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RISK ANALYSIS IN FINANCE AND INSURANCE

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To my parents

Ivea and Victor Melnikov

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Preface

This book deals with the notion of 'risk' and is devoted to analysis of risks in nance and insurance. More precisely, we study risks associated with future repayments (contingent claims), where we understand *risks* as uncertainties that may result in nancial loss and affect the ability to make repayments. Our approach to this analysis is based on the development of a methodology for estimating the present value of the future payments given current nancial, insurance and other information. Using this approach, one can adequately de ne notions of *price* of a nancial contract, of *premium* for insurance policy and of *reserve* of an insurance company. Historically, nancial risks were subject to elementary mathematics of nance and they were treated separately from insurance risks, which were analyzed in actuarial science. The development of quantitative methods based on stochastic analysis is a key achievement of modern nancial mathematics. These methods can be naturally extended and applied in the area of actuarial mathematics, which leads to uni ed methods of risk analysis and management.

The aim of this book is to give an accessible comprehensive introduction to the main ideas, methods and techniques that transform risk management into a quantitative science. Because of the interdisciplinary nature of our book, many important notions and facts from mathematics, nance and actuarial science are discussed in an appropriately simpli ed manner. Our goal is to present interconnections among these disciplines and to encourage our reader to further study of the subject. We indicate some initial directions in the Bibliographic remark.

The book contains many worked examples and exercises. It represents the content of the lecture courses 'Financial Mathematics', 'Risk Management' and 'Actuarial Mathematics' given by the author at Moscow State University and State University – Higher School of Economics (Moscow, Russia) in 1998-2001, and at University of Alberta (Edmonton, Canada) in 2002-2003.

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Introduction

Financial and insurance markets always operate under various types of uncertainties that can affect nancial positions of companies and individuals. In nancial and insurance theories these uncertainties are usually referred to as risks. Given certain states of the market, and the economy in general, one can talk about risk exposure. Any economic activities of individuals, companies and public establishments aiming for wealth accumulation assume studying risk exposure. The sequence of the corresponding actions over some period of time forms the process of risk management. Some of the main principles and ingredients of risk management are qualitative identi cation of risk; estimation of possible losses; choosing the appropriate strategies for avoiding losses and for shifting the risk to other parts of the nancial system, including analysis of the involved costs and using feedback for developing adequate controls.

The rst two chapters of the book are devoted to the (nancial) market risks. We aim to give an elementary and yet comprehensive introduction to main ideas, methods and (probabilistic) models of nancial mathematics. The probabilistic approach appears to be one of the most ef cient ways of modelling uncertainties in the nancial markets. Risks (or uncertainties of nancial market operations) are described in terms of statistically stable stochastic experiments and therefore estimation of risks is reduced to construction of nancial forecasts adapted to these experiments. Using conditional expectations, one can quantitatively describe these forecasts given the observable market prices (events). Thus, it can be possible to construct dynamic hedging strategies and those for optimal investment. The foundations of the modern methodology of quantitative nancial analysis are the main focus of Chapters 1 and 2. Probabilistic methods, rst used in nancial theory in the 1950s, have been developed extensively over the past three decades. The seminal papers in the area were published in 1973 by F. Black and M. Scholes [6] and R.C. Merton [32].

In the rst two sections, we introduce the basic notions and concepts of the theory of nance and the essential mathematical tools. Sections 1.3-1.7 are devoted to now-classical binomial model of a nancial market. In the framework of this simple model, we give a clear and accessible introduction to the essential methods used for solving the two fundamental problems of nancial mathematics: hedging contingent claims and optimal investment. In Section 2.1 we discuss the fundamental theorems on arbitrage and completeness of nancial markets. We also describe the general approach to pricing and hedging in complete and incomplete markets, which generalizes methods used in the binomial model. In Section 2.2 we investigate the structure of option prices in incomplete markets and in markets with constraints. Furthermore, we discuss various options-based investment strategies used in nancial engineering. Section 2.3 is devoted to hedging in the mean square. In Section 2.4 we study a discrete Gaussian model of a nancial market, and in particular, we derive the discrete version of the celebrated Black-Scholes formula. In Section 2.5 we discuss the transition from a discrete model of a market to a classical Black-Scholes diffusion model. We also demonstrate that the Black-Scholes formula (and the equation) can be obtained from the classical Cox-Ross-Rubinstein formula by a limiting procedure. Section 2.6 contains the rigorous and systematic treatment of the Black-Scholes model, including discussions of perfect hedging, hedging constrained by dividends and budget, and construction of the optimal investment strategy (the Merton's point) when maximizing the logarithmic utility function. Here we also study a quantile-type strategy for an imperfect hedging under budget constraints. Section 2.7 is devoted to continuous term structure models. In Section 2.8 we give an explicit solution of one particular real options problem, that illustrates the potential of using stochastic analysis for pricing and hedging long-term investment projects. Section 2.9 is concerned with technical analysis in risk management, which is a useful qualitative complement to the quantitative risk analysis discussed in the previous sections. This combination of quantitative and qualitative methods constitutes the modern shape of nancial engineering.

Insurance against possible nancial losses is one of the key ingredients of risk management. On the other hand, the insurance business is an integral part of the nancial system. The problems of managing the insurance risks are the focus of Chapter 3. In Sections 3.1 and 3.2 we describe the main approaches used to evaluate risk in both individual and collective insurance models. Furthermore, in Section 3.3 we discuss models that take into account an insurance company's nancial investment strategies. Section 3.4 is devoted to risks in life insurance; we discuss both traditional and innovative   exible methods. In Section 3.5 we study risks in reinsurance and, in particular, redistribution of risks between insurance and reinsurance companies. It is also shown that for determining the optimal number of reinsurance companies one has to use the technique of branching processes. Section 3.6 is devoted to extended analysis of insurance risks in a generalized Cramér-Lundberg model.

The book also offers the Software Supplement: Computations in Finance and Insurance (see Appendix A), which can be downloaded from

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Finally, we note that our treatment of risk management in insurance demonstrates that methods of risk evaluation and management in insurance and nance are interrelated and can be treated using a single integrated approach. Estimations of future payments and of the corresponding risks are the key operational tasks of n ancial and insurance companies. Management of these risks requires an accurate evaluation of present values of future payments, and therefore adequate modelling of (nancial and insurance) risk processes. Stochastic analysis is one of the most powerful tools for this purpose.

Chapter 1

Foundations of Financial Risk Management

1.1 Introductory concepts of the securities market. Subject of financial mathematics

The notion of an *asset* (anything of value) is one of the fundamental notions in the financial mathematics. Assets can be *risky* and *non-risky*. Here *risk* is understood as an uncertainty that can cause losses (e.g., of wealth). The most typical representatives of such assets are the following *basic securities*: *stocks* S and *bonds* (bank accounts) B. These securities constitute the basis of a *financial market* that can be understood as a space equipped with a structure for trading the assets.

Stocks are share securities issued for accumulating capital of a company for its successful operation. The stockholder gets the right to participate in the control of the company and to receive dividends. Both depend on the number of shares owned by the stockholder.

Bonds (debentures) are debt securities issued by a government or a company for accumulating capital, restructuring debts, etc. In contrast to stocks, bonds are issued for a specified period of time. The essential characteristics of a bond include the exercise (redemption) time, *face value* (redemption cost), *coupons* (payments up to redemption) and *yield* (return up to the redemption time). The zero-coupon bond is similar to a bank account and its yield corresponds to a bank interest rate.

An *interest rate* $r \geq 0$ is typically quoted by banks as an annual percentage. Suppose that a client opens an account with a deposit of B_0, then at the end of a 1-year period the client's non-risky profit is $\Delta B_1 = B_1 - B_0 = rB_0$. After n years the balance of this account will be $B_n = B_{n-1} + rB_0$, given that only the initial deposit B_0 is reinvested every year. In this case r is referred to as a *simple interest*.

Alternatively, the earned interest can be also reinvested (compounded), then at the end of n years the balance will be $B_n = B_{n-1}(1+r) = B_0(1+r)^n$. Note that here the ratio $\Delta B_n / B_{n-1}$ reflects the profitability of the investment as it is equal to r, the *compound interest*.

Now suppose that interest is compounded m times per year, then

$$B_n = B_{n-1}\left(1 + \frac{r^{(m)}}{m}\right)^m = B_0\left(1 + \frac{r^{(m)}}{m}\right)^{mn}.$$

Such rate $r^{(m)}$ is quoted as a *nominal* (annual) interest rate and the equivalent *effective* (annual) interest rate is equal to $r = \left(1 + \frac{r^{(m)}}{m}\right)^m - 1$.

Let $t \geq 0$, and consider the ratio

$$\frac{B_{t+\frac{1}{m}} - B_t}{B_t} = \frac{r^{(m)}}{m},$$

where $r^{(m)}$ is a nominal annual rate of interest compounded m times per year. Then

$$r = \lim_{m\to\infty} \frac{B_{t+\frac{1}{m}} - B_t}{\frac{1}{m}B_t} = \lim_{m\to\infty} r^{(m)} = \frac{1}{B_t}\frac{dB_t}{dt}$$

is called the nominal annual rate of interest *compounded continuously*. Clearly, $B_t = B_0 e^{rt}$.

Thus, the concept of interest is one of the essential components in the description of time evolution of 'value of money'. Now consider a series of periodic payments (deposits) $f_0, f_1, \ldots, f_n$ (*annuity*). It follows from the formula for compound interest that the present value of k-th payment is equal to $f_k\left(1+r\right)^{-k}$, and therefore the present value of the annuity is $\sum_{k=0}^{n} f_k\left(1+r\right)^{-k}$.

WORKED EXAMPLE 1.1

Let an initial deposit into a bank account be $\$10,000$. *Given that* $r^{(m)} = 0.1$, *find the account balance at the end of 2 years for* $m = 1, 3$ *and* 6. *Also find the balance at the end of each of years* 1 *and* 2 *if the interest is compounded continuously at the rate* $r = 0.1$.

SOLUTION Using the notion of compound interest, we have

$$B_2^{(1)} = 10,000\left(1 + 0.1\right)^2 = 12,100$$

for interest compounded once per year;

$$B_2^{(3)} = 10,000\left(1 + \frac{0.1}{3}\right)^{2\times 3} \approx 12,174$$

for interest compounded three times per year;

$$B_2^{(6)} = 10,000\left(1 + \frac{0.1}{6}\right)^{2\times 6} \approx 12,194$$

for interest compounded six times per year.

For interest compounded continuously we obtain

$$B_1^{(\infty)} = 10,000\,e^{0.1} \approx 11,052\,, \qquad B_2^{(\infty)} = 10,000\,e^{2\times 0.1} \approx 12,214\,.$$

$\square$

Stocks are significantly more volatile than bonds, and therefore they are characterized as *risky assets*. Similarly to bonds, one can define their *profitability* $\rho_n = \Delta S_n / S_{n-1},\ n = 1, 2, \ldots$, where S_n is the price of a stock at time n. Then we have the following discrete equation $S_n = S_{n-1}(1 + \rho_n),\ S_0 > 0$.

The mathematical model of a financial market formed by a bank account B (with an interest rate r) and a stock S (with profitabilities ρ_n) is referred to as a (B, S)-market.

The volatility of prices S_n is caused by a great variety of sources, some of which may not be easily observed. In this case, the notion of *randomness* appears to be appropriate, so that S_n, and therefore ρ_n, can be considered as *random variables*. Since at every time step n the price of a stock goes either up or down, then it is natural to assume that profitabilities ρ_n form a sequence of independent random variables $(\rho_n)_{n=1}^{\infty}$ that take values b and a ($b > a$) with probabilities p and q respectively ($p + q = 1$). Next, we can write ρ_n as a sum of its mean $\mu = bp + aq$ and a random variable $w_n = \rho_n - \mu$ whose expectation is equal to zero. Thus, profitability ρ_n can be described in terms of an 'independent random deviation' w_n from the mean profitability μ.

When the time steps become smaller, the oscillations of profitability become more chaotic. Formally the 'limit' continuous model can be written as

$$\frac{\dot{S}_t}{S_t} \equiv \frac{dS_t}{dt}\frac{1}{S_t} = \mu + \sigma \dot{w}_t\,,$$

where μ is the mean profitability, σ is the volatility of the market and $\dot{w}_t$ is the Gaussian white noise.

The formulae for compound and continuous rates of interest together with the corresponding equation for stock prices, define the binomial (Cox-Ross-Rubinstein) and the diffusion (Black-Scholes) models of the market, respectively.

A participant in a financial market usually invests free capital in various available assets that then form an *investment portfolio*. The process of building and managing such a portfolio is indeed the management of the capital. The redistribution of a portfolio with the goal of limiting or minimizing the risk in various financial transaction is usually referred to as *hedging*. The corresponding portfolio is then called a *hedging portfolio*. An investment strategy (portfolio) that may give a profit even with zero initial investment is called an *arbitrage* strategy. The presence of arbitrage reflects the instability of a financial market.

The development of a financial market offers the participants the *derivative securities*, i.e., securities that are formed on the basis of the basic securities – stocks and bonds. The derivative securities (forwards, futures, options etc.) require smaller initial investment and play the role of insurance against possible losses. Also, they increase the liquidity of the market.

For example, suppose company A plans to purchase shares of company B at the end of the year. To protect itself from a possible increase in shares prices, company A reaches an agreement with company B to buy the shares at the end of the year for a fixed (forward) price F. Such an agreement between the two companies is called a *forward contract* (or simply, *forward*).

Now suppose that company A plans to sell some shares to company B at the end of the year. To protect itself from a possible fall in price of those shares, company A buys a *put option* (seller's option), which confers the right to sell the shares at the end of the year at the fixed *strike price* K. Note that in contrast to the forwards case, a holder of an option must pay a *premium* to its issuer.

Futures contract is an agreement similar to the forward contract but the trading takes place on a *stock exchange*, a special organization that manages the trading of various goods, financial instruments and services.

Finally, we reiterate here that mathematical models of financial markets, methodologies for pricing various financial instruments and for constructing optimal (minimizing risk) investment strategies are all subject to modern financial mathematics.

1.2 Probabilistic foundations of financial modelling and pricing of contingent claims

Suppose that a non-risky asset B and a risky asset S are completely described at any time $n = 0, 1, 2, \ldots$ by their prices. Therefore, it is natural to assume that the price dynamics of these securities is the essential component of a financial market. These dynamics are represented by the following equations

$$\Delta B_n = rB_{n-1}\,, \quad B_0 = 1\,,$$
$$\Delta S_n = \rho_n S_{n-1}\,, \quad S_0 > 0\,,$$

where $\Delta B_n = B_n - B_{n-1}$, $\Delta S_n = S_n - S_{n-1}$, $n = 1, 2, \ldots$; $r \geq 0$ is a constant rate of interest and ρ_n will be specified later in this section.

Another important component of a financial market is the set of admissible actions or strategies that are allowed in dealing with assets B and S. A sequence $\pi = (\pi_n)_{n=1}^{\infty} \equiv (\beta_n, \gamma_n)_{n=1}^{\infty}$ is called an *investment strategy* (portfolio) if for any $n = 1, 2, \ldots$ the quantities β_n and γ_n are determined by prices $S_1, \ldots S_{n-1}$. In other words, $\beta_n = \beta_n(S_1, \ldots S_{n-1})$ and $\gamma_n = \gamma_n(S_1, \ldots S_{n-1})$ are functions of $S_1, \ldots S_{n-1}$ and they are interpreted as the amounts of assets B and S, respectively, at time n. The *value* of a portfolio π is

$$X_n^\pi = \beta_n B_n + \gamma_n S_n\,,$$

where $\beta_n B_n$ represents the part of the capital deposited in a bank account and $\gamma_n S_n$ represents the investment in shares. If the value of a portfolio can change only due to changes in assets prices: $\Delta X_n^\pi = X_n^\pi - X_{n-1}^\pi = \beta_n \Delta B_n + \gamma_n \Delta S_n$, then π is said to be a *self-financing* portfolio. The class of all such portfolios is denoted SF.

A common feature of all derivative securities in a (B, S)-market is their potential liability (payoff) f_N at a future time N. For example, for forwards we have $f_N = S_N - F$ and for call options $f_N = (S_N - K)^+ \equiv \max\{S_N - K, 0\}$. Such liabilities inherent in derivative securities are called *contingent claims*. One of the most important problems in the theory of contingent claims is their *pricing* at any time before the expiry date N. This problem is related to the problem of *hedging contingent claims*. A self-financing portfolio is called a *hedge* for a contingent claim f_N if $X_n^\pi \geq f_N$ for any behavior of the market. If a hedging portfolio is not unique, then it is important to find a hedge π^* with the minimum value: $X_n^{\pi^*} \leq X_n^\pi$ for any other hedge π. Hedge π^* is called the *minimal hedge*. The minimal hedge gives an obvious solution to the problem of pricing a contingent claim: the fair price of the claim is equal to the value of the minimal hedging portfolio. Furthermore, the minimal hedge manages the risk inherent in a contingent claim.

Next we introduce some basic notions from probability theory and stochastic analysis that are helpful in studying risky assets. We start with the fundamental notion of an 'experiment' when the set of possible outcomes of the experiment is known but it is not known *a priori* which of those outcomes will take place (this constitutes the *randomness* of the experiment).

Example 1.1 (Trading on a stock exchange)
A set of possible exchange rates between the dollar and the euro is always known before the beginning of trading, but not the exact value. □

Let Ω be the set of all elementary outcomes ω and let $\mathcal{F}$ be the set of all *events* (non-elementary outcomes), which contains the *impossible* event $\emptyset$ and the *certain* event Ω.

Next, suppose that after repeating an experiment n times, an event $A \in \mathcal{F}$ occurred n_A times. Let us consider experiments whose 'randomness' possesses the following property of *statistical stability*: for any event A there is a number $P(A) \in [0, 1]$ such that $n_A/n \to P(A)$ as $n \to \infty$. This number $P(A)$ is called the *probability* of event A. Probability $P : \mathcal{F} \to [0, 1]$ is a function with the following properties:

1. $P(\Omega) = 1$ and $P(\emptyset) = 0$;
2. $P\big(\cup_k A_k\big) = \sum_k P(A_k)$ for $A_i \cap A_j = \emptyset$.

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*. Every event $A \in \mathcal{F}$ can be associated with its *indicator*:

$$I_A(\omega) = \begin{cases} 1\,, \text{ if } \omega \in A \\ 0\,, \text{ if } \omega \in \Omega \setminus A \end{cases} .$$

Any measurable function $X : \Omega \to \mathbb{R}$ is called a *random variable*. An indicator is an important simplest example of a random variable. A random variable X is called *discrete* if the range of function $X(\cdot)$ is countable: $(x_k)_{k=1}^\infty$. In this case we have the following representation

$$X(\omega) = \sum_{k=1}^{\infty} x_k I_{A_k}(\omega)\,,$$

where $A_k \in \mathcal{F}$ and $\cup_k A_k = \Omega$. A discrete random variable X is called *simple* if the corresponding sum is finite. The function

$$F_X(x) := P(\{\omega : X \leq x\}), \quad x \in \mathbb{R}$$

is called the *distribution function* of X. For a discrete X we have

$$F_X(x) = \sum_{k: x_k \leq x} P(\{\omega : X = x_k\}) \equiv \sum_{k: x_k \leq x} p_k .$$

The sequence $(p_k)_{k=1}^{\infty}$ is called the *probability distribution* of a discrete random variable X. If function $F_X(\cdot)$ is continuous on $\mathbb{R}$, then the corresponding random variable X is said to be *continuous*. If there exists a non-negative function $p(\cdot)$ such that

$$F_X(x) = \int_{-\infty}^{x} p(y) dy ,$$

then X is called an *absolutely continuous* random variable and p is its *density*. The *expectation* (or *mean value*) of X in these cases is

$$E(X) = \sum_{k \geq 1} x_k p_k$$

and

$$E(X) = \int_{\mathbb{R}} x p(x) dx ,$$

respectively. Given a random variable X, for most functions $g : \mathbb{R} \to \mathbb{R}$ it is possible to define a random variable $Y = g(X)$ with expectation

$$E(Y) = \sum_{k \geq 1} g(x_k) p_k$$

in the discrete case and

$$E(Y) = \int_{\mathbb{R}} g(x) p(x) dx$$

for a continuous Y. In particular, the quantity

$$V(X) = E\Big[\big(X - E(X)\big)^2\Big]$$

is called the *variance* of X.

Example 1.2 (Examples of discrete probability distributions)

1. Bernoulli:

   $$p_0 = P(\{\omega : X = a\}) = p , \quad p_1 = P(\{\omega : X = b\}) = 1 - p ,$$

   where $p \in [0, 1]$ and $a, b \in \mathbb{R}$.

2. Binomial:

$$p_m = P(\{\omega : \ X = m\}) = \binom{n}{k} p^m (1-p)^{n-m} \,,$$

where $p \in [0,1], \ n \geq 1$ and $m = 0, 1, \ldots, n$.

3. Poisson (with parameter $\lambda > 0$):

$$p_m = P(\{\omega : \ X = m\}) = e^{-\lambda} \frac{\lambda^m}{m!}$$

for $m = 0, 1, \ldots$.

$\square$

One of the most important examples of an absolutely continuous random variable is a Gaussian (or normal) random variable with the density

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \,, \quad x, m \in \mathbb{R} \,, \ \sigma > 0 \,,$$

where $m = E(X)$ is its mean value and $\sigma^2 = V(X)$ is its variance. In this case one usually writes $X = \mathcal{N}(m, \sigma^2)$.

Consider a positive random variable $\widetilde{Z}$ on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $E(\widetilde{Z}) = 1$, then for any event $A \in \mathcal{F}$ define its new probability

$$\widetilde{P}(A) = E(\widetilde{Z} I_A) \,. \tag{1.1}$$

The expectation of a random variable X with respect to this new probability is

$$\begin{aligned}
\widetilde{E}(X) &= \sum_k x_k \widetilde{P}\big(\{\omega : \ X = x_k\}\big) = \sum_k x_k E\big(\widetilde{Z}\, I_{\{\omega:\ X = x_k\}}\big) \\
&= \sum_k E\big(\widetilde{Z}\, x_k\, I_{\{\omega:\ X = x_k\}}\big) = E\Big(\widetilde{Z} \, \sum_k x_k\, I_{\{\omega:\ X = x_k\}}\Big) \\
&= E(\widetilde{Z} X) \,.
\end{aligned}$$

The proof of this formula is based on the following simple observation

$$E\bigg(\sum_{i=1}^n c_i X_i\bigg) = \sum_{i=1}^n c_i E(X_i)$$

for real constants c_i. Random variable $\widetilde{Z}$ is called the *density* of the probability $\widetilde{P}$ with respect to P.

For the sake of simplicity, in the following discussion we restrict ourselves to the case of discrete random variables X and Y with values $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ respectively. The probabilities

$$P\big(\{\omega : \ X = x_i, \ Y = y_i\}\big) \equiv p_{ij} \,, \quad p_{ij} \geq 0, \ \sum_{i,j} p_{ij} = 1,$$

form the *joint distribution* of X and Y. Denote $p_i = \sum_j p_{ij}$ and $p_j = \sum_i p_{ij}$, then random variables X and Y are called *independent* if $p_{ij} = p_i \cdot p_j$, which implies that $E(XY) = E(X)E(Y)$.

The quantity

$$E(X|Y = y_i) := \sum_i x_i \frac{p_{ij}}{p_j}$$

is called the *conditional expectation* of X with respect to $\{Y = y_i\}$. The random variable $E(X|Y)$ is called the *conditional expectation* of X with respect to Y if $E(X|Y)$ is equal to $E(X|Y = y_i)$ on every set $\{\omega : \ Y = y_i\}$. In particular, for indicators $X = I_A$ and $Y = I_B$ we obtain

$$E(X|Y) = P(A|B) = \frac{P(AB)}{P(B)} \,.$$

We mention some properties of conditional expectations:

1. $E(X) = E\big(E(X|Y)\big)$, in particular, for $X = I_A$ and $Y = I_B$ we have $P(A) = P(B)P(A|B) + P(\Omega \setminus B)P(A|\Omega \setminus B)$;
2. if X and Y are independent, then $E(X|Y) = E(X)$;
3. since by the definition $E(X|Y)$ is a function of Y, then conditional expectation can be interpreted as a *prediction* of X given the information from the 'observed' random variable Y.

Finally, for a random variable X with values in $\{0, 1, 2, \dots\}$ we introduce the notion of a *generating function*

$$\phi_X(x) = E(z^X) = \sum_i z^i p_i \,.$$

We have

$$\phi(1) = 1 \,, \qquad \left.\frac{d^k \phi}{dx^k}\right|_{x=0} = k! p_k$$

and

$$\phi_{X_1 + \cdots + X_k}(x) = \prod_{i=1}^{k} \phi_{X_i}(x)$$

for independent random variables $X_1, \dots, X_k$.

Example 1.3 (Trading on a stock exchange: Revisited)
Consider the following time scale: $n = 0$ (present time), $\dots, n = N$ (can be one month, quarter, year etc.).

An elementary outcome can be written in the form of a sequence $\omega = (\omega_1, \dots, \omega_N)$, where ω_i is an elementary outcome representing the results of trading at time step $i = 1, \dots, N$. Now we consider a probability space

$(\Omega, \mathcal{F}_N, P)$ that contains all trading results up to time N. For any $n \le N$ we also introduce the corresponding probability space $(\Omega, \mathcal{F}_n, P)$ with elementary outcomes $(\omega_1, \dots, \omega_n) \in \mathcal{F}_n \subseteq \mathcal{F}_N$.

Thus, to describe evolution of trading on a stock exchange we need a filtered probability space $(\Omega, \mathcal{F}_N, \mathbb{F}, P)$ called a *stochastic basis*, where $\mathbb{F} = (\mathcal{F}_n)_{n \le N}$ is called a *filtration* (or *information flow*):

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_N .$$

For technical reasons, it is convenient to assume that if $A \in \mathcal{F}_n \in \mathbb{F}$, then $\mathcal{F}_n$ also contains the complement of A and is closed under taking countable unions and intersections, that is $\mathcal{F}_n$ is a σ-algebra. $\square$

Now consider a (B,S)-market. Since asset B is non-risky, we can assume that $B(\omega) \equiv B_n$ for all $\omega \in \Omega$. For a risky asset S it is natural to assume that prices $S_1, \dots, S_N$ are random variables on the stochastic basis $(\Omega, \mathcal{F}_N, \mathbb{F}, P)$. Each of S_n is completely determined by the trading results up to time $n \le N$ or in other words, by the σ-algebra of events $\mathcal{F}_n$. We also assume that the sources of trading randomness are exhausted by the stock prices, i.e. $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$ is a σ-algebra generated by random variables $S_1, \dots, S_n$.

Let us consider a specific example of a (B,S)-market. Let $\rho_1, \dots, \rho_N$ be independent random variables taking values a and b $(a < b)$ with probabilities $P(\{\omega : \ \rho_k = b\}) = p$ and $P(\{\omega : \ \rho_k = a\}) = 1 - p \equiv q$. Define the probability basis: $\Omega = \{a, b\}^N$ is the space of sequences of length N whose elements are equal to either a or b; $\mathcal{F} = 2^\Omega$ is the set of all subsets of Ω. The filtration $\mathbb{F}$ is generated by the prices (S_n) or equivalently by the sequence (ρ_n):

$$\mathcal{F}_n = \sigma(S_1, \dots, S_n) = \sigma(\rho_1, \dots, \rho_n) ,$$

which means that every random variable on the probability space $(\Omega, \mathcal{F}_n, P)$ is a function of $S_1, \dots, S_n$ or, equivalently, of $\rho_1, \dots, \rho_n$ due to relations

$$\frac{\Delta S_k}{S_{k-1}} - 1 = \rho_k , \quad k = 0, 1, \dots .$$

A financial (B,S)-market defined on this stochastic basis is called *binomial*.

Consider a contingent claim f_N. Since its repayment day is N, then in general, $f_N = f(S_1, \dots, S_N)$ is a function of all 'history' $S_1, \dots, S_N$. The key problem now is to estimate (or predict) f_N at any time $n \le N$ given the available market information $\mathcal{F}_n$. We would like these predictions $E(f_N | \mathcal{F}_n)$, $n = 0, 1, \dots, N$, to have the following intuitively natural properties:

1. $E(f_N | \mathcal{F}_n)$ is a function of $S_1, \dots, S_n$, but not of future prices $S_{n+1}, \dots, S_N$.

2. A prediction based on the *trivial* information $\mathcal{F}_0 = \{\emptyset, \Omega\}$ should coincide with the mean value of a contingent claim: $E(f_N | \mathcal{F}_0) = E(f_N)$.

3. Predictions must be compatible:

$$E(f_N|\mathcal{F}_n) = E\Big(E(f_N|\mathcal{F}_{n+1})\Big|\mathcal{F}_n\Big)\,,$$

in particular

$$E\Big(E(f_N|\mathcal{F}_n)\Big) = E\Big(E(f_N|\mathcal{F}_n)\Big|\mathcal{F}_0\Big) = E(f_N)\,.$$

4. A prediction based on all possible information $\mathcal{F}_N$ should coincide with the contingent claim : $E(f_N|\mathcal{F}_N) = f_N$.

5. Linearity:

$$E(\phi f_N + \psi g_N|\mathcal{F}_n) = \phi E(f_N|\mathcal{F}_n) + \psi E(g_N|\mathcal{F}_n)$$

for ϕ and ψ defined by the information in $\mathcal{F}_n$.

6. If f_N does not depend on the information in $\mathcal{F}_n$, then a prediction based on this information should coincide with the mean value

$$E(f_N|\mathcal{F}_n) = E(f_N)\,.$$

7. Denote $f_n = E(f_N|\mathcal{F}_n)$, then from property 3 we obtain

$$E(f_{n+1}|\mathcal{F}_n) = E\Big(E(f_N|\mathcal{F}_{n+1})\Big|\mathcal{F}_n\Big) = E(f_N|\mathcal{F}_n) = f_n$$

for all $n \leq N$. Such stochastic sequences are called *martingales*.

How to calculate predictions? Comparing the notions of a conditional expectation and a prediction, we see that a prediction of f_N based on $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ is equal to the conditional expectation of a random variable f_N with respect to random variables $S_1, \ldots, S_n$.

WORKED EXAMPLE 1.2

Suppose that the monthly price evolution of stock S is given by

$$S_n = S_{n-1}(1 + \rho_n)\,, \quad n = 1, 2, \ldots\,,$$

where profitabilities ρ_n are independent random variables taking values 0.2 and -0.1 with probabilities 0.4 and 0.6 respectively. Given that the current price $S_0 = 200$ ($), find the predicted mean price of S for the next two months.

SOLUTION Since

$$E(\rho_1) = E(\rho_2) = 0.2 \cdot 0.4 - 0.1 \cdot 0.6 = 0.02\,,$$

then

$$E\left(\frac{S_1+S_2}{2}\middle|S_0=200\right) = E\left(\frac{S_0(1+\rho_1)+S_0(1+\rho_1)(1+\rho_2)}{2}\middle|S_0=200\right)$$
$$= \frac{S_0}{2}\Big[E(1+\rho_1)+E(1+\rho_1)E(1+\rho_2)\Big]$$
$$= 100\big[1.02+1.02\cdot 1.02\big] = 206.4\ (\$)\,.$$

$\square$

We finish this section with some further notions and facts from stochastic analysis. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis. For simplicity we assume that Ω is finite. Consider a stochastic sequence $X = (X_n, \mathcal{F}_n)_{n\geq 0}$ adopted to filtration $\mathbb{F}$ and such that $E(|X_n|) < \infty$ for all n. If

$$E(X_n|\mathcal{F}_{n-1}) = X_{n-1} \quad \text{a.s.}$$

for all $n \geq 1$, then X is called a *martingale*. If

$$E(X_n|\mathcal{F}_{n-1}) \geq X_{n-1} \quad \text{a.s. or} \quad E(X_n|\mathcal{F}_{n-1}) \leq X_{n-1} \quad \text{a.s.}$$

for all $n \geq 1$, then X is called a *submartingale* or a *supermartingale*, respectively.

Let a positive random variable $\widetilde{Z}$ be the density of the probability $\widetilde{P}$ (see (1.1)) with respect to P. Consider both these probabilities on measurable spaces $(\Omega, \mathcal{F}_n),\ n \geq 0$, and denote the corresponding densities $\widetilde{Z}_n$. Then $\widetilde{Z}_n = E(\widetilde{Z}|\mathcal{F}_n)$ gives an important example of a martingale.

Any supermartingale X admits the Doob decomposition

$$X_n = M_n - A_n\,,$$

where M is a martingale and A is a non-decreasing ($\Delta A_n = A_n - A_{n-1} \geq 0$) (predictable) stochastic sequence such that $A_0 = 0$ and A_n is completely determined by $\mathcal{F}_{n-1}$.This follows from the following observation

$$\Delta X_n = \Delta M_n - \Delta A_n = \big[X_n - E(X_n|\mathcal{F}_{n-1})\big] + \big[E(X_n|\mathcal{F}_{n-1}) - X_{n-1}\big]\,.$$

Since M^2 is a submartingale, then using Doob decomposition we have

$$M_n^2 = m_n + \langle M, M\rangle_n\,,$$

where m is a martingale and $\langle M, M\rangle$ is a predictable increasing sequence called the *quadratic variation* of M. We clearly have

$$\langle M, M\rangle_n = \sum_{k=1}^{n} E\big((\Delta M_k)^2\big|\mathcal{F}_{k-1}\big)$$

and

$$E\big(M_n^2\big) = E\big(\langle M, M\rangle_n\big)\,.$$

For square-integrable martingales M and N one can define their covariance

$$\langle M, N\rangle_n = \frac{1}{4}\Big\{\langle M+N, M+N\rangle_n - \langle M-N, M-N\rangle_n\Big\}.$$

Martingales M and N are said to be *orthogonal* if $\langle M, N\rangle_n = 0$ or, equivalently, if their product $M\,N$ is a martingale.

Let M be a martingale and H be a predictable stochastic sequence. Then the quantity

$$H * m_n = \sum_{k=0}^{n} H_k \Delta m_k$$

is called a *discrete stochastic integral*. Note that

$$\langle H * m, H * m\rangle_n = \sum_{k=0}^{n} H_k^2 \Delta\langle m, m\rangle_k \,.$$

Consider a stochastic sequence $U = (U_n)_{n\geq 0}$ with $U_0 = 0$. Define new stochastic sequence X by

$$\Delta X_n = X_{n-1}\Delta U_n\,, \quad X_0 = 1\,.$$

This simple linear stochastic difference equation has an obvious solution

$$X_n = \prod_{k=1}^{n}\big(1+\Delta U_k\big) = \varepsilon_n(U)\,,$$

which is called a *stochastic exponential*.

If X is defined by a non-homogeneous equation

$$\Delta X_n = \Delta N_n + X_{n-1}\Delta U_n\,, \quad X_0 = N_0\,,$$

then it has the form

$$X_n = \varepsilon_n(U)\left[N_0 + \sum_{k=1}^{n}\frac{\Delta N_k}{\varepsilon_k(U)}\right].$$

Stochastic exponentials have the following useful properties:

1. $$\frac{1}{\varepsilon_n(U)} = \varepsilon_n(-U^*)\,,$$

   where

   $$\Delta U^* = \frac{\Delta U_n}{1+\Delta U_n}\,;$$

2. $\varepsilon(U)$ is a martingale if and only if U is a martingale;

3. $\varepsilon_n(U) = 0$ for all $n \geq \tau_0 := \inf\{k : \varepsilon_k(U) = 0\}\,;$

4.

$$\varepsilon_n(U)\varepsilon_n(V) = \varepsilon_n(U + V + [U,V])\,,$$

where

$$[U,V]_n = \sum_{k=1}^{n} \Delta U_k \Delta V_k$$

is the multiplication rule.

1.3 The binomial model of a financial market. Absence of arbitrage, uniqueness of a risk-neutral probability measure, martingale representation.

The binomial model of a (B,S)-market was introduced in the previous section. Sometimes this model is also referred to as the Cox-Ross-Rubinstein model. Recall that the dynamics of the market are represented by equations

$$\Delta B_n = rB_{n-1}\,, \quad B_0 = 1\,,$$
$$\Delta S_n = \rho_n S_{n-1}\,, \quad S_0 > 0\,,$$

where $r \geq 0$ is a constant rate of interest with $-1 < a < r < b$, and profitabilities

$$\rho_n = \begin{cases} b \text{ with probability } p \in [0,1] \\ a \text{ with probability } q = 1 - p \end{cases}, \quad n = 1,\dots,N\,,$$

form a sequence of independent identically distributed random variables. The stochastic basis in this model consists of $\Omega = \{a,b\}^N$, the space of sequences $x = (x_1,\dots,x_N)$ of length N whose elements are equal to either a or b; $\mathcal{F} = 2^\Omega$, the set of all subsets of Ω. The probability P has Bernoulli probability distribution with $p \in [0,1]$, so that

$$P\big(\{x\}\big) = p^{\sum_{i=1}^{N} I_{\{b\}}(x_i)}\,(1-p)^{\sum_{i=1}^{N} I_{\{a\}}(x_i)}\,.$$

The filtration $\mathbb{F}$ is generated by the sequence $(\rho_n)_{n\leq N}:\ \mathcal{F}_n = \sigma(\rho_1,\dots,\rho_n)$.

In the framework of this model we can specify the following notions. A predictable sequence $\pi = (\pi_n)_{n\leq N} \equiv (\beta_n,\gamma_n)_{n\leq N}$ is an *investment strategy* (portfolio). A *contingent claim* f_N is a random variable on the stochastic basis $(\Omega,\mathcal{F},\mathbb{F},P)$. *Hedge* for a contingent claim f_N is a self-financing portfolio with the terminal value $X_n^\pi \geq f_N$. A hedge π^* with the value $X_n^{\pi^*} \leq X_n^\pi$ for any other hedge π, is called the *minimal hedge*. A self-financing portfolio $\pi \in SF$ is called an *arbitrage* portfolio if

$$X_0^\pi = 0\,, \quad X_N^\pi \geq 0 \quad \text{and} \quad P\big(\{\omega:\ X_N^\pi > 0\}\big) > 0\,,$$

which can be interpreted as an opportunity of making a profit without risk.

Note that the risky nature of a (B, S)-market is associated with randomness of prices S_n. A particular choice of probability P (in terms of Bernoulli parameter p) allows one to numerically express this randomness. In general, the initial choice of P can give probabilistic properties of S such that the behavior of S is very different from the behavior of a non-risky asset B. On the other hand, it is clear that pricing of contingent claims should be *neutral to risk*. This can be achieved by introducing a new probability P^* such that the behaviors of S and B are similar under this probability: S and B are on average the same under P^*. In other words, the sequence of discounted prices $(S_n/B_n)_{n\leq N}$ must be, on average, constant with respect to probability P^*:

$$E^*\left(\frac{S_n}{B_n}\right) = E^*\left(\frac{S_0}{B_0}\right) = S_0 \quad \text{for all} \quad n = 1, \ldots, N\,.$$

For $n = 1$ this implies

$$E^*\left(\frac{S_1}{B_1}\right) = S_0\, E^*\left(\frac{1+\rho_1}{1+r}\right) = S_0\big[(1+b)p^* + (1+a)(1-p^*)\big] = S_0\,,$$

where p^* is a Bernoulli parameter that defines P^*. We have

$$p^* + bp^* + 1 + a - p^* - ap^* = 1 + r$$

and therefore

$$p^* = \frac{r-a}{b-a}\,,$$

which means that in the binomial model the risk-neutral probability P^* is *unique*, and

$$P^*\big(\{x\}\big) = (p^*)^{\sum_{i=1}^N I_{\{b\}}(x_i)}\,(1-p^*)^{\sum_{i=1}^N I_{\{a\}}(x_i)}\,.$$

Note that in this case we can find *density* Z_N^* of probability P^* with respect to probability P, i.e. a non-negative random variable such that

$$E\big(Z_N^*\big) = 1 \qquad \text{and} \qquad P^*(A) = E\big(Z_N^*\, I_A\big) \quad \text{for all} \quad A \in \mathcal{F}_N\,.$$

Since Ω is discrete, we only need to compute values of Z_N^* for every elementary event $\{x\}$. We have

$$P^*\big(\{x\}\big) = E\big(Z_N^*\, I_{\{x\}}\big) = Z_N^*(x)\, P\big(\{x\}\big)\,,$$

and hence

$$Z_N^*(x) = \frac{P^*\big(\{x\}\big)}{P\big(\{x\}\big)} = \left(\frac{p^*}{p}\right)^{\sum_{i=1}^N I_{\{b\}}(x_i)} \left(\frac{1-p^*}{1-p}\right)^{N-\sum_{i=1}^N I_{\{b\}}(x_i)}\,.$$

To describe the behavior of discounted prices S_n/B_n under the risk-neutral probability P^*, we compute the following conditional expectations for all $n \le N$:

$$\begin{aligned} E^*\left(\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right) &= E^*\left(S_0 \prod_{k=1}^{n} \frac{1+\rho_k}{1+r}\middle|\mathcal{F}_{n-1}\right) \\ &= \frac{S_0}{1+r^n}\, E^*\left(\prod_{k=1}^{n}(1+\rho_k)\big|\mathcal{F}_{n-1}\right) \\ &= \frac{S_0}{1+r^n} \prod_{k=1}^{n-1}(1+\rho_k) E^*(1+\rho_n) \\ &= \frac{S_{n-1}}{B_{n-1}}\, \frac{E^*(1+\rho_n)}{1+r} = \frac{S_{n-1}}{B_{n-1}}\, \frac{1+r}{1+r} \\ &= \frac{S_{n-1}}{B_{n-1}}\,. \end{aligned}$$

This means that the sequence $(S_n/B_n)_{n\le N}$ is a martingale with respect to the risk-neutral probability P^*. This is the reason that P^* is also referred to as a *martingale probability* (*martingale measure*).

The next important property of a binomial market is the absence of arbitrage strategies. Such a market is referred to as a *no-arbitrage market*. Consider a self-financing strategy $\pi = (\pi_n)_{n\le N} \equiv (\beta_n, \gamma_n)_{n\le N} \in SF$ with discounted values X_n^π/B_n. Using properties of martingale probability, we have that for all $n \le N$

$$\begin{aligned} E^*\left(\frac{X_n^\pi}{B_n}\middle|\mathcal{F}_{n-1}\right) &= E^*\left(\beta_n + \gamma_n\, \frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right) \\ &= E^*\big(\beta_n|\mathcal{F}_{n-1}\big) + \gamma_n\, E^*\left(\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right) \\ &= \beta_n + \gamma_n\, \frac{S_{n-1}}{B_{n-1}} = \frac{\beta_n B_{n-1} + \gamma_n S_{n-1}}{B_{n-1}} \\ &= \frac{X_{n-1}^\pi}{B_{n-1}}\,, \end{aligned}$$

which implies that the discounted value of a self-financing strategy is a martingale with respect to the risk-neutral probability P^*. This property is usually referred to as the *martingale characterization of self-financing strategies* SF.

Further, suppose there exists an arbitrage strategy $\widetilde{\pi}$. From its definition we have

$$E\left(\frac{X_N^{\widetilde{\pi}}}{B_N}\right) = \frac{E(X_N^{\widetilde{\pi}})}{B_N} > 0\,.$$

On the other hand, the martingale property of X_n^π/B_n implies

$$E^*\left(\frac{X_N^{\widetilde{\pi}}}{B_N}\right) = E^*\left(\frac{X_0^{\widetilde{\pi}}}{B_0}\right) = E^*(X_0^{\widetilde{\pi}}) = 0\,.$$

Now, for probabilities P and P^* there is a positive density Z^* so that $P^*(A) = E(Z_N^* I_A)$ for any event $A \in \mathcal{F}_N$. Therefore

$$\begin{aligned} 0 = X_0^{\widetilde{\pi}} = X_0^{\widetilde{\pi}}/B_0 &= E^*\left(\frac{X_N^{\widetilde{\pi}}}{B_N}\right) \\ &= \frac{E^*(X_N^{\widetilde{\pi}})}{B_N} = \frac{E(Z_N^* X_N^{\widetilde{\pi}})}{B_N} \\ &\geq \frac{\min_\omega \left[Z_N^*(\omega)\right] \, E(X_N^{\widetilde{\pi}})}{B_N} > 0\,, \end{aligned}$$

which contradicts the assumption of arbitrage.

Now we prove that, in the binomial market framework, any martingale can be represented in the form of a discrete stochastic integral with respect to some basic martingale. Let $(\rho_n)_{n\leq N}$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, P^*)$ defined by

$$\rho_n = \begin{cases} a \text{ with probability } p^* = \frac{r-a}{b-a} \\ b \text{ with probability } q^* = 1 - p^* \end{cases},$$

where $-1 < a < r < b$. Consider filtration $\mathbb{F}$ generated by the sequence (ρ_n) : $\mathcal{F}_n = \sigma(\rho_1, \ldots, \rho_n)$. Any martingale $(M_n)_{n\leq N}$, $M_0 = 0$, can be written in the form

$$M_n = \sum_{k=1}^{n} \phi_k \Delta m_k\,, \tag{1.2}$$

where $(\phi_n)_{n\leq N}$ is predictable sequence, and

$$\left(\sum_{k=1}^{n} \Delta m_k\right)_{n\leq N} = \left(\sum_{k=1}^{n} (\rho_k - r)\right)_{n\leq N}$$

is a ('Bernoulli') martingale.

Since σ-algebras $\mathcal{F}_n$ are generated by $\rho_1, \ldots, \rho_n$, and M_n are completely determined by $\mathcal{F}_n$, then there exist functions $f_n = f_n(x_1, \ldots, x_n)$ with x_k equal to either a or b, such that

$$M_n(\omega) = f_n(\rho_1(\omega), \ldots, \rho_n(\omega))\,, \quad n \leq N\,.$$

The required representation (1.2) can be rewritten in the form

$$\Delta M_n(\omega) = \phi_k(\omega) \Delta m_k$$

or

$$f_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), b) - f_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega)) = \phi_n(\omega)(b - r)\,,$$

$$f_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), a) - f_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega)) = \phi_n(\omega)(a - r)\,,$$

and therefore

$$\phi_n(\omega) = \frac{f_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), b) - f_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega))}{(b-r)}$$
$$= \frac{f_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), a) - f_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega))}{(a-r)},$$

which we now establish. The martingale property implies

$$E^*\Big(f_n(\rho_1, \ldots, \rho_n) - f_{n-1}(\rho_1, \ldots, \rho_{n-1})\Big|\mathcal{F}_{n-1}\Big) = 0,$$

or

$$p^* f_n(\rho_1, \ldots, \rho_{n-1}, b) - (1-p^*) f_n(\rho_1, \ldots, \rho_{n-1}, a) = f_{n-1}(\rho_1, \ldots, \rho_{n-1}).$$

Therefore

$$\frac{f_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), b) - f_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega))}{1-p^*}$$
$$= \frac{f_n(\rho_1(\omega), \ldots, \rho_{n-1}(\omega), a) - f_{n-1}(\rho_1(\omega), \ldots, \rho_{n-1}(\omega))}{p^*},$$

which in view of the choice $p^* = (r-a)/(b-a)$ proves the result.

Using the established martingale representation we now can prove the following representation for density Z_N^* of the martingale probability P^* with respect to P:

$$Z_N^* = \prod_{k=1}^{N}\left(1 - \frac{\mu - r}{\sigma^2}\,(\rho_k - \mu)\right) = \varepsilon_N\left(-\frac{\mu - r}{\sigma^2}\sum_{k=1}^{N}(\rho_k - \mu)\right),$$

where $\mu = E(\rho_k)$, $\sigma^2 = V(\rho_k)$, $k = 1, \ldots, N$.

Indeed, consider $Z_n^* = E\big(Z_N^*\big|\mathcal{F}_n\big)$, $n = 0, 1, \ldots, N$. From the properties of conditional expectations we have that $(Z_n^*)_{n\le N}$ is a martingale with respect to probability P and filtration $\mathcal{F}_n = \sigma(\rho_1, \ldots, \rho_n)$. Therefore, Z_n^* can be written in the form

$$Z_n^* = 1 + \sum_{k=1}^{n}(\rho_k - \mu)\phi_k,$$

where ϕ_k is a predictable sequence. Since $Z_n^* > 0$, we have that it satisfies the following stochastic equation

$$Z_n^* = 1 + \sum_{k=1}^{n} Z_{k-1}^*\,\frac{\phi_k}{Z_{k-1}^*}\,(\rho_k - \mu)$$
$$= 1 + \sum_{k=1}^{n} Z_{k-1}^*\,\psi_k\,(\rho_k - \mu),$$

and hence

$$Z_n^* = \prod_{k=1}^{n} \left(1 + \psi_k \left(\rho_k - \mu\right)\right).$$

Taking into account that Z_N^* is the density of a martingale probability, we can compute the coefficients $\psi_k = \phi_k / Z_{k-1}^*$. For $N = 1$ we have

$$\begin{aligned} 0 &= E^*\left((\rho_1 - r)\middle|\mathcal{F}_0\right) = E^*(\rho_1 - r) = E\big(Z_1^*(\rho_1 - r)\big) \\ &= E\Big(\left(1 + \psi_1\left(\rho_1 - \mu\right)\right)(\rho_1 - r)\Big) \\ &= (\mu - r) + \psi_1 \sigma^2 , \end{aligned}$$

thus $\psi_1 = -(\mu - r)/\sigma^2$.

Now suppose that $\psi_k = -(\mu - r)/\sigma^2$ for all $k = 1, \ldots, N-1$, then using independence of $\rho_1, \ldots, \rho_N$ we obtain

$$\begin{aligned} 0 = E^*\left((\rho_N - r)\middle|\mathcal{F}_{N-1}\right) &= \frac{E\big(Z_N^*(\rho_N - r)\big|\mathcal{F}_{N-1}\big)}{Z_{N-1}^*} \\ &= E\Big(\left(1 + \psi_N\left(\rho_N - \mu\right)\right)(\rho_N - r)\Big|\mathcal{F}_{N-1}\Big) \\ &= E\Big((\rho_N - r) + \psi_N\left(\rho_N - \mu\right)(\rho_N - r)\Big|\mathcal{F}_{N-1}\Big) \\ &= E(\rho_N - r) + \psi_N\, E\Big((\rho_N - \mu)(\rho_N - r)\Big|\mathcal{F}_{N-1}\Big) \\ &= (\mu - r) + \psi_N \sigma^2 , \end{aligned}$$

which gives $\psi_N = -(\mu - r)/\sigma^2$ and proves the claim.

1.4 Hedging contingent claims in the binomial market model. The Cox-Ross-Rubinstein formula. Forwards and futures.

In the framework of a binomial (B, S)-market we consider a financial contract associated with a contingent claim f_N with the future repayment date N.

If f_N is deterministic, then its market risk can be trivially computed since $E(f_N|\mathcal{F}_N) \equiv f_N$. In fact, there is no risk associated with the repayment of this claim as one can easily find the present value of the discounted claim f_N/B_N.

If f_N depends on the behavior of the market during the contract period $[0, N]$, then it is a random variable. The intrinsic risk in this case is related to the ability to repay f_N. To estimate and manage this risk, one should be able to predict f_N given the current market information $\mathcal{F}_n$, $n \leq N$.

We start the discussion of a methodology of pricing contingent claims with two simple examples that illustrate the essence of *hedging*.

WORKED EXAMPLE 1.3

Let $\Omega = \{\omega_1, \omega_2\}$ *and* $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \Big\{\emptyset, \{\omega_1\}, \{\omega_2\}, \Omega\Big\}$. *Consider a single-period binomial* (B, S)*-market with* $B_0 = 1\,(\$)$, $S_0 = 100\,(\$)$, $B_1 = B_0(1+r) = 1+r = 1.2\,(\$)$ *assuming that the annual rate of interest is* $r = 0.2$, *and*

$$S_1 = \begin{cases} 150\,(\$) & \textit{with probability } p = 0.4 \\ 70\,(\$) & \textit{with probability } 1-p = 0.6\,. \end{cases}$$

Find the price for a European call option $f_1 = (S_1 - K)^+ \equiv \max\{0, S_1 - K\}\,(\$)$ *with strike price* $K = 100\,(\$)$.

SOLUTION Clearly

$$f_1 = (S_1 - 100)^+ \equiv \max\{0, S_1 - 100\} = \begin{cases} 50\,(\$) & \text{with probability } 0.4 \\ 0\,(\$) & \text{with probability } 0.6\,. \end{cases}$$

The intuitive price for this option is

$$E\left(\frac{f_1}{1+r}\right) = \frac{0.4 \times 50}{1.2} = 16\,.$$

Now, using the minimal hedging approach to pricing, we construct a self-financing strategy $\pi_0 = (\beta_0, \gamma_0)$ that replicates the final value of the option: $X_1^\pi = f_1$. Since $X_1^\pi = \beta_0(1+r) + \gamma_0 S_1$, then we have

$$\beta_0\,1.2 + \gamma_0\,150 = 50\,,$$
$$\beta_0\,1.2 + \gamma_0\,70 = 0\,,$$

which gives $\beta_0 = -36.5$ and $\gamma_0 = 5/8$. Therefore, the 'minimal hedging' price is

$$X_0^\pi = \beta_0 + \gamma_0 S_0 = -36.5 + 100 \times 5/8 \approx 26\,.$$

Note that this strategy of managing risk (of repayment) assumes that the writer of the option at time 0 sells this option for 26 dollars, borrows 36.5 dollars (as β_0 is negative) and invests the obtained 62.5 dollars in 5/8 (= 62.5/100) shares of the stock S.

Alternatively, we can find a risk-neutral probability p^* from the equation

$$100 = S_0 = E^*\left(\frac{S_1}{1+r}\right) = \frac{150\,p^* + 70\,(1-p^*)}{1.2}\,.$$

So $p^* = 5/8$ and the 'risk-neutral' price is

$$E^*\left(\frac{f_1}{1+r}\right) = \frac{50 \times 5/8}{1.2} \approx 26\,.$$

$\square$

WORKED EXAMPLE 1.4

On the same market, find the price of an option with the final repayment $f_1 = \max\{S_0, S_1\} - S_1$.

SOLUTION Note that

$$f_1 = \begin{cases} 30\,(\$) \text{ with probability } 0.6 \\ 0\,(\$) \quad \text{with probability } 0.4\,. \end{cases}$$

The intuitive price for this option is

$$E\left(\frac{f_1}{1+r}\right) = \frac{0.6 \times 30}{1.2} = 15\,.$$

Using a minimal hedging self-financing strategy $\pi_0 = (\beta_0, \gamma_0)$ we have

$$\beta_0\,1.2 + \gamma_0\,150 = 0\,,$$
$$\beta_0\,1.2 + \gamma_0\,70 = 30\,,$$

hence $\gamma_0 = -3/8$ and $\beta_0 = 3/8 \times 150/1.2 = 450/96 \approx 46.8$. Therefore, the 'minimal hedging' price is

$$X_0^{\pi} = \beta_0 + \gamma_0 S_0 = 46.8 - 100 \times 3/8 = 9.3\,.$$

Finally, the 'risk-neutral' price is

$$E^*\left(\frac{f_1}{1+r}\right) = \frac{30 \times 3/8}{1.2} = \frac{90}{9.6} \approx 9.3\,.$$

In contrast to the previous example, this strategy assumes that the writer of the option at time 0 sells this option for 9.3 dollars, borrows 3/8 shares of the stock S (worth of 37.5 dollars) and invests the obtained 46.8 dollars in a bank account. $\square$

Note that in both examples the 'minimal hedging' price coincides with the 'risk-neutral' price and they differ from the intuitive price for the option. This observation leads us to a more general statement: *the price of a contingent claim is equal to the expectation of its discounted value with respect to a risk-neutral probability.*

To verify this, we consider a contingent claim f_N on a binomial (B, S)-market. The conditional expectation (with respect to a risk-neutral probability) of its discounted value

$$M_n^* = E^*\left(\frac{f_N}{B_N}\middle|\mathcal{F}_n\right), \quad n = 0, \ldots, N\,,$$

is a martingale with the boundary values $M_0^* = E^*(f_N/B_N)$ and $M_N^* = f_N/B_N$. It admits the following representation

$$M_n^* = M_0^* + \sum_{k=1}^{n} \phi_k^*\,(\rho_k - r)\,,$$

where $\phi_k^* = \phi_k^*(S_1, \dots, S_{k-1})$ are completely determined by $S_1, \dots, S_{k-1}$. Let

$$\gamma_n^* = \phi_n^* \frac{B_n}{S_{n-1}} \quad \text{and} \quad \beta_n^* = M_{n-1}^* - \gamma_n^* \frac{S_{n-1}}{B_{n-1}} \,,$$

then we obtain a strategy $\pi^* = (\pi_n^*) \equiv (\beta_n^*, \gamma_n^*)$ with values

$$X_n^{\pi^*} = \beta_n^* B_n + \gamma_n^* S_n \quad \text{and} \quad M_n^{\pi^*} = \frac{X_n^{\pi^*}}{B_n} \,, \quad n = 0, \dots, N \,.$$

In particular,

$$X_n^{\pi^*} = B_N \, M_n^{\pi^*} = \frac{B_N \, f_N}{B_N} = f_N \,,$$

which means that π^* is a hedge for f_N. For any other hedge π, from properties of conditional expectations we have

$$\frac{X_n^{\pi^*}}{B_n} = E^* \left(\frac{X_N^{\pi_N}}{B_N} \Big| \mathcal{F}_n \right) \geq E^* \left(\frac{f_N}{B_N} \Big| \mathcal{F}_n \right) = M_n^* = X_n^{\pi^*} \, B_n \,.$$

Thus π^* is the minimal hedge for a contingent claim f_N.

The initial value $C_N(f) := X_0^{\pi^*}$ of this minimal hedge is called the *price* a contingent claim f_N. As we observed before, it is equal to $E^*(f_N/B_N)$.

Now we compute the price of an arbitrary European call option on a binomial (B,S)-market. In this case $f_N = (S_N - K)^+ \equiv \max\{0, S_N - K\}$. Recall that a European call option gives its holder the right to buy shares of the stock S at a fixed strike price K (which can be distinct from the market price S_N) at time N. The writer of such an option is obliged to sell shares at this price K.

Using the described above methodology we have

$$C_N \equiv C_N\big((S_N - K)^+\big) = E^* \left(\frac{(S_N - K)^+}{(1+r)^N} \right) = \frac{E^*\big((S_N - K) \, I_{\{\omega:\, S_N \geq K\}}\big)}{(1+r)^N} \,.$$

To compute the latter expectation we write

$$\begin{aligned} E^*\big((S_N - K)^+\big) &= E\big(Z_N^* (S_N - K)^+\big) \\ &= E\left(\varepsilon_N \left(- \frac{\mu - r}{\sigma^2} \sum_{k=1}^N (\rho_k - \mu) \right) (S_N - K) \, I_{\{\omega:\, S_N \geq K\}} \right) . \end{aligned}$$

Denote

$$k_0 := \min\left\{ k \leq N \,:\, S_0 (1+b)^k (1+a)^{N-k} \geq K \right\} ,$$

then

$$k_0 = \left[\!\left[\ln \frac{K}{S_0\,(1+a)^N} \Big/ \ln \frac{1+b}{1+a} \right]\!\right] + 1 \,,$$

where $[\![x]\!]$ is the integer part of a real number x. Now since

$$p^* = \frac{r-a}{b-a} \,, \quad \mu = p\,(b-a) + a \,, \quad \sigma^2 = (b-a)^2 p\,(1-p) \,,$$

then we have

$$
\begin{aligned}
&E\Bigg(\varepsilon_N\Bigg(-\frac{\mu-r}{\sigma^2}\sum_{k=1}^{N}(\rho_k-\mu)\Bigg)\,K\,I_{\{\omega:\,S_N\geq K\}}\Bigg)\\
&\quad= K\sum_{k=k_0}^{N}\binom{N}{k}\left[1-\frac{\mu-r}{\sigma^2}(b-\mu)\right]^k p^k\left[1-\frac{\mu-r}{\sigma^2}(a-\mu)\right]^{N-k}(1-p)^{N-k}\\
&\quad= K\sum_{k=k_0}^{N}\binom{N}{k}\left[\frac{p^*}{p}\right]^k p^k\left[\frac{1-p^*}{1-p}\right]^{N-k}(1-p)^{N-k}\\
&\quad= K\sum_{k=k_0}^{N}\binom{N}{k}(p^*)^k(1-p)^{N-k}\,.
\end{aligned}
$$

Next, using properties of stochastic exponentials and the representation $S_n = S_0\,\varepsilon_N\big(\sum_{k=1}^{N}\rho_k\big)$, we obtain

$$
\begin{aligned}
&E\Bigg(\varepsilon_N\Bigg(-\frac{\mu-r}{\sigma^2}\sum_{k=1}^{N}(\rho_k-\mu)\Bigg)\,S_N\,I_{\{\omega:\,S_N\geq K\}}\Bigg)\\
&\quad= S_0\,E\Bigg(\varepsilon_N\Bigg(-\frac{\mu-r}{\sigma^2}\sum_{k=1}^{N}(\rho_k-\mu)\Bigg)\,\varepsilon_N\Big(\sum_{k=1}^{N}\rho_k\Big)\,I_{\{\omega:\,S_N\geq K\}}\Bigg)\\
&\quad= S_0\,E\Bigg(\varepsilon_N\Bigg(-\frac{\mu-r}{\sigma^2}\sum_{k=1}^{N}(\rho_k-\mu)+\sum_{k=1}^{N}\rho_k-\frac{\mu-r}{\sigma^2}\sum_{k=1}^{N}(\rho_k-\mu)\rho_k\Bigg)\,I_{\{\omega:\,S_N\geq K\}}\Bigg)\\
&\quad= S_0\sum_{k=k_0}^{N}\binom{N}{k}\left[1-\frac{\mu-r}{\sigma^2}(b-\mu)+b-\frac{\mu-r}{\sigma^2}(b-\mu)\,b\right]^k p^k\\
&\qquad\qquad\times\left[1-\frac{\mu-r}{\sigma^2}(a-\mu)+a-\frac{\mu-r}{\sigma^2}(a-\mu)\,a\right]^{N-k}(1-p)^{N-k}\\
&\quad= S_0\sum_{k=k_0}^{N}\binom{N}{k}\left[\frac{p^*}{p}(1+b)\right]^k p^k\left[\frac{1-p^*}{1-p}(1+a)\right]^{N-k}\\
&\quad= S_0\sum_{k=k_0}^{N}\binom{N}{k}\big[p^*(1+b)\big]^k\big[1-p^*(1+a)\big]^{N-k}\\
&\quad= S_0\,(1+r)^N\sum_{k=k_0}^{N}\binom{N}{k}\left[p^*\frac{1+b}{1+r}\right]^k\left[1-p^*\frac{1+a}{1+r}\right]^{N-k}.
\end{aligned}
$$

Introducing the notation

$$
\widetilde{p}:=\frac{1+b}{1+r}\,p^*\,,\quad\text{and}\quad B(j,N,p):=\sum_{k=j}^{N}\binom{N}{k}p^k\,(1-p)^{N-k}\,,
$$

we arrive to the Cox-Ross-Rubinstein formula

$$C_N = S_0\, B(k_0, N, \widetilde{p}) - K\,(1+r)^{-N}\, B(k_0, N, p^*)\,.$$

The obtained formula gives the price of the call $(S_N - K)^+$ at time 0. More generally, the price of this call at any time $n \le N$ is given by

$$C_{N,n} = S_n\, B(k_n, N-n, \widetilde{p}) - K\,(1+r)^{-(N-n)}\, B(k_n, N-n, p^*)\,, \qquad (1.3)$$

where $k_n := \min\{n \le k \le N \,:\, S_k \ge K\}$.

The price $C_{N,n}$ is equal to the value of the minimal hedge at time $n \le N$. We also observe that the risk component of the minimal hedge $\pi^* = (\beta_n^*,\, \gamma_n^*)_{n \le N}$ is related to the structure of $C_{N,n}$ in formula (1.3): $\gamma_n^* = B(k_n, N-n, \widetilde{p})$. The other component β_n^* is determined by the condition of self-financing. Thus, the Cox-Ross-Rubinstein formula gives a complete description of risk-neutral strategies for European call options.

Next, we consider a *European put option* with contingent claim $f_N = (K - S_N)^+$, which gives its holder the right to sell shares of the stock S at a fixed strike price K at time N.

Denote the price of a European put option by P_N. Taking into account the martingale property of S_N/B_N and the equality $(K - S_N)^+ = (S_N - K)^+ - S_N + K$, we obtain

$$\begin{aligned} P_N &= E^*\left(\frac{(K - S_N)^+}{(1+r)^N}\right) = E^*\left(\frac{\max\{0,\, K - S_N\}}{(1+r)^N}\right) \\ &= C_N - E^*\left(\frac{S_N}{(1+r)^N}\right) + \frac{K}{(1+r)^N} = C_N - E^*(S_0) + \frac{K}{(1+r)^N} \\ &= C_N - S_0 + \frac{K}{(1+r)^N}\,. \end{aligned}$$

This connection between the prices P_N and C_N is called the *call-put parity* relation. It obviously allows one to express the price of a European put option in terms of the price of a European call option (and vice versa). Further, we note that this holds true not just for a European put option, but also for a whole class of contingent claims of the form $f_N = g(S_N)$, where $g(\cdot)$ is a smooth function on $[0, \infty)$. Indeed, from Taylor's formula we have

$$g(x) = g(0) + g'(0)\, x + \int_0^\infty (x - y) g''(y) dy\,,$$

and therefore

$$C_N(f) = C_N\big(g(S_N)\big) = \frac{g(0)}{(1+r)^N} + S_0\, g'(0) + \int_0^\infty C_N\big((S_N - y)^+\big) g''(y) dy\,.$$

So one can use the Cox-Ross-Rubinstein formula for a European call option to find $C_N(f) = C_N\big(g(S_N)\big)$ for any smooth (twice continuously differentiable) function g.

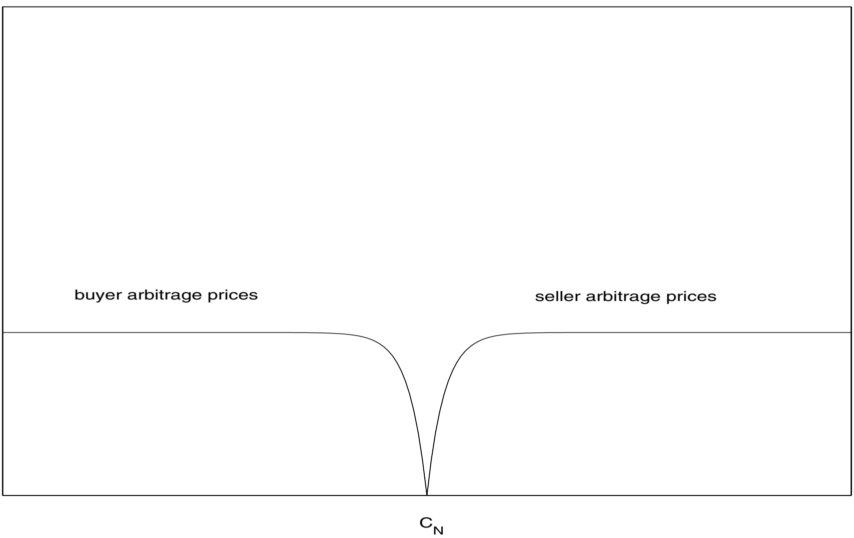
Now we can summarize that the price $C_N(f) := E^*(S_N/B_N)$ if an arbitrary contingent claim f_N has the following properties.

1. It is 'fair' both for the writer of the contract (as it is always possible to invest amount C_N in order to gain the amount f_N and to make the payment at time N) and for the holder (who pays the price that is equal to the minimal amount necessary for hedging). Note that this minimizes risk for both parties.

2. If the writer sells the contract at a price $x > C_N$, then there is an arbitrage opportunity: the amount C_N can be invested in a minimal hedge, and $x - C_N$ is a guaranteed non-risky profit.

3. Conversely, if $x < C_N$, then the holder of the contract can gain an arbitrage profit $C_N - x > 0$.

Thus, the set of all possible prices consists of two regions of arbitrage prices that are separated by C_N, which is therefore referred to as a *non-arbitrage price*.



In the following example we demonstrate an elegant application of the theory of minimal hedging and of the Cox-Ross-Rubinstein formula to pricing equity-linked life insurance contracts, where terminal payment depends on the price of a stock. This contract is attractive to a policy holder since stock may appreciate much faster than money held in a bank account. Additionally, this contract guarantees some minimal payment that protects the policy holder in the case of depreciation of stock. On the other hand, a competitive market environment encourages insurance companies to offer innovative products of this type. Thus, they face a problem of pricing such contracts.

WORKED EXAMPLE 1.5

In the framework of a binomial (B,S)-market an insurance company issues a **pure endowment assurance**. *According to this contract the policy holder is paid*

$$f_N = \max\{S_N,\, K\}$$

on survival to the time N, where S_N is the stock price and K is the guaranteed minimal payment. Find the 'fair' price for such an insurance policy.

SOLUTION Let l_x be the number of policy holders of age x. Each policy holder $i,\ i = 1, \dots, l_x$ can be characterized by a positive random variable T_i representing the time elapsed between age x and death. Denote $p_x(n) = P\big(\{\omega :\ T_i > n\}\big)$, the conditional expectation for a policy holder to survive another n years from the age of x. Suppose that $T_i,\ i = 1, \dots, l_x$, are both mutually independent and independent of $\rho_1, \dots, \rho_N$.

According to the theory developed in this section, it is natural to find the required price C by equating the sum of all premiums to the average sum of all payments:

$$C \times l_x = E^* \left(\sum_{i=1}^{l_x} \frac{f_N}{B_N}\, I_{\{\omega:\ T_i > N\}} \right),$$

where expectation E^* is taken with respect to a martingale probability.

Taking into account that $\max\{S_N,\, K\} = K + (S_N - K)^+$, and independence of T_i's and ρ_k's, we use the Cox-Ross-Rubinstein formula to obtain

$$\begin{aligned} C &= \frac{1}{l_x} \sum_{i=1}^{l_x} E^* \left(\frac{f_N}{B_N}\, I_{\{\omega:\ T_i > N\}} \right) = \frac{1}{l_x}\, l_x\, p_x(N)\, E^* \left(\frac{K + (S_N - K)^+}{B_N} \right) \\ &= p_x(N)\, \frac{K}{(1+r)^N} + p_x(N) \left[S_0\, B(k_0, N, \widetilde{p}) - \frac{K}{(1+r)^N}\, B(k_0, N, p^*) \right]. \end{aligned}$$

$\square$

Next we illustrate how arbitrage considerations can be used in pricing forward and futures contracts.

A *forward contract* is an agreement between two parties to buy or sell a specified asset S for the delivery price F at the delivery date N. Let us consider forwards as investment tools in the framework of a binomial (B,S)-market. Since such agreements can be reached at any date $n = 0, 1, \dots, N$, it is important to determine the corresponding delivery prices $F_0, \dots, F_N$. Note that we clearly have $F_0 = F$ and $F_N = S_N$.

Consider an investment portfolio $\pi = (\beta, \gamma)$ with values

$$X_n^\pi = \beta_n B_n + \gamma_n D_n\,,$$

where γ_n is the number of units of asset S, $D_k = 0$ for $n \le k \le N$, and $D_N = S_N - F_n$.

Taking into account that for a forward contract traded at time $n,\ \gamma_k = 0$ for $k \le n$ and $\gamma_k = \gamma_{n+1}$ for $k \ge n+1$, we compute the discounted value of this portfolio:

$$\Delta\left(\frac{X_k^\pi}{B_k}\right) = \gamma_k \,\frac{\Delta D_k}{B_k}\,,$$
$$\frac{X_N^\pi}{B_N} = \frac{X_n^\pi}{B_n} + \sum_{k=n+1}^{N} \gamma_k \,\frac{\Delta D_k}{B_k} = \frac{X_n^\pi}{B_n} + \gamma_{n+1}\frac{S_N - F_N}{B_N}\,.$$

Using the no-arbitrage condition for strategy π, we can now find forward price F_n:

$$0 = E^*\left(\frac{S_N - F_n}{B_N}\bigg|\mathcal{F}_n\right) = \frac{S_n}{B_n} - \frac{F_n}{B_N}\,,$$

hence

$$F_n = B_N\,\frac{S_n}{B_n}\,.$$

Therefore, we have

$$E^*\left(\frac{X_N^\pi}{B_N}\right) = E^*\left(\frac{X_n^\pi}{B_n}\right),$$

which guarantees that π is a no-arbitrage strategy.

A *futures contract* is the same agreement but the trading takes place on a stock exchange. The clearing house of the exchange opens *margin accounts* for both parties that are used for repricing the contract on daily basis.

Let $F_0^*, \ldots, F_N^*$ be futures prices. Suppose that the parties enter a futures contract on the stock S at time n with the strike price F_n^*. At time $n+1$ the clearing house announces a new quoted price F_{n+1}^*. If $F_{n+1}^* > F_n^*$, then the seller of S loses and must deposit the variational margin $F_{n+1}^* - F_n^*$. Otherwise the buyer deposits $F_n^* - F_{n+1}^*$.

Denote $\delta_0 = F_0^*$ and

$$\delta_n = F_n^* - F_{n-1}^*\,, \quad D_n = \delta_0 + \delta_1 + \cdots + \delta_n\,, \quad \Delta D_n = \delta_n$$

for $n \ge 1$. Consider an investment portfolio π with β_n representing investment in a bank account and γ_n equal to the number of shares of S traded via futures contracts. Then

$$\frac{X_N^\pi}{B_N} = \frac{X_n^\pi}{B_n} + \gamma_{n+1}\sum_{k=n+1}^{N}\frac{\Delta D_k}{B_k}\,.$$

From the no-arbitrage condition we have

$$E^*\left(\sum_{k=n+1}^{N}\frac{\Delta D_k}{B_k}\bigg|\mathcal{F}_n\right) = 0\,,$$

which is equivalent to the fact that $(D_n)_{n\le N}$ is a martingale with respect to P^*, and hence $D_n = E^*(D_N|\mathcal{F}_n)$. Taking into account the equalities $D_N = F_N^* = S_N$ and

$D_n = \delta_0 + \delta_1 + \cdots + \delta_n = F_n^*$, we obtain

$$F_n^* = E^*(S_N|\mathcal{F}_n) = B_N\, E^*\left(\frac{S_N}{B_N}\middle|\mathcal{F}_n\right)$$
$$= B_N\,\frac{S_n}{B_n} = F_n\,.$$

Thus, we arrive at the following general conclusion: *on a complete no-arbitrage binomial (B, S)-market prices of forward and futures contracts coincide.*

1.5 Pricing and hedging American options

In a binomial (B, S)-market with the time horizon N we consider a sequence of contingent claims $(f_n)_{n\leq N}$, where each f_n has the repayment date $n = 0, 1, \ldots, N$. Managing such a collection is not difficult, as we can price each claim f_n:

$$C_n(f_n) = E^*\left(\frac{f_n}{(1+r)^n}\right),$$

and therefore the whole collection:

$$C\Big((f_n)_{n\leq N}\Big) = \sum_{n=0}^{N} C_n(f_n) = E^*\left(\sum_{n=0}^{N}\frac{f_n}{(1+r)^n}\right).$$

In elementary financial mathematics a series of deterministic payments f_n is called an *annuity*. Thus, using this terminology, the latter formula gives the price of a *stochastic annuity*. Note that the linear structure of the collection of contingent claims was used in the calculation of this price. In general, the structure of a series of claims can be much more complex.

Let $(f_n)_{n=0}^N$ be a non-negative stochastic sequence adopted to filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0}^N$, where $\mathcal{F}_n = \sigma(S_0, \ldots, S_n)$. A random variable $\tau : \Omega \to \{0, 1, \ldots, N\}$ is called a *stopping time* (or a *Markov time*) if $\{\omega : \tau = n\} \in \mathcal{F}_n$, i.e., it does not depend on the future. Using sequence $(f_n)_{n=0}^N$ and a stopping time τ we define the following contingent claim

$$f_\tau(\omega) \equiv f_{\tau(\omega)}(\omega) = \sum_{n=0}^{N} f_n(\omega)\, I_{\{\omega:\ \tau=n\}}\,.$$

It is clear from the definition that this claim is determined by all trading information up to time N, but it is exercised at a random time τ, which is therefore called the *exercise time*.

According to the described earlier methodology of managing risk associated with a contingent claim in the framework of a binomial market (B, S)-market, we can price this claim using averaging with respect to a risk-neutral probability P^*:

$$C(f_\tau) = E^*\left(\frac{f_\tau}{B^\tau}\right) = E^*\left(\frac{f_\tau}{(1+r)^\tau}\right).$$

Denote $\mathcal{M}_0^N$ the collection of all stopping times, then we have a collection of contingent claims corresponding to these stopping times $\tau \in \mathcal{M}_0^N$, which is called an *American contingent claim*. Since $C(f_\tau)$ are risk-neutral predictions of future payments f_τ, then the rational price for an American claim must be

$$C_N^{\mathrm{am}} = \sup_{\tau \in \mathcal{M}_0^N} C(f_\tau) = \sup_{\tau \in \mathcal{M}_0^N} E^*\left(\frac{f_\tau}{(1+r)^\tau}\right).$$

Now, since the collection $\big(C(f_\tau)\big)_{\tau \in \mathcal{M}_0^N}$ is finite, then there exists a stopping time $\tau^* \in \mathcal{M}_0^N$ such that

$$C(f_{\tau^*}) = E^*\left(\frac{f_{\tau^*}}{(1+r)^{\tau^*}}\right) = \sup_{\tau \in \mathcal{M}_0^N} E^*\left(\frac{f_\tau}{(1+r)^\tau}\right) = C_N^{\mathrm{am}},$$

which should be the exercise time for an American contingent claim $\big(f_\tau\big)_{\tau \in \mathcal{M}_0^N}$.

Note that, from a mathematical point of view, the pair $(C_N^{\mathrm{am}}, \tau^*)$ solves the problem of finding an optimal stopping time for the stochastic sequence $\big(f_n/(1+r)^n\big)_{n=0}^N$. The financial interpretation of this mathematical problem is pricing an American contingent claim with an exercise time up to the maturity date N. More than 90 % of options traded on exchanges are of American type.

Example 1.4 (Examples of American-type options)

1. *American call and put options* are defined by the following sequences of claims:

$$f_n = (S_n - K)^+ \qquad \text{and} \qquad f_n = (K - S_n)^+, \quad n \le N,$$

respectively.

2. *Russian option* is defined by

$$f_n = \max_{k \le n} S_k.$$

□

Now we describe the *methodology* for pricing such options. As in the case of European options, we use the notion of a strategy (portfolio) $\pi = (\pi_n)_{n=0}^N = (\beta_n, \gamma_n)_{n=0}^N$ with values $X_n^\pi = \beta_n B_n + \gamma_n S_n$. A self-financing strategy is called a hedge if $X_n^\pi \ge f_n$ for all $n = 0, 1, \ldots, N$. In particular, $X_\tau^\pi \ge f_\tau$ for all stopping times $\tau \in \mathcal{M}_0^N$. A hedge π^* such that $X_n^{\pi^*} \geq X_n^{\pi}$ for all $n \leq N$ for any other hedge π, is called the minimal hedge.

Let $\mathcal{M}_n^N$, $0 \leq n \leq N$, be the collection of all stopping times with values in $\{n, \ldots, N\}$. Consider the stochastic sequence

$$Y_n := \sup_{\tau \in \mathcal{M}_n^N} E^* \left(\frac{f_\tau}{(1+r)^\tau} \middle| \mathcal{F}_n \right), \quad n = 0, 1, \ldots, N\,,$$

which has the initial value $Y_0 = C_N^{\mathrm{am}}$ and the terminal value $Y_N = f_N/(1+r)^N$. To find the structure of sequence $(Y_n)_{n=0}^N$, we write

$$Y_N = Y_{\tau_N^*} = \frac{f_N}{(1+r)^N}\,,$$

where $\tau_N^* \equiv N$ is the only stopping time in class $\mathcal{M}_N^N$. Now, for $n = N-1$ we have

$$Y_{N-1} = \begin{cases} \frac{f_{N-1}}{(1+r)^{N-1}} & \text{if} \quad \frac{f_{N-1}}{(1+r)^{N-1}} \geq E^*\left(\frac{f_N}{(1+r)^N} \middle| \mathcal{F}_{N-1}\right) \\ E^*\left(\frac{f_N}{(1+r)^N} \middle| \mathcal{F}_{N-1}\right) & \text{otherwise} \end{cases}\,,$$

which is equivalent to the formula

$$Y_{N-1} = \max\left\{ \frac{f_{N-1}}{(1+r)^{N-1}}\,,\ E^*\left(Y_N \middle| \mathcal{F}_{N-1}\right) \right\}.$$

Putting

$$\tau_{N-1}^* = \begin{cases} N-1 \text{ if} & \frac{f_{N-1}}{(1+r)^{N-1}} \geq E^*\left(\frac{f_N}{(1+r)^N} \middle| \mathcal{F}_{N-1}\right) \\ N & \text{otherwise} \end{cases}\,,$$

we obtain that $Y_{\tau_{N-1}^*}$ is equal either to

$$\frac{f_{N-1}}{(1+r)^{N-1}}$$

or

$$E^*\left(\frac{f_N}{(1+r)^N} \middle| \mathcal{F}_{N-1} \right).$$

For an arbitrary $n \leq N$ we obtain

$$Y_n = \max\left\{ \frac{f_n}{(1+r)^n}\,,\ E^*\left(Y_{n+1} \middle| \mathcal{F}_n\right) \right\}$$

and

$$\tau_n^* = \inf_{n \leq k \leq N} \left\{ k \,:\, Y_k = \frac{f_k}{(1+r)^k} \right\}.$$

Finally,

$$C_N^{\mathrm{am}} = Y_0\,, \qquad \tau^* = \tau_0^*\,.$$

Now, using sequence Y_n, we construct a hedging strategy. Since

$$Y_n \le E^*\big(Y_{n+1}\big|\mathcal{F}_n\big) \quad \text{for all} \quad n \le N-1\,,$$

then $(Y_n)_{n\le N}$ is a supermartingale that admits Doob decomposition:

$$Y_n = M_n - A_n\,,$$

where $(M_n)_{n\le N}$ is a martingale with $M_0 = Y_0$, and $(A_n)_{n\le N}$ is a predictable non-decreasing sequence with $A_0 = 0$. We also have the following martingale representation

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k^*\,\frac{S_{k-1}}{B_{k-1}}\,(\rho_k - r)\,,$$

where γ_k^* is some predictable sequence.

Using these γ_n^*'s we define a self-financing strategy $\pi^* = (\beta_n^*, \gamma_n^*)$ with values $X_n^{\pi^*} = B_n\,M_n$.

This gives us the required hedge, as for all $n \le N$

$$\begin{aligned} X_n^{\pi^*} &= M_n\,B_n = (Y_n + A_n)\,B_n \ge Y_n\,B_n = \sup_{\tau\in\mathcal{M}_n^N} E^*\left(\frac{f_\tau}{(1+r)^\tau}\middle|\mathcal{F}_n\right)B_n \\ &= \sup_{\tau\in\mathcal{M}_n^N} E^*\left(\frac{f_\tau\,B_n}{B_\tau}\middle|\mathcal{F}_n\right) \ge f_n\,, \end{aligned}$$

and

$$X_0^{\pi^*} = Y_0 = \sup_{\tau\in\mathcal{M}_0^N} E^*\left(\frac{f_\tau}{(1+r)^\tau}\right) = C_N^{\mathrm{am}}\,.$$

WORKED EXAMPLE 1.6

On a two-step (B,S)-market, price an American option with payments

$$f_0 = (S_0 - 90)^+ \quad f_1 = (S_1 - 90)^+ \quad f_2 = (S_2 - 120)^+\,,$$

where $S_0 = 100\,(\$)$, $\Delta S_i = S_{i-1}\,\rho_i$, *with*

$$\rho_i = \begin{cases} 0.5 & \textit{with probability } 0.4 \\ -0.3 & \textit{with probability } 0.6 \end{cases}, \quad i = 1, 2,$$

and annual interest rate $r = 0.2$.

SOLUTION It is clear that the risk-neutral probability is defined by Bernoulli's probability $p^* = 5/8$. We have that

$$\begin{aligned} Y_2 &= \frac{(S_2 - 120)^+}{(1+r)^2} = \frac{(S_1\,(1+\rho_2) - 120)^+}{(1.2)^2}\,, \\ Y_1 &= \max\left\{\frac{f_1}{(1+r)}\,,\, E^*(Y_2|\mathcal{F}_1)\right\}, \\ Y_0 &= \max\big\{f_0\,,\, E^*(Y_1|\mathcal{F}_0)\big\}\,. \end{aligned}$$

Computing

$$E^*(Y_2|\mathcal{F}_1) = \begin{cases} \frac{p^*(225-120)}{(1+r)^2} = \frac{5/8\times 105}{(1.2)^2} \approx 44 & \text{on the set } \{\omega : S_1 = 150\} \\ 0 & \text{on the set } \{\omega : S_1 = 70\} \end{cases},$$

we obtain

$$Y_1 = \begin{cases} \max\left\{\frac{150-90}{1.2}, \frac{5/8\times 105}{(1.2)^2}\right\} = 50 = \frac{f_1}{1+r} & \text{on the set } \{\omega : S_1 = 150\} \\ 0 & \text{on the set } \{\omega : S_1 = 70\} \end{cases}.$$

Taking into account that $E^*(Y_1|\mathcal{F}_0) = E^*(Y_1) \approx 31$, we obtain

$$Y_0 = \max\{0, 31\} = 31 \neq 10 = f_0 ,$$

and the optimal stopping time

$$\tau^* \equiv \tau_0^* \equiv \tau_1^* \equiv 1 .$$

□

We complete this section with the following general remark regarding situations when the optimal stopping time for an American option is equal to the terminal time N. Let $f_n = g(S_n)$, where g is some non-negative convex function. Suppose for simplicity that $r = 0$. We have

$$C_N^{\mathrm{am}}(f) = \sup_{\tau \in \mathcal{M}_0^N} E^*\big(f_\tau\big) = \sup_{\tau \in \mathcal{M}_0^N} E^*\big(g(S_\tau)\big) .$$

Since by Jensen's inequality $\big(g(S_\tau)\big)_{n \leq N}$ is a submartingale, then for any $\tau \leq N$

$$E^*\big(g(S_\tau)\big) \leq E^*\big(g(S_N)\big) ,$$

which implies that $\tau^* \equiv N$ is the optimal stopping time.

1.6 Utility functions and St. Petersburg's paradox. The problem of optimal investment.

In the previous sections we studied investment strategies (portfolios) from the point of view of hedging contingent claims. Another criterion for comparing investment strategies can be formulated in terms of utility functions. A continuously differentiable function $U : [0, \infty) \to \mathbb{R}$ is called a *utility function* if it is non-decreasing, concave and

$$\lim_{x \downarrow 0} U'(x) = \infty , \qquad \lim_{x \to \infty} U'(x) = 0 .$$

An investor's aim to maximize $U(X_N^\pi)$ can lead to a difficult problem, as X_N^π is a random variable. Therefore, it is natural to compare average utilities: we say that a strategy π' is preferred to strategy π if

$$E\big(U(X_N^{\pi'})\big) \geq E\big(U(X_N^\pi)\big)\,.$$

One of the fundamental notions in this area of financial mathematics is the notion of *risk aversion*. Its mathematical description is given by the Arrow-Pratt function

$$R_A(\cdot) := -\,\frac{U''(\cdot)}{U'(\cdot)}$$

(in the case when U is twice continuously differentiable). This function characterizes decreasing of risk aversion if $R'_A < 0$, and increasing of risk aversion if $R'_A > 0$.

Thus such utility functions allow one to introduce a *measure of investment preferences* for risk averse participants in a market.

Historically, the theory of optimal investment with the help of utility functions grew from the famous Bernoulli's *St. Petersburg's paradox*.

WORKED EXAMPLE 1.7 (St. Petersburg's paradox)
Peter challenges Paul to a game of coin–toss. The game ends when the tail appears for the first time. If this happens after n tosses of a coin, then Peter pays Paul 2^{n-1} dollars. What price C should Paul pay Peter for an opportunity to enter this game?

SOLUTION Let X be Paul's prize money, which is obviously a random variable. An intuitive way of finding C suggests computing the average of X:

$$E(X) = 1 \times 1/2 + 2 \times 1/4 + \cdots + 2^{n-1}/2^n + \cdots = 1/2 + \cdots + 1/2 \cdots = \infty\,.$$

Thus, since the average of Paul's prize money is infinite, then Paul can agree to any price offered by Peter, which is clearly paradoxical.

Bernoulli suggested that the price C can be found from the equation

$$E\big(\ln X\big) = \ln C\,,$$

which implies $C = 2$, as

$$\begin{aligned} E\big((\ln X)''\big) &= \sum_{n=1}^{\infty} \frac{\ln 2^{n-1}}{2^n} = \sum_{n=1}^{\infty} \frac{(n-1)\ln 2}{2^n} \\ &= \ln 2 \sum_{n=1}^{\infty} \frac{n-1}{2^n} = \ln 2 \times 1 = \ln 2\,. \end{aligned}$$

$\square$

In general, given a utility function U, consider a problem of finding a self-financing strategy π^* such that

$$\max_{\pi \in SF} E\Big(U\big(X_N^\pi(x)\big)\Big) = U\big(X_N^{\pi^*}(x)\big)\,. \tag{1.4}$$

For simplicity, let $U(x) = \ln x$. Then

$$\ln X_N^\pi(x) = \ln \frac{X_N^\pi(x)}{B_N} + \ln B_N\,,$$

and therefore, the optimization problem (1.4) reduces to finding the maximum of

$$E\bigg(\ln \frac{X_N^\pi(x)}{B_N}\bigg)$$

overall $\pi \in SF$.

Denote $Y_n(x) := X_n^\pi(x)/B_n$ the discounted value of a self-financing portfolio π. Recall that $(Y_n)_{n \le N}$ is a positive martingale with respect to a risk-neutral probability P^*. Thus, we arrive at the problem of finding a positive martingale $Y^*(x) \equiv (Y_n^*)_{n \le N}$ with $Y_0^* = x$, such that

$$\max_Y E\big(\ln Y_N(x)\big) = E\big(\ln Y_N^*(x)\big)\,,$$

where the maximum is taken over the set of all positive martingales with the initial value x.

Let $Y_N^*(x) = x/Z_N^*$, where Z_N^* is the density of the martingale probability P^*. All other values of $Y^*(x)$ are defined as the following conditional expectations with respect to P^*:

$$Y_0^* = x\,, \quad Y_n^*(x) = E^*\bigg(\frac{x}{Z_N^*}\bigg|\mathcal{F}_n\bigg)\,, \quad n = 1, \ldots, N\,.$$

For any other martingale Y we have

$$\begin{aligned}
E\big(\ln Y_N(x)\big) &= E\bigg(\ln \frac{x}{Z_N^*} + \bigg[\ln Y_N(x) - \ln \frac{x}{Z_N^*}\bigg]\bigg) \\
&\le E\bigg(\ln \frac{x}{Z_N^*}\bigg) + E\bigg(\frac{Z_N^*}{x}\bigg[Y_N(x) - \frac{x}{Z_N^*}\bigg]\bigg) \\
&= E\bigg(\ln \frac{x}{Z_N^*}\bigg) + \bigg[E^*\big(Y_N(x)\big) - E^*\bigg(\frac{x}{Z_N^*}\bigg)\bigg]\Big/x \\
&= E\bigg(\ln \frac{x}{Z_N^*}\bigg) + \frac{x - x}{x} = E\bigg(\ln \frac{x}{Z_N^*}\bigg) \\
&= E\big(\ln Y_N^*(x)\big)\,.
\end{aligned}$$

Thus, $Y^*(x)$ is an optimal martingale. Recall that, for such a martingale, $Y_N^*(x)$ necessarily coincides with the discounted value of some self-financing strategy π^*.

To find this optimal portfolio $\pi^* = (\beta_n^*\,,\,\gamma_n^*)_{n\le N}$, we introduce quantities

$$\alpha_n^* := \gamma_n^* \, \frac{S_{n-1}}{X_{n-1}^{\pi^*}}\,,$$

which represent the *proportion* of risky capital in the portfolio.

Using mathematical induction in N, we obtain

$$\frac{X_N^{\pi^*}(x)}{B_N} = x\prod_{k=1}^{N}\left(1 - \frac{\alpha_k^*}{1+r}(\rho_k - r)\right),$$

and on the other hand

$$\frac{X_N^{\pi^*}(x)}{B_N} = \frac{x}{Z_N^*} = x\prod_{k=1}^{N}\left(1 - \frac{\mu - r}{\sigma^2}(\rho_k - r)\right)^{-1},$$

where $\mu = E\big(\rho_k\big)$. This gives us the following equation for α_k^*:

$$\prod_{k=1}^{N}\left(1 - \frac{\alpha_k^*}{1+r}(\rho_k - r)\right)\times\left(1 - \frac{\mu - r}{\sigma^2}(\rho_k - r)\right) = 1\,.$$

Let $N = 1$, then the latter equation reduces to

$$\left(1 - \frac{\alpha_1^*}{1+r}(\rho_1 - r)\right)\times\left(1 - \frac{\mu - r}{\sigma^2}(\rho_1 - r)\right) = 1\,,$$

and on the set $\{\omega\,:\,\rho_1(\omega) = b\}$ we have

$$\left(1 - \frac{\alpha_1^*}{1+r}(b - r)\right)\times\left(1 - \frac{\mu - r}{\sigma^2}(b - r)\right) = 1\,,$$

which implies that

$$\alpha_1^* = \frac{(1+r)\,(\mu - r)}{(r-a)\,(b-r)}\,.$$

On the set $\{\omega\,:\,\rho_1(\omega) = b\}$, the expression for α_1^* is exactly the same. Next, suppose that $\alpha_1^* \equiv \alpha_2^* \equiv \cdots \equiv \alpha_{N-1}^*$, then by induction we obtain that α_N^* is also given by this expression.

Thus, the constant proportion of risky capital

$$\alpha^* = \frac{(1+r)\,(\mu - r)}{(r-a)\,(b-r)}\,, \tag{1.5}$$

is a characteristic property of the optimal strategy π^* that solves the optimization problem (1.4) with the logarithmic utility function. Therefore, in this case, management of the risk associated with an investment portfolio reduces to retaining the proportion of risky capital in this portfolio at the constant level (1.5).

Note that management of this type of risk differs from hedging contingent claims. To illustrate this, we revisit Worked Examples 1.3 and 1.4. Recall that in these examples we consider a single-period binomial (B, S)-market with the annual rate of interest $r = 0.2$ and with the profitability of the risky asset

$$\rho_1 = \begin{cases} 0.5 & \text{with probability } 0.4 \\ -0.3 & \text{with probability } 0.6 \end{cases} .$$

The average profitability

$$m = E(\rho_1) = 0.5 \times 0.4 - 0.3 \times 0.4 = 0.02$$

is less than $r = 0.2$, and the optimal proportion of risky capital

$$\alpha^* = \frac{1.2 \times (-0.18)}{0.5 \times 0.3} \approx -1.5$$

is negative. This indicates that an investor should prefer depositing money in a bank account.

Recall that for the contingent claim in Worked Example 1.3 we have

$$f_1 = (S_1 - 100)^+ = \begin{cases} 50 & \text{with probability } 0.4 \\ 0 & \text{with probability } 0.6 \end{cases}$$

and its minimal hedging price is 26. For the contingent claim in Worked Example 1.4 it is

$$f_1 = \max\{S_0, S_1\} - S_1 = \begin{cases} 0 & \text{with probability } 0.4 \\ 30 & \text{with probability } 0.6 \end{cases}$$

and 9.3, respectively.

Now we compute terminal values of optimal investment portfolios with $\alpha^* = -1.5$, and the initial values $X_0^{\alpha^*} = 26$:

$$\begin{aligned} X_1^{\alpha^*} &= X_0^{\alpha^*} + \Delta X_1^{\alpha^*} = X_0^{\alpha^*} + \left(r\, X_0^{\alpha^*} + \alpha^*\, X_0^{\alpha^*} (\rho_1 - r) \right)\Big|_{X_0^{\alpha^*}=26} \\ &\approx \begin{cases} 33 & \text{with probability } 0.4 \\ 52 & \text{with probability } 0.6 \end{cases} \neq \begin{cases} 50 & \text{with probability } 0.4 \\ 0 & \text{with probability } 0.6 \end{cases} , \end{aligned}$$

and $X_0^{\alpha^*} = 9.3$:

$$\begin{aligned} X_1^{\alpha^*} &= X_0^{\alpha^*} + \left(r\, X_0^{\alpha^*} + \alpha^*\, X_0^{\alpha^*} (\rho_1 - r) \right)\Big|_{X_0^{\alpha^*}=9.3} \\ &\approx \begin{cases} 4 & \text{with probability } 0.4 \\ 14 & \text{with probability } 0.6 \end{cases} \neq \begin{cases} 0 & \text{with probability } 0.4 \\ 30 & \text{with probability } 0.6 \end{cases} . \end{aligned}$$

Thus, the optimal strategy of managing the investment risk differs from both strategies of minimal hedging.

REMARK 1.1 In a single-period binomial (B, S)-market, every portfolio can be associated with the pair (α_0, α_1) of non-negative real numbers $\alpha_0, \alpha_1 \in [0,1]$, $\alpha_0 + \alpha_1 = 1$, that represent the proportions of the capital invested in assets B and S, respectively. Then the profitability of a portfolio is equal to the weighted sum of the profitabilities r and ρ_1:

$$\rho(\alpha_0, \alpha_1) = \alpha_0 \, r + \alpha_1 \, \rho_1 \, .$$

In this case, the optimal portfolio (α_0^*, α_1^*) can be found as a solution to the following optimization problem

$$E\big(\rho(\alpha_0^*, \alpha_1^*)\big) = \max_{(\alpha_0, \alpha_1)} E\big(\rho(\alpha_0, \alpha_1)\big)$$

under assumption of either

$$V\big(\rho(\alpha_0, \alpha_1)\big) \le const$$

or

$$P\Big(\big\{\rho(\alpha_0, \alpha_1) \le const\big\}\Big) \le c \, , \quad c \in (0,1) \, .$$

Solving this type of optimization problem leads to the introduction of a notion of optimal (effective) portfolio, to the Markovitz theory and to the Capital Asset Pricing model. The concept of *Value of Risk* also originates from this type of problem, and it is widely used in financial practice. □

1.7 The term structure of prices, hedging and investment strategies in the Ho-Lee model

Recall that *bonds* (debentures) are debt securities issued by a government or a company for accumulating capital. Bonds are issued for a specified period of time $[0, N]$, where N is called the *exercise (redemption) time*, and they are characterized by their *face value* (redemption cost). Payments up to redemption are called *coupons*. We consider zero-coupon bonds with face value 1. To satisfy the no-arbitrage condition one has to assume that

$$0 < B(n, N) < 1 \, , \quad n < N \, ,$$

where $B(n, N)$ is the price at time n of a bond with the exercise time N. Suppose that the evolution of these prices is described by the *Ho-Lee model*:

$$B(n+1, N) = \frac{B(n, N)}{B(n, n+1)} \, h(\xi_{n+1}; n+1, N) \, ,$$

where $(\xi_n)_{n \le N}$ is a sequence of independent random variables taking values 0 or 1 with probabilities p and $1-p$ respectively. The perturbation function h has the properties: $h(\cdot\,; N, N) = 1$ and $h(1; n, N) \ge h(0; n, N)$.

As in the case of a binomial (B,S)-market, we can take the probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{0,1\}^{N^*}$, $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$ and with probability P defined by a Bernoulli parameter $p \in [0,1]$. The family $\big(B(n,N)\big)_{n \le N^*}$ is said to be arbitrage-free if for any $n \le N^*$ the stochastic sequence

$$\big(B_n^{-1}\, B(n,N)\big)_{n \le N}\,, \quad \text{where} \quad B_n^{-1} = \prod_{k=0}^{k} B(k-1,k)\,,$$

is a martingale with respect to some probability P^*.

The no-arbitrage condition implies the existence of $p = p^*$ such that

$$p^*\, h(0;n,N) + (1-p^*)\, h(0;n,N) = 1\,.$$

Further, there is a $\delta_* > 1$ such that

$$h^{-1}(0;n,N) = p^* + (1-p^*)\delta_*^{N-n}\,,$$
$$h(1;n,N) = \delta_*^{N-n}\Big(p^* + (1-p^*)\delta_*^{N-n}\Big)^{-1}\,,$$

and

$$\delta_*^{N-n} = h(1;n,N)\, h^{-1}(0;n,N)\,.$$

Now consider the introduced family of bonds $\big(B(n,N)\big)_{n \le N^*}$ and a bank account $(B_n)_{n \le N^*}$ with the rate of interest $r \ge 0$. For a perturbation function

$$h(\xi_j; j, N) = \delta_*^{(N-n)\xi_j}\, h(0;j,N)\,,$$

we have

$$B(n,N) = \frac{B(0,N)}{B(0,n)} \prod_{j=1}^{n} h^{-1}(\xi_j; j, n)\, h(\xi_j; j, N)\,.$$

Further, introducing a new parameter $\delta = \ln \delta_*$, we can rewrite the perturbation function h in the form

$$h(\xi_n; n, N) = e^{(N-n)\xi_n \delta}\Big(p^* + (1-p^*)\, e^{(N-n)\delta}\Big)^{-1}\,.$$

We obtain the following term structure of bond prices in the Ho-Lee model:

$$\begin{aligned} B(n,N) &= B(0,N) \prod_{i=1}^{n} B(i-1,i)^{-1} \frac{\prod_{i=1}^{n} \delta_*^{(N-i)\xi_i}}{\prod_{i=1}^{n}\big(p^* + (1-p^*)\delta_*^{(N-i)}\big)} \\ &= B(0,N)\, B_n \left[e^{\delta \sum_{i=1}^{n}(N-i)\xi_i} \Big/ E^*\big(e^{\delta \sum_{i=1}^{n}(N-i)\xi_i}\big)\right] \\ &= \frac{B(0,N)}{B(0,n)}\, e^{\delta\,(N-n)\sum_{i=1}^{n}\xi_i} \prod_{i=1}^{n} \frac{p^* + (1-p^*)\delta_*^{(n-i)}}{p^* + (1-p^*)\delta_*^{(N-i)}}\,. \end{aligned}$$

Now let us choose a particular bond $\big(B(n,N^1)\big)_{n\le N^1}$ from the family $\big(B(n,N)\big)_{n\le N\le N^*}$. Using it as a risky asset, and a bank account $(B_n)_{n\le N^1}$ as a non-risky asset, we can form a financial market. A portfolio π is formed by β_n units of asset B_n and $\gamma_n(N^1)$ bonds $B(n,N^1)$ with the exercise date N^1. The values of this portfolio are

$$X_n^\pi = \beta_n\,B_n + \gamma_n(N^1)\,B(n,N^1)\,.$$

The portfolio π is self-financing if

$$\Delta X_n^\pi = \beta_n\,\Delta B_n + \gamma_n(N^1)\,\Delta B(n,N^1)\,.$$

Thus, this $\big(B_n, B(n,N^1)\big)_{n\le N^1}$-market is analogous to the binomial (B,S)-market with the unique martingale measure P^*.

Let us consider a contingent claim

$$f_N = \big(B(N,N^1) - K\big)^+\,,\quad N\le N^1\,,$$

which corresponds to the European call option. Its price is uniquely determined by

$$C_N = E^*\Big(B_N^{-1}\big(B(N,N^1)-K\big)^+\Big)\,.$$

Taking into account the term structure of bond prices, we have that $B(N,N^1)\ge K$ if not less than $k_0 := k\big(N,N^1,B(0,N),B(0,N^1)\big)$ quantities $\xi_1,\dots,\xi_N$ take value 1, where

$$\begin{aligned}
&k(t,T,B,B')\\
&\quad= \inf\left\{k\le t\,:\,k\ge \frac{1}{(T-t)\delta}\,\ln\left(K\,\frac{B}{B'}\prod_{i=1}^{t}\frac{p^*+(1-p^*)\delta^{(T-i)}}{p^*+(1-p^*)\delta^{(t-i)}}\right)\right\}.
\end{aligned}$$

Denote

$$\mathbb{B}(k_0,t,T,p) := \sum\left\{\frac{\prod_{i=1}^t \delta_*^{x_i(T-t)}}{\prod_{i=1}^t\big(p^*+(1-p^*)\delta_*^{(T-i)}\big)}\,p^{t-\sum x_i}\,(1-p)^{\sum x_i}\right\},$$

where summation is taken over all vectors $(x_1,\dots,x_t)$ consisting of 0s and 1s and such that $\sum x_i\ge k_0$.

We obtain that

$$\begin{aligned}
C_N &= E^*\Big(B_N^{-1}\,\big(B(N,N^1)-K\big)^+\Big)\\
&= E^*\Big(\big(B_N^{-1}\,B(N,N^1) - B_N^{-1}\,K\big)^+\Big)\\
&= E^*\left(\left[B(0,N^1)\,\frac{\prod_{i=1}^N\delta_*^{x_i(N^1-i)}}{\prod_{i=1}^N\big(p^*+(1-p^*)\delta_*^{(N^1-i)}\big)}\right.\right.\\
&\qquad\qquad\left.\left. -K\,B(0,N)\,\frac{\prod_{i=1}^N\delta_*^{x_i(N-i)}}{\prod_{i=1}^N\big(p^*+(1-p^*)\delta_*^{(N-i)}\big)}\right]^+\right)\\
&= B(0,N^1)\,\mathbb{B}(k_0,N,N^1,p^*) - K\,B(0,N)\,\mathbb{B}(k_0,N,N,p^*)\,.
\end{aligned}$$

Now denoting $k_n := k\big(N-n, N^1-n, B(n,N), B(n,N^1)\big)$, we obtain the structure of the minimal hedge π^*:

$$\begin{aligned} X_n^{\pi^*} &= B_n\, E^*\Big(B_N^{-1}\big(B(N,N^1)-K\big)^+\Big|\mathcal{F}_n\Big) \\ &= B(n,N^1)\,\mathbb{B}(k_n, N-n, N^1-n, p^*) \\ &\qquad\qquad\qquad -K\,B(n,N)\,\mathbb{B}(k_n, N-n, N-n, p^*)\,. \end{aligned}$$

On the same market, we now solve the optimization problem (1.4) with the logarithmic utility function. Note that the density Z_N^* if probability P^* with respect to probability P has the form

$$Z_N^* = \varepsilon_N\bigg(-\frac{p^*-p}{p\,(1-p)}\sum\big(\xi_n-(1-p)\big)\bigg)\,.$$

Hence, the discounted value of the optimal strategy $\pi^* = (\beta_n^*, \gamma_n^*)$ is

$$\frac{X_N^{\pi^*}}{B_N} = \frac{x}{Z_N^*} = x\Big/\varepsilon_N\bigg(-\frac{p^*-p}{p\,(1-p)}\sum\big(\xi_n-(1-p)\big)\bigg)\,.$$

Let the proportion of risky capital be

$$\alpha_n^*(N) = \frac{\gamma_n^*(N)\,B(n-1,N)}{X_{n-1}^{\pi^*}}\,,$$

then, since π^* is self-financing, we obtain

$$\Delta\frac{X_n^{\pi^*}}{B_n} = \gamma_n^*(N)\,\Delta\frac{B(n,N)}{B_n} = \frac{X_{n-1}^{\pi^*}\,\alpha_n^*(N)}{B(n-1,N)}\,\Delta\frac{B(n,N)}{B_n}\,.$$

Using structure of $\big(B(n,N)\big)$ we write

$$\Delta\frac{X_n^{\pi^*}}{B_n} = \frac{X_{n-1}^{\pi^*}\,\alpha_n^*(N)}{B_{n-1}}\left[\frac{\delta_*^{(N-n)\xi_n}}{p^*+(1-p^*)\,\delta_*^{N-n}}-1\right]$$

and

$$\frac{X_N^{\pi^*}}{B_N} = x\,\varepsilon_N\bigg(\sum\alpha_n^*(N)\left[\frac{\delta_*^{(N-n)\xi_n}}{p^*+(1-p^*)\,\delta_*^{N-n}}-1\right]\bigg)\,,$$

and therefore arrive at the expression

$$\alpha_n^*(N) = \frac{p^*-p}{p\,(1-p)}\;\frac{\delta_*^{(N-n)}-1}{p^*+(1-p^*)\,\delta_*^{N-n}}\,.$$

Chapter 2

Advanced Analysis of Financial Risks

2.1 Fundamental theorems on arbitrage and completeness. Pricing and hedging contingent claims in complete and incomplete markets.

Let $(\Omega, \mathcal{F}_N, \mathbb{F}, P)$ be a discrete stochastic basis with filtration $\mathbb{F} = (\mathcal{F}_n)_{n \le N}$:

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_N .$$

Consider a (B, S)-market with a non-risky asset B defined by a deterministic (or predictable) sequence of its prices $(B_n)_{n=0}^N$, $B_0 = 1$. A risky asset S is defined by a stochastic sequence (of prices) $(S_n)_{n=0}^N$ adopted to filtration $\mathbb{F}$.

Further, consider the sequence $\big(S_n/B_n\big)_{n=0}^N$. We say that a probability $\widetilde{P}$ is a *martingale probability* if $\big(S_n/B_n\big)_{n=0}^N$ is a martingale with respect to $\widetilde{P}$. The collection of all such probabilities is denoted $\mathcal{M}\big(S_n/B_n\big)$.

As in the case of a binomial (B, S)-market, one can consider the notions of a self-financing strategy, a portfolio, etc. Recall that we say that there is an *arbitrage opportunity* in this market if there exists a self-financing strategy $\widetilde{\pi} \in SF$ such that $X_0^{\widetilde{\pi}} = 0$ (a.s.), $X_N^{\widetilde{\pi}} \ge 0$ (a.s.) and $P\big(\{\omega : \ X_N^{\widetilde{\pi}} > 0\}\big) > 0$.

THEOREM 2.1 (First Fundamental Theorem of Financial Mathematics)

A (B, S)-market is arbitrage-free if and only if $\mathcal{M}\big(S_n/B_n\big) \neq \emptyset$.

PROOF We prove the 'if' part of this statement. For simplicity, suppose that $B_n \equiv 1$ for all n. Let $\widetilde{P} \in \mathcal{M}\big(S_n/B_n\big)$, then for any self-financing strategy $\pi = (\beta, \gamma)$, its discounted value

$$X_n^\pi = X_0^\pi + \sum_{k=1}^n \gamma_k \Delta S_k$$

is a martingale with respect to $\widetilde{P}$. Recall that, in the case of binomial markets, this fact is referred to as the martingale characterization of the class SF of self-financing strategies. Now suppose that π^* is an arbitrage strategy. By its definition we have $E\big(X_N^{\pi^*}\big) > 0$. On the other hand, the martingale property of $\big(X_n^{\pi^*}\big)$ implies $\widetilde{E}\big(X_N^{\pi^*}\big) = \widetilde{E}\big(X_0^{\pi^*}\big) = X_0^{\pi^*} = 0$, which contradicts to the assumption that π^* is an arbitrage strategy.

The proof of the converse is technically far more complex. It can be found, for example, in [42]. $\square$

We say that a (B,S)-market is *complete* if every contingent claim f_N can be *replicated* by some self-financing strategy, i.e., there exist $\pi \in SF$ and $x \geq 0$ such that

$$X_0^\pi = x \qquad \text{and} \qquad X_N^\pi = f_N \quad \text{(a.s.)}\,.$$

The sequence of discounted prices $\big(S_n/B_n\big)_{n=0}^N$ is a martingale with respect to any probability $\widetilde{P} \in \mathcal{M}\big(S_n/B_n\big)$. It turns out that, in the case of a complete market, it forms a basis for the space of all martingales with respect to $\widetilde{P}$: any martingale can be written in the form of a discrete stochastic integral with respect to S/B. This property of a market is called the *martingale representation* property.

PROPOSITION 2.1

A (B,S)-market is complete if and only if it possesses the martingale representation property.

PROOF For simplicity, suppose that $B_n \equiv 1$ for all n. Consider a complete (B,S)-market. Let $(X_n)_{n=0}^N$ be an arbitrary martingale and define a contingent claim by $f_N \equiv X_N$. The completeness of the market implies that there exist $\pi \in SF$ and $x \geq 0$ such that

$$X_N^\pi = f_N = X_N \qquad \text{and} \qquad X_n^\pi = x + \sum_{k=1}^n \gamma_k\, \Delta S_k \quad \text{(a.s.)}\,.$$

Since π is a self-financing strategy, the later equality means that $(X_n)_{n=0}^N$ is a martingale with respect to any probability $\widetilde{P} \in \mathcal{M}\big(S_n/B_n\big)$. Thus we have two martingales with the same terminal value f_N, and therefore for all $n = 0, 1, \ldots, N$

$$X_n^\pi = \widetilde{E}\big(f_N \big| \mathcal{F}_n\big) = X_n\,,$$

which gives us a representation of X in terms of the basis martingale S/B.

Conversely, consider a contingent claim f_N and a stochastic sequence $(X_n)_{n=0}^N$, where $X_n = \widetilde{E}\big(f_N \big| \mathcal{F}_n\big)$ for any fixed probability $\widetilde{P} \in \mathcal{M}\big(S_n/B_n\big)$. Then we can represent this martingale in the form

$$X_n = X_0 + \sum_{k=1}^n \phi_k\, \Delta S_k\,,$$

where $(\phi_k)_{k=1}^N$ is a predictable sequence. Now let

$$\gamma_n^* = \phi_n\,, \qquad \beta_n^* = X_n - \gamma_n^*\, S_n\,, \quad n \le N\,.$$

Note that

$$\begin{aligned}\beta_n^* = X_n - \gamma_n^*\, S_n &= X_0 + \sum_{k=1}^{n-1} \gamma_k^*\, S_k + \gamma_n^*(S_n - S_{n-1}) - \gamma_n^* S_n \\ &= X_0 + \sum_{k=1}^{n-1} \gamma_k^*\, S_k - \gamma_n^* S_{n-1}\end{aligned}$$

is completely determined by the information contained in $\mathcal{F}_{n-1}$, that is, $(\beta_n^*)_{n=0}^N$ is a predictable sequence. This implies that $\pi^* = (\beta_n^*, \gamma_n^*)_{n=0}^N$ is a self-financing strategy such that for all $n = 0, 1, \ldots, N$

$$X_n^{\pi^*} = X_n \quad (a.s.)\,.$$

In particular, we obtain that $X_N^{\pi^*} = X_N = f_N$, i.e., an arbitrary contingent claim f_N can be replicated and the market is complete. ∎

The essence of complete markets is characterized in the following result.

THEOREM 2.2 (Second Fundamental Theorem of Financial Mathematics)
A (B, S)-market is complete if and only if the set $\mathcal{M}\big(S_n/B_n\big) \neq \emptyset$ consists of a unique element P^*.

PROOF Consider an arbitrary event $A \in \mathcal{F}_N$ and let $f_N = I_A$. This contingent claim can be replicated: there are $x > 0$ and $\pi \in SF$ such that $X_0^\pi = x$, $X_N^\pi(x) = f_N$, and $X_n^\pi = x + \sum_{k=1}^n \gamma_k\, \Delta S_k$ for all $n = 0, \ldots, N$.

If $P_1,\ P_2 \in \mathcal{M}\big(S_n/B_n\big)$, then X_n^π form martingales with respect to both these probabilities. Therefore

$$x = X_0^\pi = E_i\big(X_N^\pi \big| \mathcal{F}_0\big) = E_i\big(X_N^\pi\big) = E_i\big(I_A\big) = P_i\big(A\big)\,, \quad i = 1, 2.$$

Hence, $P_1 = P_2$.

Now we sketch the proof of the converse. Let P^* be the unique martingale measure. Using mathematical induction, we will show that $\mathcal{F}_n = \mathcal{F}_n^S = \sigma(S_0, \ldots, S_N)$. Suppose $\mathcal{F}_{n-1} = \mathcal{F}_{n-1}^S$. Let $A \in \mathcal{F}_n$ and define a random variable

$$Z = 1 + \frac{1}{2}\Big[I_A - E\big(I_A \big| \mathcal{F}_n^S\big)\Big] > 0\,.$$

Clearly, $E^*\big(Z\big) = 1$ and $E^*\big(Z\big|\mathcal{F}_n^S\big) = 1$. Now define a new probability $P'\big(C\big) := E^*\big(Z\, I_C\big)$. We have

$$\begin{aligned} E'\big(\Delta S_n\big|\mathcal{F}_{n-1}\big) &= E^*\big(Z\Delta S_n\big|\mathcal{F}_{n-1}\big) = E^*\big(Z\Delta S_n\big|\mathcal{F}_{n-1}^S\big) \\ &= E^*\Big(E^*\big(Z\Delta S_n\big|\mathcal{F}_{n-1}\big)\Big|\mathcal{F}_{n-1}\Big) \\ &= E^*\Big(\Delta S_n E^*\big(Z\big|\mathcal{F}_{n-1}\big)\Big|\mathcal{F}_{n-1}\Big) \\ &= E^*\big(\Delta S_n\big|\mathcal{F}_{n-1}^S\big) = 0\,, \end{aligned}$$

which implies that P' is a martingale measure. Using the uniqueness of the martingale measure P^* we conclude that $Z = 1$ (a.s.). Hence, $I_A = E\big(I_A\big|\mathcal{F}_n^S\big)$ and therefore $\mathcal{F}_n = \mathcal{F}_n^S$.

Next consider the following conditional distributions

$$P^*\big(\{\omega:\ \rho_n \in dx\}\big|\mathcal{F}_{n-1}\big)\,, \qquad \text{where} \qquad \rho_n = \frac{\Delta S_n}{S_{n-1}}\,, \quad n = 1, \ldots, N\,.$$

It turns out that these distributions have the following structure: there are non-positive predictable sequence $(a_n)_{n\le N}$ and non-negative predictable sequence $(b_n)_{n\le N}$ such that

$$P^*\big(\{\omega:\ \rho_n = a_n\}\big|\mathcal{F}_{n-1}\big) + P^*\big(\{\omega:\ \rho_n = b_n\}\big|\mathcal{F}_{n-1}\big) = 1\,, \quad n \le N\,.$$

The later equality follows from the following result from the general probability theory: the set of all distributions F (on real line) with the properties

$$\int_{-\infty}^{\infty} |x|\, dF(x) < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} x\, dF(x) = 0\,,$$

consists of a unique distribution F^* if and only if there exist $a \le 0$ and $b \ge 0$ such that $F^*(\{a\}) + F^*(\{b\}) = 1$.

Now let

$$\begin{aligned} p_n^* &:= P^*\big(\{\omega:\ \rho_n = b_n\}\big|\mathcal{F}_{n-1}\big)\,, \\ 1 - p_n^* &:= P^*\big(\{\omega:\ \rho_n = a_n\}\big|\mathcal{F}_{n-1}\big)\,. \end{aligned}$$

We have $E^*\big(\rho_n\big|\mathcal{F}_{n-1}\big) = 0$, or equivalently

$$b_n(\omega)\, p_n^* + a_n(\omega)\,(1 - p_n^*) = 0\,.$$

Thus

$$\begin{aligned} p_n^* &:= \frac{-a_n(\omega)}{b_n(\omega) - a_n(\omega)}\,, \\ 1 - p_n^* &:= \frac{b_n(\omega)}{b_n(\omega) - a_n(\omega)}\,. \end{aligned}$$

Now, if $\big(X_n,\,F_n^S\big)_{n\le N}$ is a martingale with respect to P^*, then there exist functions $f_n(x_1,\ldots,x_n)$ such that

$$X_n(\omega)=f_n\big(\rho_1(\omega),\ldots,\rho_n(\omega)\big)\,,\quad n\le N\,.$$

As in the case of the binomial market, we then arrive to the following martingale representation

$$X_n=X_0+\sum_{k=1}^{n}\gamma_k\,\Delta S_k\,,$$

where $(\gamma_k)_{k\le N}$ is a predictable sequence. Since this is equivalent to the completeness of the market, the proof is completed. $\square$

Now we discuss general methodologies of pricing contingent claims in complete and incomplete markets. We start with a complete (B,S)-market that admits a unique martingale measure P^*. Let f_N be a contingent claim. Note that if the probability space $(\Omega,\mathcal{F},P)$ is not finite, then one has to assume that $E^*\big(f_N/B_N\big)<\infty$. Consider the martingale

$$M_n^*:=E^*\left(\frac{f_N}{B_N}\,\Big|\,\mathcal{F}_n\right),\quad n=0,1,\ldots,N,$$

which has the initial and terminal values

$$M_0^*=E^*\big(f_N/B_N\big)\qquad\text{and}\qquad M_N^*=f_N/B_N\,,$$

respectively. By theorem 2.2 M^* has the following representation

$$M_n^*=M_0^*+\sum_{k=1}^{n}\gamma_k^*\,\Delta\frac{S_k}{B_k}\,,\quad n=0,1,\ldots,N,$$

where $(\gamma_k)_{k\le N}$ is a predictable sequence.

Define a strategy $\pi^*=(\beta_n^*,\gamma_n^*)_{n\le N}$ with $\beta_n^*=M_n^*-\gamma_n^*\,S_n/B_n$. Then we have that

$$\begin{aligned}\beta_n^*&=M_0^*+\sum_{k=1}^{n}\gamma_k^*\,\Delta\frac{S_k}{B_k}-\gamma_n^*\,\frac{S_n}{B_n}\\&=M_0^*+\sum_{k=1}^{n-1}\gamma_k^*\,\Delta\frac{S_k}{B_k}\end{aligned}$$

is a predictable sequence. Hence, we constructed a self-financed strategy $\pi^*\in SF$ with values (a.s.) given by

$$\begin{aligned}&\frac{X_0^{\pi^*}}{B_0}=M_0^*\,,\\&\Delta\frac{X_n^{\pi^*}}{B_n}=\gamma_n^*\,\frac{S_n}{B_n}=\Delta M_n^*\,,\quad n\le N\,,\\&\frac{X_n^{\pi^*}}{B_n}=M_n^*=E^*\left(\frac{f_N}{B_N}\,\Big|\,\mathcal{F}_n\right),\end{aligned}$$

and, in particular, $f_N = B_N$ a.s.

Thus we proved the following result.

THEOREM 2.3 (Pricing Contingent Claims in Complete Markets)

Let f_N be a contingent claim in a complete (B, S)-market. Then there exists a self-financing strategy $\pi^* = (\beta^*, \gamma^*)$ so that it is a minimal hedge with the values

$$X_n^{\pi^*} = B_n\, E^*\left(\frac{f_N}{B_N}\,\Big|\,\mathcal{F}_n\right),$$

and β^*, γ^* are defined by relations

$$\begin{aligned} E^*\left(\frac{f_N}{B_N}\,\Big|\,\mathcal{F}_n\right) &= E^*\left(\frac{f_N}{B_N}\right) + \sum_{k=1}^{n} \gamma_k^*\, \Delta \frac{S_k}{B_k}\,, \\ X_n^{\pi^*} &= \beta_n^*\, B_n + \gamma_n^*\, S_n\,. \end{aligned}$$

In particular, the price of f_N is

$$C_N = E^*\left(\frac{f_N}{B_N}\right).$$

Note that the fact that π^* is the *minimal* hedge follows from the following inequalities:

$$\frac{X_n^{\pi}}{B_n} = E^*\left(\frac{X_N^{\pi}}{B_N}\,\Big|\,\mathcal{F}_n\right) \geq E^*\left(\frac{f_N}{B_N}\,\Big|\,\mathcal{F}_n\right) = \frac{X_n^{\pi^*}}{B_n} \quad \text{(a.s.)}, \quad n = 0, 1, \ldots, N\,,$$

for any other $\pi \in SF$ hedging f_N.

Now we consider incomplete markets. In this case, not every contingent claim can be replicated by self-financing strategies. Consider a strategy $\pi = (\beta_n, \gamma_n)_{n \leq N}$ that is not necessarily self-financing. We can write

$$\begin{aligned} \Delta X_n^{\pi} &= \beta_n\, \Delta B_n + \gamma_n\, \Delta S_n + B_{n-1}\, \Delta\beta_n + S_{n-1}\, \Delta\gamma_n \\ &= \beta_n\, \Delta B_n + \gamma_n\, \Delta S_n - \Delta c_n\,, \end{aligned}$$

where

$$-\Delta c_n := B_{n-1}\, \Delta\beta_n + S_{n-1}\, \Delta\gamma_n \quad n = 1, \ldots, N; \quad c_0 = 0.$$

Let $c = (c_n)_{n \leq N}$ be a non-decreasing stochastic sequence (consumption process). A class of strategies (π, c) is then called *consumption strategies*. Clearly we have

$$\begin{aligned} X_n^{\pi,c} &= X_0^{\pi,c} + \sum_{k=1}^{n} \left(\beta_k\, \Delta B_k + \gamma_k\, \Delta S_k\right) - c_n \\ &= X_0^{\pi,c} + \sum_{k=1}^{n} \left(\beta_k'\, \Delta B_k + \gamma_k\, \Delta S_k\right), \end{aligned}$$

where

$$\beta'_k := \beta_k - \frac{\Delta c_k}{B_k}$$

is not necessarily predictable since the consumption c_n is determined by the information in $\mathcal{F}_n \supseteq \mathcal{F}_{n-1}$.

The discounted value of a consumption strategy (π, c) has the following dynamics:

$$\Delta \frac{X_n^{\pi,c}}{B_n} = \gamma_n \, \Delta \frac{S_n}{B_n} - \frac{\Delta c_n}{B_{n-1}} \, .$$

Now let f_N be a contingent claim in an incomplete (B, S)-market. If it is possible to find a consumption strategy (π^*, c^*) that replicates f_N, then the value of this strategy will be the natural choice for the price of f_N.

Consider the following stochastic sequence

$$\begin{aligned}
Y_n &= \sup_{\widetilde{P} \in \mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N} \Big| \mathcal{F}_n \right), \quad n = 1, \ldots, N, \\
Y_0 &= \sup_{\widetilde{P} \in \mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N} \right), \\
Y_N &= \frac{f_N}{B_N} \quad \text{a.s.}
\end{aligned} \tag{2.1}$$

Note that here one has to assume that

$$\sup_{\widetilde{P} \in \mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N} \right) < \infty \, ,$$

which is obviously satisfied in the case of discrete markets.

It turns out that this sequence $\big(Y_n\big)_{n \le N}$ is a positive supermartingale with respect to any probability $\widetilde{P} \in \mathcal{M}(S/B)$. Therefore, for a fixed $\widetilde{P}$ we can write the Doob decomposition:

$$Y_n = Y_0 + \widetilde{M}_n - \widetilde{A}_n \, , \quad n \le N,$$

where $\widetilde{M}$ is a martingale with respect to $\widetilde{P}$ and $\widetilde{A}$ is a non-decreasing predictable sequence. Clearly, this decomposition depends on the choice of $\widetilde{P}$. The following *optional decomposition* is invariant on the class of martingale measures $\mathcal{M}(S/B)$:

$$Y_n = Y_0 + M'_n - c'_n \, , \quad n = 0, \ldots, N; \; M'_0 = c'_0 = 0,$$

where M' is a martingale with respect to any probability from $\mathcal{M}(S/B)$ and c' is a non-decreasing (but not necessarily predictable) stochastic sequence.

Furthermore, M' has the following representation

$$M'_n = \sum_{k=1}^{n} \gamma'_k \, \Delta \frac{S_k}{B_k} \, ,$$

where (γ'_k) is a predictable sequence.

Now we define a consumption strategy:

$$\gamma^*_n = \gamma'_n\,, \quad \beta^*_n = Y_n - \gamma^*_n\,\frac{S_n}{B_n}\,, \quad c^*_n = \sum_{k=1}^{n} B_{k-1}\,\Delta c'_k\,.$$

We have

$$X_0^{\pi^*,c^*} = Y_0 = \sup_{\widetilde{P}\in\mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N}\right),$$

$$\Delta\frac{X_n^{\pi^*,c^*}}{B_n} = \gamma^*_n\,\Delta\frac{S_n}{B_n} - \frac{\Delta c^*_n}{B_{n-1}} = \Delta Y_n\,, \quad n \le N.$$

Thus

$$\frac{X_N^{\pi^*,c^*}}{B_N} = Y_N = \frac{f_N}{B_N} \quad \text{a.s.},$$

which means that f_N is replicated by the consumption strategy (π^*, c^*).

We almost proved the following result.

THEOREM 2.4 (Pricing Contingent Claims in Incomplete Markets)

Let f_N be a contingent claim in an incomplete (B,S)-market. Then there exists a consumption strategy (π^*, c^*) so that it is a minimal hedge with the values

$$X_n^{\pi^*,c^*} = B_n \sup_{\widetilde{P}\in\mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N}\,\middle|\,\mathcal{F}_n\right),$$

where β^*, γ^* and c^* are defined from the optional decomposition of the positive supermartingale Y (2.1):

$$Y_n = \sup_{\widetilde{P}\in\mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N}\right) + \sum_{k=1}^{n} \gamma^*_k\,\Delta\frac{S_k}{B_k} - \sum_{k=1}^{n} \frac{\Delta c^*_k}{B_{k-1}}\,,$$

$$\beta^*_n = \frac{X_n^{\pi^*,c^*} - \gamma^*_n S_n}{B_n}\,.$$

In particular, the initial (upper) price of f_N can be defined as

$$C^*_N = \sup_{\widetilde{P}\in\mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N}\right).$$

PROOF We need to show only that hedge (π^*, c^*) is the minimal hedge. Let (π, c) be an arbitrary consumption strategy hedging f_N. Then for any $\widetilde{P} \in \mathcal{M}(S/B)$ we have

$$\frac{X_n^{\pi,c}}{B_n} \ge \widetilde{E}\left(\frac{X_N^{\pi,c}}{B_N}\,\middle|\,\mathcal{F}_n\right) \ge \widetilde{E}\left(\frac{f_N}{B_N}\,\middle|\,\mathcal{F}_n\right), \quad n \le N,$$

therefore for all $n \le N$

$$\frac{X_n^{\pi,c}}{B_n} \ge \sup_{\widetilde{P} \in \mathcal{M}(S/B)} \widetilde{E}\left(\frac{f_N}{B_N} \,\Big|\, \mathcal{F}_n \right) = \frac{X_n^{\pi^*,c^*}}{B_n} \qquad \text{a.s.},$$

which proves the claim. □

2.2 The structure of options prices in incomplete markets and in markets with constraints. Options-based investment strategies.

Consider an incomplete (B,S)-market. As we noted in the previous section, there may be more than one risk-neutral probability $\widetilde{P}$, and therefore the quantity $\widetilde{E}\big(f_N/B_N\big)$ is not unique.

In this section we discuss arbitrage-free pricing of a contingent claim f_N in the case of incomplete markets.

It is intuitively clear that any number from the interval

$$\left[\min_{\widetilde{P}} \widetilde{E}\big(f_N/B_N\big) \,,\, \max_{\widetilde{P}} \widetilde{E}\big(f_N/B_N\big) \right]$$

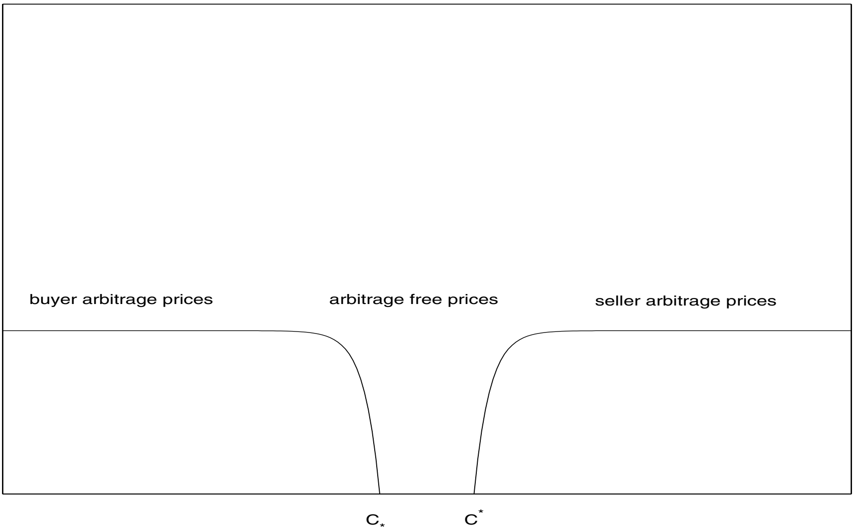
can be considered as an arbitrage-free price of a contingent claim f_N.

On the other hand, if we denote $X_N^{\pi}(x)$ the terminal value of a strategy with the initial value x, then we can define quantities

$$\begin{aligned} C^* &= \min\big\{x \;:\; X_N^{\pi}(x) \ge f_N \quad \text{for some} \quad \pi \in SF\big\}, \\ C_* &= \max\big\{x \;:\; X_N^{\pi}(x) \ge f_N \quad \text{for some} \quad \pi \in SF\big\}. \end{aligned}$$

If the set of all risk-neutral measures $\widetilde{P}$ consists only of one measure P^*, then there exists a hedge π^* with the initial value $C_N = E^*\big(f_N/B_N\big)$ and with the terminal value $X_N^{\pi^*}(C_N) = f_N$. In this case $C^* = C_* = C$.

In general, $C_* \le C^*$ and the interval $\big[C_* \,,\, C^*\big]$ contains all possible arbitrage-free prices for f_N, i.e., prices that are not risk-free for both parties involved in the contract. Note that intervals $(0, C_*)$ and (C^*, ∞) represent arbitrage prices for the buyer and for the seller of an option, respectively.



For example, if $x > C^*$, then the seller of the option can use $y \in (C^*, x)$ for building a strategy π^* with values $X_0^{\pi^*} = y$ and $X_N^{\pi^*} \geq f_N$, which is possible by the definition of C^*. Then

$$(x - f_N) + (X_N^{\pi^*} - y) = (x - y) + (X_N^{\pi^*} - f_N) \geq x - y > 0$$

is a risk-free profit of the seller.

It turns out that the intervals $\left[C_*, C^*\right]$ and

$$\left[\min_{\widetilde{P}} \widetilde{E}\big(f_N/B_N\big)\,,\ \max_{\widetilde{P}} \widetilde{E}\big(f_N/B_N\big)\right]$$

are the same, which gives a method of managing risks associated with a contingent claim f_N even in the case of incomplete markets.

Now we describe *super-hedging*, which is an effective methodology for deriving upper and lower prices C^* and C_*. Given a contingent claim f_N of possibly rather complex structure, one can consider a dominating claim $\tilde{f}_N \geq f_N$ (a.s.), that can be replicated by a self-financing strategy. Then the initial value of such a strategy can be taken as a *super-price* of f_N, which naturally may be higher than required. Next, for any martingale probability $\widetilde{P}$ we have $\widetilde{E}\tilde{f}_N \geq \widetilde{E} f_N$, which by definition of C^* and C_* implied that the quantities in the latter inequality coincide with the upper and lower super-prices respectively.

We now use the European call option to illustrate this result. We have $f_N = (S_N - K)^+$. Since S_N is non-negative, then $(S_N - K)^+ \leq S_N$. Using Jensen's inequality and the fact that S_n/B_n is a martingale with respect to any martingale probability $\widetilde{P}$, we obtain

$$\left(S_0 - \frac{K}{B_N}\right)^+ = \left(\widetilde{E}\left(\frac{S_N}{B_N}\right) - \frac{K}{B_N}\right)^+ \le \widetilde{E}\left(\frac{S_N - K}{B_N}\right)^+ \le \widetilde{E}\left(\frac{S_N}{B_N}\right)$$
$$= \widetilde{E}(S_0) = S_0 .$$

Thus

$$\left(S_0 - \frac{K}{B_N}\right)^+ \le C_* \le C^* \le S_0 ,$$

where due to the properties of the market, the first and the last inequalities become equalities and give us the lower and the upper prices of the option. The quantity $C^* - C_*$, called the *spread*, is a measure of incompleteness of the market.

Note that complete (B, S)-markets give an idealistic model of real financial markets. Incomplete markets can be regarded as a step toward more realistic models. A further step consists of introducing *markets with constraints*. Now we consider one of the simplest models of this type. We refer to it as a (B^1, B^2, S)-market:

$$\begin{aligned}
&\Delta B^i_n = r^i\, B^i_{n-1} , \quad B^i_0 = 1 , \quad i = 1, 2 ,\\
&\Delta S_n = \rho_n\, S_{n-1} , \quad S_0 \ge 0 ,\\
&-1 < a < r^1 \le r^2 < b ,
\end{aligned}$$

where (ρ_n) is a sequence of independent random variables (representing profitability of asset S) that take values b and a with probabilities p and $1 - p$ respectively.

Assets B^1 and B^2 can be interpreted as *saving* and *credit* accounts and S represents shares. It is natural to assume that $r^1 \le r^2$. If $r^1 = r^2$, then $B^1 = B^2$ and we arrive to a (B, S)-market.

A strategy (portfolio) $\pi = (\pi_n)_{n \le N}$ in a (B^1, B^2, S)-market is defined by three predictable sequences $(\beta^1_n, \beta^2_n, \gamma_n)_{n \le N}$. The values of this strategy are:

$$X^\pi_n = \beta^1_n\, B^1_n + \beta^2_n\, B^2_n + \gamma_n\, S_n .$$

A strategy π is *self-financing* if

$$\Delta X^\pi_n = \beta^1_n\, \Delta B^1_n + \beta^2_n\, \Delta B^2_n + \gamma_n\, \Delta S_n .$$

A strategy is *admissible* if its values are always non-negative. If credit and saving accounts have different rates of interest, then this creates an arbitrage opportunity. To avoid this, we assume that $\beta^1_n \ge 0$ and $\beta^2_n \le 0$.

A strategy π will be identified with the corresponding proportion of risky capital $\alpha_n = \gamma_n\, S_{n-1}/X^\pi_{n-1}$. Let $(1 - \alpha_n)^+$ and $-(1 - \alpha_n)^-$ represent an investor's deposits in savings and credit accounts, respectively. Then the dynamics of the values of such admissible strategy is described by

$$\begin{aligned}
&\Delta X^\pi_n(x) = X^\pi_{n-1}(x)\big[(1 - \alpha_n)^+\, r^1 - (1 - \alpha_n)^-\, r^2 + \alpha_n\, \rho_n\big] ,\\
&X^\pi_0 = x > 0 .
\end{aligned}$$

Recall that, in a complete arbitrage-free (B, S)-market, contingent claims can be priced uniquely. In the case of incomplete markets one can find an interval $[C_*, C^*]$ of arbitrage-free prices. The following methodology demonstrates that a similar result holds true in the case of (B^1, B^2, S)-markets.

Suppose that f_N is a contingent claim in a (B^1, B^2, S)-market. We will introduce an auxiliary complete market and find conditions that will guarantee that strategies with the same proportion of risky capital will have equal values in both markets. Let $d \in [0, r^2 - r^1]$. Define a (B^d, S)-market:

$$\Delta B_n^d = r^d \, B_{n-1}^d \,, \quad B_0^d = 1 \,,$$
$$\Delta S_n = \rho_n \, S_{n-1} \,, \quad S_0 \geq 0 \,,$$

with rate of interest $r^d = r^1 + d$.

Since the (B^d, S)-market is complete, the price of a contingent claim f_N is uniquely determined by the initial value of the minimal hedge:

$$C_N(f, r^d) = E^d \left(\frac{f_N}{B_N^d} \right) ,$$

where expectation is taken with respect to a martingale probability P^d.

Now let $\alpha = (\alpha_n)_{n \leq N}$ be the proportion of risky capital, and $\pi(\alpha)$ and $\pi(\alpha, d)$ be the corresponding strategies in the (B^1, B^2, S)-market and (B^d, S)-market, respectively.

LEMMA 2.1
Suppose $X_0^{\pi(\alpha)} = X_0^{\pi(\alpha,d)}$. Then $X_n^{\pi(\alpha)} = X_n^{\pi(\alpha,d)}$ for all $n \leq N$ if and only if

$$(r^2 - r^1 - d)\,(1 - \alpha_n)^- + d\,(1 - \alpha_n)^+ = 0 \tag{2.2}$$

for all $n \leq N$.

PROOF The dynamics of $X_n^{\pi(\alpha,d)}$ in the (B^d, S)-market are given by the following recurrence relation

$$\begin{aligned} \Delta X_n^{\pi(\alpha,d)} &= \beta_n^d \, \Delta B_n^d + \gamma_n \, \Delta S_n = \beta_n^d \, r^d \, B_{n-1}^d + \gamma_n \, \rho_n \, S_{n-1} \\ &= r^d \big(\beta_n^d \, B_{n-1}^d + \gamma_n \, S_{n-1} \big) + \gamma_n \, (\rho_n - r^d) \, S_{n-1} \\ &= r^d \, X_{n-1}^{\pi(\alpha,d)} + \gamma_n \, (\rho_n - r^d) \, S_{n-1} \\ &= r^d \, X_{n-1}^{\pi(\alpha,d)} + \alpha_n \, (\rho_n - r^d) \, X_{n-1}^{\pi(\alpha,d)} \\ &= X_{n-1}^{\pi(\alpha,d)} \big((1 - \alpha_n) \, r^d + \alpha_n \, \rho_n \big) \,. \end{aligned}$$

Similarly, in the (B^1, B^2, S)-market we obtain

$$\Delta X_n^{\pi(\alpha)} = X_{n-1}^{\pi(\alpha)} \big((1 - \alpha_n)^+ \, r^1 - (1 - \alpha_n)^- \, r^2 + \alpha_n \, \rho_n \big) \,,$$

which proves the claim. □

This result suggests the following methodology of pricing contingent claims f_N in a (B^1, B^2, S)-market. For any $d \in [0, r^2 - r^1]$ we consider a (B^d, S)-market, where one can use the initial value of the minimal hedge as a price $C_N(f, r^d)$ for f_N. Then quantities

$$\min_d C_N(f, r^d) \qquad \text{and} \qquad \max_d C_N(f, r^d)$$

are obvious natural candidates for lower and upper prices of f_N in the (B^1, B^2, S)-market.

Now we apply this methodology for pricing a European call option, i.e., $f_N = (S_N - K)^+$. The Cox-Ross-Rubinstein formula gives us prices $C_N(f, r^d)$ for $r^d \in [r^1, r^2]$. Also, it is clear that the function $C_N(f, \cdot)$ is increasing on $[r^1, r^2]$. Thus, the lower and upper prices in the (B^1, B^2, S)-market can be computed by applying the Cox-Ross-Rubinstein formula in (B^d, S)-markets with interest rates r^1 (in this case $d = 0$) and r^2 ($d = r^2 - r^1$), respectively:

$$C_N(r^i) = S_0\, B(k_0, N, \widetilde{p}_i) - K\,(1 + r^i)^{-N}\, B(k_0, N, p_i^*)\,,$$

$$p_i^* = \frac{r^i - a}{b - a}\,, \quad \widetilde{p}_i = \frac{1 + b}{1 + r^i}\, p_i^*\,, \quad i = 1, 2\,.$$

Note that prices $C_N(r^1)$ and $C_N(r^2)$ illustrate the difference of interests of a buyer and a seller in a (B^1, B^2, S)-market. Price $C_N(r^2)$ is attractive to a buyer because it is the minimal price of the option that guarantees the terminal payment. Price $C_N(r^1)$ reflects the intention of a seller to keep the option as an attractive investment instrument for a buyer.

WORKED EXAMPLE 2.1

Consider a (B^1, B^2, S)-market with $r^1 = 0$ and $r^2 = 0.2$. Suppose $S_0 = 100(\$)$ and

$$S_1 = \begin{cases} 150\,(\$) \ \textit{with probability } p = 0.4 \\ 70\,(\$) \quad \textit{with probability } 1 - p = 0.6\,. \end{cases}$$

Find the upper and lower prices for a European call option $f_1 = (S_1 - K)^+ \equiv \max\{0, S_1 - K\}$ with strike price $K = 100\,(\$)$.

SOLUTION From the Cox-Ross-Rubinstein formula we have

$$C_1(0) = S_0\, \frac{r^1 - a}{b - a}\, \frac{1 + b}{1 + r^1} - K\,(1 + r^1)^{-1}\, \frac{r^1 - a}{b - a}$$

$$= 100\, \frac{0.3}{0.8}\, \frac{1.5}{1} - 100\,(1)^{-1}\, \frac{0.3}{0.8} \approx 19\,,$$

$$C_1(0.2) = 100\, \frac{0.5}{0.8}\, \frac{1.5}{1.2} - 100\,(1.2)^{-1}\, \frac{0.5}{0.8} \approx 26\,.$$

Thus, the *spread* in such (B^1, B^2, S)-market is equal to $C_1(0.2) - C_1(0) = 26 - 19 = 7$.

Now consider the same market with $r^1 = 0.1$ and $r^2 = 0.2$. Then we compute

$$C_1(0.1) = 100 \, \frac{0.3}{0.8} \, \frac{1.5}{1.1} - 100 \, (1.1)^{-1} \, \frac{0.7}{0.8} \approx 22 \, ,$$

and the spread in this case is $C_1(0.2) - C_1(0.1) = 26 - 22 = 4$. □

Note that in this example the condition (2.2) from Lemma 2.1 is satisfied, and the example illustrates that if the gap $r^2 - r^1$ between the rates of interest on saving and credit accounts become smaller, then the spread decreases. Spread can be regarded as a measure of proximity of (B^1, B^2, S)-market to an ideal complete market.

Next we consider a problem of finding an optimal strategy (a strategy that maximizes the logarithmic utility function) in a (B^1, B^2, S)-market. By Lemma 2.1 it is equivalent to solving an optimization problem in a complete (B^d, S)-market. The optimal proportion is given by

$$\alpha_n \equiv \frac{(1 + r^d) \, (\mu - r^d)}{(r^d - a) \, (b - r^d)} \, , \qquad d \in [0, r^2 - r^1].$$

For the boundary values $d = 0$ and $d = r^2 - r^1$ we have

$$\alpha^{(i)} = \frac{(1 + r^{(i)}) \, (\mu - r^{(i)})}{(r^{(i)} - a) \, (b - r^{(i)})} \, , \qquad i = 1, 2,$$

given that $\alpha^{(1)} \leq 1$ and $\alpha^{(2)} \geq 1$.

In Worked Example 2.1 with $r^1 = 0$ and $r^2 = 0.2$ we compute

$$\alpha^{(1)} = \frac{0.02}{0.3 \cdot 0.5} \approx 0.13 \leq 1, \quad \alpha^{(2)} = \frac{-1.02 \cdot 0.18}{0.5 \cdot 0.3} < 1.$$

Thus, the optimal proportion is 0.13 and the rest of the capital must be invested in a saving account with the rate of interest r^1.

Another step in studying more realistic models of a market consists of an introduction to *transaction costs*. Consider the binomial model of a (B, S)-market:

$$\Delta B_n = r B_{n-1} \, , \quad B_0 = 1 \, ,$$
$$\Delta S_n = \rho_n S_{n-1} \, , \quad S_0 > 0 \, ,$$

where $r \geq 0$ is a constant rate of interest with $-1 < a < r < b$, and $1 \leq n \leq N$.

Now we suppose that any transaction of capital from one asset to another attracts a fee or a *transaction cost* (with a fixed parameter $\lambda \in [0, 1]$): a buyer of asset S pays $S_n \, (1 + \lambda)$ at time n, and a seller receives $S_n \, (1 - \lambda)$ accordingly.

Recall that a writer of a European call option is obliged to sell at time N one unit of asset S at a fixed price K. After receiving a premium x, the writer hedges the corresponding contingent claim by redistributing the capital between assets B and S in proportion (β, γ).

Suppose that, at terminal time N, both sell and buy prices are equal to S_N. Then the contingent claim corresponding to this option can be represented here in an appropriate two-component form:

$$f = (f^1, f^2) = \begin{cases} \big(- K/B_N, 1\big) & \text{if } S_N > K \\ (0,0) & \text{if } S_N \leq K\,, \end{cases}$$

where f^1 and f^2 represent the number of bonds and shares, respectively, necessary for making the repayment.

We claim that this model admits a unique 'fair' arbitrage-free price C_N for such an option. First, we describe a transition from portfolio $\pi = (\beta, \gamma)$ to portfolio $\pi' = (\beta', \gamma')$ at some time $n \leq N$. Clearly, there are two cases in a situation when buying and selling shares attract transaction costs:

1. If $\gamma > \gamma'$, then we have to sell $\gamma - \gamma'$ shares and use the received capital for buying the corresponding number of bonds. This leads to the following condition
$$(\beta' - \beta)\, B_n = (\gamma - \gamma')\, S_n\, (1 - \lambda)\,.$$
2. If $\gamma < \gamma'$, then we arrive to condition
$$(\beta - \beta')\, B_n = (\gamma' - \gamma)\, S_n\, (1 + \lambda)\,.$$

Combining these conditions results in

$$(\beta' - \beta)\, B_n + (\gamma' - \gamma)\, S_n = -\lambda\, |\gamma' - \gamma|\, S_n\,.$$

Our claim follows from the following theorem (see [10]).

THEOREM 2.5 (Boyle-Vorst)
In the framework of a binomial (B,S)-market with transaction costs, for any European call option there exists a unique replicating strategy. This strategy coincides with the strategy that replicates the same option in a (complete) binomial market without transaction costs, where values $\bar{b}$ and $\bar{a}$ of the profitability sequence ρ are defined by

$$1 + \bar{b} = (1+b)\,(1+\lambda) \qquad \text{and} \qquad 1 + \bar{a} = (1+a)\,(1-\lambda)\,.$$

PROOF We use the method of reverse induction. First, we introduce the following useful notations for values of $\mathcal{F}_n$-measurable quantities β_{n+1} and γ_{n+1} on sets $\{\omega:\ \rho_n = b\}$ and $\{\omega:\ \rho_n = a\}$:

$$\begin{aligned}
\beta^b_{n+1}(\rho_1, \ldots, \rho_{n-1}) &= \beta_{n+1}(\rho_1, \ldots, \rho_{n-1}, b)\,,\\
\beta^a_{n+1}(\rho_1, \ldots, \rho_{n-1}) &= \beta_{n+1}(\rho_1, \ldots, \rho_{n-1}, a)\,,\\
\gamma^b_{n+1}(\rho_1, \ldots, \rho_{n-1}) &= \gamma_{n+1}(\rho_1, \ldots, \rho_{n-1}, b)\,,\\
\gamma^a_{n+1}(\rho_1, \ldots, \rho_{n-1}) &= \gamma_{n+1}(\rho_1, \ldots, \rho_{n-1}, a)\,.
\end{aligned}$$

Then redistribution of capital can be expressed in the form:

$$
\begin{aligned}
&(\beta^b_{n+1}-\beta_n)\,B_n+(\gamma^b_{n+1}-\gamma_n)\,S_{n-1}\,(1+b)=-\lambda\,|\gamma^b_{n+1}-\gamma_n|\,S_{n-1}\,(1+b)\,,\\
&(\beta^a_{n+1}-\beta_n)\,B_n+(\gamma^a_{n+1}-\gamma_n)\,S_{n-1}\,(1+a)=-\lambda\,|\gamma^a_{n+1}-\gamma_n|\,S_{n-1}\,(1+a)\,.
\end{aligned}
\tag{2.3}
$$

Subtracting the second equation from the first, we define function

$$
\begin{aligned}
g(\gamma_n)=\;&\gamma_n\,S_{n-1}\,(b-a)-\gamma^b_{n+1}\,S_{n-1}\,(1+b)+\gamma^a_{n+1}\,S_{n-1}\,(1+a)\\
&-\beta^b_{n+1}\,B_n+\beta^a_{n+1}\,B_n\\
&-\lambda\,|\gamma_n-\gamma^b_{n+1}|\,S_{n-1}\,(1+b)+\lambda\,|\gamma_n-\gamma^a_{n+1}|\,S_{n-1}\,(1+a)\,.
\end{aligned}
$$

Thus, the problem of finding β_n and γ_n given values of β_{n+1} and γ_{n+1}, reduces to the question of solvability of system (2.3), or equivalently, to finding number if zeros of function $g(\gamma_n)$. Note that this function is continuous and linear on intervals $(-\infty,\gamma^a_{n+1})$, $(\gamma^a_{n+1},\gamma^b_{n+1})$ and (γ^b_{n+1},∞) with positive derivative equal to

$$
\Big[(1+\lambda)\,(1+b)-(1+\lambda)\,(1+a)\Big]\,S_{n-1}\,,
$$

$$
\Big[(1+\lambda)\,(1+b)-(1-\lambda)\,(1+a)\Big]\,S_{n-1}\,,
$$

and

$$
\Big[(1-\lambda)\,(1+b)-(1-\lambda)\,(1+a)\Big]\,S_{n-1}\,,
$$

respectively.

Hence $g(\gamma_n)$ is a strictly monotone, continuous, piece-wise linear function, which implies that there is a unique solution to equation $g(\gamma_n)=0$. Now we have to show that $\gamma_n\in[\gamma^a_{n+1},\gamma^b_{n+1}]$, or $g(\gamma^a_{n+1})\le 0$ and $g(\gamma^b_{n+1})\ge 0$. It is clear that

$$
g(\gamma^a_{n+1})=\left(\gamma^a_{n+1}-\gamma^b_{n+1}\right)S_{n-1}\,(1+b)\,(1+\lambda)-B_n\,\beta^b_{n+1}+B_n\,\beta^a_{n+1}\,.
$$

Taking into account

$$
\gamma^{ba}_{n+2}\le\gamma^b_{n+1}\le\gamma^{bb}_{n+2}\quad\text{and}\quad\gamma^{aa}_{n+2}\le\gamma^a_{n+1}\le\gamma^{ab}_{n+2}\le\gamma^{ba}_{n+2}\,,
$$

we can rewrite equations (2.3) in the form

$$
\begin{aligned}
\left(\beta^{ba}_{n+2}-\beta^b_{n+1}\right)B_n\,(1+r)&+\left(\gamma^{ba}_{n+2}-\gamma^b_{n+1}\right)S_{n-1}\,(1+b)\,(1+a)\\
&=\lambda\left(\gamma^b_{n+1}-\gamma^{ba}_{n+2}\right)S_{n-1}\,(1+b)\,(1+a)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\beta^{ba}_{n+2}-\beta^a_{n+1}\right)B_n\,(1+r)&+\left(\gamma^{ba}_{n+2}-\gamma^a_{n+1}\right)S_{n-1}\,(1+b)\,(1+a)\\
&=\lambda\left(\gamma^{ba}_{n+2}-\gamma^b_{n+1}\right)S_{n-1}\,(1+b)\,(1+a)\,,
\end{aligned}
$$

respectively. Subtracting the second equation from the first, we obtain

$$\left(\beta^a_{n+1} - \beta^b_{n+1}\right) B_n \,(1+r) = \left(\gamma^b_{n+1} - \gamma^a_{n+1}\right) S_{n-1}\,(1+a)\,(1+b)\,(1-\lambda)\,,$$

and hence

$$g(\gamma^a_{n+1}) = \left(\gamma^a_{n+1} - \gamma^b_{n+1}\right) S_{n-1}\,(1+b)\left[(1+\lambda) - \frac{(1+a)\,(1-r)}{1+r}\right] \le 0\,,$$

since $\gamma^b_{n+1} \ge \gamma^a_{n+1}$ and $a \le r$.

Similarly, one can prove that $g(\gamma^b_{n+1}) \ge 0$. To complete the proof, we need to check the base of induction. At the terminal time, there are two possible types of portfolios. First: $\gamma_{N+1} = 1$, $\beta_{N+1} = -K$, second: $\gamma_{N+1} = \beta_{N+1} = 0$. Note that in both cases $\gamma^b_{N+1} \ge \gamma^a_{N+1}$. Suppose $\gamma^b_{N+1} = \gamma^a_{N+1}$, then $\gamma_N = \gamma^b_{N+1}$, $\beta_N = -K/B_N$ is a unique solution of system (2.3) and $\gamma^a_{N+1} \le \gamma_N \le \gamma^b_{N+1}$.

If $\gamma^b_{N+1} = 1$ and $\gamma^a_{N+1} = 0$, then the unique solution has the form

$$\gamma_N = \frac{S_{N-1}\,(1+\bar{b}) - K}{S_{N-1}\,(1+\bar{b}) - S_{N-1}\,(1+\bar{a})}$$

with $\gamma^a_{N+1} = 0 < \gamma_N < 1 = \gamma^b_{N+1}$.

The case when $\gamma^b_{N+1} = \gamma^a_{N+1} = 0$ is trivial:

$$\gamma_N = \beta_N = 0\,, \qquad \gamma^a_{N+1} \le \gamma_N \le \gamma^b_{N+1}\,.$$

$\square$

Finally, we give examples of typical investment strategies that are based on options (see, for example, [42]).

Straddle is a combination of call and put options on the same stock with the same strike price K and the same expiry date N. The function $\mathcal{V}(S_N) := f(S_N) - C_N$ is called the *gain-loss* function. For a buyer we have

$$\mathcal{V}(S_N) = |S_N - K| - C_N\,.$$

Strangle is a combination of call and put options on the same stock with the same expiry date N but different strike prices K_1 and K_2. The gain-loss function in this case has the form

$$\mathcal{V}(S_N) = |S_N - K_2|\, I_{\{\omega:\, S_N > K_2\}} + |S_N - K_1|\, I_{\{\omega:\, S_N < K_1\}} - C_N\,.$$

Strap is a combination of one put option and two call options on the same stock with the same expiry date N but possibly different strike prices K_1 and K_2. If $K_1 = K_2$, then the gain-loss function has the form

$$\mathcal{V}(S_N) = 2\,|S_N - K|\, I_{\{\omega:\, S_N > K\}} + |S_N - K|\, I_{\{\omega:\, S_N < K\}} - C_N\,.$$

Strip is a combination of one call option and two put options on the same stock with the same expiry date N but possibly different strike prices K_1 and K_2. The gain-loss function is given by

$$\mathcal{V}(S_N) = |S_N - K_2|\, I_{\{\omega:\, S_N > K_2\}} + 2\,|S_N - K_1|\, I_{\{\omega:\, S_N < K_1\}} - C_N\,.$$

'Bull's' Spread is a strategy that consists of buying a call option with a strike price $K_2 > K_1$. In this case

$$\mathcal{V}(S_N) = |K_2 - K_1|\, I_{\{\omega:\, S_N \geq K_2\}} + |S_N - K_1|\, I_{\{\omega:\, K_1 < S_N < K_2\}} - C_N\,.$$

It is usually used in a situation when the stock price is expected to rise.

'Bear's' Spread is a strategy that consists of selling a call option with a strike price $K_2 > K_1$. Then

$$\mathcal{V}(S_N) = -|K_2 - K_1|\, I_{\{\omega:\, S_N \geq K_2\}} + |S_N - K_1|\, I_{\{\omega:\, K_1 < S_N < K_2\}} - C_N\,.$$

2.3 Hedging contingent claims in mean square

Consider an incomplete (B, S)-market with the time horizon N. As we discussed above, for (perfect) hedging of contingent claims on such markets, one has to consider strategies with consumption.

An alternative approach to hedging of contingent claims was suggested by Foellmer and Sondermann. It is a combination of the ideas of hedging and of investment portfolio with the quadratic utility function.

First we consider a one-step model. Let H be the discounted value of a contingent claim f_1. At time $n = 0$ the seller of the claim forms a portfolio $\pi_0 = (\beta_0, \gamma_0)$ with value

$$X_0^\pi = \beta_0\, B_0 + \gamma_0\, S_0,$$

and the discounted value

$$V_0^\pi = \frac{X_0^\pi}{B_0} = \beta_0 + \gamma_0\, \frac{S_0}{B_0} = \beta_0 + \gamma_0\, X_0,$$

where $X = S/B$.

At time $n = 1$ we replace β_0 with β_1, so that the value of the portfolio becomes

$$V_1^\pi = \beta_1 + \gamma_0\, X_1,$$

where β_1 is determined by the replication condition:

$$V_1^\pi = H \qquad \text{or} \qquad X_1^\pi = f_1.$$

Thus, for finding an 'optimal' strategy $\hat{\pi}$ one has to determine $\gamma_0 = \gamma$. We define the following *price sequence* C^π:

$$C_0^\pi = V_0^\pi, \quad C_1^\pi = V_1^\pi - \gamma\,(X_1 - X_0) = H - \gamma\,\Delta X_1\,.$$

This choice has an obvious interpretation: the amount $V_1^\pi = H$ must be paid to the holder of an option, and $\gamma\,\Delta X_1$ is the gain-loss implied by strategy γ. To determine an optimal γ, one has to solve the following optimization problem: find $\hat{\pi}$ such that

$$V\big(C_1^{\hat{\pi}}\big) = \inf_\pi V\big(C_1^\pi\big).$$

Further, if $V\big(\Delta X_1\big) > 0$, then the variance

$$V\big(C_1^{\hat{\pi}}\big) = V\big(H\big) - 2\gamma\,Cov\big(H, \Delta X_1\big) + \gamma^2\,V\big(\Delta X_1\big)$$

attains its unique minimum at the point

$$\hat{\gamma} = \frac{Cov\big(H, \Delta X_1\big)}{V\big(\Delta X_1\big)}\,,$$

and therefore

$$\begin{aligned} V\big(C_1^{\hat{\pi}}\big) &= V\big(H - \hat{\gamma}\,\Delta X_1\big) = V\big(H\big) - \frac{Cov^2\big(H, \Delta X_1\big)}{V\big(\Delta X_1\big)} \\ &= V\big(H\big)\big[1 - Cov^2\big(H, \Delta X_1\big)\big]\,. \end{aligned}$$

Another natural optimization problem of finding

$$\inf_\pi\; E\big(C_1^\pi - C_0^\pi\big)^2$$

is obviously solved by

$$C_0^\pi = E\big(C_1^\pi\big).$$

Now we consider an arbitrary time horizon $N \geq 1$. It is clear from the one-step case that strategies $\pi = (\beta_n, \gamma_n)_{n \leq N}$ must be such that γ_n are predictable (i.e., determined by information $\mathcal{F}_{n-1}$) and β_n are adapted (i.e., determined by information $\mathcal{F}_n$).

We have the following discounted values

$$\Delta\,V_n^\pi = V_n^\pi - V_{n-1}^\pi = X_{n-1}\,\Delta\gamma_n + \gamma_n\,\Delta X_n + \Delta\beta_n\,.$$

We say that a strategy π is *admissible* if

$$V_N^\pi = H \qquad \text{or} \qquad X_N^\pi = f_N\,.$$

The price of such a strategy at time n is

$$C_n^\pi = V_n^\pi - \sum_{k=1}^{n} \gamma_k\,\Delta X_k\,, \quad n \leq N.$$

For simplicity, we assume that the original probability P is a martingale probability i.e., $P = P^*$. Now we define the following *risk sequence*

$$R_n^\pi = E^*\Big(\big(C_N^\pi - C_n^\pi\big)^2\Big|\mathcal{F}_n\Big).$$

Its initial value

$$R_0^\pi = E^*\Big(H - \sum_{k=1}^{N}\gamma_k\,\Delta X_k - C_0^\pi\Big)^2$$

is referred to as *risk* of strategy π.

Note that discounted values of a self-financing strategy π have the form

$$V_n^\pi = V_0^\pi - \sum_{k=1}^{n}\gamma_k\,\Delta X_k,$$

which implies that it has a constant price sequence: $C_n^\pi = C_0^\pi$ for all $n \leq N$.

Now we solve the minimization problem in the class of all admissible strategies. Suppose that X is a square integrable martingale and $E^*\left(H^2\right) < \infty$ (note that in the case of a discrete probability space $(\Omega, \mathcal{F}, P^*)$, these integrability assumptions are trivially satisfied).

Consider another martingale

$$V_n^* = E^*\left(H\middle|\mathcal{F}_n\right), \quad n \leq N.$$

The following decomposition is a key technical tool for solving our problem.

LEMMA 2.2 (Kunita-Watanabe Decomposition)
The martingale V^* admits the decomposition

$$V_n^* = V_0^* + \sum_{k=1}^{n}\gamma_k^H\,\Delta X_k + L_n^H\,,$$

where $\left(\gamma_n^H\right)_{n\leq N}$ is a predictable sequence and L^H is a martingale orthogonal to X:

$$\langle X, L^H\rangle_n = 0.$$

PROOF Define sequence $\left(\gamma_n^H\right)_{n\leq N}$ by

$$\gamma_n^H = \frac{E^*\left(H\,\Delta X_n\middle|\mathcal{F}_{n-1}\right)}{E^*\left((\Delta X_n)^2\middle|\mathcal{F}_{n-1}\right)}\,, \quad n \leq N, \tag{2.4}$$

which is clearly predictable. Also let

$$L_n^H := V_n^* - V_0^* - \sum_{k=1}^{n}\gamma_k^H\,\Delta X_k,$$

which is a martingale being a difference of two martingales.

Using Cauchy-Schwartz inequality we obtain

$$E^*\big(\gamma_n^H\,\Delta X_n\big)^2 = E^*\left(E^*\left(\left[\frac{E^*\big(H\,\Delta X_n\big|\mathcal{F}_{n-1}\big)}{E^*\big((\Delta X_n)^2\big|\mathcal{F}_{n-1}\big)}\,\Delta X_n\right]^2\middle|\mathcal{F}_{n-1}\right)\right)$$

$$= E^*\left(\frac{\big(E^*\big(H\,\Delta X_n\big|\mathcal{F}_{n-1}\big)\big)^2}{E^*\big((\Delta X_n)^2\big|\mathcal{F}_{n-1}\big)}\right)$$

$$\leq E^*\big(E^*\big(H^2|\mathcal{F}_{n-1}\big)\big) = E^*\big(H^2\big) < \infty,$$

which implies that L^H is a square integrable martingale.

Now we show that the product

$$L_n^H\,X_n = \left(E^*(H|\mathcal{F}_n) - E^*(H) - \sum_{k=1}^n \gamma_k^H\,\Delta X_k\right)X_n$$

forms a martingale. This follows from the definition (2.4) of γ_n^H and the equality

$$E^*\Big(E^*(H|\mathcal{F}_n)\,X_n\big|\mathcal{F}_{n-1}\Big) - E^*\Big(\gamma_n^H\,\Delta X_n\,X_n\big|\mathcal{F}_{n-1}\Big) = E^*(H|\mathcal{F}_{n-1})\,X_{n-1}\,.$$

Since sequence γ^H is predictable and $L^H\,X$ is a martingale, then

$$E^*\Big(L_n^H\,\sum_{k=1}^n \gamma_k^H\,\Delta X_k\big|\mathcal{F}_{n-1}\Big)$$

$$= E^*\big(L_n^H|\mathcal{F}_{n-1}\big)\,\sum_{k=1}^{n-1}\gamma_k^H\,\Delta X_k + \gamma_n^H\,E^*\big(L_n^H\,\Delta X_n|\mathcal{F}_{n-1}\big)$$

$$= L_{n-1}^H\,\sum_{k=1}^{n-1}\gamma_k^H\,\Delta X_k\,.$$

Hence, L^H is orthogonal to X and $\Big(\sum_{k=1}^n \gamma_k^H\,\Delta X_k\Big)_{n\leq N}$.

Finally, we note that it is not difficult to prove the *uniqueness* of Kunita-Watanabe decomposition. □

Now, since $\sum_{k=1}^n \gamma_k\,\Delta X_k$ is a martingale, then to minimize risk R_0^π we must have $C_0^\pi = E^*(H)$. Furthermore, R_0^π does not depend on changes of β, the non-risky component of strategy π. We can rewrite R_0^π in the form

$$R_0^\pi = E^*\Big(H - \sum_{k=1}^N \gamma_k\,\Delta X_k - E^*(H)\Big)^2 = E^*\Big(\sum_{k=1}^N(\gamma_k^H - \gamma_k)\,\Delta X_k + L_N^H\Big)^2$$

$$= E^*\big(L_N^H\big)^2 + \sum_{k=1}^N E^*\Big(\big(\gamma_k^H - \gamma_k\big)^2\,(\Delta X_k)^2\Big),$$

and therefore the required risk-minimizing strategy is uniquely determined by

$$\gamma_n = \gamma_n^H, \quad n = 1, 2, \dots, N.$$

Similarly, we obtain the risk sequence

$$R_n^\pi = E^*\Big(\big(L_N^H - L_n^H\big)^2 \big| \mathcal{F}_n\Big).$$

Thus we obtain the following formulae for the optimal strategy $\hat{\pi} = (\hat{\beta}, \hat{\gamma})$:

$$\hat{\gamma}_n = \gamma_n^H, \quad \hat{\beta}_n = V_n^* - \hat{\gamma}_n X_n, \quad n \leq N.$$

The price of this strategy

$$\begin{aligned} C_n^{\hat{\pi}} &= V_n^{\hat{\pi}} - \sum_{k=1}^{n} \hat{\gamma}_k \, \Delta X_k = V_n^* - \sum_{k=1}^{n} \gamma_k^H \, \Delta X_k \\ &= E^*(H) + \sum_{k=1}^{n} \gamma_k^H \, \Delta X_k + L_n^H - \sum_{k=1}^{n} \gamma_k^H \, \Delta X_k = E^*(H) + L_n^H \end{aligned}$$

is a martingale. Such strategies $\hat{\pi}$ are referred to as *self-financing in average*.

WORKED EXAMPLE 2.2

Consider a one-step (B, S)-market with the rate of interest $r = 0.1$ and profitability

$$\rho = \rho_1 = \begin{cases} 0.2 & \textit{with probability } 0.7 \\ -0.1 & \textit{with probability } 0.3 \end{cases}.$$

Consider a pure endowment assurance with the claim

$$f_1 = \max\{1 + r, \, 1 + \rho\},$$

which is paid to the policy holder on survival to time $N = 1$ (year). Suppose that the probability of death during this year is 0.004, and let $B_0 = S_0 = 1\,(\$)$. Find γ and policy's initial price C_0.

SOLUTION Denote $C(\gamma) \equiv C^\pi$, where $\pi = (\beta, \gamma)$. We need to minimize $V\big(C_1(\gamma)\big)$ and $E\big(C_1(\gamma) - C_0(\gamma)\big)^2$. We have

$$E\big(\max\{1 + r, \, 1 + \rho\}\big) = 1.1 \cdot 0.3 + 1.2 \cdot 0.7 = 1.17$$

and

$$V(\rho - r) = V(\rho) = 0.0189\,.$$

If H is the discounted value of the payoff and $I_1 = I_A$, where A is the event of policy holder's survival for at least one year, then

$$H = I_1 \, \frac{\max\{1 + r, \, 1 + \rho\}}{1 + r}\,.$$

Further, for discounted prices of S we have

$$\frac{S_1}{B_1} - \frac{S_0}{B_0} = \frac{1+\rho}{1+r} - 1 = \frac{\rho - r}{1+r},$$

and therefore

$$\gamma = \frac{Cov\big(H, (\rho - r)/(1+r)\big)}{V\big((\rho - r)/(1+r)\big)} = \frac{Cov\big(I_1 \, \max\{1+r,\, 1+\rho\}\,,\, \rho - r\big)}{V(\rho - r)}$$

$$= 0.996 \left[0.7 \, \frac{(1.2 - 1.17)\,(0.1 - 0.01)}{0.0189} + 0.3 \, \frac{(1.1 - 1.17)\,(-0.2 - 0.01)}{0.0189} \right]$$

$$\approx 0.382\,.$$

Hence

$$C_0 = E\big(C_1(\gamma)\big) = \frac{0.996 \cdot 1.17 - 0.332 \cdot 0.01}{1.1} = 1.0624\,.$$

Note that if there is no additional source of risk related to the survival of a policy holder (i.e., if probability of policy holder's death is 0), then a replicating self-financing strategy can be easily found from the system

$$\begin{cases} 1.1 \cdot C_0 + 0.1 \cdot \gamma = 1.2 \\ 1.1 \cdot C_0 - 0.2 \cdot \gamma = 1.1 \end{cases},$$

which gives $\gamma = 0.333$ and $C_0 = 1.06$. □

2.4 Gaussian model of a financial market and pricing in flexible insurance models. Discrete version of the Black-Scholes formula.

Since prices S_n are always positive, we can write them in the *exponential* form:

$$S_n = S_0 \, E^{W_n}, \quad S_0 > 0,$$

where $W_n = \sum_{k=1}^n w_k,\ W_0 = 0$, and

$$w_n = \Delta W_n = \ln \frac{S_n}{S_{n-1}} = \ln \left(1 + \frac{\Delta S_n}{S_{n-1}} \right), \quad n = 1, \ldots, N.$$

On the other hand, from the definition of stochastic exponentials and from the recurrence relations $\Delta S_n = \rho_n \, S_{n-1}$, we obtain

$$S_n = S_0 \, \varepsilon_n \left(\sum_{k=1}^n \rho_k \right) = S_0 \, \varepsilon_n \big(V\big)\,,$$

where we introduced the notation $\sum_{k=1}^n \rho_k = V_n = \sum_{k=1}^n \Delta V_k = \sum_{k=1}^n v_k$ with $v_k = \rho_k = \Delta V_k > -1$.

Hence, we obtain the following connection between the two stochastic sequences W and V:

$$S_0\, e^{W_n} = S_n = S_0\, \varepsilon_n\big(V\big) = S_0\, e^{W_n}\, e^{-W_n} \prod_{k=1}^{n} \big(1 + \Delta V_k\big)$$
$$= S_0\, e^{W_n} \prod_{k=1}^{n} \big(1 + \Delta V_k\big)\, e^{-\Delta W_k}\,,$$

so

$$W_n = \sum_{k=1}^{n} \ln\big(1 + \Delta V_k\big)$$

and

$$V = \sum_{k=1}^{n} \Big(e^{\Delta W_k} - 1\Big) = W_n + \sum_{k=1}^{n} \Big(e^{\Delta W_k} - \Delta W_k - 1\Big)\,.$$

Now, using Doob decomposition, we can write

$$W_n = M_n + A_n, \quad n = 1, \ldots, N,$$

where $A_0 = M_0 = W_0 = 0$, $E(|\Delta W_n|) < \infty$, sequence $\Delta A_n = E\big(\Delta W_n|\mathcal{F}_{n-1}\big)$ is predictable and $\Delta M_n = \Delta W_n - E\big(\Delta W_n|\mathcal{F}_{n-1}\big)$ forms a martingale.

Thus, prices S_n can be written in the form

$$S_n = S_0\, \exp\{M_n + A_n\}$$

on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = \big(\mathcal{F}_n\big)_{n\leq N}$ is a filtration with $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$.

Suppose that sequence $(w_k)_{k\leq N}$ consists of Gaussian random variables with means μ_k and variances σ_k^2:

$$w_k \sim \mathcal{N}(\mu_k, \sigma_k^2), \quad k = 1, \ldots, N.$$

We can then write

$$w_k = \mu_k + \sigma_k\, \epsilon_k, \quad k = 1, \ldots, N,$$

where $\epsilon_k \sim \mathcal{N}(0,1)$ are standard Gaussian random variables. In this case (deterministic) sequence A_n and Gaussian martingale M_n have the form

$$A_n = \sum_{k=1}^{n} \mu_k \quad \text{and} \quad M_n = \sum_{k=1}^{n} \sigma_k\, \epsilon_k\,,$$

and quadratic variation of M is deterministic: $\langle M, M\rangle_n = \sum_{k=1}^n \sigma_k^2\,,\ n \leq N$.

We assume that $\mathcal{F} = \mathcal{F}_N = \sigma(\epsilon_1, \ldots, \epsilon_N)$. Define a stochastic sequence

$$Z_n = \exp\left\{ -\sum_{k=1}^{n} \frac{\mu_k}{\sigma_k}\,\epsilon_k - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\mu_k}{\sigma_k}\right)^2 \right\}, \quad Z_0 = 1, \quad n = 1, \ldots, N.$$

We show that $\big(Z_n, \mathcal{F}_n\big)_{n \le N}$ is a martingale with respect to the initial probability P. Indeed, taking into account independence of $\epsilon_1, \ldots, \epsilon_N$, we have

$$E\left(\exp\left\{ -\frac{\mu_k}{\sigma_k}\,\epsilon_k - \frac{1}{2}\left(\frac{\mu_k}{\sigma_k}\right)^2 \right\} \right) = 1,$$

and for all $n = 1, \ldots, N$

$$\begin{aligned} E\big(Z_n | \mathcal{F}_{n-1}\big) &= E\left(Z_{n-1}\, \exp\left\{ -\frac{\mu_k}{\sigma_k}\,\epsilon_k - \frac{1}{2}\left(\frac{\mu_k}{\sigma_k}\right)^2 \right\} \middle| \mathcal{F}_{n-1} \right) \\ &= Z_{n-1}\, E\left(\exp\left\{ -\frac{\mu_k}{\sigma_k}\,\epsilon_k - \frac{1}{2}\left(\frac{\mu_k}{\sigma_k}\right)^2 \right\} \right) \\ &= Z_{n-1} \quad \text{(a.s.)}. \end{aligned}$$

Now, since $Z_N > 0$ and $E\big(Z_N\big) = 1$, then the following probability

$$P^*(A) := E\big(Z_N\, I_A\big)\,, \quad A \in \mathcal{F}$$

is well-defined. Computing

$$\begin{aligned} E^*\left(e^{\imath\lambda w_n}\right) &= E\left(\exp\left\{ \left(\imath\lambda\sigma_n - \frac{\mu_n}{\sigma_n}\right)\epsilon_n + \imath\lambda\mu_n - \frac{1}{2}\left(\frac{\mu_n}{\sigma_n}\right)^2 \right\} \right) \\ &= E\left(\left(\imath\lambda\sigma_n - \frac{\mu_n}{\sigma_n}\right)\epsilon_n - \frac{1}{2}\left(\imath\lambda\sigma_n - \frac{\mu_n}{\sigma_n}\right)^2 \right) \\ &\qquad \times \exp\left\{ \frac{1}{2}\left(\imath\lambda\sigma_n - \frac{\mu_n}{\sigma_n}\right)^2 + \imath\lambda\mu_n - \frac{1}{2}\left(\frac{\mu_n}{\sigma_n}\right)^2 \right\} \\ &= \exp\left\{ -\frac{\lambda^2\,\sigma_n^2}{2} \right\}, \quad \lambda \ge 0 \quad n = 1, \ldots, N, \end{aligned}$$

we conclude that $(w_k)_{k \le N}$ is a sequence of Gaussian random variables with respect to P^*, with mean zero and variance σ_n^2. Note that independence of $(w_k)_{k \le N}$ follows from the equality

$$E^*\left(\exp\left\{\imath\sum_{k=1}^{N}\lambda_k w_k\right\}\right) = E^*\left(\exp\left\{\imath\sum_{k=1}^{N-1}\lambda_k w_k\right\} E^*\left(e^{\imath\lambda_N w_N}\middle|\mathcal{F}_{N-1}\right)\right)$$
$$= E^*\left(\exp\left\{\imath\sum_{k=1}^{N-1}\lambda_k w_k\right\}\right) e^{-\lambda_N^2 w_N^2} = \ldots$$
$$= \exp\left\{-\frac{1}{2}\sum_{k=1}^{N}\lambda_k^2 w_k^2\right\}.$$

As a corollary we obtain the following version of *Girsanov theorem.*

PROPOSITION 2.2

If under probability P random variables

$$w_k \sim \mathcal{N}(\mu_k, \sigma_k^2), \quad k = 1, \ldots, N$$

are independent, then they are also independent and normally distributed under probability P^*:

$$w_k \sim \mathcal{N}(0, \sigma_k^2), \quad k = 1, \ldots, N.$$

Next we consider the following discrete Gaussian (B, S)-market:

$$B_n = \prod_{k=1}^{n}(1 + r_k) = \exp\left\{\sum_{k=1}^{n}\delta_k\right\},$$
$$S_n = S_0 \exp\left\{\sum_{k=1}^{n}\mu_k + \sum_{k=1}^{n}\sigma_k\epsilon_k\right\}, \quad S_0 > 0,$$

where non-negative deterministic sequences (r_k) and (δ_k) represent the rate of interest and are such that

$$1 + r_k = e^{\delta_k}, \quad k = 1, \ldots, N.$$

Now our aim is to construct a martingale probability P^* for this market. We are looking for a probability of the form of Essher transform:

$$P^*(A) = E\big(Z_N\, I_A\big),$$

where

$$Z_N = \prod_{n=1}^{N} z_n, \quad \text{with} \quad z_n = \frac{\exp\left\{a_n(w_n - \delta_n)\right\}}{E\big(\exp\left\{a_n(w_n - \delta_n)\right\}\big)},$$

and $(a_n)_{n\leq N}$ is some deterministic sequence. To find $(a_n)_{n\leq N}$ we use the martingale property of $\big(S_n/B_n\big)_{n\leq N}$:

$$E^*\left(\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right) = \frac{S_{n-1}}{B_{n-1}}, \quad n = 1, \ldots, N,$$

which is equivalent to

$$E^*\big(\exp\{\widetilde{\mu}_n + \sigma_n\,\epsilon_n\}\big) = 1,$$

where $\widetilde{\mu}_n = \mu_n - \delta_n\,,\ n = 1,\ldots,N.$

Taking into account the expression for Z_n, we obtain

$$E\big(\exp\{a_n(\widetilde{\mu}_n + \sigma_n\,\epsilon_n) + \widetilde{\mu}_n + \sigma_n\,\epsilon_n\}\big) = E\big(\exp\{a_n(\widetilde{\mu}_n + \sigma_n\,\epsilon_n)\}\big)$$

and

$$E\big(\exp\{(a_n+1)(\widetilde{\mu}_n + \sigma_n\,\epsilon_n)\}\big) = E\big(\exp\{a_n(\widetilde{\mu}_n + \sigma_n\,\epsilon_n)\}\big)\,.$$

Since $\epsilon_n \sim \mathcal{N}(0,1)$, then

$$E\big(\exp\{a_n\,\sigma_n\,\epsilon_n\}\big) = \exp\{(a_n\,\sigma_n)^2/2\}\,,$$

which implies

$$\begin{aligned}
&\exp\{(a_n+1)\widetilde{\mu}_n\}\ \exp\{(a_n+1)^2\,\sigma_n^2/2\} = \exp\{a_n\widetilde{\mu}_n\}\ \exp\{a_n^2\,\sigma_n^2/2\}\,,\\
&\widetilde{\mu}_n + \frac{\sigma_n^2}{2} = -a_n\,\sigma_n^2.
\end{aligned}$$

Thus

$$a_n = -\frac{\widetilde{\mu}_n}{\sigma_n^2} - \frac{1}{2} = -\frac{\mu_n - \delta_n}{\sigma_n^2} - \frac{1}{2}\,.$$

Now using this a_n we compute

$$\begin{aligned}
&E\big(\exp\{a_n(w_n - \delta_n)\}\big)\\
&\quad = E\bigg(\exp\bigg\{-\Big(\frac{\widetilde{\mu}_n}{\sigma_n^2} + \frac{1}{2}\Big)(\widetilde{\mu}_n + \sigma_n\,\epsilon_n)\bigg\}\bigg)\\
&\quad = \exp\bigg\{-\frac{\widetilde{\mu}_n^2}{\sigma_n^2} - \frac{\widetilde{\mu}_n}{2}\bigg\}\,E\bigg(\exp\bigg\{-\sigma_n\Big(\frac{\widetilde{\mu}_n}{\sigma_n^2} + \frac{1}{2}\Big)\,\epsilon_n\bigg\}\bigg)\\
&\quad = \exp\bigg\{-\frac{\widetilde{\mu}_n^2}{\sigma_n^2} - \frac{\widetilde{\mu}_n}{2}\bigg\}\,\exp\bigg\{\frac{\sigma_n^2}{2}\Big(\frac{\widetilde{\mu}_n}{\sigma_n^2} + \frac{1}{2}\Big)^2\bigg\}\\
&\quad = \exp\bigg\{-\frac{\widetilde{\mu}_n^2}{\sigma_n^2} - \frac{\widetilde{\mu}_n}{2}\bigg\}\,\exp\bigg\{\frac{\sigma_n^2}{2}\Big(\frac{\widetilde{\mu}_n^2}{\sigma_n^4} + \frac{\widetilde{\mu}_n}{\sigma_n^2} + \frac{1}{4}\Big)\bigg\}\\
&\quad = \exp\bigg\{-\frac{\widetilde{\mu}_n^2}{2\,\sigma_n^2} + \frac{\sigma_n^2}{8}\bigg\},
\end{aligned}$$

which gives us

$$z_n = \exp\bigg\{-\Big(\frac{\widetilde{\mu}_n}{\sigma_n} + \frac{\sigma_n}{2}\Big)\,\epsilon_n - \frac{1}{2}\Big(\frac{\widetilde{\mu}_n}{\sigma_n} + \frac{\sigma_n}{2}\Big)^2\bigg\}$$

and

$$Z_N = \exp\bigg\{-\sum_{n=1}^{N}\bigg[\Big(\frac{\widetilde{\mu}_n}{\sigma_n} + \frac{\sigma_n}{2}\Big)\,\epsilon_n + \frac{1}{2}\Big(\frac{\widetilde{\mu}_n}{\sigma_n} + \frac{\sigma_n}{2}\Big)^2\bigg]\bigg\}.$$

Now for simplicity we consider a special case of our Gaussian (B, S)-market. Let

$$B_n = (1+r)^n = e^{\delta\, n} \quad \text{with} \quad \delta = \ln(1+r), \quad r \geq 0,$$

$$S_n = S_0\, e^{W_n}, \quad S_0 > 0.$$

Here $W_n = \sum_{k=1}^{n} w_k\,,\ w_k = \mu + \sigma\,\epsilon_k$ and $\epsilon_n \sim \mathcal{N}(0,1)$ are independent random variables on the stochastic basis

$$(\Omega_1, \mathcal{F}^1, \mathbb{F}_1, P_1)\,,$$

where $\mathbb{F}_1 = \big(\mathcal{F}^1_n\big)_{n \leq N}$ is a filtration with $\mathcal{F}^1_0 = \big\{\emptyset, \Omega\big\}\,,\ \mathcal{F}^1_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$ and $\mathcal{F}^1 = \mathcal{F}^1_N$.

WORKED EXAMPLE 2.3

As in Worked Example 1.5 we consider a pure endowment assurance issued by an insurance company. According to this contract the policy holder is paid

$$f_N = g(S_N)$$

on survival to the time N, where S_N is the stock price and g is some function specified by the contract. Suppose $E\big(g^2(S_N)\big) < \infty$. Find the 'fair' price for such insurance policy.

SOLUTION Recall that if l_x is the number of policy holders of age x, then each policy holder $i,\ i = 1, \ldots, l_x$ can be characterized by a positive random variable T_i representing the time elapsed between age x and death. Suppose that T_i's are defined on another probability space $(\Omega_2, \mathcal{F}^2, P_2)$ with the filtration $\mathcal{F}^2_n = \sigma\big(T_i \leq k,\ k \leq n,\ i = 1, \ldots, l_x\big)$.

Denote $p_x(n) = P_2\big(\{\omega:\ T_i > n\}\big),\ n = 0, 1, \ldots, N$, the conditional expectation for a policy holder to survive another n years from the age of x (clearly, $p_x(0) = 1$).

From Bayes's formula we have

$$p_{x+n}(y) = \frac{P_2\big(\{\omega:\ T_i > y + x\}\big)}{P_2\big(\{\omega:\ T_i > n\}\big)} = \frac{p_x(y+n)}{p_x(n)},$$

and hence

$$p_x(y+n) = p_x(n)\, p_{x+n}(y).$$

Denote

$$N_n := \sum_{i=1}^{l_x} I_{\{\omega:\ T_i \leq n\}}$$

the counter of deaths in the given group of policy holders. Then

$$E_2\big(l_x - N_n \big| \mathcal{F}^2_k\big) = \big(l_x - N_k\big)\, p_{x+k}(n-k)\,, \quad k \leq n \leq N.$$

Therefore the discounted value of the total payoff is given by

$$H = \sum_{k=1}^{l_x} Y_k = g(S_N)\,\frac{l_x - N_N}{B_N}\,,$$

where

$$Y_k = g(S_N)\,\frac{I_{\{\omega:\ T_k > N\}}}{B_N}\,, \quad k \le l_x.$$

Since we have to price a contingent claim with an insurance component, we introduce the following stochastic basis

$$(\Omega, \mathcal{F}, \mathbb{F}, P) = (\Omega_1 \times \Omega_2,\, \mathcal{F}^1 \times \mathcal{F}^2,\, \mathbb{F}_1 \times \mathbb{F}_2,\, P_1 \times P_2).$$

Clearly we have that stochastic sequences (ϵ_n) and (T_i) are independent on this basis.

Now, since probability P_1^* with density

$$Z_N = \exp\left\{ -\left(\frac{\mu-\delta}{\sigma} + \frac{\sigma}{2}\right)\sum_{k=1}^{N}\epsilon_k - \frac{1}{2}\left(\frac{\mu-\delta}{\sigma} + \frac{\sigma}{2}\right)^2 N \right\}$$

is a martingale probability on the (B,S)-market, then the probability $P^* := P_1^* \times P_2$ on $(\Omega, \mathcal{F}, P)$ is such that $\big(S_n/B_n\big)_{n\le N}$ is a martingale under this probability.

Next, using the methodology of hedging in mean square, we obtain

$$\begin{aligned} V_n^* &= E^*\big(H|\mathcal{F}_n\big) = E^*\big(g(S_N)B_N^{-1}|\mathcal{F}_n^1\big)\,E^*\big(l_x - N_N|\mathcal{F}_n^2\big) \\ &= E^*\big(g(S_N)|\mathcal{F}_n^1\big)\,B_N^{-1}\,\big[l_x - N_n\big]\,p_{x+n}(N-n)\,, \end{aligned}$$

and

$$\hat{\gamma}_n = \gamma_n^H = \frac{B_N^{-1}\,E^*\big(g(S_N)\,\Delta X_n|\mathcal{F}_{n-1}^1\big)\,\big[l_x - N_{N-1}\big]\,p_{x+n-1}(N-n+1)}{X_{n-1}^2\,\big[\exp\{\sigma^2\} - 1\big]}\,.$$

Here we also used independence of sequences $\big(S_n\big)_{n\le N}$ and $\big(T_k\big)_{k\le l_x}$, and the equality

$$E^*\big((\Delta X_n)^2|\mathcal{F}_{n-1}^1\big) = X_{n-1}^2\,\big[\exp\{\sigma^2\} - 1\big]\,.$$

The optimal strategy $\hat{\pi} = (\hat{\gamma}, \hat{\beta})$ and its values have the form

$$V_n^{\hat{\pi}} = V_n^*\,, \quad \hat{\beta}_n = V_n^* - \hat{\gamma}_n\,X_n, \quad n = 1, \ldots, N.$$

The quantity

$$V_0^{\hat{\pi}} = p_x(N)\,l_x\,E^*\big(g(S_N)\big)\,e^{-\delta N}$$

determines the total premium received by an insurance company.

Now we consider three particular cases of function g and compute premiums there.

Case 1. Let $g(S_N) = S_N$, then

$$\begin{aligned}
V_n^* &= \big[l_x - N_n\big]\, X_n\, p_{x+n}(N-n)\,,\\
\hat{\gamma}_n &= \big[l_x - N_{n-1}\big]\, p_{x+n-1}(N-n+1)\,,\\
\hat{\beta}_n &= X_n\Big\{\big[l_x - N_n\big]\, p_{x+n}(N-n) - \big[l_x - N_{n-1}\big]\, p_{x+n-1}(N-n+1)\Big\}\\
&= X_n\, p_{x+n}(N-n)\,\Big\{\big[l_x - N_n\big] - \big[l_x - N_{n-1}\big]\, p_{x+n-1}(1)\Big\}\\
&\quad n = 1, \ldots, N\,.
\end{aligned}$$

Premium is therefore determined by

$$V_0^{\hat{\pi}} = p_x(N)\, l_x\, S_0\,.$$

We can compute risk of such strategy:

$$R_n^{\hat{\pi}} = \Big\{\sum_{k=n+1}^{N} e^{k\,\sigma^2}\, q_{x+k-1}(1)\, p_{x+k}(N-k)\Big\} \times p_{x+n}(N-n)\,\big[l_x - N_n\big]\, e^{-n\,\sigma^2}\, X_n^2\,,$$

$$R_0^{\hat{\pi}} = \Big\{\sum_{k=1}^{N} e^{k\,\sigma^2}\, q_{x+k-1}(1)\, p_{x+k}(N-k)\Big\}\, p_x(N)\, l_x\, S_0^2\,,$$

where $q_y(1)$ is the probability of death during the year following year y. The latter formula also implies

$$\frac{\big(R_0^{\hat{\pi}}\big)^{1/2}}{l_x} \to 0 \quad \text{as} \quad l_x \to \infty\,,$$

which means that if there are enough policy holders, then the company's risk associated with this contract is infinitesimal.

Case 2. If $g(S_N) \equiv K = const$, then

$$\begin{aligned}
V_n^{\hat{\pi}} &= V_n^* = K\, e^{-\delta N}\big[l_x - N_n\big]\, p_{x+n}(N-n)\,,\\
\hat{\gamma}_n &= 0\,, \quad \hat{\beta}_n = V_n^{\hat{\pi}} \quad \text{for} \quad n = 1, \ldots, N\,;\\
V_0^{\hat{\pi}} &= K\, e^{-\delta N}\, l_x\, p_x(N)\,,
\end{aligned}$$

which indicates that in this case one has to invest money in a bank account. The risk-sequence here:

$$R_n^{\hat{\pi}} = K^2\, e^{-2\delta N}\big[l_x - N_n\big]\, p_{x+n}(N-n)\, q_{x+n}(N-n), \quad n \le N.$$

In particular

$$R_0^{\hat{\pi}} = K^2\, e^{-2\delta N}\, l_x\, p_x(N)\, q_x(N),$$

and again $\big(R_0^{\hat{\pi}}\big)^{1/2}/l_x \to 0$ as $l_x \to \infty$.

Case 3. Let $g(S_N) = \max\{S_N, K\}$. We can write

$$\max\{S_N, K\} = K + (S_N - K)^+,$$

and therefore we have to compute

$$E^*\big((S_N - K)^+ \, \Delta X_n \big| \mathcal{F}_{n-1}\big) \quad \text{and} \quad E^*\big((S_N - K)^+ \big| \mathcal{F}_n\big). \tag{2.5}$$

For the latter we have

$$\begin{aligned} &E^*\big((S_N - K)^+ \big| \mathcal{F}_n\big) \\ &= E^*\big((S_n \, e^{\delta(N-n)} \, e^{w^*_{n+1}+\cdots+w^*_N} - K)^+ \big| \mathcal{F}_n\big) \\ &= E^*\Big(\big(S_n \, e^{\delta(N-n)} \, e^{\mathcal{N}\left(-\frac{\sigma^2}{2}(N-n),\, \sigma^2\,(N-n)\right)} - K\big)^+ \Big| \mathcal{F}_n\Big), \end{aligned}$$

where $w^*_k = \mu - \delta + \sigma\,\epsilon_k \sim \mathcal{N}\big(-\frac{\sigma^2}{2}, \sigma^2\big)$ with respect to probability P^*. Note that for $\xi \sim \mathcal{N}(0,1)$ and constants a, b and K one has

$$E\Big(\big(a\, e^{b\xi - b^2/2} - K\big)^+\Big) = a\,\Phi\bigg(\frac{\ln(a/K) + b^2/2}{b}\bigg) - K\,\Phi\bigg(\frac{\ln(a/K) - b^2/2}{b}\bigg),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

is a standard normal distribution. Hence, we obtain

$$\begin{aligned} &E^*\big((S_N - K)^+ \big| \mathcal{F}_n\big) \\ &\qquad = S_n \, e^{\delta(N-n)} \, \Phi\bigg(\frac{\ln(S_n/K) + (N-n)\,(\delta + \sigma^2/2)}{\sigma\sqrt{N-n}}\bigg) \\ &\qquad\qquad - K\,\Phi\bigg(\frac{\ln(S_n/K) + (N-n)\,(\delta - \sigma^2/2)}{\sigma\sqrt{N-n}}\bigg). \end{aligned}$$

Note that for $n = 0$ we have

$$\begin{aligned} E^*\bigg(\frac{(S_N - K)^+}{e^{\delta\,N}}\bigg) &= S_0\,\Phi\bigg(\frac{\ln(S_0/K) + N\,(\delta + \sigma^2/2)}{\sigma\sqrt{N}}\bigg) \\ &\qquad - K\,e^{-\delta N}\,\Phi\bigg(\frac{\ln(S_0/K) + N\,(\delta - \sigma^2/2)}{\sigma\sqrt{N}}\bigg), \end{aligned}$$

which is the discrete version of the Black-Scholes formula for a European call option.

Now we compute the first expectation from (2.5):

$$E^*\big((S_N-K)^+ X_n \big| \mathcal{F}_{n-1}\big) = X_{n-1}\, E^*\big(e^{w_n^*}\,(S_N-K)^+\big|\mathcal{F}_{n-1}\big)$$

$$= X_{n-1}\, E^*\left(E^*\Big(e^{w_n^*}\,(S_{n-1}\, e^{\delta(N-n+1)}\, e^{w_n^*+\cdots+w_N^*} - K)^+\big|\mathcal{F}_n\Big)\Big|\mathcal{F}_{n-1}\right)$$

$$= X_{n-1}\, E^*\left(e^{w_n^*}\left[S_{n-1}\, e^{\delta(N-n+1)}\, e^{w_n^*} \right.\right.$$

$$\times\Phi\left(\frac{\ln(S_{n-1}/K) + w_n^* + (N-n)\,(\delta+\sigma^2/2)+\sigma^2}{\sigma\sqrt{N-n}}\right)$$

$$\left.\left. -K\,\Phi\left(\frac{\ln(S_{n-1}/K) + w_n^* + (N-n)\,(\delta-\sigma^2/2)+\sigma^2}{\sigma\sqrt{N-n}}\right)\right]\Bigg|\mathcal{F}_{n-1}\right).$$

Since for $\xi \sim \mathcal{N}\big(-\frac{\sigma^2}{2},\, \sigma^2\big)$ we have

$$E\big(e^{\xi}\,\Phi(x\,\xi+y)\big) = \Phi\left(\frac{y+\sigma^2\,x/2}{\sqrt{1+x^2\,\sigma^2}}\right),$$

then

$$E^*\big((S_N-K)^+ X_n\big|\mathcal{F}_{n-1}\big)$$

$$= X_{n-1}\left[X_{n-1}\, e^{\delta\, N}\, \exp\{\sigma^2\}\,\Phi\left(\frac{\ln(S_{n-1}/K)+(N-n+1)\,(\delta+\sigma^2/2)+\sigma^2}{\sigma\sqrt{N-n+1}}\right)\right.$$

$$\left. -K\,\Phi\left(\frac{\ln(S_{n-1}/K)+(N-n+1)\,(\delta-\sigma^2/2)+\sigma^2}{\sigma\sqrt{N-n+1}}\right)\right].$$

Thus, we obtain the following formulae for the optimal strategy and its capital:

$$V_n^{\hat{\pi}} = V_n^* = e^{-\delta N}\big[l_x - N_n\big]\, p_{x+n}(N-n)$$

$$\times\left[K + S_n\, e^{\delta(N-n)}\,\Phi\left(\frac{\ln(S_n/K)+(N-n)\,(\delta+\sigma^2/2)}{\sigma\sqrt{N-n}}\right)\right.$$

$$\left. -K\,\Phi\left(\frac{\ln(S_n/K)+(N-n)\,(\delta-\sigma^2/2)}{\sigma\sqrt{N-n}}\right)\right],$$

$$V_0^* = e^{-\delta N}\, l_x\, p_x(N) \Bigg[K + S_0\, e^{\delta\, N}\, \Phi\bigg(\frac{\ln(S_0/K) + N\,(\delta + \sigma^2/2)}{\sigma\,\sqrt{N}} \bigg) - K\, \Phi\bigg(\frac{\ln(S_0/K) + N\,(\delta - \sigma^2/2)}{\sigma\,\sqrt{N}} \bigg) \Bigg],$$

$$\hat{\gamma}_n = \frac{\big[l_x - N_{N-1}\big]\, p_{x+n-1}(N-n+1)}{X_{n-1}\,\big[\exp\{\sigma^2\} - 1\big]} \times \Bigg\{ X_{n-1}\, e^{\delta\, N} \Bigg[\exp\{\sigma^2\}\, \Phi\bigg(\frac{\ln(S_{n-1}/K) + (N-n+1)\,(\delta + \sigma^2/2) + \sigma^2}{\sigma\,\sqrt{N-n+1}} \bigg) - \Phi\bigg(\frac{\ln(S_{n-1}/K) + (N-n+1)\,(\delta + \sigma^2/2)}{\sigma\,\sqrt{N-n+1}} \bigg) \Bigg]$$

$$+ K \Bigg[\Phi\bigg(\frac{\ln(S_{n-1}/K) + (N-n+1)\,(\delta - \sigma^2/2)}{\sigma\,\sqrt{N-n+1}} \bigg) - \Phi\bigg(\frac{\ln(S_{n-1}/K) + (N-n+1)\,(\delta - \sigma^2/2) + \sigma^2}{\sigma\,\sqrt{N-n+1}} \bigg) \Bigg] \Bigg\},$$

$$\hat{\beta}_n = V_n^* - \hat{\gamma}_n\, X_n, \qquad n = 1, \dots, N.$$

□

2.5 The transition from the binomial model of a financial market to a continuous model. The Black-Scholes formula and equation.

In previous sections we dealt with discrete markets, where time horizon is described by integers $0, 1, \dots, N$, representing some units of time (e.g., years, months etc). Now suppose that we wish to consider a market with time horizon $[0, T]$ for some real number $T \geq 0$. We can divide this interval into m equal parts, so that we will have a time scale with the step $\tau = T/m > 0$. Thus, it is natural to consider the following (B, S, τ)-market:

$$B_t^\tau - B_{t-\tau}^\tau = r(\tau)\, B_{t-\tau}^\tau, \quad B_0^\tau > 0,\ r(\tau) > 0,$$

$$S_t^\tau - S_{t-\tau}^\tau = \rho_t(\tau)\, S_{t-\tau}^\tau, \quad S_0^\tau > 0,$$

where $\big(\rho_t(\tau)\big)$ is a stochastic sequence of profitabilities that generates the following filtration

$$\mathcal{F}_t^\tau = \sigma\big(\rho_n(\tau),\ n \le t\big), \quad t = 0, \tau, 2\tau, \ldots, (m-1)\tau, (T/\tau)\tau.$$

This discrete market can be extended to the whole of $[0, T]$: for $s \in [t, t+\tau)$, where $t = 0, \tau, \ldots, m\,\tau$, define

$$B_s^\tau \equiv B_t^\tau, \quad S_s^\tau \equiv S_t^\tau, \quad \mathcal{F}_s^\tau \equiv \mathcal{F}_t^\tau, \quad \rho_s^\tau \equiv \rho_t^\tau,$$

so that all stochastic sequences become *stochastic processes* and we obtain a (formally) continuous model of a market.

Consider a European call option on a (B, S, τ)-market. In this case $f_T = \big(S_{(T/\tau)\tau} - K\big)^+$, and let C_T^τ be its price. If we consider a one-parameter family of (B, S, τ)-markets with respect to $\tau > 0$, then we expect processes that form and characterize these markets to have 'reasonable' limits as $\tau \to \infty$.

Suppose that sequence $\big(\rho_t(\tau)\big)_{t=\tau,2\tau,\ldots}$ consists of independent random variables that assume values $a(\tau)$ and $b(\tau)$ with probabilities p_τ and $1-p_\tau$, respectively (here $a(\tau) < b(\tau)$). Let $r(\tau) = r\,\tau$ and

$$\rho_t(\tau) = \mu\,\tau + \sigma\,\Delta w_t^\tau,$$

where $E\big(\rho_t(\tau)\big) = \mu\,\tau,\ V\big(\Delta w_t^\tau\big) = \tau$ and $V\big(\rho_t(\tau)\big) = \sigma^2\,\tau$. Then we can rewrite a (B, S, τ)-market in the form

$$\begin{aligned} B_t^\tau &= r\,B_{t-}^\tau\ \tau, \\ S_t^\tau &= (\mu\,\tau + \sigma\,\Delta w_t^\tau)\,S_{t-}^\tau, \end{aligned}$$

where $B_{t-}^\tau := B_{t-\tau}^\tau,\ S_{t-}^\tau := S_{t-\tau}^\tau$, and process w_t^τ has independent increments.

If $\tau \to \infty$, we arrive to a natural *limit* model:

$$\begin{aligned} dB_t &= r\,B_t\,dt, \\ dS_t &= (\mu\,dt + \sigma\,dw_t)\,S_t, \end{aligned}$$

with differentials dt, dw_t, dB_t, dS_t being formal limits of $\tau, \Delta w_t^\tau, B_t^\tau$ and S_t^τ.

It is well known in the probability theory that Bernoulli random variables are related to two 'limit' distributions: Gaussian and Poisson. Thus, it is natural to assume that the limit process w_t has properties

$$w_0 = 0, \quad E(w_t) = 0, \quad V(w_t) = t,$$

and its independent increments are either Gaussian or Poisson. In the Gaussian case, w_t is a *Wiener process* (or *Brownian motion*), and the corresponding model of a (B, S)-market is the *Black-Scholes model.* In the Poisson case, w_t is a (centered) *Poisson process*, and this corresponds to the *Merton model.*

We now concentrate on the former case. Parameters $r,\ \mu$ and σ are usually referred to as *interest rate, drift* and *volatility*, respectively. Consider a European call option on this continuous market, with the claim $f_T = \big(S_T - K\big)^+$. We will find its price using the passage to the limit:

$$C_T = \lim_{\tau \to 0} C_T^\tau.$$

Suppose that parameters of the (B, S, τ)-market and of the limit market satisfy the relations

$$1 + r(\tau) = e^{r\,\tau}, \quad 1 + b(\tau) = e^{\sigma\sqrt{\tau}}, \quad 1 + a(\tau) = e^{-\sigma\sqrt{\tau}}, \quad \sigma > 0.$$

Using the Cox-Ross-Rubinstein formula we obtain

$$C_T^\tau = S_0\, B\big(k_0(\tau),\, m,\, \widetilde{p}_\tau\big) - K\,\big(1 + r(\tau)\big)^{-m} B\big(k_0(\tau),\, m,\, p_\tau^*\big)\,,$$

where

$$m = \frac{T}{\tau}, \qquad k_0(\tau) = 1 + \frac{\ln\Big(K/S_0\,\big(1 + a(\tau)\big)^m\Big)}{\ln\Big([1 + a(\tau)]/[1 + b(\tau)]\Big)},$$

and

$$p_\tau^* = \frac{r(\tau) - a(\tau)}{b(\tau) - a(\tau)}, \qquad \widetilde{p}_\tau = \frac{1 + b(\tau)}{1 + a(\tau)}\, p_\tau^*.$$

By the De Moivre-Laplace limit theorem we have

$$B\big(k_0(\tau),\, m,\, p_\tau^*\big) \sim \Phi\left(\frac{m\,p_\tau^* - k_0(\tau)}{\sqrt{m\,p_\tau^*\,(1 - p_\tau^*)}}\right) = \Phi\big(y_\tau^*\big)\,,$$

$$B\big(k_0(\tau),\, m,\, \widetilde{p}_\tau\big) \sim \Phi\left(\frac{m\,\widetilde{p}_\tau - k_0(\tau)}{\sqrt{m\,\widetilde{p}_\tau\,(1 - \widetilde{p}_\tau)}}\right) = \Phi\big(\widetilde{y}_\tau\big)\,.$$

Also, for $\tau \to 0$

$$k_0(\tau) \sim \frac{\ln\big(K/S_0\big) + m\,\sigma\sqrt{\tau}}{2\,\sigma\sqrt{\tau}}, \qquad \big(1 + r(\tau)\big)^{-m} \sim e^{r\,T}\,.$$

Finally, taking into account relations

$$m\,p_\tau^* \sim \frac{T\,\tau\,\big(r - \sigma^2/2\big) + T\,\sigma\sqrt{\tau}}{2\,\sigma\,\tau^{3/2}}, \qquad m\,\widetilde{p}_\tau - k_0(\tau) \sim \frac{T\,\tau\,\big(r - \sigma^2/2\big) + \tau\,\ln(S_0/K)}{2\,\sigma\,\tau^{3/2}},$$

and

$$\sqrt{m\,p_\tau^*\,(1 - p_\tau^*)} \sim \sqrt{T/4\tau},$$

we obtain

$$\lim_{\tau \to 0} \frac{m\,p_\tau^* - k_0(\tau)}{\sqrt{m\,p_\tau^*\,(1 - p_\tau^*)}} = \frac{\ln\big(S_0/K\big) + T\,\big(r - \sigma^2/2\big)}{\sigma\sqrt{T}} = y^*,$$

$$\lim_{\tau \to 0} \frac{m\,\widetilde{p}_\tau - k_0(\tau)}{\sqrt{m\,\widetilde{p}_\tau\,(1 - \widetilde{p}_\tau)}} = \frac{\ln\big(S_0/K\big) + T\,\big(r + \sigma^2/2\big)}{\sigma\sqrt{T}} = \widetilde{y}.$$

Thus, we arrive to the celebrated Black-Scholes formula:

$$\lim_{\tau\to 0} C_T^\tau = C_T = S_0\,\Phi(\widetilde{y}) - K\,e^{-r\,T}\,\Phi(y^*). \tag{2.6}$$

In general, one can replace interval $[0, T]$ with $[t, T]$, where $0 \le t \le T$. In this case we consider a contract written at time t with time to expiry $T - t$. Replacing T by $T - t$ and S_0 by $S_t =: x$ in formula (2.6), we obtain the corresponding version of the Black-Scholes formula:

$$C(x,t) = S_t\,\Phi\left(\frac{\ln\big(S_t/K\big) + (T-t)\big(r + \sigma^2/2\big)}{\sigma\sqrt{T-t}}\right)$$
$$-K\,e^{-r\,T}\,\Phi\left(\frac{\ln\big(S_t/K\big) + (T-t)\big(r - \sigma^2/2\big)}{\sigma\sqrt{T-t}}\right),$$

which also indicates that price C is a function of time and price of the asset $S_t = x$.

PROPOSITION 2.3
Suppose that function $C(\cdot,\cdot)$ is continuously differentiable in t and twice continuously differentiable in x. Then it satisfies the Black-Scholes differential equation

$$\frac{\partial C}{\partial t} + r\,x\,\frac{\partial C}{\partial x} + \frac{1}{2}\,\sigma^2\,x^2\,\frac{\partial^2 C}{\partial x^2} - r\,C = 0. \tag{2.7}$$

PROOF Consider a (B, S, τ)-market with parameters

$$1 + r(\tau) = e^{r\,\tau}, \quad 1 + b(\tau) = e^{\sigma\,\sqrt{\tau}}, \quad 1 + a(\tau) = e^{-\sigma\,\sqrt{\tau}}, \quad \sigma > 0.$$

For the martingale probability p_τ^* we have

$$p_\tau^* = \frac{\big(e^{r\tau} - 1\big) - \big(1 - e^{-\sigma\sqrt{\tau}}\big)}{e^{\sigma\sqrt{\tau}} - e^{-\sigma\sqrt{\tau}}} \sim \frac{1}{2}\left(1 + \frac{r}{\sigma}\sqrt{\tau}\right) \qquad \text{as} \quad \tau \to 0.$$

Since prices of asset S can take only two possible values on this (B, S, τ)-market, then

$$e^{r\tau}\,C(x,t) = p_\tau^*\,C(x\,e^{\sigma\sqrt{\tau}}, t+\tau) + (1 - p_\tau^*)\,C(x\,e^{-\sigma\sqrt{\tau}}, t+\tau).$$

Using Taylor's formula we can write

$$\begin{aligned}
&e^{r\tau}\,C(x,t) = (1 + r\,\tau)\,C(x,t) + o(\tau),\\
&C(x\,e^{\sigma\sqrt{\tau}}, t+\tau)\\
&\qquad = C(x,t) + \frac{\partial C}{\partial t} + x\,\sigma\sqrt{\tau}\,\frac{\partial C}{\partial x} + \frac{1}{2}\,\sigma^2\,x^2\,\tau\,\frac{\partial^2 C}{\partial x^2} + o(\tau),\\
&C(x\,e^{-\sigma\sqrt{\tau}}, t+\tau)\\
&\qquad = C(x,t) + \frac{\partial C}{\partial t} - x\,\sigma\sqrt{\tau}\,\frac{\partial C}{\partial x} + \frac{1}{2}\,\sigma^2\,x^2\,\tau\,\frac{\partial^2 C}{\partial x^2} + o(\tau)
\end{aligned}$$

for $\tau \to 0$, and hence

$$(1 + r\,\tau)\,C(x,t) = C(x,t) + \tau\,\frac{\partial C}{\partial t} + x\,r\,\tau\,\frac{\partial C}{\partial x} + \frac{1}{2}\,\sigma^2\,x^2\,\tau\,\frac{\partial^2 C}{\partial x^2} + o(\tau),$$

which implies the claim. $\square$

2.6 The Black-Scholes model. 'Greek' parameters in risk management, hedging under dividends and budget constraints. Optimal investment.

This section is devoted to the rigorous study of the Black-Scholes model of a (B,S)-market with time horizon $T < \infty$.

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis. Here filtration $\mathbb{F} = (\mathcal{F}_t)_{t \le T}$ represents a continuous information flow that is parameterized by a time parameter $t \in [0,T]$. It is natural to assume that $\mathcal{F}_t$ (being the information up to time t) is a σ-algebra, i.e.,

1. $\emptyset,\ \Omega \in \mathcal{F}_t$;
2. $A \in \mathcal{F}_t \Rightarrow \Omega \setminus A \in \mathcal{F}_t$ (closed under taking complements);
3. $(A_k)_{k=1}^{\infty} \subset \mathcal{F}_t \Rightarrow \cup_{k=1}^{\infty} A_k \in \mathcal{F}_t$ (closed under taking countable unions);
4. $(A_k)_{k=1}^{\infty} \subset \mathcal{F}_t \Rightarrow \cap_{k=1}^{\infty} A_k \in \mathcal{F}_t$ (closed under taking countable intersections).

The initial information is usually considered to be trivial: $\mathcal{F}_0 = \{\emptyset,\ \Omega\}$.

On this stochastic basis we consider a Wiener process (Brownian motion) w_t, i.e., a process with the following properties

(W1) $w_0 = 0$;

(W2) $w_t - w_s$ and $w_v - w_u$ are independent for $s < t < v < u$;

(W3) $w_t - w_s \sim \mathcal{N}(0, t-s)$.

It is assumed that all 'randomness' of the model is generated by this process, and therefore

$$\mathcal{F}_t = \sigma(w_s,\ s \le t) =: \mathcal{F}_t^w.$$

Note that any stochastic process w is a function of two variables: elementary event $\omega \in \Omega$ and time $t \le T$. For a fixed ω, the function $w(\omega, \cdot)$ is called a *trajectory*. Without loss of generality one can assume that trajectories of a Wiener process are continuous in t.

Let us divide interval $[0,T]$ into n parts: $0 = t_0 < t_1 < \ldots < t_n = T$, and define

$$\varphi(t,\omega) = \sum_{k=1}^{n} \varphi_{k-1}(\omega)\, I_{(t_{k-1},t_k]}(t), \tag{2.8}$$

where φ_{k-1} are square-integrable random variables that are completely determined by σ-algebras $\mathcal{F}_{t_{k-1}}$ (in other words, φ_{k-1} are $\mathcal{F}_{t_{k-1}}$-measurable square-integrable random variables).

Now we define a stochastic integral

$$(\varphi * w)_t \equiv \int_0^t \varphi(s,\omega)\, dw_s := \sum_{k=1}^{n} \varphi_{k-1}(\omega)\left(w_{t_k\wedge t} - w_{t_{k-1}\wedge t}\right).$$

It has the following properties

(I1) $\big((\alpha\,\varphi + \beta\,\psi) * w\big)_t = \alpha\,(\varphi * w)_t + \beta\,(\psi * w)_t$;

(I2) $E\Big((\varphi * w)_T \Big| \mathcal{F}_t\Big) = (\varphi * w)_t$;

(I3) $E\big((\varphi * w)_t \cdot (\psi * w)_t\big) = E\Big(\int_0^t \varphi(s)\psi(s)\, ds\Big)$,

for any functions φ and ψ of type (2.8), and any constants α and β.

Next, consider a stochastic function (process) $\big(\varphi_t\big)_{t\le T}$ that is *adapted* to filtration $\mathbb{F} = (\mathcal{F}_t)_{t\le T}$, i.e. $\varphi_t \equiv \varphi(t,\omega)$ is $\mathcal{F}_t$-measurable for each $t \le T$. If

$$E\Big(\int_0^t \varphi^2(s,\omega)\, ds < \infty\Big),$$

then the stochastic integral is well defined for such function φ as a mean square limit of integrals of functions of type (2.8), and properties (I1) – (I3) hold true.

Thus, one can consider stochastic processes of the following type

$$X_t = X_0 + \int_0^t b(s,\omega)\, ds + \int_0^t a(s,\omega)\, dw_s\,, \tag{2.9}$$

where $\int_0^t b(s,\omega)\, ds$ is a usual Lebesgue-type integral and $\int_0^t a(s,\omega)\, dw_s$ is a stochastic integral. Note that equation (2.9) is often formally written in the differential form

$$dX_t = b_t\, dt + a_t\, dw_t\,.$$

Let $F(t,x)$ be a real-valued function that is continuously differentiable in t and twice continuously differentiable in x. Then the process $Y_t := F(t,X_t)$ is also of type (2.9), which follows from the celebrated *Kolmogorov-Itô formula*:

$$F(t,X_t) = F(0,X_0) + \int_0^t \left[\frac{\partial F}{\partial s}(s,X_s) + b_s\,\frac{\partial F}{\partial x}(s,X_s) + \frac{1}{2}\,a_s^2\,\frac{\partial^2 F}{\partial x^2}(s,X_s)\right] ds$$
$$+ \int_0^t a_s\,\frac{\partial F}{\partial x}(s,X_s)\, dw_s\,. \tag{2.10}$$

To sketch the proof of this formula, we note that, by Taylor's formula, increments of a smooth function F can be written in the form

$$\Delta F(t,x) = \frac{\partial F}{\partial x}\,\Delta x + \frac{\partial F}{\partial t}\,\Delta t + \frac{1}{2}\,\frac{\partial^2 F}{\partial x^2}\,(\Delta x)^2 + \frac{\partial^2 F}{\partial x \partial t}\,\Delta x\,\Delta t + \frac{1}{2}\,\frac{\partial^2 F}{\partial t^2}\,(\Delta t)^2 + \dots .$$

Since $\Delta w_t \sim \mathcal{N}(0, \Delta t)$, then increments ΔX_t of process X are equivalent (in distribution) to random variable $b\,\Delta t + a\,\varepsilon\,\sqrt{\Delta t},\ \varepsilon \sim \mathcal{N}(0,1)$. Further

$$\big(\Delta X_t\big)^2 = b^2\,(\Delta t)^2 + a^2\,\varepsilon^2\,\Delta t + 2\,b\,a\,\varepsilon\,(\Delta t)^{3/2}$$

and

$$E\Big(\big(\Delta X_t\big)^2\Big) = E\big(a^2\,\varepsilon^2\,\Delta t\big) = a^2\,\Delta t\,.$$

up to terms of higher order in Δt.

Thus, we obtain the following approximation

$$\begin{aligned} \Delta F(t, X_t) &= \frac{\partial F}{\partial x}\,\Big[b\,\Delta t + a\,\varepsilon\,\sqrt{\Delta t}\Big] + \frac{\partial F}{\partial t}\,\Delta t + \frac{1}{2}\,\frac{\partial^2 F}{\partial x^2}\,a^2\,\Delta t \\ &= \left(\frac{\partial F}{\partial x}\,b + \frac{\partial F}{\partial t} + \frac{1}{2}\,\frac{\partial^2 F}{\partial x^2}\,a^2\right)\Delta t + \frac{\partial F}{\partial x}\,a\,\varepsilon\,\sqrt{\Delta t}\,, \end{aligned}$$

which implies (2.10).

We say that a process $M = (M_t, \mathcal{F}_t^w)_{t \le T}$ is a *martingale* on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ if $E\big(|M_t|\big) < \infty$, and for all $s \le t$

$$E\big(M_t | \mathcal{F}_s\big) = M_s \qquad \text{a.s.}$$

If filtration $\mathbb{F}$ is generated by a Wiener process w, then any martingale M can be written in the form

$$M_t = M_0 + \int_0^t \varphi_s\,dw_s\,, \tag{2.11}$$

for some stochastic function φ that is adapted to filtration $\mathbb{F}$ and 'stochastically' integrable with respect to w.

Note that the very construction of a stochastic integral with respect to a Wiener process w originates from the martingale property of constructed process. The martingale representation (2.11) is a subtle result saying that stochastic integrals are the only martingales that exist in this case.

Now we proceed to the Black-Scholes model. Suppose that on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ processes B_t and S_t are given by

$$\begin{aligned} B_t &= e^{rt}\,, \\ S_t &= S_0\,e^{(\mu - \sigma^2/2)t + \sigma\,w_t}\,, \quad S_0 > 0\,, \end{aligned}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$.

Applying the Kolmogorov-Itô formula to the process

$$S_t = S_0\, e^{X_t} \qquad \text{with} \qquad X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\, w_t\,,\ X_0 = 0\,,$$

we obtain

$$\begin{aligned} S_t &= S_0 + \int_0^t \left[S_0\, e^{X_u} \left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2}\, e^{X_u}\, \sigma^2 \right] du + \int_0^t \sigma\, S_0\, e^{X_u}\, du \\ &= S_0 + \int_0^t S_u\, \mu\, du + \int_0^t S_u\, \sigma\, dw_u = S_0 + \int_0^t S_u \left(\mu\, du + \sigma\, dw_u\right). \end{aligned}$$

Thus, the Black-Scholes model can be represented in the following differential form

$$\begin{aligned} dB_t &= r\, B_t\, dt\,, \\ dS_t &= S_t \left(\mu\, dt + \sigma\, dw_t\right), \quad S_0 > 0\,, \end{aligned}$$

Parameters r, μ and σ are referred to as *rate of interest, profitability* and *volatility* of the (B, S)-market. In practice, parameters μ and σ are unknown and ought to be estimated, say, from the statistics of prices S_t. If time intervals between observations are τ, then we have

$$S_t = S_{t-\tau}\, e^{R_t}\,, \qquad \text{where} \quad R_t \sim \mathcal{N}\big((\mu - \sigma^2/2)\,\tau,\ \sigma^2\,\tau\big)\,,$$

and therefore μ and σ can be estimated using the fact that process R is normally distributed with parameters $(\mu - \sigma^2/2)\,\tau$ and $\sigma^2\,\tau$.

REMARK 2.1 Consider the following linear stochastic equation

$$dX_t = X_t \left(\mu_t\, dt + \sigma_t\, dw_t\right) \equiv X_t\, dY_t\,,$$

where X_0 is a finite (a.s) random variable,

$$Y_t = \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dw_s,$$

and (in general, random) functions μ_t and σ_t satisfy some integrability conditions (e.g. μ_t and σ_t are bounded). Now we introduce the *stochastic exponential*

$$\mathcal{E}_t(Y) = \exp\left\{ Y_t - \frac{1}{2} \int_0^t \sigma_s^2\, ds \right\},$$

It is not difficult to check that

$$X_t = X_0\, \mathcal{E}_t(Y), \quad t \geq 0,$$

and that the following properties hold true (compare with the discrete case in Section 1.2):

($\mathcal{E}$1) $\mathcal{E}_t(Y) > 0$ (a.s.);

($\mathcal{E}$2) $1/\mathcal{E}_t(Y) = \mathcal{E}_t(\widetilde{Y})$, where

$$d\widetilde{Y}_t = -dY_t + \sigma_t^2\, dt\,;$$

($\mathcal{E}$3) If $\mu_t \equiv 0$ (which implies that Y_t is a martingale), then $\mathcal{E}_t(Y)$ is a martingale;

($\mathcal{E}$4) The multiplication rule:

$$\mathcal{E}_t(Y^1)\, \mathcal{E}_t(Y^2) = \mathcal{E}_t\big(Y^1 + Y^2 + [Y^1, Y^2]\big)\,,$$

where

$$dY_t^i = \mu_t^i\, dt + \sigma_t^i\, dw_t\,, \quad i = 1, 2, \quad \text{and} \quad d[Y^1, Y^2]_t = \sigma_t^1\, \sigma_t^2\, dt\,.$$

As in binomial case, we can write the Black-Scholes model in terms of stochastic exponentials:

$$B_t = B_0\, \mathcal{E}_t(r\, t)\,,$$
$$S_t = S_0\, \mathcal{E}_t(\mu\, t + \sigma\, w_t)\,.$$

This representation is useful for studying martingale properties of S_t and S_t/B_t. □

Now we introduce the standard basic notions related to a (B, S)-market. If processes $\beta = (\beta_t)_{t \le T}$ and $\gamma = (\gamma_t)_{t \le T}$ are adapted to filtration $\mathbb{F}$, then $\pi = (\pi_t)_{t \le T} := (\beta_t, \gamma_t)_{t \le T}$ is called a *portfolio* or *strategy* on a (B, S)-market. The *capital* (or *value*) of strategy π is given by

$$X_t^\pi = \beta_t\, B_t + \gamma_t\, S_t\,.$$

A *contingent claim* f_T with the repayment date T is defined to be a $\mathcal{F}_T$-measurable non-negative random variable. We say that a strategy π is *self-financing* if

$$dX_t^\pi = \beta_t\, dB_t + \gamma_t\, dS_t\,.$$

A self-financing strategy π is called a (perfect) *hedge* for a contingent claim f_T if

$$X_t^\pi \ge f_T \qquad \text{(a.s.)}.$$

We say that a strategy π *replicates* f_T if

$$X_t^\pi = f_T\,.$$

A hedge π^* is called the *minimal hedge* if for any other hedge π and for all $t \le T$

$$X_t^{\pi^*} \le X_t^\pi\,.$$

The *price* of a contingent claim f_T is defined as

$$C_T = X_0^{\pi^*}.$$

A (B, S)-market is *complete* if every contingent claim f_N can be *replicated* by some self-financing strategy. We say that a probability P^* is a *martingale probability* if S_t/B_t is a martingale with respect to P^*. Similar to the discrete case, P^* is completely determined by its density Z_T^*:

$$Z_T^* = \exp\left\{ -\frac{\mu - r}{\sigma}\, w_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T \right\}.$$

The *Girsanov Theorem* states that in this setting the process

$$w_t^* := w_t + \frac{\mu - r}{\sigma} t$$

is a Wiener process with respect to the new probability P^* and the initial filtration $\mathbb{F}$.

Let F_Y and F_Y^* be distribution functions of a random variable Y with respect to probabilities P and P^*, respectively. Then the equality

$$\begin{aligned} \mu\, T + \sigma\, w_T &= r\, T + \sigma\, w_T + (\mu - r)T = r\, T + \sigma \left(w_T + \frac{\mu - r}{\sigma} T \right) \\ &= r\, T + \sigma\, w_T^* \end{aligned}$$

implies that

$$F^*_{\mu T + \sigma w_T} = F^*_{r T + \sigma w_T^*} = F_{r T + \sigma w_T},$$

and therefore

$$F^*_{S_T} = F^*_{S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\, w_T\right\}} = F_{S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma\, w_T\right\}}.$$

From the general methodology of pricing contingent claims in complete markets we have

$$C_T = E^* \left(\frac{f_T}{B_T} \right)$$

for any claim f_T.

For a European call option with $f_t = \big(S_T - K\big)^+$, we obtain

$$\begin{aligned}
C_T = E^*\left(\frac{f_T}{B_T}\right) &= e^{-rT}\, E^*\Big(\big(S_T - K\big)^+\Big) \\
&= e^{-rT}\, E^*\left(\left(S_0 \exp\big\{\big(\mu - \frac{\sigma^2}{2}\big)T + \sigma\, w_T\big\}\right)^+\right) \\
&= e^{-rT}\, E\left(\left(S_0 \exp\big\{\big(r - \frac{\sigma^2}{2}\big)T + \sigma\, w_T\big\}\right)^+\right) \\
&= e^{-rT}\, E\left(\left(S_0 \exp\big\{\big(r - \frac{\sigma^2}{2}\big)T + \sigma\,\sqrt{T}\, w_1\big\}\right)^+\right) \\
&= e^{-rT}\, E\left(\left(S_0\, e^{rT}\, \exp\big\{ - \frac{\sigma^2}{2}\, T + \sigma\,\sqrt{T}\, w_1\big\}\right)^+\right) \\
&= e^{-rT}\, E\left(\left(a\, e^{b\xi - b^2/2} - K\right)^+\right),
\end{aligned}$$

where $a = S_0\, e^{rT},\;\; b = \sigma\,\sqrt{T}$ and $\xi \sim \mathcal{N}(0,1)$. Here we also used the following property of a Wiener process:

$$w_T = \sqrt{T}\, w_1\,.$$

Taking into account that

$$E\left(\left(a\, e^{b\xi - b^2/2} - K\right)^+\right) = a\,\Phi\left(\frac{\ln(a/K) + \frac{1}{2}b^2}{b}\right) - K\,\Phi\left(\frac{\ln(a/K) - \frac{1}{2}b^2}{b}\right),$$

we arrive to the Black-Scholes formula:

$$\begin{aligned}
C_T &= S_0\,\Phi\left(\frac{\ln(a/K) + \frac{1}{2}b^2}{b}\right) - K\, e^{-r\,T}\,\Phi\left(\frac{\ln(a/K) - \frac{1}{2}b^2}{b}\right) \\
&= S_0\,\Phi\big(y_+\big) - K\, e^{-r\,T}\,\Phi\big(y_-\big)\,,
\end{aligned}$$

with

$$y_\pm = \frac{\ln(S_0/K) + T(r \pm \sigma^2/2)}{\sigma\sqrt{T}}.$$

Thus, we found the 'fair' non-arbitrage price of a European call option. As in the case of binomial markets, we have the following *call-put parity* relation:

$$P_T = C_T - S_0 + K\, e^{-rT}\,, \tag{2.12}$$

where P_T is the price of a European put option. Relation (2.12) allows us to compute P_T:

$$\begin{aligned}
P_T &= -S_0\,\big(1 - \Phi(y_+)\big) + K\, e^{-rT}\,\big(1 - \Phi(y_-)\big) \\
&= -S_0\,\Phi(-y_+) + K\, e^{-rT}\,\Phi(-y_-)\,.
\end{aligned}$$

Note that prices C_T and P_T are functions of K, σ and S_0. Dividing both sides of the identity

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K$$

by e^{rT} and taking expectations with respect to the risk-neutral probability P^*, we obtain

$$C_T(K, \sigma, S_0) - P_T(K, \sigma, S_0) = E^*\big(S_T\big)\, e^{-rT} - K\, e^{-rT} = S_0 - K\, e^{-rT}.$$

Finally, using the Black-Scholes formula we write

$$\begin{aligned}
P_T(K, \sigma, S_0) &= C_T(K, \sigma, S_0) - S_0 + K\, e^{-rT} \\
&= -S_0\,\big(1 - \Phi(y_+)\big) + K\, e^{-rT}\,\big(1 - \Phi(y_-)\big) \\
&= (-S_0)\,\Phi\left(\frac{\ln\big(-S_0/(-K)\big) + T(r + (-\sigma)^2/2)}{-\sigma\sqrt{T}}\right) \\
&\qquad - \big(-K\, e^{-r\,T}\big)\,\Phi\left(\frac{\ln\big(-S_0/(-K)\big) + T(r - \sigma^2/2)}{-\sigma\sqrt{T}}\right) \\
&= C_T(-K, -\sigma, -S_0),
\end{aligned}$$

which represents the *duality* of prices of European call and put options.

We also can write the price of a European call option at any time $t \in [0, T]$:

$$C_T(t, S_t) = S_t\, \Phi\big(y_+(t)\big) - K\, e^{-r\,(T-t)}\, \Phi\big(y_-(t)\big)\,,$$

where

$$y_\pm(t) = \frac{\ln(S_t/K) + (T-t)(r \pm \sigma^2/2)}{\sigma\sqrt{T-t}}\,.$$

This suggests the following structure of the minimal hedge π^*:

$$\begin{aligned}
\gamma_t^* &= \Phi\big(y_+(t)\big) = \frac{\partial C_T}{\partial S}(t, S_t)\,, \\
\beta_t^* &= -K\, e^{-r\,(T)}\, \Phi\big(y_-(t)\big)\,.
\end{aligned}$$

Since the option price $C_T(t, S_t)$ is a function of time t, price S_t, rate if interest r and volatility σ, one can consider the following ‘Greeks’ often used by the risk management practitioners:

Theta:

$$\theta = \frac{\partial C_T}{\partial t} = \frac{S_t\, \sigma\, \varphi\big(y_+(t)\big)}{2\,\sqrt{T-t}} - K\, r\, e^{-r\,(T-t)}\, \Phi\big(y_-(t)\big)\,,$$

Delta:

$$\Delta = \frac{\partial C_T}{\partial S} = \Phi\big(y_+(t)\big)\,,$$

Rho:

$$\rho = \frac{\partial C_T}{\partial r} = K\,(T-t)\,e^{-r\,(T-t)}\,\Phi\big(y_-(t)\big)\,,$$

Vega:

$$\Upsilon = \frac{\partial C_T}{\partial \sigma} = S_t\,\varphi\big(y_+(t)\big)\,\sqrt{T-t}\,,$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}$. (Note that since there is no Greek letter 'vega', we use upsilon instead.)

Let $Y_t \equiv Y_t^\pi := X_t^\pi / B_t \geq 0$ be the discounted value of a portfolio π. The the Kolmogorov-Itô formula implies that

$$dY_t = \phi_t\,dw_t^* \qquad Y_0 = X_0^\pi\,,$$

where $\phi_t = \sigma\,\gamma_t\,S_t/B_t$ and $dw_t^* = dw_t + t\,(\mu - r)/\sigma$ is a Wiener process with respect to probability P^*.

The set

$$A = A(x,\pi,f_T) = \big\{\omega\,:\,X_T^\pi(x) \geq f_T\big\} = \big\{\omega\,:\,Y_T^\pi(x) \geq f_T/B_T\big\}$$

is called the *perfect hedging set* for claim f_T and strategy π with the initial wealth x.

The theory of perfect hedging that was discussed above allows one to find a hedge with the initial wealth $X_0 = E^*\big(f_T/B_T\big)$ and $P(A) = 1$. However, it is possible that an investor responsible for claim f_T may have initial budget constraints. In particular, an investor's initial capital may be less than amount X_0, which is necessary for successful hedging.

Thus, we arrive at the following problem of *quantile* hedging.

QUESTION 2.1 *Among all admissible strategies find a strategy $\tilde{\pi}$ such that*

$$P\big(A(x,\tilde{\pi},f_T)\big) = \max_\pi P\big(A(x,\pi,f_T)\big)$$

under the budget constraint

$$x \leq x_0 < E^*\left(\frac{f_T}{B_T}\right) = X_0\,,$$

where x_0 is investor's initial capital.

The following lemma addresses this problem.

LEMMA 2.3
Suppose perfect hedging set $\widetilde{A}$ is such that

$$P(\widetilde{A}) = \max_\pi P(A) \qquad \textit{where} \qquad E^*\left(\frac{f_T}{B_T}\,I_A\right) \leq x\,.$$

Then a perfect hedge $\tilde{\pi}$ for the claim $\tilde{f}_T = f_T\, I_{\widetilde{A}}$, with the initial wealth x, yields a solution for the problem of quantile hedging. Furthermore, the perfect hedging set $A(x, \tilde{\pi}, f_T)$ coincides with $\widetilde{A}$.

PROOF

Step 1. Let π be an arbitrary admissible strategy with the initial wealth

$$x \le E^*\left(\frac{f_T}{B_T}\right) = X_0\,.$$

Its discounted value

$$Y_t = x + \int_0^t \phi_s\, dw_s^*$$

is a non-negative supermartingale with respect to P^*. For a perfect hedging set $A = A(x, \pi, f_T)$ we have

$$Y_t \ge \frac{f_T}{B_T}\, I_A\,, \quad (P - \text{a.s.}).$$

Hence

$$x = E^*\left(Y_T\right) \ge E^*\left(\frac{f_T}{B_T}\, I_A\right),$$

and $P(A) \le P(\widetilde{A})$.

Step 2. Let $\tilde{\pi}$ be a perfect hedge for the claim $\tilde{f}_T = f_T\, I_{\widetilde{A}}$, with the initial wealth x satisfying the inequality

$$E^*\left(\frac{f_T}{B_T}\, I_{\widetilde{A}}\right) \le x \le x_0 < E^*\left(\frac{f_T}{B_T}\right) = X_0\,.$$

We show that this strategy is optimal for the problem of quantile hedging. Since

$$x + \int_0^t \tilde{\phi}_s\, dw_s^* \ge E^*\left(\frac{f_T}{B_T}\, I_{\widetilde{A}}\right) + \int_0^t \tilde{\phi}_s\, dw_s^* = E^*\left(\frac{f_T}{B_T}\, I_{\widetilde{A}} \Big| \mathcal{F}_t\right) \ge 0\,,$$

then $\tilde{\pi}$ is an admissible strategy. Denote

$$A' = \left\{\omega \,:\, x + \int_0^T \tilde{\phi}_s\, dw_s^* \ge f_T/B_T\right\}$$

the perfect hedging set for $\tilde{\pi}$. Since $\tilde{\pi}$ is a perfect hedge for claim $\tilde{f}_T$, we obtain

$$A' \supseteq \left\{\omega \,:\, f_T\, I_{\widetilde{A}} \ge f_T\right\} \supseteq \widetilde{A}\,,$$

and hence $P(A') \ge P(\widetilde{A})$.

Step 3. Now we observe that

$$A = \widetilde{A}\,, \quad (P - \text{a.s.}),$$

and taking into account that $\widetilde{A}$ is a perfect hedging set for $\tilde{\pi}$, we conclude that $\tilde{\pi}$ is optimal strategy for the problem of quantile hedging.

$\square$

Next, we will use the fundamental Neumann-Pearson Lemma (see [19]) for construction of a maximal perfect hedging set.

Suppose that distributions Q^* and P correspond to hypothesis' H_0 and H_1, respectively. Let $\alpha = E_{Q^*}(\phi)$ be the error of the first kind and $\beta = E_P(\phi)$ be the criterium's power corresponding to a *critical function* ϕ. The Neumann-Pearson criterium has the following structure:

$$\phi = \begin{cases} 1\,, & dP/dQ^* > c \\ 0\,, & dP/dQ^* < c \end{cases}\,,$$

and it maximizes β given that the error of the first kind does not exceed a set level α. Here c is some constant, and values 0 and 1 in the critical function ϕ indicate which of the hypothesis' H_0 or H_1 should be preferred.

If we introduce a probability Q^* by the relation

$$\frac{dQ^*}{dP^*} = \frac{f_T}{B_T\, E^*\big(f_T/B_T\big)} = \frac{f_T}{E^*\big(f_T\big)}\,,$$

then the constraint in Lemma 2.3 can be written in the form

$$Q^*(A) = \int_A \frac{dQ^*}{dP^*}\, dP^* \le \frac{x}{E^*\big(f_T/B_T\big)} = \alpha\,.$$

The solution of the corresponding optimization problem is given by

$$\widetilde{A} = \left\{\omega\,:\, \frac{dP}{dQ^*} > c\right\} = \left\{\omega\,:\, \frac{dP}{dP^*} > c\, \frac{f_T}{E^*\big(f_T\big)}\right\}, \tag{2.13}$$

where

$$c = \inf\left\{a\,:\, Q^*\left(\left\{\omega\,:\, \frac{dP}{dQ^*} > a\right\}\right) \le \alpha\right\},.$$

The proof of this claim follows from the fundamental Neumann-Pearson Lemma and from the equalities

$$\alpha = E_{Q^*}(\phi) = Q^*(\widetilde{A}) \qquad \text{and} \qquad \beta = E_P(\phi) = P(\widetilde{A}) = \max_{\pi} P(A)\,.$$

Thus we arrive at the following theorem.

THEOREM 2.6

An optimal strategy $\tilde{\pi}$ for the problem of quantile hedging coincides with the perfect hedge for the contingent claim $\tilde{f}_T = f_T\, I_{\widetilde{A}}$, where the maximal perfect hedging set $\widetilde{A}$ is given by (2.13).

Next we consider the problem of quantile hedging for a European call option with $f_T = (S_T - K)^+$. The initial value of a perfect hedge in this case is

$$X_0 = S_0\, \Phi(d_+) - K\, e^{-rT}\, \Phi(d_-)\,.$$

Suppose an investor has an initial capital $x < X_0$. By Theorem 2.6 the optimal strategy for the problem of quantile hedging coincides with the perfect hedge for the contingent claim $f_T\, I_{\widetilde{A}}$, where

$$A = \left\{\omega\,:\, \frac{dP}{dQ^*} > c\right\} = \left\{\omega\,:\, \frac{dP}{dP^*} > c_1\, f_T\, e^{-rT}\right\},$$

Since density Z_T^* has the form

$$Z_T^* = \exp\left\{-\frac{\mu-r}{\sigma}\, w_T^* + \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right\}, \tag{2.14}$$

then

$$\begin{aligned}
A &= \left\{\omega\,:\, \exp\left\{\frac{\mu-r}{\sigma}\, w_T^* - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right\} > c_1\,(S_T - K)^+\right\} \\
&= \left\{\omega\,:\, \exp\left\{\frac{\mu-r}{\sigma^2}\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \sigma\, w_T^*\right)\right\}\right. \\
&\quad \left.\times \exp\left\{-\frac{\mu-r}{\sigma^2}\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T\right) - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right\} > c_1\,(S_T - K)^+\right\} \\
&= \left\{\omega\,:\, S_T^{\,\frac{\mu-r}{\sigma^2}}\, \exp\left\{-\frac{\mu-r}{\sigma^2}\left(\ln S_0 + \frac{\mu+r-\sigma^2}{2}\right) T\right\} > c_1\,(S_T - K)^+\right\}.
\end{aligned}$$

Now we consider two cases.

Case 1. $\frac{\mu-r}{\sigma^2} \le 1$.

Set A can be written in the form

$$A = \{\omega\,:\, S_T < d\} = \{\omega\,:\, w_T^* < b\} = \left\{\omega\,:\, S_T < S_0\, \exp\left\{\left(r - \sigma^2/2\right) T + b\,\sigma\right\}\right\}$$

for some constants b and d under the constraint

$$E^*\left(\frac{f_T}{B_T}\, I_A\right) = x_0\,.$$

Taking into account that

$$S_T = S_0 \, \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) T + \sigma \, w_T^* \right\},$$

we obtain

$$P(A) = \Phi\left(\frac{b - T\,(\mu - r)/\sigma}{\sqrt{T}} \right).$$

Constant b can be found from the equality

$$x_0 = E^* \big(e^{-rT} \, f_T \big) \, I_A = e^{-rT} \, F_T^*(S_T)\,,$$

where

$$\begin{aligned} F_T^* &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b/\sqrt{T}} f\left(S_0 \, \exp\left\{ \sigma \sqrt{T}\, y + \left(r - \frac{\sigma^2}{2} \right) T \right\} \right) e^{-\frac{y^2}{2}} \, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-d_0}^{b/\sqrt{T}} \left(S_0 \, \exp\left\{ \sigma \sqrt{T}\, y + \left(r - \frac{\sigma^2}{2} \right) T \right\} - K \right)^+ e^{-\frac{y^2}{2}} \, dy\,; \\ d_0 &= \frac{\ln\big(K/S_0\big) - T\,(r - \sigma^2)/2}{\sigma \sqrt{T}}\,. \end{aligned}$$

Hence

$$\begin{aligned} x_0 &= S_0 \left[\Phi(\sigma \sqrt{T} - d_0) - \Phi\left(\sigma \sqrt{T} - \frac{b}{\sqrt{T}} \right) \right] - K\, e^{-rT} \left[\Phi(d_0) - \Phi\left(-\frac{b}{\sqrt{T}} \right) \right] \\ &= S_0 \left[\Phi(d_+) - \Phi\left(\sigma \sqrt{T} - \frac{b}{\sqrt{T}} \right) \right] - K\, e^{-rT} \left[\Phi(d_-) - \Phi\left(-\frac{b}{\sqrt{T}} \right) \right]. \end{aligned}$$

Case 2. $\frac{\mu - r}{\sigma^2} > 1$.

Set A can be written in the form

$$A = \big\{ \omega \,:\, w_T^* < b_1 \big\} \cup \big\{ \omega \,:\, w_T^* > b_2 \big\}$$

for some constants b_1 and b_2. Solving the problem of quantile hedging, we obtain

$$P(A) = \Phi\left(\frac{b_1 - T\,(\mu - r)/\sigma}{\sqrt{T}} \right) + \Phi\left(\frac{b_2 - T\,(\mu - r)/\sigma}{\sqrt{T}} \right).$$

Constants b_1 and b_2 can be found from the same equality

$$x_0 = E^* \big(e^{-rT} \, f_T \big) \, I_A = e^{-rT} \, F_T^*(S_T)\,,$$

where now

$$F_T^* = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b_1/\sqrt{T}} f\left(S_0 \, \exp\left\{\sigma\sqrt{T}\, y + \frac{r-\sigma^2}{2}\, T\right\}\right) e^{-\frac{y^2}{2}}\, dy$$

$$+\frac{1}{\sqrt{2\pi}} \int_{b_2/\sqrt{T}}^{\infty} f\left(S_0 \, \exp\left\{\sigma\sqrt{T}\, y + \frac{r-\sigma^2}{2}\, T\right\}\right) e^{-\frac{y^2}{2}}\, dy\,.$$

Similarly to Case 1,

$$x_0 = S_0\left[\Phi(d_+) - \Phi\left(\sigma\sqrt{T} - \frac{b_1}{\sqrt{T}}\right) + \Phi\left(\sigma\sqrt{T} - \frac{b_2}{\sqrt{T}}\right)\right]$$

$$-K\, e^{-rT}\left[\Phi(d_-) - \Phi\left(-\frac{b_1}{\sqrt{T}}\right) + \Phi\left(-\frac{b_2}{\sqrt{T}}\right)\right].$$

Next, we consider the case when an owner of asset S receives *dividends*. Denote $\widetilde{S}_t$ the process that represents the wealth of the owner of asset S, and let $\delta\, S_t,\ \delta \geq 0$, represent the received dividends. Then the evolution of $\widetilde{S}_t$ is described by following stochastic equation

$$d\left(\frac{\widetilde{S}_t}{B_t}\right) = d\left(\frac{S_t}{B_t}\right) + \delta\,\frac{S_t}{B_t}\, dt, \quad \delta \geq 0\,.$$

Using

$$dS_t = S_t\left(\mu\, dt + \sigma\, dw_t\right)$$

and

$$d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}\left((\mu - r)\, dt + \sigma\, dw_t\right),$$

we obtain

$$d\left(\frac{\widetilde{S}_t}{B_t}\right) = \frac{S_t}{B_t}\left((\mu - r + \delta)\, dt + \sigma\, dw_t\right).$$

We can notice the analogy of

$$\overline{w}_t := w_t + \frac{\mu - r + \delta}{\sigma}\, t \qquad \text{with} \qquad \widetilde{w}_t = w_t + \frac{\mu - r}{\sigma}\, t$$

and of

$$\overline{Z}_T := \exp\left\{-\frac{\mu - r + \delta}{\sigma}\, w_T^* + \frac{1}{2}\left(\frac{\mu - r + \delta}{\sigma}\right)^2 T\right\},$$

with Z_T^* (see (2.14)).

We now define a new probability $\overline{P}_T$ with density $\overline{Z}_T$. By Girsanov Theorem, $\left(\overline{w}_t\right)_{t \leq T}$ is a Wiener process with respect to $\overline{P}_T$. Distribution functions are given by

$$\overline{F}_{\mu T + \sigma w_T} = \overline{F}_{(r-\delta)T + \sigma \overline{w}_T} = F_{(r-\delta)T + \sigma w_T}$$

and

$$\overline{F}_{S_T} = F_{S_0 \exp\{(r-\delta-\sigma^2/2)T+\sigma w_T\}}\,.$$

We compute then the price of a European call option

$$\begin{aligned}
C_T(\delta) &= \overline{E}\left(\frac{(S_T-K)^+}{B_T}\right) = e^{-rT}\,\overline{E}\left(\left(S_0\,e^{(r-\sigma^2/2)T+\sigma w_T}-K\right)^+\right)\\
&= e^{-rT}\,E\left(\left(S_0\,e^{(r-\delta-\sigma^2/2)T+\sigma w_T}-K\right)^+\right)\\
&= e^{-rT}\,E\left(\left(S_0\,e^{(r-\delta-\sigma^2/2)T+\sigma\sqrt{T}\,w_1}-K\right)^+\right)\\
&= S_0\,e^{-\delta T}\,\Phi\left(\frac{\ln(S_0/K)+T\,(r-\delta+\sigma^2/2)}{\sigma\sqrt{T}}\right)\\
&\qquad -K\,e^{-rT}\,\Phi\left(\frac{\ln(S_0/K)+T\,(r-\delta-\sigma^2/2)}{\sigma\sqrt{T}}\right).
\end{aligned}$$

In Section 2.2 we studied the binomial model of a market with transaction costs. It was shown in Theorem 2.5 that if the terminal buy and sell prices of stock S are equal, then there exists a unique strategy that replicates the European call option. This strategy is related to a binomial market without transaction costs where values of profitability (and therefore, of volatility) are increased.

In the case of the Black-Scholes model, a similar result was proved in [25]. For simplicity, suppose that $B_t \equiv 1,\ t \le T$, and that capital in portfolio $\pi = (\beta, \gamma)$ is redistributed at discrete times $t_i = i\,T/N,\ i \le N$.

Constraints on redistribution of capital $X_t^\pi = \beta_t + \gamma_t\,S_t$ of portfolio π can be written in the form of proportional *transaction costs* with parameter $\lambda \ge 0$:

$$\Delta X_t^\pi = \gamma_t\,\Delta S_t - \lambda\,S_t\,|\Delta\gamma_t|\,.$$

Now consider a European call option that will be hedged in the class of strategies described above. Denote $\mathcal{C}^{\mathrm{BS}}(t_i, S_{t_i}),\ i \le N$, the capital of a Black-Scholes strategy. Then an appropriate hedging strategy π must have capital X_t^π such that

$$X_{t_i}^\pi = \mathcal{C}^{\mathrm{BS}}(t_i, S_{t_i}), \quad i \le N,$$

and approximately (up to infinitesimals of high order of Δt) satisfy equation

$$\frac{\partial X^\pi(t,S_t)}{\partial t} + \frac{\widetilde{\sigma}^2}{2}\,S_t^2\,\frac{\partial^2 X^\pi(t,S_t)}{\partial s^2} = 0\,,$$

with parameter

$$\widetilde{\sigma}^2 = \sigma^2\left(1 + \lambda\sqrt{\frac{8}{\sigma\,\pi\,\Delta t}}\right) > \sigma^2\,.$$

Thus, for pricing European call options in this case one can use the Black-Scholes formula with the increased volatility.

Next, we consider an optimal investment problem in the framework of the Black-Scholes model, where the optimal strategy is defined by the relation

$$\sup_{\pi\in SF} E\big(\ln X_T^{\pi}\big) = E\big(\ln X_T^{\pi^*}\big)\,, \qquad X_0^{\pi} = X_0^{\pi^*} = x.$$

We sketch the solution of this optimal investment problem: find

$$Y_T^*(x) = \sup_Y E\big(\ln Y_T(x)\big)\,,$$

where supremum is taken over the set of all positive martingales with respect to P^*, starting at x.

Let the *optimal* martingale be

$$Y_t^*(x) = E^*\left(\frac{x}{Z_T^*}\Big|\mathcal{F}_t\right), \qquad t\in[0,T]\,,$$

where

$$Z_T^* = \exp\left\{-\frac{\mu-r}{\sigma}\,w_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T\right\}$$

is the density of the unique martingale probability P^* with respect to P.

As in Section 1.6, it can be shown that

$$E\big(\ln Y_T(x)\big) \le E\big(\ln Y_T^*(x)\big)\,.$$

Then using the martingale characterization of self-financing strategies, we obtain

$$Y_t^* = \frac{X_t^{\pi^*}(x)}{B_t} = X_t^*$$

for some self-financing strategy $\pi^* = \big(\beta_t^*, \gamma_t^*\big)_{t\le T}$. Denote

$$\alpha_t^* = \frac{\gamma_t^*\,S_t}{X_t^{\pi^*}}$$

the proportion of risky capital in portfolio π^*. By the Kolmogorov-Itô formula we have

$$dX_t^* = X_t^*\,\alpha_t^*\,\sigma\,d\widetilde{w}_t\,,$$

and therefore

$$X_T^* = x\,\exp\left\{\sigma\,\alpha^*\,w_T + \alpha^*\,(\mu-r)\,T - \frac{1}{2}\,\sigma^2\,(\alpha^*)^2\,T\right\},$$

where $\alpha_t^* \equiv \alpha^*$.

On the other hand,

$$X_T^* = \frac{x}{Z_T^*} = x\,\exp\left\{\frac{\mu-r}{\sigma}\,w_T + \frac{1}{2}\Big(\frac{\mu-r}{\sigma}\Big)^2 T\right\}$$

Comparing these formulae, we deduce the expression for the optimal proportion:

$$\alpha^* = \frac{\mu-r}{\sigma^2}\,,$$

which is often referred to as the *Merton's point.*

So far we were studying markets with information flow $\mathbb{F} = \big(\mathcal{F}_t\big)_{t\le T}$ defined by prices of asset S: $\mathcal{F}_t = \sigma\big(S_0,\dots,S_t\big)$. When one wants to take into account the *non-homogeneity* of the market, this leads to the assumption that some (but not all!) of the market participants have access to a larger information flow. Mathematically this can mean, for example, that the terminal value S_T is known at time $t < T$ or that S_T will belong to some interval $[S', S'']$ etc. Let ξ be a random variable that extends market information $\mathcal{F}_t$ to $\mathcal{F}_t^\xi = \sigma\big(\mathcal{F}_t, \xi\big)$. Then $\mathbb{F}^\xi = \big(\mathcal{F}_t^\xi\big)_{t\le T}$ is called the *insider* information flow. Now we investigate how this additional information can be utilized by a market participant.

For simplicity let $r = 0$. Using the formula for Merton's point and the martingale property of the stochastic integral, we obtain that the expected utility is given by

$$\begin{aligned}
v_{\mathbb{F}}(x) &= \sup_{\pi\in SF(\mathbb{F})} E\Big(\ln X_T^\pi(x)\Big)\\
&= x + E\bigg(\int_0^T \alpha_s\,\sigma\,dw_s + \int_0^T \mu\,\alpha_s\,ds - \frac{1}{2}\int_0^T \alpha_s^2\,\sigma^2\,ds\bigg)\\
&= x + E\bigg(\frac{\mu^2}{\sigma^2}\,T - \frac{\sigma^2}{2}\,\frac{\mu^2}{\sigma^4}\,T\bigg)\\
&= x + \frac{1}{2}\,\frac{\mu^2}{\sigma^2}\,T
\end{aligned}$$

for the information flow $\mathbb{F} = \big(\mathcal{F}_t\big)_{t\le T}$.

When using the insider information flow $\mathbb{F}^\xi = \big(\mathcal{F}_t^\xi\big)_{t\le T}$, one cannot assume that the process $\big(w_t,\,\mathcal{F}_t^\xi\big)_{t\le T}$ is a Wiener process. Nevertheless, it is natural to assume that as in Girsanov theorem, there exists a $\mathbb{F}^\xi$-adapted process $\mu^\xi = \big(\mu_t^\xi\big)_{t\le T}$ such that

$$\int_0^T |\mu_s^\xi|\,ds < \infty \qquad \text{(a.s.)}\,,$$

and the process

$$\widetilde{w}_t = w_t - \int_0^t \mu_s^\xi\,ds\,, \qquad t \le T\,,$$

is a Wiener process with respect to $\mathbb{F}^\xi$.

In this case the additional utility can be expressed in terms of the 'information drift' μ^{ξ}. Indeed, for a self-financing strategy $\pi \in SF(\mathbb{F}^{\xi})$ the terminal capital can be written in the form

$$X_T^{\pi}(x) = x \,\exp\left\{ \int_0^T \alpha_s\,\sigma\, d\widetilde{w}_s - \frac{1}{2}\int_0^T \alpha_s\,\sigma^2\, ds + \int_0^T \alpha_s\,(\mu + \sigma\,\mu_s^{\xi})\, ds \right\}.$$

Taking into account that

$$E\left(\int_0^T \frac{\mu}{\sigma}\,\mu_s^{\xi}\, ds \right) = E\left(\int_0^T \frac{\mu}{\sigma}\,(dw_s - d\widetilde{w}_s) \right) = 0\,,$$

we find the expected utility

$$v_{\mathbb{F}^{\xi}}(x) = x + \frac{1}{2}\,E\left(\int_0^T \frac{(\mu + \sigma\,\mu_s^{\xi})^2}{\sigma}\, ds \right) = x + \frac{1}{2}\,E\left(\int_0^T \left[\frac{\mu^2}{\sigma^2} + (\mu_s^{\xi})^2\right] ds \right),$$

given the insider information $\mathbb{F}^{\xi}$. Thus, the additional utility is given by formula

$$\Delta\, v_{\mathbb{F}^{\xi}} = v_{\mathbb{F}^{\xi}}(x) - v_{\mathbb{F}}(x) = \frac{1}{2}\,E\left(\int_0^T \left(\mu_s^{\xi}\right)^2 ds \right),$$

which can be written in more detailed form in many particular cases (see, for example, [3]).

WORKED EXAMPLE 2.4

Find prices of European call and put options on a Black-Scholes market if $r = 0.1,\ T = 215/365,\ S_0 = 100(\$),\ K = 80(\$),\ \mu = r,\ \sigma = 0.1.$

SOLUTION By the Black-Scholes formula we have

$$\begin{aligned} C_T = C_T(K, S_0, \sigma) &= S_0\,\Phi(y_+) - K\,e^{-rT}\,\Phi(y_-) \\ &= 100\,\Phi\left(\frac{\ln(100/80) + \frac{215}{365}\left(0.1 + (0.1)^2/2\right)}{0.1\,\sqrt{215/365}} \right) \\ &\quad -80\,e^{-0.1\frac{215}{365}}\,\Phi\left(\frac{\ln(100/80) + \frac{215}{365}\left(0.1 - (0.1)^2/2\right)}{0.1\,\sqrt{215/365}} \right) \\ &= 100\,\Phi(3.177) - 80\,e^{-0.1\frac{215}{365}}\,\Phi(3.64) \approx 24.57\,. \end{aligned}$$

The call-put parity can be used now to find the price of a European put option:

$$P_T = P_T(K, S_0, \sigma) = C_T - S_0 + K\,e^{-rT} = 24.57 - 100 + 80\,e^{-0.1\frac{215}{365}} \approx 0\,.$$

If we increase the rate of interest to $r = 0.2$, then

$$C_T \approx 28.9 \qquad \text{and} \qquad P_T \approx 0\,.$$

Increasing volatility to $\sigma = 0.8$ implies higher prices:

$$C_T \approx 35.55 \qquad \text{and} \qquad P_T \approx 10.97 \qquad \text{for} \quad r = 0.1\,,$$

and

$$C_T \approx 38.05 \qquad \text{and} \qquad P_T \approx 9.16 \qquad \text{for} \quad r = 0.2\,.$$

Finally, for the model with dividends we have

$$C_T(\delta, r) = e^{-\delta T}\, C_T(0, r-\delta) \qquad \text{and} \qquad P_T(\delta, r) = e^{-\delta T}\, P_T(0, r-\delta)\,.$$

Let $\delta = 0.1$, then

$$C_T \approx 18.86 \qquad \text{and} \qquad P_T \approx 0 \qquad \text{for} \quad r = 0.1\,,$$

and

$$C_T \approx 23.17 \qquad \text{and} \qquad P_T \approx 0 \qquad \text{for} \quad r = 0.2\,.$$

For $\delta = 0.2$

$$C_T \approx 13.5 \qquad \text{and} \qquad P_T \approx 0.04 \qquad \text{for} \quad r = 0.1\,,$$

and

$$C_T \approx 17.8 \qquad \text{and} \qquad P_T \approx 0 \qquad \text{for} \quad r = 0.2\,.$$

$\square$

2.7 Assets with fixed income

Consider a zero-coupon bond maturing at time $T < T^*$, i.e., a claim that pays 1 at time T. Let $B(t,T)$ be its price at time $t \in [0,T]$. Naturally we have $B(T,T) = 1$ and $B(t,T) < 1$ for all $t \le T$.

The price $B(t,T)$ can be written in three equivalent forms:

$$B(t,T) = \exp\big\{ -r(t,T)\,(T-t)\big\}\,,$$

$$B(t,T) = \exp\left\{ -\int_t^T f(t,s)\,ds\right\},$$

$$B(t,T) = \exp\Big\{ -(T-t)\,\ln\big(1 + \rho(T-t,t)\big)\Big\}\,.$$

Functions $r(\cdot, T)$ and $\rho(T-\cdot, \cdot)$ are called *yield* and *yield to maturity*, respectively. Function $f(t, s)$, called the *forward rate*, represents the instantaneous interest rate at time $t \leq s$ for borrowing at time s.

Under some reasonable assumptions we have the following relations

$$r(t,T) = -\frac{\ln B(t,T)}{T-t}\,,$$

$$f(t,T) = -\frac{\partial}{\partial t}\ln B(t,T) = r(t,T) + (T-t)\,\frac{\partial}{\partial T}r(t,T)\,.$$

Denote $r_t = f(t,t)$ the *instantaneous short rate* at t. This rate of interest can be a stochastic process, therefore bonds must be studied as risky assets since their prices depend on interest rates.

Let $r = (r_t)_{t\geq 0}$ be a stochastic process on some stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Defining a bank account by

$$B_t = \exp\left\{\int_0^t r_s\,ds\right\},$$

we arrive at the notion of a *bonds market* as a family $\Big(B_t,\, B(t,T)\Big)_{t\leq T\leq T^*}$. As in the case of the studied above (B, S)-market ('shares' market) , we can consider discounted bond prices:

$$\overline{B}(t,T) = \frac{B(t,T)}{B_t}$$

and construct a probability P^* that is equivalent to the initial probability P and such that the process $\overline{B} = \big(\overline{B}(t,T)\big)_{t\geq 0}$ is a martingale with respect to P^*. If such probability exists, then we say that the bonds market is *arbitrage free*. We can interpret the absence of arbitrage as the impossibility of making profit without risk.

Taking into account that $B(T,T) = 1$, we obtain

$$E^*\big(B_T^{-1}\big|\mathcal{F}_t\big) = \frac{B(t,T)}{B_t}$$

and therefore we have the representation

$$B(t,T) = E^*\left(\exp\left\{-\int_0^t r_s\,ds\right\}\Big|\mathcal{F}_t\right),$$

which allows one to study the structure of prices $B(t,T)$ by specifying process $r = (r_t)_{t\geq 0}$.

Here we list some of the frequently used models.

Merton

$$dr_t = \alpha\,dt + \gamma\,dw_t\,, \qquad \alpha,\,\gamma \in \mathbb{R};$$

Vasiček

$$dr_t = (\alpha - \beta\, r_t)\, dt + \gamma\, dw_t\,, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{R};$$

Ho-Lee

$$dr_t = \alpha(t)\, dt + \gamma\, dw_t\,, \qquad \gamma \in \mathbb{R};$$

Black-Derman-Toy

$$dr_t = \alpha(t)\, dt + \gamma(t)\, dw_t\,,$$

Hull-White

$$dr_t = (\alpha(t) - \beta\, r_t)\, dt + \gamma\, dw_t\,, \qquad \beta,\ \gamma \in \mathbb{R};$$

Another way of specifying process r is given by the Schmidt model: let functions f and g be continuous, and functions T and F be continuous and strictly increasing. Then define

$$r_t = F\big(f(t) + g(t)\, w_{T(t)}\big)\,.$$

All the models listed above can be obtained from the Schmidt model by choosing appropriate functions $F,\ f,\ g$ and T.

An equivalent alternative way of describing the structure of bond prices is based on specifying the evolution of forward rate:

$$df(t,T) = \sigma^2\,(T-t)\, dt + \sigma\, dw_t\,,$$

or

$$f(t,T) = f(0,T) + \sigma^2\, t\,(T - t/2) + \sigma\, w_t\,,$$

where $f(0,T)$ is the present forward rate. This implies

$$dr_t = \left(\frac{\partial}{\partial t} f(0,t) + \sigma^2\, t\right) dt + \sigma\, dw_t\,,$$

or

$$r_t = f(0,t) + \frac{\sigma^2}{2}\, t^2 + \sigma\, w_t\,.$$

Substituting the expression for $f(t,s)$ into formula

$$B(t,T) = \exp\left\{ -\int_t^T f(t,s)\, ds \right\}, \quad t \le T\,,$$

we obtain

$$\begin{aligned}\int_t^T f(t,s)\, ds &= \int_t^T \Big[f(0,s) + \sigma^2\, t\,(s - t/2)\Big]\, ds + \sigma\,(T-t)\, w_t \\ &= \int_t^T f(0,s)\, ds + \frac{\sigma^2}{2}\, t\, T\,(T-t) + \sigma\,(T-t)\, w_t\,,\end{aligned}$$

and hence

$$B(t,T) = \exp\Big\{ - \int_t^T f(0,s)\, ds - \frac{\sigma^2}{2}\, t\, T\, (T-t) + \sigma\, (T-t)\, w_t \Big\}$$

$$= \frac{B(0,T)}{B(0,t)}\, \exp\Big\{ - \frac{\sigma^2}{2}\, t\, T\, (T-t) + \sigma\, (T-t)\, w_t \Big\}.$$

We can also rewrite it in the form

$$B(t,T) = \frac{B(0,T)}{B(0,t)}\, \exp\Big\{ (T-t)\, f(0,T) - \frac{\sigma^2}{2}\, t\, (T-t)^2 - (T-t)\, r_t \Big\}.$$

Note that this model is a particular case of the Heath-Jarrow-Morton model, and it is not difficult to check that the initial probability is a martingale probability.

Now we proceed to detailed study of the Vasiček model. According to this model the interest rate oscillates around α/β: r_t has positive drift if $r_t < \alpha/\beta$, and negative if $r_t > \alpha/\beta$. If $\alpha/\beta = 0$, then r_t is a stationary (Gaussian) Ornstein-Uhlenbeck process.

Applying the Kolmogorov-Itô formula we obtain

$$r_t = e^{-\beta t} \Big[r_0 + \int_0^t \alpha\, e^{\beta t}\, ds + \int_0^t \gamma\, e^{\beta t}\, dw_s \Big].$$

Using the Markov property of r_t, we can write

$$B(t,T) = E\Big(\exp\Big\{ - \int_t^T r_s\, ds \Big\} \Big| \mathcal{F}_t \Big) = E\Big(\exp\Big\{ - \int_t^T r_s\, ds \Big\} \Big| r_t \Big)$$

$$= \exp\Big\{ \frac{\gamma^2}{2} \int_t^T \Big(\int_s^T e^{-\beta(u-s)}\, du \Big)^2 ds$$

$$- \alpha \int_t^T \int_t^u e^{-\beta(u-s)}\, ds\, du - r_t \int_t^T e^{-\beta(u-t)}\, du \Big\}$$

$$\equiv \exp\big\{ a(t,T) - r_t\, b(t,T) \big\},$$

where

$$a(t,T) := \frac{\gamma^2}{2} \int_t^T \Big(\int_s^T e^{-\beta(u-s)}\, du \Big)^2 - \alpha \int_t^T \int_t^u e^{-\beta(u-s)}\, ds\, du$$

$$b(t,T) := \int_t^T e^{-\beta(u-t)}\, du.$$

This gives us the general structure of bond prices. Now, in the framework of the Vasiček model, we consider a European call option with the exercise date $T' \le T \le T^*$ and payoff function

$$f = \big(B(T',T) - K \big)^+,$$

where K is the strike price.

The price of this option is given by

$$C(T',T) = B(0,T)\,\Phi(d_+) - K\,B(0,T')\,\Phi(d_-)\,,$$

where

$$d_\pm = \frac{\ln \frac{B(0,T)}{K\,B(0,T')} \pm \frac{1}{2}\,\sigma^2(T',T)\left(\int_{T'}^{T} e^{-\beta(u-T')}\,du\right)^2}{\sigma(T',T)\,\int_{T'}^{T} e^{-\beta(u-T')}\,du}\,,$$

$$\sigma(T',T) = \left(\int_{T'}^{T}\left(\int_{s}^{T}\gamma\,e^{-\beta(u-s)}\,du\right)^2 ds\right)^{1/2}.$$

We need to compute

$$\begin{aligned} C(T',T) &= E\left(e^{-\int_0^{T'} r_u\,du}\left(B(T',T)-K\right)^+\right)\\ &= E\left(I_{\{\omega:\,B(T',T)>K\}}\;e^{-\int_0^{T'} r_u\,du}\,B(T',T)\right)\\ &\qquad\qquad -K\,E\left(I_{\{\omega:\,B(T',T)>K\}}\;e^{-\int_0^{T'} r_u\,du}\right) \end{aligned}$$

Note that

$$\begin{aligned} \{\omega:\;B(T',T)>K\} &= \{\omega:\;a(T',T) - r_{T'}\,b(T',T)) > \ln K\}\\ &= \{\omega:\;r_{T'} \le r'\}\,, \end{aligned}$$

where

$$r' = \frac{\ln K - a(T',T)}{-b(T',T)}\,.$$

Let

$$\xi = r_{T'} \qquad \eta = \int_0^{T} r_u\,du \qquad \zeta = \int_0^{T'} r_u\,du\,,$$

we obtain

$$C(T',T) = E\left(I_{\{\omega:\,\xi\le r'\}}\;e^{-\eta}\right) - K\,E\left(I_{\{\omega:\,\xi\le r'\}}\;e^{-\zeta}\right).$$

To find the final expression for the price, we need the following lemma.

LEMMA 2.4

Suppose X and Y are Gaussian random variables. Then

$$E\left(I_{\{\omega:\,X\le x\}}\;\exp\{-Y\}\right) = \exp\left\{\frac{\sigma_Y^2}{2} - \mu_Y\right\}\Phi(\tilde{x})\,,$$

$$E\left(I_{\{\omega:\,X\le x\}}\,X\;\exp\{-Y\}\right) = \exp\left\{\frac{\sigma_Y^2}{2} - \mu_Y\right\}\left[(\mu_X - \rho_{XY})\,\Phi(\tilde{x}) - \sigma_X\,\varphi(\tilde{x})\right],$$

where

$$\tilde{x} = \frac{x - (\mu_X - \rho_{XY})}{\sigma_X}\,,$$

σ_X^2 *and* σ_Y^2 *are variances of* X *and* Y, $\rho_{XY} = Cov(X,Y)$ *and*

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,, \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy\,.$$

Note that the quantitative characteristics of ξ, η and ζ are given by

$$\mu_\xi = E\big(r_{T'}\big) = e^{-\beta T'} \left(r_0 + \alpha \int_0^{T'} e^{-\beta s}\, ds \right),$$

$$\mu_\eta = E\bigg(\int_0^T r_u\, du \bigg) = r_0 \int_0^T e^{-\beta u}\, du + \alpha \int_0^T \int_0^u e^{-\beta(u-s)}\, ds\, du\,,$$

$$\mu_\zeta = E\bigg(\int_0^{T'} r_u\, du \bigg) = r_0 \int_0^{T'} e^{-\beta u}\, du + \alpha \int_0^{T'} \int_0^u e^{-\beta(u-s)}\, ds\, du\,,$$

$$\sigma_\xi^2 = V\big(r_{T'}\big) = \gamma^2 \int_0^{T'} e^{-2\beta(T'-s)}\, ds\,,$$

$$\sigma_\eta^2 = V\bigg(\int_0^T r_u\, du \bigg) = \gamma^2 \int_0^T \bigg(\int_s^T e^{-\beta(u-s)}\, du \bigg)^2 ds\,,$$

$$\sigma_\zeta^2 = V\bigg(\int_0^{T'} r_u\, du \bigg) = \gamma^2 \int_0^{T'} \bigg(\int_s^{T'} e^{-\beta(u-s)}\, du \bigg)^2 ds\,,$$

$$\rho_{\xi\zeta} = Cov\bigg(r_{T'},\, \int_0^{T'} r_u\, du \bigg) = \gamma^2 \int_0^{T'} e^{-\beta(T'-s)} \int_s^{T'} e^{-\beta(u-s)}\, du\, ds\,,$$

$$\rho_{\xi\eta} = Cov\bigg(r_{T'},\, \int_0^{T} r_u\, du \bigg) = \rho_{\xi\zeta} + \sigma_\xi^2 \int_{T'}^{T} e^{-\beta(u-T')}\, du\,.$$

Thus

$$C(T',T) = E\Big(I_{\{\omega:\, \xi \le r'\}}\; e^{-\eta} \Big) - K\, E\Big(I_{\{\omega:\, \xi \le r'\}}\; e^{-\zeta} \Big)$$

$$= \exp\left\{ \frac{\sigma_\eta^2}{2} - \mu_\eta \right\} \Phi\left(\frac{r' - (\mu_\xi - \rho_{\xi\eta})}{\sigma_\xi} \right)$$

$$-K\, \exp\left\{ \frac{\sigma_\zeta^2}{2} - \mu_\zeta \right\} \Phi\left(\frac{r' - (\mu_\xi - \rho_{\xi\zeta})}{\sigma_\xi} \right).$$

Substituting expressions for μ_ξ, μ_η, μ_ζ, σ_ξ^2, σ_η^2, σ_ζ^2, $\rho_{\xi\zeta}$ and $\rho_{\xi\eta}$ into the latter formula, gives us the final expression for the price $C(T',T)$.

Using the observation

$$\big(K - B(T',T)\big)^+ = \big(B(T',T) - K\big)^+ - B(T',T) + K\,,$$

we compute the price of a European put option in a $\big(B_t, B(t,T)\big)$-market:

$$P(T',T) = K\,B(0,T')\,\Phi(-d_-) - B(0,T)\,\Phi(-d_+)\,.$$

Now we discuss one of the approximation methods for pricing such assets with the fixed income. Consider a zero-coupon bond with face value 1 and terminal date $T = 1$ (say, year). For simplicity, suppose that $P^* = P$ (i.e., the initial probability is a martingale probability; see for example the Vasiček model). The bond price is given by

$$B(t,T) = E\bigg(\exp\Big\{-\int_t^T r_s\,ds\Big\}\Big|\,\mathcal{F}_t\bigg)\,.$$

In our case $t = 0$ and $T = 1$, hence

$$B(0,1) = E\bigg(r_0\,\exp\Big\{-\int_0^1 r_s\,ds\Big\}\Big|\,\mathcal{F}_0\bigg) = E\bigg(\exp\Big\{\ln r_0 - \int_0^1 r_s\,ds\Big\}\bigg)\,.$$

Suppose that the evolution of the interest rate is described by

$$r_t = r_0\,e^{at+\sigma w_t} = e^{\ln b + Y_t} = b\,e^{Y_t}\,,$$

where

$$r_0 = b \qquad \text{and} \qquad Y_t = a\,t + \sigma\,w_t\,.$$

Our further discussion is based on the following methodology, which can be found, for example, in [4]. Let $f = f(x),\ x \in \mathbb{R}$, be a convex function, $(\xi_s)_{0\le s\le 1}$ be a Gaussian process, $X := \int_0^1 e^{\xi_s}\,ds$ and $\xi \sim \mathcal{N}(0,1)$.

From Jensen's inequality we have

$$E\big(f(X)\big) = E\Big(E\big(f(X)\big)\big|\xi\Big) \ge E\Big(f\big(E(X|\xi)\big)\Big)\,.$$

Now choose

$$f(x) = e^{-bx}$$

and

$$\xi = \int_0^1 w_s\,ds\bigg/\sqrt{V\Big(\int_0^1 w_s\,ds\Big)}\,,$$

which is clearly a Gaussian random variable. Then

$$\begin{aligned} E(\xi) &= E\bigg(\int_0^1 w_s\,ds\bigg/\sqrt{V\Big(\int_0^1 w_s\,ds\Big)}\bigg) \\ &= \int_0^1 E(w_s)\,ds\bigg/\sqrt{V\Big(\int_0^1 w_s\,ds\Big)} = 0\,. \end{aligned}$$

Using the Kolmogorov-Itô formula we write

$$\left(\int_0^1 w_s\,ds\right)^2 = 2\int_0^1\int_0^t w_s\,ds\,d\Big(\int_0^t w_u\,du\Big) = 2\int_0^1\int_0^t w_s\,ds\,w_t\,dt$$
$$= 2\int_0^1\int_0^t w_t\,w_s\,ds\,dt = 2\int_0^1\int_0^t \big[w_s + (w_t - w_s)\big]\,w_s\,ds\,dt\,.$$

Since increments of w are independent and $V(w_t) = t$, then

$$E\Big(\int_0^1 w_s\,ds\Big)^2 = 2\int_0^1\int_0^t E\big(w_s^2\big)\,ds\,dt = 2\int_0^1\int_0^t s\,ds\,dt$$
$$= \int_0^1 t^2\,dt = \frac{1}{3}\,.$$

Using the Theorem on normal correlation (see [41]) we can write

$$E\big(Y_t|\xi\big) = a\,t + k_t\,\xi\,,$$

where

$$k_t = Cov(Y_t,\,\xi) = \sqrt{3}\,\sigma\,Cov\Big(w_t,\,\int_0^1 w_s\,ds\Big) = \sqrt{3}\,\sigma\int_0^1 (1-s)\,ds$$
$$= \sqrt{3}\,\sigma\left(t - \frac{t^2}{2}\right).$$

Also

$$V\big(Y_t\big|\,\xi\big) = V\big(Y_t\big) - k_t^2 = \sigma^2\left(t - 3\,t^2 + 3\,t^3 - 3\,t^4/4\right) = \nu_t\,,$$
$$Cov\big(Y_t\,Y_s\big|\,\xi\big) = \sigma^2\,\min\{t,s\} - k_t\,k_s = \nu_{ts}\,.$$

Now consider

$$h(\xi) = E\Big(\int_0^1 e^{Y_s}\,ds\Big|\,\xi\Big) = \int_0^1 e^{as + k_s\xi + \nu_s/2}\,ds\,.$$

Computing

$$LB_1 = \int_{-\infty}^{\infty} h(z)\,\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz\,,$$

gives us the lower estimate for the bond price.

To find the upper estimate UB_1 we note that there exists a random variable η such that

$$E\big(f(X)\big) = E\Big(f\big(E(X|\xi)\big)\Big) + E\Big(\big[X - E(X|\xi)\big]\,f'\big(E(X|\xi)\big)\Big)$$
$$+\frac{1}{2}\,E\Big(\big[X - E(X|\xi)\big]^2\,f''(\eta)\Big)\,.$$

This implies the estimates

$$E\big(f(X)\big) \le f\big(E(X|\xi)\big) + \frac{1}{2}\,E\Big(\big[X - E(X|\xi)\big]^2\, f''(\eta)\Big)$$

and

$$E\big(f(X)\big) \le E\Big(f\big(E(X|\xi)\big)\Big) + \frac{1}{2}\,E\Big(\big[X - E(X|\xi)\big]^2 \sup_x f''(x)\Big)\,.$$

Thus

$$UB_1 = LB_1 + \frac{1}{2}\,c^2\,E\Big(V(X|\xi)\Big)\,,$$

where

$$c^2 = \sup_x f''(x)\,.$$

One can compute LB_1 using standard approximation methods for computing integrals. Thus this methodology allows one to approximate bond prices and to compute the corresponding error estimates.

This methodology can also be used for computing prices of options. For example, for a European call option we have

$$f(x) = \big(e^{-bx} - K\big)^+\,,$$

and one has to approximate

$$LB_2 = \int_{-\infty}^{\infty} f\big(h(z)\big)\,\frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz\,.$$

2.8 Real options: pricing long-term investment projects

Long-term investment projects play a significant role in modern economy. Development of a new enterprise is a typical example of such a project. A company that plans an investment of this type is often **not obliged** to realize the project. In this sense such investment activities are similar to a call option on a financial asset. In both cases an investor has the **right** to gain some outcomes of a project in return for invested capital (e.g., buy shares at a strike price). Such investment programs in 'real economy' are referred to as *real options* (see, for example, [13]).

This similarity suggests that methods of managing risk related to contingent claims may be helpful in managing risk related to long-term investment projects.

Let us consider a project with a fixed implementation date T. As before, we will use the notion of a *basic asset*, which represents the expected result of the project. Let S_t be its price, then it is natural to expect that the price of the project is given by some function $F(S_T)$. Clearly, this quantity must reflect the discounted yield generated by the basic asset S.

Studying the *profitability* of the investment project is essential for making a decision about its realization. If I is a fixed capital of a proposed investment, then it must be compared with some *level of profitability* R, which depends on $F(S_T)$:

$$\begin{array}{lll} \text{If} & I \le R\,, & \text{then the project is accepted for realization;} \\ \text{If} & I > R\,, & \text{then the project is rejected.} \end{array}$$

How to find sensible values of R? If evolution of the basic asset is deterministic, then its price can be written in the form

$$S_t^{\text{det}} = \exp\{s_t\}\,,$$

where s_t is a deterministic function of $t \in [0, T]$. If r is the rate of interest, then the level of profitability can be defined as

$$R = R_0^{\text{det}} = e^{-rT}\,F\big(S_T^{\text{det}}\big)\,.$$

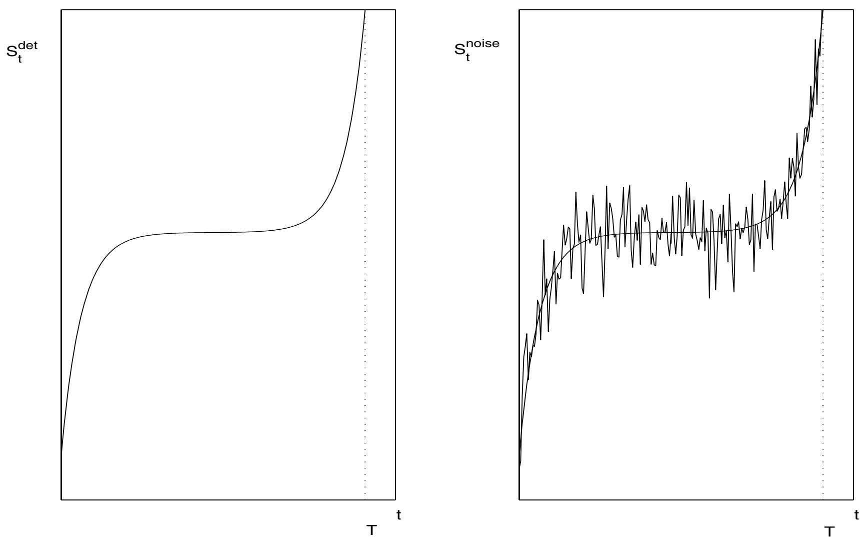
If evolution of price of S is not deterministic, then we model it in terms of S_t^{det} perturbed by a Gaussian white noise with mean zero and variance σ^2. Then the expectation of price S_t^{noise} will coincide with price's deterministic component:

$$E\big(S_t^{\text{noise}}\big) = S_t^{\text{det}} = \exp\{s_t\}\,, \quad S_0 = 1\,.$$

The evolution of prices is given by

$$S_t^{\text{noise}} = S_t^{\text{det}}\,\exp\left\{\sigma\,w_t - \frac{\sigma^2}{2}\,t\right\} = \exp\left\{s_t + \sigma\,w_t - \frac{\sigma^2}{2}\,t\right\},$$

where w is a Wiener process.



Then it is natural to define

$$
\begin{aligned}
R_\sigma^{\text{noise}} &= E\Big(e^{-rT}\,F\big(S_T^{\text{det}}\big)\Big) \\
&= e^{-rT}\int_{-\infty}^{\infty} F\Big(S_T^{\text{det}}\,\exp\Big\{\sqrt{T}\,\sigma\,x-\frac{\sigma^2}{2}\,T\Big\}\Big)\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F\Big(\exp\Big\{s_T+\sqrt{T}\,\sigma\,x-\frac{\sigma^2}{2}\,T\Big\}\Big)\,e^{-x^2/2}\,dx \\
&\to R_0^{\text{det}} \quad \text{as} \quad \sigma\to 0\,.
\end{aligned}
$$

Alternatively, one can define R as the expectation of $F\big(S_T\big)$ with respect to a risk-neutral probability P^* (see [26]) whose density with respect to P is

$$
Z_T^* = \exp\Big\{\frac{r}{\sigma}\,w_T-\frac{1}{2}\Big(\frac{r}{\sigma}\Big)^2\,T\Big\}\,.
$$

By Girsanov theorem, the process

$$
w_t^* = w_t - \frac{r}{\sigma}\,t
$$

is a Wiener process with respect to P^*. Thus, we obtain another value of R:

$$
\begin{aligned}
R = R^* &= E^*\Big(e^{-rT}\,F\big(S_T\big)\Big) = E^*\Big(e^{-rT}\,F\Big(\exp\Big\{s_T+\sigma\,w_T-\frac{\sigma^2}{2}\,T\Big\}\Big)\Big) \\
&= e^{-rT}\,E^*\Big(F\Big(e^{s_T+rT}\,\exp\Big\{\sigma\,w_T-r\,T-\frac{\sigma^2}{2}\,T\Big\}\Big)\Big) \\
&= e^{-rT}\,E^*\Big(F\Big(e^{s_T+rT}\,\exp\Big\{\sigma\,\Big(w_T-\frac{r}{\sigma}\,T\Big)-\frac{\sigma^2}{2}\,T\Big\}\Big)\Big) \\
&= e^{-rT}\,E^*\Big(F\Big(e^{s_T+rT}\,\exp\Big\{\sigma\,w_T^*-\frac{\sigma^2}{2}\,T\Big\}\Big)\Big) \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F\Big(\exp\Big\{s_T+r\,T+\sqrt{T}\,\sigma\,x-\frac{\sigma^2}{2}\,T\Big\}\Big)\,e^{-x^2/2}\,dx\,.
\end{aligned}
$$

In some types of long-term investment projects it is natural to assume that the total value of an investment and the implementation date are not known in advance. For example, investments in scientific research or in the production of energy are projects of this type.

Suppose we know the final cost of the basic asset, and let $X_t,\ t\geq 0$ be the amount of capital necessary for completion of the project. As a particular example, consider the *Pindyck model*, where random process X satisfies the following stochastic differential equation

$$
dX_t = -\alpha_t\,dt+\beta\,\sqrt{\alpha_t\,X_t}\,dw_t\,, \quad X_0 = x\,,
$$

where $\beta > 0$ and α_t is the intensity of the investment flow.

Since investment potential is often limited, and it is not possible to reverse the investment flow, then it is natural to assume that $\alpha = (\alpha_t)$ is a bounded random variable. For simplicity, say $\alpha_t \in [0,1]$. Process α plays the role of control in the process of spending the investment capital $X_t = X_t^\alpha$. Choosing α from the set of all admissible processes

$$\daleth = \Big\{\alpha : \ \alpha_t \in [0,1]\Big\}$$

implies defining a natural implementation time

$$\tau = \tau^\alpha = \inf \Big\{t : \ X_t = X_t^\alpha = 0\Big\}$$

for a project.

If V is the final cost of the project and r is the rate of interest, then the quantity

$$V\, e^{-r\tau} - \int_0^\tau \alpha_t\, e^{-rt}\, dt$$

represents the profit gained by choosing the investment strategy α. The average profit is given by

$$v^\alpha(x) = E_x\Big(V\, e^{-r\tau} - \int_0^\tau \alpha_t\, e^{-rt}\, dt\Big)\,,$$

where notation E_x for mathematical expectation indicates that the initial investment x was necessary for completion of this project.

Since all control strategies belong to the class $\daleth$, it is natural to define the *optimal strategy* α^* from

$$v(x) \equiv \sup_{\alpha \in \daleth} v^\alpha(x) = v^{\alpha^*}(x)\,. \tag{2.15}$$

Problems of this type are usually solved by the method of *dynamic programming*, where one of the main tools is the *Bellman principle*. Here we briefly sketch this method (for more details see [23]).

Suppose that the controlled process $X_t = X_t^\alpha$ satisfies the stochastic differential equation

$$dX_t \equiv dX_t^\alpha = b_\alpha(X_t^\alpha)\, dt + \sigma_\alpha(X_t^\alpha)\, dw_t\,, \quad X_0 = x\,,$$

where b_α and σ_α are some reasonable functions (for example, satisfying Lipschitz condition), and α is a control process that is adapted to a σ-algebra generated by X_t.

For estimating the quality of control α we introduce a function $f^\alpha(x)$, $\alpha \in [0,1]$, $x \in \mathbb{R}$, which is interpreted as the intensity of the profit flow. Then the total profit on interval $[0,t]$ is equal to

$$\int_0^t f^\alpha(X_s^\alpha)\, ds\,.$$

Denoting

$$v^\alpha(x) = E_x\Big(\int_0^\infty f^\alpha(X_s)\, ds\Big)$$

its expectation on $[0,\infty)$, we will find the optimal control α^* from condition (2.15):

$$v(x) \equiv \sup_{\alpha \in \beth} v^\alpha(x) = v^{\alpha^*}(x) .$$

We use the *Bellman principle*:

$$v(x) = \sup_{\alpha \in \beth} E_x \bigg(\int_0^t f^\alpha(X_s^\alpha)\, ds + v(X_t^\alpha) \bigg) , \quad t > 0 , \tag{2.16}$$

for determining the *price* $v(x)$. We briefly explain the motivation for using it. Let us write the total profit of using strategy α in the form

$$\int_0^\infty f^\alpha(X_s)\, ds = \int_0^t f^\alpha(X_s)\, ds + \int_t^\infty f^\alpha(X_s)\, ds .$$

If this strategy was used only up to time t, then the first term in the right-hand side represents the profit on interval $[0,t]$. Suppose the controlled process has value $y = X_t$ at time t. If we wish to alter the control process after time t with the aim of maximizing the profit over the whole of $[0,\infty)$, then we have to maximize the expectation

$$E_y \bigg(\int_t^\infty f^\alpha(X_s)\, ds \bigg) ,$$

where α also denotes the continuation of the control process to $[t,\infty)$. Changing variable $s = t+u,\ u \geq 0$, and using independence and stationarity of increments of the Wiener process, we obtain

$$E_{X_t} \bigg(\int_t^\infty f^\alpha(X_s)\, ds \bigg) = v^\alpha(X_t) \leq v(X_t) ,$$

Thus, a strategy that is optimal after time t, gives the average profit such that

$$E_x \bigg(\int_0^t f^\alpha(X_s^\alpha)\, ds + v(X_t^\alpha) \bigg) \geq v^\alpha(x) .$$

One can choose $\alpha_s,\ s \geq t$, so that the corresponding profit is close enough to the average profit. Hence, taking supremum of both sides of the latter inequality yields the Bellman principle (2.16). If we *a'priori* assume that the *Bellman function* is smooth enough, then the Bellman principle can be written in the following differential form

$$v(X_t^\alpha) = v(x) + \int_0^t \left[\frac{\partial v}{\partial x}\, b_\alpha(X_s^\alpha) + \frac{1}{2}\, \frac{\partial^2 v}{\partial x^2}\, \sigma_\alpha^2(X_s^\alpha) \right] ds$$
$$+ \int_0^t \frac{\partial v}{\partial x}\, \sigma_\alpha(X_s^\alpha)\, dw_s ,$$

where Kolmogorov-Itô formula was used. Since the last term in the right-hand side is a martingale, then we obtain

$$
\begin{aligned}
v(x) &= \sup_{\alpha\in\daleth} E_x\biggl(\int_0^t f^\alpha(X_s^\alpha)\,ds + v(X_t^\alpha)\biggr)\\
&= \sup_{\alpha\in\daleth} E_x\biggl(\int_0^t \left[\frac{\partial v}{\partial x}\,b_\alpha(X_s^\alpha) + \frac{1}{2}\,\frac{\partial^2 v}{\partial x^2}\,\sigma_\alpha^2(X_s^\alpha) + f^\alpha(X_s^\alpha)\right] ds + v(x)\biggr).
\end{aligned}
$$

Hence

$$
\sup_{\alpha\in\daleth}\Bigl[L_\alpha\, v(x) + f^\alpha(x)\Bigr] = 0\,,
$$

where

$$
L_\alpha\, v = \frac{\partial v}{\partial x}\,b_\alpha + \frac{1}{2}\,\frac{\partial^2 v}{\partial x^2}\,\sigma_\alpha^2\,.
$$

The latter relation is usually referred to as *Bellman differential equation.*

Note that the considered investment problem controls process X_t^α only up to the time

$$
\tau = \tau_D^\alpha
$$

of its exit from region D. Thus this problem can be written in the following general form

$$
v(x) = \sup_{\alpha\in\daleth} E_x\biggl(\int_0^{\tau_D^\alpha} f^\alpha(X_s^\alpha)\,e^{-rs}\,ds + g(X_{\tau_D^\alpha}^\alpha)\,e^{-r\tau_D^\alpha}\biggr)\,,
$$

where $g = g(x)$ is some function defined on the boundary ∂D of set D. In this case we again arrive at a Bellman differential equation

$$
\sup_{\alpha\in\daleth}\left[\frac{1}{2}\,\frac{\partial^2 v}{\partial x^2}\,\sigma_\alpha^2(x) + \frac{\partial v}{\partial x}\,b_\alpha(x) - r\,v(x) + f^\alpha(x)\right] = 0\,,
$$

which is satisfied by Bellman function v for sufficiently wide class of coefficients $b_\alpha(x)$ and $\sigma_\alpha(x),\ \alpha\in[0,1],\ x\in\mathbb{R}$.

Consider again an investment problem in Pindyck model (see [36]). We note that

$$
f^\alpha(x) = \alpha,\quad g(x) = V,\quad D = \{x:\ x>0\}\,.
$$

Then Bellman differential equation has the form

$$
r\,v(x) = \sup_{\alpha\in\daleth}\left[-\alpha - \alpha\,\frac{\partial v}{\partial x} + \frac{\beta^2}{2}\,\alpha\,x\,\frac{\partial^2 v}{\partial x^2}\right],
$$

or, taking into account linearity in α

$$
r\,v(x) = \begin{cases} -1 - \frac{\partial v}{\partial x} + \frac{\beta^2}{2}\,x\,\frac{\partial^2 v}{\partial x^2} & \text{if } -1 - \frac{\partial v}{\partial x} + \frac{\beta^2}{2}\,x\,\frac{\partial^2 v}{\partial x^2} > 0\\ 0 & \text{otherwise}\,. \end{cases}
$$

It is clear that the investment strategy

$$\alpha_t^* = \begin{cases} 1 & \text{if } X_t < x^* \\ 0 & \text{if } X_t \geq x^* \,, \end{cases}$$

where x^* is a solution of $v(x^*) = 0$, is a candidate for being optimal.

Consider differential equation

$$r\, v(x) = -1 - \frac{\partial v}{\partial x} + \frac{\beta^2}{2}\, x\, \frac{\partial^2 v}{\partial x^2} \,.$$

Its general solution has the form

$$v(x) = c_1\, x^{\nu/2}\, J_\nu\big(2 - \sqrt{b\, x}\big) + c_2\, x^{\nu/2}\, H_\nu^1\big(2\sqrt{-b\, x}\big) + (1-\nu)\, b \,,$$

where $\nu = 1 + 2/\beta^2$, $b = 2\, r/\beta^2$,

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, (x/2)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)}$$

is Bessel function of the first kind, Γ is gamma-function and $H_\nu^{(1)}$ is Hankel function of the first kind.

This solution can be also written in terms of modified Bessel functions:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)} \,,$$

$$K_\nu(x) = \frac{\pi}{2}\, \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)} \,, \qquad \nu \notin \mathbb{Z} \,,$$

$$K_n(x) = (-1)^{n+1}\, I_n(x)\, \ln(x/2) + \frac{1}{2} \sum_{k=0}^{n-1} \frac{(-1)^k\, (n-k-1)!}{k!}\, (x/2)^{2k-n}$$

$$+ \frac{(-1)^n}{2} \sum_{k=0}^{n} \frac{(x/2)^{2k+n}}{k!\, (n+k)!} \Big[\Psi(n+k+1) + \Psi(k+1)\Big] \,, \quad n \in \mathbb{Z} \,,$$

where Ψ is the logarithmic derivative of Γ. We have

$$v(x) = c_1\, (-1)^{\nu/2}\, x^{\nu/2}\, I_\nu\big(2\sqrt{b\, x}\big) + c_2\, \frac{2}{\pi}\, (-1)^{(\nu+1)/2}\, x^{\nu/2}\, K_\nu\big(2\sqrt{b\, x}\big) + (1-\nu)/b \,.$$

Since

$$x^{\nu/2}\, I_\nu\big(2\sqrt{b\, x}\big) = x^{\nu/2} \sum_{k=0}^{\infty} \frac{(b\, x)^{\nu/2+k}}{k!\, \Gamma(\nu+k+1)} \to 0 \quad \text{as } x \to 0 \,,$$

then the initial condition $v(0) = V$ allows to compute

$$c_2 = \begin{cases} \sin(\pi\nu)\, \Gamma(1-\nu)\, b^{\nu/2}\, (-1)^{-(\nu+1)/2} \left(V + \frac{1}{r}\right) & \text{if } \nu \notin \mathbb{Z} \\ \frac{\pi}{(n-1)!}\, b^{\nu/2}\, (-1)^{-(\nu+1)/2} \left(V + \frac{1}{r}\right) & \text{if } \nu \in \mathbb{Z} \,. \end{cases}$$

Note that we are solving a problem with an unknown boundary. In the theory of differential equations, such problems are referred to as Stefan problems. The methodology of dealing with such problems involves the ideas of continuity and smooth gluing on the boundary $x = x^*$:

$$v(x^*) = 0 \qquad \text{and} \qquad v'(x^*) = 0\,.$$

This implies

$$c_1 = (-1)^{-\nu/2}\,\frac{K_{\nu-1}\big(2\sqrt{b\,x^*}\big)}{I_{\nu-1}\big(2\sqrt{b\,x^*}\big)}\,,$$

$$v(x) = \frac{K_{\nu-1}\big(2\sqrt{b\,x^*}\big)}{I_{\nu-1}\big(2\sqrt{b\,x^*}\big)}\,x^{\nu/2}\,I_\nu\big(2\sqrt{b\,x}\big) + c\,K_\nu\big(2\sqrt{b\,x}\big) + (1-\nu)/b\,,$$

where $c = \frac{2}{\pi}\,c_2$.

Now we have to check that the constructed function v and control α^* indeed solve the initial investment problem. The 'verification conditions' in this case are

1) $v^\alpha(x) \le v(x)$ for any α and x;

2) $v^{\alpha^*}(x) = v(x)$ for $x \ge 0$.

Here is the sketch of this verification. From the properties of Bessel functions we have that the solution to

$$\frac{\beta^2}{2}\,x\,\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} - r\,v - 1 = 0$$

is a smooth function. Also

$$\frac{\beta^2}{2}\,\alpha\,x\,\frac{\partial^2 v}{\partial x^2} - \alpha\,\frac{\partial v}{\partial x} - r\,v - \alpha \le 0$$

for $\alpha \in [0,1]$. Further, using the Kolmogorov-Itô formula, we have

$$\begin{aligned} e^{-r(t\wedge\tau)}\,v\big(X_{t\wedge\tau}\big) &= v(x) + \int_0^{t\wedge\tau} e^{-rs}\,\beta\,\sqrt{\alpha_s\,X_s}\,v(X_s)\,dw_s \\ &\quad + \int_0^{t\wedge\tau} e^{-rs}\left[\frac{\beta^2}{2}\,\alpha_s\,X_s\,\frac{\partial^2 v}{\partial x^2}(X_s) - \alpha_s\,\frac{\partial v}{\partial x}(X_s) - r\,v(X_s)\right] ds \\ &\le v(x) + \int_0^{t\wedge\tau} e^{-rs}\,\beta\,\sqrt{\alpha_s\,X_s}\,v(X_s)\,dw_s + \int_0^{t\wedge\tau} e^{-rs}\,\alpha_s\,ds\,. \end{aligned}$$

Taking expectations and using the martingale property of stochastic integrals, we obtain

$$v(x) \ge E_x\Big(e^{-r(t\wedge\tau)}\,v\big(X_{t\wedge\tau}\big)\Big) - E_x\Big(\int_0^{t\wedge\tau} e^{-rs}\,\alpha_s\,ds\Big)$$

and hence

$$v(x) \geq v^{\alpha}(x)$$

due to convergence

$$E_x\Big(e^{-r(t\wedge\tau)}\, v\big(X_{t\wedge\tau}\big)\Big) \to V\, e^{-r\tau} \qquad \text{as} \quad t \to \infty\,.$$

Establishing second verification property, we note that it clearly holds true for $X_t \geq x^*$. For $X_t < x^*$ we use the Kolmogorov-Itô formula:

$$\begin{aligned}
v(x) &= E_x\Big(e^{-r(t\wedge\tau)}\, v\big(X_{t\wedge\tau}\big)\Big) \\
&\qquad + E_x\bigg(\int_0^{t\wedge\tau} e^{-rs}\,\beta\,\sqrt{\alpha_s\, X_s}\, v(X_s)\, dw_s\bigg) \\
&\qquad + E_x\bigg(\int_0^{t\wedge\tau} e^{-rs}\left[\frac{\beta^2}{2}\,\alpha_s\, X_s\,\frac{\partial^2 v}{\partial x^2}(X_s) - \alpha_s\,\frac{\partial v}{\partial x}(X_s) - r\, v(X_s)\right] ds\bigg) \\
&= E_x\Big(e^{-r(t\wedge\tau)}\, v\big(X_{t\wedge\tau}\big)\Big) \\
&\qquad + E_x\bigg(\int_0^{t\wedge\tau} e^{-rs}\left[\frac{\beta^2}{2}\,\alpha_s\, X_s\,\frac{\partial^2 v}{\partial x^2}(X_s) - \alpha_s\,\frac{\partial v}{\partial x}(X_s) - r\, v(X_s)\right] ds\bigg)\,.
\end{aligned}$$

Passing to the limit as $t \to \infty$ and choosing $\alpha = \alpha^*$ completes the verification.

Finally, we note that the existence of x^* as a solution to $v(x^*) = 0$, follows from analyzing this equation with the help of the following asymptotic representations of the modified Bessel functions:

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\Big(1 + \mathcal{O}\big(1/x\big)\Big) \quad \text{and} \quad K_\nu(x) = \frac{\pi}{\sqrt{2x}}\, e^{-x}\Big(1 + \mathcal{O}\big(1/x\big)\Big)$$

as $x \to \infty$.

2.9 Technical analysis in risk management

The study of *market action* is an essential part of analysis of financial markets. The collection of methods and tools of qualitative analysis of market prices forms an important part of modern *financial engineering* and is usually referred to as *technical analysis*. Recent developments in financial mathematics provide significant theoretical support to empirical methodologies of technical analysis, and hopefully will encourage development of new trends in this area.

Technical analysts believe that market prices depend on psychology of market participants, and therefore various types of financial information are often used in technical analysis. Forecasting future price *trends* is the major goal of technical analysis. All relevant current information is represented in the form of *indicators* and expressed in graphs, mnemonic rules and mathematical functions.

To make an informed investment decision, one has to identify the most probable *trends* in the market, estimate effectiveness of operations and risk of having losses, determine volumes of transactions given information on the liquidity of stocks taking into account transaction costs and other factors.

Charts are traditional forms of visualization of dynamics of prices and indices. The most widely used forms of charts are *bar charts* and (Japanese) *candlestick charts*. For example, the candlestick consists of a line that represents the price range from the low to the high, and of a rectangle that measures the difference between open and close prices: it is white if the close price is higher than the open price, and it is black otherwise.

The most important elements of a chart are *trend lines, support lines* and *resistance lines*. Uptrends have ascending sequences of local maximums and minimums, downtrends correspond to descending sequences, and sideways trends correspond to constant sequences. *Support* is represented by a horizontal line that indicates the level from which prices start growing. It 'supports' the graph of the price trend from below. *Resistance* is represented by a horizontal line that bounds the graph of the price trend from above. It indicates the price level when selling pressure overcomes buying pressure and prices start going down.

Support and resistance lines can move up and down, which corresponds to increasing or decreasing price trends. It is extremely important to identify the moments when a trend line breaks, i.e., becomes decreasing after being increasing or vice versa, since most financial gains and losses happen at such moments.

More complex patterns on charts are usually described in terms of *figures*. The most popular are *head and shoulders*, various types of *triangles* and *flags* (see, for example, [34]).

One of most essential axioms of technical analysis is that prices 'remember' their past. This makes the concept of trend the key element of technical analysis: one has to identify trends in the appropriately chosen past and use them for forecasting future prices.

Quantitative realization of these ideas is given by indicators. One of the most popular indicators is *moving average*, whose simplest and most commonly used version is defined by

$$\frac{S_{t-n} + \ldots + S_{t+n}}{2n+1},$$

where t is the current time, n defines time horizon, S_t is the price of stock S at time t.

Moving average is widely used in identifying trends, in making decisions about buying or selling stock and in constructing other indicators. If the stock's price moves above moving average, then it is recommended to buy this stock, and to sell otherwise. Thus, moving average is designed to keep one's position in the boundaries of the main trend, and parameter n must correspond to the length of the market cycle.

Another important indicator is *divergence*. Fluctuation of prices reflects the instability of the market and is represented by a sequence of rises and drops. It is essential to determine as quickly as possible which of the rises or drops indicates a change in the main trend. If the price line reaches its new peak but the indicator does not, this indicates that the market activity is becoming slow and is called *bearish divergence* (or negative divergence). The symmetric *bullish divergence* (or positive divergence) corresponds to a situation when prices continue to drop but the indicator does not.

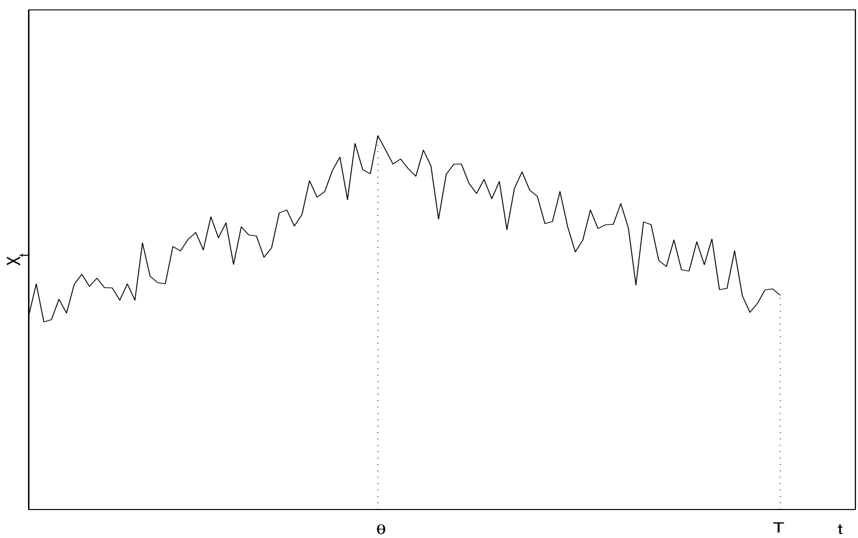
Technical analysis of various averaging indicators and individual stock prices is often complemented by the study of trading *volumes*. Volume-based indicators are based on the hypothesis that changes in trading volumes precede changes of prices. Thus, observation of a *change point* in the dynamics of a volume indicator can be naturally interpreted as a change in the price trend. One of the key indicators here is called the *accumulation-distribution* indicator, which is defined by the formula

$$\frac{S_2 - S_1}{\max S - \min S}\, V + I\,,$$

where S_1 and S_2 are open and close prices, $\max S$ and $\min S$ are price's maximum and minimum taken over a specified period of time, V is trading volume and I is the previous value of indicator.

We can summarize that one of the key problems of technical analysis consists in detection of change points in price trends. We will use quantitative methods for dealing with this problem, for which we need to introduce some notions and assumptions.

Let a stochastic process $X = \big(X_t\big)_{t\in[0,T]}$ represent the evolution of prices. We wish to identify a moment of time θ when process X changes its probabilistic characteristics. This can be a point in $[0,T]$ where X attains its maximum, i.e., prices change the ascending tendency to descending.



Then we need to choose a stopping time τ^* adapted to the observed information, such that τ^* is sufficiently close to θ and the values of X at these points are also close in some sense; for example, the variance of the difference $X_\theta - X_{\tau^*}$ is minimal. In 1900 Bachelier suggested that the evolution of prices can be modelled with the help of a standard Wiener process (Brownian motion): $X_t = w_t,\ t \in [0,T]$.

For simplicity, let $T = 1$. We construct an approximation of quantity w_θ using w_τ, where stopping time τ is adapted to filtration $\mathbb{F} = \big(\mathcal{F}_t\big)_{t\ge 0}$ generated by Wiener process w:

$$\mathcal{F}_t = \sigma(w_t), \qquad t \ge 0.$$

Let us introduce the following notation:

$$S_t := \max_{0\le s\le t} w_s\,,$$

and

$$V^* := \inf_\tau E\Big(G\big(S_1 - w_\tau\big)\Big)\,,$$

where G is an observations cost function, and the infimum is taken over all stopping times τ, i.e., over all random variables such that

$$\{\omega :\ \tau \le t\} \in \mathcal{F}_t \quad \text{for all} \quad t \ge 0\,.$$

Our aim is to find an optimal stopping time τ^*, so that

$$V^* = E\Big(G\big(S_1 - w_\tau^*\big)\Big)\,.$$

The existence and structure of quantities τ^* and V^* is given by the following theorem [43].

THEOREM 2.7

For the cost function $G(x) = x^2$ the optimal stopping time τ^* is defined by the formula

$$\tau^* = \inf\big\{t \le 1 :\ S_t - w_t \ge z^*\sqrt{1-t}\ \big\}\,.$$

Here $z^* \approx 1.12$ is a solution to equation

$$4\,\Phi(z) - 2\,z\,\phi(z) - 3 = 0\,,$$

where

$$\Phi(z) = \int_{-\infty}^{z} \phi(x)\,dx \qquad \textit{and} \qquad \phi(z) = \frac{1}{\sqrt{2\pi}}\,e^{-z^2/2}\,.$$

In this case

$$V^* = 2\,\Phi(z^*) - 1 \approx 0.73\,.$$

REMARK 2.2

1. The quantitative characteristics of the optimal stopping time are

$$E(\tau^*) = \frac{z^{*\,2}}{1+z^{*\,2}} \approx 0.55\,,$$
$$V(\tau^*) = \frac{2\,z^{*\,4}}{(1+z^{*\,2})\,(3+6\,z^{*\,2}+z^{*\,4})} \approx 0.05\,.$$

2. For an arbitrary time interval $[0,T]$ the optimal stopping time and price are given by

$$\tau^*(T) = \inf\big\{t \le T:\ S_t - w_t \ge z^*\,\sqrt{T-t}\,\big\} \quad \text{and} \quad V^*(T) = V^*\,T\,,$$

respectively.

∎

PROOF (of Theorem 2.7) From the strong Markov property of the Wiener process, for any stopping time τ and for any cost function $G = G(x)$ with $E\big(|G(S_1 - w_t)|\big) < \infty$, we have

$$\begin{aligned}
E\Big(G(S_1 - w_\tau)\big|\mathcal{F}_\tau\Big) &= E\Big(G\big(\max\big\{\max_{u\le\tau} w_u,\ \max_{\tau<u\le 1} w_u\big\} - w_\tau\big)\Big|\mathcal{F}_\tau\Big)\\
&= E\Big(G\big(\max\big\{s,\ \max_{0\le\tau\le 1-t} w_\tau + x\big\} - x\big)\Big)\Big|_{x=w_\tau,s=S_\tau,t=\tau}\\
&= E\Big(G\big(\max\big\{s,\ \eta + x\big\} - x\big)\Big)\Big|_{x=w_\tau,s=S_\tau,t=\tau}\,,
\end{aligned}$$

where random variable

$$\eta = \max_{0\le\tau\le 1-t} w_\tau$$

has the following distribution

$$dF_\eta(t,y) = 2\,\phi\Big(\frac{y}{\sqrt{1-t}}\Big)\,\frac{dy}{\sqrt{1-t}}\,.$$

Thus

$$\begin{aligned}
E\Big(G(S_1 - w_\tau)\big|\mathcal{F}_\tau\Big) &= G(s-x)\,F_\eta(t,s-x) + \int_{s-x}^{\infty} G(y)\,dF_\eta(t,y)\\
&= G(s-x) + \int_{s-x}^{\infty}\Big[G(y) - G(s-x)\Big]\,dF_\eta(t,y)\,,
\end{aligned}$$

where $x = w_\tau,\ s = S_\tau,\ t = \tau$.

∎

Using this representation, we compute

$$V^* = \inf_\tau E\Big(G(S_1 - w_\tau)\Big)$$

$$\inf_\tau E\left(G(S_\tau - w_\tau)\left[2\,\Phi\left(\frac{S_\tau - w_\tau}{\sqrt{1-\tau}}\right) - 1\right] + 2\int_{\frac{S_\tau - w_\tau}{\sqrt{1-\tau}}}^{\infty} G(z\sqrt{1-\tau})\,\phi(z)\,dz\right).$$

In particular, for $G(x) = x^2$ we have

$$V^* = \inf_\tau E\left((1-\tau)\left(\frac{S_\tau - w_\tau}{\sqrt{1-\tau}}\right)^2\left[2\,\Phi\left(\frac{S_\tau - w_\tau}{\sqrt{1-\tau}}\right) - 1\right] + 2\int_{\frac{S_\tau - w_\tau}{\sqrt{1-\tau}}}^{\infty} z^2\,\phi(z)\,dz\right).$$

Taking into account that distributions of processes $S - w$ and $|w|$ coincide, we obtain

$$\begin{aligned} V^* &\\ &= \inf_\tau E\left((1-\tau)\left(\frac{|w_\tau|}{\sqrt{1-\tau}}\right)^2\left[2\,\Phi\left(\frac{|w_\tau|}{\sqrt{1-\tau}}\right) - 1\right] + 2\int_{\frac{S_\tau - w_\tau}{\sqrt{1-\tau}}}^{\infty} z^2\,\phi(z)\,dz\right) \\ &= \inf_\tau E\left((1-\tau)\,H_2\left(\frac{|w_\tau|}{\sqrt{1-\tau}}\right)\right), \end{aligned}$$

where $H_2(z) = z^2 + 4\int_z^\infty u\,\big(1 - \Phi(u)\big)\,du$.

Let us introduce new time $s \geq 0$ by

$$1 - t = e^{-2s}\,, \qquad t \in [0,1]\,,$$

then

$$\frac{|w_\tau|}{\sqrt{1-\tau}} = e^s\,w_{1-e^{-2s}} =: Z_s\,.$$

Using the Kolmogorov-Itô formula we represent Z_s in differential form:

$$dZ_s = Z_s\,ds + \sqrt{2}\,d\beta_s\,,$$

where

$$\beta_s = \frac{1}{\sqrt{2}}\int_0^{1-e^{-2s}} \frac{dw_\tau}{\sqrt{1-\tau}}\,, \quad s \geq 0\,,$$

is a new Brownian motion.

Let $Z_0 = Z$ and

$$V^*(z) = \inf_s E\Big(e^{-2s}\,H_2\big(|Z_s|\big)\Big)\,,$$

then $V^*(0)$ is a solution to the initial problem.

The optimization problem for diffusion process Z_s is reduced to the following Stefan problem:

$$L_Z V(z) = 2\,V(z)\,, \qquad z \in (-z^*, z^*)\,, \tag{2.17}$$

$$V(\pm z^*) = H_2(z^*)\,, \quad V'(\pm z^*) = \pm H_2'(z^*)\,,$$

where $L_Z = z\,\frac{d}{dz} + \frac{d^2}{dz^2}$ is generating operator of the diffusion process $Z_s,\ s \geq 0$.

General solution of equation (2.17) is given by

$$V(z) = c_1\, e^{-z^2/2}\, M\big(3/2, 1/2; z^2/2\big) + c_2\, z\, e^{-z^2/2}\, M\big(2, 3/2; z^2/2\big)\,,$$

where

$$M(a, b; x) = 1 + \frac{a}{b}\, x + \frac{a\,(a+1)}{b\,(b+1)}\,\frac{x^2}{2!} + \dots$$

is the hypergeometric Kummer function.

Since $V^*(z)$ is even function, we have $c_2 = 0$. From boundary conditions we also have

$$c_1 = e^{z^{*2}/2}\,\frac{H_2(z^*)}{M\big(3/2, 1/2; z^{*2}/2\big)}\,,$$

where z^* is a strictly positive solution of equation

$$\frac{H_2'(z)}{H_2(z)} + z = 3\,z\,\frac{M\big(5/2, 1/2; z^2/2\big)}{M\big(3/2, 1/2; z^2/2\big)}\,.$$

Thus we obtain the following expression for the price $V^*(z)$:

$$V^*(z) = \begin{cases} H_2(z)\ \exp\left\{\frac{z^{*2} - z^2}{2}\right\} \frac{M\left(3/2, 1/2; z^2/2\right)}{M\left(3/2, 1/2; z^{*2}/2\right)}\,, & \text{if} \quad |z| \leq z^* \\ \\ H_2(|z|)\,, \quad \text{if} \quad |z| \geq z^* \end{cases}.$$

The corresponding stopping time is then defined as the first exit time from the ball

$$\sigma^* = \inf\big\{s \geq 0 :\ |Z_s| \geq z^*\big\}$$

with radius z^*.

The standard verification procedure can be used now to prove that these quantities are optimal. Then we can write solutions to the initial problem. Indeed, changing back to time $t = 1 - e^{-2s}$ and using the facts that $Z_t = w_t/\sqrt{1-t}$ and that the distributions of $S - w$ and $|w|$ coincide, we obtain an expression for the optimal stopping time for the initial problem:

$$\tau^* = \inf\big\{t \leq 1 :\ S_t - w_t \geq z^*\,\sqrt{1-t}\,\big\}\,.$$

Since evolution of prices in the problem of *quickest detection of tendencies* is described in terms of a stochastic process, one can consider a slightly different setting of that problem. Namely, one solves a problem of quickest detection of time when the probabilistic characteristics of the stochastic process change. This problem is referred to as a *change point* problem, and it was introduced by Kolmogorov and Shiryaev.

It is convenient to specify process X in the following way:

$$dX_t = \begin{cases} \sigma\, dw_t\,, & \text{if} \quad t < \theta \\ r\, dt + \sigma\, dw_t\,, & \text{if} \quad t \geq \theta \end{cases}.$$

The stopping time τ adapted to observations of process X can be interpreted as an *alarm time* by considering the following events:

$$\{\omega:\ \tau < \theta\} \qquad \text{and} \qquad \{\omega:\ \tau \geq \theta\}\,,$$

where the first event corresponds to a false alarm, and the second indicates that the change point has been passed and one has to make a decision as promptly as possible. One of the natural criteria for making such a decision can be formulated in the following form: for a fixed $c > 0$,

(a) find

$$V(c) = \inf_{\tau} \Big\{ P\big(\{\omega:\ \tau < \theta\}\big) + c\, E\big((\tau - \theta)^+\big) \Big\},$$

where τ is a stopping time adapted to filtration $\big(\mathcal{F}_t^X\big)$ generated by the observed price process X_t;

(b) find a stopping time τ^* such that

$$V(c) = P\big(\{\omega:\ \tau^* < \theta\}\big) + c\, E\big((\tau^* - \theta)^+\big)\,.$$

This criterium has a clear and natural meaning: the decision to stop is made at a time when the probability of a false alarm and the average delay after the change point θ are minimal.

Suppose that random variable θ has an exponential *a priori* distribution with parameter $\lambda > 0$:

$$P\big(\{\omega:\ \theta = 0\}\big) = \pi \in [0, 1]$$

$$P\big(\{\omega:\ \theta \geq t \,|\, \theta > 0\}\big) = e^{-\lambda t}\,.$$

Posterior distribution of θ is denoted

$$\pi_t = P\big(\{\omega:\ \theta \leq t\} \,|\, \mathcal{F}_t^X\big)\,.$$

It gives rise to a new statistic

$$\varphi_t = \frac{\pi_t}{1 - \pi_t}\,,$$

whose structure we now study.

Denote P_θ the conditional distribution of X with respect to θ. Note that P_0 corresponds to the case when $dX_t = r\, dt + \sigma\, dw_t$, and P_∞ corresponds to the case when $dX_t = \sigma\, dw_t$. Introducing statistics

$$L_t = \frac{dP_0}{dP_\infty}\,(t, X)\,,$$

we can write

$$\frac{dP_\theta}{dP_\infty} = \frac{L_t}{L_\theta}\,, \qquad \theta \le t\,.$$

By Bayes's formula we obtain

$$\begin{aligned}\varphi_t(\lambda) = \varphi_t &= \frac{P\big(\{\omega:\ \theta \le t\}\,|\,\mathcal{F}_t^X\big)}{P\big(\{\omega:\ \theta > t\}\,|\,\mathcal{F}_t^X\big)} \\ &= \frac{\pi}{1-\pi}\, e^{\lambda t}\, \frac{dP_0}{dP_\infty}\,(t,X) + e^{\lambda t} \int_0^t \frac{dP_\theta}{dP_\infty}\,(t,X)\,\lambda\, e^{-\lambda\theta}\, d\theta \\ &= \frac{\pi}{1-\pi}\, e^{\lambda t}\, L_t + \lambda\, e^{\lambda t} \int_0^t \frac{L_t}{L_\theta}\, e^{-\lambda\theta}\, d\theta\,.\end{aligned}$$

Now, taking into account

$$dL_t = \frac{r}{\sigma^2}\, L_t\, dX_t\,,$$

and using the Kolmogorov-Itô formula, we obtain

$$d\varphi_t = \lambda\,(1+\varphi_t)\,dt + \frac{r}{\sigma^2}\,\varphi_t\, dX_t\,, \quad \varphi_0 = \frac{\pi}{1-\pi}\,.$$

Taking into account the relationship between φ_t and π_t, and using the Kolmogorov-Itô formula, we arrive at a stochastic differential equation for the posterior probability π_t:

$$d\pi_t = \Big(\lambda - \frac{r}{\sigma^2}\,\pi_t^2\Big)\,(1-\pi_t)\,dt + \frac{r}{\sigma^2}\,\pi_t\,(1-\pi_t)\,dX_t\,, \quad \pi_0 = \pi\,.$$

Now we solve the problem (a)–(b) in this Bayes's setting with the *a priori* probability π. Rewrite $V(c) = V(c,\pi)$ in the form

$$V(c,\pi) = \inf_\tau E\bigg((1-\pi_t) + c\int_0^\tau \pi_s\, ds\bigg) = \rho^*(\pi)\,.$$

Consider the following *innovation* representation of process X:

$$dX_t = r\,\pi\, dt + \sigma\, d\overline{w}_t\,,$$

where $\overline{w}$ is some new Brownian motion with respect to filtration $\big(\mathcal{F}_t^X\big)$.

Using this representation we can rewrite stochastic differential equation for π_t in the form

$$d\pi_t = \lambda\,(1-\pi_t)\,dt + \frac{r}{\sigma^2}\,\pi_t\,(1-\pi_t)\,d\overline{w}_t\,.$$

Noting that

$$\int_0^t \pi_s\, ds = \frac{\pi_0 - \pi_t}{\lambda} + \frac{1}{\lambda}\,\frac{r}{\sigma}\int_0^t \pi_s\,(1-\pi_s)\,d\overline{w}_t + t\,,$$

we arrive at the following expression for the price function

$$V(c,\pi) = \rho^*(\pi) = \inf_\tau E\biggl(\Bigl(1+\frac{c}{\lambda}\,\pi\Bigr) - \Bigl(1+\frac{c}{\lambda}\Bigr)\,\pi_\tau + c\,\tau\biggr).$$

Thus, π_t is a diffusion process generated by operator

$$L = a(\pi)\,\frac{d}{d\pi} + \frac{1}{2}\,b^2(\pi)\,\frac{d^2}{d\pi^2}\,,$$

where $a(\pi) = \lambda\,(1-\pi)$ and $b(\pi) = \pi\,(1-\pi)\,r/\sigma$.

Now we can apply the standard method of solving the change point problem, which reduces to the Stefan problem

$$\begin{aligned}
&L\,\rho(\pi) = -c\,\pi\,, \quad \pi \in [0,B),\\
&\rho(B) = 1-B\,, \quad \pi \in [B,1],\\
&\rho'(B) = -1\,,\ \rho'(0) = 0\,.
\end{aligned}$$

A general solution of this problem depends on two unknown constants. Another unknown parameter is constant B, which defines the *a priori* unknown boundary of the region in this free-boundary problem. Having one boundary condition for ρ at $\pi = B$ and two conditions for derivatives $\rho'(B)$ and $\rho'(0)$ (conditions of smooth sewing of a solution), we can write solution in the explicit form:

$$\rho(\pi) = \begin{cases} (1-B^*) - \int_\pi^{B^*} y^*(x)\,dx\,, & \pi \in [0,B^*) \\ 1-\pi\,, \quad \pi \in [B^*,1]\,, \end{cases}$$

where

$$y^*(x) = -C\int_0^x e^{-\Lambda\,[G(x)-G(y)]}\,\frac{dy}{y\,(1-y)^2}$$

with

$$G(y) = \log\frac{y}{1-y} - \frac{1}{y}\,, \quad \Lambda = \frac{\lambda}{r^2/2\sigma}\,, \quad C = \frac{c}{r^2/2\sigma}\,,$$

and B^* is a solution to

$$C\int_0^{B^*} e^{-\Lambda\,[G(B^*)-G(y)]}\,\frac{dy}{y\,(1-y)^2} = 1\,.$$

The standard verification technique can be used to show that found function $\rho(\pi)$ coincides with $\rho^*(\pi)$, and

$$\tau^* = \tau^*(B) = \inf\bigl\{t\,:\,\pi_t \geq B^*\bigr\}$$

is an optimal stopping time, such that

$$\begin{aligned}
&\rho^*(\pi) = E\biggl((1-\pi_{\tau^*}) + c\int_0^{\tau^*}\pi_s\,ds\biggr)\,,\\
&V(c,\pi) = P_\pi\bigl(\{\omega\,:\,\tau^* < 0\}\bigr) + c\,E_\pi\bigl((\tau^*-\theta)^+\bigr)\,,
\end{aligned}$$

where notation P_π and E_π reflects the presence of an *a priori* distribution with $P(\{w:\ \theta = 0\}) = \pi$.

Finally, we note that the same methodology can be applied when the evolution of prices is represented by process $X_t = \mu t + w_t$. Using Girsanov theorem, we can construct a new probability P^* such that process $\mu t + w_t$ is a Brownian motion with respect to it.

Chapter 3

Insurance Risks. Foundations of Actuarial Analysis

In this chapter we discuss insurance risks, methodologies of premium calculations and of estimation of reserves. In particular, we focus on actuarial analysis of risks that takes into account the investment strategies of an insurance company.

3.1 Modelling risk in insurance and methodologies of premium calculations

Insurance is a contract (*policy*) according to which one party (a *policy holder*) pays an amount of money (*premium*) to another party (*insurer*) in return for an obligation to compensate some possible losses of the policy holder. The aim of such a contract is to provide a policy holder with some protection against certain *risks*. Death, sickness, disability, motor vehicle accident, loss of property, etc. are some typical examples of such risks. Each policy contract specifies the policy term and the method of compensation. Usually compensation is provided in the form of payment of an amount of money. Any event specified in the policy contract that takes place during its term can result in such an *insurance claim*. If none of the events specified in the policy contract happen during the policy term, then the policy holder has no monetary compensation for the paid premiums.

The problem of premium calculation is one of the key issues in the insurance business: if the premium rate is too high, an insurance company will not have enough clients for successful operation. If the premium rate is too low, the company also may not have sufficient funds to pay all the claims.

To study this problem we need the following basic notions:

- x, the initial capital of an insurance company;
- non-negative sequence of random variables $\sigma_0 = 0 \leq \sigma_1 \leq \ldots$, time moments of receiving claims. Sequence $T_n = \sigma_n - \sigma_{n-1},\ n \geq 1$, represents time intervals between claims arrivals;
- $N(t) = \sup\{n :\ \sigma_n \leq t\}$ is the total number of claims up to time t. It is

obviously connected with sequence (σ_n):

$$\big\{\omega:\ N(t)=n\big\}=\big\{\omega:\ \sigma_n\le t<\sigma_{n-1}\big\}\,;$$

- sequence of independent identically distributed random variables (X_n), where each X_n represents amount of claim at time σ_n;
- $X(t)=\sum_{i=1}^{N(t)}X_i$ is the aggregate claim amount up to time t. Usually X is referred to as a *risk process*. Note that $X(t)=0$ if $N(t)=0$;
- denote $\Pi(t)$ the total premium income up to time $t\ge 0$;
- the capital of an insurance company at time $t\ge 0$ is given by

$$R(t)=x+\Pi(t)-X(t)\,.$$

Naturally, we want to measure and to compare risks. The most common measure of risk in insurance is the *probability of bankruptcy*:

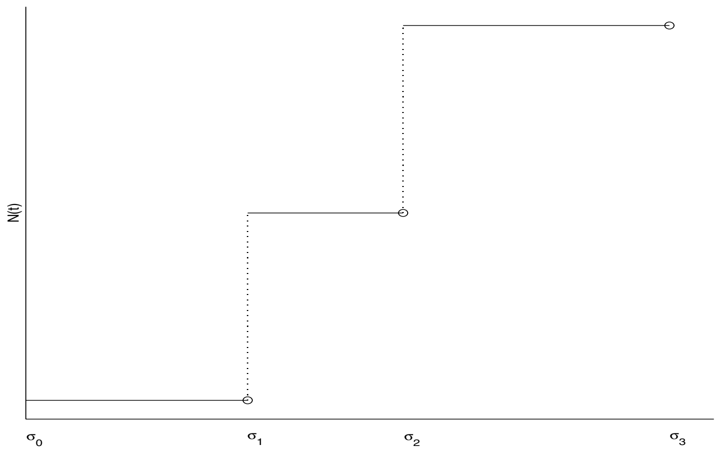
$$1-P\big(\{\omega:\ R(t)\ge 0,\ t\in[0,T]\}\big)\,,$$

where T is some time horizon.

Next we introduce some natural assumptions regarding process $N(\cdot)$:

1. $N(0)=0$;
2. $N(t)\in\{0,1,2,\ldots\}$;
3. $N(t)\le N(t+h)$.

Thus, the quantity $N(t+h)-N(t)$ describes the number of claims received during the time interval $(t,t+h)$.



Usually it is assumed that process $N(\cdot)$ can have only unit jumps, i.e., it is not possible to receive two or more claims simultaneously. Consider the distribution of $N(\cdot)$:

$$p_k(t) = P\big(\{\omega :\ N(t) \le k\}\big) = P\bigg(\Big\{\omega :\ \sum_{i=1}^{k} T_i \le t < \sum_{i=1}^{k+1} T_i\Big\}\bigg).$$

Probabilities $p_k(t)$ can be explicitly computed under the additional assumption on sequence (T_n). If (T_n) is a sequence of independent identically distributed random variables with the distribution function

$$F_T(x) = P\big(\{\omega :\ T_n \le x\}\big),$$

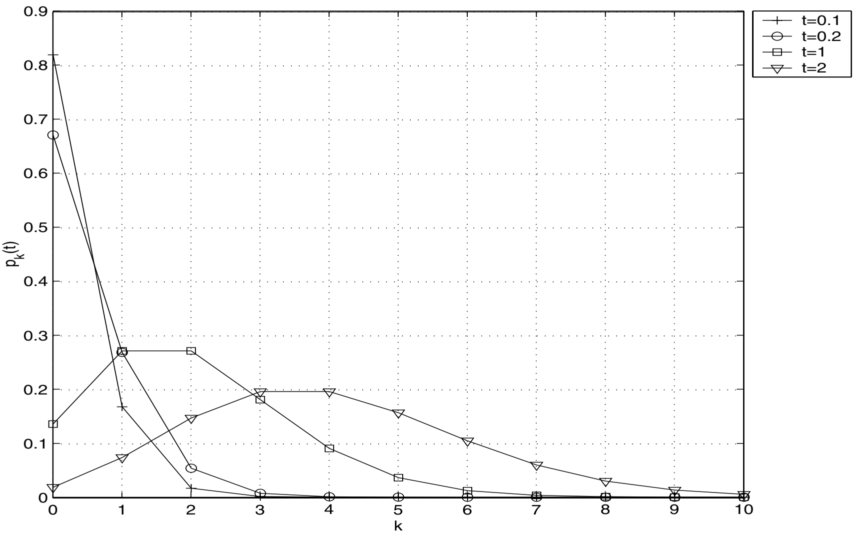
then sequence (σ_n) is called a *renewal process*. A typical example of such a process is a Poisson renewal process, when (T_n) has an exponential distribution with a parameter $\lambda > 0$, and therefore the distribution of $N(t)$ has the form

$$p_k(t) = e^{-\lambda t}\,\frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, \dots .$$

In this case $E\big(N(t)\big) = \lambda\, t,\ V\big(N(t)\big) = \lambda\, t.$

For example, if $\lambda = 2$, then we have the following values of $p_k(t)$:

k	$t = 0.1$	$t = 0.2$	$t = 1$	$t = 2$
0	0.8187	0.6703	0.1353	0.0183
1	0.1637	0.2681	0.2707	0.0733
2	0.0164	0.0536	0.2707	0.1465
3	0.0011	0.0072	0.1804	0.1954
4	0.0001	0.0007	0.0902	0.1954
5	0	0.0001	0.0361	0.1563
6	0	0	0.0120	0.1042
7	0	0	0.0034	0.0595
8	0	0	0.0009	0.0298
9	0	0	0.0002	0.0132
10	0	0	0	0.0053



We will assume that claims are paid instantaneously at the time of arrival, although in reality there may be a time delay related to estimation of the amount of a claim. Sometimes these delays can be rather significant, e.g., in insurance against catastrophic events.

The exact distribution of claims is often unknown. It is assumed that it can be described by some parametric family. Hence, one of the primary tasks in modelling insurance risks is estimating these parameters.

Here are examples of some widely used distributions

Poisson

$$P\big(\{\omega :\ X = x\}\big) = e^{-\lambda}\,\frac{\lambda^x}{x!}\,, \qquad x = 0, 1, 2, \dots\,,$$

is often used for modelling the number of claims;

Binomial

$$P\big(\{\omega :\ X = x\}\big) = \binom{m}{x}\, q^x\,(1-q)^{m-x}\,, \qquad x = 0, 1, 2, \dots, m\,,$$

represents the number of claims for a portfolio of m independent policies, where q is probability of receiving a claim (if $m = 1$, then it is called **Bernoulli** distribution);

Normal

$$P\big(\{\omega :\ X \le x\}\big) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\mu)^2/2\sigma^2}\,dx\,;$$

Exponential

$$P\big(\{\omega :\ X \le x\}\big) = 1 - e^{-\lambda x}\,, \qquad x \ge 0, \lambda > 0\,,$$

has various applications, for example, models the distribution of jumps of a Poisson process with intensity λ;

Gamma

$$P\big(\{\omega:\ X \le x\}\big) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta x}\, dx\,, \qquad \beta > 0\,;$$

Papeto

$$P\big(\{\omega:\ X \le x\}\big) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha}, \qquad x \ge 0,\ \alpha > 0,\ \lambda > 0\,,$$

has a 'heavy' tail, and hence is often used in modelling large claims;

Lognormal

$$P\big(\{\omega:\ X \le x\}\big) = \int_0^x \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(\log x - \mu)^2/2\sigma^2}\, dx\,.$$

Denote

$$F_{X(t)} = P\big(\{\omega:\ X \le x\}\big) = P\left(\left\{\omega:\ \sum_{i=1}^{N(t)} X_i \le x\right\}\right),$$

the distribution of the risk process.

To compute $F_{X(t)}(x)$ one needs some additional assumptions. Usually processes (X_n) and $N(\cdot)$ are assumed to be independent. Then we can write

$$F_{X(t)}(x) = P\big(\{\omega:\ X \le x\}\big) = \sum_{k=0}^{\infty} p_k(t)\, F_X^{*k}(x)\,,$$

where

$$F_X^{*k}(x) = P\big(\{\omega:\ X_1 + \ldots + X_k \le x\}\big)\,.$$

Premium calculation or determination of process $\Pi(t)$ is one of the most essential and complex tasks of an insurer. Premium flow must guarantee payments of claims, but on the other hand, premiums must be competitive. One of the most widely used ways of computing Π on interval $[0, t]$ is given by

$$\Pi(t) = (1 + \theta)\, E\big(N(t)\big)\, E(X),$$

where X is a random variable with the same distribution as X_i, and θ is the *security loading coefficient*. This formula says that the average premium income should be greater than the average aggregate claims payment. If they are equal, then such premium is called *net-premium* and the method of its computing is referred to as *equivalence principle*.

The *bonus-malus* system is an example of a different approach to premium calculations. In this case, all policy holders are assigned certain ratings according to their claims history, and they can be transferred from one group to another. This system is typically used by the motor vehicle insurance companies.

Calculation of adequate premium consists in construction of process $\Pi(t)$ given $F_{X(t)}$, the distribution function of the risk process. In this case we will write $\Pi(F_X)$ or simply $\Pi(X)$.

Process Π has the following properties

- $\Pi(a) = a$ for any constant a if $\theta = 0$;
- $\Pi(a\,X) = a\,\Pi(X)$ for any constant a;
- $\Pi(X+Y) \le \Pi(X) + \Pi(Y)$;
- $\Pi(X+a) = \Pi(X) + a$ for any constant a;
- if $X \le Y$, then $\Pi(X) \le \Pi(Y)$;
- for any $p \in [0,1]$ and any random variable Z

$$\Pi(X) = \Pi(Y)$$

implies that

$$\Pi\big(p\,F_X + (1-p)\,F_Z\big) = \Pi\big(p\,F_Y + (1-p)\,F_Z\big)\,.$$

We list some widely used actuarial *principles of premium calculations*:

Expectation principle

$$\Pi(X) = (1+a)\,E(X),\; a > 0;$$

Variance principle

$$\Pi(X) = E(X) + a\,V(X);$$

Standard deviation principle

$$\Pi(X) = E(X) + a\,\sqrt{V(X)};$$

Modified variance principle

$$\Pi(X) = \begin{cases} E(X) + a\,V(X)/E(X)\,, & E(X) > 0 \\ 0\,, & E(X) = 0\,; \end{cases}$$

Exponential utility principle

$$\Pi(X) = \frac{\log E(e^{aX})}{a};$$

Quantile principle

$$\Pi(X) = F_X^{-1}(1-\varepsilon);$$

Absolute deviation principle

$$\Pi(X) = E(X) + a\,\kappa_X \qquad \text{where} \quad \kappa_X = E\big(\big|X - F_X^{-1}(1/2)\big|\big);$$

Zero utility principle

$$E\big(v(\Pi(X) - X)\big) = v(0)\,,$$

where v is a given utility function.

Note that the exponential principle is a particular case of the zero utility principle with

$$v(x) = \frac{1 - e^{-ax}}{a}\,.$$

The notion of *risk* is the key ingredient of insurance theory and practice. Risk exposure gives rise to insurance companies that manage risks and provide some protection against these risks to their clients. Reinsurance companies provide similar services to insurance companies. There are several approaches to modelling the risk process.

Consider a *portfolio* that consists of n policy contracts with claim payments ('risks') $X_1, \ldots, X_n$ being independent non-negative random variables. Then the risk process

$$X^{\mathrm{ind}} = \sum_{i=1}^{n} X_i\,,$$

has distribution $F_{X_1} * \ldots * F_{X_n}$. This model of risk is referred to as *individual*.

Suppose that an insurance company issues n insurance contracts that terminate at some time t, e.g., in one year's time. Each contract allows no more than one claim. Claim payments $X_1, \ldots, X_n$ are non-negative random variables. The total amount of claims incurred over this period is represented by the risk process $X^{\mathrm{ind}} = \sum_{i=1}^{n} X_i$. It is also assumed that all claims are payable at the termination time. Therefore, the *probability of bankruptcy* (or *insolvency*) is given by

$$P\big(\{\omega :\ X^{\mathrm{ind}} > x + \Pi\}\big)\,,$$

where x is the initial capital of the company and Π is the premium income.

Thus, the model of individual risk is based on the following assumptions:

- time horizon is relatively short;
- number of insurance contracts is deterministic and fixed;
- premiums are payable at the time of contracts issue;
- the distribution of claim payments is known.

Example. Consider a model of individual risk with a sufficiently large number of insurance contracts. Since exact calculation of probability of bankruptcy is technically complicated, we use the Central Limit Theorem for its approximation.

Using the net-premium principle we have that $\Pi = X^{\text{ind}}$. Then we compute the probability of bankruptcy:

$$\begin{aligned} P\big(\{\omega :\ X^{\text{ind}} > \Pi\}\big) &= P\big(\{\omega :\ X^{\text{ind}} - E(X^{\text{ind}}) > 0\}\big) \\ &= P\left(\left\{\omega :\ \frac{X^{\text{ind}} - E(X^{\text{ind}})}{\sqrt{V(X^{\text{ind}})}} > 0\right\}\right) \\ &\approx 1 - \Phi(0) = 0.5\,, \end{aligned}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy\,.$$

This means that the net-premium principle cannot be used in this situation.

The standard deviation principle gives

$$\Pi = E(X^{\text{ind}}) + a\,\sqrt{V(X^{\text{ind}})}$$

and

$$P\big(\{\omega :\ X^{\text{ind}} > \Pi\}\big) = P\left(\left\{\omega :\ \frac{X^{\text{ind}} - E(X^{\text{ind}})}{\sqrt{V(X^{\text{ind}})}} > a\right\}\right) \approx 1 - \Phi(a)\,.$$

In this case, for any fixed level of risk ε, we can find a parameter a such that $\Phi(a) = 1 - \varepsilon$, so that the probability of bankruptcy is

$$P\big(\{\omega :\ X^{\text{ind}} > \Pi\}\big) \approx \varepsilon\,.$$

Now we consider a situation when N, the number of possible claims, is unknown. We can single out two types of insurance contracts: static and dynamic. In the static case, claims are payable at the terminal time, and therefore N is an integer-valued random variable. In the dynamic model, $N = N(t)$ is a stochastic process that counts the number of claims incurred during the time interval $[0, t]$. Both these models of risk are referred to as *collective*. The risk process has the form

$$X^{\text{col}} = \sum_{i=1}^{N} X_i\,,$$

where claims amounts X_i are positive and independent of N. Clearly, the collective model of risk is more realistic than the individual model, and it gives more flexibility in managing risk for an insurance company.

Some essential differences between the two models are summarized in the following table.

individual model	**collective model**
n, the number of insurance contracts is known *a priori* and all claims are payable at the same time	the process of receiving claims is represented by a stochastic process
each contract admits no more than one claim	there is no restrictions on number of claims per contract
all claims are assumed to be independent	it is assumed that the amounts of incurred claims are independent

WORKED EXAMPLE 3.1

Suppose an insurance company issues 1-year contracts. All policy holders are divided into four groups:

k	q_k	b_k	n_k
1	0.02	1	500
2	0.02	2	500
3	0.1	1	300
4	0.1	2	500

Here n_k is the number of policy holders in group k, q_k is the probability of making a claim by a member of this group and b_k is the amount of the corresponding claim. Using normal approximation, find the value of the security loading coefficient that will reduce the probability of insolvency to 0.05.

SOLUTION The total number of policy holders is 1800, so the total amount of claims is

$$S = X_1 + \ldots + X_{1800}\,.$$

We will find parameter θ from the equation

$$P\big(\{\omega:\ S \le (1+\theta)\,E(S)\}\big) = 0.95\,,$$

which can be written in the form

$$P\left(\left\{\omega:\ \frac{S - E(S)}{\sqrt{V(S)}} \le \frac{\theta\,E(S)}{\sqrt{V(S)}}\right\}\right) = 0.95\,.$$

Since the total number of policy holders is reasonably large, then the quantity $\big(S - E(S)\big)/\sqrt{V(S)}$ can be accurately approximated by a standard normal distribution. Hence we obtain the equation

$$\frac{\theta\,E(S)}{\sqrt{V(S)}} \approx 1.645\,.$$

The following table contains expectations $\mu_k = b_k\, q_k$ and variances $\sigma_k^2 = b_k^2\, q_k\,(1 - q_k)$ for each policy.

k	q_k	b_k	μ_k	σ_k^2	n_k
1	0.02	1	0.02	0.0196	500
2	0.02	2	0.02	0.0784	500
3	0.1	1	0.1	0.09	300
4	0.1	2	0.2	0.36	500

Thus

$$E(S) = \sum_{i=1}^{1800} E(X_i) = \sum_{k=1}^{4} n_k\, \mu_k = 160\,,$$

$$V(S) = \sum_{i=1}^{1800} V(X_i) = \sum_{k=1}^{4} n_k\, \sigma_k^2 = 256\,,$$

and

$$\theta \approx 1.645\, \frac{\sqrt{V(S)}}{E(S)} = 1.645\, \frac{16}{160} = 0.1645\,.$$

□

Note that situations where the number of claims is a random variable are typical for life insurance, and they are studied in Section 3.4.

Another example of a collective risk model is the Cramér-Lundberg model. The claims flow is modelled here as a Poisson process $N(t)$ with intensity λ, and claims amounts are independent random variables that are also independent of $N(t)$. The premium income is a linear function of time t: $\Pi(t) = c\,t$. The risk process

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

is a compound Poisson process. It turns out that if the initial probability P that describes the distribution of claims is replaced by an equivalent probability Q under which X is also a compound Poisson process, then applying the equivalence principle with respect to this new probability Q, we obtain all the above mentioned traditional principles of premium calculations.

Indeed, define a positive process

$$M_t^\beta = \exp\Big\{X^\beta(t) - \lambda\, t\, E_P\big(\exp\{\beta(X_1)\} - 1\big)\Big\}, \quad M_0^\beta = 1,\ t \in [0, T]\,,$$

where $X^\beta(t) = \sum_{k=1}^{N(t)} \beta(X_k)$, function $\beta : \mathbb{R}_+ \to \mathbb{R}$ is such that $E_P\big(\exp\{\beta(X_1)\}\big) < \infty$, and the expectation E_P is taken with respect to probability P.

Process M^β is a martingale since for $s \le t$ we have

$$\begin{aligned} &E_P\big(M_t^\beta|\mathcal{F}_s\big) \\ &\quad = M_s^\beta\, E_P\big(\exp\{X^\beta(t) - X^\beta(s)\}\big)\, e^{-\lambda(t-s)}\, E_P\big(\exp\{\beta(X_1)\} - 1\big) = M_s^\beta . \end{aligned}$$

Hence, $E_P\big(M_t^\beta\big) = 1$, and for each such function β one can define new probability Q with density M_t^β. Note that any other probability Q under which the risk process $X(t)$ is a compound Poisson process must have exactly the same structure with some appropriate function β (for details see [31]).

Thus, we can use this function β and the corresponding probability Q for calculating premium. This calculation is based on the condition that the difference

$$X(t) - c\,t$$

between the total amount of claims and the total premium income is a martingale with respect to Q. This agrees with the equivalence principle in insurance and with the no-arbitrage principle in finance.

So we obtain that

$$c = E_Q\big(X(1)\big)\,,$$

and since $X(t)$ is a compound Poisson process, then

$$E_Q\big(X(1)\big) = E_Q\big(N(1)\big)\, E_Q\big(X_1\big)\,,$$

where

$$E_Q\big(N(1)\big) = \lambda\, E_P\big(\exp\{\beta(X_1)\}\big)$$

$$E_Q\big(X_1\big) = E_P\big(X_1\, \exp\{\beta(X_1)\}\big) / E_P\big(\exp\{\beta(X_1)\}\big)\,.$$

Finally, we deduce

$$c = \lambda\, E_P\big(X_1\, \exp\{\beta(X_1)\}\big)\,.$$

Choosing appropriately β, we then obtain all the traditional actuarial principles of premium calculations. For example, the expectation principle corresponds to $\beta(x) = \ln(1+a)$, and we have

$$c = \lambda\, E_P\big(X_1\, \exp\{\beta(X_1)\}\big) = \lambda\, E_P\big(X_1\,(1+a)\big) = (1+a)\, E_P\big(X_1\big) = \Pi(X)\,.$$

3.2 Probability of bankruptcy as a measure of solvency of an insurance company

Consider a collective risk model with a *binomial* process $N(t)$ representing the total number of claims up to time t:

$$N(0) = 0, \quad N(t) = \xi_1 + \ldots + \xi_t,\ t = 1, 2, \ldots\,,$$

where $(\xi_i)_{i=1}^{\infty}$ is a sequence of independent Bernoulli random variables such that

$$P\big(\{\omega:\ \xi_i = 1\}\big) = q \qquad \text{and} \qquad P\big(\{\omega:\ \xi_i = 0\}\big) = 1 - q\,.$$

Sequence of independent identically distributed random variables $(X_i)_{i=1}^{\infty}$ with values in the set of all natural numbers $\mathbb{N}$, represents the amounts of claims. Denote

$$f_n = P\big(\{\omega:\ X_i = n\}\big)\,, \qquad \tilde{f}(z) = \sum_{n=1}^{\infty} f_n\, z^n \qquad \text{and} \qquad \mu = E(X_i)$$

the distribution, the generating function and the expectation of $(X_i)_{i=1}^{\infty}$, respectively.

Assuming that sequences $(X_i)_{i=1}^{\infty}$ and $(\xi_i)_{i=1}^{\infty}$ are independent, let

$$X(k) = X_1\,\xi_1 + \ldots + X_k\,\xi_k$$

and

$$g_n(k) = P\big(\{\omega:\ X(k) = n\}\big)\,, \quad n = 0, 1, 2, \ldots\,.$$

Then the sum

$$G_n(k) = \sum_{m=0}^{n} g_n(k)\,, \qquad n = 1, 2, \ldots$$

is the distribution function of $X(k)$, the sum of independent identically distributed random variables $X_l\,\xi_l,\ l = 1, \ldots k$ with generating functions

$$\begin{aligned}\sum_{i=0}^{\infty} P\big(\{\omega:\ X_l\,\xi_l = i\}\big)\, z^i &= 1 - q + q \sum_{i=1}^{\infty} P\big(\{\omega:\ X_l = i\}\big)\, z^i \\ &= 1 - q + q\,\tilde{f}(z)\,.\end{aligned}$$

Therefore

$$\tilde{g}(z, k) = \big[1 - q + q\,\tilde{f}(z)\big]^k\,.$$

is the generating function of $X(k)$.

Consider a stochastic sequence

$$R(k) = x + k - X(k), \quad k = 1, 2, \ldots\,, \quad R(0) = x \in \{0, 1, 2, \ldots\}\,,$$

which represents the capital of an insurance company, where x is the initial capital, k is premium income (i.e., at each time $k = 1, 2, \ldots$ the company receives the premium of 1). This model is referred to as a *compound binomial model.*

Functions

$$\phi(x, k) = P\big(\{\omega:\ R(j) \geq 0,\ j = 0, 1, \ldots, k\}\big) \quad \text{and} \quad \phi(x) = \lim_{k \to \infty} \phi(x, k),$$

are called the *probabilities of non-bankruptcy* (*probabilities of solvency*) on a finite interval $[0, k]$ and infinite interval $[0, \infty)$, respectively.

Clearly, knowing the analytical expressions for this functions, one can estimate the solvency of the company.

To find an expression for $\phi(x, k)$ we assume that the initial capital is $x - 1$, where $x \geq 1$. We also accept that the probability of solvency of a company with negative initial capital is equal to zero. Then for any integers k and x we have the following recurrence relation:

$$\begin{aligned}\phi(x-1,k) &= E\big(\phi(R(1),k-1)\big)\\ &= (1-q)\,\phi(x,k-1) + q\sum_{y=1}^{x}\phi(x-y,k-1)\,f_y\,.\end{aligned}$$

Further, using the technique of generating functions, we obtain the following expression for the probability of solvency of a company with zero initial capital (for details see Section 3.2.2: Mathematical appendix 1):

$$\phi(0,k) = \frac{\sum_{m=0}^{k}(k+1-m)\,g_m(k+1)}{(1-q)\,(k+1)}\,, \qquad k = 0,1,\ldots\,.$$

If the initial capital $x = 1, 2, \ldots$, then we have

$$\phi(x,k) = G_{x+k}(k) - (1-q)\sum_{m=0}^{k-1}\phi(0,k-1-m)\,g_{x+m+1}(m)\,,$$

for $k = 1, 2, \ldots$ (see Section 3.2.3: Mathematical appendix 2).

In the case of the infinite time interval $[0, \infty)$, we use the following formula from Section 3.2.4: Mathematical appendix 3

$$\phi(0,k) = \frac{1-q\,\mu}{1-q} + \frac{\sum_{m=k+1}^{\infty}\Big(1-G_m(k+1)\Big)}{(1-q)\,(k+1)}\,.$$

Taking limit as $k \to \infty$, we obtain (see Section 3.2.4: Mathematical appendix 3)

$$\phi(0) = \frac{1-q\,\mu}{1-q}\,.$$

Now we establish a relation between the initial capital and probabilistic characteristics of claims, which guarantees the solvency of an insurance company over the infinite period of time with the probability that corresponds to a chosen (fixed) level of risk $\varepsilon > 0$:

$$\phi(0) \geq 1 - \varepsilon\,.$$

This implies

$$\mu \leq 1 - \varepsilon + \frac{\varepsilon}{q}\,.$$

The case when the initial capital is greater than zero is illustrated by the following example.

WORKED EXAMPLE 3.2
Consider function

$$\tilde{\phi}(z) = \sum_{x=0}^{\infty} \phi(x)\, z^x\,.$$

Let $X_i \equiv 2$*, then* $f(z) = z^2$ *and* $\mu = 2$*.*

Find values of the initial capital that guarantee that the probability of insolvency is less than the chosen level of risk.

SOLUTION Note that function $\tilde{\phi}(z)$ can be written in the form (see Section 3.2.5: Mathematical appendix 4):

$$\tilde{\phi}(z) = \frac{1}{1-z}\, \frac{1-q\,\mu}{1-q\,\mu\,\tilde{b}(z)}\,,$$

where

$$\tilde{b}(z) = \frac{\tilde{g}(z,1)-1}{q\,\mu\,(z-1)} = \frac{\tilde{f}(z)-1}{\mu\,(z-1)}\,.$$

In our case $\tilde{b}(z) = (1+z)/2$, hence

$$\tilde{\phi}(z) = \frac{1}{1-z}\, \frac{q\,(1-q)^{-1}}{1-q\,z\,(1-q)^{-1}}\,.$$

Expanding $\tilde{\phi}(z)$ in powers of Z, we obtain

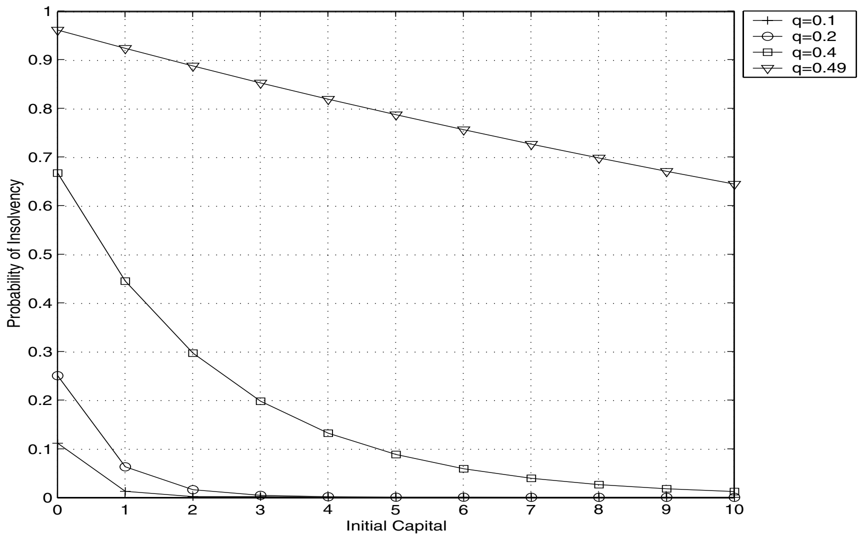
$$\phi(x) = 1 - \left(\frac{q}{1-q}\right)^{x+1}\,.$$

Positivity of the security loading coefficient $1-q\,\mu > 0$ implies

$$q < \frac{1}{2} \qquad \text{and} \qquad \frac{q}{1-q} < 1,.$$

The following table and figure give probabilities of insolvency for four different values of q with accuracy 0.0001.

Initial capital	$q = 0.1$	$q = 0.2$	$q = 0.4$	$q = 0.49$
0	0.1111	0.25	0.6667	0.9608
1	0.0123	0.0625	0.4444	0.9231
2	0.0014	0.0156	0.2963	0.8869
3	0,0002	0.0039	0.1975	0.8521
4	0	0.001	0.1317	0.8187
5	0	0.0002	0.0878	0.7866
6	0	0.0001	0.0585	0.7558
7	0	0	0.0390	0.7261
8	0	0	0.0260	0.6976
9	0	0	0.0173	0.6703
10	0	0	0.0116	0.6440



For a given level of risk ε we solve the following inequality for x

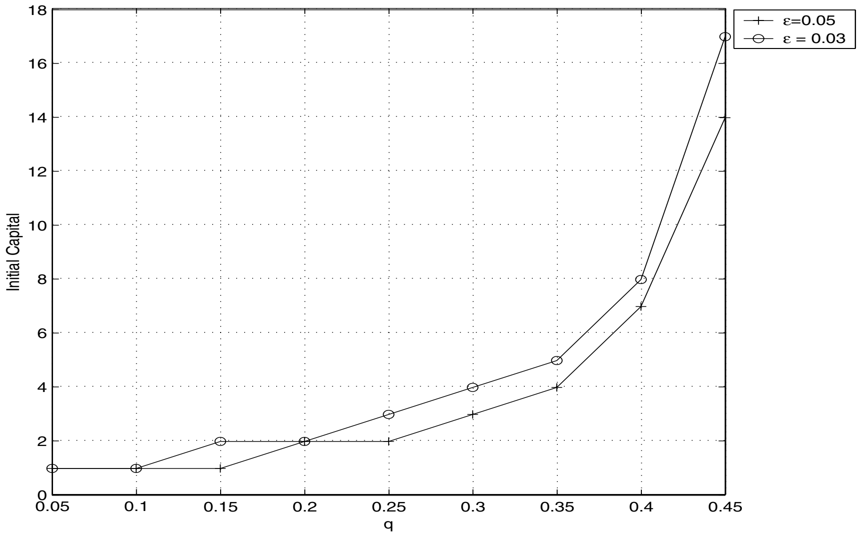
$$\phi(x) > 1 - \varepsilon\,.$$

We have

$$x > \frac{\ln(\varepsilon)}{\ln\big(q/(1-q)\big)} - 1\,.$$

The next table and figure give the minimal values of the initial capital x for various values of q and ε.

q	$\varepsilon = 0.05$	$\varepsilon = 0.03$	$\varepsilon = 0.01$
0.05	1	1	1
0.1	1	1	2
0.15	1	2	2
0.2	2	2	3
0.25	2	3	4
0.3	3	4	5
0.35	4	5	7
0.4	7	8	11
0.45	14	17	22



□

3.2.1 Cramér-Lundberg model

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider a Poisson process $N(t),\ t \geq 0$, with intensity λ: $N(0) = 0,\ E(N(t)) = \lambda t$, and an independent of N sequence $(X_i)_{i=1}^{\infty}$ of independent identically distributed random variables with expectation μ and distribution function $F(x),\ F(0) = 0$.

The claims flow (number of claims received up to time t) is represented by the Poisson process $N(t)$. The amounts of these claims are represented by sequence $(X_i)_{i=1}^{\infty}$. The premium income of an insurance company is given by $\Pi(t) = ct$, where c is a constant. If x is the initial capital of the company, then the dynamics of the company's capital is given by

$$R(t) = x + c\,t - \sum_{i=1}^{N(t)} X_i\,.$$

Since $N(t)$ and $(X_i)_{i=1}^{\infty}$ are independent, then the expectation of the risk process $X(t) = \sum_{i=1}^{N(t)} X_i$ is $E\big(X(t)\big) = \lambda\,t\,\mu$. Setting the security loading coefficient at

$$\theta = \frac{\Pi(t)}{E(X(t))} - 1 = \frac{c - \lambda\,\mu}{\lambda\,\mu}\,,$$

we obtain $c = (1+\theta)\,\lambda\,\mu$.

Now we compute the probability of solvency

$$\phi(x) = P\big(\{\omega :\ R(t) \geq 0,\ R(0) = x,\ t \geq 0\}\big)\,.$$

First we investigate smoothness of function $\phi(x)$ assuming that the distribution function $F(x)$ has density $f(x)$. Since bankruptcy cannot occur prior time T_1, when the Poisson process N has its first jump, then we can write

$$\begin{aligned}\phi(x) &= E\big(\phi(x + c\,T_1 - x_1)\big) \\ &= \int_0^\infty \lambda\, e^{-\lambda s} \int_0^{x+cs} \phi(x + c\,s - y)\, f(y)\, dy\, ds\,.\end{aligned}$$

Making a substitution $q = x - y$, we can rewrite the latter equality in the form

$$\phi(x) = \int_0^\infty \lambda\, e^{-\lambda s} \int_{-cs}^{x} \phi(q + c\,s)\, f(x - q)\, dq\, ds\,.$$

Thus, if $F(y) \in C^n[0,\infty)$, then $\phi(x) \in C^{n-1}[0,\infty)$. In further discussion we assume $F(y) \in C^3[0,\infty)$.

Using properties of the Poisson process and the formula for total probability, we obtain

$$\begin{aligned}\phi(x) = \phi(x + c\,\Delta t)\, &\big[1 - \lambda\,\Delta t + o(\Delta t)\big] \\ &+ \lambda\,\Delta t \int_0^{x+c\Delta t} \phi(x + c\,\Delta t - y)\, dF(y) + o(\Delta t)\,.\end{aligned}$$

By Taylor's formula we also have

$$\begin{aligned}\phi(x) = \Big[\phi(x) + c\,\phi'(x)\,\Delta t\Big]\, &\big[1 - \lambda\,\Delta t + o(\Delta t)\big] \\ &+ \lambda\,\Delta t \int_0^{x+c\Delta t} \phi(x + c\,\Delta t - y)\, dF(y) + o(\Delta t)\,,\end{aligned}$$

hence

$$\begin{aligned}\phi(x)\,\big[\lambda\,\Delta t + o(\Delta t)\big] = c\,\phi'(x)\,\Delta t\, &\big[1 - \lambda\,\Delta t + o(\Delta t)\big] \\ &+ \lambda\,\Delta t \int_0^{x+c\Delta t} \phi(x + c\,\Delta t - y)\, dF(y) + o(\Delta t)\,.\end{aligned}$$

Dividing the latter equality by Δt and taking limits as $\Delta t \to 0$, we obtain

$$\phi(x)\,\lambda = c\,\phi'(x) + \lambda \int_0^x \phi(x - y)\, dF(y)\,. \tag{3.1}$$

In the case of an exponential distribution function F, it is not difficult to find an explicit solution of this equation. Indeed, if $F(y) = 1 - e^{-x/\mu}$, then equation (3.1) is reduced to

$$\phi(x)\,\lambda = c\,\phi'(x) + \lambda \int_0^x \phi(x - y)\, \frac{1}{\mu}\, e^{-y/\mu}\, dy\,.$$

Differentiating and integrating by parts, we obtain

$$\begin{aligned}
\lambda\,\phi'(x) &= c\,\phi''(x) + \frac{\lambda}{\mu}\,\phi(0) + \lambda \int_0^x \phi_x'(x-y)\,\frac{1}{\mu}\,e^{-y/\mu}\,dy \\
&= c\,\phi''(x) + \frac{\lambda}{\mu}\,\phi(0) - \lambda \int_0^x \frac{1}{\mu}\,e^{-y/\mu}\,d\phi(x-y) \\
&= c\,\phi''(x) + \frac{\lambda}{\mu}\,\phi(x) + \lambda \int_0^x \phi(x-y)\,\frac{1}{\mu}\,de^{-y/\mu} \\
&= c\,\phi''(x) + \frac{\lambda}{\mu}\,\phi(x) - \frac{1}{\mu}\left[\lambda \int_0^x \phi(x-y)\,\frac{1}{\mu}\,e^{-y/\mu}\,dy\right] \\
&= c\,\phi''(x) + \frac{\lambda}{\mu}\,\phi(x) + \frac{c}{\mu}\,\phi'(x) - \frac{\lambda}{\mu}\,\phi(x) \\
&= c\,\phi''(x) + \frac{c}{\mu}\,\phi'(x)\,.
\end{aligned}$$

Thus we arrive at the following differential equation

$$\phi''(x) + \phi'(x)\left[\frac{1}{\mu} - \frac{\lambda}{c}\right] = 0\,,$$

whose general solution is of the form

$$\phi(x) = B + A\,\exp\left\{x\left[\frac{\lambda}{c} - \frac{1}{\mu}\right]\right\},$$

where A and B are some constants. The inequality

$$\frac{\lambda}{c} < \frac{1}{\mu}$$

can be written as

$$\lambda\,\mu - c < 0\,,$$

which reflects the positivity of θ, and therefore $\phi(\infty) = B$.

Unknown constants A and B can be found from the following relations

1. $\phi(\infty) = 1$;

2. Substituting $x = 0$ into equation (3.1):

$$\phi(x)\,\lambda = c\,\phi'(x) + \lambda \int_0^x \phi(x-y)\,dF(y)$$

implies

$$\phi(0)\,\lambda = c\,\phi'(0)\,.$$

Thus we arrive at the following expression

$$\phi(x) = 1 - \frac{\lambda\,\mu}{c}\,\exp\left\{x\left[\frac{\lambda}{c} - \frac{1}{\mu}\right]\right\} = 1 - \frac{1}{1+\theta}\,\exp\left\{-\frac{\theta\,x}{(1+\theta)\,\mu}\right\}.$$

In general, for an arbitrary distribution function F, it may be difficult to find an explicit expression for ϕ. In this case one can look for various estimates of $\psi(x) = 1 - \phi(x)$, the probability of insolvency. The main result here is usually referred to as the *Cramér-Lundberg inequality*:

$$\psi(x) \le e^{-R\,x}\,, \tag{3.2}$$

where R is a positive solution to the equation

$$\lambda + c\,r = \lambda \int_0^\infty e^{r\,x}\,dF(x)\,.$$

Note that so far we have dealt with the classical insurance models, where one does not take into account the investment strategies of an insurance company.

3.2.2 Mathematical appendix 1

Consider equation

$$\begin{aligned} \phi(x-1,k) &= E\big(\phi(X_1,k-1)\big) \\ &= (1-q)\,\phi(x,k-1) + q\sum_{y=1}^{x}\phi(x-y,k-1)\,f_y\,, \end{aligned} \tag{3.3}$$

and

$$\tilde{\phi}_1(z,k) = \sum_{x=0}^{\infty}\phi(x,k)\,z^x\,,$$

the generating function of $\big(\phi(x,k)\big)_{x=0}^{\infty}$.

Multiplying equation (3.3) by z^x, and summing in x from 1 to ∞, we obtain

$$\begin{aligned} z\,\tilde{\phi}_1(z,k) = (1-q)&\left[\tilde{\phi}_1(z,k-1) - \phi(0,k-1)\right] \\ &+q\,\tilde{\phi}_1(z,k-1)\,\tilde{f}(z) \end{aligned}$$

or

$$z\,\tilde{\phi}_1(z,k) = \tilde{g}(z,1)\,\tilde{\phi}_1(z,k-1) - (1-q)\,\phi(0,k-1)\,. \tag{3.4}$$

Introduce two auxiliary functions:

$$\tilde{\phi}(z,t) = \sum_{k=0}^{\infty}\tilde{\phi}(z,k)\,t^k = \sum_{k=0}^{\infty}\sum_{x=0}^{\infty}\phi(z,k)\,z^x\,t^k$$

and

$$\tilde{\phi}_0(t) = \sum_{k=0}^{\infty} \phi(0,k)\, t^k\,. \tag{3.5}$$

Multiplying equation (3.4) by t^k, and summing in k from 1 to ∞, we obtain

$$z\,\tilde{\phi}(z,k) - z\,\tilde{\phi}_1(z,0) = t\,\tilde{g}(z,1)\,\tilde{\phi}(z,k) - t\,(1-q)\,\tilde{\phi}_0(t)\,. \tag{3.6}$$

From the definition of function ϕ we have that $\phi(x,0) = 1$ for all $x = 0, 1, 2, \ldots$. Hence

$$\tilde{\phi}_1(z,0) = \frac{1}{1-z} \qquad \text{for} \quad |z| < 1\,.$$

Then equation (3.6) can be written in the form

$$\tilde{\phi}(z,k)\Big[z - t\,\tilde{g}(z,1)\Big] = \frac{z}{1-z} - t\,(1-q)\,\tilde{\phi}_0(t)\,. \tag{3.7}$$

Fix t with $|t| < 1$. Consider function

$$F(z) := z - t\,\tilde{g}(z,1)\,,$$

then

$$F(0) = 0 \qquad \text{and} \qquad F(1) = 1 - t\left[1 - q + q\sum_{n=1}^{\infty} f_n\right] = 1 - t > 0\,.$$

Also

$$F'(z) = 1 - t\,q\sum_{n=1}^{\infty} n\,f_n\,z^{n-1} > 1 - q\sum_{n=1}^{\infty} n\,f_n\,z^{n-1} > 1 - q\,\mu > 0\,.$$

The inequality $1 - q\,\mu > 0$ is equivalent to positivity of the security loading coefficient, and we assume that it is the case.

Thus, for each fixed t with $|t| < 1$ the equation

$$z = t\,\tilde{g}(z,1) \tag{3.8}$$

has a unique root $z = z(t) \in (0,1)$. Therefore function $z(t)$, $|t| < 1$, is a solution to (3.8).

Now, for any analytic function h with $h(0) = 0$, we have

$$h\big(z(t)\big) = \sum_{n=1}^{\infty} \frac{t^n}{n!}\,\frac{d^{n-1}}{ds^{n-1}}\left[h'(s)\,\big(\tilde{g}(s,1)\big)^n\right]\Bigg|_{s=0}\,, \tag{3.9}$$

where $z(t)$ is a solution of (3.8). Note that $\big(\tilde{g}(s,1)\big)^n = \tilde{g}(s,n)$.

If $h(z) = z$, then the solution to (3.8) has the form

$$z(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!}\,g(n-1,n)\,,$$

where

$$g(n-1,n) = \frac{1}{(n-1)!}\,\frac{d^{n-1}}{ds^{n-1}}\,\tilde{g}(s,n)\Big|_{s=0}$$

Substituting $h(z) = z/(1-z)$ into (3.9), we obtain

$$\frac{z(t)}{1-z(t)} = \sum_{n=1}^{\infty} \frac{t^n}{n!}\,\frac{d^{n-1}}{ds^{n-1}}\,\frac{\tilde{g}(s,n)}{(1-s)^2}\Big|_{s=0}\,. \tag{3.10}$$

For s with $|s| < 1$, we have

$$\begin{aligned}
\frac{\tilde{g}(s,n)}{(1-s)} &\equiv \frac{\sum_{k=0}^{\infty} g_k(n)\, s^k}{(1-s)} \\
&= \Big[g_0(n) + g_1(n)\, s + g_2(n)\, s^2 + \ldots\Big] \times \Big[1 + s + s^2 + \ldots\Big] \\
&= g_0(n) + s\,\big[g_1(n) + g_0(n)\big] + s^2\,\big[g_2(n) + g_1(n) + g_0(n)\big] + \ldots \\
&\quad + s^k\,\big[g_k(n) + g_{k-1}(n) + \ldots + g_0(n)\big] + \ldots\ ,
\end{aligned}$$

so the coefficient in front of s^k is

$$\sum_{m=0}^{k} g_m(n) = G_k(n)\,.$$

Similarly, for s with $|s| < 1$, we obtain that the coefficient in front of s^k in the expansion

$$\frac{\tilde{g}(s,n)}{(1-s)^2} \equiv \frac{\tilde{g}(s,n)\,(1-s)^{-1}}{(1-s)}$$

is equal $\sum_{m=0}^{k} G_m(n)$.

Thus we can write (3.10) in the form

$$\frac{z(t)}{1-z(t)} = \sum_{n=1}^{\infty} \left[\sum_{m=0}^{n-1} G_m(n)\right] \frac{t^n}{n!}\,. \tag{3.11}$$

If we substitute $z = z(t)$ in (3.7), then the left-hand side of this equation vanishes, so we can find an expression for $\tilde{\phi}_0(t)$:

$$\tilde{\phi}_0(t) = \frac{z(t)}{t\,(1-q)\,\big(1-z(t)\big)}\,,$$

which in view of (3.11) becomes

$$\tilde{\phi}_0(t) = \frac{1}{1-q} \sum_{k=0}^{\infty} \frac{t^k}{k+1} \left[\sum_{m=0}^{k} G_m(k+1)\right].$$

Since representation of $\tilde{\phi}_0(t)$ in form (3.5) is unique, then

$$\phi_0(0,k) = \frac{\sum_{m=0}^{k} G_m(k+1)}{(1-q)\,(k+1)}\,, \qquad k = 0, 1, \dots\ . \tag{3.12}$$

Finally, taking into account that

$$\sum_{m=0}^{k} G_m(k+1) = \sum_{m=0}^{k} (k+1-m)\, g_m(k+1)\,,$$

we write

$$\phi_0(0,k) = \frac{\sum_{m=0}^{k}(k+1-m)\, g_m(k+1)}{(1-q)\,(k+1)}\,, \qquad k = 0, 1, \dots\ .$$

3.2.3 Mathematical appendix 2

In the case when the initial capital is greater than zero, equation (3.7) implies

$$\tilde{\phi}(z,t) = \left(\frac{1}{1-z} - \frac{1-q}{z}\, t\, \tilde{\phi}_0(t)\right) \Bigg/ \left(1 - t\, \frac{\tilde{g}(z,1)}{z}\right). \tag{3.13}$$

To represent the right-hand side of this equality as a series in powers of t, we write

$$\begin{aligned}
\left(\frac{1}{1-z}\right) \Bigg/ \left(1 - t\, \frac{\tilde{g}(z,1)}{z}\right) &= \frac{1}{1-z}\left(1 + t\, \frac{\tilde{g}(z,1)}{z} + t^2\, \frac{\tilde{g}^2(z,1)}{z^2} + \dots\right) \\
&= \frac{1}{1-z}\left(1 + t\, \frac{\tilde{g}(z,1)}{z} + t^2\, \frac{\tilde{g}(z,2)}{z^2} + \dots + t^k\, \frac{\tilde{g}(z,k)}{z^k} + \dots\right) \\
&= \sum_{k=0}^{\infty} t^k\, \frac{\tilde{g}(z,k)}{(1-z)\, z^k}\,,
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{1-q}{z}\, t\, \tilde{\phi}_0(t)\right) \Bigg/ \left(1 - t\, \frac{\tilde{g}(z,1)}{z}\right) &= \sum_{k=0}^{\infty} \frac{1-q}{z}\, \phi_0(0,k)\, t^{k+1} \sum_{m=0}^{\infty} t^m\, \frac{\tilde{g}(z,m)}{z^m} \\
&= \sum_{l=1}^{\infty} t^l\, a_l\,,
\end{aligned}$$

where

$$a_l = (1-q) \sum_{m=0}^{l-1} \tilde{g}(z,m)\, \phi_0(l-m-1)\, z^{-m-1}\,.$$

Substituting these in (3.13) we equate the coefficients in front of $t^k,\ k \geq 1$:

$$\tilde{\phi}_1(z,k) = \frac{\tilde{g}(z,k)}{(1-z)\,z^k} - (1-q)\,\sum_{m=0}^{k-1} \tilde{g}(z,m)\,\phi_0(k-m-1)\,z^{-m-1}$$

or

$$z^k\,\tilde{\phi}_1(z,k) = \frac{\tilde{g}(z,k)}{(1-z)} - (1-q)\,\sum_{m=0}^{k-1} \tilde{g}(z,m)\,\phi_0(k-m-1)\,z^{k-m-1}\,. \quad (3.14)$$

If $k = 0$, then (3.13) reduces to

$$\tilde{\phi}_1(z,0) = \sum_{x=0}^{\infty} \phi(x,0)\,z^x = \sum_{x=0}^{\infty} z^x = \frac{1}{1-z}\,.$$

Noting that

$$z^k\,\tilde{\phi}_1(z,k) = \sum_{j=0}^{\infty} \phi(j,k)\,z^{j+k}\,,$$

and

$$\begin{aligned}
\frac{\tilde{g}(z,k)}{(1-z)} &= \sum_{i=0}^{\infty} g_i(k)\,z^i\,\sum_{j=0}^{\infty} z^j \\
&= \Big[g_0(k) + g_1(k)\,z + g_2(k)\,z^2 + \ldots\Big] \times \Big[1 + z + z^2 + \ldots\Big] \\
&= g_0(k) + z\,\big[g_1(k) + g_0(k)\big] + z^2\,\big[g_2(k) + g_1(k) + g_0(k)\big] + \ldots \\
&= \sum_{i=0}^{\infty} G_i(k)\,z^i\,,
\end{aligned}$$

and

$$\begin{aligned}
&(1-q)\,\sum_{m=0}^{k-1} \tilde{g}(z,m)\,\phi_0(k-m-1)\,z^{k-m-1} \\
&\qquad\qquad = (1-q)\,\sum_{m=0}^{k-1} \phi(0,k-m-1)\,\sum_{j=0}^{\infty} z^{j+k-m-1}\,g_j(m)\,,
\end{aligned}$$

we can rewrite (3.13) in the form

$$\begin{aligned}
\sum_{j=0}^{\infty} z^{j+k}\,\phi(j,k) &= \sum_{i=0}^{\infty} G_i(k)\,z^i \\
&\qquad - (1-q)\,\sum_{m=0}^{k-1} \phi(0,k-m-1)\,\sum_{j=0}^{\infty} z^{j+k-m-1}\,g_j(m)\,.
\end{aligned}$$

Changing summation indices to $i = j + k$ in the first sum and to $i = j + k - 1 - m$ in the last sum, we obtain

$$\sum_{i=0}^{\infty} z^i \, \phi(i-k,k) = \sum_{i=0}^{\infty} G_i(k) \, z^i \\ -(1-q) \sum_{m=0}^{k-1} \phi(0, k-m-1) \quad \sum_{i=k-1-m}^{\infty} z^i \, g_{i+m+1-k}(m) \, .$$

We rearrange the last term in the latter relation:

$$\sum_{i=0}^{\infty} z^i \, \phi(i-k,k) = \sum_{i=0}^{\infty} G_i(k) \, z^i \\ -(1-q) \sum_{m=0}^{k-1} \sum_{i=k-1-m}^{k-1} z^i \, \phi(0, k-m-1) \, g_{i+m+1-k}(m) \\ -(1-q) \sum_{m=0}^{k-1} \sum_{i=k}^{\infty} z^i \, \phi(0, k-m-1) \, g_{i+m+1-k}(m) \, ,$$

and change the order of summation:

$$\sum_{i=0}^{\infty} z^i \, \phi(i-k,k) = \sum_{i=0}^{\infty} G_i(k) \, z^i \\ -(1-q) \sum_{i=0}^{k-1} z^i \sum_{m=k-1-i}^{k-1} \phi(0, k-m-1) \, g_{i+m+1-k}(m) \\ -(1-q) \sum_{i=k}^{\infty} z^i \sum_{m=0}^{k-1} \phi(0, k-m-1) \, g_{i+m+1-k}(m) \, .$$

Equating coefficients in front of z^i, we have

$$\phi(i-k,k) = G_i(k) - (1-q) \sum_{m=0}^{k-1} \phi(0, k-m-1) \, g_{i+m+1-k}(m) \, ,$$

for $i \geq k \geq 1$. In other words, for $x = 0, 1, \ldots$ and $k = 1, 2, \ldots$

$$\phi(x,k) = G_{x+k}(k) - (1-q) \sum_{m=0}^{k-1} \phi(0, k-m-1) \, g_{x+m+1}(m) \, .$$

3.2.4 Mathematical appendix 3

Equation (3.12) implies

$$\phi(0,k)=\frac{\sum_{m=0}^{k}\Big[1-\big(1-G_m(k+1)\big)\Big]}{(1-q)\,(k+1)}=\frac{k+1-\sum_{m=0}^{k}\big(1-G_m(k+1)\big)}{(1-q)\,(k+1)}$$

$$=\frac{k+1-(k+1)\,q\,\mu+\sum_{m=k+1}^{\infty}\big(1-G_m(k+1)\big)}{(1-q)\,(k+1)}\,.$$

Here we used the relation

$$\begin{aligned}\sum_{m=0}^{\infty}\big(1-G_m(k+1)\big)&=\big(1-G_0(k+1)\big)+\big(1-G_2(k+1)\big)+\dots\\&=P\big(\{\omega:\ X(k+1)>0\}\big)+P\big(\{\omega:\ X(k+1)>1\}\big)+\dots\\&=\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}P\big(\{\omega:\ X(k+1)=i\}\big)\\&=E\big(X(k+1)\big)=(k+1)\,q\,\mu\,.\end{aligned}$$

This latter relation also implies the convergence of the series

$$\sum_{m=0}^{\infty}\big(1-G_m(k+1)\big)$$

since the sequence of its partial sums is monotonically increasing and it is bounded from above by $(k+1)\,q\,\mu$.

Thus, the probability of non-bankruptcy on $[0,k]$ has the following analytical form

$$\phi(0,k)=\frac{1-q\,\mu}{1-q}+\frac{\sum_{m=k+1}^{\infty}\big(1-G_m(k+1)\big)}{(1-q)\,(k+1)}\,.$$

An expression for the probability of non-bankruptcy on an infinite interval can be obtained directly from (3.1) by passing to the limit as $k\to\infty$:

$$\phi(j)=(1-q)\,\phi(j+1)+q\,E\big(\phi(j+1-X_1)\big)\,,\quad j=0,1,2,\dots\,,$$

or

$$\phi(j+1)-\phi(j)=q\,\Big[\phi(j+1)-E\big(\phi(j+1-X_1)\big)\Big]\,,\quad j=0,1,2,\dots\,.$$

Summing in j from 0 to $k-1$, we obtain

$$\phi(k)-\phi(0)=q\left[\sum_{j=1}^{k}\phi(j)-E\bigg(\sum_{j=1}^{k}\phi(j-X_1)\bigg)\right],\quad k=1,2,\dots$$

or

$$\phi(k) - (1-q)\,\phi(0) = q\left[\sum_{j=1}^{k}\phi(j) - E\Bigg(\sum_{j=1}^{k}\phi(j-X_1)\Bigg)\right], \quad k = 1, 2, \ldots \tag{3.15}$$

Introduce function

$$1_+(j) := \begin{cases} 1\,, & j = 0, 1, 2, \ldots \\ 0\,, & j = -1, -2, \ldots\,. \end{cases}$$

For a pair of integer-valued functions f and g we define their convolution:

$$\big(f * g\big)(j) := \sum_{i=-\infty}^{\infty} f(j-i)\, g(i)\,.$$

If $f(i) = g(i) = 0$ for $i = -1, -2, \ldots$, then

$$\big(f * g\big)(j) = \sum_{i=0}^{j} f(j-i)\, g(i)\,.$$

Now, since

$$\sum_{j=0}^{k}\phi(j) = \big(\phi * 1_+\big)(k)\,,$$

$$\sum_{j=1}^{k}\phi(j - X_1) = \sum_{j=0}^{k}\phi(j - X_1) = \big(\phi * 1_+\big)(k - X_1)\,,$$

then we can rewrite equation (3.15) in the form

$$\begin{aligned} \phi(k) - (1-q)\,\phi(0) &= q\left[\big(\phi * 1_+\big)(k) - E\Big(\big(\phi * 1_+\big)(k - X_1)\Big)\right] \qquad (3.16) \\ &= q\left[\big(\phi * 1_+\big)(k) - \big(\phi * 1_+ * f\big)(k)\right] \\ & \qquad k = 1, 2, \ldots\,;\, f(n) = f_n\,. \end{aligned}$$

Since $f(0) = 0$, then (3.16) also holds for $k = 0$. Now we can extend (3.16) to all integers k:

$$\phi(k) - (1-q)\,\phi(0)\,1_+(k) = q\left[\big(\phi * 1_+\big)(k) - \big(\phi * 1_+ * f\big)(k)\right]. \tag{3.17}$$

Introduce function

$$\delta(j) := \begin{cases} 1\,, & j = 0 \\ 0\,, & j \neq 0\,. \end{cases}$$

Then (3.17) can be written in the form

$$\phi(k) * \Big[\delta(k) - q\,\big[1_+(k) * \big(\delta(k) - f(k)\big)\big]\Big] = c\,1_+(k)\,,$$

where $c = (1-q)\,\phi(0)$.

A solution to this equation can be written in the form of the following Neumann series

$$\phi(k) = c\sum_{n=0}^{\infty} q^n \Big[\big(\delta(k) - f(k)\big)^{*n} * 1_+^{*(n+k)}(k)\Big]\,,$$

where $g^{*0} = \delta,\; g^{*n} = g^{*(n-1)} * g,\; n = 1, 2, \ldots$.

If $k \to \infty$, then (3.16) gives

$$\begin{aligned} 1 - (1-q)\,\phi(0) &= q\sum_{j=-\infty}^{\infty}\Big[1_+(j) - \big(1_+ * f\big)(j)\Big] \\ &= q\sum_{j=0}^{\infty}\Big[1 - P\big(\{\omega:\ X_1 \le j\}\big)\Big] = q\,\mu\,. \end{aligned}$$

Hence

$$\phi(0) = \frac{1 - q\,\mu}{1-q}\,.$$

3.2.5 Mathematical appendix 4

Introduce function

$$\tilde\phi(z) := \sum_{x=0}^{\infty}\phi(x)\,z^x\,.$$

Taking into account

$$\begin{aligned} \lim_{t\nearrow 1}(t-1)\,\tilde\phi(z,t) &= \lim_{t\nearrow 1}\sum_{k=0}^{\infty}\tilde\phi_1(z,k)\,t^k\,(1-t) \\ &= \lim_{t\nearrow 1}\Big[\tilde\phi_1(z,0)\,(1-t) + \tilde\phi_1(z,1)\,t\,(1-t) + \ldots + \tilde\phi_1(z,k)\,t^k\,(1-t) + \ldots\Big] \\ &= \lim_{t\nearrow 1}\Big[\tilde\phi_1(z,0) + t\,\big(\tilde\phi_1(z,1) - \tilde\phi_1(z,0)\big) + \ldots\Big] \\ &= \tilde\phi_1(z,\infty) \equiv \tilde\phi(z) \end{aligned}$$

and equation (3.13), we obtain

$$\begin{aligned} \tilde\phi(z) &= \lim_{t\nearrow 1}(t-1)\,\tilde\phi(z,t) = -\frac{1-q}{z - \tilde g\,\tilde\phi(z,1)}\;\lim_{t\nearrow 1}(t-1)\,\tilde\phi_0(t) \\ &= \frac{1-q\,\mu}{\tilde g(z,1) - z} = \frac{1}{1-z}\,\frac{1-q\,\mu}{1-q\,\mu\,\tilde b(z)}\,, \end{aligned}$$

where

$$b(z) = \frac{\tilde{g}(z,1) - 1}{q\,\mu\,(z-1)} = \frac{\tilde{f}(z) - 1}{\mu\,(z-1)}\,.$$

Also note that

$$\phi(x) = \frac{d^x \tilde{\phi}(z)}{dz^x}\,\frac{\tilde{\phi}(z)}{x!}\bigg|_{z=0}\,.$$

3.3 Solvency of an insurance company and investment portfolios

As in Chapter 1, we consider a binomial (B, S)-market. The dynamics of this market are described by equations

$$\begin{aligned} \Delta B_n &= r\,B_{n-1}, \quad B_0 > 0 \\ \Delta S_n &= \rho_n\,S_{n-1}, \quad S_0 > 0,\ n \le N\,, \end{aligned}$$

where $r \geq 0$ is a constant rate of interest with $-1 < a < r < b$, and profitabilities

$$\rho_n = \begin{cases} b \text{ with probability } p \in [0,1] \\ a \text{ with probability } q = 1 - p \end{cases}, \quad n = 1, \ldots, N\,,$$

form a sequence of independent identically distributed random variables.

Suppose that an insurance company with the initial capital $x = R_0$ forms an investment portfolio (β_1, γ_1) at time $n = 0$, so that

$$R_0 = \beta_1\,B_0 + \gamma_1\,S_0\,.$$

At time $n = 1$ the capital of the company is

$$R_1 = \beta_1\,B_1 + \gamma_1\,S_1 + c - Z_1\,,$$

where c is the premium income and Z_1 is a non-negative random variable representing total claims payments during this time period. This capital is reinvested into portfolio (β_2, γ_2):

$$R_1 = \beta_2\,B_1 + \gamma_2\,S_1\,.$$

At any time n we have

$$R_n = \beta_n\,B_n + \gamma_n\,S_n + c - Z_n\,,$$

where predictable sequence $\pi = (\beta_n, \gamma_n)_{n\geq 0}$ is an *investment strategy* and Z_n is a non-negative random variable representing total claims payments during the time step from $n-1$ to n. The distribution function of Z_n is denoted $F_{Z_n} \equiv F_Z$. It is assumed that sequence $(Z_n)_{n\geq 0}$ of independent identically distributed random variables is also independent of the dynamics of market assets B and S.

Thus, the dynamics of the capital of the insurance company have the form

$$\begin{aligned} R_{n+1} &= \beta_n\, B_{n+1} + \gamma_n\, S_{n+1} + c - Z_{n+1} \\ &= R_n\,(1+r) + \gamma_n\, S_n\,(\rho_{n+1} - r) + c - Z_{n+1}\,. \end{aligned}$$

As we discussed in the previous section, the probability of bankruptcy (or insolvency)

$$P\big(\{\omega:\ R_n < 0 \quad \text{for some} \quad n \geq 0\}\big)$$

is one of the typical measures used in the insurance risk management. Now we study this measure taking into account the investment strategies of an insurance company.

We start with the case when a company invests only in the non-risky asset B. In this case

$$R_{n+1} = R_n\,(1+r) + c - Z_{n+1}\,.$$

First we compute the probability of insolvency over the finite time interval $[0, k]$:

$$\psi_k(R_0) = P\big(\{\omega:\ R_n < 0 \quad \text{for some} \quad n \leq k\}\big)\,.$$

Note that ψ is an increasing function of k and R_0.

The probability of insolvency after one time step is given by

$$\begin{aligned} \psi_1(R_0) &= P\big(\{\omega:\ R_1 < 0\}\big) = P\big(\{\omega:\ R_0\,(1+r) + c - z_1 < 0\}\big) \\ &= P\big(\{\omega:\ z_1 > R_0\,(1+r) + c\}\big) = 1 - F_z\big(R_0(1+r) + c\big)\,. \end{aligned}$$

The probability of insolvency after two steps is

$$\begin{aligned} \psi_2(R_0) &= P\Big(\{\omega:\ R_1 < 0\} \cup \{\omega:\ R_1 > 0,\ R_2 < 0\}\Big) \\ &= P\big(\{\omega:\ R_1 < 0\}\big) + P\big(\{\omega:\ R_1 > 0,\ R_2 < 0\}\big) \\ &= \psi_1(R_0) + \int_{\{\omega:\ R_1 > 0,\, R_2 < 0\}} dF_{Z_1}\, dF_{Z_2} \\ &= \psi_1(R_0) + \int_0^{R_0(1+r)+c} \int_{R_1(1+r)+c}^{\infty} dF_{Z_2}\, dF_{Z_1} \\ &= \psi_1(R_0) + \int_0^{R_0(1+r)+c} \psi_1(R_1)\, dF_{Z_1} \\ &= \psi_1(R_0) + \int_0^{R_0(1+r)+c} \psi_1(R_0\,(1+r) + c - Z_1)\, dF_{Z_1}\,. \end{aligned}$$

And after three steps:

$$
\begin{aligned}
&\psi_3(R_0)\\
&= P\Big(\{\omega:\ R_1 < 0\} \cup \{\omega:\ R_1 > 0,\ R_2 < 0\} \cup \{\omega:\ R_1 > 0,\ R_2 > 0,\ R_3 < 0\}\Big)\\
&= \psi_1(R_0) + \int^{\{\omega:\, R_1>0,\, R_2<0\}} dF_{Z_1}\, dF_{Z_2}\\
&\qquad\qquad\qquad\qquad + \int_{\{\omega:\, R_1>0,\, R_2>0,\, R_3<0\}} dF_{Z_1}\, dF_{Z_2}\, dF_{Z_3}\\
&= \psi_1(R_0) + \int_0^{R_0(1+r)+c} \int_{R_1(1+r)+c}^{\infty} dF_{Z_2}\, dF_{Z_1}\\
&\qquad\qquad + \int_0^{R_0(1+r)+c} \int_0^{R_1(1+r)+c} \int_{R_2(1+r)+c}^{\infty} dF_{Z_3}\, dF_{Z_2}\, dF_{Z_1}\\
&= \psi_1(R_0) + \int_0^{R_0(1+r)+c} \psi_1(R_1)\, dF_{Z_1}\\
&\qquad\qquad\qquad\qquad + \int_0^{R_0(1+r)+c} \int_0^{R_1(1+r)+c} \psi_1(R_2)\, dF_{Z_2}\, dF_{Z_1}\\
&= \psi_1(R_0)\\
&\quad + \int_0^{R_0(1+r)+c} \left[\psi_1(R_1) + \int_0^{R_1(1+r)+c} \psi_1\Big(R_1\,(1+r) + c - Z_2\Big)\, dF_{Z_2}\right] dF_{Z_1}\\
&= \psi_1(R_0) + \int_0^{R_0(1+r)+c} \psi_2(R_1)\, dF_{Z_1}\\
&= \psi_1(R_0) + \int_0^{R_0(1+r)+c} \psi_2(R_0(1+r) + c - Z_1)\, dF_{Z_1}\,.
\end{aligned}
$$

Using mathematical induction we obtain that the probability of insolvency after $k+1$ steps is

$$
\psi_{k+1}(R_0) = 1 - F_Z\big(R_0(1+r) + c\big) + \int_0^{R_0(1+r)+c} \psi_k(R_0(1+r) + c - y)\, dF_y\,,
$$

with

$$
\psi_1(R_0) = 1 - F_Z\big(R_0(1+r) + c\big)\,.
$$

The *probability of solvency* over the time period $[0, n]$ is

$$\phi_n(x) = P\big(\{\omega :\ R_1 > 0,\ R_2 > 0,\ \ldots\ R_n > 0, \}\big)\,.$$

For illustration we consider a particular example of

$$F_Z(x) \equiv P\big(\{\omega :\ Z_i \le x\}\big) = 1 - e^{-\lambda\, x}\,.$$

The capital of the company can be represented in the form

$$\begin{aligned} R_n &= R_{n-1}\,(1+r) + c - Z_n \\ &= (1+r)\,\Big[R_{n-2}\,(1+r) + c - Z_{n-1}\Big] + c - Z_n \\ &= R_{n-2}\,(1+r)^2 + c\,\big[1 + (1+r)\big] - Z_{n-1}\,(1+r) - Z_n \\ &= \ldots \\ &= R_0\,(1+r)^n + c\,\big[1 + (1+r) + (1+r)^2 + \ldots + (1+r)^{n-1}\big] \\ &\quad -Z_1\,(1+r)^{n-1} - Z_2\,(1+r)^{n-2} - \ldots - Z_{n-1}\,(1+r) - Z_n \\ &= R_0\,(1+r)^n + c\,\frac{(1+r)^n - 1}{r} - S_n\,, \end{aligned}$$

where

$$S_n = Z_1\,(1+r)^{n-1} + Z_2\,(1+r)^{n-2} + \ldots + Z_{n-1}\,(1+r) + Z_n\,.$$

Note that

$$\phi_n(x) = \int_D \lambda^n\, e^{-\lambda\,(z_1 + \ldots + z_n)}\, dz_1 \ldots dz_n\,,$$

where

$$D = \Bigg\{ 0 < z_1 < R_0(1+r) + c\,, \\ 0 < z_k < R_0\,(1+r)^k + c\,\frac{(1+r)^k - 1}{r} - z_1\,(1+r)^{k-1} - \ldots - z_{k-1}\,(1+r), \\ k = 2, \ldots, n \Bigg\}\,.$$

The integral equation for the probability of insolvency has the form

$$\psi_{k+1}(x) = e^{-\lambda[x(1+r)+c]} + \int_0^{x(1+r)+c} \psi_k\big(x(1+r) + c - y\big)\,\lambda\, e^{-\lambda y}\, dy$$

with

$$\psi_1(x) = e^{-\lambda[x(1+r)+c]}\,.$$

Compute the probability of insolvency after two steps:

$$\begin{aligned}\psi_2(x) &= \psi_1(x) + \int_0^{x(1+r)+c} \psi_1\big(x(1+r)+c-y\big)\,\lambda\,e^{-\lambda y}\,dy \\ &= \psi_1(x) + \int_0^{x(1+r)+c} e^{-\lambda\left([x(1+r)+c-y](1+r)+c-y\right)}\,\lambda\,e^{-\lambda y}\,dy \\ &= \psi_1(x) + e^{-\lambda\left(x(1+r)^2+c(2+r)\right)} \int_0^{x(1+r)+c} \lambda\,e^{yr}\,dy \\ &= e^{-\lambda(x(1+r)+c)} + e^{-\lambda\left(x(1+r)^2+c(2+r)\right)}\, \frac{e^{\lambda r}}{r}\Bigg|_0^{x(1+r)+c} \\ &= e^{-\lambda(x(1+r)+c)}\left(1+\frac{e^{-\lambda c}}{r}\right) - \frac{e^{-\lambda\left(x(1+r)^2+c(1+(1+r))\right)}}{r}\,.\end{aligned}$$

For probability of insolvency after infinite number of steps we have (see Section 3.3.1: Mathematical appendix 5)

$$\begin{aligned}&\psi_\infty(x) \\ &= b\left[e^{-\lambda(x(1+r)+c)} + \sum_{m=2}^{\infty}(-1)^{m-1}\frac{e^{-\lambda\left(x(1+r)^m+c(1+(1+r)+\ldots+(1+r)^{m-1})\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^{m-1}-1]}\right],\end{aligned}$$

where

$$b = \left(1 - \sum_{m=1}^{\infty}(-1)^{m-1}\frac{e^{-\lambda c\left(1+(1+r)+\ldots+(1+r)^{m-1}\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}\right)^{-1}.$$

If the rate of interest $r=0$, then the equation for the probability of insolvency has the form

$$\tilde\psi_{k+1}(x) = \tilde\psi_1(x) + \int_0^{x+c} \tilde\psi_k(x+c-y)\,dF(y)$$

with

$$\tilde\psi_1(x) = 1 - F(x+c)\,.$$

In the case of the exponential distribution function $F(y) = 1 - e^{-\lambda y}$, we obtain

$$\tilde\psi_1(x) = e^{-\lambda(x+c)}\,,$$

and

$$\begin{aligned}\tilde\psi_2(x) &= \tilde\psi_1(x) + \int_0^{x+c} e^{-\lambda(x+2c-y)}\,\lambda\,e^{-\lambda y}\,dy \\ &= e^{-\lambda(x+c)} + e^{-\lambda(x+2c)}\,\lambda\,(x+c)\,.\end{aligned}$$

Note that these formulae can be also obtained by passing to the limit in expressions for ψ_1 and ψ_2:

$$
\begin{aligned}
\lim_{r\to 0}\psi_1(x) &= e^{-\lambda(x+c)}\,,\\
\lim_{r\to 0}\psi_2(x) &= e^{-\lambda(x+c)} - \lim_{r\to 0}\frac{e^{-\lambda\left(x(1+r)^2+c(2+r)\right)} - e^{-\lambda c}}{r}\\
&= e^{-\lambda(x+c)} - e^{-\lambda\left(x(1+r)^2+c(2+r)\right)}\Big|_{r=0}\\
&= e^{-\lambda(x+c)} + \lambda\,(x+2c)\,e^{-\lambda(x+c)} = \tilde{\psi}_2(x)\,.
\end{aligned}
$$

Next we consider the case when an insurance company invests in both risky and non-risky assets. By

$$
\alpha_n = \frac{\gamma_{n+1}\,S_n}{R_n}
$$

we denote the proportion of the risky asset in the investment portfolio. Let us consider a class of strategies with constant proportion $\alpha_n \equiv \alpha$. In the case of the exponential distribution function F, we will obtain an estimate from above for function ψ_∞, and hence for ψ_k since

$$
\psi_1(x) < \psi_2(x) < \ldots < \psi_k(x) < \ldots < \psi_\infty(x)\,.
$$

Note that γ_{n+1} is the number of units of asset S that a company buys at time n after collecting premium c and making claim payment Z_n, so that its capital is R_n.

The dynamics of the capital are given by

$$
\begin{aligned}
R_{n+1} &= R_n\,(1+r) + \gamma_n\,S_n\,(\rho_n - r) + c - Z_{n+1}\\
&= R_n\left(1 + r + \alpha\,(\rho_n - r)\right) + c - Z_{n+1}\,.
\end{aligned}
$$

Hence the probability of insolvency after one step is

$$
\begin{aligned}
\psi_1(R_0) &= P\left(\{\omega:\ R_1 < 0\}\right)\\
&= P\left(\{\omega:\ R_0\,[1+r+\alpha\,(\rho_1 - r)] + c - Z_1 < 0\}\right)\\
&= 1 - F_Z\Big(R_0\,[1+r+\alpha\,(\rho_1 - r)] + c\Big)\\
&= 1 - p\,F_Z\Big(R_0\,[1+r+\alpha\,(b - r)] + c\Big)\\
&\qquad\qquad -q\,F_Z\Big(R_0\,[1+r+\alpha\,(a - r)] + c\Big)\,.
\end{aligned}
$$

As in the previous case, we obtain the following integral equation

$$\begin{aligned}\psi_{k+1}(R_0) = 1 - p\,F_Z\Big(R_0\,[1+r+\alpha\,(b-r)]+c\Big) \\ -q\,F_Z\Big(R_0\,[1+r+\alpha\,(a-r)]+c\Big) \\ +p\int_0^{R_0\,[1+r+\alpha\,(b-r)]+c} \psi_k\Big(R_0\,[1+r+\alpha\,(b-r)]+c-y\Big)\,dF_Z(y) \\ +q\int_0^{R_0\,[1+r+\alpha\,(a-r)]+c} \psi_k\Big(R_0\,[1+r+\alpha\,(a-r)]+c-y\Big)\,dF_Z(y)\,.\end{aligned}$$

For the exponential claims distribution function $F(y) = 1 - e^{-\lambda y}$, we have the following estimate

$$\psi_\infty(x) \le \psi_1(x)\left[1 - e^{-\lambda c}\,\frac{r + p\,\alpha\,(b-r) + q\,\alpha\,(a-r)}{[r+\alpha\,(b-r)]\,[r+\alpha\,(b-r)]}\right]^{-1}$$

under condition that

$$\frac{q\,\alpha\,(b-r) + p\,\alpha\,(a-r) + r}{e^{\lambda c}\,[r+\alpha\,(b-r)]\,[r+\alpha\,(b-r)] - r - q\,\alpha\,(b-r) - p\,\alpha\,(a-r)} > 0\,.$$

In particular, for $\alpha = 0$ (i.e., when investing in non-risky asset only), we have

$$\psi_\infty(x) \le \psi_1(x)\left[1 - \frac{e^{-\lambda c}}{r}\right]^{-1}$$

under condition $r > e^{-\lambda c}$.

If $\alpha = 1$ (i.e., if investing in risky asset only), then

$$\psi_\infty(x) \le \psi_1(x)\left[1 - e^{-\lambda c}\,\frac{p\,b + q\,a}{a\,b}\right]^{-1}$$

under condition

$$\frac{q\,b + p\,a}{b\,a\,e^{\lambda c} - q\,b - p\,a} > 0\,.$$

We can give the following interpretation of these estimates. Clearly, for all k and x we have

$$\psi_1(x) < \psi_2(x) < \ldots < \psi_k(x) < \ldots < \psi_\infty(x)\,.$$

Hence

$$\psi_\infty(x) < \overline{C}\,\psi_1(x)\,,$$

where $\overline{C}$ is independent of x, i.e., the probability of insolvency after infinite number of time steps can be estimated in terms of the the probability of insolvency after one step. Note that due to our additional assumptions, constant $\overline{C}$ is always positive. And since $\psi_\infty(x)$ is less or equal to 1, then these estimates are satisfactory if

$$\overline{C}\,\psi_1(x) < 1\,,$$

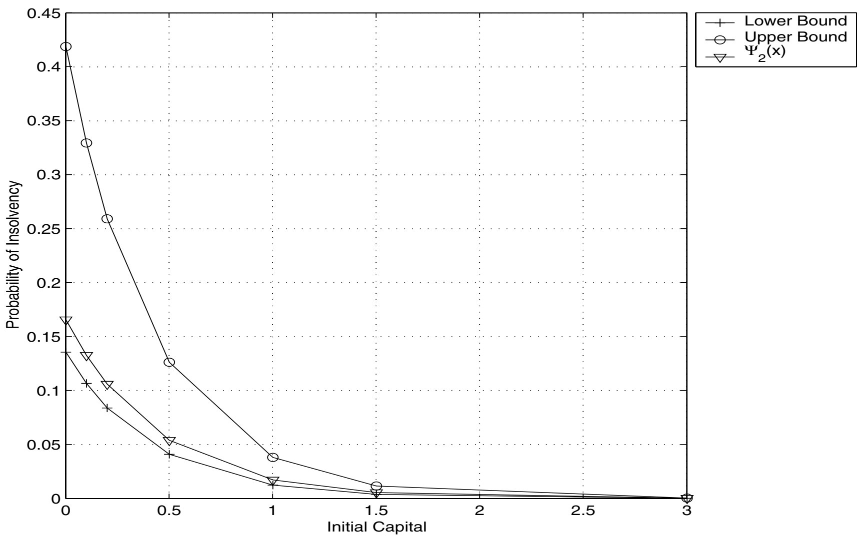
which holds true for sufficiently big initial capital x.

WORKED EXAMPLE 3.3

Let $r = 0.2,\ c = 1,\ \lambda = 2,\ \alpha = 1$. *Given values of the initial capital:* $0,\ 0.1,\ 0.2,\ 0.5,\ 1,\ 1.5,\ 3$*, compute values of* $\psi_1(x),\ \psi_2(x)$ *and upper estimate for* ψ_∞ *with accuracy* 0.0001*.*

SOLUTION The results are given in the following table and figure.

Initial capital	Lower bound	Upper bound	$\psi_2(x)$
0	0.1353	0.4186	0.1655
0.1	0.1065	0.3293	0.1325
0.2	0.0837	0.259	0.1059
0.5	0.0408	0.1261	0.0538
1	0.0123	0.03797	0.0171
1.5	0.0037	0.0114	0.0054
3	0.0001	0.0003	0.0002



∎

Consider a generalization of the Cramér-Lundberg model when it is assumed that an insurance company has an opportunity to invest in the framework of the Black-Scholes model of a (B, S)-market. Recall that the dynamics of the risky asset in this model are described by the following stochastic differential equation (see Section 2.6):

$$dS_t = S_t\big(\mu\, dt + \sigma\, dw_t\big)\,, \qquad S_0 > 0\,.$$

Then the capital of the company can be written in the form (see [17] for details):

$$R(t) = x + \mu \int_0^t R(s)\, ds + \sigma \int_0^t R(s)\, dw_s + c\, t - \sum_{k=1}^{N(t)} X_k\,.$$

In this case the probability of solvency ϕ satisfies the following integro-differential equation

$$\frac{1}{2}\,\sigma^2\, x^2\, \phi''(x) + (\mu\, x + c)\, \phi'(x) - \lambda\, \phi(x) + \lambda \int_0^x \phi(x-y)\, dF(y) = 0\,.$$

Analyzing the behavior of function ϕ as $x \to \infty$ in the case of exponential distribution function of claims

$$X_k \sim F(x) = 1 - e^{-x/\alpha} \qquad x > 0\,,$$

leads to the following result. If the profitability μ of asset S is greater than $\sigma^2/2$, where σ is the volatility of the market, then the probability of insolvency $\psi(x) = 1-\phi(x)$ converges to zero according to the following power law (not exponentially!):

$$\psi(x) = \mathcal{O}\big(x^{1-2\mu/\sigma^2}\big)\,.$$

If asset S is not profitable enough: $\mu < \sigma^2/2$, then for any initial capital $x > 0$ the probability of bankruptcy $\psi(x) = 1$.

3.3.1 Mathematical appendix 5

We will look for a solution of the form

$$\psi_k(x) = e^{-\lambda(x(1+r)+c)}\, b_1^k + \sum_{m=2}^{k} b_m^k\, \frac{e^{-\lambda\left(x(1+r)^m + c(1+(1+r)+\ldots+(1+r)^{m-1})\right)}}{r\,[(1+r)^2-1] \times \ldots \times [(1+r)^{m-1}-1]}\,,$$

where $\left(b_m^k\right)$ is a two-parameter sequence independent of x. Here parameter k corresponds to function ψ_k and parameter m corresponds to factor

$$\frac{e^{-\lambda\left(x(1+r)^m + c(1+(1+r)+\ldots+(1+r)^{m-1})\right)}}{r\,[(1+r)^2-1] \times \ldots \times [(1+r)^{m-1}-1]}\,.$$

Expressions for probabilities of insolvency after one and two time steps imply that

$$b_1^1 = 1$$

$$b_1^2 = 1 + \frac{e^{-\lambda c}}{r}\, b_2^2 = -1\,.$$

It is convenient to write sequence $\left(b_m^k\right)$ in the form of a triangular table:

b_1^1				
b_1^2	b_2^2			
b_1^3	b_2^3	b_3^3		
b_1^4	b_2^4	b_3^4	b_4^4	
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

From the recurrence equation we have

$$\psi_k(x\,(1+r)+c-y)$$

$$= \sum_{m=2}^{k} b_k^m \, \frac{e^{-\lambda\left(x(1+r)^{m+r}+c(1+(1+r)+\ldots+(1+r)^{m-1})-y\,(1+r)^m\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^{m-1}-1]}$$

$$+b_k^1\, e^{-\lambda[x(1+r)^2+c(1+(1+r))]}\,,$$

and

$$\int_0^{x\,(1+r)+c} \psi_k(x\,(1+r)+c-y)\,\lambda\, e^{-\lambda y}\, dy$$

$$= \sum_{m=2}^{k} b_k^m \, \frac{e^{-\lambda\left(x(1+r)^{m+1}+c(1+(1+r)+\ldots+(1+r)^m)\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^{m-1}-1]} \int_0^{x\,(1+r)+c} \lambda\, e^{\lambda\, y\,[(1+r)^m-1]}\, dy$$

$$+b_k^1\, e^{-\lambda[x(1+r)^2+c(1+(1+r))]} \int_0^{x\,(1+r)+c} \lambda\, e^{\lambda\, y\, r}\, dy$$

$$= \sum_{m=2}^{k} b_k^m \, \frac{e^{-\lambda\left(x(1+r)^{m+1}+c(1+(1+r)+\ldots+(1+r)^m)\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^{m-1}-1]} \times \left.\frac{e^{\lambda\,[(1+r)^m-1]}}{(1+r)^m-1}\right|_0^{x\,(1+r)+c}$$

$$+b_k^1\, e^{-\lambda[x(1+r)^2+c(1+(1+r))]}\, \frac{e^{\lambda\, r}}{r}\Bigg|_0^{x\,(1+r)+c}$$

$$= \sum_{m=2}^{k} b_k^m\, \frac{e^{-\lambda\left(x(1+r)+c(1+(1+r)+\ldots+(1+r)^{m-1}+c)\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}$$

$$-\sum_{m=2}^{k} \frac{e^{-\lambda\left(x(1+r)^{m+1}+c(1+(1+r)+\ldots+(1+r)^{m})\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}$$

$$+b_k^1\, \frac{e^{-\lambda[x(1+r)+c+c]}}{r} + b_k^1\, \frac{e^{-\lambda[x(1+r)^2+c(1+(1+r))]}}{r}\,.$$

Thus

$$\psi_{k+1}(x) = e^{-\lambda(x(1+r)+c)}$$

$$+\sum_{m=1}^{k} b_k^m\, e^{-\lambda(x(1+r)+c)}\, \frac{e^{-\lambda\, c(1+(1+r)+\ldots+(1+r)^{m-1})}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}$$

$$-\sum_{m=1}^{k} b_k^m\, \frac{e^{-\lambda\left(x(1+r)^{m+1}+c(1+(1+r)+\ldots+(1+r)^{m})\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}$$

$$= e^{-\lambda(x(1+r)+c)}\left[1+\sum_{m=1}^{k} b_k^m\, \frac{e^{-\lambda\, c(1+(1+r)+\ldots+(1+r)^{m-1})}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}\right]$$

$$-\sum_{m=1}^{k+1} b_k^{m-1}\, \frac{e^{-\lambda\left(x(1+r)^{m}+c(1+(1+r)+\ldots+(1+r)^{m-1})\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^{m-1}-1]}\,,$$

which implies

$$b_{k+1}^1 = 1+\sum_{m=1}^{k} b_k^m\, \frac{e^{-\lambda\, c(1+(1+r)+\ldots+(1+r)^{m-1})}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}\,,$$

$$b_{k+1}^m = -b_k^{m-1}\,.$$

So sequence $\left(b_m^k\right)$ has the following structure

b_1				
b_2	$-b_1$			
b_3	$-b_2$	b_1		
b_4	$-b_3$	b_2	$-b_1$	
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

where we introduced the notation: $b_i := b_i^1$ with $b_1 = 1$,

$$b_{k+1} = 1 + \sum_{m=1}^{k} (-1)^{m-1}\, b_{k-m+1}\, \frac{e^{-\lambda\, c(1+(1+r)+\ldots+(1+r)^{m-1})}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}\,,$$

Properties of

$$0 < \psi_1(x) < \psi_2(x) < \ldots < \psi_k(x) < \ldots$$

imply that sequence $\big(b_i\big)_{i=1}^{\infty}$ is positive and increasing.

Condition

$$\frac{e^{-\lambda c}}{r} < 1 \quad \text{i.e.} \quad P\big(\{\omega:\ Z_1 > c\}\big) < r \quad \text{or} \quad c > \frac{-\ln r}{\lambda}\,,$$

is sufficient for boundedness of $\big(b_i\big)_{i=1}^{\infty}$ and therefore for existence of finite $b = \lim_{i\to\infty} b_i$.

Then passing to the limit in

$$\psi_k(x) = e^{-\lambda(x(1+r)+c)}\, b_1^k + \sum_{m=2}^{k} b_m^k\, \frac{e^{-\lambda\left(x(1+r)^m + c(1+(1+r)+\ldots+(1+r)^{m-1})\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^{m-1}-1]}$$

we obtain

$$\psi_\infty(x) = b\left[e^{-\lambda(x(1+r)+c)} + \sum_{m=2}^{\infty}(-1)^{m-1}\frac{e^{-\lambda\left(x(1+r)^m + c(1+(1+r)+\ldots+(1+r)^{m-1})\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^{m-1}-1]}\right],$$

where

$$b = \left(1 - \sum_{m=1}^{\infty}(-1)^{m-1}\frac{e^{-\lambda c\left(1+(1+r)+\ldots+(1+r)^{m-1}\right)}}{r\,[(1+r)^2-1]\times\ldots\times[(1+r)^m-1]}\right)^{-1}.$$

3.4 Risks in traditional and innovative methods in life insurance

Life insurance clearly deals with various types of uncertainties, e.g. the uncertainty of future lifetimes, variable interest rates etc. Thus it is natural that stochastic methods are widely used in life insurance mathematics. In this section we discuss some survival models as one of the key ingredients of the stochastic approach.

Introduce a random variable T representing the future lifetime of a newborn individual, i.e., T is the time elapsed between birth and death. The distribution function of T is

$$F(x) = P\big(\{\omega :\ T \le x\}\big)\,, \quad x \ge 0\,.$$

Define the *survival function* as

$$s(x) = 1 - F(x) = P\big(\{\omega :\ T > x\}\big)\,, \quad x \ge 0\,.$$

In practice one usually introduces the *limiting age* (i.e., the age beyond which survival is supposed to be impossible). Traditionally it is denoted by ω. To avoid ambiguities, we will use letter ϖ instead. Thus, we have that $0 \le T \le \varpi < \infty$. Clearly, function $F(x)$ is increasing and continuous.

Next we define a random variable $T(x)$ to be the future lifetime of an individual of age x. Obviously, $T(0) = T$.

There is standard actuarial notation for probabilities in survival models: ${}_tp_x$ denotes the probability that an individual of age x survives to age $x + t$. Again, in order to avoid ambiguities, we will write $p_x(t)$ instead. Also $q_x(t) := 1 - p_x(t)$, and $p_x(1) := p_x,\ q_x(1) := q_x$.

From the definition of a conditional expectation we have

$$\begin{aligned} p_x(t) &= P\big(\{\omega :\ T(x) > t\}\big) = P\big(\{\omega :\ T > x + t \big| T > x\}\big) \\ &= \frac{p_0(x+t)}{p_0(x)} = \frac{s(x+t)}{s(x)}\,, \end{aligned}$$

and

$$q_x(t) = 1 - \frac{p_0(x+t)}{p_0(x)} = 1 - \frac{s(x+t)}{s(x)}\,.$$

One of the most widely used actuarial representations of the survival model is the *life table* (or *mortality table*). Suppose that l_0 is the number of newborn individuals, and let random variable $L(x)$ represent the number of individuals surviving to age x. The life table consists of set of expected values of $L(x)$:

$$l_x = E\big(L(x)\big) = l_0\, s(x)$$

for all $0 \le x \le \varpi$.

The following relations hold true

$$\begin{aligned} l_1 &= l_0\,(1 - q_0) = l_0\, p_0\,, \\ l_2 &= l_1\,(1 - q_0) = l_0\,(1 - q_0(2)) = l_0\, p_0\, p_1\,, \\ &\qquad\qquad \dots \\ l_x &= l_{x-1}\,(1 - q_{x-1}) = l_0\,(1 - q_0(x)) = \prod_{y=0}^{x-1} p_y = l_0\, p_0(x)\,. \end{aligned}$$

Example.

1. The probability that an individual of age 20 survives to the age of 100 is
$$p_{20}(80) = \frac{s(100)}{s(20)} = \frac{l_{100}}{l_{20}}\,.$$
2. The probability that an individual of age 20 dies before the age of 70 is
$$q_{20}(80) = \frac{s(80) - s(70)}{s(20)} = 1 - \frac{l_{70}}{l_{20}}\,.$$
3. The probability that an individual of age 20 survives to the age of 80 but dies before the age of 90 is
$$\frac{s(80) - s(90)}{s(20)} = \frac{l_{80} - l_{90}}{l_{20}}\,.$$

Introduce the notion of the *force of mortality* at age x as

$$\mu_x = \lim_{h \to 0^+} \frac{P\big(\{\omega :\ T \le x + h \,\big|\, T > x\}\big)}{h}\,, \quad 0 \le x < \varpi\,.$$

The following laws for μ_x are widely used in actuarial theory and practice

- Gompertz' formula: $\mu_x = B\,c^x$,
- Makeham's formula: $\mu_x = A + B\,c^x$.

Now we obtain an expression for density of the distribution function of $T(x)$:

$$\begin{aligned}
f_x(t) &= \frac{d}{dt} P\big(\{\omega :\ T(x) \le t\}\big) \\
&= \lim_{h \to 0^+} \frac{P\big(\{\omega :\ T(x) \le t + h\}\big) - P\big(\{\omega :\ T(x) \le t\}\big)}{h} \\
&= \lim_{h \to 0^+} \frac{P\big(\{\omega :\ T \le x + t + h \,|\, T > x\}\big) - P\big(\{\omega :\ T \le x + t \,|\, T > x\}\big)}{h} \\
&= \lim_{h \to 0^+} \left[\frac{P\big(\{\omega :\ T \le x + t + h\}\big) - P\big(\{\omega :\ T \le x\}\big)}{s(x)\,h} \right. \\
&\qquad\qquad \left. - \frac{P\big(\{\omega :\ T \le x + t\}\big) - P\big(\{\omega :\ T \le x\}\big)}{s(x)\,h} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0^+} \frac{P\big(\{\omega:\ T \le x+t+h\}\big) - P\big(\{\omega:\ T \le x+t\}\big)}{s(x)\,h} \\
&= \frac{s(x+t)}{s(x)} \lim_{h\to 0^+} \frac{P\big(\{\omega:\ T \le x+t+h\}\big) - P\big(\{\omega:\ T \le x+t\}\big)}{s(x+t)\,h} \\
&= p_x(t) \lim_{h\to 0^+} \frac{P\big(\{\omega:\ T \le x+t+h \,|\, T > x+t\}\big)}{h} \\
&= p_x(t)\,\mu_{x+t}\,, \quad 0 \le t \le \varpi - x\,.
\end{aligned}
$$

Further

$$
q_x(t) \equiv \int_0^t \frac{d}{ds}\, q_x(s)\, ds = \int_0^t f_x(s)\, ds = \int_0^t p_x(s)\, \mu_{x+s}\, ds\,,
$$

hence

$$
\frac{\partial}{\partial s} p_x(s) \equiv -\frac{\partial}{\partial s} q_x(s) = -p_x(s)\,\mu_{x+s}\,.
$$

Solving this differential equation for $p_x(t)$ with the initial condition $p_x(0) = 1$, we obtain

$$
p_x(t) = \exp\left\{ -\int_0^t \mu_{x+s}\, ds \right\}.
$$

These expressions for $q_x(t)$ and $p_x(t)$ are widely used for premium calculations in standard life insurance contracts.

We also introduce an integer-valued random variable $K(x) := [\![T(x)]\!]$, which obviously represents the number of whole years survived by an individual of age x. The set of its values is $\{0, 1, 2, \ldots, [\![\varpi - x]\!]\}$. We have

$$
\begin{aligned}
P\big(\{\omega:\ K(x) = k\}\big) &= P\big(\{\omega:\ k \le T(x) < k+1\}\big) \\
&= P\big(\{\omega:\ k < T(x) < k+1\}\big) = p_x(k)\, q_{x+k}\,.
\end{aligned}
$$

It is more convenient to use quantities $K(x)$ when using life tables.

A standard life insurance contract assumes payment of b_t at time t. If ν_t is the discount factor, then the present value (at time $t = 0$) of this payment is $z_t = b_t\,\nu_t$. Since the amount of payment b_t is set at the time of contract issue, then without loss of generality we can assume that $b_t = 1$.

First we consider contracts when benefits are paid upon the death of the insured individual (i.e., *life assured*). Let $Z = b_{T(x)}\,\nu_{T(x)}$, where x is the age of the life assured at the time of contract issue. The equivalence principle is used for premium calculations.

Term-life assurance pays a lump sum benefit upon the death of the life assured within a specified period of time, say within n-years term, i.e.,

$$b_t = \begin{cases} 1\,, & t \le n \\ 0\,, & t > n\,, \end{cases}$$

$$\nu_t = \nu^t\,, \quad t \ge 0\,,$$

$$Z = \begin{cases} \nu^{T(x)}\,, & T(x) \le n \\ 0\,, & T(x) > n\,. \end{cases}$$

The net-premium in this case is

$$\overline{A}^{\,1}_{x:\overline{n}|} = E(Z) = E(z_{T(x)}) = \int_0^\infty z_t\, f_x(t)\, dt = \int_0^n \nu^t\, p_x(t)\, \mu_{x+t}\, dt\,.$$

Whole life assurance pays a lump sum benefit upon the death of the life assured whenever it should occur:

$$b_t = 1\,, \quad t \ge 0\,,$$
$$\nu_t = \nu^t\,, \quad t \ge 0\,,$$
$$Z = \nu^{T(x)}\,, \quad T(x) \ge 0\,.$$

The net-premium is

$$\overline{A}_x = E(Z) = \int_0^\infty \nu^t\, p_x(t)\, \mu_{x+t}\, dt\,.$$

WORKED EXAMPLE 3.4

Consider 100 *whole life assurance contracts. Suppose that all life assured are of age x and the benefit payment is 10. Let discount factor be $\nu = e^{-\delta} = e^{-0.06}$ and $\mu = 0.04$. Compute the premium that guarantees the probability of solvency at* 0.95.

SOLUTION For an individual contract we have

$$b_t = 10\,, \quad t \ge 0\,,$$
$$\nu_t = \nu^t\,, \quad t \ge 0\,,$$
$$Z = 10\,\nu^{T(x)}\,, \quad T(x) \ge 0\,.$$

The risk process in this case is $S = \sum_{i=1}^{100} Z_i$.

For individual claims we have that payment amounts in the case of death are

$$\overline{A}_x = \int_0^\infty e^{-\delta t}\, e^{-\mu t}\, \mu\, dt = \frac{\mu}{\mu + \delta}$$

then

$$
E(Z) = 10\,\overline{A}_x = 10\,\frac{0.04}{0.1} = 4\,,
$$
$$
E(Z^2) = 10^2 \int_0^\infty e^{-2\delta}\, e^{-\mu t}\,\mu\, dt = 100\,\frac{0.04}{0.04 + 2\times 0.06} = 25\,,
$$

which also implies that $V(Z) = 9$.

The premium payment h can be found from the equation

$$
P\big(\{\omega:\ S \le h\}\big) = 0.95\,,
$$

which can be written in the form

$$
P\left(\left\{\omega:\ \frac{S - E(S)}{\sqrt{V(S)}} \le \frac{h - 400}{30}\right\}\right) = 0.95\,.
$$

Since random variable $\big(S - E(S)\big)/\sqrt{V(S)}$ is normal, we obtain

$$
\frac{h - 400}{30} \approx 1.645 \qquad \text{and} \qquad h \approx 449.35\,.
$$

Thus we have that the premium is higher than the expected claim payment. The corresponding security loading coefficient is

$$
\theta = \frac{h - E(S)}{E(S)} \approx 0.1234\,.
$$

□

Pure endowment assurance pays a lump sum benefit on survival of the life assured up to the end of a specified period of time, say up to the end of n years term:

$$
b_t = \begin{cases} 0\,, & t \le n \\ 1\,, & t > n\,, \end{cases}
$$

$$
\nu_t = \nu^n\,, \quad t \ge 0\,,
$$

$$
Z = \begin{cases} 0\,, & T(x) \le n \\ \nu^n\,, & T(x) > n\,. \end{cases}
$$

Net-premium is

$$
A^{\;\,1}_{x:\overline{n}|} = E(Z) = \nu^n\, p_x(n)\,.
$$

Endowment assurance pays a lump sum benefit on death of the life assured within a specified period of time, say within the n years term, or on survival of the life assured up to the end of this period:

$$b_t = 1\,,\quad t \geq 0\,,$$

$$\nu_t = \begin{cases} \nu^t\,, & t \leq n \\ \nu^n\,, & t > n\,, \end{cases}$$

$$Z = \begin{cases} \nu^{T(x)}\,, & T(x) \leq n \\ \nu^n\,, & T(x) > n\,. \end{cases}$$

This contract is obviously a combination of a pure endowment assurance and a term-life assurance:

$$Z_1 = \begin{cases} 0\,, & T(x) \leq n \\ \nu^n\,, & T(x) > n \end{cases} \qquad \text{and} \qquad Z_2 = \begin{cases} \nu^{T(x)}\,, & T(x) \leq n \\ 0\,, & T(x) > n\,, \end{cases}$$

respectively. Therefore the net-premium is

$$\overline{A}_{x:\overline{n}|} = E(Z) = E\big(Z_1 + Z_2\big) = \overline{A}^{\;\;1}_{x:\overline{n}|} + A^{1}_{x:\overline{n}|}\,.$$

Deferred whole life assurance pays a lump sum benefit upon the death of the life assured if it occurs at least, say, m years after issuing the contract:

$$b_t = \begin{cases} 1\,, & t > m \\ 0\,, & t \leq m\,, \end{cases}$$

$$\nu_t = \nu^t\,,\quad t > 0\,,$$

$$Z = \begin{cases} \nu^{T(x)}\,, & T(x) > m \\ 0\,, & T(x) \leq m\,. \end{cases}$$

The net-premium in this case is

$$_{m|}\overline{A}_x = E(Z) = \int_m^\infty \nu^t\, p_x(t)\, \mu_{x+t}\, dt\,.$$

Next we consider contracts with variable amounts of benefit paid upon the death of life assured.

Increasing whole life assurance :

$$\begin{aligned} b_t &= [\![t+1]\!]\,,\quad t \geq 0\,,\\ \nu_t &= \nu^t\,,\quad t \geq 0\,,\\ Z &= [\![T(x)+1]\!]\,\nu^{T(x)}\,,\quad T(x) \geq 0\,. \end{aligned}$$

Net-premium is

$$\big(I\overline{A}\big)_x = E(Z) = \int_0^\infty [\![t+1]\!]\,\nu^t\, p_x(t)\, \mu_{x+t}\, dt\,.$$

Decreasing term-life assurance :

$$b_t = \begin{cases} n - [\![t]\!]\,, & t \le n \\ 0\,, & t > n\,, \end{cases}$$

$$\nu_t = \nu^t\,, \quad t \ge 0\,,$$

$$Z = \begin{cases} \nu^{T(x)}\,(n - [\![T(x)]\!])\,, & T(x) \le n \\ 0\,, & T(x) > n\,. \end{cases}$$

Net-premium is

$$\left(D\overline{A}\right)^{1}_{x:\overline{n}|} = \int_0^n \nu^t\,(n - [\![t]\!])\,p_x(t)\,\mu_{x+t}\,dt\,.$$

One can consider variations of these contracts in the case when benefits are paid at the end of the year in which death occurred, i.e. at time $K(x)+1$. Some of them are presented in the following able. Note that we write k for $K(x)$ here.

Type of insurance	b_{k+1}	z_{k+1}	**Premium**
Whole life	1	ν^{k+1}	A_x
Term-life	$\begin{cases} 1\,, k \in K_0 \\ 0\,, k \in K_1 \end{cases}$	$\begin{cases} \nu^{k+1}\,, & k \in K_0 \\ 0\,, & k \in K_1 \end{cases}$	$A^{1}_{x:\overline{n}\|}$
Endowment assurance	1	$\begin{cases} \nu^{k+1}\,, & k \in K_0 \\ \nu^n\,, & k \in K_1 \end{cases}$	$A_{x:\overline{n}\|}$
Increasing term-life	$\begin{cases} n+1\,, & k \in K_0 \\ 0\,, & k \in K_1 \end{cases}$	$\begin{cases} (k+1)\,\nu^{k+1}\,, & k \in K_0 \\ 0\,, & k \in K_1 \end{cases}$	$(IA)^{1}_{x:\overline{n}\|}$
Decreasing term-life	$\begin{cases} n-1\,, & k \in K_0 \\ 0\,, & k \in K_1 \end{cases}$	$\begin{cases} (n-1)\,\nu^{k+1}\,, & k \in K_0 \\ 0\,, & k \in K_1 \end{cases}$	$(DA)^{1}_{x:\overline{n}\|}$
Increasing whole life	$k+1,\ k = 0, 1, \ldots$	$(k+1)\,\nu^{k+1}$	$(IA)_x$

Here $K_0 = \{0, \ldots, n-1\}$ and $K_1 = \{n, n+1, \ldots\}$.

Note that traditional insurance contracts considered in this section have an essential common feature with financial products studied in the first two chapters: contingent payments at some future dates. In traditional insurance theory it is assumed that the amounts of these payments are deterministic and all randomness is due to the uncertainty of future lifetimes. Due to market competition, some insurers (particularly investment companies, hedging funds, merchant bank, etc.) now offer more attractive (from the investment's point of view) 'options-type' insurance contracts whose structure depends on risky financial assets. These ideas gave rise to a new approach of *innovative methods* in insurance, which is usually referred to as *equity-linked life insurance*.

We begin our discussion of such *flexible insurance methods* by revisiting Worked Example 1.5 from Section 1.4, which is concerned with a pure endowment assurance contract in the framework of a binomial (B, S)-market.

Let $(\Omega_1, \mathcal{F}_N^1, \mathbb{F}_1, P_1)$ be a stochastic basis. Consider a binomial (B, S)-market with

$$\begin{aligned} \Delta B_n &= r\, B_{n-1}, \quad B_0 > 0 \\ \Delta S_n &= \rho_n\, S_{n-1}, \quad S_0 > 0,\ n \le N\,, \end{aligned}$$

where $r \ge 0$ is a constant rate of interest with $-1 < a < r < b$, and

$$\rho_n = \begin{cases} b \text{ with probability } p \in [0,1] \\ a \text{ with probability } q = 1 - p \end{cases}, \quad n = 1, \ldots, N\,,$$

form a sequence of independent identically distributed random variables.

Suppose that an insurance company issues l_x contracts with policy holders of age x. As before, random variable $T(x)$ represents the future lifetime of an individual of age x and $p_x(t) = P\big(\{\omega:\ T(x) > t\}\big)$.

Introduce a process

$$N_t^x = \sum_{i=1}^{l_x} I_{\{\omega:\ T_i(x) \le t\}}\,,$$

that counts the number of deaths during the time interval from 0 to t.

Random variables $T_1(x), T_2(x), \ldots T_{l_x}(x)$ are defined on a stochastic basis $(\Omega_2, \mathcal{F}_N^2, \mathbb{F}_2, P_2)$, where $\mathcal{F}_n^2 = \sigma(N_k^x,\ k \le n),\ n \le N$.

Thus, we have two sources of randomness: the future lifetime of life assured and the prices of assets of the financial market. It is natural to assume that these sources of randomness are independent. Hence, formally we have two probability spaces. One of them describes the dynamics of the market, and the other describes lifetimes of the life assured. The following stochastic basis

$$\big(\Omega_1 \otimes \Omega_2, \mathcal{F}_N^1 \otimes \mathcal{F}_N^2, \mathbb{F}_1 \otimes \mathbb{F}_2, P_1 \otimes P_2\big)\,,$$

naturally corresponds to the problem in consideration.

Consider a pure endowment assurance contract that pays a lump sum f_N upon survival of the life assured to the time N. The total amount of claims at time N is given by

$$\sum_{i=1}^{l_x} \frac{f_N}{B_N} I_{\{\omega:\, T_i(x)>N\}} .$$

Consider the case when $f_N = \max\{S_N, K\}$, where K is the guaranteed minimal payment. We wish to price this contingent claim, i.e., to calculate premium $U_x(N)$. One approach consists of applying the equivalence principle for the risk-neutral probability $P_1^* \otimes P_2$. Since S and T are independent, we have

$$\begin{aligned} U_x(N) &= \frac{1}{l_x} E^*\left(\sum_{i=1}^{l_x} \frac{f_N}{B_N} I_{\{\omega:\, T_i>N\}}\right) = p_x(N)\, E^*\left(\frac{K+(S_N-K)^+}{B_N}\right) \\ &= p_x(N)\, \frac{K}{(1+r)^N} + p_x(N)\left[S_0\, B(k_0, N, \widetilde{p}) - \frac{K}{(1+r)^N}\, B(k_0, N, p^*)\right] \end{aligned}$$

where p^* is a risk-neutral probability:

$$p^* = \frac{r-a}{b-a} \qquad \text{and} \qquad \tilde{p} = \frac{1+b}{1+a} p^* .$$

Recall (see Section 1.4) that

$$B(j, N, p) := \sum_{k=j}^{N} \binom{N}{k} p^k\, (1-p)^{N-k} ,$$

constant k_0 is defined by

$$k_0 = \min\left\{k \leqslant N:\ S_0(1+b)^k(1+a)^{N-k} \geqslant K\right\}$$

so that

$$k_0 = \left[\ln \frac{K}{S_0(1+a)^N} \Big/ \ln \frac{1+b}{1+a}\right] + 1 .$$

Alternatively, one can use hedging in mean square for computing the premium. Suppose that the discounted total amount of claims is

$$H = \sum_{i=1}^{l_x} Y_i \qquad \text{with} \qquad Y_i = \frac{g(S_N)}{B_N} I_{\{\omega:\, T_i(x)>N\}} ,$$

where function g determines amount of claim for an individual contract.

It was shown in Section 2.3 that the unique optimal (risk-minimizing) strategy $\hat{\pi} = (\hat{\beta}, \hat{\gamma})$ is given by

$$\hat{\gamma}_n = \gamma_n^H , \quad \hat{\beta}_n = V_n^* - \hat{\gamma}_n X_n , \quad n \le N ,$$

where X_n represents the capital of the portfolio, V_n^π represents the discounted capital of the portfolio, and

$$V_n^* = E^*\big(H\big|\mathcal{F}_n\big)\,, \quad n \le N\,,$$

with respect to a risk-neutral probability P^*. Sequence γ^H and martingale L^H are uniquely determined by the Kunita-Watanabe decomposition (see Lemma 2.2).

Also we have the price-sequence

$$C_n^{\hat\pi} = V_n^{\hat\pi} - \sum_{k=1}^{n} \hat\gamma_k\, \Delta X_k = E^*(H) + L_n^H\,,$$

and the risk-sequence

$$R_n^\pi = E^*\Big(\big(L_N^H - L_n^H\big)^2\big|\mathcal{F}_n\Big).$$

Note that this strategy $\hat\pi = (\hat\beta, \hat\gamma)$ is not self-financing, but it is self-financing in average.

As an illustration we consider a *lognormal* model of a financial market:

$$S_n = S_0\, e^{h_1+\ldots+h_n}, \quad h_i = \mu + \sigma\,\varepsilon_n \quad \text{and} \quad B_n = B_0\,(1+r)^n\,,$$

where ε_n are independent identically distributed normal random variables.

Let

$$g(S_N) = \max\big\{S_N,\, K\big\} = K + (S_N - K)^+\,,$$

where K is a constant.

Denote

$$h_i^* = \mu - \delta + \sigma\,\varepsilon_n \quad \text{and} \quad S_k^* = \frac{S_k}{B_k}\,,$$

where $\delta = \ln(1+r)$. The discounting factor is $\nu = 1/(1+r)$ and $N_t^x = \sum_{k=1}^{l_x} I_{\{\omega:\, T_k(x)\le t\}}$.

From properties of expectations we have

$$V_t^{\pi} = \left(l_x - N_t^x\right) B_0^{-1}\, \nu^N\, p_{x+t}(N-t)$$

$$\times \left[K + S_t\,(1+r)^{N-t}\, \Phi\left(\frac{\ln\left(S_t/K\right) + (N-t)\left(\delta + \sigma^2/2\right)}{\sigma\sqrt{N-t}} \right) \right.$$

$$\left. -K\,\Phi\left(\frac{\ln\left(S_t/K\right) + (N-t)\left(\delta - \sigma^2/2\right)}{\sigma\sqrt{N-t}} \right) \right],$$

$$V_0^{\pi} = l_x\, B_0^{-1}\, \nu^N\, p_x(N)$$

$$\times \left[K + S_0\,(1+r)^{N}\, \Phi\left(\frac{\ln\left(S_0/K\right) + N\left(\delta + \sigma^2/2\right)}{\sigma\sqrt{N}} \right) \right.$$

$$\left. -K\,\Phi\left(\frac{\ln\left(S_0/K\right) + N\left(\delta - \sigma^2/2\right)}{\sigma\sqrt{N}} \right) \right].$$

Also

$$\gamma_t^H = \frac{\left(l_x - N_t^x\right) B_0^{-1}\, \nu^N\, p_{x+t-1}(N-t+1)}{S_{t-1}^*\left(\exp\{\sigma^2\} - 1\right)}$$

$$\times \left\{ S_{t-1}^*\, B_0\,(1+r)^N \left[\Phi\left(\frac{\ln\left(S_{t-1}/K\right) + \sigma^2 + (N-t+1)\left(\delta + \sigma^2/2\right)}{\sigma\sqrt{N-t+1}} \right) \right. \right.$$

$$\left. -\Phi\left(\frac{\ln\left(S_{t-1}/K\right) + (N-t+1)\left(\delta + \sigma^2/2\right)}{\sigma\sqrt{N-t+1}} \right) \right]$$

$$+K \left[\Phi\left(\frac{\ln\left(S_{t-1}/K\right) + (N-t+1)\left(\delta - \sigma^2/2\right)}{\sigma\sqrt{N-t+1}} \right) \right.$$

$$\left. \left. -\Phi\left(\frac{\ln\left(S_{t-1}/K\right) + \sigma^2 + (N-t+1)\left(\delta - \sigma^2/2\right)}{\sigma\sqrt{N-t+1}} \right) \right] \right\},$$

and

$$\beta_t^H = V_t^\pi - \gamma_t^H \, S_t^* \,, \qquad t = 1, 2, \ldots, N \,.$$

WORKED EXAMPLE 3.5

A one-step model with $l_x = 2,\ N = 1$.

SOLUTION The contingent claim is

$$H = \frac{\max\{S_1,\, K\}\, I_{\{\omega:\ T_1 > 1\}}}{B_1} + \frac{\max\{S_1,\, K\}\, I_{\{\omega:\ T_2 > 1\}}}{B_1} \,.$$

Let

$$B_0 = 100,\ S_0 = 100,\ K = 100,\ r = 0.01,\ \mu = 5,\ \sigma = 0.5,\ p_x(1) = 0.999996,$$

then

$$\delta \approx 0.00995, \qquad \nu = \frac{100}{101} \,,$$

and

$$V_0^\pi \approx 2.383, \qquad \gamma_1^H = 1.245.$$

Note that since $\phi(\infty) = 1$ and $\phi(-\infty) = 0$, we obtain

$$V_1^\pi = (2 - N_1^x)\, \frac{1}{B_1}\, \max\{S_1,\, K\} \,.$$

Here $\max\{S_1,\, K\}$ is the amount of an individual payment, $(2 - N_1^x)$ is the number of survivors, B_1 is a discounting factor, and $\beta_1^H = V_1^\pi - \gamma_1^H \, S_1^*$.

Note that since sequence γ^H is predictable then value of γ_1^H is chosen at time 0, i.e., when the value of S_1 is unknown. The value of β_1^H depends on S_1 and therefore it is a random variable. ∎

Now we consider a pure endowment assurance contract in the framework of a continuous Black-Scholes model of a (B, S)-market.

Recall (see Section 2.6 for all details) that dynamics of asset S are described the following stochastic differential equation

$$dS_t = S_t \left(\mu \, dt + \sigma \, dw_t \right) ,$$

and for a bank account B we have

$$dB_t = r \, B_t \, dt \,, \quad B_0 = 1,\ t \le T \,.$$

As in the case of a binomial model, we assume that the Black-Scholes model of a (B, S)-market is defined on a stochastic basis $(\Omega_1, \mathcal{F}_T^1, \mathbb{F}_1, P_1)$, and random variables $T_1(x), T_2(x), \ldots T_{l_x}(x)$ are defined on a stochastic basis $(\Omega_2, \mathcal{F}_N^2, \mathbb{F}_2, P_2)$.

Then on the stochastic basis

$$\left(\Omega_1 \otimes \Omega_2, \mathcal{F}_N^1 \otimes \mathcal{F}_N^2, \mathbb{F}_1 \otimes \mathbb{F}_2, P_1 \otimes P_2\right),$$

we consider a pure endowment assurance contract with the discounted total amount of claims

$$\sum_{k=1}^{l_x} \frac{\max\{S_T,\, K\}}{B_T}\, I_{\{\omega:\, T_k(x)>T\}}\,.$$

To calculate premium $U_x(T)$, we compute the average of the latter sum with respect to probability $P^* = P_1^* \otimes P_2$:

$$\begin{aligned} U_x(T) &= \frac{1}{l_x}\, E^*\left(\sum_{k=1}^{l_x} \frac{\max\{S_T,\, K\}}{B_T}\, I_{\{\omega:\, T_k(x)>T\}}\right) \\ &= p_x(T)\, K\, e^{-rT} + p_x(T)\left[S_0\, \Phi\big(d_+(0)\big) - K\, e^{-rT}\, \Phi\big(d_-(0)\big)\right], \end{aligned} \tag{3.18}$$

where

$$d_\pm(t) = \frac{\ln\big(S_t/K\big) + (T-t)\,\big(r \pm \sigma^2/2\big)}{\sigma\sqrt{T-t}}\,,$$

and $p_x(T)$ is the probability that an individual of age x survives to age $x+T$.

This formula for premium $U_x(T)$ has the following obvious interpretation that is based on the structure of the payment $\max\{S_T,\, K\} = K + (S_T - K)^+$. The first term $p_x(T)\, K\, e^{-rT}$ reflects the obligation to pay the guaranteed amount K. Clearly, K is discounted and multiplied by the survival function. The second term takes into account both the risk of surviving and the market risk related to the payment of amount $(S_T - K)^+$. The second risk component is estimated in terms of the price of a European call option. Hence, the Black-Scholes formula is naturally used for calculating $U_x(T)$.

REMARK 3.1

1. The discrete Gaussian model of a market gives the same results as hedging in mean square in the Black-Scholes model if the discrete time $t \le N$ is replaced with the continuous time $t \le T$.

2. In practice, the premium sometimes is not paid as a lump sum at time 0, but is arranged as a periodic payment. In this case, it is natural to characterize the premium in terms of its density $p(t)$, which can be found from the following equivalence relation:

$$U_x(T) = \int_0^T p(t)\, e^{-rt}\, p_x(t)\, dt\,.$$

3. Recall that the contingent claim $\max\{S_T,\, K\}$ can be perfectly hedged in a complete market (see Section 2.1). The same hedge can be used for the mixed claim $\max\{S_T,\, K\}\, I_{\{\omega:\ T_k(x)>T\}}$, but it turns out that premium $U_x(T)$ is insufficient for perfect hedging since

$$U_x(T) = p_x(T)\, E^*\Big(\max\{S_T,\, K\}\, e^{-rT}\Big) = x_0 < E^*\Big(\max\{S_T,\, K\}\, e^{-rT}\Big)\,,$$

which can be interpreted as a budget constraint. Thus, one can use the methodology of quantile hedging (see Section 2.6) to minimize risk related to such a contract.

$\square$

The notion of *reserve* V_t is an important ingredient of actuarial mathematics. The reserve at time t is defined as the difference between the value of future claims and the value of future premiums:

$$V_t = p_{x+t}(T-t)\, E^*\Big(\max\{S_T,\, K\}\, e^{-rt}\big|\mathcal{F}_t^1\Big) - \int_t^T p(u)\, e^{-r(u-t)}\, p_{x+t}(u-t)\, du\,.$$

Assuming that

$$p_x(t) = P_2\big(\{\omega:\ T_k(x) > t\}\big) = \exp\left\{-\int_0^t \mu_{x+\tau}\, d\tau\right\},$$

and using the Kolmogorov-Itô formula, we obtain the following equation for $V_t \equiv V(t,s)$:

$$\frac{\partial V}{\partial t}(t,s) = p(t) + \big(\mu_{x+t} + r\big)\, V(t,s) - \frac{1}{2}\,\sigma^2\, s^2\, \frac{\partial^2 V}{\partial s^2}(t,s) - r\, s\, \frac{\partial V}{\partial s}(t,s). \quad (3.19)$$

Note that since V is a function of the price of asset S, then naturally the latter equation is a generalization of the Black-Scholes equation. Otherwise, it reduces to the well-known Thiele's equation

$$\frac{\partial V}{\partial t} = p(t) + \big(\mu_{x+t} + r\big)\, V\,.$$

On the other hand, if insurance characteristics $p(t) = \mu_{x+t} = 0$, then equation (3.19) reduces to the Black-Scholes equation. Thus, we can summarize that equation (3.19) for the reserve reflects the presence of both insurance risk and financial risk.

3.5 Reinsurance risks

Reinsurance is a mechanism that insurance companies use to transfer some or all of their risks to reinsurance companies. The primary aim of reinsurance is to protect the solvency of the insurance company by minimizing the probability of bankruptcy. Some typical examples of situations when such solvency protection is required include receiving very large claims (e.g., in the cases of big man-made disasters such as an aeroplane crash, etc.); receiving a large number of claims from policies affected by the same event (e.g., in the case of natural disasters such as earthquakes, hurricanes, floods, etc.); sudden changes in the premiums flow (say, due to inflation) or in the number of policy holders; the need to access some additional capital so that the insurance company can take on larger risks and therefore attract more clients.

As in Section 3.1 we consider a risk process

$$X(t) = \sum_{i=1}^{N(t)} X_i \,,$$

which represents the aggregate amount of claims up to time t.

One of the main characteristics of a reinsurance contract is quantity $h(X)$ that determines the amount of claims payment made by the insurance company. The remainder $X - h(X)$ is paid by the reinsurance company and is naturally referred to as an *insured proportion* of risk X. The insurance company pays premium to the reinsurance company in order to transfer this part of its risk.

Function $h(X)$ is called the *retention function*. It has the following properties:

(a) $h(X)$ and $X - h(X)$ are non-decreasing functions;

(b) $0 \le h(X) \le X, \quad h(0) = 0.$

There are two basic forms of reinsurance: *proportional reinsurance* and *non-proportional reinsurance*. The main types of proportional reinsurance are *quota share* and *surplus*. A quota share reinsurance transfers all risks in the same proportion, whereas in a surplus reinsurance proportions of transfer may vary. The typical examples of non-proportional reinsurance are *stop-loss* reinsurance and *excess of loss* reinsurance. They provide protection when claims exceed a certain agreed level.

The following retention functions

1. $h(x) = a\,x \quad 0 < a \le 1,$

2. $h(x) = \min\{a,\, x\} \quad a > 0,$

correspond to the quota share reinsurance and the stop-loss reinsurance, respectively.

From the point of view of the reinsurance company a reinsurance contract is just a usual insurance against risk $X - h(X)$. Hence one can calculate the corresponding premium level using the methodology described earlier in this chapter:

$$\widetilde{\Pi} = \Pi\big(X - h(X)\big)\,.$$

Let us consider a quota share reinsurance contract in the framework of the individual risk model. If X_i is the amount of an individual claim, then $a\,X_i$ is paid by the insurance company and $(1-a)\,X_i$ by the reinsurance company. Thus the total amount of claim

$$S = X_1 + \ldots + X_n$$

received by the insurance company is reduced to

$$a\,S = a\,(X_1 + \ldots + X_n)\,.$$

Suppose both insurance and reinsurance companies use the Expectation principle in the premium calculation (see Section 3.1), and the security loading coefficients are θ and θ^*, respectively.

Prior to entering the reinsurance contract the capital of the insurance company is

$$x + (1+\theta)\,E(S)\,.$$

After paying the premium $(1+\theta^*)\,(1-a)\,E(S)$, the capital becomes

$$x + \big[\theta - \theta^* + a\,(1+\theta^*)\big]\,E(S)\,.$$

Now we compare the probabilities of insolvency as measures of risk to which the insurance company is exposed when it purchases the reinsurance contract and when it does not. In the first case it is

$$P\Big(\big\{\omega:\ a\,S < x + \big[\theta - \theta^* + a\,(1+\theta^*)\big]\,E(S)\big\}\Big)$$
$$= P\left(\left\{\omega:\ S < \frac{x + \big[\theta - \theta^* + a\,(1+\theta^*)\big]\,E(S)}{a}\right\}\right),$$

and in the second:

$$P\Big(\big\{\omega:\ S < x + (1+\theta)\,E(S)\big\}\Big)\,.$$

This allows us to manage the risk of the insurance company, since if $(\theta-\theta^*)\,E(S) < x$, then the probability of bankruptcy can be reduced by purchasing the reinsurance contract.

Next we consider a stop-loss reinsurance contract with the retention level a. According to this contract, if the amount of an individual claim $X_i \le a$, then it is paid by the insurance company, otherwise the insurance company pays a and the reinsurance company pays the remainder $X_i - a$. So by purchasing such a reinsurance contract the insurance company protects itself from paying more than a per individual claim.

Suppose that the insurance company issues N identical insurance contracts, so that the independent identically distributed random variables $X_1, \ldots, X_N$ represent the amounts of corresponding claims. Under the stop-loss reinsurance contract the total amount of claim

$$S = X_1 + \ldots + X_N$$

received by the insurance company is reduced to

$$S^{(a)} = X_1^{(a)} + \ldots + X_n^{(a)}\,, \qquad \text{where} \qquad X^{(a)} = \min\{X,\, a\}\,.$$

For example, the sequence of payments made by the insurance company may look like

$$X_1,\, X_2,\, a,\, a,\, X_5, \ldots\,,$$

and the corresponding sequence of payments made by the reinsurance company is

$$0,\, 0,\, X_3 - a,\, X_4 - a,\, 0, \ldots\,.$$

Note that the number of claims paid by the reinsurance company may be less than N. Nevertheless we can represent the risk process of the reinsurance company in the form $\sum_{i=1}^{N} Z_i$, where some Z_i may be equal to zero.

Again we assume that both insurance and reinsurance companies use the Expectation principle in the premium calculation, and that the security loading coefficients are θ and θ^*, respectively.

The capital of the insurance company prior to entering the reinsurance contract is given by

$$N\,p = N\,(1+\theta)\,p_0 \equiv N\,(1+\theta)\,E(X)\,.$$

After paying the premium

$$N\,(1+\theta^*)\,\big(E(X) - E(X^{(a)})\big)\,,$$

it becomes

$$N\,(1+\theta)\,E(X) - N\,(1+\theta^*)\,\big(E(X) - E(X^{(a)})\big)$$
$$= N\,(\theta - \theta^*)\,E(X) + N\,(1+\theta^*)\,E(X^{(a)})\,.$$

Hence the probability of insolvency is

$$P\Big(\big\{\omega:\ S^{(a)} > N\,(\theta-\theta^*)\,E(X) + N\,(1+\theta^*)\,E(X^{(a)})\big\}\Big).$$

It is rather difficult to compute this probability explicitly. We use the Central Limit theorem for computing its approximation:

$$P\left(\left\{\omega:\ \frac{S^{(a)} - E\big(S^{(a)}\big)}{\sqrt{V\big(S^{(a)}\big)}} > \frac{N\,(\theta-\theta^*)\,E(X) + N\,\theta^*\,E(X^{(a)})}{\sqrt{N\,V\big(X^{(a)}\big)}}\right\}\right)$$
$$\approx 1 - \Phi\left(\sqrt{N}\ \frac{(\theta-\theta^*)\,E(X) + \theta^*\,E(X^{(a)})}{\sqrt{V\big(X^{(a)}\big)}}\right),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

is a standard normal distribution.

Suppose that the insurance company can vary the retention level a (variation of a, of course, changes the premium payable to the reinsurance company). Suppose that $\tilde{a}$ is the maximum of function

$$\varphi(a) = \frac{\Big[(\theta - \theta^*)\, E(X) + \theta^*\, E\big(\min\{X,\, a\}\big)\Big]^2}{V\big(\min\{X,\, a\}\big)}\,,$$

then the stop-loss reinsurance contract with the retention level $\tilde{a}$ minimizes the probability of insolvency.

WORKED EXAMPLE 3.6

Suppose that a random variable representing the amount of an individual claim is uniformly distributed in $[0, x]$. Consider a stop-loss reinsurance contract and find the value of $\tilde{a}$ that minimizes the probability of insolvency.

SOLUTION Clearly, we can assume that the retention level $a < x$. Then we have

$$E(X) = \frac{x}{2}\,, \quad E\big(X^2\big) = \frac{x^2}{3}\,, \quad V(X) = \frac{x^2}{12}\,,$$

and

$$E\big(X^{(a)}\big) = a - \frac{a^2}{2x}\,, \quad E\big((X^{(a)})^2\big) = a^2 - \frac{2a^3}{3x}\,, \quad V\big(X^{(a)}\big) = \frac{a^3}{3x} - \frac{a^4}{4x^2}\,,$$

which implies

$$\varphi(a) = \frac{\Big[(\theta - \theta^*)\,\frac{x}{2} + \theta^*\,\Big(a - \frac{a^2}{2x}\Big)\Big]^2}{\frac{a^3}{3x} - \frac{a^4}{4x^2}}\,.$$

Due to no-arbitrage considerations, we have that $\theta \le \theta^*$, since otherwise the insurance company can transfer all the risk to the reinsurance company and make a non-zero profit with zero initial capital. If we additionally assume that $\theta^* < 3\,\theta$, then it is not difficult to see that function $\varphi(a)$ has a unique maximum on $[0, x]$:

$$\tilde{a} = \frac{3\,x\,\big[\theta^* - \theta\big]}{2\,\theta^*}\,, \quad \text{so that} \quad \varphi(\tilde{a}) = \frac{(\theta^*)^2}{9}\,\frac{9\,\theta - \theta^*}{\theta^* - \theta}\,.$$

$\square$

There are large risks (e.g., a jumbo jet or an oil drilling platform) of a magnitude that makes it impossible for most single insurance companies to insure the whole risk without sharing this risk exposure. In this case, an insurer transfers some risk to a reinsurer. The reinsurer itself may also reinsure this risk, which is often referred to as *retrocession*. Thus, one can summarize that the insurance market has at least three levels:

1. Primary market (insurance companies)

2. Reinsurance market (reinsurance companies)

3. Retrocession market (reinsurance companies that provide insurance to other reinsurance companies)

Clearly, the retrocession market can consist of more than one level. For each n^{th} level reinsurance company, the *risk transfer time*, i.e., time between receiving a risk from a $(n-1)^{\text{st}}$-level company and passing it to a $(n+1)^{\text{st}}$-level company, is a random variable with some distribution F, and it is independent of risk transfer times of another companies. Denote $\Sigma\big(X_n\big)$ the total number of n^{th}-level companies, and $R_{n,i}$ the number n^{th}-level companies that insured the i^{th} company from $(n-1)^{\text{st}}$ level. We assume that $R_{n,i}$ are independent random variables with distribution $(p_k)_{k=0,1,\dots}$. Note that

$$X_n = 1\,, \qquad X_{n+1} = \sum_{i=1}^{X_n} R_{n,i}\,.$$

Denote $\hat{g}_{X_n}(s)$ and $\hat{g}_R(s)$ the generating functions of X_n and R, respectively.

For $|s| < 1$ and all $n \in \mathbb{N}$ we have the following relation

$$\hat{g}_{X_{n+1}}(s) = \hat{g}_{X_n}\big(\hat{g}_R(s)\big) = \hat{g}_R\big(\hat{g}_{X_n}(s)\big) = \underbrace{\hat{g}_R\Big(\hat{g}_R\big(\dots\hat{g}_R(s)\big)\Big)}_{n \text{ times}}\,,$$

which follows from the equality

$$\begin{aligned}\hat{g}_{X_{n+1}}(s) \equiv E\big(s^{X_{n+1}}\big) &= \sum_{j=0}^{\infty} E\left(s^{\sum_{i=1}^{X_n} R_{n,i}}\,\middle|\, X_n = j\right) P\big(\{\omega:\ X_n = j\}\big) \\ &= \hat{g}_{X_n}\big(\hat{g}_R(s)\big)\,,\end{aligned}$$

applied n times.

Using the latter formula and taking into account that

$$E(X_{n+1}) = E(X_n)\,E(R) \quad \text{and} \quad V(X_{n+1}) = V(X_n)\,\big(E(R)\big)^2 + E(X_n)\,V(R)\,,$$

we obtain

$$E(X_n) = \big(E(R)\big)^2 \equiv \mu_R^n\,,$$

and

$$V(X_n) = \begin{cases} \frac{\mu_R^{n-1}\,(\mu_R^n - 1)}{\mu_R - 1}\,V(R)\,, & \mu_R \neq 1 \\ n\,V(R)\,, & \mu_R = 1\,. \end{cases}$$

Let $\Sigma\big(X(t)\big)$ be the number of companies involved in the contract up to time t, and $X_{n,i}(t)$ the number auxiliary companies that insured the i^{th} company from n^{th} level after time t. It is clear that

$$X(t) = \begin{cases} 1\,, & t < T \\ \sum_{i=1}^{R} X_{2,i}(t-T)\,, & t \geq T\,. \end{cases}$$

Let $\mu(t) = E\big(X(t)\big)$, the average number of companies involved in a reinsurance project at time t. The following results hold true.

PROPOSITION 3.1
We have

$$\mu(t) = \overline{F}(t) + \mu_R \int_0^t \mu(t-\nu)\,dF(\nu)\,, \tag{3.20}$$

where $F(x)$ is the distribution function of the risk transfer time, and $\overline{F}(x) = 1 - F(x)$.

PROOF Let random variable T represent the time between issuing the primary insurance contract by a primary insurance company and the time when this risk is reinsured by a next level company. Then, using properties of conditional expectations, we obtain

$$\begin{aligned}\mu(t) &= E\Big(E\big(X(t)\,|\,T\big)\Big) = \int_0^\infty E\big(X(t)\,|\,T=\nu\big)\,dF(\nu) \\ &= \int_0^t E\big(X(t)\,|\,T=\nu\big)\,dF(\nu) + \int_t^\infty E\big(X(t)\,|\,T=\nu\big)\,dF(\nu)\,.\end{aligned}$$

Note that the conditional expectation in the second integral in the right-hand side is equal to 1. To compute first term we write

$$E\big(X(t)\,|\,T=\nu\big) = \sum_{j=0}^{\infty} p_j\,E\big(X(t)\,|\,T=\nu\,,\ R=j\big)\,,$$

where R is the number of reinsurers of the primary company. Each second-level reinsurance company generates an independent chain of reinsurers. Hence

$$\begin{aligned}E\big(X(t)\,|\,T=\nu\,,\ R=j\big) &= E\Big(\sum_{i=1}^{j} X_{2,i}(t-\nu)\,|\,T=\nu\,,\ R=j\Big) \\ &= j\,\mu(t-\nu)\,,\end{aligned}$$

which implies the result. ∎

The following result is typical for the theory of branching processes (see, for example, [40]).

PROPOSITION 3.2
If distribution F is continuous then the following statements hold true.

1. *If $\mu_R = 1$, then $\mu(t) = 1$ for all $t \geq 0$.*

2. *If $\mu_R > 1$, then*
$$\lim_{t\to\infty} \frac{\mu(t)}{e^{\gamma t}} = \frac{\mu_R - 1}{\gamma\,\mu_R^2\,|\hat{l}'_T(\gamma)|}\,,$$
where γ is the unique solution of equation
$$\hat{l}_T(y) := \int_0^\infty e^{-x\,y}\,dF(x) = \mu_R^{-1}\,.$$

3. *If $\mu_R < 1$ and there exists a positive solution γ of equation*
$$\hat{m}_T(y) := \int_0^\infty e^{y\,x}\,dF(x) = \mu_R^{-1}\,,$$
then
$$\lim_{t\to\infty} \frac{\mu(t)}{e^{-\gamma t}} = \frac{1-\mu_R}{\gamma\,\mu_R^2\,|\hat{m}'_T(\gamma)|}\,,$$
otherwise this limit is equal to zero.

PROOF We prove here only the first statement. If $\mu_R = 1$, then $\mu(t) \equiv 1$ is a solution of equation (3.20). Thus, we need only to establish the uniqueness of this solution to
$$\mu(t) = \overline{F}(t) + \int_0^t \mu(t-\nu)\,dF(\nu)\,,$$
Suppose that both $\mu_1(t)$ and $\mu_2(t)$ are solutions of this equation. Then
$$\mu_1(t) - \mu_2(t) = \int_0^t \big(\mu_1(t-\nu) - \mu_2(t-\nu)\big)\,dF(\nu)$$
if and only if
$$\mu_1(t) - \mu_2(t) = \big(\mu_1 - \mu_2\big) * F(t)\,.$$
Further
$$\begin{aligned}|\mu_1(t) - \mu_2(t)| &= |\big(\mu_1 - \mu_2\big) * F(t)| = |\big(\mu_1 - \mu_2\big) * F * F(t)| \\ &= \dots \\ &= |\big(\mu_1 - \mu_2\big) * F^{*n}(t)| \leq F^{*n}(t) \sup_{\nu\in[0,t]} |\mu_1(\nu) - \mu_2(\nu)|\,.\end{aligned}$$

Now, since $F^{*n}(t) \leq \left[F(t)\right]^n$, then

$$\lim_{n\to\infty} F^{*n}(t) = 0\,,$$

which proves the claim. $\square$

WORKED EXAMPLE 3.7

Describe the asymptotic behavior of $\mu(t)$ if the risk transfer time has an exponential distribution.

SOLUTION We have

$$\hat{l}_T(y) := \int_0^\infty e^{-x\,y}\,dF(x) = \int_0^\infty e^{-x\,y}\,\lambda\,e^{-\lambda\,x}\,dx = \frac{\lambda}{\lambda+y}$$

and

$$\hat{m}_T(y) := \int_0^\infty e^{y\,x}\,dF(x) = \int_0^\infty e^{y\,x}\,\lambda\,e^{-\lambda\,x}\,dx = \frac{\lambda}{\lambda-y}\,,$$

so that their derivatives are

$$\hat{l}'_T(y) = -\frac{\lambda}{(\lambda+y)^2} \qquad \text{and} \qquad \hat{m}'_T(y) = -\frac{\lambda}{(\lambda-y)^2}\,.$$

Since

$$\gamma = \lambda\,(\mu_R - 1) \qquad \text{is a solution to} \qquad \hat{l}_T(y) = \frac{1}{\mu_R}\,,$$

and

$$\gamma = \lambda\,(1-\mu_R) \qquad \text{is a solution to} \qquad \hat{m}_T(y) = \frac{1}{\mu_R}\,,$$

then we deduce that

1) if $\mu_R = 1$, then $\mu(t) = 1$ for all $t \geq 0$;

2) if $\mu_R > 1$, then

$$\lim_{t\to\infty} \frac{\mu(t)}{e^{\lambda\,(\mu_R-1)\,t}} = 1 \qquad \text{(exponential growth);}$$

3) if $\mu_R < 1$, then

$$\lim_{t\to\infty} \frac{\mu(t)}{e^{\lambda\,(\mu_R-1)\,t}} = 1 \qquad \text{(exponential decay).}$$

In this case we can also find an exact expression for $\mu(t)$. Substitute $F(y) = 1 - e^{-\lambda y}$ into equation (3.20):

$$\mu(t) = e^{-\lambda t} + \mu_R \int_0^t \mu(t-\nu)\,\lambda\,e^{-\lambda\nu}\,d\nu\,.$$

Differentiating in t and integrating by parts, we obtain

$$\begin{aligned}\mu'(t) &= -\lambda\, e^{-\lambda t} + \mu_R\, \mu(0)\, \lambda\, e^{-\lambda t} - \mu_R \int_0^t \lambda\, e^{-\lambda \nu}\, d\mu(t-\nu) \\ &= -\lambda\, e^{-\lambda t} + \mu_R\, \lambda\, \mu(t) - \lambda\, \mu_R \int_0^t \mu(t-\nu)\, \lambda\, e^{-\lambda \nu}\, d\nu \\ &= \mu(t)\, \lambda\, (\mu_R - 1)\,.\end{aligned}$$

The Cauchy problem

$$\mu'(t) = \mu(t)\, \lambda\, (\mu_R - 1)\,, \quad t \geq 0\,, \quad \mu(0) = 1\,,$$

has the unique solution

$$\mu(t) = e^{\lambda\, (\mu_R - 1)\, t}\,,$$

which implies the asymptotic behavior as described above. ☐

The structure of the traditional insurance market provides reasonable protection to insurance companies against 'moderately' large risks. There are events (*catastrophes*) that can give rise to giant claims, when the total claim amount can be comparable with the total premium income. For example, Hurricane Andrew (1992, USA) gave rise to the total claim equivalent to $1/5$ of the combined total premium income of the insurance market of the USA. Floods in Europe in 2002 caused a situation where insurance companies had to rely on the financial intervention of governments and other organizations in order to stay solvent. Furthermore, some catastrophes may cause insured losses that are comparable to the capacity of the whole of the insurance industry. For instance, computer simulations of some earthquakes in California suggest that they could cause losses up to $US 100 billion, whereas in the mid 1990s the combined capital of the insurance industry was less than $US 300 billion.

Risk securitization is one of the possible ways of dealing with this situation. It consists of introducing *insurance securities*: catastrophe (CAT) bonds, forwards, futures, options etc., as derivative instruments in catastrophe reinsurance. This form of reinsurance is viable due to the 1:100 ratio of insurance industry to finance industry.

The first step in the process of risk securitization is related to choosing an underlying asset that can be used to construct the corresponding CAT derivative securities. At the beginning of the 1990s it was the ISO-index (introduced by the Insurance Services Office statistical agency, USA) that reflected the losses of a pool of most significant insurance companies of the USA. From the mid 1990s the PCS-index (introduced by a non-profit organization Property Claim Service, USA) is the most commonly used. The PCS-index is based on actual losses from a catastrophic event over some period of time. During the initial *risk-period* one assesses the catastrophic event. This is followed by the period of *losses development* when the initial information is refined. At the beginning of the risk-period the PCS-index is set to be zero. Each point of it corresponds to US$ 100 million. The total amount of claim payments is restricted to US$ 50 billion. Thus, we have the following expression for a

PCS-index L at time t:

$$L_t = \frac{X_t}{C}\,,$$

where X_t is the loss process and C = US\$ 100 million. Then, given a specific structure of the loss process X_t, one can price CAT derivative securities using the pricing methods of financial risk management.

3.6 Extended analysis of insurance risks in a generalized Cramér-Lundberg model

As we discussed earlier, solvency of an insurance company is a natural characterization of its exposure to risk, and the traditional actuarial measure of such exposure in the probability of insolvency (or bankruptcy).

Representing the capital R of the insurance company as the difference between the premium process Π and risk process X, we can define the probability of solvency on finite and infinite intervals as

$$\begin{aligned} \varphi(t,x) &= P\big(\{\omega :\ R(s) > 0 \quad \text{for all} \quad s \le t\}\big)\,, \qquad R(0) = x\,,\\ \varphi(x) &= P\big(\{\omega :\ R(t) > 0 \quad \text{for all} \quad t \ge 0\}\big)\,, \qquad R(0) = x\,, \end{aligned}$$

respectively.

Suppose that

$$E\big(\Pi(t)\big) > E\big(X(t)\big)\,,$$

i.e., *pure income* is positive.

This section is devoted to generalizations of the Cramér-Lundberg model. In particular, we study the probability of insolvency as a measure of exposure to risk in situations when the premium process Π has more complex structure than in the original model. We also will take into account various factors of financial and insurance markets.

First we consider a case when premiums are received at some random times and their amounts are also random. The capital of the company has the form

$$R(t) = x + \sum_{i=1}^{N_1(t)} c_i - \sum_{i=1}^{N(t)} X_i\,,$$

where N_1 and N are independent Poisson processes with intensities λ_1 and λ, respectively, and (c_i) and (X_i) sequences of independent random variables with distribution functions $G(\cdot)$ and $F(\cdot)$, respectively.

Hence

$$\lambda_1 \, t \, E(c_i) = E\Big(\sum_{i=1}^{N_1(t)} c_i\Big) = E\big(\Pi(t)\big) > E\big(X(t)\big)$$

$$= E\Big(\sum_{i=1}^{N(t)} X_i\Big) = \lambda \, t \, E(X_i) \, .$$

Therefore the condition of positivity of pure income is reduced to inequality

$$\lambda_1 \, E(c_i) > \lambda \, E(X_i) \, .$$

Under these assumptions we have that the probability of solvency satisfies the inequality

$$\varphi(x) \geq 1 - e^{-R\,x} \, ,$$

where constant R is a solution of the characteristic equation

$$\lambda_1 \Big[E\big(e^{-R\,c_i}\big) - 1\Big] + \lambda \Big[E\big(e^{-R\,X_i}\big) - 1\Big] = 0 \, .$$

For exponential distribution functions $G(\cdot)$ (premium process) and $F(\cdot)$ (risk process) we have the following result.

PROPOSITION 3.3
If

$$G(x) = P\big(\{\omega : \; c_i \leq x\}\big) = 1 - e^{-b\,x} \, ,$$

$$F(x) = P\big(\{\omega : \; X_i \leq x\}\big) = 1 - e^{-a\,x} \, ,$$
$$a > 0, \; b > 0, \; x > 0 \, ,$$

then we have an exact expression

$$\varphi(x) = 1 - \frac{(a+b)\,\lambda}{(\lambda_1 + \lambda)\,a} \, \exp\left\{ \frac{\lambda\,b - \lambda_1\,a}{\lambda_1 + \lambda} \, x \right\} .$$

PROOF Using the independence of $\Pi(t)$ and $X(t)$, we have

$$E\Big(e^{-R\,\big[\Pi(t) - X(t)\big]}\Big)$$

$$= \left(\sum_{k=0}^{\infty} e^{-\lambda_1 t} \, \frac{(\lambda_1 \, t)^k}{k!} \, E\Big(e^{-R\,\sum_{i=1}^{k} c_i}\Big) \right) \left(\sum_{k=0}^{\infty} e^{-\lambda t} \, \frac{(\lambda \, t)^k}{k!} \, E\Big(e^{R\,\sum_{i=1}^{k} X_i}\Big) \right)$$

$$= \exp\Big\{ \lambda_1 \, t \, \big[E\big(e^{-R\,c_i}\big) - 1\big] + \lambda \, t \, \big[E\big(e^{R\,X_i}\big) - 1\big] \Big\} ,$$

for any constant R. Now let R be a solution of the characteristic equation, then for $t > s$ we have

$$E\Big(e^{-R\,\big[\Pi(t)-X(t)\big]}\Big|\mathcal{F}_s\Big)$$

$$= e^{-R\,\big[\Pi(s)-X(s)\big]}\;E\Big(e^{-R\,\big[\Pi(t)-\Pi(s)-X(t)+X(s)\big]}\Big|\mathcal{F}_s\Big)$$

$$= e^{-R\,\big[\Pi(s)-X(s)\big]}\;E\Big(e^{-R\,\big[\Pi(t-s)-X(t-s)\big]}\Big) = e^{-R\,\big[\Pi(s)-X(s)\big]},$$

where $\mathcal{F}_t$ is a σ-algebra generated by processes $\Pi(s)$ and $X(s)$ up to time t.

Hence the process

$$Y_R(t) = e^{-R\,\big[\Pi(t)-X(t)\big]}$$

is a martingale with the initial condition $Y_R(0) = 1$.

Consider the bankruptcy moment

$$\tau = \inf\{t \geq 0\ :\ R(t) < 0\}\,.$$

Since the average value of a martingale is constant, we obtain

$$\begin{aligned}1 &= E\big(Y_R(\tau\wedge t)\big) \geq E\big(Y_R(\tau\wedge t)\, I_{\{\omega:\ \tau\leq t\}}\big)\\ &= E\big(e^{-R\,\big[\Pi(\tau)-X(\tau)\big]}\;\; I_{\{\omega:\ \tau\leq t\}}\big) > e^{Rx}\, P\big(\{\omega:\ \tau\leq t\}\big)\,,\end{aligned}$$

where we also used the fact that $\Pi(\tau) - X(\tau)$ for $\tau < \infty$. Passing to the limit as $t \to \infty$, we obtain

$$\varphi(x) \geq 1 - e^{-Rx}\,.$$

Note that if distribution functions $G(\cdot)$ and $F(\cdot)$ are exponential, then the condition of positivity of pure income has the form

$$\frac{\lambda_1}{b} > \frac{\lambda}{a}\,,$$

and the characteristic equation is

$$\lambda_1\left[\frac{b}{b+R} - 1\right] + \lambda\left[\frac{a}{a-R} - 1\right] = 0\,.$$

Hence constant R is either equal to zero or to

$$\frac{\lambda_1\, a - \lambda\, b}{\lambda_1 + \lambda}\,.$$

Thus, we have that either $\varphi(\infty) = 1$ or

$$\varphi(x) > 1 - \exp\left\{\frac{\lambda\, b - \lambda_1\, a}{\lambda_1 + \lambda}\, x\right\}.$$

As in Section 3.2 one can use the formula for total probability to obtain the following integral equation for $\varphi(x)$:

$$(\lambda + \lambda_1)\,\varphi(x) = \lambda_1 \int_0^\infty \varphi(x+v)\, b\, e^{-bv}\, dv + \lambda \int_0^x \varphi(x-u)\, a\, e^{-au}\, du\,.$$

Changing variables $v_1 = v + x$, $u_1 = x - u$, we can write $\varphi(x)$ in the form

$$\varphi(x) = \frac{\lambda_1}{\lambda + \lambda_1} \int_x^\infty \varphi(v_1)\, b\, e^{-b(v_1 - x)}\, dv_1 + \frac{\lambda}{\lambda + \lambda_1} \int_0^x \varphi(u_1)\, a\, e^{-a(x-u_1)}\, du_1\,,$$

which, in particular, indicates that function φ is differentiable. Also note that

$$\left(\int_0^\infty \varphi(x+v)\, b\, e^{-bv}\, dv\right)'_x = -b\,\varphi(x) + b \int_0^\infty \varphi(x+v)\, b\, e^{-bv}\, dv\,,$$

$$\left(\int_0^x \varphi(x-u)\, a\, e^{-au}\, du\right)'_x = a\,\varphi(x) - a \int_0^x \varphi(x-u)\, a\, e^{-au}\, du\,.$$

Differentiating the equation for $\varphi(x)$, we have

$$\begin{aligned}(\lambda + \lambda_1)\,\varphi'(x) &+ (\lambda_1 b - \lambda a)\,\varphi(x)\\ &= b\,\lambda_1 \int_0^\infty \varphi(x+v)\, b\, e^{-bv}\, dv - a\,\lambda \int_0^x \varphi(x-u)\, a\, e^{-au}\, du\,.\end{aligned}$$

Differentiating second time, we obtain

$$\begin{aligned}(\lambda + \lambda_1)\,\varphi''(x) &+ (\lambda_1 b - \lambda a)\,\varphi'(x) + (\lambda_1 b^2 + \lambda a^2)\,\varphi(x)\\ &= b^2\,\lambda_1 \int_0^\infty \varphi(x+v)\, b\, e^{-bv}\, dv + a^2\,\lambda \int_0^x \varphi(x-u)\, a\, e^{-au}\, du\,.\end{aligned}$$

This implies that

$$\varphi''(x) = \frac{\lambda b - \lambda_1 a}{\lambda + \lambda_1}\,\varphi'(x)\,.$$

This equation has a solution of the form

$$\varphi(x) = C_1 + C_2\,\exp\left\{\frac{\lambda b - \lambda_1 a}{\lambda + \lambda_1}\, x\right\}.$$

It is clear from the statement of the problem that

$$C_1 = \varphi(\infty) = 1\,.$$

Substituting this expression into the initial integral equation for $\varphi(x)$, we obtain that for $x = 0$

$$(\lambda + \lambda_1)\,\varphi(0) = \lambda_1 \int_0^\infty \varphi(v)\, b\, e^{-bv}\, dv\,,$$

and hence

$$C_2 = -\frac{(a+b)\,\lambda}{(\lambda+\lambda_1)\,a}\,,$$

which completes the proof. ∎

REMARK 3.2 Similar results can be obtained for a discrete version of the Cramér-Lundberg model, when $\Pi(t)$ and $X(t)$ are independent compound binomial processes. ∎

Now we consider a generalization of the Cramér-Lundberg model that takes into account the insurance market competition. Suppose the pool of insurance companies is large enough, and each company has only limited influence on the insurance market. Then it is natural to use Gaussian diffusion for modelling the capital of an insurance company:

$$R(t) = x + \Pi(t) - X(t) + \sigma\, w_t\,,$$

where

$$\Pi(t) = \mu\, t + \sum_{i=1}^{N_1(t)} c_i\,, \quad \mu > 0\,,$$

is the premium process,

$$X(t) = \sum_{i=1}^{N(t)} X_i\,,$$

is the risk process, w_t is a standard Wiener process and $\sigma \geq 0$.

It is assumed that all processes $\Pi(t)$, $X(t)$ and w_t are independent and the condition of positivity of income:

$$\mu + \lambda_1\, E(c_i) > \lambda\, E(X_i)\,,$$

holds true.

In this case the probability of solvency again satisfies the estimate

$$\varphi(x) \geq 1 - e^{-R\,x}\,,$$

where R is a solution of the characteristic equation

$$-R\,\mu + \sigma^2\, R^2 + \lambda_1\left[\int_0^\infty e^{-R\,v}\, dG(v) - 1\right] + \lambda\left[\int_0^\infty e^{R\,y}\, dF(y) - 1\right] = 0\,.$$

Another generalization of the Cramér-Lundberg model takes into account the fact that insurance companies are active participants of the financial market. Earlier we discussed several discrete models of this type. Now we consider a version of Cramér-Lundberg model in the framework of a Black-Scholes market:

$$\begin{aligned} dB_t &= r\, B_t\, dt\,, \quad B_0 = 1\,,\\ dS_t &= S_t\,\big(\mu\, dt + \sigma\, dw_t\big)\,, \quad S_0 > 0\,. \end{aligned}$$

Suppose that the initial capital of an insurance company is x, and the capital of the investment portfolio $\pi = (\beta, \gamma)$ is

$$R(t) = \beta_t \, B_t + \gamma_t \, S_t \,,$$

whose dynamics are described by

$$dR(t) = \beta_t \, dB_t + \gamma_t \, dS_t + B_t \, d\beta_t + S_t \, d\gamma_t \,.$$

If $\Pi(t) = \sum_{i=1}^{N_1(t)} c_i$ is the premium process and $X(t) = \sum_{i=1}^{N(t)} X_i$ is the risk process, then the following constraint

$$B_t \, d\beta_t + S_t \, d\gamma_t = \sum_{i=N_1(t)}^{N_1(t+dt)} c_i - \sum_{i=N(t)}^{N(t+dt)} X_i \,,$$

is natural for the class of admissible strategies. It means that the redistribution of the capital in the portfolio happens due to premium and claim flows.

Suppose that *all capital is invested into a bank account*, then its dynamics are described by equation

$$R(t) = x + \int_0^t r \, R(s) \, ds + \sum_{i=1}^{N_1(t)} c_i - \sum_{i=1}^{N(t)} X_i \,,$$

whose solution has the form

$$R(t) = e^{rt} \left[x + \sum_{i=1}^{N_1(t)} c_i \, e^{-r\sigma_i} - \sum_{i=1}^{N(t)} X_i \, e^{-r\tau_i} \right],$$

where σ_i are jumps of process $N_1(t)$ and τ_i are jumps of $N(t)$.

Since random variable

$$\tau = \inf\{t \geq 0 \, : \; R(t) < 0\}$$

represents the bankruptcy time, then the probability of solvency

$$\varphi(x) = P\big(\{\omega : \; \tau = \infty\}\big)$$

is established in the following theorem.

THEOREM 3.1

Suppose that all capital of an insurance company is invested in a bank account, then the probability of the company's solvency satisfies the integro-differential equation

$$r \, x \, \varphi'(x) - (\lambda_1 + \lambda) \, \varphi(x) + \lambda \int_0^x \varphi(x-y) \, dF(y) + \lambda_1 \int_0^\infty \varphi(x+\nu) \, dG(\nu) = 0 \,.$$

PROOF Since for a fixed $R(t) = x$ the further evolution of the process depends neither on t nor on its history, then using the equation for $R(t)$ we can write for a small time interval Δt:

$$\varphi(x) = \big[1 - (\lambda_1 + \lambda)\big]\, \varphi(x + r\,x\,\Delta t) + \lambda_1 \int_0^\infty \varphi(x + r\,x\,\Delta t + \nu)\, dG(\nu)$$
$$+\lambda \int_0^x \varphi(x + r\,\Delta t - u)\, dF(u) + o(\Delta t)\,.$$

Since by Taylor's formula we have

$$\varphi(x + r\,x\,\Delta t) = \varphi(x) + r\,\Delta t\, \varphi'(x) + o(\Delta t)\,,$$

then dividing the latter equality by Δt and taking limits as $\Delta \to \infty$ proves the claim. ∎

To estimate the probability of solvency on a finite time interval we consider the discounted capital $\widetilde{R}(t) = R(t)\, e^{-rt}$.

Clearly, for any finite interval we have

$$P\big(\{\omega :\ \widetilde{R}(s) \geq 0 \quad \text{for all} \quad 0 \leq s \leq t\}\big)$$
$$= P\big(\{\omega :\ R(s) \geq 0 \quad \text{for all} \quad 0 \leq s \leq t\}\big) = \varphi(x,t)\,,$$

since processes $\widetilde{R}(t)$ and $R(t)$ are positive multiples of each other.

Then we have the following estimate from below.

THEOREM 3.2

For all R such that

$$f(R,t)$$
$$= \exp\left\{ \int_0^t \Big[\lambda_1 + \lambda - \lambda_1\, E\big(\exp\{-R\, c_i\, e^{-rs}\}\big) - \lambda\, E\big(\exp\{R\, X_i\, e^{-rs}\}\big)\Big]\, ds \right\}$$
$$< \infty$$

and for all $t \geq 0$, the process $e^{-R\widetilde{R}(t)} / f(R,t)$ is a martingale and

$$\varphi(x,t) \geq 1 - f(R,t)\, e^{-Rx}\,.$$

PROOF Denote $g(\widetilde{R}(t), t) = e^{-R\widetilde{R}(t)}$, and compute

$$E\big(g(x, t+\Delta t)\big) = \big[1 - (\lambda_1 + \lambda)\,\Delta t\big]\, E\big(g(x,t)\big)$$
$$+\lambda_1\, \Delta t \int_0^\infty g(x + \nu\, e^{-rt}, t)\, dG(\nu)$$
$$+\lambda\, \Delta t \int_0^\infty g(x - u\, e^{-rt}, t)\, dF(u) + o(\Delta t)\,.$$

Hence we obtain the following integro-differential equation

$$\frac{\partial}{\partial t} E\big(g(x,t)\big) + (\lambda_1 + \lambda)\, g(x,t)$$
$$= \lambda_1 \int_0^\infty g(x + \nu\, e^{-rt}, t)\, dG(\nu) + \lambda \int_0^\infty g(x - u\, e^{-rt}, t)\, dF(u)\,.$$

Let us find a solution of the form $E\big(g(x,t)\big) = b(t)\, e^{-Rx}$. We obtain that $b(t)$ satisfies the equation

$$b'(t) = b(t)\big[- \lambda_1 - \lambda + \lambda_1 E\big(\exp\{-R\, c_i\, e^{-rt}\}\big) + \lambda\, E\big(\exp\{R\, X_i\, e^{-rt}\}\big)\big]$$

with the initial condition $b(0) = 1$.

Further

$$E\bigg(\frac{e^{-R\widetilde{R}(t)}}{f(R,t)} \bigg| \mathcal{F}_s \bigg) = \frac{e^{-R\widetilde{R}(s)}}{f(R,s)}\, E\Big(e^{-R(\widetilde{R}(t) - \widetilde{R}(s))} \Big| \mathcal{F}_s \Big)\, \frac{f(R,s)}{f(R,t)}$$
$$= \frac{e^{-R\widetilde{R}(s)}}{f(R,s)}\, E\Big(e^{-R(\widetilde{R}(t) - \widetilde{R}(s))} \Big)\, \frac{f(R,s)}{f(R,t)}\,,$$

where the latter equality holds true due to the independence of increments of $\widetilde{R}(t)$. Also note that random variables $\widetilde{R}(t) - \widetilde{R}(s)$ and $\widetilde{R}(t-s)\, e^{-rs}$ have the same distribution function. Hence

$$E\Big(e^{-R(\widetilde{R}(t) - \widetilde{R}(s))} \Big) = E\Big(e^{-rs\widetilde{R}(t-s)} \Big)$$
$$= e^{-(\lambda_1 + \lambda)\,(t-s)} \exp\bigg\{ \int_0^{t-s} \Big[\lambda_1 E\big(\exp\{-R\, c_i\, e^{-r(s+l)}\}\big)$$
$$+ \lambda\, E\big(\exp\{R\, X_i\, e^{-r(s+l)}\}\big)\Big]\, dl \bigg\}$$
$$= e^{-(\lambda_1 + \lambda)\,(t-s)} \exp\bigg\{ \int_s^{t} \Big[\lambda_1 E\big(\exp\{-R\, c_i\, e^{-rl}\}\big)$$
$$+ \lambda\, E\big(\exp\{R\, X_i\, e^{-rl}\}\big)\Big]\, dl \bigg\}$$
$$= \frac{f(R,t)}{f(R,s)}\,,$$

and therefore the process $e^{-R\widetilde{R}(t)} \big/ f(R,t)$ is a martingale.

Using martingale properties we obtain

$$1 = \frac{E\big(e^{-R\widetilde{R}(t)}\big)}{f(R,t)} = \frac{E\big(e^{-R\widetilde{R}(t\wedge\tau)}\big)}{f(R, t\wedge\tau)} \geq \frac{E\big(e^{-R\widetilde{R}(t\wedge\tau)}\, I_{\{\omega:\ \tau \leq t\}}\big)}{f(R, t\wedge\tau)}$$
$$\geq \frac{e^{Rx}\, P\big(\{\omega:\ \tau \leq t\}\big)}{f(R,t)}\,,$$

which proves the result. ∎

Now suppose that all capital of an insurance company is invested in stock. The dynamics of prices of stock S are described by the Black-Scholes model (with $\beta_0 = 0$). In this case the capital of the insurance company satisfies the equation

$$R(t) = \mu \int_0^t R(s)\, ds + \sigma \int_0^t R(s)\, dw_s + \sum_{i=1}^{N_1(t)} c_i - \sum_{i=1}^{N(t)} X_i \,.$$

We have the following result.

THEOREM 3.3

Suppose that all capital of an insurance company is invested in stock, then the probability of company's solvency satisfies the integro-differential equation

$$\frac{\sigma^2}{2}\, x^2\, \varphi''(x) + \mu\, x\, \varphi'(x) - (\lambda_1 + \lambda)\, \varphi(x) \tag{3.21}$$

$$+\lambda \int_0^x \varphi(x-y)\, dF(y) + \lambda_1 \int_0^\infty \varphi(x+\nu)\, dG(\nu) = 0\,,$$

which in the case of

$$G(\nu) = 1 - e^{-b\nu} \qquad and \qquad F(u) 1 - e^{-au}\,,$$

can be reduced to a third order ordinary differential equation. For $\mu > \sigma^2/2$ we have the asymptotic behavior

$$\varphi(x) = K_1 + x^{1-2\,\mu/\sigma^2}\, \big(K_2 + o(1)\big)\,.$$

PROOF (see, for example, [17]) As in Theorem 3.1 we can write

$$\begin{aligned}\varphi(x) = \big[1 - (\lambda_1 + \lambda)\big]\, \varphi(x + \mu\, x\, \Delta t + \sigma\, x\, \Delta t) \\ +\lambda_1 \int_0^\infty \varphi(x + \mu\, x\, \Delta t + \sigma\, x\, \Delta t + \nu)\, dG(\nu) \\ +\lambda \int_0^{x+\mu\, x\, \Delta t + \sigma\, x\, \Delta t} \varphi(x + \mu\, x\, \Delta t + \sigma\, x\, \Delta t - u)\, dF(u) + o(\Delta t)\,.\end{aligned}$$

Using the Kolmogorov-Itô formula we obtain

$$\varphi(x) = \big[1 - (\lambda_1 + \lambda)\, \Delta t\big] \Big(\varphi(x) + \mu\, x\, \varphi'(x)\, \Delta t + \frac{\sigma^2}{2}\, x^2\, \varphi''(x)\, \Delta t\Big)$$

$$+\lambda_1\, \Delta t \int_0^\infty \varphi(x+\nu)\, dG(\nu) + \lambda\, \Delta t \int_0^x \varphi(x-u)\, dF(u)\,,$$

which implies (3.21).

Now consider the case of

$$G(\nu) = 1 - e^{-b\nu} \qquad \text{and} \qquad F(u) = 1 - e^{-au} .$$

Equation (3.21) becomes

$$(\lambda_1 + \lambda)\,\varphi(x) - \mu\,x\,\varphi'(x) - \frac{\sigma^2}{2}\,x^2\,\varphi''(x) = \lambda_1\,I_1 + \lambda\,I\,, \tag{3.22}$$

where

$$I_1 = \int_0^\infty \varphi(x+\nu)\,b\,e^{-b\nu}\,d\nu \qquad \text{and} \qquad I = \int_0^x \varphi(x-u)\,a\,e^{-au}\,du\,.$$

Since

$$I_1' = -b\,\varphi(x) + b\,I_1 \qquad \text{and} \qquad I' = a\,\varphi(x) - a\,I\,,$$

then differentiating (3.22) we obtain

$$(\lambda_1 + \lambda)\,\varphi'(x) - \mu\,\varphi'(x) - \mu\,x\,\varphi''(x) - \sigma^2\,x\,\varphi''(x) - \frac{\sigma^2}{2}\,x^2\,\varphi^{(3)}(x)$$
$$= a\,\lambda\,\varphi(x) - a\,\lambda\,I - b\,\lambda_1\,\varphi(x) + b\,\lambda_1\,I_1$$

or

$$(\lambda_1 + \lambda - \mu)\,\varphi'(x) - (\mu + \sigma^2)\,x\,\varphi''(x) - \frac{\sigma^2}{2}\,x^2\,\varphi^{(3)}(x) \tag{3.23}$$
$$= (a\,\lambda - b\,\lambda_1)\,\varphi(x) - a\,\lambda\,I + b\,\lambda_1\,I_1\,.$$

Further differentiation gives

$$(\lambda_1 + \lambda - \mu)\,\varphi''(x) - (\mu + \sigma^2)\,\varphi''(x)$$
$$-(\mu + \sigma^2)\,x\,\varphi^{(3)}(x) - \sigma^2\,x\,\varphi^{(3)}(x) - \frac{\sigma^2}{2}\,x^2\,\varphi^{(4)}(x)$$
$$= (a\,\lambda - b\,\lambda_1)\,\varphi'(x) - (b\,\lambda_1^2 + a\,\lambda^2)\,\varphi(x) + a^2\,\lambda\,I + b^2\,\lambda_1\,I_1$$

or

$$(\lambda_1 + \lambda - 2\,\mu - \sigma^2)\,\varphi''(x) - (\mu + 2\,\sigma^2)\,x\,\varphi^{(3)}(x) - \frac{\sigma^2}{2}\,x^2\,\varphi^{(4)}(x) \tag{3.24}$$
$$= (a\,\lambda - b\,\lambda_1)\,\varphi'(x) - (b\,\lambda_1^2 + a\,\lambda^2)\,\varphi(x) + a^2\,\lambda\,I + b^2\,\lambda_1\,I_1\,.$$

Now we multiply equation (3.22) by $(a-b)$, equation (3.23) by $a\,b$, and add both to equation (3.24):

$$\varphi^{(4)}(x) + \left[(a-b) + \frac{2\,(\mu+2\,\sigma^2)}{\sigma^2\,x}\right]\varphi^{(3)}(x)$$
$$+\left[-\,a\,b + \frac{2\,(\mu+\sigma^2)(a-b)}{\sigma^2\,x} - \frac{2\,(\lambda_1+\lambda-2\,\mu-\sigma^2)}{\sigma^2\,x^2}\right]\varphi''(x)$$
$$+\left[-\,\frac{2\,a\,b\,\mu}{\sigma^2\,x} - \frac{2\,\big(\lambda_1\,a-\lambda\,b+\mu\,(b-a)\big)}{\sigma^2\,x^2}\right]\varphi'(x) = 0\,.$$

Making substitution $G=\varphi'$, we obtain

$$G^{(3)}(x) + \left[(a-b) + \frac{2\,(\mu+2\,\sigma^2)}{\sigma^2\,x}\right]G^{(2)}(x)$$
$$+\left[-\,a\,b + \frac{2\,(\mu+\sigma^2)(a-b)}{\sigma^2\,x} - \frac{2\,(\lambda_1+\lambda-2\,\mu-\sigma^2)}{\sigma^2\,x^2}\right]G'(x)$$
$$+\left[-\,\frac{2\,a\,b\,\mu}{\sigma^2\,x} - \frac{2\,\big(\lambda_1\,a-\lambda\,b+\mu\,(b-a)\big)}{\sigma^2\,x^2}\right]G(x) = 0\,.$$

We can use standard methods of theory of ordinary differential equations (see, for example, [20]) to find the asymptotic behavior of a solution of the latter equation as $x\to\infty$. We use the substitution $G(x)=e^{\tau x}\,G_1(x)$ where τ is chosen so that the constant coefficient in front of $G_1(x)$ vanishes. This implies that τ satisfies the equation

$$\tau^3+\tau^2\,(a-b)-\tau\,a\,b=0\,,$$

which has solutions $\tau=0,\,-a,\,b$. The case of $\tau=b$ is not suitable as $\varphi(x)$ is a bounded function. If $\tau=0$, then the equation stays unchanged. Next, we use the substitution $G_1(x)=x^r\,G_2(x)$ with r such that the coefficient in front of $G_2(x)/x$ is zero. Hence r satisfies the equation

$$-r\,a\,b-\frac{2\,a\,b\,\mu}{\sigma^2}=0\,,$$

which implies $r=-2\,\mu/\sigma^2$.

Thus, we find a solution in the form of the series

$$G_2(x)=\sum_{k=0}^{\infty}\frac{c_k}{x^k}\,,$$

which, in general, may be divergent, but it gives us the following asymptotic representation

$$G_2(x)=c_0+o(1)\,.$$

In this case

$$\varphi'(x) = x^{-2\,\mu/\sigma^2}\,\big(c_0 + o(1)\big)\,.$$

If $\tau = -a$, then

$$\varphi'(x) = o\big(x^{-2\,\mu/\sigma^2}\big)\,.$$

$\square$

Note that this theorem reiterates the following important observation: if an insurance company has investments in risky assets of the financial market, then the asymptotic behavior of the probability of its solvency cannot be exponential as it was in the standard Cramér-Lundberg model.

Appendix A

Software Supplement: Computations in Finance and Insurance

The software is offered in the form of an open source code. It can be downloaded from

www.crcpress.com/e_products/downloads/download.asp?cat_no = C429

Any C++ compiler can be used under a Linux or Windows operating system. This code can be easily read and modified.

Software berprog.cpp: Forecast of stock prices in a Bernoulli market

Suppose that price dynamics of stock S in a Bernoulli market are given by the following recurrence formula

$$S_{i+1} = S_i\,(1+\rho)\,, \qquad 0 \le i \le n\,,$$

where ρ is the profitability of S and it takes values b or a with probabilities p and $(1-p)$, respectively. Assuming that the initial price is S_0, we forecast the average price of S at time n.

The inputs are:

1. S_0, the initial price of S;
2. a and b, values of possible change (as a percentage) in price of S;
3. p, the probability for ρ to take value b;
4. n, time horizon.

The forecast is based on the properties of conditional expectations $E(Y|X)$ of independent random variables. Namely, we compute

$$S_{\rm av} = E\left(\frac{S_1+S_2+\cdots+S_n}{n}\,\middle|\, S_0\right).$$

In a two-step case we have

$$
\begin{aligned}
S_{\text{av}} = E\left(\frac{S_1 + S_2}{2}\,\middle|\, S_0\right) &= E\left(\frac{S_0(1+\rho_1) + S_0(1+\rho_1)(1+\rho_2)}{2}\,\middle|\, S_0\right) \\
&= \frac{S_0}{2}\Big[E(1+\rho_1) + E(1+\rho_1)\,E(1+\rho_2)\Big] \\
&= \frac{S_0}{2}\Big[(1+a)p + (1+b)(1-p) \\
&\qquad + \big((1+a)p + (1+b)(1-p)\big)\big((1+a)p + (1+b)(1-p)\big)\Big] \\
&= \frac{S_0}{2}\Big[(1+a)p + (1+b)(1-p) + \big((1+a)p + (1+b)(1-p)\big)^2\Big]
\end{aligned}
$$

Software binoptprice.cpp:Pricing options in a binomial market

Consider a one-step binomial (B, S)-market. Dynamics of a bank account and stock price are given by

$$B_1 = B_0\,(1+r)\,, \qquad \text{and} \qquad S_1 = S_0\,(1+\rho)\,,$$

where r is a fixed rate of interest and ρ is the profitability of S that takes values b or a with probabilities p and $(1-p)$, respectively. Quantities a, b r must satisfy the inequality $-1 < a < r < b$. Suppose that $B_0 = 1$. Consider a European call option with the contingent claim

$$f_1 = (S_1 - K)^+ = \max(0, S_1 - K)\,,$$

where K is a strike price. Let $K = S_0$. The intuitive price for this option is

$$E\left(\frac{f_1}{1+r}\right) = \frac{p\,(S_0\,b)^+ + (1-p)\,(S_0\,a)^+}{1+r}\,.$$

Alternatively, using the minimal hedging approach (see Section 1.4), one can construct a strategy $\pi_0(b_0, g_0)$ such that

$$X_1^\pi\big(X_0^\pi\big) = f_1\,.$$

In this case

$$X_0^\pi = \beta_0^*\,B_0 + \gamma_0^*\,S_0\,,$$

where

$$\gamma_0^* = \frac{f_1^{(1)} - f_1^{(2)}}{S_0(\rho_1 - \rho_2)}, \quad \text{and} \quad \beta_0^* = \frac{1}{(1+r)\,B_0}\left(f_1^{(1)} - \gamma_0^* S_0(1+\rho_1)\right).$$

Finally, consider a risk-neutral probability E^* such that

$$E^*\left(\frac{S_1}{B_1}\right) = S_0.$$

This implies

$$p^* = \frac{(1+r)\,B_0 - 1 - a}{b-a}.$$

If $B_0 = 1$ then

$$p^* = \frac{r-a}{b-a}.$$

Thus, risk-neutral price is given by

$$C = E^*\left(\frac{f_1}{1+r}\right) = \frac{f_1^{(1)} p^* + f_1^{(2)}(1-p^*)}{1+r}.$$

The inputs are

1. S_0, the initial price of S;
2. a and b, values of possible change (as a percentage) in price of S;
3. p, the probability for ρ to take value b;
4. r, the rate of interest.

The outputs are

1. values of the contingent claim;
2. intuitive price;
3. initial capital of the minimal hedge;
4. risk-neutral price;
5. risk-neutral probability.

Software CRRoptprice.cpp: Call-put parity in a Cox-Ross-Rubinstein market

Now we consider a N-step binomial (B, S)-market. The Cox-Ross-Rubinstein formula

$$C_N = S_0\, B(k_0, N, \tilde{p}) - K\,(1+r)^{-N} B(k_0, N, p^*)$$

gives the risk-neutral price of a European call option. Here p^* is a risk-neutral probability:

$$p^* = \frac{r-a}{b-a} \qquad \text{and} \qquad \tilde{p} = \frac{1+b}{1+a} p^* .$$

Recall (see Section 1.4) that

$$B(j, N, p) := \sum_{k=j}^{N} \binom{N}{k} p^k \,(1-p)^{N-k} ,$$

constant k_0 is defined by

$$k_0 = \min \left\{ k \leqslant N : S_0(1+b)^k (1+a)^{N-k} \geqslant K \right\}$$

so that

$$k_0 = \left[\ln \frac{K}{S_0(1+a)^N} \Big/ \ln \frac{1+b}{1+a} \right] + 1 .$$

Using the call-put parity, we can price a European put option with the claim

$$f_N = (K - S_N)^+ .$$

Namely, price of this option is given by

$$P_N = C_N - S_0 + K(1+r)^{-N} .$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. a and b, values of possible change (as a percentage) in price of S;
4. r, the rate of interest;
5. N, the terminal time.

The output contains prices of the European call and put options.

Software amoptprice.cpp: Pricing an American call option

Consider an N-step binomial (B, S)-market. Recall (see Section 1.5) that price of an American call option with the sequence of claims

$$f_n = (S_n - K)^+ , \quad n \leq N ,$$

is defined by

$$C_N^{\mathrm{am}} = \sup_{\tau \in \mathcal{M}_0^N} C(f_\tau) = \sup_{\tau \in \mathcal{M}_0^N} E^* \left(\frac{f_\tau}{(1+r)^\tau} \right) .$$

Computing

$$Y_n = \max \left\{ \frac{f_n}{(1+r)^n} , E^* \left(Y_{n+1} \middle| \mathcal{F}_n \right) \right\}$$

we obtain

$$C_N^{\mathrm{am}} = Y_0 .$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. a and b, values of possible change (as a percentage) in price of S;
4. r, the rate of interest;
5. N, the terminal time.

The output consists of a price of the American call option.

Software spread.cpp: Computing spreads in a market with constraints

Let (B_1, B_2, S) be a Cox-Ross-Rubinstein market with constraints (see Section 2.2). We find the interval $[C_*, C^*]$ of all arbitrage-free prices, and therefore compute spread $C^* - C_*$ as a measure of incompleteness of the market.

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;

3. a and b, values of possible change (as a percentage) in price of S;
4. r_1 and r_2, the rates of interest on saving and credit accounts, respectively;
5. N, the terminal time.

The output contains prices of the European call option corresponding to given rates r_1 and r_2, and the spread of the market.

Software BandSgreek.cpp: Pricing contingent claims using the Black-Scholes formula and computing Greek parameters in the continuous case

In Section 2.6 we studied pricing of contingent claims in a Black-Scholes model of a (B,S)-market. The 'fair' arbitrage-free price of a European call option is given by the Black-Scholes formula:

$$C_T = S_0\,\Phi\big(y_+\big) - K\,e^{-r\,T}\,\Phi\big(y_-\big)\,,$$

where

$$y_\pm = \frac{\ln(S_0/K) + T(r \pm \sigma^2/2)}{\sigma\sqrt{T}}.$$

The price of a European put option:

$$P_T(K,\sigma,S_0) = C_T(-K,-\sigma,-S_0)$$

is derived from the call-put parity relation.

The following 'Greeks' are used by the risk management practitioners:

Theta:

$$\theta = \frac{\partial C_T}{\partial t} = \frac{S_t\,\sigma\,\varphi\big(y_+(t)\big)}{2\,\sqrt{T-t}} - K\,r\,e^{-r\,(T-t)}\,\Phi\big(y_-(t)\big)\,,$$

Delta:

$$\Delta = \frac{\partial C_T}{\partial S} = \Phi\big(y_+(t)\big)\,,$$

Rho:

$$\rho = \frac{\partial C_T}{\partial r} = K\,(T-t)\,e^{-r\,(T-t)}\,\Phi\big(y_-(t)\big)\,,$$

Vega:

$$\Upsilon = \frac{\partial C_T}{\partial \sigma} = S_t\,\varphi\big(y_+(t)\big)\,\sqrt{T-t}\,,$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. σ, the volatility of the market;
4. r, the rate of interest;
5. T, the terminal time;
6. t, intermediate time.

The output contains prices of the European call and put options and values of 'Greek' parameters.

Software BandSdiv.cpp: Pricing contingent claims using the Black-Scholes formula in a model with dividends

In the case when a holder of asset S receives dividends, the Black-Scholes formula gives the price of a European call option in the following form

$$C_T(\delta) = S_0\, e^{-\delta T}\, \Phi\left(\frac{\ln(S_0/K) + T\,(r - \delta + \sigma^2/2)}{\sigma\sqrt{T}}\right) \\ -K\, e^{-rT}\, \Phi\left(\frac{\ln(S_0/K) + T\,(r - \delta - \sigma^2/2)}{\sigma\sqrt{T}}\right).$$

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. σ, the volatility of the market;
4. r, the rate of interest;
5. T, the terminal time;
6. δ, the dividends rate.

The output contains price of the European call option in the market with dividends.

Software Indicators.cpp: Some indicators used in technical analysis

This software supplements Section 2.9. We compute the following indicators:

1. Simple Moving Average (SMA);
2. Weighted Moving Average (WMA);
3. Exponential Moving Average (EMA);
4. Bollinger Bands;
5. Moving Average Convergence/Divergence (MACD);
6. Relative Strength Index (RSI);
7. Parabolic Time Price system (PTP);
8. Volume Price Trend (VPT).

The user has to specify which indicator must be calculated. The software contains two data files: 'data.txt' and 'datav.txt'. File 'data.txt' contains close prices from daily bar charts for shares of Microsoft for the period from April, 2001 till April, 2002. File 'datav.txt' contains the complete information of the same bar charts: daily open, close, low, high prices and trading volumes. First six indicators use file 'data.txt', and the last two use file 'datav.txt'. The resulting text file can be processed by various graphical tools, e.g. Gnuplot, MetaStock etc.

Simple Moving Average represents the average value of a quantity during a specified period of time. At time j it is computed as

$$SMA_j = \frac{1}{N} \sum_{i=j-N}^{j} v_i \, ,$$

where $v_i, \; j-N \leq i \leq j$, are values of this quantity. Input N defines the time horizon. The output file 'series' contains the initial time series, and file 'ma' contains the time series for moving average.

Weighted Moving Average is a modification of the SMA:

$$WMA_j = \sum_{i=1}^{N} \varpi_i \, v_i \, ,$$

where ϖ_i are weights with $\sum_{i=1}^{N} \varpi_i = 1$. As above, time horizon N corresponds to some analysis time j. Usually, time points that are closer to the analysis time j, have heavier weights. We use the following formula

$$\varpi_i = \frac{i}{\sum_{k=1}^{N} k}, \qquad i = 1, \dots, N.$$

The output file 'series' contains the initial time series, and file 'wma' contains the time series for weighted moving average.

Exponential Moving Average is the most widely used modification of the WMA. It uses all preceding values, but times that are distant from the analysis time j, correspond to very small weights. The Exponential Moving Average is defined by the formula

$$EMA_j = (1-\alpha)\, EMA_{j-1} + \alpha\, v_j\,,$$

where

$$\alpha = \frac{2}{N+1}\,.$$

Clearly, this is the simplest Moving Average indicator. File 'ema' contains the time series for exponential moving average.

Bollinger Bands. First, one can use any of the Moving Average indicators to construct a Middle Band with values m_j. Then Upper and Lower Bands are defined by

$$u_j = m_j + k\,\sigma_j \qquad \text{and} \qquad l_j = m_j - k\,\sigma_j\,,$$

where

$$\sigma_j = \sqrt{\frac{\sum_{i=j-N}^{j} (v_i - m_j)^2}{N}}\,.$$

The inputs include the order of averaging N and coefficient k which reflects the sensitivity of the indicator. The output file 'bbands' contains Middle, Upper and Lower Bands.

Moving Average Convergence/Divergence is constructed in the following way:

1. compute short Moving Average;
2. compute long Moving Average;
3. compute quick line by subtracting long Moving Average from the short Moving Average;
4. compute signal line by smoothing quick line with the help of Moving Average;
5. compute MACD as the difference between signal and quick lines – this is contained in the output file 'macd'.

Relative Strength Index is computed in terms of average increase and decrease of price over some period of time.

Average increase is given by

$$U_j = \begin{cases} \frac{U_{j-1}(N-1)+(v_j - v_{j-1})}{N} & \text{if } v_j > v_{j-1} \\ \frac{U_{j-1}(N-1)}{N} & \text{if } v_j < v_{j-1} \,. \end{cases}$$

Similarly, for average decrease we have

$$D_j = \begin{cases} \frac{D_{j-1}(N-1)+(v_{j-1} - v_j)}{N} & \text{if } v_j < v_{j-1} \\ \frac{D_{j-1}(N-1)}{N} & \text{if } v_j > v_{j-1} \,. \end{cases}$$

Then

$$RSI_j = 100 - \frac{100}{1 + U_j/D_j} \,.$$

The output file 'rsi' contains the values of RSI.

Parabolic Time Price system is represented by a line that is positioned either above or below the price graph, which identifies decreasing or increasing trends, respectively. The close price C_j is determined daily by the recurrence relation

$$C_j = C_{j-1} + \mathcal{A}\,(\mathcal{E}_{j-1} - C_{j-1})\,,$$

where $\mathcal{E}$ is the critical level of daily trading: in a long position it is equal to the highest price since buying, in a short position it is the lowest price since selling. Constant $\mathcal{A}$ is the averaging factor. It determines how fast the close price should be shifted toward open position, and it depends on the number of new peaks since buying and new lows since selling. The initial value of $\mathcal{A}$ is usually set to be 0.02. Increase or decrease of the initial value respectively increases or decreases the sensitivity of PTP line. Note that in this computer version of constructing PTP lines, by open position we understand the corresponding dynamics of a trend.

This software does not have any input parameters, but the input data in file 'datav.txt' must be in the form (open, low, high, close).

Volume Price Trend reflects overbuying or overselling in the market. It is computed as

$$VPT_j = VPT_{j-1} + V_j\,\frac{P_j - P_{j-1}}{P_{j-1}}\,,$$

where V_j and P_j are values of volume and price respectively.

Software elenshure.cpp: Pure endowment assurance

This software is the computer realization of Worked Example 1.5 from Section 1.4.

The inputs are

1. S_0, the initial price of S;
2. K, the strike price;
3. a and b, values of possible change (as a percentage) in price of S;
4. r, the rate of interest;
5. N, the terminal time;
6. p_x, the probability of surviving.

The output is the price of the insurance policy.

Software var.cpp: Computing the Value at Risk

This software uses historical modelling and Monte-Carlo modelling for computing values of Value at Risk. Both approaches have similar structure, the only difference is that in historical modelling one uses the real data for determining the distribution of losses and profits, whereas, in Monte-Carlo modelling, it is assumed price movements are normally distributed.

The software computes the Value at Risk for one asset. File 'vardata.txt' contains the input data. After assessing the volume of the input data, the software asks to define the time horizon. Given this information it then determines the number of possible scenarios. Depending on the number of scenarios requested by the user, the software creates as many independent scenarios as possible and computes the distribution of profits and losses. Further, it requests the value of probability for which we compute the Value at Risk, i.e., losses that correspond to this probability will not exceed the corresponding value of the Value at Risk. Finally, since the input data is discrete in time, the software requests the size of the data confidence interval in terms of probability of being in this interval and then calculates the confidence interval for the Value at Risk.

Software RiskPremInd.cpp: Computing premiums in the model of individual risk

This software is the computer realization of Example from Section 3.1. It requests the level of bankruptcy's probability and n, the number of policies. The initial data is contained in file 'RPIdata.txt' and it consists of uniformly distributed independent random variables that are used for computing the distribution of X^{ind}, the total payoff to the policy holders. The output is the price of the insurance policy.

Appendix B

Problems and Solutions

B.1 Problems for Chapter 1

Problem B.1.1 *Suppose that the effective annual rate of interest is* 10%. *Find the present value of a 3-year bond with face value* $\$500$ *and with annual coupon payments of* $\$100$.

SOLUTION We have

$$B_0 = \frac{100}{1+0.1} + \frac{100}{(1+0.1)^2} + \frac{100}{(1+0.1)^3} + \frac{500}{(1+0.1)^3} \approx 624\,(\$)\,.$$

$\square$

Problem B.1.2 *An investor buys two European put options with strike price* $\$40$ *and one European call option with strike price* $\$50$ *on the same stock* S *with the same expiry date* N. *The total price of these options is* $\$10$. *Write down the gain-loss function and discuss the possible outcomes.*

SOLUTION If S_N is the price of the asset at time N, then the gain-loss function has the form

$$\mathcal{V}(S_N) = 2\left(40 - S_N\right)^+ + \left(S_N - 50\right)^+ - 10\,.$$

We have the following cases.

(1) If $S_N < 40$, then the put options are exercised and the call option is not. In this case

$$\mathcal{V}(S_N) = 2\left(40 - S_N\right)^+ - 10 = 70 - 2\,S_N\,,$$

and a profit is earned if $S_N < 35$.

(2) If $40 \leq S_N \leq 50$, then none of the options are exercised, and the premium is lost:

$$\mathcal{V}(S_N) = -10\,.$$

(3) If $S_N > 50$, then only the call option is exercised:

$$\mathcal{V}(S_N) = \big(S_N - 50\big)^+ - 10 = S_N - 60\,,$$

and a profit is earned if $S_N > 60$.

Thus, this investment strategy reflects the investor's expectation that the price of S will be less than \$35. Note that the maximal possible loss cannot be higher than the amount of premium. ∎

Problem B.1.3 *The joint distribution of profitabilities α and β is given in the following table.*

$\alpha \diagdown \beta$	-0.1	0	0.1
-0.2	0.1	0	0.4
0.1	0.3	0.1	0.1

Find their individual distributions, average of β and the conditional expectation $E(\beta|\alpha)$.

SOLUTION From the given table we compute:

$$\begin{aligned}
&P\big(\{\omega:\ \alpha = -0.2\}\big) = 0.1 + 0.4 = 0.5\\
&P\big(\{\omega:\ \alpha = 0.1\}\big) = 0.3 + 0.1 + 0.1 = 0.5\\
&P\big(\{\omega:\ \beta = -0.1\}\big) = 0.1 + 0.3 = 0.4\\
&P\big(\{\omega:\ \beta = 0\}\big) = 0.1\\
&P\big(\{\omega:\ \beta = 0.1\}\big) = 0.4 + 0.1 = 0.5\,,
\end{aligned}$$

which implies

$$E(\beta) = -0.1 \times 0.4 + 0.1 \times 0.5 = 0.01\,.$$

The conditional expectation $E(\beta|\alpha)$ can be written in the form (see [41], for example):

$$E(\beta|\alpha) = E\big(\beta\big|\{\omega:\ \alpha = -0.2\}\big)\, I_{\{\omega:\ \alpha=-0.2\}} + E\big(\beta\big|\{\omega:\ \alpha = 0.1\}\big)\, I_{\{\omega:\ \alpha=0.1\}}\,.$$

Computing

$$\begin{aligned}
E\big(\beta|\{\omega:\ \alpha = -0.2\}\big) &= -0.1\, P\big(\{\omega:\ \beta = -0.1\}\big|\{\omega:\ \alpha = -0.2\}\big)\\
&\qquad +0.1\, P\big(\{\omega:\ \beta = 0.1\}\big|\{\omega:\ \alpha = -0.2\}\big)\\
&= \frac{-0.1 \times 0.1 + 0.1 \times 0.4}{0.5} = 0.06\,,
\end{aligned}$$

and

$$E\big(\beta|\{\omega:\ \alpha=0.1\}\big) = -0.1\,P\big(\{\omega:\ \beta=-0.1\}\big|\{\omega:\ \alpha=0.1\}\big)$$

$$+0.1\,P\big(\{\omega:\ \beta=0.1\}\big|\{\omega:\ \alpha=0.1\}\big)$$

$$= \frac{-0.1\times 0.3+0.1\times 0.1}{0.5} = -0.04\,,$$

we obtain

$$E(\beta|\alpha) = 0.06\,I_{\{\omega:\ \alpha=-0.2\}} - 0.04\,I_{\{\omega:\ \alpha=0.1\}}\,.$$

Note that

$$E\big(E(\beta|\alpha)\big) = 0.06\times 0.5 - 0.04\times 0.5 = 0.01 = E(\beta)\,.$$

□

Problem B.1.4 *Suppose that analysis of the market data suggests that the price of a certain asset S will increase by 2% in one month's time with probability p, or will decrease by 1% with probability $1-p$. Find all values of p such that an investment in this asset will be, on average, more profitable than an investment in a bank account with effective monthly interest rate of 1%.*

SOLUTION The average monthly profitability of an investment in asset S is equal to

$$0.02\,p - 0.01\,(1-p) = 0.03\,p - 0.01\,.$$

Hence p must satisfy

$$0.03\,p - 0.01 \geq 0.01\,,$$

or $p \geq 2/3$. □

Problem B.1.5 *As in Section 1.3 we consider a binomial (B,S)-market. Suppose we are given the following values of its parameters:*

$$a=-0.4\,,\quad b=0.6\,,\quad r=0.2\,,\quad B_0=1\,,\quad S_0=200\,.$$

Find the price and the minimal hedge of a 'look back' European call option with the contingent claim

$$f_2 = \big(S_2 - K_2\big)^+\,,\quad \textit{where}\quad K_2 = \min\{S_0,\,S_1,\,S_2\}\,.$$

SOLUTION First we compute the risk-neutral probability

$$p^* = \frac{r-a}{b-r} = \frac{0.2+0.4}{0.6+0.4} = 0.6\,.$$

Now we write all possible prices of S and values of claim f_2 in the following table.

event	**probability**	ρ_1	ρ_2	S_0	S_1	S_2	K_2	f_2
ω_1	0.16	-0.4	-0.4	200	120	72	72	0
ω_2	0.24	-0.4	0.6	200	120	192	120	72
ω_3	0.24	0.6	-0.4	200	320	192	192	0
ω_4	0.36	0.6	0.6	200	320	512	200	312

We compute price of this claim:

$$C_2 = \frac{E^*(f_2)}{(1+r)^2} = \frac{0.16 \times 0 + 0.24 \times 72 + 0.24 \times 0 + 0.36 \times 312}{1.44} = 90\,.$$

Next we find the minimal hedge $\pi^* = \big(\beta_n, \gamma_n\big)_{n=1}^{2}$ that replicates claim f_2. Consider time $n = 1$. Since the value of profitability ρ_1 is known at this time, we can construct the pair $\big(\beta_2, \gamma_2\big)$. Indeed, we have that hedge π^* replicates f_2:

$$X_2^{\pi^*} = f_2\,,$$

which can be written in the form of the following system

$$\begin{cases} X_2^{\pi^*}(\omega_1) = \beta_2(\omega_1)\,(1+r)^2 + \gamma_2(\omega_1)\,S_2(\omega_1) \\ X_2^{\pi^*}(\omega_2) = \beta_2(\omega_2)\,(1+r)^2 + \gamma_2(\omega_2)\,S_2(\omega_2) \\ X_2^{\pi^*}(\omega_3) = \beta_2(\omega_3)\,(1+r)^2 + \gamma_2(\omega_3)\,S_2(\omega_3) \\ X_2^{\pi^*}(\omega_4) = \beta_2(\omega_4)\,(1+r)^2 + \gamma_2(\omega_4)\,S_2(\omega_4)\,. \end{cases}$$

Substituting all known values we obtain

$$\begin{cases} 0 = \beta_2(\omega_1)\,(1+0.2)^2 + \gamma_2(\omega_1)\,72 \\ 72 = \beta_2(\omega_2)\,(1+0.2)^2 + \gamma_2(\omega_2)\,192 \\ 0 = \beta_2(\omega_3)\,(1+0.2)^2 + \gamma_2(\omega_3)\,192 \\ 312 = \beta_2(\omega_4)\,(1+0.2)^2 + \gamma_2(\omega_4)\,512\,. \end{cases}$$

Since random variables β_2 and γ_2 do not depend on ρ_2, we also have

$$\beta_2(\omega_1) = \beta_2(\omega_2)\,, \qquad \beta_2(\omega_3) = \beta_2(\omega_4)\,,$$

and

$$\gamma_2(\omega_1) = \gamma_2(\omega_2)\,, \qquad \gamma_2(\omega_3) = \gamma_2(\omega_4)\,.$$

Hence

$$\begin{cases} \beta_2(\omega_1) = \beta_2(\omega_2) = -30 \\ \gamma_2(\omega_1) = \gamma_2(\omega_2) = 0.6 \\ \beta_2(\omega_3) = \beta_2(\omega_4) = -130 \\ \gamma_2(\omega_3) = \gamma_2(\omega_4) = 39/40\,. \end{cases}$$

The pair (β_1, γ_1) is chosen at time $n = 0$ and does not depend on prices of S. Since π^* is self-financing, we have

$$\begin{cases} \beta_1\,(1+r) + \gamma_1\, S_1(\omega_1) = \beta_2(\omega_1)\,(1+r) + \gamma_2(\omega_1)\, S_1(\omega_1) \\ \beta_1\,(1+r) + \gamma_1\, S_1(\omega_3) = \beta_2(\omega_3)\,(1+r) + \gamma_2(\omega_3)\, S_1(\omega_3)\,, \end{cases}$$

which reduces to

$$\begin{cases} \beta_1\,(1+0.2) + \gamma_1\, 120 = 36 \\ \beta_1\,(1+0.2) + \gamma_1\, 320 = 156\,. \end{cases}$$

Hence

$$\beta_1 = -30 \qquad \text{and} \qquad \gamma_1 = 0.6\,.$$

Note that the initial capital of this hedging strategy

$$X_0^{\pi^*} = -30 + 0.6 \times 200 = 90$$

coincides with the price C_2. ∎

Problem B.1.6 *Let the rate of interest be $r \geq 0$ and suppose that the price of an asset S has the following dynamics*

Ω	$n=0$	$n=1$	$n=2$
ω_1	$S_0 = 10$	$S_1 = 12$	$S_2 = 15$
ω_2	$S_0 = 10$	$S_1 = 12$	$S_2 = 10$
ω_3	$S_0 = 10$	$S_1 = 6$	$S_2 = 10$
ω_4	$S_0 = 10$	$S_1 = 6$	$S_2 = 3$

1. *Find an expression for risk-neutral probability.*
2. *Find all values of $r \geq 0$ that admit the existence of a risk-neutral probability.*
3. *Consider an American call option with the sequence of claims*

$$f_0 = (S_0 - 9)^+\,, \qquad f_1 = (S_1 - 9)^+\,, \qquad f_2 = (S_2 - 10)^+\,.$$

Price this option, find the minimal hedge and the stopping times for $r = 0$.

SOLUTION

1. An expression for risk-neutral probability $P^*(r) = \big(p_1^*(r), p_2^*(r), p_3^*(r), p_4^*(r)\big)$ can be found from the equalities

$$E^*\left(\frac{S_1}{1+r}\right) = S_0 \qquad \text{and} \qquad E^*\left(\frac{S_2}{(1+r)^2}\bigg|\sigma(S_1)\right) = \frac{S_1}{1+r}\,,$$

which reduce to the following system

$$\begin{cases} 12\,(p_1^* + p_2^*) + 6\,(p_3^* + p_4^*) = 10\,(1+r) \\ \frac{15\,p_1^* + 10\,p_2^*}{p_1^* + p_2^*} = 12\,(1+r) \\ \frac{10\,p_3^* + 3\,p_4^*}{p_3^* + p_4^*} = 6\,(1+r) \\ p_1^* + p_2^* + p_3^* + p_4^* = 1\,. \end{cases}$$

Solving this system, we obtain

$$p_1^*(r) = \frac{2+5\,r}{3}\;\frac{2+12\,r}{5}\,, \qquad p_2^*(r) = \frac{2+5\,r}{3}\;\frac{3-12\,r}{5}\,,$$

$$p_3^*(r) = \frac{1-5\,r}{3}\;\frac{3+6\,r}{7}\,, \qquad p_4^*(r) = \frac{1-5\,r}{3}\;\frac{4-6\,r}{7}\,.$$

2. The fact that all these probabilities must be strictly positive:

$$p_1^*(r) > 0\,, \quad p_2^*(r) > 0\,, \quad p_3^*(r) > 0\,, \quad p_4^*(r) > 0\,,$$

implies that $0 \le r < 0.2$.

3. Now since $P^*(0) = \big(4/15,\ 2/5,\ 1/7,\ 4/21\big)$, we can compute all possible values of our contingent claim:

event	**probability**	S_0	f_0	S_1	f_1	S_2	f_2
ω_1	4/15	10	1	12	3	15	5
ω_2	2/5	10	1	12	3	10	0
ω_3	1/7	10	1	6	0	10	0
ω_4	4/21	10	1	6	0	3	0

For pricing this option we compute

$$E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 12\}\big) = E^*\big(f_2\big|\{\omega:\ S_1 = 12\}\big) = \frac{5\,p_1^* + 0\,p_2^*}{p_1^* + p_2^*} = 2\,,$$

$$E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 6\}\big) = E^*\big(f_2\big|\{\omega:\ S_1 = 6\}\big) = \frac{0\,p_3^* + 0\,p_4^*}{p_3^* + p_4^*} = 0\,,$$

where $Y_i^{\pi^*}$ is the capital of the minimal hedge π^* at time i. Hence

$$Y_1^{\pi^*}(\omega_1) = Y_1^{\pi^*}(\omega_2) = \max\big\{f_1,\ E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 12\}\big)\big\} = 3\,,$$

$$Y_1^{\pi^*}(\omega_3) = Y_1^{\pi^*}(\omega_4) = \max\big\{f_1,\ E^*\big(Y_2^{\pi^*}\big|\{\omega:\ S_1 = 6\}\big)\big\} = 0\,.$$

This implies

$$E^*\left(Y_1^{\pi^*}\big|\mathcal{F}_0\right) = 3\,(p_1^* + p_2^*) + 0\,(p_3^* + p_4^*) = 3\left(\frac{4}{15} + \frac{2}{5}\right) = 2\,,$$

and

$$Y_0^{\pi^*} = \max\left\{f_0,\ E^*\left(Y_1^{\pi^*}\big|\mathcal{F}_0\right)\right\} = 2\,,$$

therefore the price is $C_2 = 2$.

To construct the minimal hedge π^* we first compute the stopping times:

$$\tau_n^* = \min\left\{i:\ n \le i \le 2 \quad \text{and} \quad Y_i^{\pi^*} = f_i\right\},$$

so

$$\tau^* = \tau_1^* = 1, \quad \tau_2^* = 2\,.$$

Now due to equality $Y_1^{\pi^*} = f_1$, we have

$$Y_1^{\pi^*}(\omega) = \beta_1 + \gamma_1\, S_1(\omega) = f_1(\omega)$$

or

$$\begin{cases} Y_1^{\pi^*}(\omega_1) = Y_1^{\pi^*}(\omega_2) = \beta_1 + \gamma_1\, 12 = 3 \\ \\ Y_1^{\pi^*}(\omega_3) = Y_1^{\pi^*}(\omega_4) = \beta_1 + \gamma_1\, 6 = 0\,. \end{cases}$$

Hence $\beta_1 = -3$ and $\gamma_1 = 0.5$. Also note that the initial capital $Y_0^{\pi^*} = -3 + 10 \times 0.5 = 2$ coincides with the price of the option.

$\square$

Problem B.1.7 *Consider a single-period binomial* (B, S)*-market with* $B_0 = 1$, $S_0 = 300$, $r = 0.1$ *and*

$$S_1 = \begin{cases} 350 & \textit{with probability } 0.6 \\ 250 & \textit{with probability } 0.4\,. \end{cases}$$

As in Section 1.6 use the logarithmic utility function to find an optimal strategy with the initial capital 200.

SOLUTION First we compute parameters

$$a = \frac{250 - 300}{300} = -\frac{1}{6} \quad \text{and} \quad b = \frac{350 - 300}{300} = \frac{1}{6}\,.$$

The average profitability of S with respect to the initial probability is

$$m = \frac{4}{10}\ \frac{-1}{6} + \frac{6}{10}\ \frac{1}{6} = \frac{1}{30}\,.$$

Then, according to formula (1.5), the proportion of risky capital in the required strategy must be

$$\alpha^* = \frac{(1+r)\,(m-r)}{(r-a)\,(b-r)} = -4.125\,.$$

On the other hand,

$$\alpha^* = \gamma^* \, \frac{S_0^*}{X_0^{\pi^*}}\,,$$

hence,

$$\gamma^* = -4.125\,\frac{200}{300} = -2.75\,.$$

The non-risky component β^* can be found from the condition of self-financing:

$$X_0^{\pi^*} = \beta^* + \gamma^*\, S_0\,,$$

which implies

$$\beta^* = 200 + 2.75 \times 300 = 1025\,.$$

□

Problem B.1.8 *Repeat Problem B.1.7 with* $B_0 = 1$*,* $S_0 = 100$*,* $r = 0.2$ *and*

$$S_1 = \begin{cases} 150 & \textit{with probability } 0.7 \\ 80 & \textit{with probability } 0.3\,. \end{cases}$$

SOLUTION In this case we have

$$a = \frac{80-100}{100} = -0.2 \quad \text{and} \quad b = \frac{150-100}{100} = 0.5\,.$$

Then

$$m = 0.7 \times 0.5 - 0.3 \times 0.2 = 0.29\,,$$

and

$$\alpha^* = \frac{(1+r)\,(m-r)}{(r-a)\,(b-r)} = 0.9\,.$$

Thus

$$\gamma^* = \alpha^* \, \frac{X_0^{\pi^*}}{S_0^*} = 0.9\,\frac{200}{100} = 1.8\,,$$

and

$$\beta^* = X_0^{\pi^*} - \gamma^*\, S_0 = 200 - 1.8 \times 100 = 20\,.$$

□

B.2 Problems for Chapter 2

Problem B.2.1 *Consider a single-period* (B, S)*-market with* $B_0 = 1$*,* $S_0 = 10$*,* $r = 0.2$ *and*

$$S_1(\omega_1) = 6, \quad S_1(\omega_2) = 12, \quad S_1(\omega_3) = 18\,.$$

Find risk-neutral probability P^**.*

SOLUTION An expression for P^* can be found from the equality

$$E^*\left(\frac{S_1}{1+r}\right) = S_0\,,$$

which can be written in the form

$$p_1^*\, S_1(\omega_1) + p_2^*\, S_1(\omega_2) + p_3^*\, S_1(\omega_3) = S_0\,(1+r)\,.$$

Since $p_1^* + p_2^* + p_3^* = 1$, we have

$$6\,p_1^* + 12\,p_2^* + 18\,(1 - p_1^* - p_2^*) = 12\,,$$

and therefore $p_2^* = 1 - 2\,p_1^*$. Now let $p_1^* = \lambda$, then we have

$$p_2^* = 1 - 2\,\lambda\,, \qquad p_3^* = \lambda\,.$$

Since all these probabilities must be strictly positive, this implies that

$$0 < \lambda < 1/2\,.$$

Thus, we obtain a one-parameter family of risk-neutral probabilities

$$P_\lambda^* = \big(\lambda,\, 1 - 2\,\lambda,\, \lambda\big)\,, \qquad 0 < \lambda < 1/2\,.$$

□

Problem B.2.2 *Consider a single-period* (B, S)*-market with a non-risky asset* B *and two risky assets* S^1 *and* S^2*, where*

$$\begin{aligned}
&B_0 = 1\,, \qquad r = 0.2\,,\\
&S_0^1 = 150, \quad S_1^1(\omega_1) = 200, \quad S_1^1(\omega_2) = 190, \quad S_1^1(\omega_3) = 170\,,\\
&S_0^2 = 200, \quad S_1^2(\omega_1) = 270, \quad S_1^2(\omega_2) = 250, \quad S_1^2(\omega_3) = 230\,.
\end{aligned}$$

Find risk-neutral probability P^**. If it does not exist, find an arbitrage strategy.*

SOLUTION If there exists a risk-neutral probability P^*, then we must have

$$E^*\left(\frac{S_1^1}{1+r}\right) = S_0^1 \qquad \text{and} \qquad E^*\left(\frac{S_1^2}{1+r}\right) = S_0^2\,,$$

which can be written in the form of the following system

$$\begin{cases} p_1^*\, S_1^1(\omega_1) + p_2^*\, S_1^1(\omega_2) + p_3^*\, S_1^1(\omega_3) = S_0^1\,(1+r) \\ \\ p_1^*\, S_1^2(\omega_1) + p_2^*\, S_1^2(\omega_2) + p_3^*\, S_1^2(\omega_3) = S_0^2\,(1+r)\,. \end{cases}$$

Since $p_1^* + p_2^* + p_3^* = 1$, then this system reduces to

$$\begin{cases} 200\,p_1^* + 190\,p_2^* + 170\,(1 - p_1^* - p_2^*) = 180 \\ \\ 270\,p_1^* + 250\,p_2^* + 230\,(1 - p_1^* - p_2^*) = 240\,, \end{cases}$$

or equivalently

$$\begin{cases} 30\,p_1^* + 20\,p_2^* = 10 \\ \\ 40\,p_1^* + 20\,p_2^* = 10\,, \end{cases}$$

which implies $p_1^* = 0$. This contradicts the assumption that all initial and risk-neutral probabilities must be strictly positive. Thus, there is no risk-neutral probability P^* that is equivalent to the initial probability P.

An arbitrage strategy $\pi = (\beta,\, \gamma_1,\, \gamma_2)$ can be found from the system

$$\begin{cases} \beta + \gamma_1\, S_0^1 + \gamma_2\, S_0^2 = 0 \\ \\ \beta\,(1+r) + \gamma_1\, S_1^1(\omega_1) + \gamma_2\, S_1^2(\omega_1) \geq 0 \\ \\ \beta\,(1+r) + \gamma_1\, S_1^1(\omega_2) + \gamma_2\, S_1^2(\omega_2) \geq 0 \\ \\ \beta\,(1+r) + \gamma_1\, S_1^1(\omega_3) + \gamma_2\, S_1^2(\omega_3) \geq 0\,, \end{cases}$$

that reflects the fact that a strategy with zero initial capital can have non-negative values at time 1. We are looking for strategies such that at least one of the inequalities above is strict. Substituting given data we obtain

$$\begin{cases} \beta = -150\,\gamma_1 - 200\,\gamma_2 \\ \\ 1.2\,(-150\,\gamma_1 - 200\,\gamma_2) + 200\,\gamma_1 + 270\,\gamma_2 \geq 0 \\ \\ 1.2\,(-150\,\gamma_1 - 200\,\gamma_2) + 190\,\gamma_1 + 250\,\gamma_2 \geq 0 \\ \\ 1.2\,(-150\,\gamma_1 - 200\,\gamma_2) + 170\,\gamma_1 + 230\,\gamma_2 \geq 0\,, \end{cases}$$

or

$$\begin{cases} \beta = -150\,\gamma_1 - 200\,\gamma_2 \\ 20\,\gamma_1 + 30\,\gamma_2 \geq 0 \\ 10\,\gamma_1 + 10\,\gamma_2 \geq 0 \\ -10\,\gamma_1 - 10\,\gamma_2 \geq 0\,. \end{cases}$$

This system is satisfied by

$$\beta = -50\,a, \quad \gamma_1 = -a, \quad \gamma_2 = a,$$

with any $a \geq 0$. Hence we obtain a one-parameter family of arbitrage strategies:

$$\pi = (\beta,\, \gamma_1,\, \gamma_2) = (-50\,a,\ -a,\ a)\,, \qquad a > 0\,.$$

In other words, at time 0, an investor borrows a units of asset S^1, sells them at current price, also borrows $50\,a$ (\$) from a bank and invests all this capital in asset S^2. At time 2 the investor makes a strictly positive profit in the case of ω_1 and otherwise loses nothing. ∎

Problem B.2.3 *Consider a single-period* (B, S)*-market with* $B_0 = 1$, $S_0 = 100$, $r = 0$ *and*

$$S_1(\omega_1) = 80, \quad S_1(\omega_2) = 90, \quad S_1(\omega_3) = 180\,.$$

Is there a hedging strategy for a European call option with

$$f_1 = \big(S_1 - 100\big)^+\,?$$

SOLUTION We have the following possible values of claim f_1:

$$f_1(\omega_1) = 0, \quad f_1(\omega_2) = 0, \quad f_1(\omega_3) = 80\,.$$

Suppose that $\pi = (\beta,\, \gamma)$ is a hedging strategy, then

$$\begin{cases} \beta\,(1+r) + \gamma\,S_1(\omega_1) = f_1(\omega_1) \\ \beta\,(1+r) + \gamma\,S_1(\omega_2) = f_1(\omega_2) \\ \beta\,(1+r) + \gamma\,S_1(\omega_3) = f_1(\omega_3) \end{cases}$$

or

$$\begin{cases} \beta + 80\,\gamma = 0 \\ \beta + 90\,\gamma = 0 \\ \beta + 180\,\gamma = 80\,, \end{cases}$$

which is an inconsistent system. Hence, there is no strategy that can hedge this claim. ∎

Problem B.2.4 *Consider a single-period (B, S)-market with $B_0 = 1$, $S_0 = 200$ and*

$$S_1(\omega_1) = 150, \quad S_1(\omega_2) = 190, \quad S_1(\omega_3) = 250\,.$$

Find all values of r that admit the existence of a risk-neutral probability P^*.

SOLUTION We have the equality

$$E^*\left(\frac{S_1}{1+r}\right) = S_0\,,$$

which can be written in the form

$$p_1^*\,S_1(\omega_1) + p_2^*\,S_1(\omega_2) + p_3^*\,S_1(\omega_3) = S_0\,(1+r)\,.$$

Since $p_1^* + p_2^* + p_3^* = 1$, we have

$$150\,p_1^* + 190\,p_2^* + 250\,(1 - p_1^* - p_2^*) = 200\,(1+r)\,,$$

and therefore

$$r = \frac{5 - 10\,p_1^* - 6\,p_2^*}{20}\,.$$

We also have

$$p_1^* > 0\,, \quad p_2^* > 0\,, \quad p_1^* + p_2^* < 1\,,$$

which implies

$$-\frac{1}{4} < r < \frac{1}{4}\,.$$

∎

Problem B.2.5 *As in Section 2.6 consider the Black-Scholes model of a (B, S)-market, and compare the optimal investment strategy with the minimal hedge of an European call option with $f_T = \big(S_T - K\big)^+$.*

SOLUTION As we showed in Section 2.6, the proportion of risky capital in the optimal investment strategy is given by

$$\alpha^* = \frac{\mu - r}{\sigma^2}\,.$$

Observe that if $\mu = r$, then $\gamma_t^* = 0$ for all $t \leq T$.

On the other hand, for the minimal hedge of an European call option we have

$$\gamma_t = \Phi\left(\frac{\ln(S_t/K) + (T-t)(r + \sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$

In particular, if $S_0 > K$, then

$$\gamma_0 = \Phi\left(\frac{\ln(S_0/K) + T\,(r + \sigma^2/2)}{\sigma\sqrt{T}}\right) > \frac{1}{2} > \gamma_0^* = 0\,,$$

which means that these two strategies do not coincide. ∎

Problem B.2.6 *Consider the Black-Scholes model of a* (B, S)*-market with* $T = 215/365$, $S_0 = 100$, $\mu = r$. *Calculate premium for a pure endowment assurance with a guaranteed minimal payment* $K = 80$ *in the cases when* $r = 0.1$ *or* $r = 0.2$, *and* $\sigma = 0.1$ *or* $\sigma = 0.8$.

SOLUTION In Section 3.4 we derived formula (3.18) for calculating premiums:

$$U_x(T) = p_x(T)\,K\,e^{-rT} + p_x(T)\,C_T\,,$$

where $p_x(T)$ is the probability that an individual of age x survives to age $x + T$, and

$$C_T = \Big[S_0\,\Phi\big(d_+(0)\big) - K\,e^{-rT}\,\Phi\big(d_-(0)\big)\Big]$$

is the price of a European call option with the strike price K. Recall that all required values of C_T were computed in Worked Example 2.4, Section 2.6.

Now, let $x = 30$, for example. Then from a life table one can find the value of $p_{30}(1)$, say $p_{30}(1) \approx 0.9987$. Thus, for given values of r and σ we obtain the following values of $U_{30}(1)$:

$r \diagdown \sigma$	0.1	0.8
0.1	99.87	110.78
0.2	99.88	109.01

∎

Problem B.2.7 *Repeat the previous problem for the discrete Gaussian model of a* (B, S)*-market.*

SOLUTION Using results of Sections 2.4 and 3.4 we obtain

$$U_x(T) = p_x(T)\,(1 + r)^{-T}\Big[K + S_0\,(1 + r)^T\,\Phi\big(d_+(0)\big) - K\,\Phi\big(d_-(0)\big)\Big]\,.$$

For $p_{30}(1) \approx 0.9987$ and for given values of r and σ we obtain the following values of $U_{30}(1)$:

$r \diagdown \sigma$	0.1	0.8
0.1	99.87	110.92
0.2	99.87	109.32

$\square$

Problem B.2.8 *In the framework of the Black-Scholes model of a (B, S)-market consider an investment portfolio π with the initial capital x. Estimate the asymptotic profitability of π:*

$$\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta , \qquad \delta \in (0,1] .$$

SOLUTION First we note that by Lyapunov's inequality (see, for example, [41]), we have

$$E\big(X_T^\pi(x)\big)^\delta \le \big(EX_T^\pi(x)\big)^\delta .$$

Suppose that the initial probability P is a martingale probability and that strategy π is self-financing. Then

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta \\
&\qquad \le \delta \limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big) = \delta \limsup_{T\to\infty} \frac{1}{T} \ln \left[E\left(\frac{X_T^\pi(x)}{B_T}\right) B_T\right] \\
&\qquad = \delta \limsup_{T\to\infty} \frac{1}{T} \Big[\ln a + \ln B_T\Big] = \delta \limsup_{T\to\infty} \frac{r\,T}{T} = \delta\, r ,
\end{aligned}$$

where a is some constant.

On the other hand, if we invest only in non-risky asset B, we have

$$X_T^\pi(x) = x\, B_0\, e^{rT} ,$$

so

$$\ln E\big(X_T^\pi(x)\big)^\delta = \ln \Big[x^\delta\, B_0^\delta\, e^{\delta r T}\Big] = b + \delta\, r\, T$$

for some constant b.

Thus

$$\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta \ge \limsup_{T\to\infty} \frac{1}{T} \big[b + \delta\, r\, T\big] = \delta\, r ,$$

so therefore

$$\limsup_{T\to\infty} \frac{1}{T} \ln E\big(X_T^\pi(x)\big)^\delta = \delta\, r ,$$

and it does not depend on the initial capital x. $\square$

B.3 Problems for Chapter 3

Problem B.3.1 *Consider the binomial model of a* (B, S)*-market with* $S_0 = 100,\ B_0 = 1,\ r = 0.2$ *and*

$$\rho = \begin{cases} 0.5 & \textit{with probability } 0.4 \\ -0.3 & \textit{with probability } 0.6\,. \end{cases}$$

Calculate the premium for a pure endowment assurance with a guaranteed minimal payment $K = 100$ *in the cases when* $N = 1$ *and* $N = 2$.

SOLUTION First we compute risk-neutral probability

$$p^* = \frac{0.2 + 0.3}{0.5 + 0.3} = \frac{5}{8}\,.$$

As in Problem B.2.6, suppose that the age of life assured is $x = 30$, so that the probabilities of survival to age $30 + N$ are

$$p_{30}(1) \approx 0.9987 \qquad \text{and} \qquad p_{30}(2) \approx 0.997\,,$$

for $N = 1$ and $N = 2$, respectively.

Computing

$$E^*\left(\frac{\max\{S_1\,,\ K\}}{1+r}\right) = \frac{5}{8}\ \frac{100\,(1+0.5)}{1.2} + \frac{3}{8}\ \frac{100}{1.2} = 109.375$$

and

$$\begin{aligned} &E^*\left(\frac{\max\{S_2\,,\ K\}}{(1+r)^2}\right) \\ &\quad = \left(\frac{5}{8}\right)^2 \frac{100\,(1+0.5)^2}{1.44} + 2\,\frac{5}{8}\ \frac{3}{8}\ \frac{100\,(1+0.5)\,0.7}{1.44} + \left(\frac{3}{8}\right)^2 \frac{100}{1.44} \\ &\quad \approx 87.89\,, \end{aligned}$$

we calculate the required premiums:

$$U_{30}(1) = p_{30}(1)\, E^*\left(\frac{\max\{S_1\,,\ K\}}{1+r}\right) = 0.9987 \times 109.375 \approx 109,23$$

and

$$U_{30}(2) = p_{30}(2)\, E^*\left(\frac{\max\{S_2\,,\ K\}}{(1+r)^2}\right) = 0.997 \times 87.89 \approx 87.63\,.$$

∎

Problem B.3.2 *Suppose that an insurance company issues* 90 *independent identical policies, and suppose that the average amount of claims is* \$ 300 *with standard deviation* \$ 100. *Estimate the probability of total claim amount* S *to be greater than* \$ 29,000.

SOLUTION We compute expectation and variance of S:

$$E(S) = 300 \times 90 = 27,000 \qquad \text{and} \qquad V(S) = 100 \times 100 \times 90 = 900,000\,.$$

Since S is a sum of 90 independent identically distributed random variables, then normalized random variable

$$\frac{S - E(S)}{\sqrt{V(S)}}$$

is asymptotically normal. Thus the required probability is

$$\alpha \approx 1 - \Phi\left(\frac{29,000 - 27,000}{\sqrt{900,000}}\right) \approx 0.02\,.$$

∎

Problem B.3.3 *Suppose that an insurance company issues* 100 *independent identical policies. Find probabilistic characteristics of an individual claim* X *given the following statistical data:*

	amount of claim	**number of claims**
1	$0 - 400$	2
2	$400 - 800$	24
3	$800 - 1200$	32
4	$1200 - 1600$	21
5	$1600 - 2000$	10
6	$2000 - 2400$	6
7	$2400 - 2800$	3
8	$2800 - 3200$	1
9	$3200 - 3600$	1
10	> 3600	0

SOLUTION Let us assume that both claims from the first group (0-400) were 200, all 24 claims from the second group (400-800) were 600, etc. Then we compute

$$E(X) = 200\,\frac{2}{100} + 600\,\frac{24}{100} + \ldots + 3400\,\frac{1}{100} = 1216\,,$$

and

$$V(X) = \left[200^2 \, \frac{2}{100} + 600^2 \, \frac{24}{100} + \ldots + 3400^2 \, \frac{1}{100}\right] - 1216^2 = 362944 \, .$$

$\square$

Problem B.3.4 *Suppose that an insurance company issued* 1000 *independent identical policies, and as a result,* 120 *claims were received during the last 12 months. Find the probability of not receiving a claim from an individual policy holder during the next 9 months.*

SOLUTION

1. Suppose that the number of claims received from an individual policy holder during any 3 months is modelled by a Poisson distribution with parameter q. Then, assuming that numbers of claims that correspond to non-intersecting periods of time are independent, we have that the number of claims received from an individual policy holder during any 12 months can be represented by a Poisson distribution with parameter $4\,q$. For a portfolio of 1000 policies we have a Poisson distribution with parameter $4000\,q$. Thus

$$q = \frac{120}{4000} = 0.03 \, ,$$

   which implies that the number of claims received from an individual policy holder during any 9 months has the Poisson distribution with parameter 0.09. In particular, the probability of not receiving a claim from an individual policy holder during the next 9 months is

$$\alpha = \frac{(0.09)^0}{0!} \, e^{-0.09} \approx 0.91 \, .$$

2. Alternatively, we can use the Bernoulli distribution for modelling ξ_1, the number of claims received from an individual policy holder during any 3 months:

$$\xi_1 = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \, . \end{cases}$$

   Then the number of claims received from an individual policy holder during any 12 months has binomial distribution

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 \, ,$$

   and the total number of claims has binomial distribution

$$S = \xi_1 + \xi_2 + \xi_3 + \ldots + \xi_{4000} \, ,$$

where all ξ_i are independent and distributed identically to ξ_1. Thus

$$E(S) = 4000\,p = 120 \qquad \text{and hence} \qquad p = \frac{120}{4000} = 0.03\,.$$

We then have

$$P\big(\{\omega :\ \xi_1 + \xi_2 + \xi_3 = k\}\big) = \binom{3}{k}\,(0.03)^k\,(0.97)^{3-k}\,, \qquad k = 0, 1, 2, 3\,.$$

In particular, the probability of not receiving a claim from an individual policy holder during the next 9 months is

$$P\big(\{\omega :\ \xi_1 + \xi_2 + \xi_3 = 0\}\big) = (0.97)^3 \approx 0.91\,.$$

$\square$

Problem B.3.5 *Suppose that the following table describes the frequency of receiving claims by an insurance company during one year:*

number of claims	**number of policies**
0	3288
1	642
2	66
3	4

Find the probability of receiving only one claim from two independent policies during the next year.

SOLUTION

1. Suppose that the number of claims received during the year is modelled by a Poisson distribution with parameter q. Let N be the number of claims received from one policy. Then

$$E(N) = 642\,\frac{1}{4000} + 66\,\frac{2}{4000} + 4\,\frac{3}{4000} = 0.1965\,,$$

which implies that the number of claims received from two independent policies has Poisson distribution with parameter 0.393. Therefore, the probability of receiving only one claim from two independent policies during the next year is

$$\alpha = \frac{(0.393)^1}{1!}\,e^{-0.393} \approx 0.265\,.$$

2. Alternatively, suppose that the number of claims from one policy per year has binomial distribution:

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3=k\}\big)=\binom{3}{k}\,(0.03)^k\,(0.97)^{3-k}\,,\qquad k=0,1,2,3\,,$$

where ξ_i are independent Bernoulli random variables:

$$\xi_i=\begin{cases}1 & \text{with probability } p\\ 0 & \text{with probability } 1-p\,.\end{cases}$$

Then the average number of claims from two independent policies per year is 0.393.

On the other hand,

$$E\big(\{\omega:\ \xi_1+\xi_2+\ldots+\xi_6\}\big)=6\,p\,,$$

therefore $p=0.393/6$. Hence we obtain the following distribution

$$P\big(\{\omega:\ \xi_1+\xi_2+\ldots+\xi_6=k\}\big)=\binom{6}{k}\left(\frac{0.393}{6}\right)^k\left(1-\frac{0.393}{6}\right)^{6-k},$$

$k=0,1,2,\ldots,6$. In particular, the probability of receiving only one claim from two independent policies per year is

$$P\big(\{\omega:\ \xi_1+\xi_2+\ldots+\xi_6=1\}\big)=\binom{6}{1}\left(\frac{0.393}{6}\right)^1\left(1-\frac{0.393}{6}\right)^5\approx 0.28\,.$$

$\square$

Problem B.3.6 *Suppose that an insurance company issued* 4000 *independent identical policies. Find the expected number of policies that will result in* $0,1,2$ *and* 3 *claims per year if*

(1) *the number of claims from one policy per year has Poisson distribution with parameter* 0.1965*;*

(2) *the number of claims from one policy per year has a binomial distribution with the average* 0.1965.

SOLUTION

1. We compute the probabilities of receiving $0, 1, 2$ and 3 claims from one policy per year:

$$p_0 = \frac{(0.1965)^0}{0!}\ e^{-0.1965} \approx 0.82\,,$$

$$p_1 = \frac{(0.1965)^1}{1!}\ e^{-0.1965} \approx 0.16\,,$$

$$p_2 = \frac{(0.1965)^2}{2!}\ e^{-0.1965} \approx 0.016\,,$$

$$p_3 = \frac{(0.1965)^3}{3!}\ e^{-0.1965} \approx 0.001\,.$$

Multiplying these probabilities by 4000 we arrive at the following table.

number of claims	**number of policies**
0	3280
1	640
2	64
3	4

2. In the binomial case we have

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = k\}\big) = \binom{3}{k}\left(\frac{0.393}{6}\right)^k\left(1-\frac{0.393}{6}\right)^{3-k},\ k = 0, 1, 2, 3\,,$$

so that the probabilities of receiving $0, 1, 2$ and 3 claims from one policy per year are given by

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 0\}\big) = \left(1-\frac{0.393}{6}\right)^3 \approx 0.82\,,$$

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 1\}\big) = 3\left(\frac{0.393}{6}\right)\left(1-\frac{0.393}{6}\right)^2 \approx 0.17\,,$$

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 2\}\big) = 3\left(\frac{0.393}{6}\right)^2\left(1-\frac{0.393}{6}\right)^1 \approx 0.012\,,$$

$$P\big(\{\omega:\ \xi_1+\xi_2+\xi_3 = 3\}\big) = \left(\frac{0.393}{6}\right)^3 \approx 0.0003\,.$$

Multiplying these probabilities by 4000 we obtain

number of claims	**number of policies**
0	3280
1	680
2	48
3	1

∎

Problem B.3.7 *Suppose that an insurance company issued* 1000 *independent identical policies. Further, suppose that the probability of receiving a claim from one policy is* 0.5*, and that each policy allows no more than one claim to be made. Find the probability of the total number of claims to be between* 470 *and* 530*.*

SOLUTION The required probability is equal to

$$\sum_{k=470}^{530} \binom{1000}{k} \left(\frac{1}{2}\right)^{k} \left(\frac{1}{2}\right)^{1000-k}.$$

We can use De Moivre-Laplace Limit Theorem to compute an approximate value of this expression. Let S be the total number of received claims, then

$$E(S) = 1000/2 = 500\,, \qquad \text{and} \qquad \sqrt{V(S)} = \sqrt{1000\,\frac{1}{2}\,\frac{1}{2}} \approx 15.81\,.$$

So

$$P\big(\{\omega:\ 470 \leq S \leq 530\}\big) \approx \Phi\left(\frac{530-500+0.5}{15.81}\right)\Phi\left(\frac{470-500-0.5}{15.81}\right) \approx 0.95\,.$$

∎

Problem B.3.8 *An insurance company estimated that the probability of receiving a claim from one policy during one year is* 0.01 *and the average amount of a claim is* \$980*. Suppose that the company issues* 1000 *independent identical one-year policies. Find the probability of the total amount of claims to be more than* \$14, 850*.*

SOLUTION We use the individual risk model. Let S be the total amount of claims, and suppose that its normalized value has a standard normal distribution. Then

$$E(S) = 0.01 \times 980 \times 1000 = 9800\,,$$

and

$$\sqrt{V(S)} = \sqrt{0.01 \times 0.99 \times 980 \times 980 \times 1000} \approx 3083\,.$$

Therefore

$$P\big(\{\omega:\ S > 14,850\}\big) = P\left(\left\{\omega:\ \frac{S-9800}{3083} > 1.64\right\}\right) \approx 1 - \Phi(1.64) \approx 0.05\,.$$

∎

Problem B.3.9 *Suppose that the following table describes q, the frequency of receiving claims by an insurance company during one year:*

number of claims	number of policies
0	3280
1	640
2	64
3	4

Determine a 95% confidence interval for q.

SOLUTION Let N be the number of claims. Suppose that N has the Poisson distribution with parameter q. Then

$$E(N) = 4000\,q \qquad \text{and} \qquad V(N) = 4000\,q\,.$$

Hence, the random variable

$$\frac{N - 4000\,q}{20\,\sqrt{q}}$$

is asymptotically normal. Since

$$N = 642 + 66 \times 2 + 4 \times 3 = 786\,,$$

we obtain

$$-1.96 < \frac{786 - 4000\,q}{20\,\sqrt{q}} < 1.96\,,$$

which implies

$$0.19 < q < 0.2\,.$$

∎

Problem B.3.10 *Consider three policies with claims X_1, X_2 and X_3, respectively. Suppose*

$$P(\{\omega:\ X_1=0\})=0.5\,,\quad P(\{\omega:\ X_1=100\})=0.5\,,$$
$$P(\{\omega:\ X_2=0\})=0.8\,,\quad P(\{\omega:\ X_2=250\})=0.2\,,$$
$$P(\{\omega:\ X_3=0\})=0.4\,,\quad P(\{\omega:\ X_3=100\})=0.4\,,\quad P(\{\omega:\ X_3=50\})=0.2\,.$$

Find the most and the least risky policy.

SOLUTION Clearly, the average value of each X_i is 50, so we will compare their variances:

$$V(X_1)=0.5\times 100\times 100-2500=2500\,,$$
$$V(X_2)=0.2\times 250\times 250-2500=10,000\,,$$
$$V(X_3)=0.4\times 100\times 100+0.2\times 50\times 50-2500=2000\,.$$

Thus, we conclude that the second policy is the most risky, and the third is the least risky policy. ▯

Problem B.3.11 *Consider two independent policies with the following distributions of claims*

$$P(\{\omega:\ X_1=100\})=0.6\,,\quad P(\{\omega:\ X_1=200\})=0.4\,,$$
$$P(\{\omega:\ X_2=100\})=0.7\,,\quad P(\{\omega:\ X_2=200\})=0.3\,.$$

Suppose that the probability of receiving a claim from the first policy is 0.1 *and from the second one is* 0.2*. Find the distribution of claims for the portfolio formed by these two policies.*

SOLUTION The possible amounts of claims for this portfolio are 0, 100, 200, 300 and 400. So we have

$$p_0=P(\{\omega:\ X_1+X_2=0\})=0.9\times 0.8=0.72\,,$$
$$p_{100}=P(\{\omega:\ X_1+X_2=100\})=0.9\times 0.2\times 0.7+0.1\times 0.8\times 0.6=0.174\,,$$
$$p_{200}=P(\{\omega:\ X_1+X_2=200\})$$
$$=0.1\times 0.2\times 0.6\times 0.7+0.9\times 0.2\times 0.3+0.1\times 0.8\times 0.4=0.0944\,,$$
$$p_{300}=P(\{\omega:\ X_1+X_2=300\})$$
$$=0.1\times 0.2\times 0.6\times 0.3+0.1\times 0.2\times 0.7\times 0.4=0.0092\,,$$
$$p_{400}=P(\{\omega:\ X_1+X_2=400\})=0.1\times 0.2\times 0.4\times 0.3=0.0024\,.$$

▯

Problem B.3.12 *In the framework of the individual risk model consider a portfolio of* 50 *independent identical claims. Suppose that premiums are calculated according to the Expectation principle (see Section 3.1) with the security loading coefficient* 0.1. *Assuming that exactly one claim is received from each policy holder, find the probability of solvency in the following cases:*

(a) *each claim has an exponential distribution with average* 100*;*

(b) *each claim has a normal distribution with average* 100 *and variance* 400*;*

(c) *each claim has a uniform distribution in the interval* $[70,\ 130]$.

SOLUTION First, we observe that the total premium income in each of the cases is

$$\Pi = 100\,(1+0.1)\,50 = 5500\,.$$

The total claim amounts are all $100\times 50 = 5000$ and their standard deviations are

(a) $\sigma_1 = 100\,\sqrt{50} \approx 707.1$;

(b) $\sigma_2 = 20\,\sqrt{50} \approx 141.4$;

(c) $\sigma_3 = 60\,\sqrt{50/12} \approx 122.5$.

Now, since normalized total claim amounts are asymptotically normal, then the required probabilities are

(a) $\alpha_1 \approx 1 - \Phi\left(\frac{5500-5000}{707.1}\right) \approx 1 - \Phi\big(0.707\big) \approx 0.24$;

(b) $\alpha_2 \approx 1 - \Phi\left(\frac{5500-5000}{141.4}\right) \approx 1 - \Phi\big(3.54\big) \approx 0.0002$;

(c) $\alpha_3 \approx 1 - \Phi\left(\frac{5500-5000}{122.5}\right) \approx 1 - \Phi\big(4.08\big) \approx 0.00002$.

$\square$

Problem B.3.13 *In the framework of a binomial model consider two insurance companies. Suppose that the claims of the first company are distributed according to the Poisson law with average* 2, *and that the probability of receiving a claim equal to* 0.1. *For the second company we assume the same probability of receiving a claim and the following distribution of claims:* $P\big(\{\omega :\ X = 2\}\big) = 1$. *Given that both companies receive the premium of* 1 *and have zero initial capitals, find the corresponding probabilities of solvency:* $\phi(0,1)$, $\phi(0,2)$ *and* $\phi(0)$. *(See Section 3.2 for details.)*

SOLUTION Clearly, the first company will be solvent after one time step if it receives either a claim of 1 or no claims. The second company will be solvent only if it receives no claims during this period. Hence

$$\phi(0,1) = 0.1\ \frac{2}{1}\ e^{-2} + 0.9 \approx 0.94$$

for the first company, and

$$\hat{\phi}(0,1) = 0.9$$

for the second company.

Next, the first company will stay solvent after two time steps if any of the following events will occur

A: no claims on step one, no claims on step two;

B: no claims on step one, a claim of 1 on step two;

C: no claims on step one, a claim of 2 on step two;

D: a claim of 1 on step one, a claim of 1 on step two;

E: a claim of 1 on step one, no claims on step two.

Computing the probabilities of these events:

$$\begin{aligned}
P(A) &= 0.9 \times 0.9 = 0.81\,,\\
P(B) &= 0.9 \times 0.1 \times \frac{2}{1} \times e^{-2} \approx 0.037\,,\\
P(C) &= 0.9 \times 0.1 \times \frac{2^2}{2} \times e^{-2} \approx 0.061\,,\\
P(D) &= 0.1 \times \frac{8}{1} \times e^{-8} \times 0.1 \times \frac{8}{1} \times e^{-8} \approx 0.002\,,\\
P(E) &= P(B) \approx 0.037\,,
\end{aligned}$$

we conclude that the probability of solvency after two time steps is

$$\phi(0,2) = P(A) + P(B) + P(C) + P(D) + P(E) \approx 0.91\,.$$

For the second company we have events

F: no claims on step one, no claims on step two;

G: no claims on step one, a claim of 2 on step two;

with probabilities

$$P(F) = 0.81\,, \quad \text{and} \quad P(G) = 0.9 \times 0.1 \times 1 = 0.09\,.$$

Therefore the probability of its solvency after two time steps is

$$\hat{\phi}(0,2) = P(F) + P(G) = 0.9\,.$$

Finally, we compute

$$\phi(0) = \frac{1 - 0.1 \times 2}{1 - 0.1} \approx 0.89 \quad \text{and} \quad \hat{\phi}(0) = \frac{1 - 0.1 \times 2}{1 - 0.1} \approx 0.89\,.$$

□

Problem B.3.14 *Consider the Cramér-Lundberg model (see Section 3.2) with the premium income* $\Pi(t) = t$ *and with the claims flow represented by a Poisson process with intensity* 0.5. *Suppose that the average claim amount is* 1 *with variance* 5. *Estimate the Cramér-Lundberg coefficient (see Cramér-Lundberg inequality (3.2)).*

SOLUTION We have that the Cramér-Lundberg coefficient r satisfies the equation

$$0.5 + r = 0.5 \int_0^\infty e^{r\,x}\, dF(x)\,,$$

where F satisfies the following conditions:

$$\int_0^\infty x\, dF(x) = 1 \quad \text{and} \quad \int_0^\infty x^2\, dF(x) = 5 + 1 = 6\,.$$

Hence

$$\int_0^\infty e^{r\,x}\, dF(x) \geq \int_0^\infty \left(1 + r\,x + \frac{r^2\,x^2}{2}\right) dF(x) = 1 + r + 3\,r^2\,,$$

and therefore

$$0.5 + r \geq 0.5 + 0.5\,r + 1.5\,r^2$$

or $0 \geq 3\,r^2 - r$. Since r is positive, we conclude that $r \leq 1/3$. □

Problem B.3.15 *Consider the Cramér-Lundberg model (see Section 3.2) with the premium income* $\Pi(t) = t$ *and with the claims flow represented by a Poisson process with intensity* 0.5. *Suppose that claim amounts are equal to* 1 *with probability* 1. *Find the Cramér-Lundberg coefficient.*

SOLUTION We have that the Cramér-Lundberg coefficient r satisfies the equation

$$0.5 + r = 0.5\,e^r\,,$$

which we can write in the form

$$f(r) := 0.5\,e^r - r - 0.5\,.$$

It is not difficult to find an approximate solution to this equation (using Newton's method, say): $r \approx 1.26$. □

Problem B.3.16 *Consider* 50 *independent identical insurance policies. Suppose that the average claim received from a policy during a certain time period is* 100 *with variance* 200*. Also suppose that the equivalence principle is used for premiums calculations and that all premiums income is invested in a non-risky asset with the yield rate of* 0.025 *per specified period. Estimate the probability of solvency and the expected profit.*

SOLUTION The collected premiums are $50 \times 100 = 5000$. At the end of the specified period this accumulates to 5125. Then we compute the probability of solvency:

$$P\big(\{\omega : \; S \leq 5125\}\big) \approx \Phi\left(\frac{5125 - 5000}{10\,\sqrt{2}\,\sqrt{50}}\right) \approx 0.89\,,$$

where S is the aggregate claims payment. The expected profit is the difference between the premium income and the expected aggregate claims payment: $5125 - 5000 = 125$. Note that without the investment opportunity, the probability of solvency is 0.5, which is not acceptable. ∎

Problem B.3.17 *Repeat the previous problem assuming that there is an opportunity to invest in a risky asset with profitability*

$$\rho = \begin{cases} 0.06 & \textit{with probability } 0.5 \\ -0.005 & \textit{with probability } 0.5\,. \end{cases}$$

SOLUTION We have that the collected premiums amount of 5000 accumulates to

$$\begin{cases} 5000\,(1 + 0.06) = 5300 & \text{with probability } 0.5 \\ 5000\,(1 - 0.005) = 4975 & \text{with probability } 0.5\,, \end{cases}$$

therefore the expected profit is

$$0.5\,5300 + 0.5\,4975 - 5000 = 137.5 > 125$$

and the probability of solvency is

$$0.5\,P\big(\{\omega : \; S \leq 5300\}\big) + 0.5\,P\big(\{\omega : \; S \leq 4975\}\big) \approx 0.5\,\Phi\big(3\big) + 0.5\,\Phi\big(-0.25\big) \approx 0.7\,.$$

Note that the probability of solvency in this case is less than in the previous problem in spite of the fact that the expected profit is higher. This is one of the reasons that insurance companies may have restrictions on proportions of their capital that can be invested in risky assets. ∎

Problem B.3.18 *Consider an insurance company whose annual aggregate claims payment has an exponential distribution with the average of* $40,000$*. Suppose that* *this company operates in the framework of a* (B,S)*-market, where the profitability of a risky asset is*

$$\rho = \begin{cases} 0.1 & \textit{with probability}\ 0.5 \\ 0.3 & \textit{with probability}\ 0.5\,, \end{cases}$$

and the rate of interest is 0.2*. Suppose that* $S_0 = 10$*, and that all premium income is invested in a portfolio. Find an investment strategy* $\pi = (\beta, \gamma)$ *that minimizes the probability of bankruptcy.*

SOLUTION If Π is the collected premiums income, then at time 0 we have a portfolio with

$$(\Pi - 10\,\gamma) + 10\,\gamma = \Pi\,.$$

It time 1 the value of this portfolio is

$$\begin{cases} (\Pi - 10\,\gamma)\,1.2 + 13\,\gamma = 1.2\,\Pi + \gamma & \text{with probability } 0.5 \\ \\ (\Pi - 10\,\gamma)\,1.2 + 9\,\gamma = 1.2\,\Pi - 3\,\gamma & \text{with probability } 0.5\,. \end{cases}$$

Hence the probability of bankruptcy is

$$0.5\,e^{-\lambda(1.2\,\Pi+\gamma)} + 0.5\,e^{-\lambda(1.2\,\Pi-3\,\gamma)}\,.$$

Minimizing function

$$f(\gamma) := e^{-\lambda\,\gamma} + e^{3\,\lambda\,\gamma}\,,$$

we obtain

$$\gamma = \frac{\ln\big(1/3\big)}{4\,\lambda} \approx -10{,}986\,.$$

$\square$

Problem B.3.19 *Find the probability that a newborn individual survives to the age of* 30 *if the force of mortality is constant* $\mu_x \equiv \mu = 0.001$.

SOLUTION We have (see Section 3.4)

$$p_0(30) = e^{-\int_0^{30} 0.001\,dt} = e^{-0.03} \approx 0.97\,.$$

$\square$

Problem B.3.20 *Explain why function* $(1+x)^{-2}$ *cannot be used as the force of mortality.*

SOLUTION By contradiction, suppose

$$\mu_x = \frac{1}{(1+x)^2}\,.$$

Then

$$p_0(t) = e^{-\int_0^t (1+s)^{-2}\,ds} = e^{-\left(1-\frac{1}{1+t}\right)},$$

and therefore

$$\lim_{t\to\infty} p_0(t) = e^{-1} \approx 0.37\,,$$

i.e., a newborn individual survives to any age with the positive probability 0.37. ∎

Problem B.3.21 *Consider the survival function (see Section 3.4)*

$$s(x) = 1 - \frac{x}{100}\,, \quad 0 \le x \le 100\,.$$

Find the force of mortality and the probability that a newborn individual survives to the age of 20 *but dies before the age of* 40*.*

SOLUTION We have

$$p_x(t) = \frac{1-(x+t)/100}{1-x/100} = \frac{100-x-t}{100-x} = 1 - \frac{t}{100-x}\,.$$

Then

$$-\int_0^t \mu_{x+s}\,ds = \ln\left(1 - \frac{t}{100-x}\right),$$

therefore

$$-\mu_{x+t} = -\left(\frac{1}{100-x}\right)\Big/\left(1-\frac{t}{100-x}\right)$$

and

$$\mu_x = \frac{1}{100-x}\,.$$

Finally, the required probability is

$$1 - \frac{20}{100} - 1 + \frac{40}{100} = 0.2\,.$$

∎

Problem B.3.22 *Consider the Gompertz' model with* $\mu = [\![1.1]\!]^x$*. Find* $p_0(t)$*.*

SOLUTION We have

$$p_x(t) = e^{-\int_0^t [\![1.1]\!]^{x+s}\,ds} = e^{-[\![1.1]\!]^x \frac{[\![1.1]\!]^t-1}{\ln[\![1.1]\!]}},$$

hence

$$p_0(t) = e^{-\frac{[\![1.1]\!]^t-1}{\ln[\![1.1]\!]}} \approx e^{-10.492\left([\![1.1]\!]^t-1\right)}.$$

∎

Problem B.3.23 *Consider an insurance company with the initial capital of* 250. *Suppose that the company issues* 40 *independent identical insurance policies and that the average claim amount is* 50 *per policy with standard deviation* 40. *Premiums are calculated according to the Expectation principle with the security loading coefficient* 0.1. *The company has an option of entering a quota share reinsurance contract with retention function* $h(x) = x/2$ *(see Section 3.5). The reinsurance company calculates its premium according to the Expectation principle with the security loading coefficient* 0.15. *Estimate the expected profit and the probability of bankruptcy of the (primary) insurance company in the cases when it purchases the reinsurance contract and when it does not.*

SOLUTION If S is the aggregate claims payment, then

$$E(S) = 40 \times 50 = 2000 \quad \text{and} \quad \sqrt{V(S)} = 40\sqrt{40} \approx 252.98\,.$$

Since $2000\,(0.15 - 0.1) < 250$, then the purchase of the reinsurance contract reduces the probability of bankruptcy of the insurance company. Indeed, we have that the premiums amount is

$$\Pi = 40 \times 50 \times (1 + 0.1) = 2200\,.$$

Therefore, in the case when the reinsurance contract is not purchased, the expected profit is $\Pi - E(S) = 200$ and the probability of bankruptcy is

$$P\big(\{\omega:\ S > 250 + 2000\}\big) \approx 1 - \Phi\left(\frac{250 + 2200 - 2000}{252.98}\right) \approx 0.03764\,.$$

Otherwise, the premium

$$\Pi_1 = 40 \times 50 \times (1 + 0.15) \times 0.5 = 1150$$

is paid to the reinsurance company. Hence the expected profit is $\Pi - 0.5\,E(S) - \Pi_1 = 50$ and the probability of bankruptcy is

$$P\big(\{\omega:\ 0.5\,S > 500 + 1050\}\big) \approx 1 - \Phi\left(\frac{1100}{252.98}\right) \approx 7 \times 10^{-6}\,.$$

∎

Problem B.3.24 *Suppose that annual aggregate claims payments of an insurance company are uniformly distributed in* $[0, 2000]$. *Consider a stop-loss reinsurance contract with the retention level* 1600. *Compute expectations and variances of aggregate claims payments of both insurance and insurance companies.*

SOLUTION Let S and R be the aggregate claims payments of insurance and insurance companies, respectively. Then

$$E(S) = \int_0^{1600} \frac{x}{2000}\,dx + \int_{1600}^{2000} \frac{1600}{2000}\,dx = 960\,,$$

and therefore

$$E(R) = 1000 - E(S) = 40\,.$$

Further

$$V(S) = \int_0^{1600} \frac{x^2}{2000}\,dx + \int_{1600}^{2000} \frac{1600^2}{2000}\,dx \approx 1,194,667\,,$$

and

$$V(R) = \int_{1600}^{2000} \frac{(x-1600)^2}{2000}\,dx \approx 10,666.7\,,$$

so that

$$V(S) \approx 273,066.7 \quad \text{and} \quad V(R) \approx 9066.7\,.$$

Note that variance of the risk process without the reinsurance contract is $2000 \times 2000/12 \approx 333,333 > V(S) + V(R)$. $\square$

Appendix C

Bibliographic Remark

Chapter 1

We introduce the notions of a financial market, of basic and derivative securities; we discuss the probabilistic foundations of financial modelling and general ideas of financial risk management (see [7], [22], [29], [42]).

Quantitative analysis of risks related to contingent claims and maximization of utility functions is described in the framework of the simplest (Cox-Ross-Rubinstein) model of a market (see [11]).

As in the probability theory, where many general ideas and methods are often first explained in a discrete (Bernoulli) case (see [41]), in financial mathematics binomial markets are considered to be a good starting point in studying such fundamental notions as arbitrage, completeness, hedging and optimal investment (see [1], [14], [16], [18], [24], [27], [28], [30], [35], [37], [42]).

Chapter 2

This chapter begins with a comprehensive study of discrete markets. We give proofs of two Fundamental Theorems of financial mathematics, and discuss a methodology for pricing contingent claims in complete and incomplete markets, in markets with constraints and in markets with transaction costs (see [10], [16], [29], [30], [37], [42]).

Next, we study financial risks in the framework of the Black-Scholes model [6], [32]. The celebrated Black-Scholes formula is first derived in the discrete Gaussian setting. Then we demonstrate how the Black-Scholes model, formula and equation can be obtained from the binomial model and the Cox-Ross-Rubinstein formula by limit arguments. [27].

Methods of stochastic analysis are commonly used in the analysis of risks in the Black-Scholes model: for pricing contingent claims with or without taking into account dividends and transaction costs, for various types of hedging, for solving problems of optimal investment, including the case of insider information (see [3], [5], [14], [21], [24], [25], [31], [42]).

Further, we discuss continuous models of bonds markets and pricing of options on these bonds, including computational aspects (see [4], [35], [38], [42]).

One section is devoted to real options that we associate with long-term investment projects. The Bellmann principle is one of the main tools in studying real options (see [8], [13], [23], [26], [36]).

Technical analysis (see [34]) is a very common tool in investigating the qualitative structure of risks. We demonstrate how probabilistic methods can add some quantitative aspects to technical analysis (see [43]).

Handbooks [2], [20] are the standard sources of information on special functions and differential equations that are useful for solving the Bellmann equation, optimal stopping stopping time problem, etc.

Chapter 3

Complex binomial and Poisson models are used for modelling the capital of an insurance company. Actuarial criteria in premium calculations are presented (see [9], [31], [39]).

Probability of bankruptcy is used as a measure of solvency of an insurance company. Various estimates of probability of bankruptcy are given, including the celebrated Cramér-Lundberg estimate [12], [15], [39], [44], [45].

We discuss models that take into account an insurance company's financial investment strategies (see [17], [29], [30], [31]).

Another important type of insurance that is related to combination of risks in insurance and in finance is represented by equity-linked life insurance contracts and by reinsurance with the help of derivative securities. Analysis of such mixed risks requires a combination of modern methods of financial mathematics and actuarial mathematics (see [29], [30], [31], [33]).

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Glossary of Notation

$:=$	equality by definition
a.s.	almost surely
$\emptyset$	the empty set
$\square$	the end of proof
$\{x \in A \mid Z\}$	the subset of A whose elements possess property Z
$A \times B$	the cartesian product of sets A and B
I_A	the indicator function of set A
$f\|_A$	the restriction of function $f : X \to Y$ to the subset A of X
(a_k), $(a_k)_{k=1}^{\infty}$	the sequence $a_1, \ldots, a_k, \ldots$
$\mathbb{N}, \mathbb{Z}, \mathbb{R}$	the sets of natural numbers, integers and real numbers
$\mathbb{R}^N$	the set of all real N-tupels $(r_1, \ldots, r_n)$
2^A	the set of all subsets of A
$f(x) =_{x \to a} \mathcal{O}(g(x))$	$\|f(x)\| \leq \text{const}\ \|g(x)\|$ in a neighborhood of a
$o(x)$	a function satisfying $\|o(x)/x\| \to 0$ as $x \to 0$
$[\![x]\!]$	the integer part of $x \in \mathbb{R}$
$x \wedge y$	$:= \min\{x, y\}$
$C^n[0, \infty)$	the space of n-times continuously differentiable functions on $[0, \infty)$
$P(A)$	the probability of event A
$P(A\|B)$	the conditional probability of event A assuming event B
$P(A\|\mathcal{F})$	the conditional probability of A with respect to a σ-algebra $\mathcal{F}$
$\widetilde{P}$	a martingale probability
$\mathcal{M}(S_n/B_n)$	the collection of all martingale probabilities
$E(X)$	the expectation of a random variable X
$V(X)$	the variance of a random variable X
$\mathcal{N}(m, \sigma^2)$	a Gaussian (normal) random variable with mean value m and variance σ^2

$E(X\|Y)$	the conditional expectation of a random variable X with respect to a random variable Y
$E(X\|\mathcal{F})$	the conditional expectation of a random variable X with respect to a σ-algebra $\mathcal{F}$
$Cov(X,Y)$	the covariance of X and Y
$(X)^+$	$:= \max\{X, 0\}$
$\mathbb{F}$	a filtration (information flow)
$\langle M, M\rangle$	the quadratic variation of a martingale M
$H * m_n$	a discrete stochastic integral
$(\varphi * w)_t$	a stochastic integral
$\varepsilon_n(U)$	a stochastic exponential
$\mathcal{E}_t(Y)$	a stochastic exponential
SF	the collection of all self-financing portfolios
$\mathcal{M}_0^N$	the collection of all stopping times