose work can illustrate what is meant by that. But also the 'social' continuation of Piaget's project, put forward especially by Perret-Clermont (Doise et al., 1975; Perret-Clermont, 2000; Perret-Clermont et al., 1991) must be mentioned here. 'Individual acquisition' is the centre of attention of such diverse approaches as the one from Kruteckij (1966) or the instruction oriented paradigms based on information-processing psychology (see, e.g., Anderson et al., 1996).

Distributing diverse approaches in a two-by-two grid was not the only thing that I wanted to use the grid for. I wanted to demonstrate *ad oculos* that, for a satisfactorily comprehensive theoretical approach, it is necessary to include all four fields in an analysis. It seems plausible to me to draw two conclusions from this figure. First, there are more than just two metaphors and second, for a comprehensive picture all four elements of the grid have to be included.

A corollary of this assumption is that, in my eyes, in future research and conceptual work in mathematics education, two things stick out as being of critical importance for the future development of the field: first, in addition to the proposition and elaboration of local theories of middle range (Merton, 1961), it becomes increasingly necessary to propose and discuss comprehensive, embracing theories and conceptualization; second, this theoretical work has to be balanced by the systematic development of focal problems for practice, theory and research in mathematics education.

While reading the papers, I have collected impressions of what possibly could sum up to be a preliminary list of core problems of research in mathematics education for the future. To be sure, this will not extend to mathematics education as a whole: mathematics education is too diverse a field with too many affiliated disciplines to have a list of core problems that applies in, say, history of mathematics and mathematics education as well as in computers in mathematics education. I shall restrict my commentaries to the field of classroom studies.

My comments will refer to two themes, themes that are partly touching and even overlapping: one will refer to time and change, the other to the question of ecological validity and representation.

ON TIME AND CHANGE

At the beginning of this section, I would like to put forward the following thesis. While one can observe that slow processes in mathematics education are getting more and more into the focus of research and theoretical work, empirical studies very rarely grasp the essence of these slowly developing processes. Empirical studies are usually under an enormous pressure of time and resources.

Again it is surprising to see that the present articles are very similar in their features as regards the methodological approach: on the background of having collected enormous amounts of data the authors present small snippets from transcripts, highlighting important moments in classroom learning processes, either between students or between teacher and students. The empirical data printed in the papers often represent only a tiny fraction of what has been painstakingly collected and documented by the researchers. And what is actually presented are accordingly often dramatic incidents, short interactions containing in a nutshell the drama that has been developing over a longer period of time and has now reached a certain culminating point. Often, for the researchers, it is not easy to identify those crucial moments; often there is no crucial moment in a whole series of lesson transcripts at all. The excerpts from transcripts, those seconds picked out from endless hours of recorded video footage, have to carry the weight of the whole argument of research studies. This is why the whole drama of the argument in a research paper is so constructed as to run down streams to the solution, the creation of the new in a sentence, a word, a concept, a moment. We have, so to speak, constant translations and translations from translations: what is happening is partly recorded with a video recorder (cut out, alienated), the recording then is transcribed, the transcription giving its own version of what has been happening, reconstructing it through memory and reference to the video footage, and finally the transcript is compressed in a nutshell, pulling together what has been captured in the video and pulled out of it, and giving a summary of what ‘really’ had happened. My purpose in pointing to this sequence of interpretative steps is not to call to attention that this interpretation is highly fragile, fallible and indirect. I want to call into conscious deliberation the fact that most of the time the weight of the argument has to be carried by a line or two of classroom discourse and – this is the interesting point – that we want to know if this fraction of the discourse is in any way representative of the discourse it stands for. The excerpt, the quote, is representative for the point the author wishes to make. But he or she didn’t have to be under pressure to show what the excerpt stands for, represents in the sense of being exemplary or representative of the whole discourse on a topic, during a lesson, etc. There is no need to prove that the quotes are representative without making reference to theoretical arguments. Of course, authors usually give ample proofs that the examples are theoretically representative.

In a sense, this behaviour is rather commonsensical and has been discussed very many times under the heading of the theory-loadedness of observation (see, e.g., Hempel, 1973; Kuhn, 1971; Feyerabend, 1975). What else can one do? There is no way to deal with 100 and even 1000 pages of transcripts already in smaller studies in the sense of giving the reader access to all the data, to all the details. What I want to direct our attention to is the fact that the necessity of resting theoretical argumentation on the presentation of tiny fractions of empirical material leads nearly automatically to a growing neglect of a (re)presentation of slow processes in teaching and learning.

I want to take as an example the paper by O’Connor which in many ways is an excellent example for the discussion on time and change. The paper gives an account of a theme-bound discourse on fractions and decimals which is, as I think, a discourse like it should be. It sees students deeply engaged in the discussion of the question of whether any fraction can be turned into a decimal. Even though sometimes I thought that I might have felt even more admiring if the discourse would have been at times a little bit more theoretical or conceptual. At the end of her paper, O’Connor guesses ‘that it may take several years before a teacher has a sense of control over the discussion and a full understanding of its potential for mathematical thinking’. But to make the discussion look like the one presented in the paper is not only a product of teaching skills developed over a long time span. It is also a product of students’ skills at discussing which have taken quite a long time to develop. In other words, the paper is demonstrating how brilliant, how deep can be the discourse in mathematics classrooms, while simultaneously pointing to some point back in time where the development of teacher skills to direct the discussion and student skills to lead the discussion originated. It is worthwhile to know that discourse can be so rich and fruitful, but it would also be of tremendous interest to learn how this situation, how the skills of teacher and students have developed over time. How was it possible? What were good and what were negative influences? It remains unanswered how learning skills and corresponding situations necessary for this kind of mathematical discussion evolve, and also how this special type of classroom discourse, position-driven discussion, and the language formats used with this type of discourse are especially fruitful for mathematics discussion, for example, in the sense that positions have to be maintained and defended, that 'proofs' have to be given for a position, and so on.

It seems to me that this state of affairs, deplorable as it may be, has something to do with a very general characteristic of information processing in humans. Years ago I became aware of something very interesting while listening to a conference opening by Fritz Heider, the late social psychologist renowned for his work on phenomenal causality and interpersonal perception. He was asking in that introduction what may be the reasons why newspaper and media are so full of violence and destruction. In his answer he avoided the usual explanation referring to suppressed anger and hate as the explanation. He said:

> Now, it is something specific in our environment that destruction and annihilation lead much faster to lasting changes than building up and shaping. Millions of years had been necessary to create the conditions for humans to appear – but destroying this complicated structure can be a matter of seconds. Building up normally takes a long time, destruction takes a short time – and this is equally true for an organism or a building or a machine like a clock. The step from order to chaos is short, from chaos to order is long (Heider, 1978, p. 17; my translation)

Of course, I do not want to say here that analysing transcripts and finding crucial moments is in any way akin to destruction or annihilation. What I want to say is that while concentrating on crucial moments, on unique turning points, we may lose from sight the normal developmental sequence of learning.

Correspondingly, we can hardly find studies trying to describe the gradual development towards the new. The nature of learning is by definition, so to speak, dealing with the new, and we have grown accustomed to expect sudden changes, breaches, conflicts, contradictions when we deal with the analysis of learning. But in order to be able to deal with the new, some skills have to be gradually developed. We have to deal with a characteristic-contradiction here.

Another thing that we have to know more about is how the question of slow and fast processes is related to change in classroom practice. Is it because we know relatively little about the slow processes of evolving skills in learning that it is so hard to install a teaching and learning practice which satisfies our wishes to have our kids raised in good learning environments?

In a sense, the often cited but less really used difference between implicit and explicit learning processes is touched upon here (cf., e.g., Bauersfeld, 1998). One could say that the explicit learning processes are simply the tip of an iceberg where one sixth are the visible, explicit processes and five sixths are the implicit processes remaining invisible. For learning processes it is not so much the visibility which makes up for the difference between implicit and explicit. It can be said that implicit processes are more related to the body and the slowly evolving changes, or as Bourdieu (1977) has put it in his Theory of Practice', that the body-related processes are slower and seem insignificant, but they are lasting and survive explicit conscious processes.

ON ECOLOGICAL VALIDITY AND REPRESENTATION

An issue that is closely related to the remarks in the previous section is the issue of ecological validity. 'Ecological validity' is a concept known from psychological testing, where we have content validity as the expression of how well a test item measures what it is supposed to measure. But the concept is also known from the theoretical approach of Brunswik (1939), or from Bronfenbrenner's (1979) work on learning and environment.

As part of the 'cognitive revolution' Neisser has given a meaning to ecological validity that I would like to use here (cf. Neisser, 1976). He stated that one need for a changing of the paradigm is that conclusions based on experimental data cannot be applied to settings outside the laboratory because experimental conditions do not match conditions in the real world (for a critique of this view see, e.g., Mook, 1983). I would like to put forward similar concerns for research on teaching and learning mathematics from a discourse perspective. The question is: Is research on discourse ecologically valid, i.e., does it give a fairly comprehensive and typical account of teaching and learning in the mathematics classroom?

I have given some reasons why, in my view, discourse is only one, albeit crucial, aspect of teaching and learning mathematics. But I would say that conceptual work following this question has to be fostered in two ways: horizontally, asking for other discourses that impinge on discourses in the classroom, e.g., between peers, between students and parents and so on; and vertically, asking for a comparison with other forms of mediated thinking, e.g., with tools, with texts, with representations in a very general sense.

What I feel can be an excellent completion to the discursive perspective is the concept of representation. It has become especially attractive since the study of Donald (1991) has shown that culture and representation are intimately linked in human evolution. Donald (1991) and Raeithel (1998) have presented convincing arguments for a view which connects different steps and phases in evolution primarily as related to the development and dissemination of external sign-systems. Raeithel (1998) distinguishes three larger phases in cultural evolution that correspond to the development of dramatic-mimetic self-regulation in early and middle Palaeolithic period, the development of the discursive-mythic culture in the upper Palaeolithic period and Mesolithic period, and the development of the culture of object symbol-systems that begins in the Neolithic period and extends into modern times. Donald (1991) has presented steps in the evolution of representational systems that basically express the same picture which Raeithel had given. Donald defines four larger phases in evolution according to the dominance or invention of specific representational systems, i.e., episodic, mimetic, linguistic, and theoretic culture.

The relation of 'narrative/discursive' forms of representation in the classroom to the 'theoretic' forms of representation in the mathematics classroom has always been full of tension. It is felt increasingly problematic to relate the 'conversation about mathematics' to the 'conversation like a mathematician' (see, e.g., the discussion in Sfard et al., 1998a). In a certain sense, this difference touches upon the discourse about the 'two cultures' (see Snow, 1961). Jay Lemke (1999, in press) has formulated a new version of this old problem from a semiotic perspective which may lead to a way out of the pointless and fruitless juxtaposition of science and mathematics on the one side and humanities on the other. His approach starts from the assumption of two basic types of semiosis: typological and topological. This duality of typological and topological semiosis, in my view, seems to provide a conceptual background for a discussion on graphical representations that invites clarifications especially as regards the interaction of representations in written language and mathematical representations.

If the differentiation between typological and topological semiosis will have theoretical significance cannot be finally concluded here. I want to underline here that the difference makes a lot of sense on the background of what has been said about the role of mimetic, narrative, and pictorial-theoretic culture. The particularity of mathematical experience and representation can be seen against this background as the amalgamation of typological ('discrete') meaning ruling, for example, natural language categories, and topological ('continuous') meaning ruling motor and visual representations.[1] This unique feature of mathematical meaning making is, however, also responsible for the particular problems in understanding and doing mathematics in the classroom. Mathematics teaching has to spend great care and stamina to keep a permanent balance as it relates to typological meaning making in written and spoken language on the one hand, and to topological meaning making through graphical representations. Taking care of this balance means also an exercise in the potential and boundaries of translating representation into other systems, e.g., to translate mathematical representations into natural language or to visualise or translate natural language expressions into mathematical expressions. This cross-translation above boundaries and balance-keeping between typological and topological semiosis would certainly be much supported through the imbedding of learning into meaningful contexts as it has recently been advocated by the 'authentic learning' approach.

One conclusion of the evolutionary perspective on representations is that teaching has to take into account the variety of representational systems with their different origins. But it is not enough to say that the evolutionary perspective invites a pluralism of representations (see, e.g., a recent paper by Shaffer and Kaput, 1999, that also draws on the evolutionary approach of Donald, 1991). I would rather conclude that the diversity of representational systems helps to see more clearly where the problems come from that the students often have when they are asked to switch between representational systems. Seen from the evolutionary perspective, switching between representational systems is the problem rather than the solution of teaching and learning.

BY WAY OF AN EPILOGUE

The articles in this special issue often started from the issue of the split between individual and society, finding a bridge over the gulf in the idea of discourse, often with the idea implied that through discourse individual and social are reflexively constituted. It was surprising to see that the authors did not refer more heavily to mediating approaches like the one by Vygotskij and the idea of sign-mediated activity or the one by Peirce and the idea of semiosis. In contrast to the discourse-oriented approaches, where the mediating concept is only effective when and if the split between individual and social continues to exist, we could find here truly triadic approaches. I feel that in many of the articles the split between individual and social after having been sent out of the front door returns through the backdoor much in the sense that Marková (2000) has described the issue. If we rely only on discourse as the theoretical concept for research in learning mathematics, we may fall into the trap of identifying talking about the world with making the world.

My summary in a nutshell would be: research on discourse in the mathematics classroom should proceed on the present high level of empirical analysis. It should possibly consider doing more long-term studies on the formation of proficient discursive classrooms and it should invite in one way or the other the concept of mediated activity and/or semiosis.

Notes

1. Oliver Sacks (2000) has observed that persons who are born deaf are especially good at expressing processes of growth, continuous change and gradual difference through sign language. This supports the notion that, to a degree, discrete qualities correspond more to language, while continuous qualities are more related to the visual and motor field.

References

Anderson, J.R., Reder, L.M. and Simon, H.A.: 1996. 'Situated learning and education', *Educational Researcher* 25(4), 5–11.

Bauersfeld, H.: 1993, 'Theoretical perspectives on interaction in the mathematics classroom', in R. Biehler, R.W. Scholz, R. Sträßer and B. Winkelmann (eds.), *Didactics of Mathematics as a Scientific Discipline,* Kluwer Academic Publishers, Dordrecht, pp. 133–146.

Bauersfeld, H.: 1998, 'About the notion of culture in mathematics education', in F. Seeger, J. Voigt and U. Waschescio (eds.). *The Culture of the Mathematics Classroom,* Cambridge University Press, Cambridge, pp. 375–389.

Bauersfeld, H. and Cobb, P.: 1995, *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures,* Lawrence Erlbaum, Hillsdale, NJ.

Bourdieu, P.: 1977, *Outline of a Theory of Practice,* (R. Nice, trans.). Cambridge University Press, Cambridge.

Bronfenbrenner, U.: 1979, *The Ecology of Human Development,* Harvard University Press, Cambridge, MA.

Brunswik, E.: 1939, 'The conceptual focus of some psychological systems', *Journal of Unified Science (Erkenntnis)* 8, 36–49.

Doise, W., Mugny, G. and Perret-Clermont, A.-N.: 1975. 'Social interaction and the development of cognitive operations', *European Journal of Social Psychology* 5(3), 367–383.

Donald, M.: 1991, *Origins of the Modern Mind: Three Stages in the Evolution of Culture and Cognition,* Harvard University Press, Cambridge, MA.

Feyerabend, P.K.: 1975, *Against Method: Outline of an Anarchistic Theory of Knowledge,* NLB, London.

Greeno, J.G.: 1997, 'On claims that answer the wrong questions', *Educational Researcher* 26(1), 5–17.

Heider, F.: 1978, 'Wahrnehmung und Attribution', in D. Görlitz, W.-U. Meyer and B. Weiner (eds.), *Bielefelder Symposium über Attribution,* Klett-Cotta, Stuttgart, pp. 13–18.

Hempel, C.G.: 1973, 'The meaning of theoretical terms: A critique of standard empiricist construal', in P. Suppes et al. (eds.), *Logic, Methodology and Philosophy of Science IV,* North-Holland Publ. Co., Amsterdam.

Kruteckij, V.A.: 1966, *Zur Struktur der mathematischen Fähigkeiten,* Volk und Wissen, Berlin.

Kuhn, Th.S.: 1971, *The structure of Scientific Revolutions,* 2nd edition. University of Chicago Press, Chicago.

Lemke, J.L.: 1999, 'Typological and topological meaning in diagnostic discourse', *Discourse Processes* 27(2), 173–185.

Lemke, J.L.: in press, 'Mathematics in the middle: Measure, picture, gesture, sign, and word', in M. Anderson, V. Cifarelli, A. Saenz-Ludlow and A. Vile (eds.), *Semiotic Perspectives on Mathematics Education.*

Marková, I.: 2000, 'The individual and society in psychological theory'. *Theory and Psychology* 10(1), 107–116.

Merton, R.K.: 1961, *Social Theory and Social Structure,* 4th printing. Free Press, Glencoe, IL.

Mook, D.G.: 1983, 'In defense of external invalidity', *American Psychologist* 38, 379–87.

Neisser, U.: 1976, *Cognition and Reality,* Freeman, San Francisco.

Perret-Clermont, A.-N.: 2000, *La construction de l'intelligence dans l'interaction sociale.* Lang, Bern.

Perret-Clermont, A.-N., Perret, J.-F., and Bell, N.: 1991, 'The social construction of meaning and cognitive activity in elementary school children', in L.B. Resnick, J.M. Levine and S.D. Teasley (eds.), *Perspectives on Socially Shared Cognition,* APA, Washington, DC, pp. 41–62.

Raeithel, A.: 1998, *Selbstorganisation, Kooperation, Zeichenprozeß – Arbeiten zu einer kulturwissenschaftlichen, anwendungshezogene Psychologic,* Westdeutscher Verlag, Opladen.

Sacks, O.W.: 2000, *Seeing Voices: A Journey into the World of the Deaf,* Vintage Books, New York.

Salomon, G. and Perkins, D.N.: 1998, 'Individual and social aspects of learning', in P.D. Pearson and A. Iran-Nejad (eds.). *Review of Research in Education,* American Educational Research Association, Washington, DC, Vol. 23, pp. 1–24.

Sfard, A.: 1998a, 'On two metaphors for learning and the dangers of choosing just one', *Educational Researcher* 27(2), 4–13.

Sfard, A., Nesher, P., Streefland, L., Cobb, P. and Mason, J.: 1998b, 'Learning mathematics through conversation: Is it as good as they say?' *For the Learning of Mathematic* 18(1), 41–51.

Shaffer, D.W. and Kaput, J.J.: 1999, 'Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education', *Educational Studies in Mathematics* 37, 97–119.

Snow, C.P.: 1961, *The Two Cultures and the Scientific Revolution,* Cambridge University Press, Cambridge.

Institut für Didaktik der Mathematik,
Universität Bielefeld,
Postfach 100 131,
333501 Bielefeld,
Germany,

Telephone +49 521 1065042, Fax +49 521 1062991,
E-mail: falk.seeger@uni-bielefeld.de